



# Classification of Finite Depth Objects in Bicommutant Categories via Anchored Planar Algebras

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**Abstract:** In our article [[arxiv:1511.05226](https://arxiv.org/abs/1511.05226)], we studied the commutant  $\mathcal{C}' \subset \text{Bim}(R)$  of a unitary fusion category  $\mathcal{C}$ , where  $R$  is a hyperfinite factor of type  $\text{II}_1$ ,  $\text{II}_\infty$ , or  $\text{III}_1$ , and showed that it is a bicommutant category. In other recent work [[arxiv:1607.06041](https://arxiv.org/abs/1607.06041), [arxiv:2301.11114](https://arxiv.org/abs/2301.11114)] we introduced the notion of a (unitary) anchored planar algebra in a (unitary) braided pivotal category  $\mathcal{D}$ , and showed that they classify (unitary) module tensor categories for  $\mathcal{D}$  equipped with a distinguished object. Here, we connect these two notions and show that finite depth objects of  $\mathcal{C}'$  are classified by connected finite depth unitary anchored planar algebras in  $\mathcal{Z}(\mathcal{C})$ . This extends the classification of finite depth objects of  $\text{Bim}(R)$  by connected finite depth unitary planar algebras.

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### 1. Introduction

Let  $R$  be a hyperfinite factor which is either of type  $\text{II}_1$ ,  $\text{II}_\infty$ , or  $\text{III}_1$ . And let  $X \in \text{Bim}(R)$  be a finite index (=dualizable) self-dual bimodule. Recall that  $X$  is said to have *finite depth* if the subcategory of  $\text{Bim}(R)$  that it generates is a fusion category: the object  $X$  has finite depth iff the total number of isomorphism classes of irreducible  $R$ - $R$ -bimodules which appear as summands of  $X^{\boxtimes n}$ ,  $n \in \mathbb{N}$ , is finite. The bimodule  $X$  is called *symmetrically self-dual* if it comes equipped with a unitary isomorphism  $r : X \xrightarrow{\cong} \bar{X}$ , which is fixed under the involution  $r \mapsto \bar{r}^* : \text{Hom}(X, \bar{X}) \xrightarrow{\sim}$ .

For  $R$  as above, there is a well-known correspondence (see [Pop90, Pop95, Izu17, Tom21], or [HP23, §3.2] for a review) between conjugacy classes of finite depth symmetrically self-dual bimodules, and isomorphism classes of unitary connected finite depth planar algebras:

$$\left\{ \begin{array}{l} \text{Finite depth, symm. self-dual} \\ R\text{-}R\text{-bimodules} \end{array} \right\} / \text{conj.} \leftrightarrow \left\{ \begin{array}{l} \text{Connected finite depth} \\ \text{unitary planar algebras} \end{array} \right\} / \text{iso.} \tag{1}$$

The above correspondence has been used heavily in the classification and construction of finite depth subfactors of small index [BMPS12, JMS14, AMP23]. In [HPT24, Conj. 1.2], we conjectured that this correspondence extends to a certain class of bicommutant categories, as follows:

**Conjecture 1.1.** *Let  $\mathcal{B}$  be a bicommutant category which admits an absorbing object whose endomorphism algebra is a hyperfinite  $\text{II}_\infty$  or  $\text{III}_1$  factor. Assume that  $Z(\mathcal{B}) \cong \text{Hilb}(\mathcal{V})$  for some unitary braided fusion category  $\mathcal{V}$ . Then there exists a natural bijective correspondence:*

$$\left\{ \begin{array}{l} \text{Finite depth objects of } \mathcal{B} \end{array} \right\} / \text{conj.} \leftrightarrow \left\{ \begin{array}{l} \text{Connected finite depth unitary} \\ \text{anchored planar algebras in } \mathcal{V} \end{array} \right\} / \text{iso.}$$

(Objects  $X, Y \in \mathcal{B}$  are called *conjugate* if there exists an invertible object  $U \in \mathcal{B}$  such that  $Y \cong UXU^{-1}$ .)

The case  $\mathcal{V} = \text{Hilb}_{\text{fd}}$  and  $\mathcal{B} = \text{Bim}(R)$  recovers the correspondence (1) above. (Since  $\text{Hilb}_{\text{fd}}$  is symmetric, anchored planar algebras in  $\text{Hilb}_{\text{fd}}$  are the same thing as planar algebras in  $\text{Hilb}_{\text{fd}}$ .)

The goal of this article is to provide evidence for the above conjecture, by proving it for certain bicommutant categories associated to unitary fusion categories. Let  $R$  be a hyperfinite factor of type  $\text{II}_1$ ,  $\text{II}_\infty$ , or  $\text{III}_1$ . The main result of this paper is as follows.

**Theorem A.** *Let  $\mathcal{C} \subset \text{Bim}(R)$  be a unitary fusion subcategory, embedded in  $\text{Bim}(R)$  via a fully faithful unitary tensor functor, and let  $\mathcal{C}'$  be its commutant category (see [HP17, HP23], or Definition 2.5 below). Then Conjecture 1.1 holds true for  $\mathcal{C}'$ .*

In this case,  $Z(\mathcal{C}') \cong \text{Hilb}(Z(\mathcal{C})^{\text{rev}})$  (Theorem 2.7), so finite depth objects of  $\mathcal{C}'$  correspond to connected finite depth unitary anchored planar algebras in  $Z(\mathcal{C})^{\text{rev}}$ . As one step in our proof, we prove in Theorem 3.9 that the multiplication map

$$\mathcal{C}' \boxtimes_{Z(\mathcal{C})} \mathcal{C} \xrightarrow{\cong} \text{Bim}(R)$$

is an equivalence. This result is somewhat surprising, given that its most straightforward analog one categorical level down fails.

*1.1. Applications.* One prominent example of a bicommutant category to which Conjecture 1.1 applies is  $\text{Rep}^k(\Omega G)$ , the category of locally normal representations of the based loop group  $\Omega G$  of some compact Lie group  $G$  (see [Hen17b] for the definition of locally normal representations). Here, the superscript  $k$  means that we are looking at projective representation of  $\Omega G$  with respect to a given cocycle (again, see loc. cit. for more details).

The initial impetus for the study of anchored planar algebras in [HPT23] was to study the tensor category

$$\text{Lines}(\chi \text{WZW}_{G,k}) \tag{2}$$

of unitary topological line defects in chiral WZW models (or, more generally, in any  $2d$  chiral conformal field theory), which is a (possibly non-full) subcategory of  $\text{Rep}^k(\Omega G)$ . These are topological lines defects inside the Chern-Simons/WZW bulk/boundary system that are entirely contained in the two-dimensional boundary. When restricting to those topological line defects that are compatible with the whole chiral symmetry on the boundary, the resulting category of lines has been studied in [FRS02], and turns out to just be the representation category of the affine VOA (or, more generally, the VOA which describes the given chiral conformal field theory). But if one cares about *all* unitary topological line defects, i.e., we only require compatibility with the Virasoro symmetry, then this category becomes much larger and, to our knowledge, it has never been studied in the literature. We conjecture that (2) is a bicommutant category, but that its absorbing object is not factorial (and thus does not satisfy the hypotheses of Conjecture 1.1).

A purely mathematical definition for the category (2) goes as follows. Fix a point  $p \in S^1$ . Let  $\mathcal{A}$  be the conformal net associated to the given  $2d$  chiral conformal field theory, and let  $\mathcal{A}_{\text{Vir}} \subset \mathcal{A}$  be its Virasoro sub-net. An object in (2) consists of a Hilbert space  $H$  together with actions of  $\mathcal{A}_{\text{Vir}}(I)$  for every interval  $I \subset S^1$ , and compatible actions of  $\mathcal{A}(I)$  for every interval  $I \subset S^1$  whose interior does not contain  $p$ .

We have an evident commuting diagram of unitary tensor functors

$$\begin{array}{ccc} \text{Rep}(\mathcal{A}) & \xrightarrow{\quad} & \text{Rep}^k(\Omega G) \\ & \searrow & \nearrow \\ & \text{Lines}(\chi \text{WZW}_{G,k}) & \end{array}$$

where the functors out of the unitary modular tensor  $\text{Rep}(\mathcal{A})$  are fully faithful. Here, the upwards functor just forgets the action of the algebras  $\mathcal{A}_{\text{Vir}}(I)$  for those intervals whose interior does not contain  $p$ .

As Conjecture 1.1 applies to  $\text{Rep}^k(\Omega G)$ , but not to  $\text{Lines}(\chi \text{WZW}_{G,k})$ , we hope that information gained about the classification of objects in  $\text{Rep}^k(\Omega G)$  could be leveraged to obtain some useful information about the more physically relevant problem of classifying objects in  $\text{Lines}(\chi \text{WZW}_{G,k})$ .

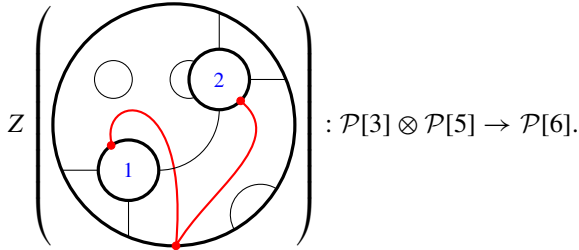
## 2. Background

In our companion paper [HPT24] we introduced the notion of a unitary anchored planar algebra, and we showed that there is a bijective correspondence between connected finite depth unitary anchored planar algebras and unitary module fusion categories equipped with the choice of a symmetrically self-dual object.

2.1. *Unitary anchored planar algebras and unitary module tensor categories.* We rapidly recall the notion of unitary anchored planar algebra from [HPT23, HPT24]. Let  $\mathcal{V}$  be a unitary ribbon fusion category. Its unique unitary spherical structure induces a twist denoted  $\theta_v : v \rightarrow v$ .

