

# SOBOLEV REGULARITY FOR CONVEX FUNCTIONALS ON BD

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ABSTRACT. We establish the first Sobolev regularity and uniqueness results for minimisers of autonomous, convex variational integrals of linear growth which depend on the symmetric rather than the full gradient. This extends the results available in the literature for the BV-setting to the case of functionals whose full gradients are a priori not known to exist as finite matrix-valued Radon measures.

## 1. INTRODUCTION

Let  $\Omega$  be an open and bounded Lipschitz domain of  $\mathbb{R}^n$ . In this work we study the regularity of minimisers of autonomous convex variational integrals of the form

$$(1.1) \quad \mathfrak{F}[u] := \int_{\Omega} f(\varepsilon(u)) \, dx, \quad u: \Omega \rightarrow \mathbb{R}^n,$$

subject to suitable Dirichlet boundary conditions. Here,  $\varepsilon(u) := \frac{1}{2}(Du + D^T u)$  denotes the *symmetric gradient* and  $f \in C^2(\mathbb{R}^{n \times n})$  is a variational integrand of *linear growth*. By this we understand that there exist two constants  $0 < c_0 \leq c_1 < \infty$  and  $c_2 \in \mathbb{R}$  such that

$$(1.2) \quad c_0|\xi| + c_2 \leq f(\xi) \leq c_1(1 + |\xi|) \quad \text{for all } \xi \in \mathbb{R}_{\text{sym}}^{n \times n}.$$

Due to the growth condition (1.2), the functional  $\mathfrak{F}$  is well-defined on Dirichlet classes  $W_{u_0}^{1,1}(\Omega; \mathbb{R}^n)$  for  $u_0 \in W^{1,1}(\Omega; \mathbb{R}^n)$ . However, as opposed to the  $p$ -growth regime for exponents  $p > 1$ , there is *no* Korn inequality on  $L^1$ . This is a well-known consequence of *Ornstein's Non-Inequality* [38] (see [15, 20, 34, 33] for more recent contributions) that in particular implies that there is *no* constant  $C > 0$  such that

$$(1.3) \quad \int_{\Omega} |Du| \, dx \leq C \int_{\Omega} |\varepsilon(u)| \, dx,$$

holds for all  $u \in C_c^1(\Omega; \mathbb{R}^n)$ . It follows that, neither  $\mathfrak{F}$  nor a suitable relaxation to the space  $BV(\Omega; \mathbb{R}^n)$  is coercive on these spaces. Note that, as a consequence of non-reflexivity of  $W^{1,1}$ , even if  $\mathfrak{F}$  had bounded minimising sequences in  $W^{1,1}$ , these could not be proven to be weakly precompact in  $W^{1,1}$ . Hence, in this case, the relaxation to the space of functions of bounded variation would be necessary.

Basically, the non-validity of estimate (1.3) is a consequence of unboundedness of singular integral operators on  $L^1$ . In this context, the appropriate substitutes are given by the spaces  $LD(\Omega)$  and  $BD(\Omega)$ . These consist of all  $u \in L^1(\Omega; \mathbb{R}^n)$  for which the distributional symmetric gradients  $\varepsilon(u)$  belong to  $L^1(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$  or can be represented by a  $\mathbb{R}_{\text{sym}}^{n \times n}$ -valued finite Radon measure  $Eu$  on  $\Omega$ , respectively; see Section 2.2. In particular, Ornstein's Non-Inequality then implies that in general  $BV(\Omega; \mathbb{R}^n) \subsetneq BD(\Omega)$  and that the *full* distributional gradients of BD-functions might not even exist as locally finite measures. Hence the chief question which we shall treat in the present work is to find conditions on the variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  such that minimisers *indeed do qualify as elements of*  $LD(\Omega)$ ,  $BV_{\text{loc}}(\Omega; \mathbb{R}^n)$  *or Sobolev spaces*  $W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^n)$ . At present, such results were only available for the full gradient

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The results of this paper appear as an extended and partially improved version of the results as presented in the first author's doctoral thesis [29]. As such, financial support through the EPSRC is gratefully acknowledged. Moreover, the authors would like to thank Gregory Seregin for discussions related to this work.

(i.e., BV-) case and, by Ornstein's Non-Inequality, do *not* apply to the situation considered here. Before we turn to a description of our results, we first introduce the concept of minima and the class of integrands we shall work with.

**1.1. Generalised Minimisers.** To define the concept of minimisers we shall work with, let  $\tilde{u}_0 \in L^1(\partial\Omega; \mathbb{R}^n)$  be a given Dirichlet datum. Since all  $W^{1,1}(\Omega; \mathbb{R}^n)$ ,  $LD(\Omega)$  and  $BD(\Omega)$  have trace space  $L^1(\partial\Omega; \mathbb{R}^n)$  (see Section 2.2 for more detail), we find  $u_0 \in LD(\Omega)$  such that  $\text{Tr}(u_0) = \tilde{u}_0$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial\Omega$ . We hence consider the variational principle

$$(1.4) \quad \text{to minimise } \mathfrak{F} \text{ within a Dirichlet class } \mathcal{D}_{u_0} := u_0 + LD_0(\Omega),$$

where  $LD_0(\Omega)$  is the closure of  $C_c^1(\Omega; \mathbb{R}^n)$  with respect to  $\|v\|_{LD} := \|v\|_{L^1} + \|\varepsilon(v)\|_{L^1}$ . By virtue of the growth condition (1.2), it is obvious that  $\inf \mathfrak{F}[\mathcal{D}_{u_0}] > -\infty$  and any minimizing sequence admits a subsequence converging to some  $u \in BD(\Omega)$  in the weak\*-sense. The latter follows from a standard compactness principle in  $BD$  that we recall for the reader's convenience in Section 2.2.

As  $\mathfrak{F}$  given by (1.1) is merely defined for elements of  $LD(\Omega)$ , it must be relaxed in order to be defined for the weak\*-limit  $u \in BD(\Omega)$ . Here we take advantage of convexity, thereby reducing to the classical theory of convex functions of measures due to GOFFMAN & SERRIN [32] and RESHETYAK [39]. To capture the behaviour of the integrand at infinity, we hereafter define the *recession function*  $f^\infty$  associated with  $f$  by

$$(1.5) \quad f^\infty(\xi) := \lim_{t \searrow 0} t f(\xi/t), \quad \xi \in \mathbb{R}_{\text{sym}}^{n \times n}.$$

Using convexity and linear growth of  $f$ , it is easy to show that  $f^\infty$  is well-defined and convex, too. Let  $u \in BD(\Omega)$  and consider the Lebesgue-Radon-Nikodým decomposition of  $E u = E^a u + E^s u$  into its absolutely continuous and singular parts with respect to  $\mathcal{L}^n$ . Then we put

$$(1.6) \quad \begin{aligned} \bar{\mathfrak{F}}_{u_0}[u] := & \int_{\Omega} f(E^a u) \, dx + \int_{\Omega} f^\infty \left( \frac{dE^s u}{d|E^s u|} \right) d|E^s u| \\ & + \int_{\partial\Omega} f^\infty((\text{Tr}(u_0) - \text{Tr}(u)) \odot \nu_{\partial\Omega}) \, d\mathcal{H}^{n-1} \end{aligned}$$

where as usual  $dE^s u / d|E^s u|$  denotes the Lebesgue density of  $E^s u$  with respect to its total variation measure  $|E^s u|$  and  $\nu_{\partial\Omega}$  is the outward unit normal to  $\partial\Omega$  (see Section 2.2 for the relevant notation). Here, the boundary integral term penalises deviations of  $u$  from the Dirichlet data  $u_0$ . Then  $\bar{\mathfrak{F}}_{u_0}$  coincides with the corresponding weak\*-lower semicontinuous envelope<sup>1</sup> of  $\mathfrak{F}$  over  $\mathcal{D}_{u_0}$ .

We then call  $u \in BD(\Omega)$  a *generalised minimiser* for  $\mathfrak{F}$  (with respect to  $u_0$ ) when

$$(1.7) \quad \bar{\mathfrak{F}}_{u_0}[u] \leq \bar{\mathfrak{F}}_{u_0}[v] \quad \text{for all } v \in BD(\Omega).$$

The set  $\text{GM}(\mathfrak{F}; u_0)$  consists of all generalised minimisers for  $\mathfrak{F}$  (with respect to  $u_0$ ). For brevity, we shall write  $\bar{\mathfrak{F}}$  instead of  $\bar{\mathfrak{F}}_{u_0}$ , tacitly assuming that  $u_0$  is fixed. Most crucially, we have  $\inf \mathfrak{F}[\mathcal{D}_{u_0}] = \min \bar{\mathfrak{F}}[BD(\Omega)]$ . Moreover, generalised minima can be conveniently characterised as those maps  $v \in BD(\Omega)$  for which there exists a minimising sequence  $(v_k) \subset \mathcal{D}_{u_0}$  that converges strongly to  $v$  in  $L^1(\Omega; \mathbb{R}^n)$ . It is difficult to give a precise reference to the literature for these results, which are entirely similar to those of the BV-case, so for the readers' convenience we have collected and briefly demonstrated them in the appendix of this paper, cf. Section 4.3.

<sup>1</sup>Here, because  $f$  is autonomous, convex with the lower bound of (1.2), it is irrelevant whether we choose the weak\*- or  $L^1$ -relaxation; indeed, they coincide in this case.

**1.2.  $\mu$ -elliptic Integrands.** Throughout the present work we shall further assume that  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  is a  $\mu$ -elliptic (and thus, in particular, convex) integrand. Reminiscent of the classical BERNSTEIN genre (cf. [48], [27, Ex. 3.2ff.] and the references therein), this notion of ellipticity had been rediscovered and studied in a series of papers by BILDHAUER, FUCHS & MINGIONE [9, 12, 25, 13] concerning minimisation problems of the type (1.1), where  $\varepsilon$  was replaced by the full gradients and which we recall here for completeness:

**Definition 1.1** ( $\mu$ -ellipticity). *Let  $1 < \mu < \infty$ . A  $C^2$ -integrand  $f: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}_{\geq 0}$  is called  $\mu$ -elliptic if and only if there exist  $0 < \lambda \leq \Lambda < \infty$  such that for all  $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{\text{sym}}^{n \times n}$  there holds*

$$(1.8) \quad \lambda \frac{|\mathbf{A}|^2}{(1 + |\mathbf{B}|^2)^{\frac{\mu}{2}}} \leq \langle f''(\mathbf{B})\mathbf{A}, \mathbf{A} \rangle \leq \Lambda \frac{|\mathbf{A}|^2}{(1 + |\mathbf{B}|^2)^{\frac{1}{2}}}.$$

As a direct consequence of the definition,  $\mu$ -elliptic integrands are automatically strictly convex. An important class of examples is provided by the one-parameter family of integrands  $\{\varphi_\mu\}_{\mu > 1}$ , given by

$$\varphi_\mu(\xi) := \int_0^{|\xi|} \int_0^s (1 + t^2)^{-\frac{\mu}{2}} dt ds, \quad \xi \in \mathbb{R}^{n \times n}.$$

Then, as shown in [9, Ex. 3.9 and 4.17], each  $\varphi_\mu$  is  $\mu$ -elliptic. Moreover, for  $\mu = 3$  we recover the usual area integrand  $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$ . Let us further note that in the definition of  $\mu$ -ellipticity we exclude the case  $\mu = 1$ . Indeed, it is easy to show that 1-elliptic  $C^2$ -integrands must be of  $L \log L$ -growth. Such integrands are easier to deal with than integrands of linear growth, since by mapping properties of singular integrals of convolution type, minimisers of (1.1) with 1-elliptic  $f$  possess full distributional gradients in  $L^1(\Omega; \mathbb{R}^{n \times n})$ . In consequence, when studying regularity properties of such minima, one may directly test with the full difference quotients and hence no modification of the common difference quotient method is required. As a characteristic feature of integrands satisfying (1.8), we note that, since  $\mu > 1$ , the growth behaviour from above and below differs *on the level of second derivatives*, a feature that is not present for functionals of controlled  $p$ -growth but rather appears in the theory of functionals with natural  $p$ -growth or  $(p, q)$ -growth, cf. [35, 36, 11, 23, 16, 18, 17].

**1.3. Background and Main Results.** Before we embark on the description of our main results, we first summarise the results available so far.

**1.3.1. Contextualisation.** To contextualise our results, let us briefly recall the results known for the full gradient case, that is, where the symmetric gradient in (1.4) is replaced by the full gradient; we are hereby lead to the Dirichlet problem in BV. Employing a vanishing viscosity approach in the spirit of SEREGIN [41, 42, 43, 44], BILDHAUER [12, 9] established the first  $W^{1,1}$ -regularity result for generalised minima for  $\mu \leq 3$ . Up to date, for *autonomous full gradient functionals* there are no Sobolev regularity results for the Dirichlet problem available beyond  $\mu = 3$ . When the integrand is allowed a smooth  $x$ -dependence the example [27, Example 3.2] shows that the minimizers are in general no better than BV.

However, even though BILDHAUER's approach leads to  $W_{\text{loc}}^{1, L \log L}$ -regularity of *one* particular generalised minimiser (namely, the limit of a special vanishing viscosity sequence) and the integrand is strictly convex, it does not rule out the possible presence of other, more irregular generalised minima. We recall that the recession function  $f^\infty$  is positively 1-homogeneous, and so degenerately convex regardless of strict convexity of  $f$ . This is an obvious source of non-uniqueness, and as long the presence of the singular part of the gradient measures of *all* generalised minima cannot be excluded, there might in fact exist other generalised minima that do not share the  $W_{\text{loc}}^{1, L \log L}$ -regularity. The uniqueness of generalised minima for the Dirichlet problem on BV has been addressed by BECK & SCHMIDT [8]. Here, the authors combine BILDHAUER's approach with the Ekeland variational principle to deduce that *all* generalised minima of  $\mu = 3$ -elliptic variational integrals share the aforementioned regularity.

However, a common difficulty in deriving a higher differentiability result for functionals of the type (1.1) under the linear growth assumption on  $f$  is that, by Ornstein's Non-Inequality, the full distributional gradients of BD-functions do not need to exist as Radon measures of finite total mass. Hence, we shall consider fractional estimates instead and utilise the fact that – as BV and BD embed into the same fractional Sobolev spaces – BD-maps behave similarly as BV on the fractional level.

**1.3.2. Main Result.** We come to the main result of the paper. In effect, it appears as the generalisation of [8, Sec. 5] to problems of the form (1.4).

**Theorem 1.2.** *Let  $n \geq 2$ . Suppose that  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  is a  $\mu$ -elliptic variational integrand of linear growth, i.e., satisfies (1.2) and (1.8) with  $1 < \mu < \frac{n+1}{n}$ . Then all generalised minimisers belong to  $W_{\text{loc}}^{1,1}(\Omega)$ . More precisely, we have for some  $p = p(\mu, n) > 1$*

$$(1.9) \quad \text{GM}(\mathfrak{F}; u_0) \subset W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^n) \cap \text{LD}(\Omega).$$

This theorem will be established in Section 3.3. As for the BV-case discussed above, a chief difficulty stems from the fact that the symmetric gradients of generalised minimisers are not a priori known to be absolutely continuous with respect to  $\mathcal{L}^n$ . Consequently, since we do not have uniqueness of generalised minima, a stabilisation procedure relying on the vanishing viscosity approach must be suitably modified. In doing so, we follow essentially the lines of BECK & SCHMIDT [8]. Here, starting from a given generalised minimiser, we construct a specific minimising sequence that weakly\*-converges to the generalised minimiser so that each of its members almost minimises an appropriately stabilised functional. Since this sequence belongs to LD, we are further in position to avoid manipulations on difference quotients of measures. In constructing the aforementioned specific minimising sequence, we make use of the Ekeland variational principle and employ it in the dual space  $(W_0^{1,\infty})^*$  to obtain perturbations that are weak enough to be dealt with using the available a priori estimates. Higher differentiability estimates can also be obtained, cf. Corollary 3.8.

