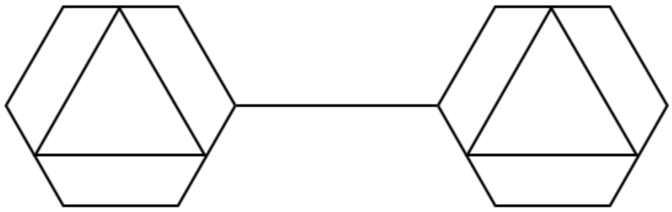
Differences In How Teachers Make Mathematical Content Available To Learners Over Time



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Abstract

The study was an investigation into the teaching decisions that mathematics teachers make over time. I view a mathematics classroom as a didactical system of teacher, learners and content within an educational institution, where content is the material that brings teachers and learners together. Within such a system I view the teacher's role as *making content available to learners*. Prior research has often investigated the teacher's role by comparing teaching practices nationally or internationally, but these comparisons have tended to use the lesson as the unit of analysis. I propose that how teachers make content available can change over the course of a series of lessons and so my study used the lesson series as the unit of analysis.

I purposefully designed the study so that it involved four cases, which allowed me to explore the role of the teacher and the topic in how content was made available. To investigate how teachers made content available to learners in each case, I developed an analytical approach from which I could study the modes of teacher interaction that featured across the lesson series, the forms of mathematical content made available and the sequencing of these forms. Attending to forms of content – rather than content itself – allowed for comparison of teaching of different topics. This original analytical approach represents a contribution to both mathematics education and mixed methods research.

Within this small sample of cases, quantifiable differences were identified in how content was made available between classwork and seatwork, from lesson to lesson and between cases. Between-case differences in the nature of teaching 'between-the-desks' during seatwork were also identified. These differences illuminated teaching decisions to which teachers and classroom researchers may not routinely attend. The findings therefore contribute – and identify additional lines of enquiry that might contribute further – to a more extensive understanding of teaching practices.

ACKNOWLEDGEMENT

I am indebted to my joint supervisors, Dr Gabriel Stylianides and Professor John Mason, whose encouragement, guidance and support has been phenomenal during this study. I would also like to thank Ashley, Bernie and Courtney and their students for welcoming me into their classrooms. Finally I offer my sincerest appreciation to family and colleagues who have supported me, and most particularly for the patience you have shown during the final stretch.

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CHAPTER 1

BACKGROUND

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1.1 – The object of enquiry

I am interested in the teaching decisions that mathematics teachers make over time. I view a mathematics classroom as a *didactical system* (Chevallard, 1982) of teacher, learners and content within an educational institution. Within such a system I view the teacher's role as *making content available to learners*. By content, I mean the 'substance or material dealt with [in teaching] as distinct from its form' (Allen, 1990; 248); it is the content that brings teachers and learners together. In the context of mathematics teaching, content is made up of the mathematical phenomena that constitute a topic. But it is precisely the 'form' of content – rather than the content itself – that is of interest to me; this is part of the 'how' in 'how teachers make content available'. *Form* brings to mind for me image, technique and application. A further aspect of how teachers make content available is the *style* of teaching. Style brings to mind exposition, discussion and exploration.

Attending to how teachers make content available to learners *over time* offers the possibility of investigating variation in the style and form of what is offered in different contexts. By different contexts, I mean different teachers, different classes, different schools and different topics. Are particular forms or styles emphasised, and how might this emphasis differ between series of lessons taught in different contexts? Are there differences in what forms or styles are emphasised from lesson to lesson? Are there differences between forms emphasised when the teacher speaks with the whole class and those emphasised during seatwork?

The notion of differences in teaching methods is well established through comparisons made internationally (for example, Stigler and Hiebert, 1999), nationally (for example, Askew et al., 1997) and locally (for example, Ayalon and Even, 2013). Furthermore, some researchers state that how mathematical content is made available to learners is an indicator of effective teaching (for example, Watson and De Geest, 2012; Askew et al., 1997). However, holistic approaches to understanding how teachers make mathematical content available to learners over time are scarce. It is this gap in our understanding that I will seek to address in the current study.

In this opening chapter, I discuss issues present in mathematics education literature relating to how teachers make mathematical content available to learners. In doing so, I offer a motivation for the current study and a more detailed explanation of the object of enquiry. At first I consider teacher decision-making and how the decisions that teachers make in the course of planning and teaching offer an account of why differences surface in how content is made available. I then discuss sources of possible variation in how content is made available, defining key terms in the questions that I posed above. I conclude the chapter by reiterating the issue on which I focus in this study, how I conceptualise it and the research questions I seek to address in order to contribute to our understanding of it.

1.2 - The origins of differences: teacher decision-making

1.2.1 – Teacher decision-making

A fundamental stance that I adopt in this study is that how teachers make content available to learners is a consequence of teacher decision-making. As will be illustrated in subsection 1.2.2, some accounts of decision-making view the mathematics teacher as problem solver, applying pedagogical and content knowledge in predictable ways in order to address concrete problems that arise in teaching. Other accounts suggest that this perspective fails to recognise the complexity of teaching and the agency of the teacher. Teacher decisions can be seen as results of individual reflection on experience influenced by collective practices within a community.

Teachers' thinking about what to do in lessons can occur before, during and after lessons and these thoughts are often labelled as *preactive*, *interactive* and *postactive* respectively (inspired by Jackson, 1968). However, in practice the postactive thoughts of one lesson and the preactive thoughts of the next are indistinct as reflection on one lesson informs thinking on the next. Frequently, thinking between lessons is described as *teacher planning* (Clark and Peterson, 1986:258). However, McNair (1978) uses the term *in-flight* rather than interactive in order to describe decisions made during the course of a lesson, and this avoids confusion with the term interaction, which I use in a particular way later. Of the range of thoughts that might come to a teacher's mind when planning or in-flight, I am concerned with those that relate to decisions about how to make content available – teaching decisions.

Early studies of teachers' in-flight thoughts sought to define clearly what was to be considered as a decision (Clark and Peterson, 1986). A distillation of these led to the following definition of an in-flight decision:

> A teacher's conscious choice between continuing to behave as before or behaving in a different way.
>
> (Clark and Peterson, 1986:273)

With this definition in mind, Clark and Peterson (1986:274) report on studies that suggest teachers make between five and seven in-flight decisions every ten minutes. While these may not all be teaching decisions relating to how content is to be made available (some may be administrative, organisational or managerial), this suggests that teachers' pre-active thoughts on how content is to be made available to learners may well be different to how content is seen to be made available in a lesson due to additional teaching decisions made in-flight.

1.2.2 – Epistemology of practice

Forms of teacher knowledge and the ways of knowing that inform decision-making may be described as the epistemology of practice. In this subsection, I investigate alternative epistemological perspectives and consider their implications for researching how teachers make content available to learners.

Schön (1983) uses the term *technical rationality* to describe a positivist perspective on professional practice. This static body of professional knowledge conflicts with his construct of the 'reflective practitioner' for whom 'knowing is inherent in intelligent action' (Schön, 1983:50) and for whom professional practice involves 'reflection-in-action' (ibid.). Technical rationality emphasises a scientific model of informing action, so in deciding how to act the teacher refers to his or her knowledge of the objective applied science of teaching. Here the teacher is a problem solver, where the solutions to problems are objective and well defined.

However, recognition of the social complexity within classrooms calls for an alternative to the model of technical rationality - or at least a search for alternative accounts of teacher actions based on more factors than technical knowledge alone. A constructivist perspective casts both teacher and learners as individuals making sense of the world, building their unique understandings shaped by their personal experiences. It is perhaps this background that leads Schön (1983) to reject the teacher as problem solver. Instead he emphasises problem setting: how a teacher needs to 'make sense of an uncertain situation' prior to deciding how to act (Schön, 1983:40). However, within the complexity of the constructivist classroom, problem setting is subjective, driven by the focus of the teacher's attention. For Schön, what the teacher attends to and how he or she decides to act is informed by experience. Reflection on experience informs action, even when this reflection may seem to be spontaneous during action, giving rise to the phrase reflection-in-action.

Dewey asserts that experience is inseparable from the environment within which it was experienced:

> An experience is always what it is because of a transaction taking place between an individual and what, at the time, constitutes his environment.
>
> (Dewey, 1938:43)

In this sense, experience is situated and teacher decision-making takes place within a situation. There is no sense here of how the teacher ought to act, but rather a pragmatic sense of the teacher acting for the best within the situation. There are parallels between this perspective and Aristotle’s notion of phronesis:

> In the case of phronesis [(practical wisdom)], one does not know in general, or by appeal to a set of rules, so much as one decides case by case – with special attention to the details of each case – what one must do.
>
> (Gallagher, 2009:36)

Schoenfeld (2011) proposes a model of teacher decision-making based on psychological choice theory, seeking to explain *why* teachers act as they do. Key elements of the model are orientations ('dispositions, beliefs, values, tastes and preferences' (Schoenfeld, 2011:29)), resources ('the information that [the teacher] has potentially available to bring to bear in order to solve problems' (Schoenfeld, 2011:25)) and goals ('something the individual wants to achieve' (Schoenfeld, 2011:20) attributed to the teacher based on observation. He segments lessons through repeated parses so as to develop a multi-layered perspective on the teacher's decision-making. In addition, Schoenfeld's model uses a decision-making mechanism based on 'subjective valuations' (Schoenfeld, 2011:37).

Schoenfeld's model is principally concerned with in-flight decisions. Central to this is the shifting of top-level goals while teaching. Goals may be taken as cognitive constructs relating to what the teacher intends to accomplish (Aguirre and Speer, 2000). However, Schoenfeld recognises goals as theoretical constructs, yet tools that are central to his modelling of the decision-making process.

Aguirre and Speer (2000) investigated the particular phenomenon of new goals taking priority. They focused on *goal shifts*, when a teacher's goal changes while teaching. Describing two case studies of lesson segments taught by two teachers, they sought to account for these shifts with reference to *belief bundles* attributed to the teachers. In doing so, they suggested a relationship between teacher's goals and their beliefs about mathematics and its teaching. Schoenfeld also contends that his broader category of *orientations* represents the fundamental variable of his model (Schoenfeld, 2011:182). However, he again recognises orientations as a theoretical construct, attributed to teachers – teachers may not 'have' beliefs, although the easiest way to account for observed acts is that teachers are guided by 'beliefs'.

Indeed there is a lack of consensus amongst researchers regarding the meaning of the term *belief* (Speer, 2005:365). Moreover, significant variation is evident in researchers' categorisations of beliefs (Mason, 2004a:347) even when there is convergence in the interpretation of the term belief itself. A socio-cultural perspective shifts the emphasis to the setting as a factor in accounting for teachers' actions (e.g. Skott, 2009). From this perspective it makes no sense to discuss practice without reference to context (Skott, 2009:28). Indeed practice and context are taken to be in flux (Davis and Sumara, 1997:111) and it is untenable that fixed beliefs will account for practice.

Speer's critique points to how researchers' methodologies may have contributed to perceptions of divergence between what teachers are heard to say about their practice (*professed beliefs*) and how they are seen to act (*attributed beliefs*) (Speer, 2005). The range of definitions of belief and the even broader range of categorisations of belief has been highlighted already and for Speer this risks compromising the shared understanding of constructs between researcher and participants. Moreover, the interpretivist nature of belief research implies that even professed beliefs are the researcher's interpretations of the teacher's narratives constructed in response to being questioned. While not professing a socio-cultural perspective, Speer nevertheless contends that the expression of belief is situated and so divergence between what teachers are heard to say and how they are seen to act may be attributed to changes in context. She therefore concludes that data on beliefs and practice needs to be collected concurrently.

However, divergence may not always be attributed to methodological limitations. Davis and Sumara (1997) provide vivid examples of their experiences of intending to change practices by co-teaching with teachers in a school. Instead they found themselves being drawn into the practices they were seeking to influence. They account for this by being caught up in a *community of practice* (Lave and Wenger, 1991). Graybeal (2010) describes how teachers may feel obliged to enact the implicit curricular messages of textbooks, assessments and professional development programmes even when this conflict with professed beliefs. Furthermore it is conceivable that teachers may feel obliged to enact the more explicit expectations of school management and local and national governing bodies.

In summary, individual teachers' beliefs regarding mathematics and its teaching are perceived by some to influence decision-making, founded on the assumption that espoused beliefs are 'a significant determiner of what gets taught [and] how it gets taught' (Wilson and Cooney, 2002:128). However, a more situated perspective rejects the teacher as an autonomous decision maker and instead emphasises the broader community of practice within which the teacher is caught (Lave and Wenger, 1991) or furthermore the embodied context of the teacher-learner-content relationship within a classroom milieu (Brousseau (1997), Cohen et al (2003), Lampert (2001), Mason (1998a)). This implies variation in what a teacher attends to in different situations – different classes, different topics, different times of the year – and consequently implies the possibility of variation in how content is made available to learners in different situations. It is this variation that I seek to illuminate and so I have purposefully designed the current study so that it involves four cases. Two cases feature the same teacher teaching different topics and two cases feature the same topic taught by different teachers. Further details of the research design are given in Section 3.1.

1.2.3 – The teacher's role in directing learners' attention

A situated perspective, as described in the final paragraph of subsection 1.2.2, leads to asserting cognition within an individual organism as being the result 'of causal processes that span the boundary' between the organism and its 'natural, social and cultural environment' (Robbins and Aydede, 2009:6). Thus cognition involves an interaction between the organism (or *agent*) and a feature of the environment (some *other system*). As mentioned previously, such a perspective is described as ecological. Watson (2004) states that within classroom research

> The ecology metaphor … focuses on the whole classroom and not on individuals within it, except to report that their activity contributes to, and develops from, the classroom as a whole.
>
> Watson (2004:24)

Greeno (1994), drawing on the work of Gibson (1954), defines the term affordance in relation to interaction involving an agent with some other system.

> In any interaction involving an agent with some other system, conditions that enable that interaction include some properties of the agent along with some properties of the other system. Consistent with his emphasis on understanding how the environment supports cognitive activity, Gibson focused on contributions of the physical system. The term affordance refers to whatever it is about the environment that contributes to the kind of interaction that occurs.
>
> (Greeno, 1994:338)

In drawing on Gibson's work, Greeno brought the field of ecological psychology to the attention of the mathematics education research community. However, since the publication of Greeno's work the definition of affordance has been contested within the ecological psychology literature (for example, Michaels (2003), Stoffregen (2003) and Chemero (2003)). Are affordances *properties* of the environment or are they *relationships* between the agent and the environment? Is *perceiving an affordance* perceiving something about oneself or perceiving a feature of the environment? My interpretation of affordances aligns most closely with Chemero's (2003). He asserts that affordances are relationships between *features* of the environment and *abilities* of agents (Chemero, 2003). From this perspective, the logical structure of affordances is

> Affords-ϕ (feature, ability), where ϕ is a behaviour.
>
> (Chemero, 2003:189)

Mason defines learning as 'the opening up of possibilities to attend in different ways in different contexts' (Mason, 2004b:30). Within the classroom environment, there are agents with abilities to learn: learners (although I would not want to discount the teacher as a learner). A feature of the classroom environment is how content is made available within the *didactical system*[1] (teacher and learners brought together by the content and the institution). With reference to the logical structure suggested by Chemero, how content is made available affords the opening up of possibilities to attend in different ways in different contexts (or *learning*) to learners, and so learning is an affordance of how content is made available. This aligns teaching, or at least aspects of teaching associated with making content available to learners, with *directing learners' attention*. What teachers say to learners and what they offer as tasks affords directing learners to particular *forms* of attention (Mason, 2004b). The forms of attention that Mason proposes are:

> Holding wholes
> Discerning details
> Recognising relationships
> Perceiving properties as being instantiated
> Deducing from definitions
>
> (Mason, 2004b:15-16)

Both Mason (2004) and Watson (2007) also recognise other frameworks as offering a structure for attention for example, the van Hiele levels as exemplified by Burger

[1] The term didactical system is borrowed from Chevallard (1982).

and Shaughnessy (1986) and Pirie and Kieren's onion-skin model (Pirie and Kieren, 1989 and 1994). These alternatives will be considered in Section 1.3.2.

It is important to be mindful that affordances are not necessarily *realised*, or in the words of Greeno:

> The presence in a situation of a system that provides an affordance for some activity does not imply that the activity will occur, although it contributes to the possibility of that activity.
>
> (Greeno, 1994:340)

So making content available to learners offers an *opportunity to learn*, but this opportunity may not be taken up. That affordances may not be realised does not invalidate their investigation. For Chemero asserts that:

> Affordances do not disappear when there is no local [agent] to perceive and take advantage of them. They are perfectly real entities that can be objectively studied and are in no way figments of the imagination of the [agent] that perceives them.
>
> Chemero (2003:193)

This, together with Watson's endorsement of focusing on the classroom as a whole and producing descriptions of 'what is available for learners to learn' as the outcome of research, provides a warrant for an ecological approach to the current study.

1.3 – Possible sources of variation in how content is made available

Reporting on French mathematics education research (*didactique des mathématiques*), Laborde identifies three sources of possible variation in how content is made available to learners:

> Choice of content to be taught
> Planning of interactions between learners and the knowledge to be learnt
> Interventions and the role of the teacher in a class situation
>
> (Laborde, 1989:32)

Because I am focusing on the form and style rather than the content of what is made available to learners, the second and third sources of variation that Laborde identifies are of particular interest. Seeing 'planning of interactions' (ibid.) as a source of variation opens up the possibility of differences arising from the mathematical forms (image, technique, application and the like) that are made available by the teacher. When 'planning of interactions' (ibid.) is viewed over time, this also opens up the possibility of differences arising in sequencing of mathematical forms. Seeing 'the role of the teacher in a class situation' (ibid.) as a source of variation opens up the possibility of differences arising from the type of exchanges that made available by the teacher.

- ▷ Differences arising from the mathematical forms that are made available by the teacher
- ▷ Differences arising from sequencing of mathematical forms
- ▷ Differences arising from the type of exchanges that are made available by the teacher

In this section, I draw on the literature that has brought me to identifying these three sources of possible variation. This is not intended to be a systematic review, but rather the literature I discuss illustrates sufficient variation along these three dimensions in order to assert that they are sources of variation present in mathematics teaching. I will return to many of the ideas raised here in Chapter 2.

1.3.1 – Differences arising from the mathematical forms that are made available by the teacher

Stigler and Hiebert (1999) identify particular between-country differences in the mathematical forms stressed overall in the videotaped lessons they observed from Japan, Germany and the United States of America. They characterise German teaching as 'developing advanced procedures' and comment

> There is concern for technique, where technique includes both the rationale that underlies the procedure and the precision with which the procedure is executed.
>
> (Stigler and Hiebert, 1999:26)

Here technical aspects of the topic seem to be emphasised as well as using procedures in order to solve problems. Yet insights into the origins of procedures seem to be offered too. This contrasts, so state the authors, with American teaching, which is characterised as 'learning terms and practicing procedures' (ibid.). A direct comparison is made between the teaching of angle in the United States and Japan. In the former what was made available was articulated definitions of particular angle types and relationships such as acute angles and supplementary angles (angles whose sum is 180°), and exercises in finding angles that are supplementary to given angles. In Japan, the teacher asked learners to investigate relationships between the angles formed by the intersection of two lines, from which they were able to derive a proof. It seems that technical aspects of the topic (definitions and simple procedures) are emphasised more in this example of how content was made available in the United States, whereas property noticing and reasoning seem to be emphasised more in the Japanese lesson.

Comparing how content is made available in these three cases highlights the emphasising of technical aspects of the topic as a dimension of possible variation. Technical aspects seem to be most emphasised in the American case and least in the Japanese. However, differences between the Japanese and German cases appear to extend beyond the relative stressing of technical aspects. Visual aspects are emphasised more in the Japanese case than seems to be the case in Germany, where what is offered appears more utilitarian. This suggests the presence of at least one other dimension of possible variation for what is emphasised over time in addition to technical aspects.

In their study, Stigler and Hiebert (1999) focus on single lessons and while the above discussion points to differences in emphasis, differences in emphasis over a series of lessons on a given topic emerge from a recent study conducted by Ayalon and Even (2013). Two teachers, Sarah and Rebecca, both taught an algebra topic with two of their respective classes.

> In general, the opportunities to engage in transformational algebraic activity related to forming and investigating algebraic expressions were found to be similar in Sarah's two classes as well as in Rebecca's two classes. Nevertheless, substantial differences were found between Rebecca's classes with regard to the opportunities to engage in transformational algebraic activity related to equivalence of algebraic expressions.
>
> (Ayalon and Even, 2013:19)

In one of Rebecca's classes, a purpose is offered for simplifying algebraic expressions, which is in order to establish the equivalence of two expressions. By contrast the second class are offered only expressions to simplify. In the former case, there are more explicit opportunities for learners to work with properties of equivalence whereas in the second technical aspects are emphasised.

Ayalon and Even's approach to comparing Sarah and Rebecca's teaching offers an effective way of discerning differences between the ways of teaching a particular topic in different contexts. However, categories are topic specific, which limit their transferability to other contexts. Further, while counts of occurrences of categories offer a way of making quantitative comparisons, I hoped to build on this approach and to find a way of marking larger differences as significant statistically when designing my analytical approach in Chapter 3.

In a study of teaching in England, Askew et al. (1997) contrast teaching that they describe as aligning most closely to a connectionist orientation with teaching that aligns to either a transmission or a discovery orientation. They exemplify the former teaching approach as likely to involve working with equivalent representations, critically comparing methods and using material world problems as sources of exploration. Teaching that aligns most closely to a transmission orientation involves emphasising methods and then offering 'contrived' (Askew et al., 1997:33) material world problems that required little reasoning in order to solve. This resonates with Skemp's notion of teaching instrumental mathematics (Skemp, 1976). Discovery teaching is described as learners generating their own methods 'based upon building up their confidence and ability in practical methods' (Askew et al., 1997:34). This approach emphasises using manipulatives, forming images and noticing properties.

These studies together illustrate how it is possible to describe teaching in terms of the mathematical forms made available: images, equivalent representations, techniques, procedures to solve problems, opportunities to notice properties, opportunities to reason. Furthermore they highlight how in different situations, some forms are more likely to be offered than others and that differences in mathematics teaching arising from the mathematical forms that are made available by the teacher are an observable phenomenon.

1.3.2 – Differences arising from sequencing of mathematical forms

Before exemplifying what I mean by the sequencing of mathematical forms, it is worth highlighting that the sequencing I have in mind differs from Rowland et al.'s (2005) use of the term within the *Knowledge Quartet*. For Rowland and his colleagues, sequencing is an aspect of *connection*: 'knowledge-in-action as revealed in deliberation and choice in planning and teaching' (Rowland et al., 2005:265).

> Within a single lesson, or across a series of lessons, the teacher unifies the subject matter and draws out coherence with respect to the relative complexity and cognitive demands of mathematical concepts and procedures, by attention to sequencing of the content.
>
> (Rowland, Huckstep and Thwaites, 2005:265)

From Rowland et al.'s perspective, what is sequenced is content, and the act of sequencing is guided by the teacher's curricular knowledge (Rowland et al., 2005:256). I am concerned with sequencing of *forms* of content, the order in which teachers make available to learners imagery, techniques, opportunities to notice properties, opportunities to reason and the like. However, literature searches revealed little insight into this particular phenomenon. So instead in this section I focus more on models of growth in understanding. The sequencing of forms of content can then be conceived as teacher actions that afford such growth. For example, Bruner (1966) discusses the developmental shift from perceptual embodiment to symbolic notation in terms of movement from enactive representation through iconic representation to symbolic representation, and in doing so emphasises the importance of the iconic both as a route to and exemplification of the symbolic. While Bruner talks of learners' cognitive development, at the same time he describes mathematical activities that afford shifts in mode of representation between the enactive, iconic and symbolic. Furthermore, he describes how the teacher may scaffold the learner's experience 'such that the [learner] could do with her what he plainly could not do without' and fade support as the learner takes control of tasks he was not able to do at first (Bruner, 1986:76), suggesting a change in the nature of the teacher's role over time.

Freudenthal (1981) takes a historical perspective on 'the growth of mathematics' (Freudenthal, 1981:137) in order to offer the following sequence:

- Knowledge acquired by insight

transformed by

- schematising and memorising (or call it, codifying)

into

- skills and insight of a higher order

(Freudenthal, 1981:137)

This perspective aligns with that of Pólya (1962), who offers a discourse of *consecutive phases* in order to describe cognitive development, from an *exploratory phase of intuition and heuristics* through a *formalising phase of concept definitions* to an *assimilation phase of absorption and utility*.

There are similarities again with Pirie and Kieren's (1989, 1994) recursive theory of understanding (see figure 1.3/a). *Primitive knowing* is described as a 'starting place for the growth of any particular mathematical understanding' (Pirie and Kieren, 1989:8), from which images are formed that can be 'examined for specific or relevant properties' (ibid.). Articulation of these properties leads to abstraction of a 'method or common quality' (Pirie and Kieren, 1994:66) or *formalising*. Growth to this point corresponds to Bruner's (1966) notion of a developmental shift from perceptual embodiment to symbolic notation. Objectification of the method or symbolically expressed quality affords co-ordination of formalising and expression of theorems (Pirie and Kieren, 1994:71) or *observing*. Gathering theorems into a unified theory involves *structuring* and being aware of constraints, which might be relaxed affording inventing or *inventising* – the exploration of 'what if not' questions (Brown and Walter, 2005).

Pirie and Kieren comment on figure 1.3/a:

> We have presented the model as a sequence of nested circles or layers, thus emphasising the fact that each layer contains all previous layers and is embedded in all succeeding layers. We see growth as represented by back and forth movement between levels and it is thus that we characterise understanding as a dynamic and organising process. We use the language of 'levels' and 'layers' and certainly there is some underlying hierarchy within the model.
>
> (Pirie and Kieren, 1994:172)

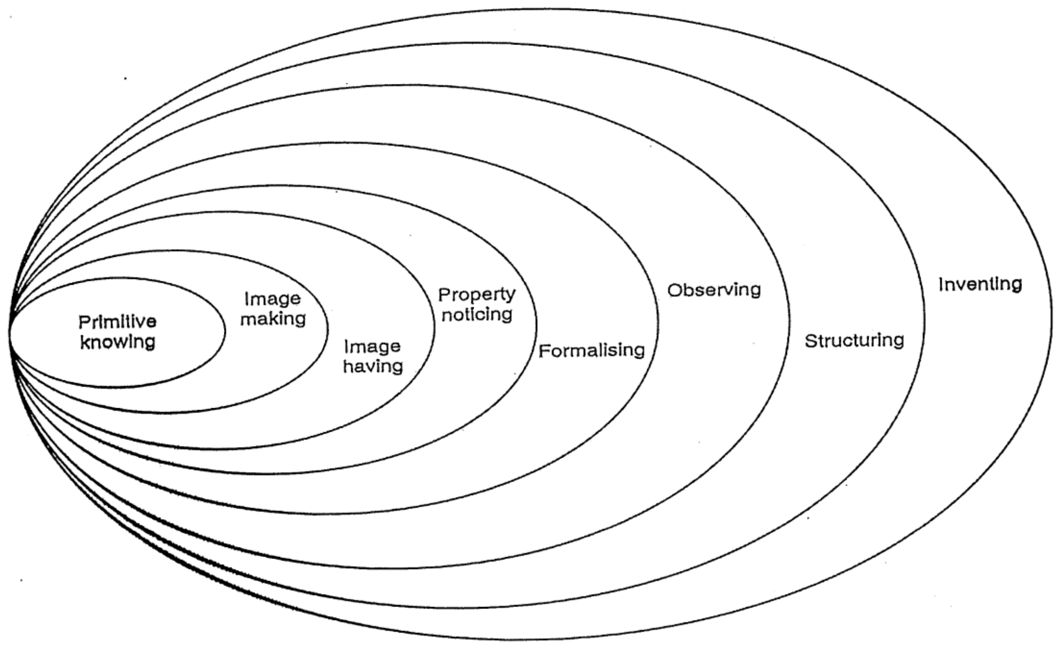

Figure 1.3/a Pirie and Kieren's (1989, 1994) model of growth in mathematical understanding.

Van Hiele-Geldorf (1957) offers an alternative hierarchical model of growth in geometrical understanding, exemplified by Burger and Shaughnessy (1986).

> Level 0 (Visualization). The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.
> Level 1 (Analysis). The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.
> Level 2 (Abstraction). The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.
> Level 3 (Deduction). The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.
> Level 4 (Rigor). The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.
>
> (Burger and Shaughnessy, 1986:31)

Once again there are links to Pirie and Kieren's model and indeed the other perspectives on growth offered in this section. Visualisation aligns with image having, analysis with property noticing, abstraction with formalising, and rigor with structuring. However, Mason (2004b) questions 'attempts to specify distinct levels' and instead asserts that the van Hiele level statements describe how 'people attend all the time, often with rapid shifts from one to another' (Mason, 2004b:27). Further, what Pirie and Kieren describe as 'back and forth movement between levels' (Pirie and Kieren, 1994:172) are shifts in the focus of attention (Mason, 2004b:27).

As mentioned above, much of the discussion in this section has centred on modelling growth in understanding rather than the sequencing of forms of content. However, if phases (Pólya, 1962), layers (Pirie and Kieren, 1989) and levels (van Hiele-Geldof, 1957) are thought of as foci of attention (Mason, 2004b), and particular forms of content afford directing learners' attention (see section 1.2.3), then the sequencing of forms can be thought of as opportunities to shift the focus of learners' attention over time. Indeed, my conceptualisation of different forms of content is linked to affordances for focusing learners' attention (full details are provided in Section 2.3).

1.3.3 – Differences arising from the type of exchanges made available by the teacher

Stigler and Hiebert (1999) note differences in the type of exchanges they saw in videotaped lessons made in Japan, Germany and United States of America:

> Differences among the lessons of Messrs. Eisner, Yoshida, and Jones seemed to lie in more than just the content. The way in which students were asked to engage in mathematics seemed to be different. In the German and U.S. lessons, students participated mostly by giving brief responses to the teacher's specific questions. Although the level of mathematics was higher in Germany, in both countries the teacher did most of the mathematical work. In Japan, the reverse seemed to be true. The typical Japanese lesson invited students to do more of the mathematical work.
>
> (Stigler and Hiebert, 1999:66)

They investigate differences in allocation of lesson time in the three countries to classwork and seatwork. Classwork is defined as the teacher 'working with all the students and, usually, orchestrating the discussion' (Stigler and Hiebert, 1999:67) and talk is public. Seatwork is 'when students work individually or in small groups on assigned tasks' and 'talk is mostly private — teacher-student or student-student' (ibid.). Stigler and Hiebert note

> Teachers in all three countries spent more time in classwork than in seatwork. In Japan and the United States 60 percent of the time was spent in classwork; in Germany, it was 70 percent. Although the overall percentage of time spent in classwork was similar, shifts within the lesson from classwork to seatwork and vice versa were considerably more frequent in Japan than in the other two countries.
>
> (Stigler and Hiebert, 1999:67)

This finding highlights the allocation of time to classwork and seatwork as a source of possible variation in how content is made available. Stigler and Hiebert's work also offers a glimpse of the structured types of exchange that are offered in Japanese lessons. Shimizu (1999) adds definition to this, identifying four distinct teacher roles within a lesson: *hatsumon*, *kikan-shido*, *neriage* and *matome* (see table 1.3/a)

Table 1.3/a Four distinct teacher roles in Japanese lessons and their descriptions taken from Shimizu (1999:110-111).

Teacher role	*Description of role*
Hatsumon	'Asking a key question that provokes students' thinking at a particular point in the lesson'
Kikan-shido	Teaching at the desks, including 'purposeful scanning by the teacher of the students' individual problem-solving processes'
Neriage	'The dynamic and collaborative nature of the whole-class discussion during the lesson'
Matome	'Summing up. Japanese teachers think that this stage is indispensable for a successful lesson'

These types of exchanges are clearly not limited to Japanese lessons, but hatsumon, neriage and matome open up the possibility of differences in the way content is made available during classwork. Clarke (2004) widens Shimizu's (1999) definition of kikan-shido to include explicitly spoken exchanges between teacher and learners during seatwork. Such exchanges may be implied in Shimizu's definition, as 'purposeful scanning' is 'included' in – rather than constitutes – kikan-shido (Shimizu, 1999).

The wider definition of kikan-shido opens up the possibility not only of differences in the type of exchanges between teacher and learners during seatwork but also differences in the form of mathematics made available through teacher talk. In particular, there is a possibility that the forms of mathematics offered through teacher talk do not align with those offered in the set problem.

Some parallels can be drawn between the observations of the previous paragraph and the findings of Henningsen and Stein (1997). They noticed that certain types of exchange appeared to contribute to learners engaging in the 'high-level mathematical thinking' afforded by particular tasks: 'scaffolding', 'modelling of high-level performance' and 'sustained press[ing] for explanation and meaning' (Henningsen and Stein, 1997:534). Particular types of exchange offered by one teacher – Mr. Hernandez – exemplified this:

> Mr. Hernandez was able to assist students as they reasoned through the problems without reducing the complexity of the task at hand.
>
> (Henningsen and Stein, 1997:540)

Whether the task built on learners' prior knowledge and sufficient time was allocated in order to accomplish the task are examples of other important influences. If these criteria were not met, then the authors suggested that engagement in high-level mathematical thinking was likely to 'decline'. I would prefer to describe 'decline' as the focus of learners' attention shifting to other features of the task, and one such shift was to focus on 'procedural thinking without connection to meaning' (Henningsen and Stein, 1997:541). In one classroom example, the authors' account for the 'decline' in mathematical thinking was the teacher (Ms. Capra) directing learners' attention to a procedural method of problem solving. Setting aside the value-laden language of 'decline', the cases of Mr. Hernandez and Ms. Capra exemplify qualitatively different types of exchanges that could be offered during seatwork: exchanges that afford focusing learners' attention on the task in hand and those that shift attention to other features of the task.

1.4 – Revisiting the object of enquiry

1.4.1 – A summary of what is already known and opportunities for contributing to this

Watson (2008) highlights the lack of detailed research addressing the specific question of how teachers structure mathematical content. The use of the verb 'structure' may evoke thoughts of control, procedure, pre-determined pathways and little learner agency. However, Watson offers an alternative perspective on structuring, which involves the teacher:

> Design[ing] the environment very carefully so that it is more likely that learners will pay attention to some aspects of mathematical objects rather than others, more likely to use certain language forms than others, and more likely to engage in some kinds of mathematical activity than others.
>
> (Watson, 2008:127)

In contrast to the lack of research that Watson (2008) identifies, much is known about differences in the general qualities of how teachers make mathematical content available to learners as I have demonstrated in Section 1.3. However, they do not specifically exemplify the issue of differences in how teachers make content available *over a series of lessons*. Less is known about this, although a recent study by Ayalon and Even (2013) compares the teaching of an algebra topic by two teachers to two pairs of classes. Specific comparisons between series of lessons on different topics are even less common, despite the fact that this is a meaningful unit of teaching for many teachers, who explicitly plan at this level.

The decisions that teachers make in the course of planning and teaching a series of mathematics lessons shape the way they act over time and are shaped by the classroom situation of which the teacher is part. That situation is in constant flux as the relationship between teacher, learners and content develops. Watson (2004) describes a classroom as a system 'which offers certain kinds of interaction as possibilities' and within which the teacher and learners 'interact in ways which themselves influence the development of the system' (Watson, 2004:24). This *ecological* perspective of the classroom to conjecture that lesson-to-lesson differences in how content is made available might be anticipated within a lesson series because of an evolving environment and that even differences between series of lessons on different topics taught by the same teacher might be anticipated. Stigler and Hiebert (1999) comment that differences in teaching between cultures 'dwarfed' between-country differences, 'differences which appear so large within our culture' (Stigler and Hiebert, 1999). Nevertheless, from an ecological perspective there is much to learn from discerning differences between series of lessons taught locally through analysis at a relatively finer grain size.

Askew et al. (1997) also identify differences in teaching but go further, asserting the teaching aligning with a connectionist orientation to be more effective. My study only focuses on what the teacher makes available, so is not concerned with measures of effectiveness through learner gains that feature in studies such as that of Askew and his colleagues. My goal is to expose and describe sources of possible variation. Mason (2004b) says that:

> Learning is … the opening up of possibilities to attend in different ways in different contexts.
>
> (Mason, 2004b:30)

Agreeing with this stance, my aim as a researcher of mathematics education is to contribute to our understanding of mathematics teaching through opening up possibilities rather than constraining possibilities by endorsing particular approaches.

1.4.2 – My thesis

In Section 1.3, I highlighted three sources of possible variation in how teachers make mathematical content available to learners. I expressed these as:

- ▷ Differences arising from the mathematical forms that are made available by the teacher
- ▷ Differences arising from sequencing of mathematical forms
- ▷ Differences arising from the type of exchanges that are made available by the teacher

I conceptualise the structure of how teachers make mathematical content available to learners as having two elements – the mathematical forms and the types of exchange that are offered. I label these elements *manifestation* and *interaction*. I conceptualise both of these two elements as having an underlying triadic structure and both can be characterised over time as a relative stressing of the three elements of their triad. I will offer a rationale for this perspective in Chapter 2. Characterising manifestation and interaction in this way over a series of lessons allows for comparisons between cases of teachers teaching a topic to a class and characterisation over a lesson allows for lesson-to-lessons comparisons within a case. This approach reveals and exemplifies possible sources of variation in how teachers make mathematical content available to learners over time, as I will demonstrate through four individual case studies (Chapters 4 to 7) and a cross-case comparison (Chapter 8). Specifically, I will address the following two research questions that follow from the discussion of sources of possible variation in how content is made available:

RQ1: Within a single series of lessons, what differences are discernible in how a teacher makes content available to learners over time?

RQ2: When comparing series of lessons, what differences are discernible between how teachers make content available to learners over time?

In order to answer these questions, I develop an analytical approach for charting and comparing how content is made available over time. This approach involves a close analysis of what the teacher says and the tasks they offer. The analytical methods I use are innovative and so, in Chapter 3, I describe in detail the way I developed and applied these methods.

My aims for this study are to:

- ▷ promote the issue of how teachers make mathematical content available to learners over time as being worthy of consideration;
- ▷ understand better how teachers make mathematical content available over time generally
- ▷ offer accounts of how teachers make mathematical content available over four specific series of lessons

Achieving these aims will mark my thesis as a contribution to our deeper understanding of how teachers make mathematics content available. But for me it will be a significant contribution if the substance of inferences abducted from analysis goes beyond features of the classroom situation to which teachers and researchers routinely attend. For if they are features to which all – and especially teachers – *could* attend, then this opens up opportunities to attend differently.

CHAPTER 2

CONCEPTUALISING HOW TEACHERS MAKE CONTENT AVAILABLE TO LEARNERS

Chapter contents

2.1 – Interaction and manifestation

In this section, I focus more closely on the phenomenon of how teachers make content available. I conceptualise the underlying structure of this phenomenon as having two elements: *interaction* and *manifestation*.

2.1.1 – The two elements of how content is made available to learners

A beginning teacher and a more experienced colleague (his mentor) were talking about the former's embryonic plans for teaching a series of lessons on a particular mathematical topic. By way of encouraging the beginning teacher to consider critically his assumptions about mathematics teaching, the mentor asked:

> "Which first: technique, exploration or application?"
>
> (De Geest and Watson, web ref.: 11)

From a teacher's perspective, teaching a topic has multiple aspects: the mentor in the above scenario emphasises technique, exploration and application, but others could have been suggested. So the teacher's role involves making decisions about the inclusion and sequencing of these aspects. Then it is through these aspects that learners come into contact with content, the mathematical objects that make up the topic. Deliberately, I use the word *aspect* as a placeholder for technique, exploration and application, and indeed other words that might be used when describing how content is made available to learners. There is an underlying similarity between technique, exploration and application, but I suggest there is a subtle but important distinction that can be drawn as well.

I consider exploration to be an example of the way in which content is made available to learners. Exploration involves the learner acting on and reacting to the mathematical content of the lesson. But the teacher has a role here too in the classroom context, bringing the learner into contact with the content but then standing back. Exploration is therefore an example of the teacher *interacting* with learners through content, and is one of six *modes of interaction* identified by Mason (1979).

A teacher interacting with learners through content is a familiar phenomenon to all who have experience of schools. We see a teacher at the front of a classroom displaying information on a board and hear them speaking to the whole class. We see and hear a teacher talking to a learner and the learner responding. We hear a teacher questioning the whole class and see learners raising their hands in order to answer. These *modes of teacher interaction* may be labelled: expounding, discussing and questioning.

On the other hand, techniques and applications are examples of the mathematical content of classroom interactions. These are the objects of explanation, discussion and the like, the forms in which mathematical phenomena are offered – or *manifested* - in the classroom.

I consider a technique and an application to be examples of features of a mathematical topic to which the teacher directs learners' attention through the *manifestation* of mathematical objects. For example, a perpendicular bisector of two points (see figure 2.1/a) is a geometrical phenomenon. This phenomenon may be manifested by its physical construction (using straight edge and compasses, for example) or by a problem involving radar navigation (where the points represent radar base stations). Additionally it may be manifested by a definition: the line perpendicular to the line segment joining two fixed points which passes through the midpoint of that line segment. Indeed, the diagram offered in figure 2.1/a is also a manifestation of a perpendicular bisector of two points. There are many more manifestations of mathematical phenomena too, but these will be considered in more detail in section 2.3.

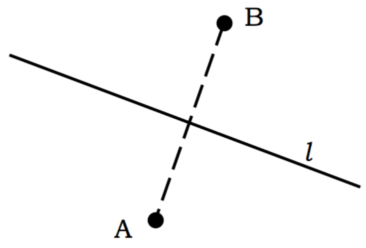

Figure 2.1/a
Line ℓ is the perpendicular bisector of the line segment AB. It is perpendicular to, and passes through the midpoint of, the line segment AB.

Thus a conceptualisation of the wholeness of how teachers make content available to learners needs to take account of both:

i. how the teacher interacts with learners and content
ii. how content is manifested.

I assert therefore that the way teachers make content available to learners has two elements: interaction and manifestation. *Interaction* refers to the way teachers *interact* with learners and content and *manifestation* refers to the way content is *manifested*.

But what *are* interaction and manifestation within my conceptualisation of how teachers make content available to learners? I view them as sets of *affordances*. Recall that Chemero (2010) posits an affordance is a relationship between an actor (in this case, the teacher) and its environment (in this case, the mathematics classroom environment), or more precisely a feature of its environment. From this perspective, the mathematics classroom environment affords various modes of teacher interaction. In particular, it is the *didactical system* (teacher and learners brought together by the mathematical content within the classroom environment and institutional requirements) that affords various modes of teacher interaction. For example, the didactical system affords exposition to the teacher. Discussion and questioning are other such affordances. Drawing on a pattern of language used by Greeno (1994:340), if a teacher is involved in making content available to learners, then the action of focusing learners' attention is a part of that activity. The teacher will therefore be attentive to features of the didactical situation that afford focusing learners' attention. It is these features that I refer to as manifestations. This having been said, for simplicity I will refer to *modes of teacher interaction* and *manifestations* rather than explicitly use the language of affordances. However, I will use the simple and technical language interchangeably as befits the point being made.

It is important to keep in mind that affordances are not necessarily *realised*, so while discussion may be an available mode of teacher interaction it is not necessary that it comes to action. However, as stated in the Section 1.2.3, such 'affordances do not disappear when [there is no teacher] to take advantage of them' but rather they 'can be objectively studied' and 'are in no way figments of the [teacher's] imagination' (Chemero, 2010:193). This warrants speaking of affordances that are realised against a backdrop of a set of affordances that includes affordances that might be perceived by the teacher but do not come to action and those that are not perceived. So for example, in a particular mathematics classroom situation discussion might come to action but this is only one of many modes of teacher interaction afforded by the mathematics classroom environment (exposition, questioning, ...). Other modes of interaction may have come to the teacher's mind (in other words, the teacher may have perceived other affordances) and some may not have done (those affordances that were not perceived).[2] Amongst the affordances that are not realised, I do not distinguish between those that may have been perceived by the teacher and those that are not. This has implications for method (see Chapter 3) as I am not focusing on the teacher's experience of affordance perception. This line of enquiry also calls for the identification of a set of affordances sufficient to account for classroom interaction and another set sufficient to account for how content is manifested. I approach this task through reflection in the

[2] Why certain manifestations are not realised is not a consideration of the current study, but for example, Schoenfeld's model of teacher decision-making based on psychological choice theory, which seeks to explain why teachers act as they do (Schoenfeld, 2011), could be applied in order to offer an account for choices that do not come to action.

following section, starting with considering other frameworks for describing how content is made available within the literature.

2.1.2 – Visual representation of how teachers make content available to learners

An image to which I regularly returned when seeking to conceptualise how teachers make content available to learners was the didactical triangle (see figure 2.1/b), which has been used by researchers in many countries for many decades now. Goodchild and Sriraman (2012) note about the didactical triangle:

> The didactical triangle in which student, teacher, and content form the vertices (or nodes) of a triangle is the classical trivium used to conceptualise teaching and learning in mathematics classrooms.
>
> (Goodchild and Sriraman, 2012:581)

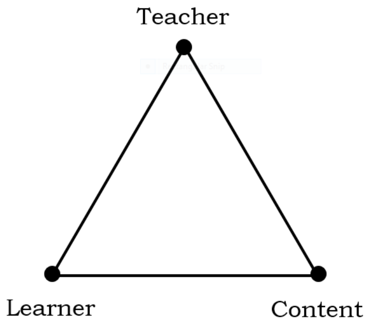

Figure 2.1/b
The didactical triangle. Learner, teacher and content form the nodes of the triangle.

However, bound in to such a conceptualisation of teaching and learning is my own thinking about the roles of learner, teacher and content and the relations between them. I seek to explain my position in this section through comparison with other perspectives.

Ruthven (2012) asserts that analysis informed by the didactical triangle of mathematics classrooms is 'shaped by the overarching "didactic"' (Ruthven, 2012:628) of the learner-teacher-content system. He cites as examples of didactics the classical German *Stoffdidaktik*, the French *didactique* (specifically Brousseau's notion of *situations didactiques* (Brousseau, 1997)), realistic mathematics education (associated with the work of Freudenthal, to which reference was made in Section 1.3.2), and the different 'cultural scripts for teaching and learning mathematics' (ibid.) highlighted by Stigler and Hiebert (1999) and which I discussed in Section 1.3. For Chevallard (1982), the didactical triangle represents the *système didactique* (didactical system) of a mathematics classroom. He emphasises how the 'overarching didactic' that Ruthven identifies – and specifically the transformation of *savoir savant* (scientific knowledge) into *savoir enseigné* (school mathematics) – is constrained by factors beyond the immediate classroom situation. He offers an image of classrooms nested within a wider regulatory/bureaucratic/political system. In particular, these external influences shape the mathematical content within didactical systems. Schoenfeld (2012) summarises somewhat critically the effect of these influences:

> The cultural arrangements that produce mathematics classrooms and the mathematics taught in them wind up transforming (one might say deforming) mathematics as practised outside the classroom by those who use mathematics in their work (including mathematicians, but considering the clients of mathematics most broadly) into a "classroom version" of mathematics that can, at times, bear at most a superficial resemblance to mathematics as practised.
>
> (Schoenfeld, 2012:588)

In response, Schoenfeld (2012) promotes a particular didactic, or 'mathematical culture of the classroom' (Schoenfeld, 2012:598), that foregrounds problematizing, learner agency and authority, learner accountability to the discipline rather than to the teacher, and availability of resources.

Ruthven concludes that:

> The didactical triangle serves, then, as a generic organiser for the analysis of didactical systems, and any such system can be characterised in terms of the way in which its overarching *didactic* frames the relations between content, student and teacher and envisages their interaction.
>
> (Ruthven, 2012:628 – my emphasis)

Ruthven (2012:628) recognises that the overarching didactic of any system might be 'overt' (such as didactics he offers as examples mentioned above) or 'tacit'. The nature of the influence an overarching didactic has on the in-flight choices enacted by a teacher within a didactical system is complex and, returning to the discussion of teacher decision making in Section 1.2, contested. My analysis is of the teacher's enacted choices relating to how content is made available to learners over time, which may bring to the surface tacit didactics characterising a didactical system (specifically a teacher working on a topic with a class over a series of lessons). So while accepting that external influences of a particular didactical system are present, I do not make these influences an object of enquiry as I am concerned with understanding rather than explaining a teacher's enacted choices.

Kansanen and Meri (1999) offer a perspective that closely parallel's my own on the teacher's role within the didactical system:

> If we describe the activities of the teacher as teaching, we would prefer to call the activities of the students as studying. It is this studying we can see and observe in the instructional process. In other words, the relation between the student and the content is visible as studying, doing something in order to achieve the aims and goals in the curriculum. The invisible part of this relation may be learning and other consequences of the instructional process.
>
> (Kansanen and Meri, 1999: 113)

With specific reference to the didactical triangle, Kansanen and Meri identify what they call the *didactical relationship* between the teacher and studying (see figure 2.1/c).

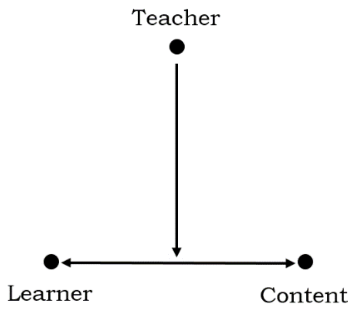

Figure 2.1/c
The didactical relationship between teacher and studying proposed by Kansanen and Meri (1999) represented within the didactical triangle.

This relationship closely parallels my conception of teachers making content available to learners (which includes both manifestation and interaction), but I would contest the unidirectional arrow representing the relationship between teacher and studying. This arrow appears to set the teacher apart from the didactical system. From my perspective, how the teacher makes content available to learners is bound in to the didactical system. This distinction is also manifested in Kansanen and Meri's perspective on the possibility of an overarching didactic:

> The didactic relation viewed as a teacher's relation to studying has some immediate consequences. It is difficult to believe that the didactic relation could be universally organised or according to certain technical rules. Each teacher is supposed to think and decide for him/herself how to handle it. It follows that every teacher has a didactics of his/her own.
>
> (Kansanen and Meri, 1999: 114)

In contrast, I assert that in-the-moment choices about how content is made available are enacted by the teacher within the context of a unique didactical system (teacher, learners and content at a particular moment in time). From this perspective, it follows that *at a sufficiently fine grain size* how content is made available to learners is likely to be different in different didactical systems. I therefore offer figure 2.1/d as an adaption of figure 2.1/c.

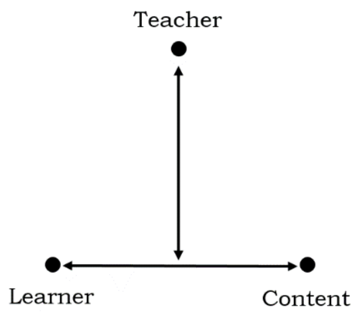

Figure 2.1/d
My conceptualisation of how teachers make content available to learners represented within the didactical triangle[3].

Steinbring (2005 and 2008) contrasts historical and contemporary paradigms within German mathematics education research (*Mathematikdidaktik*) and expresses the differences between these paradigms with respect to the didactical triangle. Historically, Stoffdidaktik focused on school mathematics (Steinbring, 2005: 314) and conceptualised mathematical content as 'uniform, objective and unchangeable', the teacher as 'conveyor of the didactically prepared content' and learners as 'passive receivers' (Steinbring, 2008: 305). The contemporary paradigm focuses on mathematical interaction (Steinbring, 2005: 314), represented by the edges joining the nodes of didactical triangle (see figure 2.1/e). Steinbring comments:

> [The didactical triangle] no longer represents an ideal paradigmatic schema … but it becomes an instrument for the analysis of [observed] mathematics instruction, in which the reciprocal interconnectedness between the three relevant elements participating in the instruction process are systematically captured.
>
> (Steinbring, 2008:309)

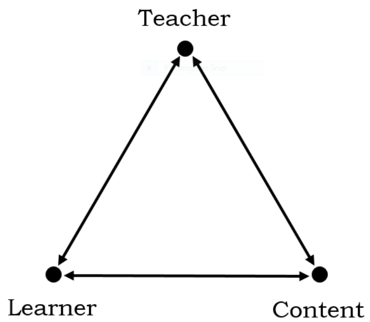

Figure 2.1/e
Representing interaction through the didactical triangle. The edges joining the nodes capture the relationships between learner, teacher and content.

[3] This image also appears within a figure in Jaworski (2012: 624), but no reference is made to why the didactical triangle is presented this way.

Within this contemporary paradigm, Steinbring's own research focuses on the 'social epistemology of mathematical knowledge', asserting that 'mathematical knowledge is constituted in classroom interaction' within lesson episodes (Steinbring, 2008: 310).

For me too, it is the edges rather than the nodes of the didactical triangle to which I attend when conceptualising *teacher interaction*. Yet my focus is not on how mathematical knowledge is interactively constructed, but rather on what the teacher makes available (through the way content is manifested) and how (through the mode of teacher interaction) that might afford knowledge construction. When I use the term 'teacher interaction' within the context of the current study, I have in mind the teacher's role within the didactical system (see Section 2.2 for further detail).

Further, I am concerned with how teachers make content available to learners *over time*, which calls for a coarser analysis than is likely to be needed in order to understand how knowledge is interactively constructed during a lesson episode. With this in mind, I now turn to consider two empirical frameworks for analysing teaching over time as I seek to continue to explain my position through comparison with other perspectives.

2.1.3 – Identifying interaction and manifestation within other frameworks

The Mathematics Education Traditions of Europe (METE) project sought to conduct a comparative study across four European countries in order to investigate 'teachers' observable didactic behaviours' and how 'teachers structure the learning opportunities they present their students' (Andrews, P. and Sayers, 2005:139). Video recordings of lessons across the four countries were analysed by the research team using a coding system developed as part of the project (Andrews, P., 2005). The smallest unit of analysis used by the team was an *episode*, a 'period of a lesson in which a teacher's observable intention remained constant' (Andrews, P. and Sayers, 2005:142), nested within an *activity*, which in turn was nested within a *lesson*. In addition to codes for 'generic mathematical learning outcomes' and 'conceptions of mathematics which underpinned mathematical tasks' (Andrews, P. and Sayers, 2005:139), ten codes were used in order to categorise how teachers made content available to learners (figure 2.1/f).

From my perspective of how teachers make content available, six of these ten codes relate to interaction: explaining, sharing, exploring, coaching, assessing or evaluating, and questioning. Furthermore, the teacher plays an observably active role – the teacher is to be seen or heard to be doing something in the classroom – in five of these six codes, with exploring being the only exception.

Activating prior knowledge	The teacher is observed to focus students' attention on mathematical content covered earlier in their careers in the form of a period of revision as preparation for activities to follow.
Exercising prior knowledge	The teacher is observed to focus students' attention on mathematical content covered earlier in their careers in the form of a period of revision unrelated to any activities that follow.
Explaining	The teacher is observed to explain an idea or solution. This may include demonstration, explicit telling or the pedagogic modelling of higher level thinking. In such instances the teacher is the informer with little or no student input.
Sharing	The teacher is observed to engage learners in a process of public sharing of ideas, solutions or answers. This may include whole-class discussions in which the teacher's role is one of manager rather than explicit informer.
Exploring	The teacher is observed to engage students in an activity, which is not teacher directed, from which a new mathematical idea is explicitly intended to emerge. Typically this activity could be an investigation or a sequence of structured problems, but in all cases students are expected to articulate their findings.
Coaching	The teacher is observed to offer hints, prompts or feedback to facilitate students' understanding of or abilities to undertake tasks or to correct errors or misunderstandings.
Assessing or evaluating	The teacher is observed to assess or evaluate students' responses to determine future direction of teaching.
Motivating	The teacher, through actions beyond those of mere personality, is observed to address students' attitudes, beliefs or emotional responses towards mathematics.
Questioning	The teacher is observed to use a sequence of questions, perhaps Socratic, which lead pupils to build up new mathematical ideas or clarify or refine existing ones.
Differentiation	The teacher is observed to treat students differently in terms of the kind of tasks or activities offered, the kind of materials provided, and/or the kind of expected outcome in order to optimise instruction to the students' characteristics and needs.

Figure 2.1/f Coding scheme from METE project (Andrews, P., 2005)
Definitions of ten 'didactic strategies' developed by the METE project team.

What may be less clear is that three of the ten codes relate to how content is manifested: activating prior knowledge, exercising prior knowledge, and motivating. In the coding scheme, the distinction between activating and exercising prior knowledge is drawn on the basis of whether that knowledge is to be emphasised subsequently in the lesson or not. Neither category stipulates the nature of interaction between teacher, learner and prior knowledge, whether this be for example through learners working on a revision exercise made available to them by the teacher, or by the teacher posing testing questions to learners (Ainley, 1987), or by the teacher suggesting that learners produce a 'mind map' as a representation of their knowledge. Rather, prior knowledge in the sense it seems to be used within the METE project – and indeed how in my experience it seems to be used regularly by teachers - is a manifestation of content: as something familiar to the learner, as something that they can manipulate with confidence, as something they use habitually, and as something that can be 'tested', 'examined' and 'assessed'.

When teaching in a way that is classified as motivating within the METE project coding scheme, the teacher addresses learners' 'attitudes, beliefs or emotional responses towards mathematics' (Andrews, P., 2005). This category does not relate to interaction, for while learners may be extrinsically motivated by particular interactions, motivation here is about how teachers make affective experiences of mathematical phenomena available to learners:

> Experiences can be natural, conflicting or alien. There is widespread agreement that appropriate disturbance, disruption of expectation, is what initiates and motivates acting in such a way as to make sense, and to learn… Some learners are motivated more by effectiveness and success in the material world (authenticity of activity); some are motivated more by aesthetics of patterns and structures.
>
> (Mason and Johnston-Wilder, 2004:114)

I describe a disturbance that initiates sense making as a *problematic origin* or *root problem* (taken from Mason's *Structure of a Topic* (Mason and Johnston-Wilder, 2004)). Such a disturbance may be encountered in the classroom, but a particular disturbance may also have been experienced in history and have led to the development of a branch of mathematics; these are *historical origins*. Problematic or historical origins and 'authentic' context are examples of manifestations of content that may be encountered in the classroom and considered to be motivating for learners.

Differentiation as a teaching strategy is neither an interaction nor a manifestation itself, but is an indicator that how teachers make content available is to be different for learners within a class. However, the nature of these differences may be described in terms of different interactions, different manifestations or different lengths of time for activities. Thus some learners may be offered different tasks or resources to others (mainly associated with different manifestations), or different levels of teacher support (mainly associated with different interactions), or more time to work on the same activity (mostly associated with the same or similar manifestations and interactions). There may also be differences in learner outcomes, but this goes beyond the consideration of how content is made available. So even within a classroom where there is explicit differentiation, how the teacher makes content available to learners may still be described in terms of interaction and manifestation. However, differentiation raises a consideration for an empirical inquiry (that I discuss in relation to Case Study 4 in Chapter 7) as to whether the unit of analysis is to involve the class, a group within the class for whom how the teacher makes content available is the same, or an individual learner.

The analytical instrument developed by Watson (2007) is set out in figure 2.1/g. The bold headings are what I would describe as interactions: the teacher making mathematical statements, directing attention, questioning, leading a discussion, and providing opportunities for learners to act mathematically. The specific words in normal text are tasks, and as such are a combined expression of interaction and manifestation. For example, 'indicate classification' is an expression of interaction (teacher indicating) and manifestation (a sense of the properties of the underlying mathematical phenomenon), and 'application to other contexts' involves on the one hand interaction (learner doing) and on the other application as a manifestation of the underlying mathematical phenomenon. Also 'prove' is a succinct expression of learner doing and justification. The words in italics may be thought of in terms of opportunities for learners to delve deeper into a particular manifestation (for example, procedural 'fluency') or to shift between manifestations (from a sense of to a 'public orientation towards' a mathematical phenomenon, for example).

An important distinction between the language in Watson's framework and the language I use to express how teachers make content available is that the former's perspective 'start[s] from mathematics rather than from teaching' and emphasises 'the aim of teaching mathematics is to enable learners to act mathematically' (Watson, 2007:118). While I would not disagree with this aim, my perspective starts from the teacher and so the manifestations of mathematical phenomena to which I refer may be somewhat different to the objects or 'that which is being worked on' (Watson, 2007:120) to which Watson refers.

Consideration of the ways in which Watson (2007) and Andrews, P. (2005) make sense of how teachers make mathematical content available to learners contributes to supporting the validity of my conceptualisation of the phenomenon in terms of interaction and manifestation. In addition, root problems, justifications, procedures and exemplifications are manifestations that arise within these frameworks and that can inform the identification of a set of affordances sufficient to account for how content is manifested.

Teacher makes or elicits declarative/nominal/factual/technical statements • Say what the lesson is about • Information giving • Define terms • Tell/know/ask facts, definitions, techniques • 'Research' facts, definitions, techniques *shift: Remember* **Learners are expected to** • Imitate method, copy object • Follow procedure • Find answer using procedure • Give answers *shift: fluency, report/record actions* **Teacher directs learner perception/attention** • Tell/show objects which are perceived as having a single feature • Tell/show objects which are perceived as having multiple features • Tell/show multiple objects • Indicate identification of characteristics/properties • Indicate classification • Indicate comparison • Indicate identification of variables and variation • Summarise what has been done *shift: public orientation towards concepts, methods, properties, relationships*	**Teacher asks for learner response** • Tells what to think about • Use prior knowledge • Find answer without known procedure • Visualise • Seek pattern • Compare, classify • Describe • Explore variation • Informal induction • Informal deduction • Create objects with one feature • Create objects with multiple features • Exemplify • Express in 'own words' *shift: personal orientation towards concepts, methods, properties, relationships* **Discuss implications** • Varying the variables deliberately • Adapting procedures • Identifying relationships • Explication/ Justification • Induction/ Prediction • Deduction *shift: analysis, focus on outcomes and relationships*	**Integrate and connect mathematical ideas** • Clarify • Associate ideas • Generalisation • Redescription • Summarise development of ideas • Abstraction • Objectification • Formalisation • New definition *shift: synthesis, connection* **Affirm/ act as if we know ...** • Explore properties of new objects • Adapt/ transform ideas • Application to more complex maths • Application to other contexts • Evaluation of development of new idea • Prove *shift: rigour, objectification, use*

Figure 2.1/g Analytical instrument focusing on the teacher's contribution to shaping the lesson (Watson, 2007). 'The bold headings are dimensions of mathematical pedagogic orientation… the words in normal text classify a range of public tasks and prompts within each overarching orientation… the italicised words are a summary of the kinds of shift a learner might be hoped to make during mathematical activity' (Watson, 2007:120).

In the following sections of this chapter I will offer ways of categorising interaction and manifestation, where each category will be an observable feature of how content is made available. In order to chart the unfolding story of the way I make sense of how teachers make content available to learners over time, at the end of each subsequent section in the current chapter I will offer an iconic representation of the fundamental elements of the phenomenon by way of summary. Literally, the picture builds as progress is made through the sections. Figure 2.1/h represents the progress thus far and shows the way in which I model how teachers make content available to learners over a lesson series in terms of interaction, manifestation and time.

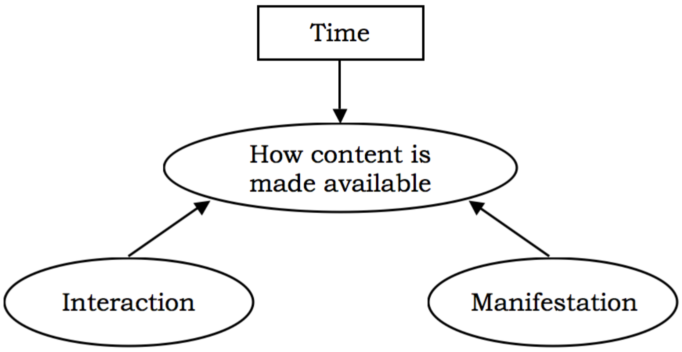

Figure 2.1/h A simple structural model of how teachers make content available to learners over time. Sections 2.2 and 2.3 will be concerned with developing a categorisation of interaction and manifestation respectively. These observable categories will then offer a characterisation of the two constructs.

2.2 – How teachers make content available: interaction

In this section I work towards a categorisation that will serve to characterise interaction over time. I start by reflecting on teachers' classroom behaviours. Then by considering the three edges of the didactical triangle, I identify three fundamental elements of interaction. I conclude by linking these fundamental elements to observable teacher actions in the classroom.

2.2.1 – The three fundamental elements of interaction

When I observe a teacher at work in a mathematics classroom, I notice a range of ways content is made available. Sometimes only the teacher speaks for a period of time; while speaking, the teacher may also write or display an image on a board or screen. On other occasions, the teacher takes turns in speaking or sharing information with learners for a period of time, perhaps posing questions and listening to learners' responses. I notice still further occasions when the teacher does not speak and is perhaps watchful while learners appear active on a task: talking, writing, calculating and drawing.

Many of these teacher behaviours can be seen in the frameworks of Andrews, P. and Sayers (2005) and Watson (2008). When 'mak[ing] or elicit[ing] declarative/nominal/technical statements', 'direct[ing] learner perception/attention' (Watson, 2008) and 'explaining' (Andrews, P., 2005) it is likely that only the teacher speaks for a period of time. When 'ask[ing] for learner response', 'discuss[ing] implications' (Watson, 2008), 'sharing', 'assessing or evaluating', 'questioning' and 'coaching' (Andrews, P., 2005) it is likely that the teacher is taking turns in speaking or sharing information with learners. When 'act[ing] as if we know' (Watson, 2008) and 'exploring' (Andrews, P., 2005) it is likely that the teacher does not speak and is perhaps watchful while learners appear active on a task.

So I notice three distinct patterns of teacher behaviour while making content available: only the teacher speaking or sharing information for a period of time, the teacher taking turns in speaking or sharing information with learners, and the teacher not speaking and perhaps being watchful while learners are active on a task. How might this teacher behaviour be described in terms of the language of interaction?

The didactical triangle was introduced in Section 2.1, and figure 2.1/e is reproduced here as figure 2.2/a.

