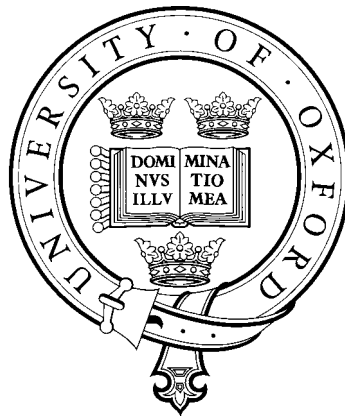


Design of Structural Mechanisms

Yan CHEN



A dissertation submitted for the degree of Doctor of Philosophy
in the Department of Engineering Science at the University of Oxford

St Hugh's College

Trinity Term 2003

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Abstract

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In this dissertation, we explore the possibilities of systematically constructing large structural mechanisms using existing spatial overconstrained linkages with only revolute joints as basic elements.

The first part of the dissertation is devoted to structural mechanisms (networks) based on the Bennett linkage, a well-known spatial $4R$ linkage. This special linkage has been used as the basic element. A particular layout of the structures has been identified allowing unlimited extension of the network by repeating elements. As a result, a family of structural mechanisms has been found which form single-layer structural mechanisms. In general, these structures deploy into profiles of cylindrical surface. Meanwhile, two special cases of the single-layer structures have been extended to form multi-layer structures. In addition, according to the mathematical derivation, the problem of connecting two similar Bennett linkages into a mobile structure, which other researchers were unable to solve, has also been solved.

A study into the existence of alternative forms of the Bennett linkage has also been done. The condition for the alternative forms to achieve the compact folding and maximum expansion has been derived. This work has resulted in the creation of the most effective deployable element based on the Bennett linkage. A simple method to build the Bennett linkage in its alternative form has been introduced and verified. The corresponding networks have been obtained following the similar layout of the original Bennett linkage.

The second effort has been made to construct large overconstrained structural mechanisms using hybrid Bricard linkages as basic elements. The hybrid Bricard linkage is a special case of the Bricard linkage, which is overconstrained and with a single degree of mobility. Starting with the derivation of the compatibility condition and the study of its deployment behaviour, it has been found that for some particular twists, the hybrid Bricard linkage can be folded completely into a bundle and deployed to a flat triangular profile. Based on this linkage, a network of hybrid Bricard linkages has been produced. Furthermore, in-depth research into the deployment characteristics, including kinematic bifurcation and the alternative forms of the hybrid Bricard linkage, has also been conducted.

The final part of the dissertation is a study into tiling techniques in order to develop a systematic approach for determining the layout of mobile assemblies. A general approach to constructing large structural mechanisms has been proposed, which can be divided into three steps: selection of suitable tilings, construction of overconstrained units and validation of compatibility. This approach has been successfully applied to the construction of the structural mechanisms based on Bennett linkages and hybrid Bricard linkages. Several possible configurations are discussed including those described previously.

All of the novel structural mechanisms presented in this dissertation contain only revolute joints, have a single degree of mobility and are geometrically overconstrained.

Research work reported in this dissertation could lead to substantial advancement in building large spatial deployable structures.

Keywords: Structural mechanism; deployable structure; 3D overconstrained linkage; network; tiling technique; Bennett linkage; hybrid Bricard linkage; alternative form.

To My Family

Preface

The study contained in this dissertation was carried out by the author in the Department of Engineering Science at the University of Oxford during the period from January 2000 to August 2003.

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Finally, I would like to thank my parents for their confidence in me, and give special thanks to my husband for all his love, patience and encouragement.

Except for commonly understood and accepted ideas, or where specific reference is made to the work of others, the contents of this report are entirely my original work and do not include any work carried out in collaboration. The contents of this dissertation have not been previously submitted, in part or in whole, to any university or institution for any degree, diploma, or other qualification.

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Notation

C :	cosine.
S :	sine.
$[I]$:	Unit matrix.
K :	Number of distinct k -uniform tilings.
L :	Actual side length of the alternative form of the linkage.
L_i :	Loop i of Bennett linkage or hybrid Bricard linkage in the possible network.
Mx_{ij} :	Moment at joint i of link ij in local coordination x_{ij} .
My_{ij} :	Moment at joint i of link ij in local coordination y_{ij} .
Mz_{ij} :	Moment at joint i of link ij in local coordination z_{ij} .
Nx_{ij} :	Force at joint i of link ij in local coordination x_{ij} .
Ny_{ij} :	Force at joint i of link ij in local coordination y_{ij} .
Nz_{ij} :	Force at joint i of link ij in local coordination z_{ij} .
R :	Radius of the deployed cylinder of the network of Bennett linkages.
R_i :	Distance from link ji to link ik positively about Z_i . Also referred as <i>offset</i> of joint i .
$[T_{ji}]$:	Transfer matrix between system of link $(j-1)j$ and system of link $(i-1)i$.
X_i :	Axis commonly normal to Z_j and Z_i , positively from joint j to joint i .
Z_i :	Axis of revolute joints i , i could be number or letter.
a :	Length of links of a Bennett linkage, or length of links of other linkages.
a_i :	Length of links of Bennett linkage i .
a_{ji} :	Distance between axes Z_j and Z_i . Also referred as <i>length</i> of link ji .

b :	Length of links of a Bennett linkage, or length of links of other linkage.
b_i :	Length of links of Bennett linkage i .
c :	Distance between joint of the original linkage and that of the alternative form of the original linkage along the axis of joint.
d :	Distance between joint of the original linkage and that of the alternative form of the original linkage along the axis of joint.
f :	Number of the kinematic variables of a joint.
k :	Number of transitivity classes with respect to the group of symmetries of the tilings.
k_B :	Bennett ratio of a Bennett joint.
k_S :	Ratio of lengths of two similar Bennett linkages.
k_i :	Ratio of lengths of two Bennett linkages, $i = 1, 2, 3, 4$.
l :	Length of links of an equilateral Bennett linkage, or length of links of a hybrid Bricard linkage.
link ji :	Link connecting joint j and joint i .
m :	Mobility of a linkage.
n :	Number of links of the linkage.
p :	Number of joints of the linkage.
t :	Ratio of two lengths of a Bennett linkage, b/a .
α :	Twist of links of Bennett linkage or the twist of links of hybrid Bricard linkage.
α_i :	Twist of a link of Bennett linkage i .
α_{ji} :	Angle of rotation angle from axes Z_j to Z_i positively about axis X_i . Also referred as <i>twist</i> of link ji .
β :	Twist of links of Bennett linkage or the twist of links of hybrid Bricard linkage.
β_i :	Twist of a link of Bennett linkage i .
δ :	Angle between two adjacent sides of the alternative form of Bennett linkage.
ε :	A small imperfection.