Theorem 1.2 comes along with higher fractional differentiability, and so we are in position to derive Hausdorff dimension bounds for the singular set of generalised minima, cf. Corollary 3.9.

As is well-known from the classical minimal surface example, another source of non-uniqueness stems from the non-attainment of boundary values; see the classical examples due to SANTI [47] or FINN [24]. In this respect, the main part of the paper is concluded by investigating the impact of regularity on the uniqueness of generalised minima, see Section 3.4 and Theorem 3.10 therein.

**1.4. Organisation of the Paper.** In Section 2 we fix notation, collect the requisite background facts regarding function spaces and record some auxiliary estimates. In Section 3, we give the proofs of the aforementioned main results regarding the regularity and uniqueness of generalised minima. Finally, the appendix in Section 4 discusses extensions of the main results and contains auxiliary material used in the main part. In particular, it covers the relaxation of the Dirichlet problem to BD and the existence of generalised minima which we tacitly assumed throughout.

**Acknowledgment.** The first author is grateful to the Hausdorff Centre in Mathematics, Bonn, for financial support. The authors moreover express their thanks to the anonymous referee, whose careful reading and suggestions led to a substantial improvement of the paper.

## 2. SETUP

**2.1. General Notation.** Unless stated otherwise, we assume  $\Omega$  to be an open and bounded Lipschitz domain in  $\mathbb{R}^n$ . Given  $x_0 \in \mathbb{R}^n$  and  $r > 0$ , we denote  $B(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$  the open ball with radius  $r$  centered at  $x_0$  and  $\langle \cdot, \cdot \rangle$  denotes the euclidean inner product on finite dimensional real vector spaces. The  $n$ -dimensional Lebesgue measure is denoted  $\mathcal{L}^n$  and the  $(n - 1)$ -dimensional Hausdorff measure is denoted  $\mathcal{H}^{n-1}$ . Given two positive, real

valued functions  $f, g$ , we indicate by  $f \lesssim g$  that  $f \leq Cg$  with a constant  $C > 0$ . If  $U \subset \mathbb{R}^n$  is measurable with  $0 < \mathcal{L}^n(U) < \infty$  and  $f \in L^1(U; \mathbb{R}^N)$ , we put as usual

$$(f)_U := \int_U f \, dx := \frac{1}{\mathcal{L}^n(U)} \int_U f \, dx.$$

For a given measurable map  $f: \Omega \rightarrow \mathbb{R}^m$ , a unit vector  $e_s$ ,  $s = 1, \dots, n$ , and a stepwidth  $h \neq 0$ , we define the *finite difference*  $\tau_{s,h}f(x)$  by

$$\tau_{s,h}f(x) := f(x + he_s) - f(x)$$

for all  $x \in \Omega$  with  $\text{dist}(x, \partial\Omega) > |h| > 0$ . Moreover, for such  $x$  we put

$$\Delta_{s,h}f(x) := \frac{\tau_{s,h}f(x)}{h}.$$

Finally, we denote by  $\mathbb{R}_{\text{sym}}^{n \times n}$  the symmetric  $n \times n$ -matrices with real entries and, given  $u, v \in \mathbb{R}^n$ , we denote their dyadic product by  $u \otimes v := uv^\top$  and their symmetric dyadic product by  $u \odot v := \frac{1}{2}(u \otimes v + v \otimes u)$ .

**2.2. Maps of Bounded Deformation.** Here we recall the space of maps of bounded deformation as introduced in [19, 50]. For more detailed background information, the reader is referred to [2, 49, 7, 26]. Let  $\Omega \subset \mathbb{R}^n$  be open. A measurable map  $u: \Omega \rightarrow \mathbb{R}^n$  belongs to  $\text{BD}(\Omega)$  if and only if  $u \in L^1(\Omega; \mathbb{R}^n)$  and its *total deformation*

$$(2.1) \quad |Eu|(\Omega) := \sup \left\{ \int_{\Omega} \langle u, \text{div}(\varphi) \rangle \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}), \|\varphi\|_{L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \leq 1 \right\}$$

is finite, where the divergence has to be understood row-wise (note that we write  $Eu$  for the distributional symmetric gradient when this is a measure and reserve  $\varepsilon(u)$  for weak symmetric gradients exclusively when  $Eu$  is absolutely continuous). The norm on  $\text{BD}(\Omega)$  is given by  $\|u\|_{\text{BD}(\Omega)} := \|u\|_{L^1(\Omega; \mathbb{R}^n)} + |Eu|(\Omega)$ , and endowed with this norm,  $\text{BD}(\Omega)$  is a Banach space. Since the norm topology is too strong for most applications, it is useful to consider the following convergences instead: We say that a sequence  $(u_k) \subset \text{BD}(\Omega)$  converges to  $u \in \text{BD}(\Omega)$  in the *weak\*-sense* provided  $u_k \rightarrow u$  strongly in  $L^1(\Omega; \mathbb{R}^n)$  and  $Eu_k \xrightarrow{*} Eu$  in the sense of  $\mathbb{R}^{n \times n}$ -valued measures as  $k \rightarrow \infty$ . Moreover, if  $(u_k)$  converges to  $u$  in the weak\*-sense and  $|Eu_k|(\Omega) \rightarrow |Eu|(\Omega)$  as  $k \rightarrow \infty$ , then we say that  $(u_k)$  converges (BD-) *strictly* to  $u$  as  $k \rightarrow \infty$ . Lastly, we say that  $(u_k)$  converges to  $u$  in the (BD-) *area-strict sense* provided  $u_k \rightarrow u$  strictly and

$$\sqrt{1 + |Eu_k|^2}(\Omega) \rightarrow \sqrt{1 + |Eu|^2}(\Omega) \quad \text{as } k \rightarrow \infty.$$

The concept of applying convex functions (so, e.g., the area-type integrand  $\sqrt{1 + |\cdot|^2}$ ) to a measure as done here will be carefully explained in Section 4.3.1 below.

Resembling the fact that  $\text{BV}(\Omega; \mathbb{R}^N)$  arises as the weak\*-closure of  $W^{1,1}(\Omega; \mathbb{R}^N)$ ,  $\text{BD}(\Omega; \mathbb{R}^N)$  is the weak\*-closure of the space

$$(2.2) \quad \text{LD}(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^n) : \varepsilon(u) \in L^1(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})\},$$

where  $\varepsilon(u)$  is the distributional symmetric gradient, and the norm on  $\text{LD}(\Omega)$  is given by  $\|u\|_{\text{LD}(\Omega)} := \|u\|_{L^1(\Omega; \mathbb{R}^n)} + \|\varepsilon(u)\|_{L^1(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})}$ . We further define  $\text{LD}_0(\Omega)$  to be the closure of  $C_c^1(\Omega; \mathbb{R}^n)$  with respect to  $\|\cdot\|_{\text{LD}(\Omega)}$ . The claimed property that  $\text{BD}(\Omega)$  is the weak\*-closure of  $\text{LD}(\Omega)$  follows from the fact that  $(\text{LD} \cap C^\infty)(\Omega)$  is dense in  $\text{BD}(\Omega)$  with respect to weak\*- and strict convergence, see [5]. If  $\Omega$  is a bounded Lipschitz subset of  $\mathbb{R}^n$ , then there exists a

- surjective trace operator  $\text{Tr}: \text{LD}(\Omega) \rightarrow L^1(\partial\Omega; \mathbb{R}^n)$  which is continuous with respect to the LD-norm;
- surjective trace operator  $\text{Tr}: \text{BD}(\Omega) \rightarrow L^1(\partial\Omega; \mathbb{R}^n)$  which is continuous with respect to strict (but not weak\*-) convergence.

Given  $u \in \text{BD}(\Omega)$  and splitting the symmetric gradient measure  $\text{Eu}$  into its absolutely continuous and singular parts with respect to Lebesgue measure,  $\text{Eu} = \text{E}^a u + \text{E}^s u$ , the above trace theorem particularly implies that the trivial extension  $\bar{u}$  of  $u$  to  $\mathbb{R}^n$  satisfies

$$\text{E}\bar{u} = \text{E}^a \bar{u} + \text{E}^s \bar{u} =: \mathcal{E}\bar{u} \mathcal{L}^n + \text{E}^s \bar{u} = \mathcal{E}u \mathcal{L}^n \llcorner \Omega + \text{E}^s u \llcorner \Omega - (\text{Tr}(u) \odot \nu_{\partial\Omega}) \mathcal{H}^{n-1} \llcorner \partial\Omega,$$

where  $\nu_{\partial\Omega}$  is the outward unit normal to the boundary of the Lipschitz set  $\Omega \subset \mathbb{R}^n$  and  $\mathcal{E}u$  is the symmetric part of the approximate gradient of  $u$ ; see [5, 2, 49, 7] for more information.

By Ornstein's Non-Inequality, we have as already mentioned,  $\text{LD}(\Omega) \not\hookrightarrow W^{1,1}(\Omega; \mathbb{R}^n)$  and  $\text{BD}(\Omega) \not\hookrightarrow \text{BV}(\Omega)$ . However, some additional information is available when passing to fractional spaces. We recall that, given  $1 \leq p < \infty$  and  $0 < \alpha < 1$ , a measurable map  $u: \Omega \rightarrow \mathbb{R}^N$  belongs to the fractional Sobolev space  $W^{\alpha,p}(\Omega; \mathbb{R}^N)$  if and only if  $u \in L^p(\Omega; \mathbb{R}^N)$  and the *Gagliardo seminorm* of  $u$  is finite, i.e.,

$$[u]_{W^{\alpha,p}(\Omega; \mathbb{R}^N)}^p := \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\alpha p}} dx dy < \infty.$$

The full norm on  $W^{\alpha,p}$  is then given by  $\|u\|_{W^{\alpha,p}} := \|u\|_{L^p} + [u]_{W^{\alpha,p}}$ . We will also need the Besov spaces to be recalled next. Let  $0 < \alpha < 1$  and  $1 \leq p, q \leq \infty$ . In this situation, we define for  $u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$  as above

$$[u]_{B_{p,q}^\alpha(\mathbb{R}^n)} := \sum_{s=1}^n \left( \int_0^\infty \left( \frac{\|\tau_{s,h} u\|_{L^p(\mathbb{R}^n)}}{h^\alpha} \right)^q \frac{dh}{h} \right)^{\frac{1}{q}} \quad \text{if } 1 \leq p, q < \infty,$$

$$[u]_{B_{p,\infty}^\alpha(\mathbb{R}^n)} := \sup_{h>0} \max_{s \in \{1, \dots, n\}} \frac{\|\tau_{s,h} u\|_{L^p(\mathbb{R}^n)}}{h^\alpha} \quad \text{if } 1 \leq p < \infty, q = \infty.$$

These quantities are referred to as  $((\alpha, p, q)$ -Besov seminorms. The full Besov norms then are given by  $\|u\|_{B_{p,q}^\alpha} := \|u\|_{L^p} + [u]_{B_{p,q}^\alpha}$ , and we say that  $u$  belongs to the Besov space  $B_{p,q}^\alpha(\mathbb{R}^n; \mathbb{R}^n)$  if and only if  $\|u\|_{B_{p,q}^\alpha} < \infty$ . For the purposes of this paper, if  $[u]_{B_{p,q}^\alpha} < \infty$ , then we say that  $u \in \dot{B}_{p,q}^\alpha$ , the corresponding homogeneous Besov space. Note that  $B_{p,p}^\alpha \simeq W^{\alpha,p}$ , and we shall sometimes call  $\mathcal{N}^{\alpha,p} := B_{p,\infty}^\alpha$  the  $(\alpha, p)$ -Nikolskiĭ space. The localised versions of these spaces are defined in the obvious manner. As a consequence of [52, Thm. 2.7.1], we obtain

**Lemma 2.1.** *Let  $0 < \alpha < 1$ . Then we have  $(B_{1,\infty}^\alpha)_{\text{loc}}(\mathbb{R}^n) \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^n)$  for any  $1 \leq q < \frac{n}{n-\alpha}$ .*

We refer the reader to [6, Chpt. 4] and [52, Chpts. 1 and 2] for more background information on these spaces. Invoking the fractional Sobolev spaces, we have that both  $\text{LD}_{\text{loc}}(\Omega)$  and  $\text{BD}_{\text{loc}}(\Omega)$  continuously embed into  $W^{\alpha,1}_{\text{loc}}(\Omega; \mathbb{R}^n)$  for any  $0 < \alpha < 1$ ; see Proposition 2.2 below. These statements in turn rest on the *Smith representation formula* [45]: Given  $u = (u^1, \dots, u^n) \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ , we may write

$$(2.3) \quad u^k = \frac{2}{n\omega_n} \sum_{i,j=1}^n \frac{\partial^2 u^k}{\partial x_i \partial x_j} * K_{ij}, \quad \text{where } K_{ij}(x) = \frac{x_i x_j}{|x|^n} \text{ for } x \in \mathbb{R}^n \setminus \{0\}.$$

Setting  $\varepsilon(u) := (\varepsilon(u)_{jk})_{jk}$ , we observe that

$$\frac{\partial^2 u^k}{\partial x_i \partial x_j} = \frac{\partial \varepsilon(u)_{jk}}{\partial x_i} - \frac{\partial \varepsilon(u)_{ij}}{\partial x_k} + \frac{\partial \varepsilon(u)_{ki}}{\partial x_j}$$

and hence, inserting this relation into (2.3), we obtain after an integration by parts

$$(2.4) \quad u^k = \frac{2}{n\omega_n} \sum_{i,j=1}^n (\varepsilon(u)_{jk} * \frac{\partial K_{ij}}{\partial x_i} - \varepsilon(u)_{ij} * \frac{\partial K_{ij}}{\partial x_k} + \varepsilon(u)_{ki} * \frac{\partial K_{ij}}{\partial x_j})$$

for all  $k = 1, \dots, n$ . This formula can be established by means of Fourier analysis and, upon differentiating, indicates the failure of Korn's inequality in  $L^1$  by the usual mapping properties of singular integrals. The following result shall be proved for the reader's convenience in the appendix, Section 4:



**Proposition 2.2** ( $\text{BD} \hookrightarrow W_{\text{loc}}^{\alpha,1}$ ). *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and  $K$  a relatively compact subset of  $\Omega$ . Then for every  $0 < \alpha < 1$  there exists a constant  $C = C(K, \alpha) > 0$  such that*

$$(2.5) \quad \|u\|_{W^{\alpha,1}(K;\mathbb{R}^n)} \leq C \|u\|_{\text{BD}(\Omega)}$$

*holds for all  $u \in \text{BD}(\Omega)$ .*

Finally, we recall that for any connected and open set  $\Omega \subset \mathbb{R}^n$ , the nullspace of  $\epsilon$  is given by the space of *rigid deformations*

$$\mathcal{R}(\Omega) := \{r: \Omega \ni x \mapsto Ax + b: A \in \mathbb{R}^{n \times n}, A^T = -A, b \in \mathbb{R}^n\}$$

As these are first order polynomials, each  $r \in \mathcal{R}(\Omega)$  clearly arises as the restriction of some  $\tilde{r} \in \mathcal{R}(\mathbb{R}^n)$ , and so we use  $\mathcal{R}(\Omega)$  and  $\mathcal{R}(\mathbb{R}^n)$  interchangeably for connected domains  $\Omega$ .

