

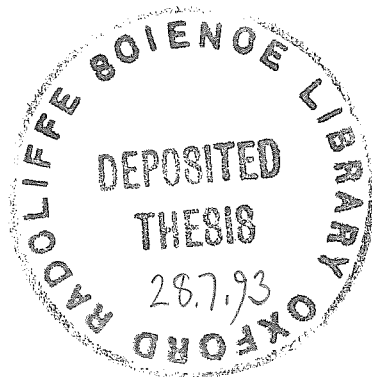
Otto Hölder and the Development of Group Theory and Galois Theory

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A thesis submitted for the degree of Doctor of Philosophy in Mathematics,
University of Oxford

Hilary Term 1993



For my Father and my Mother

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ABSTRACT

The last fifteen years of the nineteenth century saw great changes in the perception and formulation of group theory and Galois theory. At that time, the major problems which became the focus of twentieth-century finite group theory were distilled. New techniques and systematic methods were introduced as attempts were made to solve these problems. The abstraction of group theory which had taken place during the 1870s and 1880s resulted in deeper insights into Galois' theory of equations.

This thesis focuses on the algebraic studies of the German mathematician Otto Hölder (1859–1937). It consists largely of a historical and critical analysis of his original work on group theory and Galois theory, which was published in seven articles between 1889 and 1895. Hölder was instrumental in providing a clear understanding of the structure which a theory of finite groups should possess. The important mathematical problems that arise from such an understanding are formulated in his papers. He developed techniques and methods of solution which had a great influence on the direction of subsequent research on these problems.

A biography of Otto Hölder is presented in Chapter 2 of the thesis; a survey of the algebraic inheritance on which he drew forms the substance of Chapter 3. The following six chapters each examine his contribution to a specific area or problem and discuss the context of his work. His mathematics is related to and contrasted with modern techniques and methods of solution, which are included where appropriate. Each chapter concludes with some remarks on the later development of the ideas and problems Hölder considered. The subjects of these chapters are successively the re-formulation of Galois theory, the concept of quotient group, the Jordan–Hölder Theorem for finite groups, the search for simple groups, the structure theory of finite groups and group extension theory. Finally, various conclusions are drawn about Otto Hölder's achievements and his place in the history of algebra.

Acknowledgements

All your works praise you, O Lord.

Psalm 145:10

The writing of this thesis would not have been possible without the generous contributions of a great many people. I am deeply indebted to my supervisor, Peter Neumann, at whose suggestion the thesis was conceived and under whose guidance it took shape. I am most grateful for the many insights into algebra and its history that he has shared with me; it has been a great pleasure to learn from him and with him. That this thesis has been completed is due in no small measure to his constant encouragement and boundless enthusiasm.

I would like to record my gratitude to all those friends and colleagues in Oxford and elsewhere, far too many to name here, whose helpful and illuminating comments, advice and information have made this thesis so much more comprehensible and comprehensive than it would otherwise have been. My thanks go especially to those friends who have corrected and improved on my once rusty French and German and previously nonexistent Italian.

The staff of numerous libraries and archives throughout Europe have contributed, whether knowingly or unknowingly, to this study. Frau Letzel, the librarian in the *Sektion Mathematik* at the University of Leipzig, showed me much kindness and surprised me totally by bringing me volume after volume of Otto Hölder's own lecture notes. I was overwhelmed by the hospitality of Dr. and Frau Betsch in Tübingen, and am most grateful to Dr. Betsch for the generous gift of his time and effort in translating *Sütterlinschrift* into his own beautiful (and legible) German script.

The relaxed atmosphere of the algebra group at the University of Oxford has made life as a research student very enjoyable and worthwhile—my thanks to one and all, past and present. I am also aware of the support of many friends, especially that of Bill Tompson, which it is my pleasure to record with love and gratitude.

Finally I would like to say a huge Thank You to my family for their contributions, financial and otherwise, to the writing of this thesis. My father has never asked me what a group is (for which I certainly do not blame him!), but has nevertheless been happy to support me throughout my research. My mother once asked and was told (for which I commend her bravery!), and has nevertheless equally been happy to support me. I am very glad to dedicate this thesis to them.

Contents

Chapter 1	Introduction	1
Chapter 2	The Life of Otto Hölder	5
Chapter 3	Hölder's Algebraic Inheritance	27
Chapter 4	A Transitional Period for Galois Theory	37
Chapter 5	The Development and Understanding of the Concept of Quotient Group	60
Chapter 6	The Evolution of the Jordan–Hölder Theorem	84
Chapter 7	Starting the Search for Simple Groups	112
Chapter 8	The Structure Theory of Finite Groups	131
Chapter 9	The Beginnings of Group Extension Theory	165
Chapter 10	Conclusions	191
Appendix A	Career Paths of German Mathematicians in the Nineteenth Century	195
Appendix B	Obituary of Otto Hölder	204
Appendix C	Enrico Betti's Development of Group Theory	217
	References and Bibliography	246

Notation

The notation used by each author under consideration has been retained when his work is discussed, apart from a few instances in which sense or consistency have necessitated changes. Such changes are noted where they are made. The reader is warned that, although most authors were consistent in their own use of notation, many concepts were represented by a wide variety of symbols and terms throughout the nineteenth and early twentieth centuries. Some of these symbols and terms nowadays have a very definite and *different* meaning in group theory. For example, in this thesis, the symbol G' does not in general denote the commutator subgroup of the group G .

There is therefore no overall consistency of notation in the thesis. An attempt has nevertheless been made, in cases where little notation was provided by the original author, to use uppercase Roman (sometimes Greek) letters to denote groups and lowercase Roman (sometimes Greek) letters for group elements. Nineteenth-century authors often used uppercase Roman for group *elements* and uppercase Gothic or Greek for the groups themselves, reserving lowercase Roman for group orders and other numbers.

The standard modern notation for basic group theory and Galois theory has been employed throughout, when comparing nineteenth- and twentieth-century methods. It may be found in textbooks such as [Rotman 1984], [Rose 1978], [Macdonald 1968] and [Herstein 1964/1975]. The group generated by an element g is denoted by $\langle g \rangle$. The automorphism group of the group G is denoted by $\text{Aut}(G)$. Various standard (English) abbreviations for groups are employed. The cyclic group of order n is denoted by C_n , the symmetric and alternating groups of degree k by $\text{Sym}(k)$ or S_k and $\text{Alt}(k)$ or A_k , respectively. The general linear group of $n \times n$ matrices over a field F is written as $\text{GL}(n, F)$; when the field is finite of order q it becomes $\text{GL}(n, q)$. The special linear group, projective special linear group and projective general linear group are denoted, respectively, by $\text{SL}(n, F)$, $\text{PSL}(n, F)$ and $\text{PGL}(n, F)$. Notation for other groups is explained where it is introduced. The ring of integers modulo n is denoted by $\mathbb{Z}_n$; when n is prime, the multiplicative group of integers modulo n is $\mathbb{Z}_n^*$.

The German word *Satz* is translated as 'theorem', *Lehrsatz* becomes 'proposition' and *Hilfssatz* becomes 'lemma'. Many nineteenth-century authors used the term *permutation* for the modern term 'arrangement' and the term *substitution* for the modern 'permutation'. I use 'arrangement' and 'substitution' wherever possible to avoid confusion. Finally, the word 'group' is used to refer to finite groups throughout, apart from one or two occasions on which sense makes the meaning clear.

CHAPTER 1

Introduction

The history of pure group theory in the last decades of the nineteenth century has been a surprisingly neglected area of research. Literature describing and analysing the development of the subject at that time is relatively hard to find. There are of course some notable exceptions, among which is Wussing's book [1969/1984], providing a very thorough account of the genesis of the abstract group concept. Nevertheless, the fundamental changes in approach to group theory that began to take place during and after the 1880s have been very little studied. This thesis is an attempt to redress the balance to some degree, and to draw attention to a forgotten period in the history of group theory. The history of Galois theory is better documented, but again, the end of the nineteenth century has received the scantiest treatment. This time of change in Galois theory, which led to a completely new formulation of the theory, is also discussed here.

One of the mathematicians who contributed the most to the establishing of group theory in its 'twentieth-century' form is also the one whose contributions are least known nowadays. Otto Hölder's part in the development of group theory is remembered only in the name of one theorem, the Jordan–Hölder Theorem. He was, however, largely responsible for setting out the questions that have occupied finite group theorists during the twentieth century. A study of his papers on group theory and Galois theory, written between 1888 and 1898, provides a natural focus for an investigation of the history of group theory and Galois theory at that time. His work filled 350 journal pages; it was written with great clarity and thoroughness. As B.L. van der Waerden has commented, 'It is a pleasure to read his lucid papers' [1985, 155].

This thesis is an attempt to provide a critical analysis of Otto Hölder's published articles on group theory and Galois theory. There is a corresponding analysis of the context in which the papers were produced and the broader lines of development in algebra which led up to and followed from Otto Hölder's work. This task has been carried out from the viewpoint of a historian of mathematics employing the tools of modern algebra in order to interpret and evaluate Hölder's studies. The danger of this

approach, of which the author has tried to remain aware, is that modern views and insights may be wrongly imposed on Hölder's mathematics. The nineteenth-century authors must be allowed to speak for themselves, and not be squeezed into a twentieth-century mould.

The research that is discussed in this thesis is exclusively within 'pure' finite group theory. 'Applied' group theory, which I define as the theory of group characters and representations, was introduced by Frobenius, Burnside and Molien in the late 1890s, just after Hölder's group-theoretic research had been published. It seems that Hölder's interests were moving elsewhere at this time—he did not make any contribution to the theory of group representations.

The major problems of pure group theory, which have held the attention of numerous mathematicians throughout the twentieth century, were formulated by Otto Hölder during the late 1880s and early 1890s. He began work on solutions for these problems and largely set the direction of future research. He shared his very broad and deep understanding of the nature of group theory, which enabled others to carry on the investigations he had started. Thus it has happened that other mathematicians are remembered in connection with ideas that were originally discovered by Otto Hölder.

Biographical information about Otto Hölder, apart from that which is to be found in encyclopaediae, is extremely scarce. An obituary was written by B.L. van der Waerden after Hölder's death in 1937, but this contains only a short and rather factual biography. The main purpose of the obituary, originally produced for the Academy of Science at Leipzig, of which Otto Hölder had been a member since his arrival there in 1899, was to present a survey of Hölder's mathematical achievements. Van der Waerden succeeded admirably in this. His essay gives a clear account of the many and varied contributions of Otto Hölder to mathematics. Appendix B of this thesis consists of an annotated English translation of the obituary. A further account of Otto Hölder's life and work has been written by G. Eisenreich [1981], who was able to make use of documents preserved in the archives of the University of Leipzig.

The essay on the life of Otto Hölder that forms Chapter 2 of this thesis has been compiled from these and many other sources. It has proved surprisingly difficult to find or gain access to such documents as would usually be found in the *Nachlass* of a mathematician—correspondence, manuscripts and so on. There is apparently no collec-

tion of letters still in existence, apart from a small selection from Otto Hölder's student days, which are in the possession of his daughter-in-law, but as yet unavailable for inspection. This lack of biographical material is to a large extent a reflection of Hölder's character—he did not set out to be remembered but to create beautiful mathematics. His scientific achievements have indeed proved up to now to be his only memorial.

The career structure that existed for aspiring professors of mathematics in nineteenth-century Germany was in many respects different from the system that exists in academia nowadays. It is useful to have some knowledge of the older system, which is explained and discussed in Appendix A of this thesis.

Chapter 3 provides an overview of developments in Galois theory and group theory up to the time at which Hölder wrote on these subjects. He had a thorough knowledge of these developments and of the literature in which they were recorded. The way in which he used this knowledge is analysed in succeeding chapters, as his contributions to each of a series of specific areas and problems are discussed. His work is placed in context and compared and contrasted with twentieth-century ideas and techniques.

The re-formulation of Galois theory in the work of Otto Hölder and of others (Bachmann, Weber, Dedekind) during the 1880s and 1890s forms the topic of Chapter 4. Hölder's first contribution to Galois theory was the article 'Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen' ('Reduction of an arbitrary algebraic equation to a chain of equations') [1889a]. This paper is noteworthy more for the developments in pure group theory which are presented in it than for the application of these new ideas to the theory of equations. It is here that the concept of quotient group is clearly and thoroughly explained and the Jordan–Hölder Theorem for a finite group is proved. Chapters 5 and 6 deal in turn with this concept and this theorem, providing their respective histories up to the time of Otto Hölder. The substance of Chapter 5 is essentially the same as that of a published paper [Nicholson 1993].

Only the second half of Hölder's paper [1889a] is concerned with Galois theory. His insights here are explained in Chapter 4; comments are also provided on his other two notes on Galois theory. Both of these were written as a result of ideas which emerged from [1889a]. They are entitled 'Ueber den Söderberg'schen Beweis des Galois'schen Fundamentalsatzes' ('On Söderberg's proof of Galois' fundamental theorem') [1889b] and 'Ueber den Casus Irreducibilis bei der Gleichung dritten Grades' ('On the Casus

Irreducibilis for equations of degree three') [1891]. Hölder also provided an article on 'Galois'sche Theorie mit Anwendungen' ('Galois theory with applications') [1899] for the German encyclopedia of mathematics that was conceived during the 1890s.

The work of Enrico Betti during the early 1850s on substitution group theory and Galois theory prefigures the studies of Hölder, as discussed in Chapters 4, 5 and 6, in many ways. Betti's somewhat muddled exposition of group theory is explored and explained in Appendix C of this thesis.

Chapters 7, 8 and 9 focus on the central problems of pure group theory, on Otto Hölder's contributions to these problems and the influence of his work. The search for finite simple groups was begun in a systematic way with Hölder's paper 'Die einfachen Gruppen im ersten und zweiten Hundert der Ordnungszahlen' ('The simple groups of order less than two hundred') [1892]. Some ninety years later, the enormous programme of research which Hölder's article initiated was brought to completion with the classification theorem for finite simple groups.

The structure theory of finite groups is the subject of Chapter 8 of this thesis. Hölder contributed two papers, 'Die Gruppen der Ordnungen p^3 , pq^2 , pqr , p^4 ' ('The groups of orders p^3 , pq^2 , pqr , p^4 ') [1893] and 'Die Gruppen mit quadratfreier Ordnungszahl' ('The groups of squarefree order') [1895b]. His work is compared with that of other authors who published very similar studies at almost exactly the same time. This research of Hölder led him to establish the theory that is discussed in Chapter 9, the theory of group extensions. He devoted a long article to the introduction and investigation of this new theory, 'Bildung zusammengesetzter Gruppen' ('The construction of composite groups') [1895a]. It is not generally recognised that group extension theory had been studied before Otto Schreier's work of the 1920s.

The thesis concludes with some remarks about Otto Hölder's achievements. Various features of his approach to algebra and to mathematics as a whole, which have emerged in previous chapters, are drawn together here and commented on. The significance of his role in the development of Galois theory and of group theory will, the author hopes, by then have become clear.

CHAPTER 2

The Life of Otto Hölder

A complicated gentleman allow me to present,
Of all the arts and faculties the terse embodiment,
He's a great Arithmetician who can demonstrate with ease
That two and two are three, or five, or anything you please.

W.S. Gilbert: *Utopia Limited*

Otto Ludwig Hölder was born on 22nd December 1859 into a family of public officials and scholars. His father was a professor at Stuttgart's *Polytechnikum* and his grandfather had also been a teacher; his maternal grandfather had been *Oberkriegsrat* (head of the ministry for war) in Stuttgart. He had one brother Eduard, twelve years his senior, who became a university professor of law. Having been brought up in such an environment, it is not surprising that Otto Hölder also chose to pursue an academic career. In nineteenth-century Germany, university professors were accorded great respect and their profession was very highly regarded. The securing of a full professorship¹ required determination, perseverance and considerable support, largely financial, from one's family. Middle class families were generally quite content to support a son through the difficult and poorly paid years as he rose up the academic ladder. When Otto Hölder received his first full professorship he was thirty-six; before that time his salary would have been fairly small.

Student of mathematics

Hölder began his academic career in 1877, having received his school leaving certificate from the *Realgymnasium* in Stuttgart. He studied engineering science at the *Polytechnikum*², but remained there for only one year since it was suggested by a colleague of his father that he should go to Berlin if he wished to study mathematics. He took this advice and moved to Berlin, which was indeed one of the centres of mathematical

¹The structure of academic life that existed in Germany during Hölder's time is explained in Appendix A.

²This became the *technische Hochschule* in 1890.

education and research in Germany at the time. He consequently attended lectures by Weierstraß, Kronecker, Kummer, Kirchhoff and Bruns during his undergraduate days³. Since Hölder arrived late he had to catch up with the course on the theory of functions being taught by Weierstraß. In this way Hölder was inspired to pursue his own studies on the subject. Weierstraß, described by van der Waerden as Hölder's 'great teacher' [1985, 155], did in fact provide the inspiration and direction for much of Hölder's early research which was in function theory. He also acquired at this time (perhaps from Kummer) a love of number theory, a subject to which Hölder returned at the end of his career.

Here are some reminiscences written by Carl Runge, a fellow student of Hölder, about the lecturers at Berlin during this time [Lorey 1916, 161–162]:

Weierstraß' Vortrag war nicht glatt und formvollendet. Er stockte manchmal und verbesserte sich; aber im Grunde war seine Vorlesung doch sorgfältig durchdacht, wenn auch nicht im Einzelnen vorbereitet. Sie wirkte immer nachhaltig und tief, vielleicht gerade, weil sie improvisiert wurde, weil der Gedanke *in statu nascendi* einen tieferen Eindruck macht, als wenn er in vorher festgestellter Form reproduziert wird. Einige Male ist es freilich vorgekommen, daß Weierstraß einen Beweis verfehlte und sich verhedderte. Er fuhr dann in der nächsten Stunde fort, als ob er nichts zurückzunehmen hätte, und brachte dann alles in Ordnung. Ich habe ihm das einigermaßen übel genommen. Ich fand, er hätte seinen Zuhörern eingestehen müssen, daß der erste Weg falsch war, und daß er deshalb einen anderen eingeschlagen hätte. Niemals aber habe ich bemerkt, daß Weierstraß einen Schatten einer Unklarheit stehen ließ...

Weierstraß hatte Nachmittags eine Sprechstunde in seiner Wohnung, in der ich mich öfter eingefunden habe, um ihn um Aufklärung über mathematische Fragen zu bitten. Man ging immer sehr befriedigt von ihm, weil er immer sehr aufmerksam anhörte, was man zu sagen hatte, und genau darauf einging, sehr im Gegensatz zu Kronecker, der nicht zum Zuhören zu kriegen war, sondern immer gleich abschwenkte, um von seinen eigenen Arbeiten zu reden. [1]

He had more to say [Lorey 1916, 162] on the subject of Kronecker:

Seine Vorlesungen über Algebra waren sehr anregend, aber es fehlte ihnen das Fundamentale der Weierstraßschen Vorlesungen. Kronecker war persönlich weit zugänglicher als Weierstraß. Er liebte es, umgeben von einer Reihe seiner Schüler, die ihn alle um Hauptes Länge überragten, von der Universität über die Linden und durch den Tiergarten nach seiner Wohnung in der Bellevuestraße zu gehen, den ganzen Weg eifrig über Mathematik plaudernd, bisweilen stehen bleibend, um mit dem Stock in den Sand eine Figur zu zeichnen. In seinem gastfreien Hause fanden sich viele junge Leute zusammen. [2]

In the late 1870s, the university of Berlin provided a seminar for mathematics students.

³There are some twelve bound volumes of the notes Hölder made of these lectures in the library of the *Sektion Mathematik* at the University of Leipzig.

It was the first seminar to have been established in Germany exclusively for the study of mathematics. It had been created in 1861 by Weierstraß and Kummer; many ideas developed by Weierstraß' students had their origins here. Among those students were Frobenius, who contributed much original research of great importance to group theory, and Netto, whose work (in particular his textbooks) was influential in spreading the knowledge of group theory throughout Germany and abroad. Both Frobenius and Netto provided foundations on which Hölder was able to build. When Hölder arrived in Berlin, Frobenius had already left to take up a professorship in Zürich but Netto remained, and must have been in the early stages of producing his book [1882] on the theory of substitutions.

The atmosphere at Berlin during Hölder's time as a student was exceptional; after quoting from Runge's reminiscences about all the famous mathematicians who taught and worked there, Lorey concluded [1916, 163]:

Alle Namen kennzeichnen das Berlin der siebziger und achtziger Jahre als einen Mittelpunkt mathematischen Lebens, an das sich heute viele mit dankbarer Freude erinnern, die in jenen Jahren in Berlin studiert haben. [3]

As was the custom in late nineteenth-century Germany, Hölder did not remain in one place to complete his degree. After three semesters at Berlin, he moved to Tübingen (during Easter 1880) and it was there that he obtained his doctor's degree in August 1882. The degree he received was *Dr. scient. nat.*, doctor of natural sciences, rather than the usual *Dr. phil.* The university of Tübingen had decided to modernise its faculty system and in 1863 had established the first German faculty of natural sciences alongside the philosophy faculty. It was generally thought that mathematics should be part of both faculties but since that was impossible, mathematics was taught under the auspices of the natural sciences faculty. So it happened that Otto Hölder was awarded his doctorate as a natural scientist and not as a philosopher. He received the commendation *post cum laude comprobata eruditionem* ('after proven scholarship with merit'). This was the equivalent of a distinction and was the highest grade usually to be given.

Hölder wrote his doctoral dissertation under the supervision of Paul du Bois-Reymond, who had begun his career studying medicine and gradually turned to mathematical physics. This led him to the study of differential equations and to the theory of functions of real variables. At the time Hölder was with him, he was writing a book on

the fundamental concepts of mathematics (number, limit, function⁴), to which Hölder referred in his dissertation, which was entitled 'Beiträge zur Potentialtheorie' ('Contributions to Potential Theory'). He stated in the introduction that his aim was to provide rigorous function-theoretic proofs for the important theorems in potential theory: such rigour was missing from the proofs obtained previously. He also included some new ideas of his own and introduced the 'Hölder condition' which ensures that Poisson's equation holds for the potential of a continuous mass distribution. In the dissertation he referred extensively to earlier studies and results in function theory and mathematical physics, of which he obviously had a firm grasp. Hölder's dissertation was published in Stuttgart and copies sent to all German universities and various other institutions, as was customary⁵. In addition to the production of this dissertation, Hölder was required to pass oral examinations in his auxiliary subjects of physics and chemistry.

Doctor and lecturer in mathematics

Hölder then moved to Leipzig in order to work with Felix Klein and presumably to submit a *Habilitationsschrift* at his invitation. Klein had become professor at Leipzig in 1880. Hölder attended the mathematical seminar which Klein organised in his own distinctive way. He would ask the student who was to lecture at the seminar to discuss the topic and the lecture with him beforehand, and to write a report afterwards. It seems that Hölder and Klein were unable to work constructively together: their points of view were often completely different. Hölder had been trained in Weierstraß' school of mathematics at Berlin while Klein had inherited the philosophies of Plücker at Bonn and Clebsch at Göttingen. The basic approach of the Berlin school, one of rigorous deduction, was at odds with the more intuitive view that Klein held. Klein was in addition suffering during the early 1880s from exhaustion and depression brought on by an informal competition against Poincaré: they had both been aiming at the same results (a theory of automorphic functions). It is possible that Klein's subsequent lack

⁴*Die allgemeine Functionentheorie* [1882]. This book was written at the time when much thought was being given to the foundations of analysis; du Bois-Reymond was aware that his mathematical work needed a firm logical foundation on which to rest. Although he did not contribute very much to the construction of such foundations, he did discuss the philosophical problems involved and understood the strengths and weaknesses of the various philosophical viewpoints. A further contribution of his to this discussion was published posthumously [1890]. Hölder's later interest in and clear understanding of these viewpoints and his desire to see mathematics based on firm foundations must have been inspired to some degree by this work of his colleague and supervisor.

⁵The Bodleian Library in Oxford received a copy (which is now lost).

of motivation caused some of the difficulties with Hölder.

There was a further problem for Hölder in Leipzig: he was not in fact able to qualify as a lecturer (*Privatdozent*, the next step in an academic career) because he had attended the wrong sort of high school, one with a scientific rather than a classical bias⁶. So after one semester Hölder left Leipzig for Göttingen at the invitation of H.A. Schwarz (who was in sympathy with Hölder's approach to mathematics, having also learnt analysis from Weierstraß and being one of the leading exponents of his methods). Schwarz had held a professorship in Göttingen since 1875. Therefore if he had met Hölder before 1884 it would have been only for short periods during visits to each other's universities. When Hölder arrived in Göttingen, he had to be awarded a second doctorate, this time a *Dr. phil.*, which he received in July 1884. This was just a formality: the new degree (*Dr. scient. nat.*) he had been awarded in Tübingen was not recognised at Göttingen, where only a *Dr. phil.* was acceptable.

Hölder became *Privatdozent* in Göttingen on 25th July 1884; his examiners were the professors Stern, Schering and Voigt. A previously unpublished theorem of du Bois-Reymond concerning Fourier series formed the starting point for the investigations in Hölder's *Habilitationsschrift*. Having presented this piece of work, which was subsequently published in the *Mathematische Annalen*, Hölder was given the right to lecture. The lecture lists at Göttingen reveal that he usually offered one course of lectures each semester, at first in the areas of analysis and calculus (in which he was then researching), and gradually branching out to include number theory, geometry, algebraic equations and other topics. It was at this time that Hölder discovered the inequality for which he is largely remembered and which is an extension and generalisation of Schwarz' inequality (the same Schwarz who had brought Hölder to Göttingen).

In 1886 Klein arrived in Göttingen. His appointment was made by Friedrich Althoff, the minister for culture in Berlin, who was responsible for all appointments to professorships in Prussia. The move served the interests of both men. Althoff wished to set up a centre for mathematics alongside that at Berlin and had chosen Göttingen as its location. He was keen for Klein to work there. Klein himself wanted to build an internationally respected mathematical institute that would take up and work on his ideas.

⁶The Prussian educational authorities insisted on attendance at a *Gymnasium* as a prerequisite for teaching in a university. Hölder's school had been a *Realgymnasium*. The various types of high school were awarded equal status in Prussia only in 1900.

He felt that he would not be able to work them through himself since his health had suffered badly in the previous few years.

One of Klein's first lecture courses in Göttingen had for its subject 'Die Auflösung algebraischer Gleichungen' ('The solution of algebraic equations') and covered the Galois theory of equations. He gave these lectures in the summer semester of 1886, assisted by Oskar Bolza. It seems unlikely that Hölder would also have helped with the lectures but quite possible that the subject matter was discussed with him.

The coveted professorships

As early as 1886, Hölder began to be considered for assistant professorships. The death of Professor Enneper left a vacancy in the Göttingen faculty which they requested be filled by Hölder, who was described as having first class ability. However, when the proposal reached the ministry for culture in January 1887, it was rejected. Hölder had been teaching for only five semesters, which was considered insufficient. His appointment as *außerordentlicher Professor* was finally made in March 1889, but it was an unsalaried position. He moved back to Tübingen in the autumn of that year when a salaried post was offered to him. The decision to appoint Hölder as *außerordentlicher Professor* in Tübingen was made only after much deliberation on the part of the faculty and ministry for culture in Stuttgart. Here is what happened.

The process began in October 1888 when Brill, the *Dekan*⁷ of the natural science faculty and a professor of mathematics, submitted a preliminary proposal of candidates for the professorship to the academic senate. Hölder was the choice of the faculty; they praised his 'valuable papers' on the theory of functions and noted his ability as a teacher and concern for his students. Only in May 1889 was the final proposal drafted. The other two candidates put forward were Hilbert, then a *Privatdozent* of three years' standing at Königsberg, and Finsterwalder, who had only recently qualified as a lecturer at the *technischen Hochschule*, München. A report was submitted to the ministry for culture and the nomination of Otto Hölder was accepted. The ministry wrote to the chancellor of the university on 2nd July, asking him to contact Hölder and find out if he would be willing to accept the position.

⁷The full professors in each faculty chose a *Dekan* from among themselves every six or twelve months. He was responsible for the administration of the faculty.

The chancellor promptly sent a letter to Göttingen and received a reply from Erlangen, not from Otto Hölder but from his brother Eduard (a professor of law at the university there). Otto Hölder had suffered an attack of mental illness—he later referred to it himself as a 'schwere Erkrankung'. He was convalescing in a clinic in Erlangen. The faculty were confidentially informed of this unexpected state of affairs and were compelled to think again. They concluded that they should first obtain a full, official doctor's report and let any decision depend on the results of this. The report duly arrived: the director of the Erlangen clinic had diagnosed a speedy recovery and considered that a relapse was 'not at all probable' ('nicht gerade wahrscheinlich').

The faculty discussed their feelings honestly: it would be in their own interests to pass over Otto Hölder and consider another candidate, but they were not prepared to do this since they had (albeit informally) offered Hölder the job. They came to the conclusion that if the ministry would defer making a final decision for a little longer, they would suggest a provisional replacement to assume teaching duties in the winter semester 1889–1890. The minister agreed to this, asked that the faculty should nominate a suitable replacement and emphasised that the ministry did not intend to give up on the possibility of Hölder's appointment. A lecturer was found—Dr. Max Bernhard from the *Realgymnasium* in Ulm.

By the middle of August 1889, the ministry was addressing the question of Hölder's employment once again. Perhaps he could be offered a temporary job? Naturally it would be made permanent once it was certain that he had recovered his health. But would he agree to being treated in such a way?

At this point in the proceedings the chancellor received a letter from Otto Hölder. The clinic had discharged him on 14th July and he was having a further period of convalescence in Heiligenberg, a village some fifteen kilometres north of Lake Konstanz. Hölder expressed his gratitude that the faculty had not withdrawn their offer after becoming aware of his illness. The doctor expected that Hölder would regain his full capacity for work by October. He intended to remain in Heiligenberg until mid September and then stay in Stuttgart (with his family?) for the rest of the academic holiday. He was indeed willing to take the professorship under the conditions that had originally been offered, but asked for the understanding of the faculty if he was not able to work at full strength in the forthcoming winter semester.

The suggestion that Hölder should be offered temporary employment had meanwhile to be discussed with Brill (the *Dekan*), who was inconveniently on holiday at the time in Arlberg. The chancellor tried to contact him but it was eventually his wife (who appears to have stayed at home in Tübingen) who wrote to inform him of the latest developments. Brill wrote to the chancellor outlining what was the situation as far as the faculty was concerned. They were keen for Hölder and for the temporary lecturer Dr. Bernhard to be informed of their teaching duties, especially the lecture courses which would be new to them and require preparation. Brill did not seem to oppose the idea of Hölder being offered a temporary contract, so the chancellor wrote to him to explain these matters.

Hölder, who was still in his holiday village, had clearly regained some of his strength and clarity of thought. He pointed out that he was not entirely unprepared to give this 'new' lecture course, since he had recently lectured on a related subject. Although he understood the reasons for the offer of a provisional appointment, he was not prepared to accept. He was already an *Extraordinarius* in permanent employment; to resign in favour of a temporary position would be foolish. Indeed he claimed that it was not possible to leave the service of the Prussian state for anything less than a permanent professorship elsewhere.

This argument prevailed and the ministry pressed on with Hölder's appointment. It was made on 31st August 1889 by Karl I of Württemberg. Hölder accepted immediately and began negotiations about his conditions of employment (this was usual practice). He managed to have his years as *Privatdozent* and *außerordentlicher Professor* in Göttingen included in his period of pensionable service, his moving expenses were reimbursed and a lump sum (three months' salary) was paid into the widows' pension fund on his behalf. This was forward thinking for a man who was not to marry for another ten years.

It was unclear if Hölder would be fit enough to lecture in the winter semester of 1889–1890. In fact he gave no lectures in that semester but was simply required to teach in the mathematical seminar. He waited until June 1890 to give his inaugural lecture⁸ to the university. The title was 'Über Axiome und Beweise in der Mathematik' ('On axioms and proofs in mathematics'). During his time in Tübingen in the 1890s

⁸The advertisement giving notice of his lecture, signed by those who attended it, is preserved in the Universitätsarchiv Tübingen [Personalakte Hölder 126/288].

he lectured on a wide variety of subjects: geometry, number theory, algebra (including group theory and Galois theory) and calculus, to name a few.

Hölder's name appeared on shortlists for various other assistant and full professorships from the late 1880s onwards, for example in Berlin, as is recorded in the archives there⁹. Hölder was the third choice of the selection committee when Netto left his assistant professorship in Berlin in 1888. The selectors said of Hölder [Biermann 1973, 200/1988, 300]:

Derselbe, 1859 geboren, hat sich durch tüchtige wissenschaftliche Arbeiten bereits einen ganz guten Namen erworben, ist in der akademischen Lehrthätigkeit geschult und würde als Lehrer ebenfalls dem hiesigen Bedürfniß gut entsprechen. [4]

The assistant professorship was taken up by Knoblauch, the second-choice candidate.

After Fuchs's sudden death in 1902, a full professorship at Berlin became vacant and Hölder was again in third place on the shortlist. The committee were far more enthusiastic in their praise of Hilbert and Schottky, the other two nominees, than of Hölder. They noted of Hölder that his 'talent lies more in the direction of criticism than in that of creativity' ('dessen Begabung mehr in der kritischen, als in der schöpferischen Richtung liege') [Biermann 1973, 130/1988, 166]. However, when Hölder's name was sent to the ministry, he was rejected as a suitable candidate; the reason given later was that he was 'sickly' ('kränklich').

The discussion of Hölder's work in the committee's proposal for this appointment is interesting for the significance attached to the various contributions Hölder had made to mathematical research. The committee began by noting that Hölder had 'devoted a whole series of investigations to group theory' but went on to say that his work in analysis and mechanics was far more important [Biermann 1973, 211/1988, 312]. This is curious in view of the fact that the previous ten years had been taken up more with group theory than with any other subject: Hölder's work in analysis had been completed much earlier. Perhaps Frobenius' presence on the committee influenced its view of Hölder's work—the direction in which Frobenius' research in group theory had led was very different from Hölder's. Another committee member was H.A. Schwarz who had been Hölder's mentor at Göttingen and must have known and valued his work in analysis.

⁹Kurt Biermann includes the shortlists and committee reports for these posts in his book [1973/1988].

Hölder moved only twice more in his life, the first time in October 1896 to Königsberg, the second in April 1899 to Leipzig. Ernst Hölder¹⁰, in the article he wrote about his father for the *Dictionary of Scientific Biography*, is of the opinion that the first years of Otto Hölder's career up to 1896 were the most fruitful, and certainly the work for which he is remembered was completed during that time.

Otto Hölder's mathematics

Hölder's published mathematics comprises some sixty papers and two books. He wrote on a wide variety of subjects, to most of which he made original contributions: analysis, potential theory, Galois theory, group theory, geometry, number theory, mechanics, logic, philosophy of mathematics, and others. He started publishing his mathematics in 1882. Almost all his early papers appeared in *Mathematische Annalen* which was then edited by Klein. While Hölder was in Göttingen he published in the university's own journal, but after his return to Tübingen he chose to use *Mathematische Annalen* again (still under Klein's editorship). All but one of his published articles on Galois theory and group theory thus appeared in the *Annalen*; the final one in 1895 he sent to the *Nachrichten* in Göttingen.

It would be interesting to know the extent to which Hölder corresponded with other mathematicians on the subject of algebra, or indeed on any of the many subjects to which he contributed. There seems to be very little such correspondence still in existence; I have found none in the collected works of Hölder's fellow mathematicians. There are references to letters written by Hölder, to advice he gave, and to criticism he provided of other mathematicians' work in various papers and books of the time. Burnside sent Hölder a draft of part of his book on group theory [1897] and received Hölder's comments and suggestions. Jourdain quoted in a paper of 1908 from a letter Hölder had written him to clear up a dispute (this is discussed in the translation of Hölder's obituary in Appendix B). Nevertheless Hölder appears to have been a very private man [Van der Waerden 1938, 98]:

Er besuchte selten oder nie Kongresse und machte nicht das, was man heute "Betrieb" nennt. [5]

The only mention of Hölder in the records of the Colloquia held at Göttingen (he worked there from 1884 to 1889) is for June 1884, when he lectured on 'Das Verhalten

¹⁰Otto Hölder's eldest son, also a mathematician (b. 1901 Leipzig, d. 1990 Mainz)

der Modulfunctionen bei Unendlichen' ('The behaviour of the modular functions at infinity'). He lived a quiet life in his family circle and was remembered for his calm and friendly manner, the 'nobility of his thought' and his integrity. Van der Waerden wrote that Hölder's mathematical work and his family were the substance of his life [1938, 102]. In much of his work the methods he used and developed were his own and the problems he attacked were either new or approached in an original way. On the other hand he was extremely well-read in many different areas of mathematics and always gave credit to those whose results he employed.

The unfinished encyclopedia

There was one ambitious project in which Hölder took an active part at the turn of the century. This was the production of an up-to-date encyclopedia of mathematics, the *Encyklopädie der mathematischen Wissenschaften*. It proved rather too ambitious a project and was never completed, although most of pure and applied mathematics was covered in the volumes that were published. There is an account of the history of the enterprise by Walther von Dyck at the beginning of the first volume [Dyck 1898–1904], from which the following is taken.

The idea of such an encyclopedia was conceived by Felix Klein, Heinrich Weber and Franz Meyer; they first discussed it in depth in September 1894. Their aim was that the many mathematical discoveries of the previous century should be gathered together in a logical and coherent manner. They realised that such a task would require the cooperation of a large number of German mathematicians, and help from abroad 'if it should be needed'! At a meeting of the Deutsche Mathematiker-Vereinigung, also in September 1894, it was decided to adopt this plan for an encyclopedia and to ask for help, both scholarly and financial, from universities and academies. An outline plan and budget was accepted by the universities of Göttingen, München and Vienna but rejected by Leipzig (whose professors did not believe adequate resources existed). The three assenting universities appointed representatives (Klein, von Dyck and von Escherich, respectively) to draw up a precise plan and to form part of the academic commission which would be responsible for publication. The commission was strengthened by various others, including Otto Hölder, who joined just after the turn of the century when the university of Leipzig decided after all to become involved.

A separate committee was responsible for editing, which was divided up by subject area. Originally six volumes were planned: three for pure mathematics, two for applied and one for historical, philosophical and didactic themes. As the encyclopedia took shape it became clear that more volumes would be necessary.

The committee began to look for a large group of contributors who were given a set of seventeen rules (plus four extra for applied mathematics) defining the way in which the article should be constructed and written. By 1897 the volumes on pure mathematics had been planned and contributors found. Otto Hölder was asked to provide an essay on Galois theory and its applications, which appears in Part 1 of Volume I (the volume is entitled *Arithmetik und Algebra*; Hölder's article was first published in Book 4 in October 1899).

The planning of the volumes on applied mathematics involved Klein travelling throughout Europe to gain support and help. It was considered especially important, remarked von Dyck, that the contributions on applied mathematics should reflect work done outside as well as inside Germany! He gave a list [Dyck 1898–1904, XIV], with which he was obviously pleased, of those ten foreign countries that had provided contributors before 1904.

In 1900 it was agreed to produce a French translation of the encyclopedia, when only one volume of the German edition had been published. Contributors were given the opportunity to expand and revise their essays and the translators were encouraged to rewrite the articles in 'French style'. The outbreak of World War I brought this Franco-German collaboration to an end, but after the war a few books were published [Lorey 1916, 235–236].

Publication of the German edition continued until 1935. By this time six volumes had been produced: the encyclopedia was never completed. Because some forty years had passed since the first parts of the encyclopedia were written, the time had come to revise these and so a second edition was planned. The essay on group theory was assigned to Wilhelm Magnus [1939]. It has the same format as the original article by Burkhardt [1898]: short paragraphs on each concept with numerous footnotes and references. This format must have been carefully worked out as it makes the articles both very clear and very informative, a combination not always easy to achieve. Only one volume of the proposed second edition was ever published, that on Algebra and

Number Theory containing Magnus' article. Even this volume is not complete; instead, part of the material that it should have contained appeared in booklets. The Second World War prevented any publication for a while and although it was resumed in the 1950s, nothing more than individual essays came out. The last publication was made in 1967.

Teacher and administrator in Leipzig

Shortly before moving to Leipzig in 1899, Otto Hölder married Helene Lautenschlager (1871–1927) who also came from a Stuttgart family. It seems likely that they had been waiting (for a considerable time) for Hölder to be offered a full professorship before they could marry. Anything less than a professor's salary was generally considered insufficient in academic circles of the time for the support of a family. Their family soon grew; their eldest son Ernst was born in 1901 and was followed by another son, Wolfgang, and two daughters, Lotte and Irmgard. Ernst Hölder later became a colleague of his father in the mathematical circle at the university of Leipzig.

Otto Hölder's election as an *Ordinarius* at Leipzig was (as was usually the case) a long and tiresome business, involving a large number of committee meetings, reports and letters. The ministry for culture in Dresden kept an eye on proceedings within the university and made the final decision. Lie was also involved in the selection of his successor before he left for Norway in June 1898. The selection committee, consisting of professors in the university at Leipzig, recommended Hilbert (by then at Göttingen) for the vacant position. Unfortunately for them, he turned down the offer of appointment and they were required to make further suggestions. They were in favour of Weber, who was teaching in Strasbourg, but he also could not be obtained.

Their choices were always influenced by the subjects which then needed to be better taught at Leipzig: geometry was one of these. Indeed, the vacant professorship was the chair of geometry, created in 1880 and first occupied by Klein. Before that time, there had been a long neglect of geometry at Leipzig since the system of 'academic freedom' ensured that no-one had to teach it if he did not want to. The requirement that geometry be one of the responsibilities attached to this professorship has since been removed.

The committee's thoughts turned to Otto Hölder, but his selection was not unani-

mously approved. The professors Scheibner and Neumann wished to ensure that the spirit of Lie was not lost and in a separate proposal they put forward the name of Engel. Lie himself, laying aside personal satisfaction, expressed his desire that Hölder be offered the job as he was the most suitable candidate. The view of Lie and of the majority of the faculty prevailed. They noted that Hölder would in fact have been their first choice some time before as the replacement for Professor Scheibner,

... wenn nicht leider eben im Zuge der ersten Sitzung ungünstige Berichte über seinen Gesundheitszustand eingelaufen wären. [6] [Universitätsarchiv Leipzig PA 583, 9–10]

Happily they were now informed, presumably by his brother who had become professor of law at Leipzig by 1898, that Otto Hölder's health was satisfactory; therefore they saw no obstacle to his selection. They reported that

Hölder's Arbeiten ... zeichnen sich ganz besonders durch Strenge der Schlußweise und Feinheit der Untersuchungsmethoden aus. [7] [Universitätsarchiv Leipzig PA 583, 9–10]

They were reliably informed that Hölder was

... ein vorzüglicher und ungemein beliebter Docent, der auch in seinen Uebungen einen höchst anregenden Einfluß auf seine Schüler ausübt. [8] [Universitätsarchiv Leipzig PA 583, 9–10]

The selection committee were certain that Hölder would accept a call to Leipzig, which it seems he was indeed glad to do.

The appointment was officially made on 1st October 1898. Otto Hölder had written to Lie in September, expressing the hope that they would meet before Lie left for Christiania (Oslo), but by then Lie had already gone. Hölder remarked that it would be no easy task to succeed Lie, and he asked for Lie's 'advice' about the mathematical environment at Leipzig. This letter reached Lie in Norway; it is preserved in his archive there [Brevs. til Sophus Lie, no. 289]. I do not know if Lie's reply is still in existence.

Hölder began his duties in the summer semester of 1899. Two weeks after he started teaching, the faculty received a letter from the ministry in Dresden, informing them that Otto Hölder was to take over the direction of the mathematical institute and to be a joint director of the mathematical seminar [Universitätsarchiv Leipzig PA 583, 31]. Hölder appears to have taken these responsibilities seriously and did much to improve the mathematics faculty's accommodation and teaching.

The mathematical seminar at Leipzig was founded in 1881 by Klein, soon after his arrival there as professor of geometry. Five years later it became the mathematical

institute after Klein had initiated various changes, including the appointment of research assistants and more concentrated study. An ongoing problem was lack of space; the situation improved only at the beginning of the twentieth century.

In 1905 the building previously used for teaching physics became available and was designated the new *Mathematisches Institut*. This was the first time that all mathematics teaching had been brought under one roof at Leipzig. Hölder, together with the other *Ordinarien* Rohn and Herglotz, was much involved in the conception and setting up of this institute. The history of the institute, including such details as its opening hours and the capacity of each lecture room, was later written for the 500th anniversary celebrations of the university [Hölder & Rohn 1909].

Hölder held several faculty and university appointments: he was prochancellor in 1903–1904, *Dekan* of the natural science faculty 1912–1913 and *Rektor* of the university 1918–1919. The rector was elected each year by the entire university senate. The position was intended to be that of a figurehead to represent the university to the outside world, rather than a ‘head professorship’.

During his twenty-nine years as *Ordinarius* at Leipzig (and on into his retirement), Hölder often lectured on algebraic subjects, mainly algebraic equations and Galois theory. He gave some twenty-two lecture courses on these topics. Clearly his interest in the theory of groups and solution of equations did not diminish after he ceased doing original research in these areas.

What were the students’ opinions of Hölder’s lectures? Here is the view of one of his listeners at Leipzig [Lorey 1916, 368]:

Hölder befließt sich in seinen Vorlesungen einer großen Strenge der Beweisführung und tut nicht den kleinsten Schritt, ohne ihn ausführlich zu begründen. Von großer Bedeutung für das mathematische Leben sind stets sein funktionentheoretisches Seminar und die funktionentheoretischen Übungen für Fortgeschrittene. [9]

Hölder was always keen that his students should have some knowledge of the foundations of their subject, in particular the foundations of the real number system. He gave a mini-lecture on this topic from time to time for the advanced students. Eventually the lecture notes were turned into a short book [1914]. In the preface to the second edition (written in 1928), he stated that his viewpoint when discussing these questions was principally that of Weierstraß, and was thus a viewpoint Hölder had acquired during his student days.

As a full professor, Hölder was 'supervisor' and examiner to some of those writing doctoral dissertations and *Habilitationsschriften*. (Only the *Ordinarien* were allowed to perform these tasks.) The amount of supervision given was not usually very great, but it is likely that Hölder would have taken his position seriously. These students, and to a lesser extent those who attended his lectures, were those whose own work was directly influenced by Hölder.

Lorey [1916, 369] notes that there had been relatively few doctoral degrees in mathematics awarded at Leipzig in the years just before he wrote his book (although many students would have taken a teaching degree in mathematics). Between 1899 and 1909 there were only ten mathematics dissertations written. One of these was concerned with 'the Hölder function $F(u) \dots$ '. Hölder's influence seems to have been especially felt in the area of function theory, even though his own research interests had by the 1900s moved elsewhere.

Several of Hölder's inaugural lectures were produced as booklets (this was not uncommon). The lectures were given before the entire faculty and other notables of the university and were intended to give an opportunity to present one's own work to a general academic audience. This could prove a difficult task with mathematics, since it was not a suitable forum in which to include technical ideas and details. Mathematicians tended to give more philosophical lectures discussing the foundations of their subject or, as Hölder did in 1918, the relationship of mathematics with the other sciences. Hölder talked of mathematics as the science that uses 'deduction'. He claimed that in mathematics there is a specific way of thinking which is different from that used in the other sciences or in everyday life, and proceeded to elaborate on this claim and give some specific examples from various mathematical disciplines.

The lecture of 1918 was given to mark the occasion of Hölder's assuming the rectorship of the university. A ceremony took place in which the office was formally transferred; the oath was administered by the retiring rector [*Rektorwechsel an der Universität Leipzig* 1918]:

Sie schwören bei Gott, dem Allmächtigen und Allwissenden, daß Sie die Verfassung und die Gesetze der Universität treu und gewissenhaft beobachten und die Pflichten des Ihnen anvertrauten Amtes als Rektor nach bestem Wissen und Gewissen erfüllen wollen. [10]

At this time Hölder was deeply involved in the research and thought that would lead to

the publication in 1924 of *Die mathematische Methode*, his longest work with 556 pages of text. He explained in the preface to this book that he had decided some years before to make public the philosophy on which his work had rested. He claimed in particular the influence of Kant on his thought: his aim was to answer a question posed by Kant, 'Wie ist reine Mathematik möglich?' ('How is pure mathematics possible?') [1924, 3]. There had been several recent attempts to discuss the foundations of mathematics: from 1903 to 1910 works by Russell¹¹, Couturat¹² and Natorp¹³ appeared. However, Otto Hölder was not satisfied by the answers given in these studies and he resolved to write his own account. He was careful to develop the mathematical results he needed and not just make use of them; consequently, the book is rather large. It covers a wide area within pure and applied mathematics, which Hölder considered to be necessary in order for his ideas to have credibility.

The book was published at a time when the German economy was in crisis after the runaway inflation of 1923; Hölder was especially grateful to the publishers (Springer) for agreeing to produce such a book at that difficult time. Since it represented ten years of effort on his part, his thanks were most certainly sincere. In fact, the Springer cousins (Ferdinand and Julius) who directed the publishing firm had decided during World War I that they would continue to produce scientific books and journals even though the economic situation was far from secure. Their view was not shared by the Leipzig publisher, B.G. Teubner, who was keen to reduce his output and especially to hand over responsibility for the *Mathematische Annalen*. The journal was taken over by Springer in 1920, who went on to publish books and journals throughout the hard years of economic depression in Germany.

In the summer of 1922 the first post-War meeting of German scientists took place in Leipzig. Earlier in the year, there had been celebrations to mark Hilbert's sixtieth birthday. These events served to set the German mathematical community back on its feet and gave renewed impetus to mathematical research.

Retirement

Hölder retired as professor of mathematics in Leipzig in April 1928 at the age of

¹¹The *Principia Mathematica*

¹²An edition of the *Principia* in 1905 with commentary on contemporary works on the subject

¹³Natorp, P. 1910. *Die logischen Grundlagen der exakten Wissenschaften*

68 (this was the retirement age for professors at that time; see [Reid 1976, 171]). The American Mathematical Society announcement¹⁴ ran as follows: 'Professor Otto Hölder, of the University of Greifswald, has retired.' (The town of Greifswald is in the north east of modern Germany.) Hölder remained the director of the mathematical institute in Leipzig for over a year after his official retirement, since a replacement could not be found. In fact he continued to carry out the duties of a professor until the arrival of B.L. van der Waerden in 1931, and gave lectures until 1935–1936. On his 70th birthday in December 1929, the faculty sent him their 'warm good wishes' and their sincere thanks for the enormous contribution Otto Hölder had made to the study of mathematics at Leipzig:

Ihre Wissenschaft zählt Sie zu ihren Pionieren. Was Sie in langen arbeitsreichen Jahren auf den Gebieten der Potential- und der Gruppentheorie, der Analysis, der Mechanik, was Sie ferner für die Grundlagen der Geometrie und Arithmetik und ganz allgemein für die mathematische Methode und Logik zu leisten so glücklich waren, sichert Ihrem Namen eine bleibende Stätte in der Geschichte der Mathematik und hat ihn sogar mit den Grundlagen der von Ihnen gepflegten Wissenszweige unauflöslich verknüpft.

Als akademischen Lehrer haben wir Sie durch lange Jahre in weitgestecktem Wirkungskreise mit so grossem Erfolg wirken sehen, dass die Philosophische Fakultät unserer Universität dieser ihr zu reichem Nutzen zugeflossenen Tätigkeit heute ganz besonderen Dank zollen muss. Die Dankbarkeit und Anhänglichkeit Ihrer zahlreichen Schüler begleitet Sie als schönster Lohn für diese Ihre stets hilfsbereite, immer Ihr Bestes spendende Erziehung zu wissenschaftlichem Denken und Forschen.

Die akademische Jugend weiss auch, was Sie ihr als menschliches Vorbild bedeutet haben, wie auch wir in Ihnen neben dem bedeutenden Gelehrten und ausgezeichneten Lehrer nicht weniger den liebenswerten, vornehmen Kollegen verehren. Mögen Ihnen, das ist unser aufrichtiger Wunsch, noch lange Jahre des Wirkens im Dienst der deutschen Wissenschaft beschieden sein! [11] [Universitätsarchiv Leipzig PA 583, 46–47]

Hölder did indeed enjoy mathematics in his retirement: he continued to write and publish papers and lecture at the university. He requested in 1935 to be permitted to lecture again on 'The Galois Theory of Equations'. By this time the National Socialists were in power and his request had to pass through various official channels before it was accepted; emeritus professors were (by a decree of May 1935) not supposed to offer such lectures. It seems that these lectures were the last he was allowed to give.

In Hölder's letter of thanks for the faculty's birthday wishes, he expressed his gratitude that he had been appointed on his arrival in Leipzig to a position of great influence (the directorship of the mathematical institute and supervision of the seminar). He obviously

¹⁴ *Bulletin* (1928) 34, 536

took much pleasure in passing on mathematical ideas to students and was a diligent and sensitive teacher. He appreciated the efficiency of the faculty in dealing with business affairs; as director of the institute he would have been involved in such matters.

It was traditional by the 1930s in Germany for the 50th anniversary of the receiving of one's doctorate to be remembered. Otto Hölder was congratulated twice on this account since he had two doctorates. His 'golden anniversary' was recorded in the newspapers in poetic terms:

Besonders Hölder's Schüler werden wieder an die Zeit erinnert, da sie zu Füßen des Meisters der reinen Mathematik saßen, an die Zeit, wo sie in das Examen stiegen. Nichts charakterisiert das umfangreiche Wissen und Können Hölder's mehr, als sein Verlangen, daß seine Prüflinge in allen Disziplinen der Mathematik eingehend Bescheid wüßten. Insofern gehört Hölder zur alten Schule, denn sein Arbeitsgebiet war die gesamte Mathematik. [12] [Universitätsarchiv Leipzig PA 583, 58]

Hölder must have been especially gifted as a teacher—he was complimented and recommended for this ability all through his career.

Early on the morning of Sunday 29th August 1937, Otto Hölder died following an operation. His four children and three grandchildren were by this time living variously in Leipzig, Chicago and Baltimore; his wife had died ten years previously. As well as the several obituaries which appeared in German newspapers, B.L. van der Waerden was asked to produce one for the *Sächsische Akademie der Wissenschaften*, of which Otto Hölder had been a member since his arrival in Leipzig in 1899. Van der Waerden had been Hölder's colleague and successor as *Ordinarius* at the mathematical institute at Leipzig and was also an editor of *Mathematische Annalen*. An extended and more detailed version of the obituary was published by this journal in 1939. Here van der Waerden gave a comprehensive account of Hölder's work; an English translation is provided in Appendix B.

Mit seinem Hinscheiden verliert das wissenschaftliche Deutschland einen Gelehrten, dessen umfangreiches Wissen und warmherzige menschliche Art unvergessen bleiben. [13] [Newspaper cutting, Universitätsarchiv Leipzig PA 583, 71]

Notes

- 1 Weierstraß' lectures were not polished and well-constructed. He faltered sometimes and corrected himself; but basically his lectures were still carefully thought out, even if not prepared in detail. His lectures left a lasting and deep impression, perhaps pre-

cisely because they were improvised, since thought *in statu nascendi* makes a deeper impression than if it is reproduced from a preconstructed form. Admittedly it happened several times that Weierstraß missed a proof and got into a tangle. Then he continued in the next lecture as if nothing had gone amiss and put everything in its place. I was fairly annoyed with him about this. I thought that he should have admitted to his listeners that the first direction was wrong and that was why he took a different one. However, I never noticed that Weierstraß let any shadow of uncertainty remain ...

Weierstraß had an open tutorial hour in his apartment in the afternoons, which I attended fairly frequently in order to ask him for clarification about mathematical problems. One always left very satisfied, since he always listened most attentively to what one had to say. Also he really responded to the question, very much in contrast to Kronecker, who could not be made to listen, but always changed the subject straight away to talk about his own work.

2 His lectures on algebra were very stimulating but they lacked the basic quality of Weierstraß' lectures. Kronecker was personally far more approachable than Weierstraß. Surrounded by a group of his students, who all towered a head above him, he loved to walk from the university along 'Unter den Linden' and through the Tiergarten to his apartment in Bellevuestraße. The whole way he would chat avidly about mathematics, stopping from time to time in order to draw a figure in the sand with a stick. Many young people met in his hospitable home.

3 All these names characterise the Berlin of the seventies and eighties as a centre of mathematical life, which many who studied in Berlin during those years remember today with gratitude and delight.

4 The gentleman in question, born 1859, has already acquired quite a good name for himself through competent scientific works, is schooled in academic teaching and would likewise, as a teacher, answer well the local requirement.

5 He seldom or never attended congresses and was not keen on social recognition.

6 ... had not unfavourable reports about the state of his health unfortunately arrived just as the first session was taking place.

7 Hölder's work ... distinguishes itself especially by the sharpness of its conclusions and the refinement of its methods of investigation.

8 ... a first-rate and unusually popular lecturer, who also exerts a highly stimulating influence on his students in his exercises.

9 In his lectures Hölder cultivates a great strictness of reasoning and does not make the smallest step without justifying it in detail. His function-theoretic seminar and the function-theoretic exercises for advanced students are always of great significance for the mathematical life.

10 You swear by God, the almighty and all-knowing, that you will observe the constitution and the statutes of the university faithfully and conscientiously and fulfil the duties of the office entrusted to you as rector to the best of your knowledge and conscience.

11 Your science numbers you amongst its pioneers. What you were so fortunate to achieve in long and hardworking years in the fields of potential and group theory, of analysis, of mechanics, and further for the foundations of geometry and arithmetic and generally for mathematical methodology and logic, secures an abiding place for your name in the history of mathematics. It has even indissolubly joined your name with the foundations of the branches of science you served.

As academic teacher we have seen you work for long years with a wide sphere of influence, with such great success that the philosophy faculty of our university must today pay especial thanks for this truly beneficial activity. The gratitude and devotion of your numerous students accompanies you as the finest reward for your schooling them towards scientific thinking and researching. In this you were ever ready to help and always gave of your best.

The academic youth also knows what you have meant to them as a character model, just as we also honour the endearing, distinguished colleague in you no less than the important scholar and excellent teacher. Our sincere wish is that you may enjoy many more years of activity in the service of German science.

12 Above all Hölder's students will again be reminded of the time when they sat at the feet of the master of pure mathematics, of the time when they sat their examinations. Nothing characterises the extensive knowledge and ability of Hölder more than his desire that his examinees should be thoroughly well-informed in all disciplines of mathematics. Hölder belongs to the old school, inasmuch as his field of study was the

whole of mathematics.

13 With his departure the scientific community in Germany loses a scholar whose extensive knowledge and warmhearted open nature will not be forgotten.

CHAPTER 3

Hölder's Algebraic Inheritance

Books must follow sciences, and not sciences books.

Francis Bacon: *Resuscitatio* (1657)

By the late 1880s, when Otto Hölder began to think creatively about group theory and Galois theory, he had studied and assimilated a large body of material on those subjects. Much of the earlier work on which he drew is discussed in its appropriate context in subsequent chapters of the thesis. This chapter provides a survey of some of the developments in algebra during the mid and late nineteenth century which form the background to Hölder's studies. The influence of various mathematicians, for example Klein and Kronecker, is particularly evident in Hölder's work. Therefore consideration is also given here to the contributions of these mathematicians. In addition, the understanding of group theory and Galois theory as revealed in the textbooks available to Otto Hölder is outlined.

Galois theory

The creation and development of Galois theory, and the changes in the way it has been presented since Galois' original formulation of the theory during the late 1820s and early 1830s, have been discussed in particular by Kiernan [1971], Scholz [1990], Toti Rigatelli [1989], Wussing [1969/1984] and Van der Waerden [1972]. There had been a renewal of interest in Galois theory in Germany during the 1880s, in part as a result of the development of the theory of abstract groups. Before this time the basic approach to Galois theory was in essence that of Galois; now, various papers were written to bring Galois theory 'up to date'. This work led in turn to the publication of a German translation by H. Maser of the works of Abel and Galois on the solution of equations, which appeared in 1889.

The only source for Galois' papers before this time was volume 11 of the *Journal de Mathématiques Pures et Appliquées* (1846), sometimes referred to as 'Liouville's Journal'. Here Liouville provided a short introduction to Galois' writings but no commentary. Such a commentary was first attempted by Enrico Betti [1852], whose un-

derstanding of Galois' theory was somewhat muddled. This commentary appears to have had little influence; the work of Camille Jordan during the 1860s provided a more influential and comprehensible account of Galois' ideas.

The collected works of Abel had been edited by Sylow and Lie during the 1870s; the resulting two volumes were published in 1881. Hölder's first article dealing with Galois theory and group theory [1889a] was written when the German edition of Galois' papers was not yet available—they must have been in the process of translation. The translator expressed the hope that the works of Galois and Abel would inspire young mathematicians. Hölder seems to have needed no such encouragement.

The most thorough exposition of Galois theory available to Hölder in French was that included in Jordan's substantial treatise *Traité des Substitutions et des Équations Algébriques* [1870]. It is hereafter referred to simply as Jordan's *Traité*. This book also provided the best description in French of the more advanced theory of substitution groups, and the first systematic treatment of linear groups. Jordan introduced various of the 'classical linear groups' here—orthogonal and symplectic groups over particular fields. His understanding of such groups was as groups of 'linear transformations' rather than as groups of matrices. The latter approach to classical groups was not adopted until the beginning of the twentieth century.

The other standard French algebra textbook of the last half of the nineteenth century paid relatively little attention to Galois theory. It was written by Serret, and entitled *Cours d'Algèbre Supérieure* (hereafter referred to as Serret's *Cours*). The first edition was published in 1849. No Galois theory at all appeared in it until the third edition of 1866. Here Serret did little more than reproduce a paper of Galois [1846a] that is commonly known as the *Premier mémoire*. Serret provided minimal elaboration and explanation of this paper. The situation did not improve with subsequent editions of the *Cours*—they were very little revised. The most recent edition as far as Otto Hölder was concerned was the fifth of 1885. In addition to the French editions, Hölder made use of the German translation [1868] of the third edition. Although seven editions of the *Cours* were eventually produced, the last in 1928, the presentation of Galois theory hardly altered, except for some material on 'Galois imaginaries' (that is, generators for finite fields).

The only textbook on group theory and Galois theory written in German that was

available to Hölder was Eugen Netto's *Substitutionentheorie und ihre Anwendungen auf die Algebra* (The Theory of Substitutions and its Applications to Algebra) [1882], hereafter sometimes called *Substitutionentheorie*. From 1879 until 1882 Netto was an associate professor at Strassburg, then in Germany. Before and after this time he lived in Berlin, and therefore might well have been known to Hölder when he was a student. Netto referred in a preface to 'die beiden umfassenden, bisher publizierten Behandlungen der Substitutionentheorie' ('the two comprehensive treatises on the Theory of Substitutions which have thus far appeared'). These were the books of Serret and Jordan. However, Netto pointed out that his methods were so far removed from Serret's that he had been unable to make use of Serret's work. The position regarding Jordan's treatise was exactly the opposite—Netto acknowledged his employment of Jordan's 'Beweise und Gedankenfolgen' ('proofs and processes of thought'). Netto's fundamental outlook on mathematics had been formed under the inspiration and guidance of Kronecker, who later communicated the same enthusiasm to Hölder.