ϕ :	Revolute variables of a linkage.
γ_d :	Geometric parameter.
γ_f :	Geometric parameter.
φ :	Revolute variable of a linkage.
φ_d :	Revolute variable of a linkage when the alternative form of the linkage is in its fully deployed configuration.
φ_f :	Revolute variable of a linkage when the alternative form of the linkage is in its fully folded configuration.
λ :	Angle of rotation of a square cross-section bar along its central axis.
μ :	Geometric parameter.
ν :	Geometric parameter.
θ :	Revolute variable of a linkage.
θ_d :	Revolute variable of a linkage when the alternative form of the linkage is in its fully deployed configuration.
θ_f :	Revolute variable of a linkage when the alternative form of the linkage is in its fully folded configuration.
θ_i :	Revolute variable of the linkage, which is the angle of rotation from X_{i-1} to X_i positively about Z_i .
σ :	Revolute variable of the linkage.
τ :	Revolute variable of the linkage.
υ :	Revolute variable of the linkage.
ω :	Half of the angle between two adjacent sides of the alternative form of the Bennett linkage, which is $\delta / 2$.
ξ_d :	Geometric parameter.
ξ_f :	Geometric parameter.
I :	Type of similar Bennett linkages in mobile network of Bennett linkages.
I_j :	Type of similar Bennett linkages in mobile network of Bennett linkages.
Π_i :	Type of similar Bennett linkages in mobile network of Bennett linkages.
$-\Pi_i$:	Type of similar Bennett linkages in mobile network of Bennett linkages whose twists have opposite sign to those of Π_i .

Introduction

1.1 OVERCONSTRAINED MECHANISMS AND DEPLOYABLE STRUCTURES

A mechanism is commonly identified as a set of moving or working parts in a machine or other device essentially as a means of transmitting, controlling, or constraining relative movement. A mechanism is often assembled from gears, cams and linkages, though it may contain other specialised components, such as springs, ratchets, brakes, and clutches, as well. Reuleaux published the first book on theoretical kinematics of mechanisms in 1875 (Hunt, 1978). Later on the general mobility criterion of an assembly was established by Grübler in 1921 and Kutzbach in 1929, respectively (Phillips, 1984), based on the topology of the assembly.

However, it was found that this criterion is not a necessary condition. Some specific geometric condition in an assembly could make it a mechanism even though it does not obey the mobility criterion. This type of mechanisms is called an *overconstrained mechanism*. The first published research on overconstrained mechanisms can be traced back to 150 years ago when Sarrus discovered a six-bar mechanism capable of rectilinear motion. Gradually more overconstrained mechanisms were discovered by

other researchers in the next half a century. However, most overconstrained mechanisms have rarely been used in industrial applications because of the development of gears, cams and other means of transmission, except two of them: the double-Hooke's-joint linkage, which is widely applied as a transmission coupling, and the Schatz linkage, which is used as a Turbula machine for mixing fluids and powders. Over the recent half century, very few overconstrained mechanisms have been found. Most research work on overconstrained mechanisms is mainly focused on their kinematic characteristics.

During the same period, a new branch of structural engineering, deployable structures, started its rapid development. Deployable structures are a novel and unique type of engineering structure, whose geometry can be altered to meet practical requirements. Large aerospace structures, e.g. antennas and masts, are prime examples of deployable structures. Due to their size, they often need to be packaged for transportation and expanded at the time of operation. The deployment of such structures can rely on the large deformation or the concept of mechanisms, i.e. the structures are assemblies of mechanisms whose mobility is retained for the purpose of deployment. The latter are also called structural mechanisms. The key advantage of structural mechanisms is that they allow repeated deployment without inducing any strain in their structural components. In the selection of mechanisms, overconstrained geometry is preferred because it provides extra stiffness, as most such structures are for aerospace applications in which structural rigidity is one of the prime requirements. Furthermore, these structural mechanisms often have only hinged connections due to the fact that this kind of linkage provides more robust performance than sliders or other types of connections.

Research into the construction of structural mechanisms has a completely different focus from the study of mechanisms. Kinematic characteristics such as the trajectory described by a mechanism become less important. Instead, the keys to a successful concept are, first of all, to identify a robust and scalable building block made of simple mechanisms; and secondly, to develop a way by which the building blocks can be connected to form a large deployable structure while retaining the single degree of freedom. In this process, one has to ensure that the entire assembly satisfies the geometric compatibility.

Research in this area over the last three decades has primarily focused on the construction of structural mechanisms using planar mechanisms, e.g. the foldable bar structures (You and Pellegrino, 1993 and 1997) and the Pactruss structures (Rogers, et al., 1993). The building blocks involve one or more types of basic planar mechanisms of single mobility. The structures are then assembled in a way that geometric compatibility conditions are met. 3D mechanisms are rarely used, probably due to the mathematical difficulty of dealing with non-linear geometric compatibility conditions in 3D.

1.2 SCOPE AND AIM

The aim of this dissertation is to explore the possibility of constructing structural mechanisms using existing 3D overconstrained linkages, i.e. mechanisms connected by revolute joints, and the mathematical tiling technique.

In this process, we first examine the existing 3D overconstrained linkages and divide them into two groups: basic linkages and derivatives of the basic linkages. Then we concentrate on basic linkages and identify possible ways to assemble them using the mathematical tiling technique. Finally, we look into the structural mechanisms obtained, examine their profiles and explore their potential applications.

1.3 OUTLINE OF DISSERTATION

This dissertation consists of six chapters.

Chapter 2 presents a brief review of existing work related to our task, including the definitions and analysis methods for overconstrained linkages, the existing 3D overconstrained linkages, as well as the mathematical tiling technique. The reason for including tiling in this review is because it is used later in producing a suitable arrangement of basic linkages in order to build structural mechanisms.

Chapter 3 focuses on the construction of structural mechanisms using the Bennett linkage. First of all, a method to form a network of Bennett linkages is presented. This is followed by the mathematical derivation of the compact folding and maximum expanding conditions, which leads to the discovery of an alternative form of the Bennett linkage. The alternative is then extended to networks of Bennett linkages, leading to large structural mechanisms which can be compactly folded up. This chapter is ended with further discussion and conclusions.

Chapter 4 is devoted to the design of structural mechanisms using the hybrid Bricard linkage. Firstly, the construction process of a $6R$ hybrid linkage based on the Bricard linkage and its basic characteristics are described. Secondly, the deployment features and the possibilities to form networks of hybrid Bricard linkage are presented. Thirdly, the bifurcation of this linkage is studied. Furthermore alternative forms of the hybrid Bricard linkage are discussed. Finally an in-depth discussion ends this chapter.

Chapter 5 deals with the mathematical tiling technique and its application in the construction of structural mechanisms. The networks of Bennett linkages and hybrid Bricard linkages are revisited according to the tiling technique.

The main achievements of the research are summarised in Chapter 6, together with suggestions for future work, which conclude this dissertation.