**2.3. On the Space  $(W_0^{1,\infty})^*$ .** In order to work with suitably weak perturbations when applying the Ekeland variational principle, we record some properties of the dual space  $(W_0^{1,\infty})^*$  which seems natural for our purposes in the main body of the paper. A distribution  $T \in \mathcal{D}'(\Omega; \mathbb{R}^n)$  belongs to  $(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$  if and only if the norm

$$(2.6) \quad \|T\|_{(W_0^{1,\infty})^*} := \sup \{ \langle T, \varphi \rangle : \varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^n) \text{ and } \|\varphi\|_{W_0^{1,\infty}(\Omega; \mathbb{R}^n)} \leq 1 \} < \infty,$$

whenever this expression makes sense; here, we work with the gradient norm

$$\|\varphi\|_{W_0^{1,\infty}(\Omega; \mathbb{R}^n)} := \|\nabla \varphi\|_{L^\infty(\Omega; \mathbb{R}^{n \times n})} \quad \text{for } \varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^n).$$

As a dual space,  $(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$  is complete.

**Lemma 2.3.** *Let  $\Omega \subset \mathbb{R}^n$  be open and  $K \subset \Omega$  a relatively compact Lipschitz subset of  $\Omega$ . Let  $v \in L^1(\Omega; \mathbb{R}^n)$ . Then for any  $s = 1, \dots, n$  and any  $0 < |h| < \text{dist}(K, \partial\Omega)$  we have*

$$\|\Delta_{s,h} v\|_{(W^{1,\infty}(K; \mathbb{R}^n))^*} \leq \|v\|_{L^1(\Omega; \mathbb{R}^n)}.$$

*Proof.* Let  $\varphi \in W_0^{1,\infty}(K; \mathbb{R}^n)$  be arbitrary with  $\|\varphi\|_{W_0^{1,\infty}(K; \mathbb{R}^n)} \leq 1$ . Using integration by parts for difference quotients, we estimate

$$|\langle \Delta_{s,h} v, \varphi \rangle| = |\langle v, \Delta_{s,-h} \varphi \rangle| \leq \|v\|_{L^1(\Omega; \mathbb{R}^n)} \|\Delta_{s,-h} \varphi\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq \|v\|_{L^1(\Omega; \mathbb{R}^n)} \|\nabla \varphi\|_{L^\infty(\Omega; \mathbb{R}^n)}$$

and hence passing to the supremum over all admissible test maps  $\varphi$  yields the claim.  $\square$

**2.4. A  $V$ -function estimate.** We conclude this preliminary section by giving a version of an estimate for an auxiliary map in the spirit of ACERBI & FUSCO [1] that shall turn out convenient for our purposes; a proof is given in the appendix, Section 4.2. For  $\alpha > 0$  and  $M \in \mathbb{N}$ , we hereafter introduce the auxiliary map  $V_\alpha: \mathbb{R}^M \rightarrow \mathbb{R}^M$  by

$$(2.7) \quad V_\alpha(\xi) := (1 + |\xi|^2)^{\frac{1-\alpha}{2}} \xi, \quad \xi \in \mathbb{R}^M.$$

**Lemma 2.4.** *Let  $1 < \alpha < 2$  and define  $V_\alpha$  by (2.7). Then we have for any measurable map  $v: \mathbb{R}^n \rightarrow \mathbb{R}^M$ ,  $h \in \mathbb{R}$  and  $e_s \in \mathbb{R}^n$  with  $|e_s| = 1$  the estimate*

$$|\tau_{s,h} V_\alpha(v(x))| \sim (1 + |v(x + he_s)|^2 + |v(x)|^2)^{\frac{1-\alpha}{2}} |\tau_{s,h} v(x)|.$$

*Moreover, there exists a constant  $c > 0$  such that for all  $\xi \in \mathbb{R}^M$  there holds*

$$\min\{|\xi|, |\xi|^{2-\alpha}\} \leq c V_\alpha(\xi).$$

*Lastly, if  $\Omega$  is an open and bounded set and  $u: \Omega \rightarrow \mathbb{R}^M$  satisfies  $V_\alpha(u) \in L^p(\Omega; \mathbb{R}^M)$ , then we have*

$$(2.8) \quad \int_\Omega |u|^{(2-\alpha)p} dx \leq \mathcal{L}^n(\Omega) + c(p) \int_\Omega |V_\alpha(u)|^p dx,$$

*where  $c(p) > 0$  is a constant depending only on  $p$ .*

## 3. VISCOSITY APPROXIMATIONS

**3.1. The Ekeland–type Approximation.** To avoid manipulations on measures when working with the Euler–Lagrange equation satisfied by the minimiser  $u \in \text{BD}(\Omega)$ , we shall consider approximate problems which allow us to work with LD–maps first. More precisely, starting from an arbitrary minimising sequence, we shall employ Ekeland’s variational principle to construct another minimising sequence which is close to the original sequence, however, features convenient optimality properties. For the reader’s convenience, we therefore first recall

**Lemma 3.1** (Ekeland Variational Principle, [22, Thm. 6.1]). *Let  $(X, d)$  be a complete metric space and  $J: X \rightarrow \mathbb{R} \cup \{\infty\}$ ,  $J \not\equiv \infty$ , a lower semicontinuous functional which is bounded from below. Fix  $\varepsilon > 0$ . If  $u \in X$  is such that*

$$J(u) \leq \inf_X J + \varepsilon,$$

*then there exists  $v \in X$  with the following properties:  $J(v) \leq J(u)$ ,  $d(u, v) \leq \sqrt{\varepsilon}$  and for all  $w \neq v$  we have*

$$J(v) < J(w) + \sqrt{\varepsilon}d(v, w).$$

Since our strategy to prove uniform higher integrability by means of finite differences relies on suitable Nikolskiĭ–type estimates, we will need to apply Ekeland’s variational principle with respect to a metric which is considerably weaker than the symmetric–gradient metric  $d(u, v) := \|\varepsilon(u) - \varepsilon(v)\|_{L^1(\Omega; \mathbb{R}^{n \times n})}$  on appropriate Dirichlet classes. Here we again follow [8], however, invoke the metric induced by the  $(W_0^{1,\infty})^*$ –norm as discussed in Section 2.3.

**Lemma 3.2.** *Given  $p > 1$ , let  $F: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$  be a convex function such that*

$$(3.1) \quad c|\xi|^p - \vartheta \leq F(\xi) \leq \Theta(1 + |\xi|^p)$$

*holds for all  $\xi \in \mathbb{R}_{\text{sym}}^{n \times n}$  with three constants  $c, \vartheta, \Theta > 0$ . Given  $u_0 \in W^{1,p}(\Omega; \mathbb{R}^n)$ , the functional*

$$(3.2) \quad \mathcal{F}[u] := \begin{cases} \int_{\Omega} F(\varepsilon(u)) \, dx & \text{if } u \in W_{u_0}^{1,p}(\Omega; \mathbb{R}^n), \\ +\infty & \text{if } u \in ((W_0^{1,\infty})^* \setminus (W_{u_0}^{1,p}))(\Omega; \mathbb{R}^n) \end{cases}$$

*is lower semicontinuous with respect to norm convergence on  $(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$ .*

*Proof.* Let  $u, u_1, u_2, \dots \in (W_0^{1,\infty})^*(\Omega; \mathbb{R}^n)$  be such that  $u_k \rightarrow u$  strongly in  $(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$  as  $k \rightarrow \infty$ . If  $\liminf_{k \rightarrow \infty} \mathcal{F}[u_k] = \infty$ , we are done and so we may assume without loss of generality that there exists a subsequence of  $(u_k)$  (not relabelled) such that  $\lim_{k \rightarrow \infty} \mathcal{F}[u_k] = \liminf_{k \rightarrow \infty} \mathcal{F}[u_k] < \infty$ . By virtue of the coercive bound on  $F$ , we deduce that  $(\varepsilon(u_k))$  is bounded in  $L^p(\Omega; \mathbb{R}^n)$  and so, writing  $u_k = u_0 + v_k$  with  $v_k \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ , we use the Korn–Poincaré’s inequality in  $W_0^{1,p}$  to obtain

$$\int_{\Omega} |u_k|^p \, dx \lesssim \int_{\Omega} |u_0|^p + |v_k|^p \, dx \lesssim \int_{\Omega} |u_0|^p \, dx + \int_{\Omega} |\varepsilon(v_k)|^p \, dx \leq C \quad \text{for all } k \in \mathbb{N}.$$

As  $p > 1$ , we may use Korn’s inequality to deduce that  $(u_k)$  has a subsequence  $(u_{k(l)})$  which converges weakly to some  $v \in W^{1,p}(\Omega; \mathbb{R}^n)$  and, by continuity of the trace operator on  $W^{1,p}$  with respect to weak convergence,  $v \in W_{u_0}^{1,p}(\Omega; \mathbb{R}^n)$ , too. Since  $\Omega$  is assumed to be Lipschitz throughout, using the compact embedding  $W^{1,p}(\Omega; \mathbb{R}^n) \hookrightarrow L^p(\Omega; \mathbb{R}^n)$ , we may also assume that  $u_{k(l)} \rightarrow v$  in  $L^p(\Omega; \mathbb{R}^n)$  as  $l \rightarrow \infty$ . Since  $L^p(\Omega; \mathbb{R}^n) \hookrightarrow (W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$  as  $L^p \hookrightarrow L^1 \hookrightarrow (W_0^{1,\infty})^*$ , we conclude that  $u = v$   $\mathcal{L}^n$ –a.e. and that the full subsequence has the stated convergence property. Now, by virtue of the growth bound (3.1) and convexity of  $F$ , standard arguments (see for instance [22]) allow us to conclude the proof.  $\square$

Next, a lemma on linear growth integrands; recall that by *linear growth* we understand condition (1.2) throughout.



**Lemma 3.3.** *Let  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  be a convex integrand of linear growth. Then, for any  $\theta > 0$  there exists  $c_\theta, C_\theta > 0$  and  $\vartheta > 0$  such that*

$$\theta |\cdot|^2 - \vartheta \leq f + \theta |\cdot|^2 \leq C_\theta(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}_{\text{sym}}^{n \times n}.$$

The proof of this statement follows immediately from (1.2). We now come to the precise construction of a good approximation of a given generalised minimiser. Here we follow closely [8] with the requisite modifications. Let hereafter  $u \in \text{BD}(\Omega)$  be a generalised minimiser of  $\mathfrak{F}$ . Then, by (4.7), we have  $\inf \mathfrak{F}[\mathcal{D}] = \min \mathfrak{F}[\text{BD}(\Omega)]$ . Here,  $\mathcal{D} := u_0 + \text{LD}_0(\Omega)$ , and  $\mathfrak{F}$  is given by (1.6) (note that we suppress the subscript  $u_0$  for notational brevity). We then find a sequence  $(w_k) \subset \mathcal{D}$  such that

$$(3.3) \quad w_k \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^n) \text{ and } (\mathcal{L}^n, Ew_k) \rightarrow (\mathcal{L}^n, Eu) \text{ strictly}$$

as  $k \rightarrow \infty$ . By the RESHETNYAK continuity theorem (see Proposition 4.1) and  $w_k \in \text{LD}(\Omega)$  for all  $k \in \mathbb{N}$ , we deduce that  $\mathfrak{F}[w_k] \rightarrow \inf \mathfrak{F}[\mathcal{D}]$  as  $k \rightarrow \infty$  so that  $(w_k)$  indeed is a minimising sequence for  $\mathfrak{F}$ . Moreover, possibly passing to a subsequence, we may assume that

$$(3.4) \quad \mathfrak{F}[w_k] \leq \inf \mathfrak{F}[\mathcal{D}] + \frac{1}{8k^2} \quad \text{for all } k \in \mathbb{N}.$$

Next recall that, due to  $\mu$ -ellipticity and linear growth,  $f$  is Lipschitz with some Lipschitz constant  $L > 0$ . As to the Dirichlet data, we find a sequence  $(u_k^{\partial\Omega}) \subset W^{1,2}(\Omega; \mathbb{R}^n)$  satisfying

$$(3.5) \quad \|u_k^{\partial\Omega} - u_0\|_{\text{LD}(\Omega)} \leq \frac{1}{8Lk^2} \quad \text{for all } k \in \mathbb{N}$$

and thus, putting  $\mathcal{D}_k := u_k^{\partial\Omega} + W_0^{1,2}(\Omega; \mathbb{R}^n)$ , we deduce by  $w_k \in u_0 + \text{LD}_0(\Omega)$  that there exists a sequence  $(v_k) \subset \mathcal{D}_k$  such that

$$\|(v_k - u_k^{\partial\Omega}) - (w_k - u_0)\|_{\text{LD}(\Omega)} \leq \frac{1}{8Lk^2}$$

and hence

$$(3.6) \quad \|v_k - w_k\|_{\text{LD}(\Omega)} \leq \frac{1}{4Lk^2} \quad \text{for all } k \in \mathbb{N}.$$

Note that, relying on the extension results from Section 2.2, such an approximating sequence for the boundary values can be obtained by first extending the boundary values to an LD-map on the entire  $\mathbb{R}^n$  and then mollifying. Now, since  $f$  is Lipschitz with constant  $L$ , we firstly calculate for arbitrary  $\psi \in W_0^{1,2}(\Omega; \mathbb{R}^n)$

$$\begin{aligned} \inf_{u_0 + W_0^{1,2}(\Omega; \mathbb{R}^n)} \mathfrak{F} &\leq \mathfrak{F}[u_0 + \psi] = \left( \int_{\Omega} f(\varepsilon(u_0 + \psi)) - f(\varepsilon(u_k^{\partial\Omega} + \psi)) \, dx \right) + \int_{\Omega} f(\varepsilon(u_k^{\partial\Omega} + \psi)) \, dx \\ &\leq L \int_{\Omega} |\varepsilon(u_0 - u_k^{\partial\Omega})| \, dx + \int_{\Omega} f(\varepsilon(u_k^{\partial\Omega} + \psi)) \, dx \leq \frac{1}{8k^2} + \int_{\Omega} f(\varepsilon(u_k^{\partial\Omega} + \psi)) \, dx \end{aligned}$$

so that infimisation over  $\psi \in W_0^{1,2}(\Omega; \mathbb{R}^n)$  yields

$$(3.7) \quad \inf_{\mathcal{D}} \mathfrak{F} = \inf_{u_0 + W_0^{1,2}(\Omega; \mathbb{R}^n)} \mathfrak{F} \leq \inf_{\mathcal{D}_k} \mathfrak{F} + \frac{1}{8k^2}.$$

Here we have used that, by smooth approximation,  $u_0 + W_0^{1,2}(\Omega; \mathbb{R}^n)$  is norm-dense in  $\mathcal{D}$ . Thus, again using that  $f$  has Lipschitz constant  $L$  in conjunction with (3.6) in the first, (3.4) in the second and (3.7) in the last step, we eventually obtain

$$(3.8) \quad \mathfrak{F}[v_k] \leq \mathfrak{F}[w_k] + \frac{1}{4k^2} \leq \inf \mathfrak{F}[\mathcal{D}] + \frac{3}{8k^2} \leq \inf \mathfrak{F}[\mathcal{D}_k] + \frac{1}{2k^2}$$

for all  $k \in \mathbb{N}$ . Now put, for  $k \in \mathbb{N}$ ,

$$(3.9) \quad \begin{aligned} f_k(\xi) &:= f(\xi) + \frac{1}{2k^2 A_k} (1 + |\xi|^2), \quad \xi \in \mathbb{R}_{\text{sym}}^{n \times n}, \\ A_k &:= 1 + \int_{\Omega} (1 + |\varepsilon(v_k)|^2) \, dx. \end{aligned}$$