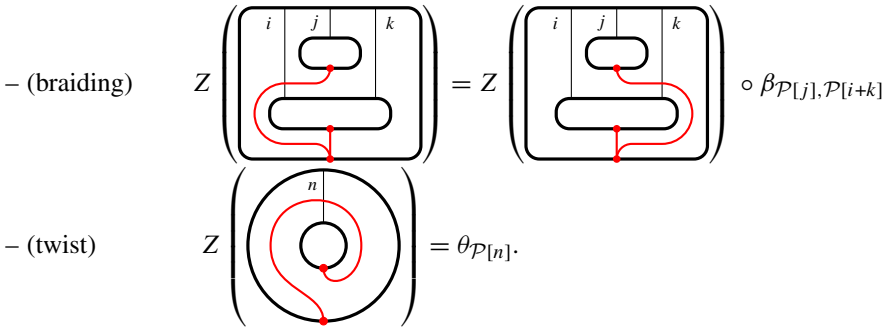
**Definition 2.1.** A unitary anchored planar algebra  $(\mathcal{P}, r, \psi)$  over  $\mathcal{V}$  consists of:

- (1) A sequence of *box objects*  $\mathcal{P}[n]$  of  $\mathcal{V}$  for each  $n \in \mathbb{N}_{\geq 0}$  together with a map  $Z(T)$  in  $\mathcal{V}$  from the tensor product of the input box objects to the output box object corresponding to each anchored planar tangle  $T$ . For example:



This data should satisfy:

- (identity) the identity anchored tangle acts as the identity morphism
- (composition) if  $S$  and  $T$  are composable anchored planar tangles at input disk  $i$  of  $S$ , then  $Z(S \circ_i T) = Z(S) \circ_i Z(T)$ .
- (anchor dependence) the following relations hold:



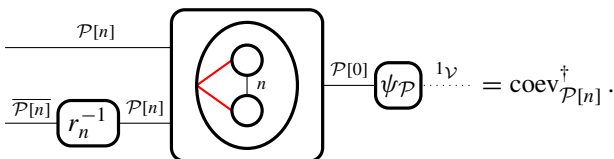
(Here, an  $n$  next to a string indicates  $n$  parallel strings.)

- (2) Real structures  $r_n : \mathcal{P}[n] \rightarrow \overline{\mathcal{P}[n]}$  satisfying for every anchored planar tangle  $T$

$$\overline{Z(T)} \circ (r \otimes \dots \otimes r) = r \circ Z(\overline{T}),$$

where  $\overline{T}$  is the reflection of  $T$ .

- (3) A morphism  $\psi_{\mathcal{P}} : \mathcal{P}[0] \rightarrow 1_{\mathcal{V}}$  satisfying  $\dots \text{[Diagram of } \mathcal{P}[0] \text{ box]} \dots = \text{id}_{1_{\mathcal{V}}}$ , and for every  $n \geq 0$ , we have an equality



We now rapidly recall the notion of a unitary module fusion category from [HPT16, HPT24]. Let  $\mathcal{V}$  be as above.

**Definition 2.2.** A unitary module fusion category  $(\mathcal{C}, \Phi^Z)$  consists of a unitary fusion category  $\mathcal{C}$  together with a pivotal braided unitary tensor functor  $\Phi^Z : \mathcal{V} \rightarrow Z(\mathcal{C})$ .

A generator of  $(\mathcal{C}, \Phi^Z)$  is an object  $c \in \mathcal{C}$  such that  $c$  and  $\Phi^Z(\mathcal{V})$  generate  $\mathcal{C}$  under the operations of taking tensor product, orthogonal direct sum, and orthogonal direct summands.

Our main theorems in [HPT23, HPT24] established an equivalence of categories

$$\left\{ \begin{array}{l} \text{Connected finite depth unitary} \\ \text{anchored planar algebras in } \mathcal{V} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Unitary module fusion categories} \\ \text{over } \mathcal{V} \text{ with a real generator} \end{array} \right\}$$

**2.2. Bicommutant categories.** The first author introduced the notion of bicommutant category in [Hen17b], and examples were constructed from unitary fusion categories and from conformal nets in [HP17] and [Hen17a], respectively. We refer the reader to [LR97, HP17, JP17] for the basics of  $C^*$ -tensor categories.

**Definition 2.3.** A bi-involutive tensor category is a  $C^*$ -tensor category  $\mathcal{C}$  equipped with a covariant anti-linear unitary functor  $\bar{\cdot} : \mathcal{C} \rightarrow \mathcal{C}$  called the conjugate. There are coherence natural isomorphisms  $\varphi_c : c \rightarrow \bar{\bar{c}}$ ,  $\nu_{a,b} : \bar{a} \otimes \bar{b} \rightarrow \bar{b} \otimes \bar{a}$  and  $r : 1 \rightarrow \bar{1}$  satisfying monoidal coherences (see [HP17, Def. 2.3]).

Basic examples include  $\text{Hilb}$ ,  $\text{Bim}(R)$  for a von Neumann algebra  $R$ , and unitary tensor categories, a.k.a. semisimple rigid  $C^*$ -tensor categories with simple unit object. We refer the reader to [HP17, p.5] or [Pen20, §3.5] for more details on this last example.

**Definition 2.4.** A bi-involutive tensor functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  between bi-involutive tensor categories is a unitary tensor functor equipped with a unitary natural isomorphism  $\chi_a : F(\bar{a}) \rightarrow \bar{F(a)}$  for  $a \in \mathcal{A}$  satisfying monoidal and involutive coherences (see [HP17, Def. 2.5] or [Pen20, Def. 3.35]).

A representation of a bi-involutive category  $\mathcal{C}$  is a von Neumann algebra  $R$  together with a bi-involutive tensor functor  $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ .

In [HP23, §5], we introduced a certain extra structure on a bi-involutive tensor category which we called a positive structure. Unitary fusion categories and  $\text{Bim}(R)$  have canonical positive structures, and every fully faithful unitary tensor functor from a unitary fusion category to  $\text{Bim}(R)$  extends uniquely to a representation which is compatible with the canonical positive structures [HP23, Thm. A]. Representations that are not compatible with the positive structures do exist, but they play no role in this paper.

**Definition 2.5.** Suppose  $(\mathcal{C}, \alpha)$  is a bi-involutive tensor category equipped with a representation into  $\text{Bim}(R)$ . The commutant category  $\mathcal{C}'$  is the unitary relative center of  $\alpha(\mathcal{C})$  inside  $\text{Bim}(R)$  [HP17, §2.3]. It has:

- objects bimodules  $X \in \text{Bim}(R)$  equipped with half-braidings  $e_X = \{e_{X,c} : X \boxtimes \alpha(c) \rightarrow \alpha(c) \boxtimes X\}_{c \in \mathcal{C}}$  satisfying the usual hexagon relation and naturality relation. Our graphical convention for the half-braiding  $e_X$  is

$$e_{X,c} = \begin{array}{c} \alpha(c) \quad X \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ \diagup \quad \diagdown \\ X \quad \alpha(c) \end{array} : X \boxtimes \alpha(c) \rightarrow \alpha(c) \boxtimes X.$$

- morphisms  $f : (X, e_X) \rightarrow (Y, e_Y)$  are bimodule maps  $f : X \rightarrow Y$  which commute with the half-braidings, i.e.,

- tensor product is given by

$$(X, e_X) \otimes (Y, e_Y) := (X \boxtimes Y, e_{X \boxtimes Y}) \quad e_{X \boxtimes Y, c} :=$$

- the bi-involutive structure is given in [HP17, §2.3].

Observe that  $\mathcal{C}'$  comes equipped with a canonical representation  $\alpha' : \mathcal{C}' \rightarrow \mathbf{Bim}(R)$  by forgetting the half-braiding. It thus makes sense to talk about the *bi-commutant*  $\mathcal{C}''$ . Observe there is a canonical bi-involutive tensor functor  $\mathcal{C} \rightarrow \mathcal{C}''$  given by  $c \mapsto (\alpha(c), \{e_{X,c}^{-1}\}_{(X, e_X) \in \mathcal{C}'})$ , where  $e_{X,c} : X \boxtimes \alpha(c) \rightarrow \alpha(c) \boxtimes X$  is the half-braiding for  $X \in \mathcal{C}'$ .

**Definition 2.6.** Let  $R$  be a von Neumann algebra. A *bicommutant category* in  $\mathbf{Bim}(R)$  is a bi-involutive tensor category  $\mathcal{B}$  equipped with a representation

$$\alpha : \mathcal{B} \rightarrow \mathbf{Bim}(R)$$

such that the canonical map  $\mathcal{B} \rightarrow \mathcal{B}''$  is an equivalence. (The category  $\mathcal{B}$  is equipped with a positive structure, inherited from  $\mathbf{Bim}(R)$ .)

**Theorem 2.7.** Suppose that  $\mathcal{B}$  is a bicommutant category in  $\mathbf{Bim}(R)$ . There is an equivalence of braided categories  $Z(\mathcal{B})^{\text{rev}} \rightarrow Z(\mathcal{B}')$  making the following diagram commute:

$$\begin{array}{ccc} Z(\mathcal{B})^{\text{rev}} & \xrightarrow{\cong} & Z(\mathcal{B}') \\ \text{Canonical} \searrow & & \swarrow \text{Forget} \\ & \mathcal{B}' & \end{array} \tag{3}$$

*Proof.* To construct a functor  $Z(\mathcal{B})^{\text{rev}} \rightarrow Z(\mathcal{B}')$ , an object

$$\underline{X} := (X, e_X = \{e_{X,Y} : X \otimes Y \rightarrow Y \otimes X\}_{Y \in \mathcal{C}})$$

in  $Z(\mathcal{B})^{\text{rev}}$  maps to

$$\underline{\underline{X}} = (\underline{X}, e_{\underline{X}}) = ((X, e_X), e_{\underline{X}})$$

in  $Z(\mathcal{B}')$ , where for  $\underline{Y} := (Y, e_Y) \in \mathcal{B}'$  we have  $e_{\underline{X}, \underline{Y}} = e_{Y, X}^{-1}$ . This is evidently a strict tensor functor. It is a braided tensor functor because the braiding on  $Z(\mathcal{B}^{\text{rev}})$  is given by

$$\beta^{Z(\mathcal{B})^{\text{rev}}}_{\underline{X}, \underline{Y}} = (\beta^{Z(\mathcal{B})}_{\underline{Y}, \underline{X}})^{-1} = e_{Y, X}^{-1}$$

which agrees with the braiding  $Z(\mathcal{B}')$ :

$$\beta_{\underline{X}, \underline{Y}}^{\mathcal{B}'} = e_{\underline{X}, \underline{Y}} = e_{Y, X}^{-1}.$$

The diagram (3) visibly commutes. It remains to show the top arrow is an equivalence.