Review of Previous Work

2.1 LINKAGES AND OVERCONSTRAINED LINKAGES

A linkage is a particular type of mechanism consisting of a number of interconnected components, individually called links. The physical connection between two links is called a joint. All joints of linkages are lower pairs, i.e. surface-contact pairs, which include spherical joints, planar joints, cylindrical joints, revolute joints, prismatic joints, and screw joints. Here we limit our attention to linkages whose links form a single loop and are connected only by revolute joints, also called rotary hinges. These joints allow one-degree-of-freedom movement between the two links that they connect. The kinematic variable for a revolute joint is the angle measured around the two links that it connects.

From classical mobility analysis of mechanisms, it is known that the mobility m of a linkage composed of n links that are connected with p joints can be determined by the Kutzbach (or Grübler) mobility criterion (Hunt, 1978):

$$m = 6(n - p - 1) + \sum f \quad (2.1.1)$$

where $\sum f$ is the sum of kinematic variables in the mechanism.

For an n -link closed loop linkage with revolute joints, $p = n$, and the kinematic variable

$\sum f = n$. Then the mobility criterion in (2.1.1) becomes

$$m = n - 6 \quad (2.1.2)$$

So in general, to obtain a mobility of *one*, a linkage with revolute joints needs at least *seven* links.

It is important to note that (2.1.2) is not a necessary condition because it considers only the topology of the assembly. There are linkages with full-range mobility even though they do not meet the mobility criterion. These linkages are called overconstrained linkages. Their mobility is due to the existence of special geometry conditions among the links and joint axes that are called overconstrained conditions.

Denavit and Hartenberg (Beggs, 1966) set forth a standard approach to the analysis of linkages, where the geometric conditions are taken into account. They pointed out that, for a closed loop in a linkage, the necessary and sufficient mobility condition is that the product of the transform matrices equals the unit matrix, i.e.,

$$[T_{n1}] \dots [T_{34}][T_{23}][T_{12}] = [I] \quad (2.1.3)$$

where $[T_{i(i+1)}]$ is the transfer matrix between the system of link $(i-1)i$ and the system of link $i(i+1)$, see Fig. 2.1.1,

$$[T_{i(i+1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{i(i+1)} & C\theta_i & S\theta_i & 0 \\ -R_i S\alpha_{i(i+1)} & -C\alpha_{i(i+1)} S\theta_i & C\alpha_{i(i+1)} C\theta_i & S\alpha_{i(i+1)} \\ -R_i C\alpha_{i(i+1)} & S\alpha_{i(i+1)} S\theta_i & -S\alpha_{i(i+1)} C\theta_i & C\alpha_{i(i+1)} \end{bmatrix} \quad (2.1.4)$$

When $i+1 > n$, $i+1$ is replaced by 1.

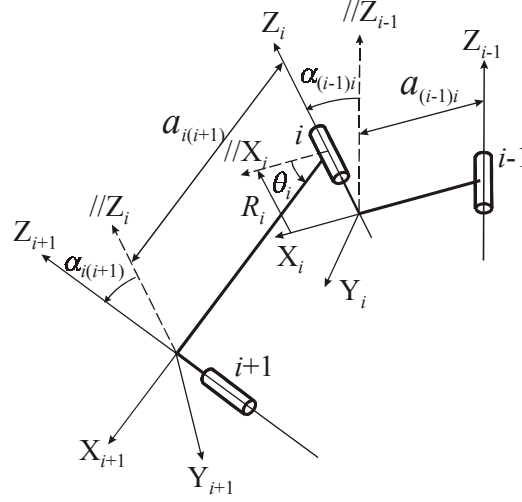


Fig. 2.1.1 Coordinate systems for two links connected by a revolute joint.

Note that the transfer matrix between the system of link $i(i+1)$ and the system of link $(i-1)i$ is the inverse of $[T_{i(i+1)}]$. That is

$$[T_{(i+1)i}] = [T_{i(i+1)}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{i(i+1)}C\theta_i & C\theta_i & -C\alpha_{i(i+1)}S\theta_i & S\alpha_{i(i+1)}S\theta_i \\ a_{i(i+1)}S\theta_i & S\theta_i & C\alpha_{i(i+1)}C\theta_i & -S\alpha_{i(i+1)}C\theta_i \\ R_i & 0 & -S\alpha_{i(i+1)} & C\alpha_{i(i+1)} \end{bmatrix} \quad (2.1.5)$$

2.2 3D OVERCONSTRAINED LINKAGES

The minimum number of links to construct a mobile loop with revolute joints is four as a loop with three links and three revolute joints is either a rigid structure or an infinitesimal mechanism when all three revolute axes are coplanar and intersect at a single point (Phillips, 1990). So 3D overconstrained linkages can have four, five or six

links. When these linkages consist of only revolute joints, they are called $4R$, $5R$ or $6R$ linkages.

The first overconstrained mechanism which appeared in the literature was proposed by Sarrus (1853). Since then, other overconstrained mechanisms have been proposed by various researchers. Of special interest are those proposed by Bennett (1903), Delassus (1922), Bricard (1927), Myard (1931), Goldberg (1943), Waldron (1967, 1968 and 1969), Wohlhart (1987, 1991 and 1993) and Dietmaier (1995). Phillips (1984, 1990) summarised all of the known overconstrained mechanisms in his two-volume book. However, the most detailed studies of the subject of overconstraint in mechanisms are due to Baker (1980, 1984, etc.).

2.2.1 $4R$ Linkage - Bennett Linkage

Common $4R$ mobile loops can normally be classified into two types: the axes of rotation are all parallel to one another, or they are concurrent, i.e. they intersect at a point, leading to 2D $4R$ or spherical $4R$ linkages, respectively. Any disposition of the axes different from these two special arrangements is known usually to be a chain of four pieces which is, in general, completely rigid and so furnishes no mechanism at all. But there is an exception, which is the Bennett linkage (Bennett, 1903).

The Bennett linkage is a skewed linkage of four pieces having the axes of revolute joints neither parallel nor concurrent. Figure 2.2.1 shows the original model made by Bennett. This linkage was also found independently by Borel (Bennett, 1914). Its

behaviour is better illustrated by the schematic diagram shown in Fig. 2.2.2. The four links are connected by revolute joints, each of which has axis perpendicular to the two adjacent links connected by it. The lengths of the links are given alongside the links, and the twists are indicated at each joint. Bennett (1914) identified the conditions for the linkage to have a single degree of mobility as follows.