We define

$$\mathfrak{F}_k[w] := \begin{cases} \int_{\Omega} f_k(\varepsilon(w)) \, dx & \text{provided } w \in \mathcal{D}_k \\ +\infty & \text{provided } w \in (W_0^{1,\infty}(\Omega; \mathbb{R}^n))^* \setminus \mathcal{D}_k. \end{cases}$$

Now we aim to apply Lemma 3.2 for each  $k \in \mathbb{N}$  to the particular choice  $p = 2$  and  $F = f_k$ . In combination with Lemma 3.3, it is then routine to check that the assumptions of Lemma 3.2 are in fact satisfied and hence each  $\mathfrak{F}_k$  is lower semicontinuous with respect to norm convergence in  $(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$ . Moreover, we find because of  $v_k \in \mathcal{D}_k$  in the first, (3.8) in the second and by definition of  $\mathfrak{F}_k$  in the third step

$$(3.10) \quad \mathfrak{F}_k[v_k] \leq \mathfrak{F}[v_k] + \frac{1}{2k^2} \stackrel{(3.8)}{\leq} \inf \mathfrak{F}[\mathcal{D}_k] + \frac{1}{k^2} \leq \inf \mathfrak{F}_k[(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*] + \frac{1}{k^2}.$$

We are now in position to apply Ekeland's variational principle, Lemma 3.1, to find a sequence  $(u_k) \subset (W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$  such that

$$(3.11) \quad \begin{aligned} \|u_k - v_k\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*} &\leq \frac{1}{k}, \\ \mathfrak{F}_k[u_k] &\leq \mathfrak{F}_k[w] + \frac{1}{k} \|w - u_k\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*} \quad \text{for all } w \in (W_0^{1,\infty}(\Omega; \mathbb{R}^n))^* \text{ and } k \in \mathbb{N}. \end{aligned}$$

Applying the second part of (3.11) to  $w = v_k$ , we then find

$$(3.12) \quad \begin{aligned} \int_{\Omega} |\varepsilon(u_k)| \, dx &\stackrel{(1.2)}{\leq} \frac{1}{c_0} (\mathfrak{F}[u_k] + c_2) \leq \frac{1}{c_0} (\mathfrak{F}_k[u_k] + c_2) \\ &\stackrel{(3.11)_2}{\leq} \frac{1}{c_0} (\mathfrak{F}_k[v_k] + \frac{1}{k} \|v_k - u_k\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*} + c_2) \\ &\stackrel{(3.11)_1}{\leq} \frac{1}{c_0} (\mathfrak{F}_k[v_k] + \frac{1}{k^2} + c_2) \\ &\stackrel{(3.10)}{\leq} \frac{1}{c_0} (\inf \mathfrak{F}_k[(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*] + \frac{2}{k^2} + c_2) \leq C, \end{aligned}$$

where  $C > 0$  is a finite constant independent of  $k \in \mathbb{N}$ ; note that we clearly have that  $\inf \mathfrak{F}_k[(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*] < \infty$ . In particular, we deduce that

$$(3.13) \quad (u_k) \text{ is uniformly bounded in } LD(\Omega).$$

Finally, we record the following lemma on the perturbed Euler–Lagrange equations satisfied by the individual  $u_k$ 's.

**Lemma 3.4** (Approximate Euler–Lagrange Equation). *Let  $f_k$  and  $u_k$  be defined as above. Then for all  $k \in \mathbb{N}$  we have*

$$(3.14) \quad \left| \int_{\Omega} \langle f'_k(\varepsilon(u_k)), \varepsilon(\varphi) \rangle \, dx \right| \leq \frac{1}{k} \|\varphi\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*}$$

for all  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^n)$ .

*Proof.* Fix  $k \in \mathbb{N}$  and let  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^n)$  be arbitrary. Then for every  $\varepsilon > 0$  we have  $u_k \pm \varepsilon \varphi \in \mathcal{D}_k$ . Consequently, we obtain by the second line of (3.11)

$$\mathfrak{F}_k[u_k] - \mathfrak{F}_k[u_k \pm \varepsilon \varphi] \leq \frac{\varepsilon}{k} \|\varphi\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*}.$$

This gives

$$-\frac{1}{k}\|\varphi\|_{(W_0^{1,\infty}(\Omega;\mathbb{R}^n))^*} \leq \frac{\mathfrak{F}_k[u_k \pm \varepsilon\varphi] - \mathfrak{F}_k[u_k]}{\varepsilon} \xrightarrow{\varepsilon \searrow 0} \pm \left( \int_{\Omega} \langle f'_k(\varepsilon(u_k)), \varepsilon(\varphi) \rangle dx \right)$$

from which (3.14) follows at once.  $\square$

To explain the advantages of the technically slightly intricate construction of the particular minimising sequence  $(u_k)$ , let us first make the following

**Remark 3.5** (Euler–Lagrange for Measures). It is possible to directly work on the Euler–Lagrange equation satisfied by a generalised minimiser  $u \in \text{GM}(\mathfrak{F})$ . Indeed, transferring ANZELLOTTI’S work [4] to functionals of type (1.1), one is able to show that

$$\begin{aligned} \int_{\Omega} \langle f'(E^a u), E^a \varphi \rangle dx + \int_{\Omega} \left\langle (f^\infty)' \left( \frac{dE^s u}{d|E^s u|} \right), \frac{dE^s \varphi}{d|E^s \varphi|} \right\rangle d|E^s \varphi| \\ = \int_{\partial\Omega} \left\langle (f^\infty)' \left( \frac{\text{Tr}(u_0 - u)}{|\text{Tr}(u_0 - u)|} \odot \nu_{\partial\Omega} \right), \varphi \odot \nu_{\partial\Omega} \right\rangle d\mathcal{H}^{n-1}, \end{aligned}$$

for all  $\varphi \in \text{BD}(\Omega)$  with  $|E^s \varphi| \ll |E^s u|$  such that  $\varphi(x) = 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\{x \in \partial\Omega : u(x) = u_0(x)\}$ , where  $\nu_{\partial\Omega}$  is the outward unit normal to  $\partial\Omega$ . However, it seems difficult to apply the difference quotient technique directly on the Euler–Lagrange equation for measures so that we rather choose approximation procedures.

To conclude, note that Lemma 3.4 enables us to work with difference quotients applied to functions and to eventually deduce uniform estimates for the  $u_k$ ’s. In particular, by arbitrariness of the generalised minimiser  $u \in \text{GM}(\mathfrak{F})$  as was assumed in this section, we have constructed a sequence  $(u_k)$  converging to  $u$  in  $(W_0^{1,\infty}(\Omega;\mathbb{R}^n))^*$  as a consequence of (3.3), (3.6) and (3.11)<sub>1</sub>. On the other hand, (3.13) allows to extract a subsequence that converges in the weak\*-sense to some  $v \in \text{BD}(\Omega)$ , and in this situation  $u_k \rightarrow u$  in  $(W_0^{1,\infty}(\Omega;\mathbb{R}^n))^*$  identifies  $u$  as the corresponding weak\*-limit. As a consequence, suitable uniform estimates on the  $u_k$ ’s will be inherited by  $u$ . Note that by starting from an arbitrary  $u \in \text{GM}(\mathfrak{F})$ , regularity for all generalised minimisers will hereby be established.

**3.2. On Projections onto  $\mathcal{R}$ .** In this intermediate section, we give a technical result which might be clear to the experts though hard to trace in the literature.

**Lemma 3.6.** *Let  $U \subset \mathbb{R}^n$  be an open, bounded and connected set with Lipschitz boundary. Let  $1 < p < \infty$ . Then for every  $1 \leq q \leq p$  there exists a finite constant  $c_{q,U} > 0$  such that for all  $u \in W^{1,p}(U;\mathbb{R}^n)$  there exists  $b \in \mathcal{R}(U)$  such that*

$$(3.15) \quad \int_U |u - b|^q dx \leq c_{q,U} \int_U |\varepsilon(u)|^q dx \quad \text{and} \quad \int_U |D(u - b)|^p dx \leq c_{p,U} \int_U |\varepsilon(u)|^p dx.$$

The key in this lemma is that we can choose *one* particular rigid deformation to validate both inequalities.

*Proof.* For an open and bounded Lipschitz set  $U \subset \mathbb{R}^n$ , denote  $X_s(U)$  either  $W^{1,s}(U;\mathbb{R}^n)$  provided  $1 < s < \infty$  or  $\text{LD}(U)$  provided  $s = 1$  and put

$$(3.16) \quad \mathcal{R}_{X_s}^\perp(U) := \left\{ \varphi \in X_s(U) : \int_U \langle \varphi, \psi \rangle dx = 0 \text{ for all } \psi \in \mathcal{R}(U) \right\}.$$

Then, by straightforward adaptation of [26, Eq. (3.25)ff.], we find that for every  $1 \leq s < \infty$  there exists  $c_s > 0$  such that  $\|v\|_{L^s(U;\mathbb{R}^n)} \leq c_s \|\varepsilon(v)\|_{L^s(U;\mathbb{R}^{n \times n})}$  holds for all  $\mathcal{R}_{X_s}^\perp(U)$ . We now consider an  $L^2$ -orthonormal basis  $\{b_1, \dots, b_m\}$  of the finite dimensional space  $\mathcal{R}(U)$ . We then define the  $L^2$ -orthogonal projections  $\Pi: L^2(U;\mathbb{R}^n) \rightarrow \mathcal{R}(U)$  by

$$(3.17) \quad \Pi\varphi := \sum_{j=1}^m \langle b_j, \varphi \rangle_{L^2(U;\mathbb{R}^n)} b_j, \quad \varphi \in L^2(U;\mathbb{R}^n).$$

Note that, since  $\mathcal{R}(U)$  consists of affine-linear polynomials and so  $\mathcal{R}(U) \subset L^\infty(U; \mathbb{R}^n)$ ,  $\Pi\varphi$  is also well-defined for  $\varphi \in L^1(U; \mathbb{R}^n)$ . We moreover have for all  $1 \leq s < \infty$

$$\left( \int_U |\Pi\varphi|^s dx \right)^{\frac{1}{s}} \leq \left( \sum_{j=1}^m \|b_j\|_{L^\infty(U; \mathbb{R}^n)}^2 \right) \left( \int_U |\varphi|^s dx \right)^{\frac{1}{s}},$$

and from here we see that  $\Pi$  is indeed  $L^s$ -stable for all  $1 \leq s < \infty$ . Let now  $u \in W^{1,p}(U; \mathbb{R}^n)$  for  $1 < p < \infty$ , so that, in particular,  $u \in \text{LD}(U)$ . We then have  $u - \Pi u \in \mathcal{R}_{X_p}^\perp(U)$  regardless of  $p$  and hence deduce the first part of (3.15). For the second one, recall that by Korn's inequality,  $\|Du\|_{L^p(U; \mathbb{R}^{n \times n})} \leq C(\|u\|_{L^p(U; \mathbb{R}^n)} + \|\varepsilon(u)\|_{L^p(U; \mathbb{R}^n)})$  with  $C$  depending on  $p$  and  $U$  only. Replacing  $u$  by  $u - \Pi u$  in this inequality and invoking the first part of (3.15) with  $q = p$ , we establish the second part of (3.15) and the proof is complete.  $\square$

**3.3. Proof of Theorem 1.2.** We now turn to the proof of Theorem 1.2 and begin with the following auxiliary lemma; recall that the finite difference  $\tau_{s,h}$  was defined in Section 2.1.

**Lemma 3.7.** *Let  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  be a  $\mu$ -elliptic integrand,  $1 < \mu < \infty$ , and define for  $1 < \alpha < 2$  the auxiliary map  $V_\alpha$  by (2.7). Then there exists a constant  $C = C(\alpha, n) > 0$  such that for all  $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^n)$ , all relatively compact Lipschitz subsets  $K \Subset \Omega$ , all  $h \in \mathbb{R}$  with  $|h| < \text{dist}(K, \partial\Omega)$ ,  $s \in \{1, \dots, n\}$  and  $\mathcal{L}^n$ -a.e.  $x \in K$  there holds*

$$(3.18) \quad \frac{|\tau_{s,h} V_\alpha(u(x))|^2}{(1 + |u(x + he_s)|^2 + |u(x)|^2)^{\frac{2(1-\alpha)+\mu}{2}}} \leq C \frac{|\tau_{s,h} u(x)|^2}{(1 + |u(x)|^2 + |u(x + he_s)|^2)^{\frac{\mu}{2}}}.$$

*Proof.* We now use the auxiliary estimate given by Lemma 2.4 to conclude for the auxiliary function  $V_\alpha(\xi) := (1 + |\xi|^2)^{\frac{1-\alpha}{2}} \xi$  with  $1 < \alpha < 2$  that

$$\frac{|V_\alpha(\xi) - V_\alpha(\eta)|^2}{(1 + |\xi|^2 + |\eta|^2)^{\frac{\mu+2(1-\alpha)}{2}}} \leq \frac{|\xi - \eta|^2}{(1 + |\eta|^2 + |\xi|^2)^{\frac{\mu}{2}}}.$$

Applying this to  $\xi = u(x + he_s)$  and  $\eta = u(x)$ , we conclude.  $\square$

After these preparations, we now come to the

*Proof of Theorem 1.2.* Let  $\mu$  be as in the theorem. We then fix an arbitrary generalised minimiser  $u \in \text{GM}(\mathfrak{F}; u_0)$  and consider the sequence  $(u_k)$  constructed in Section 3.1, cf. (3.11). Let  $k \in \mathbb{N}$  be arbitrary but fixed; then  $u_k$  satisfies the approximate Euler-Lagrange equation (3.14). Let  $x_0 \in \Omega$ ,  $0 < r < R < \text{dist}(x_0, \partial\Omega)$  and pick  $\rho \in C_c^1(B(x_0, R); [0, 1])$  with  $\mathbb{1}_{B(x_0, r)} \leq \rho \leq \mathbb{1}_{B(x_0, R)}$ . Then, let  $\Omega_1$  be the connected component of  $\Omega$  that contains  $x_0$ ; we may assume that  $\Omega_1$  itself has Lipschitz boundary. Due to Lemma 3.6 and the fact that  $u_k|_{\Omega_1} \in W^{1,2}(\Omega_1; \mathbb{R}^n)$ , we first choose a rigid deformation  $b_k \in \mathcal{R}(\Omega_1)$  such that with  $c = c(\Omega_1)$

$$(3.19) \quad \int_{\Omega_1} |u_k - b_k| dx \leq c \int_{\Omega_1} |\varepsilon(u_k)| dx \quad \text{and} \quad \int_{\Omega_1} |D(u_k - b_k)|^2 dx \leq c \int_{\Omega_1} |\varepsilon(u_k)|^2 dx.$$

For  $0 < |h| < \text{dist}(\partial B(x_0, R); \partial\Omega_1)$  and  $s \in \{1, \dots, n\}$ , we then choose  $\varphi := \tau_{s,-h}(\rho^2 \tau_{s,h}(u_k - b_k)) \in W^{1,2}(\Omega; \mathbb{R}^n) \hookrightarrow (W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*$  as a test map in (3.14). Since  $\varepsilon$  and  $\tau_{s,h}$  commute, this yields with  $\tilde{u}_k := u_k - b_k$

$$(3.20) \quad \left| \int_{\Omega} \langle f'_k(\varepsilon(u_k)), \tau_{s,-h}(\varepsilon(\rho^2 \tau_{s,h} \tilde{u}_k)) \rangle dx \right| \leq \frac{1}{k} \|\tau_{s,-h}(\rho^2 \tau_{s,h} \tilde{u}_k)\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*}.$$

We now proceed in three steps.