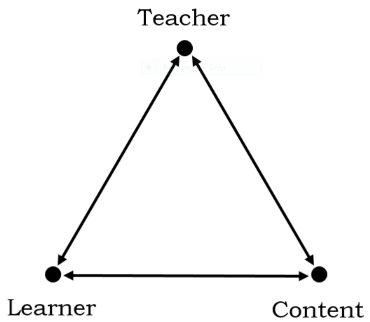

Figure 2.2/a
The didactical triangle demonstrating three elemental interactions between teacher, learner and content

As discussed previously in Section 2.1, when considering interactions within a didactical system, it is the connections between the nodes rather than the nodes themselves that are of significance, but the three nodes serve as a reminder that it is the third node that brings or holds the other two together within the didactical system. As Herbst and Chazan (2012) note, drawing on Brousseau's notion of the didactic contract (Brousseau, 1997):

> The teacher, the student and the mathematics to be studied are kept together by virtue of an implicit contract that identifies the content as something that exists outside the student and that the student must come to know with the assistance of the teacher.
>
> (Herbst and Chazan, 2012:605)

The specific terms of any such contract may vary and relate to the overarching didactic (Ruthven, 2012). Herbst and Chazan conceptualise these specific terms as 'norms that teacher and students are expected to follow as they play out their roles' (Herbst and Chazan, 2012:605). However from the authors' perspective, the norms do not define the teacher's practice as he or she may act in ways that either align with or contradict these norms by 'expressing their individual skill or personality' (Herbst and Chazan, 2012:605).

From my perspective too it is the teacher's enacted choices that define his or her practice and the teacher's role within the didactical system is constituted by these choices. But rather than express these choices in terms of satisfying or breaching pre-determined norms, I characterise them as a relative stressing of three *fundamental elements* of teacher interaction. I label these three fundamental elements: teacher-content, teacher-learner and learner-content interaction. The stressing of each of these fundamental elements individually is associated with a profoundly different interpretation of the didactic contract (see table 2.2/a). These possible interpretations of the didactic contract in turn may be associated with the three distinct patterns of teacher behaviour while making content available (only the teacher speaking for a period of time, the teacher taking turns in speaking with learners, and the teacher not speaking and perhaps being watchful while learners are active on a task) discussed above. Further exemplification of actions associated with each fundamental mode of teacher interaction follow table 2.2/a.

Table 2.2/a Linking possible interpretations of the didactic contract to fundamental elements of interaction

Possible interpretation of the didactic contract	*Description of interaction in terms of the didactical system*	*Fundamental element of interaction stressed*
The teacher's role is to present the content through exposition. The learners' role is to be an empty vessel to be filled	The teacher interacts with content but it is the learners who bring the two into contact	Teacher-content interaction
The teacher and learners' roles are to interactively construct mathematical knowledge through discussion	The teacher interacts with the learners but it is content that brings them into contact	Teacher-learner interaction
The teacher's role is to offer tasks that provide opportunities for learners to independently construct mathematical knowledge	The learners interact with content but it is the teacher who brings the learners into contact with the content	Learner-content interaction

Teacher-content interaction

When teacher-content interaction is stressed the teacher may engage in telling: articulating a perspective on the content with the intention of developing learners' understanding. Telling captures the language of transferring and conveying (Fox, 1983). The teacher may intend to demonstrate, to expound, or to present. Mason (1979) acknowledges that exposition 'has had a bad press' but points out that learners' 'recent relevant experience' can 'resonate and [become] organised by what is said' and moreover, through expounding, the teacher can 'get in touch with the content in a vigorous way' (Mason, 1998a:5). In an extreme case, the teacher is 'authoritative' with respect to the mathematical content and concentrates on 'telling others what to do and how to do it' (Mason, 1998a:5).

Teacher-learner interaction

When teacher-learner interaction is stressed, the teacher seeks to take account of learners as individuals and to consider their prior understanding of the subject through discussion. Discussion may be initiated by the teacher and involve questioning, with the teacher prompting, probing, opening possibilities or constraining them through funnelling (Bauersfeld, 1988; Holt, 1964). There may be evidence of the teacher scaffolding (Wood, Bruner and Ross, 1976) or structuring a learner's thinking in order for him or her to make better sense of a phenomenon. In an extreme case, the teacher may focus on a deconstruction of the subject matter on the assumption that 'saying more, breaking down into smaller steps, and simplifying' provides a more accessible explanation for learners (Mason, 1998a:5). However, Mason argues that this assumption leads to the teacher taking 'full responsibility for acts which are necessarily the students' (ibid.) and it is unlikely that knowledge is interactively constructed.

There are several purposes to opportunities for the learner to initiate interaction with the teacher. Through posing questions to or seeking clarification from the teacher, the learner tests his or her understanding against that of the teacher. The learner seeks affirmation of him or herself as a mathematician. Simultaneously, the teacher is provided with insights into the learner's understanding through this feedback. But feedback enables more than an assessment of learning. It contributes to assessment for learning and in turn informs subsequent teaching. Furthermore, feedback allows the teacher to respond to learners, to help refine their oral expression of mathematics and to support their reflection on progress.

Learner-content interaction

When learner-content interaction is stressed, learners are afforded the opportunity to act mathematically with greater independence from the teacher than is afforded by either teacher-content or teacher-learner interaction. The teacher facilitates learners in their exploration and consolidation of content, not by offering scaffolding (which is linked to teacher-learner interaction) but instead fading support (Brown et al., 1989) that affords learners taking the initiative. The teacher may make available opportunities for learners to explore, classify, record, conjecture, refine or consolidate. In the extreme, there may be opportunities for the learner independently to identify problems for exploration or consolidation. Such problems may arise through learners noticing something about the content.

The teacher may make available opportunities for learners to discover, to express, or to experience cognitive conflict. The intention here is often to prompt a cognitive response from learners. In more extreme cases, the teacher is passive, placing a greater emphasis on discovery rather than rediscovery.

The interaction triad

The three fundamental elements of interaction discussed above form a triadic system, represented by the image in figure 2.2/b. Of course, the above discussion does not offer an exhaustive list of teacher behaviours, but now I intend to show that other observed behaviours can be thought of as a mix of two or more of these fundamental elements of interaction.

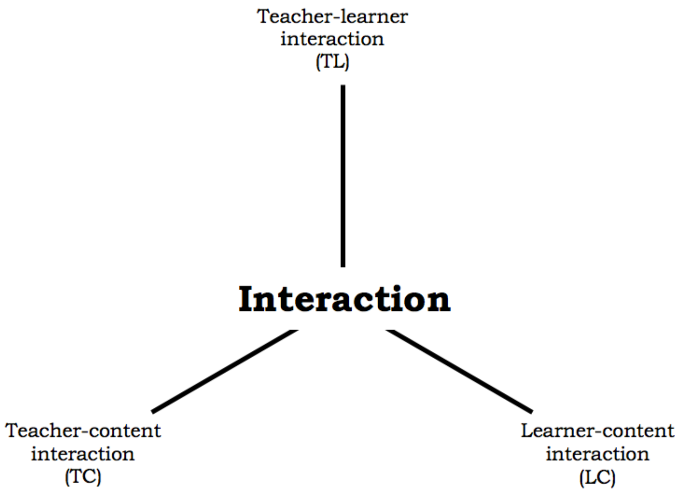

Figure 2.2/b
The interaction triad.
There are three fundamental interaction elements within a didactical system: teacher-content interaction, teacher-learner interaction and learner-content interaction.

2.2.2 – Categorisation of interaction

Mason identifies six modes of interaction within a classroom environment (Mason, 1979). His use of interaction draws on Bennett's use of the word within systematics (Bennett, 1993), and refers to the 'redistribution of things and energies' (Bennett, 1993:49). Using the language of systematics, the nodes of the didactical triangle would be described as impulses, and interaction is initiated by an affirming impulse and is an action on the receptive impulse through the reconciling impulse. Mason's six modes of interaction arise through permuting teacher, learner and content within the three impulse placeholders, and are referred to as expressing, expounding, explaining, examining, exploring and exercising (Mason, 1979).

While I initially considered Mason's six modes as a possible categorisation of interaction, I struggled to apply it to classroom observations that were focused on the actions of the teacher alone. Instead, I returned to the three fundamental elements of interaction and reflected on how these and their combinations might connect with the observable teacher behaviours I described at the start of this subsection.

When the teacher speaks to the whole class during classwork for a period of time in the form of a statement – perhaps also writing or displaying an image on a board or screen – and learners are (taken to be) watchful, I associate this with teacher-content interaction being stressed. I label this mode of teacher interaction *telling-from-the-front*. While the teacher may not literally be speaking from the front of the classroom – and indeed the classroom may not have a front as such – I use this phrase in order to emphasise that the teacher is addressing the whole class.

Using a similar pattern of language, when the teacher takes turns in speaking with learners during classwork I label this mode of teacher interaction as *discussing-from-the-front*. However, my experience of classrooms suggests that there is a further mode of teacher interaction associated with the teacher speaking to or with learners from-the-front that involves rapid switches between telling and discussing. Mason offers an example of this third mode:

> As soon as the teacher in [discussing] mode thinks they know where the student is 'going wrong' or is 'missing' an idea or a technique, there is a strong pull to jump into expository mode, to drag the student back into the teacher's world.
>
> (Mason, 1998a:7)

A further example of this third mode of teacher interaction associated with teaching-from-the-front is an Initiation-Response-Evaluation (IRE) sequence, which Schoenfeld defines as:

> The teacher initiates the sequence with a question, a student responds, and then the teacher evaluates the student's answer.
>
> (Schoenfeld, 2011:77)

The initiation and response turns follow the pattern of discussing-from-the-front, whereas the evaluation turn – particularly if it aligns with '[dragging] the student back into the teacher's world' (Mason, 1998a:7) – follows the pattern of telling-from-the-front. I label this third mode *explaining-from-the-front* and associate it with both teacher-content and teacher-learner interaction being stressed.

The three modes of teacher interaction associated with the teacher speaking to or with learners from-the-front are summarised and exemplified in table 2.2/b.

Table 2.2/b Linking observable teacher behaviours to modes of teacher interaction *from-the-front.*

Label attached to mode of teacher interaction	*Fundamental elements of interaction stressed*	*Example of observable behaviour within the classroom*
Telling-from-the-front	Teacher-content interaction	During classwork, the teacher speaks to the whole class for a period of time in the form of a statement, perhaps also writing or displaying an image on a board or screen
Discussing-from-the-front	Teacher-learner interaction	During classwork, the teacher takes turns in speaking with learners for a period of time, perhaps posing questions and listening to learners' responses
Explaining-from-the-front	Teacher-content and teacher-learner interaction	During classwork, the taking of turns in speaking is interrupted by a more prolonged teacher utterance in the form of a statement

Each teacher action from-the-front (telling, discussing or explaining) may also be seen *between-the-desks* during seatwork. I borrow the phrase 'between-the-desks' from the Japanese label for this type of interaction (*kikan-shido* – mentioned in Chapter 1). This gives rise to three further modes of teacher interaction: telling-between-the-desks, discussing-between-the-desks and explaining-between-the-desks.

Internationally, there appears not to be a consensus as to whether teaching described as kikan-shido features teacher-talk. Shimizu (1999) defines it as the teacher '[moving] about the classroom, silently monitoring students' activities' (Shimizu, 1999:110) in order to assess progress and inform the *neriage* phase[4] of the lesson. Clarke (2004) on the other hand associates kikan-shido with the teacher 'speak[ing] or otherwise interact[ing] with the students' (Clarke, 2004:234) as well as observing during seatwork. Shimizu's (1999) definition of kikan-shido aligns more closely with my notion of learner-content interaction being stressed (see table 2.2/a). More generally, I label the mode of teacher interaction associated with the teacher being watchful while learners appear active on a task as *being-watchful.* This definition of being-watchful includes the teacher observing without talking and also includes the teacher neither walking around the classroom not observing – nor even being present in the classroom – during seatwork.

Table 2.2/c extends table 2.2/b in order to summarise and exemplify the seven modes of teacher interaction that have been identified in this section.

[4] Neriage features alternative approaches to problem solving being discussed from-the-front.

Table 2.2/c Linking observable teacher behaviours to all seven modes of teacher interaction.

Label attached to mode of teacher interaction	*Fundamental elements of interaction stressed*	*Example of observable behaviour within the classroom*
Telling-from-the-front	Teacher-content interaction	During classwork, the teacher speaks to the whole class for a period of time in the form of a statement, perhaps also writing or displaying an image on a board or screen
Discussing-from-the-front	Teacher-learner interaction	During classwork, the teacher takes turns in speaking with learners for a period of time, perhaps posing questions and listening to learners' responses
Explaining-from-the-front	Teacher-content and teacher-learner interaction	During classwork, the taking of turns in speaking in a whole class setting is interrupted by a more prolonged teacher utterance in the form of a statement
Being-watchful	Learner-content interaction	During seatwork, the teacher is watchful while learners appear active on a task
Telling-between-the-desks	Teacher-content and learner-content interaction	During seatwork, the teacher speaks to a learner for a period of time in the form of a statement
Discussing-between-the-desks	Teacher-learner and learner-content interaction	During seatwork, the teacher takes turns in speaking with a learner for a period of time
Explaining-between-the-desks	Teacher-content, teacher-learner and learner-content interaction	During seatwork, the taking of turns in speaking with a learner is interrupted by a more prolonged teacher utterance in the form of a statement

These seven modes of teacher interaction are sufficient in order to account for all classroom interaction over time as I illustrate in figure 2.2/c. During a period of time, the teacher either speaks or does not speak. If the teacher speaks, then the teacher is either addressing all learners in the class or a sub-group of learners. When the teacher speaks, he or she either speaks for a period of time in the form of a statement (telling), takes turns in speaking with learners (discussing), or a mix of both (explaining). Focusing on teacher actions, no other scenarios are possible.

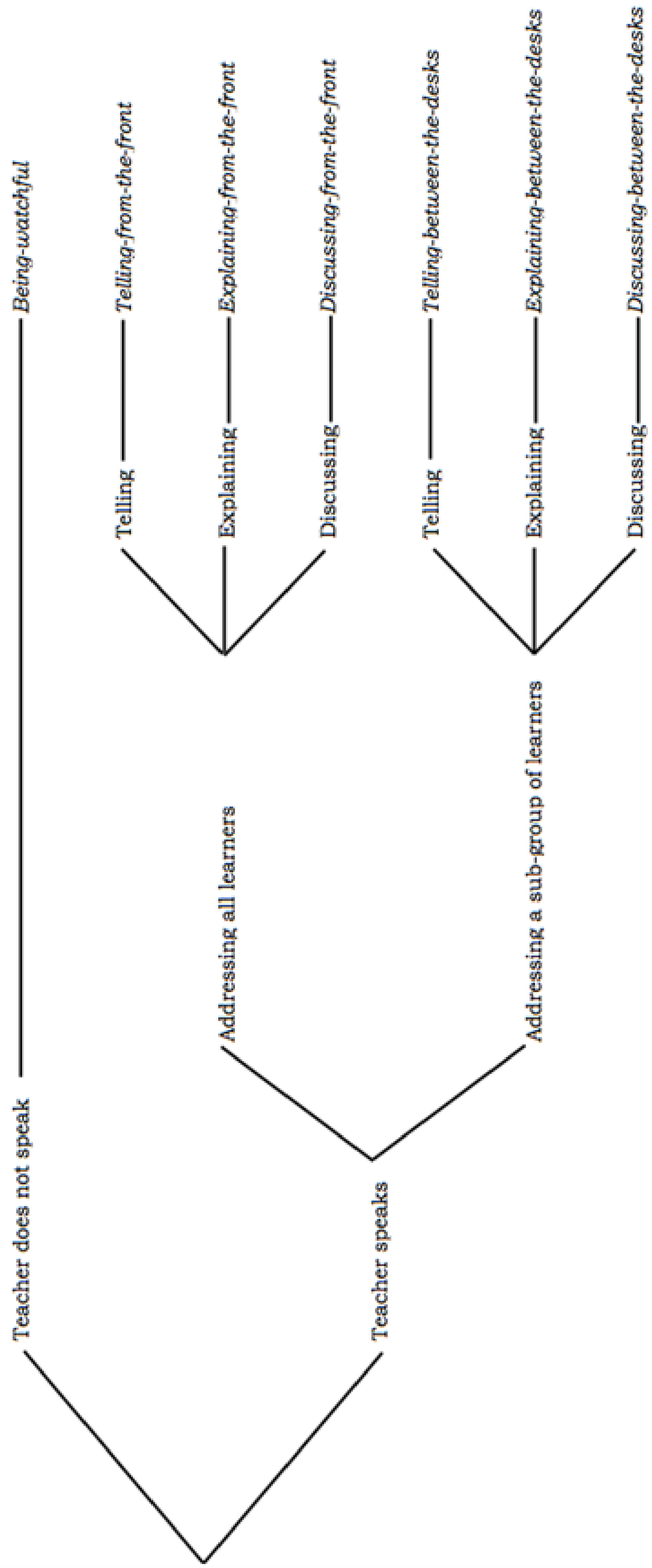

Figure 2.2/c Tree diagram for modes of teacher interaction, which emphasises how the seven modes of teacher interaction are sufficient to account for all interactions viewed from the perspective of the teacher.

2.2.3 – A structural model of interaction

In section 2.2, I have emphasised the underlying triadic nature of interaction within a classroom environment. Through considering combinations of these three fundamental interactions, I have identified a set of seven affordances sufficient to account for all interactions viewed from the teacher's perspective. These seven affordances therefore offer an exhaustive categorisation of interaction. In figure 2.2/d, I present this as a structural model. The seven observable affordances or modes of teacher interaction are presented in rectangles. The presence of one of these modes of teacher interaction at any point in time is associated with the stressing of one or more of the fundamental elements of interaction (represented by the arrows from the modes of teacher interaction). Changes in mode of teacher interaction over time result in a relative stressing of the three fundamental elements. This changing stressing offers a characterisation of interaction over the time period (represented by the arrows from the fundamental elements).

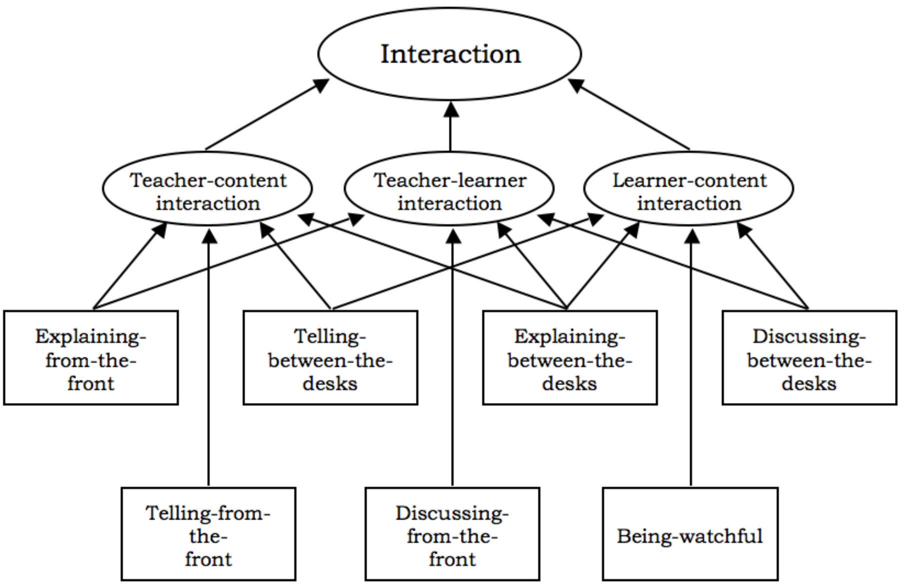

Figure 2.2/d A structural model of interaction.
The seven modes of teacher interaction contribute to a stressing of the three fundamental elements. The relative stressing of these elements then characterises interaction over time.

2.3 – How teachers make content available: manifestation

2.3.1 – Tasks and talk

Teachers frequently make content available to learners through tasks. Content can alternatively be made available through what teachers say or present from-the-front. A third possibility is that content is made available to learners by means of a combination of task and talk through teaching-between-the-desks. It is important to recognise that the stressing of a particular manifestation through the phrasing of a task does not necessarily correspond with manifestations that are stressed when this task is played out in the classroom (Stylianides and Stylianides, 2008). This can be largely due to what is stressed through what is said to learners by the teacher. So when considering how content is made available to learners it is necessary to consider both task and talk, and when using the term ‘task’ it is sensible to distinguish between the task as given and the task as played out. With this in mind, I use the terms *given-task* and *teacher-talk* when describing how content is made available.

When teaching-from-the-front, content is made available through *teacher-talk* and when the teacher is being-watchful, in almost all cases content is made available through a *given-task*. When teaching-between-the-desks, the teacher makes content available through a combination of given-task and teacher-talk. How content is manifested through teacher-talk may align with how it is manifested through the given-task, but not necessarily so. It is particularly in the latter case that the task as played out may differ from the task as given.

For the rest of Section 2.3, I will consider how manifestations might be categorised and will not necessarily distinguish between whether content is made available through a given-task or teacher-talk, although often the examples I offer will be associated with given-tasks.

<u>*2.3.2 – The three fundamental elements of manifestation*</u>

Consider the following three tasks, each of which could feature as the content of a lesson on geometrical constructions. Tasks 1 and 2 are adaptations of a problem taken from SMP 11-16 Book Y1 (1985). Task 3 was inspired by the book *Geometrical Images* (Beeney *et al.*, 1982).

<u>*Task 1*</u>

There are three ice cream sellers (called Alfie, Billy and Charlie) who set their stalls up on a beach, a plan of which is provided in figure 2.3/a. People on the beach go to the nearest seller in order to buy their ice creams, and this divides the beach up into Alfie's territory, Billy's territory and Charlie's territory. Mark the three territories on the plan.

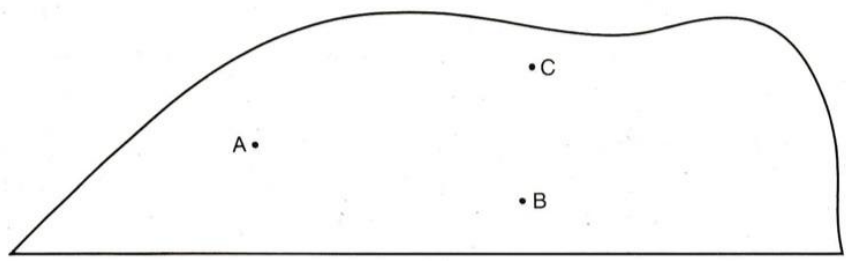

Figure 2.3/a Plan of the beach for task 1.

<u>*Task 2*</u>

Referring to figure 2.3/b,

i. Construct the perpendicular bisector of AB.
ii. Construct the perpendicular bisector of AC.
iii. Shade the locus of points within the marked region that are further from point A than point B.
iv. Shade the locus of points within the marked region that are further from point A than from point C.

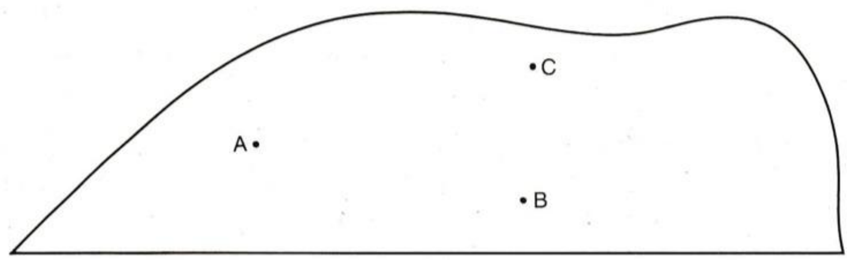

Figure 2.3/b Diagram for task 2.

Task 3 (read aloud by the teacher)

Think of two dots. Hold them still for a moment… Move the dots closer together… Hold one dot still and move the other dot away, and then bring it back again so that you can see both dots… Hold the dots still and imagine a line passing through the gap between the two dots… Move the line around… See the line and the dots… Move the line so that one point on the line is the same distance from one dot as it is from the other… Move the line so that every point on the line is the same distance from one dot as it is from another… Hold one dot still and move the other dot and the line until you find a new arrangement where every point on the line is the same distance from one dot as it is from another… And another.

The three tasks are qualitatively different to one another, yet they share one important feature. They are all manifestations of the same geometrical phenomenon: the perpendicular bisector of two points.

Task 1 manifests the perpendicular bisector as a phenomenon of the material world. The accomplishment of the task is likely to involve geometrical reasoning and the repeated application of a geometrical construction; it calls for the learner to function mathematically and draws on knowing when to use a method as well as knowing how. I label task 1 as a *functional* manifestation of a perpendicular bisector.

The accomplishment of task 2 calls for technical know-how as well as knowledge of technical language such as *perpendicular bisector* and *locus*. The task manifests the perpendicular bisector as an abstract phenomenon and calls for minimal geometrical reasoning (deciding which side of the line represents the required locus). There is also no expectation of interpretation of the overall solution (such as the unshaded region is the locus of points that are closer to point A than they are to either point B or C). I label task 2 as a *technical* manifestation of a perpendicular bisector.

Task 3 involves visualisation and dynamic imagery. The imagery points to the property of a perpendicular bisector of points A and B being the locus of points equidistant from A and B without fully defining it as such. As with task 2, this task manifests the perpendicular bisector as an abstract phenomenon. It calls for minimal geometrical reasoning and no technical knowledge of the phenomenon. I label task 3 as a *visual* manifestation of a perpendicular bisector.

I take these three manifestations – visual, functional and technical – to be fundamental elements that form a triadic system representing the structure of how content is manifested. This is represented by the image in figure 2.3/c.

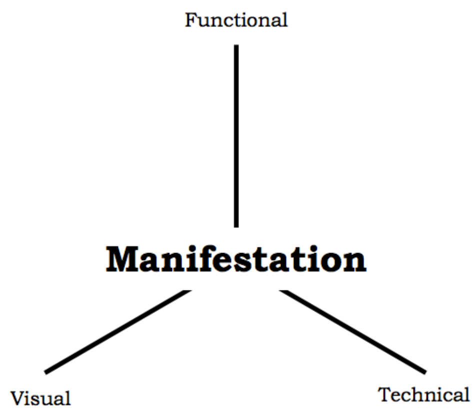

Figure 2.3/c
The manifestation triad.
In making content available to learners, the teacher manifests content. There are three fundamental elements to how content is manifested within a classroom environment: visual, functional and technical.

Arriving at an articulation of the fundamental elements of manifestation has taken time and has involved close consideration of research (see Section 2.3.3), classroom observations during this project and reflection on my classroom experience as learner, teacher and teacher educator. From before the start of this project I had a strong sense that the structure of how content is manifested is triadic - a relative stressing of three fundamental elements. During the project, I used placeholders to express the three elements of the triad. Each set of placeholders drew on established constructs within mathematics education literature and the action of working with each set of terms was within itself formative through the consideration of what may be evoked by these words in addition to their intended role here. Thus each shift away from one set of placeholders to a new set represented an important developmental point in the project.

In table 2.3/a, I offer some key mathematical terms associated with each of the three fundamental elements of manifestation; these terms feature in the descriptions of the above tasks. This triadic system aligns to some extent with other perspectives on the structure of mathematical topics (for example, Mason *et al.* (2005)) and these terms provide a succinct point of comparison. I offer further rationale in support of this triadic structure in the following section.

Table 2.3/a Mathematical terms associated with each of the three fundamental elements of the visual-functional-technical triad.

Fundamental element of manifestation	*Associated mathematical terms*
Functional	Application, reasoning
Visual	Visualisation, imagery
Technical	Definition, techniques

2.3.3 – Grounding of the three fundamental elements of manifestation within literature

Manipulating, getting a sense of and articulating

Bruner casts teaching as 'an effort to assist or to shape growth' (Bruner, 1966:1) and while much of his work related to the cognitive development of young children, this definition need not be restricted to the teaching of pre-adolescents. However, the rate of cognitive growth of adolescents is likely to be greater and shifts between representations of experience more dynamic than may be the case with younger children. Some of the examples of growth that Bruner provides play out over years (Bruner, 1966:12-14) whereas a teacher of adolescents might anticipate discernible mathematical cognitive development within a lesson, a segment of a lesson or less. For example, Mason (1980) sees potential for learners' cognitive growth through mathematical classroom activities that offer opportunities for specialising, generalising and reasoning. Thus through analysis at a finer grain-size, the theory of instruction that Bruner sets out need not be restricted to the cognitive growth of young children.

Mason (1980) develops Bruner's enactive-iconic-symbolic triad (Bruner, 1966) so as to associate each representation with a relationship between a learner and a mathematical phenomenon:

> Enactive - confidently manipulable
> Iconic - having a sense of
> Symbolic - having an articulation of
>
> (Mason, 1980:11)

A further variation of the enactive-iconic-symbolic triad is the manipulating-getting a sense of-articulation (MGA) triad proposed by Mason, Burton and Stacey (1982). A spiral based on this triad (figure 2.3/d, from Mason et al. (1982:156)) illustrates how a sequence of moves between these modes may represent learner growth.

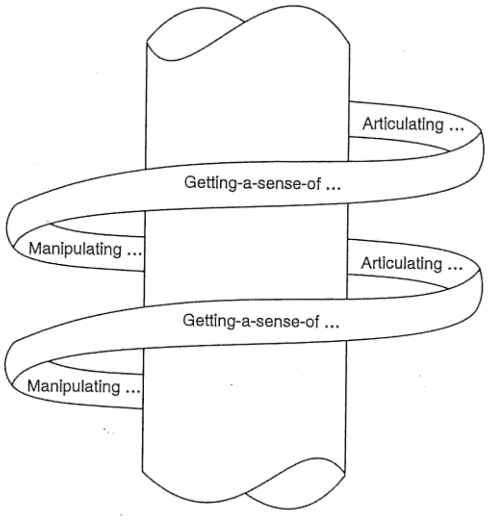

Figure 2.3/d
The manipulating-getting a sense of-articulating (MGA) spiral proposed by Mason et al. (1982), illustrating how learner growth may be represented by a sequence of moves between these three essential modes.

Both Bruner and Mason use the enactive-iconic-symbolic triad in order to express a feature of the learner: the mode of representation within which they dwell. These are not features of how teachers make content available to learners, so these three elements would be inappropriate as a triad for how content is manifested, but nevertheless evoke a sense of what such a triad might be.

Emotion, awareness and techniques

Mason also views a topic as having three strands (the 'Structure of a Topic'): emotion, awareness and techniques (in Mason, (1998a), Mason (1998b), Mason and Johnston-Wilder (2004) and Mason, Graham and Johnston-Wilder, (2005)). The emotional strand encompasses historical or 'problematic situations that originally gave rise to the topic' and mathematical or material world 'situations in which the topic ... is likely to arise' (Mason, Graham and Johnston-Wilder, 2005:280). The awareness strand includes 'images, diagrams and the like' as well as 'the classic errors that learners make', be they 'slips when not paying sufficient attention' or 'more deeply based misconstruals' (*ibid.*). The technical strand includes 'techniques and methods' as well as technical language (*ibid.*). These three strands again emphasise a relationship between the didactical situation and the learner. However, it was the relationship between the didactical situation and the teacher that I sought to capture through the concept of manifestation. This was achieved through small adaptations to the awareness-emotional-technical triad and a relabelling of the fundamental elements as visual, functional and technical (VFT).

Image having, procedural fluency and confident manipulation

Image having is a term used by Pirie and Kieren (1989, 1994) in order to distinguish between images tied to action, which are evoked through image making, and abstracted images that are no longer tied to action. I conceptualise focusing learners' attention on image having as an affordance of visual manifestation of a mathematical phenomenon. For example, task 3 above in Section 2.3.2 is concerned with an abstract image of a perpendicular bisector of two points.

Technical manifestation of a mathematical phenomenon affords focusing learners' attention on *procedural fluency*, a term I borrow from Kilpatrick et al. (2001). They describe procedural fluency as 'carrying out procedures flexibly, accurately, efficiently and appropriately' (Kilpatrick et al., 2001:116). It can be seen that task 2 above in Section 2.3.2 is concerned with carrying out geometrical constructions and interpreting technical terms such as 'perpendicular bisector' and 'locus'.

Functional manifestation affords focusing learners' attention on *confident manipulation* (Mason, 1980). Mason's term works well here, as manipulation captures the idea of working *with* a mathematical object. But the nature of mathematics is such that confident manipulation of a mathematical object opens up possibilities of awareness of other phenomena, for example confident manipulation of integer addition opens up the possibility of decimal addition. The confident manipulation associated with one phenomenon is the *primitive knowing* (Pirie and Kieren, 1994) of another. In the authors' own words:

> We noted that primitive knowing was the background mathematical understanding needed to build an understanding of some particular concept. It is therefore possible that a full or partial understanding of that concept could then, in turn, be observed as the primitive knowing for a new mathematical exploration.
>
> (Pirie and Kieren, 1994:68)

Subtle shifts in the focus of learner's attention may be afforded by each of these three elemental manifestations. For technical manifestation affords focusing learners' attention on procedural fluency but also could afford focusing attention on precisely those procedural steps that are not attended to once fluency has been achieved (Hewitt, 1996). For example, a technical manifestation of linear equations that arises in one of my case studies is an exercise in solving linear equations with the unknown on both sides such as:

$$2x - 1 = x + 3 \qquad \text{and} \qquad 2x - 3 = 7 - 3x$$

If previous experiences of linear equations have only had the unknown on one side of the equals sign, this manifestation affords focusing learners' attention on the action of transforming

$$2x - 1 = x + 3 \qquad \text{into} \qquad x - 1 = 3$$

$$\text{and} \quad 2x - 3 = 7 - 3x \qquad \text{into} \qquad 5x - 3 = 7.$$

Consideration of such subtle shifts in the focus on learners' attention is possible but this generally represents a finer grain analysis than my own and is less likely to be appropriate for analysing series of lessons. The exception to this, as arose in the case study I mention, is when teacher-talk when teaching-between-the-desks affords focusing learners' attention on this transformation. So I do not distinguish between an exercise featuring a set of linear equations to be solved that is offered to a class of learners already fluent in solving equations and the same exercise offered to a class of learners who are not yet fluent; it is the same given-task. What may well be different in the two situations is teacher-talk. Factoring in both the given-task and teacher-talk is a way of capturing what are likely to be different experiences of the exercise in the two classrooms.

Visual, functional and technical manifestations and their affordances

In this section I have illustrated how Mason's triads (Mason (1980, 1998a, 1998b), Mason et al. (1982), Mason and Johnston-Wilder (2004), Mason, Graham and Johnston-wilder (2005)) and Pirie and Kieren's model (Pirie and Kieren (1989, 1994)) lead to the manifestation triad. Table 2.3/b is an extension of table 2.3/a, summarising the links between these fundamental elements of manifestation and their affordances for directing learners' attention.

Table 2.3/b Linking the visual, functional and technical elements of how content is manifested with their affordances for directing learners' attention.

Manifestation	*Associated mathematical terms*	*The manifestation affords focusing learners' attention on…*
Visual	Visualisation, imagery	Image having
Functional	Application, reasoning	Confident manipulation
Technical	Definition, techniques	Procedural fluency

2.3.4 – Inner and outer aspects of tasks

Tahta (1981) distinguishes between the outer aspect of a task (the overt activity) and the inner aspect (any implicit activity that may be encountered by learners through engaging with the offered task). The difference between the inner and outer aspects of a task potentially offers a powerful learning experience; awareness of the inner aspect may support learners' responses to the outer task or accomplishing the outer aspect may offer a sense of the inner aspect. In section 2.3.2, I identified the following task as a functional manifestation of a perpendicular bisector of two points affording opportunities to focus learners' attention on confident manipulation.

Ice cream seller problem (functional)

There are three ice cream sellers (called Alfie, Billy and Charlie) who set their stalls up on a beach, a plan of which is provided in figure 2.3/e. People on the beach go to the nearest seller in order to buy their ice creams, and this divides the beach up into Alfie's territory, Billy's territory and Charlie's territory. Mark the three territories on the plan.

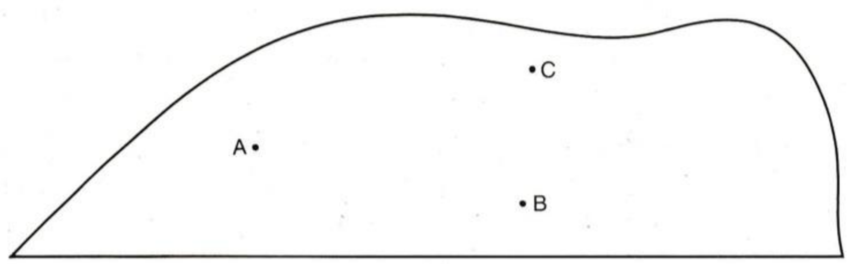

Figure 2.3/e Plan of the beach for the *Ice cream seller problem (functional)*.

A formal accomplishment of this task involves modelling this situation geometrically and then constructing the perpendicular bisectors of AB, BC and AC. These two steps are inner aspects of this task. If either of these inner aspects is revealed to learners then the way in which the perpendicular bisector is manifested is altered. If both are revealed, the task is equivalent to task 2 in Section 2.3.2 and becomes a technical manifestation.

Revealing only one of the inner aspects of the original task could be associated with scaffolding, providing an opportunity for learners to solve a problem that they would not otherwise have been able to (paraphrasing Bruner (1986:76)). A teacher action such as this affords shifting the focus of learners' attention from procedural fluency (having a technical method available to solve the problem) towards confident manipulation (knowing to use the technical method in order to solve the problem). The resulting task is a *functional-technical* manifestation of a perpendicular bisector (see figure 2.3/f). I refer to such tasks as *utilisation problems*, since the accomplishment of the task calls for *using* a particular technique or technical language.

Ice cream seller problem (functional-technical version 1)

Shade the locus of points within the marked region of figure 2.3/f that are closer to point B than points A or C.

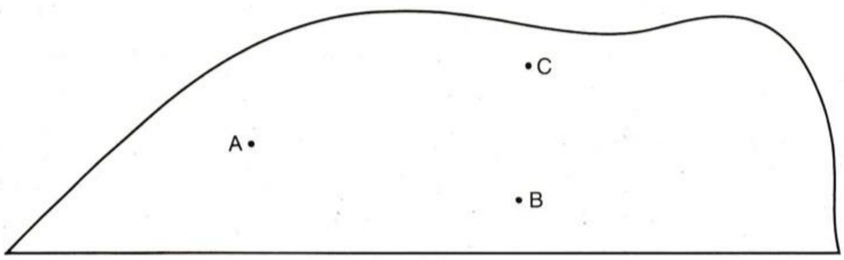

Figure 2.3/f Diagram for *Ice cream seller problem (functional-technical version 1).*

Functional-technical manifestation affords focusing learners' attention on *adaptive using*. This is a label that is partially borrowed from Kilpatrick et al.'s (2001) notion of mathematical proficiency. Such a manifestation is adaptive in the sense that the action of carrying out a procedure associated with the inner aspect is adjusted so as to be purposeful in accomplishing the outer aspect. In this sense, adaptive using builds on learners' procedural fluency. However, through adaptive using learners may also be working on their procedural fluency: procedural fluency is subordinate to adaptive using (Hewitt, 1996). Hewitt suggests that

> Children become good walkers precisely because they do not spend their time attending to the act of walking along flat floors and nothing else. It is because they place their attention in tasks which subordinate walking: they try to walk up stairs; they try to walk along walls; they try to run.
>
> (Hewitt, 1996:29)

Using Hewitt's pattern of language, functional-technical manifestation then, as opposed to technical manifestation, affords focusing learners' attention on adaptive using rather than procedural fluency and yet affords procedural fluency precisely because learners are not spending their time attending to procedures and nothing else.

A further example of a task that is a functional-technical manifestation of a perpendicular bisector is given in figure 2.3/g. Here the inner aspect (using the formal term 'equidistant') is a technical manifestation and the outer functional aspect (identifying the required points on the beach and using geometrical reasoning) is accomplished with recourse to this inner aspect.

Ice cream seller problem (functional-technical version 2)

Figure 2.3/g is a plan of the territories when there are six sellers in the positions marked A, B, C, D, E and F. Which point is equidistant from (i) A, D and F, (ii) B, E and C? The point Q is equidistant from three sellers. Which are they?

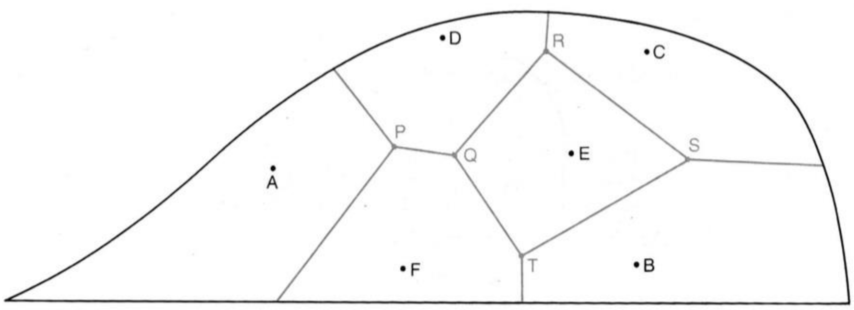

Figure 2.3/g Plan of the beach for the *Ice cream seller problem (functional-technical version 2).* Adapted from SMP 11-16 Book Y1 (1985).

As well as functional-technical manifestation of a mathematical phenomenon, *visual-functional* and *visual-technical* manifestations are also afforded to the teacher. For tasks associated with these manifestations, distinct inner and outer aspects can be identified and the tasks are such that the inner aspect is accomplished through recourse to the outer aspect or vice versa. Visual-functional and visual-technical manifestations are exemplified by the following examples.

> Ice cream seller problem (visual-functional)
>
> There are two ice cream sellers (Alfie and Billy) who set their stalls up on a beach. People on the beach go to the nearest seller in order to buy their ice creams. Figure 2.3/h is a plan of the beach with the positions of Alfie and Billie marked along with the positions of five people labelled P, Q, R, S and T. Which seller does each one go to? Divide the beach into two parts, so that people in one part go to Alfie and people in the other part go to Billy.

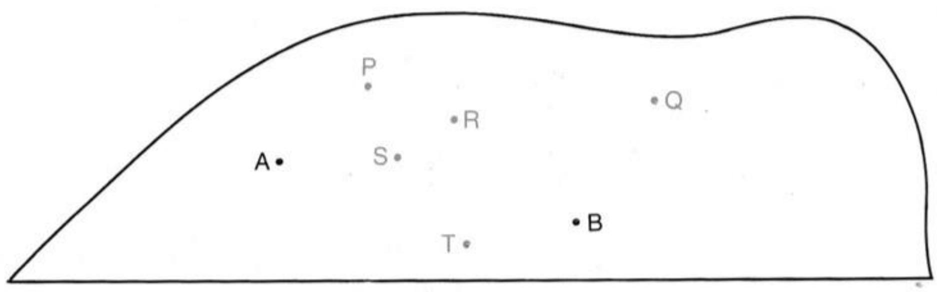

Figure 2.3/h Plan of the beach for the *Ice cream seller problem (visual-functional).* Adapted from SMP 11-16 Book Y1 (1985).

The outer aspect of this task (the relative proximity of the points P, Q, R, S and T to the points A and B) is distinct from the inner aspect (the boundary of the territories being the perpendicular bisector of AB). The inner aspect may be accomplished informally – or a sense of the inner aspect may be glimpsed – through engaging in the outer aspect. This offers learners the opportunity to shift from what may be familiar-to-the-learner actions of comparing distances by measuring or judging them by eye to having an image – but not a formal articulation – of the perpendicular bisector of AB as a line of points equidistant from A and B. This task may also be described as a *root problem*, since it offers learners an initial inner experience – or mental image – of a mathematical phenomenon in a way that reflects the phenomenon's mathematical origins. In this sense, visual-functional manifestation affords focusing learners' attention on image making. Roots problems align closely with ways of making content available to learners advocated by Freudenthal through his work on *didactical phenomenology* (Freudenthal, 1983).

Visual-technical manifestation affords focusing learners' attention on *property noticing*, as exemplified in the task below.

> <u>*Concentric circle animation (visual-technical)*</u>
>
> Using dynamic geometry computer software, fix two points A and B. Construct two circles centred on A and B with equal radius. Make sure the radius is larger than half the distance AB. Mark the points where the circles intersect. Now animate the circles so that their common radius increases from 0 and trace the position of the points of intersection.

The dynamic imagery described in this task is hard to replicate on paper, but figure 2.3/i illustrates examples of circles centred on A and B of equal radius and the trace of their points of intersection.

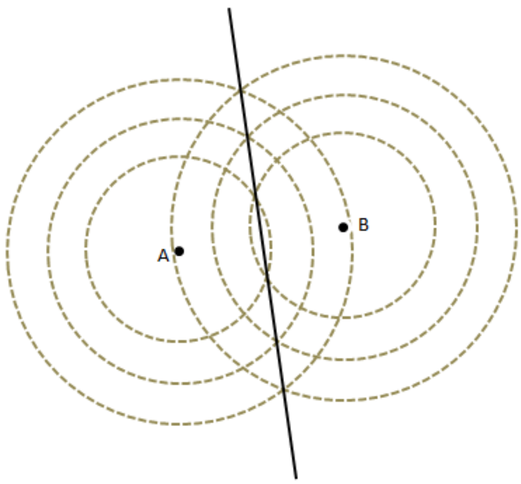

Figure 2.3/i A static representation of the dynamic imagery of the *Concentric circle animation.*

The outer aspect of this task is a visual manifestation of a perpendicular bisector through dynamic imagery. The task has an inner aspect that is a technical manifestation: a method for constructing a perpendicular bisector using straight edge and compass. This method arises from noticing that every point on the line is the intersection of two circles of equal radius centred on A and B and that just two points are sufficient in order to determine the line. The image suggests that just one pair of circles centred on A and B respectively is sufficient in order to determine the position of the line.

2.3.5 – Categorisation of manifestation

In Section 2.3.3 I associated visual, technical and functional manifestations of mathematical phenomena with opportunities to focus learners' attention on image having, procedural fluency and confident manipulation. In Section 2.3.4 I have also proposed visual-functional, visual-technical and functional-technical manifestations, which I associate with opportunities to shift the focus of learners' attention between confident manipulation and image having, image having and procedural fluency, and procedural fluency and confident manipulation respectively.

Manifestations that afford focusing learners' attention also offer opportunities for learners' mathematical behaviour to be trained and powers to be sharpened through practice. On the other hand, manifestations that afford shifting the focus of learners' attention are likely to be more formative, offering new ways in which to encounter a mathematical phenomenon but working from what is already familiar. Both have their place and one may be seen to build on the other, reinforcing the significance of considering the sequencing of tasks within a series of lessons on a particular topic.

What about a visual-functional-technical manifestation of a mathematical phenomenon? It is convenient to describe a complex task in this way if the constituent parts are different manifestations. For example, if the first phase of a task is a visual-functional manifestation of a mathematical phenomenon and the second is visual-technical, it is convenient to describe the whole task as a visual-functional-technical manifestation. Also, if a teacher is to add to a given-task through teacher-talk (as discussed in Section 2.3.1), the overall manifestation may be described as visual-functional-technical if, for example, the task is a technical manifestation and the teacher-talk is a visual-functional manifestation. This is an example of *switching*, where the manifestation of a mathematical phenomenon through teacher-talk does not align with the manifestation through a given-task. In this example, switching affords shifting a learner's attention from procedural fluency to property noticing. Switching to a previously offered manifestation affords *folding back*, a term borrowed from Pirie and Kieren (1994). The combination of manifestations is a more significant methodological issue when categorising how content is made available in the complexity of the classroom, but this is a matter that will be addressed further in Chapter 3.

Overall I have identified seven qualitatively different manifestations of a mathematical phenomenon. I summarise these in table 2.3/c linking each manifestation with affordances for focusing or shifting the focus of learners' attention.

Table 2.3/c Linking the seven categories of how content is manifested with their affordances for directing learners' attention.

Manifestation	*The manifestation affords directing learners' attention to…*
Visual	Image having
Functional	Confident manipulation
Technical	Procedural fluency
Visual-functional	Image making
Visual-technical	Property noticing
Functional-technical	Adaptive using
Visual-functional-technical	Folding back[5]

[5] Directing learners' attention to folding back is one – but not the only – example of what may be afforded by a visual-functional-technical manifestation. Such a situation might arise when a given-task is functional-technical and teaching-between-the-desks aligns with shifting towards a more visual-technical manifestation. I associate such a switch with an explicit opportunity for learners to fold back to property noticing.

2.3.6 – A structural model of interaction

In section 2.3, I have emphasised the underlying triadic nature of manifestation within a classroom environment. Through considering combinations of these three fundamental manifestations, I have identified a set of seven affordances sufficient to account for how content is manifested. These seven affordances therefore offer an exhaustive categorisation of manifestation. In figure 2.3/j, I present this as a structural model. The seven observable affordances or manifestations are presented in rectangles. The presence of one of these manifestations at any point in time within a given-task or teacher-talk is associated with the stressing of one or more of the fundamental elements of manifestation (represented by the arrows from the manifestations). Changes in manifestation over time in both given-tasks and teacher-talk result in a relative stressing of the three fundamental elements. This changing stressing offers a characterisation of manifestation over the time period.

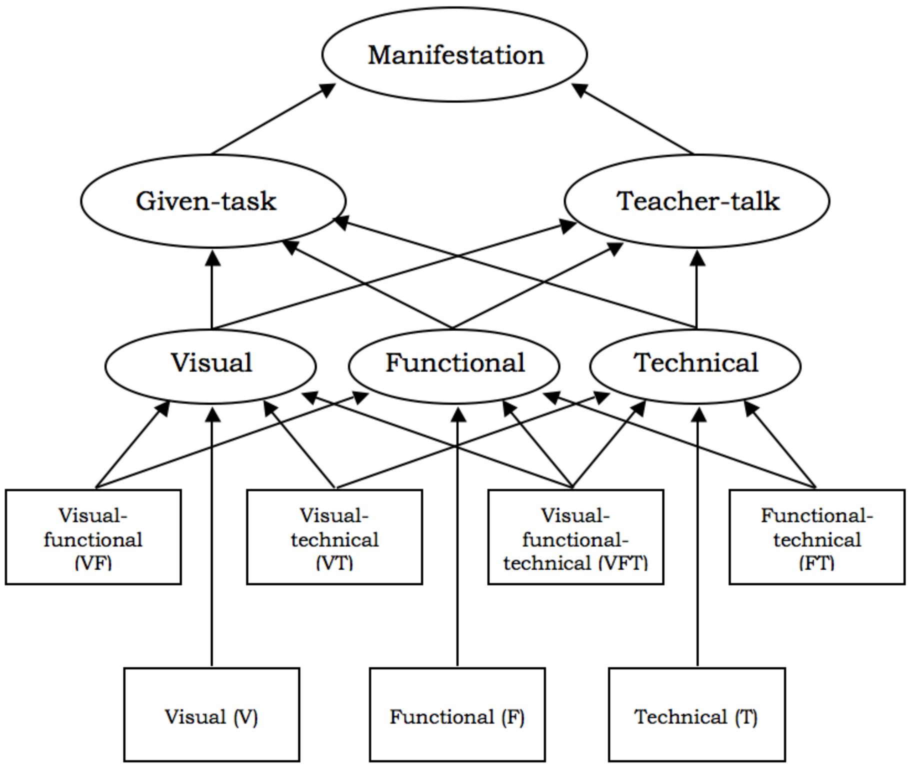

Figure 2.3/j A structural model of manifestation.
The seven modes of manifestation contribute to a stressing of the three fundamental elements. The relative stressing of these elements then characterises manifestation over time.

2.4 – Structural model of how teachers make content available over time

Figure 2.4/a overleaf combines figures 2.1/h, 2.2/d and 2.3/j into an overall structural model of how teachers make mathematical content available to learners over time. Reference back to previous sections in this chapter provides an explanation of the overall model.

> I model how teachers make content available to learners over a lesson series in terms of interaction, manifestation and time.
>
> (Section 2.1)

> The presence of one of the seven modes of teacher interaction at any point in time is associated with the stressing of one or more of the fundamental elements of interaction (represented by the arrows from the modes of teacher interaction). Changes in mode of teacher interaction over time result in a relative stressing of the three fundamental elements. This stressing offers a characterisation of interaction over the time period (represented by the arrows from the fundamental elements).
>
> (Section 2.2)

> The presence of one of the seven manifestation categories at any point in time within a given-task or teacher-talk is associated with the stressing of one or more of the fundamental elements of manifestation (represented by the arrows from the manifestations). Changes in manifestation over time in both given-tasks and teacher-talk result in a relative stressing of the three fundamental elements. This stressing offers a characterisation of manifestation over the time period.
>
> (Section 2.3)

Figure 2.4/a A structural model of how teachers make mathematical content available to learners over time.

Chapter 3

Methodology

Chapter contents

3.1 – Foundations of the research design

In order to investigate differences in how teachers make content available to learners, I will use a multiple case study design. Each case will be investigated using the same methods. These methods will be clearly defined in order to ensure that how content is made available in each series of lessons is analysed consistently. The reliability of the application of the analytical methods contributes to the validity of what is inferred about each case and what can be inferred from comparing cases. In this section, I detail the foundations of my research design particularly drawing on the work of Krippendorff (2013) and Yin (2003).

3.1.1 – Background

One of the aims of this study is to develop an empirical approach that can be used in order to discern differences in how teachers make content available to learners over time. I am sensitive to the possibility that how teachers make content available is unlikely to be easily categorised, since broad categorisation is not intended to account for 'complex, diffuse and messy' aspects of the social world (Law, 2004). This is not to say that similar ways of making content available are unlikely and do not represent interesting findings. However, the challenge here is to offer a meaningful way of representing difference, recognising messiness but also emphasising sources of possible variation.

Both planning choices and in-flight choices to act in the moment contribute to the overall way teachers make content available over time, but often these two categories of choices are of differing grain size. Planning choices that are enacted tend to shape how content is sequenced and made available across a lesson or across a series of lessons. In particular planning choices are associated with *given-task* selection. In-flight choices may shape how content is made available across a lesson segment, for example a teacher choosing to introduce an impromptu activity, but more frequently shape how content is made available through teaching from-the-front or between-the-desks; so generally in-flight choices are associated with *teacher-talk*.

Any empirical approach that might discern differences in how content is made available over time must therefore be sensitive to both coarser and finer grain choices that are enacted. So a central consideration of my research design is how to work with multiple units of analysis, and I address this issue explicitly in Section 3.3.1. However, as well as this having implications for analysis it also has implications for method. Given the frequency of in-the-moment teacher utterances across a lesson (Clark and Peterson, 1986), it is impractical to ask a teacher retrospectively about each of these. Moreover, drawing on the decision making literature reviewed in Section 1.2, what is available to come to mind when planning or teaching is a function of the situation in which the teacher finds him or herself: the features to which the teacher attends and the affordances of these features. While each of the sources of possible variation in how content is made available I am looking to investigate *could* be the focus of a teacher's attention when deciding how to act, it is not to say that they will. So what a teacher emphasises retrospectively when talking about their teaching may not align with what they were attending to during teaching (Speer, 2005; Mason and Davis, 2013), particularly if a researcher's prompts and questions shift the focus of the teacher's attention to other features. This suggests that interviewing or methods involving teacher self-reporting are unlikely to offer the type of data I am seeking in order to address my research questions.

With my study focusing on teachers making content available rather than how learners take up these opportunities to learn, methods that are available and appropriate include analysis of given-tasks and teacher-talk. Given the exploratory and developmental nature of this study, I deem the opportunity to observe lessons and have out-of-lesson conversations with teachers to be important features of the research design that might serve as a check on the validity of any inferences drawn.

The above deliberations contribute to identifying what is available to analyse, but not how it might be analysed. In a previous study (Andrews, N., 2011), I noticed a shift over time in the possibilities for classroom interaction that came to mind for pre-service teachers, as revealed through their lesson plans. I developed the *tri-polar attention space*, a triadic tool with the attractive forces of teacher-content, learner-content and teacher-learner interaction at the three poles in order to chart the shifts in attention I had noticed. Figure 3.1/a illustrates how points nearer a vertex correspond to an increase in the relative attention to that pole. The focus of each pre-service teacher's attention was a relative mix of the three forces, and so I represented each focus as a point within the interior of the tri-polar attention space. This allowed for the charting of individual shifts in attention. Furthermore at each time point I was able to represent the spread of foci of attention for all participants using the space and determine a central point.

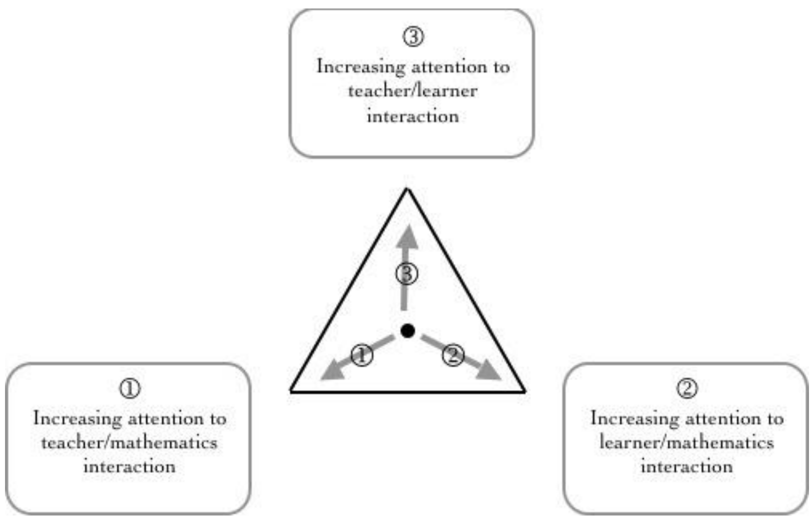

Figure 3.1/a The Tri-Polar Attention Space (Andrews, N., 2011), which will be adapted for use in the current study.

I describe in more detail the affordances of this tool for analysing data in Section 3.2. But it is important to note here how I am minded, given the underlying triadic nature of manifestation and interaction described in Chapter 2, to adapt the tri-polar attention space for the current study in order to represent graphically differences in how content is made available over time. My choice to use this analytical tool calls for a method of converting teacher-talk and documents associated with given-tasks into points within the triangular space, a conversion labelled quantitising[6] within the Mixed Methods literature (Teddlie and Tashakkori, 2003; Sandelowski et al., 2009; Seltzer-Kelly et al., 2012). I provide background information on quantitising in Section 3.1.2.

3.1.2 – Quantitising

Quantitising is often taken as a conversion from qualitative data to quantitative data, but this simple characterisation of data as either qualitative or quantitative risks sidestepping a more fundamental question of what constitutes data in the first place (Sandelowski et al., 2009). A 'data-as-given' perspective – that is a view of data being in the social world 'awaiting collection and analysis' (Sandelowski et al., 2009:209) – is common, but is countered by the perspective of 'data-as-taken' – a product of a researcher's move to present experience as data.

The data-as-taken interpretivist perspective has informed the current study. This methodological perspective aligns with the theoretical stance on teaching and learning espoused in Chapter 2, which integrates realist ontology with constructivist epistemology. Just as mathematical phenomena are taken to exist independently of our perception of them and are taken as objects that afford enactive, iconic or symbolic representation within our consciousness, classroom phenomena are also taken to exist independently of our perception of them and are taken as objects that afford representation as data through text or numbers:

> Qualitative and quantitative data are not so much different kinds of data as these data are experiences formed into, for example, words or numbers, respectively.
>
> (Sandelowski et al., 2009:209)

[6] Although in all sources to which I have referred, the American spelling 'quantitizing' is used.

From this perspective, quantitative analysis of particular classroom phenomena represented as numbers is not incompatible alongside qualitative analysis of the same classroom phenomena represented as text, and indeed together the 'complimentary strengths' (Teddlie and Tashakkori, 2003:19) of the two may be exploited. Quantitising particularly affords identification of patterns that might otherwise be overshadowed within the richness of textural data, but attending to quantitised data alone risks noticing numerical patterns that do not align with experience. In the current study, the analytical approach I will develop in order to discern differences in how content is made available over time will strongly feature quantitative analysis of numerical data. But in order to establish the validity of the approach it will be important to consider alignment between the inferences drawn from quantitative analysis and those drawn from a qualitative analysis of textural data. However, it should be stressed that quantitative and qualitative data are both representations of experience and that one is not necessarily a more valid representation than the other. Either may be warranted, and I see it as my role as researcher explicitly to articulate such warrants. This responsibility guides what I address in the rest of Chapter 3.

<u>*3.1.3 – Content analysis*</u>

The numerical representation of particular classroom phenomena may be achieved through quantitising textural representations. One such approach is content analysis (CA). CA is most succinctly defined in this quotation from Bryman (2008):

> An approach to the analysis of documents and text that seeks to quantify content in terms of predetermined categories and in a systematic and replicable manner.
>
> (Bryman, 2008:275)

Content analysis involves drawing inferences that are not available to be drawn through observation alone, often because the object of inquiry is people's intentions or is hidden within the complexity of a situation. It is particularly the latter that offers a rationale for content analysis featuring in the current study because, as stated in Chapter 1, my investigation into sources of possible variation in how content is made available goes beyond features of the classroom situation to which teachers routinely attend but are nevertheless features to which they *could* attend.

Krippendorff is regarded as a leading scholar of CA (Weber, 1990). His perspective contrasts with earlier content analysts in that content 'emerge[s] in the process of a researcher analysing a text' as opposed to it being 'contained in a text' or being 'a property of the source of a text' (Krippendorff, 2013:25). Thus Krippendorff's perspective aligns more closely to a data-as-taken, constructivist stance. He asserts that 'texts have no objective qualities' and that 'the meanings of a text are always brought to it by someone' (Krippendorff, 2013:25). The distinction between readers and analysts is that they 'merely read differently' (Krippendorff, 2013:26). Indeed, readers read differently too so that the notion of singular or even shared fine-grain meanings of texts are epistemologically questionable. Krippendorff emphasises a constructivist perspective when offering a rationale for multiple interpretations of texts: in making sense of texts, readers make connections with experiences beyond the texts themselves. Yet readers, and in particular content analysts, may be attuned to making sense of texts in particular ways and coarser-grain meanings may be taken-as-shared (Cobb et al. 2001). Krippendorff describes these attunements as *contexts*:

> Once content analysts have chosen the context within which they intend to make sense of a given text, the diversity of interpretations may well be reduced to a manageable number, sometimes to one.
>
> (Krippendorff, 2013:29)

Another cornerstone of Krippendorff's perspective is *inference*. Krippendorff asserts that that logic of inference in CA is *abductive*, since analysts 'draw inferences about phenomena that are not directly observable' (Krippendorff, 2013:42). The logic of abduction runs as follows:

> A is an occurrence that is not directly observable.
> B is an occurrence that is observable such that if A then B.
> Then if B occurs, inferring A occurs is reasonable.

It is important to emphasise that there may be exceptions to an abducted inference, but within CA a strong warrant for an inference is provided by a researcher's analytical construct which is backed by his or her deep understanding of the context.

Krippendorff highlights that as content analysis is founded upon abductive reasoning, inferences are more likely to proceed 'from particulars of one kind to particulars of another kind' (Krippendorff, 2013:41) rather than from generalisations to particulars (as with deductive inferences) or particulars to generalisations (as with inductive inferences). Content analysis is therefore appropriate to addressing research questions that do not generalise to texts other than those analysed. That having been said, sampling, and applying inductive reasoning, is an approach used by some content analysts in order to generalise their abductive inferences beyond the analysed texts. In the context of the current study, content analysis offers a way of abducting inferences about how content is made available within a particular series of lessons. Comparing the inferences abducted from each of several different lesson series allows for generalisation (in the form of conjecture) beyond the analysed series.

Issues of units of analysis, replicability and validity are of concern to content analysts as they are to researchers generally, but these are fine-grain details of the research design that I wish to address in due course (Section 3.2). But for now I intend to demonstrate how the coarser-grain methodology of CA aligns with my study's objectives.

3.1.4 – 'Making content available' conceptualised as a text

Krippendorff habitually makes reference to texts as being the sources from which inferences are drawn, but then swiftly adds 'other meaningful matter' as also being appropriate materials for CA. Thus 'text' can be interpreted in its widest sense, and historically content analyses have been conducted on political speeches, folktales and television advertisements.

In the current study, I have conceptualised what the teacher makes available to learners over a series of lessons as a text. This text includes everything the teacher is heard to say (which is converted into written text by audio recording the teacher's voice and producing a transcript from this recording), all images he or she is seen to offer and all printed resources he or she provides over the series of lessons that pertain to the topic at hand. As such, this text might be interpreted as mapping what the teacher is attending to within the classroom situation while teaching. It represents the *union* of opportunities to learn, because it is recognised that not all teacher-talk is available to all learners (for example, it might be spoken privately to individual learners) and that not all given-tasks may be available to all learners (for example, this might be within the teacher's plan, or the given-task is not accessible to the learner or the learner chooses not to engage with the given-task). The analysis of this text therefore does not focus on how teacher and learners together construct meaning (such as social constructivist or conversation analysis) or how phenomena are produced (discourse analysis) through talk; the language of the text itself is not the focus of the current inquiry.

In justifying the focus on the teacher's text, I return again to the theory of affordances and the analytical model of Chapter 2, which are the *context* for my analysis. This context is one of many possible interpretations of the text (which include individual learner's interpretations), but is one that is intended to address my objective of discerning differences in the shape of choices over time. Within my context, manifestation is a feature of the didactical situation that affords focusing learners' attention. The didactical situation also affords various modes of teacher interaction. Together manifestation and interaction produce opportunities to learn within the classroom environment, which in turn afford understanding to learners. The teacher's text represents these opportunities to learn and it is through the analysis of this text that *inferences* may be abducted about the shape of interaction and manifestation choices over time.

Figure 3.1/b offers a diagrammatic representation of the research design. This diagram in based on Krippendorff's framework for CA (Krippendorff, 2013:35).