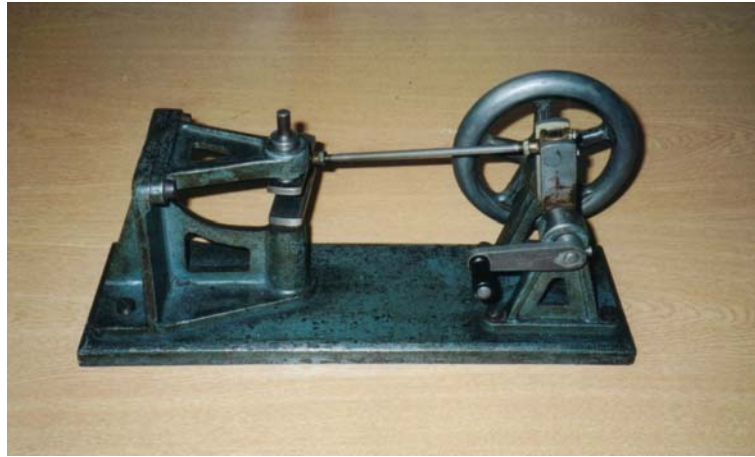


Fig. 2.2.1 Original model of the Bennett linkage.

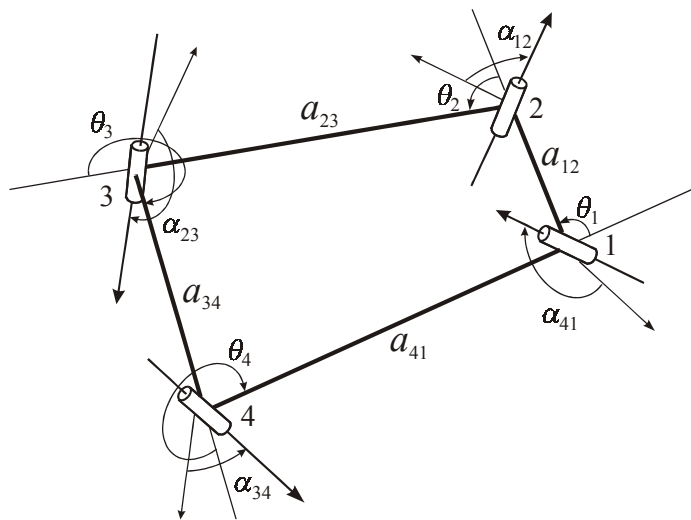


Fig. 2.2.2 A schematic diagram of the Bennett linkage.

Thick lines represent four links.

(a) Two alternate links have the same length and the same twist, i.e.

$$a_{12} = a_{34} = a \quad (2.2.1a)$$

$$a_{23} = a_{41} = b \quad (2.2.1b)$$

$$\alpha_{12} = \alpha_{34} = \alpha \quad (2.2.1c)$$

$$\alpha_{23} = \alpha_{41} = \beta \quad (2.2.1d)$$

(b) Lengths and twists should satisfy the condition

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} \quad (2.2.2)$$

(c) Offsets are zero, i.e.,

$$R_i = 0 \quad (i = 1, 2, 3, 4) \quad (2.2.3)$$

The values of the revolute variables, θ_1 , θ_2 , θ_3 and θ_4 , vary when the linkage moves, but

$$\theta_1 + \theta_3 = 2\pi \quad (2.2.4)$$

$$\theta_2 + \theta_4 = 2\pi$$

and

$$\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{\sin \frac{1}{2}(\alpha_{23} + \alpha_{12})}{\sin \frac{1}{2}(\alpha_{23} - \alpha_{12})} \quad (2.2.5)$$

These three closure equations ensure that only one of the θ 's is independent, so the linkage has a single degree of mobility (Baker, 1979).

Taking

$$\theta_1 = \theta \quad \text{and} \quad \theta_2 = \varphi,$$

(2.2.5) becomes

$$\tan \frac{\theta}{2} \tan \frac{\varphi}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \quad (2.2.6)$$

Bennett (1914) also identified some special cases.

- (a) An equilateral linkage is obtained if $\alpha + \beta = \pi$ and $a = b$. (2.2.6) then becomes

$$\tan \frac{\theta}{2} \tan \frac{\varphi}{2} = \frac{1}{\cos \alpha} \quad (2.2.7)$$

- (b) If $\alpha = \beta$ and $a = b$, the four links are congruent. The motion is discontinuous: $\theta = \pi$ allows any value for φ and $\varphi = \pi$ allows any value for θ .
- (c) If $\alpha = \beta = 0$, the linkage is a 2D crossed isogram.
- (d) If $\alpha = 0$ and $\beta = \pi$, the linkage becomes a 2D parallelogram.
- (e) If $a = b = 0$, the linkage is a spherical 4R linkage (Phillips, 1990).

Because the Bennett linkage uses a minimum number of links, it has attracted an enormous amount among attention of kinematicians. Fresh contributions have continuously been made to the abundant literature concerning this remarkable 4R linkage. Most of the research work was on the mathematical description and the kinematic characters of the Bennett linkage. Ho (1978) presented an approach to establish the geometric criteria for the existence of the Bennett linkage through the use of tensor analysis. Bennett (1914) proved that all four hinge axes of the Bennett linkage can be regarded as generators of the same regulus on a certain hyperboloid at any

configuration of the linkage. So a number of papers studied the geometry of the Bennett linkage from the viewpoint of a quadric surface. Yu (1981a) found that the links of the Bennett linkage are in fact two pairs of equal and opposite sides of a line-symmetrical tetrahedron. Both the equation of the hyperboloid and the geometry of the quadric surface can be obtained based on the analogy. The major axis of the central elliptical section of the hyperboloid is simply the line of symmetry of the tetrahedron. So the geometry of the hyperboloid for different configurations of the linkage can be readily visualised by observing the tetrahedron. Later, Yu (1987) reported that the quadric surface is a sphere that passes through the vertices of the Bennett linkage. Based on his previous work, the position and radius of the sphere are derived with respect to the tetrahedron, and a spherical $4R$ linkage can be obtained that coincides with the spherical indicatrix of the Bennett linkage. Thus, for each Bennett linkage, there is always a corresponding configuration of a spherical $4R$ linkage with the same twists as those of the Bennett linkage, such that the angles respectively defined by the four pairs of adjacent links of the Bennett linkage are the supplementary angles of those of the spherical linkage, or vice versa. Baker (1988) investigated the J -hyperboloid defined by the joint-axes of the Bennett linkage and the L -hyperboloid defined by the links of the Bennett linkage. He was able to relate directly three independent parameters and a suitable single joint variable of the linkage with the relevant quantities of the sphere and the two forms of hyperboloid associated with it. Huang (1997) explored the finite kinematic geometry of the Bennett linkage rather than the instantaneous kinematic geometry of the linkage. Since the Bennett linkage can be regarded as a combination of two R - R dyads, it is natural to consider its finite motion as the intersection of the screw systems associated with the corresponding R - R dyads. It was shown that the screw axes

of all possible coupler displacements of the linkage from any given configuration form a cylindroid, which is a ruled surface formed by the axes of a real linear combination of two screws. The axis of the cylindroid, which is the common perpendicular to the axes of two basis screws, coincides with the line of symmetry of the Bennett linkage. Recently, Baker (1998) paid attention to the relative motion between opposite links in relation to the line-symmetric character of the Bennett linkage. He discovered that the relative motion could be neither purely rotational nor purely translational at any time. Furthermore Baker (2001) examined the axode of the motion of Bennett linkage. He derived the centrode's equation of the planar Bennett linkage (special case (c) and (d) in page 12) and the axode of the spatial loop. The fixed axode of the linkage is the ruled surface traced out by the instantaneous screw axis (ISA) of one pair of opposite links. He established the relationships between the ruled surface and the corresponding centrode of the linkage.