*Step 1. Recasting (3.20).* By discrete integration by parts in (3.20), we find by the product rule for  $\varepsilon$  that

$$\begin{aligned}
 \mathbf{I} &:= \int_{\Omega} \langle \tau_{s,h} f'_k(\varepsilon(u_k)), \rho^2 \tau_{s,h} \varepsilon(u_k) \rangle dx \\
 &\leq \left| \int_{\Omega} \langle \tau_{s,h} f'_k(\varepsilon(u_k)), 2\rho \nabla \rho \odot \tau_{s,h} \tilde{u}_k \rangle dx \right| + \frac{1}{k} \|\tau_{s,-h}(\rho^2 \tau_{s,h} \tilde{u}_k)\|_{(W_0^{1,\infty})^*} \\
 (3.21) \quad &\leq \left| \int_{\Omega} \langle \tau_{s,h} f'_k(\varepsilon(u_k)), 2\rho \nabla \rho \odot \tau_{s,h} \tilde{u}_k \rangle dx \right| + \frac{1}{A_k k^2} \left| \int_{\Omega} \langle \tau_{s,h} \varepsilon(u_k), 2\rho \nabla \rho \odot \tau_{s,h} \tilde{u}_k \rangle dx \right| \\
 &\quad + \frac{1}{k} \|\tau_{s,-h}(\rho^2 \tau_{s,h} \tilde{u}_k)\|_{(W_0^{1,\infty})^*} \\
 &=: \mathbf{II} + \mathbf{III} + \mathbf{IV},
 \end{aligned}$$

where  $A_k$  is defined by (3.9).

*Step 2. Key Estimates.* We now estimate each of the terms  $\mathbf{I}, \dots, \mathbf{IV}$ . As to  $\mathbf{I}$ , we introduce the bilinear forms

$$\mathcal{B}_{k,h}(x)(\xi, \zeta) := \int_0^1 \langle f''_k(\varepsilon(u_k) + t\tau_{s,h}\varepsilon(u_k))\xi, \zeta \rangle dt, \quad \xi, \zeta \in \mathbb{R}_{\text{sym}}^{n \times n}.$$

Consequently, by the fundamental theorem of calculus we deduce

$$\begin{aligned}
 \mathbf{I} &= \int_{\Omega} \langle f'_k(\varepsilon(u_k(x + h e_s))) - f'_k(\varepsilon(u_k(x))), \rho^2 \varepsilon(u_k) \rangle dx \\
 &= \int_{\Omega} \left\langle \int_0^1 \frac{d}{dt} f'_k(\varepsilon(u_k) + t\tau_{s,h}\varepsilon(u_k)) dt, \rho^2 \tau_{s,h} \varepsilon(u_k) \right\rangle dx \\
 &= \int_{\Omega} \mathcal{B}_{k,h}(\rho \tau_{s,h} \varepsilon(u_k), \rho \tau_{s,h} \varepsilon(u_k)) dx.
 \end{aligned}$$

By  $\mu$ -ellipticity and the definition of  $f_k$ , these are strongly elliptic bilinear forms. We briefly pause to comment on the strategy. Usually, one would now apply the Cauchy-Schwarz inequality to the first term of the right hand side of the first inequality in (3.21) and then conveniently absorb, but this we *do not pursue here*. In fact, this would leave us with the task of bounding the term

$$(3.22) \quad 4 \int_{\Omega} \mathcal{B}_{k,h}(\rho \nabla \rho \odot \tau_{s,h} u_k, \rho \nabla \rho \odot \tau_{s,h} u_k) dx$$

and it is unclear to us how to do that in terms of the estimates available so far. Instead, we use the fact that by Lipschitz continuity of  $f, f'$  is bounded and so, in particular  $|\tau_{s,h} f'(\varepsilon(u_k))| \leq M$  for some  $M > 0$ . We now go back to the local embedding provided by Proposition 2.2 and hence obtain for every  $0 < \beta < 1$  the embedding  $\text{LD}(\Omega) \hookrightarrow W_{\text{loc}}^{\beta,1}(\Omega; \mathbb{R}^n)$ . By standard means, this yields  $\text{LD}(\Omega_1) \hookrightarrow W^{\beta,1}(\Omega_1; \mathbb{R}^n) \hookrightarrow (B_{1,\infty}^{\beta})_{\text{loc}}(\Omega_1; \mathbb{R}^n)$ . In particular, we obtain by Lemma 3.6

$$\begin{aligned}
 (3.23) \quad &\sup_{\substack{s \in \{1, \dots, n\} \\ 0 < |h| < \text{dist}(B(x_0, R), \partial \Omega_1)}} \int_{B(x_0, R)} \frac{|\tau_{s,h} \tilde{u}_k|}{|h|^{\beta}} dx \leq c(\Omega_1) \iint_{\Omega_1 \times \Omega_1} \frac{|\tilde{u}_k(x) - \tilde{u}_k(y)|}{|x - y|^{n+\beta}} dx dy \\
 &\leq c(\Omega_1, \beta) \|\tilde{u}_k\|_{\text{LD}(\Omega_1)} \\
 &= c(\Omega_1, \beta) (\|\tilde{u}_k\|_{L^1(\Omega_1; \mathbb{R}^n)} + \|\varepsilon(u_k)\|_{L^1(\Omega_1; \mathbb{R}^{n \times n})}) \\
 &\leq C(\Omega_1, \beta) \|\varepsilon(u_k)\|_{L^1(\Omega_1; \mathbb{R}^{n \times n})} \quad (\text{by (3.19)}).
 \end{aligned}$$

In consequence, for every  $0 < \beta < 1$  (to be fixed later) we find  $C(\beta, \rho, \Omega_1) > 0$  such that

$$\begin{aligned}
 \mathbf{II} &\leq C(\rho, \Omega_1) M \int_{B(x_0, R)} |\tau_{s,h} \tilde{u}_k| dx = C(\rho, \Omega_1) M |h|^{\beta} \int_{B(x_0, R)} \frac{|\tau_{s,h} \tilde{u}_k|}{h^{\beta}} dx \\
 &\leq C(\beta, \rho, \Omega_1) M |h|^{\beta} \|\varepsilon(u_k)\|_{L^1(\Omega; \mathbb{R}^{n \times n})}
 \end{aligned}$$

so that by (3.13) and possibly enlarging  $C(\beta, \rho, \Omega_1)$ , we end up with

$$(3.24) \quad \mathbf{II} \leq C(\beta, \rho, \Omega_1) M |h|^\beta.$$

As to  $\mathbf{III}$ , we apply the Cauchy-Schwarz inequality to find for  $\delta > 0$  sufficiently small

$$\begin{aligned} \mathbf{III} &\leq \frac{\delta}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx + \frac{C(\delta)}{A_k k^2} \int_{\Omega} |\nabla \rho \odot \tau_{s,h} \tilde{u}_k|^2 dx \\ &= \frac{\delta}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx + \frac{C(\delta, \rho) h^2}{A_k k^2} \int_{B(x_0, R)} |\Delta_{s,h} \tilde{u}_k|^2 dx \\ &\leq \frac{\delta}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx + \frac{C(\delta, \rho) h^2}{A_k k^2} \int_{\Omega_1} |\partial_s \tilde{u}_k|^2 dx. \end{aligned}$$

The ultimate term now is controlled by Korn's inequality. To be precise, we have by (3.19)

$$\int_{\Omega_1} |\partial_s \tilde{u}_k|^2 dx \leq \int_{\Omega_1} |\nabla \tilde{u}_k|^2 dx \leq c(\Omega_1) \int_{\Omega_1} |\varepsilon(u_k)|^2 dx \leq c(\Omega_1) \int_{\Omega} |\varepsilon(u_k)|^2 dx.$$

Working from here, we invoke the definition of  $\mathfrak{F}_k$  and (3.11)<sub>2</sub> with  $w = v_k$  to deduce

$$\begin{aligned} \frac{1}{2k^2 A_k} \int_{\Omega_1} |\partial_s \tilde{u}_k|^2 dx &\leq \frac{c(\Omega_1)}{2k^2 A_k} \int_{\Omega} (1 + |\varepsilon(u_k)|^2) dx \\ &\leq c(\Omega_1) \mathfrak{F}_k[u_k] \\ &\stackrel{(3.11)_2}{\leq} c(\Omega_1) (\mathfrak{F}_k[v_k] + \frac{1}{k} \|v_k - u_k\|_{(W_0^{1,\infty}(\Omega; \mathbb{R}^n))^*}) \\ &\leq C(\Omega_1) < \infty, \end{aligned}$$

where the last inequality follows in the same way as in (3.12). In consequence, we find

$$(3.25) \quad \mathbf{III} \leq \frac{\delta}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx + \frac{C(\delta, \rho, \Omega_1) h^2}{k^2} =: \mathbf{III}_1^{(\delta)} + \mathbf{III}_2^{(\delta)}.$$

Ad  $\mathbf{IV}$ . Here we estimate for  $0 < \gamma < 1$  to be specified later

$$\begin{aligned} \mathbf{IV} &= \frac{1}{k} \|\tau_{s,-h}(\rho^2 \tau_{s,h} \tilde{u}_k)\|_{(W_0^{1,\infty})^*} = \frac{|h|^{1+\gamma}}{k} \|\Delta_{s,-h}(\rho^2 \frac{\tau_{s,h}}{|h|^\gamma} \tilde{u}_k)\|_{(W_0^{1,\infty})^*} \\ (3.26) \quad &\leq \frac{|h|^{1+\gamma}}{k} \|\rho^2 \frac{\tau_{s,h}}{|h|^\gamma} \tilde{u}_k\|_{L^1} \quad (\text{by Lemma 2.3}) \\ &\leq c \frac{|h|^{1+\gamma}}{k} \|\tilde{u}_k\|_{W^{\gamma,1}(\Omega_1; \mathbb{R}^n)} \leq c \frac{|h|^{1+\gamma}}{k} \|\tilde{u}_k\|_{LD(\Omega_1)} \stackrel{(3.13)}{\leq} c \frac{|h|^{1+\gamma}}{k}, \end{aligned}$$

where we have employed a similar argument as in (3.23).

In an intermediate step, let  $0 \leq t \leq 1$  and  $a, b \in \mathbb{R}^{n \times n}$  be arbitrary. There holds (with some fixed  $C > 0$  independent of  $t, a$  and  $b$ )

$$(3.27) \quad (1 + |a + tb|^2)^{\frac{1}{2}} \leq C(1 + |a|^2 + |b|^2)^{\frac{1}{2}}.$$

Now, we estimate from below by virtue of  $\mu$ -ellipticity of  $f$  and the definition of  $f_k$

$$\begin{aligned} \mathbf{I} &\geq \int_{\Omega} \int_0^1 \langle f''(\varepsilon(u_k) + t \tau_{s,h} \varepsilon(u_k)) \rho \tau_{s,h} \varepsilon(u_k), \rho \tau_{s,h} \varepsilon(u_k) \rangle dt dx + \frac{1}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx \\ &\geq \lambda \int_{\Omega} \int_0^1 \frac{|\rho \tau_{s,h} \varepsilon(u_k)|^2}{(1 + |\varepsilon(u_k) + t \tau_{s,h} \varepsilon(u_k)|^2)^{\frac{n}{2}}} dt dx + \frac{1}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx \\ &\stackrel{(3.27)}{\geq} \tilde{\lambda} \int_{\Omega} \frac{|\rho \tau_{s,h} \varepsilon(u_k)|^2}{(1 + |\varepsilon(u_k)(x)|^2 + |\varepsilon(u_k)(x + h e_s)|^2)^{\frac{n}{2}}} dx + \frac{1}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx =: \mathbf{I}' \end{aligned}$$



after diminishing  $\lambda > 0$  to  $\tilde{\lambda} > 0$  if necessary. Now let  $1 < \alpha < 2$  to be fixed later. Then Lemma 3.7 applied to  $\varepsilon(u_k)$  yields

$$(3.28) \quad \begin{aligned} \mathbf{I} \geq \mathbf{I}' &\geq c(\tilde{\lambda}) \int_{\Omega} \rho \frac{|\tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))|^2}{(1 + |\varepsilon(u_k)(x + h e_s)|^2 + |\varepsilon(u_k)(x)|^2)^{\frac{2(1-\alpha)+\mu}{2}}} dx \\ &\quad + \frac{1}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx =: \mathbf{I}'_1 + \mathbf{I}'_2. \end{aligned}$$

We now gather the estimates given so far and put

$$(3.29) \quad \omega_{k,h,s}(x) := \frac{1}{(1 + |\varepsilon(u_k(x))|^2 + |\varepsilon(u_k(x + h e_s))|^2)^{\frac{\mu+2(1-\alpha)}{2}}}$$

for brevity. If  $\delta < 1$  we may absorb  $\mathbf{III}_1^{(\delta)}$  into  $\mathbf{I}'_2$  in the overall inequality. Hence, by (3.21), (3.28), (3.24), (3.25) and (3.26) we invoke (3.13) to end up with

$$(3.30) \quad \begin{aligned} c(\tilde{\lambda}) \int_{\Omega} |\rho \tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))|^2 \omega_{k,h,s}(x) dx &+ \frac{1-\delta}{A_k k^2} \int_{\Omega} |\rho \tau_{s,h} \varepsilon(u_k)|^2 dx \\ &\leq C(\beta, \rho, \Omega_1) M |h|^{\beta} \\ &\quad + \frac{C(\delta, \rho, \Omega) h^2}{k^2} + C(\rho) \frac{|h|^{1+\gamma}}{k}. \end{aligned}$$

Since we may assume without loss of generality that  $|h| < 1$  and by positivity of the second term on the left hand side of the previous inequality, we find by dividing the previous inequality by  $|h|^{\beta}$

$$(3.31) \quad \sup_{k \in \mathbb{N}} \int_{\Omega} \rho^2 \left| \frac{\tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))}{|h|^{\frac{\beta}{2}}} \right|^2 \omega_{k,h,s}(x) dx < \infty.$$

*Step 3. Conclusion.* We go back to (3.31) and deduce by Young's inequality that

$$\begin{aligned} \int_{\Omega} \rho \left| \frac{\tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))}{|h|^{\frac{\beta}{2}}} \right| dx &= \int_{\Omega} \rho \left| \frac{\tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))}{|h|^{\frac{\beta}{2}}} \right| \omega_{k,h,s}^{\frac{1}{2}} \frac{dx}{\omega_{k,h,s}^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \int_{\Omega} \rho^2 \left| \frac{\tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))}{|h|^{\frac{\beta}{2}}} \right|^2 \omega_{k,h,s} dx + \frac{1}{2} \int_{B(x_0, R)} \frac{dx}{\omega_{k,h,s}(x)} \\ &=: \mathbf{V} + \mathbf{VI}. \end{aligned}$$

The term  $\mathbf{V}$  is bounded by (3.31). As to  $\mathbf{VI}$ , we recall that by (3.13),  $(u_k)$  is uniformly bounded in  $\text{LD}(\Omega)$ . In consequence,  $\mathbf{VI}$  is uniformly bounded in  $k$  and  $h$  provided

$$(3.32) \quad \mu + 2(1 - \alpha) \leq 1, \quad \text{that is,} \quad \frac{\mu + 1}{2} \leq \alpha.$$