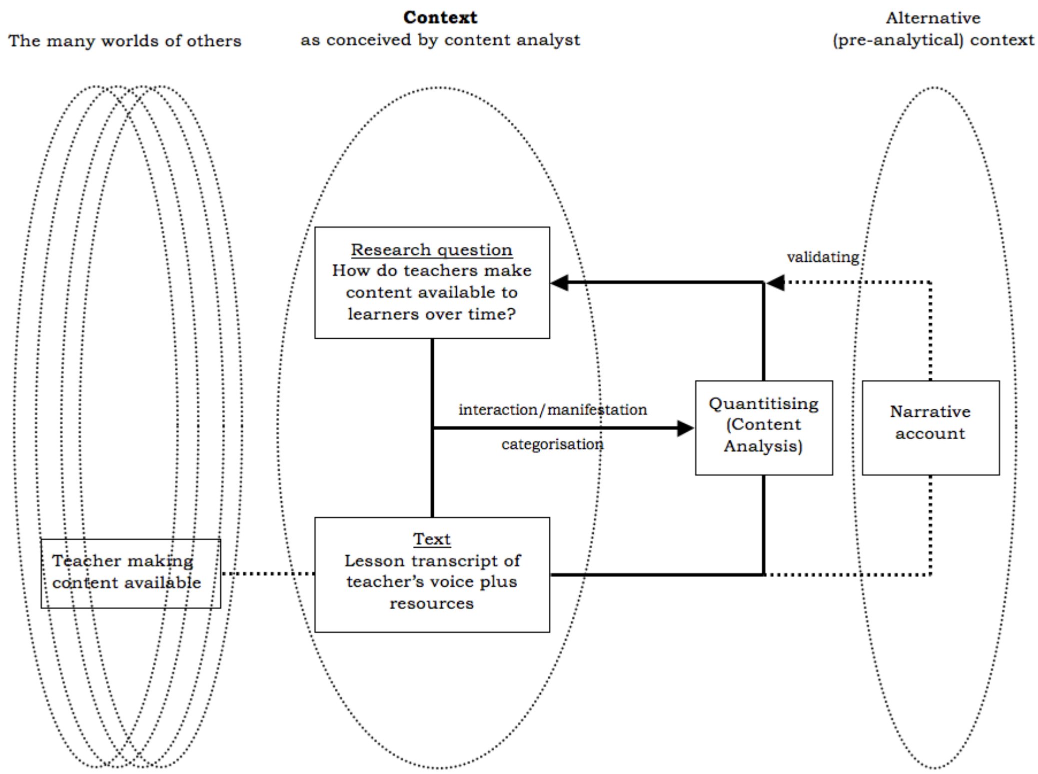

Figure 3.1/b Diagrammatic representation of my research design.

My context, which casts how content is made available in terms of interaction and manifestation, is one of many possible contexts for making sense of the text. Another such context offers a narrative account of how content is made available, which when compared to my context can be thought of as a pre-analytical account. However, a narrative account offers validation of inferences abducted from analysis of teacher-talk and given-tasks through triangulation. Triangulation in this way is somewhat problematic when what is inferred is not directly observable. However, in the current study what is inferred could be observable if attention were suitably focussed. For example, differences in foregrounding of particular modes of teacher interaction when two lessons are compared may be abducted from analysis within my context of teacher-talk and given-tasks. Such differences may also be discernible through observation or from a transcript of the teacher's utterances if the observer or reader's attention is focussed on interaction. Therefore inferences drawn from my analysis may be validated through noticing alignment within the narrative account.

3.1.5 – Multiple cases and participant selection

Yin (2003) promotes a case study strategy in order to address 'how?' questions that focus on contemporary events over which the researcher 'has little or no control' (Yin, 2003:9). Yin suggests that a reason for this lack of control may be that 'the boundaries between phenomenon and context are not clearly evident' (Yin, 2003:13). Thus a case study strategy is consistent with the use of content analysis in order to draw inferences relative to a particular context.

As discussed in Chapter 1, how teachers make content available to learners is a function of the classroom situation. Hoping to capture a sense of this, I choose to conduct multiple case studies, where cases are selected so as to ensure that there will be both a degree of perceived commonality and difference between cases. When identifying multiple cases, Yin (2003) rejects a 'sampling logic' in which cases are (almost) randomly selected and promotes a 'replication logic' in which selection is strategic, showing an awareness of dimensions of variation in cases.

> Each case must be carefully selected so that it either (a) predicts similar results (a literal replication) or (b) predicts contrasting results but for predictable reasons (a theoretical replication)
>
> (Yin, 2003:47)

As researcher I do not wish to exercise 'control' (Yin, 2003:9) over *how* content is made available within a lesson series. However, I have some degree of control over the selection of cases along certain dimensions of variation: school, teacher, class, topic and time of year. Variation along any one of these dimensions is expected to have implications for how content is made available as they are key features of the classroom environment. Asserting that differences in how content is made available

over time between any two cases would be discernible implies that a literal replication is not possible, but theoretical replication is through systematic variation along these dimensions - keeping some dimensions 'constant' while varying others.
I choose to investigate how *experienced* teachers make content available, as I am interested in understanding well-developed practices, since I conjecture that such practices are likely to feature greater diversity and so offer more opportunities to test my analytical approach. Also participants for this study are local teachers with experience of working with trainee teachers. In some respects this represented a convenient sample, but the mentoring experience of these teachers suggests that they might be more used to articulating and explaining the decisions they make while teaching than other teachers.

Given the constraint of observing teachers over a sequence of lessons, it is not practical to conduct data collection for two cases simultaneously and so the time of year dimension cannot be varied systematically. However, the school, teacher, class and topic can be varied. Figure 3.1/c represents the selection of three participants (Ashley, Bernie and Courtney) from two schools (King's Meadow School and Bishop Langton School)[7] that generate the four case studies for my investigation.

Figure 3.1/c also shows that classes are from school years 9 to 11 in England (learners aged between 13 and 16). Classes are set by prior attainment in both schools, with the average teacher-predicted attainment of learners in each class by the age of 16 being in line with the national average for mathematics. The broad topics that are the focus of each lesson series are also given in figure 3.1/c. Thus cases involving Ashley and Bernie have the school context in common, the two cases involving Bernie have the teacher in common and the two cases involving algebra have age and predicted attainment of students and topic in common. For convenience, I generally refer to cases through teacher and topic, but it is important to have in mind the significance of all four dimensions of variation and not let this

[7] Teacher and school names are pseudonyms.

convenient notation obscure the contribution of school and class to the classroom environment.

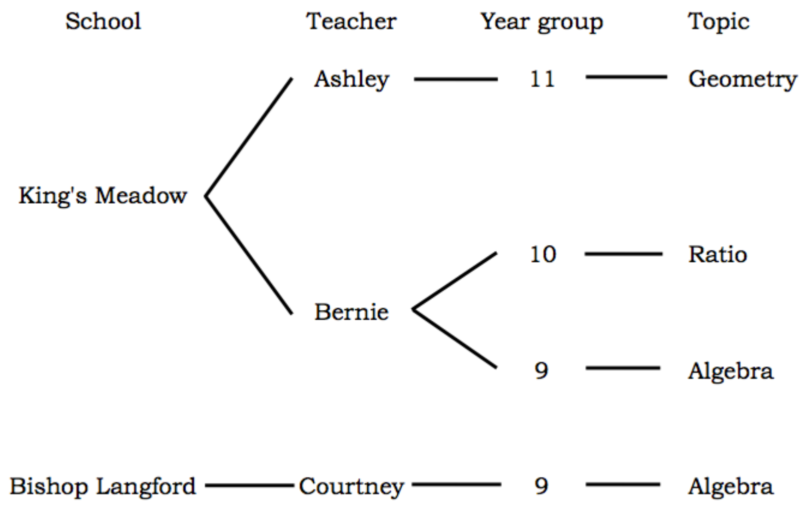

Figure 3.1/c Tree diagram representation of my multiple case study design

The multiple case studies provide broad exemplifications of teachers making content available to learners, so offer multiple responses to research question 1 (RQ1): 'Within a single series of lessons, what differences are discernible in how a teacher makes content available to learners over time?' They also allow for comparison between cases, thereby addressing research question 2 (RQ2): 'When comparing series of lessons, what differences are discernible between how teachers make content available to learners over time?'

3.1.6 – Participant induction, data collection and ethical considerations

I aim to gather sufficient data so as to insightfully represent how content is made available in a way that causes minimal disturbance to the teachers and no disturbance to learners. In practice, it is important for me to observe every lesson in the series and, in order to be able to represent how the teacher made content available to learners, it is necessary to have a record of what the teacher says, writes on the board and provides as a resource. But I want my presence as a researcher in the classroom to be as unobtrusive as possible. While videoing lessons is an option, I consider that it would be too conspicuous. I therefore choose to audio record the teacher's voice instead via a lapel microphone. Also during lessons, I sit at the side or the back of the classroom, avoiding being in lines of sight between learners and the front of the classroom (the front was taken to be where the interactive whiteboard is positioned and from where the teacher generally addresses the whole class).

While I do not want to influence the way in which the teacher makes content available to learners, I do want to gain an insight into the teacher's perspective on how this is to be done in each lesson. I want a consistent method for this that can be applied in all cases I am investigating as I intend this to support categorisation of manifestation and interaction (see in due course Sections 3.2.3 and 3.2.4). Furthermore, any such method needs to be minimally intrusive on the teachers' time. It is also important for the teachers to understand the purpose of my research and to have a sense of the aspects of their teaching on which I am to focus.

Pre-piloting suggested that an effective approach to participant induction is to hold an initial meeting with the teacher in which I share information about the research and some of the technical terms I use[8] (see Appendix 1). The teacher is then invited

[8] The language used during induction was similar but not identical to the language introduced in Chapter 2. This was because my conceptualisation of how teachers make content available to learners developed over the course of the study. However, the information I provided was sufficiently similar to the final

to bring to mind from experience a specific example of a lesson segment they have taught and talk with me about how they made content available, referencing the language I had introduced. When pre-piloting I found that these terms were not unfamiliar to teachers, and that they quickly and consistently attached meaning to them.

Given the constraints on teachers' time, it is not practically possible to conduct detailed pre- and post-lesson interviews. Instead, in advance of the lesson, the teacher is asked to identify on a simple pro-forma planned lesson segments and provide information on the planned modes of teacher interaction (whole-class exposition or discussion, or seatwork) and what aspects of the topic are planned to be emphasised (mental imagery, common misconceptions, techniques, technical language, applications, or root problems). In a brief conversation after a lesson, I ask the teacher to highlight any perceived differences between what was enacted and what was planned by annotating the pro-forma. If decisions are made in-flight to introduce new segments, then additional information is sought for these. I refer to the information collected from the teacher when observing lessons in order to help make sense in-the-moment of how the teacher is making content available within the particular classroom situation. These observations are recorded in my field notes. The field notes then support my subsequent categorisation of manifestation and interaction.

I observe all lessons within the series, taking field notes as mentioned above and recording the teacher's voice. Observation is particularly necessary in order to see how tasks set by the teacher play out in the lesson. For while the phrasing of written tasks (worksheets, problem sheets, exercises displayed on the board) give an indication of how content is manifested, this does not necessarily correspond to how content was observed to be manifested in the lesson due to the implementation of

model (figure 2.4/a) so as not to be misleading about the purpose of my research. Similarly, the information that teachers provided about their teaching was still useful when categorising manifestation and interaction.

the task[9]. I also collect any printed resources the teacher makes available for learners, including screen shots of the interactive whiteboard.

A transcript of the teacher's voice is made from the audio recording. This transcript excludes learners arriving to and departing from the lesson unless this arrival or departure is part of how content is made available (such as an 'exit question'). Similarly, any activities that could not be construed as mathematical are also excluded from the transcript, such as taking the register or dealing with an incident that interrupts mathematical activity for the whole class.

The approaches described above are informed by British Educational Research Association guidelines (BERA, 2011) and received ethical approval from the Central University Research Ethics Committee (CUREC) prior to being carried out. I gained the written consent of the teacher-participants and the Head Teachers of the schools involved in this research study. My research operates within CUREC protocol SSD/IDREC/2009/P15.1, which means that it is not necessary to seek the assent of learners and the consent of their parents or carers. An important code of behaviour that I followed during participant induction and subsequent communication with the teacher is to emphasise that there are no expectations about how content 'should' be made available and that I am genuinely interested in the ways in which teachers decided to make content available in the context of teaching the particular topic to the particular class at the particular time of year.

[9] The adaptations of the ice cream seller problem I offered in Section 2.3 exemplify this.

3.2 – Quantitising the text of how content is made available

> Content analysis is a research technique for making replicable and valid inferences from texts (or other meaningful matter) to the contexts of their use.
>
> (Krippendorff, 2013:24)

This section is concerned with how text – that is, experience formed as words (Sandelowski et al., 2009) – is made over into numerical data – that is, experience formed as numbers (ibid.). At first, I will offer a rationale for why quantitising will contribute to my analysis of how content is made available before addressing issues of replicability. Replicable quantitising in the context of my research calls for very detailed methods of analysis, and so exemplification of these methods will also be offered.

3.2.1 – Tri-polar spaces

In Section 3.1.1, I discussed how I had previously used the tri-polar attention space in order to chart shifts in attention over time. A more general *tri-polar space* may be used to chart any action with an underlying triadic structure, where the relative stressing of the three elements of a triad {X, Y, Z} can be represented by a point within the interior of a triangle (see figure 3.2/a). The vertices of the triangle can be thought of as attracting poles representing the three elements: the more the element is stressed, the greater attraction there is to the pole representing that element, so the closer the point is to that pole. The centroid of the triangle represents a perfect balance of the three elements.

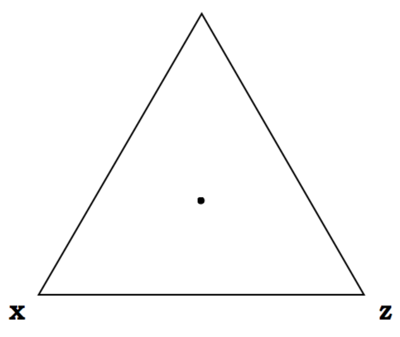

Figure 3.2/a
Tri-polar space for the triad {X, Y, Z}, within which each point represents a relative stressing of the elements X, Y and Z.

Figure 3.2/b illustrates how meaning can be attached to regions of the space through the language of relative stressing of the three elements.

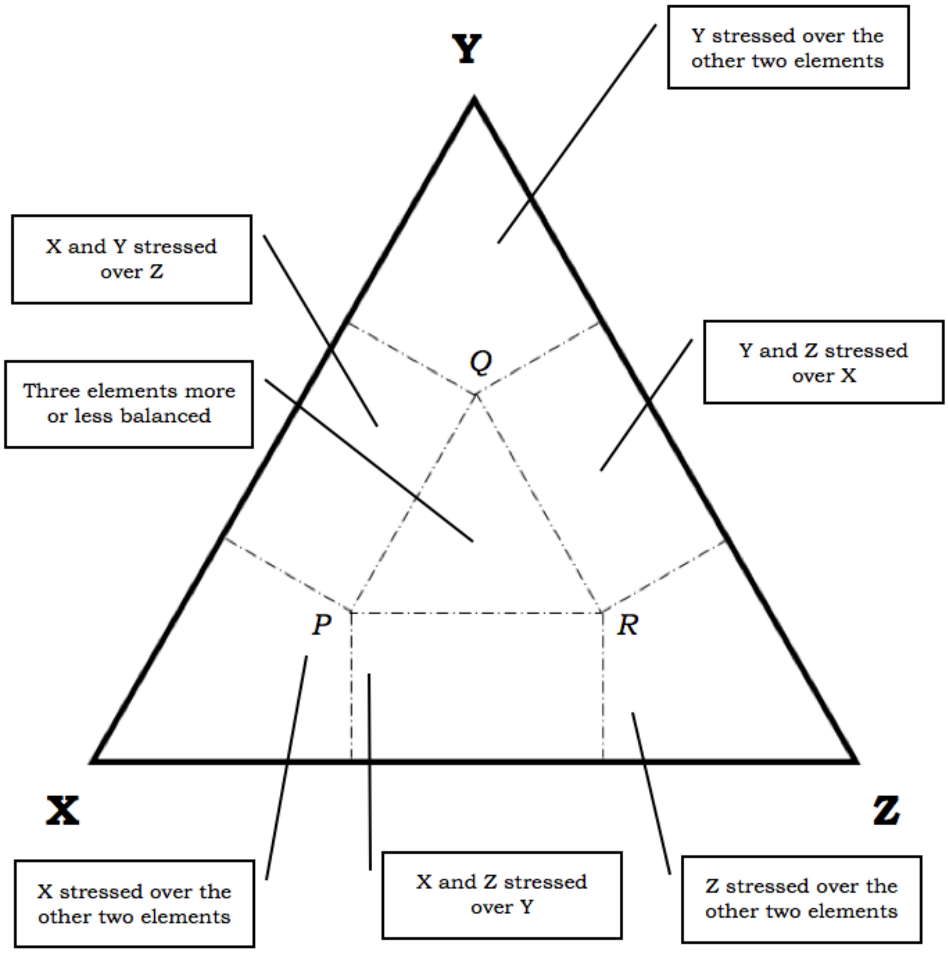

Figure 3.2/b Regions of a generalised tri-polar space based on a triad {X, Y, Z}.

The precise positioning of the dotted lines in figure 3.2/b built on my previous work with tri-polar spaces (Andrews, N., 2011). I chose to divide each of the three edges *XY*, *XZ* and *YZ* into thirds and construct at each of the six points of trisection a perpendicular to its respective edge. Adjacent pairs of perpendiculars intersect to produce triangle *PQR* at the centre of the space. The geometrical properties of the construction mean that the perpendicular height of triangle *PQR* is one-third the height of triangle *XYZ* and the perpendicular distance between lines *PR* and *XZ* is two-ninths of the height of triangle *XYZ*. The significance of these properties to my analysis will be revealed in due course in table 3.2/a and figure 3.2/e.

Barycentric co-ordinates

Barycentric co-ordinates provide a way of identifying a point within the interior of a tri-polar space. The point with co-ordinates (x, y, z) corresponds to the relative stressing of the three elements of the triad {X, Y, Z} in the ratio $x : y : z$. Thus for any chosen constant c, the vertices of the tri-polar space have co-ordinates (c, 0, 0), (0, c, 0) and (0, 0, c), the centroid of the space has co-ordinates ($^c/_3$, $^c/_3$, $^c/_3$) and for all points (x, y, z), $x + y + z = c$. Throughout this study, I have chosen to work with *normalised* Barycentric co-ordinates, in other words the family of Barycentric co-ordinates where $c = 1$.

In figure 3.2/c, I mark the three vertices and the points P, Q and R that featured in figure 3.2/b along with the three midpoints of pairs of vertices, and the centroid O of the general tri-polar space. These are special points in terms of the relative stressing of the three elements {X, Y, Z} as I highlight in table 3.2/a.

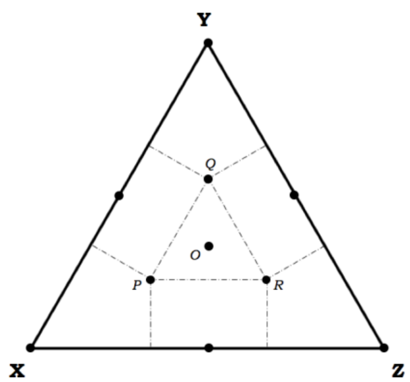

Figure 3.2/c
The vertices, midpoints of pairs of vertices and the centroid of the general tri-polar space. These are special points in terms of the relative stressing of the three elements {X, Y, Z}.

Table 3.2/a Linking the marked points in figure 3.2/d with the way in which they characterise the relative stressing of the three elements of the triad {X, Y, Z} and their Barycentric co-ordinates.

Point	*Stressing of the three elements of the triad {X, Y, Z}*	*Barycentric co-ordinate*
Vertex *X*	Element X stressed, Y and Z not stressed	(1, 0, 0)
Vertex *Y*	Element Y stressed, X and Z not stressed	(0, 1, 0)
Vertex *Z*	Element Z stressed, X and Y not stressed	(0, 0, 1)
Midpoint of vertices *X* and *Y*	Elements X and Y stressed equally, Z not stressed	($^1/_2$, $^1/_2$, 0)
Midpoint of vertices *X* and *Z*	Elements X and Z stressed equally, Y not stressed	($^1/_2$, 0, $^1/_2$)
Midpoint of vertices *Y* and *Z*	Elements Y and Z stressed equally, X not stressed	(0, $^1/_2$, $^1/_2$)
Centroid *O*	Elements X, Y and Z stressed equally	($^1/_3$, $^1/_3$, $^1/_3$)
Point *P*	Elements Y and Z stressed equally, more emphasis placed on X[10]	($^5/_9$, $^2/_9$, $^2/_9$)
Point *Q*	Elements X and Z stressed equally, more emphasis placed on Y[11]	($^2/_9$, $^5/_9$, $^2/_9$)
Point *R*	Elements X and Y stressed equally, more emphasis placed on Z[12]	($^2/_9$, $^2/_9$, $^5/_9$)

Table 3.2/a illustrates how the statement 'X stressed, Y and Z not stressed' can be represented by the point (1, 0, 0) and how 'X and Y stressed equally, Z not stressed' can be represented by the point (½, ½, 0). This observation will have important implications for the methods of quantitising I develop in Section 3.2.3 and 3.2.4.

[10] Point *P* is one of an infinite number of points on the line segment *OX* for which this statement holds.
[11] Point *Q* is one of an infinite number of points on the line segment *OY* for which this statement holds.
[12] Point *R* is one of an infinite number of points on the line segment *OZ* for which this statement holds.

The seven regions of the generalised tri-polar space based on a triad {X, Y, Z} illustrated in figure 3.2/b can also be represented as inequalities expressed in terms of a general point (x, y, z) within the space. Figure 3.2/d links these iconic and symbolic representations of the regions. By determining in which of the seven regions a point (x, y, z) representing the relative stressing of the elements of the triad {X, Y, Z} lies, data expressed numerically as co-ordinates can be *qualitised* as categorical data. This property is exploited in my analytical approach and will be discussed in more detail in Sections 3.2.3 and 3.2.4.

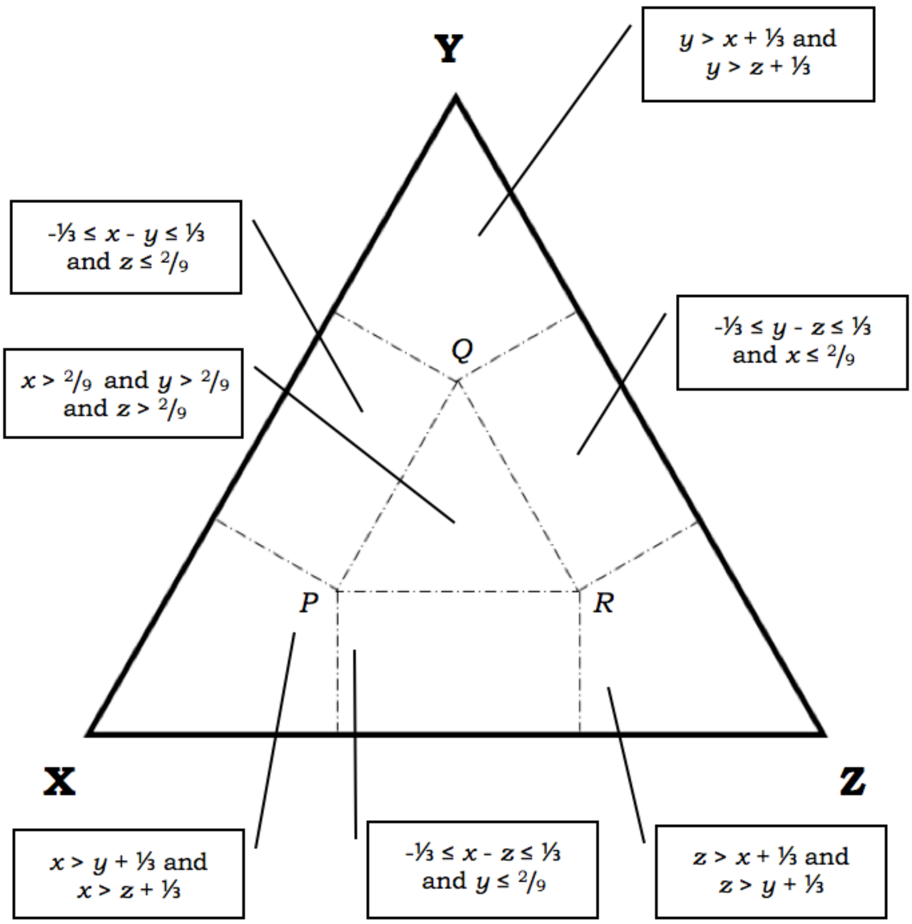

Figure 3.2/d Regions of a generalised tri-polar space based on a triad {X, Y, Z} expressed symbolically as inequalities for a point (x, y, z).

For example for the point (0.20, 0.35, 0.45), $x = 0.20$ and $y - z = -0.10$. This point therefore satisfies the inequalities $x \le {}^2/_9$ and $-{}^1/_3 \le y - z \le {}^1/_3$ and lies in the region where Y and Z are stressed over X.

Barycentric co-ordinates provide a means of representing numerically the relative stressing of the three elements of a phenomenon whose underlying structure is triadic. This numerical representation of the phenomenon can be decomposed into its three elements, affording statistical investigation of each element independently (with the limitation that the behaviour of the third element is a function of the behaviour of the other two, since the three elements are not independent). **It should be noted that the numerical values associated with one element are ordinal data, and as such only non-parametric statistical investigations are appropriate** (Field, 2009:133). Furthermore, the validity of any inferences drawn from such investigations needs to be considered.

The use of Barycentric co-ordinates also allows for calculation of a measure of central tendency (only the mean is permissible for this, given that it is the only measure of central tendency whose value is guaranteed to uphold the condition that the sum of the co-ordinates must be *c*). An application of this is that a central point of a set of relative stressings of the three elements of a triad may be identified. So for example, the relative stressing of the three elements over a longer time period may be represented quantitatively as the average of shorter time intervals that make up the longer period.

<u>*Opportunities and limitations of using tri-polar spaces*</u>

The tri-polar space has three particular features that make it a powerful research tool for charting actions with an underlying triadic structure:

1. The space contained within the triangle is infinite and so, in principle, infinite variation in the relative stressing of the three elements may be charted.

2. The space can chart the relative stressing of the three elements over any time interval or other stratification of lessons, for example an utterance, an activity or a lesson.

3. By representing points within the space numerically using co-ordinates, the relative stressing of the three elements over a longer time interval may be presented quantitatively as the average point of shorter intervals.

The first feature is also a potential limitation of the space in research terms. With an infinite number of ways in which a phenomenon might be represented as a point within the space, the only hope of genuine reliability is a well-defined one-to-one correspondence between experience and the points within the space. It also raises questions about validity and whether a point (x, y, z) is indeed a more valid representation of experience than another point in its proximity $(x+\delta, y-\delta, z)$. The third feature may also be problematic in terms of validity. Firstly, any method that looks to exploit the third feature needs to be built upon reliable and valid representation of the relative stressing of the three elements of a phenomenon with an underlying triadic structure over a short time period. Secondly, let us assume that the quantitative representations of the relative stressing of the three elements over shorter time intervals are a valid reflection of experience. Then is the quantitative representation of the relative stressing of the three elements over the longer time interval (the union of the shorter time intervals) a valid representation of experience over that longer time? These are important considerations for my research design and more details of how they will be addressed are given in Section 3.4 when discussing the writing of case studies

3.2.2 – Method of text analysis: reliability and units

When discussing content analysis in Section 3.1.3, I emphasised two aspects: context and inference. In this section, I consider two other further aspects of the research design: reliability and units. These two aspects are connected, as will be seen, because unit size influences reliability. First, I will focus on reliability. Krippendorff (2013) defines reliability as a property of analytical methods:

> A research procedure is reliable when it responds to the same phenomena in the same way regardless of the circumstances of its implementation.
>
> (Krippendorff, 2013:267)

Reliability is important within context analysis for if reliability is poor then the basis from which inferences are abducted is insecure and this risks these inferences being invalid. As Krippendorff puts it, ’reliability is a necessary, but not a sufficient, condition for validity’ (Krippendorff, 2013:270). Weber (1990) identifies three types of reliability (see table 3.2/b).

Table 3.2/b Three types of reliability and what each one measures. Taken from Weber (1990:17).

Type of reliability	*Measure of reliability: the extent to which classification of text…*
Stability	… is invariant over time when coded more than once by the same coder.
Reproducibility	… produces the same results when the text is coded by more than one coder.
Accuracy	… corresponds to a standard or norm.

In order to address issues of reliability, I developed a systematic approach to coding, including having a strong sense of each code and writing coding rules for situations where the allocation of codes is complex (see Section 3.2.3 and 3.2.4). I double coded whole lessons that featured such complex situations and resolved any discrepancies. I trialled various unit sizes in order to determine an appropriate unit that might support reliable coding; however, as I discuss below, reliability is not the only consideration when determining a unit size. I was therefore able to take steps in order to ensure that my coding was stable.

I have no evidence for my coding method's reproducibility since, given the constraints on the scope of this study, I was unable to recruit another coder. As I explain in Section 3.2.3, it is important that any coder also experiences how content is made available in the classroom. A second coder would therefore need to be trained, have the commitment to attend all lessons across all cases and be prepared to discuss coding discrepancies.

Weber (1990) comments that 'accuracy is a seldom-used form of reliability'. Given that I am developing a new analytical approach for a little researched phenomenon, accuracy as a measure of reliability is not pertinent to my study.

My categorisation rules determine the presence or absence of a particular code. 'Presence' and 'absence' however need to be interpreted within the context of my analysis. Presence is associated with the text matching my criteria, absence with it not. As Sandelowski and colleagues comment,

> Presence is… better understood as a function of what a researcher is prepared and inclined to see, whereas absence is better conceived as a function of what a researcher is prepared and inclined to see but does not.
>
> (Sandelowski, Voils and Knafl, 2009:217)

So presence based on rules of categorisation of a text may not correspond to 'present in experience'. The more narrow my criteria, the more likely it is that something 'present in experience' is not coded as present since it does not exactly match the criteria. Syntax is foregrounded over semantics. For example, if I am coding for the presence of teacher questions and also teacher statements, narrow criteria might lead to the teacher utterance, "Two add three is five, isn't it?" being coded (syntactically) as a question, whereas it is essentially a statement.

There is a balance to be made, because foregrounding stability – having narrow rules for categorisation and small text units – risks compromising validity. Krippendorff observes

> The best content analyses define their context units as large as is meaningful (adding to their validity) and as small as is feasible (adding to their reliability).
>
> (Krippendorff, 2013: 102)

The initial coding rules for interaction and manifestation I developed in order to categorise the text of my first case study evolved during the process and continued to evolve as I categorised further cases. Having coded all four cases using the evolved coding rules, I recoded a sample of one lesson from each case. This highlighted poor stability; my second coding of the sample texts differed from their original coding. I identified particular coding difficulties (which I discuss in Sections 3.2.3 and 3.2.4) and addressed these by adding to my coding rulebook. Further sampling indicated that these measures improved stability.

Text units or time units?

Text units for content analyses are often sentences or paragraphs rather than what is said within a particular period of time. Krippendorff asserts that 'seconds of talk are unnatural units, much as are lines of printed text' (Krippendorff, 2013:109) but acknowledges that time units are 'common in studies of interpersonal behaviour'

(Krippendorff, 2013:104). Time units were more appropriate to the context of my study because content is made available over time rather than through units of text. In this sub-section, I will define the various periods of time over which content is made available. In increasing size these are: interval, sub-segment, segment, lesson and series. I will offer a rationale for why I chose the interval as the time unit of analysis.

A mathematics lesson is a well-defined unit of time (often determined for organisational purposes within educational establishments) within which mathematical activities occur. The union of lessons on a common topic forms a lesson series. A lesson is formed by the union of segments, where a segment is a sub-division of a lesson within which a well-defined mathematical activity occurs, for example activity prompted by a classroom task (Watson and Sullivan, 2008:109). In the current study, I do not dictate the size of a segment, nor how many segments form a lesson; these are determined by what happens in the classroom. Generally the teacher determines segments in planning, but impromptu segments arise in lessons as a consequence of in-flight decisions. However, segments are always contained within a single lesson: if an activity started in one lesson is to be completed in the next, this is taken to be two segments, as how content is made available at the start of one lesson could be subtly different to how it is made available at the end of the previous lesson.

I recognise that not all learners may be ‘on task’ for a whole segment, but by focusing on what is available I overlook individual learners’ off-task behaviour. Further, there are some parts of lessons in which the teacher does not make content available to learners: when carrying out administrative tasks or addressing learners’ misbehaviour at a whole class level. These are excluded from segments. So strictly speaking, what I take to be a lesson series in this study is the union of segments rather than the union of lessons and in this sense a lesson series is ‘full’ of mathematical activity.

Although experienced teachers may not consciously plan to a smaller grain-size than a lesson segment, typically mathematical activities are initiated, sustained and concluded and so lesson segments can be sub-divided into sub-segments on this basis. I discern sub-segments through noticing discrete shifts in how content is made available to learners, and so within sub-segments the diversity of how content is made available is less than over larger time periods. However, even at the sub-segment level there is a considerable range in how content is made available. So a finer grain size still is called for in order to form the basis of a reliable method for charting how content is made available. A half-minute interval of a lesson was chosen as the standard minimum unit of time for each case study. This unit size is large enough so as to be able to represent the complex nature of the how content is made available but small enough for reliable coding of the presence of particular manifestations and modes of teacher interaction.

In summary, the units of analysis that define each case study are a teacher making content available to a group (usually a class) of learners within a particular classroom environment over one of the following periods of time: interval, sub-segment, segment, lesson and series. Yin (2003) describes these as embedded units of analysis. My method of text analysis focuses on the smallest unit – the half-minute interval. In the following sections (Section 3.2.3 and 3.2.4) I set out methods for categorising and quantitising the text of how content is made available at this grain size. I will also explain how analysis at this smallest grain size leads to inferences being drawn regarding how content is made available at the larger grain sizes.

3.2.3 – Method of text analysis: manifestation

When inferring how content was manifested during a lesson, I consider the mathematics-related utterances made by the teacher from-the-front or between-the-desks, which together I refer to as *teacher-talk*. I also consider whether learners were working on a particular task, which might have been made available for example through a worksheet or spoken instructions from the teacher. I refer to such tasks as *given-tasks*. Alignment between teacher-talk and given-tasks is significant when drawing inferences about teaching-between-the-desks. If the way in which mathematics was manifested through a given-task was different to how it was manifested through teacher-talk while teaching-between-the-desks (i.e., the given-task and teacher-talk did not align), then this indicates something important about the role of the teacher. For example, through teacher-talk a more visual manifestation of the topic might be offered in comparison to the technical manifestation of the given-task. Inferences might then be drawn from the frequency of such 'switches' in manifestation.

<u>*Coding how content was manifested*</u>

My goal in coding how content was manifested is to achieve for each half-minute interval both a categorical code (that I will refer to as *CAT code*) and a numerical code (that I will refer to as *NUM code*) for given-task, teacher-talk and overall manifestation. CAT codes are based on the categorisation of manifestation (discussed in Section 2.3.5), while NUM codes are Barycentric co-ordinates (see Section 3.2.1). Methods for determining CAT and NUM codes will follow, but first I will briefly offer a rationale for these different representations of half-minute intervals. As mentioned in the paragraph above, coding given-task and teacher-talk separately affords consideration of alignment between the two and CAT codes are particularly appropriate for this (more details will be provided in Section 3.3.3). NUM codes allow for the detection of smaller variations in how content is manifested overall (through the combination of given-task and teacher-task NUM codes) than would be possible with CAT codes. NUM codes may also be aggregated over time in order to represent longer time periods (see Section 3.2.1) and to discern differences between lessons and lesson series statistically (methods of statistical analysis will be addressed in Section 3.3.2).

The relationship between NUM and CAT codes can be visualised through the *tri-polar space for manifestation* (figure 3.2/e). A NUM code is the Barycentric co-ordinate of a point within the space, representing a relative stressing of the three elements of the manifestation triad {visual, functional, technical}. A CAT code is associated with one of the regions of the space. For convenience, one-, two- and three-letter codes are used in order to label the regions: V, F, T, VF. VT. FT and VFT.

An important step in the analytical approach is *qualitising* a NUM code in order to determine a CAT code. Using the tri-polar space in order to qualitise Barycentric co-ordinates was discussed previously in general terms with reference to figure 3.2/d. Figure 3.2/e sets out how each of the seven regions of the tri-polar space for manifestation can be represented symbolically through inequalities and how these in turn relate to CAT codes for manifestation.

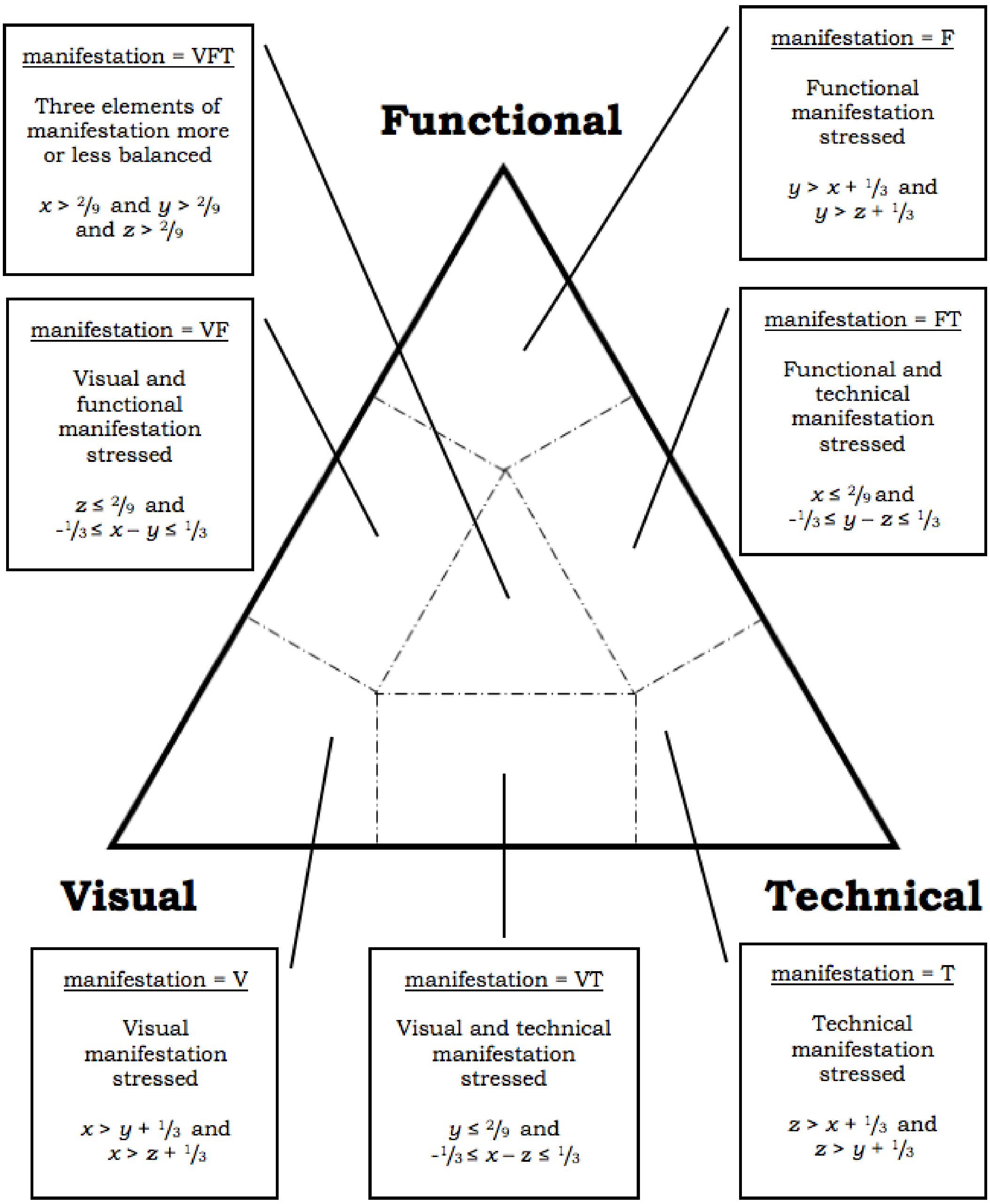

Figure 3.2/e Regions of the tri-polar space for manifestation. The three attracting poles are the three elements of the manifestation triad: Visual (V), Functional (F) and Technical (T).

A systematic approach to coding how content was manifested was developed during the project and all four cases were coded using this final system. This approach combines two coding methods: a method for coding given-task or teacher-talk and a method for coding how content is manifested overall across a half-minute interval. I will explain these two methods separately.

Method for coding given-task or teacher-talk

The procedure for coding given-task or teacher-talk is presented as a flowchart in figure 3.2/f[13].

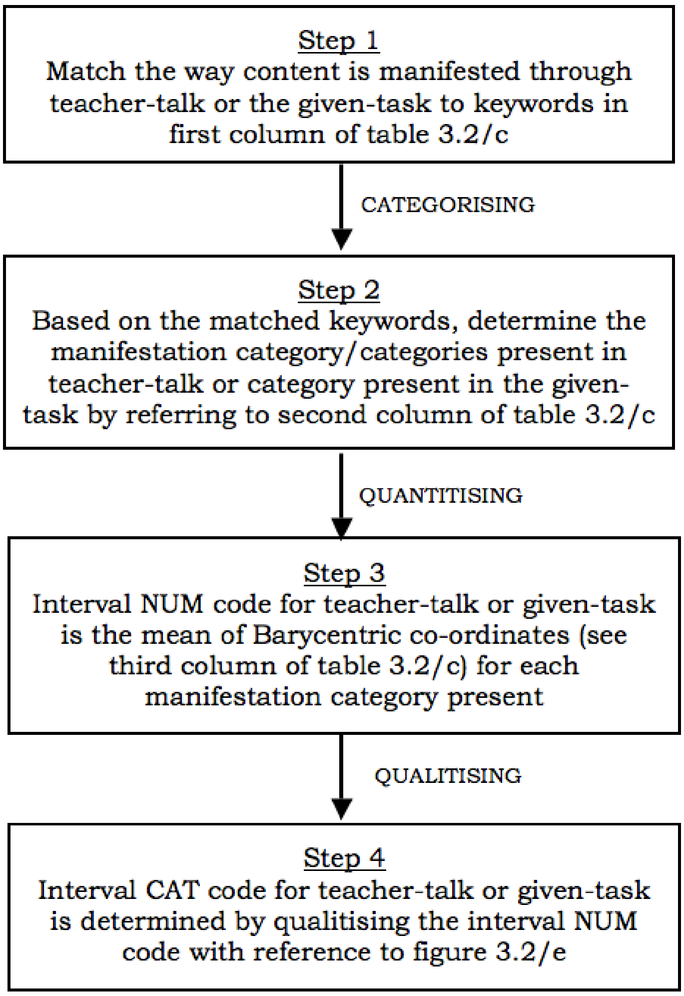

Figure 3.2/f Flowchart representing the method for coding given-task or teacher-talk across a half-minute interval.

[13] Note that there are caveats to this procedure when both given-tasks and teacher-talk feature in the interval. This is addressed when discussing the method for coding how content is manifested overall across the half-minute interval.

The transcript of the teacher's voice, field notes, screen-shots of information displayed on boards, hand-outs, worksheets and text books are all used in order to infer how the teacher made content available to learners. Determining the categorisation of a given-task or a stretch of teacher-talk for which only one category of manifestation is present is done in two mutually supportive ways. Firstly, I consider the given-task or teacher-talk in terms of the relative stressing of the three elements of the manifestation triad {visual, functional, technical}. Secondly, I consider the given-task or teacher-talk in relation to the keywords in table 3.2/c below. This list of keywords grew over time as more intervals were coded; the keywords are examples of, rather than conditions for, the presence of particular manifestations. The combination of having a sense of the relative stressing of the three elements and the act of articulating this sense as a keyword contributes to a more meaningful way of determining how the teacher manifests content than one of these approaches alone. My justification for this is that while the identification of keywords contributes to being able to categorise in a reliable way, the sense of the relative stressing of the three elements contributes to ensuring that categorisation reflects what I experienced in the classroom. With this in mind, it is not meaningful to seek independent confirmation of my categorisation, because 'being there' – experiencing how the teacher made content available – contributes to categorising given-tasks and teacher-talk.

Once the manifestation categories present in either the given-task or teacher-talk across an interval are determined, an interval NUM code is calculated. This NUM code is a Barycentric co-ordinate (x, y, z). Referring back to figure 3.2/e, the NUM code can be qualitised into a CAT code by determining in which region of the tri-polar space for manifestation the point (x, y, z) lies. Using this method, the available interval CAT codes for given-task or teacher-talk are V, F, T, VF, VT, FT and VFT.

Table 3.2/c Keywords, manifestation categorises and Barycentric co-ordinates used in association with the flowchart in figure 3.2/f in order to code given-tasks and teacher-talk across a half-minute interval.

Keywords associated with each manifestation category	*Categorisation of how content is manifested*	*Barycentric co-ordinate*
Application to a material world context, application within mathematics, proof, affording confident manipulation	Functional	(0, 1, 0)
Root problem, affording image making	Visual-functional	(½, ½, 0)
Imagery, visualisation, affording image having	Visual	(1, 0, 0)
Physical representation, verbal representation, example generation, visual method, affording property noticing	Visual-technical	(½, 0, ½)
Technique, technical language, affording procedural fluency	Technical	(0, 0, 1)
Utilisation, affording adaptive using	Functional-technical	(0, ½, ½)

Note: Additional categorisation guidance is provided in tables A1/a and A1/b in Appendix 1.

It should be noted that the CAT code VFT is only available for given-tasks or teacher-talk when two particular manifestation categories are present (e.g. visual-functional and technical). The additional CAT code Ø is used to code the given-task during classwork (in other words, when there is no given-task), and to code teacher-talk during seatwork when there is either no teacher-talk or teacher-talk aligns with the given-task[14]. Further, it should be noted that manifestation categories are either present or absent, so even if teacher-talk is matched with many keywords associated with the same manifestation category, this category is taken as present just as if there had only been one keyword.

[14] This allows for the clear identification of instances of the teacher *switching* through teaching-between-the-desks to manifestations other than those associated with the given-task. Instances of switching are coded as teacher-talk = M, where M is one of the manifestation CAT codes other than the given-task CAT code; instances of not switching are coded teacher-talk = Ø.

<u>*Coding how content is manifested overall across a half-minute interval*</u>

The procedure for coding how content is manifested overall across a half-minute interval is presented as a flowchart in figure 3.2/g. This procedure is exemplified below.

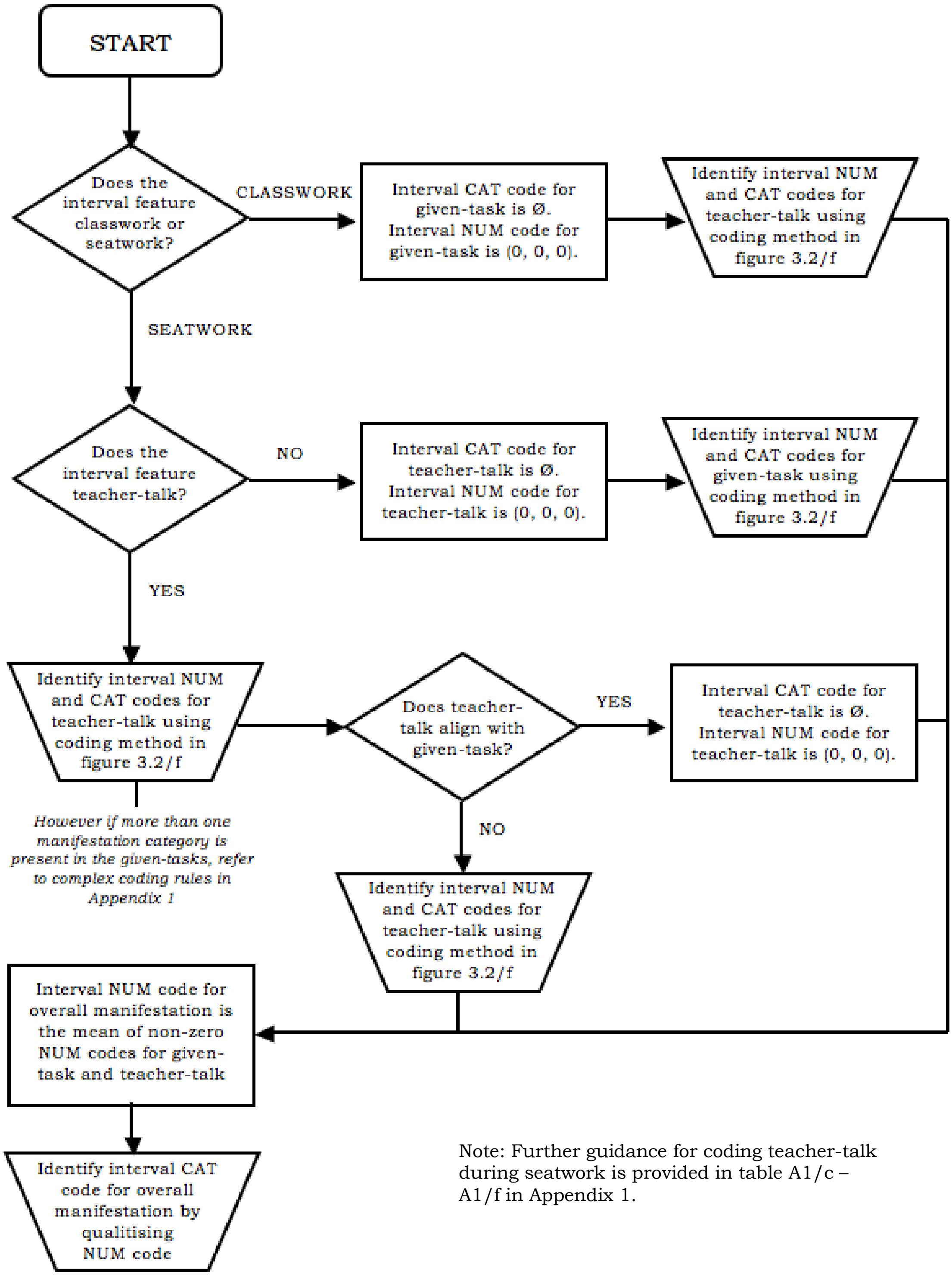

Figure 3.2/g Flowchart representing the systematic approach to determining interval NUM and CAT codes for overall manifestation.

<u>*Example of coding for manifestation*</u>

Imagine a half-minute interval of a lesson featuring the teacher explaining technical language to a group of learners and then making reference to a visual method with another learner while the rest of the class works on using techniques in order to solve material world problems. Teacher-talk is associated with the keywords 'technical language' and 'visual method' and the given-task is associated with the keyword 'utilisation' (see table 3.2/c). The keywords 'technical language' and 'visual method' indicate the presence of technical and visual-technical manifestations within the teacher-talk for the interval. The keyword 'utilisation' indicates the presence of functional-technical manifestation within the given-task.

Following the flowchart in figure 3.2/g, the interval is a classwork situation in which there is teacher-talk that does not align with the given-task. The given-task features a single manifestation category (functional-technical) and so the interval NUM code for the given-task from table 3.2/c is (0, ½, ½). Referring to figure 3.2/e the interval CAT code for given-task is FT. Teacher-talk features two manifestation categories (technical and visual-technical). Referring to table 3.2/c, the Barycentric co-ordinate for technical manifestation is (0, 0, 1) and for visual-technical manifestation is (½, 0, ½). The interval NUM code for teacher-talk is therefore the mean of these two points, namely (0+½, 0+0, 1+½) ÷ 2 = (¼, 0, ¾). Qualitising by referring to figure 3.2/e, the CAT code for teacher-talk is T[15]. The interval NUM code for overall manifestation is the mean of the given-task and teacher-talk NUM codes, namely (0+¼, ½+0, ½+¾) ÷ 2 = (⅛, ¼, ⅝). Qualitising by referring again to figure 3.2/e, the interval CAT code for overall manifestation is T[16].

[15] Since both ¼+⅓ and 0+⅓ are less than ¾.
[16] Since both ⅛+⅓ and ¼+⅓ are less than ⅝.

<u>*Coding difficulties; manifestation*</u>

In analysing the data it became clear that a reliable way of coding is needed when there are concurrent given-tasks, for example when the teacher offers learners a choice of given-tasks with different manifestations. Firstly it is important that both given-tasks are recognised. In the spirit of the coding rules that had already been established and the principle of focusing on what is available to learners rather than what is taken up, both given-tasks contribute equally to determining the presence of manifestations made available by the teacher. This is regardless of the proportion of learners that were observed to be working on each task. Secondly, it is necessary to consider whether the way of coding teacher-talk in order to represent switches in manifestation associated with teaching-between-the-desks during a single given-task was appropriate for concurrent given-tasks. Thirdly, a way is needed of recognising the additional emphasis resulting from teacher-talk that focuses on one of the given-tasks, which avoids indicating a switch in manifestation. Specific coding rules for these situations are provided in Appendix 1.

As stated in Section 3.2.2, while these rules emerged during the initial coding of the transcripts for each of the four cases, lessons featuring complex coding decisions were double coded and discrepancies resolved in order to ensure the reliable representation of classroom experience of the teacher making content available over a half-minute interval as categorical and numerical data.

In order to support making sense of these abstract coding rules, I will offer exemplification of the coding of a lesson segment in Section 3.2.4, once similar rules for interaction have been stated.

Points of information relating to coding for manifestation

The interval NUM code for overall manifestation is a Barycentric co-ordinate (x, y, z) with the property that $x + y + z = 1$. I refer to this as the *interval centre for manifestation* and the values x, y and z as the *ζ values* for the visual, functional and technical elements of manifestation respectively.

When recording interval CAT codes, it is often convenient to express how content is manifested across an interval by writing 'manifestation = M', where M stands for one of the eight CAT codes (V, F, T, VF, VT, FT, VFT and Ø).

Categorisation of time units larger than a half-minute interval: manifestation

In Section 3.2.1, I described how the aggregation of interval centres could be used in order to determine a central point for a longer time interval. This process is used in order to determine subsegment, segment, lesson and lesson series centres for manifestation. Each time unit centre for manifestation characterises the relative stressing of visual, technical and functional manifestations of the topic over that unit of time (ζ values). An alternative characterisation of a time unit is provided by considering the proportion of half-minute intervals coded as V, F, T, VF, VT, FT or VFT respectively across that unit of time. I refer to these proportions, generally expressed as percentages, as *κ values*. These two characterisations based on ζ and κ values are not independent of each other as both are derived from coding, but they afford the abducting of slightly different inferences. For example, ζ values may indicate a greater stressing of functional and technical elements of manifestation over a particular lesson segment when compared to another segment, but κ values may indicate that this is associated either with emphasising functional and technical manifestations, functional-technical manifestation or an emphasis of all three. There is therefore not a unique correspondence between the stressing of elements of the manifestation triad and emphasised manifestations.

<u>*3.2.4 – Method of text analysis: interaction*</u>

In Chapter 2, I identified seven modes of teacher interaction: telling-from-the-front, explaining-from-the-front, discussing-from-the-front, telling-between-the-desks, explaining-between-the-desks, discussing-between-the-desks and being-watchful. This section is concerned with how the lesson transcript is used in order to code interaction reliably.

My goal in coding interaction is to achieve for each half-minute interval both a categorical code (CAT code) and a numerical code (NUM code). Categorical coding of intervals for interaction allows, for example, analysis of how content is manifested during different types of exchanges. Numerical coding of intervals for interaction affords, for example, aggregation over time in order to represent longer time periods and to discern statistically differences in the type of exchanges offered between lessons and between lesson series (see Section 3.3.2).

As with manifestation, the relationship between NUM and CAT codes can be visualised through the tri-polar space for interaction (figure 3.2/h). A NUM code is the Barycentric co-ordinate of a point within the space, representing a relative stressing of the three elements of the interaction triad {teacher-content interaction, teacher-learner interaction, learner-content interaction}. A CAT code is associated with one of the regions of the space. For convenience two-letter codes are used in order to label the regions: TC, TT, TL, LL, LC, CC and OO. Figure 3.2/h also sets out how each of the seven regions can be represented symbolically through inequalities. Reference to these inequalities allows for NUM codes to be qualitised as CAT codes as was the case with manifestation.

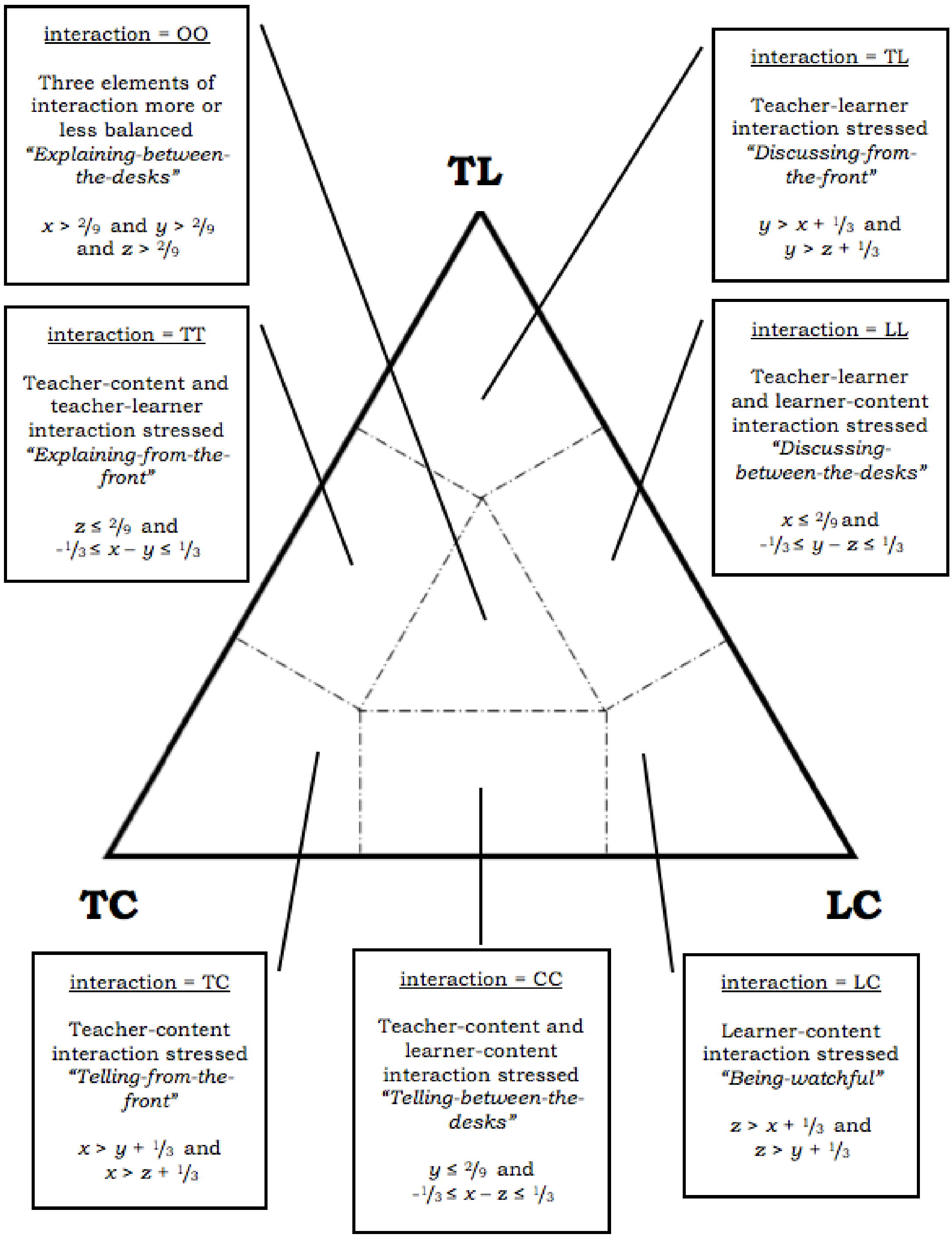

Figure 3.2/h Regions of the tri-polar space for interaction. The three poles are the three elements of the interaction triad: teacher-content (TC), teacher-learner (TL) and learner-content (LC) interaction.

Coding interaction across a half-minute interval

The procedure for interaction across a half-minute interval is presented as a flowchart in figure 3.2/i. In the four cases of this study, the overwhelming majority of intervals featured only a single mode of teacher interaction. As combinations of modes of manifestation are so rare[17], coding rules for these situations are presented in table A1/g in Appendix 1 rather than the current chapter.

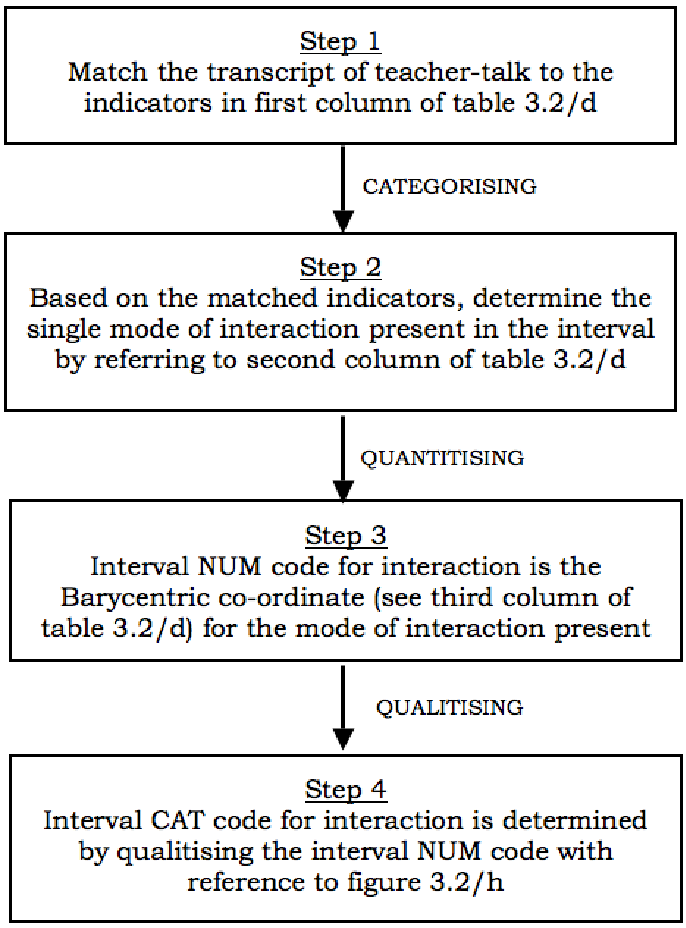

Figure 3.2/i Flowchart representing the systematic approach to determining interval NUM and CAT codes for interaction.

[17] Combinations of modes of teacher interaction only arise when there is a transition between seatwork and classwork during a half-minute interval.

Table 3.2/d Indicators, modes of teacher interaction and Barycentric co-ordinates used in association with the flowchart in figure 3.2/i in order to code interaction across a half-minute interval.

Indicators associated with each mode of teacher interaction	*Mode of teacher interaction*	*Barycentric co-ordinate*
Classwork – teacher addressing whole class (indicated in field notes). Declarative statements, rhetorical questions, no learner-talk (from transcript).	Telling-from-the-front	(1, 0, 0)
Classwork – teacher addressing whole class (indicated in field notes). Questions, likely to be learner-talk (from transcript).	Discussing-from-the-front	(0, 1, 0)
Classwork – teacher addressing whole class (indicated in field notes). Mix of telling and discussing: questions AND declarative statements, likely to be learner talk (from transcript).	Explaining-from-the-front	(½, ½, 0)
Seatwork – teacher addressing individual or small group (indicated in field notes). Declarative statements, rhetorical questions, no learner-talk, possibly some periods of no teacher-talk (from transcript).	Telling-between-the-desks	(½, 0, ½)
Seatwork – teacher addressing individual or small group (indicated in field notes). Questions, likely to be learner talk, possibly some periods of no teacher-talk (from transcript).	Discussing-between-the-desks	(0, ½, ½)
Seatwork – teacher addressing individual or small group (indicated in field notes). Mix of telling and discussing: questions AND declarative statements, likely to be learner talk, possibly periods of no teacher-talk (from transcript).	Explaining-between-the-desks	(⅓, ⅓, ⅓)
Teacher instructing the whole class during seatwork (indicated in field notes). Declarative statements, instructions, but also possibly periods of no teacher-talk (from transcript).	Dictating	(¼, 0, ¾)
Seatwork (indicated in field notes). Absence of teacher-talk. Possibility of learner-talk (from transcript).	Being-watchful	(0, 0, 1)

Each mathematics-related utterance by the teacher recorded during a half-minute interval is considered in order to determine the nature of interaction; if there are no such utterances then this is indicative of the teacher being-watchful (see first column of table 3.2/d). Firstly the text of the transcript is analysed in order to determine the syntactic form of what is said by the teacher: questions, rhetorical questions, declarative statements or no mathematics-related teacher-talk. Then, with reference to field notes and the recording of the teacher's voice, I determine the context of the teacher utterance: whether it was made during classwork or seatwork. Other teacher utterances considered non-mathematical, such as words of encouragement, are noted but these do not contribute to the coding of the interval.

Once a mode of teacher interaction is identified, an interval NUM code is determined by referring to the third column of table 3.2/d. This NUM code is a Barycentric co-ordinate (x, y, z). Referring back to figure 3.2/h, the NUM code can be quantitised into a CAT code by determining in which region of the tri-polar space for interaction the point (x, y, z) lies. Using this method, the available CAT codes for interaction are TC, TT, TL, LL, LC, CC and OO. The interval NUM code for interaction is also referred to as the interval centre for interaction and the values x, y and z as the *ζ values* for the teacher-content, teacher-learner and learner-content elements of interaction respectively.

Included in the modes of teacher interaction in table 3.2/d is *dictating*. This is not one of the original seven modes of teacher interaction set out in Section 2.2. However, it was a type of exchange that featured on a number of occasions in one of the four cases. It features the teacher giving instructions orally that learners are tasked with following which serves as a way of initiating a given-task. In order to code it consistently, I decided to recognise dictating as a mode of teacher interaction and associated it with the Barycentric co-ordinate (¼, 0, ¾). This represents a stressing of learner-content interaction (as learners are tasked with working on a problem) but with recognition of some emphasis on teacher-content interaction (associated with the teacher telling-from-the-front).

In the course of analysing interaction, a need arose for grouping intervals coded as telling or explaining (both between-the-desks and from-the-front). I chose to code such intervals as *instructing*. Instructing and discussing are therefore mutually exclusive modes of teacher interaction.

Categorisation of time units larger than a half-minute interval; interaction

As described in section 3.2.1, aggregation of interval centres can be used in order to determine subsegment, segment, lesson and lesson series centres for interaction. Each time unit centre for interaction provides a characterisation of the relative stressing of teacher-content, teacher-learner and learner-content interaction over that unit of time (ζ values). An alternative characterisation of a time unit is provided by considering the proportion of half-minute intervals (κ values) categorised as TC, TT, TL, CC, OO, LL and LC across that unit of time. These two characterisations are not independent of each other as both are derived from coding, but they afford the abducting of slightly different inferences. For example, ζ values may indicate a greater stressing of teacher-learner and learner-content interaction over a particular lesson segment when compared to another segment, but κ values may indicate that this is associated either with foregrounding discussing-from-the-front and being-watchful, discussing-between–the-desks or foregrounding all three. There is therefore not a unique correspondence between the stressing of elements of the interaction triad and foregrounded modes of teacher interaction.