It is interesting to note that the Bennett linkage is the only $4R$ linkage with revolute connections (Waldron, 1968; Savage, 1972; Baker, 1975). Attempts have also been made to build $5R$ or $6R$ 3D linkages based on the Bennett Linkage. Detail of these linkages will be the subject of the next section. It should be pointed out that most of the research was concentrated on building basic mobile units rather than exploring the possibility for construction of large structural mechanisms. The only exception was Baker and Hu's (1986) unsuccessful attempt to connect two Bennett linkages.

2.2.2 5R Linkages

Some five-bar linkages have been found (Pamidi et al., 1973), of which the only ones that are connected with revolute joints are the Goldberg 5R linkage and Myard linkage.

The Goldberg 5R linkage (Goldberg, 1943) is obtained by combining a pair of Bennett linkages in such a way that a link common to both is removed and a pair of adjacent links are rigidly attached to each other. The techniques he developed can be summarised as the *summation* of two Bennett loops to produce a 5R linkage, or the *subtraction* of a primary composite loop from another Bennett chain to form a syncopated linkage, see Fig. 2.2.3.

Prior to Goldberg, Myard (1931) produced an overconstrained 5R linkage as shown in Fig. 2.2.4. It is a plane-symmetric 5R and has later been re-classified as a special case of the Goldberg 5R linkage, for which the two ‘rectangular’ Bennett chains, whose one pair of twists are $\pi/2$, are symmetrically disposed before combining them. Two Bennett linkages are mirror images of each other, the mirror being coincident with the plane of symmetry of the resultant linkage (Baker, 1979). The conditions on its geometric parameters are as follows.

$$\begin{aligned}
 a_{34} &= 0, \quad a_{12} = a_{51}, \quad a_{23} = a_{45} \\
 \alpha_{23} = \alpha_{45} &= \frac{\pi}{2}, \quad \alpha_{51} = \pi - \alpha_{12}, \quad \alpha_{34} = \pi - 2\alpha_{12} \\
 R_3 &= R_4 = 0
 \end{aligned} \tag{2.2.8}$$

Although both 5R linkages have received some analytical attention (Baker, 1978), only the Goldberg 5R linkage has been thoroughly studied.

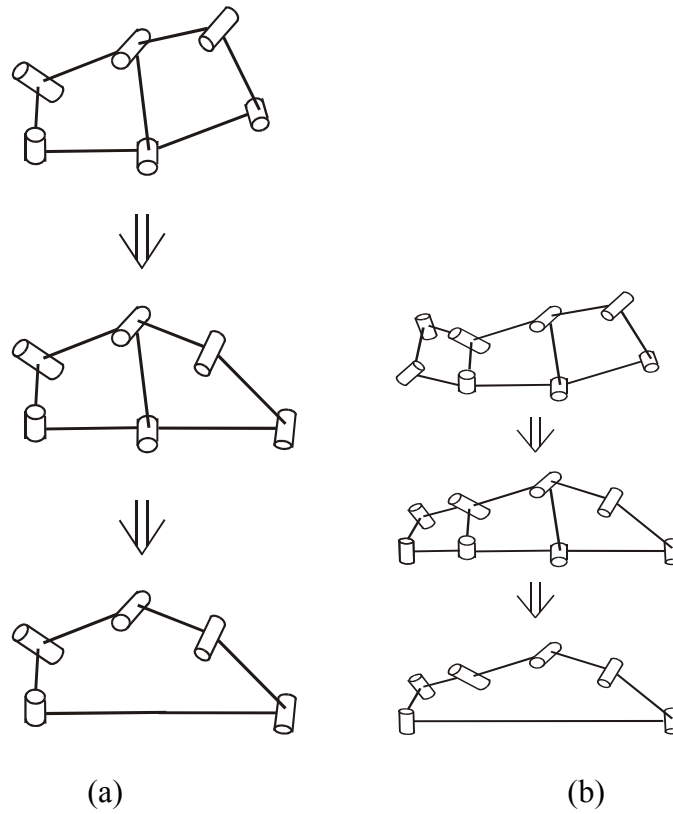


Fig. 2.2.3 Goldberg 5R Linkages. (a) Summation; (b) subtraction.

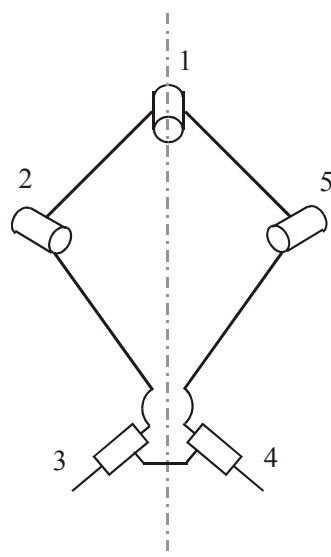


Fig. 2.2.4 Myard Linkage.

2.2.3 6R Linkages

Several 6R linkages have been discovered or synthesised over the last two centuries. In quasi-chronological order, they are as follows.

- Double-Hooke's-joint linkage

The double-Hooke's-joint linkage (Baker, 2002) is obtained from two spherical 4R linkages. Figure 2.2.5(a) shows the arrangement of two spherical 4R linkages, which consist of joints 5, 6, 1 and 0, and joints 2, 3, 4 and 0, respectively, and have different centres. The axis of joint 6 is perpendicular to the axes of both joints 1 and 5, the axis of joint 3 is perpendicular to the axes of both joints 2 and 4, and the axis of joint 0 is perpendicular to the axes of both joints 1 and 2. Replacing the common joint 0 with link 12, and adding link 45 between joints 5 and 4 result in the 6R linkage with the following parametric properties:

$$\begin{aligned}
 a_{23} &= a_{34} = a_{56} = a_{61} = 0 \\
 \alpha_{23} &= \alpha_{34} = \alpha_{56} = \alpha_{61} = \frac{\pi}{2} \\
 R_1 &= R_2 = R_3 = R_6 = 0
 \end{aligned} \tag{2.2.9}$$

The double-Hooke's-joint linkage has been widely used as a transmission coupling, see Fig. 2.2.5(b). It transmits rotation with a constant angular velocity ratio provided the transmission angles, i.e. the angle between joints 2, 4 and the angle between joints 1, 5, are equal (Phillips, 1990). Baker (2002) derived its input-output equation.