At this stage, let us recall that this appears subject to the condition  $1 < \alpha < 2$  from Lemma 3.7. Since  $\mu > 1$ , the lower bound is satisfied in any case, but the upper bound requires  $\mu < 3$  which is satisfied as well for the growth regime we are considering. Now, summarising, we obtain

$$(3.33) \quad \sup_{k \in \mathbb{N}} \int_{\Omega} \rho \left| \frac{\tau_{s,h} V_{\alpha}(\varepsilon(u_k(x)))}{|h|^{\frac{\beta}{2}}} \right| dx < \infty$$

and hence, by arbitrariness of  $x_0$ ,  $\rho$  and direction  $1 \leq s \leq n$ , infer that  $(V_{\alpha}(\varepsilon(u_k)))$  is locally uniformly bounded in  $B_{1,\infty}^{\beta/2}$ . By Lemma 2.1, we obtain that for any  $0 < \delta' < \frac{n}{n-\beta/2}$ ,  $(V_{\alpha}(\varepsilon(u_k)))$  is locally uniformly bounded in  $L^{\frac{2n}{2n-\beta}-\delta'}$ . Now we invoke Lemma 2.4, cf. (2.8), to deduce that for any relatively compact Lipschitz set  $K \Subset \Omega$  there exists

$$(3.34) \quad \sup_{k \in \mathbb{N}} \int_K |\varepsilon(u_k)|^{(2-\alpha)\left(\frac{2n}{2n-\beta}-\delta'\right)} dx = C(\alpha, \delta', \beta) < \infty,$$

and we now choose  $\beta, \alpha$  and  $\delta'$  in a suitable way. First we note that

$$(3.35) \quad \mu < 1 + \frac{1}{n} \implies \mu + 1 < 2 + \frac{1}{n} \implies \frac{\mu + 1}{2} < 1 + \frac{1}{2n}$$

and so we find and fix  $\alpha$  such that

$$(3.36) \quad \frac{\mu + 1}{2} < \alpha < 1 + \frac{1}{2n}.$$

From the previous inequality, we deduce

$$\alpha < 1 + \frac{1}{2n} \implies 1 - \alpha > -\frac{1}{2n} \implies 2 - \alpha > 1 - \frac{1}{2n} = \frac{2n - 1}{2n} \implies (2 - \alpha) \frac{2n}{2n - 1} > 1.$$

Now we may send  $\beta \nearrow 1$  and  $\delta' \searrow 0$  to deduce that there exists  $\beta < 1$  and  $\delta' > 0$  such that

$$(3.37) \quad p := (2 - \alpha) \left( \frac{2n}{2n - \beta} - \delta' \right) > 1$$

as well. Then we infer from (3.34) that for every ball  $B \Subset \Omega$  there holds

$$\sup_{k \in \mathbb{N}} \int_B |\varepsilon(u_k)|^p dx < \infty.$$

Then, by Poincaré's inequality, we find rigid deformations  $d_k \in \mathcal{R}(B)$  such that the sequence  $(u_k - d_k)|_B$  is uniformly bounded in  $L^p(B; \mathbb{R}^n)$ . Since  $1 < p < \infty$ , Korn's inequality and reflexivity of  $W^{1,p}$  allow to extract a subsequence  $(u_{k(l)})|_K$  which converges weakly to some  $v \in W^{1,p}(B; \mathbb{R}^n)$ . Since  $u_k|_B \xrightarrow{*} u|_B$ , we conclude that  $v = u|_B$  and  $E^s u$  must vanish on  $B$ . The proof is complete.  $\square$

Even though briefly mentioned in the proof, let us stress again that it is precisely the presence of the term (3.22) that forces the proof to deviate from the BV-case. If we worked with  $\mu$ -elliptic functionals (1.4) where  $\varepsilon$  is replaced by  $D$  and we thus are in the BV-framework, the suitable adaptation of the approximation procedure outlined in Section 3.1 (cf. [8, Sec. 5]) yields that the constructed sequence  $(u_k)$  is uniformly bounded in  $BV(\Omega; \mathbb{R}^n)$ . Then, by the upper bound provided by the  $\mu$ -ellipticity (cf. (1.8)) the term from (3.22) would be controlled by

$$\int_{\Omega} \mathcal{B}_{k,h}(\rho \nabla \rho \odot \tau_{s,h} u_k, \rho \nabla \rho \odot \tau_{s,h} u_k) dx \leq h^2 \int_{\Omega} \frac{|\rho \Delta_{s,h} u_k|^2}{(1 + |\varepsilon(u_k)|^2)^{\frac{1}{2}}} dx,$$

and as in [8, Lem. 5.3], the last term can be controlled as  $(u_k)$  then would be bounded in  $W^{1,1}(\Omega; \mathbb{R}^n)$ .

As we mentioned above, Theorem 1.2 implies that  $\text{GM}(\mathfrak{F}; u_0) \subset W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^n) \cap \text{LD}(\Omega)$  and so we may now use the additional integrability information to amplify the regularity of generalised minimisers. Here, as  $V_{\alpha}$  has a regularising effect on  $\varepsilon(u_k)$ , we directly work on  $\varepsilon(u_k)$ . Let  $1 < \mu < 1 + \frac{1}{n}$ . Going back to (3.37) subject to (3.36), optimising  $p$  yields that  $\text{GM}(\mathfrak{F}; u_0) \subset \text{LD}(\Omega) \cap W_{\text{loc}}^{1,q}(\Omega; \mathbb{R}^n)$  for all

$$(3.38) \quad 1 \leq q < \frac{(3 - \mu)n}{2n - 1}.$$

Exemplarily, we show how for a certain range of ellipticities we can even obtain second derivative estimates.

**Corollary 3.8.** *Suppose that  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  is a  $\mu$ -elliptic integrand of linear growth with  $1 < \mu < \frac{3n}{3n-1}$ . Then for all generalised minimisers  $u \in \text{GM}(\mathfrak{F}; u_0)$  we have  $u \in W_{\text{loc}}^{2,Q}(\Omega; \mathbb{R}_{\text{sym}}^n)$  for some  $Q = Q(\mu) > 1$ .*

*Proof.* Let us firstly note that the condition on  $\mu$  implies

$$(3.39) \quad \mu < \frac{3n}{3n-1} \implies 2n\mu - \mu < 3n - n\mu = (3 - \mu)n \implies \mu < \frac{(3 - \mu)n}{2n - 1}.$$

and so we deduce that  $u \in \text{LD}(\Omega) \cap W_{\text{loc}}^{1,\mu}(\Omega; \mathbb{R}^n)$  by the above argument. Denote  $(u_k)$  the Ekeland approximation sequence as above. We now go back to the proof of Theorem 1.2, step 1, let  $x_0 \in \Omega$  be arbitrary and choose  $0 < r < R < \text{dist}(\partial\Omega; B(x_0, R))$  together with a localisation function  $\rho \in C_c^2(B(x_0, R); [0, 1])$ . We then put, for  $s \in \{1, \dots, n\}$  and  $|h|$  sufficiently small,  $\varphi := \Delta_{s,-h}(\rho^2 \Delta_{s,h} \tilde{u}_k)$ . Then we insert  $\varphi$  into (3.20), write

$$\varepsilon(\varphi) = \Delta_{s,-h}(\rho^2 \Delta_{s,h} \varepsilon(u_k)) + \Delta_{s,-h}(2\rho \nabla \rho \odot \Delta_{s,h} \tilde{u}_k)$$

and thereby end up with

$$(3.40) \quad \mathbf{J}_1 := \left| \int_{\Omega} \langle \Delta_{s,h} f'_k(\varepsilon(u_k)), \rho^2 \Delta_{s,h} \varepsilon(u_k) \rangle dx \right| \leq \left| \int_{\Omega} \langle f'_k(\varepsilon(u_k)), \Delta_{s,-h}(2\rho \nabla \rho \odot \Delta_{s,h} \tilde{u}_k) \rangle dx \right| + \frac{1}{k} \|\rho^2 \Delta_{s,h} \tilde{u}_k\|_{L^1(\Omega; \mathbb{R}^n)} =: \mathbf{J}_2 + \mathbf{J}_3.$$

Similarly as in the proof of Theorem 1.2, we find that for some  $c > 0$

$$(3.41) \quad c \int_{\Omega} \frac{|\rho \Delta_{s,h} \varepsilon(u_k)|^2}{(1 + |\varepsilon(u_k)(x + h e_s)|^2 + |\varepsilon(u_k)|^2)^{\frac{\mu}{2}}} dx + \frac{1}{k^2 A_k} \int_{\Omega} |\rho \Delta_{s,h} \varepsilon(u_k)|^2 dx \leq \mathbf{J}_1.$$

Now, as to  $\mathbf{J}_2$ , we split and estimate by Lipschitz continuity of  $f$  in the first inequality and  $g_k := f_k - f$ ,

$$\mathbf{J}_2 \leq C \int_{\Omega} |\Delta_{s,-h}(\rho \nabla \rho \odot \Delta_{s,h} \tilde{u}_k)| dx + \left| \int_{\Omega} \langle \Delta_{s,h} g'_k(\varepsilon(u_k)), 2\rho \nabla \rho \odot \Delta_{s,h} \tilde{u}_k \rangle dx \right| =: \mathbf{J}_2^{(1)} + \mathbf{J}_2^{(2)}.$$

Ad  $\mathbf{J}_2^{(1)}$ . Let  $1 < \tilde{q} < 2$  to be fixed later. We find by recalling the local uniform boundedness of  $(\Delta_{s,h} u_k)$  in  $L^1$ , estimating difference quotients against differentials and Young's inequality

$$(3.42) \quad \begin{aligned} \mathbf{J}_2^{(1)} &\leq C \int_{\Omega} |\partial_s(\nabla \rho \odot \rho \Delta_{s,h} \tilde{u}_k)| dx \\ &\leq C(\rho) + C(\rho) \int_{\Omega} |\partial_s(\rho \Delta_{s,h} \tilde{u}_k)| dx \\ &\leq C(\rho) + C(\rho) \mathcal{L}^n(\Omega)^{\tilde{q}'} + C(\rho) \int_{\Omega} |\partial_s(\rho \Delta_{s,h} \tilde{u}_k)|^{\tilde{q}} dx \\ &\leq C(\rho) + C(\rho) \mathcal{L}^n(\Omega)^{\tilde{q}'} + C(\rho, \tilde{q}) \int_{\Omega} |\varepsilon(\rho \Delta_{s,h} \tilde{u}_k)|^{\tilde{q}} dx \quad (\text{by Korn}) \\ &\leq C(\rho, \Omega, \tilde{n}) + C(\rho, \tilde{q}) \int_{B(x_0, R)} |\Delta_{s,h} \tilde{u}_k|^{\tilde{q}} dx + C(\rho, \tilde{q}) \int_{\Omega} |\rho \Delta_{s,h} \varepsilon(u_k)|^{\tilde{q}} dx \\ &=: \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3. \end{aligned}$$

If we choose  $\tilde{q}$  sufficiently close to 1, then we are in position to utilise the fact that  $u_k \in W_{\text{loc}}^{1,q}$  uniformly in  $k$  with  $q$  provided by (3.38) and hence can assume without loss of generality that  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are uniformly bounded with respect to  $k$ . At this stage we fix  $\tilde{q}$  as follows. Because of  $\frac{\tilde{q}}{2-\tilde{q}} \searrow 1$  as  $\tilde{q} \searrow 1$  and (3.39), we find  $\tilde{q} > 1$  such that for some  $q$

$$(3.43) \quad \mu \leq \mu \frac{\tilde{q}}{2-\tilde{q}} < q < (3-\mu) \frac{n}{2n-1}.$$

Then, by (3.38), we have local uniform boundedness of  $(\varepsilon(v_k))$  in  $L^q$  and thus in  $L^{\mu\tilde{q}/(2-\tilde{q})}$ . Since  $2/(2-\tilde{q})$  is the Hölder conjugate of  $\frac{2}{\tilde{q}}$ , we then find by Young's inequality for  $\delta > 0$

$$(3.44) \quad \begin{aligned} \mathbf{K}_3 &\leq \delta C(\rho, \tilde{q}) \int_{\Omega} \frac{|\rho \Delta_{s,h} \varepsilon(u_k)|^2}{(1 + |\varepsilon(u_k)(x + h e_s)|^2 + |\varepsilon(u_k)|^2)^{\frac{\mu}{2}}} dx \\ &\quad + C(\delta, \rho, \tilde{q}) \int_{B(x_0, R)} (1 + |\varepsilon(u_k)(x + h e_s)|^2 + |\varepsilon(u_k)|^2)^{\frac{\mu}{2} \frac{\tilde{q}}{2-\tilde{q}}} dx. \end{aligned}$$

Now observe that by (3.43) and the remark afterwards, the ultimate term can be bounded independently of  $k$ . Then we choose  $0 < \delta < c$ , where  $c > 0$  now is the constant on the left hand side of (3.41) and absorb it into the left hand side of (3.41).

Ad  $\mathbf{J}_2^{(2)}$ . By definition of  $g_k$ , we consequently obtain by Young's inequality

$$\begin{aligned} \mathbf{J}_2^{(2)} &\leq \frac{1}{k^2 A_k} \int_{\Omega} |\langle \Delta_{s,h} \varepsilon(u_k), 2\rho \nabla \rho \odot \Delta_{s,h} \tilde{u}_k \rangle| dx \\ &\leq \frac{1}{2k^2 A_k} \int_{\Omega} |\rho \Delta_{s,h} \varepsilon(u_k)|^2 dx + \frac{2C(\rho)}{k^2 A_k} \int_{B(x_0, R)} |\Delta_{s,h} \tilde{u}_k|^2 dx. \end{aligned}$$

We then absorb the first term on the very right hand side into the left hand side of (3.41). Moreover, by Korn's inequality, we see similarly as in the proof of Theorem 1.2 that the second term on the right hand side of the previous inequality is bounded uniformly in  $k$ . Finally, for  $\mathbf{J}_3$  we recall the fact that because  $(u_k)$  is locally uniformly bounded in  $W^{1,q}$  with  $q > 1$ , it is locally uniformly bounded in  $W^{1,1}$ . This yields uniform boundedness of  $\mathbf{J}_3$ . Summarising the estimates obtained so far, we come up with

$$(3.45) \quad \sup_{k \in \mathbb{N}} \int_{\Omega} \frac{|\rho \Delta_{s,h} \varepsilon(u_k)|^2}{(1 + |\varepsilon(u_k)(x + h e_s)|^2 + |\varepsilon(u_k)|^2)^{\frac{\mu}{2}}} dx < \infty.$$

Eventually, repeating the argument that lead to (3.44), we easily find that for some  $Q > 1$  there holds  $\sup_{k \in \mathbb{N}} \|\Delta_{s,h} \varepsilon(u_k)\|_{L^Q(B(x_0, r); \mathbb{R}^{n \times n})} < \infty$ . From here and the arbitrariness of  $x_0$  and  $r$  we deduce by standard means that  $\varepsilon(\partial_s u) = \partial_s \varepsilon(u) \in L_{\text{loc}}^Q(\Omega; \mathbb{R}^{n \times n})$ . Then Korn's inequality yields that  $\partial_s u \in W_{\text{loc}}^{1,Q}(\Omega; \mathbb{R}^n)$  and so, by arbitrariness of  $s \in \{1, \dots, n\}$ ,  $u \in W_{\text{loc}}^{2,Q}(\Omega; \mathbb{R}^n)$ . The proof is complete.  $\square$

A standard application of the measure density lemma [28, Prop. 2.7] then yields the following bound on the Hausdorff dimension of the set of non-Lebesgue points of  $\varepsilon(u)$  as will be needed in a forthcoming study [30]:

**Corollary 3.9.** *Let  $1 < \mu < \frac{3n}{3n-1}$  and let  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  be a  $\mu$ -elliptic integrand of linear growth. Then for any  $u \in \text{GM}(\mathfrak{F}; u_0)$  there holds  $\dim_{\mathcal{H}}(\Sigma_u) < n - 1$ , where*

$$(3.46) \quad \Sigma_u := \left\{ x_0 \in \Omega : \limsup_{r \searrow 0} \int_{B(x_0, r)} |\varepsilon(u) - z| d\mathcal{L}^n > 0 \text{ for all } z \in \mathbb{R}_{\text{sym}}^{n \times n} \right\}.$$