3.2.5 – Exemplification of method of analysis of text over an interval

In this section, I provide a transcript of a lesson segment taken from Ashley's lesson series on geometry in order to illustrate the way I chart how content was made available. Each teacher utterance is numbered: those in underlined text were utterances directed to the whole class while those in normal text were directed to individuals or small groups of learners. As I did not attend to learner articulations, these are marked simply as 'student response' and these, along with non-verbal teacher actions, are also numbered and recorded in italicised text. The method for representing how content was made available includes the parsing of segments into half-minute intervals, and the horizontal lines between bodies of text in the transcript highlight these intervals.

This segment has been selected in order to exemplify the method of analysis due to the challenge it provides; many of the coding difficulties that I have referred to in this chapter (Sections 3.2.2, 3.2.3 and 3.2.4) feature. In this sense this segment is unrepresentative as it is denser in technical coding decisions than is typical and inferences the reader might draw about the general complexity of the analysis should not be based on this example alone.

The segment opens with Ashley moving on from the previous segment and setting a task. This following excerpt is the first two half-minute intervals of the segment.

1.	<u>We've got another different one to do.</u>
2.	<u>This time what I want you to do is to draw a line that's not horizontal or vertical and then somewhere that's not on the line I want you to put a dot. Go.</u>
3.	*No teacher talk for 5 seconds.*
4.	*No teacher talk for 20 seconds.*
5.	<u>OK, I'd like you to measure with your ruler the distance from the dot to the line.</u>

In lines 1 through to 5, Ashley's actions aligned with tapping in to 'what there is in the child's present that is usable with reference to [the subject matter]' (Dewey, 1902:23): drawing a line, marking a point, measuring a distance. This afforded learners seeing potential ambiguity in 'the distance' of the point from the line. The outer aspect of the task (Ashley's utterance) is distinct from the inner aspect (the shortest distance of a point from a line being the length of the perpendicular from the point to the line) and this affords shifting the focus of learners' attention from confident manipulation (of measuring) towards image having (seeing the shortest distance as the length of the perpendicular). The given-task is a root problem (manifestation = VF), and this covers both half-minute intervals of the excerpt. Any teacher-talk during these intervals focuses on the given-task (manifestation = Ø). Following the method set out in Section 3.2.3, for both intervals the NUM code for manifestation is (½, ½, 0) and the CAT code is VF.

In line 1, Ashley was sign-posting that the class was about to start a new task. In line 2, Ashley dictated the task. Thus dictating was the only category of mathematical teacher-talk that featured in first half-minute interval and following the method set out in Section 3.2.4, the interval NUM code for interaction is (¼, 0, ¾) and the CAT code is LC.

By line 4, learners are taken to be engaging with the given-task, with Ashley being-watchful. Then at line 5, Ashley adds to the task through dictation. Following the method set out in Section 3.2.4, an interval featuring Ashley dictating and a period of no teacher-talk is categorised as dictating, so the interval NUM code for interaction for the second half-minute is also (¼, 0.00, ¾) and the CAT code is LC.

The segment continues:

6.	*No teacher talk for 15 seconds.*
7.	I'd like you to measure the distance from the dot to the line.
8.	What did you get?
9.	*Student response.*
10.	What did you get?
11.	*Student response.*
12.	What did you get?
13.	*Student response.*

This interval still relates to the given-task that I took to be a *root problem* (manifestation = VF) and all teacher-talk focuses on the given-task (manifestation = Ø), so the interval NUM code for manifestation is again (½, ½, 0) and the CAT code is VF. At line 7, Ashley repeats the question from before, but this time as a prelude to a sequence of *prompting* questions asked from-the-front. So this interval includes both a period of no teacher-talk and Ashley discussing-from-the-front. Its interval NUM code for interaction is therefore (0, ½, ½) and the CAT code is LL. This concludes the first subsegment, marked by the transition from seatwork to teaching-from-the-front. The subsegment centre for manifestation is (½, ½, 0), qualitised as manifestation = VF, and for interaction is (¼+¼+0, 0+0+½, ¾+¾+½) ÷ 3 = (⅙, ⅙, ⅔), qualitised as interaction = LC. Looking back over the transcript, there is a strong sense of learner-content interaction being foregrounded (as the interval largely featured seatwork), but an emphasis on teacher-content interaction (dictating) and teacher-learner interaction (discussion) are also evident. So there is alignment between experience and the numerical representation of the subsegment. Regularly performing this 'reality check' gives me confidence in the validity of the inferences I abduct from my content analysis.

Ashley then delivered a note of surprise, acting as if every length was meant to be 8cm:

14.	I disagree with all of you….
15.	Do you think you are right?
16.	*Student response.*
17.	No? Look if I measure the distance of the point from the line it's 8cm.
18.	*No teacher talk for 10 seconds. Teacher measuring.*
19.	Ah, 8cm!
20.	*No teacher talk for 4 seconds.*
21.	*No teacher talk for 6 seconds. Teacher measuring.*
22.	8cm from the point to the line.
23.	How did I get 8cm for each one?
24.	*Student response.*
25.	Do it from a different point on the line.
26.	I didn't say what point on the line you had to use, did I?
27.	My ruler's not very good…

Both intervals are again part of the *root problem* but this time they are made available through teacher-talk (manifestation = VF) rather than a given-task (manifestation = Ø), so the interval NUM code for manifestation is again (½, ½, 0). The teacher-talk is a mix of declarative statements (lines 14, 17 and 22), rhetorical questions (line 26), prompting questions (lines 15 and 23) and affirming learner responses (line 25). So both intervals are coded as explaining-from-the-front with interval NUM for interaction (½, ½, 0) and CAT code TT.

Next Ashley began to draw on the board:

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28. *No teacher talk for 7 seconds. Drawing a point and a line on the board.*
29. So, as long as you… got a certain… it's long enough you could have any number, so that's 33cm… 54cm… 64cm… 35cm… *(Measuring on the board).*
30. So there's lots of different distances.
31. What do you think, um…?

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32. Could there be any number that's an answer?
33. So could I give you any number?
34. If I said it has to be 3cm, would that be right?
35. Is that possible?
36. *Student response.*
37. I'd have to move the dot as well.
38. Why wouldn't 3cm work for me?
39. *Student response.*
40. It's too far.
41. So as long as it's a certain amount then it would be possible because I can put it somewhere else on the line, but there is a distance that it…

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42. … can't go below, and that's the shortest distance which is what I think a lot of you have measured.
43. It is probably the shortest distance.

Line 43 (partway through the eighth interval) in practice marks the end of the second subsegment, as there is a transition from teaching-from-the-front to seatwork. But adhering to the rules for unitising, subsegments are to contain full intervals only, so line 41 is taken as marking the end.

Both of the whole intervals I take to be part of the root problem are again made available through teacher-talk (manifestation = VF) rather than a given-task (manifestation = Ø) and so their interval NUM codes for manifestation are again (½, ½, 0½) and CAT codes are VF. Lines 28 to 31 involve Ashley making declarative statements categorised as telling-from-the-front, so the interval NUM code for interaction for the first interval of the excerpt is (1, 0, 0) and the CAT code is TC. Then lines 32 to 35 and line 38 involve Ashley probing, and lines 37 and 40 affirming learners' responses, all of which are categorised as discussing-from-the-front. Lines 41 to 43 are taken as Ashley summarising through making declarative statements, categorised as telling-from-the-front. So the second interval is categorised as explaining-from-the-front (CAT code = TT) with interval NUM code ($\frac{1}{2}$, $\frac{1}{2}$, 0).

As with the first subsegment, the second subsegment centre for manifestation is ($\frac{1}{2}$, $\frac{1}{2}$, 0), qualitised as manifestation = VF. The subsegment centre for interaction is ($\frac{1}{2}+\frac{1}{2}+1+\frac{1}{2}$, $\frac{1}{2}+\frac{1}{2}+0+\frac{1}{2}$, 0) ÷ 4 = ($\frac{5}{8}$, $\frac{3}{8}$, 0). Returning to the transcript again highlights how teacher-content interaction is indeed foregrounded throughout the interval, with the prevalence of explanation and summarisation.

As well as a transition from teaching-from-the-front to seatwork between the second and third subsegment there is also a shift in manifestation from the root problem of the first two subsegments to more technical aspects in the third subsegment.

42. <u>… can't go below, and that's the shortest distance which is what I think a lot of you have measured.</u>
43. <u>It is probably the shortest distance.</u>
44. <u>What I am going to ask you to do is to use your compasses to construct the line that goes from the point to the line with the shortest possible distance.</u>
45. <u>So get your compasses…Put the point on the dot.</u>

Recall that the transition point was taken to be the end of the sixth interval of the segment, so that the first interval of the third subsegment includes some teacher-talk (lines 42 and 43) associated with second subsegment. These lines are taken as Ashley telling-from-the-front and are associated with the original root problem (manifestation = VF). Line 44 is also taken as Ashley telling-from-the-front, offering a verbal representation (manifestation = VT) of the geometrical object that was about to be constructed. Line 45 is taken as Ashley dictating, beginning to describe the method of construction (manifestation = T). So this interval features a combination of manifestations (visual-functional, visual-technical and technical). Its interval NUM code is ($\frac{1}{2}+\frac{1}{2}+0$, $\frac{1}{2}+0+0$, $0+\frac{1}{2}+1$) ÷ 3 = ($\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{2}$) and CAT code is VT. As this interval features a transition from teaching-from-the-front (interaction = TC) to seatwork (interaction = LC) both contribute to the coding (see table A1/g in Appendix 1) and its interval NUM code for interaction is (½, 0, ½) and CAT code is CC.

The next four intervals are taken as Ashley dictating the task; the interval NUM code for each is (¼, 0, ¾) and the CAT code is LC. The four intervals provide a good example of coding both given-tasks and teacher-talk. When the task is dictated manifestation is coded as a given-task. In the first three intervals below, the given-task involves construction (manifestation = T). The fourth interval also involves construction (manifestation = T) but an additional given-task is introduced at line 57 concerned with noticing properties (manifestation = VF).

46.	<u>And then make sure it is long enough so that it at least overlaps your line</u> *(demonstrating on the board).*
47.	*No teacher talk for 10 seconds.*
48.	<u>And you should be able to create two arcs on your line that look like that.</u>
49.	*No teacher talk for 7 seconds.*
50.	*No teacher talk for 4 seconds.*
51.	<u>But that's not much help because that's just kind of creating a triangle.</u>
52.	<u>So keeping your compasses the same length put a point now where you've made an intersection with your circle and the line...</u> *(Drawing on the board)...*
53.	<u>You put the point on the intersection and draw a circular arc.</u>
54.	<u>And you do the same thing keeping it exactly the same length there, and you'll create another arc and that shows you where you need to draw your line to create the shortest possible distance.</u>
55.	*No teacher talk for 10 seconds.*
56.	*No teacher talk for 8 seconds.*
57.	<u>When you've done that, I want you to tell me what angle that makes with the original line.</u>
58.	<u>You can use your protractor to measure it if you want.</u>
59.	*No teacher talk for 12 seconds.*

In addition to the given-task of construction, at line 51 Ashley offers a connection between an aspect of construction and an image (manifestation = VT) through teacher-talk. So in the first of these four intervals (lines 46 to 49) the given-task is coded as manifestation = T, teacher-talk is coded as manifestation = Ø and so overall the interval NUM code for manifestation is (1, 0, 0) and CAT code is T. In the second interval (lines 50 to 53) the given-task is coded as manifestation = T, teacher-talk is coded as manifestation = VT and so overall the interval NUM code is (¼, 0, ¾) and CAT code is T. The third interval (lines 54 and 55) is coded in the same way as the first. In the fourth interval (lines 56 to 59), in which the additional given-task is introduced, the NUM code for the given-task is (½+0, ½+0, 0+1) ÷ 2 = (¼, ¼, ½) and the CAT code is VFT. Teacher-talk is coded as manifestation = Ø and so overall the interval NUM code is (¼, ¼, ½) and the CAT code is VFT.

The given-tasks of constructing a line segment representing the shortest distance of a point from a line and noticing properties of this line segment in relation to the line continued over the next two intervals.

60. *No teacher talk for 16 seconds.*
61. Where was your original point?
62. *Student response.*
63. Right ok, so we're doing this one now.
64. *Student comment.*
65. OK. You draw a line from the point in that direction.

66. It doesn't necessarily have to go past the line.
67. But that shows… where they cross is where you need to join them up… That's yours yeah? OK.
68. First of all it was too short and you didn't get two places to do it.
69. That's good.
70. Um… What is the angle between the two lines?
71. *Student response.*
72. That angle's 90, what about this angle here?
73. *Student response.*

In the first interval (lines 60 to 65) Ashley is explaining-between-the-desks through a mix of questioning (line 61) and making a declarative statement (line 65) as well as making a non-mathematical task management related comment (line 63). So the interval NUM code for interaction is (¼, ¼, ½) and the CAT code is OO. The second interval features a transition from seatwork (lines 66 to 69) to teaching-from-the-front (lines 70 to 73). Lines 66 to 69 are categorised as Ashley telling-between-the-desks and lines 70 to 73 are coded as Ashley discussing-from-the-front. Referring to table A1/g in Appendix 1, the interval NUM code for interaction is $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and the CAT code is OO.

At lines 61 and 65 Ashley is focusing on the given-task of constructing the line segment (coded as manifestation = T). So following the complex coding rules in Appendix 1, the given-task NUM code is $(\frac{1}{2}+2\times0, \frac{1}{2}+2\times0, 0+2\times1) \div 3 = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ and CAT code is T; the CAT code for teacher-talk is Ø. The interval NUM code for manifestation is therefore $(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ and CAT code is T. From lines 66 to 69 Ashley again focuses on the given-task of constructing the line segment (coded as manifestation = T) but lines 70 to 73 are taken as focusing on the given-task of noticing properties of the line segment in relation to the line. Following the complex coding rules in Appendix 1, the given-task NUM code is $(2\times\frac{1}{2}+2\times0, 2\times\frac{1}{2}+2\times0, 2\times0+2\times1) \div 4 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and the CAT code is VFT; the teacher-talk CAT code is Ø. So the interval NUM code for manifestation is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and the CAT code is VFT.

This subsegment continued through the next interval and is concluded in practice at the end of line 80, partway into the following interval. So for the purposes of analysis, the end of the third subsegment is taken to line 79.

74. <u>So what can you say about the two lines?</u>
75. <u>What word do we use for two lines that meet at right angles?</u>
76. *Student response.*
77. <u>Perpendicular.</u>
78. <u>So actually the shortest distance you can make is always created by a perpendicular line.</u>
79. <u>So we'd call it the perpendicular</u> (*writing on the board*) <u>from a point to a line</u>
80. <u>which always gives you the shortest possible distance.</u>
81. <u>OK, could you just do that again for me please?</u>

Lines 74 and 75 are taken as Ashley questioning, line 77 as affirming a learner's response and lines 78 and 79 as making declarative statements. This interval is therefore taken as Ashley explaining-from-the-front with interval NUM code for interaction at (½, ½, 0) and CAT code TT.

As this interval is made available through teaching-from-the-front, the given-task is coded manifestation = Ø. Lines 74 to 77 are taken as teacher-talk relating to technical factual knowledge (manifestation = T) while lines 78 and 79 are taken as a verbal description of a geometrical property (manifestation = VT). So overall the NUM code for teacher-talk is $(0+\frac{1}{2}, 0+0, 1+\frac{1}{2}) \div 2 = (\frac{1}{4}, 0, \frac{3}{4})$ and CAT code is T. So the overall interval NUM code for manifestation is also (¼, 0, ¾) and CAT code is T.

For this third subsegment, the centre for manifestation is (0.19, 0.10, 0.71), qualitised as manifestation = T. The foregrounding of technical aspects suggested by this categorisation aligns with emphasis on a method of construction evident in the transcript. The centre for interaction is (0.31, 0.16, 0.53), qualitised as interaction = CC, which aligns with the periods of Ashley dictating and seatwork followed by feedback on what learners had noticed that can be seen in the transcript.

The given-task for the final subsegment calls for learners to generate their own examples of a perpendicular from a point to a line (CAT code for given-task is VT) and all teacher-talk focuses on this given-task (CAT code for teacher-talk is Ø). The interval NUM code for manifestation for each interval, and therefore the subsegment centre, is (½, 0, ½), qualitised as manifestation = VT.

80. which always gives you the shortest possible distance.
81. OK, could you just do that again for me please?
82. *No teacher talk for 22 seconds.*
83. *No teacher talk for 21 seconds.*
84. And when you've done it, check…
85. Measure the distance with your ruler and check it is the shortest possible distance you could get.
86. Because you want to verify these things
87. for yourself. We want to be sure that it actually worked…
88. Good.
89. *No teacher talk for 18 seconds.*
90. That's amazing, isn't it? Look at how accurate that is… without using a protractor ~
91. ~ or anything.
92. That's pretty good, isn't it?
93. *No teacher talk for 10 seconds.*
94. Is it 90?
95. It's good isn't it!
96. *No teacher talk for 4 seconds.*
97. Have you measured the angle?
98. *Student response.*
99. Could it be any closer to 90?
100. *Student response.*
101. Yeah it's… good. Without using a protractor at all, or measuring around a corner.
102. You know if you did the corner of the table it would be bendy because it curves.
103. You don't need any of that!
104. *No teacher talk for 14 seconds.*
105. *No teacher talk for 30 seconds.*
106. *No teacher talk for 20 seconds.*
107. OK… Can you stop please?
108. Those are two things that you need to be able to do that we have just done today.
109. The first one was the recap from yesterday

Lines 81 and 84 to 86 are taken as Ashley stating a given-task verbally to the class (equivalent to dictating) and lines 82 and 83 represent a period of no teacher-talk. The NUM code for interaction of the first two intervals is therefore (¼, 0, ¾) and the CAT code is LC. Lines 87 and 88 are taken as encouragement (line 87 is not taken as contributing to the verbal statement of the given-task) and lines 90 to 92, 94, 95, 97 and 99 are taken as Ashley offering both encouragement and affirmation of the learners' example, categorised as discussing-between-the-desks. The NUM code for interaction of these two intervals is therefore (0, ½, ½) and the CAT code is LL. Line 101 is also taken as Ashley offering both encouragement and affirmation (categorised as discussing-between-the-desks), while lines 102 and 103 are taken as Ashley making declarative statements (categorised as telling-between-the-desks). The NUM code for interaction of this interval is therefore (¼, ¼, ½) and the CAT code is OO. Lines 105 and 106 indicate Ashley being-watchful over the next two intervals, with lines 107 to 109 taken as organisational, non-mathematical statements. The NUM codes for interaction of these last two intervals are therefore (0, 0, 1) and the CAT codes are LC. The centre for interaction for the final subsegment is (0.11, 0.18, 0.71), qualitised as interaction = LC. This categorisation aligns with the emphasis on seatwork over the subsegment.

The segment centre for manifestation is (0.38, 0.20, 0.42), capturing the stressing of visual-functional aspects of the topic in the first two sub-segments, technical aspects in the third and visual-technical aspects in the fourth. The segment centre for interaction is (0.28, 0.21, 0.51), capturing the foregrounding of learner-content interaction (associated with seatwork) in all but the second subsegment and the emphasis on teacher-content and teacher-learner interaction (associated with dictation and discussion respectively) in all but the final subsegment.

3.3 – Statistical methods for abducting inferences

Having developed methods for making over text into numerical data, attention turns to how that numerical data might serve to address questions about differences in how teachers make content available to learners. I will pose sub-questions to my main research questions, which may be answered through statistical analysis. Answering the sub-questions calls for innovative statistical methods given the non-parametric nature of the data. These answers will inform responses to the main questions.

3.3.1 – Operationalising the research questions

In Chapter 1, I set out the two main research questions:

> RQ1: Within a single series of lessons, what differences are discernible in how a teacher makes content available to learners over time?
> RQ2: When comparing series of lessons, what differences are discernible between how teachers makes content available to learners over time?

These questions cannot be answered directly. It is necessary to pose further questions that can be addressed within the analytical context of this study. Then inferences can be abducted from the responses to these questions in order to address RQ1 and RQ2. This draws on the logic of content analysis.

RQ1 concerns single cases of a teacher teaching a series of lessons on a given topic. The available data for statistical analysis generated by the coding of the text of the lesson series are:

- ▷ interval-level CAT codes for given-task, teacher-talk, manifestation and interaction
- ▷ interval centres for manifestation and interaction
- ▷ interval-level ordinal data for each element of the manifestation and interaction triads
- ▷ subsegment, segment, lesson and lesson series centres for manifestation and interaction

I investigated differences at the lesson level. One benefit of this approach is that of the larger units (subsegment, segment, and lesson) the lesson is of the most consistent size, allowing for 'like-for-like' comparison. Despite this, at first I had considered investigating differences at the segment level but this produced an analysis too dense in order to be reported within the constraints of the current study.

Descriptive statistics (frequency, mean) are readily available based on the numerical data and these statistics are traceable from narrative descriptions of lessons, so the validity of any inferences drawn can easily be established. So descriptive statistics (ζ and κ values) can be used in order to characterise how the teacher made content available to learners over various time periods. In particular, ζ and κ values for manifestation for the lesson series as a whole can address:

> RQ1a: What can be inferred from analysis of teacher-talk and given-tasks about the particular forms of content the teacher made available to learners across the lesson series as a whole?

Descriptive statistics also offer an informal comparison of how content is made available from lesson to lesson. However, I wanted a more quantitative way of discerning differences than making qualitative comparisons based on descriptive statistics that might unambiguously address:

> RQ1b: What can be inferred from analysis of teacher-talk and given-tasks about between-lesson differences in the forms of content the teacher made available to learners?

The statistical approach I developed in order to address this question is described in Section 3.3.2.

As well as investigating differences in the forms of content made available by attending to manifestation, the types of exchanges offered can also be studied by attending to interaction. In particular, ζ and κ values for interaction for the lesson series as a whole and individual lessons can address:

> RQ1c: What can be inferred from analysis of teacher-talk and given tasks about the types of exchanges the teacher offered across the lesson series?

The available data offers a third line of investigation into a source of possible variation in how content is made available, which is how content is manifested during classwork when compared to during seatwork. In addition, variation in how content is manifested during seatwork can be investigated. Initial analysis of teaching-between-the-desks suggested differences in the manifestation categorisation of teacher-talk when the mode of teacher interaction was discussion-between-the-desks when compared with telling or explaining-between-the-desks. Is this a significant source of possible variation? These two lines of investigation come together in order to address the question:

> RQ1d: What can be inferred from analysis of teacher-talk and given-tasks about the forms of content the teacher made available to learners during classwork and seatwork?

Again, a statistical approach is needed in order to address this question and this will be described in Section 3.3.3. Responses to RQ1a-d, inferred responses to RQ1 and considerations of the validity of these inferred responses structure the writing of case studies (see Section 3.4).

RQ2 concerns comparing cases and this brings to the surface sources of possible variation in how content is made available. I compare cases in two ways. Firstly, by applying the statistical method I developed in order to address RQ1b (see section 3.3.2), I identify differences at the lesson series level in the relative stressing of the three elements of the interaction and manifestation triads. Secondly, cross-tabulating κ values for manifestations or modes of teacher interaction at the series level by case identifies whether particular distributions of manifestations or modes of teacher interaction are associated with particular cases. This quantitative approach affords addressing the following questions:

> RQ2a: What can be inferred from analysis of teacher-talk and given-tasks about between-case differences in the particular forms of content made available to learners?
>
> RQ2b: What can be inferred from analysis of teacher-talk and given-tasks about between-case differences in the types of exchanges offered to learners?
>
> RQ2c: What can be inferred from analysis of teacher-talk and given-tasks about between-case differences in how content was made available to learners during seatwork?

These sub-questions correspond to three of the four sub-questions of RQ1. I was unable to develop a quantitative approach to discerning between-case differences in the particular forms of content made available to learners from lesson-to-lesson. This meant that the sequencing of manifestations could not be compared across the four lesson series. A significant obstacle to this was the variable length of lesson series (something I wanted to leave to the teacher to decide), making meaningful like-for-like comparisons of sequencing unlikely.

3.3.2 – Discerning differences between lessons and lesson series

In section 3.2.1, I described how Barycentric co-ordinates (x, y, z) provide a means of representing numerically the relative stressing of the three elements {X, Y, Z} of a phenomenon whose underlying structure is triadic and how this numerical representation of the phenomenon could be decomposed into its three elements, affording statistical investigation of each element independently. In particular, these data offer a way of making statistical comparisons between lessons.

Let (x_{1i}, y_{1i}, z_{1i}) (i = 1 to N_1) represent the N_1 interval centres of Lesson 1 and (x_{2i}, y_{2i}, z_{2i}) (i = 1 to N_2) represent the N_2 interval centres of Lesson 2. The basis for a statistical comparison is a null hypothesis, which in this case is that there are no discernible differences between Lesson 1 and Lesson 2. This implies that there are no discernible differences between the x_{1i} and the x_{2i}, no discernible differences between the y_{1i} and the y_{2i}, and no discernible differences between the z_{1i} and the z_{2i}. So if discernible differences were detected between, for example, the x_{1i} and the x_{2i}, then the null hypothesis would have to be rejected and it would be concluded that there were discernible differences between Lesson 1 and Lesson 2.

I observed in section 3.2.1 that the x_{1i} and the x_{2i} were ordinal data, and as such only non-parametric approaches to discerning differences were appropriate. I select the Wilcoxon rank-sum test (Wilcoxon, 1945) for this purpose, described as a 'non-parametric equivalent of the independent t-test' (Field 2009:540). The idea behind the test is fairly simple. If there are no discernible differences between the x_{1i} and the x_{2i} and $N_1 \approx N_2$, then when all the x_{1i} and x_{2i} are mixed and then ranked smallest to largest, the sum of the x_{1i} ranks should approximately equal the sum of the x_{2i} ranks. The Wilcoxon rank-sum test generalises this for when $N_1 \neq N_2$, accounts for when ranks are tied[18] and provides a confidence interval within which the sum of

[18] Accounting for tied ranks was particularly important, as in practice this was a regular occurrence with this data (Wilcoxon, 1945).

ranks can be taken as 'approximately equal'. I designed a spreadsheet in order to carry out the test.

The Wilcoxon rank-sum test therefore offers a way of discerning statistically significant differences between lessons. However, this did not distinguish between differences that were small and differences that were larger; this required the calculation of an effect size, r (Field, 2009:550). r = .10 is taken as small effect, r = .30 as a medium effect and r = .50 as a large effect (Field, 2009:57). This informed my definition of material difference.

> Let $\{X, Y, Z\}$ be the three elements of the manifestation or interaction triad and let (x_{1i}, y_{1i}, z_{1i}) (i = 1 to N_1) represent the N_1 interval centres of Lesson 1 and (x_{2i}, y_{2i}, z_{2i}) (i = 1 to N_2) represent the N_2 interval centres of Lesson 2. Then the stressing of element X is **materially different** in Lesson 1 compared to Lesson 2 if the effect size r resulting from conducting the Wilcoxon rank-sum test on the x_{1i} and the x_{2i} is such that r ≥ .20.

The selection of .20 is arbitrary, but this ensured that differences were always statistically significant at the 1% level. I will return to this choice when reflecting on the overall analytical approach in Section 8.3.

3.3.3 – Investigating the nature of teaching-between-the-desks

When the teacher chooses to teach-between-the-desks, how content is manifested through teacher-talk may or may not align with how content is manifested through the given-task. Less formally, I was interested in when teacher-talk either did or did not align with the given-task. I conjectured that when teacher-talk did not align with the given-task, then its form was likely to go beyond questioning and affirming because the teacher was probably engaged in trying to shift the focus of a learner's attention from outer aspects of the given-task to inner aspects. I therefore anticipated an association between whether teacher-talk aligned with the given-task or not and whether the teacher was discussing-between-the-desks or instructing-between-the-desks.

In order to investigate this within a single case, I identify all the half-minute intervals across the lesson series that feature teaching-between-the-desks. For each of these intervals, teacher-talk is coded as manifestation = Ø if it aligns with the given-task and manifestation = V, F, T, VF, VT, FT or VFT otherwise. Each interval is also coded as interaction = CC, OO or LL. Associations between two categorical variables can be established statistically through applying Pearson's chi-square test. In practice over the four cases, teacher-talk while teaching-between-the-desks is most commonly coded as manifestation = Ø, T or VT and other codes, when they arise, are so small as to contradict the assumption of the chi-square test that 'expected frequencies should be greater than 5' (Field, 2009:692). Therefore for each case, a chi-square test is conducted on contingency table 3.3/a. The frequency of intervals coded as instructing-between-the-desks is simply the sum of the frequencies of intervals coded as interaction = OO and interaction = CC.

Table 3.3/a Contingency table used within each case study in order to investigate an association between the teacher's role in interaction when teaching-at-the-desks and how content is manifested through teacher-talk.

Teacher's role in teaching-between-the-desks	*Categorisation of teacher-talk* Ø	VT	T	Total
Discussing-between-the-desks (LL)				
Instructing-between-the-desks (CC/OO)				
All teaching-between-the-desks				

The strength of any association identified is determined by calculating the effect size (Cramer's *V* was selected as the effect size (Field, 2009:698)) and this is reported along with the test statistic x^2, the degrees of freedom (always 2 as the contingency table was 3 by 2) and the significance value ($p<.05$, $p<.01$ or $p<.001$).

In conducting these investigations, I am mindful of the following:

> Because content analysts control the definitions of their variables, there is always the danger that the statistical relations within content analysis results are artefacts of the recording instrument.
>
> (Krippendorff, 2013:194)

So would any associations actually be a feature of my analytical method rather than a classroom phenomenon? Or perhaps I come to expect the association, so when coding teacher-talk associated with teaching-between-the-desks as manifestation = Ø, am I more inclined to code the interval as interaction = LL because of this? And when coding teacher-talk as something other than manifestation = Ø, am I more inclined to code the interval as interaction = CC or OO? On all counts I would like to think not and would point to counter-examples of this in coding and the relative weakness of the associations as supporting evidence. However, this is an issue I will return to when evaluating the analytical method in Section 8.3.

3.4 – Writing case studies

Case studies are developed in four sections. My intention is to offer narrative reporting of classroom experience alongside reporting findings within the context of my analytical framework before abducting inferences from these findings back to the classroom context. The purpose of the initial narrative reporting is to provide evidence against which the validity of the abducted inferences can be evaluated. As Weber asserts,

> Interpretations of statistical manipulations based on quantified text must be validated by the text itself.
>
> (Weber, 1990:62)

Further, specific to the current study, it is important to establish that inferences at the segment, lesson and lesson series level that are drawn from aggregating the categorisations of half-minute intervals do parallel, or at least do not contradict, what was experienced in the classroom. It is not my intention to emphasise explicitly validity issues throughout, as this would become tiresome for the reader, but rather to offer clear pre-analytical and analytical accounts of classroom experience side-by-side.

In section 1, I offer brief contextual details of the school, the class and the topic and a timeline of the lesson series. Section 2 contains a segment-by-segment account of the lesson series. Each segment is represented through a narrative account that provides a pre-analytical description of classroom experience, marking phases of the segment. This is followed by a structural (post-analytical) account, which sets out chronologically the phases as sub-segments and categorises them using the categorical model of how content is made available. A structural account of each lesson is offered, including the distribution of manifestations and modes of teacher interaction, and the nature of teaching-between-the-desks. Section 2 contributes to monitoring the validity of inferences drawn from the analysis as it illustrates the alignment between the narrative description of lessons based on what I observed and the representation of the lessons through the lens of the categorical model.

Section 3 reports inferences drawn from statistical analysis regarding how content was made available and addresses research questions RQ1a-d. I present findings through a combination of figures of tri-polar spaces, tables and text. I place a particular emphasis on using tri-polar spaces since these are a distinctive feature of my analytical approach. A discussion follows each of these inferences, linking them to features of the structural account from Section 2.

In summary, each case study individually offers:

- ▷ understanding of how content was made available over time in the specific context of school, teacher, class and topic,
- ▷ exemplification of manifestations, modes of teacher interaction and my single case analytical approach,
- ▷ validation of this analytical approach through triangulating examples teacher utterances, narrative description of how content was made available and structural characterisations of how content was made available.

Together, the case studies offer:

- ▷ understanding of sources of variation in how content was made available,
- ▷ exemplification of manifestations, modes of teacher interaction and my multiple case study approach,
- ▷ validation of this analytical approach by triangulating multiple pre-analytical descriptions suggesting qualitative between-case differences in how content was made available and post-analytical characterisations of differences.

CHAPTER 4

CASE STUDY 1

Chapter contents

4.1 – Background to the case

Ashley taught a series of lessons at Kings Meadow School to a Year 11 class (learners aged 15 and 16) set by prior mathematical attainment. Learners were, on average, predicted to achieve GCSE Mathematics grades in line with the national average by the age of 16. This was a series of five lessons on geometrical constructions and was the last topic Ashley was due to teach the class before they embarked on revision ahead of their GCSE examinations. Some learners may have encountered some aspects of the topic previously, but they had not done so as part of their GCSE course. Figure 4.1/a presents how the lesson series was parsed into lessons and segments (segments were identified from Ashley's lesson plans).

Lesson 1	Segment 1 (15m)
	Segment 2 (25m)
	Segment 3 (16m)
Lesson 2	Segment 4 (58m)
Lesson 3	Segment 5 (8m)
	Segment 6 (12m)
	Segment 7 (38m)
Lesson 4	Segment 8 (57m)
Lesson 5	Segment 9 (29m)

Figure 4.1/a The parsing of Ashley's Year 11 lesson series on geometry into lessons and segments

Notes:
1. The length of each segment is given in minutes. Lessons were nominally 60 minutes long, but not all lesson time was devoted to mathematical activity. So the sum of the lengths of the segments making up a lesson may not equal 60 minutes.
2. The second half of Lesson 5 featured a general assessment activity, which was taken as being outside the teaching of the current topic.

4.2 – How content was made available at the segment and lesson level

This section contains a lesson-by-lesson and segment-by-segment report of the lesson series. There is a narrative, pre-analytical overview of each lesson including details of the segment(s) constituting that lesson. Each segment is represented in two ways. Firstly, there is a narrative account that provides a pre-analytical description, marking phases of the segment (which are labelled in **bold font**). This includes quotations from the teacher. The quoted talk that occurred between-the-desks is numbered, allowing subsequent referencing. Talk that occurred from-the-front is not numbered as no subsequent reference is made to it. This is followed by a structural account (presented in *italicised font*), which sets out chronologically the phases as sub-segments and categorises them using the categorical model of how content is made available. Recall that the region of the tri-polar space within which a sub-segment centre lies determines its categorisation. Each lesson report is concluded with a structural account of the lesson as a whole (presented in *italicised font*) using the language of the categorical model. This whole section has great significance for the validity of the study as it illustrates the alignment between the narrative description of lessons based on what I observed and the representation of the lessons through the lens of the categorical model.

LESSON 1

In Lesson 1, Ashley provided opportunities for learners to appreciate each of the circle (Segment 1), the perpendicular bisector (Segment 2) and the angle bisector (Segment 3) as a locus of points constrained by a particular rule.

Segment 1 (15 minutes)

Ashley provided opportunities for learners to appreciate the circle as the locus of points a fixed distance from a given point. Learners were initially instructed to plot points a fixed distance from a given point, affording them to **notice** that this results in a circle:

> On a piece of blank paper put… a small cross, in the middle… I would now like you to mark a dot 5cm away from your cross… I would like you to draw another dot 5cm away from the cross. Then I'd like you to draw a third dot 5cm away from the cross. In fact I'd like you to draw ten thousand dots 5cm away from the cross.
>
> (Ashley, Year 11 Geometry Lesson 1)

During this noticing phase, Ashley spent a little time teaching-between-the-desks. This included the following excerpt of teacher-talk, which I took as Ashley telling the learner how to perform a procedure (constructing a circle):

> **A1.1:** Push down with your left hand on the paper so it doesn't move at all and just… you want to get it so it just, so it's a really light line so it just feathers on the paper and you can actually turn the paper with your left hand whilst you are doing it.
>
> (Ashley, Year 11 Geometry Lesson 1)

The noticing phase allowed for a **discussion** about the definition of a circle before learners were tasked with **copying** a definition of a circle into their books.

I parsed Segment 1 into three sub-segments relating to the distinct noticing, discussing and copying phases highlighted above. The noticing phase involved a root problem leading to a diagram (manifestation = VFT); there was an emphasis here on Ashley giving verbal instructions and being-watchful (interaction = LC). This phase included some teaching-between-the-desks, which generally aligned with a switch to a more technical manifestation of the topic (teacher-talk = T) as in excerpt A1.1 (interaction = CC). The discussion phase (interaction = TL) involved descriptions and diagrams (manifestation = VT), while the copying phase (interaction = LC) involved a formal definition of a circle (manifestation = T).

<u>Segment 2 (25 minutes)</u>

Ashley provided opportunities for learners to appreciate the perpendicular bisector of AB as the locus of points equidistant from A and B. Learners were initially instructed to plot points equidistant from given points A and B, affording them to **notice** that this results in the perpendicular bisector of AB:

> In the top left-hand corner I would like you to... do a cross and mark it A... and then somewhere [near] the middle of the piece of paper I'd like you to do another cross and mark it B... Now I would like you to mark one point that is exactly the same distance away from A as it is from B... I would now like you to mark a second point in a different place that is the same distance from A as it is from B... When you've drawn your second point that is the same distance from A as it is from B, I'd like you to draw a third point. In fact, let's do ten thousand points that are the same distance from A as they are from B.
>
> (Ashley, Year 11 Geometry Lesson 1)

During this noticing phase, Ashley spent time teaching-between-the-desks. This included the following excerpts of teacher-talk, which I took as Ashley engaging in discussion with the learners. I read the first excerpt as Ashley focusing the learner's attention on the above task and the second as supporting the learner in moving towards a method of construction for the perpendicular bisector:

A2.1: OK, let's measure how far that [point] is from B first of all. How far is it from B? *Pause*. How far is it from B? Measure it. How far is it from A? *Learner response (without measuring).* No, that point is not 14.5cm away from A. *Learner comment.* No, what I said is that the points have to be the same distance away from A as they are from B.

(Ashley, Year 11 Geometry Lesson 1)

A2.2: You've got quite a few arcs there. Why have you drawn two circles that way and two circles that way? *Learner response*. You didn't go far enough. So what do you have to think when you are doing it? How do you make sure it goes…? *Learner response*. Make [the radius] as big as possible.

(Ashley, Year 11 Geometry Lesson 1)

The technique for constructing a perpendicular bisector was then **demonstrated**. Learners were then tasked with **generating** examples of perpendicular bisectors. During this generating phase, Ashley spent time teaching-between-the-desks. This included the following excerpts of teacher-talk, which I took as Ashley engaging in discussion with the learners. I read both excerpts as Ashley focusing the learners' attention on the construction of the perpendicular bisector, but prompting in the first and probing in the second:

A2.3: What's wrong with your one? What's not worked?

(Ashley, Year 11 Geometry Lesson 1)

A2.4: Can you make a perpendicular bisector that is hard to do on your paper? Can you create a harder example maybe? Make it harder… how could you make it harder? Because you obviously know how to do it. *Learner response*. Oh, that would be interesting…

(Ashley, Year 11 Geometry Lesson 1)

I parsed Segment 2 into three sub-segments relating to the distinct noticing, demonstrating and generating phases highlighted above. The noticing phase involved *a root problem (manifestation = VF), with Ashley being-watchful and discussing-between-the-desks (interaction = LL). Some of the teaching-between-the-desks within this phase aligned with focusing on the visual-functional manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpt A2.1 (interaction = LL), while some aligned with switching to a more visual-technical manifestation of the topic (teacher-talk = VT) as in excerpt A2.2 (interaction = LL). The demonstrating phase (interaction = TT) and generating phase (interaction = LL) involved descriptions and diagrams (manifestation = VT). The teaching-between-the-desks in this practising phase generally aligned with focusing on the visual-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpts A2.3 and A2.4 (interaction = LL).*

Segment 3 (16 minutes)

Ashley provided opportunities for learners to appreciate the angle bisector of PQR as the locus of points equidistant from PQ and QR. Learners were initially instructed to plot points equidistant from non-perpendicular lines PQ and QR affording them to **notice** that this results in the angle bisector of PQR.

During the noticing phase, Ashley spent time teaching-between-the-desks. This included the following excerpts of teacher-talk, which I took as Ashley engaging in discussion and focusing the learners' attention on their accuracy when plotting points equidistant from two lines:

> **A3.1:** 1.7... and... 1.6 ish, that's pretty good. If I pick a point, just to be difficult, if I pick that one there it looks like 2mm and that one looks like 3. So you've got them in the right kind of area, but I'm not sure it's quite accurate enough.
>
> (Ashley, Year 11 Geometry Lesson 1)
>
> **A3.2:** Have you got one do you think? You reckon? Let's have a go... 4.4cm... 4.3cm. Not bad. No cigar!
>
> (Ashley, Year 11 Geometry Lesson 1)

The importance of not using a protractor to bisect an angle was **discussed** and the method of construction with straight edge and compasses was **demonstrated** through a video animation. Learners then tasked with **generating** examples of bisected angles.

I parsed Segment 3 into four sub-segments relating to the distinct noticing, discussing, demonstrating and generating phases highlighted above. The noticing phase involved a root problem (manifestation = VF), with an emphasis on Ashley giving verbal instructions and being-watchful (interaction = LC). This phase featured some teaching-between-the-desks, which aligned with focusing on the visual-functional manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpts A3.1 and A3.2 (interaction = LL). The discussion phase (interaction = TL) involved a mixture of a root problem, descriptions and diagrams (manifestation = VFT). The demonstrating phase (interaction = TC) focused on a method of construction (manifestation = T), while the generating phase – which featured Ashley largely being-watchful (interaction = LC) – involved learner-generated diagrams (manifestation = VT).

Structural account of Lesson 1

Table 4.2/a The spread of manifestation choices across Lesson 1 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 1 (N = 104)	2	0	18	36	36	0	8

Note: The figures in the table are κ values (percentages).

Table 4.2/a shows the spread of manifestation choices across the lesson based on my analysis. Ashley offered a mix of root problems (manifestation = VF) and learner-generated examples (manifestation = VT) in each segment of Lesson 1, along with instances when formal methods of construction were emphasised (manifestation = T).

These choices aligned with greater stressing of the visual element of the manifestation triad (ζ = 0.43)[19] over the technical (ζ = 0.37) and functional (ζ = 0.20) elements.

Table 4.2/b The spread of interaction choices across Lesson 1 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 1 (N = 104)	11	6	13	12	4	23	31

Note: The figures in the table are κ values (percentages).

Table 4.2/b shows the spread of interaction choices across the lesson based on my analysis. Each segment featured periods of Ashley making content available through largely being-watchful (interaction = LC), particularly the initiating and concluding phases of each segment. Content was also made available through teaching-between-the-desks (particularly interaction = LL and CC) and through teaching-from-the-front (particularly interaction = TL and TC). These interaction choices aligned with learner-content interaction being stressed (ζ = 0.46) over teacher-learner (ζ = 0.29) and teacher-content (ζ = 0.25) interaction.

Ashley's teaching-between-the-desks in this lesson generally aligned with focusing on the visual-functional or visual-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) largely through discussion (interaction = LL).

[19] Recall that ζ is the mean relative stressing of an element of the manifestation or interaction triad over a given time period ($0 \leq \zeta \leq 1$). Here, 0.43 is the mean relative stressing of the visual element over Lesson 1.

LESSON 2

In Lesson 2, Ashley provided opportunities for learners to develop their geometrical construction skills further and to become familiar with the technical language associated with the topic. This lesson featured a single segment (Segment 4).

Segment 4 (58 minutes)

Ashley provided opportunities for learners to **generate** further examples of the ruler and compass constructions met so far. Ashley set learners the task of constructing:

1. A circle of radius 6cm.
2. The perpendicular bisector of two points of their choice.
3. The angle bisector of: (a) an acute angle, (b) an obtuse angle, (c) a reflex angle, (d) a 180° angle (as a way of constructing a perpendicular from a point on a line)
4. An angle of 22.5°.

During this practice time, Ashley engaged in teaching-between-the-desks. This included the following excerpt of teacher-talk, which I took as discussion, advising the learner on the accuracy of their constructions:

> **A4.1:** How's your acute angle? *Learner response*. Do you think the angle on this side is the same as the angle on that side? *Learner response*. It's quite close, you've done a very, very, very acute angle, so try something a bit wider and it might be a bit easier.
>
> (Ashley, Year 11 Geometry Lesson 2)

I also took the following excerpt of teacher-talk as discussion, probing the learner on the constructions they had drawn:

> **A4.2:** OK, so how did you do that [angle bisector]? *Learner response*. So how did you know where to draw the line? *Learner response*. Can you point to where the circles met? *Learner response*. So this is the reflex angle you split, this one on this side. What angle is that on that side? What type of angle?
>
> (Ashley, Year 11 Geometry Lesson 2)

With some learners, Ashley addressed explicitly the construction of the bisector of a reflex angle. The following excerpt of teacher-talk related to emphasising the method of construction and included a mix of questioning and telling:

> **A4.3:** So why have you got stuck on the reflex one? *Learner response*. Well this is where the angle is isn't it, so it's got to be splitting this angle in half. The problem is they've met there haven't they? The reason they do that is your circles sort of meet where you started off... Make your radius bigger, like that, and use the same radius on that point...
>
> (Ashley, Year 11 Geometry Lesson 2)

Later, Ashley offered a sequence of prompting questions in order to guide a learner towards constructing an angle of 22.5°:

> **A4.4:** What angles do you know you could draw? If I asked you to draw an angle what would you be able to definitely do? *Learner response*. 180 and 90, OK so one of those might help you.
>
> (Ashley, Year 11 Geometry Lesson 2)

Partway through the segment, Ashley **discussed** the underlying constructions with the whole class:

> What did the perpendicular bisector work out for us? If you remember the first thing I asked you to do, what does the perpendicular bisector give you? If you've got two points... A and B, what does the perpendicular bisector give you?
>
> (Ashley, Year 11 Geometry Lesson 2)

Towards the end of the segment, specific constructions were **explained** to the whole class, with a particular emphasis on 3c and 3d:

> So if I'm given a line and somebody said construct a perpendicular to that line at point A, what it is asking you to do is to bisect the angle 180 degrees at A and draw a line. And the way we've done it is exactly the same way we have bisected our reflex angle... We've put two points on the line, we've extended our compasses a little bit... then we've made them meet further away from the line and [when we join that point up to point A, the angle the two lines make] should be 90 degrees.
>
> (Ashley, Year 11 Geometry Lesson 2)

I parsed Segment 4 into five subsegments: three relating to generating phases of the segment, one relating to the discussion phase and one relating to the explanation phase. Diagrams and descriptions (manifestation = VT) featured throughout most of the segment. The initial generating phase involved Ashley being-watchful (interaction = LC), but a move to teaching-between-the-desks marked the start of the second sub-segment (interaction = OO). Some of this second phase aligned with focusing on the visual-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through discussion-between-the-desks as in excerpts A4.1 and A4.2 (interaction = LL), and some which aligned with switching to a more technical manifestation of the topic (teacher-talk = T) through questioning and telling as in excerpt A4.3 (interaction = OO). The discussion phase (interaction = TL) followed in which technical language was emphasised (manifestation = T), before a further generating phase (interaction = LC). This generating phase also included teaching-between-the-desks, some of which aligned with focusing on the manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpt A4.4 (interaction = LL). The segment was concluded with the explanation phase (interaction = TT), which recapped the constructions met in the lesson (manifestation = VT).

<u>Structural account of Lesson 2</u>

Table 4.2/c The spread of manifestation choices across Lesson 2 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 2 (N = 112)	1	0	28	0	53	6	12

Note: The figures in the table are κ values (percentages).

Table 4.2/c shows the spread of manifestation choices across the lesson based on my analysis. Throughout Lesson 2, Ashley offered opportunities for learners to generate examples of the geometrical constructions encountered in Lesson 1 (manifestation = VT), which at times also involved emphasising methods of construction (manifestation = T). These manifestation choices aligned with greater stressing of the visual (ζ = 0.37) and particularly the technical element (ζ = 0.57) of the manifestation triad.

Table 4.2/d The spread of interaction choices across Lesson 2 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 2 (N = 112)	8	2	8	11	15	15	41

Note: The figures in the table are κ values (percentages).

Table 4.2/d shows the spread of interaction choices across the lesson based on my analysis. Content was made available largely through Ashley being-watchful (interaction = LC), discussing-between-the-desks (interaction = LL), and instructing-between-the-desks (interaction = CC and OO). However, there were brief periods of teaching-from-the-front (interaction = TC, TT and TL). These interaction choices again aligned with learner-content interaction being stressed (ζ = 0.61) over teacher-learner (ζ = 0.20) and teacher-content (ζ = 0.19) interaction, as in Lesson 1.

Ashley's teaching-between-the-desks in this lesson aligned with either focusing on the visual-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through discussion (interaction = LL) or switching to more technical manifestation (teacher-talk = T) through instruction (interaction = CC/OO).

In comparison to the previous lesson in the series, technical manifestation of the topic were emphasised more and functional manifestation less in Lesson 2 as learners generated their own examples of geometrical constructions (manifestation = VT). By its nature, this activity featured more individual work than Lesson 1, highlighted by the greater stressing of learner-content interaction.

LESSON 3

At the start of Lesson 3, Ashley recapped a construction the class worked on towards the end of Lesson 2 (Segment 5) before going on to introduce two further constructions: the perpendicular to a line passing through a given point (Segment 6) and the triangle (given the lengths of its three sides – Segment 7).

Segment 5 (8 minutes)

Ashley provided opportunities for learners to **appreciate** the perpendicular from a point on a line as the angle bisector of a 180° angle. This picked up on an idea from Segment 4. Learners were initially tasked with constructing a perpendicular from a point on a line. The equivalence of this construction and the construction of an angle bisector of 180° was briefly stated. They were then tasked with practising this construction through additional examples.

Ashley spent a little time teaching-between-the-desks. This included the following excerpt of teacher-talk, which I took as Ashley telling a learner how to carry out the construction:

> **A5.1:** To split this angle into two equal parts you get your compasses, you do that (*teacher demonstrating the construction*) and you do that. Then you move [the compass point] there and you do that and you do that and it shows you where they cross is where you draw to split the angle in half. With this one it is different because actually what we are saying is it like a 180 degree angle... So what you do first... (*teacher demonstrating*) is draw one mark on there and one mark on there that is the same distance. *Learner question.* It doesn't matter so long as it's the same [distance] on both sides. Then the problem is, unlike with the other one where they met somewhere else, if you put [the compass point] on there it just goes back to where you started, which is no good. So what he did is he extended his compasses… and did it further away. Well as long as it is the same from both of those, they'll meet somewhere else. So have a go at that.
>
> (Ashley, Year 11 Geometry Lesson 3)

Segment 5 featured a short task that I did not parse into sub-segments. Solving a problem (constructing a perpendicular from a point on a line) through recognising its inner aspect (bisection of a 180° angle) featured throughout the segment (manifestation = FT) with Ashley largely being-watchful (interaction = LC). This segment included a little teaching-between-the-desks, which mostly aligned with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpt A5.1 (interaction = CC) but also included instances of focussing on the visual-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through discussion (interaction = LL).

<u>*Segment 6 (12 minutes)*</u>

Ashley provided opportunities for learners to appreciate the shortest distance of a point from a line as the length of the perpendicular from the point to the line. Learners were tasked with **measuring** the distance of a point from a line before the ambiguity of 'distance' of a point from a line was **highlighted**, prompting a definition based on the shortest distance:

> I want you… to draw a line that's not horizontal or vertical and then somewhere that's not on the line I want you to put a dot. I'd like you to measure with your ruler the distance from the dot to the line.
>
> What did you get? *Learner A response*. What did you get? *Learner B response*. What did you get? *Learner C response*. I disagree with all of you…. Look if I measure the distance of the point from the line (*teacher measuring*) it's 8cm. (*More teacher measuring*) Ah, 8cm! (*More teacher measuring*) 8cm from the point to the line.
>
> How did I get 8cm for each one? *Learner response*. Do it from a different point on the line. I didn't say what point on the line you had to use did I? (*Drawing on the board*) So, as long as [your line is] long enough you could have any number, so (*measuring on the board*) that's 33cm… 54cm… 64cm… 35cm…. So there [are] lots of different distances.
>
> (Ashley, Year 11 Geometry Lesson 3)

Ashley continued:

> What do you think, could there be any number that's an answer? So could I give you any number? If I said it has to be 3cm, would that be right? Is that possible? *Learner response*. I'd have to move the dot as well. Why wouldn't 3cm work for me? *Learner response*. It's too far. So as long as it's a certain amount then it would be possible because I can put it somewhere else on the line, but there is a distance that it can't go below, and that's the shortest distance which is what I think a lot of you have measured… What I am going to ask you to do is to use your compasses to construct the line that goes from the point to the line with the shortest possible distance.
>
> (Ashley, Year 11 Geometry Lesson 3)

The method for constructing a perpendicular from a point to a line was then **modelled** by Ashley before learners were tasked with **generating** their own examples.

I parsed Segment 6 into four sub-segments relating to the distinct measuring, highlighting, modelling and generating phases highlighted above. The measuring and highlighting phases involved a root problem (manifestation = VF), made available through Ashley giving verbal instructions and being-watchful (interaction = LC) and discussion (interaction = TL) respectively. The modelling (interaction = CC) phase emphasised a method for construction (manifestation = T), while learner-generated diagrams (manifestation = VT) featured throughout the generating phase (interaction = LC).

<u>*Segment 7 (38 minutes)*</u>

Ashley provided opportunities for learners to **appreciate** the role of compasses in constructing a triangle given the lengths of its three sides. They were set the challenge of drawing given triangles accurately. Ruler only constructions were anticipated at first and Ashley tested learners' (lack of) accuracy by measuring through discussing-between-the-desks:

> **A7.1:** Can I borrow your ruler please? Thank you very much. (*Teacher measuring*) How long's that side? *Learner response*. OK, that's not what I asked for… It's close. It's a little bit too… long. Why don't you go and ask how he did it?
>
> (Ashley, Year 11 Geometry Lesson 3)

If learners had encountered the use of compasses to aid accurate construction previously then Ashley encouraged them to share this technique once others became aware that their measuring approach was deficient. Further **practice** was provided in using this method of construction, before a case where the sum of the two shorter sides was less than the length of the third side was included as a **non-example** of a triangle. This non-example phase featured some teaching-between-the-desks. The following excerpt of teacher-talk, which I took as discussion, related to the non-example mentioned above:

> **A7.2:** You've done all the other ones but the last one defeated you. What if you do a different side? You drew the 10.5 first; you don't have to use 10.5 first.
>
> (Ashley, Year 11 Geometry Lesson 3)

The case of the non-example was **discussed** at the end of the segment.

I parsed Segment 7 into four sub-segments relating to the distinct appreciating, practising, non-example and discussing phases highlighted above, the first three of which were made available largely through Ashley being-watchful (interaction = LC). The appreciation phase featured a root problem (manifestation = VF). This phase included some teaching-between-the-desks, which aligned with focusing on the manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpt A7.1 (interaction = LL). Following the opportunity to find measuring only approaches for constructing triangles to be inaccurate, the practising phase featured an emphasis on technique (manifestation = T). In the non-example phase, the emphasis on technique continued along with a root problem offering opportunities to appreciate the constraints placed on the technique (manifestation = VFT). This phase *also included some teaching-between-the-desks, which aligned with focusing attention on the manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpt A7.2 (interaction = LL). Ashley returned to this root problem (manifestation = VF) in the discussion phase (interaction = TL).*

<u>Structural account of Lesson 3</u>

Table 4.2/e The spread of manifestation choices across Lesson 3 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 3 (N = 107)	1	0	17	21	11	8	42

Note: The figures in the table are κ values (percentages).

Table 4.2/e shows the spread of manifestation choices across the lesson based on my analysis. Lesson 3 featured three quite different segments, but Segment 7 was considerably longer than the other two so contributed more to the overall shape of manifestation choices over the lesson. So, as with Segment 7, in Lesson 3 there was a foregrounding of root problems (manifestation = VF), methods of construction (manifestation = T) and a mix of the two (manifestation = VFT). There were also opportunities for learners to generate their own examples of geometrical constructions (manifestation = VT) and use methods of construction to solve problems (manifestation = FT) in Segments 5 and 6. These manifestation choices aligned with greater stressing of the technical element of the manifestation triad (ζ = 0.45) over the visual (ζ = 0.28) and functional (ζ = 0.27) elements.

Table 4.2/f The spread of interaction choices across Lesson 3 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 3 (N = 107)	2	4	4	5	8	22	55

Note: The figures in the table are κ values (percentages).

Table 4.2/f shows the spread of interaction choices across the lesson based on my analysis. When compared to the manifestation choices across Lesson 3, there was less segment-to-segment variation in interaction choices that came to action. While there were short periods of teaching-from-the-front (interaction = TC, TT and TL) content was generally made available through written or spoken instructions coupled with Ashley being-watchful (interaction = LC) or discussing-between-the-desks (interaction = LL). These interaction choices aligned with greater stressing of learner-content interaction (ζ = 0.70).

Ashley's teaching-between-the-desks in this lesson generally aligned with focusing on the given-task (teacher-talk = Ø) through discussion (interaction = LL).

In comparison with the previous lessons in the series, the stressing of the three manifestation elements was similar to Lesson 1 as again geometrical constructions were introduced – so functional manifestation of the topic emphasised more and technical manifestation less than in Lesson 2. However, rather than the sequential root problem (manifestation = VF) followed by formal method of construction (manifestation = T) of Lesson 1, these were offered more concurrently in Lesson 3 (manifestation = VFT) and with less teaching-from-the-front. Learner-content interaction was therefore emphasised more in Lesson 3 than Lesson 1, but also more than in Lesson 2 as the latter featured more instructing-between-the-desks (interaction = CC/OO).

LESSON 4

Throughout Lesson 4, Ashley provided opportunities for learners to use ruler and compass constructions met so far in order to solve contextual problems. This lesson featured a single segment (Segment 8).

Segment 8 (57 minutes)

Learners were provided with two worksheets. Ashley considered[20] the appropriate constructions to be fairly evident on the first worksheet, but identifying the required constructions was considered to be more challenging on the second worksheet. The following is an example from the second worksheet:

> ABCD is a rectangular field [AB is 85 metres, AD is 45 metres]. Treasure is hidden in the field! The treasure is at equal distance from the sides AB and AD… [and] is also 60 metres from the corner C… Mark the position of the treasure with an x.
>
> (Worksheet, Year 11 Geometry Lesson 4)

Ashley spent time throughout the lesson teaching-between-the-desks. This included the following excerpts of teacher-talk, which I took as Ashley engaging in discussion with the learners. I read the first two excerpts as Ashley focusing the learner's attention on the given-task, the third as emphasising the method of construction and the fourth as supporting seeing which is the appropriate construction:

> **A8.1:** What do you need to do to finish that one?
>
> **A8.2:** So you are saying that the points in here are the same distance from AB as they are from AD?
>
> **A8.3:** So you've done an angle bisector of that angle there?
>
> (Ashley, Year 11 Geometry Lesson 4)

[20] Ashley revealed this information in a conversation with the researcher following the lesson.

> **A8.4:** You need to decide what is the right one. It's not a case of guessing. Which is the one where you are the same distance from two points? *Learner comment.* It is a perpendicular bisector.
>
> (Ashley, Year 11 Geometry Lesson 4)

Ashley's teaching-between-the-desks also included the following excerpts of teacher-talk, which I took as involving Ashley to some extent engaging in telling. I read the first excerpt as Ashley emphasising the choice of the appropriate construction and the second as explaining the method of construction.

> **A8.5:** You need to do the angle bisector of the lines AB and AD.
>
> **A8.6:** You've drawn those arcs there. Then what did you do next? *Learner comment.* So how did you know where to join it on that line? You need to do the other point on the other side so you put the compass there and you draw the arcs over there so you can see where to go. You've done it pretty close, but be sure you're accurate… Good.
>
> (Ashley, Year 11 Geometry Lesson 4)

Segment 8 was parsed into five sub-segments. For the first sub-segment, learners were working on the first utilisation task (manifestation = FT). By the second sub-segment, some learners were embarking on the second utilisation task (manifestation = FT). Ashley remained largely watchful through both sub-segments, but there was a little teaching-between-the-desks. Some of the teaching-between-the-desks aligned with focusing on the manifestation of the topic associated with the given-task (teacher-talk = Ø) as in excerpts A8.1 and A8.2 (interaction = LL), whereas some aligned with switching to more technical (teacher-talk = T) or visual-technical manifestation of the topic (teacher-talk = VT) through discussion (interaction = LL) as in excerpts A8.3 and A8.4 respectively. By the fourth sub-segment all learners were working on the second utilisation task (manifestation = FT) with Ashley more active, largely discussing at the desks (interaction = LL), focussing on the functional-technical manifestation of the given-task (teacher-talk = Ø) and switching to a more visual-technical manifestation *(teacher-talk = VT). However, some teaching-between-the-desks in this phase aligned more with telling and with switching to a more visual-technical manifestation as in excerpt A8.5 (interaction = CC, teacher-talk = VT) and to a more technical manifestation as in excerpt A8.6 (interaction = OO, teacher-talk = T). There were two phases when Ashley spoke to the whole class: sub-segments 3 and 5. The first of these involved Ashley explaining to the class (interaction = TT) about the appropriate construction coming to mind (manifestation = VT) while the second involved a discussion (interaction = TL) featuring descriptions of the approaches to solving utilisation problems (manifestation = VFT).*

Structural account of Lesson 4

Table 4.2/g The spread of manifestation choices across Lesson 4 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 4 (N = 109)	0	0	26	0	3	43	28

Note: The figures in the table are κ values (percentages).

Table 4.2/g shows the spread of manifestation choices across the lesson based on my analysis. Predominantly throughout Lesson 4, Ashley offered learners utilisation problems to solve (manifestation = FT). At times these problems were combined with teacher-talk that emphasised methods of construction[21] *(manifestation = T) or matching problems to appropriate constructions*[22] *(manifestation = VFT). These manifestation choices aligned with greater stressing of the technical (ζ = 0.56) and functional (ζ = 0.35) elements of the manifestation triad.*

[21] Here, given-task = FT and teacher-talk = T, so overall manifestation = T.
[22] Here, given-task = FT and teacher-talk = VT, so overall manifestation = VFT.

Table 4.2/h The spread of interaction choices across Lesson 4 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 4 (N = 109)	4	0	9	13	17	31	26

Note: The figures in the table are κ values (percentages).

Table 4.2/h shows the spread of interaction choices across the lesson based on my analysis. In addition to a couple of periods of teaching-from-the-front (manifestation = TC, TT and TL), content was made available through two task sheets coupled with Ashley being-watchful (interaction = LC), discussing-between-the-desks (interaction = LL) or instructing-between-the-desks (interaction = CC and OO). These interaction choices aligned with greater stressing of learner-content (ζ = 0.56) and teacher-learner (ζ = 0.29) interaction.

Ashley's teaching-between-the-desks in this lesson aligned with either focusing on the functional-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through discussion (interaction = LL) or with switching to a more technical manifestation (teacher-talk = T) through instruction (interaction = CC/OO). Some teaching-between-the-desks also aligned with switching to a more visual-technical manifestation (teacher-talk = VT) through discussion (interaction = LL) or instruction (interaction = CC/OO).

In comparison to previous lessons, learners using geometrical constructions in order to solve problems was emphasised more in Lesson 4 and this was highlighted in the analysis by the greater stressing of functional aspects of the topic. There were more instances of teaching-between-the-desks in this lesson as Ashley supported learners in identifying appropriate methods in order to solve problems (teacher-talk = VT) and communicating their solutions through accurate constructions (teacher-talk = T). So *while the vast majority of content was made available to learners between-the-desks, the analysis highlighted that learner-content interaction was stressed less and teacher-learner interaction stressed more than in the previous two lessons, which can be attributed to the frequency of Ashley's choices to discuss-between-the-desks (interaction = LL).*

LESSON 5

Throughout Lesson 5, Ashley continued to provide opportunities for learners to utilise ruler and compass constructions met so far in order to solve contextual problems. This lesson featured a single segment (Segment 9).

Segment 9

Segment 9 was a continuation of Segment 8. Again it featured Ashley teaching-between-the-desks. This included the following excerpts of teacher-talk, the first of which I took as Ashley explaining to the learners the method of construction. I read the second excerpt as Ashley focusing the learner's attention on the given-task through explanation and the third as supporting the learner in selecting the appropriate method of construction through discussion:

> **A9.1:** If you want the locus of points that is the same distance from each of those, what we do... [is] you draw your circle around this point here, I'll do it a bit further out. Then you do the same size circle around here. Then where they cross over... you join up.
>
> **A9.2:** What do you need to do to finish this one off? *Learner comment*. Yes. And then it says... what does the question actually say? It says sketch the region in which T would be. So you just need to indicate which side of the line it is, actually answering the question. So which side could T be in? Is it above the line or below the line?
>
> **A9.3:** What's this question asking us to do? It's asking us to draw the set of points that are the same distance from this line as from that line. If you look in your book we learnt how to do that. Which one of those things is the same distance from two lines and what do we call that, what's that one called? *Learner response*. Angle bisector, so you need to do the angle bisector of this angle here to get a line of those points... OK?
>
> (Ashley, Year 11 Geometry Lesson 5)

Segment 9 featured a single sub-segment. All learners were by this time working on the second utilisation task (manifestation = FT) with Ashley largely being-watchful (interaction = LC). There were instances of teaching-between-the-desks, some of which aligned with focusing on the functional-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through explanation as in excerpt A9.2 (interaction = OO), and some which aligned with switching to a more technical manifestation of the topic (teacher-talk = T) through some explanation as in excerpt A9.1 (interaction = OO) or a more visual-technical manifestation of the topic (teacher-talk = VT) through discussion as in excerpt A9.3 (interaction = LL).

Structural account of Lesson 5

Table 4.2/i The spread of manifestation choices across Lesson 5 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 5 (N = 55)	0	0	6	0	0	76	18

Note: The figures in the table are κ values (percentages).