The Galois theory of algebraic equations forms the substance of the second half of *Substitutionentheorie*. Netto considered that he was applying the theory of substitution groups to Galois theory, but his explanations nevertheless focus largely on equations and not on their groups. He treated particular cases that had previously received special attention and for which satisfactory results had been obtained, for example, cyclotomic and abelian equations. He defined the Galois group of an equation and set out Galois' criterion for the solubility of an equation by radicals fairly clearly, but his explanation did not go any deeper than that.

In the expositions of Jordan, Serret and Netto, and indeed in all writing on Galois theory before the 1890s, the Galois group of an equation was considered as a group of substitutions of the roots of the equation¹. The concept of field, introduced principally through research in number theory by Kronecker, Kummer and Dedekind during the 1870s and 1880s, was used only to specify which quantities were 'known' at any stage of the solution process. Netto used the concept in this way in [1882]. It was not until after Hölder's original research in Galois theory had been published, in the 1890s, that the theory began to be presented in terms of field extensions and their automorphisms.

¹The group consists of those substitutions of the roots under which any expression in the roots that lies in the ground field is fixed.

One of those, besides Netto, who incorporated the field concept into his treatment of Galois theory was Paul Bachmann. His approach to the idea of a field followed that of Dedekind rather than that of Kronecker. Bachmann's article [1881] was an attempt to present Galois theory in a way that made as little use of substitution group theory as possible. He considered the latter to be 'very abstract'. Instead his explanation relies, as he remarked, on the concepts of field (*Zahlenkörper*, after Dedekind) and irreducibility. He believed that this approach provided a far more 'concrete' view of the solution process of an equation. The quantities that are 'rationally known' during the solution procedure are to belong to certain fields. Bachmann discussed *conjugirte Körper* (*conjugate fields*), each of which is obtained by adjoining one root of an irreducible equation to the original field, and the *Normalkörper* (*normal field*) that is produced if all the conjugate fields are identical. He gave the name *Galois'sche Körper* to the splitting field of the equation and constructed it by the only method known at that time (essentially Galois'), by means of a 'Galois equation'. This is nowadays called a Galois resolvent; the technique corresponds to the 'theorem of the primitive element'. Bachmann showed that the Galois field is a normal field. He defined the Galois group of an equation in the traditional way, as a group of substitutions of the roots of the equation. Thereafter he explained the solution theory in the standard nineteenth-century manner, except that he concentrated on equations of prime degree. He did not give the general criterion for an algebraic equation to be soluble by radicals.

Group theory

The study of group theory, unlike Galois theory, had progressed and expanded steadily throughout the mid and late nineteenth century. The concept of group had been introduced independently by Cauchy in 1845–1846 and by Galois, whose writings of the late 1820s and early 1830s were published by Liouville in 1846. Before the 1880s research had concentrated on substitution groups and groups of transformations; after 1880 the emphasis, especially in Germany, shifted to the 'abstract' consideration of groups. This change of approach and the developments in algebra that made it possible are considered in detail in Wussing's book *Die Genesis des Abstracten Gruppenbegriffes* [1969/1984]. Wussing has thereby provided an account of many of the important advances in group theory that took place throughout the nineteenth century.

Thus, by the time Otto Hölder began his mathematical research, an abstract ap-

proach to group theory was available. Hölder was able to assimilate this new viewpoint immediately and successfully developed the insights which it gave him. The algebra textbooks on which Hölder could draw, those of Jordan, Serret and Netto, offered him only the earlier conception of group theory as the theory of substitutions and transformations. These more 'concrete' theories were nevertheless very important to Hölder; they provided him with examples and results which he could conveniently use within his broader and more abstract conception of group theory.

Jordan's *Traité*, as noted earlier, contains most of the theory of substitution groups as known in 1870. Among the concepts and results he included are Lagrange's and Cauchy's Theorems, the simplicity of the alternating groups A_n for $n \geq 5$, composition series and part of what is nowadays known as the Jordan-Hölder Theorem, the ideas of isomorphism and homomorphism, and many theorems on transitive and primitive groups. His study of linear groups enabled him to reduce a problem which particularly interested him, that of finding which groups are soluble, to various more specific questions. The latter part of the *Traité*, some 180 pages, is devoted to this investigation.

The group theory in Serret's *Cours*, even in later editions, is very 'traditional' and follows Cauchy's early ideas of 1845 and 1846. As with Galois theory, there is no discussion of substitution group theory at all until the third edition of 1866. Serret chose to use Cauchy's term *système de substitutions conjuguées* to refer to a group. The word *groupe* had more-or-less become the standard term by 1870, but Serret did not adopt it. He preserved the distinction between a *système de substitutions conjuguées* and a *groupe de permutations* which, following Galois, is a collection of arrangements of certain quantities (originally of the roots of an equation). The relationship between these concepts is explained in the discussion of Betti's work in Appendix C.

Shortly after the appearance of Jordan's *Traité*, a short but highly significant paper of Sylow was published [1872]. Here he presented, in terms of substitution groups, those theorems which have come to be known as Sylow's Theorems. His discovery of the theorems, and the many later proofs that were found, have been documented by Scharlau [1988], Waterhouse [1980] and Casadio & Zappa [1990].

Netto's *Substitutionentheorie*, although published in 1882 just when the theory of abstract groups was producing new results and ideas, nevertheless dealt exclusively with the theory of substitutions. He gathered together the main results and lines of devel-

opment in which substitutions played an important part. The book contains sections on such topics as the 'number of values of a function'², Cauchy's and Sylow's Theorems, transitivity and primitivity, simple and composite groups, homomorphism, and investigations of various types of group, both substitution and linear groups.

Netto also devoted a section of his book to abelian groups (*Gruppen von vertauschbaren Substitutionen*³). It is almost entirely lifted from an article of Kronecker [1870], as Netto acknowledged. Kronecker's paper was very frequently referred to and cited by later authors for the definition of (abstract) abelian groups and of various concepts associated with them. The paper follows from some work of Kummer; its subject is not group theory at all, but algebraic number theory. The term 'group' does not appear in it in a technical sense. In the first part of his article, Kronecker proved the fundamental theorem of finite abelian groups; that is, the existence of a fundamental system of elements $\theta_1, \theta_2, \theta_3, \dots$, which have the property that every element of the group can be expressed as

$$\theta_1^{h_1} \theta_2^{h_2} \theta_3^{h_3} \dots \quad (h_i = 1, 2, \dots, n_i)$$

where $\theta_1, \theta_2, \dots$ have orders $n_1, n_2, \dots$ and n_i is divisible by n_{i+1} for all i . This theorem and proof is discussed in more detail in Chapter 5.

In the second part of Kronecker's paper, the theorem was applied to the ideal class group of an algebraic number field, a group which is finite and abelian. The order of the group is the *Klassenanzahl* (*class number*) mentioned in the title of Kronecker's article. The finiteness of the class number was proved, in the case of cyclotomic fields, by Kummer in 1847. The usual proof presented nowadays for the general case is due to

²The investigation of the number of values of a function was one of the problems that originally gave rise to the theory of substitution groups, in particular in Cauchy's work of 1845 and 1846. A function of n variables ($n \in \mathbb{N}$) is given. The variables are permuted in all possible ways. The number of values of the function is defined to be the number of essentially different functions that are obtained in this way. For example, let $f(x_1, x_2, x_3) = x_1 x_2 + x_3$. Then we may form three essentially different functions, $x_1 x_2 + x_3$, $x_1 x_3 + x_2$ and $x_2 x_3 + x_1$. Therefore the number of values of the function $f(x_1, x_2, x_3)$ is three. This corresponds to the index of the subgroup $S := \{1, (x_1 x_2)\}$ in $\text{Sym}(x_1, x_2, x_3)$. The group $\text{Sym}(x_1, x_2, x_3)$ is the group of all substitutions of x_1, x_2, x_3 and the subgroup S is that which fixes the function f .

³The term 'Abelian group' appears to have first been used by Weber [1893] and Frobenius [1893a]. It is curious that the term was not introduced earlier. Kronecker had defined Abelian equations as early as 1853; as one would expect (and anachronistically), an Abelian equation is one whose Galois group is an abelian group. The extension of the term 'Abelian' to cover the groups as well as the equations seems fairly natural.

Minkowski and is from a much later date, after Kronecker. There is a more thorough account of some of these developments in algebraic number theory in Wussing's book [1969/1984].

Netto chose in [1882] to re-formulate Kronecker's proof of the fundamental theorem so that it was explicitly about abelian groups. He added a paragraph to show the uniqueness of the expression for each element, which result Kronecker had not proved. It had been established by Frobenius and Stickelberger, who published their proof of the fundamental theorem of abelian groups, including uniqueness, in 1879.

Frobenius

Frobenius' interest in group theory began in earnest in the 1880s when he found two new methods of proof for Sylow's Theorems. He developed a proof that works for any finite group and does not rely on substitution group theory. Frobenius, as Hölder, was comfortable with the concept of an abstract group and conducted his group-theoretic research from a purely abstract viewpoint. In fact Sylow's original proof in [1872] may be applied to abstract groups simply by explaining it in terms of a group action rather than in terms of substitutions. The paper containing Frobenius' first proof of the theorem was written in 1884 but not published until 1887. Another paper published in 1887 provided an exposition of double cosets, which Frobenius used to produce another proof of Sylow's Theorems. During the 1890s he examined soluble groups and considered various problems in the structure theory of finite groups. Unlike Hölder, who devoted himself exclusively to algebra during the early 1890s, Frobenius interspersed work on group theory with other research. In 1893 he presented his inaugural lecture to the Berlin Academy of Sciences. He summarised his attitude to algebra as follows [1893b, 627/574]:

Die Behandlung algebraischer Fragen übte von Anfang an einen besondern Reiz auf mich aus, und zu ihnen bin ich mit Vorliebe immer wieder zurückgekehrt, wenn ich nach anstrengenden analytischen Arbeiten einer Ruhepause bedurfte. [1]

Both Frobenius and Hölder were introduced to algebra at the University of Berlin by Kronecker and Kummer. This may account for the similarities in their overall approach to the subject and, perhaps, for the small overlap in their choices of research topic within group theory. Their presentation of group theory is in many respects quite different; in Frobenius' papers, the meticulous attention to detail that characterises Hölder's work

is lacking. Frobenius' style is not as 'crystal-clear' as Hölder's, but in some respects Frobenius' papers are easier to read and to appreciate quickly.

Although Hölder's academic path never crossed that of Frobenius, it is clear that Frobenius' algebraic and other research provided some inspiration and direction to Hölder. Indeed, in the early 1880s Hölder had generalised a result of Leibnitz', proved by Frobenius, to obtain the concept of Hölder-summability [1882b]. Conversely, Frobenius took up Hölder's (algebraic and other) ideas and used them where they proved helpful. The two mathematicians worked on pure group theory and on Galois theory during roughly the same period—the (late) 1880s and early 1890s. Frobenius' most important work in group theory was carried out from the late 1890s onwards, in his creation and development of group character theory and representation theory,⁴ and his application of these ideas to various problems in pure group theory.

Kronecker

Ernst Hölder asserts in the *Dictionary of Scientific Biography* essay about his father that Otto Hölder's interest in algebra was due 'primarily to Kronecker'. Kronecker had been known to Otto Hölder since his first days as a student in Berlin in the late 1870s; he would almost certainly have heard Kronecker's lectures on the theory of algebraic equations⁵, amongst other topics. This subject was one which Kronecker researched throughout his career. Thus, Hölder would presumably have been taught the ideas of Abel and Galois and the later development of these ideas, a development in which Kronecker played a large part. Kronecker always preferred the 'practical' side of the problem—the discovery of procedures that allow one to write down expressions for the roots of an equation.

Klein and Dyck

In the introduction to the first of his algebraic studies, Hölder stated [1889a, 27] that he was following a suggestion made to him by Felix Klein. Hölder made no further remarks about this and gave no further indication of the way in which he had become interested in algebraic questions. It is, however, clear that the influence of Klein through personal contact, his lectures and writings, and through the work of Dyck, his student

⁴For an account of this development refer to the articles by Hawkins, [1971, 1972, 1974].

⁵The lectures are mentioned by Kronecker in a paper of his: see his *Werke*, Volume IV, p. 199.

and later his assistant, was considerable.

Klein had first understood in depth the concepts of group theory and of Galois theory through his contact with Sophus Lie and Camille Jordan in 1870. Klein had travelled with Lie to Paris just after Jordan's *Traité* had been published. Courant's colourful assessment of the effect of this book on Klein was as follows [Courant 1925, 199]:

Klein und Lie stürzen sich mit leidenschaftlichem Eifer auf das schwierige und ihnen zunächst fast unverständliche Werk; sie fühlen, daß hier unermeßliche Schätze zu heben sind. Als Klein nach einem Aufenthalt von $2\frac{1}{2}$ Monaten infolge des Kriegausbruches Paris verlassen muß, trägt er den Stein der Weisen mit sich; er hat den Gruppenbegriff aufs tiefste erfaßt, den Wegweiser, der ihn hinfort mit unfehlbarer Sicherheit auf seinem wissenschaftlichen Lebenswege weiterführte. [2]

Klein's deep understanding of the group concept led to his well-known investigation of the relationship between groups and geometry during the 1870s, usually referred to as the *Erlanger Programm*. Under Klein's influence, Dyck became familiar with the applications of group theory to geometry, to number theory and to the theory of algebraic equations, a familiarity evident in his work. Dyck developed Klein's results concerning Riemann surfaces and from this research he was able to distill the concept of a group presentation using generators and relations, in [1882] and [1883]. These two papers mark the beginning of combinatorial group theory; they are discussed at length in [Chandler & Magnus 1982]. Dyck produced generators and relations for various specific groups, including the 'group of the icosahedron', which is isomorphic to $\text{Alt}(5)$, and the 'group of the modular equation for the 7th order transformation of elliptic functions', which is otherwise known as $\text{PSL}(2, 7)$. He proved the simplicity of $\text{Alt}(5)$ by means of his presentation for it. Hölder later borrowed these presentations that Dyck had calculated and put them to good use.

A special interest of Klein was the solution of the general quintic equation. He drew together the concepts used in the theory of its solution with ideas from geometry and analysis in order to reveal the unifying power of group theory in *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* (Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree) [1884]. The book will be referred to as *Icosahedron*. It proved extremely influential and quickly became considered a 'classic'. An English translation was made without much delay [1888]. *Icosahedron* was written during the early 1880s when Klein's health was bad, in an attempt to re-awaken the enthusiasm for mathematics which he had temporarily lost.

It contains many insights into the theory of groups and suggestions for further research on the subject. Hölder seems later to have taken up some of these suggestions, as will become apparent. In his algebraic papers Hölder frequently referred to these various studies of Klein and Dyck.

The monographs of Jordan, Serret, Netto and Klein were extremely influential in the nineteenth century and reached a wide audience. Netto's book was translated into Italian in 1885 and a revised version translated into English in 1892. There is a great and fundamental difference between the approach to group theory and Galois theory presented in these studies and the approach taken by Otto Hölder. Admittedly, textbooks and monographs cannot present the 'state of the art' for any length of time, unless they are regularly and comprehensively revised. The books available to Hölder were written very much from the traditional nineteenth-century viewpoint on algebra. Hölder had acquired another point of view, one much more akin to that of twentieth-century algebraists. It is my hope that his new approach will be clearly recognisable and something of its significance appreciated in the chapters that follow.

Notes

1 The treatment of algebraic questions had a particular attraction for me from the beginning, and I have returned with preference to it again and again if I have needed a break after strenuous analytic work.

2 Klein and Lie plunge into the difficult work, at first nearly unintelligible to them, with passionate zeal; they feel that treasures beyond measure are to be dug up here. When Klein, after a stay of $2\frac{1}{2}$ months, has to leave Paris as a result of the outbreak of war, he bears with him the philosopher's stone. He has obtained the most profound grasp of the group concept; it is the signpost which led him onwards on the paths of his scientific life with infallible certainty.

CHAPTER 4

A Transitional Period for Galois Theory

L'Algèbre est, à proprement parler, l'Analyse des équations ...

J.-A. Serret: *Cours d'Algèbre Supérieure* (1866)

In the last two decades of the nineteenth century, the initial period of growth and development of Galois theory came to an end and a time of change began. The main concepts of the theory had by then been explored and understood, and proofs had been provided with a degree of rigour that they had originally lacked. There were nonetheless several important questions which had not received a satisfactory answer, and areas in which the theory had not been worked through to its logical conclusion. In addition, the introduction of abstract ideas into group theory begged the question of how such ideas could also be incorporated into Galois theory. Otto Hölder was one of those whose work forms part of the actual transition from the old nineteenth-century approach to the very different formulation of Galois theory which is commonly known today. He picked up and tied together some loose ends of the theory, and in so doing helped prepare for the introduction of a new way of thinking about it. There were others with a similar influence, including Weber, Dedekind, Kronecker and Hilbert. They all had the ability to assimilate what had already been discovered and to point out some part of what would eventually be recognised as the way forward.

The period 1880–1900 was also the time when various inconsistencies were removed from the existing formulation of Galois theory, and the theory was consolidated and crystallised. There had been some variety in the definitions of certain concepts. For example, the solubility of an equation by radicals could be understood in slightly different ways (as will be seen); this had led to emphases on different parts of the theory and consequently to some new insights. Proofs of theorems were being revised and simplified. It also became possible to define precisely the quantities that were 'known' at each stage in the solution process for an algebraic equation by introducing the concepts of field and field extension. This was the general level of activity at the time when Hölder became interested in the theory towards the end of the 1880s.

Otto Hölder wrote four articles on Galois theory; they were published between 1889 and 1899. The last of them is a survey essay rather than a paper containing an original contribution to the theory. The two significant concepts that Hölder gave to algebra through this research are group-theoretic and form the subjects of Chapters 5 and 6 of this thesis. In the present chapter, Hölder's contributions to Galois theory itself are considered.

The results of Hölder's earliest algebraic studies were presented in the paper [1889a], 'Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen'. He wrote this while he was a lecturer (*Privatdozent*) at Göttingen, although it must have been completed on a visit to his home town of Stuttgart in October 1888, since this is the address given at the end of the paper. It deals, as the title states, with the reduction of an arbitrary algebraic equation to a chain of auxiliary equations, the solution of which is equivalent to the solution of the given equation. Hölder aimed to treat this subject in 'full generality'. He therefore assumed nothing more of the given equation than that it was separable (had no repeated roots in a splitting field). In particular, the results he obtained are true for equations that cannot be solved by radicals. His results represent genuinely new insights into the theory since they depend on the new material Hölder presented in the group-theoretic part of the paper, that is, on the concept of quotient group and the statement and proof of what has come to be known as the Jordan–Hölder Theorem.

There is a difference between Hölder's definition of 'full generality' and a modern one, a difference which he was not in a position to mention. Before the 1890s, Galois theory was a 'characteristic zero' theory. That is, the field over which an equation was to be solved was assumed to have characteristic zero; it was implicitly the rationals or some subfield of the complex numbers. Nowadays, a 'general' Galois theory would be able to treat equations over fields of prime characteristic as well. This extension of the theory was not recognised until just after the time at which Hölder's original work on the subject was published.

The background of Hölder's work

Hölder's aim when he began to explore Galois theory, the successful achievement of which is recorded in [1889a], was to generalise a theorem of Abel. In doing so he was able

to introduce the use of quotient groups into the formulation of Galois theory and explain how this concept enabled much that was previously obscure in the theory to be made clear. The Jordan–Hölder Theorem which he had proved was a necessary prerequisite for his developments. He never actually mentioned this theorem in the later part of the paper, but used it implicitly.

The theorem of Abel which Hölder took as his starting point was the following:

Theorem 4.1 [Abel 1826, 73/75]

If an equation, whose coefficients are independent variables, can be solved by radicals, then the solution can always be given in such a form that all the necessary radicals are rational functions of the roots of the given equation.

A ‘rational function of the roots of the given equation’ is, in modern language, an element of the splitting field of the equation. Abel had proved this theorem before the invention of group theory; as Hölder mentioned [1889a, 26], the theorem formed the basis of Abel’s proof of the nonsolubility of the general equation of degree ≥ 5 . The theorem tells us that it is possible to choose auxiliary equations in such a way that the collection of their roots provide *exactly* the radicals required to solve the given equation. Nothing else need be included to arrive at a splitting field.

The new insights which Hölder provided had come to him as he considered the way in which Abel’s theorem had needed to be modified in order to deal with *algebraic* equations, that is, those with constant coefficients. This modification had involved the introduction of the concept of ‘known quantities’, which crystallised into the idea of a field in the 1870s and 1880s. Hölder’s conception of a field followed that of Kronecker, and Hölder borrowed his term *Rationalitätsbereich* for the idea. Kronecker had incorporated his understanding of the concept of field (when this understanding was still not fully developed) into the modified version of Abel’s theorem in [1879, 208/77]. The theorem in its ‘constant coefficient’ version states that if an algebraic equation is soluble by radicals, then these radicals can be chosen to be rational functions of the roots of the given equation and of *certain roots of unity*. In other words, the radicals lie in an extension by some roots of unity of the splitting field. (Recall that the field in which the coefficients lie and over which the equation is to be solved was implicitly assumed at this time to be a subfield of the complex numbers.) The modified theorem was later referred to as ‘Abel’s theorem’, for example by Dickson [Miller, Blichfeldt & Dickson

1916, 320].

Hölder was not satisfied with this theorem; he explained the problem in the following way [1889a, 27]:

Man muss also Einheitswurzeln mit herbeiziehen; diese letzteren sind nicht immer in den Wurzeln der ursprünglichen Gleichung rational. Darin liegt ein Misstand. [1]

Thus the fact that the roots of unity which one must introduce in order to solve the equation are not always in the splitting field of the equation was, as far as Hölder was concerned, a defect (*Misstand*) in the previous formulation of the theory. This defect can, however, be removed. The key to its removal lies in the definition of ‘solubility by radicals’ of an algebraic equation. In approaching the problem by means of this key, Hölder was acting on a suggestion which Felix Klein, a colleague of his at Göttingen since 1886, had made to him in conversation. Hölder’s ideas followed naturally from this suggestion, which simply entailed replacing the standard definition of ‘solubility by radicals’ by an equivalent one.

The nineteenth-century definition (see, for example, [Jordan 1870, 385]), which remains standard nowadays, is:

Definition 4.2

An algebraic equation is soluble by radicals if all its roots can be obtained as ‘known quantities’ by the adjunction of the roots of a series of binomial equations to the existing collection of ‘known quantities’.

The modern formulation of this is the following: define the field R to be a *radical extension* of the field K if there exists a series of field extensions

$$K = R_0 \subset R_1 \subset \dots \subset R_t = R$$

such that $R_i := R_{i-1}(\alpha_i)$ for $i = 1, 2, \dots, t$ where $\alpha_i^{m_i} = a_i$ for some $a_i \in R_{i-1}$ and $m_i \in \mathbb{N}$. Then

Definition 4.2a

An algebraic equation¹ $f(x) = 0$ over a field K is soluble by radicals if its splitting field is contained in a radical extension of K .

¹Hölder used $F(x) = 0$ to denote the equation. I choose to replace this by $f(x) = 0$ since the letter F is commonly used in English to refer to a field, and might have caused confusion in various places.

The splitting field is likely not to be a radical extension itself. It is, however, a *normal* extension, which is not necessarily true of the radical extension R . If R is not normal, then its Galois group over K (defined as the group of automorphisms of R which fix K) does not behave in a civilised fashion. In order to obtain a ‘Galois correspondence’ between groups and field extensions, the field R must be extended to its *normal closure* L . Then L is also a radical extension of K . Therefore the Galois group of L over K , $\text{Gal}(L/K)$, is soluble, and $\text{Gal}(f(x))$ is a homomorphic image of $\text{Gal}(L/K)$.

At Klein’s suggestion, Hölder used an equivalent definition of solubility:

Definition 4.3

An algebraic equation is soluble by radicals if all its roots can be obtained as ‘known quantities’ by the adjunction of the roots of a series of abelian equations of prime degree to the existing collection of ‘known quantities’.

An abelian equation is one whose Galois group is abelian. Each auxiliary equation has prime degree p and we may take it to be irreducible (since if it is not, each irreducible factor may be used in turn). Therefore its Galois group is transitive; this group is actually C_p , the cyclic group² of order p . So this definition translates into modern terms as:

Definition 4.3a

An algebraic equation $f(x) = 0$ over a field K is soluble by radicals if its splitting field is contained in a field K_t , where

$$K = K_0 \subset K_1 \subset \dots \subset K_t$$

is a series of field extensions with the properties that each is normal over its predecessor and $\text{Gal}(K_i/K_{i-1}) \cong C_{p_i}$ for some primes p_i and for $i = 1, 2, \dots, t$.

This definition has several advantages over Definition 4.2. As Hölder noted [1889a, 27], ‘Es werden bei dieser Betrachtungsweise die Gleichungen der Galois’schen Methode

²In twentieth century language, the reason for this is that the Galois group acts faithfully and transitively on the roots and is abelian. The subgroup stabilising one root is conjugate to the subgroup stabilising any other root, but all these subgroups must be identical since the group is abelian. So the stabiliser of any root is $\{1\}$ and the group acts regularly. A group G acting regularly on a set Ω of size n has order n . This is true because, for $g \in G$ and $\alpha \in \Omega$, the identification $g \mapsto \alpha g$ is a one-one onto mapping $G \rightarrow \Omega$. In fact it provides the identification of Ω with G which is associated with Cayley’s Theorem representing G as a subgroup of $\text{Sym}(G)$. Therefore since $|G| = p$ for some prime p , it must be the case that $G = C_p$.

directer zugänglich'. 'The equations of Galois' method are more directly approached by this way of looking at things.' The new definition illuminates Galois' original formulation of the theory. Galois had found that the solubility of an algebraic equation depends on the orders of the groups we now call the composition factors of the Galois group of the equation (to him, the composition factors were the 'groups of the auxiliary equations'). If these groups all have prime order, the given equation is soluble by radicals [1832, 409/26/175]. In other words, the splitting field of the equation is reached by a series of field extensions as in Definition 4.3a. We have now come across the other advantage of this definition: it is possible to arrange the field extensions so that K_t is *precisely* the splitting field Σ of the equation. We are therefore able to avoid having to introduce roots of unity, which might lie outside the splitting field. All the auxiliary quantities used in the solution of the equation may be taken to be rational functions of the roots of the equation. Thus they are 'natural irrationalities' as defined by Klein in *Icosahedron* [1884, 157/1888, 172]. Klein used the term 'accessory irrationality' for the opposite of a 'natural' one, that is, for a quantity not in the splitting field of the equation.

Definition 4.2 allowed us to conclude only that Σ is *contained in* R , which might itself be much larger. (We require that everything in Σ is expressible in terms of the radicals α_i which are adjoined to form R , but it would be unreasonable to expect that everything which is expressible in terms of these same radicals lies in Σ . That is why Σ is in general not a radical extension.) The relationship of Σ to the intermediate fields was not known and was not immediately obvious.

The simplification produced by employing Definition 4.3 is perhaps best understood by examining how the two definitions of solubility are equivalent. What is the relationship between them? It is not too difficult to see that 4.3a follows from 4.2a: if any of the m_i in 4.2a are not prime, the binomial equations $x^{m_i} - a_i = 0$ may be replaced by a series of such equations that do have prime degree. Then if any of the R_i are not normal extensions, they may be replaced by their normal closures³. When this has been done, the Galois groups $\text{Gal}(R_i/R_{i-1})$ for $i = 1, 2, \dots, t$ are isomorphic either to C_{p_i} for some primes p_i or to the identity. By refining the series to omit those stages that produce the identity, a series of field extensions which satisfy the conditions of 4.3a is obtained.

³If $R_i = R_{i-1}(\alpha_i)$, where α_i is a root of the equation $x^{p_i} = a_i$, the other roots are given by $\alpha\omega, \alpha\omega^2, \dots, \alpha\omega^{p_i-1}$ where ω is a primitive p_i^{th} root of unity. Therefore if ω is already adjoined, the adjunction of α produces a normal field extension (the splitting field for $x^{p_i} - a_i = 0$).

How does one reverse this argument to go from 4.3a to 4.2a? In other words, how does one show that if $K \subset L$ is a normal field extension with Galois group C_p , it is contained in a radical extension of K ?

Suppose that $\text{Gal}(L/K)$ is generated by σ . Choose $\alpha \in L - K$ and let ω be a primitive p^{th} root of unity. Then σ can be extended to an automorphism of $L(\omega)$ over $K(\omega)$. In $L(\omega)$, define

$$\beta := \alpha + \omega^{-1}\alpha^\sigma + \dots + \omega^{-p+1}\alpha^{\sigma^{p-1}}.$$

Then

$$\begin{aligned}\beta^\sigma &= \alpha^\sigma + \omega^{-1}\alpha^{\sigma^2} + \dots + \omega^{-p+1}\alpha^{\sigma^p} \\ &= \omega\beta,\end{aligned}$$

since σ^p is the identity automorphism of $L(\omega)$ over $K(\omega)$, and $\omega^p = 1$. Now because $\beta^\sigma \neq \beta$, we know that $\beta \notin K(\omega)$ and so, since $|L(\omega) : K(\omega)| = p$, we must have $L(\omega) = K(\omega, \beta)$.

But $K(\omega, \beta)$ is a radical extension of K . This is true because $\omega^p = 1 \in K$, and since $\beta^\sigma = \omega\beta$, we have $(\beta^p)^\sigma = \beta^p$, so that β^p must lie in $K(\omega)$ (it is fixed by the Galois group of $L(\omega)$ over $K(\omega)$). Therefore L is contained in a radical extension of K , as required.

One of the simplest cases in which the splitting field of an equation is *not* a radical extension and in which it is necessary to introduce 'extraneous' roots of unity is the *casus irreducibilis* for cubic equations. Hölder dealt with just such equations in a later paper [1891]; it is likely that his interest in them was aroused because of this earlier work. The *casus irreducibilis* describes the situation in which one has a cubic equation $g(x) = 0$ to be solved over the field $\mathbb{Q}$ or over some subfield of $\mathbb{R}$; the equation is irreducible and has three real roots. It is nevertheless necessary to introduce complex roots of unity in order to produce a solution of this equation by radicals. The problem is considered in more detail later in this chapter.

Thus, the important fact which Hölder realised was that if the solubility of an algebraic equation by radicals is defined in the terms of 4.3, the clause 'and certain roots of unity' may be removed from the statement of the modified theorem of Abel. We obtain the following:

Theorem 4.4 [Hölder 1889a]

The solution of any algebraic equation $f(x) = 0$ can be reduced to that of a series of auxiliary equations with simple Galois groups, in such a way that the roots of the auxiliary equations are all expressible as rational functions of the roots of $f(x) = 0$ (with coefficients from the ground field).

In other words, the quantities used in the solution of $f(x) = 0$ are once more contained in the splitting field of the equation.

Theorem 4.4 holds quite generally, and not just for equations that are soluble by radicals. In the special case where $f = 0$ is soluble by radicals, the simple groups occurring as Galois groups of the auxiliary equations are cyclic of prime order, so the auxiliary equations are abelian and of prime degree. Hölder explained [1889a, 27] that the formulation of ‘soluble by radicals’ given in Definition 4.2 may be recaptured by the adjunction of roots of unity to these auxiliary equations.

He distinguished a second task to be performed with equations that are not soluble by radicals: each of the simple auxiliary equations is to be reduced to a ‘normal equation’ with the same Galois group. He remarked that this task is to be treated as a problem in its own right, and had already been dealt with in detail for equations of degree 5. He was referring here to the fact that it is always possible to transform a quintic equation into one with a certain normal form. The solution of the equation can be written down from this normal form, by using formulae involving elliptic functions; this method of solution was discovered by Hermite and others. A detailed explanation of the techniques that had been developed for solving quintics was given by Klein in [1884]. Hölder seems to have been conjecturing that this idea might also apply to equations of degrees greater than 5: that all insoluble equations of a given degree n are equivalent to one and the same equation in normal form, and this equation may be solved somehow using a similar technique to that involving elliptic functions for quintics. Hölder should have been (and perhaps was) thinking in terms of one normal form for each distinct simple group to which the Galois group of an insoluble equation reduces. The Galois group of an irreducible and insoluble equation of degree 6 may reduce either to $\text{Alt}(6)$ or to $\text{Alt}(5)$. It is not at all likely that one normal form would cover both these cases.

Hölder was perhaps rather too optimistic in his conjecture. Indeed, the thirteenth problem which Hilbert discussed at the International Mathematical Congress of 1900

[1901, 219–221/313–314] was to prove the ‘Unmöglichkeit der Lösung der allgemeinen Gleichung 7. Grades mittels Funktionen von nur 2 Argumenten’ (‘Impossibility of the solution of the general equation of degree 7 by means of functions of only 2 arguments’). A solution for equations of degrees 5 and 6 had been obtained, Hilbert noted, by employing elliptic functions of two variables. However, he was not hopeful that the same idea could be applied to equations of degree 7.

Subsequent work of Kolmogorov has resolved a problem closely related to Hilbert’s. Kolmogorov proved in [1957] that there exist continuous functions of three variables not representable by continuous functions of two variables. This work is discussed by Lorentz in [1976] and [1990]. It remains unclear, however, whether any function of three variables that may be used in the solution of the general equation of degree 7 is representable by functions of two variables. Therefore Hilbert’s problem and Hölder’s conjecture about a normal form, which is rather more subtle, appear to be unsolved.

Hölder’s method

Hölder began to explain the general procedure that is used in the ‘reduction of an arbitrary algebraic equation’. In modern language, it involves the successive extension of the ground field of the given equation by adjoining the roots of certain auxiliary equations [1889a, 27–28]:

Man stellt zunächst eine *einfache* Hilfsgleichung auf, deren Coefficienten dem ursprünglich gegebenen Rationalitätsbereich angehören. Wenn man nun sämtliche Wurzeln dieser Hilfsgleichung adjungirt, so entsteht ein neuer Rationalitätsbereich. Dieser wird zu Grund gelegt für eine zweite Hilfsgleichung, welche auch eine einfache Gruppe haben soll. Nachher wird die Gesamtheit der Wurzeln der zweiten Hilfsgleichung noch dazu adjungirt und so fortgefahren. Wenn man schliesslich die Wurzeln der letzten Hilfsgleichung adjungirt hat, so sollen die Wurzeln der ursprünglichen Gleichung nunmehr alle rational geworden sein. [2]

Thus at each stage, all the roots of an auxiliary equation with simple Galois group are adjoined and the larger field which is produced becomes the ground field for the next auxiliary equation. After the roots of all the auxiliary equations have been adjoined, a field should have been reached that contains all the roots of the equation to be solved or, in other words, that contains the splitting field of the equation.

This may be expressed in terms of the Galois group Γ of the equation. After the first step in the procedure, the group Γ is reduced to a maximal normal subgroup Γ' ,

which in turn is reduced to a maximal normal subgroup Γ'' , and so on until the identity subgroup is reached. That is, a composition series for Γ has been exploited:

$$\Gamma \supset \Gamma' \supset \Gamma'' \supset \dots \supset \{1\}.$$

Various questions naturally arise in connection with this procedure:

- (1) What are the Galois groups of the auxiliary equations?
- (2) To what extent are these groups determined? (In particular, are they altered when accessory irrationalities are introduced?)
- (3) How many auxiliary equations are needed?
- (4) How are the roots of the auxiliary equations obtained?

Hölder had posed himself these questions, and was able to answer them precisely because he saw how the concept of quotient group is present in Galois theory. This recognition had enabled him to incorporate the Jordan–Hölder Theorem into the theory in a new way. He showed that the group of any auxiliary equation employed in the solution of $f(x) = 0$ is isomorphic to some quotient group of the group Γ of the equation $f(x) = 0$. He realised that there are certain simple quotient groups which must always be included in order to obtain a solution; they are the composition factors (*Factorgruppen*) of the group Γ . They occur in the solution regardless of the method used to reach it; they appear even if ‘accessory irrationalities’ are used to generate field extensions. Hölder explained in the latter part of his paper exactly how one extracts a chain of auxiliary equations whose groups are (isomorphic to) the composition factors of Γ , and in particular how this is achieved without the introduction of accessory irrationalities.

Hölder was able to conclude that the number of composition factors of the group Γ is equal to the minimum number of auxiliary equations that will be required to solve $f(x) = 0$. This number is well-defined by reason of the Jordan–Hölder Theorem. He proved in addition that when there are as few auxiliary equations as possible, the desired conclusion holds: the roots of these equations all lie in the splitting field of the original equation.

The proof that Hölder gave of all these results rested on an extension which he made of a theorem proved by Jordan [1870, 269–270]. The theorem is concerned with the situation in which the roots of a first equation are adjoined to the ground field of a second equation, and vice versa. Hölder simply re-worked this theorem to include the concept of quotient group, as we shall discover.

Reduction to a chain of equations

The content of [1889a] is here considered in more detail in order to convey the outlines of Hölder's method of proof and to give an idea of the style in which he presented Galois theory. The impression which his paper produces is that he wished to draw together all the older conceptions of the theory by employing a more abstract approach to it, and to provide ideas for ways in which the theory could be extended and re-formulated. One may ask the question: how great an influence did his work actually have on the subsequent developments in Galois theory? Let us look at some aspects of his work before attempting to answer that question.

As mentioned earlier, Hölder laid down necessary background material from group theory in the first part of [1889a], including an explanation of the concept of quotient group and proof of the Jordan–Hölder Theorem. The second part of the paper begins with a reminder of some basic Galois theory and the establishing of notation⁴.

§13 Definitions

An equation $f(x) = 0$ of degree n over a field (τ) is given. (The field is a *Rationalitätsbereich*, hence it is denoted by τ .) The equation is assumed to be separable, that is, to have no repeated roots in a splitting field. The characteristic of the field is (implicitly) zero. The distinct roots of $f(x) = 0$ are $\xi_1, \xi_2, \dots, \xi_n$.

The Galois group of f over (τ) is viewed as the group of all substitutions of the roots of $f = 0$ that fix the numerical value of every rational function of the roots which has a rational value. In other words, such substitutions fix the value of an expression in the roots of $f = 0$ if this expression actually lies in the ground field of the equation. Thus a substitution of this type corresponds to the idea of an automorphism of the splitting field that fixes elements of the ground field. Toti Rigatelli comments [1989, 123] that Hölder began to employ the term *Galois'sche Gruppe* here in the way that it is used today. He was perhaps the first to do this. Previously, for example in Jordan's work, the Galois group was known simply as the 'group of the equation'. Hölder was not consistent in his terminology; he made use of both terms.

§§ 14–15 Formal and numerical invariance

These paragraphs form an aside which Hölder included to make clear the difference

⁴The section numbers that follow are those of Hölder's paper.

between 'numerical' and 'formal' invariance of functions. Perhaps he was intending that his remarks should be noted by a certain Herr Söderberg. At this time Hölder had the impression, in which he turned out to be mistaken, that Söderberg did not appreciate the difference (see below). The distinction which must be observed is as follows: those substitutions of the roots of an algebraic equation which leave a given rational function of the roots *formally* unchanged form a group, but those substitutions which leave such a rational function *numerically* unchanged do not in general form a group. It seems Hölder had believed there was some confusion about this, in particular because of the corresponding result for the general equation of degree n . This is the equation $G(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, where $a_{n-1}, a_{n-2}, \dots, a_1, a_0$ are algebraically independent transcendentals over the ground field. In this case the same result holds: the substitutions of the roots of $G(x) = 0$ which leave any given function of the roots unchanged always form a group. There is not the same distinction to be made between formal and numerical invariance.

Hölder stressed that it was always numerical invariance being considered here. He showed [1889a, 41] that the collection of all those substitutions of the *Galois group* of an algebraic equation that leave a rational function of the roots of the equation numerically invariant forms a group. Using the notation set out earlier, if $\phi(\xi_1, \xi_2, \dots, \xi_n)$ is a rational function of the roots of $f(x) = 0$, then

$$\{S \in \Gamma : \phi(\xi_1, \xi_2, \dots, \xi_n) = \phi(\xi_1 S, \xi_2 S, \dots, \xi_n S)\}$$

forms a group. He explained that this is not always true of the collection of *all* substitutions that leave a function numerically invariant. In a footnote he provided an example:

Let $f(x) = x^{n-1} + x^{n-2} + \dots + x + 1$ and set $\xi_\nu = e^{\frac{2\pi i \nu}{n}}$. Then, if $n > 5$, the collection of substitutions C that leave $\phi = \xi_1 \xi_{n-1}$ numerically invariant need not form a group. Hölder did not show exactly why this is so; let us consider his example further. We need to find a substitution $S \in C$ that is not in the Galois group of $f = 0$ and such that some power of S or some combination of S with other members of C is not itself in C .

Let us simplify a little more and assume that n is prime. Then $f(x)$ is the n^{th} cyclotomic polynomial; its Galois group over $\mathbb{Q}$ is the cyclic group of order $n - 1$ and consists of the substitutions $\xi_\nu \mapsto \xi_\nu^k$ for $k = 1, 2, \dots, n - 1$. We need to ensure that S is not one of these in order to obtain a non-closed collection of substitutions.

Take S to be the substitution $(\xi_1 \xi_2 \xi_{n-1} \xi_{n-2} \xi_3)$. Then

$$S(\xi_1 \xi_{n-1}) = \xi_2 \xi_{n-2} = 1 = \xi_1 \xi_{n-1},$$

therefore S is a member of the collection C that leaves ϕ invariant, but

$$\begin{aligned} S^2(\xi_1 \xi_{n-1}) &= S(\xi_2 \xi_{n-2}) = \xi_{n-1} \xi_3 \\ &= \xi_2 \neq 1. \end{aligned}$$

Thus the collection C is not closed and does not form a group.

Herr Söderberg's 'false' proof

Hölder provided this example 'weil in diesem Punkt schon gefehlt worden ist' ('since a mistake has already been made on this point'). The supposed mistake in question, claimed Hölder [1889a, 42], had been made by J.T. Söderberg and published in a paper of 1888, in which he presented a new proof of the existence and defining property of the Galois group of an equation. Söderberg's proof had first appeared in Swedish in his doctoral thesis of 1886; it was republished in a slightly modified form and in French in [1888]. He claimed that the proof was 'new and very simple'. Hölder was quite insistent that the proof was false because it rested on the implicit assumption that the collection of all substitutions which leave a function of the roots numerically invariant forms a group.

However, it transpired that Hölder was overly hasty in his remarks and Söderberg's proof was indeed correct. Hölder must have lost no time in writing an apology, and in providing some further explanation, once he had been informed of his error. His note [1889b], dated June 1889, appeared in the same volume of *Mathematische Annalen* as his original criticism in [1889a]. He recorded that Herr Engel 'hat die Freundlichkeit gehabt, mich darauf aufmerksam zu machen, dass ich mich hier in einem Irrthum befunden habe' ('has had the kindness to point out to me that I have fallen into error on this matter').

Hölder explained how he had arrived at his previous erroneous assessment of Söderberg's proof. He had believed that the proof rested on an incorrect development of a result of Lagrange. In fact it was Lagrange's own work, and not Söderberg's, which contained gaps that needed to be plugged. Hölder showed how this could be done; the 'obvious' way to proceed was actually not the easiest. He ended with the remark [1889b, 462]:

Man kann also in der That auf einfache Weise von dem Ausgangspunkt von Lagrange aus zur Galois'schen Theorie gelangen. [3]

The chain of equations again

Let us return to Hölder's earlier paper. His aside on formal versus numerical invariance had shown that, as long as only the substitutions in the Galois group are employed, one may deal with quantities rational in the roots of an algebraic equation just as if one were dealing with functions of independent variables. That is, those things which are true in the case of the general equation of degree n are equally true in the case of an equation with constant coefficients.

Hölder went on to consider the *conjugates* of a quantity g_1 rational in the roots of $f(x) = 0$. In modern terms, the quantity g_1 is an element of the splitting field of the equation, and its conjugates $g_2, g_3, \dots, g_k$ generate conjugate extension fields over the ground field of the equation. He showed that if g_1 is fixed by the substitutions of a subgroup H of Γ , then the subgroup fixing g_i for $i = 2, 3, \dots, k$ is a conjugate of H and each conjugate subgroup of H fixes some g_i . The subgroup H that fixes g_1 is, as Hölder showed a little further on, the Galois group of the splitting field over the subfield generated by g_1 . He then formed the group of permutations of $g_1, g_2, \dots, g_k$ induced by Γ , and denoted it by G . He stated that G is 'isomorphic' (that is, homomorphic) to Γ and that this homomorphism may be one-one or many-one.

There is one situation of especial importance [1889a, 43]; it is the situation in which H is normal in Γ . Then the substitutions of Γ are split into equivalence classes with respect to H (this was one of the ways in which Hölder had earlier explained the concept of quotient group) and $G \cong \Gamma/H$.

Next Hölder considered, as Galois had done, what happens to the group Γ of an equation when one 'adjoins' an irrationality to the field; in other words, when one extends the field by adding some quantity. First, he dealt with a natural irrationality g_1 (an element of the splitting field). He proved that the Galois group Γ of the equation reduces, after the adjunction of g_1 , to the subgroup H formed of precisely those substitutions that fix g_1 . He then addressed the questions: what is the equation that determines the quantity g_1 and what is the group of this equation? He showed that the group of such an equation is that which was earlier called G . Therefore if the subgroup H is normal,

this group G is isomorphic to Γ/H . Thus Hölder found the relationship between the group of an auxiliary equation and the group of the original equation, in answer to one of the questions he had posed in the introduction.

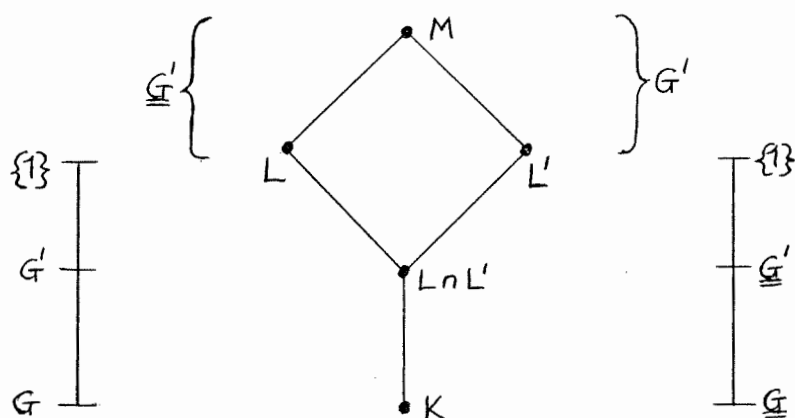
Hölder explained, following the approach of Galois, that the solution of an equation with group Γ may be split into two stages if Γ has a normal subgroup H . Repetition of this process of dividing up the group leads eventually to a chain of equations with simple groups. Galois had realised that this was the case: Hölder was able to recognise more clearly the relationship of these groups with the original group Γ . The simple groups are in fact the *Factorgruppen* (composition factors) of Γ which Hölder had defined earlier. Moreover, the roots of these auxiliary equations are rational in the roots of the given equation, that is, they lie in the splitting field of the equation. This remains true no matter what type of equation was originally given, whether soluble by radicals or not.

Another of Hölder's questions had been: what happens to the auxiliary equations and their groups when one allows the inclusion of accessory irrationalities? Therefore he next aimed to prove [1889a, §§21–22] that these groups are not changed, regardless of the sort of quantities that are adjoined. To this end, he proved an extended version of a theorem from Jordan's *Traité* [1870, 269–270]:

Theorem 4.5 [Hölder 1889a, 47]

If the group G of an equation $f(x) = 0$ reduces to the group G' under the adjunction of all the roots of a second equation $\mathfrak{F}(x) = 0$, then also the group $\mathfrak{G}$ of the second equation reduces to a group $\mathfrak{G}'$ under the adjunction of all the roots of the first equation. The groups G' and $\mathfrak{G}'$ are normal subgroups of G and $\mathfrak{G}$, respectively. Moreover, the groups represented by G/G' and $\mathfrak{G}/\mathfrak{G}'$ are (holoedrally) isomorphic.

The ground field for both equations is assumed to be the same. The result is best explained by drawing a diagram. Let the ground field for both equations be K ; let L be the splitting field of $f(x) = 0$ and L' that of $\mathfrak{F}(x) = 0$ and M that of $f\mathfrak{F} = 0$.



Then $G' \leq G$ and $G' = \text{Gal}(M/L') \cong \text{Gal}(L/L \cap L')$.

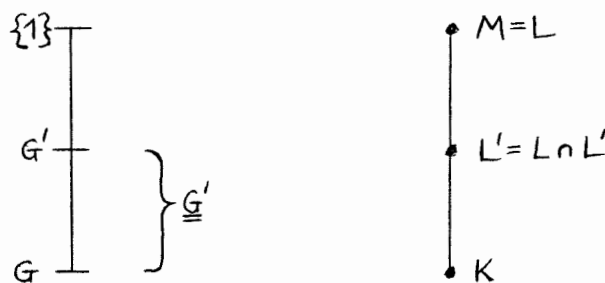
Similarly $\mathfrak{G}' \leq \mathfrak{G}$ and $\mathfrak{G}' = \text{Gal}(M/L) \cong \text{Gal}(L'/L \cap L')$.

Hölder proved the theorem by introducing quotient groups into the usual method of reasoning, which involved functions of the roots of the equation fixed only by the substitutions in a subgroup of the Galois group. He also extended a corollary of the theorem which Jordan had simply stated [1870, 270]:

Corollary 4.6 [Hölder 1889a, 49]

If $\mathfrak{G}$ is a simple group and the adjunction of the roots of $f(x) = 0$ reduces it, then the group is reduced to the identity. In this case $G/G' \cong \mathfrak{G}$ and $L' \subseteq L$. Therefore the roots of $\mathfrak{F}(x) = 0$, which lie in L' , are all contained in the splitting field L of the equation $f(x) = 0$.

The diagram has become:



Solution of the original problem: obtaining the chain of equations

The remaining sections of the paper are devoted to the construction of a chain of

auxiliary equations for any given equation $f(x) = 0$ with the particular properties described in the introduction: the auxiliary equations are to have simple groups which are to be precisely the composition factors of the group of $f(x) = 0$. Furthermore, this is to be achieved without leaving the splitting field, that is, without introducing 'accessory irrationalities'. Hölder arrived at such a chain of equations after having constructed various more general series. First, he simply required a set of auxiliary equations to be given, the roots of which would provide all the radicals needed to solve $f(x) = 0$. Therefore in particular some of these roots might lie outside the splitting field of $f = 0$. He showed how this set may be refined using Theorem 4.5 and Corollary 4.6 in order to produce exactly the required chain of equations. His method proved in addition that if there are as few auxiliary equations as possible, then their groups are precisely the composition factors of Γ , and their roots all lie in the splitting field of $f(x) = 0$.

The last sentence of the paper states that a similar but not quite as simple theorem can be formulated when one wishes to determine just one root of the equation $f(x) = 0$ by means of a chain of auxiliary equations. It is difficult to imagine what Hölder had in mind when he wrote this. The extension obtained by adjoining one root, ξ_1 say, to the ground field (τ) , is not in general a normal extension. Without the condition of normality, it is not possible to prove any 'unique factorisation' result similar to the Jordan-Hölder Theorem.

As with several ideas developed by Hölder at this time, the motivation to prove the results of [1889a] might have come partly from studying Klein's book on the icosahedron [1884]. Klein mentioned a fact which Kronecker had noted in [1861]: one may dispense with the accessory irrationalities when giving the solution of an algebraic equation that *can* be solved by radicals. Kronecker had conjectured the same was true for 'higher equations', that is, for those that cannot be solved by radicals. He had posed the question: how far can one go in the solution of such equations using only natural irrationalities? Hölder answered this question by proving that one can go as far as the composition factors of the group of the equation: the auxiliary equations corresponding to these are the smallest 'natural' building blocks in the solution of the equation.

The *casus irreducibilis* for cubic equations

Two years passed between the publication of Hölder's first studies in Galois theory and the appearance of his next note on the subject. This note [1891] concerns a famous problem in the theory of algebraic equations, mentioned briefly already in this chapter. The *casus irreducibilis* had been the subject of much deliberation ever since formulae for the solution of cubic equations were discovered and published in the fifteenth and sixteenth centuries by Scipione del Ferro, Niccolo Tartaglia and Gerolamo Cardano. The question which had been asked was the following:

Is it possible to give a formula for the solution (implicitly over the field $\mathbb{Q}$) of irreducible cubic equations with three real roots that involves only real radicals? Or does one have to introduce complex numbers?

The problem of an irreducible cubic with three real roots had come to be known as the *casus irreducibilis*, the irreducible case, because all attempts to find a solution using only real roots simply led back to another equation equivalent to the original one. Hölder gave a definitive solution to the problem: one always requires complex radicals. This result follows without too much difficulty from the developments Hölder presented in [1889a]. Here is a modern treatment of the situation:

Let $g(x) = 0$ be an irreducible cubic equation over $\mathbb{Q}$ which has three real roots α , β and γ in a splitting field Σ . Define $\Delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$. Then the Galois group Γ of $g = 0$ over $\mathbb{Q}$ is either $\text{Alt}(3)$ or $\text{Sym}(3)$, depending upon whether $\Delta \in \mathbb{Q}$ or not. (If $\Delta \in \mathbb{Q}$ then Δ is fixed by all elements of the Galois group Γ , therefore Γ can contain no transposition.) So we know that the degree of the normal field extension $\Sigma \supset \mathbb{Q}$ is either 3 or 6.

The crucial point is that Σ is not a radical extension. In order to make a radical extension, it is necessary to add a complex cube root of 1. This is shown as follows:

First, in order to simplify matters, let us assume that Δ has been adjoined to $\mathbb{Q}$ if it is not already there. Since Δ^2 , the discriminant of $g = 0$, is always an element of $\mathbb{Q}$, we know that $\mathbb{Q}(\Delta)$ is a normal radical extension of $\mathbb{Q}$ of degree 2. It is also a *real* extension: an irreducible cubic over a subfield of the real numbers has three real roots if and only if Δ^2 is positive, therefore Δ is real. The Galois group of g over $\mathbb{Q}(\Delta)$ is $\text{Alt}(3)$ and since $\text{Alt}(3)$ is transitive, the equation $g = 0$ remains irreducible over $\mathbb{Q}(\Delta)$.

Suppose $\Sigma \supset \mathbb{Q}(\Delta)$ were a radical extension. We know it has degree 3. Then it must be the case that $\Sigma = \mathbb{Q}(\Delta)(\epsilon)$, where $\epsilon^m = a$ for some $a \in \mathbb{Q}(\Delta)$ and some m . Further argument shows that m must be 3 and the equation $x^m - a = 0$ must be irreducible. But then the algebraic conjugates of ϵ are ϵ , $\omega\epsilon$ and $\omega^2\epsilon$, where ω is a complex cube root of 1. These are elements of the splitting field and are not all real, contrary to assumption. Thus Σ is not a radical extension.

Therefore in order to obtain a radical extension of $\mathbb{Q}$ containing Σ , it is necessary to adjoin a complex cube root of 1.

Hölder took up the *casus irreducibilis* question because he was not aware that any rigorous proof existed that one does in fact always require complex radicals. At about the same time as Hölder's paper, several other proofs were found and published by Kneser, Mollame and Gegenbauer. These proofs were all summarised by Netto [1900, 453–460]; he described Hölder's proof as 'one of the first'. Hölder noted [1891, 307] that a proof of this result can be deduced from Abel's proof of the nonsolvability of the general quintic. He chose, however, to present a different and more general proof that holds for each *particular* irreducible cubic with three real roots, rather than for the 'general' cubic of this type⁵. His argument, based on the same principle as that given above, follows from the theorem [1891, 309]:

Theorem 4.7

An irreducible equation of degree n over a real field, with one real root and with the property that each of its roots is rational in the others, cannot become reducible by the adjunction of a real radical with prime exponent unless the exponent is exactly 2, in which case n must also be divisible by 2.

It is not hard to obtain a proof that the *casus irreducibilis* cannot be solved by real radicals when the tools of Galois theory are at one's disposal. The result had, however, remained a conjecture before the introduction of Galois theory and, indeed, for nearly fifty years after the publication of Galois' papers. The problem had aroused considerable interest from time to time before the nineteenth century. It played an important part in the acceptance of the concept of complex number, since the introduction of complex

⁵In this 'general' cubic equation, the coefficients may take any real values as long as the discriminant Δ^2 is positive. A positive discriminant, as noted before, is a necessary and sufficient condition for the equation to have three real roots.

numbers seemed to provide a genuine method of obtaining a formula for the real roots of the cubic equation, roots which certainly 'existed'. In 1781 a prize was offered by the scientific academy at Padua for a proof of the irreducibility of Cardano's formulae for these cubics. The surge of interest in the subject of the prize question led to the appearance of various books and papers attempting to deal with it. Not everyone who considered the problem believed that the *casus irreducibilis* really was irreducible.

Reactions to Hölder's research in Galois theory

In the years after the publication of Hölder's papers, the style in which Galois theory was presented began to change significantly. Weber gave a new formulation of the theory [1893], in which he set out axioms for abstract groups and fields, and provided a rigorous treatment of transcendental and algebraic field extensions. The aim of this new formulation, as Weber remarked, was to establish Galois' theory in such a way that it encompasses all possible applications. To this end, he developed the theory for infinite and finite fields. Weber made no reference to any of Hölder's papers on Galois theory, but did refer to a later paper of his [1892] on simple groups. It is therefore likely that Weber also knew of Hölder's earlier work in Galois theory.

Weber provided another, simpler explanation of Galois theory in his *Lehrbuch der Algebra* [1895], in which he defined normal fields, conjugate fields, the degree of a field extension and other terms and concepts familiar to a twentieth-century student of the theory. He discovered various complications that arise in the theory, for example, when treating inseparable field extensions. He moved away from the old conception of the Galois group of an equation as a group of substitutions on the roots of the equation, and towards the concept of a group of automorphisms of a field extension (without, however, actually reaching this idea).

The English translation [1892] of Netto's book on substitution theory was reviewed in 1893 by Otto Bolza. He objected to Netto's exposition of Galois theory because the difficulties of passing from the general equation of degree n to one with constant coefficients were not sufficiently considered [1893, 96]. This was precisely the problem that Hölder had tackled in [1889a]. Bolza pointed out some of the difficulties: the question of algebraic solubility for a particular equation depends on the field over which one is operating (and which must therefore be defined), one must distinguish between

formal and numerical invariance of functions, and so on. These are just the points which Hölder had tried to clarify; it seems fairly clear that Bolza was influenced by Hölder's work, and that Netto's revision of his book should have been so influenced.

In 1894 the fourth edition of Dirichlet's *Vorlesungen über Zahlentheorie*, edited and with supplements by Dedekind, appeared. Dedekind took this opportunity to include in Supplement XI a much revised account of his development of the theory of ideals and the recovery of unique factorisation in algebraic number fields. In the course of this account, he gave a systematic description of field extensions, as Weber had done. Dedekind's formulation was quite different from Weber's. Dedekind considered further the idea of a *Permutation* of a field (this would nowadays be called an isomorphism; the term *Permutation* was also used by Weber in this sense). He showed that all 'permutations' of any field of characteristic zero fix the rational numbers. He mentioned the identity permutation, proved that permutations have inverses, that the set of permutations of a field is closed under composition and that composition of permutations is associative. Without explicitly saying so, he had found what amounts to the automorphism group of a field of characteristic zero.

He went on [Dirichlet 1894, 463] to consider the relationship between permutations of a field and of its subfields: given any set of permutations of a field M , the collection of all elements of M fixed by this set forms a subfield A (in modern terms, the *fixed field* of the set). This is beginning to read rather more like the description of Galois theory familiar to students today. Later [Dirichlet 1894, 482] Dedekind did briefly introduce the concept of group into his essay, and considered groups of field permutations. He discussed how the intermediate fields in a field extension may be found as fixed fields of subgroups of what would now be called the Galois group of the extension. He also employed Bachmann's term from [1881], *Normalkörper* (normal field), which has become standard.

Hilbert's contribution was not to Galois theory directly, but lay in his providing a survey and analysis of the theory of algebraic number fields in a report for the *Deutsche Mathematiker-Vereinigung* known as the 'Zahlbericht' [1897]. He treated 'Galois number fields', which are the same thing as Bachmann's and Dedekind's *Normalkörper*, as well as quadratic fields, cyclotomic fields and so on. Insights into the concepts of algebraic and transcendental field extensions had appeared in Kronecker's work much earlier,

when in [1882] he provided a thorough exposition of the concept of *Rationalitätsbereich*.

This interest in the structure of (both finite and infinite) fields and extension fields and the relationships between them gives an indication of the direction which research in Galois theory was beginning to take in the 1890s. At the end of the decade, Hölder was asked to provide a survey article on the subject for the newly planned *Encyklopädie der mathematischen Wissenschaften*. His article first appeared in 1899 as part of volume I,1 which was published in sections between 1898 and 1904.

Considering the new formulations of the concepts of field and field extension just discussed, Hölder's essay is rather traditional. He was not, it must be said, expected to provide comments about the possible future of research in Galois theory, but to give a clear picture of the current state of things. Nevertheless, his picture did not really take into account the insights of Weber and Dedekind, although he certainly mentioned and drew on their work extensively. He also drew on his own work and in particular on the explanation of Galois theory given in [1889a]. The article is therefore a clear and readable account of what may be described as 'old style' Galois theory. This old style was to remain the only style generally presented in textbooks for many years. Not until the 1930s was a completely new approach made available through the lectures and work of Artin; this approach grew out of the studies of Hilbert, Noether and others during the early twentieth century.

Modern perceptions of late nineteenth-century Galois theory

Hölder's work is placed by Kiernan [1971] in the category of 'classical Galois theory', in contrast to the contributions of Weber [1893 & 1895] which are to be considered, according to Kiernan, as marking a change of approach to the theory. He discusses Hölder's proof of the Jordan-Hölder Theorem as an 'important contribution, closing the last major gap in the group-theoretical aspects of Galois Theory' [1971, 133]. Scholz has the following to say [1990, 396] on the subject of the result Hölder obtained by extending Jordan's theorem to produce the 'Jordan-Hölder Theorem':

Inhaltlich war das natürlich kein spektakulärer Schritt; Hölders Beitrag machte aber doch deutlich, wie durch eine abstraktere Auffassung der Grundbegriffe die Ideen der Auflösungstheorie deutlicher hervorgehoben werden können. Die Auflösbarkeit einer Gruppe wurde nun durch die Bedingung formulierbar, daß alle Kompositionsfaktoren abelsche Gruppen (von Primzahlgrad) sind. Damit war die (vorher auch schon sichtbare) Brücke zu den abelschen Gleichungen deutlich gemacht. [4]

Some aspects of the development of Galois theory are covered by B.L. van der Waerden in a wide-ranging article 'Die Algebra seit Galois' [1966]. He discusses the results of Hölder's first article on the subject [1889a] in more detail than one would expect in the context of such a general paper, and obviously considers this work to be significant. He also wrote more specifically in [1972] on the neglected period in the development of Galois theory between the time of Weber and Artin. During this time, there *were* important advances in the theory which were made and assimilated by Hilbert and his school. The foundations for such advances, and for the modern presentation of Galois theory to which they eventually led, had been laid to a large extent by Hölder, Weber and Dedekind in the 1880s and 1890s.