The same construction replacing  $\mathcal{B}$  with  $\mathcal{B}'$  gives a strict braided tensor functor  $Z(\mathcal{B}')^{\text{rev}} \rightarrow Z(\mathcal{B}'')$ . Hence we also get a strict braided tensor functor  $Z(\mathcal{B}') \rightarrow Z(\mathcal{B}'')^{\text{rev}}$  by taking the reverse braiding on each side. The composite map  $Z(\mathcal{B})^{\text{rev}} \rightarrow Z(\mathcal{B}') \rightarrow Z(\mathcal{B}'')^{\text{rev}}$  is the map induced from the equivalence  $\mathcal{B} \rightarrow \mathcal{B}''$ . Again, replacing  $\mathcal{B}$  with  $\mathcal{B}'$ , the composite map  $Z(\mathcal{B}') \rightarrow Z(\mathcal{B}'')^{\text{rev}} \rightarrow Z(\mathcal{B}''')$  is also the map induced from the equivalence  $\mathcal{B}' \rightarrow \mathcal{B}'''$ . This completes the proof.  $\square$

We now prove some useful results in the case that  $\mathcal{C}$  is a unitary fusion category fully faithfully embedded in  $\text{Bim}(R)$ , for a von Neumann factor  $R$  not of type I. In [HP17], we proved that both  $\mathcal{C}'' = \text{Hilb}(\mathcal{C}) = \mathcal{C} \otimes_{\text{Hilb}_{\text{id}}} \text{Hilb}$  and  $\mathcal{C}'$  are bicommutant categories.

**Lemma 2.8.** *If  $\mathcal{C}$  is a unitary fusion category, then  $Z(\text{Hilb}(\mathcal{C})) \cong \text{Hilb}(Z(\mathcal{C}))$ .*

*Proof.* The canonical functor  $\text{Hilb}(Z(\mathcal{C})) \rightarrow Z_{\mathcal{C}}(\text{Hilb}(\mathcal{C})) = Z(\text{Hilb}(\mathcal{C}))$  is visibly fully faithful, but it is not obvious that it is essentially surjective.

The induction functor  $I : \mathcal{C} \rightarrow Z(\mathcal{C})$  (adjoint to the forgetful functor) is such that every object  $(X, e_X) \in Z(\mathcal{C})$  is a direct summand of  $I(X)$ . The same property holds true for the corresponding functor  $\text{Hilb}(I) : \text{Hilb}(\mathcal{C}) \rightarrow \text{Hilb}(Z(\mathcal{C}))$ . Every  $(X, e_X) \in Z(\text{Hilb}(\mathcal{C}))$  is a direct summand of  $\text{Hilb}(I)(X) \in \text{Hilb}(Z(\mathcal{C}))$ , hence lives in  $\text{Hilb}(Z(\mathcal{C}))$ .  $\square$

**Proposition 2.9.** *The functors  $Z(\mathcal{C}) \rightarrow \mathcal{C}'$  and  $Z(\mathcal{C}') \rightarrow \mathcal{C}'$  are fully faithful.*

*Proof.* We first show that  $Z(\mathcal{C}) \rightarrow \mathcal{C}'$  is fully faithful. Let  $(X, e_X), (Y, e_Y) \in Z(\mathcal{C})$ . Every morphism  $(X, e_X) \rightarrow (Y, e_Y)$  in  $\mathcal{C}'$  is a morphism  $X \rightarrow Y$  in  $\mathcal{C} \subset \text{Bim}(R)$  compatible with the half-braidings. This is exactly the definition of a morphism in  $Z(\mathcal{C})$ .

By Lemma 2.8, it follows that  $\text{Hilb}(Z(\mathcal{C})) \cong Z(\text{Hilb}(\mathcal{C})) \rightarrow \mathcal{C}'$  is also fully faithful. Using that  $\text{Hilb}(\mathcal{C})$  is a bicommutant category, the result now follows by commutativity of (3) for  $\mathcal{B} = \text{Hilb}(\mathcal{C})$ .  $\square$

We will need the following corollary in our construction later on.

**Corollary 2.10.** *Suppose  $\mathcal{M} \subset \mathcal{C}'$  is a full tensor subcategory containing the image of  $Z(\mathcal{C})$  in  $\mathcal{C}'$ . Then there is a fully faithful braided tensor functor  $Z(\mathcal{C})^{\text{rev}} \rightarrow Z(\mathcal{M})$  such that the following diagram commutes:*

$$\begin{array}{ccc} Z(\mathcal{C})^{\text{rev}} & \xrightarrow{\quad} & Z(\mathcal{M}) \\ & \searrow & \swarrow \\ & \mathcal{M} & \end{array}$$

*Proof.* The image of  $Z(\mathcal{C})$  in  $\mathcal{C}'$  lifts to a commutative diagram

$$\begin{array}{ccccc} Z(\mathcal{C})^{\text{rev}} & \hookrightarrow & Z(\text{Hilb}(\mathcal{C}))^{\text{rev}} & \xrightarrow{\cong} & Z(\mathcal{C}') \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{C}' & & \end{array}$$

in which the horizontal arrows are braided tensor functors. The image of  $Z(\mathcal{C})^{\text{rev}} \hookrightarrow Z(\mathcal{C}')$  lies in  $Z(\mathcal{M})$  because a half-braiding with  $\mathcal{C}'$  restricts to a half-braiding with  $\mathcal{M}$ .

Finally, the braided tensor functor  $Z(\mathcal{C})^{\text{rev}} \rightarrow Z(\mathcal{M})$  is automatically fully faithful as  $Z(\mathcal{C})^{\text{rev}}$  is modular [DMNO13, Cor. 3.26].  $\square$

### 3. Relative Tensor Product of Module Tensor Categories

Let  $\mathcal{V}$  be a unitary ribbon fusion category which is semisimple, and whose unit object is simple. In this section, we analyze the relative tensor product  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  of a  $\mathcal{V}$ -module tensor category  $(\mathcal{D}, \Phi^Z : \mathcal{V} \rightarrow \mathcal{D})$  and a  $\mathcal{V}^{\text{rev}}$ -module tensor category  $(\mathcal{C}, \Psi^Z : \mathcal{V} \rightarrow \mathcal{C})$ . Here,  $\Psi^Z$  is a reverse-braided functor, meaning that it satisfies  $\Psi^Z(\beta_{u,v}) = \beta_{\Psi^Z(v), \Psi^Z(u)}$  for  $u, v \in \mathcal{V}$ . We shall use the notational convention  $\Phi := \text{Forget} \circ \Phi^Z$  and  $\Psi := \text{Forget} \circ \Psi^Z$ .

The monoidal category  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is defined via the universal property which states that for every tensor category  $\mathcal{E}$ , the data of a  $\mathcal{V}$ -balanced tensor functor  $B : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{E}$  is equivalent to the data of a tensor functor  $B' : \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D} \rightarrow \mathcal{E}$ , via the commutative diagram

$$\begin{array}{ccc} \mathcal{C} \boxtimes \mathcal{D} & \xrightarrow{-\boxtimes_{\mathcal{V}}-} & \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D} \\ & \searrow B & \swarrow B' \\ & \mathcal{E} & \end{array}$$

Here, a tensor functor  $\mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{E}$  is called  $\mathcal{V}$ -balanced if it comes with monoidal natural isomorphisms

$$\eta_{c,v,d} : B((c \otimes \Psi v) \boxtimes d) \rightarrow B(c \boxtimes (\Phi v \otimes d))$$

for  $c \in \mathcal{C}$ ,  $d \in \mathcal{D}$ , and  $v \in \mathcal{V}$ , satisfying the coherence which states that passing  $v_1 \otimes v_2$  from one side to the other is the same as first passing  $v_2$  and then passing  $v_1$ . Note that the monoidal coherence involves the two half-braidings (the bottom vertical arrows):

$$\begin{array}{ccc} B((c_1 \otimes \Psi v_1) \boxtimes d_1) \otimes B((c_2 \otimes \Psi v_2) \boxtimes d_2) & \xrightarrow{\eta \otimes \eta} & B(c_1 \boxtimes (\Phi v_1 \otimes d_1)) \otimes B(c_2 \boxtimes (\Phi v_2 \otimes d_2)) \\ \downarrow & & \downarrow \\ B((c_1 \otimes \Psi v_1) \boxtimes d_1) \otimes ((c_2 \otimes \Psi v_2) \boxtimes d_2) & & B(c_1 \boxtimes (\Phi v_1 \otimes d_1)) \otimes (c_2 \boxtimes (\Phi v_2 \otimes d_2)) \\ \parallel & & \parallel \\ B((c_1 \otimes \Psi v_1 \otimes c_2 \otimes \Psi v_2) \boxtimes (d_1 \otimes d_2)) & & B((c_1 \otimes c_2) \boxtimes (\Phi v_1 \otimes d_1 \otimes \Phi v_2 \otimes d_2)) \\ \downarrow & & \downarrow \\ B((c_1 \otimes c_2 \otimes \Psi v_1 \otimes \Psi v_2) \boxtimes (d_1 \otimes d_2)) & \xrightarrow{\eta} & B((c_1 \otimes c_2) \boxtimes (\Phi v_1 \otimes \Phi v_2 \otimes d_1 \otimes d_2)) \end{array} \tag{4}$$

We write  $\zeta_v$  for the isomorphism  $\eta_{1,v,1} : \Phi v \boxtimes_{\mathcal{V}} 1_{\mathcal{D}} \rightarrow 1_{\mathcal{C}} \boxtimes_{\mathcal{V}} \Phi v$ .

*Remark 3.1.* When the braided category  $\mathcal{V}$  is fusion and when  $\mathcal{C}$  and  $\mathcal{D}$  are multifusion, then the balanced tensor product  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  agrees with the composition of 1-morphisms  $\text{Vec} \xrightarrow{\mathcal{C}} \mathcal{V} \xrightarrow{\mathcal{D}} \text{Vec} := \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  in the 4-category of braided fusion categories [Hau17, JFS17, BJS21, JMPP22].