Table 4.2/i shows the spread of manifestation choices across the lesson based on my analysis. Ashley continued to offer utilisation problems in Lesson 5 (manifestation = FT), aligned with greater stressing of the technical (ζ = 0.51) and functional (ζ = 0.44) elements of the manifestation triad. But the slightly greater emphasis on the functional element was one of several noticeable differences between this lesson and Lesson 4. Referring to table 4.2/j, content was again made available through a task sheet but coupled with relatively more instances of Ashley being-watchful (interaction = LC) and relatively fewer instances of instructing-between-the-desks (interaction = CC and OO). These interaction choices aligned with a greater stressing of learner-content (ζ = 0.74) over teacher-learner (ζ = 0.21) interaction than in Lesson 4.

Table 4.2/j The spread of interaction choices across Lesson 5 of Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 5 (N = 55)	0	0	4	2	16	27	51

Note: The figures in the table are κ values (percentages).

When teaching-between-the-desks, Ashley's choices that came to action aligned with either focusing on the functional-technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through discussion (interaction = LL) or with switching either to a more visual-technical manifestation (teacher-talk = VT) through discussion (interaction = LL) or explanation (interaction = OO).

4.3 – What differences are discernible in how Ashley made content available to learners over time?

In this section I present the findings from my statistical analysis addressing the following research questions:

> RQ1a: What can be inferred from analysis of teacher-talk and given-tasks about the particular forms of content Ashley made available to learners across the lesson series as a whole?
>
> RQ1b: What can be inferred from analysis of teacher-talk and given-tasks about between-lesson differences in the forms of content Ashley made available to learners?
>
> RQ1c: What can be inferred from analysis of teacher-talk and given tasks about the types of exchanges Ashley offered across the lesson series?
>
> RQ1d: What can be inferred from analysis of teacher-talk and given-tasks about the forms of content Ashley made available to learners during classwork and seatwork?

I present findings through a combination of figures of tri-polar spaces, tables and text. As previously stated, I place a particular emphasis on using tri-polar spaces since these are a distinctive feature of my analytical approach. Having presented results, I discuss the response to each question in relation to the particular range of opportunities for learning afforded during the lesson series.

4.3.1 – Particular forms of content Ashley made available across the lesson series

Ashley's manifestation choices over time (figure 4.3/a) aligned with greater stressing of the technical element of the manifestation triad (ζ = 0.50) over the visual (ζ = 0.26) and functional (ζ = 0.24) elements.

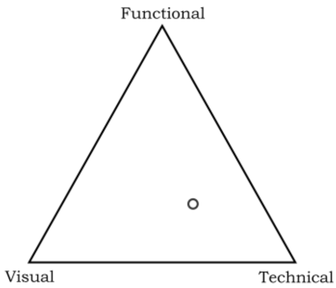


Figure 4.3/a
Series centre for manifestation for Ashley's Year 11 lesson series on geometry.

This stressing of the technical element played out in the classroom through Ashley offering learners technical, visual-technical, functional-technical and visual-functional-technical manifestations of the topic (see table 4.3/a), accounting for 87% of lesson time.

Table 4.3/a The spread of manifestation choices across Ashley's Year 11 lesson series on geometry, based on the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 487)	1	0	20	12	23	22	22

Note: The figures in the table are κ values (percentages).

In the discussion that follows, I consider each manifestation category of my model and discuss the likely implications for learning afforded by it within the context of Ashley's Year 11 lesson series on geometry. As well as viewing these affordances through the lens of shifts in the focus of learners' attention described in Chapter 2 (particularly as summarised in table 2.3/c) – and because this was a lesson series on geometry – I will also make links to the van Hiele levels (van Hiele-Geldof, 1957) and specifically Burger and Shaughnessy's (1986) exemplification of these.

Visual manifestation

Geometrical topics afford visual manifestation to the teacher; indeed the developmental origins of geometrical concepts are generally visual (van Hiele-Geldof, 1957). There were examples of these affordances being realised by Ashley in this lesson series when offering visual-functional and visual-technical manifestations of the topic.

Visual-functional manifestation of the topic was made available to learners during Lessons 1 and 3. An example of this was a line emerging from plotting points equidistant from two given points. This afforded (in the language of Mason and Pimm (1984)) opportunities for learners to see the general (a summary image of a geometrical object) in the specific (a line of points adhering to a rule on a piece of paper). A further root problem was attempting to construct a triangle from three lengths where the sum of the two shorter lengths was less than the longest length.

No purely visual manifestation of the topic (such as 'Task 3' in Section 2.3.2) was offered in this lesson series. Such visual manifestation would have afforded heightening learners' awareness of the qualities of a geometrical object by focusing attention on *image having*. Here image having refers to attending to the qualities of a geometrical object involving 'discerning details' (Mason, 2004b:15) but 'without explicit regard to properties of its components' (Burger and Shaughnessy, 1986:31).

Emphasising visual-technical manifestation of the topic, such as generating examples of perpendicular bisectors and distinguishing the properties of perpendicular bisectors from the properties of other constructions such as angle bisectors, afforded focusing learners' attention on property noticing: attending to the necessary properties of the concept (Burger and Shaughnessy, 1986). Such opportunities in this lesson series involved describing geometrical objects through explicit references to their properties (Lesson 1 and 3), and evoking images for and generating examples of formally named geometrical objects (Lesson 2). In Lesson 1 and 3, the opportunities Ashley offered afforded shifting, for a singular geometrical object, the focus of learners' attention from image having (as described above) towards procedural fluency. Here procedural fluency refers to the learner's awareness of a formal method in order to construct a geometrical object with ruler and compass. I use the word 'towards' here too in order to emphasise that secure procedural fluency may well not be an affordance of visual-technical manifestation that is *realised* by learners. In contrast to Lessons 1 and 3, opportunities in Lesson 2 afforded shifting the focus of learners' attention from procedural fluency (discrete awareness of a number of different geometrical constructions) to image having (awareness of which construction produces a particular asked-for geometrical object).

Technical manifestation

Geometrical topics afford technical manifestation to the teacher; indeed some of the historical origins of geometry are strongly technical (for example, Euclid's *Elements*). There were examples of these affordances being realised by Ashley in this lesson series when offering visual-technical (as discussed above), technical and functional-technical manifestations of the topic.

Technical manifestation, such as ruler and compass construction of perpendicular bisectors, was only made explicitly available to learners during Lesson 1. The majority of the opportunities that afforded focusing learners' attention on procedural fluency arose while Ashley was teaching-between-the-desks, returning to technical manifestation away from the visual-technical (particularly in Lesson 2) or functional-technical manifestation (particularly in Lesson 4 and 5) of content associated with the given-task. This lack of explicit opportunities to focus learners' attention on procedural fluency may be taken as *economical* (Hewitt, 1994) since learners could practise the techniques of performing geometrical constructions whilst generating their own examples (Lesson 2) and solving utilisation problems (Lesson 4 and 5).

Functional-technical manifestation of the topic, such as using geometrical constructions in order to solve problems, was particularly emphasised in Lesson 4 and 5 but it also played an important role in Lesson 2 and 3. These afforded focusing learners' attention on adaptive using, shifting from procedural fluency (awareness of a method of construction) towards confident manipulation (solving material world problems and applying rigorous geometrical reasoning). They were opportunities to *specialise*: apply a general method of construction in order to solve a particular problem. In the context of this lesson series, functional-technical manifestation also afforded forming 'correct informal deductive arguments' such as 'implicitly using the chain rule' (Burger and Shaughnessy, 1986:44). For example, shading the locus of points inside a rectangle within a fixed distance of one vertex affords the following (possibly implicit) chain of logical thought: in order to identify the required locus I need to identify points a fixed distance from the vertex; in order to identify points a fixed distance from the vertex I need to construct a circle centred on the vertex; therefore in order to identify the required locus I need to construct a circle centred on the vertex.

Functional manifestation

Jones (2010:208) promotes 'the power of geometry to bring contemporary mathematics to life'. Geometry can be used to model phenomena of the material world from bubbles to the movement of flags in the wind (Jones, 2010: 209) and contributes to scientific and technological advances such as whole-body tomography and computer animation (The Royal Society, 2001: vii). Geometrical reasoning can be applied in order to prove theorems formally by deduction. Geometrical topics therefore afford functional manifestation to the teacher. There were examples of these affordances being realised by Ashley in this lesson series when offering visual-functional and functional-technical manifestations of the topic (discussed above). However, no purely functional manifestation of the topic was offered in this lesson series.

<u>*4.3.2 – Between-lesson differences in the forms of content Ashley made available*</u>

Lesson-to-lesson differences were apparent in Ashley's manifestation choices (see figure 4.3/b). My analysis of the numerical data[23] suggested that each lesson was materially different to every other lesson in the series.

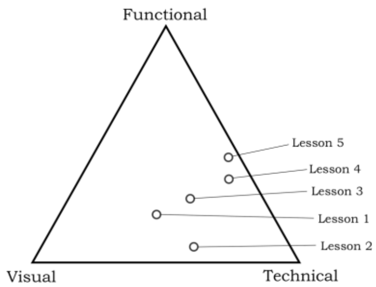


Figure 4.3/b
Lesson centres for manifestation for Ashley's Year 11 lesson series on geometry.

Figure 4.3/b also conveys that from lesson to lesson there was an underlying trend in Ashley's manifestation choices aligning with a decrease in the stressing of the visual element of the manifestation triad over time.

Table 4.3/b summarises the nature of material differences between the lessons in Ashley's series on geometry.

Table 4.3/b Material differences between the five lessons of Ashley's series on geometry in terms of how content was manifested overall.

Lesson	Difference comparison	Nature of material difference
Lesson 1	Other 4 lessons	Relative over-stressing of visual element
Lesson 2	Other 4 lessons	Relative under-stressing of functional element
Lesson 3	Lesson 4	Relative over-stressing of visual element
Lesson 4	Lesson 5	Over-stressing of technical element
Lesson 5	Other 4 lessons	Relative over-stressing of functional element

[23] See table A2/c in Appendix 2.

Having quantified lesson-to-lesson differences it was possible to account for these with reference to the lesson descriptions of Section 4.2. Lesson 1 and 3 were largely concerned with offering learners root problems from which formal methods followed (manifestation = VFT), although Lesson 3 also included learners using a construction in order to solve problems (Segment 5, manifestation = FT). Lesson 2 mainly featured learners generating their own examples of constructions (manifestation = VT). In Lessons 4 and 5 they used these constructions in order to solve problems (manifestation = FT), although Lesson 4 featured more teacher-talk from-the-front and between-the-desks. This teacher-talk addressed seeing which construction was appropriate in a given situation (manifestation = VT) and how to perform a given construction (manifestation = T).

Opportunities for learning afforded by the sequencing of manifestations

Referring back to the segment descriptions in Section 4.2, the general sequencing of Ashley's manifestation choices over time was:

from visual-functional manifestation

> (the noticing phases initiating Segments 1, 2 and 3, and the measuring/highlighting phases initiating Segment 6)

to a mix of technical and visual-technical manifestation

> (the demonstrating, modelling, practising and generating phases that sustained and concluded Segments 1, 2, 3, 4 and 7)

and then to functional-technical manifestation

> (the utilisation problems posed in Segments 8 and 9)

I represent this general sequencing of manifestations made available to learners in figure 4.3/c.

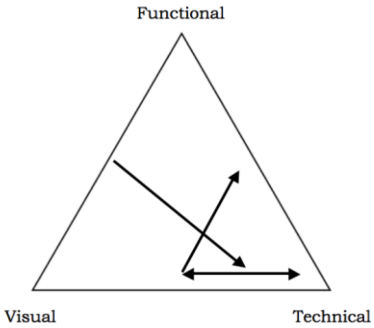


Figure 4.3/c
The sequencing of manifestation choices over time inferred from Ashley's Year 11 lesson series on geometry.

With reference to the teaching of geometry, Jones (2002) emphasises the consequences of what is made available by the teacher bypassing particular van Hiele levels. From this perspective, offering technical manifestation of a topic does not afford *abstraction* (Level 2) without the foundations of *visualisation* (Level 0) and *analysis* (Level 1), and without these foundations offering technical manifestation may be taken as teaching *instrumentally* (Skemp, 1976). This was not the case here, as Ashley only offered technical manifestation of the topic after visual-functional manifestation and generally after visual-technical manifestation too. Thus the general sequencing of manifestations of the topic aligned with progression through van Hiele Levels 0 to 2 and then on towards Level 3 (*deduction*) in Lessons 4 and 5. This sequencing resonated with Pólya's (1962) notion of *consecutive phases* in cognitive development and Freudenthal's (1981) perspective on the growth of mathematics (see Section 1.3.2).

Further analysis of teaching-between-the-desks revealed a more complex picture that was not clearly evident from observation. My analysis suggested that during seatwork Ashley's teaching-between-the-desks would sometimes align with focusing on the way content was manifested in the given-task. But at other times Ashley switched to other manifestations. My reading of those actions aligned to switching was almost always as a return to previously offered manifestations of the topic, which I associated with making available explicit opportunities to fold back (Pirie and Kieren, 1989).

The evidence of folding back afforded by Ashley's teaching-between-the-desks suggested that manifestations of the topic aligning with focusing learners' attention on procedural fluency and adaptive using – which may be associated with van Hiele Levels 2 (*abstraction*) and 3 (*deduction*) – were made available before all learners were secure at Level 1 (*analysis*). However, this is not a criticism of Ashley's teaching if, as Mason (2004b) asserts, the van Hiele level descriptors are better understood as possible foci of attention when considering moment-to-moment classroom action. From this perspective, 'shifts back and forth' (Mason, 2004b: 27) are not unexpected. And indeed a picture of switching back and forth between manifestations of the topic comes to life (see figure 4.3/d) when the effect of teacher-talk between-the-desks is added to figure 4.3/c. The dotted lines converging on the technical pole of figure 4.3/d emphasise how in the midst of given-tasks categorised as visual-technical (most particularly in Lesson 2) or functional-technical (in Lessons 4 and 5) Ashley provided explicit opportunities for the focus of learners' attention to be shifted towards procedural fluency.

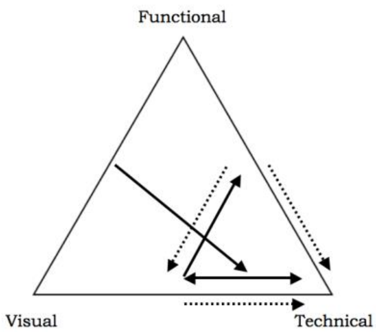


Figure 4.3/d
The sequencing of manifestation choices over time (solid line) with switches associated with teaching-between-the-desks (dotted line) inferred from Ashley's Year 11 lesson series on geometry.

Therefore focussing learners' attention on procedural fluency was not limited to a single period of the lesson series, and a similar inference was drawn regarding property noticing. Opportunities for learners first to discern the necessary properties of, for example, a perpendicular bisector were provided in Lesson 1. Yet such opportunities were not constrained to this lesson alone. In the process of generating examples of perpendicular bisectors in Lesson 2 and solving problems requiring the construction of same in Lessons 4 and 5, further opportunities to heighten awareness of properties were afforded to learners (there were explicit opportunities in Lesson 4, highlighted by the dotted line heading to the midpoint of the visual-technical edge of figure 4.3/d).

4.3.3 – Types of exchanges Ashley offered across the lesson series

Overall characterisation of the types of exchanges offered

Throughout the lesson series, Ashley's choices regarding the way content was made available aligned with greater stressing of learner-content interaction (ζ = 0.60), but also teacher-learner interaction was often emphasised (ζ = 0.24 – see figure 4.3/e).

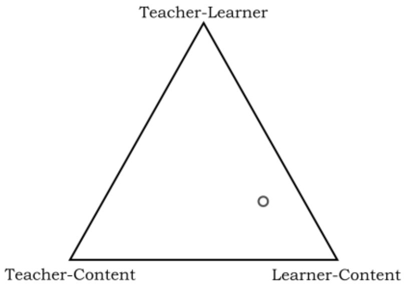


Figure 4.3/e
Series centre for interaction for Ashley's Year 11 lesson series on geometry.

Table 4.3/c shows that this stressing of learner-content interaction played out in the classroom with seatwork being foregrounded over classwork. I inferred that a range of modes of teacher interaction afforded to a teacher by the didactical situation were realised in the context of Ashley teaching the year 11 class. These included Ashley teaching-from-the-front (κ = 16%) and between-the-desks (κ = 44%) as well as being-watchful (κ = 40%). Seatwork therefore accounted for 84% of teaching time, which was by far in excess of 30-40 % observed in Stigler and Hiebert's (1999) videotaped lesson study.

Table 4.3/c The spread of interaction choices across Ashley's Year 11 lesson series on geometry, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 487)	5	3	8	9	12	23	40

Note: The figures in the table are κ values (percentages).

<u>*Comparing the types of exchanges offered from lesson to lesson*</u>

Some lesson-to-lesson differences were also apparent in Ashley's interaction choices (see figure 4.3/f).

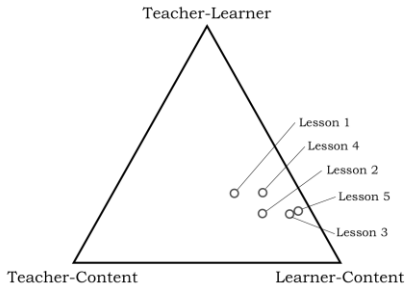


Figure 4.3/f
Lesson centres for interaction for Ashley's Year 11 lesson series on geometry.

My analysis of the numerical data[24] suggested Lesson 5 was materially different to Lessons 1, 2 and 4, most notably because Ashley's choices that came to action aligned with a relative under-stressing of teacher-content interaction compared to Lessons 1 and 2 and a relative over-stressing of learner-content interaction compared to Lessons 1 and 4. Lesson 3, with a relative over-stressing of learner-content interaction, was different to Lessons 1 and 4.

The description in Section 4.2 of how content was made available in lessons offered an account for these quantified lesson-to-lesson differences. In Lessons 2, 4 and 5, learners were largely engaged with given-tasks, but Lesson 5 featured Ashley more frequently being-watchful (interaction = LC). Furthermore, Lesson 2 featured more instructing-between-the-desks (interaction = CC/OO) than Lesson 5 and Lesson 4 featured more discussing-between-the-desks (interaction = LL). It was perhaps not surprising that by Lesson 5 in the series learners were working more independently of the teacher. In Lessons 1 and 3 learners were often engaged in discovery, but with Ashley more frequently being-watchful (interaction = LC) in Lesson 3.

[24] See table A2/f in Appendix 2.

4.3.4 – Forms of content Ashley made available during classwork and seatwork

Comparing forms of content made available during classwork and seatwork

Across the lesson series as a whole, differences were apparent in what forms of content were made available to learners during classwork (teaching-from-the-front) when compared to seatwork (teaching-between-the-desks and being-watchful). My analysis of the numerical data[25] suggested that Ashley's choices to teach-from-the-front aligned with emphasising visual aspects materially more and functional aspects materially less (see figure 4.3/g).

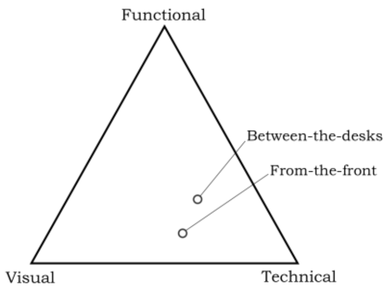


Figure 4.3/g
Comparison of manifestation centres for teaching-from-the-front and teaching-between-the-desks across Ashley's Year 11 lesson series on geometry.

This finding related to a less observable feature of how content was made available. However, it was possible to account partially for these differences by referring to the lesson descriptions of section 4.2. Lesson 1 featured the highest proportion of teaching-from-the-front, involving discussion of observed phenomena and diagrams (manifestation = VT), while root problems and methods of construction were more emphasised during seatwork (manifestation = VFT). Thus functional aspects were emphasised more during seatwork in this lesson. Lesson 4 largely involved learners using geometrical constructions in order to solve problems (manifestation = FT), whereas seeing which construction was appropriate in a given situation (manifestation = VT) was emphasised from-the-front. Thus functional aspects were emphasised more and visual aspects less during seatwork when compared to teaching-from-the-front in this lesson. These two lessons contributed most to observed differences. There was little difference in what was made available from-

[25] See tables A2/h and A2/i in Appendix 2.

the-front compared to between-the-desks in Lesson 2, and Lesson 3 and 5 featured very little teaching-from-the-front.

A feature of the lesson series was how Ashley offered opportunities for learners to encounter visual-functional, visual-technical and functional-technical manifestations first through individually tackling problems. Ashley provided initial prompts, which aligned with *psychologising the subject matter* (Dewey, 1902): working from what learners already knew in order to access the topic. Selected problems (particularly visual-technical manifestation of the topic) were subsequently addressed from-the-front, through discussion and Ashley telling.

There were opportunities for learners to generate their own examples of geometrical objects and to spend time using methods of construction in order to solve material world problems. This overall approach aligned with Stigler and Hiebert's description of Japanese lessons: 'structured problem solving' and 'inviting students to do more of the mathematical work' than the teacher (Stigler and Hiebert, 1999).

Comparing forms of content made available during seatwork

Attending more closely to the forms of content made available during seatwork, my analysis of the categorical data for interaction and manifestation arising from my classification of half-minute intervals suggested that, across the lesson series, there was a strong association[26] between Ashley's role in teaching-between-the-desks and how content was manifested through teacher-talk.

Table 4.3/d Ashley's role in teaching-between-the-desks and how content was manifested through teacher-talk across Ashley's Year 11 lesson series on geometry.

Teacher's role in teaching-between-the-desks	*Categorisation of teacher-talk*				
	Ø	VT	T	Total	
Discussing-between-the-desks	**68.4**	17.6	14.0	100.0	(114)
Instructing-between-the-desks	20.6	22.5	**56.9**	100.0	(102)
All teaching-between-the-desks	45.8	19.9	34.3	100.0	(216)
Chi Square	56.37 ***				
*** p<.001					

Note: Values in the table are percentages. Numbers in parentheses are the frequencies of half-minute intervals for each mode of teacher interaction. Significant values, referenced in the text, are highlighted in bold. Teacher-talk in one half-minute interval was coded VFT, and this was excluded from the analysis in order to comply with the assumptions of Pearson's chi-square test (Field, 2009).

Table 4.3/d conveys that Ashley's choices to discuss-between-the-desks were associated[27] with focusing on the way content was manifested in the given-task (teacher-talk = Ø). Typically in this lesson series, discussing-between-the-desks featured Ashley posing prompting (such as excerpt A4.4 in section 4.2), probing (such as excerpt A2.3) or opening questions (such as excerpt A2.4). Table 4.3/d also conveys that Ashley's choices to instruct-between-the-desks were associated[28] with switching to a more technical manifestation of the topic. In section 4.3.2, I described

[26] Cramer's V = .511. It is unlikely for this strong association not to be a feature of the shape of choices over time (χ^2 (2) = 45.39, p<.001).
[27] It is unlikely for this inferred association not to be a feature of the shape of choices over time (p<.001).
[28] It is unlikely for this inferred association not to be a feature of the shape of choices over time (p<.001).

how switching manifestation generally corresponded to returning to previously offered manifestations. Table 4.3/d shows that returns to technical manifestation were typically made available through instruction, whereas returns to visual-technical manifestation were more likely to be prompted through either discussion or instruction.

4.4 – Case summary

Ashley's interaction choices generally aligned with emphasising learner-content interaction. The types of exchanges offered in lessons varied, generally indicated by differences in stressing of learner-content interaction. Seatwork featured least in Lesson 1 of the series and most in Lesson 5. There were differences in how Ashley made content available from-the-front when compared with seatwork: visual manifestation was emphasised more and functional manifestation less when teaching-from-the-front. Just over half of the time allocated to seatwork featured Ashley teaching-between-the-desks. Discussion-between-the-desks generally afforded focusing learners' attention on the outer aspect of given-tasks. However Ashley's instructing-between-the-desks was associated with switching manifestation and afforded focusing learners' attention on the technical inner aspect of given-tasks.

I inferred from my analysis that there were material differences in how Ashley made content available across the lesson series. I associated these differences with planned opportunities to shift the focus of learners' attention over time. Broadly the sequencing of Ashley's manifestation choices afforded learners progression from primitive knowing towards adaptive using. However, at times Ashley switched to previously offered visual-technical or technical manifestation of the topic through teaching-between-the-desks, affording a shift in the focus of learners' attention that reflects Pirie and Kieren's notion of folding back (Pirie and Kieren, 1989). These in-flight choices to teach-between-the-desks offered learners many pathways from primitive knowing towards adaptive using. I concluded that Ashley's manifestation choices over the lesson series offered multiple alternative opportunities to focus learners' attention on image making, property noticing, procedural fluency and adaptive reasoning.

CHAPTER 5

CASE STUDY 2

Chapter contents

5.1 – Background to the case

Bernie taught a series of lessons at Kings Meadow School to a year 10 class (learners aged 14 and 15) set by prior mathematical attainment. Learners were, on average, predicted to achieve GCSE Mathematics grades in line with the national average by the age of 16. This was a series of four lessons on ratio, being taught partway through the school year. All learners were likely to have encountered some aspects of the topic previously, but they had not done so as part of their GCSE course. Figure 5.1/a illustrates how the lesson series was parsed into lessons and segments (segments were identified from Bernie’s lesson plans).

Lesson 1	Segment 1 (39m)
Lesson 2	Segment 2 (17m)
	Segment 3 (36m)
Lesson 3	Segment 4 (23m)
	Segment 5 (34m)
Lesson 4	Segment 6 (21m)
	Segment 7 (35m)

Figure 5.1/a The parsing of Bernie’s year 10 lesson series on ratio into lessons and segments.

Note:
1. The length of each segment is given in minutes. Lessons were nominally 60 minutes long, but not all lesson time was devoted to mathematical activity. So the sum of the lengths of the segments making up a lesson may not equal 60 minutes.
2. The first half of Lesson 1 featured a general assessment activity, which was taken as being outside the teaching of the current topic.

5.2 – How content was made available at the segment and lesson level

This section contains a lesson-by-lesson and segment-by-segment report of the lesson series. There is a narrative, pre-analytical overview of each lesson including details of the segment(s) constituting that lesson. Each segment is represented in two ways. Firstly, there is a narrative account that provides a pre-analytical description, marking phases of the segment (which are labelled in **bold font**). This includes quotations from the teacher. The quoted talk that occurred between-the-desks is numbered, allowing subsequent referencing. Talk that occurred from-the-front is not numbered as no subsequent reference is made to it. This is followed by a structural account (presented in *italicised font*), which sets out chronologically the phases as sub-segments and categorises them using the categorical model of how content is made available. Recall that the region of the tri-polar space within which a sub-segment centre lies determines its categorisation. Each lesson report is concluded with a structural account of the lesson as a whole (presented in *italicised font*) using the language of the categorical model. This whole section has great significance for the validity of the study as it illustrates the alignment between the narrative description of lessons based on what I observed and the representation of the lessons through the lens of the categorical model.

LESSON 1

In Lesson 1, Bernie provided opportunities for learners to begin dividing quantities in a given ratio (Segment 1).

Segment 1 (39 minutes)

Initially, Bernie provided a problem on the interactive whiteboard that learners were tasked with **considering** individually (figure 5.2/a). The problem was presented along with twenty circles representing the twenty £1 coins that were to be shared between the two people. After allowing a brief period for individual consideration, Bernie led a whole-class **sharing** of ways to solve this problem. As part of this, Bernie demonstrated how the 20 coins might be shared in the ratio two to three by counting them out two, then three, then two, then three, and so on (figure 5.2/b).

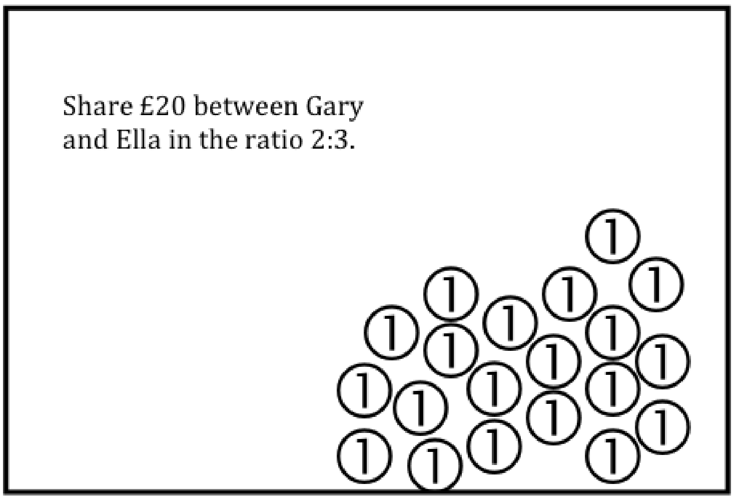


Figure 5.2/a Screenshot of the problem for learners to consider individually in Bernie's Year 10 Ratio Lesson 1

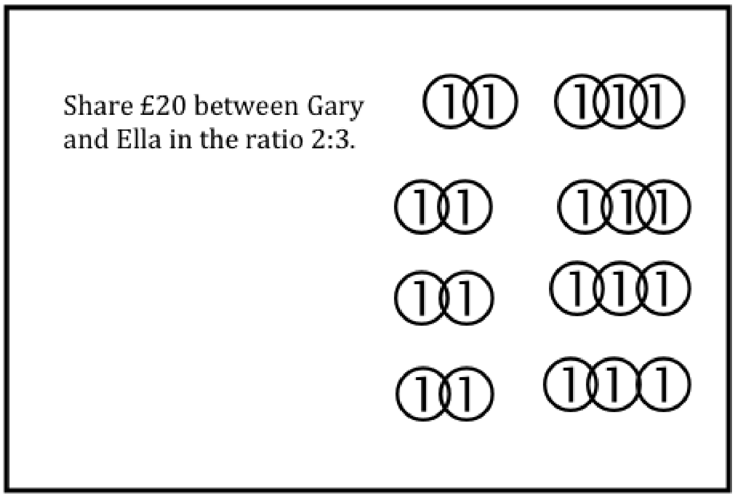


Figure 5.2/b Screenshot of how Bernie grouped the coins so as to illustrate the sharing of £20 in the ratio 2 to 3 in Year 10 Ratio Lesson 1.

Learners were then provided with the opportunity to **practise** sharing quantities of money in given ratios, for example:

> Divide £20 in the ratio 4 : 1
> Divide £14 in the ratio 5 : 2
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 1)

During this practising phase, Bernie spent time teaching-between-the-desks. This included the following three excerpts of teacher talk, the first of which I took as Bernie posing a probing question, the second as a mix of Bernie prompting responses from the learner and telling and the third as telling only. The first emphasised a technique while the other two emphasised imagery from which a technique might be abstracted.

> **$B_1$1.1:** You are going to share £10 in the ratio 3 to 2. How are you going to do that?
>
> **$B_1$1.2:** How many bags have we got altogether? *Learner response.* I'm going to spit the £40 equally into those 5 bags. So how much will be in each bag? *Learner response.* So how much is that person going to get?
>
> **$B_1$1.3:** In effect you've got 3 pots of money and 2 pots of money, which altogether is 5 pots of money. If you share your £10 equally into those 5 pots you'll have £2 in each pot won't you? That person gets 3 pots [and] that person gets 2 pots, yes?
>
> (Bernie, Year 10 Ratio Lesson 1)

I parsed Segment 1 into three sub-segments relating to the distinct considering, sharing and practising phases highlighted above. The considering phase involved a utilisation problem (manifestation = FT) made available through a given-task coupled with Bernie being-watchful (interaction = LC). The sharing phase (interaction = TT) included a mix of discussion, teacher explanation and teacher exposition, offering connections between images of sharing out coins into piles and calculations from which a technique emerged (manifestation = VT). The practising phase (interaction = LC) involved problems that were generally simple, abstract examples of sharing in a given ratio (manifestation = T). This phase included teaching-between-the-desks, some of which aligned with focusing on the technical manifestation of the topic associated with the given-task as in excerpt $B_1$1.1 (interaction = LL), while some aligned with switching to a more visual-technical manifestation of the topic (teacher-talk = VT) as in excerpts $B_1$1.2 (interaction = LL) and $B_1$1.3 (interaction = CC).

Structural account of Lesson 1

Table 5.2/a The spread of manifestation choices across Lesson 1 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 1 (N = 73)	0	0	77	0	16	6	1

Note: The figures in the table are κ values (percentages).

Table 5.2/a shows the spread of manifestation choices across the lesson based on my analysis. Bernie offered a mix of utilisation problems (manifestation = FT), visual techniques (manifestation = VT) and an emphasis on formal methods (manifestation = T) in Lesson 1. These manifestation choices aligned with a greater stressing of the technical element of the manifestation triad (ζ = 0.75).

Table 5.2/b The spread of interaction choices across Lesson 1 of Bernie's Year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 1 (N = 73)	6	15	8	8	28	12	23

Note: The figures in the table are κ values (percentages).

Table 5.2/b shows the spread of interaction choices across the lesson based on my analysis. Bernie made content available at times through a given-task coupled with being-watchful (interaction = LC) or teaching-between-the-desks (particularly interaction = OO), but at others teaching-from-the-front (particularly interaction = TT). These interaction choices aligned with learner-content interaction being stressed (ζ = 0.47) over teacher-learner (ζ = 0.29) and teacher-content (ζ = 0.24) interaction.

Bernie's teaching-between-the-desks in this lesson generally aligned with either focusing on the technical manifestation of the topic associated with the given-task (teacher-talk = Ø) or switching to a more visual-technical manifestation (teacher-talk = VT), the latter generally through explanation.

LESSON 2

At the start of Lesson 2, Bernie provided opportunities for learners to simplify ratios using the $a : b$ notation (Segment 2). The main part of the lesson was concerned with further opportunities for sharing quantities in a given ratio (Segment 3).

Segment 2

Initially Bernie provided an **exercise** for learners in simplifying ratios. This included examples such as:

> Cancel down fully the ratio 16 : 12.
> Cancel down fully the ratio 4 : 8 : 14.
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 2)

During this exercise phase, Bernie spent time teaching-between-the-desks. This included the following two excerpts of teacher talk, the first of which I took as Bernie prompting responses from the learner and the second as telling. Both emphasised methods of simplification.

> **$B_1$2.1:** (*Simplifying the ratio 10:5.*) Five goes into both of them, yes. So you've got 2 fives in 10. How many fives in 5?
>
> **$B_1$2.2:** (*Saying why trying to simplify a ratio by dividing by 1 was unnecessary.)* I said there was no point dividing by 1, because if you divide by 1 you just end up with the same number, don't you?
>
> (Bernie, Year 10 Ratio Lesson 2)

This was followed by a **feedback** session in which Bernie asked individual learners for their answers to the exercise. On occasions Bernie prompted these responses. For example, the following exchange related to simplifying the ratio 8 to 10:

> What can you divide 8 and 10 by? What number goes into 8 and 10? *Learner response*. Two. So how many twos in eight? *Learner response*. Four. And how many twos in ten? *Learner response*. Five. So [the answer is] 4 : 5.
>
> (Bernie, Year 10 Ratio Lesson 2)

I parsed Segment 2 into two sub-segments relating to the distinct exercise and feedback phases highlighted above. The exercise phase (interaction = LC) involved abstract examples of ratios to be simplified (manifestation = T). This featured some teaching-between-the-desks that aligned with focusing on the technical manifestation of the topic associated with the given-task, much of which was discussing-between-the-desks (interaction = LL); excerpt $B_1$2.1 was an example of this. Excerpt $B_1$2.2 was an example that aligned with telling-between-the-desks (interaction = CC). The feedback phase (interaction = TL) emphasised routes to simplification (manifestation = T).

<u>Segment 3</u>

Building on Segment 1, Bernie provided an opportunity to **recap** methods for sharing quantities in a given ratio. The first example offered was:

> Share £140 in the ratio 3 to 4.
>
> (Bernie, Year 10 Ratio Lesson 2)

Three methods were suggested, the third of which was a simple integer 'scaling up' method:

$$4 : 3 \equiv 40 : 30 \equiv 80 : 60 \text{ and } 80+60=140$$

The limitations of this method were revealed by the second example:

> Share £45 in the ratio 2 : 3 : 7.
>
> (Bernie, Year 10 Ratio Lesson 2)

Following this, Bernie endorsed a single method, the first step of which was dividing the quantity by the total of the parts.

Learners were then provided with the opportunity to **practise** sharing quantities in a given ratio. Examples were offered in simple material world contexts such as:

> Pocket money is split between Pete, Alan and Helen in the ratio 2 : 3 : 4. Dad pays out £36, how much does each person get?
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 2)

During this practising phase, Bernie spent time teaching-between-the-desks. This included the following two excerpts of teacher talk relating to sharing £272 in the ratio 3 to 5, the both of which I took as a mix of Bernie prompting responses from the learner and telling. Both emphasised imagery from which a technique might be abstracted.

> **B$_1$3.1:** Basically you've got £34 in each bag and that makes up your £272 altogether. Now Dr Barnados are going to have those three bags. So how much money is that altogether?
>
> **B$_1$3.2:** So it was decided to divide the money between Dr Barnados... and the RSPCA ... [in] the ratio 3 to 5. So Dr Barnados is first, they're going to get the 3 bags. RSPCA is second. They're going to get the 5 bags. So how many bags have we got altogether? *Learner response*. 8. So we've got 3 for Dr Barnados haven't we? And we've got 5 for the RSPCA. Agreed? So we've now got to take our £272 and we've got to put it into these pots equally so we've got the same amount in each pot. How would we do that? *Learner response*. Right the total amount is 272. We're going to share that equally into these 8 bags. So you've got to do 272 shared basically by 8. So how do you do that on your calculator? *Learner response*. 34. So each of these bags is going to have £34 in. So Dr Barnados are going to have 3 lots of the bags so they're going to have 3 lots of £34, so how much is that? *Learner response*. It is 102. So Dr Barnados is going to get £102, that's absolutely right.
>
> (Bernie, Year 10 Ratio Lesson 2)

I parsed Segment 3 into two sub-segments relating to the distinct recapping and practising phases highlighted above. The recapping phase involved emphasising methods (manifestation = T) and this was made available largely through whole-class discussion (interaction = TL). The practising phase involved utilisation problems (manifestation = FT), and there was an emphasis on Bernie being-watchful (interaction = LC). This phase included some teaching-between-the-desks, which generally aligned with switching to a more visual-technical manifestation of the topic (teacher-talk = VT) as in excerpts $B_1$3.1 and $B_1$3.2 (interaction = OO).

Structural account of Lesson 2

Table 5.2/c The spread of manifestation choices across Lesson 2 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 2 (N = 99)	0	0	63	0	0	20	17

Note: The figures in the table are κ values (percentages).

Table 5.2/c shows the spread of manifestation choices across the lesson based on my analysis. In Lesson 2, Bernie largely offered formal methods (manifestation = T), but with some utilisation problems (manifestation = FT). These manifestation choices aligned with greater stressing of technical manifestation of the topic (ζ = 0.74).

Table 5.2/d The spread of interaction choices across Lesson 2 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 2 (N = 99)	4	9	19	6	29	12	21

Note: The figures in the table are κ values (percentages).

Table 5.2/d shows the spread of interaction choices across the lesson based on my analysis. Content was made available through some teaching-from-the-front (largely interaction = TL) and also through two given-tasks coupled with Bernie being-watchful (interaction = LC), discussing-between-the-desks (interaction = LL), and explaining-between-the-desks (interaction = OO). These interaction choices aligned with greater stressing of learner-content (ζ = 0.44) and teacher-learner interaction (ζ = 0.37).

Bernie's teaching-between-the-desks in this lesson partly aligned with focusing on the technical or functional-technical manifestation of the topic respectively associated with the two task-sheets (teacher-talk = Ø) through discussion. However, some teaching-between-the-desks aligned with switching to a more visual-technical manifestation (teacher-talk = VT) through explanation. At these times Bernie generally made reference to the image of money bags introduced in Lesson 1.

In comparison to the previous lesson in the series, functional-technical manifestation of the topic were emphasised more in Lesson 2, although visual-technical manifestation continued to be stressed through teaching-between-the-desks. The balance between teaching-from-the-front and teaching-between-the desks was similar in both lessons. However, teaching-from-the-front featured a greater emphasis on discussion in Lesson 2, whereas explanation was more stressed in Lesson 1.

LESSON 3

Lesson 3 followed a similar pattern to Lesson 2. At first, Bernie provided opportunities for learners to express simple material world situations involving part-to-part comparisons as ratios in their simplest terms (Segment 4). The main part of the lesson provided yet further opportunity for sharing quantities in a given ratio (Segment 5).

Segment 4

Building on Segment 2, Bernie provided an **exercise** for learners in which simple material world situations involving ratio were to be expressed in their simplest terms. For example:

> This month there were 12 sunny days and 18 wet days. Find the ratio of sunny days to wet days.
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 3)

During this exercise phase, Bernie spent time teaching-between-the-desks. This included the following two excerpts of teacher talk, both of which I took as a mix of Bernie prompting responses from the learner and telling, that emphasised a method. I took the second excerpt as also emphasising alternative ways of seeing a way towards a solution.

> **$B_1$4.1:** In a bag of sweets 9 are red and 15 are blue. Find the ratio of red to blue. So how many red ones are there? *Learner response*. 9. So you write down 9 colon… How many blue ones are there? *Learner response*. 15. So 9 colon 15. And then if you can cancel that down…
>
> **$B_1$4.2:** Ok, 14 girls 16 boys. [What is the ratio] of boys to girls? How many boys are there? *Learner response*. Careful, how many *boys* are there? *Learner response*. 16. So you do 16 colon… How many girls are there? *Learner response*. 14. 16 colon 14 and then cancel it down if you can… *Learner comment*. What do you mean you don't do it that way? *Learner comment*. Ok, so you are cancelling it down first and then writing it as a ratio. That's fine. Personally it's more obvious to me to do it the other way round, but that's great.
>
> (Bernie, Year 10 Ratio Lesson 3)

This was followed by a **feedback** session in which Bernie asked individual learners for their answers to the exercise, on occasions correcting these responses. For example, the following comment related to the ratio of boys to girls in a class with 14 girls and 16 boys:

> You have to read really carefully because it says the ratio of boys to girls. So we want the number of boys first and then the number of girls. So it should have been 16 to 14. [Which simplifies to] 8 to 7. Be careful about the wording.
>
> (Bernie, Year 10 Ratio Lesson 3)

I parsed Segment 4 into two sub-segments relating to the distinct exercise and feedback phases highlighted above. The exercise phase involved simple examples of material world ratios to be interpreted and simplified (manifestation = FT) with an emphasis on Bernie being-watchful (interaction = LC). This phase included some teaching-between-the-desks, which generally aligned with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpts $B_1$4.1[29] *and $B_1$4.2*

[29] While excerpt B14.2 aligned overall with a switch to stressing more technical aspects of the topic, this excerpt aligned with an initial switch to more technical aspects – "16 colon 14 and then cancel it down if you can" – followed by a switch to more visual-technical aspects – "What do you mean you don't do it that way?".

(both interaction = OO). The feedback phase (interaction = TL) emphasised solutions to the problems (manifestation = T).

<u>*Segment 5*</u>

Learners continued to be tasked with **practising** sharing quantities in a given ratio, with Bernie making the worksheet from Segment 3 available once more for this purpose. This included the following excerpt of teacher talk, which I took as a mix of Bernie prompting responses from the learner and telling that emphasised imagery from which a technique might be abstracted:

> **$B_1$5.1:** He wants to make 276g [of butter cream by mixing fat and icing sugar in the ratio 1:2], so if you like he's going to put... his fat and his icing sugar into piles. How many piles is he going to have? *Learner response*. No he's got one pile there, two plies there. How many piles altogether? *Learner response*. 3. One of the piles is going to be fat. Two of the piles are going to be icing sugar. But they've got to have exactly the same weight in them. So how much weight would be in each of the piles? So if you like you've got a pile of fat and two piles of icing sugar. So you've got to split your 276g equally into those 3 piles. *Learner comment*. Ok. So what would that give you? 276 divided by 3. Use you calculator. *Learner comment*. Ok, sorry. Don't use your calculator! *Learner response*. 92. So you are going to have 92g in each of those piles. Therefore you need 92g of fat and you need two lots of 92g of your icing sugar. Ok? That's all you need to do. So it's exactly the same but it's not money any more is it?
>
> (Bernie, Year 10 Ratio Lesson 3)

Partway through the segment Bernie offered a different scenario that involved ratio, **extending** the range of problems that learners might solve:

> I am going to give money to Oxfam and Children in Need in the ratio 5 : 7. I give £75 to Oxfam. How much do Children in Need get?
>
> (Bernie, Year 10 Ratio Lesson 3)

Further examples of this type of problem were provided on a worksheet. During this extending phase, Bernie spent time teaching-between-the-desks. This included the following excerpt of teacher talk, which I took as a mix of Bernie prompting responses from the learner and telling, that emphasised imagery from which a technique might be abstracted:

> **$B_1$5.2:** They are slightly different these questions. Now you're going to give money to Oxfam and Children in Need in the ratio 5 to 7. So 1, 2, 3, 4, 5 for Oxfam. 1, 2, 3, 4, 5, 6, 7 for Children in Need. Now it then says I give £75 to Oxfam. So that means those 5 bags of money there must have £75 in. Do you see? *Learner response*. Alright… You know that all of those 5 bags add up to 75. How much will you have in each of them do you think? *Learner response*. Other way around 75 divided by 5. *Learner comment*. It matters which way round you divide doesn't it? What's 75 divided by 5? *Learner response*. Brilliant. So each of those (*for Oxfam*) has got 15 in. Which means each of these (*for Children in Need*) has got 15 in as well. So you've got 7 bags then with 15 in. So how much will be in all of those? *Learner response*. Yes it is, so do you see the difference?
>
> (Bernie, Year 10 Ratio Lesson 3)

I parsed Segment 5 into two sub-segments relating to the distinct practising and extending phases highlighted above. Both the practising and extending phases involved utilisation problems (manifestation = FT), with an emphasis on Bernie being-watchful (interaction = LC). Both phases (and particularly the second) included some teaching-between-the-desks, which generally aligned with switching to a more visual-technical manifestation of the topic (teacher-talk = VT) as in excerpts $B_1$5.1 and $B_1$5.2 (both interaction = OO).

Structural account of Lesson 3

Table 5.2/e The spread of manifestation choices across Lesson 3 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 3 (N = 111)	0	0	29	0	0	44	27

Note: The figures in the table are κ values (percentages).

Table 5.2/e shows the spread of manifestation choices across the lesson based on my analysis. Throughout Lesson 3, Bernie offered utilisation problems (manifestation = FT), but this included an emphasis on formal methods (manifestation = T). These manifestation choices aligned with greater stressing of functional (ζ = 0.35) and particularly technical (ζ = 0.58) manifestations of the topic.

Table 5.2/f The spread of interaction choices across Lesson 3 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 3 (N = 111)	1	6	5	10	32	12	34

Note: The figures in the table are κ values (percentages).

Table 5.2/f shows the spread of interaction choices across the lesson based on my analysis. While there was a short period of discussion and explanation from-the-front (manifestation = TL and TT), content was generally made available through two given-tasks coupled with Bernie being-watchful (interaction = LC) or explaining-between-the-desks (interaction = OO). These interaction choices aligned with greater stressing of learner-content interaction (ζ = 0.61).

Bernie's teaching-between-the-desks in this lesson generally aligned with switching from functional-technical manifestation of the topic associated with the two task-sheets through explanation. During Segment 4, these switches were mainly to more technical (teacher-talk = T) manifestations of the topic as Bernie emphasised methods for simplifying ratios. In Segment 5 switches were mainly to visual-technical (teacher-talk = VT) manifestations as in Lesson 2 as Bernie made reference to the image of moneybags.

In comparison to the previous lessons in the series, functional-technical manifestation of the topic were emphasised more in Lesson 3 although visual-technical manifestation continued to be stressed through teaching-between-the-desks. Lesson 3 also featured less teaching-from-the front than the previous two lessons.

Lesson 4

At the start of Lesson 4, Bernie provided opportunities for learners to express ratios comparing part-to-part as fractions comparing part-to-whole (Segment 6). The lesson series was concluded with a loop card activity[30] (Segment 7), which addressed the full range of ratio problems encountered across the four lessons.

Segment 6

Bernie initially provided an **exercise** for learners in which simple material world situations involving ratio were to be expressed as fractions, or vice versa. For example:

> In a grove of trees the ratio of hazelnut trees to apple trees is 5 : 2. What is the fraction of apple trees?
>
> There are two types of elephant in a zoo, African and Indian. ¾ of the elephants are Indian. What is the ratio of Indian to African elephants?
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 4)

During this exercise phase, Bernie spent time teaching-between-the-desks. This included the following excerpt of teacher talk, which I took as a mix of Bernie prompting responses from the learner and telling, that emphasised a method:

> **$B_1$6.1:** If half of the people are female, what fraction is male? *Learner response*. Half. So the fraction of men to women is half to a half. Which if you multiply both of those by two, you end up with 1 to 1 ratio. One part men to one part women. You've got equal parts.
>
> (Bernie, Year 10 Ratio Lesson 4)

[30] This consisted of a set of numbered cards each featuring a ratio problem where the solution of the problem on one card matched the number on another card, thus indicating which problem was to be solved next.

This was followed by a **feedback** session in which Bernie asked individual learners for their answers to the exercise, on occasions probing or elaborating on these responses. For example, the following exchange related to the fraction of romance books stocked in a bookshop where the ratio of crime, romance and cookery books was 9 : 5 : 2. Here Bernie probed the learner's response of five sixteenths:

> So you've got 5 parts romance. Why 16? *Learner response.* Yes, ok. Because that's your total number of parts. So 5 out of 16.
>
> (Bernie, Year 10 Ratio Lesson 4)

A further example related to the ratio of carrot cakes to other varieties in a café where one quarter of the cakes were carrot cakes. Here Bernie elaborated on the learner's response:

> One way of looking at it is that you've got a quarter carrot cakes to three quarters of the other varieties. If you multiply both of those by 4 you end up with 1 : 3.
>
> (Bernie, Year 10 Ratio Lesson 4)

I parsed Segment 6 into two sub-segments relating to the distinct exercise and feedback phases highlighted above. The exercise phase involved simple examples of material world situations expressed as ratios from which the appropriate fraction was to be abstracted, or vice versa (manifestation = FT). There was much teaching-between-the-desks, which generally aligned with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpt $B_1$6.1 (interaction = OO) above. The feedback phase (interaction = TL) emphasised the method of identification of the part and whole values from which a fraction could be obtained and expressing a fraction and its compliment of 1 as a ratio (manifestation = T).

Segment 7

Bernie provided opportunities for learners to use the methods of solving ratio problems met so far in the lesson series through a loop card activity. This included problems such as:

> The angles of a triangle are in the ratio 2 : 5 : 8. What is the size of the largest angle?
>
> Dev and Eric divide the profits of their project in the ratio 5 : 11. If Eric's share of the profits is £385 how much profit did they make altogether?
>
> The scale of a map is 1 : 250000. The actual distance between Grindle and Hopp is 150km. What is the distance, in cm, between these two places on the map?
>
> (Loop card activity, Bernie's Year 10 Ratio Lesson 4)

Bernie spent time teaching-between-the-desks. This included the following excerpt of teacher talk, which I took as a mix of Bernie prompting responses from the learner and telling. This related to the map problem given above. While overall a method was being emphasised here, Bernie appealed to verbal representations of powers of ten relating to other metric units in this exchange.

$B_1$7.1: You've got 150km. How do you convert that into metres? *Learner comment.* How many metres in a kilometre? *Learner response.* No... If you see kilometre or kilo what does that mean? *Learner response.* It does mean 1000. So you'll have 150000m. Now to turn that into centimetres, what do you times by? How many centimetres in a metre? *Learner response.* No cent, remember cent. *Learner response.* 100. So you are going to times that by 100 to change it into centimetres. So you'll have 150000 and then times by 100. Now that number then. If you divide that by this number – the 250000 – that will tell you how many cm on the map.

(Bernie, Year 10 Ratio Lesson 4)

Segment 7 featured a single sub-segment. The loop card activity was a utilisation task (manifestation = FT). There was much teaching-between-the-desks, which generally aligned with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpt $B_1$7.1 (interaction = OO) above.

Structural account of Lesson 4

Table 5.2/g The spread of manifestation choices across Lesson 4 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 4 (N = 100)	0	0	51	0	0	41	8

Note: The figures in the table are κ values (percentages).

Table 5.2/g shows the spread of manifestation choices across the lesson based on my analysis. In Lesson 4, Bernie mainly offered utilisation problems (manifestation = FT), along with an emphasis on formal methods (manifestation = T). These manifestation choices aligned with greater stressing of functional (ζ = 0.34) and particularly technical (ζ = 0.62) manifestations of the topic.

Table 5.2/h The spread of interaction choices across Lesson 4 of Bernie's year 10 lesson series on ratio, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 4 (N = 100)	0	10	5	8	39	12	26

Note: The figures in the table are κ values (percentages).

Table 5.2/h shows the spread of interaction choices across the lesson based on my analysis. In addition to a period of discussion and explaining-from-the-front (interaction = TL and TT), content was made available through two given-tasks coupled with Bernie being-watchful (interaction = LC) or explaining-between-the-desks (interaction = OO). These interaction choices aligned with greater stressing of teacher-learner (ζ = 0.26) and particularly learner-content (ζ = 0.55) interaction.

Bernie's teaching-between-the-desks in this lesson generally aligned with switching from functional-technical manifestation of the topic associated with the two task-sheets to a more technical manifestation (teacher-talk = T) through explanation.

The emphasis on functional-technical manifestation of the topic in Lesson 4 was similar to Lesson 3, but technical rather than visual-technical manifestation was stressed more through explaining-between-the-desks. This was associated with less reference being made to the moneybag image. As with Lesson 3, there was less emphasis on teaching-from-the-front than the first two lessons in the series.

5.3 – What differences are discernible in how Bernie made content available to learners over time?

In this section I present the findings from my statistical analysis addressing the following research questions:

> RQ1a: What can be inferred from analysis of teacher-talk and given-tasks about the particular forms of content Bernie made available to learners across the lesson series as a whole?
>
> RQ1b: What can be inferred from analysis of teacher-talk and given-tasks about between-lesson differences in the forms of content Bernie made available to learners?
>
> RQ1c: What can be inferred from analysis of teacher-talk and given tasks about the types of exchanges Bernie offered across the lesson series?
>
> RQ1d: What can be inferred from analysis of teacher-talk and given-tasks about the forms of content Bernie made available to learners during classwork and seatwork?

I present findings through a combination of figures of tri-polar spaces, tables and text. As previously stated, I place a particular emphasis on using tri-polar spaces since these are a distinctive feature of my analytical approach. Having presented results, I discuss the response to each question in relation to the particular range of opportunities for learning afforded during the lesson series.

5.3.1 – Particular forms of content Bernie made available across the lesson series

Bernie's manifestation choices over time (figure 5.3/a) aligned with greater stressing of technical manifestation (ζ = 0.67), but functional manifestation was also emphasised (ζ = 0.25).

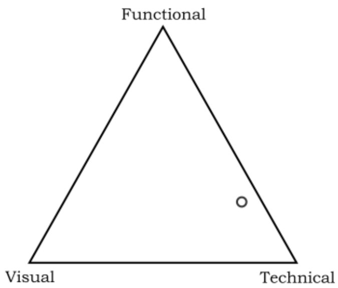


Figure 5.3/a
Series centre for manifestation for Bernie's year 10 lesson series on ratio.

This stressing of the technical element played out in the classroom through Bernie offering learners technical, visual-technical, functional-technical and visual-functional-technical manifestations of the topic (see table 5.3/a). Indeed, all intervals across the lesson series featured some stressing of the technical element of manifestation.

Table 5.3/a The spread of manifestation choices across Bernie's Year 10 lesson series on ratio, based on the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 383)	0	0	52	0	3	30	15

Note: The figures in the table are κ values (percentages).

In the discussion that follows, I consider each manifestation category of my model and discuss the likely implications for learning afforded by it within the context of Bernie's Year 10 lesson series on ratio. As well as viewing these affordances through the lens of shifts in the focus of learners' attention described in Chapter 2 (particularly as summarised in table 2.3/c) I will also make links with research on the teaching of ratio.

Visual manifestation

Ratio affords visual manifestation to the teacher in the form of 'having a sense of' multiplicative relationships. Watson et al. (2013) describe learners having a sense of 'so many of this equal so many of that' and making judgements about 'continuous quantities such as stronger or weaker fruit drinks' (Watson, Jones and Pratt, 2013:53). Griffin (1988) asserts that a 'dominant image of ratio for most pupils is that of similarity'. For example, learners recognise similarity between objects in the landscape and photographs of those objects and notice distortions. In this lesson series, visual-technical manifestation of ratio was made available to learners, but no visual-functional or purely visual manifestations of the topic were offered.

When sharing money in the ratio $a : b$, Bernie evoked an image of $a + b$ moneybags. Within the imagery, a of these moneybags belonged to one person and b of them belonged to another. This encouraged learners to imagine sharing the money equally between all the money bags, identify how much money was in every bag and then visualise collecting together the money in the a bags in order to identify the first person's share. Opportunities to play out these steps mentally would have been categorised as visual manifestation. However, the diagrammatic way Bernie offered these images in Lesson 1 provided opportunities for learners to generalise a method from the imagery and afforded shifting the focus of learners' attention from image having towards procedural fluency.

Technical manifestation

Ratio affords technical manifestation to the teacher through manipulation of the relationship $\frac{a}{b} = \frac{c}{d}$, use of the ratio notation and technical language (phrases such as 'divide in the ratio…'). There were examples of these affordances being realised by Bernie in this lesson series when offering visual-technical (as discussed above), technical and functional-technical manifestation.

Technical manifestation in this series focused on methods for dividing in a given ratio and expressing ratios in their simplest terms. Methods for dividing in a given ratio were made available during classwork in Lessons 1 and 2 and through a given-task in Lesson 1. These opportunities afforded procedural fluency. Other such opportunities arose when Bernie was teaching-between-the-desks in Lessons 2, 3 and 4, switching from functional-technical manifestation and returning to technical manifestation.

Functional-technical manifestation of the topic, such as using methods for dividing in a given ratio in order to solve simple money problems, were made available from Lesson 2. Generally there was little variation between the problems within each given-task, with the exception of the loop card activity in Lesson 4. So while there were opportunities for focusing learners' attention on adaptive using, learners could practise methods at the same time as they were solving utilisation problems.

Functional manifestation

Watson et al. (2013) highlight how ratio and proportional reasoning 'appear everywhere in the curriculum but are often implicit' (Watson, Jones and Pratt, 2013:65). This emphasises how ratio affords functional manifestation to the teacher through applications within mathematics as well as applications to the material world. There were examples of these affordances being realised by Bernie in this lesson series when offering functional-technical manifestation. However, no visual-functional or purely functional manifestations of the topic were offered in this series.

5.3.2 – Between-lesson differences in the forms of content Ashley made available

Lesson-to-lesson differences were apparent in Bernie's manifestation choices (see figure 5.3/b). My analysis of the numerical data[31] suggested that each lesson was materially different to every other lesson in the series.

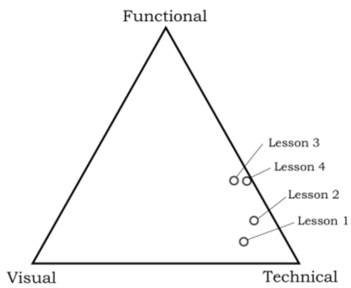


Figure 5.3/b
Lesson centres for manifestation for Bernie's year 10 lesson series on ratio.

Figure 5.3/b also conveyed that from lesson to lesson there was an underlying trend in Bernie's manifestation choices aligning with decreasing emphasis on visual manifestation of the topic.

[31] See table A3/c in Appendix 3.

Table 5.3/b summarises the nature of material differences between the four lessons in Bernie's series on ratio.

Table 5.3/b Material differences between the four lessons of Bernie's Year 10 series on ratio in terms of how content was manifested overall.

Lesson	Difference comparison	Nature of material difference
Lesson 1	Other 3 lessons	Relative over-stressing of visual element Relative under-stressing of functional element
Lesson 2	Lesson 3 and 4	Relative over-stressing of technical element Relative under-stressing of functional element
Lesson 3	Lesson 4	Relative over-stressing of visual element

Having quantified lesson-to-lesson differences it was possible to account for these with reference to the narrative descriptions. Lesson 1 was concerned with establishing methods (manifestation = T) for solving money problems (manifestation = FT) involving sharing in a given ratio, drawing on the image of money bags in order to represent a unitary share (manifestation = VT). This imagery was drawn on subsequently when Bernie was teaching-between-the-desks in Lessons 2 and 3, but this visual-technical manifestation of the topic was most emphasised in the first lesson of the series. Along with the emphasis on establishing formal methods, Lesson 1 featured a greater stressing of visual and technical elements of manifestation than the following lessons. All of the given-tasks in Lessons 3 and 4 involved solving material world problems (manifestation = FT), whereas Lesson 2 also featured a technical task in ratio simplification (manifestation = T); this accounted for the greater stressing of technical manifestation in Lesson 2. The difference in emphasis between Lesson 3 and 4 can be attributed to Bernie continuing to offer the moneybag image in Lesson 3.

<u>*Opportunities for learning afforded by the sequencing of manifestations*</u>

Referring back to the segment descriptions in Section 5.2, the general sequencing of Bernie's manifestation choices over time was:

from visual-technical manifestation

(the sharing phase of Segment 1)

to technical manifestation

(the practising phases of Segments 1 and 2, and the recapping phase of Segment 3)

and then to functional-technical manifestation

(the practising phase of Segment 3 and the majority of Segments 4 to 7)

I represent this general sequencing of manifestations made available to learners in figure 5.3/c. I associated this sequencing of manifestations with explicit opportunities for shifting the focus of learners' attention from image having to procedural fluency towards adaptive using. Visual-functional manifestation of ratio was not offered, which I associated with an absence of explicit opportunities for shifting the focus of learners' attention from primitive knowing to image having. This might have been expected as learners had encountered ratio as a topic in their previous schooling.

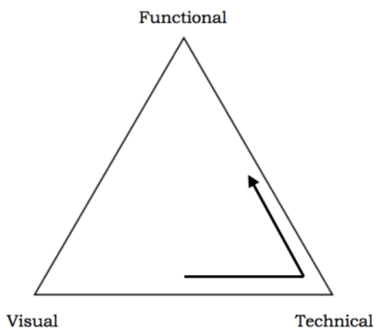


Figure 5.3/c
The sequencing of manifestation choices over time inferred from Bernie's Year 10 lesson series on ratio.

Analysis of teaching-between-the-desks revealed a slightly more complex picture that was not clearly evident from observation. My analysis suggested that during seatwork Bernie's teaching-between-the-desks often aligned with switching from the way content was manifested in the given-task. My reading of these actions was as a return to previously offered manifestations of ratio, which I associated with offering learners explicit opportunity to fold back (Pirie and Kieren, 1989) to previous foci of attention. In particular, when learners were tasked with working on utilisation problems in Lessons 2 and 3, Bernie's choices that came to action frequently aligned with returning to visual-technical manifestation of ratio, which I associated with offering learners explicit opportunity to fold back to secure image having (see figure 5.3/d). Whereas in Lesson 4, Bernie's enacted choices frequently aligned with returning to technical manifestation of ratio, which I associated with offering learners explicit opportunity to fold back to procedural fluency (see figure 5.3/d). This suggested a progression from Lesson 3 to Lesson 4, as in Lesson 4 there was less recourse to the imagery of moneybags in Bernie's teaching-between-the-desks. In Lesson 1, Bernie's enacted choices also aligned with returning to visual-technical manifestation of ratio, which I again associated with offering explicit opportunity to fold back and affording learners' attention to be focused on image having.

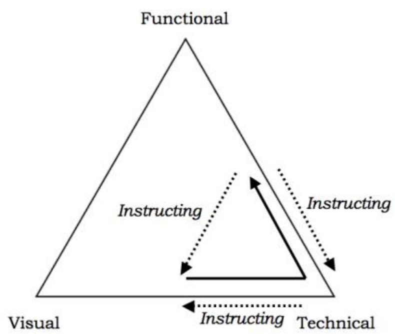


Figure 5.3/d
The sequencing of manifestation choices over time (solid line) and switches associated with teaching-between-the-desks (dotted line) inferred from Bernie's Year 10 series on ratio.

The dotted lines converging on the technical pole of figure 5.3/d emphasise how in the midst of given-tasks categorised as functional-technical Bernie provided explicit opportunities for the focus of learners' attention to be shifted towards procedural fluency. Therefore actions affording focusing learners' attention on procedural fluency were not limited to a single period of the lesson series, and a similar inference was drawn regarding property noticing. Opportunities for learners first to shift attention from image having towards procedural fluency were provided in Lesson 1. Yet such opportunities were not constrained to this lesson alone, as further opportunities were made available in the process of solving problems in Lessons 2 and 3.

5.3.3 – Types of exchanges Bernie offered across the lesson series

Overall characterisation of the types of exchanges offered

Throughout the lesson series, Bernie's choices regarding the way content was made available aligned with greater stressing of learner-content interaction (ζ = 0.53), but also teacher-learner (ζ = 0.28) and teacher-content (ζ = 0.19) interaction were emphasised (see figure 5.3/e).

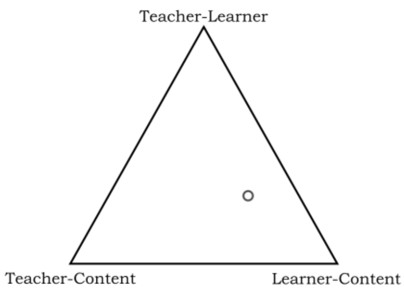


Figure 5.3/e
Series centre for interaction for Bernie's year 10 lesson series on ratio.

Table 5.3/c shows that this stressing of learner-content interaction played out in the classroom with seatwork being foregrounded over classwork. I inferred that a range of modes of teacher interaction afforded to a teacher by the didactical situation were realised in the context of Bernie teaching the year 10 class. These included Bernie teaching-from-the-front (κ = 21%) and between-the-desks (κ = 52%) as well as being-watchful (κ = 27%). Seatwork therefore accounted for 79% of teaching time, which was by far in excess of 30-40 % observed in Stigler and Hiebert's (1999) videotaped lesson study.

Table 5.3/c The spread of interaction choices across Bernie's Year 1o lesson series on ratio, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 383)	2	10	9	8	32	12	27

Note: The figures in the table are κ values (percentages).

<u>*Comparing the types of exchanges offered from lesson to lesson*</u>

Only subtle lesson-to-lesson differences were apparent in Bernie's interaction choices (see figure 5.3/f), with each lesson featuring a mix of classwork and seatwork. My analysis of the numerical data[32] suggested material differences between only two of the lessons: Lesson 2 was different to Lesson 3, as Bernie's choices aligned with a materially greater stressing of teacher-learner interaction in the former and learner-content interaction in the latter.