Notes

- 1 Therefore one must introduce roots of unity here; but these latter are not always rational in the roots of the original equation. In this there lies a defect.
- 2 First one establishes a *simple* auxiliary equation, whose coefficients belong to the originally given field. If one now adjoins all the roots of this auxiliary equation, a new field results. This will be the foundation for a second auxiliary equation, which is also to have a simple group. Afterwards, all the roots of the second auxiliary equation will again be adjoined, and the process repeated. When one has finally adjoined the roots of the last auxiliary equation, then the roots of the original equation should by this stage all have become rational.
- 3 Thus one can indeed reach Galois' theory in a simple way from the starting point of Lagrange.
- 4 As regards content it was of course no spectacular step forward. Hölder's contribution did nevertheless make clear how, through a more abstract view of the fundamental concepts, the ideas of solubility theory could be more clearly set out. The solubility of a group could now be formulated by means of the condition that all composition factors should be abelian groups (of prime degree). In this way the (previously already visible) connection to abelian equations was clearly made.

CHAPTER 5

The Development and Understanding of the Concept of Quotient Group

Where we tend to go wrong is in assuming that if there is to be a correspondence between two systems it must be a one for one correspondence—that A in the one system must be represented by a in the other, and so on ... And there never could be correspondence of that sort where the one system was really richer than the other. If the richer system is to be represented in the poorer at all, this can only be by giving each element in the poorer system more than one meaning. The transposition of the richer into the poorer must, so to speak, be algebraical, not arithmetical ...

C.S. Lewis, from a sermon 'Transposition' given at Mansfield College, Oxford,
Reproduced in *Screwtape Proposes a Toast* (1965)

Although the concept of quotient group¹ is now considered to be fundamental to the study of groups, it is a concept which was unknown to early group theorists. It emerged relatively late in the history of the subject: Hölder's paper [1889a] may be reckoned as marking the end of the period of development of this idea. The main reason for the delay in its evolution is that, in order to give a recognisably modern definition of a quotient group, one must have the ability to conceive of groups in an abstract way. Therefore the development of the concept of quotient group is closely linked with the abstraction of group theory. This process of abstraction took place largely during the period 1870–1890 and was carried out almost exclusively by German mathematicians. It was within this environment that a clear understanding of the notion of quotient group was finally obtained.

The contributions of seven mathematicians to the evolution of this notion are considered in the present chapter. An early understanding of the concept within Galois theory and permutation group theory may be found in the writings of Galois, Betti and Jordan (research which falls within the period 1829–1873). Some studies by Dedekind during the 1850s reveal his remarkable grasp of abstract group theory and of the concept of quotient group. In the work of Dyck, Frobenius, and finally Hölder, the concept was

¹I choose to use the term *quotient group* rather than *factor group* throughout; the reason becomes apparent later.

explored during the 1880s within abstract group theory; after this time it rapidly gained acceptance in the mathematical community.

Definitions

What is a quotient group? Is there in fact only one correct definition? If there is, it should be possible to trace the development of the concept with this definition in mind and to measure early attempts at a definition against it. There is certainly a *standard* modern definition, employed by such textbooks as [Rose 1978, 42–43], [Herstein 1975, 51–52] and [Macdonald 1968, 56–57]:

For a group G , the quotient group G/H is the set of cosets Hx ($x \in G$) of the normal subgroup H of G , with multiplication given by $Hx_1Hx_2 = Hx_1x_2$ ($x_1, x_2 \in G$).

The definition could have been given in terms of left cosets xH , as it is in Rose's book, but such a distinction is redundant since a normal subgroup H of G is a subgroup for which $Hx = xH$ for all $x \in G$.

Why has this definition become standard? How does it encapsulate the concept of quotient group well? I suggest that it does so in two ways. First, the definition makes use only of the elements of the group G itself; these elements are simply combined in a particular way. One need not introduce any concepts from 'outside' the group. Secondly, the definition does not depend on the way in which the group G is presented. It is an 'abstract' definition that can be applied to any group, whether a group of substitutions or of transformations or of abstract elements. The disadvantage of such a definition of the concept of quotient group is that it cannot easily be generalised to encompass other algebraic structures. However, a moment's thought reveals that the basic idea behind this definition is that of equivalence. If one can separate the elements of any algebraic structure into equivalence classes and produce a well-defined 'multiplication' of these classes, one has formed a quotient structure.

What must be understood in order that a definition of quotient group satisfying the two criteria mentioned above may be given? The first criterion necessitates a recognition that the elements in a quotient group are not of the same type as the elements of the original group. They are *sets* of the original elements in the definition stated above: these sets are the normal subgroup H and its cosets in G . An understanding of the abstract notion of a group is (obviously) required in order that the second criterion is met.

Various other notions proved helpful in the development of the concept of quotient group. Earliest among these is the idea of an auxiliary equation in Galois theory. Later appear in addition the notions of homomorphism and of several permutation representations for a group; these latter provide a description of the concept of quotient group within permutation group theory. Indeed, one now recognises that such a permutation representation is in fact a homomorphism, just as cosets are equivalence classes. The underlying ideas here, those of equivalence relation and homomorphism, are themselves connected by the first isomorphism theorem. This tells us that the quotient of a group by the kernel of a homomorphism is isomorphic to the image of the homomorphism. Therefore, as Rotman remarks [1984, 22],

The first isomorphism theorem thus says that there is no significant difference between a quotient group and a homomorphic image.

By the end of the last century, the standard definition of the concept of quotient group that has been discussed had been formulated and was in use alongside earlier, more permutation-theoretic, attempts at a definition. My intention is to survey the development of the concept up to the 1890s, concluding with the contribution of Otto Hölder, in the light of the two criteria for definition mentioned above. I do not, however, intend to dismiss in any way the insights of those mathematicians whose attempts to explain the concept do not meet my criteria. This was sometimes the case by their own choice and sometimes because of the constraints of the methods available to them. First let us look at the prehistory of this development.

Implicit use of the concept: Galois

With the benefit of hindsight, one recognises that the concept of quotient group was present in many situations before it was defined explicitly. The Galois theory of algebraic equations provides such a situation. In Galois' original explanation of the theory, he crystallised for the first time the concepts of *groupe* and normality of a subgroup (*décomposition propre*). He discussed how the solution process for a given equation corresponds to an investigation of the structure of its associated group. Galois used the term *groupe* to refer to a set of arrangements of the roots of the equation under consideration, rather than to a set of permutations of these arrangements (which was the later understanding of the term). He recognised, however, that it is the permutations which have the 'group structure'. Thus he was able to write [1832, 409/26/175]:

... quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations: l'une aura un groupe de M permutations, l'autre un de N permutations. [1]

In modern terms, this remark becomes: if the group G of an equation has a normal subgroup H , the equation can be solved by means of two equations whose groups are nowadays known as G/H and H . Galois had no concept of quotient group, so the group that is now written G/H was, in his mind, the group associated with the 'auxiliary equation'. This remarkable insight lies at the heart of Galois theory; the passage quoted above is a particularly clear formulation of Galois' proposition III in the *Premier mémoire* [1846a, 425/41/57]. Galois' thoughts were focused on the discovery of a method by which one could establish whether or not an equation was soluble by radicals. Therefore he did not need to investigate the way in which the group of an auxiliary equation arises from the group of the original equation.

Early understanding of the concept: Betti

The first extensive commentary on Galois' writings to be published was attempted by Betti [1852]. The two halves of his paper are devoted to the theory of substitutions and of substitution groups and to Galois' theory of equations. Betti was aware in 1852 of the advances in algebra as set out in Serret's textbook [1849], but did not mention Cauchy's work of 1845 and 1846 on substitution group theory. Betti's contributions to group theory are discussed in detail in Appendix C; here a summary of his work as it relates to the concept of quotient group is given.

Betti sought to explain Galois' results from the viewpoint of substitution group theory. Central to his approach to Galois' theory was an investigation of the way in which the group of an auxiliary equation is related to the group of the original equation. The first section of his paper was partially aimed at the treatment of this question within the framework of substitution group theory and not of Galois theory. That is to say, Betti searched for some way to explain the fact that a group with a normal subgroup automatically gives rise to another group, which is now understood as a quotient group. Although he made considerable progress in his search, he did not seem to understand the problem in full generality. At first his method only treats groups that are split extensions; essentially, any quotient group must be isomorphic to a subgroup. Later he

realised that it is in fact possible to develop the theory even when this is not the case.

The difficulties in understanding Betti's paper are compounded by his notation and terminology, which are often highly suggestive to a twentieth-century reader and easily open to over-interpretation. Betti followed Galois in using the term *gruppo* for a set of *arrangements* (in Betti's understanding, the arrangements are of arbitrary 'quantities', not just of the roots of an equation). His ideas were all expressed in terms of groups of arrangements. However, he clearly recognised, as had Galois, the group structure which exists on the substitutions of these arrangements. (Refer to Appendix C for further explanation.) I use the word 'group' for the substitutions, not for the arrangements, and explain Betti's ideas in terms of groups of substitutions.

Early in the paper Betti introduced the idea of conjugation of substitutions [1852, 55–58/36–38]. He explained briefly what is meant by a *gruppo di permutazioni* (*group of arrangements*) and its *sostituzioni* (*substitutions*), and then he explored conjugation of groups. He considered two groups $\hat{G}$ and $\hat{\Gamma}$, both acting on the same collection of 'quantities'; the group $\hat{G}$ is conjugated by the substitutions of $\hat{\Gamma}$. So he restricted the theory to the situation in which the 'conjugating' substitutions form a group.

Betti examined two cases: the first occurs when none of the (nonidentity) substitutions of $\hat{\Gamma}$ normalises $\hat{G}$, the second when $\hat{\Gamma}$ is contained in the normaliser of $\hat{G}$. No intermediate case was discussed. In both situations Betti constructed the 'product' $\hat{G}\hat{\Gamma} := \{\theta_j\psi_i \mid \theta_j \in \hat{G}, \psi_i \in \hat{\Gamma}\}$ and considered what would happen if the collection of substitutions in $\hat{G}\hat{\Gamma}$ formed a group $\hat{H}$. A group is certainly formed in the second case. The group $\hat{G}$ is a normal subgroup of $\hat{H}$, which becomes the split extension of $\hat{G}$ by $\hat{\Gamma}$ if $\hat{G} \cap \hat{\Gamma} = \{1\}$. Betti specified here that $\hat{G} \cap \hat{\Gamma} = \{1\}$; later he dealt with the situation when the substitutions of these groups are not distinct. He referred to $\hat{\Gamma}$ as a *moltiplicatore* (*multiplier*) of $\hat{G}$ and as a *divisore* (*divisor*) of $\hat{H}$. His choice of these terms must have reflected to some extent the way in which he viewed the relationship between $\hat{\Gamma}$ and $\hat{H}$. It is nowadays known that in this split extension $\hat{\Gamma}$ is isomorphic to the quotient group $\hat{H}/\hat{G}$. But $\hat{\Gamma}$ is also a subgroup of $\hat{H}$. How did Betti conceive their relationship?

He developed the theory further to form a group which is now recognised as the image of the permutation representation of $\hat{H}$ on the right cosets of $\hat{G}$. The name Betti provided for this group (he defined it as a collection K of arrangements) was the *gruppo delle permutazioni sopra i derivati* (*group of arrangements on the conjugates*). When

the substitutions of $\hat{\Gamma}$ normalise $\hat{G}$ (and Betti specified that the substitutions of these two groups are distinct), the image $\hat{K}$ of this permutation representation is (isomorphic to) the quotient group $\hat{H}/\hat{G}$. Thus it is isomorphic to $\hat{\Gamma}$. Betti's confusion in developing his theory only in the situation when every quotient group of a given group is isomorphic to a subgroup of that group could therefore have been avoided, had he made use of $\hat{K}$ rather than $\hat{\Gamma}$ when investigating the relationship between a group $\hat{H}$ and its normal subgroup $\hat{G}$.

A few pages later on in his paper Betti encountered the concept of quotient group again. He considered [1852, 64–65/43] what would happen if one were to form the product of two groups whose intersection is not trivial. He changed his notation: the group $\hat{\Gamma}$ became $\hat{\Gamma}_n$. Betti wrote that two substitutions of $\hat{\Gamma}_n$ which, in modern language, are in the same coset of $\hat{G}$, have now to be considered equal to each other in $\hat{\Gamma}_n$. He was in effect trying to define an equivalence relation on the substitutions of $\hat{\Gamma}_n$. Thus he stipulated that the order of $\hat{\Gamma}_n$ is to be calculated as the number of substitutions in it which are in different cosets of $\hat{G}$. Using modern notation, the order of $\hat{\Gamma}_n$ is given by $[\hat{\Gamma}_n : \hat{G} \cap \hat{\Gamma}_n]$. Nowadays we understand the equivalence classes defined here as elements of the quotient group $\hat{\Gamma}_n/(\hat{G} \cap \hat{\Gamma}_n)$. In fact this was the approach that Jordan took to introduce the concept of quotient group twenty years after Betti's work. Betti, however, did not seem to think that he was dealing with a new group but rather including an extra condition in the definition of $\hat{\Gamma}_n$.

Betti then explained that (in modern terms) if a set of coset representatives is required for the cosets of $\hat{G}$ (these cosets are found when $\hat{G}$ is multiplied by the substitutions of $\hat{\Gamma}_n$), one may not be able to choose the representatives in such a way that they form a group. Betti seems finally to have been acknowledging that not all group extensions split.

Therefore it appears that Betti understood some but not all of the ideas behind that of quotient group. He realised that in special cases one is able to 'divide' a group by another group to produce a third group. However, he did not fully appreciate that, unlike division in the real numbers where the result also lies in the real numbers, the result of this 'group division' is a group which acts in a new way. It cannot be compared with the original two groups (as groups of substitutions acting on a set of arrangements) in a simple manner.

Mammone [1989, §2] has shown how Betti introduced the set K , his *group of arrangements on the conjugates*. Mammone also explains the way in which this gives rise to a quotient group. At the end of [1989] he considers Betti's formulation of Galois' proposition III. Betti had realised that, under the hypothesis of proposition III, the Galois group of an equation is built up from smaller groups. However, Betti oversimplified and again dealt only with the case when the group is a split extension—a 'product' in his terminology. Mammone rightly points out that this nevertheless reveals Betti's concern to establish the relationship between the group of the given equation and that of the auxiliary equation.

Betti later began to develop and modify his ideas [1855]. He no longer insisted that the 'conjugating' substitutions form a group but he explained this no further. It does not seem that his papers exerted much influence on the subsequent development of Galois theory, and even less on that of substitution group theory. The scope of Betti's work was certainly known to Jordan, who wrote in the preface to his *Traité* [1870] that Betti was the first to establish rigorously the complete sequence of Galois' theorems. Jordan referred to Betti's paper [1852] on the subject as a *Mémoire important*. However, there is no evidence that Jordan had read or understood Betti's work in any great detail, and it appears that Jordan did not consider the *mémoire* important enough for him to be influenced by it in his own approach to the theory.

A systematic approach to the concept: Jordan

We have reached the next mathematician in the story, Camille Jordan, to whom several commentators have attributed the first explicit definition of a quotient group. In particular Gaston Julia, the writer of the preface to Jordan's *Oeuvres*, commented [Jordan 1961, V]:

Bien que, comme ses contemporains, Jordan ne considère guère que des groupes de permutations ..., c'est lui cependant qui dégage la notion "abstraite" de groupe quotient.
[2]

It is true that Jordan only dealt with groups as groups of substitutions or transformations, even in work as late as 1917, although he was no doubt familiar with the developments in abstract group theory. It is equally correct to say that he defined a group which is certainly isomorphic to that given by our 'standard' definition. Thus in one sense Jordan did 'bring out the "abstract" notion of quotient group', in that he un-

derstood how the “abstract” concept is present in permutation group theory. However, his definition, having been formulated in the context of his research into that theory, is not abstract (in the sense in which abstraction is nowadays understood); in Jordan’s day the definition would indeed have been considered abstract.

Dieudonné produced some notes on Jordan’s research in finite group theory; these were published at the beginning of the *Oeuvres* [1961]. He discussed the theorem of Jordan which Hölder later extended in [1889a] to become what is now known as the Jordan–Hölder Theorem. The evolution of this theorem is the subject of the next chapter of this thesis. The theorem states that for a finite group G , any two composition series for G have the same length and their composition factors, apart from the order in which they occur, are isomorphic. Jordan proved that the *orders* of these composition factors (thought of as the ratios of the orders of successive groups in a composition series) are the same for any two composition series. The proof of this theorem first appeared in Jordan’s *Traité* [1870]. Dieudonné wrote [Jordan 1961, XVIII]:

A ce moment, la notion de groupe quotient n’est pas encore conçue clairement, bien que Jordan utilise couramment, dès ses premiers travaux, le calcul “modulo” un sous-groupe distingué; ce n’est qu’en 1873 qu’il définit explicitement la notion de groupe quotient et considère les groupes quotients successifs d’une suite de composition; mais on sait que c’est seulement avec Hölder (en 1889) que le théorème d’invariance prendra sa forme définitive.
[3]

This idea of calculating ‘modulo’ a normal subgroup is indeed the idea which gave rise to Jordan’s concept of quotient group. In his approach there is an obvious parallel with Gauss’s work on arithmetic congruences in 1801, which Jordan undoubtedly had in mind. Jordan employed the symbol $\equiv$ which Gauss had introduced to denote congruence. Dieudonné commented of the Jordan–Hölder Theorem that it would assume its definitive form only with Hölder. This might also be said of the concept of quotient group: in both cases Jordan’s ideas were early formulations which later gave way to the accepted standard forms.

Jordan worked on the development of Galois’ ideas during the 1860s; the concept of quotient group thus appeared implicitly in his research as the group of an auxiliary equation. Jordan understood and explained the methods of Galois theory more clearly than earlier mathematicians, and correspondingly developed the theory of substitution groups. However, he made no attempt to produce a quotient group explicitly. It was only later, in 1873, when Jordan extended some results of Mathieu on the limit of

transitivity of groups, by way of Sylow's Theorems, that he used the idea of congruence of group elements to produce a quotient group structure [1873a, 45-47/370-372]. He remarked that the developments and definitions he would give seemed to him to make the proofs of several important propositions considerably simpler. The first definition is:

Deux substitutions s et t , permutables à un groupe H , sont dites *congrues suivant le groupe H* , si l'on a une égalité de la forme

$$s = th,$$

h étant une substitution de H .

On peut exprimer cette relation par une formule analogue à celle des congruences ordinaires

$$s \equiv t \pmod{H}. [4]$$

The phrase 'permutables à un groupe H ' translates into modern terms as $sH = Hs$ and $tH = Ht$, that is, s and t are in the normaliser of H . The 'congruences ordinaires' are congruences in the integers, as explained by Gauss.

Jordan then showed that congruences may be multiplied together member by member: if $s \equiv t \pmod{H}$ and $s' \equiv t' \pmod{H}$, then $ss' \equiv tt' \pmod{H}$. This proves that multiplication in the quotient group structure is well-defined. Next he defined this new structure:

On dira qu'une suite de substitutions $s_1, s_2, \dots$ (toutes permutables à un même groupe H) forme un *groupe suivant le module H* , si l'on a pour toutes valeurs de α et β une relation de la forme

$$s_\alpha s_\beta \equiv s_\gamma \pmod{H}. [5]$$

Therefore the structure is defined by imposing the condition of closure on its elements. The order of the *groupe suivant le module H* is defined to be the number of different substitutions in it which are not congruent mod H . If G is the group generated by $s_1, s_2, \dots$ in the normal way, then Jordan denoted by $\frac{G}{H}$ the group that they form mod H . Since H is not necessarily contained in G , the order of G is given by $|G| = \left| \frac{G}{H} \right| |G \cap H|$, using modern notation.

Thus Jordan's quotient group structure $\frac{G}{H}$ consists of congruence classes of the elements $s_1, s_2, \dots$ from which G is formed. His notation was organised so that, in effect, the quotient group consists of one representative from each conjugacy class, although these representatives do not necessarily form a group themselves. In this way he was

able to remain within the limits of substitution group theory while embracing a concept which cannot easily be adequately described in this theory. He did exactly what Betti seems to have been unable to grasp that he should do: Jordan took a *set* of substitutions in the normaliser of H and combined them to form a group of a different type, a group formed 'with respect to' H . He continued by noting that all the principal definitions relating to 'ordinary' groups (meaning groups of substitutions) can be carried over to groups 'formed with respect to the modulo H ': commutativity, membership of a normaliser, conjugation of elements, isomorphism and homomorphism.

Jordan made use of these new quotient groups in another paper of 1873, 'Mémoire sur les groupes primitifs' [1873b]. In this long article, he linked the degree of certain primitive substitution groups with properties of a particular substitution in each group. It seems unlikely that the references to quotient groups, which are embedded deep in this paper, would have had much influence on Jordan's contemporaries. This article appears to be the last in which Jordan used the new concept. Nevertheless, the first of these two papers marks significant progress in its development. The paper was cited several times by Frobenius in his later work on finite groups (see [1887a, 180/302; 1887b, 273/304; 1895a, 86/637]) and also by Burkhardt [1898, 219].

Abstraction and the concept of equivalence

After these papers of Jordan, the development becomes less easy to trace. The concept of quotient group began to be approached from new and different angles. This process happened as 'abstract' ideas were introduced into the study of groups. The old theories were gradually being re-formulated and extended by means of these new ideas. As these great changes were taking place, the usefulness of the concept of equivalence was more and more clearly recognised. This concept was certainly known to Dedekind, Kronecker, Frobenius, Cantor and Hölder, and also to Frege, whose approach was somewhat more philosophical. Here I intend only to provide a few indications of the way in which equivalence appeared in their work.

Frege's book *Die Grundlagen der Arithmetik* [1884] includes a detailed discussion of the equality of numbers (§§62–69), which he defined by means of one-one correspondence. The definition depends on the fact that one-one correspondence is an equivalence relation and, although Frege did not give an explicit statement of the concept, he cer-

tainly recognised its importance.

Dedekind came upon the concept of equivalence as a result of his investigations into the foundations of analysis and therefore into the real number system. In the introduction to *Stetigkeit und Irrationale Zahlen* [1872] (in which he explained the 'Dedekind cut'²), he remarked that his attention was first directed towards these matters in the autumn of 1858. Not until 1888 did he publish the results of his study into the nature and meaning of numbers (*Was sind und was sollen die Zahlen?*). In sections II and III of this essay, he developed the theory of functions and the idea of two systems of elements being *similar* (when there is a bijection between them). In this way, he stated, we can separate all systems into *classes*

... by putting into a determinate class all systems $Q, R, S, \dots$, and only those, that are similar to a determinate system R , the *representative* of the class; ... the class is not changed by taking as representative any other system belonging to it. [1888, 55]

Thus he showed that similarity is an equivalence relation and formed its equivalence classes.

The ideas expressed in Dedekind's essay do not seem to have been universally understood, not because they were unclear but because, in the case of equivalence, they were unfamiliar. J. van Heijenoort includes in his book [1967] a description of the correspondence which ensued between Dedekind and an *Oberlehrer* in Hamburg named Hans Keferstein as a result of this essay. Keferstein had published a paper in 1890 in which he made some suggestions for amending Dedekind's essay. One of his suggestions came about because he confused the equivalence relation between two sets with their identity.

Cantor also discovered the concept of equivalence because of his work on transfinite numbers begun in 1870 and finally summed up in two memoirs [1895&1897]. He made use of one-one correspondence to define 'equivalence' of sets and by means of this concept to define cardinal numbers³.

It may be that several of these mathematicians came across the idea of equivalence independently. However, there was much communication between them and they fre-

²Dedekind's definition of a 'cut' has to do with a quite different sort of equivalence from that under examination here. He did not define a real number as a pair of sets or cut but rather stated that to each cut there *corresponds* such a number.

³As with Dedekind and his 'cuts', this is a different concept to that used by Cantor in his definition of real numbers by Cauchy sequences. There, a symbol is associated with a Cauchy sequence and this symbol is the 'real number'.

quently refer to one another's work in their writings, and therefore the development of these techniques cannot be attributed to any one person. Nevertheless the result of their investigations was to produce a mathematical climate in which the concepts of equivalence and equivalence classes, and the possibility of definition by means of these concepts, could be freely employed. Thus this development in the understanding of the role of equivalence is reflected in the development of the concept of quotient group⁴.

Pioneer of the concept: Dedekind

Dedekind appears to have understood the role of equivalence at a much earlier period, in particular in his work during the years 1855–1858. He explored the theory of groups at that time; his writings remained unpublished until after his death in 1916. They were discovered in his *Nachlass* and included in his *Gesammelte Mathematische Werke* (as [1932]). During his lifetime he had done no more than mention this work to Frobenius in a letter of 1895. Dedekind examined the concept of homomorphism in a section entitled 'Äquivalenz von Gruppen'. He formed a homomorphic image M_1 of a group M by letting each element θ of M 'correspond' to an element θ_1 of M_1 , with certain conditions which we now recognise as the conditions for homomorphism. He proved that M_1 is a group and that those elements of M which 'correspond' to the identity in M_1 form a subgroup N of M . He went on to discover the concept of quotient group, as follows.

He expressed the group M in terms of the subgroup N and its cosets, stated that a 'composition' of cosets can be defined and that in this way the cosets (he referred to them simply as *Komplexe*, that is, *sets*) form a group. There is a correspondence between the cosets and the elements of M_1 : to each coset corresponds one element of M_1 , and to each element of M_1 corresponds one coset. This would nowadays be expressed by the statement that the group M_1 and the group of cosets are isomorphic. Dedekind gave no name either to the concept of homomorphism or to that of quotient group.

It is also known that during the years 1856–1858 Dedekind lectured on algebra at

⁴It is interesting to note the way in which attitudes have changed since the end of the last century. A footnote in Birkhoff and Mac Lane's textbook on algebra [1953, 153] provides a good example of modern understanding of the concept of equivalence: 'The procedure of treating sets as elements is not sophisticated; thus, one commonly speaks of a "regiment", meaning a certain set of men, or of a "molecule", meaning a certain set of atoms.'

the university of Göttingen. A manuscript published for the first time in a book commemorating the 150th anniversary of his birth [Scharlau 1981, 59–100] contains what is almost certainly the text of these lectures or his writings immediately following them⁵. In the manuscript Dedekind treated the concept of quotient group in much the same way as above but without the parallel development of the notion of homomorphism. The original lectures attracted a total audience of four and appear to have had little influence. In his discussion of the manuscript (see [1981, 107–108]), Scharlau records that one of those who attended the lectures, P. Bachmann, later told Dedekind that he had understood very little of them at the time!

Emmy Noether commented in her note on the 1855–1858 manuscript that one can see how completely Dedekind was in possession of concepts and methods of abstract group theory, even as early as the 1850s. We might also apply her well-known motto ‘Es steht alles schon bei Dedekind’ (‘It is already all there in Dedekind’).

The influences of abstraction: Dyck and Frobenius

We return now to the later period, to the year 1882, when a paper by Dyck, ‘Gruppentheoretische Studien’, appeared in the *Mathematische Annalen*. This paper, which marks the beginning of combinatorial group theory, has been discussed by Chandler and Magnus [1982]. It begins with the abstract construction of the free group, G say, on elements $A_1, A_2, \dots, A_m$. Dyck was concerned in section 4 to explore the way in which any group $\bar{G}$, generated by elements $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_m$ (which are defined by ‘some predetermined process’), is related to the original free group G . Dyck’s notation here already suggests a hidden assumption that $\bar{G}$ is a homomorphic image of G under the homomorphism taking $A_i \rightarrow \bar{A}_i$. Indeed, a few sentences later Dyck proceeded to show that this is indeed so; there are two cases to consider.

The first case occurs when the groups are isomorphic; the second occurs when any one element of $\bar{G}$ corresponds to infinitely many elements of G . In the second case, Dyck found the elements of G which correspond to the identity in $\bar{G}$ and showed that they form a normal subgroup H of G . He investigated the structure of this normal subgroup and then stated that the relationship between G and $\bar{G}$ may now be set out [1882, 14]:

Die isomorphe Zuordnung der Gruppen G und $\bar{G}$ spaltet die Gruppe G in zwei Factoren: In die Gruppe der Substitutionen, welche in den Substitutionen $\bar{A}_i$ geschrieben *verschieden*

⁵See W. Purkert’s interesting article [1976] for the evidence supporting this.

sind, d. h. die Gruppe $\overline{G}$ selbst—und in die Gruppe H derjenigen Substitutionen, welche in den Substitutionen von $\overline{G}$ geschrieben *der Identitätäquivalent* sind. Die letztere Gruppe H ist dabei in G ausgezeichnet enthalten und folgt aus ihr “durch Adjunction von $\overline{G}$ ”. [6]

The phrase “durch Adjunction von $\overline{G}$ ” seems to have been borrowed from Galois theory, since one reduces the Galois group of an equation to a normal subgroup by *adjoining* elements to the field, and these elements are the roots of an equation whose Galois group is $\overline{G}$. Therefore the group G can be considered to have two factors: its normal subgroup H and the homomorphic image $\overline{G}$ which now, of course, is also known as the quotient group G/H .

Later in 1882 Dyck wrote a further paper on group theory, ‘Gruppentheoretische Studien II’ [1883], based in part on some lectures he had recently given. As in [1882], he stressed that it is the ‘abstract’ properties of any group that are the important ones. He investigated the concepts of primitivity and transitivity in this article. He found it convenient [1883, 84–86] to consider the regular permutation representation of a group G of order N , and he investigated the relationship between subgroups and imprimitivity: that if the group has a subgroup then the regular representation is imprimitive, and vice versa⁶.

Next [1883, 90] one is to assume that G has a subgroup G' , where G' has order μ and $N = \mu\nu$. Dyck denoted G' and its right cosets by $T_1, T_2, \dots, T_\nu$, respectively. Then he considered how these ‘systems’ T_k are permuted amongst themselves by the elements of G . Thus he formed the representation of G on the right cosets of G' , an idea that Betti had happened upon. Dyck stated that this representation is isomorphic to the regular representation of G as long as G' is not normal in G and does not contain any normal subgroup of G . He finally investigated [1883, 96] the case when G' is normal in G ; he worked out how the table giving the regular representation of G splits into blocks which are permuted as a whole by the elements of G . However, this is all that he did: there is no explanation of the way in which G is homomorphic to this representation on the cosets of G' .

The fact that Dyck stopped where he did is rather puzzling, in view of the explanation of homomorphism he had given at the beginning of [1882]. He approached the concept of quotient group in two completely different ways but was not able to connect the

⁶Kiernan [1971, 133] appears to have misunderstood this section of Dyck’s paper—Dyck was not dealing here with quotient groups.

two. Had he done so, he might well have realised that one can dispense with the homomorphism and the representation on the cosets and can look within the group itself for the concept of quotient group. After all, he had explained in the introduction to [1883] that he aimed to take an abstract approach in order to discover the extent to which the properties of groups remain invariant, in all their different modes of representation.

This was not the first time that Dyck's work had touched on quotient groups. Two years previously they had appeared implicitly in a paper on Riemann surfaces [1880, 480ff]. He made use of composition series and wrote of the simple groups out of which a composition series is made (in modern language, the composition factors).

During the late 1870s and the 1880s Frobenius was also led to consider the idea of equivalence of group elements. His paper written with Stickelberger on abelian groups [1879] contains an early formulation of this idea. The fundamental theorem of abelian groups was proved here using what are effectively equivalence classes with respect to subgroups of a given abelian group (since it is abelian all subgroups are normal). This viewpoint was borrowed from Kronecker's article [1870], in which the first proof of the fundamental theorem of abelian groups had appeared (without, however, a proof of the uniqueness of the decomposition). The article makes no mention of the term group: Kronecker was nevertheless investigating the structure of what is now known as the ideal class group of an algebraic number field, a group that is finite and abelian. In his proof of the fundamental theorem he considered only a set of 'elements' and a well-defined binary operation on the set, and specified other conditions which are exactly those for an abelian group. He found it necessary to extend the concept of equality of elements, which he called 'equivalence' and denoted by $\sim$, so that [1870, 884/277]

... zwei Elemente θ' und θ'' als "relativ äquivalent" angesehen werden, wenn für irgend eine ganze Zahl k :

$$\theta' \cdot \theta_1^k \sim \theta''$$

ist. [7]

He went on to remark that if a complete set of relatively nonequivalent elements are chosen, they satisfy the same conditions as the original set. Thus in effect this new set is formed of coset representatives for the cosets of the subgroup generated by θ_1 ; it is isomorphic to the quotient group produced by this subgroup. The proof of the fundamental theorem of abelian groups given by Kronecker was repeated almost verbatim, but explicitly in terms of abelian groups, by Netto in his book on the theory

of substitutions [1882, 144ff/1892, 156ff]. For more information on Kronecker's article refer to Wussing [1969, 44–48/1984, 63–67].

Frobenius developed a new proof of Sylow's Theorems [1887a], which, although published in 1887, is dated 1884. In this short paper Frobenius followed Jordan's approach to the concept of quotient group but in the setting of abstract group theory. Frobenius proved that if the order of a group is divisible by p^ν where p is prime, then the group has a subgroup of order p^ν .

Let $\mathfrak{H}$ be a group of order h , where p^ν (or a higher power of p) divides h . Assume that the theorem is true for groups of order less than h . Form the centre of $\mathfrak{H}$ —it is a subgroup $\mathfrak{G}$ of $\mathfrak{H}$ whose order g will therefore divide h . Two cases are distinguished: either p divides g or it does not. The first case gives rise to a quotient group, as follows.

Frobenius first proved Cauchy's Theorem for abelian groups: the group $\mathfrak{G}$ contains a nonidentity element of order p , which he denoted by P . Now consider two elements of $\mathfrak{H}$ as (relatively) equal if they differ only by a power of P . Here Frobenius cited Kronecker's paper [1870, 884/277] and Jordan's paper [1873a, 46/371]. (The facts that Frobenius needed to cite the papers containing this definition, and that they had both been published over a decade earlier, suggest that the concept of 'equivalence' of group elements had not been influential or widely recognised.) Then the conditions for a group are fulfilled for this broader idea of equality, since each power of the element P commutes with each element of $\mathfrak{H}$. The 'relatively different' elements of $\mathfrak{H}$ form a group, whose order is $\frac{h}{p} \dots$

The proof of the Sylow's Theorem for this case follows after a few more lines of reasoning from Frobenius' inductive hypothesis. In modern terms, he had formed the quotient of $\mathfrak{H}$ by the cyclic normal subgroup generated by P .

Sylow's own proof of this theorem in [1872] includes an idea from Galois theory which established the same result that is now deduced using the concept of quotient group. Sylow introduced a function y_0 of the letters on which the group G acts; he required that y_0 remain invariant only under the substitutions of a subgroup G of order n^α , where n is prime and G has no subgroup of order n^β for $\beta > \alpha$. He considered how the distinct values of this function (corresponding to cosets of the subgroup) are permuted by the substitutions in the normaliser γ of g . A substitution group γ' is obtained in this way:

it is transitive and homomorphic to γ . We would nowadays write

$$\gamma' \cong \frac{\gamma}{g}.$$

Sylow's method is equivalent to the formation of the permutation representation on the cosets of g in its normaliser γ .

A further paper should be mentioned here; it was written by Capelli in 1878, not long after Sylow's Theorems were published. Capelli specifically aimed to show the importance of isomorphism (a term which then covered both isomorphism and homomorphism) in the theory of substitution groups. In the course of the paper he proved most of Sylow's Theorems (he had obviously not read Sylow's article), investigated properties of groups of prime power order and presented a new proof of the Jordan-Hölder Theorem. His exploration of the concept of isomorphism brought him to define a permutation representation in the same way as Sylow had done, using a function fixed by the substitutions of a normal subgroup.

Another paper of Frobenius dealing with congruence of group elements [1887b] was published soon after his note on Sylow's Theorem. In this later paper Frobenius applied the ideas of equivalence relations and equivalence classes in order to define double cosets, again citing Kronecker and Jordan for the concept of equivalence of group elements. He investigated the properties which such cosets possess, including the fact that the number of equivalence classes does not change when a common normal subgroup $\mathfrak{N}$ is factored out. His use of the concept of quotient group here follows Jordan's definition and notation of 1873, apart from Frobenius' statement that each set of elements congruent mod $\mathfrak{N}$ is to be considered as *one* element. Then these 'complexen Elemente' form a group—the quotient by the normal subgroup $\mathfrak{N}$.

The fact that Frobenius was able to think of each congruence class as *one* element was an important step forward—his understanding of the abstract approach finally permitted it. Later, in another paper on finite groups [1895a, 86/637], Frobenius cited both Jordan and Hölder when making use of quotient groups and attributed the definition to them equally. Frobenius seems to have been feeling his way between the concepts of these two mathematicians: his idea obviously followed on from Jordan's and he later recognised in Hölder's paper a satisfactory abstract definition of quotient group. He does not seem to have considered himself responsible for anything but the *development* of the concept.

The standard definition: Hölder

The final stage of development came with Hölder's paper [1889a]. We have seen that the questions which Hölder wished to answer here are those prompted by his taking a fresh look at Galois theory in the light of abstract group theory. Which groups correspond to the 'auxiliary equations'? To what extent are these groups defined? How many are there? The natural way to answer these questions is to employ the concept of quotient group. In the introduction Hölder discussed the simple groups arising from a composition series for a finite group: he named these *Factorgruppen*, and noted that this concept of *Factorgruppe* is 'ein bis jetzt nicht hinreichend gewürdigten gruppentheoretischen Begriff' ('a group-theoretic idea that has until now not been adequately appreciated'). He stated that he was to set out only the most elementary group-theoretic ideas in the discussion that followed. It seems that Hölder did not consider the concept of quotient group to be either a new or a difficult one.

In the first part of the paper, the group-theoretic section, Hölder gave axioms for a finite group and mentioned normal subgroups and composition series. He then explained *Factoren der Composition*: a *Factor der Composition* is the index of a group in a composition series in the preceding group of the series, as defined by Jordan. In modern terms it is the order of a composition factor. Hölder wrote [1889a, 30]:

Diese Theorie von den Factoren der Zusammensetzung [he reverted to the German word] muss aber dahin vertieft werden, dass die Factoren als *Gruppen* aufgefasst werden.

Es wird im nächsten Paragraphen gezeigt werden, dass durch das Verhältniss einer Gruppe zu einer in ihr ausgezeichnet enthaltenen Untergruppe stets eine neue Gruppe von im Allgemeinen anderen Operationen definiert ist. Diese letztere Gruppe ist völlig bestimmt von dem abstracten Standpunkt aus, welcher von dem Inhalt der Operationen absieht ... [8]

So he came to define the *Quotient* of a group by a normal subgroup. He showed how the elements of a group G may be divided amongst the cosets of any subgroup H and proved that if H is normal, the multiplication of any two elements from two cosets always gives an element in one and the same coset. In other words, in such a case one can define multiplication of cosets and it is well-defined. Hölder continued [1889a, 31]:

Man erhält so neue Operationen, welche gleichfalls eine Gruppe bilden. Diese vollständig bestimmte Gruppe ist es, welche in die Betrachtung eingeführt werden soll. Man könnte sie den *Quotienten* der Gruppen G und H nennen, dieselbe soll im Folgenden mit

$$G|H$$

bezeichnet werden. [9]

Hölder showed in the following paragraphs that this concept can be expressed in terms of equivalence of group elements. Call two elements of G *equivalent* if they can be transformed into one another under multiplication by an element of the normal subgroup H . Then the equivalence classes form a group. Hölder's explanation of this is very reminiscent of Frobenius' approach in [1887b]: he used the same terms *aequivalent* and *Classe*. The definition given by Hölder was therefore that of Kronecker in [1870] or Jordan in [1873a], as cited by Frobenius, but formulated here from an abstract viewpoint.

Hölder then showed that this quotient group is a homomorphic image of G and mentioned Dyck's comments [1882, 14] on splitting a group into two factors: he recognised that one of these may be thought of as a quotient group rather than as a homomorphic image. He realised that one obtains the same result, whether one sets out with a normal subgroup or a homomorphism. He went on to investigate direct products of groups and led up to a proof of the Jordan–Hölder Theorem.

In the second section of the paper Hölder returned to Galois' theory of algebraic equations, from where we began, and showed how the *Factorgruppen* are the groups of particular auxiliary equations. Referring to the group of any auxiliary equation and the corresponding quotient group of the Galois group, he remarked [1889a, 43] 'Vom abstracten Standpunkt aus, sind diese Gruppen also als identisch zu betrachten' ('From an abstract standpoint these groups are therefore to be considered identical').

Thus in this paper the concept of quotient group was systematically and explicitly defined and its previously implicit appearance in Galois theory was recognised. Hölder introduced the terms *Quotient* and *Factorgruppe* and the notation $G|H$: the terms have remained but the notation has been combined with Jordan's to produce the modern G/H .

Wider recognition of the concept: the 1890s

It was in the 1890s, after this paper of Hölder, that the notion of quotient group began to be incorporated into monographs and textbooks. Netto's book on the theory of substitutions and its applications, first published in 1882, was later revised by Netto and translated into English by F.N. Cole. This English edition 'differs from the German edition in many important particulars', as Netto remarked in an addition to the Preface.

He added that he had taken into account 'the whole material which has accumulated in the course of time since the first appearance of the book'. In particular, the English edition includes the concept of quotient group [1892, 95–96], which is defined as a permutation representation on the cosets of a normal subgroup, as one would expect in a book devoted to substitutions⁷. Netto denoted his quotient group by $G : H$. He made no reference to any previous papers when stating his definition, but later cited Hölder's 1889 paper when defining the 'factor groups' of a group G (that is, the composition factors).

The second volume of Weber's *Lehrbuch der Algebra* [1896] begins with an abstract definition of a group and, after sections on subgroups and normal subgroups, there is a discussion of the properties of subgroups and cosets and the concept of quotient group is presented. It is defined as the group of cosets of a normal subgroup; Weber carefully proved that these cosets do indeed form a group. He went on to prove the Jordan–Hölder Theorem in the same way that Hölder had proved it in 1889, using quotient groups.

In 1897 an English monograph on group theory appeared, Burnside's *Theory of Groups of Finite Order*. Burnside approached the concept of quotient group of a group G by way of 'multiple isomorphism', that is, homomorphism, and in this he followed Dyck [1882, 11–14]. Burnside considered [1897, 35ff] a group G' to which a homomorphism from G can be defined: the operations of G which correspond to the identity in G' form a normal subgroup Γ of G . Having investigated, as Dyck had done, the ways in which G' and Γ are related to and determined by one another, Burnside explained that the relation of 'multiple isomorphism' can be presented in a rather different manner. Following Hölder, he explained how the cosets of a normal subgroup form a group. He then drew these two methods of approach together: the group G' is 'simply isomorphic' (that is, isomorphic) with this group of cosets. He remarked that the group G' is completely defined when G and Γ are given. He continued [1897, 38]:

This being so it is natural to use a symbol to denote directly the group thus defined in terms of G and Γ . Herr Hölder [1889a] has introduced the symbol

$$\frac{G}{\Gamma}$$

to represent this group; he calls it the *quotient* of G by Γ , and a *factor-group* of G .

Hölder did in fact reserve the name *Factorgruppen* for the composition factors: perhaps

⁷The German edition does not mention quotient groups—Kiernan is mistaken here [1971, 133].

Burnside's misreading is the reason why both the name 'quotient group' and the name 'factor group' are now in common usage. J.J. Rotman preserves Hölder's original distinction in his textbook [1984, 21 & 75] but sadly he appears to be in the minority in doing so.

It is interesting to note which papers and mathematicians are referred to by those who made use of quotient groups in the 1890s. In [1887a] and [1887b] Frobenius had cited Jordan but in [1895a], after Hölder's definition had appeared, he attributed the definition of quotient group to both Jordan and Hölder. Burkhardt devoted only two sentences to the definition of quotient group in the 'Normalteiler' (normal subgroup) section of his encyclopedia article [1898]. He defined it in an unusual way but his choice of approach becomes understandable when we look at his sources: he attributed the discovery of quotient groups to Betti [1852], with the reservation 'wo aber alles nicht klar ist' ('where however everything is not clear'). Burkhardt's explanation *is* clear:

Let G be a substitution group of order mn . The objects it permutes are arranged in mn ways by the action of G . A normal subgroup H of G , of index m and order n , will split these arrangements into m systems each containing n arrangements. The operations of H permute the arrangements in each individual system; the remaining operations of G permute the systems as a whole. The different permutations of the systems form a group of order m .

Burkhardt added that this group, to which he gave no name, is denoted by G/H , and here he referred to Jordan and Hölder. It is remarkable that an encyclopedia article written at the turn of the century should have put such emphasis on substitution groups and passed up a more abstract definition of quotient group, at the very time when abstraction was flourishing.

In 1893 Cayley wrote a 'Note on the so-called Quotient G/H in the theory of groups', in which he cited Hölder's paper of 1889 and referred to a paper of Young [1893], who, in his turn, attributed the definition of quotient group and the term 'quotient' to Hölder. In a paper of his own [1893, 317], Hölder cited Jordan, saying that quotient groups had already appeared in Jordan's work and that he (Hölder) drew up their theory anew in 1889.

Conclusions

Thus it is clear that one cannot attribute the development of the concept of quotient group to any particular person or period of time. Like almost all mathematical ideas, the period from its first occurrence in primitive form to its full understanding and acceptance as commonplace is a long one, and includes the contributions of many mathematicians. The opinions of modern commentators on the matter seem to favour their cultural and mathematical backgrounds. In Bourbaki it is stated that Jordan introduced the notion [1960, 76], while Van der Waerden writes that the modern understanding of quotient group is due to Hölder and that Jordan had it implicitly [1985, 121]. Wussing [1969, 183/1984, 244] takes the view that the idea of quotient group was a deduction from the abstract group concept. He then notes that Hölder put an abstract definition of a group at the beginning of his 1889 paper, which suggests that he considers Hölder to have introduced the concept of quotient group.

Returning finally to this link with abstract group theory it does seem that, without an abstract setting for his work, Hölder could not have defined quotient groups as he did. As mentioned before, a recognisably modern definition of a quotient group depends on such an abstract setting. The abstraction process was well advanced by 1880 and yet at that time only Jordan's idea of quotient group was available. One last thread had still to be woven in to complete the picture—that of using equivalence to define new types of objects. Thus, once this technique had received acceptance, it was not at all unnatural for Hölder to define quotient groups as he did in 1889. Sixteen years earlier, in 1873, such ideas would have been far removed from Jordan's understanding. Although it is extremely unlikely that Jordan's substitution-theoretic concept of quotient group contributed actively to the abstraction process for group theory, the need to extend and deepen this concept could certainly have provided one of the forces moving group theory towards abstraction. The evolution of abstract group theory and the development of the concept of quotient group are interdependent, and thus it is that this theory and this concept emerged during the same period and in the work of the same mathematicians.

Notes

1 ... when the group of an equation admits a proper decomposition, in such a way that it divides into M groups of N permutations, one will be able to solve the given equation

by means of two equations: one of them will have a group of M permutations, the other a group of N permutations.

2 Although, like his contemporaries, Jordan hardly considers anything but groups of permutations ..., it is, however, he who brings out the "abstract" notion of quotient group.

3 At this time there is still no clear conception of the notion of quotient group, although Jordan freely uses, from his earliest works, calculation "modulo" a normal subgroup; it is only in 1873 that he defines explicitly the notion of quotient group and considers the successive quotient groups of a composition series; but we know that the invariance theorem will assume its definitive form only with Hölder (in 1889).

4 Two substitutions s and t which commute with a group H are said to be *congruent with respect to the group H* if one has an identity of the form

$$s = th,$$

h being a substitution in H .

One can express this relation by a formula analogous to that of ordinary congruences

$$s \equiv t \pmod{H}.$$

5 We will say that a series of substitutions $s_1, s_2, \dots$ (which all commute with the same group H) form a *group with respect to the modulo H* if for all values of α and β we have a relation of the form

$$s_\alpha s_\beta \equiv s_\gamma \pmod{H}.$$

6 The relationship of isomorphism between the groups G and $\overline{G}$ splits the group G into two factors: into the group of substitutions which are *different* when written in terms of the substitutions $\overline{A}_i$, that is, the group $\overline{G}$ itself—and into the group H of those substitutions which, when written in terms of the substitutions of $\overline{G}$, are *equivalent to the identity*. The latter group H is then contained as a normal subgroup in G and comes from it "by adjunction of $\overline{G}$ ".

7 ... two elements θ' and θ'' will be considered as "relatively equivalent" if

$$\theta' . \theta_1^k \sim \theta''$$

for some integer k .

8 This theory of the composition factors must however be deepened, in the sense that the factors are to be understood as *groups*.

It will be shown in the next paragraphs that through the relationship of a group to a normal subgroup contained in it, a new group of (usually) other operations is always defined. This latter group is completely determined from the abstract standpoint, which disregards the nature of the operations ...

9 In this way one obtains new operations, which likewise form a group. It is this well-defined group that is to be brought into consideration. One could call it the *quotient* of the groups G and H and it will from now on be denoted by

$$G|H.$$

CHAPTER 6

The Evolution of the Jordan–Hölder Theorem

There is an enormous literature on this result.

G. Birkhoff: *Lattice Theory* (1948)

The Jordan–Hölder Theorem for a finite group G states that any two composition series for G have the same length and their composition factors, apart from the order in which they occur, are isomorphic. A composition series is a series of subgroups

$$G = G_0 \supset G_1 \supset \dots \supset G_n = \{1\}$$

such that each is maximal normal in the preceding one; the composition factors are the quotient groups G_i/G_{i+1} ($i = 0, 1, \dots, n-1$) formed by successive terms in the series. The theorem and the ideas surrounding it evolved during the process of making rigorous and coherent the theory of algebraic equations developed by Galois. As this process took place, there was a simultaneous advance in the understanding of group theory. I trace in this chapter the development of the Jordan–Hölder Theorem from its beginnings until the year 1900, which corresponds with the period of evolution within group theory. During the twentieth century the theorem has appeared in a wider context as various generalisations have been made, and it has assumed its place as a fundamental uniqueness result in the study of many algebraic structures. The more recent developments lie beyond the scope of this thesis.

The publications of Camille Jordan throughout the 1860s helped in large measure to render Galois' theory of algebraic equations widely available and comprehensible. Jordan gradually turned from a consideration principally of the equations to focus on their associated groups of substitutions. He extended the body of knowledge on substitution groups, both within his researches into Galois theory and apart from these studies. In so doing, he became the first mathematician to realise the need for a 'Jordan–Hölder Theorem', or at least for the part of this theorem which he actually proved. He established that, for a finite group G , any two composition series for G have the same length and

the collection of indices $[G_i : G_{i+1}]$ of each group in the series in the preceding one is the same for any composition series. I refer to this result as Jordan's Theorem. Jordan had discovered a proof of it by 1869, but did not publish one until 1870 in his *Traité*. The theorem is stated here in terms of substitution groups (without any mention of equations or Galois theory), and proved using purely group-theoretic arguments.

During the 1870s the importance of Jordan's uniqueness theorem and of the ideas of simple and composite groups were further acknowledged, not least by Eugen Netto, who published a simplified proof of the theorem in 1874. An Italian, Alfredo Capelli, also attempted in 1878 to prove the theorem in a way which would provide a greater understanding into the reasons why the theorem holds. It was another eleven years before the result was stated and proved in a form recognisable to mathematicians today, when Otto Hölder drew together the idea of an abstract group and the concept of quotient group with this theorem in his paper [1889a]. Thus the Jordan–Hölder Theorem does in fact bear the names of the two mathematicians who were responsible for it.

Background: Galois and the theory of algebraic equations

Nowhere in Galois' writings is there evidence of anything similar to a Jordan–Hölder Theorem. The reason for this lack seems to be that he did not need such a theorem. It is likely that, had Galois realised there was a problem of 'uniqueness', he would have attempted to solve it, or at least would have mentioned the problem. Galois' attention was directed at soluble groups, and the Jordan–Hölder Theorem for soluble groups follows implicitly from his method of solution. Indeed, his method tries to preclude any question of difficulty over uniqueness.

In the *Premier mémoire* [1846a], Galois posed the problem 'Dans quels cas une équation est-elle soluble par de simples radicaux?' ('In which cases is an equation soluble by simple radicals?'). He continued '... pour résoudre une équation, il faut successivement abaisser son groupe jusqu'à ne contenir plus qu'une seule permutation' ('... in order to solve an equation, one must successively reduce its group until it contains only a single permutation'). Recall that the word permutation here refers to an arrangement of the roots of an equation: Galois' group was a group of arrangements. Therefore Galois wished to find the condition that the group of an equation must satisfy in order that it may be reduced by adjoining 'radical quantities' to the ground field. One

may assume that these 'radical quantities' are of prime degrees, that is, are the roots of equations of the form $x^p = a$, where p is prime. Then when the first radical which the method of solution provides is adjoined, one of two situations occurs. Either the group of the equation remains the same, or it is reduced and one is left with a subgroup of the group. If the equation is soluble by radicals, the group is always reduced after a certain finite number of extractions of roots. If the p^{th} roots of unity are already available as known quantities (Galois assumed that they were) then the group is reduced to a *normal* subgroup of index p . Thus, provided enough roots of unity are given, the adjunction of a sequence of radicals which ultimately allow the equation to be solved produces a sequence of subgroups stretching from G to $\{1\}$, each normal and of prime index in its predecessor. Therefore a composition series for G is produced.

Galois realised that when one follows the method of solution, one might reach a stage at which there is more than one possible way to proceed. Thus he stated [1846a, 426/43/95]:

Si, arrivé à ce point, il y avait plusieurs manières de diminuer le groupe de l'équation proposée par une simple extraction de racine, il faudrait, pour ce que nous allons dire, considérer seulement un radical du degré le moins haut possible parmi tous les simples radicaux, qui sont tels que la connaissance de chacun d'eux diminue le groupe de l'équation.

[1]

In this way Galois' method provided for the possibility that the group of a soluble equation may have more than one composition series: he specified the series that is to be chosen. He stipulated that the radical of *lowest* (prime) degree p should be chosen because he wished to assume that the p^{th} roots of unity were already adjoined. These may be obtained by extractions of roots of degrees *less* than p , and therefore when they are included, the group of the equation is not affected in any way. The only difficulty may occur if there is more than one radical of degree p to be included. In such a case, one simply chooses any of these radicals, and then at the next stage chooses another, and so on, until one has exhausted all the possibilities.

It is easy to see that the Jordan-Hölder Theorem holds for soluble groups, since the orders of the composition factors of such a group are just the prime divisors of the order of the group, and the composition factors themselves are all cyclic. In order to find the roots of an equation soluble by radicals, one may use operations of the form $\sqrt[q]{a}$, where q is prime. The same primes always occur, even though different solutions may require

the extraction of roots in a different order.

Galois gave a brief summary of his ideas concerning the solution of equations by radicals in the letter he wrote to his friend Auguste Chevalier [1832] on the night before he died. One can see that Galois' thinking had developed further in the *Lettre* than it had when he wrote the *Premier mémoire*. He was able to express the theory in terms of the properties of the *group* of an equation, rather than in terms of adjoining radicals. First he noted that there is a great difference between adjoining one root of an auxiliary equation and adjoining all its roots. In the latter case the group is reduced to a *normal* subgroup: he called this case *la décomposition propre*. He continued [1832, 409/26/175]:

Lors donc qu'on aura épuisé sur le groupe d'une équation tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrivera à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en même nombre.

Si ces groupes ont chacun un nombre premier de permutations, l'équation sera soluble par radicaux; sinon, non. [2]

This is a more general but far less explicit statement than that in the *Premier mémoire*; in modern terms, it corresponds to the formation of a composition series for the group of the equation. The composition factors are the 'groups which we can transform, but which always have the same number of permutations'—in other words, they are simple groups. It is then easy to discover whether the equation is soluble by radicals: one looks at the orders of the composition factors. Galois wrote exactly this, and it did not matter (as far as he was concerned) whether the orders of these factors are the same for any composition series when the group is *not* soluble.

The other mathematician involved in the introduction of group theory, A.-L. Cauchy, did not even recognise the concept of normal subgroup in the research he published during 1845 and 1846. The context of his work provided no motivation to consider the property of normality. Although he investigated conjugate substitutions and proved that two substitutions in the symmetric group of degree n have the same cycle structure if and only if they are conjugate, he did not consider the conjugation of groups of substitutions. Thus he did not even take the first step towards the formulation of a Jordan-Hölder Theorem.

This research of Cauchy was largely ignored during his lifetime, even by Serret when writing his *Cours* [1849]. Cauchy's death in 1857 brought his work on group theory into circulation, and Serret did include some of Cauchy's ideas in the third edition of

his book [1866]. In particular Serret made use of Cauchy's term *système de substitutions conjuguées* instead of Galois' *groupe*, and included some of Cauchy's results on substitution group theory, with the emphasis on their application to the problem of the 'number of values of a function'. Jordan was also exposed to Cauchy's ideas in a course of lectures given by Bertrand after Cauchy's death.

Background: Betti's development of Galois theory

Betti's commentary [1852] on the theory of Galois contains the concepts involved in the Jordan-Hölder Theorem (normal subgroups, composition series) in a muddled but nevertheless recognisable form. I give an outline here of the relevant sections of his work, which is discussed thoroughly in Appendix C.

In the first part of the paper Betti defined conjugation of groups of substitutions, noted the importance of normality of a subgroup, constructed a permutation representation which is nowadays recognised to correspond to a quotient group, and investigated normalisers of subgroups in the symmetric group. In so doing he sometimes worked with groups which he assumed (not always correctly in his context) to have the form $G = A.B$ where $A \trianglelefteq G$, $B \leq G$ and $A \cap B = \{1\}$. These groups G would now be described as split extensions (in this case of A by B). Betti referred to this construction as the *prodotto* (*product*) of two groups; this particular product is isomorphic to a semidirect product. He extended the idea of product to any finite number of groups, and conversely investigated how a group may be built up as the product of smaller groups. He unfortunately became confused and so developed his theory initially only for extensions that split.

Betti constructed a series for groups of this type, a series in which each term is normal in the whole group. Such a series is not in general a composition series. However, in the following section of his paper, Betti investigated the way in which a group is formed from simple groups. Here he dealt with the theory in full generality, not simply in the case of split extensions. He formed a series for groups of arrangements which corresponds to a composition series for the associated group of substitutions [1852, 64-65/43-44].

Betti gave an indication that he believed a group has only one composition series. He wrote down a group of arrangements as the product of its 'divisors' [1852, 65/44]; this corresponds to the construction of the associated group of substitutions as a product of quotient groups. Now these 'divisors' are all simple groups, since they are obtained from

a composition series. The reason that they are simple, according to Betti, is that in forming the series we have considered *all* the possible normal subgroups. Betti's statement seems to imply that at each stage there is only *one* maximal normal subgroup to choose, and consequently only one composition series for the group. If this is what Betti had in mind, he would not have recognised the need for a theorem about uniqueness properties of composition series. Alternatively, Betti was simply repeating Galois' statement [1832, 409/26/175] about performing all possible proper decompositions on a group. Betti may or may not have understood this in the same way that Galois did.

The second part of Betti's paper was based on Galois' *Premier mémoire*, which was greatly elaborated and expanded upon. Betti had a much more ambitious aim than Galois. He explained it to Professor Tortolini in a letter [1851b]:

Ora sto scrivendo una Memoria: *Sulla risoluzione delle equazioni algebriche*. I problemi che ne fanno soggetto sono i seguenti:

Determinare in generale le condizioni necessarie e sufficienti affinché una equazione irriducibile qualunque sia risolubile

- (1) per radicali;
- (2) per radici di equazioni di grado inferiore;
- (3) quando una equazione non sia risolubile nè per radicali, nè per radici di equazioni di grado inferiore, determinare le equazioni che definiscono i più semplici irrazionali, per i quali possano esprimersi le radici della proposta. [3]

This sort of classification of equations at once raises the question of uniqueness which is the subject of the Jordan–Hölder Theorem. The theorem ensures that any equation can be put into a well-defined class since its solution always consists of the same basic steps, although perhaps performed in a different order. Did Betti, however, understand that there was a problem? It seems that he did not, since, like Galois, he did not mention it. Unlike Galois, he needed such a uniqueness theorem to justify his classification of equations.

Betti explained [1852, 98–99/68–69] how, in theory, one solves (not necessarily by radicals) an equation whose group is not simple; in effect this is achieved by using roots of equations whose groups are the composition factors of the given group. He called these roots *irrazionali primitivi* (*primitive irrationals*). They fall into different classes, the simplest of which contains radicals of the form $\sqrt[p]{n}$, where p is prime and n is a positive integer. Therefore if all the 'primitive irrationals' are of this form, the equation is soluble by radicals. As noted above, the Jordan–Hölder Theorem is not needed in this case. Betti also seemed to believe, however, that in the general case the classes of the

irrationals must be uniquely determined. Indeed they are, but he gave no justification for this assumption and perhaps believed that none was required, because of his earlier confusion over the uniqueness of composition series.

Three years later Betti produced a paper [1855] that is concerned mainly with two problems relating to cyclic groups of nonprime order, which should not be a difficult subject. However, he confused the cyclic group of order p^ν (where p is prime) with the elementary abelian group of order p^ν —his theory is to a great extent incorrect. He was aiming to look for the way in which these cyclic groups are built up as a ‘product’ of normal subgroups. He realised that if the group is of prime power order p^ν then it may be regarded as the ‘product’ of ν (simple) groups of order p , but his definition of the simple groups is not correct. In modern terms there is just one composition series for such a group. However, the collections of arrangements which correspond to each group in the composition series are not unique, and so he proceeded to find them all [1855, 12–19/105–113]. If the group is not of prime power order then there is more than one composition series. In this case Betti, once more confusing the cyclic and elementary abelian groups of prime power order, only showed that such a group (of order $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\nu^{\alpha_\nu}$) is the ‘product’ of groups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_\nu^{\alpha_\nu}$. Therefore he reduced the problem to the previous situation. He continued [1855, 22/114]:

Poichè non abbiamo fatta nessuna ipotesi sulla grandezza relativa dei fattori $p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_\nu^{\alpha_\nu}$, l'ordine dei gruppi primi nel prodotto ... sarà indifferente. [4]

Thus Betti knew that the simple groups may appear in any order, that is, there are many composition series in this case. But, as before, these simple groups are already completely determined, so there is no need for a Jordan–Hölder Theorem.