**3.4. Uniqueness of Generalised Minimisers.** A consequence of Theorem 1.2 is the following result on the uniqueness of generalised minimisers. Similarly to functionals of linear growth depending on the gradient (see [8, Sec. 5]), uniqueness of generalised minimisers can only be obtained modulo rigid deformations, that is, elements of the nullspace of  $\varepsilon$ :

**Theorem 3.10 (Uniqueness).** *Let  $f: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$  be a  $\mu$ -elliptic integrand of linear growth with  $1 < \mu < \frac{n+1}{n}$ . Suppose that  $\Omega$  is an open, bounded and connected Lipschitz subset of  $\mathbb{R}^n$ . Then any two generalised minimisers  $u, v \in \text{GM}(\mathfrak{F}; u_0)$  differ by a rigid deformation, that is, there exists  $R \in \mathcal{R}(\Omega)$  such that  $u = v + R$  holds  $\mathcal{L}^n$ -a.e. in  $\Omega$ .*

*Proof.* Let  $u, v \in \text{GM}(\mathfrak{F}; u_0)$  be two generalised minimisers with respect to a prescribed Dirichlet class  $\mathcal{D}_{u_0} := u_0 + \text{LD}_0(\Omega)$ . Since  $f$  is  $\mu$ -elliptic with  $1 < \mu \leq 1 + \frac{1}{n}$ , both  $u$  and  $v$  belong to  $\text{LD}(\Omega)$  by Theorem 1.2. We will show  $\varepsilon(u) = \varepsilon(v)$ , and this will imply the claim: Indeed, since  $\varepsilon(w) = 0$  is equivalent to  $w \in \mathcal{R}(\Omega)$  provided  $\Omega$  is connected, we deduce that there exists  $R \in \mathcal{R}(\Omega)$  such that  $u = v + R$ . To prove the claim, suppose that  $\varepsilon(u) \neq \varepsilon(v)$  on a measurable set  $U$  with  $\mathcal{L}^n(U) > 0$ . Then we obtain, using that  $f$  is strictly convex and both  $E^s u$  and  $E^s v$  vanish identically in  $\Omega$ ,

$$\bar{\mathfrak{F}}\left[\frac{1}{2}(u+v)\right] < \frac{1}{2}(\bar{\mathfrak{F}}[u] + \bar{\mathfrak{F}}[v]) = \min \bar{\mathfrak{F}}[\text{BD}(\Omega)],$$

an obvious contradiction. The proof is complete.  $\square$

Building on the results of the previous sections, particularly to the proof of the higher Sobolev regularity of generalised minimisers, we now briefly comment on the uniqueness issues addressed in the introduction. In general, the failure of uniqueness of minima of variational integrals (1.4) is mostly due to two reasons (compare [8]): Going back to the relaxed functional  $\tilde{\mathfrak{F}}$  given by (1.6), positive homogeneity of  $f^\infty$  implies that  $f^\infty$  is not strictly convex even if  $f$  is. Thus a possible reason for non-uniqueness is the presence of the singular part of minimisers which genuinely only effects the recession parts of  $\tilde{\mathfrak{F}}$ . The second reason for non-uniqueness is a possible non-attainment of the correct boundary values which is partly addressed in

**Proposition 3.11.** *Let  $\Omega$  be a convex Lipschitz subset of  $\mathbb{R}^n$ . Suppose that generalised minima of the variational integral  $\tilde{\mathfrak{F}}$  given by (1.4) are unique modulo rigid deformations. If one generalised minimiser  $u$  attains the correct boundary values in the sense that  $\text{Tr}(u - u_0) = 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial\Omega$ , then  $\text{GM}(\tilde{\mathfrak{F}}; u_0) = \{u\}$ .*

*Proof.* Let  $R \in \mathcal{R}(\Omega) \setminus \{0\}$  be an arbitrary non-zero rigid deformation and denote  $\bar{R}$  its continuous extension to  $\bar{\Omega}$ . Then we have

$$(3.47) \quad \tilde{\mathfrak{F}}[u + R] = \tilde{\mathfrak{F}}[u] + \int_{\partial\Omega} f^\infty(-\bar{R} \odot \nu_{\partial\Omega}) d\mathcal{H}^{n-1}$$

because  $\text{Tr}(u - u_0) = 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial\Omega$ . Since the mapping  $T: \partial\Omega \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$  given by  $T(x) := -\bar{R} \odot \nu_{\partial\Omega}$  for  $x \in \partial\Omega$  is continuous and  $f^\infty: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}_{\geq 0}$  is continuous too, it suffices to show that there exists  $z \in \partial\Omega$  such that  $|\bar{R}(z) \odot \nu_{\partial\Omega}(z)| > 0$ . Indeed, in this case we conclude by homogeneity of  $f^\infty$  and positivity of  $f$  that the boundary integral on the right side of (3.47) is strictly positive so that  $u + R$  is not a minimiser of  $\tilde{\mathfrak{F}}$  over  $\text{BD}(\Omega)$ . The proof is then concluded by Proposition 4.2 which provides the required characterisation of generalised minima in terms of  $\tilde{\mathfrak{F}}$ . For simplicity, we shall argue for the unit ball  $\Omega = B$  and only sketch the respective generalisation to arbitrary open Lipschitz domains  $\Omega$  below. Write  $\bar{R}(z) = Az + b$ . If  $|\bar{R}(z) \odot \nu_{\partial B}(z)| = 0$  for all  $z \in \partial B$ , then  $Az \odot \nu_{\partial B}(z) = -b \odot \nu_{\partial B}(z)$  for all  $z \in \partial B$ . Since  $\nu_{\partial B}(z) = z$  for any  $z \in \partial B$ , this particularly implies  $Ae_k \odot e_k = -b \odot e_k$  for all  $k = 1, \dots, n$ . These identities imply

$$Ae_k \odot e_k = \frac{1}{2} \begin{pmatrix} 0 & \dots & a_{1k} & 0 & \dots \\ 0 & \dots & \vdots & 0 & \dots \\ a_{1k} & \dots & 2a_{kk} & \dots & a_{nk} \\ 0 & \dots & \vdots & 0 & \dots \\ 0 & \dots & a_{nk} & 0 & \dots \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & \dots & b_1 & 0 & \dots \\ 0 & \dots & \vdots & 0 & \dots \\ b_1 & \dots & 2b_k & \dots & b_n \\ 0 & \dots & \vdots & 0 & \dots \\ 0 & \dots & b_n & 0 & \dots \end{pmatrix} = -b \odot e_k.$$

and hence  $a_{jk} = -b_j$  for all  $j, k = 1, \dots, n$ . In consequence,  $a_{jj} = -b_j$  for all  $j = 1, \dots, n$ , but by scew-symmetry of  $A$ ,  $a_{jj} = 0$  and thus  $b_j = 0$  for all  $j = 1, \dots, n$ . This further implies  $a_{jk} = 0$  for all  $j, k = 1, \dots, n$  and thus  $\bar{R} \equiv 0$ . If  $\Omega$  is not a ball, then one may argue similarly, now using the fact that for any open, bounded and convex Lipschitz subset  $\Omega$  of  $\mathbb{R}^n$  there exist linearly independent  $z_1, \dots, z_n \in \partial\Omega$  such that  $\nu_{\partial\Omega}(z_1), \dots, \nu_{\partial\Omega}(z_n)$  are linearly independent too. The details are left to the interested reader.  $\square$

The previous lemma is an adaptation of [8, Lem. 5.5] to the symmetric gradient situation. Finally, the second possible source of non-uniqueness is given by the boundary behaviour of generalised minima. This is in the spirit of SANTI's example [47] which has been revisited and adapted to the vectorial case by BECK & SCHMIDT (cf. [8, Thm. 1.17]). As such, we believe that is possible by a similar adaptation as has been given in Proposition 3.11 above to generalise [8, Thm. 1.16] to the symmetric gradient situation. More precisely, we conjecture that if  $f: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$  is a convex integrand with (1.2) such that for every  $\eta \in \mathbb{R}^n \setminus \{0\}$ ,  $f_\eta: \xi \mapsto f^\infty(\eta \odot \xi)$  is a strictly convex norm<sup>2</sup> and if generalised minima are unique modulo

<sup>2</sup>in the sense that if  $f_\eta(\xi_1) = f_\eta(\xi_2) = f_\eta(\lambda\xi_1 + (1-\lambda)\xi_2)$  for  $\xi_1, \xi_2 \in \mathbb{R}^n$  and  $0 < \lambda < 1$ , then  $\xi_1 = \xi_2$ .

rigid deformations, then the set of all of generalised minima can be written as

$$\text{GM}(\mathfrak{F}; u_0) = \{u + \lambda R : -1 \leq \lambda \leq 1\}$$

for some fixed  $u \in \text{GM}(\mathfrak{F}; u_0)$  and  $R \in \mathcal{R}(\Omega)$ . However, the verification of this is beyond the scope of this paper and shall be addressed in a future work.

#### 4. APPENDIX

**4.1. Extensions of Theorem 1.2 to nonautonomous problems.** Let us now briefly comment on the situation where  $f$  has additional  $x$ -dependence. If  $f \in C^2(\bar{\Omega} \times \mathbb{R}_{\text{sym}}^{n \times n})$  is an integrand that satisfies essentially the assumptions of [9, Ass. 4.22], that is,  $f$  satisfies (1.2) uniformly in  $x$  together with

$$(4.1) \quad \begin{cases} \sup_{x \in \bar{\Omega}} \sup_{\xi \in \mathbb{R}_{\text{sym}}^{n \times n}} \max\{|\mathcal{D}_\xi f(x, \xi)|, |\mathcal{D}_x^2 \mathcal{D}_\xi f(x, \xi)|, |\mathcal{D}_x \mathcal{D}_\xi f(x, \xi)|\} < \infty, \\ \lambda \frac{|\xi|^2}{(1+|z|^2)^{\mu/2}} \leq \langle \mathcal{D}_\xi^2 f(z) \xi, \xi \rangle \leq \Lambda \frac{|\xi|^2}{(1+|z|^2)^{1/2}}, \\ |\langle \mathcal{D}_x \mathcal{D}_\xi^2 f(x, \xi) \eta, \eta' \rangle| \leq C(|\langle \mathcal{D}_\xi^2 f(x, \xi) \eta, \eta' \rangle| + |\eta| |\eta'| / (1 + |\xi|^2)) \end{cases}$$

for all  $x \in \bar{\Omega}$ ,  $\eta, \eta', \xi, z \in \mathbb{R}_{\text{sym}}^{n \times n}$ , then the results of this paper carry over in a straightforward manner to the situation of interest; in fact, as we work with finite differences, these assumptions can even be weakened, but this is left to the interested reader; also see the discussion in [8, App. C]. If the smoothness of the  $x$ -dependence is diminished, a merger of the arguments outlined in this work with [37] leads to the correspondingly modified theorems.

**4.2. Proofs of auxiliary results.** In this section, we provide the proofs of minor auxiliary results used in the main body of the paper. We begin with the

*Proof of Proposition 2.2.* Since the embedding we aim for is of local nature, it suffices to give estimates for maps  $u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  which are compactly supported in a given ball  $B = B(x_0, R)$ ; the full statement then follows by smooth approximation and is left to the reader. Let  $0 < \alpha < 1$  be given. We have for all  $0 < s < 1$

$$[u]_{W^{s, \frac{n}{n-1+s}}(\mathbb{R}^n; \mathbb{R}^n)} \leq C(n, s) \|\varepsilon(u)\|_{L^1(\mathbb{R}^n; \mathbb{R}_{\text{sym}}^{n \times n})},$$

since the symmetric gradient operator is elliptic and cancelling, cf. [53, Thm. 8.1, Prop. 6.4]. Now, for all  $h \in \mathbb{R}^n$  with  $|h| < 1$  sufficiently small,  $u(x+h)$  is supported in  $B(x_0, 2R)$ . Then, by Hölder's inequality and some  $\alpha < t < 1$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x+h) - u(x)| dx &\leq c(R, n, t) \left( \int_{B(x_0, 2R)} |u(x+h) - u(x)|^{\frac{n}{n-1+t}} dx \right)^{\frac{n-1+t}{n}} \\ &\leq C(R, n, t) \|u\|_{B_{\frac{n}{n-1+t}, \infty}(\mathbb{R}^n; \mathbb{R}^n)} |h|^t \\ &\leq C(R, n, t) \|u\|_{W^{t, \frac{n}{n-1+t}}(\mathbb{R}^n; \mathbb{R}^n)} |h|^t \\ &\leq C(R, n, t) \|\varepsilon(u)\|_{L^1(\mathbb{R}^n; \mathbb{R}_{\text{sym}}^{n \times n})} |h|^t. \end{aligned}$$

But since  $t > \alpha$  and  $|h| < 1$ , we obtain  $u \in B_{1, \infty}^t(\mathbb{R}^n; \mathbb{R}^n)$  and standard inclusions between Besov and Sobolev-Slobodeckij spaces imply that  $[u]_{W^{\alpha, 1}(\mathbb{R}^n; \mathbb{R}^n)} \leq C \|\varepsilon(u)\|_{L^1(\mathbb{R}^n; \mathbb{R}_{\text{sym}}^{n \times n})}$  and the proof is complete.  $\square$

*Proof of Lemma 2.4.* By [1, Lemma 2.2], for every  $-\frac{1}{2} < \gamma < 0$  there exists a constant  $c = c(M) > 0$  such that for all  $\xi, \eta \in \mathbb{R}^M$  there holds

$$(2\gamma + 1)|\xi - \eta| \leq \frac{|(1 + |\xi|^2)^\gamma \xi - (1 + |\eta|^2)^\gamma \eta|}{(1 + |\xi|^2 + |\eta|^2)^\gamma} \leq \frac{c(M)}{2\gamma + 1} |\xi - \eta|.$$

Applying this to  $\gamma = (1 - \alpha)/2$  yields the claim as  $-\frac{1}{2} < \gamma < 0$  if and only if  $1 < \alpha < 2$ .