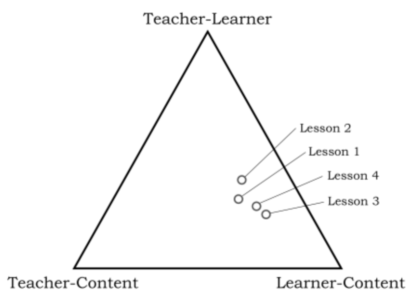


Figure 5.3/f
Lesson centres for interaction for Bernie's year 10 lesson series on ratio.

Lesson 3 and 4 followed a very similar format of task-feedback-task, so it was unsurprising that the quantitative analysis did not highlight differences. Lesson 2 followed a slightly different format (task-feedback-discussion-task), which resulted in more teaching-from-the-front (especially interaction = TL). Although Lesson 1 was very different to Lessons 3 and 4 with respect to manifestation, it too followed a task-feedback-task format (although teacher-content interaction was emphasised slightly more during the feedback) and so again it was not unexpected that the quantitative analysis did not highlight differences.

[32] See table A3/f in Appendix 3.

5.3.4 – Differences in how Bernie made content available to learners through teaching-from-the-front and through teaching-between-the-desks

Across the lesson series as a whole, I inferred that there were differences in what forms of content were made available to learners during classwork when compared to seatwork. My analysis of the numerical data[33] suggested that Bernie's choices to teach-from-the-front were associated with a material over-stressing of technical manifestation and a material under-stressing of functional manifestation (see figure 5.3/g).

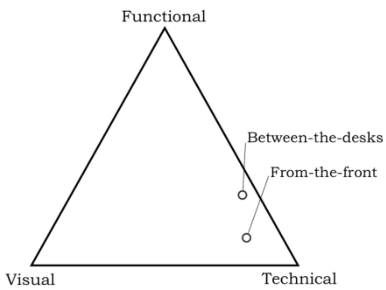


Figure 5.3/g
Comparison of manifestation centres for teaching-from-the-front and teaching-between-the-desks across Bernie's year 10 lesson series on ratio.

The first two lessons in the series featured the highest proportion of teaching-from-the-front, with formal methods for sharing in a given ratio arising from imagery in Lesson 1 (manifestation = VT) and a recap of formal methods in Lesson 2 (manifestation = T). Meanwhile, the solving of abstract and material world problems was more emphasised during seatwork (manifestation = T or FT). Lessons 3 and 4 largely involved learners using methods in order to solve contextual problems (manifestation = FT), whereas the methods used (manifestation = T) were emphasised from-the-front during feedback sessions.

Typically Bernie offered opportunities for learners to encounter visual-technical, technical and functional-technical manifestations first through problems set between-the-desks. Often these problems were subsequently addressed from-the-front, largely through discussion. I associated this with a sense of shared ownership of the mathematics. Bernie often provided direction, but from my observations I

[33] See tables A3/h and A3/i in Appendix 3.

concluded that typically this was sensitive to learners' perceived mathematical needs (informed by their responses to the problems set between-the-desks) and teaching plans were adapted.

Comparing forms of content made available during seatwork

My analysis of the categorical data for manifestation and interaction arising from my classification of half-minute intervals suggested that, across the lesson series, there was a medium association[34] between Bernie's role in teaching-between-the-desks and switches in how content was manifested. Table 5.3/d conveys that Bernie's choices to discuss-between-the-desks were associated[35] with focusing on the way content was manifested in the given-task (teacher-talk = Ø). Typically in this lesson series, such discussing-between-the-desks featured Bernie posing prompting or probing questions (such as excerpts $B_1$1.1 and $B_1$2.1 in section 5.2).

Table 5.3/d Bernie's role in teaching-between-the-desks and how mathematics was manifested as content through teacher-talk across the year 10 lesson series on ratio.

Teacher's role in teaching-between-the-desks	*Categorisation of teacher-talk* Ø	VT	T	Total	
Discussing-between-the-desks (LL)	**57.8**	<u>20.0</u>	22.2	100.0	(45)
Instructing-between-the-desks (CC/OO)	<u>18.5</u>	44.5	37.0	100.0	(146)
All teaching-between-the-desks	27.8	38.7	33.5	100.0	(191)
Chi Square	26.71 ***				
*** p<.001					

Note: Values in the table are percentages. Numbers in parentheses are the frequencies of half-minute intervals for each manifestation. Significantly larger values are highlighted in bold; significantly smaller values are underlined.

[34] Cramer's V = 0.374. It is very unlikely for this medium association not to be a feature of the shape of choices over time (χ^2 (2) = 26.71, p<.001).
[35] It is unlikely for this inferred association not to be a feature of the shape of choices over time (p<.001).

Table 5.3/d also conveyed that the majority of intervals involving Bernie teaching-between-the-desks were categorised as aligning with instruction and the majority of these intervals were categorised as aligning with switching from the way content was manifested in the given-task. In Section 5.3.2, I described how switching manifestation generally corresponded to returning to previously offered manifestations. Table 5.3/d shows that these returns to technical or visual-technical manifestation were typically made available through instruction.

5.4 – Case summary

Bernie's interaction choices generally aligned with emphasising learner-content interaction. There was little lesson-to-lesson variation in the types of exchanges offered. Seatwork featured least in Lesson 2 of the series and most in Lesson 3. There were differences in how Bernie made content available from-the-front when compared with seatwork: technical manifestation was emphasised more and functional manifestation less when teaching-from-the-front. Nearly two-thirds of the time allocated to seatwork featured Bernie teaching-between-the-desks. Discussion-between-the-desks generally afforded focusing learners' attention on the outer aspect of given-tasks. However Bernie's explaining-between-the-desks was associated with switching manifestation and afforded focusing learners' attention on technical or visual-technical inner aspects of given-tasks.

I inferred from my analysis that there were material differences in how Bernie made content available across the lesson series. I associated these differences with planned opportunities to shift the focus of learners' attention over time. Broadly the sequencing of Bernie's manifestation choices afforded learners progression from image having towards adaptive using. However, at times Bernie switched to previously offered visual-technical or technical manifestation of the topic through teaching-between-the-desks, affording a shift in the focus of learners' attention that reflects Pirie and Kieren's notion of folding back (Pirie and Kieren, 1989). These in-flight choices to teach-between-the-desks offered learners many pathways from image having towards adaptive using. I concluded that Bernie's manifestation choices over the lesson series offered multiple opportunities to focus learners' attention on property noticing, procedural fluency and adaptive reasoning.

CHAPTER 6

CASE STUDY 3

Chapter contents

6.1 – Background to the case

Bernie taught a series of lessons at Kings Meadow School to a Year 9 class (learners aged 13 and 14) set by prior mathematical attainment. Learners were, on average, predicted to achieve GCSE Mathematics grades in line with the national average by the age of 16. This was a series of three lessons on linear equations, being taught towards the end of the school year. All learners in the class had encountered the topic previously in Year 9, and Bernie was looking to build on this work. Figure 6.1/a illustrates how the lesson series was parsed into lessons and segments (segments were identified from Bernie's lesson plans).

Lesson 1	Segment 1 (11m)
	Segment 2 (7m)
	Segment 3 (33m)
Lesson 2	Segment 4 (29m)
	Segment 5 (31m)
Lesson 3	Segment 6 (7m)
	Segment 7 (13m)
	Segment 8 (34m)

Figure 6.1/a The parsing of Bernie's Year 9 lesson series on linear equations into lessons and segments.

Note: The length of each segment is given in minutes. Lessons were nominally 60 minutes long, but not all lesson time was devoted to mathematical activity. So the sum of the lengths of the segments making up a lesson may not equal 60 minutes.

6.2 – How content was made available at the segment and lesson level

This section contains a lesson-by-lesson and segment-by-segment report of the lesson series. There is a narrative, pre-analytical overview of each lesson including details of the segment(s) constituting that lesson. Each segment is represented in two ways. Firstly, there is a narrative account that provides a pre-analytical description, marking phases of the segment (which are labelled in **bold font**). This includes quotations from the teacher. The quoted talk that occurred between-the-desks is numbered, allowing subsequent referencing. Talk that occurred from-the-front is not numbered as no subsequent reference is made to it. This is followed by a structural account (presented in *italicised font*), which sets out chronologically the phases as sub-segments and categorises them using the categorical model of how content is made available. Recall that the region of the tri-polar space within which a sub-segment centre lies determines its categorisation. Each lesson report is concluded with a structural account of the lesson as a whole (presented in *italicised font*) using the language of the categorical model. This whole section has great significance for the validity of the study as it illustrates the alignment between the narrative description of lessons based on what I observed and the representation of the lessons through the lens of the categorical model.

LESSON 1

Bernie stated to me in advance of the lesson series that learners had worked previously on solving linear equations and they were generally able to solve equations featuring an unknown on one side of the equation if the solution was a positive integer. Bernie further stated that learners often solved such equations through ‘inspection’ rather than by a ‘balancing’ or ‘undoing’ method. In Lesson 1, Bernie provided opportunities for learners to appreciate that the solutions to linear equations need not only be positive integers (as previously encountered). Their solution space was extended to include all rationals (Segment 1 – explicitly including negative integers). Opportunities to reinforce this were then provided by learners generating their own linear equations, starting from a rational solution (Segment 2 and 3). Sharing these equations around the class provided opportunities to solve equations.

Segment 1 (11 minutes)

Bernie provided examples of linear equations featuring an unknown on one side of the equation, some of which had integer solutions others that did not. These examples afforded learners the opportunity to notice that some equations could be solved using familiar methods, while others could not:

$x + 11 = 20$	$3x + 1 = 15$
$4x + 3 = 27$	$2t + 8 = 3$

(Examples of equations[36], Bernie’s Year 9 Linear Equations Lesson 1)

[36] These four examples were representative of the seven equations Bernie provided. I have arranged them in two columns for the benefit of the reader, with the ones on the left likely to be solvable through methods familiar to all learners in the class. Bernie presented the equations in a single column, so offered no such cues to categorisation.

Learners were tasked initially with **solving independently**, albeit with occasional comments from Bernie:

> **B21.1:** So is that one you can't solve then you think?
> **B21.2:** So you'll either have to decide can we solve it or not?
> **B21.3:** Someone is saying it's impossible... Okay, we'll see.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

Subsequently Bernie selected learners to feedback their solutions to the whole class. In this phase, Bernie probed learners in order to draw out their methods:

> Can you solve the first one ($x + 11 = 20$), and if you can what is the answer? *Learner comment.* x is 9. How do you know it is, what did you do? *Learner comment.* So you thought about doing the opposite of adding 11 and subtracting it and you got the 9.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

> Number 4? ($4x + 3 = 27$) *Learner comment.* Right, how did you work out that it was 6? What was your thinking? *Learner comment.* So what did you think - I needed a number to fit in there, let me try different numbers out and 6 worked? *Learner comment.* Is that your method? Yes, ok. That's fine. And so x is 6 in that one. Is every happy with that? Because 4 times 6 is 24, plus 3 is 27.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

The method offered under probing for solving $x + 11 = 20$ seemed to align with an undoing method ('so you thought about doing the *opposite* of adding 11' – my emphasis), suggesting that such methods were familiar to some learners. However, the method offered under probing for solving $4x + 3 = 27$ seemed to align with the inspection method Bernie had anticipated.

During the feedback phase, Bernie also emphasised the categorisation of equations based on the nature of their solutions:

> What about the next one? ($3x + 1 = 15$) *Learner comment*. You haven't finished it. [You think] it's impossible. *Learner comment*. It's not possible if you... unless you use a decimal. So is that a valid thing to do then in an equation, to use a decimal number? *Group response*. Yeah? I will tell you the reasons why we are doing this work today is... we did do some work on these equations if you remember back in term 2. And I did an assessment and all of you got the questions right where the answer was a whole number. That wasn't a problem. If it was a positive whole number, you found them straightforward. When you got to ones like this where the answer isn't as obvious you couldn't do it, you weren't able to do it. Apart from a very few of you.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

> Anybody done number 6? ($2t + 8 = 3$) *Learner comment*. Yes? What did you get for that one? *Learner comment*. Minus 2.5. So you were able to solve it even though it came up with a negative answer? Actually negative and decimal as well.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

I parsed Segment 1 into two sub-segments relating to the distinct independent solving and feedback phases highlighted above. The independent solving phase (interaction = LC) involved learners discerning what was familiar and what was not (manifestation = VF). The feedback phase included some discussion with the whole class about learners' methods but was largely Bernie explaining-from-the-front (interaction = TT) about the categorisation of equations based on their solutions (manifestation = VT).

Segment 2 (7 minutes)

Bernie provided opportunities for learners to appreciate how a linear equation could be built up from a solution and how reversing the process offered a method for solving equations. Throughout this segment Bernie **modelled** the building up and solving of linear equations. At first, an equation with a positive integral solution was generated (see figure 6.2/a):

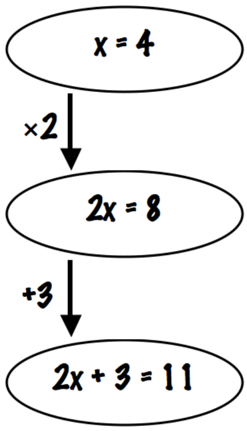


Figure 6.2/a
Screen shot of how Bernie built up a linear equation from the solution x = 4 in Year 9 Linear Equations Lesson 1.

> I'm going to start from the solution and build up the equation. Okay, so I'm starting with x =4 as my solution. I'm going to times by 2. Because it's an equation, to keep the balance right I've got to times both sides of my equation by 2, so I end up then with 2x is equal to 8. I've made our solution, our original equation, more complicated by timsing by 2. Then, I'm going to add on 3. And again, I've got to do that to both sides. So the left hand side will become 2x+3 and the right hand side is 11 (*see figure 6.2/a*). Okay. Now what you're going to get presented with to solve is something looking like this (*indicating* $2x + 3 = 11$) and you've got to get back to this (*indicating* $x = 4$). What we are doing here is a reverse. So if I start here and go up with my arrow (*see figure 6.2/b*), I've got to do the opposite of the adding 3 that we did here. *Learner comment.* Take away three... Then if I start with my equation, I've then got to take away 3 from both sides of my equation to get this one and then divide both sides by 2 and I get back to my solution.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

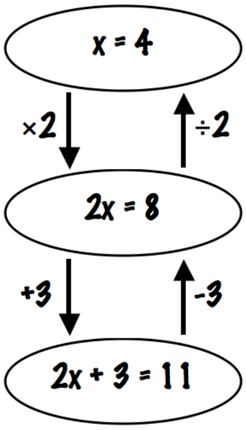


Figure 6.2/b
Screen shot of how Bernie built up a linear equation from the solution $x = 4$ and then reversed the steps in order to demonstrate a method for solving in Year 9 Linear Equations Lesson 1.

Bernie modelled two further examples: the first ($3x - 1 = 9.5$) built from a positive non-integral solution ($x = 3.5$) and the second ($4x + 6 = 2$) built from a negative integral solution ($x = -1$).

Segment 2 featured a single modelling phase (interaction = TT), which I did not parse into sub-segments. In this phase, techniques for generating and solving linear equations were foregrounded along with some emphasis on diagrammatic representations of these techniques (manifestation = T).

<u>*Segment 3 (33 minutes)*</u>

Having modelled generating and solving linear equations in the previous segment, Bernie provided opportunities for learners to **generate** their own examples of linear equations, with each equation being written on a card. These cards were then passed around the class so as to provide opportunities for learners to practise **solving** linear equations.

During the generation phase, Bernie spent time teaching-between-the-desks. This included the following four excerpts of teacher talk, the first two of which I took as Bernie posing prompting questions and focusing on the generation task at hand:

> **$B_2$3.1:** *x* is equal to what? Decimal or negative number. Or both… Come on, anything!
>
> **$B_2$3.2:** What could you do to both sides of that equation? You could times by…?
>
> (Bernie, Year 9 Linear Equations Lesson 1)

I took this third excerpt as Bernie engaging in discussion, validating the technical steps the learner had taken:

> **$B_2$3.3:** (*Checking an example a learner has generated*) *x* is 2.5, so you've times by 2, 2x is 5. Take away 1. Very nice. I'm impressed!
>
> (Bernie, Year 9 Linear Equations Lesson 1)

The fourth excerpt I took as a mix of Bernie posing prompting questions and telling, switching to the technical steps that might generate an example of a linear equation.

> **B$_2$3.4:** So what did you do? What did you start with for this one? What was your solution? *Learner comment. x* equals 2.5? So what did you then decide to do? Times by 4? You've got 4*x* there, did you times by 4? *Learner comment.* Right, so that would give you 4*x* = 10 as you are timesing both of those by 4. Then it looks to me that you are taking away 7.5… So 4*x* - 7.5 = 2.5. And that's your equation there.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

During the solution phase, Bernie also spent time explaining-between-the-desks, returning to how the examples were generated (in both excerpts below) and emphasising specific technical steps (in the second excerpt):

> **B$_2$3.5:** Remember how you built up your equations? You times by something, then you added or subtracted something - remember? So you're going in the opposite way around now to solve this one.
>
> **B$_2$3.6:** Do you remember how we built this up? We started with this, x is -4. We then timesed it by 2, so instead of having x we had 2x is equal to -8. We then added on 10 didn't we? So going backwards, you'd subtract 10… and you'd divide by 2… How's that equation (2*x* + 10 = 4) been built up? Did they add the 10 before they timesed by 2? No. Can I borrow your pen? So imagine (*drawing a diagrammatic representation*) they started down here with x equals something, which is what you are trying to find out. You can see to get to here they timesed by 2 haven't they? So they've got 2x equals something - you don't know what it is yet. Okay? And then to get to the 2x+10 they must have added on 10.
>
> (Bernie, Year 9 Linear Equations Lesson 1)

I parsed Segment 3 into three sub-segments: the first and third relating to the distinct generating and solving phases respectively highlighted above, and the second a crossover phase in which some learners continued to generate examples while others were solving equations. Each phase featured an emphasis on Bernie being-watchful (interaction = LC). The generating phase (manifestation = VT) included teaching-between-the-desks, some of which aligned with focusing on the visual-technical manifestation of the topic associated with the given task through discussion as in excerpts $B_2$3.1 and $B_2$3.2 (interaction = LL) and some which aligned with switching to a more technical manifestation of the topic (teacher-talk = T) through discussion (interaction = LL, as in excerpt $B_2$3.3) or explanation (interaction = OO, as in excerpt $B_2$3.4). The crossover and solving phases largely involved an emphasis on methods (manifestation = T) and featured Bernie explaining-between-the-desks (interaction = OO), some of which aligned with switching to a more visual-technical manifestation of the topic as in excerpt $B_2$3.5 (teacher-talk = VT) and some which aligned with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpt $B_2$3.6.

<u>Structural account of Lesson 1</u>

Table 6.2/a The spread of manifestation choices across Lesson 1 of Bernie's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 1 (N = 97)	0	0	68	8	19	0	5

Note: The figures in the table are κ values (percentages).

Table 6.2/a shows the spread of manifestation choices across the lesson based on my analysis. Bernie offered a mix of root problems (manifestation = VF), opportunities for learners to categorise and generate their own examples of equations (manifestation = VT) and a particular emphasis on formal methods of solving equations (manifestation = T) in Lesson 1. These manifestation choices aligned with greater stressing of technical manifestation of the topic (ζ = 0.66), but with some stressing of visual manifestation (ζ = 0.29).

Table 6.2/b The spread of interaction choices across Lesson 1 of Bernie's Year 10 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 1 (N = 97)	10	10	9	3	22	14	32

Note: The figures in the table are κ values (percentages).

Table 6.2/b shows the spread of interaction choices across the lesson based on my analysis. Bernie made content available at times through written questions and orally given instructions coupled with being-watchful (interaction = LC) or teaching-between-the-desks (particularly interaction = OO or LL), but at others teaching-from-the-front (interaction = TC, TT or TL). These interaction choices aligned with learner-content interaction being stressed (ζ = 0.51) over teacher-learner (ζ = 0.27) and teacher-content (ζ = 0.22) interaction.

Some of Bernie's teaching-between-the-desks in this lesson aligned with focusing on the visual-technical or technical manifestation of the topic associated with the given-task (teacher-talk = Ø) largely through discussion, while some aligned with switching to a more visual-technical (teacher-talk = VT) or technical (teacher-talk = T) manifestation of the topic through explanation.

LESSON 2

In Lesson 2, Bernie provided opportunities for learners to develop their linear equation solving skills further (Segment 4), including generating examples of – and solving – equations with an unknown on both sides (Segment 5).

Segment 4 (29 minutes)

Bernie provided opportunities for learners to become more familiar with methods for solving linear equations with rational solutions. A worked solution (to $7x + 4 = 5$) for learners to **read** through individually was offered before Bernie **talked** through the steps of the solution:

> Following yesterday's method, we take 4 away from both sides leaving 7x=1. Then divide both sides by 7 leaving x is equal to… Then you've got the problem of 1 being divided by 7. Now if you've got a calculator you can write the answer as 0.143, that's rounded to 3 decimals places. Or, you can just leave the answer as a fraction: 1 over 7 is the same as 1 divided by 7.
>
> (Bernie, Year 9 Linear Equations Lesson 2)

Bernie then **recapped** the links between the methods of generating examples of and solving equations emphasised in Lesson 1:

> Remember we started with a solution ($x = 4$). We timesed that solution by 2 both sides to get $2x = 8$. We then added 3 onto both sides to get this equation ($2x + 3 = 11$). The point of this was to help you to be able to take an equation apart so you can solve it. So I said to you that… you do the opposite of adding the 3, which is to take 3 away. Opposite of timesing by 2 is to divide by 2, gets you back to the solution again, okay? So yesterday you were creating equations… and then you were learning to take those apart… [to get] back to the solution.
>
> (Bernie, Year 9 Linear Equations Lesson 2)

Learners were then tasked with continuing to **solve** the equations generated during Segment 3. During this solving phase, Bernie spent no time teaching-between-the-desks but rather was engaged in managing the organisation of the task.

I parsed Segment 4 into four sub-segments relating to the distinct reading, talking, recapping and solving phases highlighted above. An emphasis on methods for solving linear equations (manifestation = T) featured throughout the segment. The initial reading phase involved Bernie being-watchful (interaction = LC), while the talking and recapping phases involved Bernie telling-from-the-front (interaction = TC). The solving phase again involved Bernie being-watchful (interaction = LC) and included no teaching-between-the-desks.

<u>Segment 5 (31 minutes)</u>

Bernie provided opportunities for learners to appreciate how a linear equation with unknowns on both sides could be built up from a solution and how reversing the process offered a method for solving equations. At first Bernie **modelled** the building up of an equation (see figure 6.2/c):

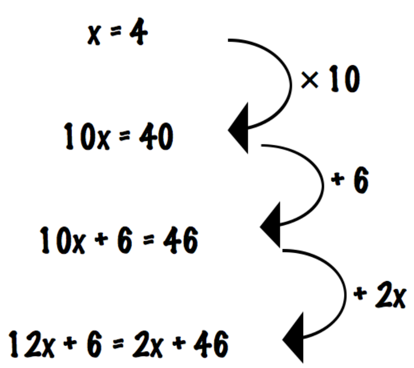


Figure 6.2/c
Screen shot of how Bernie built up a linear equation with unknowns on both sides from the solution $x = 4$ in Year 9 Linear Equations Lesson 2.

Bernie then stated how the steps could be reversed in order to solve the equation $12x + 6 = 2x + 46$:

> Using yesterday's method, we're going to do the opposite of adding the $2x$ to get back from here ($12x + 6 = 2x + 46$). The opposite of adding the $2x$ is to take away the $2x$ to get back to this stage ($10x + 6 = 46$). The opposite then to get back to that stage ($10x = 40$) is to take away 6. Then the opposite there to get back to our solution is to divide by 10.
>
> (Bernie, Year 9 Linear Equations Lesson 2)

Having given learners the opportunity to **copy** down this example, Bernie **guided** learners through the steps they could take in order to generate their own example of a linear equation with unknowns on both sides:

> In your book, I want you to write down a number that is either negative or decimal or both. So I'm going to write down $x = 6.2$. But you choose any number that is either negative, decimal or both… Right, next thing you are going to do now is you are going to multiply both sides of your equation with a number. I'm going to multiply by 4, so I'm going to get $4x$ is equal to… 24.8… Now I want you to add or subtract a number form your equation, so for example I'm going to add on 10… Now I would like you… to add or subtract a multiple of x, what I mean by that is for example, I am going to add on $3x$ to both sides of my equation.
>
> (Bernie, Year 9 Linear Equations Lesson 2)

Bernie then provided opportunities for learners to **generate** their own examples of linear equations with unknowns on both sides, with each equation being written on a card. These cards were then passed around the class so as to provide opportunities for learners to practise **solving** linear equations of this type.

During the generating and solving phases, Bernie spent time teaching-between-the-desks. The generating phase included the following excerpt of teacher talk as Bernie checked an example a learner had generated, which I took as Bernie emphasising the precision of algebraic manipulation called for at each step:

> **B$_2$5.1:** x is -10. Times 5... Where's your $5x$ gone? You've timesed by 5 so you should have $5x$ = -50... Then you are taking away 50 from both sides, so you'll have $5x$-50 is equal to -50 take away another 50 which is -100, okay? Then you are adding on and I wanted a multiple of x here. Are we adding on $50x$? *Learner comment.* So on the left hand side you've got $50x$ and $5x$ is $55x$ minus 50 is equal to -100 plus $50x$, and that's your equation ($55x - 50 = -100 + 50x$).
>
> (Bernie, Year 9 Linear Equations Lesson 2)

The solving phase included the following excerpt of teacher talk, which I took as Bernie returning to the image of building up an equation:

> **B$_2$5.2:** (*Solving* $6x + 3.5 = 2x + 29.5$) The last thing they would have done would be to take $2x$ away from both sides. So you do the opposite of taking away $2x$ which is what? *Learner comment.* Right, so $4x + 3.5$ is equal to 29.5. Let's write that down in here... Now the stage before that was what? *Learner comment.* Adding the... No we've dealt with that bit now. The next stage working backwards would have been adding the 3.5 on. So they've got to do the opposite of adding the 3.5 which is to take away the 3.5 isn't it? So $4x$ will be equal to that (29.5) take away 3.5 which is 26 isn't it? And then the last thing is to divide by 4 and you've got their answer.
>
> (Bernie, Year 9 Linear Equations Lesson 2)

I parsed Segment 5 into five sub-segments relating to the distinct modelling, copying, guiding, generating and solving phases highlighted above. An emphasis on methods for solving linear equations with unknowns on both sides (manifestation = T) featured throughout the segment. The modelling phase involved Bernie telling-from-the-front (interaction = TC), while the copying phase involved Bernie being-watchful (interaction = LC). The guiding phase involved a mix of Bernie telling-from-the-front and being-watchful, akin to telling-between-the-desks (interaction = CC). Bernie was teaching-between-the-desks throughout the generating phase, featuring a mix of telling and discussing (interaction = OO), aligning with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpt $B_2$5.1 (where interaction = CC). The solving phase was made available largely through Bernie being-watchful (interaction = LC). This phase included some teaching-between-the-desks, which aligned with switching to a more visual-technical manifestation of the topic (teacher-talk = VT) as in excerpt $B_2$5.2 (interaction = OO).

<u>*Structural account of Lesson 2*</u>

Table 6.2/c The spread of manifestation choices across Lesson 2 of Bernie's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 2 (N = 92)	0	0	88	0	12	0	0

Note: The figures in the table are κ values (percentages).

Table 6.2/c shows the spread of manifestation choices across the lesson based on my analysis. Throughout Lesson 2, formal methods of solving equations were emphasised (manifestation = T) with Bernie also providing opportunities for learners to generate their own examples of equations with unknowns on both sides (manifestation = VT).

These manifestation choices aligned with greater stressing of technical manifestation of the topic (ζ = 0.85) over visual manifestation (ζ = 0.15).

Table 6.2/d The spread of interaction choices across Lesson 2 of Bernie's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 2 (N = 92)	21	8	3	10	15	14	29

Note: The figures in the table are κ values (percentages).

Table 6.2/d shows the spread of interaction choices across the lesson based on my analysis. Content was made available through a mix of Bernie remaining watchful (interaction = LC), teaching-between-the-desks (interaction = LL, OO or CC), and teaching-from-the-front (particularly interaction = TC). These interaction choices again aligned with learner-content interaction being stressed (ζ = 0.50) over teacher-content (ζ = 0.32) and teacher-learner (ζ = 0.18) interaction.

Bernie's teaching-between-the-desks in this lesson generally aligned with switching either to a more technical manifestation (teacher-talk = T) or to more visual-technical manifestation (teacher-talk = VT) of the topic through instruction.

In comparison to the previous lesson in the series, technical manifestation of the topic were emphasised more and visual-technical manifestation less in Lesson 2 through the solution of linear equations. While there was a similar balance between teaching-from-the-front and between-the-desks in both lessons, there was a greater stressing of telling-from-the-front in Lesson 2.

LESSON 3

In Lesson 3, Bernie provided opportunities for learners to recall and rehearse how to expand brackets (Segment 6) before going on to provide opportunities for learners to develop their linear equation solving skills further, including solving equations which featured brackets (Segment 7).

Segment 6 (7 minutes)

Bernie provided a **demonstration** of how a grid method could be used for expanding brackets, using the example 3(9a + 9) (see figure 6.2/d):

> The number outside the bracket goes here. What's inside the bracket – the 9a and the 9 – go here. We're multiplying, so 3 times 9a is 27a. 3 times 9 is 27. You end up with 27a + 27.
>
> (Bernie, Year 9 Linear Equations Lesson 3)

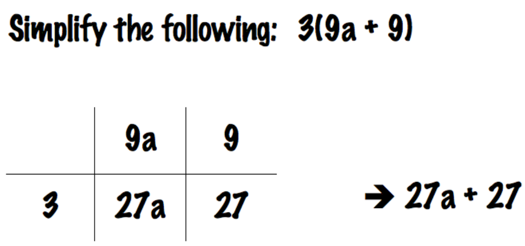


Figure 6.2/d
Screen shot of the grid method for expanding brackets that Bernie offered in Year 9 Linear Equations Lesson 3.

Bernie then set an **exercise** for learners to attempt individually and without teacher support, featuring expressions to be expanded such as 3(4a + 4) and 5(4a – 6).

I parsed Segment 6 into two sub-segments relating to the distinct demonstration and exercise phases highlighted above. The demonstration phase involved a diagrammatic representation of a method (manifestation = VT), made available through Bernie telling-from-the-front (interaction = TC). The exercise phase emphasised technical manifestation of the topic (manifestation = T), made available through Bernie being-watchful (interaction = LC).

Segment 7 (13 minutes)

Bernie provided opportunities for learners to **recall** the methods of generating and solving linear equations with an unknown on one side met in Lesson 1 through a whole class discussion. Bernie repeatedly emphasised actions and inverse actions through posing prompting questions to learners, as in the following excerpts:

> x is equal to 5 is our solution. So to get from here ($x = 5$) to here ($7x = 35$), what have I done? … So to get back again what would I have to do?
>
> To get from here ($7x = 35$) to here ($7x - 4 = 31$) I took away 4 so to get from here ($7x - 4 = 31$) to here ($7x = 35$) what have I got to do?
>
> So if I'm at that equation ($7x - 4 = 31$) and I want to find the solution, what have I got to do to get from here ($7x - 4 = 31$) all the way to there ($x = 5$)? *Learner comment.* Add 4 [and then] divide by 7. Good one.
>
> (Bernie, Year 9 Linear Equations Lesson 3)

Through posing a series of probing questions and telling-from-the-front, Bernie then **demonstrated** a method for solving linear equations featuring brackets:

> (*Equation:* $5(4x + 3) = 29$) Anybody got any ideas what we would do first? *Learner comment.* Expand the brackets first. So expand the brackets for me.
>
> So then once we've expanded the brackets, we're then at the same stage as we were in with the other equations… So what do we do next?
>
> Ok, so remembering what we were just doing… we take away 15 first. So take away 15 from both of those ($20x + 15$ and 29), what do I get?
>
> From here we've got $20x = 14$ and we're trying to find out what x is…
>
> What would we have done to get from there (x) to there ($20x$)? *Learner comment.* We'd have timesed by 20… So to get from here (x) to here ($20x$) we would have had to times by 20 so to get from there ($20x$) to there (x) we've got to divide by 20.
>
> (Bernie, Year 9 Linear Equations Lesson 3)

I parsed Segment 7 into two sub-segments relating to the distinct recalling and demonstrating phases highlighted above. An emphasis on methods for solving linear equations (manifestation = T) featured throughout the segment. The recalling phase involved Bernie discussing-from-the-front (interaction = TL), whereas the demonstrating phase involved a greater emphasis on Bernie explaining-from-the-front (interaction = TT).

<u>Segment 8 (34 minutes)</u>

Bernie provided learners with a worksheet exercise on solving linear equations involving brackets. This included examples such as:

$3(2x - 1) = 21$
$3(2x + 9) = 39$
$3(3x - 2) = 66$
$2(4x - 5) = 14$

(Worksheet[37], Bernie's Year 9 Linear Equations Lesson 3)

Bernie spent time throughout the segment teaching-between-the-desks. This included the following excerpt of teacher talk, which I took as Bernie engaging in discussion, aligning with focusing on the given task:

> **$B_2$8.1:** Are you saying 12 is the answer for that question [$3(2x - 1) = 21$]? *Learner comment.* Right okay, where does 12 come from? Do you want to explain to me how you got it?
>
> (Bernie, Year 9 Linear Equations Lesson 3)

[37] All of these examples in this exercise were of the form $a(bx + c) = d$, with d divisible by a. Therefore rather than expanding brackets first, both sides of the equation could have been divided by a. Bernie continued to emphasise the expansion method through this segment, although there was a suggestion that some learners were able to solve some of these equations by 'inspection'. For example, from the transcript it was evident that a learner 'thinks' the solutions to $2(4x - 5) = 14$ was 3 (I conjectured that the learner could see that $4x - 5 = 7$ and that $4 \times 3 - 5 = 7$), but claimed not to know how to solve $3(3x - 2) = 66$.

However, much of the teaching-between-the-desks in this segment I took as involving Bernie instructing, with the choices that came to action aligning with repeatedly switching between an overview of the method and technical details of algebraic manipulation, as in the following excerpt:

> **$B_2$8.2:** First thing I want to do is to get rid of the brackets, so I want to expand that expression [$3(2x - 1)$] and replace it with another expression that hasn't got brackets in. So can you expand that bracket for me? *Learner comment.* No. The first thing we did in the lesson we were expanding brackets weren't we.... and I said use the grid method... *Learner comment.* No not quite. So you've got 3 outside your bracket that you're timesing... 2x you're right is 6x, but then if you do 3 times -1 what do you get? *Learner comment.* No. What's 3 times 1? Ignore the minus. *Learner comment.* 3 times 1 is 3. Stick a minus in front. So that bit you can replace with $6x$ - 3 equals 21. Alright? That 21 stays the same because all you've done is to replace an expression with brackets with something that's the same but without the brackets in. So can you now go on and solve that? Can you unpick that equation? *Learner comment.* What we are trying to get to is 6x equals something. *Learner comment.* Yes you add it. Because to get from there ($6x$) to there ($6x - 3$) you would have subtracted 3 so going back this way we'd add 3. Now we're going to x equals aren't we? From x to 6x we'd times by 6, wouldn't we? [Doing the] opposite, 24 divide by 6 is? *Learner comment.* There's your answer.
>
> (Bernie, Year 9 Linear Equations Lesson 3)

Segment 8 featured a single sub-segment with an emphasis on methods for solving linear equations (manifestation = T) throughout and made available through Bernie largely being-watchful (interaction = LC). There were instances of teaching-between-the-desks, some of which aligned with focusing on the given task through discussion as in excerpt $B_2$8.1 (interaction = LL), but much more which aligned with switching to slightly a more visual-technical manifestation of the topic (teacher-talk = VT) through explaining-between-the-desks (interaction = OO) as in excerpt $B_2$8.2.

<u>*Structural account of Lesson 3*</u>

Table 6.2/e The spread of manifestation choices across Lesson 3 of Bernie's year 10 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 3 (N = 100)	0	0	94	0	6	0	0

Note: The figures in the table are κ values (percentages).

Table 6.2/e shows the spread of manifestation choices across the lesson based on my analysis. In Lesson 3, formal methods of solving equations were again emphasised (manifestation = T) with Bernie also offering diagrammatic representations of methods for learners (manifestation = VT). As in Lesson 2, these manifestation choices aligned with greater stressing of technical manifestation (ζ = 0.89) of the topic over visual manifestation (ζ = 0.11).

Table 6.2/f The spread of interaction choices across Lesson 3 of Bernie's year 10 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 3 (N = 100)	8	11	8	6	37	11	19

Note: The figures in the table are κ values (percentages).

Table 6.2/f shows the spread of interaction choices across the lesson based on my analysis. A smaller proportion of time to the other lessons in the series was spent teaching-from-the-front (manifestation = TC, TT or TL). Content was also made available through given-tasks coupled with Bernie being-watchful (interaction = LC), discussing-between-the-desks (interaction = LL) and particularly explaining-between-the-desks (interaction = OO). These interaction choices aligned with greater stressing *of learner-content interaction (ζ = 0.46), but with some emphasis on teacher-content (ζ = 0.26) and teacher-learner (ζ = 0.28) interaction too.*

Bernie's teaching-between-the-desks in this lesson generally aligned with switching to a more visual-technical manifestation of the topic (teacher-talk = VT) through instruction. In comparison to the previous lessons, there was a greater emphasis on explaining-between-the-desks in Lesson 3. This resulted in a similar balance between the three interaction elements as Lesson 1.

6.3 – What differences are discernible in how Bernie made content available to learners over time?

In this section I present the findings from my statistical analysis in order to address the following research questions:

> RQ1a: What can be inferred from analysis of teacher-talk and given-tasks about the particular forms of content Bernie made available to learners across the lesson series as a whole?
>
> RQ1b: What can be inferred from analysis of teacher-talk and given-tasks about between-lesson differences in the forms of content Bernie made available to learners?
>
> RQ1c: What can be inferred from analysis of teacher-talk and given tasks about the types of exchanges Bernie offered across the lesson series?
>
> RQ1d: What can be inferred from analysis of teacher-talk and given-tasks about the forms of content Bernie made available to learners during classwork and seatwork?

I present findings through a combination of figures of tri-polar spaces, tables and text. As previously stated, I place a particular emphasis on using tri-polar spaces since these are a distinctive feature of my analytical approach. Having presented results, I discuss the response to each question in relation to the particular range of opportunities for learning afforded during the lesson series.

6.3.1 – Particular forms of content Bernie made available across the lesson series

Bernie's manifestation choices over time (figure 6.3/a) aligned with greater stressing of technical manifestation (ζ = 0.80). Visual manifestation was also emphasised at times (ζ = 0.18), but the emphasis placed on functional manifestation was negligible (ζ = 0.02).

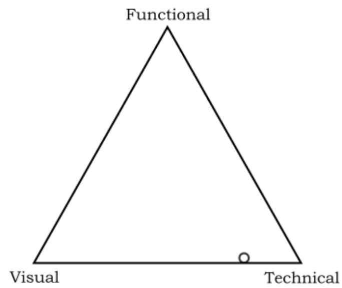


Figure 6.3/a
Series centre for manifestation for Bernie's year 9 lesson series on linear equations.

This stressing of the technical element played out in the classroom through Bernie offering learners technical, visual-technical and visual-functional-technical manifestations of the topic (see table 6.3/a), accounting for 97% of lesson time.

Table 6.3/a The spread of manifestation choices across Bernie's Year 11 lesson series on geometry, based on the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 289)	0	0	83	3	12	0	2

Note: The figures in the table are κ values (percentages).

In the discussion that follows, I consider each manifestation category of my model and discuss the likely implications for learning afforded by it within the context of Bernie's Year 9 lesson series on linear equations. As well as viewing these affordances through the lens of shifts in the focus of learners' attention described in Chapter 2 (particularly as summarised in table 2.3/c) – and because this was a lesson series on algebra – I will also draw some parallels with Kieran's (1996) classification of algebraic activity. This model draws a distinction between *generational activities*, *transformational activities*, and *global meta-level activities* (Kieran, 1996:141-142).

Visual manifestation

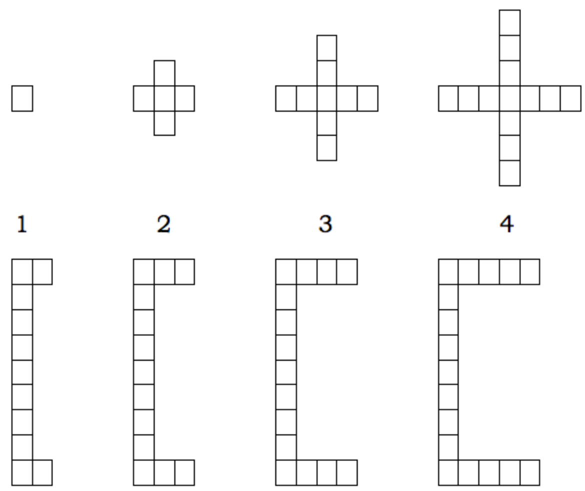
Some aspects of generational activities (Kieran, 1996) are associated with visual manifestations afforded by algebraic topics. These activities include 'expressions of generality arising from geometric patterns' and 'expressions of the rules governing numerical relationships' (Kieran, 1996:142). Therefore algebra affords visual manifestation to the teacher. There were examples of these affordances being realised by Bernie in this lesson series when offering visual-functional and visual-technical manifestations of the topic.

Offering a range of linear equations, some of which could be solved using familiar methods and others that could not, afforded shifting the focus of learners' attention from primitive knowing (knowing how to solve equations with positive integral solutions, maybe by 'inspection') to image having (a sense of there being equations that could not be solved through familiar methods, thus prompting a need for refined or additional methods). This was an example of visual-functional manifestation.

Bernie offered diagrammatic representations of methods, which were examples of visual-technical manifestation. The predominant image Bernie offered was of the processes of generating examples of equations and solving equations as being inverses, and representing these processes as flow diagrams. These were opportunities that afforded shifting the focus of learners' attention from image having (the image of equations as flow diagrams) towards procedural fluency (solving equations using flow diagrams). But other opportunities also afforded folding back, perhaps to consolidate, from insecure procedural fluency (some familiarity with solving equations) to a more secure property noticing (generating examples of equations and their solutions).

No purely visual manifestation of the topic was offered in this lesson series. Such visual manifestation would have provided opportunities for learners to heighten their sense of equations. For example, the following is based on Griffin and Hirst (1989:15):

Consider these two sequences of pictures:



Is there a point where the corresponding terms in these two sequences contain the same number of squares?

Technical manifestation

Parallels may be drawn between some transformational activities (Kieran, 1996) and technical manifestations afforded by algebraic topics. Such activities include 'collecting like terms, factoring, expanding' and 'solving equations' (Kieran, 1996:142). There were examples of these affordances being realised by Bernie in this lesson series when offering visual-technical (as discussed above) and technical manifestations of the topic. No functional-technical manifestation was made available.

Technical manifestation, such as algebraic manipulation and methods of solving linear equations, was made available across the lesson series, but particularly in Lessons 2 and 3. Thus Bernie provided extensive opportunities that afforded procedural fluency.

Functional manifestation

Certain global meta-level activities (Kieran, 1996) are associated with functional manifestations afforded by algebraic topics. Such activities include 'problem solving, modelling … [and] proving' (Kieran, 1996:142). There were examples of these affordances being realised by Bernie in this lesson series when offering visual-functional manifestation (as discussed above). No purely functional or functional-technical manifestations were made available.

<u>*6.3.2 – Between-lesson differences in how content was made available to learners*</u>

Some lesson-to-lesson differences were apparent in Bernie's manifestation choices (see figure 6.3/b). My analysis of the numerical data[38] suggested Lesson 1 was different to the other two lessons, most notably because Bernie's choices that came to action aligned with a materially greater stressing of visual manifestation of the topic (and functional manifestation, as these were absent in Lessons 2 and 3). Lessons 2 and 3 were not materially different. Figure 6.3/b also conveyed that from lesson to lesson there was an underlying trend in Bernie's manifestation choices aligning with increasing emphasis on technical manifestation and decreasing emphasis on visual manifestation of the topic.

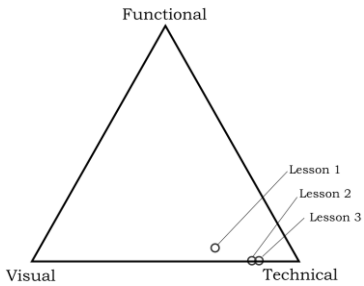


Figure 6.3/b
Lesson centres for manifestation for Bernie's year 9 lesson series on linear equations.

The difference between Lesson 1 and Lessons 2 and 3 can be accounted for with reference to the narrative descriptions of Section 6.2. As well as the emphasis on formal methods (manifestation = T) throughout, each lesson featured visual representations of methods (manifestation = VT) from the function machines of Lessons 1 and 2 to the grids for expanding brackets in Lesson 3. However, categorisation of linear equations based on whether or not their solutions were positive integers (manifestation = VF/VT) was also stressed in Lesson 1.

[38] See table A4/c in Appendix 4.

Opportunities for learning afforded by the sequencing of manifestations

Referring back to the segment descriptions in Section 6.2, following an initial emphasis on visual-functional manifestation (the independent solving phase of Segment 1) the general sequencing of Bernie's manifestation choices over time was cycles of switching between visual-technical (use of diagrammatic representations of methods and generation of examples of linear equations) and technical manifestation (solving linear equations). I represent this general sequencing of manifestations made available to learners in figure 6.3/c. I associated this sequencing of manifestations with explicit opportunities for shifting the focus of learners' attention from primitive knowing through image having to procedural fluency.

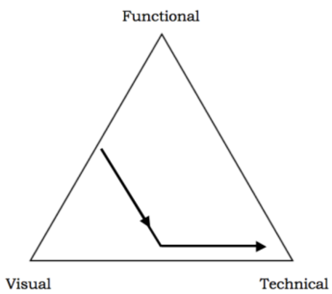


Figure 6.3/c
The sequencing of manifestation choices over time inferred from Bernie's Year 9 lesson series on linear equations.

My analysis suggested that during seatwork Bernie's teaching-between-the-desks would sometimes align with focusing on the way content was manifested in the given-task. At other times Bernie returned to previously offered manifestations of the topic, which I associated with offering learners explicit opportunity to fold back (Pirie and Kieren, 1989) to previous foci of attention. In particular, Bernie's enacted choices frequently aligned with returning to visual-technical manifestation of equations (the image of 'building up' an equation), which I associated with offering learners explicit opportunity to fold back to property noticing (see figure 6.3/d). In Lessons 1 and 2, Bernie's enacted choices frequently aligned with a return to technical manifestation, which I associated with actions affording focusing learners' attention on algebraic manipulation.

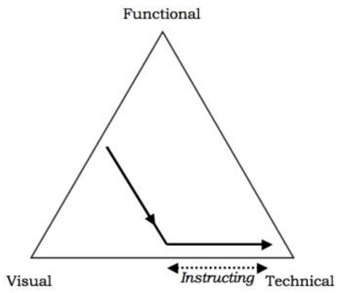


Figure 6.3/d
The sequencing of manifestation choices over time (solid line) and switches associated with teaching-between-the-desks (dotted line) inferred from Bernie's Year 9 lesson series on linear equations...

The dotted line in figure 6.3/d emphasises how in the midst of given-tasks categorised as technical Bernie provided explicit opportunities for the focus of learners' attention to be shifted towards property noticing (for example in Segments 5 and 8). Similarly, in the midst of given-tasks categorised as visual-technical Bernie provided explicit opportunities for the focus of learners' attention to be shifted to procedural fluency (for example in Segment 3). Therefore actions affording focusing learners' attention on property noticing were not limited to a single period of the lesson series, and the same inference was drawn regarding procedural fluency.

6.3.3 – Types of exchanges Bernie offered across the lesson series

Overall characterisation of the types of exchanges offered

Throughout the lesson series, Bernie's choices regarding the way content was made available aligned with greater stressing of learner-content interaction ($\zeta = 0.49$) over teacher-content ($\zeta = 0.27$) and teacher-learner ($\zeta = 0.24$) interaction.

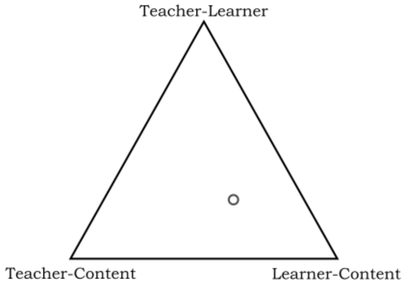


Figure 6.3/e
Series centre for interaction for Bernie's year 9 lesson series on linear equations.

Table 6.3/b shows that this stressing of learner-content interaction played out in the classroom with seatwork being foregrounded over classwork. I inferred that a range of modes of teacher interaction afforded to a teacher by the didactical situation were realised in the context of Bernie teaching the year 9 class. These included Bernie teaching-from-the-front (κ = 28%) and between-the-desks (κ = 44%) as well as being-watchful (κ = 28%). Seatwork therefore accounted for 72% of teaching time, which was by far in excess of 30-40 % observed in Stigler and Hiebert's (1999) videotaped lesson study.

Table 6.3/b The spread of interaction choices across Bernie's Year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 289)	11	10	7	7	26	11	28

Note: The figures in the table are κ values (percentages).

<u>*Comparing the types of exchanges offered from lesson to lesson*</u>

Each lesson in the series featured a mix of classwork and seatwork. The only lesson-to-lesson difference that was apparent in was between Lessons 2 and 3 (see figure 6.3/f), with Bernie's interaction choices aligning with a relative over-stressing of teacher-learner interaction[39] in Lesson 3.

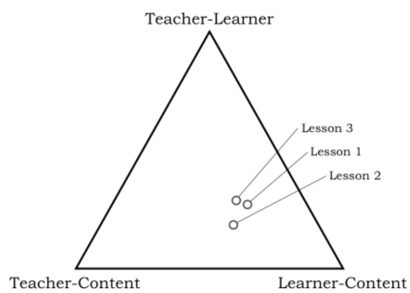


Figure 6.3/f
Lesson centres for interaction Bernie's year 9 lesson series on linear equations.

[39] See table A4/f in Appendix 4.

That the analysis indicated a lack of difference was unsurprising given the similar balance between teaching-from-the-front and between-the-desks in Lessons 1 and 2. Also in Lesson 3 there was a slightly greater emphasis on teaching-between-the-desks that balanced the greater emphasis on teaching-from-the-front and being-watchful in the first two lessons.

Bernie offered opportunities for learners to encounter visual-functional manifestation first through a seatwork problem, but typically opportunities to encounter visual-technical and technical manifestations were first offered from-the-front, through instruction. I associated this with some sense of shared ownership of the mathematics, but with Bernie taking an authoritative role in directing learners' attention towards particular features of the topic.

6.3.4 – Forms of content Bernie made available during classwork and seatwork

Comparing forms of content made available during classwork and seatwork

Across the lesson series as a whole, I inferred that there were no material differences in what forms of content were made available to learners during classwork when compared to seatwork (see figure 6.3/g). My analysis suggested there may have been a small but statistically significant association between seatwork and technical manifestation of the topic, but the effect size of this association meant differences could not be classified as material.

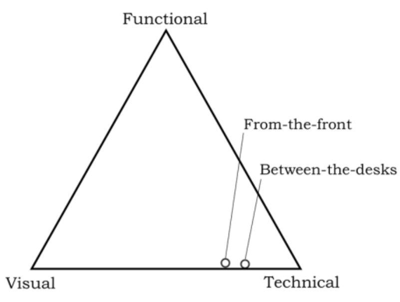


Figure 6.3/g
Comparison of manifestation centres for teaching-from-the-front and teaching-between-the-desks across Bernie's year 9 lesson series on linear equations.

A possible explanation for alignment between the forms of content offered during classwork and seatwork was that much of Bernie's teaching-from-the-front modelled the mathematical activity in which learners were intended to engage.

Comparing forms of content made available during seatwork

My analysis of the categorical data for interaction and manifestation arising from my classification of half-minute intervals suggested that, across the lesson series, there was a medium association[40] between Bernie's role in teaching-between-the-desks and how content was manifested through teacher-talk.

Table 6.3/c Bernie's role in teaching-between-the-desks and how mathematics was manifested as content through teacher-talk across the year 9 lesson series on linear equations.

Teacher's role in teaching-between-the-desks	*Categorisation of teacher-talk*				
	Ø	VT	T	Total	
Discussing-between-the-desks (LL)	**66.7**	3.0	30.3	100.0	(33)
Instructing-between-the-desks (CC/OO)	21.3	44.7	34.0	100.0	(94)
All teaching-between-the-desks	33.1	33.8	33.1	100.0	(127)
Chi Square	27.83 ***				
*** p<.001					

Note: Values in the table are percentages. Numbers in parentheses are the frequencies of half-minute intervals for each manifestation. Significantly larger values are highlighted in bold; significantly smaller values are underlined.

Table 6.3/c conveys that Bernie's choices to discuss-between-the-desks were associated[41] with focusing on the way content was manifested in the given-task (teacher-talk = Ø). Typically in this lesson series, discussing-between-the-desks featured Bernie posing prompting (such as excerpts $B_2$1.2 and $B_2$3.1 in section 6.2) or probing questions (such as excerpts $B_2$1.1 and $B_2$8.1), or offering affirmation

[40] Cramer's V = 0.468. It is unlikely for this medium association not to be a feature of the shape of choices over time (χ^2 (2) = 27.83, p<0.001).
[41] It is unlikely for this inferred association not to be a feature of the shape of choices over time (p<0.001).

(such as excerpt $B_2$3.3). Table 6.3/c also conveyed that Bernie's choices to instruct-between-the-desks were associated[42] with switching to more technical or visual-technical manifestation of the topic. In Section 6.3.2, I described how switching to visual-technical manifestation generally corresponded to returning to previously offered manifestations. Table 6.3/c shows that these returns were typically made available through instruction.

[42] Indeed there was a significant *negative* association between switching to more visual-technical manifestation of the topic and discussing-between-the-desks ($p<0.01$)

6.4 – Case summary

Bernie's interaction choices generally aligned with emphasising learner-content interaction. There was little lesson-to-lesson variation in the types of exchanges offered and a similar proportion of each lesson was devoted to seatwork. Further, there was little difference in the forms of content Bernie made available during classwork when compared with seatwork: technical manifestation was emphasised in both. Nearly two-thirds of the time allocated to seatwork featured Bernie teaching-between-the-desks. There was a small amount of discussion-between-the-desks that generally afforded focusing learners' attention on the outer aspect of given-tasks. However there were more instances of Bernie explaining-between-the-desks and these were generally associated with switching manifestation and afforded focusing learners' attention on technical or visual-technical inner aspects of given-tasks.

I inferred from my analysis that there was only one lesson that was materially different to the other two in the lesson series. I associated this difference with a planned opportunity to shift the focus of learners' attention from image having towards procedural fluency. However, at times Bernie switched to previously offered visual-technical manifestation of the topic through teaching-between-the-desks, affording a shift in the focus of learners' attention that reflects Pirie and Kieren's notion of folding back (Pirie and Kieren, 1989). These in-flight choices to teach-between-the-desks offered learners many pathways from image having towards procedural fluency.

CHAPTER 7

CASE STUDY 4

Chapter contents

7.1 – Background to the case

Courtney taught a series of lessons at Bishop Langton School[43] to a Year 9 class (students aged 13 and 14) set by prior mathematical attainment. Learners were, on average, predicted to achieve GCSE Mathematics grades in line with the national average by the age of 16. This was a series of four lessons on solving linear equations, being taught at the end of the school year. All learners had encountered some aspects of the topic previously, and in Lesson 1 Courtney made reference to learners' performance on a recent test that featured questions related to this topic.

> My job is to help the people here who struggled with bits. All of you did well on that test - you may not think so but I did. You all did well on that test but there are a very few gaps for each of you but each of you has slightly different gaps. So we are going to be doing the same thing, but there will be lots of extra work if you are finding it easy.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

This case raised some methodological issues for me, as throughout the series of lessons Courtney provided an alternative pathway for four learners who had performed particularly well on the test. These learners were grouped together and worked independently of the rest of the class. Through recording in the field notes when Courtney spoke to the group, I was able to distinguish in the transcript teacher talk addressed to this group from talk addressed to the rest of the class. For the purposes of analysis, I chose to focus only on how content was made available to the majority of the class and not analyse the way content was made available to this small group. Similarly, there was one absentee for part of the lesson series and Courtney provided an alternative pathway for this learner too, which was not included in my analysis.

[43] Bishop Langton School is a single-sex school.

On several occasions, Courtney offered learners a choice of two tasks. Typically the way the topic was manifested in these tasks was different, but the mode of teacher interaction was the same. At these times, I included both tasks within my analysis. So if one task would have been categorised as technical (manifestation = T) and the other as visual-technical (manifestation = VT), what Courtney offered was taken as a mix of the two.

Figure 7.1/a illustrates how the lesson series was parsed into lessons and segments (segments were identified from Courtney's lesson plans).

Lesson 1	Segment 1 (15m)
	Segment 2 (12m)
	Segment 3 (29m)
Lesson 2	Segment 4 (33m)
	Segment 5 (23m)
Lesson 3	Segment 6 (7m)
	Segment 7 (44m)
Lesson 4	Segment 8 (29m)
	Segment 9 (29m)

Figure 7.1/a The parsing of Courtney's year 9 lesson series on linear equations into lessons and segments.

Note: The length of each segment is given in minutes. Lessons were nominally 60 minutes long, but not all lesson time was devoted to mathematical activity. So the sum of the lengths of the segments making up a lesson may not equal 60 minutes.

7.2 – How content was made available at the segment and lesson level

This section contains a lesson-by-lesson and segment-by-segment report of the lesson series. There is a narrative, pre-analytical overview of each lesson including details of the segment(s) constituting that lesson. Each segment is represented in two ways. Firstly, there is a narrative account that provides a pre-analytical description, marking phases of the segment (which are labelled in **bold font**). This includes quotations from the teacher. The quoted talk that occurred between-the-desks is numbered, allowing subsequent referencing. Talk that occurred from-the-front is not numbered as no subsequent reference is made to it. This is followed by a structural account (presented in *italicised font*), which sets out chronologically the phases as sub-segments and categorises them using the categorical model of how content is made available. Recall that the region of the tri-polar space within which a sub-segment centre lies determines its categorisation. Each lesson report is concluded with a structural account of the lesson as a whole (presented in *italicised font*) using the language of the categorical model. This whole section has great significance for the validity of the study as it illustrates the alignment between the narrative description of lessons based on what I observed and the representation of the lessons through the lens of the categorical model.

<u>LESSON 1</u>

In Lesson 1, at first Courtney provided opportunities for learners to work on a task in which the need for algebraic manipulation and solving of equations arose (Segment 1). Then Courtney offered a diagrammatic representation for expanding brackets (Segment 2) and provided learners with an opportunity to practise bracket expansion (Segment 3).

<u>Segment 1 (15 minutes)</u>

Initially, Courtney provided an **exercise** featuring 'algebra pyramids' which called for learners to use their powers of algebraic manipulation in order to combine and simplify like terms. By way of introduction, Courtney drew learners' attention to a worked example (see figure 7.2/a). Two examples of problems set in this exercise are given in figure 7.2/b.

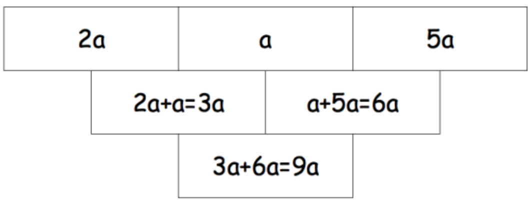


Figure 7.2/a Worked example of the 'algebra pyramids' exercise from Courtney's Year 9 Linear Equations Lesson 1.

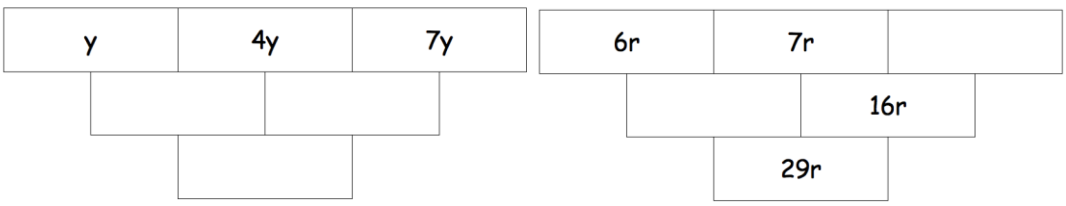


Figure 7.2/b Examples of problems from the 'algebra pyramids' exercise from Courtney's Year 9 Linear Equations Lesson 1.

The exercise also included examples with mixed terms (unknowns and constants) in which the value of the unknown was to be calculated (see figure 7.2/c).

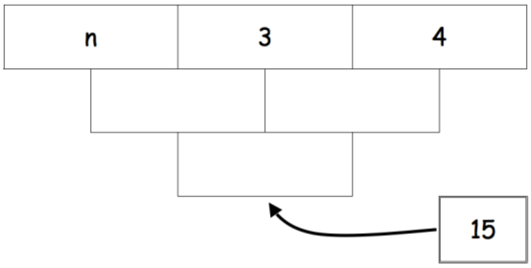


Figure 7.2/c An example of a problem from the 'algebra pyramids' exercise in which the value of the unknown was to be calculated from Courtney's Year 9 Linear Equations Lesson 1.

During this exercise phase, Courtney spent a little time teaching-between-the-desks. This included the following three excerpts of teacher talk, the first two of which I took as Courtney discussing the purpose of the task. The third excerpt I took as Courtney explaining how to represent the problem using formal algebraic notation.

> **C1.1:** It's just about simplifying.
>
> **C1.2:** (*Referring to an example such as the second in figure 7.2/b*) It's just getting you to think about subtraction as well.
>
> **C1.3:** *(Referring to the example in figure 7.2/c)* You simplify to the bottom square as you did on the other page. So what would you have here *(in the box below n and 3)*? What goes in the first square? *Learner comment.* You're [meant to be] adding. *Learner comment.* *n*+3. In the second square? *Learner comment.* 7, which is … 7. 3 plus 4 is 7, yeah? And then in the bottom you've got *n*+3+7, so in the bottom you've got? *Learner comment.* 10 plus *n*, good. 10 plus *n* is equal to 15. 10 plus *n* equals 15 (*Courtney writing*). So then you have to solve it.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

Following on from the exercise Courtney led a **discussion**, which at first drew attention to how learners had approached the example in figure 7.2/c in which the value of the unknown was to be calculated:

> So you know that is 15... *Learner comment.* So you're doing what [someone else] did, which is to say I already know that this is 15 and that is 7, so what must I have added to 7 to make 15? And you saw that as 8, and therefore $n = 5$.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

Courtney then drew attention to differences in how learners approached these examples (working from the bottom up) to those examples that featured like terms only (working from the top down), and began to suggest a top down approach for the example in figure 7.2/c too:

> Who can tell me something different other than 8 that they filled in for that square (*below n and 3 in figure 7.2/c*)? *Learner comment.* $n+3$. What made this pyramid different from the ones you were doing before...? *Learner comment.* It's all letters, yes. This is saying (*the first example in figure 7.2/b*) I've got 4 lots of y, and all of you were happy to recognise that you've got 4 lots of y, and you've got one more y, so how many y have you got? You've got 5y. Here, you mixing terms. So you are mixing letters and numbers.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

I parsed Segment 1 into two sub-segments relating to the distinct exercise and discussion phases highlighted above. The exercise phase involved explicit use of algebraic manipulation and equation solving in a synthetic[44] context (manifestation = FT) made available through a task sheet coupled with Courtney being-watchful (interaction = LC). This phase included teaching-between-the-desks, some of which aligned with focusing on the task in hand through discussion (interaction = LL) as in excerpts C1.1 and C1.2, while some aligned with switching to a more technical manifestation of the topic (teacher-talk = T) through explanation (interaction = OO) as in excerpt C1.3. In the discussion phase (interaction = TL) Bernie offered alternative

[44] The outer aspect here (filling in the grids) is distinct from the inner aspect (algebraic manipulation and equation solving), but *synthetic* has been used to describe the somewhat abstract context as opposed to *material world* contexts that sometimes feature in utilisation problems.

ways of accomplishing the inner aspect of the task and highlighted distinctions between expressions featuring like terms and expressions featuring mixed terms (manifestation = VT).

Segment 2 (12 minutes)

Courtney led a whole-class **discussion** about expanding brackets. This started with Courtney directing learners to use a grid method when expanding brackets[45] and demonstrating this approach through the example 3(t + 8). The demonstration was accomplished through Courtney asking a sequence of prompting questions:

> I want you to use the grid to expand, to multiply out, these single brackets [6(*t* + 8)]. What does the number on the outside of the brackets mean? *Learner comment*. It means multiply. So can you think of a way of putting that into a sentence? *Learner comment*. 6 lots of *t* plus 8. 6 multiplied by *t* + 8. Excellent, so in your grid method (*see figure 7.2/d*)... what's going to go in here (*bottom left hand cell*)? *Learner comment*. I'm going to put a 6 there and I'm going to put... what's inside the brackets across the top. So... what's going to go here (*top right hand cell*)? *Learner comment*. Please copy down the sign as well. It will help you when you collect your answer back together, ok? What's going to go here (*bottom middle cell*)? *Learner comment*. What I'm saying, whatever's here (*top middle cell*) is multiplied by here (*bottom left hand cell*) to get this one (*bottom middle cell*). So I look above and to the side. *Learner comment*. 6t, perfect. What goes here (*bottom right hand cell*)? *Learner comment*. Good. Just 48? *Learner comment*. Thank you, excellent. What's in the centre is your answer, ok?
>
> (Courtney, Year 9 Linear Equations Lesson 1)

[45] In the lead up to this discussion, Courtney stated to the class, "I'm going to show you a method and you are going to practise it. This is the method I want you to use. If you know a different method and it works every time, fantastic you can use it. But if you didn't get both of the questions on the test right, then I would like you to use my method." This excerpt of teacher talk was categorised as *sign-posting* rather than Courtney making content available, so has not been included within the description of how content was made available, but was included here in order to qualify the extent to which learners were directed.