This paper deals very inaccurately with a relatively trivial subject and it is not surprising that it seems to have been almost completely ignored by Betti's contemporaries.

Jordan: Developments in the 1860s

Camille Jordan completed his doctoral thesis in 1860, in time to submit it for the Paris Academy's *Grand Prix de Mathématiques*. The subject of the competition was the ‘number of values of a function’, which Jordan (and the other competitors, Mathieu and Kirkman) interpreted as the problem of finding the subgroups of the symmetric groups. Before his thesis was submitted for the prize he added a *Supplément* of eight pages,

which is concerned entirely with Galois theory. Jordan apparently began to acquire a grasp of Galois' theory of algebraic equations shortly before he published his thesis. The way in which he understood the application of group theory to the study of equations and of irrationals may be deduced from the introduction to the thesis [1861, 113/1]:

Galois, dans un admirable Mémoire, en a fait dépendre, non-seulement les conditions de la résolution algébrique, mais la théorie entière des équations, considérée sous son point de vue le plus général, et la classification des irrationnelles algébriques. [5]

Jordan discussed Galois' fundamental theorem in the *Supplément* and mentioned that because of this theorem, the study of equations and of substitution groups is basically one and the same problem. Jordan was aiming to prove a theorem of Galois' but he, as Galois, made a mistake in his proof. The theorem concerns equations soluble by radicals (and so the Jordan–Hölder Theorem holds automatically). Jordan did, however, mention composition series here, in a very rudimentary form. He introduced [1861, 191/79] a 'ladder' of substitutions satisfying certain relations. He then considered a series of groups which begins with $\{1\}$, and successive terms of which are obtained by taking the next substitution in the ladder and forming the group generated by this substitution and all those which precede it. He stated Galois' criterion for solubility of an algebraic equation in terms of this new concept. An algebraic equation is soluble by radicals if and only if the group of the equation is generated by an ordered sequence of substitutions satisfying:

- (1) for each substitution in the sequence, the least positive power of this substitution which lies in the group generated by all the preceding substitutions has prime exponent;
- (2) each substitution of the sequence normalises the group generated by all those that precede it.

In modern terms this may be written: an algebraic equation is soluble by radicals if and only if its group G is equal to $\langle g_1, g_2, \dots, g_k \rangle$ and, if $G_i := \langle g_1, g_2, \dots, g_i \rangle$, then for all i

- (1) $g_{i+1}^{p_{i+1}} \in G_i$ for some prime p_{i+1} ;
- (2) g_{i+1} normalises G_i .

This clearly corresponds to a composition series for G with composition factors of prime order.

Among the many papers that Jordan published during the 1860s in which he devel-

oped the Galois theory of algebraic equations and the theory of substitution groups, two in particular were specifically written as commentaries on Galois' work. Jordan had a clear understanding of this work and had been able to overcome the considerable difficulties which the reading of Galois' papers presents. There were others who studied or wrote or lectured on Galois theory before and during the time Jordan was concerned with it, amongst whom were Dedekind, Kronecker, Hermite, Bertrand, Liouville and of course Betti.

In his *Traité*, Jordan claimed inspiration to study algebra from Serret, and in particular from Serret's *Cours*. Serret did not include a section on Galois theory until the 1866 edition of his book (and here there is little more than an elaboration of Galois' original ideas), by which time Jordan's work on Galois theory was well-developed. Serret had adopted the notation and terminology and way of thinking that Cauchy had developed in 1815 for the 'calculus of substitutions' and at first Jordan, motivated by Serret, followed this approach. However, Jordan soon abandoned these methods and developed his own, which are much closer to those of Galois. It is possible that he learnt something of these methods from Hermite and Bertrand: Lebesgue wrote, in the obituary of Jordan which he produced [1926], that Jordan was Bertrand's pupil before becoming his colleague. As noted in the previous chapter, although Jordan appears to have read at least something of Betti's work, there is no evidence that he was influenced by it.

The Jordan-Hölder Theorem is a theorem about composition series. It is possible to define many different types of series for a (finite) group, each series consisting of subgroups satisfying certain properties. These series have been developed with specific aims and needs in mind, whether of classifying groups or of giving an insight into their structure. The idea of a composition series for a group G , that is, a series of subgroups

$$G = G_0 \supset G_1 \supset \dots \supset G_n = \{1\}$$

such that each is maximal normal in the preceding one, is a particularly useful concept since every finite group has such a series. A composition series provides a natural way to express Galois' criterion for the solubility of an algebraic equation by radicals. Thus, with the understanding and development of Galois Theory came the emergence of the idea of a composition series. This idea was not made explicit until Galois' theory was formulated in terms of groups and not of equations. The concept of normal subgroup is

an essential prerequisite for the definition of such a series, but the concept of quotient group is not; composition series were investigated without the use of quotient groups for almost thirty years.

It was Jordan who gave a precise and explicit definition of such a series and pointed out the way in which it can be used, together with other types of series for finite groups, as he brought clarity to the work of Galois. It seems that Jordan gradually began to realise that the usefulness of these series depends on their possession of certain uniqueness properties. In this way he was able to formulate and prove the theorem that, for a given finite group, the lengths of all composition series are the same, and the set of ratios of the orders of consecutive groups is the same in any such series. Without the concept of quotient group, it was not possible for him to do more. Indeed, Jordan's Theorem is all that is needed to make rigorous the Galois theory of algebraic equations.

Jordan seems to have started thinking about these questions of uniqueness as his attention moved away from soluble equations and groups to a more general situation. Before this shift of attention, in a short paper [1864], 'Mémoire sur les groupes des équations résolubles par radicaux', Jordan revealed that his ideas on soluble equations had developed considerably. He was by this time no longer employing Cauchy's methods, the generating substitutions of 1861 have disappeared, and his 'fundamental theorem' has become [1864, 963/83]:

Pour qu'un groupe L appartienne à une équation résoluble par radicaux, il faut et il suffit qu'il soit le dernier terme d'une série de groupes partiels $I, F, G, H, \dots$ jouissant des propriétés suivantes:

- (1) chacun de ces groupes contient le précédent;
- (2) ses substitutions sont échangeables entre elles, aux substitutions près du précédent;
- (3) toutes les substitutions de L lui sont permutable. [6]

Condition (2) states, in modern terms, that successive quotient groups are abelian; condition (3) that each group is normal in L . Therefore Jordan proved that a group is soluble if and only if it has a normal series with abelian factors. This is not a composition series except in special cases: Jordan's attention was on *soluble* groups and he chose his series accordingly.

A year later Jordan published his first commentary on Galois' work, which consists of thirteen theorems stated without proof [1865]. The commentary sets out the main ideas of Galois' theory very concisely and coherently. The role of groups is emphasised and

the importance of normal subgroups is highlighted. In the last theorem of the paper Jordan gave conditions under which an equation is soluble by radicals, in terms of a series of substitutions and the groups generated by these substitutions. This is similar to his 1861 approach, but the series formed is a subnormal series with cyclic factors, which are not necessarily of prime orders, so it is not necessarily a composition series. He also made the distinction in this paper between *équations simples* and *équations composées*, which as one would expect are those whose associated groups are simple and composite, respectively.

In 1866 Jordan published a 'Note sur les irrationnelles algébriques'. He pointed out that if an equation (any equation, not necessarily soluble by radicals) is not simple, its solution can be reduced to that of a series of simple equations. However, he had by this time realised what Betti did not seem to see at all clearly, namely that 'il existera souvent un grand nombre de manières entièrement différentes d'opérer cette réduction' ('there will often exist a large number of completely different ways to perform this reduction'). On comparing these different methods, he obtained the result that the number μ of equations in this series is constant, no matter how the series is chosen. In terms of groups, he proved that the length of any composition series of a finite group is the same. (No proof was provided in this paper.) He noted the usefulness of this theorem for the classification of algebraic irrationals: one can define μ to be the *degré d'irrationalité* of the roots of the equation. In an announcement which appeared in the *Comptes rendus* [1867a], 'Mémoire sur la résolution algébrique des équations', he stated the condition under which a group is soluble by radicals in exactly the same way as in [1864]. The article to which the announcement refers was published in Liouville's Journal [1867b]. It covers in greater detail material on the series of subgroups forming the composition series of a soluble group.

By 1869, Jordan's understanding was much more complete. He announced some new theorems in a note [1869a]. It is here that Jordan's Theorem was first stated, in terms of equations and not of groups [1869a, 257/197]:

La résolution d'une équation composée se ramène (souvent de plusieurs manières) à celle d'une suite d'équations simples. L'ordre de la proposée est égal au produit des ordres de ces équations auxiliaires; on peut appeler ces derniers nombres *les facteurs de composition* de la proposée. *Ces facteurs sont toujours les mêmes*, de quelque manière que la réduction de l'équation soit opérée; mais ils pourront se présenter dans un ordre de succession différent.

[7]

Jordan's term *facteurs de composition* refers to the orders of the groups that are nowadays known as composition factors. The theorem that the composition factors of a subgroup divide those of the group also appeared here, without proof. The paper to which this note refers appeared in Liouville's Journal [1869b]; a footnote states that it is an extract from the *Traité*, then in the process of publication.

Later in 1869 came a long and detailed commentary on Galois theory [1869c], published in the first volume of the *Mathematische Annalen*. Here Jordan presented his own development of the ideas in Galois' memoir. The emphasis is on the role of groups. Jordan examined the link between maximal normal subgroups and simple groups: the quotient of a group by a maximal normal subgroup is a simple group. He understood this simple group not as a quotient but as the group associated with a certain 'auxiliary equation'. He then proceeded to define a composition series (for any group which is associated with an equation). In Theorem IX he proved that the solubility of a given equation, or correspondingly of its associated group, depends on that of the simple groups now called the composition factors. He went on [1869c, 152/222]:

Ce théorème suggère naturellement l'idée de classer les équations à groupe composé d'après le nombre et la valeur des *facteurs de composition* $\nu, \nu_1, \dots$: mais la suite des groupes $G, I, I_1, \dots$ peut souvent être déterminée de plusieurs manières: il est donc absolument nécessaire, pour justifier cette classification, de prouver que, *de quelque manière que l'on choisisse cette suite, on trouvera toujours, à l'ordre près, les mêmes facteurs*. Mais nous supprimons, pour abréger, la démonstration de ce théorème. [8]

The suppression of the proof did indeed save space: in the *Traité* it takes up six full pages.

Jordan's proof

All this material, which had evolved since his doctoral thesis, was reorganised by Jordan for inclusion in the *Traité* [1870]. In addition, the treatise contains previously unpublished research and many other results and ideas beginning to be known at the time. The group theory is dealt with as a separate topic in its own right, and then applied to the theory of algebraic equations.

Jordan's Theorem is stated and proved in book II of the *Traité* on pages 41–48. The theorem is treated as a result in pure group theory and its applications dealt with later. The proof is long (in comparison with a modern one) and very intricate, and is therefore difficult to reproduce in any sort of concise form. The reason for this is largely that it

relies on Jordan's ability to manipulate substitutions. He did this with extreme care and attention to detail.

As mentioned before, Jordan used the term 'composition factors' for numbers that are the *orders* of the groups nowadays called composition factors. This practice did not die out with the discovery that the factors may be considered as *groups*. Dickson was still using this terminology when he wrote *Modern Algebraic Theories* [1926]. It must be remembered that Jordan was not aware of the concept of quotient group at this time.

The following explanation and proof does not claim to be a reproduction of Jordan's. Instead, it identifies the basic steps of Jordan's proof and in doing so attempts to give a 'flavour' of the original. The conventions and notation used by Jordan are followed in most places here, with the addition of many more recently introduced notational symbols.

For any composite group G , one can determine a series of groups

$$G, H, H', H'', \dots, I, I', \dots, K, \dots, 1 \quad (1)$$

which satisfy:

- (1) each group is maximal normal in the preceding one;
- (2) the last group of the series consists of only the identity.

Jordan never gave a name to this type of series but, following modern terminology, I refer to it as a composition series. Let the composition factors of this series be $\lambda, \mu, \mu', \dots, \nu, \nu', \dots$, respectively.

Note: The unprimed letters $H, I, K, \dots$ refer to those groups in the series which are normal in the *whole* of G .

Theorem

If

$$G, \mathfrak{H}, \mathfrak{H}', \dots, \mathfrak{I}, \dots, 1 \quad (2)$$

is another composition series whose composition factors are, respectively, $l, m, m', \dots$, then these factors are identical to $\lambda, \mu, \mu', \dots$ (up to order).

Proof

The proof is by 'induction': one shows that the theorem is true for G if it is true for

$\mathfrak{H}$. Likewise, it holds for $\mathfrak{H}$ if it holds for $\mathfrak{H}'$, and so on. The penultimate group in the series is simple, and the theorem is clearly true for simple groups.

The proof consists of the construction of a second composition series for $\mathfrak{H}$ whose composition factors comprise all but one of the original factors $\lambda, \mu, \dots$. Then, after showing that the missing factor is the index of $\mathfrak{H}$ in G , one obtains two composition series for G , both starting $G, \mathfrak{H}, \dots$, whose composition factors are those of (1) and (2). By the inductive hypothesis, the uniqueness follows.

1. Assume, without loss of generality, that K is the first group of the series $G, H, I, K, \dots$ which is contained in $\mathfrak{H}$ (the last group 1 is certainly contained in $\mathfrak{H}$), and that the series (1) contains no terms between G and I other than H, H' . (If there are intermediate terms $H'', H''', \dots$, the method can clearly be extended to include them.) Jordan seems to have assumed that $H \neq \mathfrak{H}$, as is certainly permissible, since otherwise the theorem follows immediately from the inductive hypothesis.

2. *Claim:* $K = I \cap \mathfrak{H}$.

Let $K' = I \cap \mathfrak{H}$. Then K' is normal in G , since I and $\mathfrak{H}$ are normal in G . Also, $K \subseteq K'$. Since K is by hypothesis (Jordan says) *maximal* amongst groups normal in G and contained in I , one concludes that $K' = K$.

Jordan's argument is not quite right because the hypothesis does not in fact imply that K is maximal amongst groups normal in G and contained in I . It implies only that K is the group with this property in the particular composition series that has been chosen. It is quite possible that there exists another group X such that $I > X > K$ and $X \trianglelefteq G$. The direct product of two symmetric groups of degree 3 provides a simple example of this situation [9]. The slip is, however, not hard to correct for one may assume that the group K possesses the required property. By the inductive hypothesis, since I has smaller order than G , the theorem holds for I and all its composition series are equivalent. Then it may be assumed that the part of the series below I is a series for I which is obtained by refining the series $I > K' > \{1\}$. This ensures that K' appears in the series (and so $K = K'$).

Claim: $G = I\mathfrak{H} = \mathfrak{H}I$.

Let $\Gamma = I\mathfrak{H}$. Then Γ is normal in G , and $\mathfrak{H} \subset \Gamma$. But since $\mathfrak{H}$ is maximal normal in G , $\Gamma = G$.

3. Two elements of $\mathfrak{H}$ in the same coset of I are also in the same coset of K .

Take two elements $\mathfrak{h}_1$ and $\mathfrak{h}_2$ in the same I -coset. Then $\mathfrak{h}_1 I = \mathfrak{h}_2 I$, so $\mathfrak{h}_2^{-1} \mathfrak{h}_1 \in I$. Since $\mathfrak{h}_2^{-1} \mathfrak{h}_1 \in \mathfrak{H}$, one has $\mathfrak{h}_2^{-1} \mathfrak{h}_1 \in I \cap \mathfrak{H} = K$. Therefore $\mathfrak{h}_1 K = \mathfrak{h}_2 K$.

$$4. |\mathfrak{H}| = \frac{|G||K|}{|I|} \quad \text{since } G = \mathfrak{H}I, \text{ so } |G| = \frac{|\mathfrak{H}||I|}{|\mathfrak{H} \cap I|} = \frac{|\mathfrak{H}||I|}{|K|}.$$

5. It is now known that $G = I\mathfrak{H}$. Let the elements of K be denoted by k_α .

Let coset representatives for K -cosets in I be denoted by i_β ;

coset representatives for I -cosets in H' be denoted by $h'_{\gamma'}$;

coset representatives for H' -cosets in H be denoted by h_γ ;

coset representatives for H -cosets in G be denoted by g_δ .

Jordan wrote [1870, 42: last line] that the i_β are substitutions in $\mathfrak{H}$, but clearly this cannot be the case, and $\mathfrak{H}$ seems to be a misprint for I . Then any element of G can be written as $g_\delta h_\gamma h'_{\gamma'} i_\beta k_\alpha$. By Jordan's construction this representation of an element of G is unique.

Now denote by $(h'_{\gamma'})$ the μ' substitutions of the form $h'_{\gamma'} i_{\beta_{\gamma'}}$ that lie in $\mathfrak{H}$;

by (h_γ) the μ substitutions of the form $h_\gamma i_{\beta_{\gamma}}$ that lie in $\mathfrak{H}$;

by (g_δ) the λ substitutions of the form $g_\delta i_{\beta_\delta}$ that lie in $\mathfrak{H}$.

Then

$$\mathfrak{h} \in \mathfrak{H} \iff \text{there exist } \delta, \gamma, \gamma', \alpha \text{ such that } \mathfrak{h} = (g_\delta)(h_\gamma)(h'_{\gamma'})k_\alpha;$$

$$g \in G \iff \text{there exist } \beta, \delta, \gamma, \gamma', \alpha \text{ such that } g = i_\beta(g_\delta)(h_\gamma)(h'_{\gamma'})k_\alpha;$$

$$h \in H \iff \text{there exist } \beta, \gamma, \gamma', \alpha \text{ such that } h = i_\beta(h_\gamma)(h'_{\gamma'})k_\alpha;$$

$$h' \in H' \iff \text{there exist } \beta, \gamma', \alpha \text{ such that } h' = i_\beta(h'_{\gamma'})k_\alpha.$$

Also, for any $i \in I$ and $\mathfrak{h} \in \mathfrak{H}$ one has $i\mathfrak{h} = \mathfrak{h}ik_\alpha$ for some $k_\alpha \in K$.

6. Jordan formed his new composition series

$$G, \mathfrak{H}, J, J', K, \dots, 1$$

as follows:

$$g \in G \iff g = i_\beta(g_\delta)(h_\gamma)(h'_{\gamma'})k_\alpha;$$

$$\mathfrak{h} \in \mathfrak{H} \iff \mathfrak{h} = (g_\delta)(h_\gamma)(h'_{\gamma'})k_\alpha;$$

$$j \in J \iff j = (h_\gamma)(h'_{\gamma'})k_\alpha;$$

$$j' \in J' \iff j' = (h'_{\gamma'})k_\alpha.$$

Perhaps it is not immediately obvious that J and J' are actually groups. Jordan did not mention this, but in fact it is not hard to see that $J = \mathfrak{H} \cap H$ and $J' = \mathfrak{H} \cap H'$. The groups $G, \mathfrak{H}, J, J'$ have orders $|G|, \frac{|G||K|}{|I|}, \frac{|H||K|}{|I|}, \frac{|H'||K|}{|I|}$.

The proof of the theorem concludes by showing that:

- (i) in the series $G, H, H', I, \dots, K, \dots, 1$, there is no intermediate group between I and K ;
- (ii) the series $G, \mathfrak{H}, J, J', K, \dots, 1$ is a composition series for G .

7. Suppose there is an intermediate group I' between I and K , with $I' \neq K$. Then, by definition, $I' \trianglelefteq I$. Now take $\mathfrak{h} \in \mathfrak{H}$ and $i' \in I'$. Then

$$\begin{aligned} \mathfrak{h}^{-1}i'\mathfrak{h} &= i'i'^{-1}\mathfrak{h}^{-1}i'\mathfrak{h} \\ &= i'[i', \mathfrak{h}] \\ &= i'k_\alpha \in I' \end{aligned}$$

for some $k_\alpha \in K$. So $\mathfrak{h}^{-1}I'\mathfrak{h} = I'$. It is known that $I' \trianglelefteq I$ and $\mathfrak{H}$ is in the normaliser of I' , so $I\mathfrak{H} = G$ is in the normaliser of I' , that is, $I' \trianglelefteq G$. But this cannot happen unless $I' = I$.

8. Consider any group of the new series, J' for example. First Jordan proved that $J' \trianglelefteq J$. This is essentially his method:

Clearly J' is in the normaliser of J' . And J is formed by combining the substitutions of J' with the (h_γ) . Now the (h_γ) are in the normaliser of H' . Jordan then showed that the (h_γ) are in the normaliser of J' , so that J is in the normaliser of J' , as required.

Secondly he proved that J' is *maximal* normal in J .

Assume that it is not: there exists J'' such that $J' < J'' \triangleleft J$. Let Γ be the subgroup formed by combining the substitutions of J'' with the i_β , so $J'' = \mathfrak{H} \cap \Gamma$. Then $H' < \Gamma$ and $\Gamma \triangleleft H$. That is, if J'' exists, then there exists Γ such that $H' \triangleleft \Gamma \triangleleft H$, which is by definition impossible.

9. The proof is now completed, since two composition series for G have been obtained,

$$\begin{aligned} &G, \mathfrak{H}, J, J', \dots, 1 \\ &\text{and } G, \mathfrak{H}, \mathfrak{H}', \mathfrak{J}, \dots, 1. \end{aligned}$$

The composition factors for the series

$$\mathfrak{H}, J, J', \dots, 1$$

$$\text{and } \mathfrak{H}, \mathfrak{H}', \mathfrak{J}, \dots, 1$$

are $\lambda, \mu, \dots$ and $m, m', \dots$, respectively. By the inductive hypothesis, these are identical up to order. In addition it is known that $|G|/|\mathfrak{H}| = |I|/|K|$, so $l = \nu$. The theorem follows.

The proof of the theorem given by Jordan in the *Traité* is difficult to read because he used little terminology and much notation. He did not have a name for composition series and consequently he had to repeat the whole definition several times. Similarly he gave a definition rather than a name for the property of normality: a normal subgroup is one that is *permutable* (commutes) with the substitutions of the group. Jordan also required the use of large numbers of subscripts to distinguish different types of substitutions.

A few pages after the proof of the theorem, the concepts of homomorphism and isomorphism were introduced, and Jordan did provide names for these. He stated that it is easy to see that two isomorphic groups are either both simple or both composite, and have the same composition factors.

From Jordan to Hölder

Jordan continued his investigations into the theory of substitution groups and into Galois theory after 1870. In the meantime, the *Traité* was reviewed by a German mathematician, Eugen Netto [1871]. The review was not particularly complimentary; Netto commented that there were in the *Traité* 'manche Lücken auszufüllen, manche Weitschweifigkeiten zu entfernen, manche Beweise zu berichtigen' ('some gaps to fill, some long passages to clarify, some proofs to correct'). Then in 1874 Netto published a paper giving new proofs of some of the theorems in the *Traité* concerning composite groups. Referring to Jordan's Theorem, he asserted that Jordan's proof was not sufficient. Netto claimed that Jordan had introduced 'systems of substitutions' $h_\gamma h'_{\gamma'} k_\alpha$ and had treated them as groups without having proved that they were indeed groups. In fact, as noted before, it is easy to see that the substitutions $h_\gamma h'_{\gamma'} k_\alpha$ (those belonging to J) form a group, since $J = \mathfrak{H} \cap H$. Jordan pointed this out in a note at the end of a paper published the following year [1875]. He continued:

Nous ne saurions d'ailleurs avoir la prétention de n'avoir laissé se glisser aucune inexactitude dans un ouvrage aussi étendu que le nôtre, et qui traite un sujet nouveau et difficile; mais nous sommes persuadé qu'elles y sont en petit nombre. Nous en avons reconnu deux, que nous nous sommes empressé de rectifier; et nous saurions beaucoup de gré aux géomètres qui pourraient nous en indiquer de nouvelles. [10]

Jordan also remarked here that Netto's proofs are only 'a little different' from his, and still rest on the same principles. In fact Netto's proof of Jordan's Theorem reads much more like a modern proof, and although it still involves some manipulation of substitutions, it is far easier to read than Jordan's. Netto included this theorem and the same proof in his book on the theory of substitutions [1882]. By this time his attitude towards Jordan had become more friendly. (Perhaps this could in part be due to the fact that France and Germany were not at war at this time, as they had been in 1870!) Netto acknowledged the importance of the *Traité* and of Jordan's other work on the theory of substitutions.

I give a brief summary of Netto's proof; the notation is that used in the paper of 1874. This simplified proof rests on the following lemma [1874, 84]. Netto used the definite article for composition series all through his work, but by this he meant *any* composition series.

Folgen in der Reihe von G auf eine Gruppe H zwei Gruppen J und J' mit den Factoren der Zusammensetzung q und q' , so kann man auf jede dieser beiden ein und dieselbe bestimmte Gruppe M folgen lassen mit den Factoren q' bezüglich q ; diese Gruppe ist mit den Substitutionen von H permutabel. [11]

That is, if J and J' are maximal normal in H , then, writing $M = J \cap J'$, one has that

- (1) M is maximal normal in J and J' ;
- (2) $q = \frac{|H|}{|J|} = \frac{|J'|}{|M|}$ and $q' = \frac{|H|}{|J'|} = \frac{|J|}{|M|}$;
- (3) $h^{-1}Mh = M$ for all $h \in H$.

The proof of Jordan's Theorem follows without difficulty, and is by a sort of induction on the length of the composition series. One is given two composition series for a group G ,

$$G, H, J, K, \dots, P, Q, 1, \tag{1}$$

$$G, H', J', K', L', \dots \tag{1'}$$

It should be noted that Netto did not keep to Jordan's notation of unprimed letters for groups normal in the whole of G . If $H = H'$ but $J \neq J'$, one can form two composition

series

$$G, H, J, K'', L'', \dots, \quad (10)$$

$$G, H', J', K'', L'', \dots, \quad (10')$$

where $\frac{|H|}{|J|} = \frac{|J'|}{|K''|}$ and $\frac{|H|}{|J'|} = \frac{|J|}{|K''|}$.

Then (1) and (10) are identical in three terms and three composition factors. Similarly (1') and (10') are identical in three terms and three composition factors. And (10) and (10') have identical composition factors up to order. Therefore if the theorem holds for (1) and (10), and for (1') and (10'), it holds for (1) and (1').

Now assume $H \neq H'$. Then one can form two new series which each have four composition factors identical with those of (1) and (1'). Since (1) has only a finite number of terms, carrying on in this way one finally obtains series with all composition factors identical. Thus the theorem is proved.

By 1882 Netto's notation had changed and he had begun to use subscripts to distinguish between the groups of a composition series, as is usual today. Netto also provided the name *Reihe der Zusammensetzung* (*composition series*) here; in the earlier paper he had simply called it a *zu G gehörige Reihe* (*series belonging to G*). Later, in the English edition of his book [1892], Netto included the notion of quotient group, as has been noted in Chapter 5, but he did not change the statement or proof of Jordan's Theorem.

There was another published proof of Jordan's theorem during the 1870s. It subsequently went unnoticed by Hölder. This is not altogether surprising: the proof is at the end of a detailed 56-page paper which includes various concepts and results of substitution group theory that are linked by the common theme of homomorphism¹. The paper was written by Alfredo Capelli [1878] and was mentioned in Chapter 5. It has previously received some attention by virtue of the fact that Capelli independently proved some of Sylow's Theorems (see [Waterhouse 1980], [Casadio & Zappa 1990]).

Capelli managed to establish Jordan's Theorem on composition factors 'con semplici considerazioni di Meriedria' ('by simple considerations of the meriedric', that is, the homomorphic). He had previously developed some ideas concerning homomorphisms

¹The term generally used in the nineteenth century was *isomorphism*: it covered one-one and many-one mappings.

which he realised would apply to this theorem. He was one of the first after Jordan to recognise the importance of homomorphism: '... mi parve di ravvisare in essa la chiave naturale per lo studio delle sostituzioni ...' ('... in my opinion we can recognise in it the natural key to the study of substitutions ...') [1878, 32].

He also appreciated the significance of Jordan's Theorem and had attempted to understand the way in which Jordan's proof worked. However [1878, 82],

La dimostrazione dell'importantissimo teorema dei fattori di Composizione quale è data nell'eccellente trattato di Jordan (n° 54-58) mi parve ancora così malagevole e poco atta a dare un'idea adeguata della cosa, che, non appena ebbi rilevato la stretta relazione che hanno fra loro le proprietà della Composizione dei gruppi e quelle dell'isomorfismo meriedrico, venni naturalmente nell'idea di cercarne io stesso una nuova fondandomi sulla nozione dell'isomorfismo. [12]

Capelli aimed to remedy this situation. His reasoning follows, in essence, the same line as Jordan's, with the introduction of a result that corresponds to Netto's lemma.

Take two maximal normal subgroups P^0 and Q^0 of a finite group G , subgroups which appear as the second terms of two composition series for G , and form their intersection I_{00} . The group G may be represented in a table using I_{00} and the cosets of this subgroup relative to P^0 and Q^0 . Capelli had earlier worked out an isomorphism which holds in such a situation; it is now expressed as

$$\frac{P^0}{P^0 \cap Q^0} \cong \frac{P^0 Q^0}{Q^0}.$$

He had shown that normal subgroups of P^0 containing $P^0 \cap Q^0$ correspond to normal subgroups of $P^0 Q^0 (= G)$ containing Q^0 . He then formed a third composition series as Jordan had done and used these isomorphisms to show it has the required properties. Then a sort of induction, similar to Jordan's, completes the proof.

The introduction of homomorphisms certainly makes Capelli's proof easier than Jordan's for a modern reader to unravel, but it is debatable whether Capelli's exposition actually contributed to an improved understanding of the theorem for those without the benefit of hindsight. His study of homomorphism throughout the paper is, not surprisingly, rather more of an exploration of the concept—in parts he seems to have had no fixed aim in mind. He does not appear to have read Netto's simplified version of Jordan's proof. Likewise, there is no mention of Capelli in Hölder's paper of 1889.

Capelli had the good idea of using superscripts to distinguish between members of a composition series in the same way that Netto used subscripts. The many innovations in

Capelli's paper went largely unnoticed at this time when France and Germany were still considered (and considered themselves) to be the countries in which serious mathematics was pursued and published.

It is particularly interesting to see that in 1873, three years after the publication of the *Traité*, Jordan was actually in a position to formulate a Jordan's Theorem with quotient groups. It has been seen that he used the idea of congruence of group elements modulo a normal subgroup to produce a quotient group [1873a, 45–47/370–372]. He went on to consider a composition series for a group G and formed the quotient groups we now call the composition factors (he named them the *groups composants de G*). He proved a result concerning the *groups composants* of an intransitive group. But he did not see the possibility of extending his uniqueness theorem to these *groups composants*.

Hölder: Extending the theorem

The problem that forms the subject of Hölder's paper [1889a] is precisely that which first uncovered the need for a Jordan–Hölder Theorem. We have seen that Klein's influence on Hölder was strong at this time. Particularly relevant here is a passage in Klein's *Icosahedron* [1884, 7/1888, 7–8]. He discussed the decomposition of composite groups, which is achieved by producing a composition series for the group. He remarked: 'It need hardly be said that this decomposition process admits of much variation according to circumstances.' This statement, which Klein considered to be so obvious, may nevertheless have aroused Hölder's curiosity, since he must have been aware by that time of Jordan's Theorem and Netto's simplified proof.

The 'chain of equations' in the title of Hölder's paper are auxiliary equations whose Galois groups, as we have discovered, are certain simple groups—the composition factors of the group of the original equation. In this way Hölder was brought to consider the unique definition of these composition factors (as *groups*). In order to prove their uniqueness, he used Netto's simplified proof of Jordan's Theorem as a framework, and inserted the concepts of quotient group and isomorphism. As he remarked [1889a, 34], 'Man wird im Folgenden unschwer eine Modification des von Herrn Netto benutzten Gedankenganges erkennen' ('It will not be hard to recognise in what follows a modification of the line of thought used by Mr. Netto').

The Jordan–Hölder Theorem appears near the beginning of the paper. Hölder noted

that the results he would develop hold for *finite* groups. He discussed composition series, composition factors and Jordan's Theorem. He explained that Jordan's Theorem had to be extended to include the concept of quotient group, which he then introduced. He proved that, if $H \trianglelefteq G$, then G/H is simple if and only if H is a maximal normal subgroup of G .

The Theorem

Hölder's proof of the invariance of the composition factors (this phrase now refers to the *groups*, not to their orders) reads very much like a modern proof. It consists of preliminary 'lemmas' from which the theorem can easily be deduced. Jordan's part of the theorem is of course a trivial corollary of this result.

In Hölder's terminology, the composition factors of a group G are the *Factorgruppen* of G . He showed in some preliminary working how a group G can be split into two 'factors' by a normal subgroup H : one has

$$|G| = |G/H||H|.$$

One can divide the factors further until they are all simple groups. 'Man gelangt so zugleich zum Begriff eines Products aus mehreren Gruppen'. ('Thus one immediately reaches the idea of a product of many groups.') Hölder noted that these simple factor groups are also obtained from a composition series by taking the quotients of successive terms. He stated [1889a, 33]:

Es muss nur im Folgenden noch gezeigt werden, dass diese Aggregate einfacher Gruppen alle identisch sind, d. h. dass alle Reihen der Zusammensetzung, abgesehen von der Reihenfolge, dieselben Factorgruppen ergeben. [13]

He mentioned the work of Jordan and Netto, and gave credit to Dyck for having dealt with these factor groups already, but without proving their uniqueness. In [1880, 482ff] Dyck did indeed consider the simple groups out of which a composition series is made, but he did not recognise them as quotient groups.

The main steps in Hölder's proof of the theorem are as follows:

1. Given any finite group G , suppose that it has two *different* maximal normal subgroups H and H' , and let $\Gamma = H \cap H'$. Then $\Gamma \triangleleft G$ since H and H' are normal in G , and so $\Gamma \triangleleft H$ and $\Gamma \triangleleft H'$. It must be the case that $G = HH' (= H'H)$, since otherwise the group HH' would be such that $H \triangleleft HH' \triangleleft G$, which is not allowed.

2. One can form the quotient groups G/Γ , H/Γ and H'/Γ . Denote these by G_0 , H_0 and H'_0 respectively.

3. The homomorphisms between the groups and quotient groups show that

$$G_0 = H_0 H'_0 (= H'_0 H_0)$$

and that H_0 and H'_0 are maximal normal in G_0 . Also since $H \cap H' = \Gamma$, it follows that $H_0 \cap H'_0 = \{1\}$, where 1 is the identity in G_0 (which is the group Γ).

4. The elements of H_0 and H'_0 commute, since these groups are both normal in G_0 . (If $S \in H_0, T \in H'_0$ then the commutator $[T, S] := T^{-1}S^{-1}TS$ lies both in H_0 and in H'_0 and hence is 1.) Any element in G_0 can be written uniquely as ST for $S \in H_0, T \in H'_0$. Moreover, all possible combinations of S and T are used in doing this.

5. One can form the quotient group G_0/H_0 , and as a consequence of step 4, $G_0/H_0 \cong H'_0$. At this point Hölder concluded [1889a, 36]:

Die Gruppe G_0 ist das Product der Gruppen G_0/H_0 und H_0 , d. h. der Gruppen H'_0 und H_0 ... Es würde vielleicht nützlich sein, für diesen Fall einen Ausdruck zu besitzen und ein solches Product etwa das *directe* zu nennen. [14]

Hölder mentioned that this direct product is uniquely defined by its two factors, and he noted that the idea of direct product can be extended immediately to more than two factors.

6. Hölder had shown how to form quotient groups by defining an equivalence relation on the group elements, which produces the elements of the quotient group as equivalence classes. We may form G/H in this way and also G_0/H_0 (which is a 'quotient group of quotient groups'): define an equivalence relation $\sim$ on G by

$$C \sim C_1 \iff C^{-1}C_1 \in H \quad \text{for } C, C_1 \in G$$

and an equivalence relation $\approx$ on G/Γ by

$$C\Gamma \approx C_1\Gamma \iff C^{-1}C_1\Gamma \in H/\Gamma.$$

But since $C^{-1}C_1 \in H \iff C^{-1}C_1\Gamma \in H/\Gamma$ the two equivalence relations are themselves equivalent² and so $G/H \cong G_0/H_0$. Thus

$$G/H \cong G_0/H_0 \cong H'_0 = H'/\Gamma.$$

²This is one of the group isomorphism theorems formulated explicitly during the 1920s.

Hölder next gave a summary of these preliminary results: If a group G contains two different maximal normal subgroups H and H' , and if $\Gamma = H \cap H'$, then

$$G/H \cong H'/\Gamma,$$

$$G/H' \cong H/\Gamma,$$

and G/Γ is the direct product of H/Γ and H'/Γ . Since G/H and G/H' are simple, it follows that H'/Γ and H/Γ are simple. So Γ is a maximal normal subgroup of H and H' .

7. The proof of the uniqueness of the composition factors is again by induction on the order of the group. The theorem is clearly true for groups of the smallest orders, and he showed that if it is true for groups of order $\leq n$, then it is true for groups of order $\leq 2n$.

Assume that the group G , whose order is $\leq 2n$, has two composition series

$$G, H, K, \dots, I, \tag{1}$$

$$G, H', K', \dots, I. \tag{2}$$

Hölder used I to denote the identity subgroup. Assume further that H and H' are different, and form $\Gamma = H \cap H'$. One can then form the series

$$G, H, \Gamma, \dots, I, \tag{3}$$

$$G, H', \Gamma, \dots, I, \tag{4}$$

such that the series are identical after Γ , and both are composition series for G .

By the preliminary results, the series (3) and (4) give the same composition factors.

Now

$$H, K, \dots, I,$$

$$H, \Gamma, \dots, I$$

are both composition series for H , and H has order $\leq n$, so by the inductive hypothesis, the above composition series have the same composition factors (up to order). So the series (1) and (3) have the same composition factors.

A similar argument with H' shows that (2) and (4) have the same composition factors.

Therefore the series (1) and (2) have the same composition factors, up to order.

Hölder noted that:

- (i) If H and H' are identical, the series (1) and (2) are immediately seen to have the same composition factors, since the two series for H and H' do, by the inductive hypothesis;
- (ii) One quotient group can appear repeatedly, in which case it must come into both composition series equally often.

The explanation given by Hölder in the second part of the paper of the relationship between a chain of auxiliary equations and a composition series for the group of the original equation rests on the uniqueness properties proved in the Jordan–Hölder Theorem. The extended version of the theorem ensures that a given finite group is formed from a set of simple groups, which are the ‘building blocks’ of the group, and there is only one such set. The Jordan–Hölder Theorem is to group theory what the Fundamental Theorem of Arithmetic is to number theory.

From 1889 to 1900

As with the concept of quotient group, the importance of the Jordan–Hölder Theorem became generally known after Hölder’s paper was published, and attempts were then made to simplify the proof. In the first *Bulletin of the American Mathematical Society*, there appeared a note on the Jordan–Hölder Theorem written by G.L. Brown [1895]. This note explains how steps 1–6 of Hölder’s proof may be greatly simplified by making use of a theorem due to Giudice giving the order of the ‘product’ of two ~~abelian groups~~ groups that commute with each other. A year later Weber included the theorem, proved by the same method that Hölder used, in the second volume of his *Lehrbuch der Algebra* [1896]. Frobenius dealt with the group-theoretic background and methods involved in proving the Jordan–Hölder Theorem in [1895a].

However, when Netto’s book on the theory of substitutions appeared in its English translation in 1892, he included only Jordan’s statement of the theorem, as has been noted, and the proof which he (Netto) had first published in 1874. His unwillingness to state the theorem as Hölder had formulated it is curious, especially as he did include the concept of quotient group and explained what are the *factor groups* (composition factors) of a group G , giving reference to Hölder’s paper.

A chapter of Burnside’s book [1897] is devoted to the composition series of a finite

group. His proof of the Jordan–Hölder Theorem, as he says, is similar to Hölder’s; Burnside also remarked that the result ‘is of great importance in the subsequent theory’. In Burkhardt’s encyclopedia article [1898], there is a short section on composition series in which the Jordan–Hölder Theorem is stated with references to Hölder, Frobenius, Jordan and Netto.

The period up to 1900 may be described as the early history of the Jordan–Hölder Theorem. During this century the theorem has been generalised to many other algebraic structures (for example, groups with operators, quasigroups, lattices) and new methods of proof have been developed. Suffice it to say that within group theory, there are now two standard methods of proof. The first is Hölder’s; it rests on what is sometimes referred to as the ‘diamond’ lemma and on induction. The second is an ‘all-in-one’ method; it is based on Schreier’s ‘refinement theorem’, published in 1928, and a result known as Zassenhaus’ Lemma, published in 1934. Both proofs are therefore over fifty years old.

Notes

1 If, having reached this point, there were several ways to reduce the group of the given equation by the simple extraction of a root, it would be necessary, for the following analysis, to consider only a radical of lowest degree possible among all the simple radicals which have the property that knowledge of any one of them reduces the group of the equation.

2 When therefore we have performed on the group of an equation all the proper decompositions which are possible on this group, we arrive at groups which we can transform, but which always have the same number of permutations.

If these groups each have a prime number of permutations, the equation will be soluble by radicals; if not, not.

3 I am presently writing a memoir: *On the Solution of Algebraic Equations*. The problems which it addresses are the following:

Determine in general the necessary and sufficient conditions for any irreducible equation to be soluble

(1) by radicals;

(2) by roots of equations of lower degree;

- (3) when an equation is soluble neither by radicals, nor by roots of equations of lower degree, determine the equations which define the simplest irrationals, in terms of which the roots of the given equation can be expressed.
- 4 Because we have made no assumptions on the relative size of the factors $p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_v^{\alpha_v}$, the order of the simple groups in the product ... will not matter.
- 5 In an admirable Memoir, Galois has made not only the conditions for the algebraic solution of equations depend on it [group theory], but the entire theory of equations, considered from the most general point of view, and the classification of algebraic irrationals.
- 6 In order that a group L may belong to an equation soluble by radicals, it is necessary and sufficient that it is the final term of a series of subgroups $I, F, G, H, \dots$ having the following properties:
- (1) each of these groups contains the preceding one;
 - (2) the substitutions of each group commute with one another, up to [that is, 'modulo'] substitutions of the preceding one;
 - (3) all the substitutions of L permute with them.
- 7 The solution of a composite equation reduces (often in several ways) to that of a series of simple equations. The order of the given equation is equal to the product of the orders of these auxiliary equations; we can call these last numbers the *composition factors* of the given equation. *These factors are always the same*, in whatever way the reduction of the equation is performed; but they may appear in a different order.
- 8 This theorem naturally suggests the idea of classifying equations with composite group according to the number and the value of their *composition factors* $\nu, \nu_1, \dots$: but the series of groups $G, I, I_1, \dots$ can often be determined in several ways: it is therefore absolutely necessary, in order to justify this classification, to prove that, *in whatever way one may choose this series, one will always find, up to order, the same factors*. We will however leave out the proof of this theorem to save space.

9 Let

$$G := \text{Sym}(3) \times \text{Sym}(3)$$

$$= \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \quad \text{where } a_1^3 = a_2^3 = 1 \text{ and } b_1^2 = b_2^2 = 1.$$

Consider the series

$$G \supset \text{Sym}(3) \times \langle a_2 \rangle \supset \langle a_1 \rangle \times \langle a_2 \rangle \supset \langle (a_1, a_2) \rangle \supset \{1\},$$

where $\langle (a_1, a_2) \rangle$ is the diagonal subgroup of $\langle a_1 \rangle \times \langle a_2 \rangle$. This is a composition series for G which corresponds to

$$G \supset H \supset I \supset I' \supset K.$$

The group $\langle (a_1, a_2) \rangle$ is not normal in G because the conjugate of (a_1, a_2) by $(b_1, 1)$ is $a_1^{-1}a_2$. Take $X = \langle a_1 \rangle \times \{1\}$, which is a normal subgroup of G and has the property that $I > X > K$.

10 In no way would we claim to have let no inaccuracies slip into a work as extensive as ours, and which treats a new and difficult subject; but we are persuaded that they are few in number. We have identified two, which we have been swift to rectify; and we would be very grateful to mathematicians who could point out further mistakes to us.

11 If two groups J and J' with composition factors q and q' follow a group H in the [a composition] series for G , then one can have one and the same determinate group M follow on from each of these two with factors q' and q respectively; this group is permutable [commutes] with the substitutions of H .

12 The proof of the extremely important theorem on the composition factors, as presented in the excellent treatise of Jordan (nos. 54–58), seemed to me still so cumbersome and ill-fitted to give an adequate idea of the thing that, as soon as I realised the close relation between the composition property of groups and that of meriedric isomorphism, I naturally came to the idea of trying to find a new proof myself, based on the notion of isomorphism.

13 All that needs to be shown in what follows is that these collections of simple groups are all identical, that is, that all composition series give the same factor groups up to order.

14 The group G_0 is the product of the groups G_0/H_0 and H_0 , that is, of the groups H'_0 and $H_0 \dots$ It would perhaps be useful to coin an expression in this case, and name such a product the *direct* product.

CHAPTER 7

Starting the Search for Simple Groups

"Now," said Rabbit, "This is a Search, and I've Organised it—"

"Done what to it?" said Pooh.

"Organised it. Which means—well, it's what you do to a Search, when you don't all look in the same place at once."

A.A. Milne: *The House at Pooh Corner*, Chapter 3

Hölder's contribution to the study of simple groups, 'Die einfachen Gruppen im ersten und zweiten Hundert der Ordnungszahlen' ('The simple groups of order less than two hundred'), begins as follows [1892, 55]:

Es wäre von dem grössten Interesse, wenn eine Uebersicht der sämtlichen einfachen Gruppen von einer endlichen Zahl von Operationen gegeben werden könnte. [1]

The Jordan–Hölder Theorem of 1889 had made it clear that an investigation of finite simple groups was highly desirable; they were now recognised as the building blocks for all finite groups. Hölder did not mention the idea of a search for simple groups in 1889 but since it follows so naturally from the theorem he proved there, it is not at all surprising that he chose to take up the problem. The results of his meditations were written by August 1891 and published a year later [1892]. He filled thirty-three pages in proving that the only nonabelian simple groups of order ≤ 200 were the two groups known previously, in fact known to Galois some sixty years before.

So what is the significance of a paper in which no new concepts or groups are discovered? Hölder's proof of the absence of possible further simple groups with order ≤ 200 was actually both a new and a significant result. The importance of his paper rests both on the result he proved and on two aspects of his research that are original:

- (1) His approach—the overall view he had of the problem;
- (2) His methods—the way in which he attacked the problem.

Hölder chose to begin at the opposite end of the problem from that at which research had been directed before. Up to the 1890s, the interest lay in discovering new groups and investigating them, with the result that some were found to be simple. There was no (published) thought directed to the questions: what makes it possible for a group

to be simple, what sort of structure does a simple group have, what kinds of properties do simple groups possess? The idea of classifying simple groups had not received the attention it deserved. Enrico Betti was one of the few to take this problem seriously [1852, 77/52]:

La classificazione dei gruppi primi farà soggetto di un altro mio lavoro, se avrò agio a seguitare con risultato le ricerche intraprese. [2]

Unfortunately he was somewhat optimistic about the scale and complexity of the task, as well as rather confused about this aspect of group theory. Hölder, on the other hand, could see that a systematic approach was needed which would both find new simple groups and prove that in certain cases they could not be found. His approach was also to be abstract, not relying on the contexts in which simple groups had previously been discovered (as groups of substitutions and so on). Thus the huge programme of research which eventually led to the classification of all finite simple groups was initiated by Hölder in the 1890s.

He saw that it was necessary to develop methods that would prove, for any given order, whether a simple group of that order existed. He therefore considered the prime decomposition of the group order, beginning with the easiest cases: when the group order is divisible by only one, two or three (not necessarily distinct) primes. He could then dismiss a large number of possible orders of simple groups at once. It was already known that there is no simple group of order p^2 or pq ; he discovered that there are no simple groups of orders p^2q and pqr ~~and only one of order p^2qr~~ , where p, q and r are distinct primes. It quickly became apparent that Sylow's Theorems were the most important tools in this line of investigation and provided various lemmas proving nonsimplicity. Hölder also made use of results from the theory of substitution groups, since considerably more structure theory existed in this area (see Chapter 8 for more details).

He presented his methods as they applied to the search for simple groups up to (and including) order 200. It presumably seemed natural to start with the lowest group orders and work upwards. The order 200 is a convenient point at which to stop: up to order 100 there is no particular difficulty, but between 100 and 200 there are a few numbers which require some thought, especially with the limited amount of theory available to Hölder. He was keen to point out that his methods 'may be of some interest' and could

be applied to groups of higher order, and in this respect were important in themselves and worthy of publication. Indeed they formed the basis for subsequent papers on the existence or nonexistence of simple groups throughout the 1890s: the beginnings of the search rested on the foundations Hölder had laid with such care. He was nevertheless aware that the method would not do in general; something much more powerful was needed. His would suffice 'so lange man nicht über eine bessere verfügt' [1892, 56] ('as long as we do not have a better one at our disposal').

What, then, was the background to Hölder's paper and the basis from which he was working? Firstly he remarked [1892, 55] that groups are to be considered in an abstract way. Therefore isomorphic (*holoedrisch isomorphe*¹) groups would appear identical. This viewpoint did not preclude his using substitution groups and other concepts which he did not reckon as 'abstract'. He chose to call group elements *Operationen*, as Cayley had done before him and Burnside did later, but this did not imply that the elements of the group were supposed to be 'operating' on something. Secondly, there already existed a list of known types of finite simple groups, on which Hölder commented. The list contained six types, all with infinitely many members. Hölder wrote of these types [1892, 55]:

Dahin gehören einmal die Gruppen von Primzahlordnung, die überhaupt keine Untergruppe besitzen; dann die alternirende Gruppe, d.h. die Gesamtheit der geraden Vertauschungen von n Dingen, die einfach ist, wenn $n > 4$. Ausserdem hat Herr C. Jordan noch andere einfache Gruppen aufgestellt, die aus der "linearen Gruppe", der "abelschen" und den beiden "hypoabelschen" Gruppen durch Zerlegung hervorgehen. [3]

With modern notation² (where p is prime and $n \in \mathbb{N}$) these six types of simple groups comprise:

- (1) cyclic groups of prime order C_p (which, commented Hölder, have no subgroups at all, so they certainly have no normal subgroups);
- (2) alternating groups A_n for $n \geq 5$;
- (3) projective special linear groups $\text{PSL}(2, p)$ for $p > 3$ and $\text{PSL}(n, p)$ for $n > 2$ (although the groups $\text{PSL}(2, p^m)$ were known, their simplicity had not been investigated);

¹This is Jordan's term: the concept was defined in chapter II of the *Traité*.

²I have chosen one of the notations that reflects the way the groups were conceived in the nineteenth century.

- (4) projective symplectic³ groups $\mathrm{PSp}(2n, p)$ for $n > 1$;
- (5) the simple subgroup of the 1st orthogonal⁴ group $O^+(n, 2)$;
- (6) the simple subgroup of the 2nd orthogonal group $O^-(n, 2)$.

The last four types were introduced in Jordan's *Traité* [1870, 91–213]. The orthogonal simple groups, remarked Hölder, come from the orthogonal groups by decomposition (*Zerlegung*); that is, they are composition factors, just as $\mathrm{PSL}(n, p)$ is a composition factor for $\mathrm{GL}(n, p)$. There were in addition the groups Mathieu had discovered, M_{11} , M_{12} , M_{22} , M_{23} and M_{24} , each of which is a simple subgroup of the alternating group with degree equal to the subscript. However, the simplicity of these groups was not formally noted or proved until after Hölder's paper of 1892, by Cole [1895] and Miller⁵ [1900].

By this time, no great fuss was made about the fact that the Mathieu groups were simple. Indeed, their simplicity was all but implicit in Mathieu's work. It is impossible to establish, but nevertheless likely, that Mathieu could have given a proof of simplicity in a couple of hours' work if he had been asked to do so. The simplicity of M_{11} and M_{23} also follows from the final pages of Sylow's paper [1872]: a transitive substitution group of prime degree n has a simple normal subgroup formed from the Sylow n -subgroups.

Hölder [1892, 55] recognised that there is a question to be raised at this point, even though the above list of simple groups is likely to be incomplete: which of these groups are isomorphic? He pointed out that

$$A_5 \cong \mathrm{PSL}(2, 5)$$

and that $\mathrm{PSL}(2, 7)$ and $\mathrm{PSL}(3, 2)$ both have order 168. He showed later that there is only one simple group of order 168 and therefore these two are isomorphic. Jordan had considered isomorphisms between simple groups in [1870, end of book II]. This isomorphism problem for simple groups was finally settled only with the completion of the classification theorem ninety years later. It had been observed at the beginning of the

³Symplectic groups were called 'abelian' by Jordan when they were first introduced. This is because they were found to be Galois groups for the problem of division of periods of abelian functions. The term 'abelian' prevailed into the twentieth century until the word 'complex' was suggested by Weyl [1946]. Since 'complex' already had various other meanings and confusion was arising, Weyl replaced 'complex' with its Greek equivalent, 'symplectic'.

⁴These were originally called 'hypoabelian' since they are subgroups of the corresponding 'abelian' group.

⁵In fact Miller had previously 'proved' [1898] that M_{24} does not exist.

twentieth century that there is an infinite list of group orders which yield two nonisomorphic simple groups⁶. The list nevertheless contains just a small part of the collection of all finite simple groups. Furthermore, two is the maximum number of nonisomorphic groups possible. The classification theorem is required in order to establish this fact. The first example of this phenomenon was given by Ida Schottensfeld in 1900 when she proved 'by direct calculation' [Dickson 1901] that $A_8 \not\cong \text{PSL}(3, 4)$ but both have order $8!/2$.

The third factor which gave direction to Hölder's research was the 'unproved remark of Galois' stating that the smallest composite order of a simple group is 60 [Galois 1832, 409/26/175]. This was in part the source of Hölder's motivation to search 'from order to order'. He considered that he had continued the investigation begun by Galois, to whom he gave credit for having been the true instigator of the search.

The study of simple groups before 1892

Various surveys exist of the history of the search for simple groups, in particular Silvestri [1979] covers the nineteenth century and Feit [1981] the twentieth century. As Jordan remarked in the preface to his *Traité*, it was Galois who recognised the fundamental distinction between simple and composite (nonsimple) groups. Galois also discovered the family we know as $\text{PSL}(2, p)$, the projective special linear groups of 2×2 matrices over the field of p elements, and knew that these groups are simple for $p \geq 5$.

Further simple groups appeared in the work of Mathieu and Jordan. Their research in group theory was prompted by the *Grand Prix de Mathématiques* announced by the Académie des Sciences in Paris in 1857 and 1858. Jordan's substantial discussion of linear groups in the *Traité* was where various of the 'classical' simple groups, formed from symplectic and orthogonal groups, were first introduced. Mathieu published his discoveries of the first 'sporadic' simple groups⁷ in articles of 1861 and 1873. Neither Mathieu nor Jordan was particularly interested in the simplicity of the groups he discovered. Indeed, the simplicity of the Mathieu groups was not recognised until the 1890s. Mathieu had been concerned with the formation of multiply transitive groups, and it just happened that some of the groups he constructed were simple. The groups $\text{PSL}(2, p^m)$ of 2×2 matrices of determinant 1, modulo plus and minus the identity, over the

⁶These are certain families of symplectic and orthogonal groups.

⁷The term 'sporadic' seems to have been coined by Burnside to describe Mathieu's groups.

field with p^m elements also emerged (not thought of in this way) in Mathieu's doctoral thesis of 1859, published in 1860. Again, he did not ask the question 'Are these groups simple?' In fact they are if $p^m \geq 4$. By 1900 it was known that $\text{PSL}(d, p^m)$ is simple for $d \geq 3$ and for $d = 2$, $p^m \geq 4$, where $d \in \mathbb{N}$. Dickson wrote down a proof of this but the result was not his—it had emerged during the late nineteenth century.

Jordan's *Traité* includes the first proof of the simplicity of the alternating group A_5 . The extension to a proof for A_n where $n > 5$ is not difficult; it usually uses the fact that A_n is generated by certain of its 3-cycles. This fact was proved by Cauchy in 1815; a simplicity proof of this sort for A_n was provided, for example, by Netto [1882, 91–92]. Jordan had given an incorrect proof in the *Traité* (see Silvestri [1979, 325]), based on the idea of multiple transitivity. The simplicity of the alternating groups of degree ≥ 5 , as that of the Mathieu groups, had been 'known' before. However, an actual method of proof had never been written down or clearly established before the publication of the *Traité*. That Jordan made a mistake in his proof is not as surprising as it might at first seem.

As well as Sylow's Theorem, Hölder made use of results on transitive and primitive substitution groups in his paper. These had been treated in particular by Cauchy in 1845 and Mathieu and Jordan from the 1860s onwards.

The impetus, given largely by Klein, to study groups from a geometric viewpoint led to new insights connected with simple groups. A well-known example of the results of such an approach is Klein's book on the icosahedron [1884/1888], which showed a variety of situations in which the simple group of order 60 proves to be of central importance. Klein's student Dyck provided generators and relations for A_5 and $\text{PSL}(2, 7)$ in [1882]. Another topic of interest since the time of Galois had been the study of the 'group of the modular equation' which is isomorphic to $\text{PSL}(2, p)$. It was investigated by Hermite, Kronecker, Bianchi, Jordan and others, and analysed in great detail during the 1880s by Klein, Gierster and Gordan.

Some consequences of Sylow's Theorems

Hölder began his survey in 1892 with some preliminary lemmas which follow from or rely on Sylow's Theorems. He did not even mention the fact that if a group G has a unique Sylow p -subgroup P , then $P \trianglelefteq G$. Instead he established a corollary more useful

for his purposes, that also proves nonsimplicity:

Lemma I

Let the order n of a group G be a composite number divisible by the prime number p and let s be the largest divisor of n which is $\equiv 1 \pmod{p}$. Then, provided that $s!$ is not divisible by n , the group is composite. [1892, 57]

This is proved by considering the action of the group G on the set of Sylow p -subgroups $H_1, H_2, \dots, H_r$. We obtain a transitive⁸ permutation representation of G of degree r , which must be an isomorphism if G is to be simple (since otherwise its kernel would be a normal subgroup of G). Therefore $|G|$ must divide $r!$. We might not know exactly what r is, but it is no larger than the 'largest divisor of n which is $\equiv 1 \pmod{p}$ '. Hölder's reasoning was just the same as this; the only difference was that he did not express it in terms of a group action but showed instead that there was a correspondence between elements of G and permutations of the H_i .

There is another result which follows naturally [1892, 57]:

Lemma II

If the order n of a simple group G is divisible by the prime p and if, among the divisors of n which are $\equiv 1 \pmod{p}$, there is precisely one t for which $t!$ is a multiple of n , then the group can be faithfully represented as a transitive permutation group on t letters.

This is true because, if G is to be simple, its order n must divide $r!$, where r is the number of Sylow p -subgroups of G . Also $r \equiv 1 \pmod{p}$. Therefore if there is only one t satisfying the conditions above, $t = r$ and G acts on the t Sylow p -subgroups to form a permutation group of degree t .

Lemma II was applied later in the paper to groups whose orders were not susceptible to treatment by Lemma I. The final result proving nonsimplicity which Hölder discovered was:

Lemma III

Let the order n of a group G be divisible by the prime power p^o without being

⁸It is transitive since the Sylow subgroups are conjugate. Hölder's word for 'conjugate' is *gleichberechtigt* ('having equal rights') and was borrowed from Klein, who defined it in [1884, 7].

divisible by $p^{\rho+1}$ and let $n = p^{\rho}m$ where $m > 1$. If the group contains an operation of order p^{ρ} , and p^{ρ} is larger than the largest divisor of m different from m , then the group is composite. [1892, 58]

Thus if G has a cyclic Sylow p -subgroup whose order p^{ρ} is such that $p^{\rho} > m_1$ whenever $m_1|m$ ($m_1 \neq m$), then G is not simple. Hölder's proof of this takes three pages. It is not unlike a modern proof. First Hölder showed that there must be exactly m Sylow p -subgroups, $H_1, H_2, \dots, H_m$, if the group is to be simple. Then he investigated to what extent the H_i can intersect one another. He found that it is not possible for them to intersect in anything larger than $\{1\}$ if the group is simple. When the H_i are all distinct, we count group elements to discover that there are only m elements not included in the Sylow p -subgroups. If $m = q^{\tau}$, Sylow's Theorem tells us that there is a normal subgroup Γ of order m .

Otherwise, m is divisible by the primes $q, q_1, \dots$. Assume that q is the largest of these. Cauchy's Theorem provides elements of order $q, q_1, \dots$ in G . We would like to show that the m elements whose orders are divisors of m do indeed form a group, which must be normal in G since it can have no conjugates.

How many elements of order q are contained in G ? There must be fewer than m , and therefore there are fewer than $m/(q-1)$ subgroups of order q . Let K denote one of these subgroups. Then K has a conjugates where $a < m/(q-1)$, therefore $a < m/q_1$. So a is less than the largest divisor of m different from m . As before, either G is composite or it can be faithfully represented as a transitive permutation group on a letters. The latter hypothesis is not possible since there would be no way to represent an element of order p^{ρ} . Therefore the theorem is proved.

Having reached the stage where the H_i are all distinct, we would now note that Burnside's Transfer Theorem applies to provide a normal p -complement. Hölder's proof requires far more detailed investigation into the structure of the group than would be undertaken in a modern proof, but the underlying ideas are similar.

These three lemmas, along with Sylow's result that a group of prime power order is soluble, enabled Hölder to conclude (Proposition IV) that if the order of a group is divisible by at most three (equal or distinct) prime numbers then the group is not simple, and must therefore be soluble.

Results from the theory of substitution groups

The fact that we can use Sylow's Theorems to obtain transitive permutation representations for a group resulted in the need for some substitution group theory. First Hölder noted that a simple permutation group of order > 2 must contain only even substitutions (otherwise the even substitutions would form a normal subgroup). He went on to give three results:

Proposition V

A transitive substitution group which can be generated by some 3-cycles is the alternating group.

Lemma VI

A simple transitive substitution group of degree m which contains a 3-cycle is the alternating group of order $m!/2$.

Lemma VII

A simple transitive substitution group of degree 6 is either A_6 or is isomorphic to A_5 .

Proposition V was dealt with as follows: Hölder proved first that all possible 3-cycles on the letters a, b, c, d can be derived from any two of them. Next he showed that, given two 3-cycles with one letter in common, the same result applies: all possible 3-cycles on the letters can be derived from these two. Now if the 3-cycles $C_1, C_2, C_3, \dots$ are to generate a transitive substitution group G on m letters, at least two of them, C_1 and C_2 , must have either one or two letters in common. Therefore all possible 3-cycles on these letters lie in G . The group G must also contain a 3-cycle involving one of these letters and a new letter (if such exists), since G is transitive. But then it contains all possible 3-cycles on this larger set of letters. By repeating this argument we see that G must contain all 3-cycles on the m letters, and therefore it is the alternating group.