**3.1. The ladder category model.** One model for  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is the Cauchy completion of the ladder category [MW12, BBJ19],<sup>1</sup> which we denote by  $\mathcal{C} \square_{\mathcal{V}} \mathcal{D}$ . The tensor category  $\mathcal{C} \square_{\mathcal{V}} \mathcal{D}$  is known to satisfy the universal property that characterises  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$ . That universal property is phrased in terms of (non-monoidal)  $\mathcal{V}$ -balanced functors from  $\mathcal{C} \boxtimes \mathcal{D}$ . By [Lur17, Prop 3.2.3.1.(4)],  $\mathcal{C} \square_{\mathcal{V}} \mathcal{D}$  also satisfies another, almost identical, universal property, which is now phrased in terms of  $\mathcal{V}$ -balanced *tensor* functors from  $\mathcal{C} \boxtimes \mathcal{D}$ .

**Definition 3.2.** Objects in the ladder category are of the form  $a \square d$  where  $a \in \mathcal{C}$  and  $d \in \mathcal{D}$ , and the morphism spaces are given by

$$(\mathcal{C} \square_{\mathcal{V}} \mathcal{D})(a \square c \rightarrow b \square d) := \bigoplus_{c \in \text{Irr}(\mathcal{V})} \mathcal{C}(a \rightarrow b \otimes \Psi v) \otimes \mathcal{D}(\Phi v \otimes c \rightarrow d).$$

We refer the reader to [BBJ19, Def. 7] for the remainder of the definition as a linear category and to [Row19, §3.2] for the monoidal structure.

**Lemma 3.3.** *Consider the canonical functors*

$$\begin{aligned} F : \mathcal{C} &\rightarrow \mathcal{C} \boxtimes 1_{\mathcal{D}} \subset \mathcal{C} \boxtimes \mathcal{D} \xrightarrow{-\boxtimes_{\mathcal{V}} -} \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D} \\ G : \mathcal{D} &\rightarrow 1_{\mathcal{C}} \boxtimes \mathcal{D} \subset \mathcal{C} \boxtimes \mathcal{D} \xrightarrow{-\boxtimes_{\mathcal{V}} -} \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}. \end{aligned}$$

- (1) *If  $\Phi : \mathcal{V} \rightarrow \mathcal{D}$  is fully faithful, then  $F : \mathcal{C} \rightarrow \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is fully faithful.*  
(2) *If  $\Psi : \mathcal{V} \rightarrow \mathcal{C}$  is fully faithful, then  $G : \mathcal{D} \rightarrow \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is fully faithful.*

*Proof.* We only prove (1) as (2) is similar. We use the ladder category model  $\mathcal{C} \square_{\mathcal{V}} \mathcal{D}$  for  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$ . Observe that

$$\begin{aligned} (\mathcal{C} \square_{\mathcal{V}} \mathcal{D})(F(a) \rightarrow F(b)) &= (\mathcal{C} \square_{\mathcal{V}} \mathcal{D})(a \square 1_{\mathcal{D}} \rightarrow b \square 1_{\mathcal{D}}) \\ &= \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{C}(a \rightarrow b \otimes \Psi v) \otimes \mathcal{D}(\Phi v \rightarrow 1_{\mathcal{D}}). \end{aligned}$$

If  $\Phi$  is fully faithful, the only  $v \in \text{Irr}(\mathcal{V})$  with  $\mathcal{D}(\Phi v \rightarrow 1_{\mathcal{D}}) \neq 0$  is  $1_{\mathcal{V}}$ , and moreover,  $\mathcal{D}(\Phi 1_{\mathcal{V}} \rightarrow 1_{\mathcal{D}}) = \mathcal{D}(\Phi 1_{\mathcal{V}} \rightarrow \Phi 1_{\mathcal{V}}) \cong \mathcal{V}(1_{\mathcal{V}} \rightarrow 1_{\mathcal{V}}) = \mathbb{C}$ . The result follows.  $\square$

**Corollary 3.4.** *Suppose  $\mathcal{C}, \mathcal{D}, \mathcal{V}$  are all fusion. If  $\Psi : \mathcal{V} \rightarrow \mathcal{C}$  or  $\Phi : \mathcal{V} \rightarrow \mathcal{D}$  is fully faithful, then  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is fusion.*

*Proof.* When  $\mathcal{C}, \mathcal{D}, \mathcal{V}$  are all fusion, then  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is always multifusion as it is 1-composition in the 4-category of braided fusion categories (see Remark 3.1). When  $\Psi : \mathcal{V} \rightarrow \mathcal{C}$  or  $\Phi : \mathcal{V} \rightarrow \mathcal{D}$  is fully faithful, then  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  has simple unit object by Lemma 3.3, and is thus fusion.  $\square$

<sup>1</sup> The ladder category construction is the analog of the balanced tensor product for non-Cauchy complete categories.

**3.2. Canonical centralizing structure.** We now discuss the notion of *centralizing structure* for two tensor categories  $\mathcal{A}, \mathcal{B}$  equipped with tensor functors to some other tensor category  $\mathcal{C}$  – see [CJP24, Def. 3.24].

**Definition 3.5.** Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be tensor categories, and suppose we have tensor functors  $F : \mathcal{A} \rightarrow \mathcal{C}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$ . A *centralizing structure* is a family of natural isomorphisms  $\{\sigma_{a,b} : F(a) \otimes G(b) \rightarrow G(b) \otimes F(a)\}_{a \in \mathcal{A}, b \in \mathcal{B}}$  satisfying the following conditions, where coherence isomorphisms have been suppressed:

- For  $a \in \mathcal{A}$  and  $b, b' \in \mathcal{B}$ ,  $(\text{id}_{G(b)} \otimes \sigma_{a,b'}) \circ (\sigma_{a,b} \otimes \text{id}_{G(b')}) = \sigma_{a,b \otimes b'}$
- For  $a, a' \in \mathcal{A}$  and  $b \in \mathcal{B}$ ,  $(\sigma_{a,b} \otimes \text{id}_{F(a')}) \circ (\text{id}_{F(a)} \otimes \sigma_{a',b}) = \sigma_{a \otimes a', b}$ .

A centralizing structure for  $F : \mathcal{A} \rightarrow \mathcal{C}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$  is equivalent to the data needed to promote  $F \boxtimes G : \mathcal{A} \boxtimes \mathcal{B} \rightarrow \mathcal{C}$  to a tensor functor.

**Corollary 3.6.** *By universality of  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  for  $\mathcal{V}$ -balanced functors out of  $\mathcal{C} \boxtimes \mathcal{D}$ ,  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  comes equipped with a canonical centralizing structure for the canonical functors from Lemma 3.3. Indeed, since*

$$(c \boxtimes 1) \otimes (1 \boxtimes d) = c \boxtimes d = (1 \boxtimes d) \otimes (c \boxtimes 1)$$

in  $\mathcal{C} \boxtimes \mathcal{D}$  for all  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$ , we get a canonical isomorphism  $\sigma_{c,d} : F(c) \otimes G(d) \rightarrow G(d) \otimes F(c)$  from the tensorator of  $- \boxtimes_{\mathcal{V}} - : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$ :

$$(c \boxtimes_{\mathcal{V}} 1) \otimes (1 \boxtimes_{\mathcal{V}} d) \xrightarrow{\cong} c \boxtimes_{\mathcal{V}} d \xrightarrow{\cong} (1 \boxtimes_{\mathcal{V}} d) \otimes (c \boxtimes_{\mathcal{V}} 1).$$

The centralizing axioms are satisfied by associativity of the tensorator.

Moreover, we have the following additional property of this centralizing structure: whenever  $c$  or  $d$  is in the image of  $\mathcal{V}$ , the centralizing structure  $\sigma$  is compatible with the half-braiding coming from the image of  $\mathcal{V}$  in  $Z(\mathcal{C})$  or  $Z(\mathcal{D})$  respectively. For example, when  $c = \Psi v$ , the following diagram commutes:

$$\begin{array}{ccccc} (\Psi v \boxtimes_{\mathcal{V}} 1) \otimes (1 \boxtimes_{\mathcal{V}} d) & \xrightarrow{\zeta_v \otimes \text{id}} & (1 \boxtimes_{\mathcal{V}} \Psi v) \otimes (1 \boxtimes_{\mathcal{V}} d) & \xrightarrow{\cong} & 1 \boxtimes_{\mathcal{V}} (\Psi v \otimes d) \\ \downarrow \sigma_{\Psi v, d} & \searrow \cong & \Psi v \boxtimes_{\mathcal{V}} d & \xrightarrow{\eta_{1, v, d}} & \downarrow \text{id} \boxtimes_{\mathcal{V}} e_{\Psi v, d} \\ (1 \boxtimes_{\mathcal{V}} d) \otimes (\Psi v \boxtimes_{\mathcal{V}} 1) & \xrightarrow{\text{id} \otimes \zeta_v} & (1 \boxtimes_{\mathcal{V}} d) \otimes (1 \boxtimes_{\mathcal{V}} \Psi v) & \xrightarrow{\cong} & 1 \boxtimes_{\mathcal{V}} (d \otimes \Psi v) \end{array}$$

The top square is naturality of  $\eta$  (recall  $\zeta_v = \eta_{1, v, 1}$ ), the left triangle is the definition of  $\sigma$ , and the bottom right pentagon is (4) for  $B = - \boxtimes_{\mathcal{V}} -$  setting  $c_1 = c_2 = 1_{\mathcal{C}}$ ,  $v_1 = 1_{\mathcal{V}}$  and  $v_2 = v$ , and  $d_1 = d$  and  $d_2 = 1_{\mathcal{D}}$ . Similarly, when  $d = \Phi v$ , the following diagram commutes

$$\begin{array}{ccccc} (c \otimes \Psi v) \boxtimes 1 & \xrightarrow{\cong} & (c \boxtimes_{\mathcal{V}} 1) \otimes (\Psi v \boxtimes_{\mathcal{V}} 1) & \xrightarrow{\text{id} \otimes \zeta_v} & (c \boxtimes_{\mathcal{V}} 1) \otimes (1 \boxtimes_{\mathcal{V}} \Phi v) \\ \uparrow e_{\Psi v, c} \boxtimes_{\mathcal{V}} \text{id} & \searrow \eta_{c, v, 1} & c \boxtimes_{\mathcal{V}} \Phi v & \xrightarrow{\cong} & \downarrow \sigma_{c, \Phi v} \\ (\Psi v \otimes c) \boxtimes 1 & \xrightarrow{\cong} & (\Psi v \boxtimes_{\mathcal{V}} 1) \otimes (c \boxtimes_{\mathcal{V}} 1) & \xrightarrow{\zeta_v \otimes \text{id}} & (1 \boxtimes_{\mathcal{V}} \Phi v) \otimes (c \boxtimes_{\mathcal{V}} 1). \end{array} \quad (5)$$