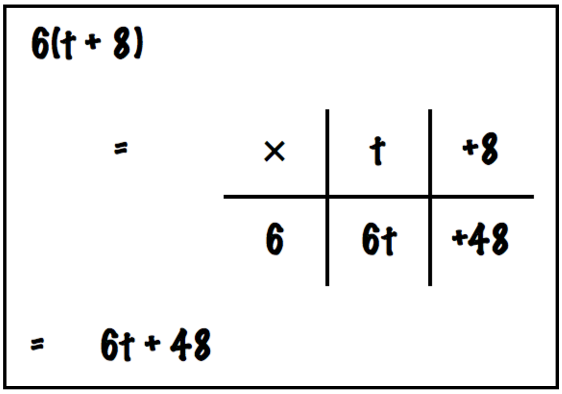


Figure 7.2/d Screen shot of the grid method used in order to expand 6(t + 8) from Courtney's year 9 linear equations Lesson 1.

Courtney then offered a second example: m (2 – m). A comparison was made with the previous example:

> What makes this one more difficult? *Learner comment.* Multiplying a letter on the outside and… *Learner comment.* There's a negative in there.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

The product of m and $-m$ produced a range of responses from learners, which Courtney listed, at first without comment: $-m$, $-2m$, $-m^2$, 0 and m. Courtney probed one learner who had offered 0 as the product of m and $-m$:

> You chose 0. If it was 0, what would you have to be multiplying… what would one of the parts that you are multiplying by have to be? *Group response.* 0. So can it be 0? No.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

Courtney then repeated one learner's justification for why the product of m and $-m$ was $-m^2$:

> Ok, m times m gives you m^2. *Learner comment.* One is positive, one is negative. If one number is positive and one number is negative, then you've got what as your answer? We're talking about the signs, not… the answer. *Learner comment.* It's going to be negative.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

Segment 2 featured a single discussion phase (interaction = TL), which I did not parse into sub-segments. In this phase, there was some emphasis on techniques for expanding brackets, but diagrammatic representations of these techniques and meanings attached to algebraic terms were foregrounded (manifestation = VT).

Segment 3 (29 minutes)

Courtney offered learners a choice of activity in order to **practise** expanding brackets. In one task, learners were provided with a sequence of expressions involving brackets to expand. In the alternative task, learners were tasked with matching elements from a set of expressions involving brackets and with elements from a set of equivalent expanded expressions.

During this practising phase, Courtney spent time teaching-between-the-desks. This included the following excerpt of discussion, which I took as Courtney focusing attention on using the diagrammatic representation of bracket expansion:

> **C3.1:** (*Courtney writing the grid for* $2(x+y+z)$) And your answer is? *Learner comment.* Good. What have you said that you haven't written down? *Learner comment.* Yes [the signs]. So can you see why it's easier to include [the signs] in your working out?
>
> (Courtney, Year 9 Linear Equations Lesson 1)

I took other instances of teaching-between-the-desks as Courtney emphasising algebraic manipulation, for example in the first excerpt below through telling and in the second through explanation:

> **C3.2:** Each question, the number on the outside multiplies each part on the inside in turn.
>
> **C3.3:** What goes in here? *Learner comment.* That's positive 3 and that's minus z. Is your answer going to be positive or negative? *Learner comment.* Good. Now you've worked out what sign it is, what happens when you multiply the 3 by z? *Learner comment.* z cubed would be z times z times z. That would be z cubed. Here I'm saying 3 times z. *Learner comment.* It would be $3z$. We know it's minus and we know it's $3z$.
>
> (Courtney, Year 9 Linear Equations Lesson 1)

The segment was concluded with Courtney briefly offering the whole class the opportunity to **consider** how they might simplify the expression $2+6m+4(m+3)$.

I parsed Segment 3 into two sub-segments relating to the distinct practising and considering phases highlighted above. The practising phase involved a choice of two tasks, one emphasising methods and the other emphasising alternative representations of algebraic expressions; the main emphasis was taken to be on methods for expanding brackets (manifestation = T) and this was made available through Courtney largely being-watchful (interaction = LC). This phase included teaching-between-the-desks, some of which aligned with focusing on the task in hand through discussion (interaction = LL) as in excerpt C3.1, and some which aligned with switching to a more technical manifestation of the topic (teacher-talk = T) as in excerpts C3.2 (interaction = CC) and C3.3 (interaction = OO). The brief considering phase foregrounded methods for expanding brackets and simplifying expressions (manifestation = T) and was made available through discussion (interaction = TL).

<u>*Structural account of Lesson 1*</u>

Table 7.2/a The spread of manifestation choices across Lesson 1 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 1 (N = 104)	0	0	71	0	19	8	2

Note: The figures in the table are κ values (percentages).

Table 7.2/a shows the spread of manifestation choices across the lesson based on my analysis. Courtney offered a mix of utilisation problems (manifestation = FT), visual representations (manifestation = VT) and an emphasis on formal methods (manifestation = T) in Lesson 1. These manifestation choices aligned with greater stressing of technical manifestation of the topic (ζ = 0.69), but with some stressing of visual manifestation (ζ = 0.24).

Table 7.2/b The spread of interaction choices across Lesson 1 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 1 (N = 104)	1	9	16	7	18	12	37

Note: The figures in the table are κ values (percentages).

Table 7.2/b shows the spread of interaction choices across the lesson based on my analysis. Courtney made content available at times through a given-task coupled with being-watchful (interaction = LC) or teaching-between-the-desks (particularly interaction = OO and LL), but at others teaching-from-the-front (particularly interaction = TL). These interaction choices aligned with learner-content interaction being stressed (ζ = 0.56) over teacher-learner (ζ = 0.31) and teacher-content (ζ = 0.13) interaction.

Courtney's teaching-between-the-desks in this lesson generally aligned with either focusing on the functional-technical or technical manifestation of the topic associated with the given-task (teacher-talk = Ø) largely through discussion or switching to a more technical manifestation (teacher-talk = T) largely through instruction.

LESSON 2

At the start of Lesson 2, Courtney provided opportunities for learners who had encountered obstacles when expanding brackets in Lesson 1 to practise further[46]. Opportunities to link expanding brackets with solving equations were provided for the rest of the class. The main part of the lesson was concerned with developing a method for solving linear equations involving brackets (Segment 4). The lesson concluded with a game built around solving equations (Segment 5).

Segment 4 (33 minutes)

Segment 4 opened with Courtney providing two alternative activities for learners. Those learners who had encountered obstacles when expanding brackets in Lesson 1 were tasked with continuing the exercise first met in Segment 3, offering further practice with **expanding** expressions such as $4(x + 6)$. For the rest of the class, Courtney provided a 'true/false' activity involving attempts at **solving** equations featuring brackets. For example, Courtney provided the three attempts at a solution to the equation $5(a + 3) = 10$ (see figure 7.2/e), calling for learners to determine whether the attempted solution was valid ('true') or not ('false'). During this phase Courtney spent a little time teaching-between-the-desks, mainly offering quick feedback to learners working on the 'true/false' activity:

[46] Courtney identified these learners to the researcher.

> **C4.1:** You've got one too many in your false column. You've got one in your false column that's true.
>
> (Courtney, Year 9 Linear Equations Lesson 2)

$5(a + 3) = 10$	$5(a + 3) = 10$	$5(a + 3) = 10$
$5a + 3 = 10$	$5a + 15 = 10$	$a + 3 = 2$
$5a = 7$	$5a = -5$	$a = -1$
$a = 1\,^2/_5$	$a = -1$	

Figure 7.2/e Examples of attempts at solving $5(a + 3) = 10$ from the true/false activity (the one on the left is false, the other two are true).

Next, Courtney asked for **feedback** on the 'true/false' activity, emphasising two possible approaches to solving equations such as $5(a + 3) = 10$.

> I think there are 3 which start 5(a+3). Can you find the one that you think is correct, so it's got -15 and then ÷5 on the left hand side showing you what's been done... What method has been used to find the solution? So this is our starting point [$5(a + 3) = 10$]. What method has been used...? Let me phrase this a different way. Is there another card which starts 5 lots of a + 3 that is also true, or is there only one true card? *Learner comment*. There's another one that's true. So we've got two cards that are true, and two different methods for solving them. So which method has been used here...? *Learner comment*. So they've expanded out the bracket first.
>
> (Courtney, Year 9 Linear Equations Lesson 2)

Discussing an alternative method involved more teacher talk, particularly when Courtney was emphasising the result of dividing $5(a + 3)$ by 5. Courtney recorded the following algebraic notation on the screen (see figure 7.2/f) and explained how this equation was equivalent to $a + 3 = 2$.

$$\frac{5 \times (a + 3)}{5} = \frac{10}{5}$$

Figure 7.2/f Screen shot of the algebraic notation Courtney used so as to explain the 'dividing first' method from linear equations Lesson 1.

> [He] chose 5 as a division first because that is what is on the outside of the brackets. Okay, so that makes sense? So if I'm dividing by 5, what have I got…? This 5 on the outside is saying 5 lots of, 5 times everything inside. I've said this before and I will keep saying it until it really really sticks. If I'm multiplying by something and dividing by the same thing, what happens…? So if I multiply by 2 and divide by 2, what happens? *Learner comment.* It stays the same. They cancel. So if I'm multiplying by 5 and then dividing by 5, which bit of that [$5(a + 3) \div 5$] is going to stay? So if I've got multiply by 5 on the outside and I'm saying I'm going to divide through by 5, so I've got 5 times $a + 3$ all divided by 5 equals 10 divided by 5, what happens – what stays, what cancels – there…? *Learner comment.* The stuff inside the brackets… Just leaving us with what? *Learner comment.* $a + 3$ equals 10 divided by 5, which I can simplify…? *Learner comment.* To 2, good. Okay? Two different methods for solving the same thing.
>
> (Courtney, Year 9 Linear Equations Lesson 2)

Courtney provided full solutions to $5(a + 3) = 10$ on the screen using both the 'brackets first' and 'dividing first' methods. Learners were invited to choose their preferred method and **copy** the solution into their books.

Finally in this segment, Courtney prompted a **consideration** about what constituted a solution to an equation:

> What makes an answer a final answer rather than part of your working out…? If I've got -12x = 3, if I draw a circle around it, does that make it a final answer…? What makes a final answer for an equation? Give me one thing. *Learner comment.* A single value of x, which must be… *Learner comment.* Positive. And it must be equal to a number.
>
> (Courtney, Year 9 Linear Equations Lesson 2)

I parsed Segment 4 into four sub-segments relating to the distinct expanding/solving, feedback, copying and considering phases highlighted above. The expanding/solving phase involved two alternative tasks, one emphasising methods and the other emphasising alternative methods and invalid algebraic manipulation; the overall *emphasis was taken to be on methods (manifestation = T) and this was made available through Courtney largely being-watchful (interaction = LC). This phase included a little teaching-between-the-desks, which generally aligned with focusing on the task in hand through discussion (interaction = LL) as in excerpt C4.1. The feedback (interaction = TT) and copying (interaction = LC) phases both involved methods for solving linear equations involving brackets (manifestation = T). The considering phase (interaction = TT) emphasised sense making of what constituted the solution of an equation (manifestation = VT).*

Segment 5 (23 minutes)

Courtney provided a 'top trumps' card **game** to be played in pairs, the inner aspect of which called for learners to solve linear equations involving brackets. Each card featured four equations, and to 'win' a learner needed to select an equation whose solution was greater than the solution to the equation chosen by their partner. This segment included an extended period of one-to-one teaching-between-the-desks with an individual learner[47], which I took as a mix of Courtney prompting responses from the student and telling that emphasised using and making sense of algebraic notation:

> **C5.1:** This is saying 2 lots of *x*. And this is saying add 1. You are saying I'm going to take 1 away. That 1, you're not saying 1*x* you are saying 1. 1 number. There is only one number here, so you are going to take that number away. Okay? You've got 2*x*s add 1 number. You are going to subtract one number. What are you keeping and what are you loosing on this side? *Learner comment.* Good, you are keeping the 2*x*s. If you are subtracting 1 from both sides, what happens on this side…? *Learner comment.* Okay, I've got 2*x*s equals 8… 2 lots of *x*. 2 times *x*. *x* add *x*. All the same thing… If I've 2*x*s are worth 8, what is 1 *x* worth? *Learner comment.* I'm going to half this. I'm going to divide by 2 to get 1*x* on its own. Can you see why?
>
> (Courtney, Year 9 Linear Equations Lesson 2)

[47] Excerpts C5.1 and C5.2 lasted for 5 minutes of lesson time in total. Extended periods of teaching-between-the-desks such as this featured in the latter stages of the lesson series. I included others examples in Segment 7 (C7.1) and Segment 9 (C9.2).

At this point, a choice that came to action was for Courtney to switch from a balancing method to an inverse operation method of solving equations. I took this excerpt of teacher talk as Courtney prompting responses from the learner, emphasising the inverse operation method:

> **C5.2:** If this is like [a foreign language] to you, then how about we look at it in terms of function machines? So you've got $2x+1$. If we start with x, what is the first thing that happened to x? Think about BIDMAS here. Is it multiplication or division? *Learner comment.* Okay, so what was x multiplied by? *Learner comment.* Okay. Next thing that happened to x? *Learner comment.* What was added? *Learner comment.* What is the answer? *Learner comment.* I want to go back the other way now. I want to do the inverse operations. I want to undo everything that was done to x. I'm going to start with 9, what is the opposite of adding 1? *Learner comment.* Which gives me 8. What is the opposite of multiplying by 2? *Learner comment.* Which gives me 4. So x is equal to 4. *Learner comment.* I've been hiding that from you as I like to make life as complicated as possible! That's much easier is it?
>
> (Courtney, Year 9 Linear Equations Lesson 2)

Segment 5 featured a single game phase (manifestation = FT), which was made available through Courtney largely being-watchful (interaction = LC). This phase included some teaching-between-the-desks, which generally aligned with switching to a more technical (teacher-talk = T) or visual-technical (teacher-talk = VT) manifestation of the topic through a mix of telling, explaining and discussing (interaction = CC, OO and LL) as in excerpts C5.1 and C5.2.

<u>Structural account of Lesson 2</u>

Table 7.2/c The spread of manifestation choices across Lesson 2 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 2 (N = 108)	0	0	57	0	15	23	5

Note: The figures in the table are κ values (percentages).

Table 7.2/c shows the spread of manifestation choices across the lesson based on my analysis. In Lesson 2, Courtney largely offered formal methods (manifestation = T), but with some utilisation problems (manifestation = FT). These manifestation choices aligned with greater stressing of technical manifestation of the topic (ζ = 0.66) over visual (ζ = 0.19) and functional (ζ = 0.15) manifestations.

Table 5.2/d The spread of interaction choices across Lesson 2 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 2 (N = 108)	3	16	7	7	12	16	39

Note: The figures in the table are κ values (percentages).

Table 7.2/d shows the spread of interaction choices across the lesson based on my analysis. While there were periods of teaching-from-the-front (particularly interaction = TT and TL) content was generally made available through given-tasks coupled with Courtney being-watchful (interaction = LC) or teaching-between-the-desks (particularly interaction = LL and OO). These interaction choices aligned with greater stressing of learner-content (ζ = 0.57) and to a lesser extent teacher-learner interaction (ζ = 0.26).

Courtney's teaching-between-the-desks in this lesson generally aligned with either focusing on the technical manifestation of the topic associated with the given-task (teacher-talk = Ø) through short episodes of discussion or switching to a more technical (teacher-talk = T) or visual-technical manifestation (teacher-talk = VT) through more extended episodes featuring both discussion and instruction.

LESSON 3

At first, Courtney built on the discussion that had taken place in Lesson 2 about what constituted the solution to an equation (Segment 6). The main part of Lesson 3 was concerned with solving linear equations in which the coefficient of the unknown was not an integer (Segment 7). Courtney encouraged learners to use a function machine approach when solving such equations.

Segment 6 (7 minutes)

Courtney initially provided a **quiz** for learners about what constituted the solution to an equation. A sequence of equations were displayed such as $y = 12$, $–x = 5$ and 5/t = 10. Learners were tasked with indicating if they thought it was true that the displayed equation correctly constituted a solution or false. A discussion was had about each example:

> Why is that correct ($y = 12$)? *Learner comment.* Because it's a single letter and…? *Learner comment.* It's positive.
>
> So that's wrong ($-x = 5$). Why is it wrong? *Learner comment.* Good, well done. You can never have a negative letter, than you.
>
> Why is it false ($5/t = 10$)? *Learner comment.* Because there's *t* and something. You are right, it is *t* and something, it's not *t* on its own. It's not a single letter, it's 5 divided by *t*.
>
> (Courtney, Year 9 Linear Equations Lesson 3)

Segment 6 featured a single quiz phase (interaction = TL), which I did not parse into sub-segments. In this phase, there was an emphasis on learners noticing what constituted the solution of an equation (manifestation = VT).

<u>*Segment 7 (44 minutes)*</u>

Courtney led a **discussion** about how to solve linear equations in which the coefficient of the unknown was not an integer (or as they were described in the lesson 'equations with any kind of division'). Initially this took the form of an explanation, in which Courtney introduced the idea of function machines:

> (*Referring to the equation p/5 = 9*) When you are talking about equations and solving equations what you are actually talking about is undoing the things you have done. So what is the opposite of dividing by 5? *Learner comment.* Multiplying by 5, good. If we put 9 in and multiply by 5, we then know that p is equal to what? *Learner comment.* 45. So *p* = 45. Function machines I think are the best way of solving equations with any kind of division. It's getting the order of the division or the order of what happened to the letter correct, isn't it?
>
> (Courtney, Year 9 Linear Equations Lesson 3)

Courtney made other examples available through discussion, while offering diagrammatic representations of equations by drawing function machines:

> (*Referring to the equation (x/3) + 5 = 7*) So, [he] is saying you divide by 3 first on here and then do what? *Learner comment.* If we've agreed you divide by 3 first, what is the next thing that happened? *Learner comment.* Plus 5. So we put x in and we got 7 out (*see figure 7.2/g*). What do you think is going to happen next? *Learner comment.* Take away 5, times 3. Superb. So if 7 goes in... what is the answer going to be? *Learner comment.* 6. So x is going to be equal to 6.
>
> (Courtney, Year 9 Linear Equations Lesson 3)

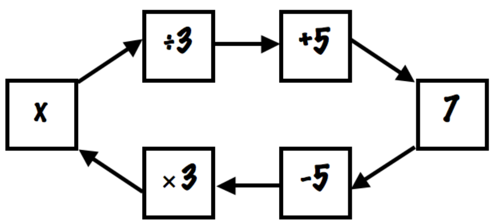


Figure 7.2/g Screen shot of the function machines Courtney used so as to represent and solve the equation (x/3) + 5 = 7 diagrammatically in linear equations Lesson 3.

Learners were then set an **exercise** on solving linear equations in which the coefficient of the unknown was not an integer. Courtney provided a resource sheet for learners featuring multiple blank versions of the diagrammatic representation of an equation offered in the discussion phase. There were two instances of Courtney teaching-between-the-desks in this phase, both of which were associated with the equation $4 - (t/2) = 1$. I took the instance below as Courtney explaining, emphasising the use of algebraic notation:

> **C7.1:** Imagine it as -t/2 + 4... That would be -t is divided by 2. If you wanted a positive t... what would you need to divide by to make minus half t? What would you need to divide by to make negative long significant pause half t? *Learner comment.* Minus...? *Learner comment.* But it has to be negative, so it has to be divide by -2. Because you start with positive t. Or if you don't do that, when you come back you'll think you've got a positive value for t, but actually you will have a negative value for t.
>
> (Courtney, Year 9 Linear Equations Lesson 3)

The segment was concluded with a **feedback** session on the exercise. The solution to the equation –n/3 = 6 was discussed in some detail, with Courtney highlighting once again the form of the unknown – 'We want a single positive value of n' – as in Segment 6. I took the following excerpt as Courtney emphasising a diagrammatic representation of the equation and increasingly emphasising algebraic manipulation through explanation:

> The equation is -*n* divided by 3 equals 6. But when we start, we start with what we want to end up with. We start with a single positive value of *n*. We want a single positive value of *n*. Which means that's what we need to start with. What could *n* – or must *n* – have been divided by if you end up with minus *n* over 3? So at this point we have minus *n* over 3. What happened to *n*...? *Learner comment.* It was divided by 3, but what would it have to have been divided... what else...? *Learner comment.* Minus 3, good. Because otherwise it would have been positive *n* over 3.
>
> (Courtney, Year 9 Linear Equations Lesson 3)

I parsed Segment 7 into three sub-segments relating to the distinct discussion, exercise and feedback phases highlighted above. A method for solving linear equations in which the coefficient of the unknown was not an integer was foregrounded throughout (manifestation = T), with a diagrammatic representation of this method also being emphasised. The discussion and feedback phases both largely featured explanation-from-the-front (interaction = TT). The exercise phase made available through a task sheet coupled with Courtney being-watchful (interaction = LC). This phase included some teaching-between-the-desks, which aligned with switching to even a more technical manifestation of the topic (teacher-talk = T) through explanation (interaction = OO) as in excerpt C7.1.

<u>Structural account of Lesson 3</u>

Table 7.2/e The spread of manifestation choices across Lesson 3 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 3 (N = 97)	0	0	78	0	22	0	0

Note: The figures in the table are κ values (percentages).

Table 7.2/e shows the spread of manifestation choices across the lesson based on my analysis. In Lesson 3, Courtney offered opportunities for classification (manifestation = VT) and then an emphasis on formal methods (manifestation = T). These manifestation choices aligned with foregrounding technical (ζ = 0.72) and to a lesser extent visual-technical (ζ = 0.28) manifestations of the topic.

Table 7.2/f The spread of interaction choices across Lesson 3 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 3 (N = 97)	2	24	21	4	8	3	38

Note: The figures in the table are κ values (percentages).

Table 7.2/f shows the spread of interaction choices across the lesson based on my analysis. Almost half the lesson featured teaching-from-the-front, particularly explanation (interaction = TT) and discussion (interaction = TL). Otherwise content was generally made available through a task sheet coupled with Courtney being-watchful (interaction = LC). These interaction choices aligned with foregrounding learner-content (ζ = 0.46) and to a lesser extent teacher-learner interaction (ζ = 0.36).

Courtney's teaching-between-the-desks was more scarce in this lesson and generally aligned with switching to a more technical manifestation (teacher-talk = T) through instruction.

LESSON 4

Lesson 4 was concerned with solving equations in which the unknown appeared on both sides. Initially Courtney provided opportunities for learners to appreciate how these equations could be transformed into more familiar equations with an unknown on one side (Segment 8). Opportunities for learners to practise solving examples from the new class of equations were then provided (Segment 9).

Segment 8 (29 minutes)

Courtney provided opportunities for learners to **appreciate** how linear equations in which the unknown appeared on both sides could be transformed into more familiar equations with an unknown on one side:

> (*Referring to* $2x + 4 = x + 11$) There are four parts to this equation, okay? One (*pointing at* $2x$), two (*pointing at* $+4$), three (*pointing at* x), four (*pointing at* $+11$). Which one part of those four would you most like to go so that you can solve it using methods you already know?
>
> (Courtney, Year 9 Linear Equations Lesson 4)

In the ensuing discussion, Courtney confirmed that learners were generally confident in solving the equation[48] produced by removing the *x*. The notion of 'removing the *x*' was then **formalised** through further discussion. At first, a learner suggested[49] adding an x:

> But if I add an *x* to this side, what would I have to do to this side? *Learner comment.* What is the basic tenet of solving equations? What you do to one side, you must do to the other. So if I add an *x* to this side, what must I do to the other side? *Learner comment.* Add an *x*. Does that solve it? Does that get rid of that *x*? Does that simplify my equation?
>
> (Courtney, Year 9 Linear Equations Lesson 4)

[48] It should be noted that at this stage Courtney was asking learners for an assessment of their confidence in solving $2x + 4 = 11$ (so with the *x* removed) rather than $x + 4 = 11$ (the result of subtracting *x* from both sides). So there was no emphasis on formal methods at this stage.
[49] This unfolding discussion was also recorded in my field notes.

A second learner then suggested subtracting x from both sides, and the equation was solved. Courtney offered a second example: $8y - 3 = 5y + 6$. Through Courtney posing a sequence of prompting questions, it was suggested that $5y$ be subtracted from both sides, leaving $3y - 3 = 6$. When asked for the next step, one learner suggested subtracting y, and asserted that $3y - y$ was equivalent to 3. To Courtney's surprise this was not openly refuted:

> I'm in a state of shock. I'd actually thought we had done quite well. I think we have done quite well. But when faced with a bold statement from someone who says, "Yes 3y take away y gives me 3" and you are all like "I can live with that, that's alright"... in my head I'm going "Arghhh!"
>
> (Courtney, Year 9 Linear Equations Lesson 4)

Courtney then set learners to **solve** the equation $3y - 3 = 6$ independently in order to avoid, in Courtney's words:

> One person's misconceptions compound[ing] another person's misconceptions.
>
> (Courtney, Year 9 Linear Equations Lesson 4)

Courtney sought **feedback** on learners' solutions to the equation $3y - 3 = 6$:

> What would be the next steps? *Learner comment.* Okay, so we are adding 3 to get $3y = 9$. And [the] last step? *Learner comment.* Divide by 3. Thank you. $y = 3$.
>
> (Courtney, Year 9 Linear Equations Lesson 4)

As a final response to the assertion that $3y - y$ was equivalent to 3, Courtney sought from learners examples of expressions equivalent to $3y$. $y\times3$, $6y/2$, $2y + y$, $4y - y$ and $y + y + y$ were all suggested, with Courtney using the final expression in order to **explain** why $3y - y$ was not equivalent to 3, but $2y$.

Having concluded the diversion, Courtney **recapped** the method of solving linear equations in which the unknown appeared on both sides introduced earlier in the segment by offering a further example ($2m - 3 = m + 2$) and explaining how to decide on the first step:

> Which would you rather get rid of m or $2m$? *Learner comment.* Why would you rather get rid of m rather than $2m$? Why is getting rid of m better than getting rid of $2m$? *Learner comment.* I can take away $2m$ from both sides, I can actually do anything I like but if I take away $2m$ from both sides what will that leave me? *Learner comment.* Perfect. This is the most common mistake when you have [unknowns] on both sides is to take away the first letter that you see. Actually you want to take away what? *Learner comment.* The smallest, to leave you with a positive value of m.
>
> (Courtney, Year 9 Linear Equations Lesson 4)

I parsed Segment 8 into six sub-segments relating to the distinct appreciating, formalising, solving, feedback, explaining and recapping phases highlighted above. Courtney's choices that came to action in-flight regarding the introduction of the solving, feedback and explaining phases were taken to be in response to the erroneous assertion made by a learner. The appreciating phase featured a root problem (using what was already familiar in order to get a sense of how to approach an unfamiliar problem – manifestation = VF) made available through a mix of telling, explaining and discussing-from-the-front (interaction = TT). The formalising phase (manifestation = T) was made available through a discussion (interaction = TL), which afforded learners the opportunity to put forward their assertions. One such assertion prompted the diversion into the solving (interaction = LC) and feedback (interaction = TL) phases, which foregrounded methods of solving equations (manifestation = T), and the explaining phase (interaction = TT), which drew on learner-generated examples of equivalent algebraic expressions (manifestation = VT). During the recapping phase, Courtney explained (interaction = TT) a way of seeing a possible first step when attempting to solve an equation in which the unknown appeared on both sides (manifestation = VT).

Segment 9 (29 minutes)

Courtney offered learners a choice of activity, providing alternative opportunities for them to engage in **solving** equations in which the unknown appeared on both sides. In one task, learners were provided with a sequence of linear equations to solve, starting with examples such as $2x - 1 = x + 3$ with positive coefficients of x on both sides of the equation, then examples featuring brackets on one side such as $2(x + 3) = x + 7$, then examples such as $2x - 3 = 7 - 3x$ featuring negative coefficients of x, and finally examples featuring brackets on both sides of the equation such as $7(x - 2) = 3(2x - 7)$. The alternative task featured a similar range of examples but in the form of a loop card activity[50].

During this equation-solving phase, I took Courtney's teaching-between-the-desks to be largely affirming learners' methods for solving equations, as in the following excerpt that also featured prompts regarding how the non-integer solution to the equation might be expressed:

> **C9.1:** So you have subtracted x from both sides, very nice. $3x + 1 = 8$, very nice. Subtracted 1 from both sides $3x = 7$. So x is going to be? What's 7 divided by 3? How could you write 7 divided by 3?
>
> (Courtney, Year 9 Linear Equations Lesson 4)

[50] This consisted of a set of numbered cards each featuring a linear equation where the solution of the equation on one card matched the number on another card, thus indicating which equation was to be solved next.

To **conclude** the lesson series, Courtney again offered learners a choice[51]:

> If you would like to practise for 10 minutes more, please start doing that now. If you would like to reflect on what you have learnt over the last 3 or 4 lessons I would like a true/false card - remember the cards I hand out to you? One is true one is false – both have workings out. I would like you to make two cards, one of which is true, one of which is false for either expanding brackets, equations with fractions or xs on both sides, okay?
>
> (Courtney, Year 9 Linear Equations Lesson 4)

During this concluding phase, Courtney spent an extended period teaching-between-the-desks with one learner[52], focusing on a solution to the equation $3x + 6 = 5(x - 2)$. I took this to be a step-by-step explanation of a method, combining two techniques Courtney had made available during the lesson series: expanding brackets (using a grid method – see figure 7.2/h) and solving linear equations in which the unknown appeared on both sides. I took the explanation to feature a mix of Courtney asking prompting questions and telling.

	x	-2
5	5x	-10

Figure 7.2/h Grid method for expanding $5(x - 2)$ featured in excerpt C9.2 from Courtney's linear equations Lesson 4.

[51] This excerpt of teacher talk was categorised as sign-posting rather than Courtney making content available, so was not included within the analysis of how content was made available, but was included here in order to describe the format of the true/false activity in Courtney's words.
[52] This learner was tasked with continuing to practise solving equations rather than produce true/false cards.

C9.2: So when you have brackets as well you need to expand the brackets first. So what will the right hand side be of that equation? What's going to be on the right hand side? Expand those brackets. What will you get? *Learner comment*. Okay, grid method. What do I put into my grid (*see figure 7.2/h*)? *Learner comment*. I'm going to put my 5 here. What's going to go here? *Learner comment*. (...) What's my answer to 5 multiplied by -2? *Learner comment*. And what goes in here? *Learner comment*. 5x, good. So I know that on the right hand side I've got 5x - 10 which equals 3x + 6. This equation would be easier without that. What I'm trying to get you to do or what I was trying to get you to do at the beginning of the lesson was to identify which part of the equation you liked the least. Which part it would be easier without. And it is the smaller number of xs. If you have x only on one side you know how to solve those kind of equations don't you? Do you? Is that fair to say? I'll tell you how to… Because I've got plus 3x, I'm going to do the same… the same term but with the opposite sign. So instead of plus 3x I'm going to subtract 3x from both sides. That's how I'm turning that – puff! – to zero. Okay…? So 'puff' – it's a very technical… No I'm talking about evaporating things – puff. I'm going to subtract 3x from both sides. 5x take away 3x gives me…? *Learner comment*. Okay. I've got 2x - 10, because that hasn't changed, equals… That is now zero, because 3x - 3x is zero. And that is going to be 6. Now having identified what needed to be taken away and how to do it, is that type of equation [2x – 10 = 6] solvable? Easier? How would you do it? *Learner comment*. Plus 10. *Learner comment*. Perfect. So x = 8.

(Courtney, Year 9 Linear Equations Lesson 4)

I parsed Segment 9 into two sub-segments relating to the distinct solving and concluding phases highlighted above. A method for solving linear equations in which the unknown appeared on both sides was foregrounded throughout (manifestation = T), with other methods that were offered during the lesson series also being emphasised at times. There was also some emphasis on learner generation of examples of linear equations in the concluding phase. Both phases featured tasks coupled with Courtney being-watchful (interaction = LC) and included teaching-between-the-desks, a little of which aligned with focusing on the given-task through *discussion (interaction = LL) as in excerpt C9.1 and more of which aligned with switching to even a more technical manifestation of the topic (teacher-talk = T) through a combination of discussion (interaction = LL) and explanation (interaction = OO) as in excerpt C9.2.*

<u>Structural account of Lesson 4</u>

Table 7.2/g The spread of manifestation choices across Lesson 4 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Lesson 4 (N = 107)	0	0	78	9	11	0	2

Note: The figures in the table are κ values (percentages).

In Lesson 4, Courtney offered a root problem (manifestation = VF) and opportunities for learners to generate examples (manifestation = VT) along with an emphasis on formal methods (manifestation = T). These manifestation choices aligned with foregrounding technical (ζ = 0.78) over visual (ζ = 0.17) manifestations of the topic.

Table 7.2/h The spread of interaction choices across Lesson 4 of Courtney's year 9 lesson series on linear equations, based on the categorisation of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Lesson 4 (N = 107)	4	22	17	7	12	9	29

Note: The figures in the table are κ values (percentages).

Table 7.2/h shows the spread of interaction choices across the lesson based on my analysis. As in Lesson 3, both discussing (interaction = TL) and explaining-from-the-front (interaction = TT) were emphasised, with just under half the lesson featuring teaching-from-the-front. Otherwise content was generally made available through a *choice of task sheets coupled with Courtney being-watchful (interaction = LC) or explaining-between-the-desks (interaction = OO). These interaction choices aligned with foregrounding learner-content interaction (ζ = 0.43) over teacher-learner (ζ = 0.36) and teacher-content (ζ = 0.21) interaction.*

Courtney's teaching-at-the-desks in this lesson generally aligned with switching to a more technical manifestation (teacher-talk = T) through instruction.

7.3 – What differences are discernible in how Courtney made content available to learners over time?

In this section I present the findings from my statistical analysis in order to address the following research questions:

> RQ1a: What can be inferred from analysis of teacher-talk and given-tasks about the particular forms of content Courtney made available to learners across the lesson series as a whole?
>
> RQ1b: What can be inferred from analysis of teacher-talk and given-tasks about between-lesson differences in the forms of content Courtney made available to learners?
>
> RQ1c: What can be inferred from analysis of teacher-talk and given tasks about the types of exchanges Courtney offered across the lesson series?
>
> RQ1d: What can be inferred from analysis of teacher-talk and given-tasks about the forms of content Courtney made available to learners during classwork and seatwork?

I present findings through a combination of figures of tri-polar spaces, tables and text. As previously stated, I place a particular emphasis on using tri-polar spaces since these are a distinctive feature of my analytical approach. Having presented results, I discuss the response to each question in relation to the particular range of opportunities for learning afforded during the lesson series.

7.3.1 – Particular forms of content Courtney made available across the lesson series

Courtney's manifestation choices over time (figure 7.3/a) aligned with greater stressing of technical manifestation (ζ = 0.71), but visual manifestation was also emphasised (ζ = 0.22).

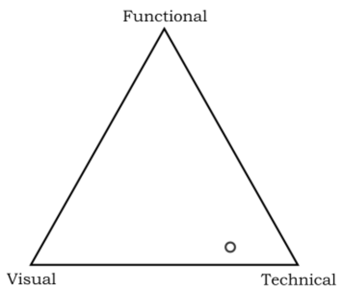


Figure 7.3/a
Series centre for manifestation for Courtney's year 9 lesson series on linear equations.

This stressing of the technical element played out in the classroom through Courtney offering learners technical, visual-technical, functional-technical and visual-functional-technical manifestations of the topic (see table 7.3/a), accounting for 98% of lesson time.

Table 7.3/a The spread of manifestation choices across Courtney's Year 9 lesson series on linear equations, based on the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 416)	0	0	71	2	17	8	2

Note: The figures in the table are κ values (percentages).

In the discussion that follows, I consider each manifestation category of my model and discuss the likely implications for learning afforded by it within the context of Courtney's Year 9 lesson series on linear equations. These affordances are viewed through the lens of shifts in the focus of learners' attention described in Chapter 2 (particularly as summarised in table 2.3/c). Some parallels have already been drawn between Kieran's (1996) classification of algebraic activity visual, functional and technical manifestations in Section 6.3.1, so these will not be repeated here.

Visual manifestation

Through offering visual-functional manifestation of the topic in Lesson 4, Courtney provided opportunities for learners to have a general sense of transforming equations with the unknown on both sides into more familiar equations with the unknown on one side. Specific examples of equations with the unknown on both sides were offered and Courtney asked learners what they wanted to 'remove' in order to transform the equations into a more solvable form. These opportunities afforded focusing learners' attention on image making, an image of 'removing x's' from one side of the equation.

Courtney also offered visual-technical manifestations of the topic. Such opportunities involved diagrammatic representations of processes (using a grid for expanding brackets in Lesson 1 and function machines to support solving equations involving non-integer coefficients of the unknown in Lesson 3), noticing properties of methods (Lesson 1, 2 and 3) and generating examples of equations and their solutions (Lesson 4). These opportunities afforded shifting the focus of learners' attention from image having (the association of distributivity with the partitioning of areas; the image of operations as function machines) to procedural fluency (expanding brackets using a grid method; solving equations using function machines). But other opportunities also afforded folding back, perhaps to consolidate, from insecure procedural fluency (some familiarity with the method of expanding brackets; some familiarity with solving equations) to more secure property noticing (awareness of common errors such as writing $a[b+c]$ as $ab+c$ rather than $ab+ac$; generating examples of equations and their solutions).

No purely visual manifestation of the topic was offered in this lesson series. Such visual manifestation would have provided opportunities for learners to heighten their sense of equations (see Section 6.3.1 for an example).

Technical manifestation

Technical manifestation, such as methods for expanding brackets and solving linear equations, was made available across the whole lesson series. This provided extensive opportunities for learners' attention to be focussed on procedural fluency.

Puzzles and games with mathematical inner aspects afforded shifting the focus of learners' attention from procedural fluency towards adaptive using (demonstrated through application). These functional-technical manifestations were opportunities to specialise: apply the general (method) *to* the particular (puzzle or game).

Functional manifestation

Algebraic topics afford functional manifestation (see Section 6.3.1) and there were examples of these affordances being realised by Courtney in this lesson series when offering visual-functional and functional-technical manifestation. However, no purely functional manifestation was made available.

7.3.2 – Between-lesson differences in the forms of content Courtney made available

Lesson-to-lesson differences were apparent in Courtney's manifestation choices at the lesson level (see figure 7.3/b). My analysis of the numerical data[53] suggested that each lesson was materially different to every other lesson in the series.

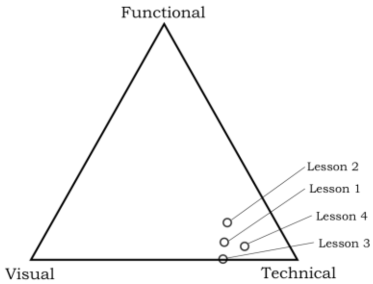


Figure 7.3/b
Lesson centres for manifestation for Courtney's year 9 lesson series on linear equations.

Table 7.3/b summarises the nature of material differences between the lessons in Courtney's series on linear equations.

Table 7.3/b Material differences between the four lessons in terms of how content was manifested overall.

Lesson	Difference comparison	Nature of material difference
Lesson 1	Lesson 3	Relative over-stressing of functional element
Lesson 2	Other 3 lessons	Relative over-stressing of functional element
Lesson 3	Lessons 2 and 4	Relative over-stressing of visual element
Lesson 4	Other 3 lessons	Relative over-stressing of technical element

[53] See table A5/c in Appendix 5.

Opportunities for learning afforded by the sequencing of manifestations

Referring back to the segment descriptions in Section 7.2, a general pattern emerged of Courtney offering three cycles (Lesson 1/2, Lesson 3 and then Lesson 4) of visual-technical and then technical manifestation of linear equations. I associated this sequencing of manifestations with explicit opportunities for shifting the focus of learners' attention from property noticing towards procedural fluency. However, even when viewed at a coarse grain size, the shape of Courtney's manifestation choices over time was more complex than this (see figure 7.3/c). Functional-technical manifestation was offered in Lesson 2 at the end of the first cycle (associated with solving linear equations involving brackets, affording shifting the focus of learners' attention from procedural fluency towards adaptive using) and visual-functional manifestation was offered in Lesson 4 at the start of the third cycle (associated with solving linear equations in which the unknown appeared on both sides, affording focusing learners' attention on image making). Furthermore, I associated three instances of Courtney offering visual-technical manifestation during classwork (the discussion phase of Segment 1, the consideration phase of Segment 4, and Segment 6) with opportunities that afforded folding back from insecure procedural fluency to property noticing. My interpretation of these three instances and the overall way content was made available over time was that Courtney sought to emphasise opportunities that afforded sense making of methods for solving linear equations.

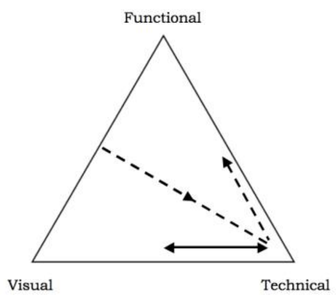


Figure 7.3/c
The sequencing of manifestation choices over time inferred from Courtney's year 9 lesson series on linear equations. The solid line represents consistent features of how content was manifested emphasised throughout the lesson series while the dashed line represents occasional features.

My analysis suggested that during seatwork Courtney's teaching-between-the-desks would sometimes align with focusing on the way content was manifested in the given-task. But at other times Courtney returned to previously offered manifestations of the topic, which I associated with offering learners explicit opportunities to fold back (Pirie and Kieren, 1989) to previous foci of attention (see figure 7.3/d). In particular, Courtney's enacted choices sometimes aligned with a return to visual-technical manifestation (typically diagrammatic representations such as grid for expanding brackets or function machines for solving equations), which I associated with offering learners explicit opportunity to fold back from insecure procedural fluency to more secure property noticing. At other times, Courtney's enacted choices aligned with a return to technical manifestation, which I associated with offering learners explicit opportunity to fold back to procedural fluency. A further feature of Courtney's teaching-between-the-desks was the time spent with individual learners, offering both visual-technical and technical manifestation during extended explanations.

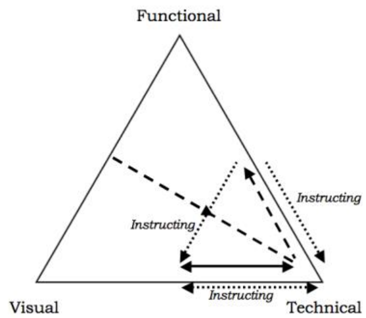


Figure 7.3/d
The sequencing of manifestation choices over time (solid and dashed lines) and switches associated with teaching-between-the-desks (dotted line) inferred from Courtney's year 9 lesson series on linear equations.

7.3.3 – Types of exchanges Courtney offered across the lesson series

Overall characterisation of the types of exchanges offered

Throughout the lesson series, Courtney's choices regarding the way content was made available aligned with greater stressing of learner-content interaction (ζ = 0.50), but also teacher-learner interaction was often emphasised (ζ = 0.32).

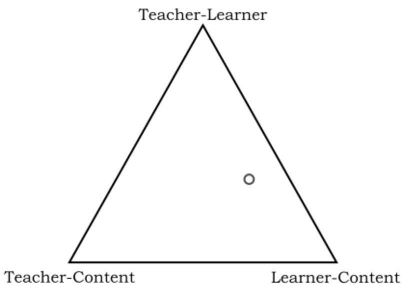


Figure 7.3/e
Series centre for interaction for Courtney's lesson series on linear equations.

Table 7.3/c shows that this stressing of learner-content interaction played out in the classroom with seatwork being slightly foregrounded over classwork. I inferred that a range of modes of teacher interaction afforded to a teacher by the didactical situation were realised in the context of Courtney teaching the year 9 class. These included Courtney teaching-from-the-front (κ = 35%) and between-the-desks (κ = 29%) as well as being-watchful (κ = 36%). Seatwork therefore accounted for 64% of teaching time, which was by far in excess of 30-40 % observed in Stigler and Hiebert's (1999) videotaped lesson study.

Table 7.3/c The spread of interaction choices across Courtney's year 9 lesson series on linear equations, based on the coding of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 416)	2	18	15	6	13	10	36

Note: The figures in the table are κ values (percentages).

<u>*Comparing the types of exchanges offered from lesson to lesson*</u>

Only subtle lesson-to-lesson differences were apparent in Courtney's interaction choices at the lesson level (see figure 7.3/f), with each lesson featuring a mix of classwork and seatwork. My analysis of the numerical data[54] suggested no material differences between any of the lessons.

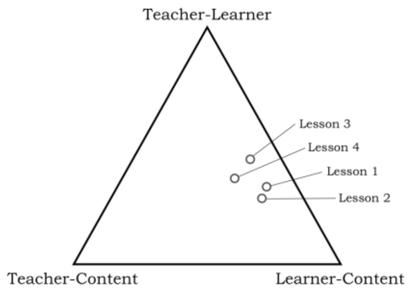


Figure 7.3/f
Lesson centres for interaction.

Each of the three cycles mentioned previously was initiated through Courtney teaching-from-the-front. In subsequent segments of each cycle, Courtney tended to offer learners opportunities to encounter technical manifestation of the topic through seatwork problems. Often these problems were subsequently addressed from-the-front, largely through discussion. I associated this with a sense of shared ownership of the mathematics. Courtney often provided direction, but from my observations I concluded that typically this was sensitive to learners' perceived mathematical needs (informed by their responses to seatwork problems) and teaching plans were adapted.

[54] See table A5/f in Appendix 5.

7.3.4 – Forms of content Courtney made available during classwork and seatwork

Comparing forms of content made available during classwork and seatwork

Across the lesson series as a whole, differences were apparent in what forms of content were made available to learners during classwork when compared to seatwork. My analysis of the numerical data[55] suggested that Courtney's choices to teach-from-the-front aligned with a material over-stressing of visual aspects and material under-stressing of functional aspects (see figure 7.3/g).

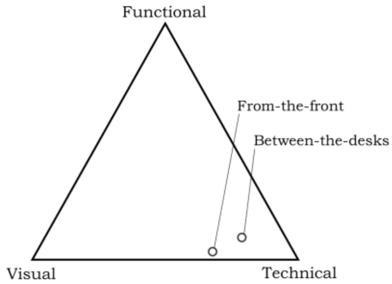


Figure 7.3/g
Comparison of manifestation centres for classwork and seatwork across Courtney's year 9 lesson series on linear equations.

This finding can be related to the narrative descriptions of lessons by attending to teaching-from-the-front and given-tasks. Teaching-from-the-front and being-watchful significantly outweighed teaching-between-the-desks in Courtney's lesson series. Also both teaching-from-the-front and given-tasks generally emphasised technical aspects of the topic. However, teaching-from-the-front also emphasised visual aspects (visual-technical and visual-functional manifestation) whereas given-tasks also emphasised functional aspects (functional-technical manifestation).

[55] See tables A5/h and A5/i in Appendix 5.

Comparing forms of content made available during seatwork

My analysis of the categorical data for manifestation and interaction arising from my classification of half-minute intervals suggested that, across the lesson series, there was a weak association[56] between Courtney's role in teaching-between-the-desks and how content was manifested through teacher-talk.

Table 7.3/d Courtney's role in teaching-between-the-desks and how mathematics was manifested as content through teacher-talk across the year 9 lesson series on linear equations.

Teacher's role in teaching-between-the-desks	*Categorisation of teacher-talk* Ø	VT	T	Total	
Discussing-between-the-desks (LL)	69.0	11.9	19.1	100.0	(42)
Instructing-between-the-desks (CC/OO)	45.4	23.4	31.2	100.0	(77)
All teaching-between-the-desks	53.8	19.3	26.9	100.0	(119)
Chi Square	6.14 *				
* p<.05					

N.B. Values in the table are percentages. Numbers in parentheses are the frequencies of half-minute intervals for each manifestation. While overall there was a weak association between the teacher's role in teaching-between-the-desks and how content was manifested through teacher-talk, analysis indicated that none of the individual observed frequencies in the cells of the table were significantly larger or smaller compared to the individual expected frequencies that would be predicted under the null hypothesis of no association between the teacher's role and teacher-talk.

Table 7.3/d also conveys that Courtney's choices that aligned with focusing on the way content was manifested in the given-task were often offered through discussing-between-the-desks. However, this was not a statistically significant association.

[56] Cramer's V = 0.227. It is unlikely for this weak association not to be a feature of the shape of choices over time (χ^2 (2) = 6.15, $p<0.05$).

7.4 – Case summary

Courtney's interaction choices generally aligned with emphasising learner-content interaction. The types of exchanges offered in lessons varied, generally indicated by differences in stressing of learner-content interaction. Seatwork featured most in Lessons 1 and 2, while over 40% of Lessons 3 and 4 was devoted to classwork. There were differences in how Courtney made content available during classwork when compared with seatwork: visual manifestation was emphasised more and functional manifestation less when teaching-from-the-front. Under half of the time allocated to seatwork featured Courtney teaching-between-the-desks. There were only weak associations between how content was manifested through teacher-talk and the types of exchanges offered when teaching-between-the desks. However, a notable feature of teaching-between-the-desks in which Courtney did participate was that it tended to involve prolonged exchanges with individual learners.

I inferred from my analysis that there were material differences in how Courtney made content available across the lesson series. I associated these differences with three cycles within the lesson series associated with the three types of equation that were addressed. Slightly different forms of content were made available in each cycle: visual-technical and technical featured in each, but the first (which addressed previously encountered types of equation) also featured functional-technical manifestation and the third (which addressed types of equations that were much less familiar to learners). I concluded that Courtney's manifestation choices over each cycle within the lesson series afforded focusing learners' attention on property noticing and procedural fluency, with the final cycle (that addressed types of equations that were much less familiar to learners) also affording image making.

CHAPTER 8

SOURCES OF VARIATION IN HOW CONTENT WAS MADE AVAILABLE OVER TIME

Chapter contents

8.1 – Between-case comparisons of how content was made available

I now turn my attention to the second research question:

> RQ2: When comparing series of lessons, what differences are discernible between how teachers made content available to learners over time?

In this section I present the findings from my statistical analysis that address the following research questions:

> RQ2a: What can be inferred from analysis of teacher-talk and given-tasks about between-case differences in the particular forms of content made available to learners?
>
> RQ2b: What can be inferred from analysis of teacher-talk and given-tasks about between-case differences in the types of exchanges made available to learners?
>
> RQ2c: What can be inferred from analysis of teacher-talk and given-tasks about between-case differences in how content was made available to learners during seatwork?

I present findings through a combination of figures of tri-polar spaces, tables and text. As previously stated, I place a particular emphasis on using tri-polar spaces since these are a distinctive feature of my analytical approach.

Within the confines of this study, I associate inferred differences in how content was made available over time with the four case dimensions (school, teacher, topic and class). I do not assert this as a causal relationship, nor is it offered as an externally valid generalisation. However, these associations open up the possibilities of further research of a wider sample of cases. Table 8.1/a presents the conditions for between-case differences to be associated with differences in school, teacher and topic respectively. All other between-case differences are associated with differences in class.

Table 8.1/a Material differences between the four cases in terms of how content was manifested overall.

No material differences between…	These series materially different to…	Variation by…
Ashley (Geometry), Bernie (Ratio) and Bernie (Algebra)	Courtney (Algebra)	School
Bernie (Ratio) and Bernie (Algebra)	Ashley (Geometry) and Courtney (Algebra)	Teacher
Bernie (Algebra) and Courtney (Algebra)	Ashley (Geometry) and Bernie (Ratio)	Topic

8.1.1 – Between-case differences in how content was manifested

The series centres for manifestation highlighted differences in how content was manifested across each of the four lesson series. Figure 8.1/a presents the four series centres (individually presented previously in figures 4.3/a, 5.3/a, 6.3/a and 7.3/a) in a single tri-polar space.

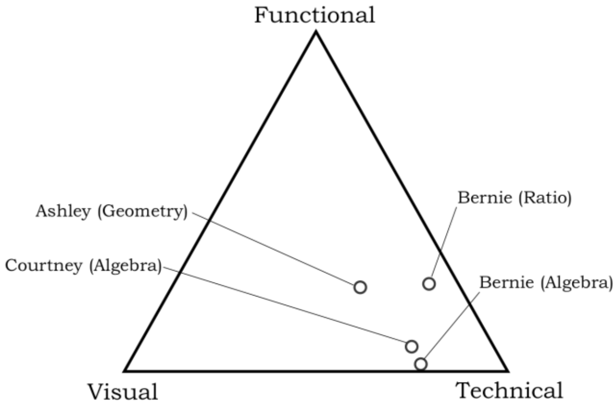


Figure 8.1/a
The four series centres for manifestation.

In order to assert material difference, I analysed the ζ values of interval centres for manifestation across all four cases[57]. The material differences between cases are presented in table 8.1/b.

Table 8.1/b Material differences between the four cases in terms of how content was manifested overall.

Case	Difference comparison	Nature of material difference
Ashley (Geometry)	Other 3 cases	Under-stressing of technical element
Ashley (Geometry)	Bernie (Algebra) Courtney (Algebra)	Over-stressing of functional element
Bernie (Ratio)	Other 3 cases	Under-stressing of visual element
Bernie (Ratio)	Bernie (Algebra) Courtney (Algebra)	Over-stressing of functional element
Bernie (Ratio)	Bernie (Algebra)	Under-stressing of technical element
Bernie (Algebra)	Courtney (Algebra)	Under-stressing of functional element

[57] See table A6/a in Appendix 6.

Figure 8.1/a and table 8.1/b show that the three elements of the manifestation triad were most evenly stressed in Ashley's series on geometry. Functional and particularly technical elements were stressed in Bernie's series on ratio, while visual and particularly technical elements were stressed in the two series on algebra (Bernie and Courtney). I inferred that each lesson series was materially different to the other three with respect to manifestation. The least different series were the two on algebra as these were only just materially different (effect size = 0.204) with respect to the functional element of manifestation.

As well as attending to the relative stressing of the three elements of the manifestation triad in order to establish between-case differences, the categorical data (categorisation codes of each half-minute interval) also highlighted differences[58]. Analysis indicated that how content was manifested across the lesson series was strongly associated[59] with the particular lesson series. Visual-functional, visual-technical or visual-functional-technical manifestations were more features of Ashley's series on geometry than the other three series. Functional-technical manifestation was a significant feature of Bernie's series on ratio and technical manifestation featured more prominently in the two series on algebra. Thus the series of lessons taught by Ashley on geometry and Bernie on ratio were marked out as different to the two series on algebra, and different to each other. However, this analysis did not highlight significant differences between the two series on algebra. This, along with the similarity of the relative stressing of the three elements of manifestation in the two series on algebra, led me to infer that within this study **the forms of content made available during a lesson series appeared to vary by topic.**

[58] See table A6/b in Appendix 6.
[59] Cramer's V = 0.582. It is unlikely for this strong association not to be a feature of how content was made available over time (χ^2 (15) = 531.29, $p<0.001$).

8.1.2 – Between-case differences in interaction

Figure 8.1/b presents the four series centres for interaction (individually presented previously in figures 4.3/e, 5.3/e, 6.3/e and 7.3/e) in a single tri-polar space.

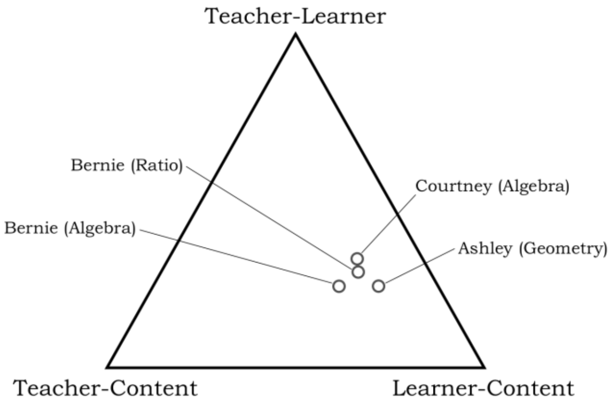


Figure 8.1/b
The four series centres for interaction.

Figure 8.1/b suggests only small between-case differences, but in order to assert material difference, I analysed the ζ values of interval centres for interaction across all four cases[60]. This confirmed that there were no *material* differences series-to-series in interaction. However, there were small, statistically significant differences[61] detected by the analysis (see table 8.1/c), which corresponded to the relative positions of the series centres within the tri-polar space.

Table 8.1/c Small, non-material, but statistically significant differences between the four cases with respect to interaction.

Case	Difference comparison	Nature of difference
Ashley (Geometry)	Other 3 cases	Over-stressing of learner-content interaction
Ashley (Geometry)	Bernie (Ratio) Bernie (Algebra)	Under-stressing of teacher-content interaction
Ashley (Geometry)	Courtney (Algebra)	Under-stressing of teacher-learner interaction
Bernie (Algebra)	Courtney (Algebra)	Over-stressing of teacher-content interaction

[60] See table A6/c in Appendix 6.
[61] Effect size 0.1 to 0.2. An effect size between 0.1 and 0.3 is usually described as small, but 0.2 is the threshold for my definition of material difference. Confidence level for statistical significance here is 99.9%.

While the relative stressing of the three elements of the interaction triad indicated no material series-to-series differences, the categorical data (categorisation codes of each half-minute interval) did highlight differences between the types of exchanges made available in the four lesson series[62]. Analysis indicated that the types of exchanges that were made available across the lesson series were associated[63] with the individual cases. Teaching-from-the-front was less a feature of Ashley's lesson series on geometry than the other cases. Teaching-between-the-desks was a significant feature of Bernie's series on ratio, while teaching-from-the-front featured more prominently and teaching-between-the-desks much less in Courtney's series on algebra. In summary, **the overall relative stressing of the three elements of the interaction triad was similar across the four series of lessons, but the balance between teaching-from-the-front and teaching-between-the-desks appeared to vary by class.**

A deeper understanding of interaction in the four series of lessons followed from attending to classwork and seatwork separately. The series centres for each case are presented in figure 8.1/c.

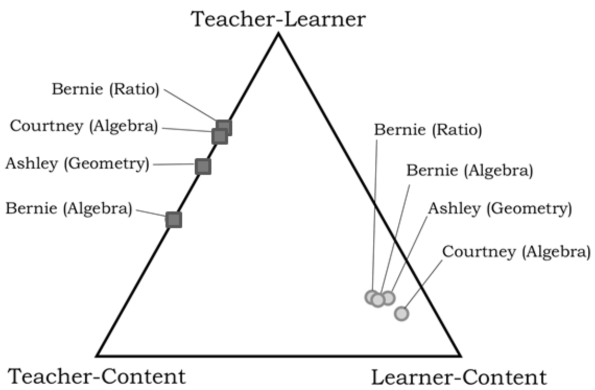


Figure 8.1/c
The four series centres for interaction during classwork (squares) and seatwork (circles).

By definition, the stressing of learner-content interaction during classwork is zero, which constrains these series centres to the edge connecting the teacher-content and teacher-learner interaction poles. The types of exchanges offered during

[62] See table A6/d in Appendix 6.
[63] Cramer's V = 0.222. It is unlikely for this weak to medium association not to be a feature of how content was made available over time (χ^2 (8) = 77.65, p<0.001).

classwork in Bernie's series on algebra were materially different to the same teacher's series on ratio and Courtney's series on the same topic, with an over-stressing of teacher-content interaction in the former[64]. There were no other material differences. My analysis of the categorical data for interaction arising from my classification of half-minute intervals given over to classwork suggested that there was a medium association[65] between the teacher's role in teaching-from-the-front and case (see table 8.1/d).

Table 8.1/d Contingency table of types of exchanges during classwork by case.

Teacher's role in teaching-from-the-front	*Case* Ashley (Geometry)	Bernie (Ratio)	Bernie (Algebra)	Courtney (Algebra)	Total	
Telling-from-the-front	**33.3**	11.6	**42.3**	12.8	100.0	(78)
Explaining-from-the-front	<u>8.0</u>	24.7	18.7	**48.6**	100.0	(150)
Discussing-from-the-front	24.7	22.8	<u>12.6</u>	39.9	100.0	(158)
All teaching-from-the-front	20.0	21.2	21.0	37.8	100.0	(386)

Chi Square 64.21 ***

*** p<.001

Note: Values in the table are percentages. Numbers in parentheses are the frequencies of half-minute intervals for each mode of teacher interaction. Significantly larger values are highlighted in bold; significantly smaller values are underlined.

Table 8.1/d also conveys that a mix of telling and discussing-from-the-front was characteristic of Ashley's series on geometry, whereas a mix of explaining and discussing-from-the-front was more representative of classwork in Courtney's series on algebra. Telling-from-the-front featured prominently in Bernie's series on algebra, but much less so in the same teacher's series on ratio. Within this study, **the types of exchanges offered during classwork appeared to vary more by class than by teacher or topic**.

[64] See table A6/e in Appendix 6.
[65] Cramer's V = .407. It is unlikely for this medium association not to be a feature of how content was made available over time (χ^2 (6) = 64.21, p<.001).

The types of exchanges offered during seatwork in the four lesson series resulted in a similar stressing of the three elements of interaction in each case, although analysis indicated that there was a material over stressing of learner-content interaction in Courtney's series on algebra when compared to Bernie's series on ratio[66]. My analysis of the categorical data for interaction arising from my classification of half-minute intervals given over to seatwork suggested that there was a weak association[67] between the teacher's role in teaching-from-the-front and case (see table 8.1/e).

Table 8.1/e Contingency table of types of exchanges during seatwork by case.

Teacher's role in teaching-between-the-desks	*Case* Ashley (Geometry)	Bernie (Ratio)	Bernie (Algebra)	Courtney (Algebra)	Total	
Telling-between-the-desks	36.9	25.4	16.4	21.3	100.0	(122)
Explaining-between-the-desks	18.9	**39.7**	**24.1**	17.3	100.0	(307)
Discussing-between-the-desks	**48.5**	19.6	14.0	17.9	100.0	(235)
Being-watchful	36.8	19.4	15.4	**28.4**	100.0	(525)
All teaching-from-the-front	34.5	25.3	17.5	22.7	100.0	(1189)
Chi Square		96.13 ***				
*** p<.001						

Note: Values in the table are percentages. Numbers in parentheses are the frequencies of half-minute intervals for each mode of teacher interaction. Significantly larger values are highlighted in bold; significantly smaller values are underlined.

[66] See table A6/f in Appendix 6.
[67] Cramer's V = .284. It is unlikely for this weak association not to be a feature of how content was made available over time (χ^2 (9) = 96.13, p<.001).

Table 8.1/e also conveys that being-watchful was a significant feature of Courtney's lesson series on algebra. Meanwhile, explaining-between-the-desks was a significant feature of both of Bernie's lesson series and discussing-between-the-desks featured more prominently in Ashley's series on geometry. Within this study, **the types of exchanges offered during seatwork appeared to vary more by teacher than by class or topic**.

<u>*8.1.3 – Between-case differences in how content was made available during seatwork*</u>

My analysis of the categorical data for interaction and manifestation arising from classification of half-minute intervals suggested that, across the four series of lessons, there was a medium association[68] between the nature of teaching-between-the-desks and case (see table 8.1/f).

Table 8.1/f Contingency table of all teaching-between-the-desks by case, showing that there was a medium association between the nature of teaching-between-the-desks and case.

Teaching-between-the-desks (Teacher-talk ~ interaction)	*Case* Ashley (Geometry)	Bernie (Ratio)	Bernie (Algebra)	Courtney (Algebra)	Total	
Ø ~ Discussion	**50.3** ***	<u>16.8</u> **	14.2	18.7	100.0	(155)
Ø ~ Instruction	<u>20.4</u> *	26.2	19.4	**34.0** ***	100.0	(103)
VT ~ Discussion	**57.1** *	25.7	<u>2.9</u> *	14.3	100.0	(35)
VT ~ Instruction	<u>15.5</u> ***	**43.9** ***	**28.4** *	12.2	100.0	(148)
T ~ Discussion	36.4	22.7	22.7	18.2	100.0	(44)
T ~ Instruction	34.5	32.1	19.1	14.3	100.0	(168)
All manifestations	33.1	29.3	19.4	18.2	100.0	(653)

Chi Square 92.06 ***

*** p<.001 ** p<.01 * p<.05

Discussing-between-the-desks and focusing on how content was manifested in the given-task was more a feature of Ashley's series on geometry than the other three series. Focusing on how content was manifested in the given-task through instruction was a significant feature of Courtney's series on algebra. Switching to visual-technical manifestation through instruction featured more prominently in the two series taught by Bernie. Thus within this study **the nature of teaching-between-the-desks appeared to vary by teacher.**

[68] Cramer's V = .375. It is unlikely for this medium association not to be a feature of how content was made available over time (χ^2 (15) = 92.06, p<.001).

8.2 – Discussion

In this discussion, I integrate the findings from each case study that were expressed in the previous four chapters with the findings from the cross-case analysis reported in the previous section. The current section is organised around two themes: the forms of content made available in each lesson series and how content was made available during classwork and seatwork. Within each theme I draw comparisons between the four cases, identifying each theme as a source of variation in how content was made available to learners over time.

8.2.1 – The forms of content made available

In Section 8.1.1, I inferred that, within the confines of this study, the forms of content made available during a lesson series appeared to vary most by topic. The current section is therefore organised with this in mind, with the two series on algebra discussed together.

Ashley's series on geometry

Ashley's enacted choices over the lesson series foregrounded technical aspects of geometry over visual and functional aspects. However, it was noted that this over-stressing of technical aspects did not correspond to devoting lesson time to learners purely practising methods of construction. Rather, there was a range of opportunities offered such as describing images of geometrical objects, presenting methods of construction, learners generating examples of geometrical constructions and using constructions in order to solve contextual problems, including identifying the appropriate construction for each problem. These were visual-technical, technical and functional-technical manifestations of geometry and afforded property noticing, procedural fluency and adaptive using. I suggested that Ashley's approach might be described as *economical* (Hewitt, 1994) since opportunities to practise construction techniques whilst learners generated examples and solved problems using geometrical constructions were made available.

Through root problems, Ashley also offered learners opportunities to appreciate geometrical objects as a locus of points constrained by a rule, the shortest distance of a point from a line as the length of the perpendicular from the point to the line, the role of compasses in constructing a triangle given the lengths of its three sides and the constraints placed on the technique of triangle construction. These were examples of visual-functional manifestation and were particularly made available when a geometrical object was first introduced. These afforded image making, which generally preceded opportunities to focus learners' attention on property noticing and procedural fluency.

Bernie's series on ratio

In Bernie's series on ratio, technical aspects of the topic were particularly foregrounded but there was also some emphasis on functional aspects. A central thread to the lesson series was the development and use of a method for dividing quantities in a given ratio. This started with offering opportunities for learners to make connections between images of sharing out coins into piles and calculations, from which a technique emerged. Opportunities to practise this technique were made available through abstract examples of sharing in a given ratio and solving material world problems that generally involved sharing out money. These were visual-technical, technical and functional-technical manifestations of ratio and afforded property noticing, procedural fluency and adaptive using.