Lemma VI simply follows from the fact that the 3-cycle and its conjugates would generate a normal subgroup, which must therefore be equal to the whole group.

The proof of Lemma VII included the application of a result from Serret's *Cours* [1885]: any simple transitive group G of degree 6 has order divisible by 6, and so it contains an element of order 3. If this is a 3-cycle then $G \cong A_6$. If not, it is of the form

$(abc)(def)$. But the symmetric group S_6 on a, b, c, d, e, f has an outer automorphism taking the above substitution to one of the form (xyz) . Let the group G be mapped to a group J under this automorphism: the group J must be simple but is not necessarily transitive. If J is transitive we return to the previous situation and $G \cong A_6$. If J is intransitive the only possibility is $G \cong A_5$.

Hölder generally referred to A_5 as the 'icosahedral' group. This terminology became fairly standard after the appearance of Klein's book [1884], in which it is shown that the group of rotations of the icosahedron is isomorphic to A_5 . The group A_5 can be represented as a simple transitive substitution group of degree 6 because it has 6 Sylow 5-subgroups. This corresponds to the fact that the icosahedron has 12 vertices, which can be grouped into 6 pairs of opposite vertices. Such pairs are fixed under any rotation of the icosahedron, so the rotation group can be thought of as a substitution group on these 6 objects. The substitution group is transitive because any pair of opposite vertices can be mapped to any other pair. Further, the stabiliser of a pair of vertices is a cyclic group of order 5 (a Sylow 5-subgroup)⁹.

Possible orders of simple groups up to 200

Having mentioned groups of prime order, most possible (composite) orders could be discounted straight away using Proposition IV and Sylow's result that groups of prime power order are soluble. Of the remaining orders, nearly all are excluded by Lemma I and a couple more are dealt with by Lemma III. Hölder wished his readers to be aware of 'die Fruchtbarkeit der Sylow'schen Sätze' ('the fruitfulness of Sylow's Theorems') so he deliberately used Lemma I in every case where it worked.

There remained only seven possibilities which were undecided:

$$60, 90, 112, 120, 144, 168, 180.$$

The first four of these are still straightforward. Lemma II with $p = 5$ can be applied to groups of order 60, 90 and 120 to find that they can be represented as transitive substitution groups of degree 6. Then Lemma VII shows that A_5 is the only simple group of order 60, while simple groups of orders 90 and 120 do not exist. The order

⁹In case this sort of reasoning proved too difficult for the English-speaking readers of *Icosahedron* to visualise, the translator G.G. Morrice provided a solution [1888, 291]. 'Glass models of the Icosahedron are sold as letterweights by stationers for eighteen pence. They may be obtained at No. 355 Strand.'

$112 = 7 \cdot 2^4$ is also dealt with by Lemma II with $p = 2$: such a simple group could be faithfully represented as a substitution group of degree 7. What would a Sylow 2-subgroup, of order 16, look like? Cauchy constructed such a group of order 16 and degree 7 and it contains transpositions. This implies that the permutation representation of G contains odd substitutions and therefore it cannot be simple¹⁰.

The last three undecided orders required rather more effort. In fact Hölder took twenty pages over them. His task would have been much simpler if more of the information on substitution groups of low degrees that was gathered throughout the 1890s had been available to him. Such groups had been classified up to degree ten before 1900 (more details are found in Chapter 8). Indeed, Hölder had read the relevant papers published on this subject before 1892, but stated that he would make use of such classifications as little as possible (especially as they were by no means universally correct).

Hölder dealt with order $144 = 2^4 \cdot 3^2$ by representing a simple group G of this order as a transitive permutation group of degree 9 and considering how an element of order 2 in G can be represented. He obtained the representation of G as follows¹¹:

First he showed that at least one conjugacy class of elements in G must contain an odd number of elements, and this number is 9. Then the centraliser of an element in this class has order 16, so it is a group containing an element of order 2. This element ($T_1^{2^k-1}$ in Hölder's notation) of order 2 is also in a conjugacy class of 9 elements. Therefore G can be faithfully represented as a transitive permutation group of degree 9 (act by conjugation on these 9 elements $\Theta_1, \Theta_2, \dots, \Theta_9$ of order 2).

Then, asked Hölder, what form must be taken by the substitution B_1 corresponding to one of these elements Θ_1 ? It must fix Θ_1 , it is an even substitution and of order 2. There are two essentially different possibilities:

$$\text{I} \quad B_1 = (\Theta_1)(\Theta_2)(\Theta_3)(\Theta_4)(\Theta_5)(\Theta_6\Theta_7)(\Theta_8\Theta_9)$$

$$\text{II} \quad B_1 = (\Theta_1)(\Theta_2\Theta_3)(\Theta_4\Theta_5)(\Theta_6\Theta_7)(\Theta_8\Theta_9)$$

Case I is dealt with by a theorem of Jordan, but Hölder preferred to give his own

¹⁰In fact there is an easier way. Since the permutation representation of G must contain no odd substitutions, there must be an isomorphism from G into A_7 . But the order of A_7 is not divisible by 2^4 , so this is not possible.

¹¹This is not the most obvious way to produce a representation of degree 9, but the method of construction is important and necessary to the rest of the proof. (A simple group G of order 144 must have 9 Sylow 2-subgroups; the obvious representation is that obtained when G acts by conjugation on these subgroups.)

proof that such a group of order 144 cannot be simple. Again this proof relied on the construction of further permutation representations, which he obtained by a detailed investigation of the correspondence already defined between elements of G and their representations as substitutions. Case II was discounted in a similar way.

The rejection of 144 as the possible order of a simple group is made much easier by an application of Burnside's Transfer Theorem:

Let P be a Sylow p -subgroup of the finite group G . If P lies in the centre of its normaliser then G has a normal p -complement, that is, a subgroup K such that $K \trianglelefteq G$, $K \cap P = \{1\}$ and $G = KP$.

The Transfer Theorem is so called because it is nowadays proved using the transfer homomorphism, a concept unknown in the mid 1890s when Burnside first established the theorem using a counting argument. Once the result is available, order 144 can be dealt with as follows:

A simple group G of order 144 must have 16 Sylow 3-subgroups of order 9. Therefore these subgroups are identical to their normalisers. Since a group of order 9 is abelian, a Sylow 3-subgroup lies in the centre of its normaliser and Burnside's Transfer Theorem applies to produce a nontrivial normal subgroup of G .

Hölder knew that there was at least one simple group of order 168, which he referred to as the 'group of the modular equation for the seventh order transformation of elliptic functions'. This group is $\text{PSL}(2, 7)$. However, he did not want to assume this knowledge, but instead proved that such a group exists in his abstract framework, showed that it was unique and concluded that the simple groups of order 168 already discovered in various contexts were in fact isomorphic. Once more he employed permutation representations: Lemma II implies that a simple group of order 168 can be represented as a transitive permutation group of degree 8. He decided to determine all such groups by working out what their structure must be in some detail and eventually finding generators for them. There are two cases, one leading to a 2-transitive soluble permutation group, the other giving the simple group. Generators and relations for the simple group had already been worked out by Dyck [1882, 40–41] but again Hölder chose to develop his method rather than take this shortcut. Hölder proved simplicity by noting that a normal subgroup consists of complete conjugacy classes and showing that no combination of orders of conjugacy classes divided 168. This method had been used by Klein [1884, 17–18/1888,

19] to prove the simplicity of A_5 .

Finally the order $180 = 5 \cdot 2^2 \cdot 3^2$ was rejected. If a simple group of this order existed, by Sylow's Theorem and Lemma VII it would have 36 Sylow 5-subgroups. These account for 144 nonidentity elements, so there are 35 remaining. There are only two possibilities for orders of conjugacy classes of the remaining elements: $20 + 15$ or $15 + 10 + 10$. These correspond to an element of order three being conjugate or not conjugate to its square. In both cases an application of Lemma II with $p = 3$ shows that G can be represented as a transitive substitution group of degree 10. As before, Hölder considered how particular elements of G are represented as substitutions. In every possible case, Hölder produced substitutions which could not be there if the group were simple.

Once it is established that there are 36 Sylow 5-subgroups, the argument involving Burnside's Transfer Theorem can be used in exactly the same way as for order 144 to obtain a normal 5-complement and therefore prove nonsimplicity.

Consequences of Hölder's paper and further progress in the search

In fact Hölder's methods for searching for simple groups soon get out of hand when applied to groups of larger and larger order. The method was used up to order 2,000 by Cole, Burnside and Miller. By the time Siceloff dealt with orders 2,001 to 3,640 in 1912, many new techniques existed and the old ones had been greatly extended and adapted. Twelve years later, Cole returned to the problem and continued the 'exhaustive enumeration' to order 6,232. All this work did not even enable the smallest of the Mathieu groups (of order 7,920) to be reached.

F.N. Cole took up the problem in the 1890s at order 201 where Hölder had finished. His approach (as Hölder's) was from the viewpoint of abstract group theory with frequent recourse to substitution groups. Cole had in fact taken part in enumerating transitive substitution groups of certain low degrees and was that much more familiar with them than Hölder had been. Cole's survey was completed in June 1892. He must have worked intensely on the problem for no more than a few months, since volume 40 of the *Mathematische Annalen* (containing Hölder's paper) was published only in the early months of 1892. Cole's survey was, however, not quite comprehensive; the orders 432 and 360 proved too difficult. A year later (May 1893) he conquered them: the alternating group A_6 is the only simple group of order 360 and there is none of order 432. In the meantime Frobenius had begun to study the solubility of groups and

had proved a general theorem [1893a, 344–345/572–573] to the effect that no group $\mathfrak{H}$ of order $h = p^4 q^\beta$ can be simple if p and q are primes such that $p < q$. This theorem applies to the group order $432 = 2^4 3^3$, as Frobenius noted, referring to Cole's paper.

By the time of his second paper on the subject, Cole had increased the limit of the investigation to 660, the order of $\text{PSL}(2, 11)$, and in doing so he discovered a simple group of order 504. It turned out to be $\text{PSL}(2, 2^3)$: it had been studied by Mathieu and Kirkman but they had not considered its simplicity or otherwise. This result prompted E.H. Moore to study 'linear fractional groups', which are the equivalent of Jordan's linear groups but with coefficients from any finite field of order p^n where p is prime and $n > 1$. It is slightly odd that Jordan did not treat such groups (over an arbitrary finite field) in the *Traité*, in view of the fact that the groups $\text{PSL}(2, p^n)$ had appeared in Mathieu's doctoral thesis [1860] and in another of his papers [1861]. The fact, which seems to be fairly obvious, that the generalisation to arbitrary finite fields can be performed for all the types of linear groups that were known, was apparently not properly appreciated until the mid 1890s.

Moore's study led to his discovery of another family of simple groups in [1893]; Burnside considered their structure in greater detail. He went on [1895, Note IX] to push the limit of investigation of simple groups up to order 1,092 (there is one of this order, namely $\text{PSL}(2, 13)$). By this time there was more information at his disposal on the enumeration of substitution groups. He proved several new theorems discounting the existence of a simple group whose order had a certain prime decomposition. In particular he showed that A_5 is the only nonabelian simple group whose order is the product of four or fewer primes [1894a, Note II]. He acknowledged Hölder's work and proved Hölder's result that a group whose order is the product of three primes cannot be simple, but without using substitution groups (he did not consider they should appear at all in a paper written from an abstract viewpoint).

It was at this time that Burnside began proving (and publishing) results that had been known to Frobenius (and published by him) some years previously. Burnside's proof of Sylow's Theorems [1894a, 9–12] was essentially that of Frobenius, published in two papers in 1887. Burnside did apologise for having presented the proof as new when he later found out about Frobenius' work. The two men found themselves working on more-or-less the same questions: their rivalry became more friendly as time passed and

each gained the respect of the other.

In 1895 Burnside proved that a simple group of even order must have order divisible by $12 = 2^2 \cdot 3$, $16 = 2^4$ or $56 = 2^3 \cdot 7$. He proved this by combinatorial arguments; now it is far more easily proved using the transfer homomorphism, which is an important tool in the search for simple groups. Burnside was partly responsible for the introduction of the transfer, which he produced in 'character-theoretic' form in [1901a], after having studied the representation and character theory of groups of odd order. The other source of the transfer, in a form that is now more usual, is a paper [1902] of Schur. Burnside showed by means of the transfer that if p is the smallest prime dividing the order n of a simple group G , then either $p^3 \mid n$ or $12 \mid n$. Then he was able to prove that a simple group of odd order must have at least seven prime factors. He found [1901b] that there could be no simple group of odd order less than 40,000. Indeed, simple groups of odd composite order were beginning to prove especially elusive, and the existing results had shown that the order of such a group would already have to be much larger than the limit to which groups of even order had been taken. It was not altogether surprising, although still quite a major step, when Burnside suggested in an appendix to the second edition of his book [1911] that there is *no* simple group of odd composite order. In [1897, 379] he had recommended the investigation of this problem to his readers. In fact it was over sixty years before the question was definitively (and negatively) answered [Feit & Thompson 1963].

Burnside devoted the whole of the last chapter of the first edition of his book [1897, Ch. 15] to the question of simple groups and presented together all the theorems he and others had discovered which established nonsimplicity. In the Preface [1897, x] he had written: 'I wish to express my grateful thanks also to Prof. O. Hölder of Königsberg who very kindly read and criticized parts of the last chapter'. Burnside was familiar with Hölder's group-theoretic work; all of Hölder's papers on the subject were referred to at some point in the book. Hölder's results on simple groups of low order were verified by Burnside [1897, 371ff], who also made mention of the 'very interesting and suggestive methods' of Hölder's paper [1895a] on extension theory. Burnside set several of Hölder's results as exercises in the text.

By 1897, work was well under way towards proving that all groups of order $p^\alpha q^\beta$ (where p and q are distinct primes) are soluble, and Burnside presented all the partial

results that had been obtained in this connection. He was also able to prove that there are only four simple groups whose orders are divisible by four or five (not necessarily distinct) primes.

Burnside had remarked at the beginning of the chapter:

If the results appear fragmentary, it must be remembered that this branch of the subject has only recently received attention: it should be regarded rather as a promising field of investigation than as one which is thoroughly explored. [1897, 344]

Later he gave his own opinion on the possible outcome of the search for simple groups. 'A complete solution of this . . . problem is not to be expected . . . ' [1897, 370]. Although he was not optimistic with regard to the general problem, he was nevertheless certain that it was important to discover as much as possible, and that the research instigated by Hölder was of great significance.

The general search progressed up to order 2,001 with a paper of G.A. Miller and of his student G.H. Ling in 1900 (dated July 1899). They found no simple group with order lying between 1,092 and 2,001. The paper contains a summary of previous results and methods: in it are stated the principal theorems of Hölder, Cole and Burnside and further results of Frobenius, who studied groups according to the prime factorisation of their order and proved solubility in certain cases.

Dickson found some new simple groups at the end of the nineteenth century by recognising that there were finite groups corresponding to some of the infinite groups investigated by Lie, Cartan and others. The motivation to do this came as it became apparent that there was an analogy between Jordan's and Mathieu's simple groups and the infinite Lie groups, if these latter were arranged in a particular way. Some Lie groups did not yet have finite analogues, so it was sensible to look for them. Dickson and others discovered many of the cases in which these finite groups existed (and were simple). In Dickson's treatment of linear groups, he began to use the concepts of linear transformations and matrices to represent the groups, rather than thinking of them as 'linear substitutions', as Jordan had done.

Much later, in the 1950s, the terms 'groups of Lie type' and 'Chevalley groups' were coined to describe the 'classical' simple groups. This happened after a paper of Chevalley [1955] appeared, in which he showed how finite analogues of all the simple Lie groups could be made, and after Steinberg, Suzuki, Ree, Tits and others showed how to produce

'twisted versions' of the groups, including the unitary groups (which are *not* complex Lie groups) and certain missing orthogonal groups, and then some new simple groups.

Dickson presented all his results in a book [1901] which has recently been discussed by K.H. Parshall [1991]. I cannot find a single mention of Otto Hölder in the book, which is not really surprising, since Dickson's approach to simple groups was purely through linear groups. As well as the simple groups that he discovered as finite analogues of Lie groups, he came across another family of simple groups in a different way: the projective unitary groups $U(n, p^{2m})$. Unitary ('hyperorthogonal') groups were described by Dickson as being formed from those substitutions in linear groups that leave a certain function invariant. In fact they were later recognised as the direct analogues of the complex unitary groups (which are real, but *not* complex, Lie groups).

At the end of the book [1901, 307–310] Dickson gave a summary of the known systems of simple groups and listed the isomorphisms that had by then been discovered between them. He used these results to list all the simple groups he could find of order less than one million. The list is almost complete. There are only three missing groups, the smallest Suzuki group $Sz(8)$ and two sporadic groups, the Janko groups J_1 and J_2 , of orders 29,120, 175,560 and 604,800, respectively. These groups were discovered only in the 1960s. Suzuki was presumably not intending to produce a new family of simple groups when he was working on the classification of Zassenhaus groups¹², but it transpired that the groups he created were indeed simple. The Janko groups appeared as part of the continuing search for simple groups. They are constructed by considering centralisers of involutions (elements of order 2, known to exist in every nonabelian finite simple group since the Feit–Thompson Theorem [1963]). The existence of the first Janko group was informally announced in 1964; the announcement was then published [Janko 1965]. It was another year before full details of the group were provided [Janko 1966]. The second Janko group J_2 was subsequently conjectured but could not immediately be proved to exist. Proof was eventually given by Hall & Wales [1968].

It happened that the task of classifying simple groups of order less than one million was also taken up again in the 1960s by Marshall Hall. He produced reports [1969 & 1972] on the progress of this classification, the first given at Oxford in the summer of

¹² A Zassenhaus group is a doubly transitive permutation group in which any nonidentity element fixes at most two points.

1967. By this time all the relevant simple groups had in fact been found, but it could nevertheless not yet be proved that the list was complete.

An important result that drew together work of Frobenius, Jordan and Burnside was the $p^\alpha q^\beta$ theorem [Burnside 1904] which says that every group whose order is of this form (where p and q are distinct primes) is soluble, or equivalently, there is no simple group of such an order unless the order is p or q . Burnside proved this after a great deal of effort in developing combinatorial methods; his proof worked independently of the previous partial results that had been obtained. It used the character theory which had been introduced in the late 1890s by Frobenius. A character-free proof was not formulated until the 1970s, by which time the contributions of various authors enabled such a proof to be constructed. A more polished version of this proof is presented by Blackburn and Huppert [1982, 11–19]. The proof involving characters is nevertheless considerably simpler.

Various other ideas were introduced at the beginning of the twentieth century, in particular the theory of ‘Frobenius groups’. This contributed some nonsimplicity theorems of a new kind to the methods available in the search.

It was realised in the early 1900s that an approach other than Hölder’s was needed to tackle the problem of simple groups. The importance of this initial step in the search is not, however, to be underestimated. Without it, the extreme difficulties in proving whether or not a simple group of a particular order may exist would not have been so readily seen. The classification actually proceeded along a very different line: when a fairly large supply of simple groups had been found, the task of proving that these are the only ones was begun, not proceeding order by order but by looking into the structure of a simple group.

The situation reached by 1905 is summed up by Cartwright [1985, 28]. There was a list of known simple groups, which remained the same for the next forty years, although many advances took place in group theory that later contributed to the search. The subject became fashionable again in the late 1940s and new results soon began to appear regularly and frequently. Further details can be found in the various survey papers which were published as the search progressed and in other articles, for example [Feit 1981].

Over one hundred mathematicians have contributed to the classification of finite sim-

ple groups since Hölder in 1892. Work continues today (and has been in progress since well before the classification was completed) to reduce the 10,000 pages devoted to the proof of the theorem to a more manageable 3,000 pages. Hölder, with his desire for conciseness and clarity, would have approved. Little could he have foreseen the depth and scope of research needed to produce the 'survey of all [finite] simple groups' he was so interested to see in the 1890s.

Notes

- 1 It would be of the greatest interest if a survey of all simple groups with a finite number of operations could be given.
- 2 The classification of simple groups will be the subject of another of my works if I find it easy to obtain results from the research I am undertaking.
- 3 There belong first the groups of prime order, which possess no subgroups at all; then the alternating group, that is, the totality of even permutations of n objects, which is simple if $n > 4$. Mr. C. Jordan has in addition established yet other simple groups, which emerge through decomposition from the "linear group", the "abelian" and both the "hypoabelian" groups.

CHAPTER 8

The Structure Theory of Finite Groups

If I should count them, they are more in number than the sand.

Psalm 139:18

The situation before the 1890s

The ideal aim of research in finite group theory would be to ‘find’ or ‘classify’ all finite groups. The more that is known about finite groups, the more it becomes clear that this is a vast project and a seemingly unachievable goal. Even if it should be achieved, the complexity of the answer would render it practically useless. In 1878 Cayley did not have such doubts. He was able to say with confidence that ‘the general problem is to find all the groups of a given order n ’ [1878, 51/402]. He began to attack the problem by listing all groups of orders up to six. The list included three distinct groups of order six since he had forgotten (having correctly given two groups of order six in [1854a]) that $C_3 \times C_2 \cong C_6$.

Despite such unpromising origins, the task of classifying groups of a given order or groups with a certain property gained momentum during the late nineteenth century. The ‘general problem of groups’ was broken down into simpler ones. Some research was directed at groups of low order, other at groups whose order has a particularly simple prime decomposition, other at groups of substitutions of low degree, and so on. Then another route was opened by Otto Hölder in 1889 with the proof of the Jordan–Hölder Theorem—one should study the simple groups and the way in which they can be combined. This approach provided a general framework within which the more specialised problems that had been attacked before could be reconsidered. Otto Hölder was one of those who guided the transition in group theory from concern with specific problems to an understanding of overall patterns. This chapter focuses on the way in which problems in the structure theory of finite groups were re-formulated and partially or completely solved as a result of a new general understanding of the theory. Otto Hölder’s two papers dealing with group structure theory, [1893] and [1895b], are discussed and compared with other articles on the subject that appeared almost simultaneously.

It happened that research into group structure had in general taken place within the theory of groups of substitutions even after an abstract approach to group theory became available. By the 1880s a large amount of structure theory existed for substitution groups, which may have lessened the motivation to transfer such research to abstract groups. There was also an established notation which enabled substitution groups to be 'written down'. Such a notation for abstract groups was introduced principally by Dyck in 1882; his idea was to give what is now called a presentation for a group using generators and relations. This method of specifying a group has the advantage that large and complicated groups can be expressed relatively concisely¹. Group presentations were not widely employed before the 1890s, at which time Hölder and others recognised their usefulness.

The earlier research within substitution group theory had been aimed at investigations of the structure of various types of substitution groups, sometimes in great detail. In particular, the concepts of transitivity and primitivity enabled difficult questions to be reduced to a series of possibly easier ones. The mathematicians involved in the classification of substitution groups of different types included Cauchy, Serret, Jordan, Mathieu and Sylow.

The techniques they used and results they obtained fall into two categories. The first consists of those results and techniques which are naturally theorems and tools of abstract group theory, although originally conceived in terms of substitution groups. Sylow's Theorems are examples of these. The second category contains ideas and theorems which are best formulated and proved within substitution group theory, and naturally remained within that theory after an abstract approach became available. Examples here are theorems on the limit of transitivity of a substitution group.

Some theorems of Sylow

The significance of Sylow's Theorems as tools for investigating the structure of a group became apparent very rapidly after they appeared in print in 1872. Sylow showed existence of what came to be known as Sylow p -subgroups² in a finite group G . He

¹It contrasts with other much 'messier' ways of representing a group, for example Cayley's colourgroups, now known as Cayley graphs.

²A Sylow p -subgroup of a finite group G of order $p^\alpha m$, where p is prime and not a divisor of m , is a subgroup of order p^α .

proved that all such subgroups are conjugate and that the number of them is congruent to 1 mod p . He also showed that any p -subgroup³ of G has a conjugate contained in a given Sylow p -subgroup. Although Sylow was working within substitution group theory, his methods of proof apply more-or-less as they stand to groups considered abstractly. Many different techniques for proving these results were known by 1900: see [Waterhouse 1980] and [Casadio & Zappa 1990] for details.

It is less well-known that Sylow proved various results concerning groups of prime power order in this paper. The investigation of such groups followed naturally from the theorems he had just established. He discovered [1872, 587–588] that a nontrivial p -group g must have a nontrivial centre; his reasoning, although in nineteenth-century language, was close to a modern approach to a proof. Here is a ‘translation’ into modern terms. The group g (of order n^α in Sylow’s notation, where n is prime; for consistency I use p instead of n) acts on itself by conjugation. The orbits of the action are conjugacy classes of elements in g . The size of a conjugacy class must therefore divide the order of g . If the order of g is written as the sum of orders of conjugacy classes, there must be at least p classes of size 1 (since there is certainly one such class—that consisting of the identity element). The elements in these classes are in the centre of g .

Sylow next discovered the idea of what is nowadays called a power commutator presentation for a p -group. Such a presentation can be very simply defined if quotient groups are understood—as has been shown, this was not the case in 1872. Sylow therefore used the corresponding idea from Galois theory, that of a rational function of the ‘letters’ which are permuted amongst themselves by the group g . The centre of g contains an element θ_0 of order p . There always exists a rational function of the letters which is fixed when and only when the substitutions θ_0^i permute them. The various values $y_0, y_1, y_2, \dots$ which this function may take then correspond to the cosets in g of the subgroup $\langle \theta_0 \rangle$. The substitutions of g acting on the letters in the y_j permute the y_j amongst themselves. These permutations form a group which is homomorphic to g . It is in fact (isomorphic to) the quotient group $g/\langle \theta_0 \rangle$ which has order $p^{\alpha-1}$.

This argument is repeated another $(\alpha - 1)$ times to produce a series of substitutions $\theta_0, \theta_1, \dots, \theta_{\alpha-1}$, each central and of order p modulo the group generated by the

³A subgroup whose order is a power of the prime p

preceding substitutions in the series. Any element θ of g can be written

$$\theta = \theta_0^i \theta_1^k \theta_2^l \dots \theta_{\alpha-1}^r.$$

If the exponents $i, k, l, \dots, r$ are chosen to lie between 0 and $p-1$, then the expression for θ is unique. Sylow did not actually mention this explicitly, but it lies just under the surface of his explanation of how to obtain a power commutator presentation. It is quite possible that he did not consider it necessary to point out the uniqueness of such an expression for a group element.

A corollary of the fact that this presentation exists is that all p -groups are soluble, since their composition factors all have order p .

A power commutator presentation for a p -group may be obtained from its upper central series by inserting extra terms⁴. Both these concepts were recognised as having an important part to play in the study of p -groups when a sudden surge of interest in structure theory took place in the 1890s.

Sylow next related the results he had produced to the solution of algebraic equations and used his various theorems to prove that an equation is soluble by radicals if its group G has order

$$p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \dots,$$

where $p_0, p_1, p_2, \dots$ are primes and $p_i > p_{i+1}^{\alpha_{i+1}} p_{i+2}^{\alpha_{i+2}} \dots$. There was a corresponding surge of interest during the 1890s in the problem of the solubility of groups with particular prime decompositions: this problem was pursued especially by Frobenius and Burnside. Thus Sylow had anticipated research in two areas that were later seen as significant.

The remainder of his paper is concerned with theorems about transitive substitution groups. He found conditions on the possible order of a transitive group of given degree. In this he extended some research of Mathieu. The paper concludes with a substantial contribution to the study of transitive substitution groups of *prime* degree.

Results in the structure theory of groups up to the 1890s

What was actually achieved before the 1890s towards the classification of groups? A start was made on various more specific problems:

- (1) The classification of substitution groups by degree: those of low degree were

⁴See the explanation of Young's paper later in this chapter.

completely determined;

- (2) The classification of substitution groups that are transitive or multiply transitive or primitive or soluble: general results were obtained and exact enumerations given for groups of low degree and other particular cases;
- (3) The classification of groups with simple prime decomposition (considered as substitution groups): the groups of orders p , p^2 and pq , where p and q are distinct primes, were determined, and insights obtained into the structure of groups of prime power order;
- (4) The classification of groups by order (considered as substitution groups): those with small order were enumerated.

The first problem and much of the second is naturally part of substitution group theory; the latter two were attacked in an ever more abstract framework during the late nineteenth century. Let us consider them one by one.

(1) In 1893, when Bolza reviewed the English translation of Netto's *Substitutionentheorie*, he explained how the investigation of the 'number of values' of a function of n variables leads to the study of substitution groups—in particular to the problem of determining those substitution groups whose degree is n . He noted, as Netto had remarked before, that this general problem of substitution group theory 'presents difficulties as yet insuperable' and yet 'a great number of theorems concerning the construction of groups are known'. Perhaps it should be pointed out here, as it was in Bolza's review, that the problem under consideration was that of finding all substitution groups of an *indeterminate* degree n . It is true, as Bolza commented, that the answer may (in theory) always be found for a given value of n , although the calculations involved may be unreasonably long. Netto had provided a series of known theorems on the construction and structure of substitution groups in Chapter II of his book.

The enumeration of substitution groups by degree was begun as early as 1845 by Cauchy, who reached as far as degree six. He expressed the theory in terms of the number of values of a function of n variables and produced lists of all the subgroups of the symmetric groups of degrees ≤ 6 . Bertrand contributed some general theorems about possible orders of substitution groups of given degree; Serret also tried his hand at such an enumeration [1850] and reached degree five. The task was taken up again by Mathieu in the late 1850s when Cauchy's influence led to the problem of the 'number

of values of a function' being proposed for the Paris *Grand Prix de Mathématiques*. Mathieu recalled some theorems of Bertrand and Serret on the subject and proceeded to deal with degrees 7 and 8. Jordan attacked the general problem without enumerating particular degrees, and produced various new theorems.

There was another attempt to enumerate substitution groups during the 1860s, also inspired by the Paris *Grand Prix*. Although highly original, it was completely unimportant, not least because it was written in a specially designed 'new' language for group theory. The author was T.P. Kirkman. He developed ideas such as direct and semidirect products and transitive extension, and used them to make a list of substitution groups of degree up to ten.

There was a long pause before interest in this problem was renewed at the beginning of the 1890s. A sudden increase of activity established results up to $n = 10$. Interest was greatest in England and in the United States, where the problem was energetically tackled by Askwith, Cayley, Cole, Miller and others. Askwith [1890a] recalculated the groups of degree three to seven, having developed some new theorems in a more general framework than had earlier been adopted, and without considering the old question of the 'number of values of a function'. He quickly went on to degree eight [1890b] whilst correcting and adding to his previous results. It was not generally the case that a complete and correct list was given on one's first attempt.

Cayley joined the search in 1890–91 by repeating the investigation for degrees up to 8 and revising Askwith's lists, but his enumeration is also incorrect. In the meantime Askwith was attacking degree 9; he published his results in 1893. These papers had aroused the interest of F.N. Cole in the United States, who made his own study [1893a] for $n = 6, 7$ and 8 and found 'over forty omitted groups'. He moved on to degree 9 and again improved on Askwith's list [1893b & 1893c]. No correct enumeration had yet appeared for degrees 8 and 9. G.A. Miller attempted to put the matter straight in 1894 by pointing out a missing group of degree 8 and giving yet another enumeration. Interest in the problem gradually decreased during the late 1890s as the focus of research in group theory changed and substitution groups were no longer viewed as forming the centre of the theory.

(2) Cauchy recognised that the classification of substitution groups reduces to the

classification of those of a given degree that are transitive⁵. Mathieu took up this idea in 1858 when investigating groups of degrees 7 and 8. Later Jordan realised the importance of the further reduction, which is always possible, to the case of primitive groups⁶. He studied the notion of primitivity more extensively than Cauchy had done. The notion had appeared in a confused form as early as the 1790s in Ruffini's work. Galois had also considered primitivity, but only with respect to soluble groups.

Various aims in the study of transitive, multiply transitive and primitive groups began to be formulated; work continued on these problems throughout the second half of the nineteenth century. Jordan was particularly interested in such questions and studied the 'limit of transitivity' of a substitution group not containing the alternating group of its degree. He conjectured in 1873 that there might be an absolute bound for this limit. In fact there is such a bound: a substitution group not containing the alternating group of its degree cannot be more than fivefold transitive. This result may be deduced from the classification of finite simple groups. The problem was considered in the nineteenth century by Bochert [1892] as well as by Jordan, and in the twentieth century by Weiss and Wielandt. In 1960 Wielandt proved, subject to 'Schreier's Conjecture', that the limit of transitivity must be ≤ 7 . Schreier's Conjecture was that the outer automorphism group of any finite simple group is soluble; the truth of this statement can also be concluded from the classification of finite simple groups.

Jordan began a classification of primitive groups of low degrees, and had reached degree 17 in 1872. He simply announced the number of such groups of degree 17, without providing any details. The primitive groups were especially emphasised in his *Traité* as a result of his investigation of the question 'Which equations are soluble by radicals?' He undertook a study of maximal soluble primitive substitution groups. The classification of primitive groups was brought to completion in 1980 with the O'Nan-Scott Theorem, by which such groups are classified according to the structure of minimal normal subgroups, and thus the problem is essentially reduced to one about simple groups.

⁵An intransitive group can always be expressed as a subdirect product of substitution groups of smaller degrees (these degrees are the lengths of the orbits). In 1899 a paper of Bolza, 'On the Construction of Intransitive Groups', appeared in the *American Journal of Mathematics*. He tackled the problem of forming an intransitive group which is made up from given transitive groups; he confirmed the usefulness of being able to reduce a problem about groups to one about transitive groups.

⁶An imprimitive group can be expressed as a subgroup of a wreath product of groups of smaller degree.

(3) The first systematic examination of the groups of orders p^2 and pq (where p and q are distinct primes) seems to have been provided by Netto in his book [1882, 133–137]. The book is concerned with substitution groups, therefore he dealt with this general question in a particular way. What are the groups whose order, pq or p^2 , equals their degree? He answered the question correctly. If the order is pq with $p > q$, there is certainly a cyclic group C_{pq} . He referred to such cyclic groups as the type of group of order pq that 'ist uns bereits bekannt' ('is already known to us'), implying that any other type of group of order pq had not previously been known. If another type does exist Netto discovered, by counting elements and using Sylow's Theorem, that one is forced to impose the condition $p \equiv 1 \pmod q$. When this is the case he recognised that there is just one group (up to isomorphism), a semidirect product $C_p.C_q$. For the order p^2 , apart from the cyclic group C_{p^2} there is one other, $C_p \times C_p$, since any two elements that generate the group must commute. Netto's arguments require some substitution group theory, but not a significant amount, to make them carry through.

A hint of how to deal with groups of order pq was given by Sylow [1872, 589–590], when he noted that if the order of a group is $n^\alpha m$, where n is prime and $m < n$, then the subgroup of order n^α is normal. He formulated the general result of which this is a special case in Theorem V.

The study of groups of prime power order, begun by Sylow in 1872, was ignored for the following twenty years, with the exception of a few special cases. Netto examined what are nowadays called Sylow subgroups of the symmetric groups in his book [1882, 40–42/1892, 41–43]. He proved that these subgroups exist without invoking Sylow's Theorems.

(4) Cayley showed, albeit without giving a formal proof, that any finite group can be represented as a group of substitutions in [1878, 52/403]; the result was prefigured in his earlier studies in group theory [1854a, 41/124]. Therefore the classification of all (abstract) finite groups and all finite groups of substitutions amounts essentially to the same problem. Thus the substantial collection of theorems concerning substitution groups brought important contributions to the 'general problem of groups'. However, Cayley also remarked that

... it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions. [1878, 52/403]

His words went largely unheeded before the 1890s, when they were finally acted upon simultaneously by mathematicians in Europe and America. Cayley himself had made a start during the 1850s on the classification of groups by order; he reached order eight [1854a & 1859]. He held a more abstract view of the group concept than as simply a group of substitutions. Later, in 1886, Kempe investigated 'Groups containing from 1 to 12 units' in a 'Memoir on the theory of mathematical form'. Three years afterwards, Cayley corrected Kempe's enumeration for order 12 and re-introduced a pictorial method of representing a group, now known as the Cayley graph. This was as far as the enumeration of finite groups had reached at the beginning of the 1890s.

Useful tools

Nearly without exception, those interested in classifying groups began their papers by stating that Sylow's Theorems were the most important aids in their research. As the theory of groups was approached in an ever more abstract way, it was realised that Sylow's Theorems are essentially theorems not of substitution group theory but of abstract group theory. The theorems and their proofs were re-formulated accordingly.

Various other helpful techniques were available by the 1890s. The concept of the direct product of two groups gradually crystallised out of the work of Jordan, Dyck, Netto and others, who recognised and used the idea without giving a precise definition. Such a definition (and the name) were provided by Hölder in 1889. Semidirect products had appeared by accident and in very confusing terms in Betti's research, and in the work of Kirkman in the 1860s, but his work was completely uninfluential.

The 1890s: Otto Hölder enters the scene

The interest which arose in the 1890s in the structure of finite groups had its first climax in 1893; three substantial papers on the subject appeared during the course of that year. Otto Hölder in Tübingen and Cole & Glover at Ann Arbor, Michigan published their respective classifications of groups of orders pq^2 , pqr , p^3 and p^4 where p , q and r are distinct primes⁷. Young at Clark University gave the first systematic discussion of p -group structure since Sylow's paper of 1872 and classified groups of orders p^3 and p^4 .

⁷Only Hölder dealt with order p^4 .

Hölder, Cole & Glover and Young all attacked their tasks as problems of *abstract* group theory. Therefore the problems were viewed at last in the most sensible and natural way. That is, the number and type of groups of a given order n depend on the prime decomposition of n and on relationships between the various primes in the decomposition. Thus the structure of groups whose orders have particularly straightforward prime decompositions was naturally investigated first. Once suitable methods had been found to analyse such groups, the methods might be generalised to deal with more complicated cases and give insights into the structure problem in general.

It would perhaps be easier to gain an appreciation of Hölder's contribution to the structure problem, and especially of the deep and extensive insights he had into the theory of groups as a whole, by comparing his work on structure theory with that of the Americans mentioned above. It seems that Cole & Glover, Young and Hölder had worked independently of one another. It is nevertheless worthwhile and interesting to ask why they all took up the same problem in similar ways simultaneously. Was there a common source of inspiration and motivation? Or was there in fact some more direct influence of one mathematician on another?

Later [1895a, 322–323] Hölder put the record straight about the chronology of the three papers and the possibility of plagiarism. His paper, he pointed out, had been sent to the editors of the *Mathematische Annalen* before he first saw the other articles as they appeared in successive volumes of the *American Journal of Mathematics*. Therefore it is unlikely that Hölder had knowingly borrowed ideas from Cole & Glover and Young. It is equally improbable that he had directly influenced the Americans, since he seems to have worked alone on each problem he tackled until he reached a satisfactory conclusion. In addition, it was the case at this time that the German mathematical community regarded the growing mathematics departments at American universities with some suspicion and was inclined not to treat them too seriously.[⊗] Of course it is impossible to say to what extent Otto Hölder held this view.

Other hypotheses may be put forward in order to explain the coincidences of research. The structure theory problem may have been 'in the mathematical air' at the time and simply required someone's concentrated attention. Hölder and the Americans may all have been influenced by the same earlier work or by ideas from a particular mathematician. The latter hypothesis seems most likely. The influence of Felix Klein through his

⊗ The actual relationship between German and American mathematics departments was far more complex than this. A number of U.S. students spent time in Germany, and a number of German mathematicians, accepted professorships in the U.S.

lectures, informal comments and in conversation may well have turned the thoughts of all these mathematicians to similar questions. During the 1880s both Cole and Young had made their way to Germany in order to study.

Frank Nelson Cole was two years Hölder's junior. After a year as a graduate student spent at Harvard, he had been awarded a travelling fellowship for the period 1883–1885 in order to broaden his mathematical knowledge. The fellowship provided an opportunity to visit Leipzig, where he became a member of Klein's mathematical seminar. It is just possible that Cole met Otto Hölder during his stay in Germany. He would certainly have become aware of areas of current interest in mathematics and of Klein's *Icosahedron*, published in 1884. Cole's attitude on returning to Harvard in 1885 is reported in [Fiske 1927, 773]; he 'was aglow with the enthusiasm which Felix Klein inspired in his students.' Soon afterwards he wrote a long review [1887] of *Icosahedron* and began work on a translation of Netto's *Substitutionentheorie* [1882]. This English edition was published at precisely the time when Cole was engaged in research with Glover on groups whose orders are divisible by three prime numbers. The appearance of a group theory text translated into English provided a great deal of impetus to study the subject in the United States and the United Kingdom. It is very likely that Cole did not lose contact after 1885 with German mathematicians and with their latest ideas. In the summer of 1888 he returned to Germany to marry Martha Marie Streiff from Hamburg; he took his bride back to live in the United States.

Jacob William Albert Young studied in Berlin during 1888 and 1889. He must have come into contact with Weierstraß, Kronecker and Netto and many other prominent mathematicians. He would have acquired something of the spirit of the 'Berlin school', as Hölder had done in the 1870s. He returned to the States in 1889 and continued his studies for the PhD as a Fellow at Clark University, Worcester, under the supervision of Oskar Bolza. Young's published paper on p -groups is essentially his PhD thesis which he completed in 1892. The subject of his thesis may have been suggested to him in Germany or by Bolza in the United States. Young wrote [1893, 125]:

I take this opportunity to express my obligation and warmest thanks to Dr. Oskar Bolza, who, throughout the entire course of the investigation, very kindly and materially assisted me with suggestions and counsel ...

Bolza had in his turn obtained his PhD at Göttingen in 1886 under Klein's supervision. Bolza must have known Otto Hölder, who was then a *Privatdozent* there, although

Hölder was not yet researching into group theory. However, both Bolza and Hölder had the opportunity to hear Klein lecturing on and discussing group theory and Galois theory.

The most plausible answer to the question of motivation and inspiration is that, at least in the cases of Hölder and Cole, their interest in the structure theory of finite groups followed from the work on simple groups in which they had both been engaged. Hölder had proved [1892] that all groups whose orders are divisible by at most three (not necessarily distinct) primes are soluble. He had not, however, produced any insights into the exact structure of such a group. The fact that a normal subgroup always exists provided a natural reason to consider the way in which such groups are built up from their normal subgroups. Cole had not been able to complete his survey of simple groups up to order 500 when his first paper on the subject was published in June 1892. He must have realised that an investigation of the structure of groups with straightforward prime decompositions would help him to understand better the structure of simple groups. Only three months passed before he and Glover completed their paper, but a further eight months went by before he managed to solve the simple groups problem up to order 660.

Hölder versus Cole & Glover and Young

What was the particular contribution to group structure theory made by Otto Hölder? What was different about his work that makes it especially worthy of note? There are four factors which must surely be significant:

- (1) His abstract approach;
- (2) The new concepts he introduced and methods he developed;
- (3) His exact classifications, rigorously proved;
- (4) His very clear and broad overall view of the problem.

The significance of these four points becomes apparent as we look at Hölder's paper, 'Die Gruppen der Ordnungen p^3 , pq^2 , pqr , p^4 ' [1893], in more detail and relate it to the studies of Cole, Glover and Young. In the introduction to his article [1893, 301] Hölder stressed once more that the structure problem would be considered within an abstract framework. He was at pains to make this clear—the operations of the group are 'lediglich Symbole' ('merely symbols') for which a law of combination is defined,

and are no longer permutations, rotations or collineations (that is, automorphisms of a (projective) space that preserve the lines of the space). Hölder scrupulously avoided using results about substitution groups, in contrast to his earlier approach to simple groups.

Cole & Glover [1893, 191] similarly laid emphasis on 'regarding the group from its purely formal side, as a group in the abstract'. They attributed the fact that the structure theory of groups had received so little attention to another fact—that the importance of such a theory is not as paramount in substitution group theory, in which groups are primarily classified according to degree and not according to order. They noted the theorem proved by Hölder in [1892], ensuring the existence of a normal subgroup in every group that would appear in their paper. Their aim was to present the structure of all possible types of group whose order is divisible by at most three primes, corresponding exactly to Hölder's result that all such groups are soluble.

Young mentioned briefly in his introductory remarks that his approach to the problem of p -groups would be abstract. He had found a method for assigning a 'type' to each group of order p^k . In modern terms, two p -groups are of the same type if the quotient groups in their upper central series are the same. This classification into types, claimed Young, enables 'the question of the identity of two groups' to be reduced 'to that of "equivalence" of two sets of integers'. We shall see later what Young had in mind.

In all three articles, Netto's results for (substitution) groups of orders p^2 and pq , where p and q are distinct primes, were noted in the introduction. These results were later re-established by both Hölder and Cole & Glover without using substitution group theory because they wished their papers to be self-contained. They were also keen to show the usefulness of various methods and theorems.

The easiest and most readily understandable way to write down a classification of groups with orders p^3 , pq^2 and so on is to use generators and relations. Hölder remarked [1893, 302] that he had in most cases made use of them. Cole & Glover, however, did not seem to appreciate this labour-saving device. They did in effect give presentations of groups in this way, but the conciseness of the method was lost. Young discussed Dyck's 'generators and relations' paper [1882] at length but, as Cole & Glover, did not express the groups he discovered as clearly as he could have done.

A task undertaken by Hölder but not attempted by Cole & Glover was the develop-

ment of the actual *theory* by means of which the structure of groups is investigated. Hölder drew attention to this theory at the beginning of his paper and pointed out two paragraphs in his article [1893, §§18 & 20] of which he wanted his readers to take especial note. Here we find the first clear explanation of the formation of semidirect products and of group extensions by a cyclic group. In fact, Hölder devoted a large section of his paper to the consideration and creation of general methods for the study of group structure. The paper splits into five sections:

- (1) Groups of orders p^2 and pq ;
- (2) General aids for the construction of groups;
- (3) Groups of order pq^2 ;
- (4) Groups of order pqr ;
- (5) Groups of orders p^3 and p^4 .

There are various digressions within the article: if a new idea needed to be explained or expanded, or was significant or merely interesting, Hölder simply inserted an extra paragraph about it. Nevertheless, the reader does not feel that Hölder lost sight of his goal. His digressions serve to make the general outlines of structure theory more clearly discernible.

Introduction to Hölder's method

Hölder's approach to the classification of groups was to use various of the theorems proved by Sylow in order to find normal subgroups. The possible normal structures he discovered for a particular group order enabled the possible groups of that order to be put into categories. Then the group is reconstructed from its normal subgroups and expressed by means of generators and relations with some indices undetermined. The group multiplication is investigated to find conditions on the unknown indices. In modern terms, the conditions in most cases enable the group to be written as a semidirect product $G = H\Gamma$ that does not decompose into a direct product, that is, in which there is a nontrivial action of the subgroup H on the normal subgroup Γ by conjugation. Finally, the conditions on the indices are imposed and the resulting generators and relations are studied—there may be a contradiction, in which case no such group exists, or the group may not have the required order. If there is a group or groups of the correct order then they are investigated further to determine if they are

abelian or are direct products of subgroups or have other easily recognisable properties.

The important step in this procedure in the case of finding all groups of a given order is the calculation of number-theoretic conditions on the unknowns. Hölder was very careful to do this accurately and comprehensively, and consequently he managed to give correct classifications for all the group orders he considered. This cannot be said of Cole & Glover, however, whose paper contains several errors because they did not check their results as thoroughly as could have been wished. Hölder remarked on this [1895a, 323] whilst justifying the appearance of his paper so soon after Cole & Glover's.

When the general structure theory of groups is under consideration, the important steps in Hölder's method become the search for normal subgroups and the reconstruction of a group from its normal subgroups. Hölder devoted some space in his article to an exploration of these tasks. That of finding normal subgroups turned out to be easy for the group orders he had chosen. He simply invoked Sylow's Theorems in each case and discovered that there was a unique (and therefore normal) Sylow subgroup corresponding to at least one of the primes dividing the order of the group. The other problem, that of extending normal subgroups to form larger groups, did not prove as easy to solve. Hölder recognised certain recurring patterns, which he distilled in §§18 and 20, as mentioned above. However, beyond this, Hölder realised that there was a general process at work which required a new theory to deal with it. He began to explore the beginnings of the theory in 1893 but only gave it his full attention after *Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4* was published. This group extension theory forms the subject of Chapter 9 of this thesis.

Groups of orders p^2 and pq

If a group G of order p^2 does not contain an element of order p^2 , then it is the group $C_p \times C_p$, the direct product of two cyclic groups of order p . Hölder and Cole & Glover proved this, in similar manners, by applying a theorem of Sylow in two different ways, both of which ensure the existence of a normal subgroup. Hölder [1893, 302] applied the result: a p -group has a nontrivial centre [Sylow 1872, 588] to obtain an element Θ in the centre of G . Then if Θ_1 is any element of G that is not a power of Θ , these relations hold:

$$\Theta^p = 1, \Theta_1^p = 1, \Theta_1^{-1}\Theta\Theta_1 = \Theta.$$

Hölder showed that the group consists of the p^2 elements $\Theta_1^\alpha \Theta^\beta$ ($\alpha, \beta = 0, 1, \dots, p-1$) and gave the rule for multiplying elements together. This method of writing generators and relations for the group echoes the power commutator presentation which Sylow introduced (see earlier in this chapter).

Cole & Glover [1893, 192] applied the result: a group of order p^a always contains a normal subgroup of order p^{a-1} . They claimed that this had been proved by Sylow, but in fact it is only implicit in his paper [1872]. The existence of a subgroup of order p^{a-1} in a group of order p^a follows from the power commutator presentation for a p -group that Sylow introduced. It also follows from this presentation that the normaliser of such a subgroup of order p^{a-1} is strictly larger than the subgroup, and must therefore equal the whole group. Thus, any subgroup of order p^{a-1} is normal.

This result enables generators s and t for the group G to be found with relations

$$s^p = 1, \quad t^p = 1, \quad t^{-1}st = s^\mu$$

and it must then be proved that $\mu = 1$. Therefore Hölder's method turns out to be the most concise. Although the case of groups of order p^2 is not particularly demanding, the approaches used here give an introduction to the method to be employed in the investigation of more complicated groups.

The possibility of a noncyclic group G of order pq ($p > q$) is considered by Hölder and Cole & Glover in the same way: Sylow's Theorem provides a normal subgroup of order p , generated by T . In fact Sylow's Theorem is a more sophisticated tool than is needed here. Cauchy's Theorem provides an element T of order p . If there were two different subgroups X_1, X_2 of order p , then $X_1 \cap X_2 = \{1\}$. Hence $|X_1 X_2| = p^2 > pq$, which is impossible. So $\langle T \rangle$, the subgroup generated by T , is the unique and therefore normal subgroup of order p .

Then if S is an element of order q , we know that $S^{-1}TS = T^a$ holds for some a . If $a = 1$, then S and T commute and the group becomes $C_p \times C_q = C_{pq}$. Therefore the problem is to discover when other values of a are permissible and what these values are. In other words, when is G a semidirect product of $\langle T \rangle$ by $\langle S \rangle$ with a nontrivial homomorphism $\phi : \langle S \rangle \longrightarrow \text{Aut}(\langle T \rangle)$? Both Hölder and Cole & Glover showed that $a^q \equiv 1 \pmod{p}$ is a necessary condition on the value of a . This congruence only has roots other than 1 when q divides $p-1$ (since the condition $a^q \equiv 1 \pmod{p}$ means that a has

order q as an element of the multiplicative group $\mathbb{Z}_p^*$, and the group has order $p - 1$ (which the order of any element must divide). Cole & Glover went on to prove that any root of the congruence not equal to 1 produces the same group, which may be written

$$T^p = 1, S^q = 1, S^{-1}TS = T^a,$$

where $a^q \equiv 1 \pmod{p}$ and $a \neq 1$ (here, Cole & Glover's parameter ρ has been replaced by Hölder's parameter a).

Hölder also proved this eventually, but first established some more theory. He was careful to note that, at this stage, one does not know whether groups of the above type exist, and it must not be assumed that they do.

The general idea was to find out what form the multiplication in the group G must take. Hölder showed that the group consists of the pq elements $S^\alpha T^\beta$ ($\alpha = 0, 1, \dots, q-1$; $\beta = 0, 1, \dots, p-1$) and that multiplication is given by

$$(S^\alpha T^\beta)(S^\gamma T^\delta) = S^{\alpha+\gamma} T^{\beta a^\gamma + \delta}, \quad (3)$$

where exponents of S and T are calculated mod q and mod p , respectively⁸.

Now one is to think of the $S^\alpha T^\beta$ not as products of the elements S^α and T^β but simply as abstract 'symbols' representing pq different objects, with the above rule of combination of symbols. (In this way Hölder ensured that he did not proceed from the assumption that the groups he wished to classify already existed.) He formulated a new question: when do the rules of group theory hold for these objects under the given method of combination?

Hölder discussed this question [1893, 305–307] and listed two properties that are required:

- (1) The multiplication is associative;
- (2) If A, B and C are objects, then $A \neq B$ implies $AC \neq BC$ and $CA \neq CB$.

These two properties imply (for a finite set of objects) that there is an identity object and that each object has an inverse.

Next he imposed the conditions (1) and (2) on the multiplication defined by (3). The associative condition forces $a^q \equiv 1 \pmod{p}$, which we already knew to be the case;

⁸The numbering of equations follows that in Hölder's article [1893].

conversely, this congruence implies associativity. Therefore property (2) is necessary and sufficient to ensure that the multiplication (3) is group-theoretic; the property holds when a is not divisible by p , which must certainly be true if the above congruence is satisfied. Then a group of order pq has been found, with identity S^0T^0 and with two operations $S^1T^0 := A$ and $S^0T^1 := B$ which generate the group and satisfy

$$A^q = 1, A^{-1}BA = B^a, B^p = 1. \quad (6)$$

There relations are (fortunately) just the same as those discovered by Cole & Glover: the elements S and T have become A and B and the second and third equations have been interchanged. On balance, Cole & Glover's treatment of groups of order pq is better than Hölder's has been so far, especially in terms of readability, even though they were somewhat cavalier about the existence question, which Hölder most certainly was not.

Hölder followed this result with another digression to explain the concept of 'defining relations' introduced by Dyck [1882 & 1883]. Hölder dealt with them in a completely formal way, showing how such relations define (in modern terms) an equivalence relation on the generating elements of the group. These elements under the 'equivalence' generate a finite or infinite group; in modern terms it is a quotient group of the free group on the generating elements. Hölder explained how to write group elements in a 'normal form' if the relations are of a particular kind. So, for example, any element of the group defined by the equations (6) may be expressed as $A^\alpha B^\beta$ ($\alpha = 0, 1, \dots, q-1$; $\beta = 0, 1, \dots, p-1$). This can be achieved because B and A generate a normal series for the group. It follows from the existence of such a normal series that an upper bound may be given for the order of the group.

Hölder at last returned [1893, 312] to groups of order pq . His digressions had shown that all groups of this order are found by substituting the roots of the congruence $a^q \equiv 1 \pmod{p}$ into equations (6). If $a = 1$ a cyclic group is produced. Hölder showed that the other roots of the congruence lead to only one group, up to isomorphism. Thus one obtains two different groups in total. It took Hölder ten pages to reach this conclusion.

Innovations

The second section of Hölder's paper introduced the tools he would later need to

use. Much more significantly, new concepts which form the basis of group extension theory are presented here: automorphisms, factor sets and so on. There is nothing at all to correspond to this development of theory in Cole & Glover's article. They limited themselves at the beginning of their article to the special task of classifying groups of order pq . Hölder, on the other hand, evolved general methods out of those which he had used for this special task. In so doing, he opened up an enormously important area of group theory—his contributions to group extension theory are considered in more detail in the following chapter.

Hölder had realised that in order to construct a group G from a group Γ that is to be a normal subgroup of G , one requires that the way in which the elements of G are to conjugate the elements of Γ be specified. The substitution of the elements of Γ that takes each such element to its conjugate by an element S in G is an *Isomorphismus der Gruppe Γ in sich* (*isomorphism of the group Γ into itself*); it is nowadays called an automorphism of Γ . Hölder then defined an automorphism of a group Γ without reference to any larger group G in which Γ is contained, and he showed that the set of automorphisms of a group Γ also forms a group. He defined *cogredient* (*inner*) and *contragredient* (*outer*) automorphisms.

He returned to the case of a group G with normal subgroup $\Gamma := \{T_0, T_1, \dots, T_{n-1}\}$ and with a set of coset representatives $\{S_0, S_1, \dots, S_{m-1}\}$ (now called a *transversal*) for Γ in G . He investigated how multiplication in G is defined and realised that it depends on certain functions: one of these (in Hölder's notation, $\psi(a, b)$) is now called a *factor set*, another (denoted $\omega(\alpha, a)$) fixes the way that the coset representatives act by conjugation on elements of Γ to produce automorphisms of Γ . He then considered the multiplication defined in this manner as an abstract method of combination of symbols, as he had done before, and found the conditions that the functions (by which it is defined) must satisfy in order that the multiplication is group-theoretic.

Modern tradition in group theory ascribes these concepts and this analysis to Schreier, whose results appeared some thirty years after Hölder's. A comparison of the achievements of Hölder and Schreier is given at the end of Chapter 9.

Semidirect products and extensions by a cyclic group

‘Die entwickelten Principien führen über das Ziel dieser Abhandlung hinaus. Ich

hebe hier nur zwei besondere Verfahren hervor' [1893, 329]. ('The principles that have been developed are leading beyond the object of this paper. I shall only point out two particular methods here.') Thus Hölder introduced his 'particular methods', which he explained in sections 18 and 20 of his article.

The first method of constructing a group becomes possible when the coset representatives S_a themselves form a group. This is achieved by specifying that $\psi(a, b) = 0$ for all a, b . Then the group of the S_a is isomorphic to $H := G/\Gamma$ and G becomes a *semidirect product* (the term is not Hölder's). Hölder noted that it is not always possible to form a semidirect product with two given groups: the process of construction might degenerate into the direct product of the groups.

Cole & Glover recognised the distinguishing property of a semidirect product:

... the characteristic feature of the not cyclical group of order pq is that it contains one subgroup of one order and a system of conjugate subgroups of another order, the latter being joined by transformation with respect to the operations of the former. It will at once suggest itself that we may, by way of trial, replace the subgroup of order p by a group of a higher type, for example, p^2 , at the same time properly increasing the number of subgroups of order q , or replacing these also by a proper number of groups of a higher type. [1893, 195–196]

However, they did nothing to make the idea precise or to understand how it can be used systematically.

The second method of group construction applies to the case when G/Γ is cyclic, of order m ; Hölder calculated the forms that are then taken by the defining relations for G . He noted that, given a group Γ which has an automorphism whose m^{th} power is an inner automorphism, one may construct a group G with $\Gamma \trianglelefteq G$ and $G/\Gamma \cong C_m$. This latter procedure proved to be the most useful in the remainder of Hölder's paper.

Groups of order pq^2

The groups of this order which are most easily recognised are the two abelian groups, C_{pq^2} and $C_{pq} \times C_q$. Hölder wrote these as $[p][q^2]$ and $[p][q][q]$, so that $[m]$ denotes the cyclic group of order m and $[m][n]$ denotes the direct product of the two groups. A group which can be written as the direct product of two groups of smaller order is called *zerfallend* (*decomposable*) [1893, 335].

The abelian groups form a 'first category' of groups of order pq^2 . Hölder distinguished other categories by using Sylow's Theorem. Any nonabelian group G of order pq^2 must

lie in one and only one of the following categories:

- (2) G has a normal Sylow q -subgroup isomorphic to C_{q^2} ;
- (3) G has a normal Sylow q -subgroup isomorphic to $C_q \times C_q$ and a normal subgroup of order q ;
- (4) G has a normal Sylow q -subgroup isomorphic to $C_q \times C_q$ but no normal subgroup of order q ;
- (5) G has p Sylow q -subgroups isomorphic to C_{q^2} and a normal Sylow p -subgroup;
- (6) G has p Sylow q -subgroups isomorphic to $C_q \times C_q$ and a normal Sylow p -subgroup.

Each category was investigated by writing down generating elements and relations, in which various indices might still be unknown, for the possible groups in the category. Hölder calculated the values that the indices could in fact take in order that a group of the required order be constructed. There are basically two types of group, depending upon which prime number corresponds to a normal Sylow subgroup. All the possible groups may be formed as semidirect products; this in general implies that one of the conditions $q \equiv 1 \pmod{p}$, $q \equiv -1 \pmod{p}$ or $p \equiv 1 \pmod{q}$ holds, in order that the semidirect product does not degenerate into a direct product.

There is one group (up to isomorphism) in each of categories (2) and (4), and there are $(p+3)/2$ groups in category (3) if p is an odd prime and two groups if $p=2$. Category (6) always contains just one group; category (5) contains one if $p \equiv 1 \pmod{q}$ but $p \not\equiv 1 \pmod{q^2}$, and it contains two if $p \equiv 1 \pmod{q^2}$.

Cole & Glover assumed throughout their paper that $p > q$ and so they discussed groups of order pq^2 in two parts: first, groups of order p^2q and second, groups of order pq^2 . Within each part they distinguished categories into which the possible groups would fall, simply on the basis of all the possibilities for the number of Sylow p - and q -subgroups. They had six cases for p^2q and four for pq^2 . Hölder had at least done some work at this preliminary stage to establish the categories more precisely.

The method Cole & Glover used to find groups is superficially the same as Hölder's: they produced generators and relations for each case and worked out the additional conditions which allow values to be given for various unknown indices. In each case, they provided tables that show how many elements of each order the group contains, and they calculated the number of subgroups and normal subgroups of each order. They also produced a substitution group (of the lowest permissible order, for example, order

18 for p^2q) that is isomorphic to each type of group. However, what Cole & Glover gained in brevity, they lost in rigour and in a deeper understanding of the theory. They never attempted to prove that the groups they obtained were of the required order, or to give insights into the processes by which the groups were formed.

Groups of order pqr

Fortunately for us, both Hölder and Cole & Glover assumed that $p > q > r$. After dealing with the one possible abelian group, Hölder applied Sylow's Theorem and investigated further to discover that *every group of order pqr in which p is the largest prime has a normal subgroup of order p* [1893, 364]⁹. He then established two categories for the nonabelian groups:

- (2) The group of order pqr contains an element of order qr ;
- (3) The group does not contain an element of order qr .

Cole & Glover similarly employed Sylow's Theorem to discover that a group G of order pqr has only one subgroup of order p (which, since it is unique, must be normal, but they did not mention this). They used this fact to prove that G has a normal subgroup of order pq . Their conception of how to obtain the structure of a group involved working from the whole group 'down' to the identity and consequently they wished to find a *large* normal subgroup, whereas Hölder began at the identity and worked 'upwards'. In Cole & Glover's approach there were two cases: the subgroup H of order pq is or is not cyclic.