**Lemma 3.7.** *If  $\Psi^Z : \mathcal{V} \rightarrow Z(\mathcal{C})$  is fully faithful, then the functor which maps  $\mathcal{D}$  to the relative commutant  $Z_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D})$  of  $\mathcal{C}$  inside  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is fully faithful.*

*Proof.* Using the ladder category model  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$ ,

$$(\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D})(1 \boxtimes d_1 \rightarrow 1 \boxtimes d_2) = \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{C}(1 \rightarrow 1 \otimes \Psi v) \otimes \mathcal{D}(\Phi v \otimes d_1 \rightarrow d_2).$$

A morphism  $f : 1 \boxtimes d_1 \rightarrow 1 \boxtimes d_2$  lies in  $Z_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D})$  exactly when it is compatible with the half-braidings, which are induced from the centralizing structure. Writing  $f = \sum_{v \in \text{Irr}(\mathcal{V}), i} g_{v,i} \otimes h_{v,i}$  as a sum of its  $v$ -components, where  $g_{v,i} : 1_{\mathcal{C}} \rightarrow \Psi(v)$  and  $h_{v,i} : \Phi(v) \otimes d_1 \rightarrow d_2$ , where the  $h_{v,i}$  form a basis of  $\mathcal{D}(\Phi v \otimes d_1 \rightarrow d_2)$ , this compatibility reduces via (5) to

$$e_{\Psi v, c} \circ (g_{v,i} \otimes \text{id}_c) = \text{id}_c \otimes g_{v,i} \quad \forall c \in \mathcal{C}, \forall v \in \text{Irr}(\mathcal{V}).$$

This implies that  $g_{v,i} \in \mathcal{C}(1 \rightarrow \Psi v)$  is actually a morphism in  $Z(\mathcal{C})$  and thus  $g_{v,i} \in Z(\mathcal{C})(1 \rightarrow \Psi^Z v)$ . Since  $\Psi^Z$  was assumed to be fully faithful,  $g_{v,i} = 0$  unless  $v = 1_{\mathcal{V}}$ , in which case each  $g_{1,i} : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$  is a scalar. We conclude that  $Z_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D})(1 \boxtimes d_1 \rightarrow 1 \boxtimes d_2) \cong \mathcal{D}(d_1 \rightarrow d_2)$ .  $\square$

**Proposition 3.8.** *Suppose  $\mathcal{C}$  is a spherical fusion category,  $\mathcal{E}$  is any tensor category with simple unit, and  $\mathcal{C} \hookrightarrow \mathcal{E}$  is a fully-faithful embedding. The tensor product map  $\otimes : Z_{\mathcal{C}}(\mathcal{E}) \boxtimes_{Z(\mathcal{C})} \mathcal{C} \rightarrow \mathcal{E}$  is fully faithful.*

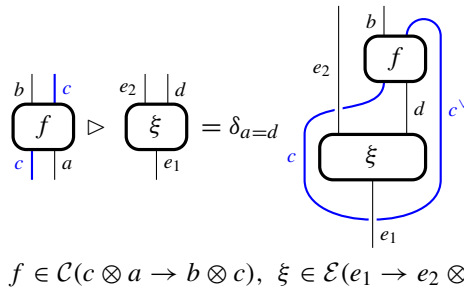
*Proof.* We must show that whenever  $c_1, c_2 \in \mathcal{C}$  and  $e_1, e_2 \in Z_{\mathcal{C}}(\mathcal{E})$ , we have

$$\bigoplus_{z \in \text{Irr}(Z(\mathcal{C}))} Z_{\mathcal{C}}(\mathcal{E})(e_1 \rightarrow e_2 \otimes z) \otimes \mathcal{C}(z \otimes c_1 \rightarrow c_2) \cong \mathcal{E}(e_1 \otimes c_1 \rightarrow e_2 \otimes c_2).$$

As  $\mathcal{C}$  is rigid and semisimple, we may assume  $c_1 = 1$  and  $c_2 = X := \bigoplus_{c \in \text{Irr}(\mathcal{C})} c$ . Our question becomes:

$$\bigoplus_{z \in \text{Irr}(Z(\mathcal{C}))} Z_{\mathcal{C}}(\mathcal{E})(e_1 \rightarrow e_2 \otimes z) \otimes \mathcal{C}(z \rightarrow X) \stackrel{?}{\cong} \mathcal{E}(e_1 \rightarrow e_2 \otimes X).$$

The right hand side  $\mathcal{E}(e_1 \rightarrow e_2 \otimes X)$  carries a canonical action of Ocneanu’s tube algebra  $\text{Tube}(\mathcal{C})$ :



We can decompose the above representation into irreps of  $\text{Tube}(\mathcal{C})$ . Recall from [Izu00, Müg03] that  $\text{Rep}(\text{Tube}(\mathcal{C})) \cong Z(\mathcal{C})^{\text{op}}$ , and every irrep is of the form  $H_z = \mathcal{C}(z \rightarrow X)$

where  $z \in \text{Irr}(Z(\mathcal{C}))$ , which carries a similar  $\text{Tube}(\mathcal{C})$ -action as above. Hence we can decompose

$$\mathcal{E}(e_1 \rightarrow e_2 \otimes X) \cong \bigoplus_{z \in \text{Irr}(Z(\mathcal{C}))} M_z \otimes H_z = \bigoplus_{z \in \text{Irr}(Z(\mathcal{C}))} M_z \otimes \mathcal{C}(z \rightarrow X)$$

where  $M_z$  is a multiplicity space. It remains to identify  $M_z$  with  $Z_{\mathcal{C}}(\mathcal{E})(e_1 \rightarrow e_2 \otimes z)$ . Since  $\mathcal{C} \hookrightarrow \mathcal{E}$  fully faithfully,  $H_z = \mathcal{C}(z \rightarrow X) \cong \mathcal{E}(z \rightarrow X)$ , so

$$M_z \cong \text{Rep}(\text{Tube}(\mathcal{C}))(\mathcal{E}(z \rightarrow X) \rightarrow \mathcal{E}(e_1 \rightarrow e_2 \otimes X)).$$

Observe that  $\mathcal{E}(z \rightarrow -)$  and  $\mathcal{E}(e_1 \rightarrow e_2 \otimes -)$  are both functors  $\mathcal{E} \rightarrow \mathbf{Vec}$ . By the Yoneda Lemma,  $\text{Hom}(\mathcal{E}(z \rightarrow -) \rightarrow \mathcal{E}(e_1 \rightarrow e_2 \otimes -)) \cong \mathcal{E}(e_1 \rightarrow e_2 \otimes z)$  canonically. Since maps of  $\text{Tube}(\mathcal{C})$  representations are maps between the underlying vector spaces which intertwine the  $\text{Tube}(\mathcal{C})$ -actions, we see that  $M_z$  is exactly the subspace of  $\mathcal{E}(e_1 \rightarrow e_2 \otimes z)$  which intertwines the two  $\text{Tube}(\mathcal{C})$ -actions, i.e.,  $Z_{\mathcal{C}}(\mathcal{E})(e_1 \rightarrow e_2 \otimes z)$ .  $\square$

Using the fact that  $\mathcal{C}' = Z_{\mathcal{C}}(\text{Bim}(R))$ , we have the following immediate result.

**Theorem 3.9.**  $\text{Bim}(R) \cong \mathcal{C}' \boxtimes_{Z(\mathcal{C})} \mathcal{C}$ .

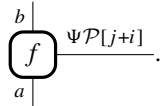
*Proof.* After applying Proposition 3.8, the only thing that remains to show is that the functor  $\mathcal{C}' \boxtimes_{Z(\mathcal{C})} \mathcal{C} \rightarrow \text{Bim}(R)$  is dominant. Indeed, for any object  $X \in \text{Bim}(R)$ ,  $X$  is a summand of  $\bigoplus_{c \in \text{Irr}(\mathcal{C})} c \otimes X \otimes \bar{c} \in \mathcal{C}'$  by [HP23, Lem. 6.3].  $\square$

**3.3. The anchored planar algebra model for  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$ .** In this section, we give a model  $\mathcal{E}$  for  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  using anchored planar algebras in the setting that  $\Phi : \mathcal{V} \rightarrow \mathcal{D}$  admits a right adjoint (so that there is an anchored planar algebra associated to  $\mathcal{D}$ ). We remind the reader that  $\Psi^Z : \mathcal{V} \rightarrow Z(\mathcal{C})$  is reverse braided. We begin by defining a full subcategory  $\mathcal{E}_0$ , and  $\mathcal{E}$  is the (unitary) Cauchy completion.

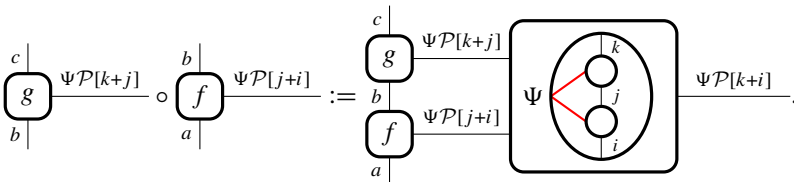
- Objects in  $\mathcal{E}_0$  are formal symbols  $c \otimes x^i$  for  $c \in \mathcal{C}$  and  $i \geq 0$ . By convention,  $x^0 = 1$ .
- Morphism spaces are defined by

$$\mathcal{E}_0(a \otimes x^i \rightarrow b \otimes x^j) := \mathcal{C}(a \rightarrow b \otimes \Psi \mathcal{P}[j+i]),$$

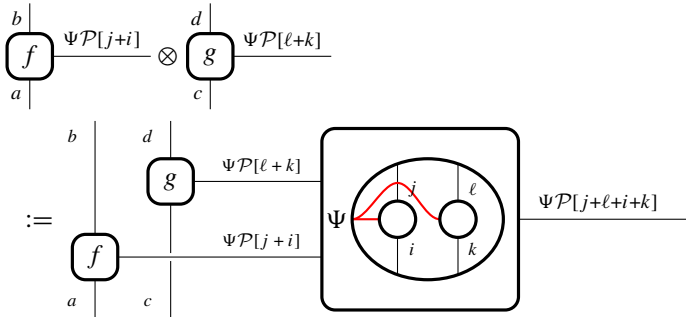
where  $\mathcal{P}$  is the anchored planar algebra in  $\mathcal{V}$  corresponding to  $(\mathcal{D}, \Phi^Z : \mathcal{V} \rightarrow Z(\mathcal{D}), x)$ . We represent morphisms in the graphical calculus by