Now let  $\xi \in \mathbb{R}^M$  with  $|\xi| \geq 1$  and let  $1 < \alpha < 2$ . Then  $(1 - \alpha)/2 < 0$ . Hence, since  $t \mapsto t^{(1-\alpha)/2}$  is monotonically decreasing on  $(0, \infty)$ ,

$$\begin{aligned} |\xi| \geq 1 &\Rightarrow 2|\xi|^2 \geq 1 + |\xi|^2 \Rightarrow 2^{\frac{1-\alpha}{2}} |\xi|^{1-\alpha} \leq (1 + |\xi|^2)^{\frac{1-\alpha}{2}} \\ &\Rightarrow |\xi|^{2-\alpha} \leq 2^{\frac{\alpha-1}{2}} |V_\alpha(\xi)| \stackrel{1 < \alpha < 2}{\leq} \sqrt{2} |V_\alpha(\xi)|. \end{aligned}$$

Since  $1 < \alpha < 2$  and  $|\xi| \geq 1$ , we have  $|\xi|^{2-\alpha} \leq |\xi|$  and thus  $\min\{|\xi|, |\xi|^{2-\alpha}\} \leq \sqrt{2} |V_\alpha(\xi)|$  in this case. Now, if  $|\xi| < 1$ , then

$$2^{\frac{1-\alpha}{2}} \leq (1 + |\xi|^2)^{\frac{1-\alpha}{2}} \Rightarrow 2^{\frac{1-\alpha}{2}} |\xi| \leq |V_\alpha(\xi)| \Rightarrow |\xi| \leq \sqrt{2} |V_\alpha(\xi)|,$$

and hence  $\min\{|\xi|, |\xi|^{2-\alpha}\} \leq \sqrt{2} |V_\alpha(\xi)|$  holds, too. Therefore  $\min\{|\xi|, |\xi|^{2-\alpha}\} \leq \sqrt{2} |V_\alpha(\xi)|$  holds for all  $\xi \in \mathbb{R}^M$ . Lastly if the measurable map  $u: \Omega \rightarrow \mathbb{R}^M$  is such that  $V_\alpha(u) \in L^p(\Omega; \mathbb{R}^m)$ , then we have

$$\begin{aligned} \int_{\Omega} |u|^{(2-\alpha)p} dx &= \int_{\Omega \cap \{|u| \leq 1\}} |u|^{(2-\alpha)p} dx + \int_{\Omega \cap \{|u| > 1\}} |u|^{(2-\alpha)p} dx \\ &\leq \mathcal{L}^n(\Omega) + c(p) \int_{\Omega} |V_\alpha(u)|^p dx \end{aligned}$$

The proof is complete.  $\square$

**4.3. Relaxation.** As mentioned in the introduction, we now give justification for some results used in the main part of the paper. The primary aim of this section is to explain formula (1.6) and the existence of generalised minima. We thus recap the requisite foundational theory of functions of measures as exposed in [3, Section 2.6], see also [21, 4].

**4.3.1. Convex Functions of Measures.** Given  $m \in \mathbb{N}$ , let  $f: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$  be a convex function of linear growth, i.e.,  $f$  satisfies (1.2) with the obvious modifications. In this situation, it can be shown that  $f^\infty$  defined by (1.5) is well-defined, convex and positively 1-homogeneous. Let  $\lambda$  be an  $\mathbb{R}^m$ -valued Radon measure of finite total variation on an open and bounded set  $\Omega \subset \mathbb{R}^n$ . We denote

$$\lambda = \lambda^a + \lambda^s = \frac{d\lambda}{d\mathcal{L}^n} \mathcal{L}^n + \frac{d\lambda}{d|\lambda^s|} |\lambda^s|$$

its Radon–Nikodým decomposition into its absolutely continuous and singular parts  $\lambda^a, \lambda^s$  with respect to Lebesgue measure, where  $|\lambda^s|$  denotes the total variation measure of  $\lambda^s$ . We then define a new Radon measure  $f[\lambda]$  by

$$(4.2) \quad f[\lambda](A) := \int_A f \left( \frac{d\lambda}{d\mathcal{L}^n} \right) d\mathcal{L}^n + \int_A f^\infty \left( \frac{d\lambda}{d|\lambda^s|} \right) d|\lambda^s|, \quad A \in \mathcal{B}(\Omega),$$

where  $\mathcal{B}(\Omega)$  denotes the Borel– $\sigma$ -algebra on  $\Omega$ . We note that, by positive 1-homogeneity of  $f^\infty$ , this gives rise to a well-defined measure indeed. Linking this to the area functional as required for the definition of area-strict convergence, for a given map  $u \in \text{BD}(\Omega)$ , we have with  $f := \sqrt{1 + |\cdot|^2}$  that  $\sqrt{1 + |Eu|^2}(\Omega) := f[Eu](\Omega)$ .

We turn to formula (1.6) for the relaxed functional as given for BV-functions in [31] and find by a straightforward applications of the results of GOFFMAN & SERRIN [32] that, given an open and bounded Lipschitz subset  $\Omega$  of  $\mathbb{R}^n$  together with a Dirichlet datum  $u_0 \in \text{LD}(\Omega)$ , we have

$$(4.3) \quad \bar{\mathfrak{F}}_{u_0}[u] = \inf \left\{ \liminf_{k \rightarrow \infty} \mathfrak{F}[u_k] : \begin{array}{l} (u_k) \subset \mathcal{D}_{u_0} := u_0 + \text{LD}_0(\Omega) \\ u_k \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^n) \end{array} \right\}.$$

We pick a ball  $B = B(z, R)$  with  $\Omega \Subset B$ . By surjectivity of the trace operator  $\text{Tr}: \text{LD}(U) \rightarrow L^1_{\mathcal{H}^{n-1}}(\partial U; \mathbb{R}^n)$  on bounded Lipschitz subsets  $U$  of  $\mathbb{R}^n$  (see Section 2.2) that there exists  $v_0 \in \text{LD}(B \setminus \bar{\Omega})$  such that  $\text{Tr}(v_0)|_{\partial B} = 0$  and  $\text{Tr}(v_0)|_{\partial \Omega} = \text{Tr}(u_0)|_{\partial \Omega}$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial B$  or  $\partial \Omega$ , respectively.

Given  $u \in \text{BD}(\Omega)$ , we put

$$(4.4) \quad \tilde{u}(x) := \begin{cases} u(x) & \text{for } x \in \Omega, \\ v_0(x) & \text{for } x \in B \setminus \bar{\Omega}. \end{cases}$$

Then there holds  $\tilde{u} \in \text{BD}(B)$ , and by the interior trace theorem as recalled in Section 2.2 we have

$$\begin{aligned} E\tilde{u} &= E^a\tilde{u} + E^s\tilde{u} = E^a u \llcorner \Omega + E^s u \llcorner \Omega + E^s \tilde{u} \llcorner \partial\Omega + E^a \tilde{u} \llcorner (B \setminus \bar{\Omega}) \\ &= \mathcal{E} u \mathcal{L}^n \llcorner \Omega + \frac{dEu}{d|E^s u|} |E^s u| + \text{Tr}(v_0 - u) \odot \nu_{\partial\Omega} \mathcal{H}^{n-1} \llcorner \partial\Omega + \mathcal{E} v_0 \mathcal{L}^n \llcorner (B \setminus \bar{\Omega}). \end{aligned}$$

We insert this expression for  $\lambda = E\tilde{u}$  with  $A = B$  into (4.2) (which equally holds for measures  $\lambda$  on  $\mathbb{R}^n$  subject to the obvious modifications) and obtain

$$(4.5) \quad \begin{aligned} f[E\tilde{u}](B) &= \int_{\Omega} f(\mathcal{E} u) d\mathcal{L}^n + \int_{\Omega} f\left(\frac{dEu}{d|E^s u|}\right) d|E^s u| \\ &\quad + \int_{\partial\Omega} f^\infty(\text{Tr}(v_0 - u) \odot \nu_{\partial\Omega}) d\mathcal{H}^{n-1} + \int_{B \setminus \bar{\Omega}} f(\mathcal{E} v_0) d\mathcal{L}^n. \end{aligned}$$

If we then aim for minimising  $f[E\tilde{u}](B)$  over all  $u \in \text{BD}(\Omega)$ , we see by constancy of the very last term of the preceding expression that it does not affect the minimiser  $v \in \text{BD}(\Omega)$ , and thus a function  $v \in \text{BD}(\Omega)$  minimises  $f[E\tilde{u}](B)$  if and only if it minimises the relaxed functional given by (1.6).

We conclude this section by recalling two results due to Reshetnyak concerning the (lower semi-)continuity of convex functions of measures in the version as given in [8].

**Proposition 4.1** (RESHETNYAK, [39]). *Let  $m \in \mathbb{N}$ ,  $\Omega \subset \mathbb{R}^n$  open and let  $(\lambda_k)$  be a sequence of  $\mathbb{R}^m$ -valued Radon measures of finite total variation which converges to a  $\mathbb{R}^m$ -valued Radon measure of finite total variation  $\lambda$  on  $\Omega$  in the weak\*-sense. Moreover, assume that all measures  $\lambda_k$  and  $\lambda$  take values in some closed convex cone  $K \subset \mathbb{R}^m$ . Then the following holds:*

- (a) *Lower Semicontinuity. If  $\tilde{f}: K \rightarrow [0, \infty]$  is a lower semicontinuous function, then there holds*

$$\int_{\Omega} \tilde{f}\left(\frac{d\lambda}{d|\lambda|}\right) d|\lambda| \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \tilde{f}\left(\frac{d\lambda_k}{d|\lambda_k|}\right) d|\lambda_k|.$$

- (b) *If  $\lambda_k \rightarrow \lambda$  strictly<sup>3</sup> as  $k \rightarrow \infty$  and  $\tilde{f}: K \rightarrow [0, \infty)$  is a continuous and 1-homogeneous function, then there holds*

$$\int_{\Omega} \tilde{f}\left(\frac{d\lambda}{d|\lambda|}\right) d|\lambda| = \lim_{k \rightarrow \infty} \int_{\Omega} \tilde{f}\left(\frac{d\lambda_k}{d|\lambda_k|}\right) d|\lambda_k|.$$

4.3.2. *Generalised Minima: Existence and Characterisations.* We now pass on to the verification of (4.3) and establish the existence of generalised minima.

**Proposition 4.2.** *Let  $\Omega$  be an open and bounded Lipschitz subset of  $\mathbb{R}^n$ . Given a convex integrand  $f: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$  with (1.2) and a boundary datum  $u_0 \in \text{LD}(\Omega)$ , define  $\mathfrak{F}_{u_0}$  by (1.6). Then there exists a generalised minimiser of  $\mathfrak{F}$  in the sense of (1.7).*

*Moreover, the following are equivalent for  $u \in \text{BD}(\Omega)$ :*

- (a)  *$u$  is a generalised minimiser in the sense of (1.7).*
- (b)  *$u$  is the weak\*-limit of an  $\mathfrak{F}$ -minimising sequence  $(u_k) \subset \mathcal{D}_{u_0} (:= u_0 + \text{LD}_0(\Omega))$ .*
- (c)  *$u$  is the strong  $L^1$ -limit of an  $\mathfrak{F}$ -minimising sequence  $(u_k) \subset \mathcal{D}_{u_0}$ .*

<sup>3</sup>In the sense that  $\lambda_k \xrightarrow{*} \lambda$  and  $|\lambda_k|(\Omega) \rightarrow |\lambda|(\Omega)$  as  $k \rightarrow \infty$ .

*Proof.* We begin with a preparatory remark. We choose an open and bounded Lipschitz subset  $\tilde{\Omega} \subset \mathbb{R}^n$  with  $\Omega \Subset \tilde{\Omega}$ . Given  $u_0 \in \text{LD}(\Omega)$ , by surjectivity of the trace operator on LD (see Section 2.2), we may extend  $u_0$  by some  $v_0 \in \text{LD}(B \setminus \tilde{\Omega})$  to a function  $\tilde{u}_0 \in \text{LD}_0(\tilde{\Omega})$ . We now invoke the straightforward generalisation of [9, Chpt. 2.3.1] whose proof we leave to the interested reader:

*Given  $\tilde{\Omega}$  and  $u_0$  as above, let  $u \in \text{BD}(\Omega)$  and denote its extension to  $\tilde{\Omega}$  via  $\tilde{u}_0$  by  $\tilde{u}$ . Then there exists  $(u_k) \subset u_0 + C_c^\infty(\Omega; \mathbb{R}^n)$  such that  $\tilde{u}_k \rightarrow \tilde{u}$  area-strictly in  $\tilde{\Omega}$  as  $k \rightarrow \infty$ , where  $\tilde{u}_k, \tilde{u}$  denote the extensions of  $u_k, u$  to  $\tilde{\Omega}$  by  $\tilde{u}_0$ , respectively.*

We turn to the actual proof, and choose  $\tilde{\Omega} \equiv B$  as above before (4.5).

*Step 1. Existence of a generalised minimiser.* By (1.2) we have  $m := \inf_{u \in \text{BD}(\Omega)} f[E\tilde{u}](B) > -\infty$  and so there exists a sequence  $(u_k) \subset \text{BD}(\Omega)$  and  $v \in \text{BD}(B)$  such that  $\tilde{u}_k \xrightarrow{*} v$  in  $\text{BD}(B)$  as  $k \rightarrow \infty$ . By Proposition 4.1(a),  $f[E\tilde{u}](B) \leq \liminf_{k \rightarrow \infty} f[E\tilde{u}_k](B) = m$ . Since  $\tilde{u}_k|_{B \setminus \tilde{\Omega}} = v_0$ ,  $v|_{B \setminus \tilde{\Omega}} = v_0$  and so we conclude from (4.5) that  $u := v|_\Omega$  is a generalised minimiser in the sense of (1.7). Now, since  $\mathcal{D}_{u_0} \subset \text{BD}(\Omega)$  and  $\tilde{\mathfrak{F}}_{u_0}|_{\mathcal{D}_{u_0}} = \mathfrak{F}|_{\mathcal{D}_{u_0}}$ , we have  $\inf_{\text{BD}(\Omega)} \tilde{\mathfrak{F}}_{u_0} \leq \inf_{\mathcal{D}_{u_0}} \mathfrak{F}$ . On the other hand, let  $u \in \text{GM}(\mathfrak{F}; u_0)$  and apply the above area-strict approximation strategy to obtain a sequence  $(u_k) \subset u_0 + C_c^\infty(\Omega; \mathbb{R}^n)$  such that  $\tilde{u}_k \rightarrow \tilde{u}$  area-strictly as  $k \rightarrow \infty$ . Then we obtain by (4.1) – as the ultimate term on the right side of (4.5) is constant –

$$\begin{aligned} \tilde{\mathfrak{F}}_{u_0}[u] &= f[E\tilde{u}](B) - \int_{B \setminus \tilde{\Omega}} f(\mathcal{E}v_0) \, dx \\ (4.6) \quad &= \lim_{k \rightarrow \infty} f[E\tilde{u}_k](B) - \int_{B \setminus \tilde{\Omega}} f(\mathcal{E}v_0) \, dx = \lim_{k \rightarrow \infty} \mathfrak{F}[u_k] \geq \inf_{\mathcal{D}_{u_0}} \mathfrak{F}. \end{aligned}$$

Altogether, we have therefore established the absence of gaps, i.e.,

$$(4.7) \quad \min_{\text{BD}(\Omega)} \tilde{\mathfrak{F}}_{u_0} = \inf_{\text{BD}(\Omega)} \tilde{\mathfrak{F}}_{u_0} = \inf_{\mathcal{D}_{u_0}} \mathfrak{F}.$$

*Step 2. Proof of the claimed equivalences.* Ad (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c). Let  $u \in \text{GM}(\mathfrak{F}; u_0)$  and choose an area-strictly approximating sequence  $(u_k) \subset u_0 + C_c^\infty(\Omega; \mathbb{R}^n)$  as indicated. Then, employing formula (4.5) with  $\tilde{\Omega} \equiv B$ , we obtain  $f[E\tilde{u}_k](B) \rightarrow f[E\tilde{u}](B)$  by virtue of Proposition 4.1. By constancy of the ultimate term in (4.5) and the fact that area-strict convergence implies both  $L^1$ - and weak\*-convergence, we conclude by means of (4.7). Ad (b) $\Rightarrow$ (c). This follows trivially as weak\*-convergence implies strong  $L^1$ -convergence. Ad (c) $\Rightarrow$ (a). Let  $(u_k) \subset \mathcal{D}_{u_0}$  be an  $\mathfrak{F}$ -minimising sequence. By (4.3), we obtain for all  $v \in \text{BD}(\Omega)$

$$(4.8) \quad \tilde{\mathfrak{F}}_{u_0}[u] \leq \liminf_{k \rightarrow \infty} \mathfrak{F}[u_k] = \inf_{\mathcal{D}_{u_0}} \mathfrak{F} \stackrel{(4.7)}{=} \min_{\text{BD}(\Omega)} \tilde{\mathfrak{F}}_{u_0}.$$

The proof is complete.  $\square$

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