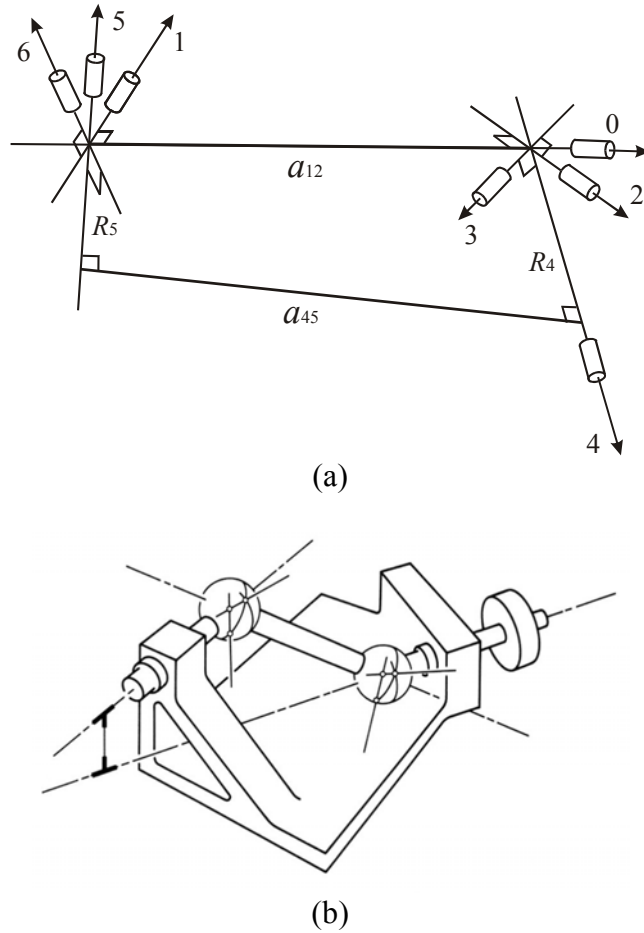


Fig. 2.2.5 Double-Hooke's-joint linkage.

(a) A schematic diagram; (b) sketch of a practical model.

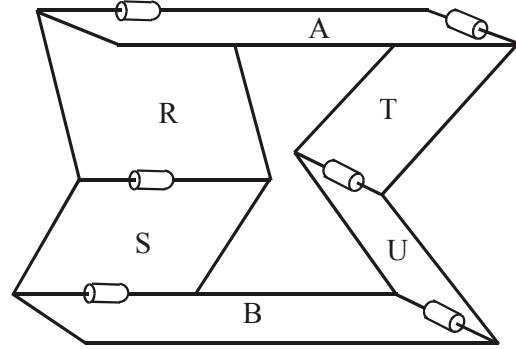
- The Sarrus linkage

The Sarrus linkage is the first 3D overconstrained linkage that has ever been published (Sarrus, 1853). Bennett (1905) also studied this linkage. A model made by him is shown in Fig. 2.2.6(a), together with a schematic diagram in Fig. 2.2.6(b). The four links A, R, S, and B are consecutively hinged by three parallel horizontal hinges, as are the links A, T, U, and B. The directions of the two sets of hinges are different. With such an

arrangement, link A can have rectilinear motion, vertically up and down, relative to link B.



(a)



(b)

Fig. 2.2.6 Sarrus linkage. (a) Model by Bennett; (b) a schematic diagram.

- The Bennett 6R hybrid linkage

The 6R linkage shown in Fig. 2.2.7 was discovered by Bennett (1905). Hinges 1, 2 and 3 consecutively connect links A, R, S, and B, and intersect at point X, while hinges 4, 5 and 6 connect links A, T, U, and B, and intersect at point Y. X and Y are not the same point. The link A, relative to link B, has a motion of pure rotation about the line XY, which is denoted as hinge 0. This linkage may also be regarded as composed of two spherical 4R linkages, each one placed with different centres. One consists of a closed chain of four links, A, R, S, and B with hinges 1, 2, 3, and 0, and the other consists of a closed chain of four links, A, T, U, and B with hinges 4, 5, 6, and 0. The links A and B and hinge 0 are common to both spherical 4R linkages. The Bennett 6R hybrid linkage can be obtained by removing the redundant hinge 0.

Note that the double-Hooke's-joint linkage is actually a special case of the Bennett 6R hybrid linkage. Two further special forms can be derived by taking one or both of the points X and Y to infinity. The former forms a Bennett plano-spherical hybrid linkage. A model made by Bennett himself is shown in Fig. 2.2.8. The latter produces a Sarrus linkage.

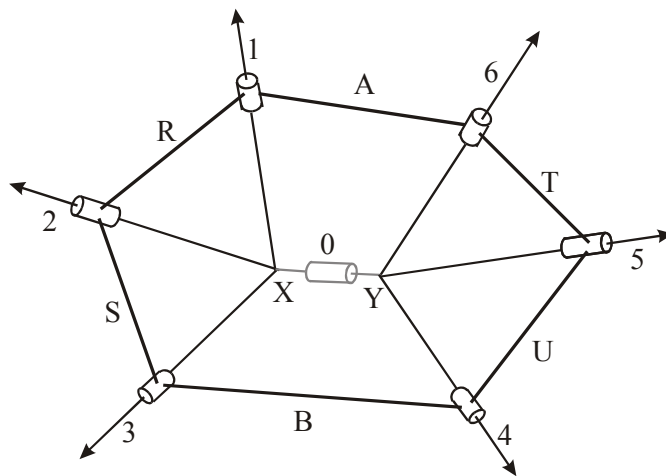


Fig. 2.2.7 Bennett 6R hybrid linkage.

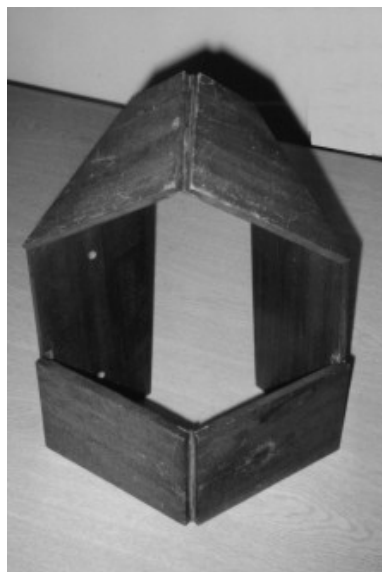


Fig. 2.2.8 Bennett plano-spherical hybrid linkage.

- The Bricard linkages

Six distinct types of mobile 6R linkages were discovered and reported by Bricard (1927). They can be summarised as follows.