The methods used to discover what are the possible groups were the same as before, resulting in a presentation of each group by generators and relations. Hölder's category (2) gives rise to three distinct groups if $p \equiv 1 \pmod{qr}$ but only one group if just one of the congruences $p \equiv 1 \pmod{q}$, $p \equiv 1 \pmod{r}$ holds. There are no groups otherwise. Category (3) produces groups only if $q \equiv 1 \pmod{r}$. If in addition $p \equiv 1 \pmod{r}$ then there are r distinct groups, of which one is decomposable. If this is not the case, there is just one group, which decomposes. Cole & Glover made a mistake here: their Case A.4 should produce the $(r-1)$ nondecomposable groups [1893, 217–218], but they seemed to think that there is only one. They omitted to check that there are $(r-1)$ essentially different actions by conjugation possible on the normal subgroup of order pq .

⁹ ... jede Gruppe von der Ordnung pqr enthält eine einzige und deshalb ausgezeichnete Untergruppe von der Ordnung p , wenn p die grösste der Primzahlen p, q, r bedeutet (italics Hölder's).

The structure of p -groups

The final group orders considered by Hölder were p^3 and p^4 . Cole & Glover dealt only with p^3 . The difference in depth of insight into group theory between the two papers is most clearly noticeable in their respective treatments of p -groups. Hölder had realised that special techniques were applicable in the study of p -groups, and accordingly treated their classification in a different way. Cole & Glover simply applied the same method to groups of order p^3 as they had done to the groups of other orders. They concluded from Sylow's work that a group G of order p^3 always has a normal subgroup of order p^2 . They proved that if G contains an element of order p^2 , then it contains a normal cyclic subgroup of order p^2 . Thus there are two classes of groups here:

- (A) Groups containing an element of order p^2 ;
- (B) Groups not containing an element of order p^2 .

For both classes of groups, they considered how (in modern terms) the elements of G act by conjugation on the normal subgroup of order p^2 . They correctly discovered three abelian and two nonabelian groups for any prime number p ; the nonabelian groups must be discussed separately for $p = 2$.

Hölder found these results in another way, and here his approach was paralleled by that of Young [1893]. I consider Hölder's work first. He applied to a p -group G the theorem that a group of prime power order has a nontrivial element Θ in its centre. One may assume that Θ has order p (if it does not, it can be replaced by one of its powers which does have order p). Then in the quotient group $G/\langle\Theta\rangle$ there is also a nontrivial central element of order p ; let S be an element of G corresponding to it. Therefore $S^p = \Theta^a$ for some a , and $V^{-1}SV = S\Theta^\alpha$ for some α and all $V \in G$. By continuing in this way (with the quotient group $G/\langle\Theta, S\rangle$ and so on) one obtains a series of elements which generate G . This is the power commutator presentation that Sylow had discovered. It enables the generators and relations for a p -group to be written in a particularly convenient way.

Hölder wrote down such generators and relations for groups of orders p^3 and p^4 [1893, 373], and afterwards calculated the possible values of the unknown indices a , α etc. Some preliminary investigations for the group order p^4 established various number-theoretic conditions on the unknowns, which must be satisfied in order that a group exists [1893, 376–381]. By studying the conditions in more detail they can be reduced

to essentially two options [1893, 384–387]. When each of the two options is substituted into the defining relations, three categories of group G of order p^4 are produced besides category (1) containing the five abelian groups [1893, 387–392]:

- (2) The centre $Z(G)$ is isomorphic to $C_p \times C_p$; the quotient group $G/Z(G)$ is also isomorphic to $C_p \times C_p$;
- (3) The centre $Z(G)$ is isomorphic to C_{p^2} ; the quotient group $G/Z(G)$ is isomorphic to $C_p \times C_p$;
- (4) The centre $Z(G)$ is isomorphic to C_p ; the quotient group $G/Z(G)$ is isomorphic to a semidirect product of C_p by $C_p \times C_p$.

Again, the case $p = 2$ must be treated separately.

Hölder calculated for category (2) the possible structure of the commutator and p^{th} power subgroups of G . The commutator subgroup H has order p , the p^{th} power subgroup K may have order 1, p or p^2 . He pointed out that unless two given groups have commutator and p^{th} power subgroups that are, respectively, identical, the groups cannot be isomorphic. Therefore the four types of group obtained with all possible choices for H and K must be different from each other. Hölder went on to show that in fact there is just one group of each type. He proved separately that $p = 2$ also provides four groups. Category (3) required less than two pages [1893, 397–398]: there are two groups here, for both $p = 2$ and p odd. One of them has the property that it cannot be generated by any two of its elements, as Hölder showed.

The defining relations for category (4) are the most complicated. Hölder worked out the number-theoretic conditions which imply that two groups in (4) are isomorphic [1893, 401–405]. He used these to find four nonisomorphic groups for $p > 3$; there are also four groups for $p = 3$ but they required special treatment. The groups of order 16 in this category were the last to be found—there are three of them. Therefore in total there are 15 groups of order p^4 when p is an odd prime, and 14 groups when $p = 2$. Hölder's paper concludes with a four-page enumeration of all the groups he had discussed. It suffers somewhat from the lack of a concise notation to express the semidirect product of two groups, although he introduced such a notation for the particular case when both the groups are cyclic.

On the determination of groups whose order is the power of a prime

This is the title of Young's article, and suggests that he wrote from the point of view of one concerned to develop a general theory and then apply it in particular cases. He actually determined the groups of orders p^3 and p^4 , but additionally discovered certain situations in which he could give a presentation for a group of order p^κ . He was greatly helped in being able to concentrate specifically on p -groups and was consequently able to see how their theory is best approached. After presenting the theory and enumerating particular cases, he gave some examples using 'colourgroups' (Cayley graphs) and substitution groups.

He began by reminding his readers of some basic group theory and of Dyck's generators and relations. Then the study of p -groups was introduced by quoting Sylow's theorem giving the power commutator presentation for such a group G of order p^κ . There are many exponents in the presentation which are undetermined; if we assign all possible values to them, we obtain all the groups of order p^κ . However, how does one determine when two such groups are isomorphic? The problem is that there is too large a number of unknown exponents. Therefore, Young created a 'modified classification' which considerably reduced the number of unknowns. This classification is based on the upper central series for the group G , which Young introduced [1893, 129–130], albeit not in the form in which it is expressed today.

Form the centre $\Gamma^{(1)}$ of G ; it has order p^{κ_1} where $\kappa_1 > 0$. Construct the quotient $G^{(1)} := G/\Gamma^{(1)}$, of order $p^{\kappa - \kappa_1}$. 'To the group $G^{(1)}$... the previous reasoning can be applied, and thus by successive steps we pass unambiguously from one quotient group to another, until finally a group $G^{(\mu)} = 1$ is reached' [1893, 130]. Therefore the groups $G^{(i)}$ are 'quotients of quotients of quotients ...'. The Third Isomorphism Theorem tells us that they are in fact simply quotients of G (see below); Young seems to have recognised this is the context provided by his work.

In modern terms, the upper central series of a p -group G is the series

$$\{1\} \leq Z_1(G) \leq Z_2(G) \leq \dots \leq Z_\mu(G) = G$$

where $\frac{Z_{i+1}(G)}{Z_i(G)} = Z\left(\frac{G}{Z_i(G)}\right)$ for $i = 1, 2, \dots, \mu - 1$ and $Z\left(\frac{G}{Z_i(G)}\right)$ denotes the centre of $G/Z_i(G)$. So $Z_1(G)$ is the centre of G and $Z_2(G)$ consists of those elements of G

which are central modulo $Z_1(G)$, and so on. Young's series¹⁰ is related to this series by the following:

$$\Gamma^{(1)} = Z_1(G), \Gamma^{(2)} = \frac{Z_2(G)}{Z_1(G)}, \dots, \Gamma^{(i)} = \frac{Z_i(G)}{Z_{i-1}(G)} = Z\left(\frac{G}{Z_{i-1}(G)}\right), \dots$$

and

$$G^{(1)} = \frac{G}{Z_1(G)}, G^{(2)} = \frac{G/Z_1(G)}{Z_2(G)/Z_1(G)} \cong \frac{G}{Z_2(G)}, \dots, \\ G^{(i)} = \frac{G/Z_{i-1}(G)}{Z_i(G)/Z_{i-1}(G)} \cong \frac{G}{Z_i(G)}, \dots$$

Young noted [1893, 130]: 'If two groups, G_1 and G_2 , are identical, their subgroups $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$... must be identical, and the quotient groups $G_1^{(1)}$ and $G_2^{(1)}$ must also be identical. The converse does not necessarily hold.' Therefore if G_1 and G_2 are to be identical, it is necessary (but not sufficient) that $\Gamma_1^{(i)} = \Gamma_2^{(i)}$ for $i = 1, 2, \dots, \mu$.

Each of the $\Gamma^{(i)}$ is abelian and may therefore be completely defined by a set of integers κ_{ij} giving the orders of elements that generate its cyclic direct factors, as Frobenius and Stickelberger had proved in [1879]. These sets of integers for all the $\Gamma^{(i)}$ must be identical if the groups G_1 and G_2 are isomorphic. This is nevertheless not a sufficient condition for isomorphism, but it provides the basis for Young's classification of p -groups.

We shall say that all the groups leading to the same set of values κ_{ij} constitute a *type*, and denote the type by the symbol

$$(\kappa_{11} \kappa_{12} \dots \kappa_{1l_1})(\kappa_{21} \kappa_{22} \dots \kappa_{2l_2}) \dots (\kappa_{\mu 1} \kappa_{\mu 2} \dots \kappa_{\mu l_\mu}).$$

[The *American Journal of Mathematics* has j here, instead of μ which seems to agree better with earlier notation.] A type is of as many *steps* as there are parentheses in its symbol. The commutative groups are groups of one step. [1893, 132]

Therefore the groups in a *type* are those whose upper central series have identical factors. The number of steps is the length of the upper central series, which in the case of p -groups is nowadays known as the *nilpotency class*.

Young proved one final result before investigating groups of orders p^2 , p^3 and p^4 . It helps to reduce the number of possibilities to be considered. *In a group of more than*

¹⁰Young used notation which is faintly confusing to a modern reader. It is now customary to use Γ_i or γ_i for the i^{th} term of the *lower* central series (the descending central series defined by iterated commutators) and Z_i for the *upper* central series, as here.

one step, the last group $\Gamma^{(\mu)}$ cannot be a cyclic group. In other words, if a group H is such that $H/Z(H)$ is cyclic, then H is abelian.

Young's treatment of groups of orders p^2 , p^3 and p^4 is in the same style as Hölder's, but begins by ordering the possible groups according to type, using the 'final result' above to discount certain cases. He produced generators and relations for each type, proved that the groups were of the required order and determined which values of the exponents in the relations produce isomorphic groups. Finally, the results (which are correct) are summarised [1893, 159–164].

At the end of the paper [1893, 168–178] Young dealt with two other 'types' of group. The first was denoted by $(\kappa)(1\ 1\ \dots\ 1)$, where there are s units in the second parenthesis. This is the p -group G with centre isomorphic to C_{p^κ} such that $G/Z(G)$ is isomorphic to $C_p \times C_p \times \dots \times C_p$ (with s factors). He proved that such a group exists only if s is even, and there are two nonisomorphic groups of this type¹¹. The second type he discussed was $(1)(1)(1\ 1\ \dots\ 1)$ with $s = 2\rho$ units in the last parenthesis, but a little work produced the fact that there are no groups of this type if $\rho > 1$.

The three papers compared

An obvious remark to make is that Hölder's paper is by far the longest of the three. It filled 111 pages of the *Mathematische Annalen*. There was no detail that Hölder chose to omit, which may have inconvenienced the printers but is certainly convenient for any conscientious and patient reader. In studying the paper, one hardly ever needs to supply missing lines of reasoning. Additionally, Hölder included a great deal more calculation with group elements than did the others, which enabled him to present a detailed picture of the structure of the group. This is in part a consequence of his general method. He focused in his calculations on various important concepts such as the normal structure of the group, whereas Cole & Glover chose obvious but, with hindsight, perhaps less significant details to include such as the number of elements of each order.

Hölder was the most thorough in checking every detail and each step of the method, and in proving that groups of each order he considered actually exist, and that the defining relations produce groups of the correct order. This checking is rather tedious

¹¹In modern terms these are the 'amalgamated direct product' of C_{p^κ} with E (amalgamating the centre of E with the subgroup C_p in C_{p^κ}), where E is one of the two 'extraspecial' groups of order p^{1+s} .

but he nevertheless considered it necessary to put into print. It did have the advantage that he made no errors in the calculations!

Along with his eye for detail, Hölder seems to have had a far broader overall view of the problem of classifying groups than Cole & Glover. Similarly, Young had a good general idea of the direction in which the study of p -groups should proceed. Both Hölder and Young were motivated and able enough to develop the theory, in contrast to Cole & Glover who chose to use only known results in a fairly simple way. The advantages of the latter approach are brevity and ease of reading. However, Cole & Glover's paper is not inspiring to read, in the way that Hölder's and Young's papers are. It does not give insights into the formation of group structure, into those constructions that are permissible and those that are not. It simply presents results without explaining any of the ideas on which the results rest. Sadly, it gives no motivation for further thought on the subject.

Hölder had the rare capacity to present new ideas, as they were needed throughout his paper, in a sufficiently polished form that they are immediately understandable. One also sees at once how they relate to the problem under consideration. He did not rush to publish ideas that he had not completely thought through; he had both the ability to create and the sense to organise.

The depth of insight into structure theory possessed by each of the authors was reflected in the way abelian groups were treated in each paper. There is a great difference in the emphasis placed on them. Young and Hölder merely mentioned their existence for each group order, gave a simple notation to describe them and moved on to the nonabelian groups. Cole & Glover, however, did not define a separate category for the abelian groups of each order. Such groups appeared and were investigated along with the nonabelian ones, an investigation which included the enumeration of subgroups, counting of elements of each order and so on. The fact that the abelian groups of a given order are easily recognised did not influence their approach.

Hölder's broad overall view of the problems of structure theory is perhaps the most significant factor which distinguishes his work from that of Cole, Glover and Young. He seems to have recognised general patterns in the structure of groups very clearly. He was able to distil these ideas by proving the Jordan–Hölder Theorem, investigating simple groups and now by exploring extension theory. He had realised that results

for groups with complicated prime decompositions would have to be found by other methods than Sylow's Theorems. Therefore he pointed out the most general way in which such an 'extension theory' of groups could be developed. He only had to describe semidirect products and extensions by a ^{cyclic} ~~quotient~~ group explicitly, since no other types of extension were required in his paper. Nevertheless, the general theory interested him sufficiently that he carried on investigating it and working on particular cases for a further two years, and eventually published his results in [1895a].

Later developments

All the groups considered in this chapter so far have been soluble, and therefore their normal subgroups provided a 'way in' to an investigation of their structure. What happened when the field of study was widened and the group orders under consideration gave rise to nonsoluble groups? This occurred as soon as groups whose orders are divisible by four or more (equal or distinct) primes were studied; the first to tackle them was Frobenius in 1893. He proved that the simple group of order $60 = 2^2 \cdot 3 \cdot 5$ is the only nonsoluble group whose order is divisible by four primes. Burnside continued this investigation in 1894–1895, discovering that there are only three simple groups whose orders are the product of five primes. They have orders 168, 660 and 1,092 and can be written as the projective special linear groups $\text{PSL}(2, 7)$, $\text{PSL}(2, 11)$ and $\text{PSL}(2, 13)$. Frobenius had expressed his belief that this was the case in 1893. The case of six or more primes remained unsolved in the nineteenth century.

The problem for six or more primes is, however, not a group-theoretic but a number-theoretic problem. The group theory stops at the time of these papers in the 1890s. The point is this. The simple group $\text{PSL}(2, p)$ has order $p(p-1)(p+1)/2$ if p is odd. This may perfectly well be a product of six primes. For example, it is so if $p = 2q + 1 = 12r - 1$, where p, q, r are all prime numbers (take $p = 23, q = 11, r = 2$ or $p = 59, q = 29, r = 5, \dots$). The conjecture that there are infinitely many such triads of primes is similar to the twin-prime conjecture. Thus, although only finitely many (though many) simple groups whose order is a product of six primes are 'known', there is every reason to believe that there are actually infinitely many such groups.

In his 'extension theory' paper [1895a, 420], Hölder enumerated all the nonsoluble groups of order less than 480. The accumulated results of Hölder, Young and Cole &

Glover were repeated in a somewhat more concise form by Burnside [1897, 81–89 & 132–137]. He based his notation on Young's and his method and philosophy of investigation on Hölder's, both of whose papers he recommended to his readers.

The enumeration of p -groups was taken up by Bagnera [1898, 137–228]. He attempted to classify groups of order p^5 , but Miller [1899, 246–247/342] was not convinced that he obtained correct results. The order p^6 was studied by Potron in his thesis [1904a]; later [1904b] he published some corrections to his enumeration. In his 'Second report on recent progress in the theory of groups of finite order' [1902], Miller noted the work of Levavasseur [1901] in the enumeration of abstract groups. He had begun to deal with the groups whose orders are divisible by four primes, giving exact classifications rather than looking at general properties as Frobenius had done. Miller mentioned Western [1899] who considered groups of order p^3q . The missing case, groups of order p^2qr , was treated by Glenn [1906, 137].

The concepts of commutators, the commutator subgroup and derived series for a group were explored at the suggestion of Lie by Miller, who showed [1896b, 266/158] that if the derived series of a group reaches the identity after a finite number of steps, then the group is soluble. It was Frobenius who made the most important contributions to the study of soluble groups in the early 1890s. In 1893 he had proved that a group whose order is not divisible by the square of a prime number has a normal subgroup whose order is that of the largest prime divisor of the order of the group. Burnside proved this result at almost the same time. This implies that all such groups are soluble (since every normal subgroup and quotient group also has squarefree order). But was it possible to classify such groups exactly? Otto Hölder showed that it was in [1895b].

The groups of squarefree order

Hölder, working from the results of Frobenius, had found the structure of any group of squarefree order. 'Dabei hat sich das überraschende Resultat ergeben, daß alle diese Gruppen durch metacyklische Buchstabenvertauschungen dargestellt werden können, somit von einer bekannten Form sind' [1895b, 211–212]. ('This has produced the surprising result that all these groups can be represented as metacyclic substitution groups, and thus are of a known form.') Hölder discovered that every group of squarefree order has a cyclic normal subgroup with cyclic quotient group. This must indeed have been

a surprising result at a time when the study of groups was producing more and more complicated structures.

There are basically two methods of dealing with a group G of squarefree order. Either one can work up from the identity to the whole of G using the Fitting subgroup¹², or one begins with G and considers its derived series, which ends with $\{1\}$. As in his earlier paper, Hölder chose to build up the group from $\{1\}$. He applied two theorems proved by Frobenius in 1893:

Theorem 1

If $p_1, p_2, \dots, p_r$ are distinct primes where we take $p_1 > p_2 > \dots > p_r$, then a group of order $p_1 p_2 \dots p_r$ contains one and only one subgroup of order $p_1 p_2 \dots p_\lambda$ for each λ .

Theorem 2

Let $\mathfrak{G} \leq G$ and let the order of $\mathfrak{G}$ be g and the order of G be g . Suppose g and g/g are coprime. Then $\mathfrak{G}$ consists of all elements of G whose order is a divisor of g .

Theorem 2 is not nearly as deep a result as Theorem 1, but it seems not to have been crystallised or explicitly proved before the 1890s. Hölder used these theorems to prove the existence of what is now called the Fitting subgroup of G . Theorem 1 establishes the existence of a normal subgroup of prime order in G . Assume that there are s normal subgroups of prime orders $q_1, q_2, \dots, q_s$, generated by the elements $T_1, T_2, \dots, T_s$, respectively. Then the group $H := \langle T_1, T_2, \dots, T_s \rangle$ is a cyclic normal subgroup of G that contains every other cyclic normal subgroup.

Let $T = T_1 T_2 \dots T_s$. Then T is a generator for H . Hölder proved that no element of G commutes with T apart from its own powers, that is, the centraliser of H in G is H itself [1895b, 214–215].

Now let $|G| := mn$ where $|H| := n$. It is possible to find r elements $S_1, S_2, \dots, S_r$ of G which have orders $p_1, p_2, \dots, p_r$, respectively, where $m = p_1 p_2 \dots p_r$. Hölder proved that each pair of elements S_i and S_j commute mod H , and used this result to show by induction that all the elements S_α commute mod H . Therefore $S' = S_1 S_2 \dots S_r$ mod H is an element of order m which generates the whole of G/H . Hölder had shown [1895b, 218]:

¹²The maximal nilpotent normal subgroup of a finite group; it is unique.

Ist G eine Gruppe, deren Ordnung kein Quadrat enthält, und ist H ihre größte ausgezeichnete zyklische Untergruppe, so ist G/H eine zyklische Gruppe. [1]

A modern proof of this theorem might proceed in the same way as Hölder's until the stage where it has been shown that $C_G(H) = H$, that is, the centraliser in G of H is H . Then we can use the fact that $N_G(H)/C_G(H) \leq \text{Aut}(H)$. This becomes $G/H \leq \text{Aut}(H)$. Since H is cyclic, $\text{Aut}(H)$ is abelian. Therefore G/H is also abelian, it has squarefree order, and so it is cyclic.

Defining relations

A suitable power of S' , denoted by S , has order m since m divides the order of S' . The element S also generates G/H . We therefore obtain these relations:

$$S^m = 1, S^{-1}TS = T^a, T^n = 1. \quad (4)$$

Conjugate T by S repeatedly to obtain the condition

$$a^m \equiv 1 \pmod{n}. \quad (5)$$

No smaller power of a is allowed to be congruent to 1 mod n . The above condition implies that m must be a divisor of $\phi(n)$, that is, each of the primes $p_1, p_2, \dots, p_r$ must divide at least one of $q_1 - 1, q_2 - 1, \dots, q_s - 1$. Hölder went on to prove that (4) are indeed defining relations for the group G .

Such relations show that G can be represented by metacyclic substitutions: take n 'letters' x_ξ where ξ runs through the residues mod n . Then the elements of G may be represented by the mn substitutions

$$\begin{pmatrix} \xi \\ a^h \xi + k \end{pmatrix}$$

where $h = 0, 1, \dots, m-1$; $k = 0, 1, \dots, n-1$.

Next Hölder established a division of groups of squarefree order according to their *Gattung* (*genus*). Two groups with the same values for m and n are said to be of the same genus. Then all groups of a particular genus are obtained on substituting in (4) all values of a for which $a^m \equiv 1 \pmod{n}$. Two such values of a that are powers of each other mod n produce identical groups. Therefore each set of $\phi(m)$ incongruent residues mod n leads to the same group.

The number of groups in a genus is the number of residues $a \pmod{n}$ such that $a^m \equiv 1 \pmod{n}$, divided by $\phi(m)$. Suppose c_α denotes the number of the $q_i - 1$ that have p_α as a factor. Then Hölder proved [1895b, 222] that

Die Anzahl der Gruppen in der durch die Zahlen m und n characterisirten Gattung ist

$$\prod_{\alpha=1}^{\alpha=r} \frac{p_\alpha^{c_\alpha} - 1}{p_\alpha - 1}. \quad [2]$$

More detailed classifications

Each genus of groups can be further divided into *Arten* (*species*). This division depends on the following definition:

Two primes p_α and q_ν are called *conjugirt* (*conjugate*) in the group G if $a_\alpha^{p_\alpha} \equiv 1 \pmod{q_\nu}$. Therefore p_α is conjugate to q_ν when p_α divides $q_\nu - 1$. In order for the group G to exist, each p_α must be conjugate to at least one of the primes $q_1, q_2, \dots, q_s$, as Hölder had noted earlier. This ensures that the action of S_α , the element of order p_α in G , on the normal subgroup H by conjugation is nontrivial. If the action were trivial then S_α would have to be contained in H .

Two groups of the same genus are of the same *species* if identical prime pairs are conjugate in both groups. Hölder represented this symbolically, writing the primes dividing $|G|$ as

$$p_1, p_2, \dots, p_r; q_1, q_2, \dots, q_s$$

and connecting conjugate primes by drawing a line between them.

The next question which arises is: how does one find all groups of a given species? Hölder showed how to do this by presenting an example [1895b, 223–224]. He followed it with an example of the classification of all groups of a given (squarefree) order, and noted the general formulae which give values for the number of species of group in a given genus and the number of groups in a given species.

The paper concludes with an examination of the conditions under which a group of squarefree order is decomposable (the direct product of groups of smaller orders), and with some properties of group elements.

Hölder's classification of groups of squarefree order did not bring the study of such groups to an end. They have been investigated periodically throughout the twentieth

century, in particular by Balash [1966] and Murty & Murty [1984a, 1984b]. These investigations concentrate on obtaining an idea of the number of nonisomorphic groups of any squarefree order. The articles deal essentially with number theory; the initial group-theoretic insights had been provided by Hölder.

Notes

- 1 If G is a group whose order is squarefree and if H is its largest cyclic normal subgroup, then G/H is a cyclic group.
- 2 The number of groups in the genus characterised by the numbers m and n is

$$\prod_{\alpha=1}^{\alpha=r} \frac{p_{\alpha}^{c_{\alpha}} - 1}{p_{\alpha} - 1}.$$

CHAPTER 9

The Beginnings of Group Extension Theory

The extension problem for finite groups may be stated as follows: given two finite groups N and Q , determine all groups G such that $N \trianglelefteq G$ and $G/N \cong Q$. I have chosen the letters N and Q to reflect the fact that N is to be a normal subgroup and Q a quotient group of G . This problem is usually associated with the name of Otto Schreier and perhaps also with those of Baer and Eilenberg & Mac Lane, but not immediately with that of Otto Hölder.

Indeed one sometimes reads, for example in [Zassenhaus 1949, 94], that Schreier's paper of 1926 presents a solution of the extension problem. This is, however, true only in one particular sense. It would be fairer to say that Schreier *set out* the theory, but that the problem remains unsolved, despite much effort in subsequent years, including especially the development of the cohomology of groups.

Without wishing to belittle in any way the important contribution of Schreier, it has to be said that no serious consideration has yet been given to the possibility that the concepts of group extension theory may be traced back before 1926, and even into the nineteenth century. Was Schreier in fact the first to set out elements of the theory as, for example, Marshall Hall asserted [1959, 218]? It is shown in this chapter that much of Schreier's theory had been prefigured in work of Otto Hölder during the 1890s. Schreier [1926a, 165 & 166] mentioned this research of Hölder, of which he was certainly aware even if he had not read it in detail. There is a great deal of similarity in the ideas and approaches to the theory presented in the papers of these two mathematicians.

A question immediately arises on reading a statement of the extension problem: what is meant by 'determining' all the groups G ? Two answers may be given to this question. The first is that the multiplication tables for all possible G should be constructed. This is essentially the task that Schreier's solution (in theory) enables one to perform. The second answer is that the isomorphism classes of groups G should be characterised. This problem is much more difficult and is today unsolved; it is unlikely that a meaningful solution could be given. Nevertheless, partial solutions and insights into general methods

of attack remain worthwhile goals of investigation.

The construction of groups before 1900

It was in fact Otto Hölder who first posed the extension problem in full generality and began a systematic attack on it [1895a], having laid foundations for such a theory in [1893]. However, this does not imply that the process of 'building groups' had not been investigated previously. We have seen various particular cases that were discovered or happened upon: direct products (Dyck), semidirect products (Kirkman). There were intrinsic obstacles which prevented significant progress on the extension problem within the theory of substitution groups or of groups of transformations; a proper appreciation of the problem relies largely on an abstract conception of groups. The abstract approach enables all non-essential information about a group to be discarded and thus provides a clear view of the way in which the elements of two groups may interact to form a larger group.

It is in addition essential to have grasped the concept of quotient group in order to formulate the group extension problem. As has been shown in Chapter 5, Hölder had a firm grasp of the concept by the end of the 1880s. Any sort of understanding of group extensions is very difficult to achieve without this concept. This was precisely the problem that Betti had in 1852 and which led to his accidental discovery of semidirect products. He seems to have wished to explore a more general 'theory of extensions', but was limited by his treatment of group theory as the theory of permutations acting on a prescribed set of objects. The reader is invited to refer to Appendix C for further details of the way in which Betti developed the theory and of the concepts that appear in his work.

Jordan similarly encountered the problem of building larger groups from smaller in his work in the 1860s and 1870s. He searched for a method of constructing all finite soluble groups. Kiernan describes [1971, 121–122] the difficulties such a search entailed with regard to the formation of 'products' of groups. Jordan came upon the same obstacles that Betti had discovered when he attempted to 'combine' the substitutions of two groups.

Klein's *Icosahedron*

It is by now clear that Klein's book *Vorlesungen über das Ikosaeder und die Auflösung*

der Gleichungen vom fünften Grade [1884] exerted a great deal of influence on Hölder in the early years of his research career. Perhaps this was because Hölder studied (if only for a short time) with Klein in Leipzig at precisely the time when the book was being completed and published. *Icosahedron* draws together ideas from algebra, analysis and geometry; the central concept of the book is the 'group of the icosahedron'. Thus Klein was able to include a fair number of comments and suggestions concerning various problems in group theory¹.

He discussed [1884, 8–10/1888, 8–9] the group of rotations of a regular n -gon, which is the cyclic group C_n . He went on to investigate the dihedral groups D_{2n} , formed when reflections as well as rotations of a regular n -gon are allowed. He was quite insistent that his readers should draw diagrams or construct models to help visualise this situation. The investigation concludes with a proof that any given dihedral group is isomorphic to a metacyclic group of substitutions.

Later, Klein included a section on 'The Extended Groups' [1884, 23–24/1888, 24–26]. The group of rotations of a regular n -gon or of a Platonic solid can be *extended* by including reflections. In the former case the group C_n is extended to produce D_{2n} . In every case the extended group contains double the number of operations of the original group. Klein wrote [1884, 24/1888, 25]:

Wir würden jetzt unsere neuen, die *erweiterten* Gruppen, in demselben Sinne gruppentheoretisch untersuchen können, wie wir dies in den vorhergehenden Paragraphen bei den ursprünglichen Gruppen ausgeführt haben. Ich möchte eine solche Discussion dem Leser als eine geeignete Übungsaufgabe empfehlen und beschränke mich hier in dieser Richtung nur auf folgende Angabe: Selbstverständlich ist innerhalb der erweiterten Gruppe die ursprüngliche Gruppe jedesmal ausgezeichnet enthalten. [1]

It seems that Hölder took up Klein's recommendation, whether knowingly or unknowingly. He certainly discussed the extensions of A_5 , the group of the icosahedron, by a cyclic group of order 2 in [1895a]. Some time before, in [1889a], he had recognised the significance of the fact that the original group is a normal subgroup in the extended group.

¹During the first appearance of the group of the icosahedron [1884, 16–19/1888, 17–20], Klein proved that the group is simple and isomorphic to A_5 . The translator G.G. Morrice was not as familiar with the terminology of group theory as he ought to have been, and rendered *einfach* (simple) as 'primitive'.

The awakening of Hölder's interest

How did Hölder become involved in group extension theory? This question has substantially been answered in previous chapters: the construction of groups became necessary to Hölder's research and he therefore devoted a separate study to 'Bildung zusammengesetzter Gruppen' ('The construction of composite groups'). He had realised by the late 1880s that this process required investigation. After stating in the first of his algebraic papers that a group G with a normal subgroup H may be considered as having two factors, the groups G/H and H , he continued [1889a, 33]:

Zu der Aufgabe, eine Gruppe in zwei Factoren zu spalten, kann man sich die umgekehrte stellen: Gegeben zwei Gruppen als Factoren, aus denselben eine Gruppe als Product zusammenzusetzen. Diese Aufgabe lässt manchmal mehrere Lösungen zu, ich hoffe dieselbe bei einer anderen Gelegenheit zu behandeln. [2]

The opportunity was provided in the years that followed and culminated in the treatment of the question in two long articles [1893] and [1895a]. The first of these includes a section in which some general theory for constructing groups is established, and in particular the notion of automorphism is introduced. The second article is largely concerned with the production of specific examples of group extensions and the calculation of automorphism groups, although Hölder did not neglect to investigate and develop the theory further.

The theory of group extensions

At the beginning of [1895a] Hölder gave a clear explanation of the task which was to be undertaken. If one wishes to construct finite groups, he began, one can first determine the simple groups and then build up the composite groups from their simple components. This building process is characterised by a composition series for the group. The Jordan–Hölder Theorem ensures that the set of factor-groups (that is, the composition factors) of a given group is well-defined.

Man kann jetzt durch Umkehrung der soeben angestellten Betrachtung die Aufgabe formulieren, eine Gruppe zu bilden, welche gegebene Factorgruppen hat. Diese Aufgabe lässt vielfach mehrere Lösungen zu. Die Factorgruppen sind stets einfache Gruppen. Es genügt häufig, statt der Factorgruppen die Ordnungen der Factorgruppen, also bloss Zahlen anzugeben. Die Angabe dieser Zahlen leistet dasselbe wie die Angabe der Factorgruppen, wenn man weiss, dass jede dieser Zahlen nur einer einzigen einfachen Gruppe als Ordnung angehört. [3]

In the examples Hölder was to give, he specified the simple factor groups by means of

their order alone where he or others had proved that there was a single simple group of that order. It is conceivable, although not very likely, that L.E. Dickson or Ida May Schottenfels had this passage of Hölder in mind when they considered the question of whether a simple group is determined by its order (and Ida Schottenfels [1900] proved this to be false for order $8!/2$).

The task of forming a group that has given composition factors is then reduced to that of repeatedly solving the problem [1895a, 322]:

... eine Gruppe Δ zu bilden, welche eine gegebene Gruppe Γ auf die Weise ausgezeichnet enthält, dass zugleich Δ/Γ mit einer gegebenen Gruppe übereinstimmt. [4, italics Hölder's]

When this method of group building is applied to a composition series, the group Δ/Γ is always a simple group. Hölder considered various cases in which Δ/Γ was sometimes simple and sometimes composite. At the end of the introduction, he mentioned the cases for which the extension problem had been solved: solutions were given by Cole & Glover, by Young and by Hölder for groups with three composition factors all of prime order, and for groups with four composition factors all of the same prime order. Hölder was to extend this list to include certain groups that have A_5 or $\text{PSL}(2, 7)$, the simple groups of orders 60 and 168, as composition factors.

The mathematics in Hölder's paper of 1895 was introduced by a reference to his earlier paper [1893] in which, Hölder reminded his readers, '... einige zur Lösung dieser Aufgabe dienende Mittel habe ich ... entwickelt' ('I have developed some methods to serve in the solution of this problem'). What exactly were these methods?

Automorphisms

The first concept Hölder had introduced [1893, 313–314] was that of *automorphism*. An automorphism is *ein Isomorphismus einer Gruppe in sich* (an isomorphism of a group into itself). The name 'automorphism' was not used in the nineteenth century. Hölder came upon the concept when investigating how multiplication is defined in a group if a normal subgroup of the group is already known. Let G be a group with normal subgroup $\Gamma := \{T_0, T_1, \dots, T_{n-1}\}$. When one conjugates the operations of Γ by means of $S \in G$, one obtains the operations of Γ again but (possibly) in another order. The substitution σ taking T_i to $S^{-1}T_iS$ has the property that $\sigma(T_iT_j) = \sigma(T_i)\sigma(T_j)$, and it is therefore an isomorphism of Γ into itself. Thus all operations of G produce such an automorphism of Γ .

Now consider the group Γ without reference to any larger group G in which it is contained. In general, the substitution

$$J = \begin{pmatrix} T_0 & T_1 & \dots & T_{n-1} \\ T_{\omega(0)} & T_{\omega(1)} & \dots & T_{\omega(n-1)} \end{pmatrix}$$

represents an automorphism of Γ if, whenever $T_\alpha T_\beta = T_\gamma$, then also $T_{\omega(\alpha)} T_{\omega(\beta)} = T_{\omega(\gamma)}$. Since there may be many such substitutions, one must reckon on the existence of many automorphisms. When two substitutions of this type are composed, they produce another, therefore automorphisms can be ‘multiplied’ together. Hölder had proved [1893, 314]:

Die sämtlichen möglichen Isomorphismen einer Gruppe Γ in sich bilden selbst eine Gruppe. [5, italics Hölder’s]

He noted that the identity automorphism must be included in order for this statement to be true.

The notion of automorphism was independently discovered by E.H. Moore in Chicago and published in a paper of 1895. Moore referred to an automorphism as ‘a holoedric transformation into itself of a given group’. The phrase suggests that Moore considered automorphisms to be substitutions, as indeed he did. According to Moore, the automorphism group of a finite group of order n is a substitution group of degree n (it is actually, as he realised, of degree $n - 1$, since the identity element of the group is always fixed). At the end of his paper Moore wrote [1895, 66]: ‘The idea of the group Γ^{n-1} of holoedric transformation into itself of a given group G_n is, so far as I know, new’. He added a footnote to the effect that he had presented the concept to the Mathematical Club of the University of Chicago in a paper on January 19th 1893. Thus, he had studied the concept at least as early as Hölder.

Moore later discovered Hölder’s article and explained to his readers in [1896], which is a continuation of the earlier paper, that the two men had indeed ‘hit on the notion independently of each other’. In fact Moore had earlier spent a year in Germany (1885–1886) after receiving his doctorate at Yale. He studied in Göttingen for one semester, where he must have made the acquaintance of Otto Hölder (neither were actively interested in algebra at that time). He spent his winter semester in Berlin and came under the influence of Weierstraß and Kronecker, as Hölder had done when a student. Thus the two mathematicians had not had such different mathematical backgrounds as one might have expected. In Moore’s papers it was proved, amongst other things, that the

automorphism group of an elementary abelian p -group of order p^n is the general linear group $GL(n, p)$.

Some years later, Dickson commented on the introduction of the concept of automorphism in [1900]. Having mentioned Moore and Hölder, he felt it necessary to refer to Betti's paper [1855]. Here Betti 'obtains a group equivalent to the homogeneous linear group as the "massimo Multiplicatore" [sic] of a cyclic group of order p^ν '. It was actually the elementary abelian group, which I denote by $\hat{G}$ when considered as a group of substitutions, and not the cyclic group of order p^ν that Betti studied (for details, refer to Appendix C). He discovered that the substitutions of $GL(\nu, p)$ normalise this group, and that the normaliser can be written as the semidirect product of $\hat{G}$ with $GL(\nu, p)$. In fact this last statement is true only because Betti considered $\hat{G}$ as a *regular* substitution group². When this is the case, the normaliser of $\hat{G}$ in the symmetric group is (abstractly) the semidirect product of $\hat{G}$ with $\text{Aut}(\hat{G})$, the *holomorph* of $\hat{G}$. Since $\hat{G}$ is abelian, $\text{Inn}(\hat{G}) = \{1\}$ and substitutions that normalise $\hat{G}$ correspond to elements of $\text{Out}(\hat{G}) \cong \text{Aut}(\hat{G})$. This seems to be the cause of Dickson's confusion: he claimed further that Galois also 'had the conception' of the automorphism group of the elementary abelian group of order p^ν , but Galois, as Betti, had been investigating normalisers.

Although Hölder defined an automorphism originally by means of a substitution, he seems to have had a more abstract conception of the notion than did Moore. Hölder's definition was re-stated by Frobenius [1895a, 184/653]: the substitution *gives rise to* the automorphism—it is not the automorphism itself.

Hölder made the distinction [1893, 314] between what are nowadays called *inner* and *outer* automorphisms. He named them *cogredient* and *contragredient*: a cogredient automorphism of the group Γ is one which arises under conjugation by an element of Γ . All other automorphisms are contragredient. Hölder explained that he had borrowed these terms from Klein, who had defined them in a different and more particular context in connection with the group of the icosahedron. Klein wrote [1884, 232/1888, 257]:

... es handelt sich also in letzter Linie um die Aufgabe, zu untersuchen, auf wie viel verschiedene Weisen man eine Gruppe von 60 Ikosaedersubstitutionen ... holoeidrisch isomorph sich selbst zuordnen kann. Zwei Arten dieser Zuordnung sind durch Cogredienz und Contragredienz gegeben; wir wollen zeigen, dass auf sie alle anderen im Wesentlichen zurückkommen. [6]

²A regular substitution group H acting on a set Ω is one that is transitive on Ω and such that no non-identity substitution of H fixes any point of Ω .

Hölder noted in passing that the group of inner automorphisms is always a normal subgroup of the whole automorphism group³.

A group with a normal subgroup

Now Hölder investigated how the structure of a group G of order mn is built up when a normal subgroup Γ of the group is known. The elements of Γ are denoted by $T_0, T_1, \dots, T_{n-1}$ as before, where T_0 denotes the identity. One can choose a further m elements $S_0, S_1, \dots, S_{m-1}$ of G (coset representatives for the cosets of Γ in G) so that all the elements of G are contained in the table⁴

$$\begin{array}{cccc} S_0 T_0, & S_0 T_1, & \dots & S_0 T_{n-1}, \\ S_1 T_0, & S_1 T_1, & \dots & S_1 T_{n-1}, \\ \vdots & \vdots & \ddots & \vdots \\ S_{m-1} T_0, & S_{m-1} T_1, & \dots & S_{m-1} T_{n-1}. \end{array} \quad (12)$$

Again, S_0 denotes the identity element.

Multiplication in G is determined once it is known how the S and T interact with one another. Hölder proceeded to explore such interactions and to set out the various conditions which must be satisfied in order that G may be a group. The first point he made was that, since the T themselves form a group, they satisfy equations of the form

$$T_\alpha T_\beta = T_{\chi(\alpha, \beta)} \quad (\alpha, \beta = 0, 1, \dots, n-1). \quad (13)$$

Therefore the function χ determines the multiplication in Γ .

Next he considered the form that a product of two of the S must take. The product is one of the entries in the table (12), therefore

$$S_a S_b = S_{\phi(a, b)} T_{\psi(a, b)} \quad (a, b = 0, 1, \dots, m-1). \quad (15)$$

Some conditions which the functions ϕ and ψ must satisfy are given by putting S_a and then S_b equal to the identity. There is a further conclusion that can be drawn, since Γ is a *normal* subgroup. Each horizontal row of (12) is an element of the quotient group G/Γ . Denote the rows by $\Sigma_0, \Sigma_1, \dots, \Sigma_{m-1}$. Then the multiplication $\Sigma_a \Sigma_b = \Sigma_{\phi(a, b)}$ is group-theoretic, and therefore satisfies certain conditions. The function ψ measures

³The term *Automorphismus* and the adjectives *innere* and *äussere* were invented by Frobenius [1901, 1324/204].

⁴The numbering of tables and equations follows that in Hölder's papers.

how close the coset representatives S come to forming a group themselves. If they were to form a group, then $\psi(a, b) = 0$ for all a, b . This function is now known as a *factor set* for the extension.

Conjugate Γ by S_a to obtain the automorphism

$$\begin{pmatrix} T_\alpha \\ S_a^{-1} T_\alpha S_a \end{pmatrix}.$$

Write $S_a^{-1} T_\alpha S_a = T_{\omega(\alpha, a)}$; this can be rewritten as

$$T_\alpha S_a = S_a T_{\omega(\alpha, a)} \quad (18)$$

and thus expresses the interaction of S and T . Put T_α and then S_a equal to the identity to discover two conditions on the function ω .

Hölder denoted the automorphisms of Γ produced under conjugation by $S_a, S_b, S_{\phi(a, b)}$ and $T_{\psi(a, b)}$ by $J_a, J_b, J_{\phi(a, b)}$ and $C_{\psi(a, b)}$, respectively. Then he showed that the automorphisms $J_a \times J_b$ (that is, J_a followed by J_b) and $J_{\phi(a, b)} \times C_{\psi(a, b)}$ are obtained under conjugation by $S_a S_b$ and $S_{\phi(a, b)} T_{\psi(a, b)}$, respectively. Equation (15) implies that

$$J_a J_b = J_{\phi(a, b)} C_{\psi(a, b)}. \quad (20)$$

Finally, the equations (13), (15) and (18) can be viewed as the defining relations for the group G .

Inverting the problem

Hölder had begun his investigation by assuming the existence of a group G . He now assumed instead that we are given the functions ϕ, χ, ψ, ω and wish to find the group that they represent, in other words, we wish to solve the extension problem. The equations (13), (15) and (18) tell us that multiplication is given by the equation

$$(S_a T_\alpha) \times (S_b T_\beta) = S_{\phi(a, b)} T_{\psi(a, b)} T_{\omega(\alpha, b)} T_\beta.$$

Consider the expressions $S_a T_\alpha$ not as products of group elements but as symbols to denote mn different objects, with the above law of multiplication. The multiplication is well-defined; we require also that it is group-theoretic.

We know already that the n symbols $S_0 T_\alpha$ with multiplication defined by the function χ must form a group. Denote the group (as before) by Γ and the symbol $S_0 T_\alpha$ simply by T_α .

Hölder next began to investigate [1893, 320ff] the conditions that must be imposed on ϕ , ψ and ω in order that a group is produced. The functions ϕ and ψ determine how the S multiply together, and ω determines how the elements T of Γ are acted on by the S . The substitutions defined by $\sigma_a : T_\alpha \rightarrow T_{\omega(\alpha,a)}$ must all represent automorphisms of the group Γ . Further, the function that takes each S_a to the corresponding σ_a must be a homomorphism. Hölder stated (wisely) that he was not going to discuss the general conditions under which such a homomorphism existed. In each of the particular cases he considered, he was able to find the possible homomorphisms.

He went on to list various relations which hold between the ‘symbols’ $S_a T_\alpha$ [1893, 324]. It cannot quite be proved that the associative law holds for the multiplication defined above, but only that it holds when at least one of the three symbols is an element of Γ .

This problem is caused because it is not known whether the symbols S satisfy the associative law. If they do, Hölder calculated that the general symbols $S_a T_\alpha$ also satisfy this law. He did not actually work out the conditions on the function ψ which ensure that multiplication of the S is associative. This is a curious omission: Hölder was usually very thorough about such details.

One final condition was required to ensure that the multiplication of the symbols $S_a T_\alpha$ is group-theoretic. The fact that the symbols Σ_a form a group⁵ was enough to prove that the condition holds.

Hölder summed up all this working [1893, 327]: if the various conditions and equations he had given were satisfied, then a group G of order mn had been proved to exist.

Aus diesen Gleichungen folgt nun wiederum, dass die aus den Operationen T bestehende Gruppe Γ in der Gruppe G ausgezeichnet enthalten ist, und dass die aus G und Γ durch einen bekannten Process entstehende Gruppe G/Γ diejenige ist, deren Multiplicationsgesetz durch die Function ϕ vermöge der Gleichung

$$\Sigma_a \Sigma_b = \Sigma_{\phi(a,b)}$$

beschrieben wird. [7]

He then drew together the theory, repeating its basic outlines [1893, 327–329], and produced the two particular cases that have been mentioned in Chapter 8: semidirect products and extensions by a cyclic group. A semidirect product is formed when $\psi(a, b) = 0$, so that the S multiply according to the rule

$$S_a S_b = S_{\phi(a,b)}.$$

⁵In other words, the group Γ is being extended by another *group* and not just by a collection of elements.

Hölder stated this theorem [1893, 329–330]:

Man wähle n Symbole $T_0, T_1, \dots, T_{n-1}$, welche eine Gruppe Γ bilden, und m Symbole $S_0, S_1, \dots, S_{m-1}$, welche unter sich auch eine Gruppe ausmachen. Jedem Symbol S_a werde ein Isomorphismus J_a der Gruppe Γ in sich zugeordnet, wobei aber verschiedenen Symbolen S unter Umständen derselbe Isomorphismus entsprechen kann. Ausserdem wird gefordert, dass für je drei Indices a, b, c , für die $S_a S_b = S_c$ ist, auch die Gleichung $J_a J_b = J_c$ besteht. Wird nun die Function $\omega(\alpha, a)$ dadurch erklärt, dass

$$J_a = \begin{pmatrix} T_0, & T_1, & \dots & T_{n-1} \\ T_{\omega(0,a)}, & T_{\omega(1,a)}, & \dots & T_{\omega(n-1,a)} \end{pmatrix},$$

so bestimmen die definirenden Relationen

$$\begin{aligned} T_\alpha T_\beta &= T_{\chi(\alpha, \beta)}, \\ S_a S_b &= S_{\phi(a, b)}, \\ T_\alpha S_a &= S_a T_{\omega(\alpha, a)} \end{aligned}$$

eine Gruppe von der Ordnung mn mit der ausgezeichneten Untergruppe Γ . [8]

Hölder was not able to formulate this theorem in the same way as it is stated nowadays. The correspondence between the S and automorphisms of Γ is now expressed: there exists a homomorphism of the group $\{S_0, S_1, \dots, S_{m-1}\}$ into $\text{Aut}(\Gamma)$.

Hölder noted that if the functions χ and ϕ are given, it is always possible to determine the function ω in accordance with the prescribed conditions, but in many cases the only way that this can be done is to set $\omega(\alpha, a) = \alpha$ for all α, a . This means that the elements S and T commute. ‘Ich nenne in diesem Fall die gebildete Gruppe das *directe Product* der beiden zur Bildung verwendeten Gruppen.’ (‘In this case I call the group that has been constructed the *direct product* of the two groups used in its construction.’)

An example

After so much theory, the discussion of a particular situation comes as something of a relief. Hölder chose the Klein 4-group (*Vierergruppe*) as normal subgroup: it can be expressed as $\langle T_1, T_2 : T_1^2 = 1 = T_2^2, T_1 T_2 = T_2 T_1 \rangle$. In order to find an automorphism of the group, wrote Hölder, simply permute the three nontrivial elements in any way. He found an automorphism J of order 3 which fixes no nontrivial element (there are also automorphisms of order 2 which fix one nontrivial element; the complete automorphism group is isomorphic to S_3). If J is to be the automorphism with which the elements S of another group are to correspond, then such an element S_a must have order divisible by 3. Further investigation produced the theorem:

Wenn m durch 3 theilbar ist, so definiren die Relationen

$$S^m = 1, S^{-1}T_1S = T_2, S^{-1}T_2S = T_1T_2, T_1^2 = 1$$

eine Gruppe $4m^{\text{ter}}$ Ordnung. [9]

Thus if $m = 3$ we obtain the group A_4 , the 'group of the tetrahedron'. Of course this example just shows *one* of the ways in which a semidirect product with the Klein 4-group as normal subgroup can be obtained; if the automorphism of order two had been chosen, a different sort of group would have been produced⁶.

The other special case to which Hölder drew attention is that in which the group of the symbols Σ is cyclic of order m . Here, an extension is obtained when the normal subgroup Γ has an automorphism whose m^{th} power is an inner automorphism. Hölder provided defining relations for the extension [1893, 334].

More theory in 1895: complete groups

Once the foundations of the theory had been laid in 1893, Hölder began to concentrate on exploring more specific areas of it, areas which were immediately helpful when considering the formation of a group from its composition factors. Firstly, he looked at the problem from the opposite end. Suppose that the large group (the extended group) Δ is given, with normal subgroup Γ and a set of coset representatives for Γ in Δ . What happens if Γ has only inner (cogredient) automorphisms? In such a situation, conjugation of Γ by S_a must give the same result as conjugation by an element T_{β_a} of Γ . Replace S_a by $S'_a := S_a T_{\beta_a}^{-1}$, as one is allowed to do since S_a and S'_a both serve equally well as coset representatives, and we find that conjugation by S'_a produces the identity automorphism on Γ .

This replacement can be performed as necessary on every element S . Now denote the new elements S' by S again. Then each element of Δ is represented in one way by $S_a T_\alpha$, as before, but in addition the relation $T_\alpha S_a = S_a T_\alpha$ holds for all α, a .

Now $S_a S_b$ is an element of Δ , therefore

$$S_a S_b = S_{\phi(a,b)} T_{\psi(a,b)}. \quad (2)$$

We know that S_a , S_b and $S_{\phi(a,b)}$ commute with each element of Γ . Therefore the elements in the factor set $T_{\psi(a,b)}$ must also commute with the elements of Γ , that is, they are in the centre of Γ .

⁶The group of this type with smallest order is the dihedral group of order 8.

If Γ has trivial centre, then $T_{\psi(a,b)} = 1$ for all a, b and the elements S form a group. The elements S and T are distinct and they commute, so Δ becomes a *direct product* of Γ with the group of the S [1895a, 325].

Es entsteht also in diesem Fall die Gruppe Δ durch eine triviale Combination der Gruppe Γ mit einer andern Gruppe; wir wollen dies dadurch ausdrücken, dass wir sagen, die Gruppe Γ *spalte* sich von der Gruppe Δ *ab*. Zum Zweck einer bequemeren Ausdrucksweise mache ich noch die folgende Festsetzung:

Definition. Eine Gruppe, die nur *cogrediente Isomorphismen* in sich zulässt und keine ausgezeichnete Operation, ausser der identischen, besitzt, heisst *vollkommen*. [10]

Thus the result Hölder obtained can be expressed as

Proposition I

A complete group splits off [as a direct factor of] every group in which it is contained as a normal subgroup.

The term *complete* has today the same meaning as that which Hölder originally gave to it. However, the idea of a 'split extension' now includes semidirect products as well as direct products, but presumably the use of the term in its modern sense has developed from Hölder's definition given here.

Automorphisms revisited and extensions by a simple group

Assume now that Γ may once more contain outer automorphisms. Hölder had mentioned in 1893 that the group of inner automorphisms is a normal subgroup of the whole automorphism group; in modern notation $\text{Inn}(\Gamma) \trianglelefteq \text{Aut}(\Gamma)$. He gave a proof here, and went on to form $\text{Out}(\Gamma)$, the outer automorphism class group, as the quotient group $\text{Aut}(\Gamma)/\text{Inn}(\Gamma)$.

Next he worked out that there is a relationship between the quotient group Δ/Γ and $\text{Out}(\Gamma)$. The automorphisms of Γ produced by the n elements $S_a T_\beta$ (where a is fixed and $\beta = 0, 1, \dots, n-1$) are all in the same class of outer automorphisms. Thus each element of Δ/Γ corresponds to a single element of $\text{Out}(\Gamma)$. Further, the automorphism classes found by this correspondence form a group Λ , a subgroup of $\text{Out}(\Gamma)$. Hölder discovered that there is in fact a homomorphism from Δ/Γ to Λ , so those elements of Δ/Γ which correspond to the identity of Λ (that is, to inner automorphisms) form a normal subgroup of Δ/Γ .

If a group is being formed from its composition series, the group Δ/Γ is simple.

Assume this is the case and one of two situations must occur [1895a, 327]:

- (1) All elements of Δ/Γ correspond to the identity of Λ ;
- (2) Only the identity of Δ/Γ corresponds to the identity of Λ .

The first situation was treated before: if it is assumed that Γ has trivial centre, then the group Δ is the direct product of Γ and Δ/Γ . What happens in case (2) if we assume that Γ has trivial centre? (This is a sensible assumption to make when many of the particular groups to be considered, being part of a composition series, are simple.) It transpires that there is a one-one correspondence between the automorphisms of Γ and the elements of Δ , so Δ can in fact be represented as a subgroup of $\text{Aut}(\Gamma)$ containing $\text{Inn}(\Gamma)$.

Hölder remarked that it is sometimes easy to recognise which of the above situations is present, especially when dealing with simple groups. He formulated two theorems [1895a, 328]:

Theorem II

Let the group Δ contain a maximal normal subgroup Γ of composite index, and suppose that $\text{Out}(\Gamma)$ is abelian. Then the elements of Δ induce only inner automorphisms on Γ .

Theorem III

Suppose the maximal normal subgroup Γ of the group Δ has trivial centre. Then either Γ splits in Δ [that is, the group Γ is a direct factor of Δ] or Δ is isomorphic to a subgroup of $\text{Aut}(\Gamma)$ containing $\text{Inn}(\Gamma)$.

More investigation was required in order to apply these theorems. If the centre of Γ is the identity, then Γ is isomorphic to a subgroup of $L := \text{Aut}(\Gamma)$; in fact Γ is isomorphic to $N := \text{Inn}(\Gamma)$. When one forgets that L and N were produced as groups of automorphisms and simply considers the relationship between them, the following can be proved [1895a, 331]:

Proposition IV

Suppose that $N \trianglelefteq L$ and L contains no subgroup isomorphic to N apart from N . Suppose further that the centraliser of N in L is $\{1\}$ and all automorphisms of N are obtained under conjugation by the elements of L . Then two subgroups of L

which both contain N can only be isomorphic if they are conjugate in L . Under these assumptions, the group L is complete.

This proposition has a

Corollary

Such a subgroup of L that contains N and is larger than N cannot be expressed as the direct product of a group isomorphic to N and another group.

More can be said about recognising when N is 'the only subgroup of its kind' in the group L . The final theorem that Hölder established before moving on to specific examples was [1895a, 332–333]:

Theorem V

If $N \leq L$ and the product of $|N|$ with the order of some simple group does not divide $|L|$, then L can contain a group isomorphic to this simple group only within N .

When the simple group in question is identical to N , the theorem becomes:

If $N \leq L$ is a simple group and $|L|$ is not divisible by the square of $|N|$, then the group N is the only subgroup of its kind in L .

1895: Extensions in practice

The remaining 89 pages of Hölder's article were given over to examples, or rather to the application of the theory in particular situations. Hölder picked ten different groups to be the normal subgroup Γ and explored the possible outcomes. The normal subgroups chosen were:

- §2 A_n for $n \geq 5$;
- §3 $\text{PSL}(2, p)$ for $p \geq 5$;
- §4 C_n for $n \in \mathbb{N}$;
- §5 $C_p \times C_p$ for p prime;
- §6 A metacyclic group;
- §7 $\mathfrak{R}_{60}^{(m)}$ where $C_m \leq \mathfrak{R}_{60}^{(m)}$ and $\mathfrak{R}_{60}^{(m)}/C_m = A_5$;
- §8 $\mathfrak{R}_{168}^{(m)}$ where $C_m \leq \mathfrak{R}_{168}^{(m)}$ and $\mathfrak{R}_{168}^{(m)}/C_m = \text{PSL}(2, 7)$;
- §9 $\mathfrak{Z}_{60}^{(2)} = A_5 \times A_5$;
- §10 $A_5 \times C_p$;

§11 $\text{PSL}(2, 7) \times C_p$.

In choosing these, Hölder attempted to put the principles of group-building into action by beginning with simple groups and gradually adding more and more simple components.

The first task in each case was to calculate the automorphism group of Γ . This was accomplished mostly by considering the effect of an automorphism on a set of generating elements for Γ . For example in [1895a, 363–364] Hölder treated the group $C_p \times C_p$. It has generators and relations

$$T_1^p = 1, T_2^p = 1, T_1 T_2 = T_2 T_1.$$

An automorphism of $C_p \times C_p$ replaces $T_1^\alpha T_2^\beta$ by $T_1^{a\alpha+b\beta} T_2^{c\alpha+d\beta}$ where $ad - bc \not\equiv 0 \pmod p$ (this condition ensures that the mapping is one-one and $C_p \times C_p$ does not degenerate into C_p). This automorphism may be expressed by the substitution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod p.$$

Hölder showed how composition of automorphisms corresponds to multiplication of the matrices representing them, and so the automorphism group is isomorphic to the general linear group $\text{GL}(2, p)$, which Hölder considered as a group of linear substitutions, following Galois and Jordan. Indeed, one is reminded of the work of Galois in the *Second mémoire* on primitive equations of degree p^2 [1846b, 436ff/53ff/135ff] when reading Hölder's calculation and result here.

Once the automorphisms of Γ had been found, the group was considered to be a maximal normal subgroup of the group Δ , and the possibilities for Δ were investigated using the theorems proved in the first section. Sometimes the quotient group Δ/Γ was also specified, or various types of simple groups were in turn to be chosen for Δ/Γ .

Let us look at another example. Take the cyclic group of order m (where m is not prime), generated by Θ , as normal subgroup Γ . Hölder searched for groups Δ such that $\Delta/\Gamma \cong A_5$ [1895a, 350–355]. He found that the automorphism group of Γ is abelian and has order $\phi(m)$, which is the number of integers less than m that are coprime to m . The identity is the only inner automorphism. Then Theorem II can be applied to show that all elements of Δ produce inner automorphisms of Γ , and therefore fix the elements of Γ . So all elements of Δ commute with Θ .

Dyck had provided a presentation of A_5 by generating elements A_1 and A_3 and relations:

$$A_1^5 = 1, A_3^2 = 1, (A_1 A_3)^3 = 1.$$

Therefore there must be two elements S and T in Δ such that

$$S^5 = \Theta^a, T^2 = \Theta^b, (ST)^3 = \Theta^c, \Theta^m = 1$$

and Θ commutes with S and T . These relations imply that there are $60m$ possible expressions which can be formed using S, T and Θ , so if Δ exists it has the above generators and relations, and order $60m$.

The following question must now be answered: what are the conditions on a, b, c and m which ensure that the relations do define a group of order $60m$? Hölder had had experience in [1893] of working out such conditions. It must be ensured that no lower power of Θ than the m^{th} power equals the identity. This will be the case if and only if

$$2(6a + 15b - 10c) \equiv 0 \pmod{m},$$

as Hölder proceeded to prove. The system of generators and relations simplifies [1895a, 353] to:

$$S'^5 = 1, T'^2 = \Theta^{km/2}, (S'T')^3 = 1, \Theta^m = 1,$$

where Θ again commutes with S' and T' , and $k = 0$ if m is odd and $k = 0, 1$ if m is even.

If $k = 0$, the group Δ becomes the direct product $C_m \times A_5$. What happens if m is even and $k = 1$? First assume that $m = 2$. We obtain the relations:

$$S'^5 = 1, T'^2 = \Theta, (S'T')^3 = 1, \Theta^2 = 1,$$

where Θ commutes with S' and T' . 'Diese Gleichungen definiren in der That eine bereits bekannte Gruppe 120^{ter} Ordnung ...' ('These equations in fact define a previously known group of order 120 ...') This group, which Hölder later investigated, is the special linear group $SL(2, 5)$, sometimes called the 'binary icosahedral group'. Hölder denoted it by $\mathfrak{R}_{60}^{(2)}$.

Similarly, a group $\mathfrak{R}_{60}^{(m)}$ also exists for any even m . Hölder later showed that such a group is not a semidirect product. In fact the group $\mathfrak{R}_{60}^{(m)}$ for $m > 2$ would now be

constructed as an amalgamated direct product of $\text{SL}(2, 5)$ with C_m , by identifying the central subgroup of order two in both groups.