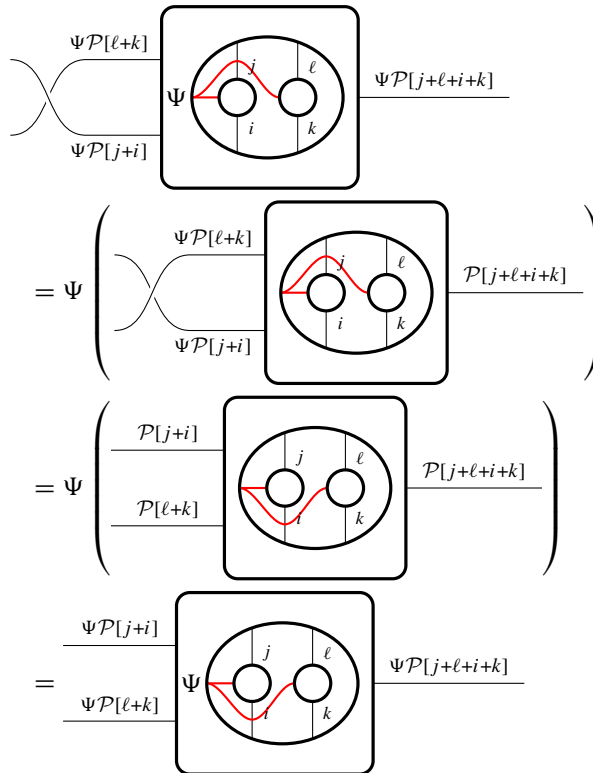
- Composition of morphisms is defined by



- Tensor product is given on objects by  $(a \otimes x^i) \otimes (b \otimes x^j) := a \otimes b \otimes x^{i+j}$ , and on morphisms by



where the crossing is the half-braiding of  $\Psi^Z \mathcal{P}[j+i] \in Z(\mathcal{C})$  with  $c \in \mathcal{C}$  (this choice ensures that the box labelled  $g$  can pass under the crossing). Since  $\Psi$  is reverse-braided, we have



ensuring that the exchange axiom  $(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2)$  is satisfied (see [HPT23, §6.3] for a similar computation).

The associators are induced from those of  $\mathcal{C}$ :

$$\alpha_{a \otimes x^i, b \otimes x^j, c \otimes x^k} := \begin{array}{c} a \otimes (b \otimes c) \\ | \\ \alpha \\ | \\ (a \otimes b) \otimes c \end{array} \otimes \begin{array}{c} \Psi \\ | \\ \ell \end{array} \xrightarrow{\Psi \mathcal{P}[2r]} \quad \ell := i + j + k,$$

and similarly for the unitors.

- $\mathcal{E}_0$  is rigid with duals given by  $(c \otimes x^i)^\vee := c^\vee \otimes x^i$ , and evaluation and coevaluation are given by

$$\text{ev}_{c \otimes x^i} = \begin{array}{c} \text{ev}_c \\ \cup \\ c^\vee \quad c \end{array} \quad \boxed{\Psi} \begin{array}{c} \bullet \\ \circ \\ n \end{array} \xrightarrow{\Psi \mathcal{P}[2i]} \quad \text{coev}_{c \otimes x^i} = \begin{array}{c} c \quad c^\vee \\ \cup \\ \text{coev}_c \end{array} \quad \boxed{\Psi} \begin{array}{c} \bullet \\ \circ \\ n \end{array} \xrightarrow{\Psi \mathcal{P}[2i]}.$$

In the unitary setting, the evaluation and coevaluation in  $\mathcal{C}$  come from our chosen unitary dual functor on  $\mathcal{C}$ .

- The pivotal structure  $\varphi^\mathcal{E} : c \otimes x^{\otimes i} \rightarrow (c \otimes x^i)^{\vee\vee}$  is given by

$$\begin{array}{c} c^{\vee\vee} \\ | \\ \boxed{\varphi_c} \quad \boxed{\Psi} \begin{array}{c} \bullet \\ \circ \\ i \end{array} \xrightarrow{\Psi \mathcal{P}[2i]} \\ | \\ c \end{array}.$$

*Remark 3.10.* When  $\Phi : \mathcal{V} \rightarrow \mathcal{D}$  is fully faithful and  $\mathcal{C}$  has simple unit object, then the unit of  $\mathcal{E}_0$  is also simple. Indeed, in that case,  $\mathcal{P}[0] = 1_{\mathcal{V}}$ , and hence

$$\text{End}_{\mathcal{E}_0}(1_{\mathcal{C}} \otimes x^0) = \mathcal{C}(1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}} \otimes \Psi \mathcal{P}[0]) \cong \mathcal{C}(1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}) \cong \mathbb{C}.$$

*Remark 3.11.* When  $\Psi : \mathcal{V} \rightarrow \mathcal{C}$  is dominant and admits a left adjoint  $I$ , as in [Row19],  $\mathcal{E}$  can be identified with  $\text{coMod}_{\mathcal{D}}(\Phi A)$  for  $A = I(1_{\mathcal{C}})$ . Indeed,

$$\begin{aligned} \mathcal{E}_0(\Psi u \otimes x^i \rightarrow \Psi v \otimes x^j) &= \mathcal{C}(\Psi u \rightarrow \Psi v \otimes \Psi \mathcal{P}[j+i]) \\ &\cong \mathcal{C}(1_{\mathcal{C}} \rightarrow \Psi(\bar{u} \otimes v \otimes \mathcal{P}[j+i])) \\ &\cong \mathcal{V}(A \rightarrow \bar{u} \otimes v \otimes \text{Tr}_{\mathcal{V}}(x^{j+i})) \\ &\cong \mathcal{V}(A \rightarrow \text{Tr}_{\mathcal{V}}(x^i \otimes \Phi \bar{u} \otimes \Phi v \otimes x^j)) \\ &= \mathcal{V}(A \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(\Phi u \otimes x^i \rightarrow \Phi v \otimes x^j)) \\ &= \mathcal{D}(\Phi u \otimes x^i \otimes \Phi A \rightarrow \Phi v \otimes x^j) \\ &\cong \text{coMod}_{\mathcal{D}}(A)(\Phi u \otimes x^i \otimes \Phi A \rightarrow \Phi v \otimes x^j \otimes \Phi A) \end{aligned}$$

where the  $x$  which appears in the left hand side is a formal symbol, and the  $x$  which appears in the right hand side is our chosen generator of  $\mathcal{D}$ .

**Proposition 3.12.** *The monoidal category  $\mathcal{E}$  constructed above is canonically equivalent, as monoidal category, to the balanced tensor product  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$ .*

*Proof.* The category  $\mathcal{E}$  is the idempotent completion of  $\mathcal{E}_0$ , and the balanced tensor product  $\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}$  is canonically equivalent to the idempotent completion of the ladder category  $\mathcal{C} \boxdot_{\mathcal{V}} \mathcal{D}_0$ , where  $\mathcal{D}_0 \subset \mathcal{D}$  is the full subcategory on the objects of the form  $\Phi v \otimes x^i$ . So it is enough to construct an equivalence  $F_0$  of monoidal categories from  $\mathcal{E}_0$  to  $\mathcal{C} \boxdot_{\mathcal{V}} \mathcal{D}_0$ . We define  $F_0$  by

$$F_0(c \otimes x^i) := c \boxdot x^i.$$

On morphism spaces, we define  $F_0$  by the following sequence of isomorphisms:

$$\begin{aligned}
 \mathcal{E}_0(a \otimes x^i \rightarrow b \otimes x^j) &= \mathcal{C}(a \rightarrow b \otimes \Psi\mathcal{P}[j+i]) \\
 &\cong \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{C}(a \rightarrow b \otimes \Psi v) \otimes \mathcal{V}(v \rightarrow \mathcal{P}[j+i]) \\
 &= \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{C}(a \rightarrow b \otimes \Psi v) \otimes \mathcal{V}(v \rightarrow \text{Tr}_{\mathcal{V}}(x^{j+i})) \\
 &\cong \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{C}(a \rightarrow b \otimes \Psi v) \otimes \mathcal{D}(\Phi v \rightarrow x^{j+i}) \\
 &\cong \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{C}(a \rightarrow b \otimes \Psi v) \otimes \mathcal{D}(\Phi v \otimes x^i \rightarrow x^j) \\
 &= (\mathcal{C} \boxtimes_{\mathcal{V}} \mathcal{D}_0)(a \boxtimes x^i \rightarrow b \boxtimes x^j).
 \end{aligned}$$

The first isomorphism in the second line above uses the fact that each object  $w \in \mathcal{V}$  can be canonically decomposed into simple objects as  $w = \bigoplus_{v \in \text{Irr}(\mathcal{V})} \mathcal{V}(v \rightarrow w) \otimes v$ . It is a somewhat tricky exercise to show that  $F_0$  is a tensor functor, which is then automatically fully faithful. Essential surjectivity follows since  $a \boxtimes (\Phi v \otimes x^i) \cong (a \otimes \Psi v) \boxtimes x^i$ .  $\square$

*3.4. The unitary setting.* We now consider the case that  $\mathcal{V}$  is a unitary tensor category equipped with a unitary dual functor  $\vee$ . Suppose we have a unitary  $\mathcal{V}$ -module multitensor category  $(\mathcal{D}, \Phi^Z : \mathcal{V} \rightarrow Z(\mathcal{D}), x, \vee_{\mathcal{D}}, \psi_{\mathcal{D}})$  equipped with a real generator  $x \in \mathcal{D}$  and a unitary  $\mathcal{V}^{\text{rev}}$ -module multitensor category  $(\mathcal{C}, \Psi^Z : \mathcal{V} \rightarrow Z(\mathcal{C}), \vee_{\mathcal{C}}, \psi_{\mathcal{C}})$  where  $\Psi^Z$  is a reverse-braided functor. We further assume that  $\Phi : \mathcal{V} \rightarrow \mathcal{D}$  admits a unitary adjoint [HPT24, §2.1].