(a) The general line-symmetric case

$$a_{12} = a_{45}, a_{23} = a_{56}, a_{34} = a_{61}$$

$$\alpha_{12} = \alpha_{45}, \alpha_{23} = \alpha_{56}, \alpha_{34} = \alpha_{61} \quad (2.2.10a)$$

$$R_1 = R_4, R_2 = R_5, R_3 = R_6$$

(b) The general plane-symmetric case

$$a_{12} = a_{61}, a_{23} = a_{56}, a_{34} = a_{45}$$

$$\alpha_{12} + \alpha_{61} = \pi, \alpha_{23} + \alpha_{56} = \pi, \alpha_{34} + \alpha_{45} = \pi \quad (2.2.10b)$$

$$R_1 = R_4 = 0, R_2 = R_6, R_3 = R_5$$

(c) The trihedral case

$$a_{12}^2 + a_{34}^2 + a_{56}^2 = a_{23}^2 + a_{45}^2 + a_{61}^2$$

$$\alpha_{12} = \alpha_{34} = \alpha_{56} = \frac{\pi}{2}, \alpha_{23} = \alpha_{45} = \alpha_{61} = \frac{3\pi}{2} \quad (2.2.10c)$$

$$R_i = 0 \quad (i = 1, 2, \dots, 6)$$

(d) The line-symmetric octahedral case

$$a_{12} = a_{23} = a_{34} = a_{45} = a_{56} = a_{61} = 0$$

$$R_1 + R_4 = R_2 + R_5 = R_3 + R_6 = 0$$

(2.2.10d)

(e) The plane-symmetric octahedral case

$$a_{12} = a_{23} = a_{34} = a_{45} = a_{56} = a_{61} = 0$$

$$R_4 = -R_1, R_2 = -R_1 \frac{\sin \alpha_{34}}{\sin(\alpha_{12} + \alpha_{34})}, R_5 = R_1 \frac{\sin \alpha_{61}}{\sin(\alpha_{45} + \alpha_{61})}, \quad (2.2.10e)$$

$$R_3 = R_1 \frac{\sin \alpha_{12}}{\sin(\alpha_{12} + \alpha_{34})}, R_6 = -R_1 \frac{\sin \alpha_{45}}{\sin(\alpha_{45} + \alpha_{61})}$$

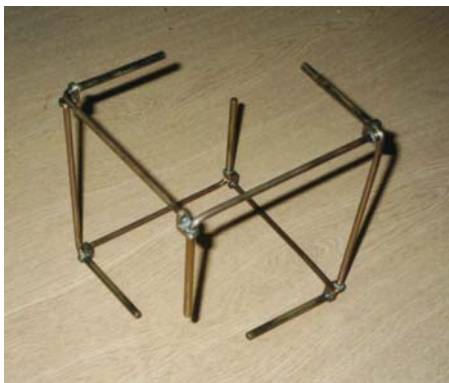
(f) The doubly collapsible octahedral case

$$a_{12} = a_{23} = a_{34} = a_{45} = a_{56} = a_{61} = 0 \quad (2.2.10f)$$

$$R_1 R_3 R_5 + R_2 R_4 R_6 = 0$$

Models based on cases (c) and (d) are shown in Fig. 2.2.9.

Bennett (1911) studied the geometry of the three types of deformable octahedron, and presented the kinematic properties of the mechanisms. But a thorough analysis on all six Bricard linkages was done by Baker (1980), delineating them by appropriate sets of independent closure equations. Interestingly Baker (1986) found that the stationary configurations of a special line-symmetric octahedral case of Bricard linkage are precisely equivalent to the minimum energy conformations of the flexing molecule.



(a)



(b)

Fig. 2.2.9 Bricard linkages. (a) Trihedral case; (b) line-symmetric octahedral case.

Yu (1981b) pointed out that the links and hinge axes of the trihedral Bricard linkage form the twelve edges of a hexahedron. He investigated two geometrical aspects of the hexahedron, namely, its circumscribed sphere and its associated quadric surface. The former leads to a solution for its angular relationships; the latter offers a physical interpretation of the central axis of the special complex to which the six hinge axes belong. Wohlhart (1993) studied one of the Bricard linkages, the orthogonal Bricard linkage, and showed that two clearly distinct types of this linkage exist, and that only one of them can be characterised by a set of system parameters proposed by Baker (1980).

The trihedral case of the Bricard linkage also appears as a particular type of linkage popularly known as the kaleidocycle. A kaleidocycle is a 3D ring made from a chain of identical tetrahedra. As shown in Fig. 2.2.10, each tetrahedron is linked to an adjoining one along an edge. When the chain of tetrahedra is long enough, the ends can be brought together to form a closed loop. The ring can be turned through its centre in a continuous motion. In order to form a closed ring, at least six tetrahedra are required (Schattschneider and Walker, 1977). In fact when the number of tetrahedra is six, the loop becomes a trihedral case of the Bricard linkage. Its geometrical properties are as follows.

$$\begin{aligned}
 a_{12} &= a_{23} = a_{34} = a_{45} = a_{56} = a_{61} \\
 \alpha_{12} &= \alpha_{34} = \alpha_{56} = \frac{\pi}{2}, \quad \alpha_{23} = \alpha_{45} = \alpha_{61} = \frac{3\pi}{2} \\
 R_i &= 0 \quad (i = 1, 2, \dots, 6)
 \end{aligned} \tag{2.2.11}$$

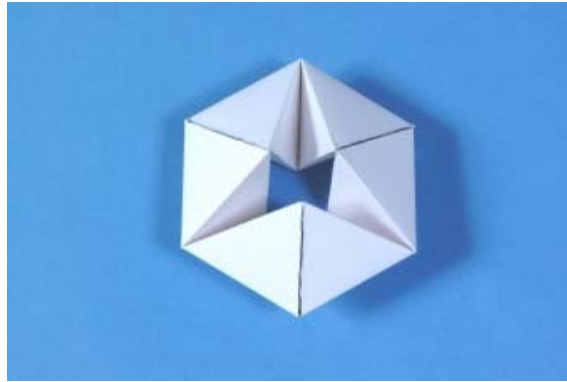


Fig. 2.2.10 A kaleidocycle made of six tetrahedra.

- Goldberg $6R$ linkage

Similar to the Goldberg $5R$ linkage, the Goldberg $6R$ linkage (Goldberg, 1943) is also produced by combining Bennett linkages. There are four types of Goldberg $6R$ linkages, as shown in Fig. 2.2.11.

The first Goldberg $6R$ linkage is formed by arranging three Bennett linkages in series. The first two Bennett linkages have a link in common, and the opposite link of one of them is common with a link of a third Bennett linkage. The second Goldberg $6R$ linkage is the *subtraction* of the first Goldberg $6R$ linkage from another Bennett linkage resulting in a *syncopated* linkage. The third Goldberg $6R$ linkage is built by an L-shaped arrangement of three Bennett linkages. The fourth Goldberg $6R$ linkage is produced by subtracting the Goldberg L-shaped $6R$ linkage from another Bennett linkage.