Amalgamated direct products

Although Hölder did not construct $\mathfrak{R}_{60}^{(m)}$ as an amalgamated direct product, he did introduce the concept (without giving it a name) and constructed other groups in this way. He proved the following theorem [1895a, 384–385]:

Wenn zwei Gruppen G und H von der $m\mu^{\text{ten}}$ und $n\mu^{\text{ten}}$ Ordnung je eine aus ausgezeichneten Operationen bestehende Untergruppe μ^{ter} Ordnung enthalten, und wenn diese Untergruppen miteinander holoeidrisch isomorph sind, so lässt sich stets eine Gruppe $mn\mu^{\text{ter}}$ Ordnung herstellen. Man nimmt zu diesem Zweck die Operationen der Gruppe G mit denen der Gruppe H vertauschbar an und identificirt die sich entsprechenden Operationen der beiden Untergruppen. [11]

Hölder's method of proof reads just like a modern proof. The operations S_{0k} and T_{0k} ($k = 0, 1, \dots, \mu - 1$) form central subgroups of G and H , respectively, and there is an isomorphism which maps S_{0k} to T_{0k} for each value of k . The direct product $G \times H$ has order $mn\mu^2$. This group has a normal subgroup Γ of order μ , formed by the operations $S_{0k}^{-1}T_{0k}$. Then the amalgamated direct product is the group $(G \times H)/\Gamma$.

Hölder applied this construction to two groups of the kind $\mathfrak{R}_{60}^{(2)} \cong \text{SL}(2, 5)$. He produced 'die einzige nichtzerfallende Gruppe, welche die Gruppe $\mathfrak{R}_{60}^{(2)}$ als ausgezeichnete Maximaluntergruppe vom Index 60 enthält' ('the only non-decomposable group which contains the group $\mathfrak{R}_{60}^{(2)}$ as maximal normal subgroup of index 60').

There is a related process of group construction which Hölder also introduced [1895a, 396–398]: the 'upside-down' version of the amalgamated direct product, in which it is a quotient group and not a normal subgroup that is amalgamated. That is, assume we have three groups G , H and K , and surjective homomorphisms $\mu : G \rightarrow K$ and $\nu : H \rightarrow K$. Then the direct product with amalgamated quotient group, which is a subgroup of $G \times H$, can be defined as

$$\{(U, V) : U \in G, V \in H, U^\mu = V^\nu\}.$$

This idea is nowadays often expressed in terms of a 'pullback'.

Hölder used this process of construction to give insights into the structure of various group extensions, as he had done with the other amalgamated direct product.

The symmetric and alternating groups

The first particular type of group that Hölder chose to study [1895a, 333–345] was the alternating group A_n , where $n \geq 5$ in order that the group is simple. He established several new results and developed ideas that explained some of the properties which the symmetric and alternating groups were already known to possess.

Hölder began as usual by searching for the automorphisms of A_n . Klein had in fact dealt with the ‘isomorphisms of A_5 into itself’ in [1884, 232–233/1888, 257–258]: there are two classes of isomorphisms, which implies that $\text{Out}(A_5) = C_2$. Hölder proceeded to show that the same is true for A_n if $n > 6$. The automorphisms of A_n are obtained under conjugation by all the substitutions of S_n . The case $n = 6$, as he noted, forms an exception.

He proved these results by considering what might be the image of a 3-cycle in A_n under an automorphism of A_n . If $n \neq 6$, the image must also be a 3-cycle, and therefore all the possible automorphisms are formed when one conjugates A_n by any element of S_n (since there exists an element of S_n that conjugates a given 3-cycle to another given 3-cycle). The case $n = 6$ is exceptional because there are in A_6 equal numbers of 3-cycles (abc) and double 3-cycles $(\alpha\beta\gamma)(\delta\epsilon\zeta)$. Besides automorphisms formed under conjugation by elements of S_6 , an automorphism of A_6 can also be found which maps the first type of substitution (above) to the second. ‘... Es beruht dies darauf, dass sechswerthige Functionen von sechs Veränderlichen existiren, die nicht in Beziehung auf fünf von den Veränderlichen symmetrisch sind.’ (‘... This is related to the fact that six-valued functions of six variables exist that are not symmetric with respect to five of the variables.’) Hölder gave an example of such a function [1895a, 338–339], and used it to produce the new type of automorphism. The existence of such functions of six variables goes back to Bertrand [1848] and was well-known by Hölder’s time.

In terms of group theory, the group S_6 has a subgroup H of order 120, isomorphic to S_5 , that is *not* the stabiliser of one of the letters on which the group S_6 acts. Serret gave a proof of this in the *Cours*; see, for example, Volume II of the fourth edition [1879, 335–340]. In fact H is isomorphic to $\text{PGL}(2, 5)$, the projective general linear group, in its natural permutation representation of degree six (it acts faithfully on the projective line $\text{PG}(1, 5)$ which has six members). The subgroup H has six cosets. Therefore the representation of S_6 on the cosets of H provides a homomorphism $\sigma : S_6 \rightarrow S_6$.

This is actually an automorphism of S_6 (since $\text{Ker } \sigma$ is a normal subgroup of S_6 and one can prove that it equals neither S_6 nor A_6). The important point is that σ is not an inner automorphism. If it were, it would preserve the cycle structure of every substitution in S_6 , but Hölder showed that it takes transpositions to products of three transpositions [1895a, 339]. Therefore S_6 has an outer automorphism which Hölder showed is essentially unique.

This investigation had uncovered the result that there are four classes of automorphisms of A_6 , and $\text{Out}(A_6) = C_2 \times C_2$.

Now consider the alternating group A_n ($n > 4$) as a maximal normal subgroup Γ of a group Δ . An application of Theorem III gives the result: if the group Δ is not a direct product, it is isomorphic to a subgroup of $\text{Aut}(\Gamma)$ containing $\text{Inn}(\Gamma)$. When $n = 5$ or $n > 6$, the only possibility is that $\Delta \cong S_n$. However, if $n = 6$, there are three essentially different choices for Δ , depending upon which class of outer automorphisms is included to form Δ . One of these possibilities is again the group S_6 .

The last result Hölder proved in this section follows from Theorem V and Proposition IV, putting $N := A_n$ and $L := S_n$. These together imply that if $n = 5$ or $n > 6$, the symmetric group S_n is complete.

At the end of his article, Hölder provided, as was his custom, a summary of the results he had obtained. He had discovered the structure of many groups with two or three composition factors of orders 60, 168 or a prime number, and had found all the isomorphisms between these groups, so that his table is of isomorphism classes of extensions. He also listed all the nonsoluble groups of order ≤ 480 , which his previous investigations had enabled him to do.

Group extensions in the twentieth century

We have seen that Hölder made a very promising start on the study of group extensions during the 1890s. Nevertheless, no-one else took up this work for another thirty years and the ideas Hölder had developed were all but forgotten. In the meantime, a few related concepts were introduced into group theory. One of these was that of the Schur multiplier or multiplicator.

Schur began to investigate group representation theory in the early 1900s, carrying on the work of his teacher Frobenius. He extended the ideas of Frobenius to include

projective representations for a finite group G in [1904]. In so doing, he came across the notion of a *central* extension [1904, 22/88], that is, one in which the normal subgroup is to be in the centre of the extended group. In relation to this, he defined the *Multiplikator* of a group, nowadays usually known as the Schur multiplier⁷.

Later in the paper he employed the transfer homomorphism in a form recognisable nowadays as such. A corollary of his investigation (which, however, remained implicit in his article) is that if the order of an abelian group C in the centre of G is coprime to its index, then G is the split extension of C by a subgroup isomorphic to G/C . This is a particular case of the 'Schur-Zassenhaus Theorem', which is really rather more Zassenhaus' than Schur's. Zassenhaus generously attributed the theorem to Schur in the case of an abelian extension, but in fact Zassenhaus extended it himself to deal with an arbitrary abelian normal subgroup, and then went on to prove the general theorem [1949, 132].

Schur's article [1904] also contains the concept of a 'factor set' in connection with group extensions; he found various of the conditions that are satisfied by the elements of a factor set in a central extension. In later papers Schur referred to Hölder for the name *complete group* [1907, 86/199] and provided the name *factor system* [1919, 8/277].

Schreier and the rebirth of group extension theory

The two papers on extension theory published by Schreier in 1926 are reproductions of the first and second halves of his doctoral dissertation, completed at Vienna in 1923. He was supervised by Furtwängler, a number theorist, at whose suggestion Schreier had taken up the question of building groups. His papers are entitled 'Über die Erweiterung von Gruppen' ('On the extension of groups'). He began by stating the problem he had been considering [1926a, 165]:

Die abstrakten Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ seien vorgelegt. Es soll die Konstitution aller Gruppen $\overline{\mathfrak{B}}$ untersucht werden, die $\mathfrak{A}$ als Normalteiler enthalten, so daß $\overline{\mathfrak{B}}/\mathfrak{A}$ mit $\mathfrak{B}$ einstufig isomorph ist, oder, wie wir kurz sagen wollen, die Konstitution aller Erweiterungen von $\mathfrak{A}$ mit Hilfe von $\mathfrak{B}$. [12]

A footnote tells us that this question was first raised by Otto Hölder and especially investigated by him in [1893]. Therefore Schreier knew at least a little of Hölder's work

⁷The *Multiplikator* of a group H is the unique largest abelian group A such that there is a central extension G of A by H with $A \leq G'$ (where G' denotes the commutator subgroup of G).

on the subject. It becomes clear that he had actually read Hölder's articles in some detail and had grasped various aspects of Hölder's approach, in that he wrote [1926a, 166]: 'Diese Auffassung geht im wesentlichen auf O. Hölder zurück' ('This view in essence dates back to O. Hölder'). In fact Schreier's approach to and development of the theory of extensions seems to follow on directly from that of Otto Hölder: much of Hölder's methodology was borrowed by Schreier. The difference between their work is that Schreier drew out each idea to its logical conclusion, whereas Hölder had been motivated initially by the wish to classify particular sorts of groups and therefore developed theory with this fixed aim in mind.

The change in outlook from the nineteenth to the twentieth centuries can clearly be seen in Schreier's papers: he approached the problem from a completely abstract standpoint. There is not a single worked example in [1926a]. This has the effect of making the paper much harder to read than Hölder's. Schreier's articles also suffer from an unfortunate choice of notation: he used superscripts to denote both the different elements of a group (for example, $A, A', A'', \dots$) and also to denote the image of functions acting on these elements (for example, A^B is the element obtained when B acts on A). It is especially confusing when they appear together, as in A''^B .

What was Schreier's achievement towards the establishing of a theory of group extensions? His first paper [1926a] has three parts:

- (1) Extending a group by means of a given group: the general theory was worked through and conditions found which must be satisfied by the factor set ($A_{B', B''}$ in Schreier's notation) and, in modern terms, by the homomorphism from $\mathfrak{B}$ to $\text{Aut}(\mathfrak{A})$: Schreier denotes the action of elements of $\mathfrak{B}$ on elements of $\mathfrak{A}$ by A^B .
- (2) Extending a group by means of a group that is a direct product: Schreier found the conditions under which this is possible, calculated the defining relations for such an extension, and then generalised from the direct product of two groups to the direct product of any number (not necessarily finite).
- (3) Extending a group by means of an abelian group: this is investigated by assuming that the factors of a direct product from part (2) are cyclic (of finite or infinite order). As before, Schreier calculated the conditions under which this may happen and the defining relations for the extension.

The second paper [1926b] specialised further: the normal subgroup $\mathfrak{A}$ is to be finite

abelian, and the quotient group also abelian. The theory is greatly simplified by this assumption, since elements of the factor set now commute and the homomorphism is from $\mathfrak{B}$ to $\text{Out}(\mathfrak{A})$. Schreier applied his results to groups of prime power order (which he called *p-Gruppen*). He classified various types of *p*-groups and, in particular, was able to deal with groups of order p^3 , p^4 and p^5 with relative ease.

The result of Schreier's investigation in (1) above was given in his Theorem I [1926a, 168], which sets out precisely when an extension of $\mathfrak{A}$ by $\mathfrak{B}$ is possible, and the necessary conditions on the factor set and homomorphism. All the relations and conditions which contribute to Theorem I had been worked out by Hölder, with the exception of those conditions that ensure multiplication in the extended group is associative. Hölder simply required that multiplication of the elements he denoted by S (they are elements of a transversal for the normal subgroup) is associative [1893, 325]. That is not to say that he assumed the product of S_a and S_b is precisely S_c , but that the multiplication of these elements in the whole extended group must satisfy $(S_a S_b) S_c = S_a (S_b S_c)$. If this is so, the multiplication of any three elements in the extended group is also associative. Schreier calculated the actual conditions on the factor set which ensure associativity of the elements S (these are B in his notation).

Part (1) of [1926a] concludes with the remark that the various conditions which a factor set and homomorphism must satisfy are certainly not independent of one another. 'Doch scheint es schwierig zu sein, sie durch weniger Gleichungen zu ersetzen, so lange über die Gruppe $\mathfrak{B}$ keine Voraussetzungen gemacht werden.' ('All the same, it seems to be difficult to replace them by fewer equations, as long as no assumptions are made about the group $\mathfrak{B}$.') Schreier was right to recognise that the general approach he had taken could now give useful information only by restricting to particular cases: the equations for the general case were too many in number to provide further helpful insights.

Extensions after Schreier

A short quote from Zassenhaus' book *The Theory of Groups* [1949, 94] is indicative of the view commonly held of extension theory by the 1930s: 'The extension problem posed and solved by Otto Schreier reads ... ' Zassenhaus wrote this even though he knew enough about Hölder's work to attribute to him the theorem giving generators

and relations for an extension of one cyclic group by another. By the time Zassenhaus produced his book, extension theory had begun to be recognised as an independent field of study, largely due to the influence of Schreier's papers. In 1934 Reinhold Baer carried on the work of Schreier in studying extensions of abelian groups, and proved that the extension classes themselves form an abelian group. Then Marshall Hall Jr. [1938] investigated what he referred to as central extensions⁸ and dealt with the equivalence of factor sets, amongst other things.

Group extension theory was transformed during the 1940s by the introduction of ideas that had their conception in algebraic topology. This study of *cohomological* extension theory was begun by Eilenberg and Mac Lane, whose results were first published in [1942]. 'This paper originated from an accidental observation that the groups obtained by Steenrod were identical with some groups that occur in the purely algebraic theory of *extensions of groups* ...' [1942, 758]. Saunders Mac Lane has recently written about his involvement in 'Group extensions for 45 years' [1988], and described how he came to work together with Eilenberg (whom Mac Lane met at a lecture he gave in 1941). Further discussion of various aspects of the development of group extension theory can be found in [Beyl & Tappe 1982].

It is unlikely that Hölder would recognise a modern (cohomological) introduction to this theory as the fruition of the ideas he had during the 1890s. However, it would undoubtedly give him much pleasure and satisfaction to see the wealth of new concepts that have become associated with extension theory, and to discover how it has enabled links between previously separate areas of mathematics to be uncovered. The interest of a great many mathematicians in developing the theory during the twentieth century is ample testimony to its attractiveness and abiding importance.

Notes

1 We might here have investigated our new groups, *the extended groups*, in the same sense by the theory of groups, as we have done for the original groups in the preceding paragraphs. I should like to recommend such a discussion to the reader as an appropriate exercise, and limit myself here in this direction to the following statement:—The original

⁸ A central extension, according to Hall, is an extension of a group A by a group H in which the factor set is in the centre of A . Therefore the case in which A is abelian is included.

group is in every case manifestly self-conjugate within the extended group.

2 One can pose the inverse problem to that of splitting a group into two factors: given two groups as factors, put together a group as their product. This task sometimes allows of many solutions; I hope to deal with this at another opportunity.

3 One can now, by turning round the investigation just employed, formulate the task of forming a group which has given factor groups. This task admits of very many solutions. The factor groups are always simple groups. It frequently suffices, instead of the factor groups, to give the order of the factor groups, that is, the mere numbers. The specification of these numbers achieves the same thing as the specification of the factor groups, if one knows that each of these numbers is the order of only a single simple group.

4 ... form a group Δ which contains a given group Γ as a normal subgroup in such a way that Δ/Γ coincides with a given group.

5 All possible isomorphisms of a group Γ into itself themselves form a group.

6 ... we have finally to deal with the problem of investigating in how many different ways the group of 60 ikosahedral substitutions ... can be co-ordinated to itself in holohedric isomorphism. Two sorts of this co-ordination are given by cogredience and contragredience; we will show that all others are essentially reduced to these.

7 Again it now follows from these equations, that the group Γ consisting of the operations T is contained as a normal subgroup in the group G and that the group G/Γ , arising from G and Γ through a familiar process, is that group whose law of multiplication is described by the function ϕ according to the equation

$$\Sigma_a \Sigma_b = \Sigma_{\phi(a,b)}.$$

8 Choose n symbols $T_0, T_1, \dots, T_{n-1}$ which form a group Γ and m symbols $S_0, S_1, \dots, S_{m-1}$ which also make up a group. Each symbol S_a will determine an isomorphism J_a of Γ into itself where, however, different symbols S may possibly correspond to the same isomorphism. In addition it is required that for any three indices a, b, c for which $S_a S_b = S_c$, then the equation $J_a J_b = J_c$ also holds. Define the function $\omega(a, a)$ according to the rule that

$$J_a = \begin{pmatrix} T_0, & T_1, & \dots & T_{n-1} \\ T_{\omega(0,a)}, & T_{\omega(1,a)}, & \dots & T_{\omega(n-1,a)} \end{pmatrix}.$$

Then the defining relations

$$\begin{aligned}T_{\alpha}T_{\beta} &= T_{\chi(\alpha,\beta)} \\S_aS_b &= S_{\phi(a,b)} \\T_{\alpha}S_a &= S_aT_{\omega(\alpha,a)}\end{aligned}$$

determine a group of order mn with normal subgroup Γ .

9 If m is divisible by 3, then the relations

$$S^m = 1, S^{-1}T_1S = T_2, S^{-1}T_2S = T_1T_2, T_1^2 = 1$$

define a group of order $4m$.

10 Thus in this case the group Δ is formed by a trivial combination of the group Γ with another group; we shall express it in this way, namely by saying that the group Γ *splits off* from the group Δ . With the aim of a more convenient mode of expression I now fix the following:

Definition. A group which admits only cogredient isomorphisms into itself and contains no normal operation apart from the identity is called complete.

11 If two groups G and H of orders $m\mu$ and $n\mu$ each contain a central subgroup of order μ and if these subgroups are isomorphic to one another, then a group of order $mn\mu$ can always be manufactured. To this end one postulates that the operations of the groups G and H commute and identifies the corresponding operations of the two subgroups.

12 Let the abstract groups $\mathfrak{A}$ and $\mathfrak{B}$ be given. We seek the constitution of all groups $\overline{\mathfrak{B}}$ which contain $\mathfrak{A}$ as normal subgroup in such a way that $\overline{\mathfrak{B}}/\mathfrak{A}$ is isomorphic with $\mathfrak{B}$, or, in short, the constitution of all extensions of $\mathfrak{A}$ with the help of $\mathfrak{B}$.

CHAPTER 10

Conclusions

To appreciate the living spirit rather than the dry bones of mathematics, it is necessary to inspect the work of a master at first hand.

E.T. Bell, *Mathematics: Queen and Servant of Science* (1951)

The reader should by now have become aware and convinced (the author would like to hope) of the important position held by Otto Hölder in the history of group theory. Hölder's original contributions to the subject, which he expressed in 350 pages of clear and precise German text, uncovered what have proved to be the major themes of twentieth-century pure (finite) group theory: the search for simple groups, the investigation of the structure of different types of group and the development of group extension theory. He wrote in a style unlike that of most of his predecessors, a style which is immediately accessible to a modern reader. In both of these respects his work properly belongs to the twentieth and not to the nineteenth century.

How, then, was Otto Hölder in a position to make such contributions? The problems which he formulated and researched would eventually have been recognised and brought into prominence without his help, since they are the basic questions of any theory of finite groups. They were distilled so readily by Hölder because he had been able to assimilate previous research both in group theory and Galois theory, and more widely in large parts of pure mathematics. Thus his study is very much an outworking of the earlier developments in pure mathematics of the nineteenth century. He brought his own distinctive insights to these developments and thereby was able to express them, and his outworking of them, in a new form—a 'twentieth-century' form.

Hölder had recognised the fundamental structure which a theory of finite groups should be given. This recognition seems to have come to him fairly early in his dealings with the subject; by the time he wrote his first article [1889a] on group theory and Galois theory, he had it in mind to explore the 'construction of groups', that is, group extension theory. Before he finally wrote on extension theory, he had dealt with other major themes in finite group theory—the search for simple groups and the investigation of group structure.

In Hölder's original contributions to group theory and Galois theory, a good balance is preserved between the development of concepts with a particular application in mind and the exploration of new ideas for their own sake. Hölder possessed an *abstract* view of the underlying theory but this was linked to a *practical* concern to show how the theory could be applied to specific types of group or specific situations. He provided numerous 'concrete' examples throughout his papers.

On some occasions, for example in the treatment of group extensions in [1893], Hölder worked out the theory to a much greater extent than was required for the solution of the problem under consideration. On other occasions he allowed himself to be sidetracked inasmuch as it seemed worthwhile and fruitful: an example here is the development of the idea of amalgamated direct product [1895a, 384–385]. However, he never lost sight of the task in hand, and was able to incorporate such diversions successfully precisely because he recognised their place in the overall picture.

One of the reasons that Hölder's group-theoretic papers are accessible to a modern reader is that he introduced various mathematical terms and notation (for example the term quotient group and symbol $G|H$) which have become standard and familiar expressions. His terms have been retained since they are the most appropriate; they are appropriate because he invented them within a very broad and deep understanding of group theory, an understanding that enabled him to recognise exactly where each idea fits in.

Underlying all of Hölder's research, not only that in algebra, was the philosophy of mathematics that he had inherited principally from his teachers, Weierstraß, Kronecker and Kummer. Hölder thought deeply and extensively about the foundations of mathematics; his ideas grew and developed to such an extent that he felt the need to gather them together in the form of a book. Here he could set out his philosophy systematically and comprehensively. His basic thesis was that it is not possible to embrace the whole of mathematics within any one logical framework, since the terms in which the framework itself is expressed are necessarily both part of mathematics and yet go beyond it. ~~This reasoning was justified by Gödel in the 'Incompleteness Theorem' published in 1931, some years after the appearance of Hölder's book.~~

Hölder's philosophy worked itself out in his own studies and teaching in a great desire for logical accuracy, both in the ideas under consideration and the way in which they

were expressed. He was of the firm opinion that a mathematician should not make a single step of reasoning, however small, without being able to justify it, and should preferably do so explicitly. In this way, Hölder's creativity was properly channeled and controlled. His viewpoint led to a natural reluctance to make conjectures—the few which do appear in his algebraic studies have more of a feel of 'general optimism' on some question than of conjecture. His pupils certainly seem to have appreciated the rigour and thoroughness of the exercises he set them and the lectures he gave.

It would be very gratifying to have one's curiosity satisfied as to the significance Hölder attached to his group-theoretic research towards the end of his life, at which time many of the problems he had studied were taken up again in earnest. It would be most interesting to know the extent to which he was aware of and active in encouraging this new research. He does not seem to have been tempted to return to it himself, or else he showed great wisdom and restraint in leaving it to younger mathematicians, which is on reflection the only sensible course of action after a thirty-year break from the subject.

It is difficult to imagine that Hölder could have realised how fundamental were the problems which he had posed, and how far-reaching had been his insights into their solutions. The only memorial that mathematics has kept to his research in group theory and Galois theory is in the name 'Jordan-Hölder Theorem', which was admittedly the theorem out of which so many of his ideas grew. But how often is Hölder's name associated with any of these ideas? His glory lies not in being remembered for his achievements but rather in providing a clear understanding of important problems and giving outlines, which would be developed by others, of some methods of solution. In this he was a very generous scholar.

He seems to have taken great pleasure in exploring all sorts of mathematical concepts throughout his life. It is almost as if, having realised there was far too much mathematics for him to be able to comprehend all of it in depth, he determined instead to study a very wide range of concepts in sufficient depth that he was able to make original and sometimes very significant contributions to them. A glance at the obituary written by B.L. van der Waerden [1939/Appendix B] reveals the breadth of his research interests.

Hölder's ability to turn from one area of mathematics to another was ably and concisely expressed by Van der Waerden [1939, 157]: Hölder was 'von jeder Einseitigkeit

frei' ('free of all one-sidedness'). Van der Waerden also commented on Hölder's gift of uncovering the essential patterns, concepts and lines of development of a subject, a gift that was not confined to group theory and Galois theory. In Van der Waerden's opinion, which the author shares, Hölder was one of those who gave modern mathematics its direction at the beginning of the twentieth century. He was able to move from concern with actual calculations to a consideration of concepts and to a critical perception of mathematical theories. In this he stands alongside some of the mathematical 'greats' of the late nineteenth and early twentieth centuries: his work is much more akin to that of Dedekind and Hilbert, for example, than to that of his teachers and those who influenced him directly.

He was nevertheless greatly influenced by his teachers and by his older colleagues, especially by Klein with regard to group theory and Galois theory. He was glad to accept and act on their suggestions and ideas. His success in developing these ideas may be attributed to his independent and original mind. Once he had taken hold of a concept or theory, he explored it thoroughly himself and published only the polished results of his meditations. He never produced revised versions of any of his original work, and none of his mathematical correspondence seems still to be in existence. This makes the historian's task at once much easier and much more difficult. It is all but impossible to reconstruct the *development* of Hölder's thoughts on a subject. It is comparatively simple, and has been a great pleasure, to give an account of some of his ideas as they appear in final published form.

APPENDIX A

Career Paths of German Mathematicians in the Nineteenth Century

The structure of academic life that existed in Germany during Hölder's working years had been developed in the early part of the nineteenth century. It consisted of various stages through which an aspiring academic would have to pass in order to qualify for appointment as a professor. The system is not easily understandable on one's initial contact with it. There exists a comprehensive account of it in German, written by Wilhelm Lorey¹ [1916], but there does not appear to be any published description of the system in English.

German academic reform in the early nineteenth century

Before 1800, the mathematical life in Germany and throughout Europe was divided into two parts: that at the universities was oriented towards teaching and that in the academies of science towards research. Mathematics in universities came under the jurisdiction of the philosophy faculties, which by the nineteenth century had begun to specialise in training schoolteachers for *Gymnasien* and *Realschulen*. At this time also, researchers from the academies became university teachers but were still responsible for carrying out research. The divide between teaching and research was gradually removed; individual universities and faculties nevertheless stressed one or the other of these functions. The development of mathematics at universities followed the pattern for arts rather than science subjects.

It was decided to introduce a career structure for those who wished to become academics in order to eliminate the haphazard method of appointing professors that had existed previously. Such a structure would also ensure that self-taught scientists were brought within the state system. The training of scientists would take place in much the same way as that of schoolteachers. Various levels were laid down through which

¹Lorey's book was written under the encouragement and guidance of Felix Klein, who was interested to see an account of how the study of pure and applied mathematics in Germany had changed during the nineteenth century, a change resulting in a great neglect of applied mathematics.

one would have to pass: *Doktor* (= postdoctoral research student), *Privatdozent* (= unsalaried lecturer), *Extraordinarius* (= assistant/associate professor) and *Ordinarius* (= full professor).

In the late eighteenth century there was a 'classical revival' in the German states and principalities². The ideals of ancient Greece were brought back into prominence, with their emphasis on morality and one's duty to the state. The school curriculum was revised accordingly (in various of the principalities) in order to base education on a solid grounding in Latin and Greek language, literature and thought. A new school-leaving examination, the *Abitur* (Latin for 'leaving'), was introduced in 1812 in Prussia. The schools at which it was possible to take this examination were designated *Gymnasien*.

In order to become a high school teacher or *Oberlehrer* one was required to pass a very difficult university examination. The profession was accorded much respect, and admittance to the rank of *Oberlehrer* was valued above receipt of a PhD. A prospective *Oberlehrer* in mathematics would have to be examined in both mathematics and physical science, a condition which was imposed until the end of the nineteenth century.

There were other types of secondary school with a more practical bias, the *Realschulen*. They gave an education including modern languages, practical mathematics and technology. Those who had attended a *Realgymnasium* (which taught Latin but not Greek) were not allowed to attend university courses in Prussia before 1870, and after then would only be qualified to teach in the *Realschulen*. The university-level equivalent of the *Realschulen* were the *technische Hochschulen* (institutes of technology, the majority of which were founded between 1800 and 1850), which provided more practical courses for those who would enter business or industry.

The classical bias of education meant that pure mathematical research was considered superior to applied research. Mathematics was to be pursued for its own sake. In the 1820s the idea was conceived of introducing seminars to encourage research and give further training to natural science teachers. These seminars were in operation in most German universities towards the end of the nineteenth century. They were made necessary partly by the expansion of the school system, which took place around 1870.

²Before Napoleon became (in effect) master of western Europe in 1807, there were several large German states such as Prussia, and hundreds of tiny principalities, of which very few had their own universities or academies. From 1815 (the year in which Napoleon was exiled to St. Helena) to 1866, Germany consisted of thirty-six states. During that time, Prussia had six universities and Bavaria had three.

They took different forms but were all intended to provide a forum for the more active participation of students in learning and research. The students in the first years of their studies were set exercises; those at the end of their degrees would lecture on their own research or that of others. Indeed it was assumed that those who wished to teach mathematics in school would have an aptitude for original research.

The expansion of the school system caused problems ten years later in the 1880s when the universities were overflowing with students. It was decided to reduce the intake of students and allow fewer to participate in lectures and seminars; this contraction took place at the end of the decade. It caused some stagnation at the universities since the students were drawn instead to *technische Hochschulen*, whose student intake came to equal that at the universities.

The grip of the traditional methods of teaching and the philosophy of education embodied in the *Gymnasien* was eventually broken at the end of the nineteenth century. The other educational establishments were accorded equal status and the value of practical and applied learning finally recognised.

The academic ladder

Undergraduate life

In order to enter university one was required to sit the *Abitur*—the qualifying examination—when eighteen or nineteen years of age, at the end of one's time at high school (*Gymnasium*). This is still the practice in Germany. It was possible to do some intermediate study at a college in order to be better prepared for university. Dedekind did so for two years at the Collegium Carolinum in Brunswick (whilst giving private mathematics lessons to a fellow student).

Having matriculated at a university (which was usually either the local one, if such existed, or one of the centres for mathematics such as Berlin or, later, Göttingen), one discovered that university life imposed hardly any restrictions. This academic freedom (for both students and teachers) was one of the principles of the German philosophy of university education. It was often challenged but was never overturned. The professors received their appointments (which were for life) from the sovereign of the state, the assistant professors from the minister for culture. This meant that they were *Staatsbeamte*, civil servants. They were free to teach what they wished in the way that suited

them. There was usually no more than a requirement to teach for a certain number of hours per week. It was equally understood that the professors were responsible not only for the academic progress of their students but for their general growth of character. Consequently they were expected to provide a good example and exert a positive influence on those they taught.

The students in turn could choose to attend any or no lectures and learn what suited them. There were no minimum requirements and no compulsory examinations until the end of the course. The course ended when the student felt prepared to be examined. The usual undergraduate life consisted initially of more play than work. It would eventually settle down to attendance at two or three lecture courses in each semester and a seminar, if there was one in a relevant subject.

The majority of university students were aiming to take the *Staatsexamen* that would qualify them to become teachers (and was, as the name implies, awarded by the state). Only a small percentage worked towards the *Doktorprüfung*. There were intermediate examinations in both courses of study, which were taken by those students with scholarships who needed a report to show they had been diligent in their studies. In general, these examinations were not compulsory and 'academic freedom' prevailed.

The academic year consisted of two semesters. The summer semester began in April and ended in August, with a holiday at Whitsun. The winter semester stretched from October to March, with two or three weeks' holiday at Christmas. There were other holidays: the Kaiser's birthday, the day on which the rector of the university was installed, and so on.

It was common practice after 1850 to attend various different universities during the course of one's degree. Hilbert spent time in Königsberg and Heidelberg, Hölder in Stuttgart, Berlin and Tübingen, Kronecker in Berlin, Bonn and Breslau. At the beginning of the nineteenth century, one had to complete six semesters (three years) in order to be eligible for a doctor's degree. In 1852 Dedekind completed his doctoral work with Gauss in only four semesters. In practice it usually took longer than three years, especially towards the end of the century, and so the rules were changed to give an eight-semester minimum. The dissertation required for a doctor's degree was expected to include some original research. Each student had a supervisor who had to approve the finished dissertation, which would be some 40 (octavo) pages. The student was then

required to have his dissertation published at his own expense and to send a copy to every German university. In addition, copies were usually sent abroad to well-known centres of learning.

Next there was an oral examination to pass, usually in front of three examiners and lasting several hours. The requirements for the writing and defence of one's dissertation were set down by the individual faculties and varied considerably. In practice it was one's supervisor or 'promoter' whose approval mattered. At some universities, the candidate was also required to be examined (orally) in his auxiliary subjects. This was the case at Tübingen in the early 1880s when Otto Hölder passed examinations in physics and chemistry. In 1845 Kronecker was asked questions on the application of the calculus of probabilities to observation, the method of least squares, definite integrals, series, differential equations, Greek and the history of legal philosophy. In 1909 Courant should have been asked questions on mathematics, physics and philosophy, but the examination became a complete fiasco when two of the examiners did not arrive. (He was awarded the degree anyway.)

There was usually a final hurdle: one's public 'promotion' (receiving a doctoral degree in Germany is called *Promotion*). In Hilbert's case this took place in the university hall (the university in question is Königsberg, the date 1885). Two fellow students were asked to engage in debate with Hilbert on subjects of his choice. This was not intended to be a serious competition but to establish that Hilbert could think clearly and understand what were the fundamental issues of the subject. The debate concluded successfully with the official degree ceremony. Hilbert was asked to swear this oath (see [Reid 1970, 17]):

I ask you solemnly whether by the given oath you undertake to promise and confirm most conscientiously that you will defend in a manly way true science, extend and embellish it, not for gain's sake or for attaining a vain shine of glory, but in order that the light of God's truth shine bright and expand.

The obtaining of a PhD in nineteenth century Germany was in fact not as difficult as it might seem. Pyenson [1983] is of the opinion that 'almost any determined *Gymnasium* graduate' would have been capable of it. The *Staatsexamen* for *Oberlehrer* were a different matter: determination was only one of the qualities needed to pass them.

Becoming a lecturer

Reception of the title *Doktor* was just the first step on the path to a full professorship. Unfortunately for the aspiring mathematician, the next step on the path was one he could not choose to take himself—it was in the gift of the faculty. In the meantime, the postdoctoral students were often encouraged to travel and visit mathematicians at other universities, especially the centres for mathematics such as Paris, Berlin and Göttingen. They would have the opportunity to talk to well-known mathematicians, attend lectures, discuss their own work and discover which were the main areas of current interest at other universities. Sometimes they came back inspired and full of ideas which would form the basis of their subsequent research, as did Klein and Lie from Paris in 1870.

In case a *Doktor* did not rise any further up the academic ladder, he could take the *Staatsexamen* in addition to his doctoral examinations and qualify as an *Oberlehrer*. Although this was a sensible course of action it did not happen often, since the lure of a professorship was so great. Hilbert was one of the few who took and passed the *Staatsexamen* (only three months after becoming a *Doktor*).

Somehow a postdoctoral student had to get himself noticed by a professor, which required more than academic ability alone. Constance Reid quotes Courant on the subject of 'getting oneself noticed':

... there is no way to come in contact with the instructors and the professors except by doing something which makes you stand out from the great impersonal mass of students.
[Reid 1976, 16]

The student who stood out in this way would be approached by a professor with the suggestion that he work under the professor's supervision to produce a *Habilitationsschrift*. This had to be a substantial original contribution to some area of mathematics. Since the number of students allowed to carry on in this way was limited, a gifted student (Otto Hölder, for example) might find there was no opportunity to do so at his university, but would be contacted by a professor from elsewhere.

The *Habilitationsschrift* had to be examined and accepted by the faculty. Afterwards the student would be given the right to lecture (*venia legendi*) and became a *Privatdozent*. He was a *private* lecturer since he did not receive a salary from the university, but was permitted to give lectures which the students would pay to hear. The lecture fees were generally his only source of income. Even this was hardly lucrative since the

compulsory and standard courses were always taught by the professors. A *Privatdozent* might lecture to five or six students—if numbers fell to three or four, the course might not be allowed to run. Thus lecturers had to choose their subjects carefully, but it inevitably happened that not all their proposed lectures would take place. They could expect to give one or two courses per semester; popular subjects were often repeated.

The *Privatdozenten* did not have an easy life financially, even though their number was strictly regulated by the faculty. If the maximum number of lecturers had been reached, there was no possibility of obtaining such a position until one of them moved elsewhere or was promoted. Their financial problems were generally relieved by family support. It would have been virtually impossible to carry on otherwise. The prestige attached to a full professorship was a considerable incentive for a family to support even a middle-aged son.

The start of one's lecturing career was marked by the inaugural lecture, which was always treated as a significant event and had to be very carefully prepared. The lecture was intended to give an opportunity to prove oneself before the faculty. The topic would usually be chosen from an area of current research (not necessarily one's own). An ambitious mathematician had to find an important problem whose significance would easily be recognised and present it in such a way as to make a favourable impression. In particular, the faculty had to be satisfied with the lecturer's ability in order for him to teach lecture courses in the university. By Courant's time (1912) in Göttingen it had become the custom to give this lecture without notes.

Professorships

The length of time spent as a *Privatdozent* varied enormously. Hilbert lectured in this capacity for seven years, Hölder for five and Klein for only one. All the time one hoped that one's name would begin to appear on shortlists for assistant/associate professorships (the title given was either *Extraordinarius* or *außerordentlicher Professor*; sometimes the two terms distinguished salaried and nonsalaried posts but there was often no difference between them). Inclusion on a shortlist usually ensured that an appointment was not far away. Although some of these professorships were nonsalaried, the majority provided a respectable income. The assistant professors were appointed to supplement the teaching of the full professors. Frequently an *außerordentlicher* professorship would be created in a newly developing branch of mathematics, in order that

suitable emphasis be placed on its teaching and study. However, the assistant professors had no influence at all on faculty matters, although the situation was improving during the early twentieth century.

Appointments to professorships, both assistant and full, were made in consultation with the local Ministry for Culture. The name is rather misleading: the *Kultusministerium* had nothing to do with culture but was the Ministry of Spiritual, Educational, and Medicinal Affairs³.

The transition from assistant to full professor was by no means assured and many scholars chose to take non-university academic jobs if such were offered to them. It was in fact often the case that a *Privatdozent* or *Extraordinarius* would be offered a position as teacher in a *Gymnasium* or in some college of education which did not have university status. Although such positions could still be steps on the way to a professorship, some academics preferred to remain outside university life. Dedekind worked at the *Polytechnikum* in Brunswick for thirty-two years. This institution had been the Collegium Carolinum (where he studied before university) and became a 'technical university' while he was its director.

The number of professors was limited: in the 1880s there were three at Berlin (the acknowledged centre for mathematics in Germany) and two at most other Prussian universities. There had been only 42 mathematicians employed in 1864 at the universities which were later within the German Reich; in 1900 there were 82. In comparison, this number reached 1,243 plus 1,497 assistants (with contracts for a restricted time) for the Federal Republic of Germany in 1976 [Gross 1981, 180–181]. Once a full professorship had been secured, it was very unlikely that an academic would move again unless another job with much improved conditions of employment were available at a particularly desirable university. The professors were in a position to negotiate with the Ministry for Culture and obtain extra benefits with their appointment (indeed, this was considered the normal procedure).

By the beginning of this century the mathematics professors at Göttingen were accustomed to having assistants. Although professors in the experimental sciences had previously had assistants who cared for equipment and set up experiments, Göttingen was the only German university to provide assistants for its mathematics professors. It

³Ministerium der geistlichen, Unterrichts- und Medizinalangelegenheiten

did so on Klein's insistence in the 1880s. The assistants (who were students or *Privatdozenten*) were paid a small salary, in return for which they attended the professor's lectures, took notes and prepared a neat copy to be put in the library. They could expect to spend much more time with the professor than the other students, and could consequently learn a great deal whilst ensuring that their work was noticed. When Courant was Hilbert's assistant he would research the literature for Hilbert's lectures and write reports on relevant points. He also read some of the papers submitted to Hilbert as one of the editors of the *Mathematische Annalen* and helped prepare them for publication.

The establishing of *Ordinarien* at the top of the academic ladder goes back to the sixteenth and seventeenth centuries in the German states. They were *professores publici ordinarii*, full members of the senate of their university. They were responsible for electing a rector each year who was the university's representative to the outside world. Sometimes the local nobleman would always be the rector but an election was still made to the post of *Prorektor*. The *Ordinarien* in each faculty chose an administrator from among themselves who occupied the post of *Dekan* for six months or a year.

APPENDIX B

Obituary of Otto Hölder

There follows an annotated English translation of B.L. van der Waerden's 'Nachruf auf Otto Hölder', *Mathematische Annalen* **116** (1939), 157–165. Van der Waerden wrote in the style which was usual at that time in Germany. Long sentences and complicated grammatical constructions were favoured. Consequently the translation has had to be rather free, but I have endeavoured to retain something of the style of the original. Use of the present tense to describe Hölder's work has mostly been replaced by use of the past tense, as is more usual in English.

The biographical section of the obituary is very short and factual, with no discussion of family background or children. It seems as though the information given is simply that which van der Waerden had to hand. Perhaps the impending war made it difficult to research the obituary properly. In particular, the important events in Otto Hölder's career are barely mentioned: to say 'er habilitierte sich 1884 in Göttingen' hardly does justice to the process. *Habilitation* was a very significant step and not to be treated lightly. In addition to the writing of a dissertation, Hölder was required to present an inaugural lecture in the presence of the entire faculty. He had somehow had to obtain the support of a faculty member (Schwarz) in order to be allowed to write a *Habilitationsschrift* at all. This was no easy task since the supply of doctoral students eager to become lecturers far outstripped demand.

The bibliography of Otto Hölder's published work that appeared in the *Mathematische Annalen* is also reproduced here, with various additions.

I am indebted to Professor B.H. Neumann for his friendly assistance with the preparation of this translation and for answering my questions so patiently.

Obituary of Otto Hölder

The Leipzig mathematician Otto Hölder died on 29th August 1937 at the age of 77. Not only did he serve as a member of the editorial team of the *Mathematische Annalen* from 1908 to 1928, but above all he contributed to our periodical a series of articles of

fundamental significance. With him the editorial staff have lost a loyal and involved colleague.

A truly great scientist has passed on, one of those men who, at the turn of the century, gave modern mathematics its direction: a movement from the formal to the critical, from calculation to the conceptual. He always aimed at strict logical accuracy in thought and expression. His papers are crystal-clear; reading them is again and again a profound intellectual treat.

He was, like all great exponents of our science, free of all one-sidedness. His work crossed new frontiers in algebra and large parts of analysis, but geometry, logic and number theory also owe to him valuable insights.

Otto Hölder was born in Stuttgart on 22nd December 1859, the son of Professor Otto Hölder who taught French at Stuttgart's *Polytechnikum*. He studied first in Stuttgart, then in Berlin with Weierstraß who left a lasting impression on his whole way of thinking, and finally in Tübingen with P. du Bois-Reymond. After gaining his doctorate in Tübingen in 1882 he went first to Felix Klein in Leipzig, but Klein's attitude and mode of thought differed too much from Hölder's for them to be able to cooperate fruitfully at that time; only later in Göttingen did their common interest in group theory bring the two men together. Hölder therefore left Leipzig and acquired the *venia legendi*¹ in Göttingen in 1884, was appointed assistant professor (*außerordentlicher Professor*) there in 1889 and received an associate professorship (*Extraordinariat*) in Tübingen in the same year. In 1896 he was invited to Königsberg² as successor to Minkowski and in 1899 succeeded to the chair for mathematics previously occupied by Lie³ at the university of Leipzig.

Shortly before the move to Leipzig he married Helene Lautenschläger from Stuttgart. From that time onwards he lived a quiet, withdrawn scholarly life within his family circle⁴. In 1899 he was elected to membership of the Saxon Academy of Science⁵. Up

¹The permission to lecture without salary to student classes in the university. It was obtained via the *Habilitation*, a further strict examination which involved producing a major piece of original research. This has lately been formalised as a second doctorate, the *Dr. habil.*

²now Kaliningrad

³Lie left Leipzig in June 1898 to return to Norway, where he died in 1899.

⁴His older brother Eduard was professor of Roman law at Leipzig up to his death in 1911.

⁵He was also a member of the Prince Jablonowski Scientific Society from 1914 and served as president from 1928. This society had been founded in 1774 by a Polish prince who wished to promote scientific

to the last years of his life he remained untiringly active in the pursuit of mathematical knowledge.

Those who knew him well valued his high intellect and integrity, his upright character and his kind nature.

The conceptual and not formulistic⁶ nature of his thinking is already evident in the formulation of the question which was the subject of his doctoral dissertation [1]: What conditions must be imposed in addition to the continuity of a mass distribution in order that the Laplace equation⁷

$$\Delta V = -4\pi\kappa$$

should hold for the potential of this distribution? To us such a question may appear natural, but at a time when the existence of [continuous] non-differentiable functions had only just been heard of, it was quite novel. The answer is equally original: the well-known "Hölder condition"

$$|\kappa(x, y, z) - \kappa(a, b, c)| < Ar^\mu \quad (\mu > 0).$$

In the same careful way, after a discussion of the concepts of surface and surface area, he investigated the potential of a continuous distribution on a surface. He proved the existence of the internal and external normal derivatives of the potential at a point on the surface and he calculated the discontinuity in the normal derivative. Hölder's results on the behaviour of the potential at the boundary of a bounded surface with a mass distribution are new.

The method Hölder created for investigating potential theory was further developed by Lichtenstein, Petrini and others (see the encyclopedia account by L. Lichtenstein). Even Hölder himself later produced further beautiful contributions to potential theory [30, 40, 44].

Hölder then turned to function theory under the influence of Weierstraß. The well-known theorem that an analytic function comes arbitrarily close to every value in the

research; a prize question was set every three or four years in some area of mathematics, with the incentive of the publication of the winning entry (as well as financial gain). In 1927 Otto Hölder became a corresponding member of the Bavarian Academy of Sciences. The king of Saxony bestowed on him the title *Geheimer Hofrat* ('privy councillor') in the spring of 1909. He joined the *Deutsche Mathematiker-Vereinigung* in 1891 soon after it was founded.

⁶*formelmäßig*: 'not measured by formulae'

⁷This is now known as Poisson's equation, of which Laplace's equation $\Delta V = 0$ is a special case.

neighbourhood of an isolated essential singularity [2] is due to him. Weierstraß had only dealt with the case of a function meromorphic in the whole [complex] plane, by difficult arguments using the product representation; Hölder however gave a direct, completely general proof.

The paper [3] has become fundamentally important for the theory of divergent series: the method of summation which has become known as Hölder summation⁸ is introduced there and it is shown that Abel's Limit Theorem remains valid for all Hölder-summable series. In his dissertation [5] for *Habilitation* and following on from du Bois-Reymond, Hölder investigated the question: under what conditions can the coefficients of a Fourier series which represent a function that is not necessarily continuous be represented in the familiar way as integrals? In order to be able to answer this question also for unbounded functions, the (improper) integral of such a function must first be defined, which Hölder accomplished in his own accurate way. Following this he established [6] a new sufficient condition for the representability of a continuous function by means of a Fourier series, which says: if one divides the interval into subintervals and approximates the function $f(x)$ in each subinterval by a linear function $g(x)$, then the sum of the quotients of the integrals $\int |f(x) - g(x)| dx$ divided by the respective interval lengths should tend to zero as the division is indefinitely refined⁹.

For a long time Hölder had struggled without success to find an algebraic differential equation for the gamma function, until the hopelessness of the undertaking gradually dawned on him¹⁰. However, as a seeker after complete clarity, he was not satisfied until

⁸This technique is equivalent to Cesàro summation; the equivalence was shown by Schnee [1909].

⁹Let $x_1 < x_2 < \dots < x_k$ be a partition of the interval and denote it by $P(x)$. Then call the partition $y_1 < y_2 < \dots < y_l$ a refinement of $P(x)$ if $x_1 = y_1$, $x_k = y_l$, and $x_i = y_{j_i}$ ($2 \leq i \leq k-1$) for some j_i . So the y -partition is obtained by adding extra division points to the x -partition. For a partition $P(x)$ define

$$A(P) := \sum_{i=1}^{k-1} \frac{\int_{x_i}^{x_{i+1}} |f(x) - g(x)| dx}{x_{i+1} - x_i}.$$

The condition that $A(P)$ tends to zero as the partition is refined becomes: for any $\epsilon > 0$ there exists a partition $x_1 < x_2 < \dots < x_k$ such that for all refinements Q we have $A(Q) < \epsilon$. Hölder seems to have been feeling his way towards the space now known as L_1 ; he approximated with linear functions and not step functions. In [6] Hölder thanked du Bois-Reymond for making known to him this theorem: it had been established by Weierstraß but had remained unpublished. Hölder's aim was to prove and extend it. Indeed Weierstraß wrote a generous footnote for Hölder's paper in which he gave Hölder full credit for his proof of the theorem, and emphasised that Hölder's work had been completely independent of his.

¹⁰In [1930] Lichtenstein asserts that Weierstraß had already conjectured that it was impossible.

he had completed a rigorous proof that such a differential equation is impossible [9]. By the same method he constructed a function which satisfies no algebraic functional equation. These two results together with investigations by Liouville have given rise to a series of studies by other researchers, on which Bieberbach gave a survey lecture at the Zurich Congress¹¹ of 1932.

At the end of his time in Göttingen, Hölder began his now classical investigations in group theory. Proceeding from a problem in Galois theory, namely that of the reduction of an algebraic equation to a chain of equations with simple groups without introducing accessory irrationalities¹², he arrived at the question of the uniqueness of these simple groups. Jordan had already proved that in a composition series of a finite group the indices (that is, the ratios of successive orders) are determined uniquely up to the order in which they appear. Hölder now [11] formulated the concept of factor group which has been fundamental for the development of group theory since that time and he proved the uniqueness of the composition factors up to isomorphism. Nowadays, one can hardly conceive of a group theory without factor groups and a Galois theory without the Jordan–Hölder theorem, so great has been the penetration throughout algebra of the concepts created by Hölder. He was also the first to prove [15] that a cubic equation cannot be solved in terms of real radicals in the *casus irreducibilis*.

In the papers which followed Hölder investigated, quite systematically, the structure of finite groups of given order. Since the simple groups form the building blocks of all groups, he considered these first [16]. Of the composite¹³ groups, those of orders p^2 and pq presented no difficulty so he considered [17] the groups of orders p^3 , pq^2 , pqr and p^4 . However, in order to proceed further, one first needs to know how to put together a composite group from a given factor group and a given normal subgroup. This question was thoroughly dealt with in the substantial article [18]. In order to be able to apply the results of this investigation in a concrete case one must know the automorphisms of the given normal subgroup. Accordingly, in the same article the automorphism groups of a collection of important groups were thoroughly discussed. Among other results this led incidentally to the theorem that the symmetric group is complete for $n \neq 6$, that is,

¹¹International Congress of Mathematicians

¹²That is, quantities that cannot be expressed as rational functions of the roots of the equation with coefficients from a given ground field.

¹³nonsimple

it has trivial centre and is isomorphic to its own group of automorphisms. The results of this investigation permitted the enumeration of all nonsoluble groups of orders < 200 . [In fact Hölder enumerated them up to order 480.]

Hölder also completely determined the structure of groups of squarefree order in that he proved [19] that each such group contains a cyclic normal subgroup with cyclic factor group.

In the intricate investigation [20] on the principles of mechanics a complex question¹⁴ was definitively settled, a question on which Jourdain and Réthy (*Math. Annalen* **48**, p. 514 [not 513]) had been unable to reach agreement¹⁵. Next there followed a series of papers on the foundations of geometry which took their starting point from his inaugural lecture at Leipzig, a lecture tending more towards the philosophical. These beautiful papers have sadly not received the attention which they deserve. In [25] Hölder investigated the question: under what conditions can a system of mutually comparable and additive quantities be numerically measured? The particular case of line segments subject to the axioms of congruence and the archimedean postulate was treated in full detail. In [26] he carried out the numerical parametrisation of the projective line, without reference to any embedding space, on the basis simply of the axioms of [point pairs in] harmonic position and the continuity axioms. The principal tool here is the infinitely continued harmonic bisection of a line segment. Finally, in [28] the calculation of coordinates in an affine or projective Desarguan geometry was given a new foundation without recourse to continuity and without the archimedean postulate.

In the short memoir [29] Hölder returned to the theory of Fourier series. In an elegant way he established necessary and sufficient conditions which the coefficients of a Fourier series must satisfy in order that the series represents a real analytic function.

In the elementary algebraic note [33] he calculated the value of a determinant in which the same number a appears everywhere above the main diagonal and the same number b everywhere below this diagonal. In [35] he gave an easier proof of the same formula,

¹⁴Are Hamilton's variational principles valid for nonholonomic motions?

¹⁵Their dispute occurred after Hölder's article had been published. Jourdain followed Hölder's method for deriving equations in generalised coordinates which are applicable to nonholonomic systems. Réthy gave an alternative derivation to which Jourdain objected. The argument is carried out in papers published in *Mathematische Annalen* **62**, **64** & **65** from 1906 to 1908. Finally Hölder wrote to Jourdain to clear up the matter by explaining that there were two different kinds of 'variation' under consideration.

a proof due to Hurwitz.

In the period 1914–1923 Hölder dealt mainly with logical and philosophical investigations into the foundations of mathematics. His attention was held by the question of the way in which mathematical concepts are formed and how such formation is possible, and this question never left him. In his *Programmabhandlung* [34] he justified the view that arithmetic needs no axioms but is based only on the logically consistent construction of its concepts. As he recognised however, the investigation must be carried out from an even broader basis. In his great treatise on mathematical methodology [38], Hölder systematically investigated logic, arithmetic, geometry, mechanics and a part of physics to discover their own peculiar deductive schemes and basic assumptions. Here I shall highlight only a few of the most important basic ideas of this work.

According to Hölder one of the essential features of mathematical methodology consists in building up a layer of concepts of a higher level onto the concepts already present; more specifically, the concepts and methods of proof of one level are taken as objects of mathematical investigation at the next level. For example, one first develops a method of proof and afterwards counts its steps, allocates them to other objects, or connects them together by means of relations. From this point of view Hölder concluded (and more recent logical investigations of Gödel's completely confirm this[⊕]) that one can never grasp the whole of mathematics in a single logical formalism precisely because the logical considerations which one applies to the forms of the formalism itself necessarily transcend the formalism but nevertheless belong to mathematics.

On the basis of logical considerations regarding the definition of subsets, Hölder rejected the concept of the power set, that is, the set of all subsets of a given set. It followed that the theory of the continuum founded on the Dedekind cut collapses. Hölder then saw that he was compelled to postulate the existence of the continuum by means of special axioms. This point has been attacked many times.

The fundamental concepts and axioms of geometry, just as those of mechanics and physics, were, according to Hölder's point of view which follows that of Helmholtz, abstracted from experience. Hölder dismissed the opposite philosophical viewpoint by detailed discussion. He justified his rejection of Weyl's intuitionism¹⁶ in a later note [43].

¹⁶Now better known by Brouwer's name.

⊕ In fact, Gödel's 'Incompleteness Theorem', published in 1931, is concerned with truth and provability in formal systems within mathematics, whereas Hölder was discussing meta-mathematics. Thus Hölder's view was not justified by Gödel.

Of his later investigations on special analytic functions two should be mentioned: [45] and [48], in which, following some investigations of Abel's, he justified a limit process and proved a functional equation for the function defined by

$$\Psi(x) = - \int \frac{\ln(1-x)}{x} dx.$$

Hölder had always loved the theory of numbers; in the last years of his life he could dedicate himself fully to this passion. In the notes [53, 57, 60, 61, 63] certain reciprocity laws for summation functions were established, first in special cases, then with increasing generality. The majority of them have the form

$$\begin{aligned} \sum_{\alpha < a \leq \alpha'} \phi(a) \Psi(f(a)) - \sum_{\beta' < b \leq \beta} \psi(b) \Phi(g(b)) \\ = \Phi(\alpha') \Psi(\beta') - \Phi(\alpha) \Psi(\beta) \quad [\beta = f(\alpha); \beta' = f(\alpha')], \end{aligned}$$

where Φ and Ψ are the summation functions of ϕ and ψ , and g is the inverse function of the monotone function f .

In the notes [54] and [58], Hölder started from known asymptotic forms for elementary number-theoretic functions and derived other such forms.

A general principle for the derivation of inversion formulas for number-theoretic functions was given in [59]:

$$G(x) = \sum_{n \leq x} a_n F\left(\frac{x}{n}\right) \iff F(x) = \sum_{n \leq x} b_n G\left(\frac{x}{n}\right).$$

Special attention should still be drawn to the note [65], in which a simpler expression was given for the "Ramanujan sum", that is, for the sum of the n -th powers of the primitive m -th roots of unity

$$\sum \zeta^n = \frac{\phi(m)}{\phi(m')} \mu(m') \quad \left[d = (m, n); m' = \frac{m}{d} \right].$$

It has not been possible to draw attention to every individual result in Hölder's rich output. It must suffice to have uncovered the principal lines of the conceptual development which runs through his whole life's work. He always sought after the fundamental ideas upon which the structure of mathematics is built; he pursued these ideas to their final consequences and tried to clarify them down to their deepest foundations. The totality of his thought has become the inalienable possession of science.

Otto Hölder's Publications

At the end of each entry is provided a reference by year and page number to the review of the paper or book in the *Jahrbuch über die Fortschritte der Mathematik*, abbreviated to F.

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APPENDIX C

Enrico Betti's Development of Group Theory

The name of Enrico Betti is usually associated with topology and not with algebra, partly because his name is remembered in the term 'Betti numbers' which Poincaré introduced. Betti, however, devoted much of his early study to algebra (during the period 1850–1860) and in particular to a commentary on the work of Galois. These studies have recently received attention in a book on the development of Galois theory [Toti Rigatelli 1989] and a paper by Mammone [1989]. Betti's research in Galois' theory of equations and in group theory has been discussed before: see, for example, [Wussing 1969], [Kiernan 1971] and [Bottazzini 1980]. The two more recent works appear, however, to be the first in which a detailed study of some aspects of Betti's work on group theory is carried out and an attempt is made to grasp the essential steps in Betti's approach to the theory. This has been done because the development of Galois' theory of equations presented by Betti rests on his formulation of group theory. Here, in his development of a theory of substitution groups, Betti was often not at all clear or rigorous. Thus Mammone asks several times in his paper if Betti's statements are correct or meaningful.

In order to make sense of Betti's explanation of Galois' theory, it is therefore necessary to unravel some of the confusion and difficulties in Betti's group theory. I attempt here to deal with some of the difficulties and to show that he was at least more-or-less consistent, if not always entirely correct. The interpretation of his ideas which follows still leaves several unanswered questions concerning the way in which, and the extent to which, Betti understood what he was doing and understood how to apply his results to the theory of equations. Betti seems to have been somewhat confused over various aspects of group theory. He certainly recognised the importance of developing such a theory, but did not have the special insight given to Galois, which enabled him to grasp the essential relationship between different concepts.

Betti nevertheless explored a wide range of ideas in substitution group theory. Here is a selection of the more significant of these ideas, translated into modern terms:

- (1) Conjugation of substitutions and groups;
- (2) The order of a substitution of a group is a divisor of the order of the group;
- (3) Normal subgroups;
- (4) The permutation representation on the cosets of a subgroup;
- (5) A primitive understanding of the concept of quotient group;
- (6) Split extensions;
- (7) Normal series and composition series for a group;
- (8) The construction of groups as the 'product' of smaller groups;
- (9) Normalisers of subgroups in the symmetric groups.

In addition, throughout his papers on substitution group theory and Galois theory, Betti analysed the structure and properties of various specific groups.

It is not possible to obtain such a translation of Betti's ideas into modern terms very rapidly. Wussing relates the story [1969, 94/1984, 128–129] that Liouville once said Galois' proofs are easy to understand—if one studies only them and nothing else for a couple of months. One encounters a similar problem (but for different reasons) in the case of Betti's theory, and consequently an understanding of Betti takes some time to form. My notation was therefore developed independently of Mammone's and Toti Rigatelli's; theirs has been adopted as far as possible.

There are several points to bear in mind concerning notation and terminology. Betti often expressed himself in words where symbols would now be used, or repeated a definition where we would make use of a standard term, as did many authors in the nineteenth century. It seems sensible to supplement his notation and make use of modern terminology. The only case where the modern has simply replaced Betti's own symbols and terms here is that of conjugation. His notation and terminology can also be confusing. As far as the group theory is concerned, this is mainly because Betti considered individual *substitutions* but collections (called 'groups', following Galois) of *arrangements*. The concepts he discussed would have made much better sense and are more naturally expressed using groups of *substitutions*. Betti, however, was following Galois and, as Mammone comments [1989, 145], Betti preferred to retain Galois' method.

The notation and terminology employed by Betti frequently suggest ideas to a twentieth century reader which Betti almost certainly did not have. He chose terms that now have a definite meaning within group theory (not necessarily that which he gave

to them), and that are consequently open to over-interpretation. However, the facts that Betti expressed his theory in this way and that the same sort of notation and terminology is employed nowadays suggest that he understood the general direction in which research should proceed.

The main differences in notation and terminology are set out below for ease of reference:

Betti [1852]	Mammone [1989]	here
gruppo di permutazioni G	groupe d'arrangements G	group of arrangements G
sostituzioni del gruppo G	groupe de substitutions $\hat{G}$	group of substitutions $\hat{G}$
derivato	dérivé	conjugate
$\theta_0, \theta_1, \dots, \theta_{p-1}$	$1, \psi_1, \dots, \psi_{m-1}$	$1, \theta_1, \dots, \theta_{p-1}$
permutazioni di G	$A_0, A_1, \dots, A_{m-1}$	$A\theta_0, A\theta_1, \dots, A\theta_{p-1}$
$\psi_0, \psi_1, \dots, \psi_{n-1}$	$1, \gamma_1, \dots, \gamma_{n-1}$	$1, \psi_1, \dots, \psi_{n-1}$
sostituzioni del gruppo Γ	$\hat{\Gamma}$	$\hat{\Gamma}$
$D_{\psi_i}G$	$D_{\psi_i}G$	H_i
sostituzioni del gruppo $D_{\psi_i}G$	$(D_{\psi_i}G)^\wedge$	$\psi_i^{-1}\hat{G}\psi_i$ or $\hat{H}_i$
sostituzioni del gruppo H	$\hat{H}$	$\hat{H}$
sostituzioni del gruppo K	$\hat{K}$	$\hat{K}$

Some biographical notes

Betti's contributions to mathematics and education in Italy are recounted in the numerous obituaries which appeared during 1892 and 1893, for example [Basso 1892], [Volterra 1892] and [Padova 1893]. Betti had been a student at the university of Pisa; he graduated in pure and applied mathematics in 1846. He subsequently worked as a schoolteacher at Pistoia, his home town in Tuscany, and then at Firenze. Before beginning his teaching career he had fought during 1848 in the Italian war of independence. He was appointed professor at his old university in 1857, aged thirty-four. Later he became director of the teachers' college[⊗] in Pisa.