Other opportunities to practise and then apply techniques of simplifying ratios and expressing a fractional part of a whole and its compliment as a ratio were also provided. These were technical and functional-technical manifestations of ratio, largely affording focusing learners' attention on procedural fluency. Unlike Ashley's series, visual aspects were less emphasised and no explicit opportunities for learners' image making were offered. Rather, images such as bags representing shares of money were given.

Bernie's series and Courtney's series on algebra

Technical aspects of the topic were also foregrounded in the two series on algebra, but there was some emphasis on visual aspects in these cases too. There were differences in the sequencing of manifestations in these two series on algebra, although less difference in how content was manifested overall. Bernie generally offered shifts in the focus of learners' attention from property noticing to procedural fluency, whereas Courtney's sequencing also afforded re-focusing learner's attention on property noticing.

Both Bernie and Courtney offered opportunities for learners to solve linear equations with an unknown on one side, expand products of a constant and a linear expression (simply referred to as 'expanding brackets'), solve linear equations involving brackets and solve linear equations with an unknown on both side of the equation. These were technical manifestations of algebra, affording procedural fluency.

Ahead of solving equations, Bernie offered categorisation of equations based on their solutions, diagrams to represent building up of an equation from its solution and opportunities for learners to generate their own examples of equations. Courtney offered diagrams to scaffold solving linear equations in which the coefficient of the unknown was not an integer. Both offered diagrams to scaffold expanding brackets. These were visual-technical manifestations affording focusing learner's attention on property noticing before shifting attention to procedural fluency. Courtney also offered comparison of equivalent expressions, identification of invalid algebraic expression and opportunities to articulate what constituted the solution of an equation. Having already focused attention on procedural fluency, these were visual-technical manifestations affording re-focusing learner's attention on property noticing.

Both series featured a single root problem, affording image making. Bernie's very first task afforded noticing that some equations could be solved using familiar methods (those with positive integer solutions), while others could not. This sense of a further class of equations then motivated the study of linear equations with non-integer solutions. At the start of the final lesson of the series, Courtney offered learners opportunity to appreciate how linear equations in which the unknown appeared on both sides could be transformed into more familiar equations with an unknown on one side. Again, this sense of a further class of equations that could be solved motivated the study – this time – of linear equations with an unknown on both sides.

Summary

These descriptions illustrate differences in how content was manifested in Ashley's series on geometry, Bernie's series on ratio and the two series on algebra. As Ayalon and Even (2013) noted in their study of lesson series on algebra, differences in how content is made available over time can be characterised by differences in the forms of content offered. While differences have been discerned, there is some alignment between all four lesson series and Stigler and Hiebert's (1999) characterisation of German teaching, which emphasises 'the rationale that underlies the procedure' and 'the precision with which the procedure is executed' (Stigler and Hiebert, 1999:26). But given the between-nation differences in teaching highlighted by Stigler and Hiebert (1999), the size of differences in how content was manifested between the four cases of the current study is likely to be smaller than the size of differences in a much wider sample.

What was absent from the four series of lessons was functional manifestation and in particular opportunities for learners to shift attention from confident manipulation of one topic to primitive knowing, image making and beyond of another without explicit teacher intervention. This is Pirie and Kieren's (1994) notion of *inventising* and resonates with Brown and Walter's (2005) 'what if not' questions (both discussed in Section 1.3.2), as well as paralleling the spiral image of Mason et al.'s (1982) manipulating-getting a sense of-articulating triad (see figure 2.3/b in Section 2.3.3). Examples of 'what if not' style questions relevant to the four lesson series are included in Appendix 7. It was beyond the scope of this study to investigate the absence of these manifestations, but this may be a product of external influences, such as a 'fragmented mathematics curriculum' (Watson 2002:156) and the pressures of examination expectations.

8.2.2 – How content was made available during classwork and seatwork

In Sections 8.1.2 and 8.1.3, I inferred that, within the confines of this study, how content was made available during classwork and seatwork appeared to vary most by teacher. The current section is therefore organised with this in mind, with the two series taught by Bernie discussed together.

Ashley's series on geometry

My analysis characterised Ashley's enacted interaction choices as foregrounding discussing-between-the-desks and being-watchful. Seatwork was emphasised over classwork the most across the three teachers: 84% of lesson time was seatwork. Just over half of the time allocated to seatwork featured Ashley teaching-between-the-desks. Typical exchanges were Ashley posing questions and affirming or probing learners' solutions to problems. These exchanges did not involve directing learners' attention to particular inner features of tasks. Like Mr. Hernandez in Henningsen and Stein's (1997) study, Ashley's actions generally afforded supporting learners' problem solving 'without reducing the complexity of the task at hand' (Henningsen and Stein, 1997:540). There were instances of Ashley telling or explaining between-the-desks, but these were less common. More than any of the other series in the study, Ashley's series on geometry afforded learners opportunity to work independently; 40% of lesson time overall was categorised as Ashley being-watchful.

Periods of seatwork were frequently followed by shorter periods of classwork, aligning with the practice of *hatsumon*, *kikan-shido* and *neriage* described by Shimizu (1999). Classwork was not evenly distributed across the five lessons; 30% of the first lesson was taught from-the-front, yet only 4% of the final lesson. Typical exchanges during classwork were either Ashley expounding, for example stating methods of construction in Lesson 1, or posing probing questions such as, "What is a circle?" or testing questions such as "What do you need to use from our list of things we have learnt about to do number 4?" Such exchanges generally emphasised visual and technical aspects of geometry, while functional aspects were generally emphasised during seatwork.

Bernie's series on ratio and algebra

Bernie's enacted interaction choices in both lesson series also foregrounded seatwork over classwork: 79% of lesson time in the series on ratio and 70% in the series on algebra was seatwork. Comparison of Bernie's two lesson series revealed similarities in the types of exchanges offered during seatwork but differences in those offered during classwork.

Explaining-between-the-desks was a particular feature of both series. Typically these exchanges involved Bernie summarising some key information about the problem and then posing a question or sequence of questions. The summarising frequently afforded focusing a learner's attention on a technical or more visual inner aspect of the problem, which could be interpreted as reducing the complexity of the problem (Henningsen and Stein, 1997) or more positively as adapting the problem so that together with the teacher it could be solved in a way that was not possible for the learner – at that stage at least – to do alone (Bruner, 1986). Of the four cases, there were fewer instances of learners working independently in Bernie's two series of lessons; just over a quarter of lesson time overall in each series was categorised as Bernie being-watchful.

Differences emerged between the two series of lessons taught by Bernie with respect to the types of exchanges offered during classwork. In the first two lessons of the series on ratio and throughout the series on algebra, around 30% of each lesson featured classwork; however, the final two lessons in the series on ratio featured much less classwork. Explaining or discussing from-the-front was more a feature of the series on ratio, and generally classwork in the form of sharing learners' solutions followed on from periods of seatwork. On the other hand, telling-from-the-front was a feature of the series on algebra, and generally whole-class demonstrations of methods preceded seatwork.

Courtney's series on algebra

As in the other three series of lessons, seatwork was emphasised over classwork in Courtney's series on algebra. However, this difference was less marked when compared to the other series: 65% of lesson time was seatwork. The proportion of lesson time given over to classwork was much higher (just under half) in the final two lessons of the series as new content was introduced. Telling-from-the-front hardly featured in the series at all. Rather, discussing and explaining from-the-front were particular features of Courtney's lessons. Typically these exchanges involved Courtney asking questions and building on learners' responses.

The greater emphasis on teaching-from-the-front in Courtney's series on algebra was balanced by much less teaching-between-the-desks. In comparison to Ashley and Bernie, Courtney engaged in much fewer mathematical exchanges during seatwork. However, it was noted during observation that Courtney was more likely to engage in extended exchanges with individual learners.

There was less regularity in the sequencing of classwork and seatwork in Courtney's series than the other three. A cycle of whole-class discussion followed by seatwork followed by whole-class feedback featured three times across the series, but there were also instances of whole-class feedback following on from seatwork and seatwork following on from whole-class explanation.

Summary

The theme of how content was made available during classwork and seatwork drew together a number of sources of variation. It took account of differences between classwork and seatwork along three dimensions: the proportion of lesson time allocated to each, the types of exchanges offered within each, and the forms of content made available within each. In the current study there was little variation in the types of exchanges offered across the four series of lessons overall, yet closer analysis of classwork and seatwork revealed differences.

Attending to this theme highlighted how Ashley offered structured exchanges (Watson, 2008 – see Section 1.4.1) with the whole class: planned cycles of problem posing, problem solving and solution sharing. This approach was reminiscent of Japanese teaching described by Stigler and Hiebert (1999). Courtney also offered exchanges with the whole class, placing a greater emphasis on teaching-from-the-front. Yet whole-class teaching plans were adapted in response to perceived developmental needs of learners; I characterised Courtney's approach to making content available as more responsive than structured. Bernie's approach might also be described as responsive in both series, but within overall structures that were different for the two classes. Bernie was more likely to address perceived developmental needs individually through an emphasis on explaining-between-the-desks. This contrasted with Courtney's more collective responses and Ashley's more pre-emptive approach.

8.3 – Methodological issues

I finally focus attention on methodological issues that have arisen around the analysis of the four cases. Four of these issues were identified in Chapter 3 in advance of the analysis: the interplay between the numerical and categorical data, the independence of coding interaction and manifestation for teaching-between-the-desks, detection of material differences, and coding when alternative pathways were offered. One further issue arose following analysis, which was the coding of functional-technical manifestation.

8.3.1 – Analysis of numerical and categorical data

Analysis of the numerical data of this study provided a means of expressing the relative stressing of the three elements of either the manifestation or interaction triad. It also provided a means of identifying material difference. Analysis of the categorical data offered different insights into the forms of content and types of exchanges made available. Thus the combination of the two analyses was an important feature of my analytical method, as I will now exemplify.

With regards to manifestation, the categorical data suggested that the visual element was over-stressed in Ashley's series on geometry. This was because the three visual categories (manifestation = VF, VT, VFT) were more likely to feature in this series on geometry than the other series on ratio and algebra. However, analysis of the numerical data revealed no material difference in the stressing of the visual element in Ashley's series and the two series on algebra. This did not represent inconsistency in the findings, as I will explain. Many of intervals coded as technical in the two series on algebra featured some stressing of the visual element (10% of all intervals in Courtney's series and 50% of all intervals in Bernie's series). Reflecting on my lesson observations, I would not say that imagery was emphasised *more* in Ashley's series (aligning with the inference drawn from attending to the relative stressing of the three elements) but that it was emphasised *differently* (aligning with the inference drawn from attending to the categorical data). Indeed, the interval data indicated Ashley's series featured more time allocated to given-tasks categorised as either VF or VT than the two series on algebra.

Turning to interaction, the stressing of teacher-learner and learner-content interaction may have corresponded to foregrounding both discussing-from-the-front and being-watchful or foregrounding discussing-between-the-desks or a mix of all three. In these four cases, analysis of the categorical data appeared more sensitive in detecting differences in interaction than analysis of the numerical data. This would not have been the case if the effect size threshold for material difference had been lower (see Section 8.3.3), but of course a lower threshold would have increased the risk of inferring an effect when there was not one.

8.3.2 – Independence of coding for interaction and manifestation

In Section 3.3.3, I questioned whether an association might be 'a feature of my analytical method rather than a classroom phenomenon' (p. 136). I was also concerned that a given coding of manifestation through teacher-talk might prompt me to code interaction in a particular way rather than attending closely to the coding rules. The combination of three cases where an association between discussing-between-the-desks and actions affording focusing learners' attention was inferred and one when it was not offers evidence against this concern. For if an association were a feature of either the analytical method or consistent slips in attending to coding rules, the association would surely be inferred in *every* case.

<u>*8.3.3 – Detection of material differences*</u>

Differences in how content was manifested from lesson to lesson were revealed in each case through:

- ▷ the changing relative stressing of the three elements of the manifestation triad (attending to the ζ values of the lesson centres for manifestation)

and

- ▷ variation in the distribution of manifestations offered (attending to the κ values for the seven manifestation categories at the lesson level)

Revealed differences were marked as material through statistical analysis of the numerical codes for each half-minute interval (the ζ values of the interval centres for manifestation). This analysis tested the null hypothesis that there were no lesson-to-lesson differences in the ζ values for the visual, functional and technical element of the manifestation triad respectively. A rejection of the null hypothesis at the 5% level combined with an effect size greater than 0.2 for any one of these elements led to lessons being classed as materially different. Without the effect size constraint, the statistical test would only have detected difference rather than detecting differences large enough to correspond to qualitative differences in how content was manifested. The selection of an effect size of 0.2 was arbitrary, although informed by the convention that an effect size between 0.1 and 0.3 is considered a small effect. In the context of lesson-to-lesson differences, an effect size of 0.2 was small enough in order to detect observed qualitative differences in how content was manifested (in other words ensured face validity) but large enough in order to ensure sensitivity.

The same approach was used in order to detect material differences between pairs of lesson series. The requirement of an effect size of at least 0.2 in order to mark pairs of lesson series as materially different based on the coding of half-minute intervals again seemed sensible when the series centres for manifestation are compared. Figure 8.3/a reproduces figure 8.1/a but with lines marked highlighting equal stressing of the visual (green), functional (red) and technical elements of manifestation.

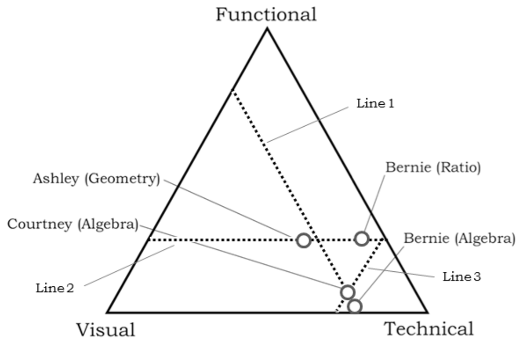


Figure 8.3/a
The four series centres for manifestation and lines of equal stressing of visual (line 1), functional (line 2) and technical (line 3) elements of the manifestation triad.

Clear visual differences in the position of series centres corresponded to material differences revealed by the analysis. The line 1 in figure 8.3/a suggests that the visual element was similarly stressed in the two series on algebra and Ashley's series on geometry; the analysis suggested that only Bernie's series on ratio was materially different to the other three series with respect to the visual element of manifestation. The line 2 in figure 8.3/a suggests that the functional element was similarly stressed in Ashley's series and Bernie's series on ratio; this was the only pair of series that the analysis revealed as not materially different with respect to the functional element. The line 3 in figure 8.3/a suggests that the technical element was slightly more stressed in Bernie's series on algebra and slightly less stressed in Bernie's series on ratio when compared to Courtney's series; the analysis revealed Bernie's two series not to be materially different to Courtney's with respect to the technical element (although Bernie's two series were revealed to be materially different to each other). The material difference analysis affords detection of differences in central tendency and spread, however a feature of the numerical data for the relative stressing of each element of the manifestation triad at the series level was that series similarly centred with respect to one of the elements were also similarly distributed. Therefore in this particular situation material difference was effectively a quantification of difference in the series centres.

8.3.4 – Coding when alternative pathways were offered

Courtney's series on algebra presented a particular coding challenge, as in three segments of lessons alternative pathways for learners were offered through the teacher making available different given-tasks. In one segment this turned out not to be at all problematic, since both given-tasks were coded as technical manifestation. In each of the other two segments, one given-task was an exercise in expanding brackets (coded as technical manifestation) and the other afforded attending to properties of solutions (coded as visual-technical manifestation). In both cases the affordance of attending to properties of solutions might not have been realised if learners approached the task procedurally, and indeed my observations suggested that this appeared to be the case for some learners. The overall coding – technical manifestation – for the combined given-tasks in each segment therefore appeared appropriate in this situation. At the same time, the affordance of attending to properties of solutions was recognised by the numerical coding (the numerical half-minute interval code for the combined given-tasks was W_V=3, W_F=0, W_T=9 – see Section 3.2.3). I was therefore satisfied that the combination of categorical and numerical codes appropriately characterised the alternative pathways Courtney offered.

I am sceptical that my analytical approach would characterise ways of making content available that featured multiple pathways of different forms of content, such as individualised approaches that involved extensive differentiation by task, in a meaningful way. I suggest that my approach is most appropriate for understanding how content is made available during whole-class teaching. By 'whole-class' I mean teaching in which classwork is directed to the whole class and seatwork involves all learners working on the same or similar given-tasks.

8.3.5 – Coding functional-technical manifestation

A limitation of the coding method was highlighted by the between-case analysis. A finding presented in Section 8.1.1 was that functional-technical manifestation was most likely to feature in Bernie's series on ratio, although it featured strongly in Ashley's series too. However, the given-tasks categorised as functional-technical in Bernie's series afforded focusing learners' attention on adaptive using differently to those in Ashley's series[69]. For example, the following pairs of problems were taken from given-tasks offered by Ashley and Bernie respectively:

> ABCD is a rectangular field [AB is 85 metres, AD is 45 metres]. Treasure is hidden in the field! The treasure is at equal distance from the sides AB and AD… [and] is also 60 metres from the corner C… Mark the position of the treasure with an x.
>
> Two radio stations 70 km apart broadcast over distances of 45 km and 55 km. Using a scale of 1 cm to 10 km draw the loci of the limits of their broadcasting ranges, shade in the area where both stations can be heard.
>
> (Worksheet, Ashley's Year 11 Geometry Lesson 4)

> Pocket money is split between Pete, Alan and Helen in the ratio 2 : 3 : 4. Dad pays out £36, how much does each person get?
>
> Mr. Allen, Mr. Book and Mr. Collins own 2, 5 and 7 parts of a business respectively. The business makes £420 profit a week. How much does each man get?
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 2)

All four problems were coded FT because their outer aspects (relating to fields, radio stations, pocket money and profit) were distinct from inner technical aspects (constructing angle bisectors and circles, sharing in a given ratio). The examples from Ashley's given-task called for representation of the problem (by recognising the technical inner aspect) and the selection of an appropriate method in order to solve

[69] It is worth stating that this discussion is a critique of the coding method and not of the given-task offered by Bernie.

the problem. The representation aspect was complex and called for reasoning. There were very good reasons for coding this given-task FT and it afforded focusing learners' attention on adaptive using.

As part of a series of similar problems within the given-task, the problems offered by Bernie afford simple (as opposed to complex) representation. The amount to be shared was marked in pounds and the relative shares were listed as numbers: the problems afforded solving without attending to their wording. Indeed, the profit-sharing problem failed to specify that the £420 was to be shared in the ratio of the parts of the business owned. Yet there was no evidence from the transcript of this being questioned either by Bernie or the learners, so there was no evidence to contradict the idea that learners followed the pattern of the exercise and shared 420 in the ratio 2 : 5 : 7. There were good reasons why this given-task was not coded T, but the extent to which this manifestation afforded focusing learners' attention on *adaptive* using of methods (rather than just using) was questionable. That having been said, Bernie's teaching-between-the-desks did often focus on representing the problems visually first (returning to the image of moneybags), which suggests that representation of the problem was not straightforward for some learners. This may particularly have been the case for those who did not attend to the pattern of problems in the exercise.

This discussion will prompt further reflection on the coding method in the future. This will focus on the possibility of breaking down the FT code further and assigning different numerical codes (the functional-technical category of manifestation would remain as a useful label). A reliable distinction could be drawn between given-tasks featuring problems where the inner aspect is a single method (such as Bernie's task) and those where the inner task could be one of a range of methods (such as Ashley's task).

CHAPTER 9

CONCLUSION

Chapter contents

9.1 – Conclusion

In this final chapter by way of a summary, I offer an overview of the study and succinctly bring together the key findings. I reiterate the issue I was investigating and the methodological/analytical approach I developed in order to study it. I suggest ways in which aspects of this approach might have applications beyond the current study. I state what I have learnt through this investigation about differences in how teachers make content available to learners and discuss the implications of these findings for future research and teacher education.

9.1.1 – The object of enquiry

This study has been an exploration of how teachers make content available to learners over time. In particular it was an investigation into the types of exchanges – or *modes of teacher interaction* – teachers offer in the classroom and the forms of mathematical content – or *manifestations* – they make available. Moreover, it was an investigation into *differences* in how teachers make content available over time. In selecting this object of enquiry, I was making a conscious choice to focus my attention on the actions of the teacher and discern variation in these actions. Attending to teacher and learner actions risked introducing too many dimensions of variation in order to sustain focused attention. Yet overlooking that teachers act within a didactical system would risk oversimplifying the complexity of the classroom environment. The 'middle way' was to adopt an ecological perspective and to focus on what was afforded to the teacher by the classroom environment.

Prior empirical research that informed this study compared teaching practices internationally (such as Stigler and Hiebert (1999) and Andrews, P. and Sayers (2005)), nationally (such as Askew et al. (1997)) or locally (such as Ayalon and Even (2013) and Watson and De Geest (2012)). These comparisons tended to use either the lesson as the unit of analysis, or compare teaching of the same topic, or account for differences with reference to teacher beliefs, cultural diversity or both. I proposed that how teachers make content available to learners can change from lesson to lesson, so my study used the lesson series as unit of analysis. I also wanted to make comparisons between how content was made available when teaching different topics. Furthermore, an ecological perspective of decision-making suggested different affordances being perceived by the same teacher in different classroom situations. In particular this highlighted the possibility of differences in how the same teacher makes content available when working with different classes.

9.1.2 – Analytical approach

In order to investigate how teachers make content available to learners, I developed an original analytical approach. I conceptualised both interaction and manifestation as having an underlying triadic structure. The types of exchanges offered over time were characterised as a relative stressing of teacher-content, teacher-learner and learner-content interaction. The forms of content made available over time were characterised as a relative stressing of visual, functional and technical manifestation. Consideration of the over-stressing of one or two of the elements of each triad or a balance of all three led to the identification of seven categories of interaction and a further seven of manifestation. A strength of this way of conceptualising teaching practices is that it provides a common vocabulary for describing teaching decisions at a range of grain sizes. For example, at the series or lesson level, choices about sequencing of forms of content, including exploiting opportunities to return to forms of content offered previously in the lesson series perhaps when teaching-between-the-desks, can be articulated with reference to manifestation categories. At the lesson or segment level, choices about modes of teacher interaction can be articulated with reference to interaction categories. Finer-grain in-flight choices to teach between-the-desks can be articulated with reference to switching between manifestation categories and also the mode of teacher interaction. Furthermore, attending to forms of content – rather than content itself – provides a vocabulary for describing the teaching of any topic.

Teacher-talk and given-tasks constituted the data for how content was made available. The categorical model offered a way of coding this data. In order to ensure that data was coded reliably, every half-minute interval of lesson time in which content was made available to learners was closely analysed. Each interval was coded for interaction and manifestation, and categorical codes were quantitised as numerical codes. Analysis of the categorical codes afforded lesson-to-lesson and series-to-series comparisons using Pearson's chi-square test. Equivalent comparisons were made through Wilcoxon's rank-sum test being applied to the numerical codes. Interrogation of comparisons based on the categorical and numerical codes offered deeper insights than consideration of the two separately. This approach to analysis represents an important contribution to research in this field, as it is sufficiently sensitive in order for subtle differences in enacted teaching decisions to be inferred. It allowed me to identify sources of variation in how content was made available that the teacher may not have – but could have – attended to when planning or in-flight while teaching.

The analytical approach also makes a methodological contribution to mixed methods research as it offers a way of combining qualitative and quantitative methods in order to characterise over time any phenomenon with an underlying triadic structure as a relative stressing of the three elements of the triad. Mathematics Education Research offers a number of such triadic structures for characterising teaching and learning such as the 'didactical triangle' (discussed in Chapter 2), the 'epistemological triangle' (Steinbring, 2008), the 'structure of a topic' (Mason and Johnston-Wilder, 2004) and the 'teaching triad' (Jaworski, 1994).

As an example of a potential application of my approach, I refer to a study by Potari and Jaworski (2002) that uses the teaching triad both as an analytical and teaching development tool. The three elements of the teaching triad in that study are *management of learning*, *sensitivity to students* and *mathematical challenge*. The precise meanings attached by the authors to these three elements is not central to the current discussion, but details can be found in Potari and Jaworski (2002). Accounts of two episodes within the same lesson series are provided, the first of which demonstrates 'the three elements of the teaching triad in harmony' (Potari and Jaworski, 2002:365) while the second features a 'lack of balance' in which 'management of learning dominates' and 'mathematical challenge seems to disappear' (ibid, p.368). The authors choose to compare these two episodes on the basis of a close qualitative analysis, but a different analysis could include quantitising. The relative emphasising of the three elements of the teaching triad could then be expressed as a Barycentric co-ordinate for each episode, providing a quantitative comparison[70]. The mixing of qualitative and quantitative analysis affords insight into the teacher's practice that may not be afforded by qualitative analysis alone (Teddlie and Tashakkori, 2003).

Returning to the current study, analysis brought to the surface two particular limitations. Firstly, while provision was made for coding two or more concurrent given-tasks, I recognised that the analytical approach was most suited to being applied to series of lessons that featured whole-class teaching, or at least that featured seatwork during which all learners worked on the same or similar given-tasks. Secondly, I became aware that the categorical codes for manifestation were broad, albeit well defined. This meant that subtly different given-tasks were assigned the same code. This raised the possibility of assigning multiple numerical codes to the same categorical code when using this analytical approach in the future.

[70] These points could be plotted in a tri-polar space with poles labelled management of learning, sensitivity to students and mathematical challenge. Assuming alignment between qualitative and quantitative analysis of the episodes, the first episode may well be characterised by a point within the central region of the tri-polar space, whereas the second may well be characterised by a point near to the 'management of learning' pole.

9.1.3 – Case studies

In order to create a richer context for the investigation, I purposefully designed the study so that it would involve four cases. Three were series of lessons taught in the same school; two cases featured the same teacher teaching different topics; two cases featured different teachers teaching the same topic. The three teachers were experienced and the predicted average attainment at age 16 of all four classes was similar. Both schools served similar catchment areas within the same county. Yet how content was made available to learners over the four series of lessons was found to be different in many respects.

In each case study I offered a pre-analytical narrative account of the lesson series along with a post-analytical structural account. The juxtaposition of narrative and structural accounts allowed for validation of inferences drawn from analysis of each case and from cross-case comparisons. The structural accounts in themselves represent an important contribution to research within this field since they offer a non-content-specific language for describing teaching. Findings that resulted from statistical analysis of categorical and numerical codes offered further contributions to the field by identifying six sources of variation in how content was made available to learners.

9.1.4 – Findings

1. Each lesson series in the study was materially different to the other three with respect to forms of content made available and amount of lesson time given over to them. The least different of the four lesson series were the two where the common feature was the topic.

2. Within each lesson series there were lessons that were materially different with respect to forms of content made available and amount of lesson time given over to them. Each lesson was materially different to every other lesson in the same series in three out of the four cases[71].

3. There was no difference in the relative emphasis each teacher placed on teacher-content, teacher-learner and learner-content interaction across the four lesson series as a whole. There were also no lesson-to-lesson differences in the relative emphasis each teacher placed on teacher-content, teacher-learner and learner-content interaction in the two series on algebra.

4. However, there were differences between cases in the particular types of exchanges offered and amount of lesson time given over to them across the lesson series. In particular, there were differences between cases in the particular types of exchanges offered during classwork (telling, explaining and discussing) and differences between the three teachers in the particular types of exchanges they offered during seatwork (explaining, discussing and being-watchful).

[71] A corollary of this finding is that a single lesson or part of a lesson is unlikely to be representative of how the teacher makes content available generally. This may have consequences for research in this field in which the unit of analysis is a lesson or smaller.

5. There were material differences in the forms of content made available during classwork and seatwork in three out of four cases. In each of these three cases, functional aspects of the topic were more emphasised during seatwork. Visual aspects were emphasised more during classwork in two of the cases.

6. There were differences between the three teachers' teaching-between-the-desks. One teacher engaged in relatively little teaching-between-the-desks, and was more likely to address issues relating to a given-task collectively from-the-front; one engaged in more discussion, associated with focusing learners' attention on the outer aspect of the given-task; one engaged in more explanation, associated with shifting learners' attention to inner aspects of the given-task.

9.1.5 – Implications of findings for research and teacher education

The explorative nature of this investigation has opened up the possibility of a number of further lines of enquiry that were beyond the scope of the current study. Not all manifestations afforded to the three teachers were realised (Finding 1). Whether these affordances were perceived or not and, if they were, why these manifestation choices were not enacted could be investigated. Differences in the forms of mathematical content offered during classwork and seatwork (Finding 5) gets to the heart of the purpose of these two types of engagement from a didactic perspective. From other perspectives, classwork may have a specific purpose such as offering learners extrinsic motivation to engage in mathematical activity. But what forms of content is it necessary for the teacher to make available through classwork rather than seatwork, if any? And what forms of content is it necessary for the teacher to make available through seatwork rather than classwork, if any? The small sample in this study raised the conjecture that making available visual manifestation might be better served by classwork rather than seatwork. This is no more than a conjecture, but one that might be explored empirically through the analytical approach used in this study. Teaching-between-the-desks (Finding 6) is under-researched as an object of enquiry in its own right. I emphasise 'in its own right' because many studies do analyse the exchanges enacted when teaching-between-the-desks in the course of investigating different phenomena. The different ways in which Ashley, Bernie and Courtney made content available between-the-desks across their series of lessons contributes to our limited understanding of this issue and opens up possibilities of further research.

Together the six findings contribute to how the practice of each of these teachers might be understood. However, they have wider implications than this. They highlight that the forms of content that a teacher makes available from lesson to lesson, the types of exchanges offered, the forms of content made available during classwork and seatwork, and the nature of teaching-between-the-desks are sources of variation that contribute to differences in how content is made available over time. These four phenomena may not be features of the classroom situation to which teachers and classroom researchers routinely attend. However, they are features to which all *could* attend. Exploring these phenomena further affords contributing to a more extensive understanding of the range of possible ways of making content available. Such research could also contribute to teacher education through opening up possibilities for a teacher to choose to act differently, particularly when comparisons are made between how they make content available when teaching a topic and (i) how a colleague makes the same content available, or (ii) how they make the same content available when teaching a different class, or (iii) how they make content available when teaching a different topic.

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APPENDIX

APPENDIX 1

A TECHNICAL OVERVIEW OF THE STUDY FOR TEACHER PARTICIPANTS

(a) This study is concerned with how teachers *make content available* to students during a *sequence of lessons* on a particular *topic*.

(b) *Mathematical classroom activity* involves the coming together of teacher(s), student(s) and mathematics within a classroom.

(c) A mathematics *lesson* is a well-defined unit of time within which *mathematical classroom activity* occurs. A *lesson* is formed by the union of *segments*, where a *segment* is a sub-division of a lesson within which a well-defined *mathematical classroom activity* occurs, for example teacher exposition or activity prompted by a *task*. The union of *lessons* on a particular topic forms a *sequence of lessons*.

(d) A *topic* is taken to be a sub-division of mathematics. The curriculum is organised around topics, with the intention that aspects of general mathematical proficiency are brought to the surface through topic-based study. The part of a *topic* that forms the object of a *segment*, *lesson* or *sequence of lessons* is referred to as the *subject matter*.

(e) Teachers *transform* the subject matter into *content*: something that has pedagogical potential. Even if this involves the selection of a textbook exercise, this decision has nevertheless been privileged over other choices on the basis of the specific teacher-student-mathematics situation. In *transforming* the subject matter, the teacher makes *particular forms of content available*: for example mental imagery, common misconceptions, techniques, technical language, applications, or root problems.

(f) *Making content available* to students also involves the teacher privileging certain types of interaction over others. Examples of ways in which students may encounter the *content* include: through a teacher presentation, through a discussion with the teacher, in providing feedback to or seeking affirmation from the teacher, or by engaging with a mathematical task.

Coding Rule Book

Manifestation - Additional categorisation guidance

Table A1/a Categorising teacher-talk during classwork

Teacher-talk is categorised…	… when it concerns:
Visual	Dynamic imagery Visual justification
Functional	Applications within mathematics
Technical	Specific actions (specialising, using a technique) Directing attention to specific actions when using visual methods (e.g. to fill in this cell I do…?) Definitions Addressing technical errors Using formal language and definitions Emphasising relevant algebraic manipulation Emphasising relevant calculations Emphasising inner method when reviewing a given-task with distinct outer aspect (FT, VT)
Visual-technical	Representations (e.g. do we treat numbers and letters differently?) Generalised actions (e.g. categorising, obstacles to generating examples) Verbal descriptions of objects and features Using visual representations in order to perform techniques Verbal representations (e.g. 'for every…' in ratio) Generating examples Comparing alternatives (e.g. offering a range of possible solutions for consideration) Making sense of a method
Functional-technical	Interpreting results in context (e.g. money 1.2 interpreted as £1.20) Reviewing a utility given-task without emphasising method

Table A1/b Categorising given-tasks during seatwork

Given-tasks are categorised…	… when they concern:
Functional-technical	Generating examples where the example-of-what is not explicitly stated Drawing on familiar **formal** methods in order to address an unfamiliar problem
Visual-functional	Drawing on **informal** methods in order to address an unfamiliar problem Some learners using formal methods but this is not highlighted in what the teacher says to the class
Visual-technical	Generating examples

Table A1/c Coding teacher-talk during seatwork when the given-task is categorised as functional-technical

CAT code for teacher-talk is…	… when it concerns:
Ø	Talking about aspects of solving the problem not directly relevant to the underlying topic (e.g. suggesting a scale when constructing) Drawing attention to the problem (e.g. the question says…, we want…)
T	Emphasising an inner aspect related to a specific action (e.g. you need to perform this step) Returning to previously-offered technical manifestations Emphasising symbolic notation Representing the wording of a problem symbolically
VT	Emphasising an inner aspect related to a general action (e.g. it's about simplifying, what method did you use?) Comparing (e.g. it's like…, it's kind of…, what type of thing…?) Addressing misconceptions Emphasising visual methods Emphasising representations and images Returning to previously-offered visual-technical manifestations Representing the solution to a problem visually

Table A1/d Coding teacher-talk during seatwork when the given-task is categorised as visual-functional

CAT code for teacher-talk is…	… when it concerns:
T	Emphasising a technical aspect or formal method related to accomplishing the task
VT	Emphasising a representation of the emerging mathematical phenomenon

Table A1/e Coding teacher-talk during seatwork when the given-task is categorised as visual-technical

CAT code for teacher-talk is…	… when it concerns:
T	Emphasising a formal method Directing attention to specific actions when using diagrammatic representations

Table A1/f Coding teacher-talk during seatwork when the given-task is categorised as technical

CAT code for teacher-talk is…	… when it concerns:
FT	Drawing on a familiar formal method in an unfamiliar situation
VT	Emphasising visual methods Visual representations and images Returning to previously offered visual-technical manifestations

COMPLEX CODING RULES

The following coding rules were adopted for concurrent given-tasks categorised X and Y (where X and Y might be any of the categories in table 3.2/c):

1. If an interval concerns the teacher ensuring learners are on task or engaging in non-task-specific checking, the interval NUM code for given-task is the mean of the Barycentric co-ordinates for X and Y (see table 3.2/d) and the interval NUM code for teacher-talk is (0, 0, 0).

2. If there is teacher-talk between-the-desks (categorised as manifestation = Z) representing a *switch* in manifestation relating to one of the concurrent tasks, the interval NUM code for given-task is the mean of the Barycentric co-ordinates for X and Y (see table 3.2/d) and the NUM code for teacher-talk is the Barycentric co-ordinate for Z.

3. If there is teacher-talk between-the-desks *focusing* on one of the concurrent tasks (the one categorised as X) rather than representing a switch in manifestation, the interval NUM code for given-task is a third of twice the Barycentric co-ordinate of X plus the Barycentric co-ordinate of Y (see table 3.2/d) and the interval NUM code for teacher-talk is (0, 0, 0).

Two further situations concerning the coding of given-tasks and teacher-talk arose during analysis that called for further rules to be defined. The first concerns the teacher introducing an additional given-task during seatwork and the second concerns the teacher concluding a period of seatwork through reviewing a given-task from-the-front. The following coding rules were adopted for these situations:

4. If a new given-task (categorised as manifestation = Y) is introduced verbally by the teacher between-the-desks in addition to an ongoing given-task (categorised as manifestation = X), the interval NUM code for given-task is the mean Barycentric co-ordinate of X and Y (see table 3.2/d) and the interval NUM code for teacher-talk is (0, 0, 0).

5. When reviewing a given-task from-the-front, only teacher-talk within the interval is coded. There is no retrospective acknowledgement of the given-task unless the teacher draws attention to it (‘drawing attention’ includes for example the teacher asking for and being provided with simple answers to exercises).

Finally, when an interval features both classwork and seatwork, the classwork and seatwork elements are coded separately following the usual coding rules in order to arrive at an interval NUM code for overall manifestation during classwork and another for seatwork. The interval centre for manifestation is then the mean of these two co-ordinates.

Interaction

Table A1/g The rules for categorising half-minute intervals for interaction for intervals that feature a transition from teaching-from-the-front to seatwork or vice versa.

Mode of teacher interaction present in the interval		Interval CAT code	Interval NUM code
and	telling-from-the-front telling-between-the-desks	TC	(¾, 0, ¼)
and	telling-from-the-front being-watchful	CC	(½, 0, ½)
and	telling-from-the-front explaining-between-the-desks	CC	(½, ⅙, ⅓)
and	telling-from-the-front discussing-between-the-desks	OO	(½, ¼, ¼)
and	explaining-from-the-front telling-between-the-desks	OO	(½, ¼, ¼)
and	explaining-from-the-front explaining-between-the-desks	OO	(⅓, ⅓, ⅓)
and	explaining-from-the-front discussing-between-the-desks	OO	(¼, ½, ¼)
and	explaining-from-the-front being-watchful	OO	(¼, ¼, ½)
and	discussing-from-the-front telling-between-the-desks	OO	(¼, ½, ¼)
and	discussing-from-the-front explaining-between-the-desks	LL	(⅙, ½, ⅓)
and	discussing-from-the-front being-watchful	LL	(0, ½, ½)
and	discussing-from-the-front discussing-between-the-desks	TL	(0, ¾, ¼)

APPENDIX 2: Case study 1 – Results in tabular form

A2.1 – Manifestation

Table A2/a The relative stressing of the three elements of the manifestation triad across Ashley's Year 11 lesson series on geometry.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 487)	0.26	0.24	0.50
Lesson 1 (N = 104)	0.43	0.20	0.37
Lesson 2 (N = 112)	0.37	0.06	0.57
Lesson 3 (N = 107)	0.28	0.27	0.45
Lesson 4 (N = 109)	0.09	0.35	0.56
Lesson 5 (N = 55)	0.05	0.44	0.51

Note: The figures in the table are ζ values.

Table A2/b The spread of manifestation choices across Ashley's Year 11 lesson series on geometry, represented by the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 487)	1	0	20	12	23	22	22
Lesson 1 (N = 104)	2	0	18	36	36	0	8
Lesson 2 (N = 112)	1	0	28	0	53	6	12
Lesson 3 (N = 107)	1	0	17	21	11	8	42
Lesson 4 (N = 109)	0	0	26	0	3	43	28
Lesson 5 (N = 55)	0	0	6	0	0	76	18

Note: The figures in the table are κ values (percentages).

Table A2/c Comparison of lessons within Ashley's Year 11 lesson series on geometry based on the emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
L1 × L2 (104, 112)	DIFF W = 12791.5 *** r = 0.258	DIFF W = 12897.5 *** r = 0.282	DIFF W = 9085 *** r = -0.361
L1 × L3 (104, 107)	DIFF W = 13631.5 *** r = 0.444	ND W = 9922.5 ** r = -0.181	ND W = 10188.5 * r = -0.141
L1 × L4 (104, 109)	DIFF W = 15816 *** r = 0.761	DIFF W = 9016.5 *** r = -0.341	DIFF W = 9054 *** r = -0.353
L1 × L5 (104, 55)	DIFF W = 10845 *** r = 0.792	DIFF W = 6776.5 *** r = -0.487	DIFF W = 7554.5 ** r = -0.248
L2 × L3 (112, 107)	DIFF W = 13807.5 *** r = 0.228	DIFF W = 8628 *** r = -0.571	DIFF W = 13801.5 *** r = 0.254
L2 × L4 (112, 109)	DIFF W = 17084.5 *** r = 0.689	DIFF W = 7364.5 *** r = -0.755	ND W = 12555.5 r = 0.022
L2 × L5 (112, 55)	DIFF W = 12068 *** r = 0.731	DIFF W = 6503.5 *** r = -0.826	DIFF W = 10066.5 ** r = 0.243
L3 × L4 (107, 109)	DIFF W = 14817.5 *** r = 0.510	DIFF W = 10120 *** r = -0.241	DIFF W = 10244.5 *** r = -0.243
L3 × L5 (107, 55)	DIFF W = 10688 *** r = 0.582	DIFF W = 7144 *** r = -0.474	ND W = 8443 r = -0.098
L4 × L5 (109, 55)	ND W = 9465 * r = 0.162	DIFF W = 7957 *** r = -0.321	DIFF W = 9594.5 ** r = 0.239

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

A2.2 – Interaction

Table A2/d The relative stressing of the three elements of the interaction triad across Ashley's Year 11 lesson series on geometry.

	Mode of teacher interaction		
	Teacher-Content	Teacher-Learner	Learner-Content
Series (N = 487)	0.16	0.24	0.60
Lesson 1 (N = 104)	0.25	0.29	0.46
Lesson 2 (N = 112)	0.19	0.20	0.61
Lesson 3 (N = 107)	0.11	0.19	0.70
Lesson 4 (N = 109)	0.15	0.29	0.56
Lesson 5 (N = 55)	0.05	0.21	0.74

Note: The figures in the table are ζ values.

Table A2/e The spread of interaction choices across Ashley's Year 11 lesson series on geometry, represented by the coding of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 487)	5	3	8	9	12	23	40
Lesson 1 (N = 104)	11	6	13	12	4	23	31
Lesson 2 (N = 112)	8	2	8	11	15	15	41
Lesson 3 (N = 107)	2	4	4	5	8	22	55
Lesson 4 (N = 109)	4	0	9	13	17	31	26
Lesson 5 (N = 55)	0	0	4	2	16	27	51

Note: The figures in the table are κ values (percentages).

Table A2/f Comparison of lessons within Ashley's Year 11 lesson series on geometry based on the emphasis placed upon the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner	Learner-Content
L1 × L2	ND	ND	ND
	W = 12006	W = 11948	W = 10008 **
(104, 112)	r = 0.118	r = 0.11	r = -0.199
L1 × L3	DIFF	ND	DIFF
	W = 12451.5 ***	W = 11713	W = 8920 ***
(104, 107)	r = 0.252	r = 0.121	r = -0.341
L1 × L4	ND	ND	ND
	W = 12209.5 **	W = 10886	W = 10008 **
(104, 109)	r = 0.185	r = -0.04	r = -0.199
L1 × L5	DIFF	ND	DIFF
	W = 9386.5 ***	W = 8527	W = 7131.5 ***
(104, 55)	r = 0.348	r = 0.065	r = -0.359
L2 × L3	ND	ND	ND
	W = 13079	W = 12355.5	W = 11485
(112, 107)	r = 0.130	r = 0.006	r = -0.129
L2 × L4	ND	ND	ND
	W = 12822.5	W = 11306.5 **	W = 12967
(112, 109)	r = 0.064	r = -0.173	r = 0.083
L2 × L5	DIFF	ND	ND
	W = 10155.5 **	W = 9219.5	W = 8845 *
(112, 55)	r = 0.238	r = -0.056	r = -0.161
L3 × L4	ND	ND	DIFF
	W = 11240	W = 10534 *	W = 13070.5 ***
(107, 109)	r = -0.067	r = -0.174	r = 0.236
L3 × L5	ND	ND	ND
	W = 9078.5	W = 8538.5	W = 8591.5
(107, 55)	r = 0.130	r = -0.057	r = -0.039
L4 × L5	ND	ND	DIFF
	W = 9538.5 *	W = 9385.5	W = 8128.5 ***
(109, 55)	r = 0.186	r = 0.114	r = -0.265

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

Table A2/g Comparison of the relative stressing of learner-content interaction in the first sub-segment and the segment as a whole across Ashley's Year 11 lesson series on geometry.

Stressing of learner-content interaction in the first sub-segment is…	Percentage (N=9)
Greater than the segment as a whole	67 %
Equal to the segment as a whole	22 %
Less then the segment as a whole	11 %

Table A2/h The relative stressing of the three elements of the manifestation triad across Ashley's Year 11 lesson series on geometry, grouped by intervals categorised as classwork and seatwork.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 487)	0.26	0.24	0.50
Classwork (N = 77)	0.09	0.35	0.56
Seatwork (N = 410)	0.05	0.44	0.51

Table A2/i Comparison of seatwork and classwork across Ashley's Year 11 lesson series on geometry based on the emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
Seatwork × Classwork	DIFF	DIFF	ND
	W = 95168 ***	W = 106131 ***	W = 99712
(410, 77)	r = -0.205	r = 0.256	r = -0.016

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals given over to seatwork and classwork.

APPENDIX 3: Case study 2 – Results in tabular form

A3.1 – Manifestation

Table A3/a How the shape of choices over the lesson series aligned with a relative stressing of the three elements of the manifestation triad.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 383)	0.08	0.25	0.67
Lesson 1 (N = 73)	0.16	0.09	0.75
Lesson 2 (N = 99)	0.08	0.18	0.74
Lesson 3 (N = 111)	0.07	0.35	0.58
Lesson 4 (N = 100)	0.02	0.35	0.63

Note: The figures in the table are ζ values.

Table A3/b The spread of manifestation choices over the lesson series, represented by the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 383)	0	0	52	0	3	30	15
Lesson 1 (N = 73)	0	0	77	0	16	6	1
Lesson 2 (N = 99)	0	0	63	0	0	20	17
Lesson 3 (N = 111)	0	0	29	0	0	44	27
Lesson 4 (N = 100)	0	0	51	0	0	41	8

Note: The figures in the table are κ values (percentages).

Table A3/c Comparison of lessons within the lesson series based on the attributed emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
L1 × L2 (73, 99)	DIFF W = 7120.5 ** r = 0.217	DIFF W = 5419 ** r = -0.235	ND W = 6366.5 r = 0.013
L1 × L3 (73, 111)	DIFF W = 7799.5 *** r = 0.253	DIFF W = 3686.5 *** r = -0.671	DIFF W = 8784.5 *** r = 0.464
L1 × L4 (73, 100)	DIFF W = 7937 *** r = 0.47	DIFF W = 3525 *** r = -0.694	DIFF W = 7558 *** r = 0.303
L2 × L3 (99, 111)	ND W = 10652.5 r = 0.041	DIFF W = 7841 *** r = -0.435	DIFF W = 12705 *** r = 0.394
L2 × L4 (99, 100)	DIFF W = 11032 *** r = 0.278	DIFF W = 7536 *** r = -0.44	DIFF W = 11284 *** r = 0.26
L3 × L4 (111, 100)	DIFF W = 12833 *** r = 0.238	ND W = 11855 r = 0.016	DIFF W = 10609 ** r = -0.209

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

A3.2 – Interaction

Table A3/d How the shape of choices over the lesson series aligned with a relative stressing of the three elements of the interaction triad.

	Mode of teacher interaction		
	Teacher-Content	Teacher-Learner	Learner-Content
Series (N = 383)	0.19	0.28	0.53
Lesson 1 (N = 73)	0.24	0.29	0.47
Lesson 2 (N = 99)	0.19	0.37	0.44
Lesson 3 (N = 111)	0.17	0.22	0.61
Lesson 4 (N = 100)	0.19	0.26	0.55

Note: The figures in the table are ζ values.

Table A3/e The spread of interaction choices over the lesson series, represented by the coding of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 383)	2	10	9	8	32	12	27
Lesson 1 (N = 73)	6	15	8	8	28	12	23
Lesson 2 (N = 99)	4	9	19	6	29	12	21
Lesson 3 (N = 111)	1	6	5	10	32	12	34
Lesson 4 (N = 100)	0	10	5	8	39	12	26

Note: The figures in the table are κ values (percentages).

Table A3/f Comparison of lessons within the lesson series based on the attributed emphasis placed upon the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner	Learner-Content
L1 × L2 (73, 99)	ND W = 6728.5 r = 0.106	ND W = 5946.5 r = -0.09	ND W = 6466.5 r = 0.039
L1 × L3 (73, 111)	ND W = 7262 r = 0.115	ND W = 7274 r = 0.115	ND W = 5934.5 * r = -0.187
L1 × L4 (73, 100)	ND W = 6586 r = 0.059	ND W = 6495 r = 0.035	ND W = 5895 r = -0.118
L2 × L3 (99, 111)	ND W = 10475 r = 0.005	DIFF W = 11662 ** r = 0.201	DIFF W = 9107 *** r = -0.229
L2 × L4 (99, 100)	ND W = 9583 r = -0.06	ND W = 10591.5 r = 0.126	ND W = 9064 * r = -0.16
L3 × L4 (111, 100)	ND W = 11367 r = -0.067	ND W = 11203.5 r = -0.093	ND W = 12252.5 r = 0.085

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

Table A3/g Comparison of the attributed relative stressing of learner-content interaction associated with Bernie's choices that came to action in the first sub-segment and the segment as a whole.

Stressing of learner-content interaction in the first sub-segment is…	Percentage (N=7)
Greater than the segment as a whole	72 %
Equal to the segment as a whole	14 %
Less then the segment as a whole	14 %

Table A3/h The relative stressing of the three elements of the manifestation triad across Bernie's Year 10 lesson series on ratio, grouped by intervals categorised as classwork and seatwork.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 383)	0.08	0.25	0.67
Classwork (N = 82)	0.14	0.11	0.75
Seatwork (N = 301)	0.06	0.29	0.65

Table A3/i Comparison of seatwork and classwork across Bernie's Year 10 lesson series on ratio based on the emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
Seatwork × Classwork	ND	DIFF	DIFF
	W = 55464.5 **	W = 64099 ***	W = 53885 ***
(301, 82)	r = -0.168	r = 0.384	r = -0.243

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals given over to seatwork and classwork.

APPENDIX 4: Case study 3 – Results in tabular form

A4.1 – Manifestation

Table A4/a How the shape of choices over the lesson series aligned with a relative stressing of the three elements of the manifestation triad.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 289)	0.18	0.02	0.80
Lesson 1 (N = 97)	0.29	0.05	0.66
Lesson 2 (N = 92)	0.15	0.00	0.85
Lesson 3 (N = 100)	0.11	0.00	0.89

Note: The figures in the table are ζ values.

Table A4/b The spread of manifestation choices over the lesson series, represented by the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 289)	0	0	83	3	12	0	2
Lesson 1 (N = 97)	0	0	68	8	19	0	5
Lesson 2 (N = 92)	0	0	88	0	12	0	0
Lesson 3 (N = 100)	0	0	94	0	6	0	0

Note: The figures in the table are κ values (percentages).

Table A4/c Comparison of lessons within the lesson series based on the attributed emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
L1 × L2 (97, 92)	DIFF W = 10939.5 *** r = 0.352	DIFF W = 9813 *** r = 0.258	DIFF W = 7331.5 *** r = -0.381
L1 × L3 (97, 100)	DIFF W = 12105.5 *** r = 0.466	DIFF W = 10253 *** r = 0.263	DIFF W = 6986.5 *** r = -0.484
L2 × L3 (92, 100)	ND W = 9331.5 r = 0.089	ND W = 8878 r = 0	ND W = 8424.5 r = -0.089

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

A4.2 – Interaction

Table A4/d How the shape of choices over the lesson series aligned with a relative stressing of the three elements of the interaction triad.

	Mode of teacher interaction		
	Teacher-Content	Teacher-Learner	Learner-Content
Series (N = 289)	0.27	0.24	0.49
Lesson 1 (N = 97)	0.22	0.27	0.51
Lesson 2 (N = 92)	0.32	0.18	0.50
Lesson 3 (N = 100)	0.26	0.28	0.46

Note: The figures in the table are ζ values.

Table A4/e The spread of interaction choices over the lesson series, represented by the coding of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 289)	13	10	7	6	25	13	26
Lesson 1 (N = 97)	10	10	9	3	22	14	32
Lesson 2 (N = 92)	21	8	3	10	15	14	29
Lesson 3 (N = 100)	8	11	8	6	37	11	19

Note: The figures in the table are κ values (percentages).

Table A4/f Comparison of lessons within the lesson series based on the attributed emphasis placed upon the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner	Learner-Content
L1 × L2 (97, 92)	ND W = 8615.5 r = -0.125	ND W = 9896.5 * r = 0.144	ND W = 9306.5 r = 0.019
L1 × L3 (97, 100)	ND W = 8992 r = -0.116	ND W = 9287.5 r = -0.059	ND W = 9306.5 r = 0.019
L2 × L3 (92, 100)	ND W = 9057 r = 0.035	DIFF W = 7814.5 ** r = -0.213	ND W = 9084 r = 0.041

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

Table A4/g Comparison of the attributed relative stressing of learner-content interaction associated with Bernie's choices that came to action in the first sub-segment and the segment as a whole.

Stressing of learner-content interaction in the first sub-segment is…	Percentage (N=8)
Greater than the segment as a whole	25 %
Equal to the segment as a whole	37.5 %
Less then the segment as a whole	37.5 %

Table A4/h The relative stressing of the three elements of the manifestation triad across Bernie's Year 9 lesson series on linear equations, grouped by intervals categorised as classwork and seatwork.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 289)	0.22	0.02	0.76
Classwork (N = 81)	0.27	0.02	0.71
Seatwork (N = 208)	0.20	0.02	0.78

Table A4/i Comparison of seatwork and classwork across Bernie's Year 9 lesson series on linear equations based on the emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
Seatwork × Classwork	ND	ND	ND
	W = 28320 **	W = 29934	W = 32118.5 **
(208, 81)	r = -0.182	r = -0.059	r = 0.193

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals given over to seatwork and classwork.

APPENDIX 5: Case study 4 – Results in tabular form

A5.1 – Manifestation

Table A5/a How the shape of choices over the lesson series aligned with a relative stressing of the three elements of the manifestation triad.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 416)	0.22	0.07	0.71
Lesson 1 (N = 104)	0.24	0.07	0.69
Lesson 2 (N = 108)	0.19	0.15	0.66
Lesson 3 (N = 97)	0.28	0.00	0.72
Lesson 4 (N = 107)	0.17	0.05	0.78

Note: The figures in the table are ζ values.

Table A5/b The spread of manifestation choices over the lesson series, represented by the coding of half-minute intervals.

	How content was manifested						
	V	F	T	VF	VT	FT	VFT
Series (N = 416)	0	0	71	2	17	8	2
Lesson 1 (N = 104)	0	0	71	0	19	8	2
Lesson 2 (N = 108)	0	0	57	0	15	23	5
Lesson 3 (N = 97)	0	0	78	0	22	0	0
Lesson 4 (N = 107)	0	0	78	9	11	0	2

Note: The figures in the table are κ values (percentages).

Table A5/c Comparison of lessons within the lesson series based on the attributed emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
L1 × L2 (104, 108)	ND W = 12133.5 * r = 0.171	DIFF W = 9863 *** r = -0.231	ND W = 11869 r = 0.13
L1 × L3 (104, 97)	ND W = 9826 r = -0.125	DIFF W = 11474 *** r = 0.316	ND W = 10021 r = -0.091
L1 × L4 (104, 107)	DIFF W = 12540.5 *** r = 0.245	ND W = 11417 r = 0.097	DIFF W = 8690 *** r = -0.374
L2 × L3 (108, 97)	DIFF W = 9482.5 *** r = -0.287	DIFF W = 13209.5 *** r = 0.481	DIFF W = 9912 ** r = -0.214
L2 × L4 (108, 107)	ND W = 12138 r = 0.075	DIFF W = 13246.5 *** r = 0.309	DIFF W = 9186 *** r = -0.383
L3 × L4 (97, 107)	DIFF W = 11980 *** r = 0.353	DIFF W = 9360.5 *** r = -0.232	DIFF W = 8094 *** r = -0.319

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

A5.2 – Interaction

Table A5/d How the shape of choices over the lesson series aligned with a relative stressing of the three elements of the interaction triad.

	Mode of teacher interaction		
	Teacher-Content	Teacher-Learner	Learner-Content
Series (N = 416)	0.18	0.32	0.50
Lesson 1 (N = 104)	0.13	0.31	0.56
Lesson 2 (N = 108)	0.17	0.26	0.57
Lesson 3 (N = 97)	0.18	0.36	0.46
Lesson 4 (N = 107)	0.21	0.36	0.43

Note: The figures in the table are ζ values.

Table A5/e The spread of interaction choices over the lesson series, represented by the coding of half-minute intervals.

	Mode of teacher interaction						
	TC	TT	TL	CC	OO	LL	LC
Series (N = 416)	2	18	15	6	13	10	36
Lesson 1 (N = 104)	1	9	16	7	18	12	37
Lesson 2 (N = 108)	3	16	7	7	12	16	39
Lesson 3 (N = 97)	2	24	21	4	8	3	38
Lesson 4 (N = 107)	4	22	17	7	12	9	29

Note: The figures in the table are κ values (percentages).

Table A5/f Comparison of lessons within the lesson series based on the attributed emphasis placed upon the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner	Learner-Content
L1 × L2 (104, 108)	ND W = 10721 r = -0.064	ND W = 11347 r = 0.045	ND W = 11017 r = -0.01
L1 × L3 (104, 97)	ND W = 10116 r = -0.078	ND W = 10186.5 r = -0.058	ND W = 11124 r = 0.113
L1 × L4 (104, 107)	ND W = 10215.5 * r = -0.143	ND W = 10546.5 r = -0.078	ND W = 11967.5 * r = 0.155
L2 × L3 (108, 97)	ND W = 11053.5 r = -0.013	ND W = 10508.5 r = -0.109	ND W = 11808 r = 0.12
L2 × L4 (108, 107)	ND W = 11216.5 r = -0.076	ND W = 10861.5 r = -0.128	ND W = 12691 * r = 0.163
L3 × L4 (97, 107)	ND W = 9619.5 r = -0.061	ND W = 9876 r = -0.012	ND W = 10088.5 r = 0.026

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each lesson.

Table A5/g Comparison of the attributed relative stressing of learner-content interaction associated with Courtney's choices that came to action in the first sub-segment and the segment as a whole.

Stressing of learner-content interaction in the first sub-segment is...	Percentage (N=9)
Greater than the segment as a whole	45 %
Equal to the segment as a whole	33 %
Less then the segment as a whole	22 %

Table A5/h The relative stressing of the three elements of the manifestation triad across Courtney's Year 9 lesson series on linear equations, grouped by intervals categorised as classwork and seatwork.

	How content was manifested		
	Visual	Functional	Technical
Series (N = 416)	0.22	0.07	0.71
Classwork (N = 146)	0.31	0.03	0.66
Seatwork (N = 270)	0.17	0.09	0.74

Table A5/i Comparison of classwork and seatwork across Courtney's Year 9 lesson series on linear equations based on the emphasis placed upon the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
Seatwork × Classwork	DIFF W = 48540.5 ***	DIFF W = 59599 ***	ND W = 60250 ***
(270, 146)	r = -0.339	r = 0.206	r = 0.173

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals given over to seatwork and classwork.

APPENDIX 6: Cross-case comparisons – results in tabular form

Table A6/a Comparison of overall teaching across the four cases based on the relative stressing of the three elements of the manifestation triad.

Comparison	Visual	Functional	Technical
C1 × C2 (488, 383)	DIFF W = 257336.5 *** r = 0.45	ND W = 209260 r = -0.027	DIFF W = 174953 *** r = -0.383
C1 × C3 (488, 289)	ND W = 197394.5 ** r = 0.102	DIFF W = 232281.5 *** r = 0.573	DIFF W = 144031.5 *** r = -0.574
C1 × C4 (488, 416)	ND W = 232781.5 *** r = 0.113	DIFF W = 268087 *** r = 0.455	DIFF W = 164834 *** r = -0.503
C2 × C3 (383, 289)	DIFF W = 104330.5 *** r = -0.422	DIFF W = 166309 *** r = 0.648	DIFF W = 110763.5 *** r = -0.297
C2 × C4 (383, 416)	DIFF W = 118736.5 *** r = -0.406	DIFF W = 195459 *** r = 0.51	ND W = 138198 *** r = -0.172
C3 × C4 (289, 416)	ND W = 102948 r = 0.014	DIFF W = 93764 *** r = -0.204	ND W = 111292.5 *** r = 0.138

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each case.

Table A6/b Contingency table of all manifestations by case, showing that there was a strong association between how content was manifested across the lesson series and the particular classroom context

Manifestation	Ashley (Geometry)	Bernie (Ratio)	Bernie (Algebra)	Courtney (Algebra)	Total	
Visual-Functional	**75.7**	0.0	11.5	12.8	100.0	(78)
Visual-Technical	**47.1**	5.0	18.9	29.0	100.0	(238)
Technical	12.0	24.3	**28.0**	**35.7**	100.0	(827)
Functional-Technical	**41.7**	**45.2**	0.0	13.1	100.0	(252)
Visual-Functional-Technical	**61.4**	31.8	1.7	5.1	100.0	(176)
All manifestations	30.7	24.4	18.4	26.5	100.0	(1571)

Chi Square 531.29 ***

*** p<.001

Table A6/c Comparison of overall teaching across the four cases based on the relative stressing of the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner	Learner-Content
C1 × C2	ND	ND	ND
	W = 199293 ***	W = 202176.5 **	W = 223613 ***
(488, 383)	r = -0.132	r = -0.098	r = 0.115
C1 × C3	ND	ND	ND
	W = 174993.5 ***	W = 187301.5	W = 200209 ***
(488, 289)	r = -0.189	r = -0.025	r = 0.14
C1 × C4	ND	ND	ND
	W = 215264.5	W = 207827.5 ***	W = 232575.5 ***
(488, 416)	r = -0.048	r = -0.113	r = 0.112
C2 × C3	ND	ND	ND
	W = 124376.5	W = 133680.5 *	W = 131163.5
(383, 289)	r = -0.075	r = 0.079	r = 0.038
C2 × C4	ND	ND	ND
	W = 159066 *	W = 150842.5	W = 155448.5
(383, 416)	r = 0.07	r = -0.027	r = 0.026
C3 × C4	ND	ND	ND
	W = 110919.5 ***	W = 95759 *	W = 101407
(289, 416)	r = 0.139	r = -0.094	r = -0.009

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals constituting each case.

Table A6/d Contingency table of types of exchanges by case.

	Case					
Interaction	Ashley (Geometry)	Bernie (Ratio)	Bernie (Algebra)	Courtney (Algebra)	Total	
Teaching-from-the-front	20.0	21.2	21.0	**37.8**	100.0	(386)
Teaching-between-the-desks	32.7	**30.0**	19.1	18.2	100.0	(664)
Being-watchful	**36.8**	19.4	15.4	28.4	100.0	(525)
All exchanges	30.9	24.3	18.4	26.4	100.0	(1575)

Chi Square 77.65 ***

*** p<.001

Table A6/e Comparison of teaching-from-the-front across the four cases based on the relative stressing of the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner
C1 × C2 (77, 82)	ND W = 6527 r = 0.109	ND W = 5793 r = -0.109
C1 × C3 (77, 81)	ND W = 5478 * r = -0.19	ND W = 6765 * r = 0.19
C1 × C4 (77, 146)	ND W = 9091.5 r = 0.074	ND W = 8156.5 r = -0.074
C2 × C3 (82, 81)	DIFF W = 5457.5 *** r = -0.351	DIFF W = 7990.5 *** r = 0.351
C2 × C4 (82, 146)	ND W = 9079.5 r = -0.048	ND W = 9698.5 r = 0.048
C3 × C4 (81, 146)	DIFF W = 11390 *** r = 0.326	DIFF W = 7078 *** r = -0.326

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals given over to teaching-from-the-front in each case.

Table A6/f Comparison of seatwork across the four cases based on the relative stressing of the three elements of the interaction triad.

Comparison	Teacher-Content	Teacher-Learner	Learner-Content
C1 × C2 (410, 301)	ND W = 136397.5 *** r = -0.152	ND W = 143110 r = -0.043	ND W = 152314 ** r = 0.102
C1 × C3 (410, 208)	ND W = 121816.5 ** r = -0.113	ND W = 126043.5 r = -0.018	ND W = 129348 r = 0.054
C1 × C4 (410, 270)	ND W = 141779.5 r = 0.041	ND W = 145200.5 * r = 0.098	ND W = 132596 ** r = -0.121
C2 × C3 (301, 208)	ND W = 77922.5 r = 0.035	ND W = 77746 r = 0.029	ND W = 75376.5 r = -0.045
C2 × C4 (301, 270)	ND W = 94087 *** r = 0.195	ND W = 93125.5 *** r = 0.166	DIFF W = 77431.5 *** r = -0.214
C3 × C4 (208, 270)	ND W = 54333 *** r = 0.162	ND W = 53789 ** r = 0.137	ND W = 44957 *** r = -0.171

Note: *W* is the Wilcoxon rank-sum test statistic (significance *** p<.001, ** p<.01, * p<.05) and *r* is the effect size. DIFF indicates material difference between lessons (absolute value of *r* greater than 0.2) and ND indicates no material difference. The figures in parentheses indicate the number of half-minute intervals given over to seatwork each case.

APPENDIX 7 – Examples of 'what if not' problems

When working with the perpendicular bisector of given points A and B in the series on geometry, removing the constraint that the distance of a point from A and the distance of that point from B must be in the ratio 1:1 switches the topic from geometric constructions to conic sections.

In Bernie's series on ratio, the following problem was posed:

> The sides of a triangle are in the ratio 2 : 3 : 5. The perimeter is 35 cm.
> find the length of each side.
>
> (Worksheet, Bernie's Year 10 Ratio Lesson 2)

Asking 'what if not' this particular ratio firstly offers an alternative insight into a problem set by Ashley, when learners were asked to attempt the impossible construction a triangle with sides 5.1 cm, 4.5 cm and 10.1 cm. This could be worded:

> The sides of a triangle are in the ratio $a : b : c$. The perimeter is 35 cm.
> For different values of a, b and c, find the length of each side. Are there any limits on the values of a, b, and c you could choose?

This switches the topic to triangle construction. If the values of a, b and c are constrained to satisfy $a^2 + b^2 = c^2$, then the topic switches to Pythagoras' theorem.