As in [HPT24, §5.2.1], we can promote  $\mathcal{E}$  to a unitary multitensor category equipped with a unitary dual functor and canonical state. This amounts to the following tasks:

- (E1) construct a  $\dagger$ -structure on  $\mathcal{E}_0$  under which is is a unitary multitensor category,
- (E2) construct a faithful state  $\psi_{\mathcal{E}}$  on  $\text{End}_{\mathcal{E}}(1_{\mathcal{E}})$ , and
- (E3) check  $\vee_{\mathcal{E}}$  is unitary, and the canonical unitary pivotal structure induced by  $\vee_{\mathcal{E}}$  is  $\varphi^{\mathcal{E}}$ .

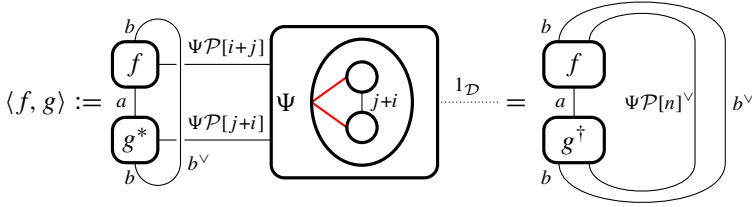
To accomplish (E1) above, we define  $\dagger$  on  $\mathcal{E}_0$  as in [HPT24, (25)], but we apply  $\Psi$  when necessary to push objects and morphisms from  $\mathcal{V}$  into  $\mathcal{C}$ :

$$\left( \begin{array}{c} b \\ | \\ \boxed{f} \\ | \\ a \end{array} \xrightarrow{\Psi\mathcal{P}[n+k]} \right)^* = \begin{array}{c} a \\ | \\ \boxed{f^*} \\ | \\ b \end{array} \xrightarrow{\Psi\mathcal{P}[n+k]} := \begin{array}{c} a \\ | \\ \boxed{f^\dagger} \\ | \\ b \end{array} \begin{array}{c} \boxed{\Psi_{n+k}^{-1}} \\ \xrightarrow{\Psi\mathcal{P}[n+k]} \end{array} \begin{array}{c} \Psi\mathcal{P}[k+n] \\ \curvearrowright \end{array} .$$

The proof that this defines a dagger structure on  $\mathcal{E}_0$  is entirely similar to the one in [HPT24].

Since  $\Psi$  is a unitary pivotal tensor functor between unitary multitensor categories equipped with unitary dual functors, it automatically preserves the unitary pivotal structure, and thus  $\Psi \text{coev}_{\mathcal{P}[i+j]}^\dagger = \text{coev}_{\Psi\mathcal{P}[i+j]}^\dagger$ . Thus by [HPT24, Lem. 5.9], the sesquilinear

form on  $\mathcal{E}_0(a \otimes x^i \rightarrow b \otimes x^j)$



is positive definite. This immediately implies as in [HPT24, Prop. 5.10] that the  $2 \times 2$  linking algebras

$$L := \begin{pmatrix} \mathcal{E}_0(a \otimes x^i \rightarrow a \otimes x^i) & \mathcal{E}_0(b \otimes x^j \rightarrow a \otimes x^i) \\ \mathcal{E}_0(a \otimes x^i \rightarrow b \otimes x^j) & \mathcal{E}_0(b \otimes x^j \rightarrow b \otimes x^j) \end{pmatrix}$$

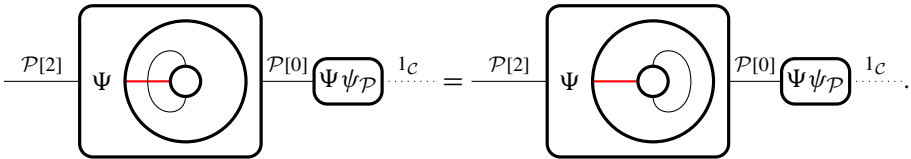
are finite dimensional  $C^*$ -algebras, proving  $\mathcal{E}_0$  is  $C^*$ . We have thus established (E1).

The construction of a faithful state  $\psi_{\mathcal{E}}$  on  $\text{End}_{\mathcal{E}}(1_{\mathcal{E}})$  is similar to [HPT24, Cor. 5.12]

$$\psi_{\mathcal{E}} \left( \begin{array}{c} 1_c \\ \vdots \\ \boxed{f} \xrightarrow{\Psi \mathcal{P}[0]} \\ \vdots \\ 1_c \end{array} \right) := \begin{array}{c} 1_c \\ \vdots \\ \boxed{f} \xrightarrow{\Psi \mathcal{P}[0]} \boxed{\Psi \psi_{\mathcal{P}}} \xrightarrow{1_c} \\ \vdots \\ 1_c \end{array},$$

establishing (E2). Finally, that  $\vee_{\mathcal{E}}$  is unitary and induces  $\varphi^{\mathcal{E}}$  is similar to [HPT24, Prop. 5.13] adding  $\Psi$ s when necessary, establishing (E3).

By the same argument as [HPT24, Prop. 5.15], when  $\vee_{\mathcal{V}}$  and  $\psi_{\mathcal{P}}$  are spherical, then so is  $\psi_{\mathcal{E}}$ . In fact, instead of demanding that  $\psi_{\mathcal{P}}$  be spherical, the following weaker condition is sufficient (in addition to sphericity of  $\vee_{\mathcal{V}}$ ):



#### 4. Classification of Finite Depth Objects in $\mathcal{C}'$

Let  $\mathcal{C}$  be a unitary fusion category fully faithfully embedded in  $\text{Bim}(R)$  where  $R$  is a hyperfinite factor of type  $\text{II}_1$ ,  $\text{II}_{\infty}$ , or  $\text{III}_1$ . We denote by  $\mathcal{C}'$  its commutant category defined in §2.2 above.

Recall that our main goal is to prove Conjecture 1.1 in the case that  $\mathcal{B} = \mathcal{C}'$ : finite depth real (symmetrically self-dual) objects of  $\mathcal{C}'$  up to conjugation by invertible elements of  $\mathcal{C}'$  are in bijective correspondence with connected finite depth unitary anchored planar algebras in  $\mathcal{Z}(\mathcal{C})^{\text{rev}}$  up to isomorphism.

4.1. *From finite depth objects to anchored planar algebras.* We first fix a symmetrically self-dual object  $m \in \mathcal{C}'$ , and we construct a connected unitary anchored planar algebra  $\mathcal{P}$  in  $\mathcal{Z}(\mathcal{C})^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}')$ . Since  $m \in \mathcal{C}'$ , it is equipped with a half-braiding, which we'll call  $\{\eta_{m,d} : m \otimes d \rightarrow d \otimes m\}_{d \in \mathcal{C}}$ .

Let  $\mathcal{M} := \langle m, Z(\mathcal{C}) \rangle$  be the tensor  $\mathbb{C}^*$ -subcategory of  $\mathcal{C}'$  generated by  $m$  and the image of  $Z(\mathcal{C})$  in  $\mathcal{C}'$  (under the operations of tensor product, orthogonal direct sums, and orthogonal direct summands). By Corollary 2.10, there is a canonical  $Z(\mathcal{C})^{\text{rev}}$ -module tensor category structure on  $\mathcal{M}$ . We equip  $\mathcal{M}$  with its unique spherical unitary dual functor. Finally, we obtain a unitary anchored planar algebra  $\mathcal{P}$  from the unitary pivotal  $Z(\mathcal{C})$ -module tensor category  $(\mathcal{M}, m)$  by [HPT24].

Observe that when  $m$  has finite depth, then  $\mathcal{M}$  is fusion by Lemma 4.1 below. In this case,  $\mathcal{P}$  is finite depth.

**Lemma 4.1.** *Let  $\mathcal{M}$  be a semisimple rigid tensor category with simple unit object, and suppose  $\mathcal{C}, \mathcal{D} \subset \mathcal{M}$  are fusion subcategories that generate  $\mathcal{M}$  under tensor products and subobjects. If  $c \otimes d \cong d \otimes c$  for every  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$ , then  $\mathcal{M}$  is again fusion.*

*Proof.* A complete set of simples for  $\mathcal{M}$  is obtained by taking all distinct simple summands of the finitely many objects of the form  $c \otimes d$  where  $c \in \text{Irr}(\mathcal{C})$  and  $d \in \text{Irr}(\mathcal{D})$ .  $\square$

4.2. *From anchored planar algebras to finite depth objects.* Suppose  $\mathcal{P}$  is a connected unitary anchored planar algebra in  $Z(\mathcal{C})^{\text{rev}}$ . By the main result of our previous paper [HPT24],  $\mathcal{P}$  corresponds to a unitary module tensor category with real generator  $(\mathcal{M}, m)$ . By the construction from §3, noting that the forgetful functor  $Z(\mathcal{C})^{\text{rev}} \rightarrow \mathcal{C}$  is reverse braided, we can then form the balanced tensor product  $\mathcal{E} := \mathcal{C} \boxtimes_{Z(\mathcal{C})^{\text{rev}}} \mathcal{M}$ . It will be convenient to also think of  $\mathcal{E} = \mathcal{M} \boxtimes_{Z(\mathcal{C})} \mathcal{C}$  during this construction. By Construction 3.6,  $\mathcal{E}$  can be equipped with a canonical centralizing structure  $\{\sigma_{m,c} : m \otimes c \rightarrow c \otimes m\}_{m \in \mathcal{M}, c \in \mathcal{C}}$ .