His first paper, published in 1850, was on the subject of liquid flow through a tiny aperture. His attention turned after this to the study of algebra. In addition to his work on Galois' theory of equations, he studied the analytic solution of equations, involving elliptic functions and their generalisations. He came close to producing the solution of

⊗ the Scuola Normale Superiore, similar to the École Normale in Paris

the general quintic using elliptic functions (such a solution was obtained by Hermite in 1858). More studies of algebraic functions of a complex variable followed.

He returned during the 1860s to the more applied side of mathematics: potential theory, hydrodynamics, mechanics, elasticity theory and static electricity were among his interests. He had translated Bertrand's elementary algebra textbook into Italian in 1856; eleven years later an Italian edition of Euclid's *Elements* for use in schools was published at his instigation and under his and Brioschi's editorship. Betti had a long and fruitful career as a teacher, as is made clear by Ernesto Padova [1893, 620]:

L'affetto e l'ammirazione che gli scolari del Betti hanno sempre conservato per lui, dopo compiuti gli studi universitari, sono la più bella testimonianza del suo valore come insegnante. [1]

Betti was involved not only in research and education but also in politics. He served as a member of parliament, as Under Secretary of State for Public Education and eventually became a Senator in 1884. Nevertheless, his heart seems to have remained firmly in mathematical research. Soon after his death in 1892, it was suggested by Volterra (a pupil of Betti) that an edition of his collected works be produced; the first of two volumes appeared in 1903.

1852: The commentary on Galois

I concentrate on Betti's *Sulla risoluzione delle equazioni algebriche* [1852], which provided the first extensive commentary on the work of Galois on the solution of algebraic equations to be published. It appeared in a newly founded journal, in fact the first Italian periodical devoted only to mathematics and physics, the *Annali di Scienze matematiche e fisiche*. In 1852 Betti was at the beginning of his career, an unknown schoolteacher in Pistoia, and it therefore seems likely that he was keen to publish. This may account for some of his lack of clarity and his failure to develop ideas fully. One instance of such a failure to think through an idea resulted, as Kiernan has pointed out [1971, 105] and as is shown later, in the implicit (but false) assumption that every quotient group of a group G is isomorphic to a subgroup of G .

The mathematician Guglielmo Libri, who had been at the centre of a scandal during the 1840s involving books and manuscripts missing from French libraries, became interested in Betti's work on Galois theory during 1851. Betti sent him a copy of his first short note [1851a] on the subject, and there ensued a detailed correspondence between

them, in which Libri questioned Galois' results and Betti attempted to explain them¹. Betti must have appreciated very much the interest of such a famous mathematician, whose innocence over the missing manuscripts had always been asserted in his native Italy. Libri's not-very-positive attitude to Galois theory had presumably been formed when Galois' writings were published by Liouville, Libri's arch enemy, and it is unlikely that Betti was able to improve this attitude by showing that the theory was in fact correct (as Liouville had been so pleased to confirm). Indeed, Libri had tried to convince Betti to give up on this work: such questions, he was sure, must appear 'oscuri ed infeconde' ('obscure and fruitless') in Parisian circles.

Betti was not, however, to be persuaded and continued working on his commentary on Galois. Without the benefit of access to modern Galois theory textbooks, Galois' original papers must certainly have been considered obscure and all but impossible to explain coherently. There is ample evidence of this in that neither Liouville, nor any of the circle of mathematicians who were exposed to Galois' work in Paris seminars after 1846, had managed to produce and publish a commentary. It seems that Betti was not even aware that such seminars on Galois' theory existed (see [Bottazzini 1980, 27]).

Betti wrote his commentary of 1852 in two sections: the first half is an exposition of the theory of substitutions and of substitution groups and the second half deals with Galois' theory of equations. Betti sought in this way to emphasise the importance of group theory in mathematics, an importance that he realised was not confined to the theory of equations. He was influenced by the work of Cauchy [1815] on the 'number of values of a function'; a theory of substitutions is developed in this paper. He was familiar with Serret's *Cours d'Algèbre Supérieure* [1849]. He did not apparently know of Cauchy's studies of 1845 and 1846 in which the theory of substitution *groups* is developed. These studies were not widely known until Bertrand lectured on them after Cauchy's death in 1857. Betti was of course also influenced by the 1846 publication of Galois' papers by Liouville although, as will become clear, Betti's understanding of them was rather muddled.

Betti stated in the preface to his paper that the important theorems of Abel and Galois were, to a large extent, lacking in proof. He later remarked that the conditions for solubility by radicals of equations of nonprime degree had been only partly announced

¹Some of their letters are reproduced in [Procissi 1953].

by them, and without proof. Both these statements seem rather bold; nevertheless, Betti obviously thought that he was filling in some substantial gaps in the theory which had not been dealt with previously. Peter Neumann points out in his review [1991] of Mammone's paper that Galois' *Premier mémoire* provided a *general* theory which can be applied to equations of any degree.

Introduction to group theory

The notation that Betti used for substitutions, which is based on that developed by Cauchy and Galois, has been discussed by Wussing [1969, 92/1984, 125–126] and Kiernan [1971, 104]. Betti spent several pages towards the beginning of the paper defining and explaining conjugation of substitutions [1852, 55–58/36–38]. His term for this operation translates as *derivation*, so that a conjugate substitution is a *sostituzione derivata*. He went on to give a definition of a group of arrangements (*gruppo di permutazioni*) and its associated substitutions (see also [Mammone 1989, 144–145]).

A *substitution* (the common nineteenth-century term) or *permutation* (the usual modern term) is a bijective mapping that permutes a set of objects amongst themselves. In Galois' theory of equations these objects are the roots $x_1, x_2, \dots, x_\nu$ of the equation under consideration. Suppose the roots are arranged as follows:

$$x_1 \ x_2 \ \dots \ x_\nu.$$

This arrangement of the roots may be permuted to give another arrangement

$$x_{\phi(1)} \ x_{\phi(2)} \ \dots \ x_{\phi(\nu)},$$

where ϕ is some bijective mapping of the indices. The substitution which takes the first arrangement to the second is denoted by

$$\begin{pmatrix} x_i \\ x_{\phi(i)} \end{pmatrix}.$$

In fact ϕ may be viewed as the substitution itself: Betti seems to have preserved some distinction between the two, but referred to this substitution as ϕ 'for brevity'.

Now suppose we have a collection of p ($p \in \mathbb{N}$) arrangements of the roots:

x_1	x_2	$\dots$	x_ν	A_1
$x_{\phi(1)}$	$x_{\phi(2)}$	$\dots$	$x_{\phi(\nu)}$	A_2
$x_{\psi(1)}$	$x_{\psi(2)}$	$\dots$	$x_{\psi(\nu)}$	A_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_{\theta(1)}$	$x_{\theta(2)}$	$\dots$	$x_{\theta(\nu)}$	A_p

I denote the first arrangement by A_1 , the second by A_2 , and so on. These form a *group of arrangements* G if the following is satisfied:

Let χ_{ij} be the substitution that takes A_i to A_j , that is, $A_i\chi_{ij} = A_j$. Then it is required that $G\chi_{ij} := \{A_1\chi_{ij}, A_2\chi_{ij}, \dots, A_p\chi_{ij}\}$ is equal to the original collection $G := \{A_1, A_2, \dots, A_p\}$ for all possible choices of i and j . This means that no new arrangements are created when a substitution of the form χ_{ij} is applied to all the existing arrangements.

The *substitutions of the group of arrangements* are defined to be those substitutions that take one arrangement of G to all the others. If $A_1\psi_i = A_i$, then $\psi_2, \psi_3, \dots, \psi_p$ are the substitutions of G . It follows that $\psi_r\psi_s = \psi_t$ for any choices of r and s , that is, this set of substitutions is closed under composition. Consequently the identity substitution must be included among them. The set of substitutions forms what we think of nowadays as a permutation group; I follow Mammone in denoting it by $\hat{G}$.

Both Kiernan and Mammone comment that Betti did not consider the identity substitution to be a substitution. When Betti wrote down the substitutions of a group, he usually gave $n - 1$ or $p - 1$ substitutions and not n or p as one might expect. However, on the first few occasions when such an enumeration was necessary [1852, 59–60/39–40], in particular when he defined what he meant by the substitutions of a group of arrangements, he gave exactly p of these substitutions. It is therefore unclear whether he had in fact realised that an identity substitution was needed or if he simply forgot to be consistent and overlooked the resulting mistake in his statement.

Perhaps an example might be useful towards making these ideas clear. Here is a group of arrangements of the quantities a, b, c, d :

$$\begin{array}{cccc} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{array}$$

If we take the first arrangement (the top line) as point of reference, the substitutions

of this group of arrangements are, respectively,

$$\begin{aligned} &1 \\ &(ab)(cd) \\ &(ac)(bd) \\ &(ad)(bc) \end{aligned}$$

The same substitutions are produced by taking any of the other arrangements as point of reference. Thus the group of arrangements corresponds to the substitution group V_4 , the Klein-4 group.

The order of groups and substitutions

After giving his definition of groups of arrangements and their substitutions, Betti attempted to prove [1852, 59–60/39–40] that the order of a substitution of a group is always a divisor of the number of arrangements of the group². This fact, nowadays often proved as a corollary of Lagrange's Theorem, was not new in Betti's day. It can be traced back to Ruffini (1799) and Abbati (1803), and a clear proof of it was given by Cauchy [1815, 5–6/68–69]. Betti's supposed proof of this statement is, however, almost completely nonsensical. He was trying to show that if q is the order of a substitution ψ_0 of a group, then the number of arrangements of the group (the order of the group) is divisible by q .

The first part of his proof appears to be 'proof by contradiction'. If q is strictly less than the order of the group, then there must be other substitutions not contained in the subgroup generated by ψ_0 . Let one such be ψ_1 of order n . Then, claimed Betti, these two substitutions give rise to qn different arrangements (and therefore qn different substitutions). This is clearly not true in general. Betti's argument purports to show that if there are not qn different arrangements obtainable from the substitutions $\psi_0^r \psi_1^s$ then ψ_1 is, after all, a power of ψ_0 (and there remain only q arrangements). His line of reasoning depends, however, on the unstated (and generally unwarranted) assumption that q and n are coprime.

It is difficult to make sense of the second part of his proof as it stands. A coherent argument can be extracted from it under the assumption that there are two printing

²The number of arrangements of the group equals the number of substitutions of the group.

errors here. Betti's text reads as follows:

Se q non è eguale al numero delle permutazioni del gruppo, si avrà anche un'altra sostituzione ψ_2 , colla quale si otterranno altre qn permutazioni, tutte differenti dalle precedenti: poichè supponiamo

$$\psi_0^r \psi_1^s = \psi_0^u \psi_2,$$

si avrebbe

$$\psi_2 = \psi_0^{r-u} \psi_1^s,$$

ossia ψ_2 eguale a una delle precedenti sostituzioni, contro il supposto. [2]

If the letters q and qn are reversed, then the first part of the proof becomes 'proof by contradiction'. Thus it is now assumed that there are qn distinct arrangements of the form $\psi_0^r \psi_1^s$. If these still do not make up the entire group of arrangements, Betti asserted that there must be another substitution ψ_2 not contained amongst the elements $\psi_0^r \psi_1^s$. Then, as he showed, neither is any substitution of the form $\psi_0^u \psi_2$ contained amongst the $\psi_0^r \psi_1^s$. Therefore

$$\psi_2, \psi_0 \psi_2, \psi_0^2 \psi_2, \dots, \psi_0^{q-1} \psi_2$$

are q new substitutions which, added to those obtained already, give a total of $q(n+1)$ substitutions. Since the argument does not depend on the fact that the first qn substitutions are of the form $\psi_0^r \psi_1^s$ (Betti ought to have taken them to be of the form $\psi_0^r \psi_i$ where $\psi_1, \psi_2, \dots, \psi_n$ are chosen one by one), any other substitutions that are included give rise, in the same way, to a multiple of q as the total number. Therefore q divides the order of the whole group.

This proof bears some resemblance to the standard modern proof, in which the elements of the subgroup generated by ψ_0 are written down:

$$1, \psi_0, \psi_0^2, \dots, \psi_0^{q-1}$$

and other substitutions (not already present) are added one by one until the group is complete:

$$\begin{array}{ccccccc} 1 & \psi_0 & \psi_0^2 & \dots & \psi_0^{q-1} & & \\ \psi_1 & \psi_0 \psi_1 & \psi_0^2 \psi_1 & \dots & \psi_0^{q-1} \psi_1 & & \\ \psi_2 & \psi_0 \psi_2 & \psi_0^2 \psi_2 & \dots & \psi_0^{q-1} \psi_2 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \end{array}$$

This seems to be the line of reasoning towards which Betti was moving.

Conjugate groups

Betti next turned to consider the conjugation of groups. This is where his conclusion that every quotient group of a group is isomorphic to a subgroup of that group had

its origins. Quite correctly, he realised that conjugation works for groups as well as for substitutions, but he took the generalisation one step further than is actually required. He insisted that not only are we to conjugate a group of substitutions, but also that we are to conjugate *by means of* a group of substitutions. That is, the ‘conjugating’ substitutions must as a whole form a group. He did not consider the possibility here of conjugating a group by a single substitution.

His explanation [1852, 60–61/40] refers to the arrangements as well as to the substitutions acting on those arrangements (his notation has been supplemented for the sake of clarity):

Let G be a group of arrangements of the quantities $x_1, x_2, \dots, x_\nu$. Let one of the arrangements in the group be denoted by A . Let $\theta_1, \theta_2, \dots, \theta_{p-1}$ be the substitutions which take this arrangement of the quantities to the others in the group. Then the other arrangements may be written as $A\theta_1, A\theta_2, \dots, A\theta_{p-1}$. Borrowing Mammone’s notation, I write $\hat{G} := \{1, \theta_1, \theta_2, \dots, \theta_{p-1}\}$ for the group of substitutions corresponding to the group G of arrangements. Similarly, let Γ be another group of arrangements of $x_1, x_2, \dots, x_\nu$; it also has a corresponding group of substitutions $\hat{\Gamma} := \{1, \psi_1, \psi_2, \dots, \psi_{n-1}\}$.

It has been assumed (as can be seen by comparing Betti’s with my notation) that $\theta_0 = 1$ and $\psi_0 = 1$. This seems to be consistent with Betti’s definition of the substitutions of a group [1852, 59/39], as noted above. Later in his paper [1852, 64/43 ff], Betti seems to have omitted to include the identity substitution and the number of substitutions in his groups was one fewer.

Another group is obtained by means of the groups $\hat{G}$ and $\hat{\Gamma}$ by acting on the arrangements of G with a substitution $\psi_i \in \hat{\Gamma}$, so that this new group of substitutions corresponds to the group of arrangements

$$\{A\psi_i, A\theta_1\psi_i, A\theta_2\psi_i, \dots, A\theta_{p-1}\psi_i\} := H_i.$$

Then the new group of substitutions $\hat{H}_i$ is in fact $\psi_i^{-1}\hat{G}\psi_i$, the conjugate of $\hat{G}$ by ψ_i . This is the case because

$$(A\psi_i)(\psi_i^{-1}\theta_j\psi_i) = A\theta_j\psi_i$$

for $j = 0, 1, \dots, p-1$. Therefore in order to transform any arrangement of H_i into any other, a substitution of the form $\psi_i^{-1}\theta_j\psi_i$ is required.

Next Betti asserted that each conjugate group of arrangements H_i (for $i = 1, 2, \dots, n-1$) is either *eguale* (*equal*) or *simile* (*similar*) to the original group of arrangements G , and to the other conjugate groups H_i . He used the word *eguale* in an unconventional sense: two groups of arrangements G_1 and G_2 are equal if the *substitutions* of both groups are the same ($\hat{G}_1 = \hat{G}_2$), although the arrangements themselves may be different. The term *simile* is applied when the substitutions of the groups are different ($\hat{G}_1 \neq \hat{G}_2$), but $\hat{G}_1$ and $\hat{G}_2$ both contain the same number of substitutions of each order. (This implies, as Betti noted, that G_1 and G_2 contain the same number of arrangements.) The property of similarity is rather redundant in the present context, since (as Mammone points out [1989, 146]) similarity is a much weaker attribute than conjugacy, and seems to serve little purpose beyond what could be expressed with the term ‘not equal’. Indeed, it leaves out of account the important quality of conjugate groups—that their *structure* is the same. Mammone provides an example [1989, 146] of two similar groups that are not conjugate. He also notes that similarity does not even imply isomorphism, and mentions two families of similar but nonisomorphic groups.

The term *semblable* was used by Cauchy in his researches of 1845–1846 with the same meaning that the term ‘conjugate’ has nowadays. He proved that two substitutions in the symmetric group are *semblable* if and only if they have the same cycle structure. It is fairly certain, however, that Betti had not discovered this work before or during the production of his paper of 1852. Cauchy had not defined the term *semblable* in the earlier paper [1815] of which Betti did have knowledge in 1852. Galois had also employed the term in the *Premier mémoire* [1846a, 430/47/65] without any explanation of what he meant by it. He applied it to the conjugates of a normal subgroup (which are not identical as groups of arrangements), but in this instance he seems to have used it as an ordinary word of the French language rather than a technical mathematical term. It is just possible that Betti took over the adjective *simili* from Galois’ *semblable* without realising the meaning, or rather the lack of meaning, that Galois had given to it.

Products of groups without the condition of normality

Betti considered the situation in which none of the conjugate groups of arrangements is ‘equal’ to another, that is, in which none of the (nonidentity) substitutions of $\hat{\Gamma}$ normalises $\hat{G}$. He produced the ‘sum’ of the conjugate groups, which (since he was thinking in terms of arrangements) was formed by taking the set-theoretic union of all

the conjugate groups of arrangements, that is, $H := \bigcup_i H_i = \{A\theta_j\psi_i : j = 0, \dots, p-1; i = 0, \dots, n-1\}$. Betti investigated the case when H is not simply a collection but a *group* of arrangements, or equivalently, when the substitutions of H form a group $\hat{H}$.

The substitutions in $\hat{H}$ are all those of the form $\theta_j\psi_i$. This is because the arrangement A is a member of H and the substitutions required are precisely those which take A to any other member of H . The condition which must be imposed in order that H is a *group* of arrangements is that $\hat{G}$ and $\hat{\Gamma}$ commute with one another (as groups). Betti did not, however, state this condition. It follows from the theorem: Given a group L , and $M, N \leq L$, then MN is a group if and only if $MN = NM$. The group L in this case is simply the symmetric group of an appropriate degree (the degree is the number ν of quantities in each arrangement). Mammone remarks [1989, 148] that H is a group of arrangements if $\hat{\Gamma}$ normalises $\hat{G}$, which is a special case.

The substitutions of $\hat{H} = \{\theta_j\psi_i : j = 0, \dots, p-1; i = 0, \dots, n-1\}$ permute the conjugate groups of arrangements H_i amongst themselves. Betti proved this by showing that if $\phi \in \hat{H}$ transforms an arrangement in G into an arrangement in one of its conjugates H_r , then it transforms the whole of G into this conjugate group H_r . Now denote the arrangement $H_0, H_1, \dots, H_{n-1}$ of the conjugate groups by D_0 . This is indeed an arrangement (that is, it has no repetitions) since none of the nonidentity substitutions of $\hat{\Gamma}$ normalises $\hat{G}$, which certainly implies that $\hat{G} \cap \hat{\Gamma} = \{1\}$, therefore the H_i are all different. Then by means of this action of $\hat{H}$ we obtain a group of arrangements of D_0 . Betti called this the *gruppo delle permutazioni sopra i derivati* (*group of arrangements on the conjugates*) and denoted it by K . Thus $K := \{D_0\theta_j\psi_i : \theta_j\psi_i \in \hat{H}\}$. The substitutions of K form a group $\hat{K}$, which is a group of substitutions of the H_i .

Betti claimed that $\hat{K}$ contains the same number of substitutions as $\hat{H}$, and in addition that these two groups contain the same number of substitutions of each order. They are, in his terminology, 'similar' to one another. This claim is easier to understand if we deal with substitutions rather than arrangements³:

For each value of i , the set H_i of arrangements considered as a subset of H corresponds to the set of substitutions $\{\theta_1\psi_i, \theta_2\psi_i, \dots, \theta_{p-1}\psi_i\}$ in the group $\hat{H}$, that is, to the right coset $\hat{G}\psi_i$. Therefore the above action of $\hat{H}$ on the set $\{H_0, H_1, \dots, H_{n-1}\}$ of sets of

³Mammone says [1989, 149] that he is unsure of what Betti meant here. The following explanation seems to be consistent with Betti's paper.

arrangements H_i is equivalent to the action of $\hat{H}$ by right multiplication on the set $\{\hat{G}, \hat{G}\psi_1, \dots, \hat{G}\psi_{n-1}\}$ of right cosets of $\hat{G}$. Denote the set of right cosets by X . Then, for $\phi \in \hat{H}$, we have a mapping $\rho_\phi : X \rightarrow X$ given by

$$\rho_\phi(\hat{G}\psi_i) = \hat{G}\psi_i\phi.$$

This mapping ρ_ϕ is a permutation of X , that is, $\rho_\phi \in \text{Sym}(X)$, where $\text{Sym}(X)$ is the group of all permutations of the set X . If we define $\rho(\phi) = \rho_\phi$, then $\rho : \hat{H} \rightarrow \text{Sym}(X)$ is a homomorphism—the permutation representation of $\hat{H}$ on the right cosets of $\hat{G}$. Betti's group $\hat{K} \leq \text{Sym}(D_0)$ is then isomorphic to $\text{Im}\rho \leq \text{Sym}(X)$, that is, to the image of this permutation representation. The condition required to ensure that $\hat{K}$ and $\hat{H}$ have equal order is that ρ should be injective. Therefore in modern terms, Betti's claim is that $\text{Ker}\rho = \{1\}$.

Now

$$\begin{aligned} \text{Ker}\rho &= \{\phi \in \hat{H} : \rho(\phi) = 1\} \\ &= \{\phi \in \hat{H} : \hat{G}\psi_i\phi = \hat{G}\psi_i \text{ for } 0 \leq i \leq n-1\} \\ &= \{\phi \in \hat{H} : \hat{G}\psi_i\phi\psi_i^{-1} = \hat{G} \text{ for } 0 \leq i \leq n-1\} \\ &= \{\phi \in \hat{H} : \psi_i\phi\psi_i^{-1} \in \hat{G} \text{ for } 0 \leq i \leq n-1\} \\ &= \{\phi \in \hat{H} : \phi \in \psi_i^{-1}\hat{G}\psi_i \text{ for } 0 \leq i \leq n-1\} \\ &= \bigcap_{i=0}^{n-1} \psi_i^{-1}\hat{G}\psi_i. \end{aligned}$$

The conjugate groups of arrangements H_i are 'similar' but not 'equal', that is, $\psi_i^{-1}\hat{G}\psi_i \neq \hat{G}$ for $1 \leq i \leq n-1$. Therefore Betti claimed that, in any such situation, $\bigcap_{i=0}^{n-1} \psi_i^{-1}\hat{G}\psi_i = \{1\}$. However, this is not necessarily true (a counterexample is provided in [Neumann 1991]). It is quite possible that $\hat{G}$ contains a nontrivial normal subgroup of $\hat{H}$; this subgroup then forms the kernel of the homomorphism ρ . If no such subgroup exists, then ρ is injective and $\hat{H}$ and $\text{Im}\rho$ certainly contain the same number of substitutions of each order, as Betti wrote: in fact, they are isomorphic.

To a twentieth century reader this isomorphism means that the two groups may be considered as having the same substitutions. Perhaps we would say that they possess Betti's property of equality. Betti, however, did not state that H and K are 'equal' and on reflection this is not at all surprising. The two groups of arrangements H and

K are arrangements of two distinct sets: on the one hand, the original set of quantities $x_1, x_2, \dots, x_\nu$ and on the other hand, the set of conjugate groups of arrangements $H_0, H_1, \dots, H_{n-1}$. The fact that the group $\hat{H}$ is (or, rather, might be) isomorphic to $\hat{K}$ is a fact which Betti's language and thinking did not seem to have the capacity to assimilate. What Betti was able to recognise were precisely those conditions which he had made into his definition of 'similar': that the substitutions of the groups are equal in number and of equal orders. How then did he conceive the group K , which we now understand as a permutation representation? If he did not realise that it is a 'representation' of H , how were the two groups connected in his mind?

Semidirect products and normal series

Betti turned next to the situation in which the conjugate groups of arrangements H_i are all 'equal', that is, in which the substitutions of $\hat{\Gamma}$ normalise $\hat{G}$. He did not seem to allow for any intermediate case, when some (but not all) of the nonidentity substitutions of $\hat{\Gamma}$ normalise $\hat{G}$. He assumed here in addition that the substitutions of $\hat{G}$ and $\hat{\Gamma}$ are distinct: he specified [1852, 62/41] that the conjugate groups of arrangements are to be equal *but not identical* to each other. That is, the substitutions of the conjugate groups are to be the same but their arrangements are to be different. Betti expressed this idea of normality by writing $\bar{G} = D_{\psi_0} G = D_{\psi_1} G = \dots = D_{\psi_{n-1}} G$, which in my notation is $G = H_0 = H_1 = \dots = H_{n-1}$. He used the sign '=' as a symbol for his term 'equal', even though the equality is of substitutions and not of arrangements. At a first reading it appears as if he was contradicting himself and expressed exactly the opposite idea in symbols of that which he had just put in words. The possible confusion here seems, however, to be caused by his lack of notation for groups of substitutions.

The situation described above occurs precisely when $\hat{\Gamma}$ normalises $\hat{G}$ and $\hat{G} \cap \hat{\Gamma} = \{1\}$. This is because the fact that the groups of arrangements $H_0 = G, H_1, \dots, H_{n-1}$ are equal is expressed by putting $\psi_i^{-1} \hat{G} \psi_i = \hat{G}$ for $1 \leq i \leq n-1$, which simply says that $\hat{\Gamma}$ normalises $\hat{G}$. If $\hat{G} \cap \hat{\Gamma} \neq \{1\}$, there exist j_1 and i_1 such that $\theta_{j_1} = \psi_{i_1} \neq 1$, and so

$$\begin{aligned} G &= G\theta_{j_1} \quad (\text{by definition of a group of arrangements}) \\ &= G\psi_{i_1}. \end{aligned}$$

Therefore the conjugate groups of arrangements G and H_{i_1} are identical, which is not allowed. This point was brought up in [Neumann 1991]: Betti's theory requires mod-

ification if $\hat{G} \cap \hat{\Gamma} \neq \{1\}$. Later [1852, 65/43] Betti dealt with the situation when the intersection is allowed to be nontrivial (this is considered in the next section of this appendix).

If $\hat{\Gamma}$ normalises $\hat{G}$, the group $\hat{H}$ may be formed as before: its substitutions are again given by $\theta_j \psi_i$ (or by $\psi_i \theta_j$, since $\hat{G}\hat{\Gamma} = \hat{\Gamma}\hat{G}$). But now we have come across a particular type of group product, familiar to a twentieth-century reader: the group $\hat{G}$ is a normal subgroup of $\hat{H}$, and $\hat{H}$ is the *split extension* of $\hat{G}$ by $\hat{\Gamma}$. The group $\hat{H}$ may also be treated as the semidirect product of $\hat{G}$ by $\hat{\Gamma}$, with multiplication given by

$$(\psi_{i_1} \theta_{j_1})(\psi_{i_2} \theta_{j_2}) = \psi_{i_1} \psi_{i_2} \theta_{j_1}^{\psi_{i_2}} \theta_{j_2}$$

where $\theta_{j_1}^{\psi_{i_2}} = \psi_{i_2}^{-1} \theta_{j_1} \psi_{i_2}$, which is in $\hat{G}$.

Betti gave a name to the group formed in this situation by specifying that H is to be called the *product* of G by Γ , which he expressed symbolically as $H = G\Gamma$. His use of groups of arrangements is again rather misleading, since one cannot ‘multiply’ arrangements by one another. Betti understood that it is the substitutions *corresponding* to these arrangements which are multiplied together to form the ‘product’, but he had no notation by which to express this, since all his notation was based on his consideration of a ‘group’ as a group of arrangements. He also referred to Γ as a *moltiplicatore* (*multiplier*) of G and as a *divisore* (*divisor*) of H . However, since $\hat{\Gamma}$ is both a subgroup of $\hat{H}$ and is isomorphic to $\hat{H}/\hat{G}$, it is difficult to tell exactly what Betti intended these terms to mean.

Having described how to construct a larger group from two smaller groups, Betti immediately went on to consider how one of the smaller groups, Γ , is itself built up as a ‘product’. These remarks of his form an introduction to his discussion of normalisers in the next section of the paper. He appears to have wished to show how the normaliser of a group can be formed from the group itself in several stages, by using his product construction repeatedly. Thus he wrote [1852, 62/41]:

Se [If]

$$\Gamma = G_1 \Gamma_1, \quad \Gamma_1 = G_2 \Gamma_2, \quad \dots, \quad \Gamma_{n-2} = G_{n-2} G_{n-1}$$

sarà [it will be]

$$H = G G_1 G_2 \dots G_{n-1}.$$

Betti included a proviso: *if* Γ splits in this way, then H will be a product of many groups. He was either acknowledging that Γ may not be formed by means of repeated semidirect

products, or the word 'if' means 'if we write' and refers only to the introduction of new notation (in which case he assumed that Γ can be formed in this way).

What is the meaning of this statement of Betti? A translation into substitution group language helps to shed light on it. The groups of substitutions corresponding to the groups of arrangements G_i and Γ_i ($i = 1, 2, \dots, n-1$) will be denoted as usual by $\hat{G}_i$ and $\hat{\Gamma}_i$, respectively. Then his statement may be interpreted as follows: the group $\hat{\Gamma}$ has a normal subgroup $\hat{G}_1$ and a subgroup $\hat{\Gamma}_1$ isomorphic to $\hat{\Gamma}/\hat{G}_1$. The subgroup $\hat{\Gamma}_1$ has a normal subgroup $\hat{G}_2$ and a subgroup $\hat{\Gamma}_2$ isomorphic to $\hat{\Gamma}_1/\hat{G}_2$, and so on. Betti's statement may simply be read in terms of substitutions rather than of arrangements:

$$\hat{\Gamma} = \hat{G}_1 \hat{\Gamma}_1, \quad \hat{\Gamma}_1 = \hat{G}_2 \hat{\Gamma}_2, \quad \hat{\Gamma}_2 = \hat{G}_3 \hat{\Gamma}_3, \quad \dots, \quad \hat{\Gamma}_{n-2} = \hat{G}_{n-2} \hat{\Gamma}_{n-1}.$$

Therefore the group $\hat{G}_1$ appears to be a normal subgroup of $\hat{\Gamma}$ of small order. The group $\hat{G}_2$ is then a normal subgroup ^{cf} contained in a complement subgroup to $\hat{G}_1$. Thus $\hat{G}_1 \hat{G}_2$ is also a normal subgroup of $\hat{\Gamma}$, and so on. When such a situation occurs, a normal series for $\hat{\Gamma}$ is obtained:

$$\{1\} \leq \hat{G}_1 \leq \hat{G}_1 \hat{G}_2 \leq \hat{G}_1 \hat{G}_2 \hat{G}_3 \leq \dots \leq \hat{G}_1 \hat{G}_2 \dots \hat{G}_{n-1} = \hat{\Gamma}.$$

So the group $\hat{H} = \hat{G} \hat{\Gamma}$ also has a normal series:

$$\{1\} \leq \hat{G} \leq \hat{G} \hat{G}_1 \leq \hat{G} \hat{G}_1 \hat{G}_2 \leq \hat{G} \hat{G}_1 \hat{G}_2 \hat{G}_3 \leq \dots \leq \hat{G} \hat{G}_1 \hat{G}_2 \dots \hat{G}_{n-1} = \hat{H}.$$

A refinement of maximal length of such a normal series is nowadays called a chief series⁴. Betti did not, however, require that the series should contain as many nontrivial terms as possible (or equivalently, that the $\hat{G}_i$ are to be *minimal* normal subgroups). He allowed for $n-1$ normal subgroups $\hat{G}_i$ in the decomposition of the group $\hat{\Gamma}$. He had already employed n to denote the order of $\hat{\Gamma}$; he appears to have overlooked this conflict of notation. (It is extremely unlikely that the notation is consistent, since $n-1$ is a ridiculously high upper bound for the length of a normal series in a group of order n .)

Betti continued by remarking that the order in which the groups of arrangements $G_1, G_2, \dots, G_{n-1}$ appear in the series is important, and he established a method for numbering them (for example, G_{n-1} is called the first divisor of H and the $(n-1)^{\text{th}}$ multiplier of G). He then defined Γ to be the *massimo moltiplicatore* (*maximal multiplier*)

⁴In general, this does not coincide with a composition series for the group.

of G if there cannot be other multipliers after it. Translating into groups of substitutions, this definition states that $\hat{H} = \hat{G}\hat{G}_1\hat{G}_2\ldots\hat{G}_{n-1}$ and $\hat{H}$ cannot be enlarged by adding on an extra group $\hat{G}_n$ if the group $\hat{G}$ is to remain a normal subgroup of $\hat{H}$. All the groups under consideration are themselves subgroups of the symmetric group $\text{Sym}(\nu)$ of substitutions on the original quantities $x_1, x_2, \ldots, x_\nu$. Therefore the 'maximal multiplier' of a group $\hat{G}$ corresponds to the largest subgroup of $\text{Sym}(\nu)$ in which $\hat{G}$ is a normal subgroup, that is, to the normaliser of $\hat{G}$ in this symmetric group. Betti did not, however, include the group $\hat{G}$ itself in his definition (the maximal multiplier is the group of arrangements Γ and not $G\Gamma$).

The next definition which Betti made was that a group which admits no 'divisor' is called *prime*. It is likely that this choice of terminology was prompted by the analogy with division and primes in the set of natural numbers. Unfortunately, it is clear that Betti thought 'prime' means 'simple' (see later for justification). He was seemingly not aware that there exist plenty of extensions which do not split: a group might very well admit no 'divisor' but still not be simple. The way in which he had developed the theory had overlooked this point.

Betti continued, having introduced the term 'prime', as follows [1852, 62–63/41–42]:

Poichè del gruppo H le sole sostituzioni che ha comuni con Γ permutano tra loro i derivati di G ; si può stabilire che *il gruppo delle permutazioni sopra i derivati per mezzo di un moltiplicatore di essi è simile a questo moltiplicatore*. [3]

How are we to understand this? First, recall that the group K of arrangements on the conjugates (*gruppo delle permutazioni sopra i derivati*) is the set of all the arrangements of $\{H_0, H_1, \ldots, H_{n-1}\}$ one obtains when the H_i are permuted amongst themselves under the action of $\hat{H}$. The *moltiplicatore* is the group of arrangements Γ . Then if $\hat{\Gamma}$ normalises $\hat{G}$, the substitutions of $\hat{G}$ do not have any effect on the arrangement of the H_i , since each H_i itself has $\hat{G}$ as its group of substitutions ($\hat{H}_i = \hat{G}$). More precisely, for $\theta_j\psi_i \in \hat{H}$ where $\theta_j \in \hat{G}$ and $\psi_i \in \hat{\Gamma}$,

$$\begin{aligned} H_r\theta_j\psi_i &= G\psi_r\theta_j\psi_i && \text{by definition} \\ &= G\theta_k\psi_r\psi_i && \text{by normality of } \hat{G} \\ &= G\psi_r\psi_i && \text{since } G\theta_k = G \text{ by definition} \\ &= H_r\psi_i. \end{aligned}$$

So the way in which the substitutions of $\hat{H}$ permute the H_i is equivalent to and may

be characterised by the way in which the substitutions of $\hat{\Gamma}$ permute them. (It is here that the condition $\hat{G} \cap \hat{\Gamma} = \{1\}$ is required.) Therefore $\hat{\Gamma}$ is isomorphic to the group of permutations $\hat{K}$ of the H_i : Betti again used the term *simili*, which is much weaker than ‘isomorphic’, to describe this relationship⁵.

We discovered earlier that the group $\hat{K}$ is, in modern terms, the image of the representation of $\hat{H}$ on the set of (right) cosets of $\hat{G}$. If $\hat{G}$ is normal in $\hat{H}$, this representation becomes the regular representation of the quotient group $\hat{H}/\hat{G}$. Therefore Betti could have avoided falling into error over split extensions if he had used $\hat{K}$ rather than $\hat{\Gamma}$ to explain how a group is built up from a normal subgroup contained in it. It is, however, unlikely that such a thing would have been possible for him to understand, since $\hat{K}$ is a group of substitutions which acts on *sets* of the original arrangements rather than on those arrangements themselves. Since Betti’s conception of group theory relied so heavily on the arrangements, it would have seemed to him that the difference between the group $\hat{K}$ and the groups $\hat{G}$, $\hat{\Gamma}$ and $\hat{H}$ is much greater than we now understand it to be.

Normalisers and composition series

Chapter III of Betti’s paper [1852, 63–85/42–58] deals with the determination of the ‘multipliers’ and ‘divisors’ of a group. In other words, let a group of arrangements G be given. Then Betti was interested in finding groups whose substitutions normalise those of G , and in finding normal subgroups of $\hat{G}$ and their complement subgroups (they all have such complements, according to his theory so far). He finally considered the problem of forming the product of two groups whose intersection is not trivial [1852, 64–65/43]. His notation changed (as it not infrequently did): the group Γ became Γ_n , so its group of substitutions is denoted by $\hat{\Gamma}_n$. Betti stated that in order for a group of arrangements H to be the product of two groups of arrangements G and Γ_n , it is necessary and sufficient that the substitutions of G really do form a group and in addition that the derivatives (conjugates) of G by Γ_n are ‘equal’, that is, the substitutions of $\hat{\Gamma}_n$ normalise $\hat{G}$.

He noted that the substitutions of Γ_n which are the product of one and the same substitution by two different substitutions of G produce identical conjugate groups of

⁵Mammone has also shown [1989, 150–151] that $\hat{\Gamma} = \hat{K}$.

arrangements. That is, if two substitutions in $\hat{\Gamma}_n$ can be written in the forms $\phi\theta_{a_i}$ and $\phi\theta_{a_j}$, where ϕ is some substitution and $\theta_{a_i}, \theta_{a_j} \in \hat{G}$ (this was Betti's new notation for the substitutions of $\hat{G}$), then not only are the conjugates of $\hat{G}$ by these two members of $\hat{\Gamma}_n$ identical, but also their associated groups of arrangements are identical.⁶ In this situation, the two substitutions of $\hat{\Gamma}_n$ must be treated as equal in $\hat{\Gamma}_n$. Betti appears to have been defining a sort of equivalence relation on the substitutions of $\hat{\Gamma}_n$. The equivalence classes that are formed in this way would nowadays be understood, as has been noted in Chapter 5, as elements of the quotient group $\hat{\Gamma}_n/(\hat{G} \cap \hat{\Gamma}_n)$.

Betti next admitted that the substitutions of Γ_n now under consideration (those which produce different conjugate groups of arrangements) may not themselves form a group. In other words, there may be no set of coset representatives for the cosets of the normal subgroup $\hat{G}$ in $\hat{H}$ which themselves form a subgroup of $\hat{H}$. Nevertheless, Betti recognised that an absence of split extensions did not matter: he claimed that, under this more general definition of Γ_n , his previous theorem still held. The theorem stated that the group of arrangements Γ_n is 'similar' to K , the group of arrangements on the conjugates. The substitutions of K correspond to those of the quotient group $\hat{H}/\hat{G}$, as explained earlier. In this case it is indeed true that $\hat{\Gamma}_n/(\hat{G} \cap \hat{\Gamma}_n) \cong \hat{H}/\hat{G}$, so the theorem holds. This interpretation seems to fit in with Betti's ideas. Betti also remarked that the product of the orders of the divisors of a group is equal to the order of the group itself. In the light of Betti's recognition that group extensions do not always split, this statement is true. It would nowadays be expressed as follows: given a normal series for a group M , the order of M is equal to the product of the orders of the quotient groups formed from successive terms of the series.

Betti next attacked the problem of forming a series for the group of arrangements H ; it corresponds to a composition series for the group $\hat{H}$. He worked from the whole of the group 'downwards': in terms of substitutions, his method consists of first finding a maximal normal subgroup $\hat{G}$ of $\hat{H}$, and then finding a maximal normal subgroup of $\hat{G}$, and so on until a normal subgroup is reached that is simple. Betti expressed this as follows:

I valori più generali che soddisfanno le equazioni (15) e (17) danno le sostituzioni del secondo divisore G , sul quale operando come sopra H , se ne potrà ottenere un terzo G_1 ,

⁶This is equivalent to the assertion that if one substitution of $\hat{\Gamma}_n$ is the product of another one by a substitution of $\hat{G}$ ($\gamma_1 = g\gamma_2$ for $g \in \hat{G}$ and $\gamma_1, \gamma_2 \in \hat{\Gamma}_n$), then γ_1 and γ_2 are in the same coset of $\hat{G}$.

e così seguitando si giungerà finalmente a un ultimo divisore primo Γ_1 . [4]

The equations (15) and (17) are satisfied by any collection of substitutions which form a group that is normal in $\hat{H}$. Therefore the most general values which satisfy them give a maximal normal subgroup of $\hat{H}$, which is denoted by $\hat{G}$. If one then operates on the group $\hat{G}$ in the same way, one may be able to obtain a maximal normal subgroup of $\hat{G}$, denoted by $\hat{G}_1$, and so on. Betti thus formed a composition series for $\hat{H}$,

$$\hat{H} \supseteq \hat{G} \supseteq \hat{G}_1 \supseteq \dots \supseteq \hat{G}_{n-2} \supseteq \hat{\Gamma}_1 (\supseteq \{1\}).$$

He then worked 'upwards' again to construct the group $\hat{H}$ as the product of smaller groups. He wrote:

Le sostituzioni dell'ultimo gruppo G_{n-2} di cui è divisore Γ_1 , private dei fattori eguali a qualcuna delle sostituzioni di Γ_1 , daranno un primo moltiplicatore $\Gamma_2 \dots$ [5]

In terms of groups of substitutions, remove from the group $\hat{G}_{n-2}$ those of its substitutions that are equal to any of the substitutions of $\hat{\Gamma}_1$. Betti was in effect factoring out the group $\hat{G}_{n-2}$ by the substitutions of $\hat{\Gamma}_1$ to form the quotient group $\hat{G}_{n-2}/\hat{\Gamma}_1 := \hat{\Gamma}_2$. But clearly Betti did not understand his construction in this way. The 'multiplier' Γ_2 seems to have for its substitutions a set of coset representatives for the cosets of $\hat{\Gamma}_1$ in $\hat{G}_{n-2}$. If one continues to work 'upwards', one obtains a series of groups of arrangements Γ_i . Betti wrote

$$H = \Gamma_1 \Gamma_2 \Gamma_3 \dots \Gamma_n.$$

If one thinks of the substitutions of $\Gamma_2, \Gamma_3 \dots, \Gamma_n$ as coset representatives, then it is true that the substitutions of H can be expressed as the product of substitutions of the Γ_i . Betti again expressed this in terms of arrangements. All the 'divisors' Γ_i in the series are simple, he added, because otherwise one would not have considered *all* the maximal normal subgroups. The groups of arrangements Γ_i therefore correspond in some way to the series

$$\hat{\Gamma}_1, \hat{G}_{n-2}/\hat{\Gamma}_1, \hat{G}_{n-3}/\hat{G}_{n-2}, \dots, \hat{G}/\hat{G}_1, \hat{H}/\hat{G}$$

of composition factors of $\hat{H}$.

In a situation when a group $\hat{H}$ has no *normal* subgroup but does have a subgroup, $\hat{J}$ say, Betti remarked that the group of arrangements H is the sum of 'similar' conjugate groups, as he had discussed much earlier. These conjugate groups of arrangements are formed from the arrangements of H not contained in the subgroup $\hat{J}$.

It was noted earlier that Betti's term for 'simple' is 'prime'. This is confirmed by the following statement [1852, 66/44]:

La decomposizione di un gruppo nei suoi divisori primi corrisponde a quella che Galois chiamava *decomposizione propria*. [6]

In other words, Betti thought that the 'prime divisors' of a group have the same function as those groups which we now call the composition factors.

Some examples

The remainder of the first half of his paper was devoted to the calculation of normalisers, normal subgroups and composition series in various particular cases. Section 24 [1852, 66–67/44–45] was inspired by the end of Galois' *Premier mémoire* [1846a]. There Galois had investigated soluble (irreducible) equations of prime degree, which correspond to soluble groups of prime degree. He proved that an irreducible equation of prime degree p is soluble by radicals if and only if the group of this equation contains only substitutions of the form $\begin{pmatrix} k \\ ak + b \end{pmatrix}$ where k, a, b lie in $\mathbb{Z}_p$, a and b are constants and arithmetic is to be done modulo p . Betti found the normaliser of a cyclic group $\hat{G}$ of prime order p in $\text{Sym}(p)$: if the substitutions of $\hat{G}$ are the powers of $\begin{pmatrix} i \\ i + 1 \end{pmatrix}$ where i lies in $\mathbb{Z}_p$ and arithmetic is to be done modulo p as before, then those of the normaliser are of the form $\begin{pmatrix} i \\ ai + b \end{pmatrix}$.

Subsequent sections of Betti's paper were inspired by part of Galois' *Lettre à Chevalier* [1832] and by the *Second mémoire* [1846b]. In the *Second mémoire* Galois discussed primitive equations that are soluble by radicals. There are three pages of general theory and seven pages devoted to primitive equations of degree p^2 , including demonstrations that the order of the general linear group $\text{GL}(2, p)$ is $(p^2 - 1)(p^2 - p)$ and the order of the affine group $\text{AGL}(2, p)$ is $p^2(p^2 - 1)(p^2 - p)$. Betti reworked this theory and related it to his concepts of 'multipliers' and 'divisors' of a group. He calculated [1852, 71–76/48–52] that the normaliser of an elementary abelian group of order p^ν in $\text{Sym}(p^\nu)$ is the affine group $\text{AGL}(\nu, p)$. He repeated this section in [1855]; his proof has been discussed by Mammone [1991, 152–155]. Next Betti sought the 'prime divisors' of $\text{AGL}(\nu, p)$, which correspond to its composition factors, but there are various problems with his theory. He wrote down the group as a 'product' of smaller groups but in general this is not possible, since the extensions here do not all split. The groups he obtained are not all simple; he claimed that they were. The largest group in the product is presumably

intended to be $\text{PSL}(\nu, p)$ but he calculated its order incorrectly, and did not even begin to try to prove that it is simple.

Lastly [1852, 83–84/57–58] he considered the group $\text{AFL}(1, p^\nu)$ of semilinear maps of the line over the field of p^ν elements. He found the decomposition of this group into ‘prime divisors’ which are all cyclic, so the group is soluble. He provided an example of a particular group, $\text{AFL}(1, 2^2)$, which happens to be the same as $\text{Sym}(4)$. He determined a ‘composition series’ for this group and gave a diagram of the decomposition of the corresponding group of arrangements H . Betti asserted that H is a product of groups $GG_1G_2G_3$. He specified what are the substitutions of each of these groups of arrangements. The group $\hat{G}$ contains the substitution $(03)(12)$ in addition to the identity. The group $\hat{G}_1$ contains $(01)(23)$ and the identity i , the group $\hat{G}_2$ contains i , (123) and therefore also (132) , and the group $\hat{G}_3$ contains i and the substitution (12) . Then using modern notation, the group $\hat{G}$ is isomorphic to C_2 , the cyclic group of order 2, the group $\hat{G}\hat{G}_1$ is isomorphic to V_4 , the Klein-4 group, the group $\hat{G}\hat{G}_1\hat{G}_2$ is isomorphic to $\text{Alt}(4)$, the alternating group on four letters, and $\hat{G}\hat{G}_1\hat{G}_2\hat{G}_3$ is isomorphic to $\text{Sym}(4)$.

As before when Betti first discussed normal subgroups and products, we have a series for $\text{Sym}(4)$, but this time it is a *subnormal* series

$$\{1\} \trianglelefteq \hat{G} \trianglelefteq \hat{G}\hat{G}_1 \trianglelefteq \hat{G}\hat{G}_1\hat{G}_2 \trianglelefteq \hat{G}\hat{G}_1\hat{G}_2\hat{G}_3 = \text{Sym}(4)$$

or equivalently

$$\{1\} \trianglelefteq C_2 \trianglelefteq V_4 \trianglelefteq \text{Alt}(4) \trianglelefteq \text{Sym}(4).$$

This is a composition series for $\text{Sym}(4) \cong \text{AFL}(1, 2^2)$. Betti stated that the group of arrangements H decomposes into groups ‘of the first class’, which means that the composition factors of $\text{AFL}(1, 2^2)$ are of prime order and therefore cyclic, as expected.

Applications to Galois theory: Galois’ propositions II and III

Chapter 1 of the second half of Betti’s paper is based on Galois’ *Premier mémoire*, to which Betti attempted to apply the group theory he had developed. I consider two points which arise from Mammone’s article and which concern Betti’s treatment of Galois’ propositions II and III.

Proposition II deals with the result of adjoining a single root of an auxiliary equation to the field of ‘known quantities’. As Mammone notes [1989, 162–163], Betti added

an extra assertion to this proposition, namely that the group of the auxiliary equation is 'similar' to that of the original equation. Recall that 'similar' groups contain the same number of substitutions of each order. Then, if Betti's assertion is true, it follows [Mammone 1989, 165] that the splitting fields for the auxiliary equation and the original equation are the same.

Now this is obviously not generally the case, and indeed if it does happen, the chosen auxiliary equation is either not a particularly helpful one or there was in fact no difficulty in solving the original equation. What then did Betti have in mind when he made this assertion? It becomes clear when one reads to the end of his proof, which concludes [1852, 92/63]:

... il gruppo della equazione (10) dovrà essere eguale a quello delle permutazioni sopra i derivati, e simile a quello della proposta. [7]

The equation (10) is the auxiliary equation. The 'group of permutations on the derivatives' is, as he defined much earlier, equivalent to what we now call the permutation representation on the cosets. Here the cosets are those of the subgroup $\hat{G}$ of the Galois group $\hat{H}$ of the original equation. The group $\hat{G}$ is defined to be the Galois group of this equation $f(x) = 0$ after a root of the auxiliary equation has been adjoined. (The notation employed here accords with that used earlier.) Mammone shows [1989, 164–165] that the group of the auxiliary equation is indeed 'similar', in fact is isomorphic, to the 'group of permutations on the derivatives'. However, this permutation representation is itself isomorphic to the original group only under certain conditions, a fact which Betti had overlooked. As noted before, the required condition is that the subgroup $\hat{G}$ must contain no nontrivial subgroup normal in the whole Galois group $\hat{H}$. What form must be taken by the auxiliary equation in order to produce a subgroup with this property?

The group of arrangements H is partitioned into smaller groups of arrangements $H_0, H_1, \dots, H_{n-1}$ by an auxiliary equation. Each H_i becomes the Galois group of the original equation $f(x) = 0$ when one root of the auxiliary equation $\phi(r) = 0$ is adjoined. The groups of arrangements $H_0, H_1, \dots, H_{n-1}$ are 'gruppi derivati', that is, their corresponding groups of substitutions $\hat{G} = \hat{G}_0, \hat{G}_1, \dots, \hat{G}_{n-1}$ are conjugate subgroups which together form the whole of $\hat{H}$. Therefore in order for Betti's assertion to be true, it is required that there are no substitutions common to all the $\hat{G}_i$. (This ensures that there is no nontrivial normal subgroup of $\hat{G}$ in $\hat{H}$.) Such a requirement

supposes that the roots of the auxiliary equation must be completely unrelated to each other so that there is no overlap of their respective splitting fields. In general such a situation is neither likely to present itself nor is it desirable. Thus Betti's assertion is in most cases incorrect and the group of the auxiliary equation has *smaller* order than the original Galois group $\hat{H}$. It is (in modern terms) isomorphic to a quotient group of $\hat{H}$.

Galois' proposition III is concerned with the adjunction of *all* the roots of an auxiliary equation. Betti again added to the statement of the proposition. He claimed that, in this case, the Galois group $\hat{H}$ is the 'product' of two groups: of the group $\hat{G}$ of the equation $f(x) = 0$ after all the roots of the auxiliary equation have been adjoined and of a group $\hat{\Gamma}$ that is 'similar' to the group of the auxiliary equation. He showed that the group $\hat{G}$ is a normal subgroup of the Galois group $\hat{H}$ and continued [1852, 93/64]:

... il gruppo dell'ausiliaria ... è simile a quello per il quale bisogna moltiplicare uno dei derivati per ottenere il gruppo stesso che appartiene alla proposta quando non siano aggiunte le (15) ... [8]

The (15) are the roots of the auxiliary equation. In terms of groups of substitutions, the group $\hat{\Gamma}$ of the auxiliary equation is 'similar' (in fact is isomorphic) to a group, $\hat{\Delta}$ say, which has the property that the entire Galois group $\hat{H}$ is produced when one forms the product of the normal subgroup $\hat{G}$ with $\hat{\Delta}$. Mammone is certainly right in saying [1989, 167] that since Betti's definition of 'product' cannot be interpreted as a quotient group construction, his claim is again not quite correct. Betti had forgotten that the substitutions of the 'multiplier' may not themselves form a group, despite the fact that he had dealt with this situation earlier in his paper.

1855: Betti's return to group theory

A comparison of the article [1852] with Betti's paper of three years later [1855] reveals that his understanding of various concepts had developed. He had neatened and modified several of his ideas. He introduced yet more new notation and terminology, in most cases simply to replace that in the earlier paper. He abandoned the term *simile* which had been rather superfluous in [1852] because it did not express quite what was needed. The term *permutazione* was replaced by *disposizione* to mean arrangement. Betti no longer required that the 'conjugating' substitutions form a group (see [Mammone 1989, 151–152] and [Toti Rigatelli 1989, 62]), but he explained this no further. The paper is

on balance, however, rather irrelevant in that it deals badly with a subject that should present little difficulty: the determination of composition series and normalisers for cyclic groups of nonprime order. As mentioned in Chapter 6, Betti managed to confuse the cyclic group of order p^ν (where p is prime) with the elementary abelian group of order p^ν ; this has the effect of rendering much of his theory incorrect. He succeeded in finding the substitutions that normalise this elementary abelian group $\hat{\Gamma}$ of order p^ν . He discovered [1855, 27/118] that there are $(p^\nu - 1)(p^\nu - p) \dots (p^\nu - p^{\nu-1})$ of them and (in modern terms) that they form the group $\text{GL}(\nu, p)$. Despite the presence of this result, the paper seems to have had no influence at the time of its publication or later.

Reflections on Betti's group theory

The first and most obvious conclusion to draw from a study of Betti's work is that it is not easy to understand. The effort required of a twentieth-century reader in order to gain an understanding of it is quite possibly as great as that which went into its writing. Betti was certainly an inventive mathematician and was not afraid to introduce new concepts and new approaches to old concepts. However, in his enthusiasm to follow a particular line of enquiry, he consistently failed to explore fully the ideas he discovered. This led to various omissions and gave the impression that the theory is less complicated than is actually the case. Betti tended to consider only extremes and left no thought for intermediate cases. For example, in his discussion of conjugate groups and normalisers [1852, 60–62/40–41], he investigated only two cases. The first occurs when none of the conjugate groups is equal to another and the second when they are all equal. He seems to have mentioned the general situation only in passing [1852, 61/40], when he stated that the conjugate groups of arrangements will be either 'similar' or 'equal' to the original group of arrangements and to each other. This is his acknowledgement that each individual substitution of Γ may or may not normalise the substitutions of G .

Betti attempted several times [1852, 62–63/41–42 & 64–65/43–44] to grasp the concept of quotient group. He was prevented from gaining a clear understanding of it by his insistence that a group is a group of arrangements, which made it very difficult to recognise the relationship between a quotient group and the original group. He seems to have realised that there was an idea here that was important but that he could not express. His failure to express it correctly led to the consideration only of groups whose quotient groups are isomorphic to subgroups, that is, of groups which are split exten-

sions. Later [1852, 65/43] he acknowledged that extensions do not necessarily split, but he was not able to formulate this new idea precisely.

Kiernan states [1971, 105] that in Betti's work it is often impossible to distinguish a normal subgroup of a group from the corresponding quotient group. He follows this remark with the assertion that Betti developed a concept which one can interpret either as a composition series for a group or as the corresponding series of quotient groups (the composition factors). This assertion does not seem to be altogether true. Betti's series always consisted of groups of arrangements, and it is fairly clear when these correspond to subgroups of the group and when they correspond to the 'multipliers' by which one subgroup is enlarged to form another.

In spite of the difficulties it presents to the reader, Betti's work seems to have had a considerable (and positive) reputation in the nineteenth century. On what was this reputation based? One wonders how many mathematicians had actually read and understood Betti's papers. They certainly seem to have exerted very little influence on the subsequent development of group theory and of Galois theory.

Bottazzini records [1980, 27] that Tortolini, the editor of the *Annali di Scienze matematiche e fisiche*, had sent excerpts of Betti's 1852 commentary to many well-known European mathematicians, including Gauss, Kummer, Dirichlet, Serret, Bertrand, Liouville, Hermite and Cauchy. Liouville seems to have been aware of Betti's work soon after it was published, in that the remark 'Lire Betti' appears in notes on Galois theory which Liouville wrote during 1851–1852 (see [Lützen 1990, 579]). Serret, however, did not include an exposition of Galois theory in the second edition of his textbook on higher algebra [1854]—he ignored Betti's work, although he must have seen it by that time and in addition met Betti when he visited Paris in 1853. Tortolini was surprised at Serret's omission and sent him a further extract from Betti's paper [Bottazzini 1980, 28]. By the time Serret produced the third edition of his textbook [1866] he was more conversant with Betti's papers. He mentioned Betti's terminology for 'conjugate' (*derivato*) when introducing the idea of conjugation [1866 Vol. II, 255]. He does not, however, appear to have taken up any of Betti's ideas or to have used any of his terminology or notation.

In the introduction to [1855] Betti mentioned the 'illustre Geometra' who had advised him to publish these additions to his earlier paper. Toti Rigatelli suggests [1989, 61] that this was very probably Sylvester, who was in Italy at that time and who must have

been at least aware of the obscurities in Betti's commentary, even if he had not read it himself. Jordan drew attention to the contribution of Betti towards the understanding of Galois theory in the preface to his *Traité* [1870]. There is nothing to suggest that Jordan had read Betti's papers in any detail. Later in the preface, Jordan gave a list of those mathematicians who had contributed to the advancement of the theory of substitution groups, a list which includes Betti's fellow-countryman Brioschi but not Betti himself.

In 1854 Kronecker wrote to Betti to compliment him on his recent work [Toti Rigatelli 1989, 66]. It also appears that Dedekind had studied Betti's papers by the time he prepared lectures on Galois theory in the late 1850s [Scharlau 1981, 106]. However, Dedekind's lectures were hardly influential (he had four listeners) and his notes on Galois theory remained unpublished in the nineteenth century. Dedekind's exposition of the theory does not seem to bear any obvious relation to Betti's.

The encyclopedia article on finite discrete groups written by Burkhardt [1898, 219] refers to Betti's paper of 1852 for the definition of the concept of quotient group. Burkhardt defined the concept as the permutation representation on the cosets of a normal subgroup, which was the idea Betti recognised only vaguely and which none of his contemporaries or immediate followers seem to have adopted. Burkhardt appears to be alone in attributing this concept to Betti.

The friends, colleagues and pupils of Betti who wrote obituaries about him were full of appreciation and praise of his algebraic studies. Brioschi said [1892, 256/41]:

... egli, pel primo, penetrando negli alti ed astrusi concepimenti di Galois sulla teoria delle equazioni algebriche, pose in luce la esattezza delle asserzioni contenute nella celebre lettera a Chevalier, iniziando così il movimento in quelle ricerche alle quali oggi ancora è dedicata l'attività di eminenti geometri. [9]

Volterra, who had been Betti's pupil, was of the opinion [1892, 6] that

Le opere del Betti su questo argomento hanno già da lungo tempo preso posto accanto ai classici lavori dei più chiari matematici dei nostri giorni, e costituiscono un monumento del sapere e della forza intellettuale del giovane matematico. [10]

Without doubt Betti did possess considerable 'force of intellect', but it seems to have taken some time for him to control and channel it effectively. Thus his early papers on group theory and Galois theory reflect a creative but somewhat hasty approach to two subjects still very much in their infancy. Perhaps Pierpont's comment [1898, 339–340]

might have been more moderately expressed:

Thus in 1852, twenty years after Galois' death, his theory of the resolution of algebraic equations was for the first time made intelligible to the general public and established with complete rigor.

Notes

1 The affection and the admiration which Betti's pupils have always retained for him after they completed their university studies are the finest testimony to his worth as a teacher.

2 If q is not equal to the number of permutations of the group, one will also have another substitution ψ_2 , with which one will obtain another qn permutations, all different from the previous ones: since suppose

$$\psi_0^r \psi_1^s = \psi_0^u \psi_2,$$

then one would have

$$\psi_2 = \psi_0^{r-u} \psi_1^s,$$

that is, ψ_2 equal to one of the previous substitutions, contrary to assumption.

3 The substitutions which the group H has in common with Γ suffice to characterise the way in which the substitutions of H permute the conjugates of G among themselves. One can therefore establish that *the group of arrangements on the conjugates obtained by means of a multiplier of them is similar to this multiplier*.

4 The more general values which satisfy the equations (15) and (17) give the substitutions of the second divisor G . Operating on this as on H , if one can obtain a third divisor G_1 of H , and continuing in this way, one finally arrives at a last prime divisor Γ_1 .

5 The substitutions of the last group G_{n-2} of which Γ_1 is divisor, with factors equal to any of the substitutions of Γ_1 removed, give a first multiplier $\Gamma_2 \dots$

6 The decomposition of a group into its prime divisors corresponds to the process which Galois called *proper decomposition*.

7 ... the group of the equation (10) must be equal to that of the permutations on the derivatives, and similar to that of the original equation.

8 ... the group of the auxiliary equation ... is similar to that by which one must multiply one of the derivatives in order to obtain the group itself which belongs to the original equation when the (15) are not adjoined ...

9 ... he was the first who, by penetrating the difficult and obscure concepts of Galois on the theory of algebraic equations, made clear the essential nature of the assertions contained in the famous letter to Chevalier. In this way he initiated the advance in this research to which the activity of prominent mathematicians is dedicated even today.

10 The works of Betti on this subject have long since taken their place beside the classical studies of the most famous mathematicians of our day, and constitute a monument to the young mathematician's learning and force of intellect.

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