Since  $\mathcal{C}$  is fusion, it has simple unit object. Since  $\mathcal{P}$  is connected,  $\Phi : Z(\mathcal{C}) \rightarrow \mathcal{M}$  is fully faithful. Thus  $\mathcal{E}$  has simple unit object and  $\mathcal{C} \hookrightarrow \mathcal{E} = \mathcal{M} \boxtimes_{Z(\mathcal{C})} \mathcal{C}$  is fully faithful by Lemma 3.3. When  $\mathcal{P}$  has finite depth,  $\mathcal{M}$  is fusion, and thus  $\mathcal{E}$  is also fusion by Corollary 3.4.

$$\begin{array}{ccccc}
 & & Z(\mathcal{C})^{\text{rev}} & \xrightarrow{\text{Forget}} & \mathcal{C} \\
 & & \downarrow \Phi & & \downarrow \searrow \\
 \{m\} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{E} \xleftarrow{\alpha} \text{Bim}(R)
 \end{array}$$

As reviewed in [HP23, §3.2], by [Pop90, Pop95, Izu17, Tom21], there is a fully faithful unitary tensor functor  $\alpha : \mathcal{E} \rightarrow \text{Bim}(R)$ , which is unique up to conjugation by an invertible object of  $\text{Bim}(R)$ . The restriction of  $\alpha|_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Bim}(R)$  is similarly unique up to conjugation by an invertible object of  $\text{Bim}(R)$ . Since  $\mathcal{C}$  was already given to us as a full subcategory of  $\text{Bim}(R)$ , conjugating by a suitable invertible object of  $\text{Bim}(R)$ , we may assume that  $\alpha|_{\mathcal{C}}$  agrees with our initial presentation of  $\mathcal{C}$  (that is,  $\alpha|_{\mathcal{C}} = \text{id}_{\mathcal{C}}$ ). Such tensor functors  $\alpha : \mathcal{E} \rightarrow \text{Bim}(R)$  satisfying  $\alpha|_{\mathcal{C}} = \text{id}_{\mathcal{C}}$  are unique up to conjugation by an invertible object of  $\mathcal{C}'$ .

Since  $\mathcal{C} \hookrightarrow \mathcal{E}$  is fully faithful, our centralizing structure  $\sigma$  canonically promotes the image of each object  $n \in \mathcal{M}$  in  $\mathcal{E}$  to the relative center  $Z_{\mathcal{C}}(\mathcal{E})$ , i.e., objects in  $\mathcal{E}$  which are equipped with half-braidings with objects in  $\mathcal{C}$ . Taking the image of  $\sigma$  in  $\text{Bim}(R)$ ,

we immediately get a unitary tensor functor  $\mathcal{M} \rightarrow \mathcal{C}'$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 Z(\mathcal{C})^{\text{rev}} & \xrightarrow{\text{Forget}} & \mathcal{C} & & \\
 \downarrow \Phi & & \downarrow & \searrow & \\
 \{m\} \longrightarrow \mathcal{M} & \longrightarrow & \mathcal{E} & \xleftarrow{\alpha} & \mathbf{Bim}(R) \\
 & \searrow & \uparrow \text{Forget} & & \uparrow \text{Forget} \\
 & & Z_{\mathcal{C}}(\mathcal{E}) & \hookrightarrow & \mathcal{C}'
 \end{array}$$

Now the image of  $m$  in  $\mathcal{C}'$  is our desired symmetrically self-dual finite depth object.

The functor  $\mathcal{M} \hookrightarrow Z_{\mathcal{C}}(\mathcal{E})$  is fully faithful by Lemma 3.7. It follows that  $\mathcal{M} \rightarrow \mathcal{C}'$  is also fully faithful.

**4.3. Uniqueness of  $\mathcal{M} \rightarrow \mathcal{C}'$ .** Let  $\mathcal{M}$  be a unitary  $Z(\mathcal{C})^{\text{rev}}$ -module tensor category such that the composite unitary tensor functor  $\Phi : Z(\mathcal{C}) \rightarrow \mathcal{M}$  is fully faithful (in terms of the associated anchored planar algebra, this is the condition that  $\mathcal{P}[0]$  is connected). Our next goal is to show that given two fully faithful  $Z(\mathcal{C})^{\text{rev}}$ -central unitary tensor functors  $\mathcal{M} \rightarrow \mathcal{C}'$  (a.k.a. functors of  $Z(\mathcal{C})^{\text{rev}}$ -module tensor categories), there exists an invertible object of  $\mathcal{C}'$  that conjugates one into the other.

Consider  $\mathcal{C} \subset \mathbf{Bim}(R)$ . We claim that the unitary tensor functor  $\mathcal{M} \hookrightarrow \mathcal{C}' \rightarrow \mathbf{Bim}(R)$  and the inclusion  $\mathcal{C} \subset \mathbf{Bim}(R)$  assemble to a fully faithful unitary tensor functor  $\beta : \mathcal{E} = \mathcal{M} \boxtimes_{Z(\mathcal{C})} \mathcal{C} \rightarrow \mathbf{Bim}(R)$  such that  $\beta|_{\mathcal{C}} = \text{id}_{\mathcal{C}}$ .

Observe that the underlying  $R - R$  bimodule of an object of  $Z(\mathcal{C})^{\text{rev}}$  lives in  $\mathcal{C} \subset \mathbf{Bim}(R)$ . Moreover, our tensor functor  $G : \mathcal{M} \rightarrow \mathcal{C}'$  is a morphism of unitary  $Z(\mathcal{C})^{\text{rev}}$ -module tensor categories. This means that we have a unitary action-coherence morphism  $\gamma : \Phi_2 \Rightarrow G\Phi_1$  where  $\Phi_1 : Z(\mathcal{C})^{\text{rev}} \rightarrow \mathcal{M}$  and  $\Phi_2 : Z(\mathcal{C})^{\text{rev}} \rightarrow \mathcal{C}'$  satisfying the coherence conditions from [HPT23, Def. 3.2] (see also [HPT24, Def. 3.3]). Since  $\text{Forget} \circ \Phi_2$  is identically the forgetful functor  $Z(\mathcal{C}) \rightarrow \mathcal{C}$ , the external square of the following digram

$$\begin{array}{ccc}
 Z(\mathcal{C})^{\text{rev}} & \xrightarrow{\text{Forget}} & \mathcal{C} \\
 \Phi_1 \nearrow \gamma & \Downarrow & \downarrow \Phi_2 \\
 \mathcal{M} & \xrightarrow{G} & \mathcal{C}' \xrightarrow{\text{Forget}} \mathbf{Bim}(R)
 \end{array}$$

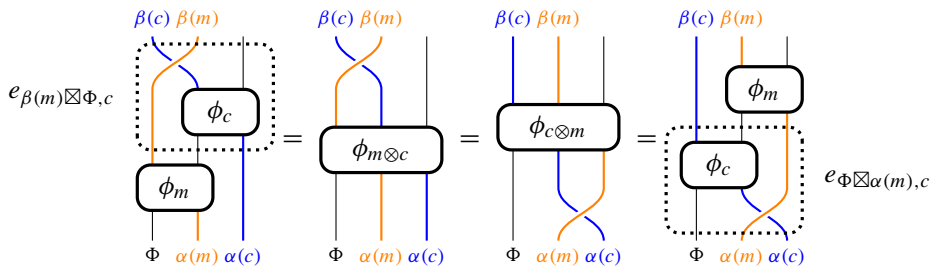
commutes on the nose. Thus the image of  $Z(\mathcal{C})^{\text{rev}}$  inside the images of  $\mathcal{C}$  and  $\mathcal{M}$  in  $\mathbf{Bim}(R)$  are identical. This equips the canonical map  $\mathcal{C} \boxtimes \mathcal{M} \rightarrow \mathbf{Bim}(R)$  (which is the identity on  $\mathcal{C}$ ) with a  $Z(\mathcal{C})^{\text{rev}}$ -balancing structure.

This map thus descends to a map  $\beta : \mathcal{C} \boxtimes_{Z(\mathcal{C})^{\text{rev}}} \mathcal{M} \rightarrow \mathbf{Bim}(R)$  (which is still the identity on  $\mathcal{C}$ ) by the universal property of the relative tensor product. As in the previous section, we identify  $\mathcal{C} \boxtimes_{Z(\mathcal{C})^{\text{rev}}} \mathcal{M} = \mathcal{M} \boxtimes_{Z(\mathcal{C})} \mathcal{C} = \mathcal{E}$ . Since the inclusion  $\mathcal{M} \rightarrow \mathcal{C}'$  is fully faithful,  $\mathcal{E}$  is a full subcategory of  $\mathcal{C}' \boxtimes_{Z(\mathcal{C})} \mathcal{C}$ , the latter of which is equivalent to  $\mathbf{Bim}(R)$  by Theorem 3.9. We conclude the map  $\beta : \mathcal{E} \rightarrow \mathbf{Bim}(R)$  is fully faithful.

We now have two fully faithful unitary tensor functors  $\alpha, \beta : \mathcal{E} \rightarrow \mathbf{Bim}(R)$ . As reviewed in [HP23, §3.2], by [Pop90, Pop95, Izu17, Tom21], there is an isomorphism of representations  $(\Phi, \phi) : \alpha \rightarrow \beta$ , i.e., an invertible  $\Phi \in \mathbf{Bim}(R)$  equipped with a natural family of unitary isomorphisms

$$\{\phi_e : \Phi \boxtimes \alpha(e) \rightarrow \beta(e) \boxtimes \Phi\}_{e \in \mathcal{E}}$$

satisfying the coherence condition [HP23, (7)] ( $(\Phi, \phi)$  is a natural transformation between 2-functors). Since  $\alpha|_{\mathcal{C}} = \beta|_{\mathcal{C}} = \text{id}_{\mathcal{C}}$ , we see that  $\phi|_{\mathcal{C}}$  promotes  $\Phi$  to an object of  $\mathcal{C}'$ . We also claim each  $\phi_m$  is morphism in  $\mathcal{C}'$ . Indeed, representing the canonical centralizing structure for objects  $c \in \mathcal{C}$  and  $m \in \mathcal{M}$  by a crossing, we have



We conclude that the two fully faithful unitary tensor functors  $\mathcal{M} \leftrightarrow \mathcal{C}'$  are conjugate by an invertible object in  $\mathcal{C}'$ .

Using the equivalence between unitary anchored planar algebras and unitary module tensor categories equipped with a real generator, the above argument shows that the composite

$$\left\{ \begin{array}{l} \text{Finite depth} \\ \text{objects of } \mathcal{C}' \end{array} \right\} / \text{conj.} \rightarrow \left\{ \begin{array}{l} \text{Connected finite depth} \\ \text{unitary APAs in } Z(\mathcal{C})^{\text{rev}} \end{array} \right\} / \text{iso.} \rightarrow \left\{ \begin{array}{l} \text{Finite depth} \\ \text{objects of } \mathcal{C}' \end{array} \right\} / \text{conj.}$$

is the identity.

The other composite, from unitary anchored planar algebras to finite depth objects of  $\mathcal{C}'$  back to unitary anchored planar algebras, produces an isomorphic unitary anchored planar algebra because unitarily equivalent unitary module tensor categories with chosen real generators give equivalent unitary anchored planar algebras by [HPT24, Thm. A].

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**Code availability** Not applicable, as no code was created or analyzed.

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