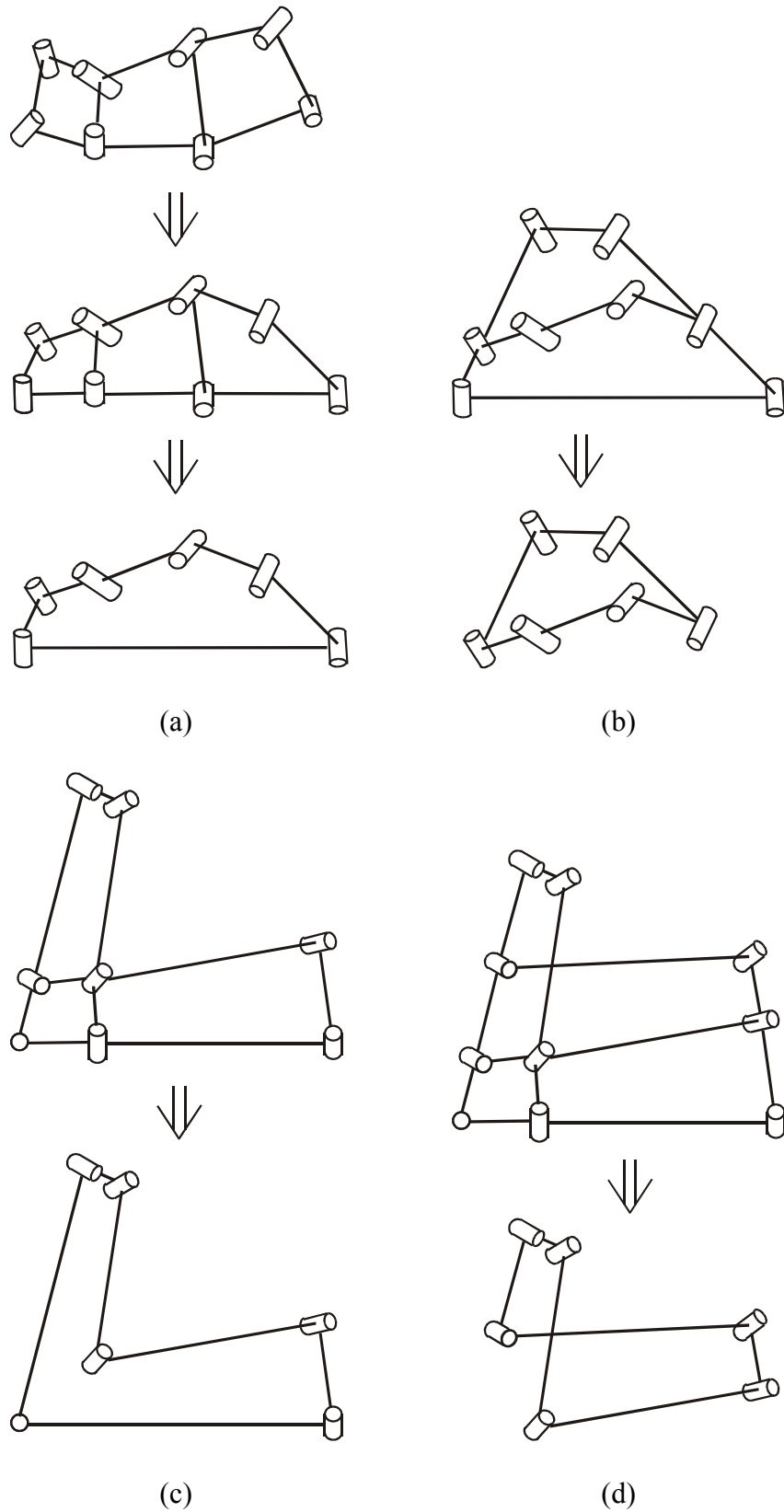


Fig. 2.2.11 Goldberg 6R linkages.

(a) Combination; (b) subtraction; (c) L-shaped; (d) crossing-shaped.

- Altmann linkage

Altmann (1954) presented a 6R linkage as shown in Fig. 2.2.12, which actually is a special case of the Bricard line-symmetric linkage. Its dimensional conditions can be expressed as follows:

$$\begin{aligned}
 a_{12} = a_{45} = a, \quad a_{23} = a_{56} = 0, \quad a_{34} = a_{61} = b \\
 \alpha_{12} = \alpha_{45} = \frac{\pi}{2}, \quad \alpha_{23} = \alpha_{56} = \frac{\pi}{2}, \quad \alpha_{34} = \alpha_{61} = \frac{3\pi}{2} \\
 R_i = 0 \quad (i = 1, 2, \dots, 6)
 \end{aligned} \tag{2.2.12}$$

Baker (1993) explored the algebraic representation of the linkage using the technique of the screw theory.

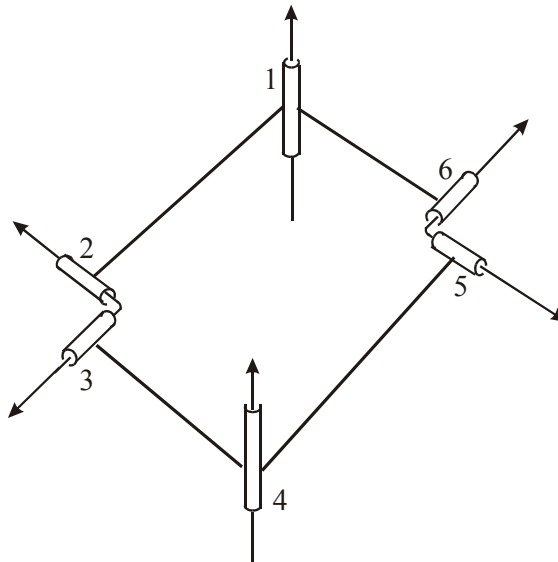


Fig. 2.2.12 Altmann linkage.

- Waldron hybrid linkage

Waldron (1968) drew attention to a class of mobile six-bar linkages with only lower pairs which include helical, cylinder and prism joints in addition to revolute joints. He suggested that any two single-loop linkages with a single degree of freedom could be arranged in space to make them share a common axis. After this common joint is removed, the resulting linkage certainly remains mobile. The condition for the resultant linkage to have mobility of one is that the equivalent screw systems of the original linkages shall intersect only in the screw axis of the common joint when they are placed. Waldron listed all six-bar linkages, which can be formed from two four-bar linkages with lower joints. It is obvious that the double-Hooke's-joint linkage and the Bennett $6R$ linkages that we discussed above belong to this linkage family.

An example of the Waldron hybrid linkage with revolute joints is illustrated here. It is made from two Bennett linkages connected in such a way that two revolute axes, one from each Bennett linkage, are collinear, see Fig. 2.2.13. Then the old links protruding from this shared axis are replaced by the common-perpendicular links between axes 1, 6 and axes 3, 4. The redundant axis and links are then removed to form a $6R$ overconstrained linkage.

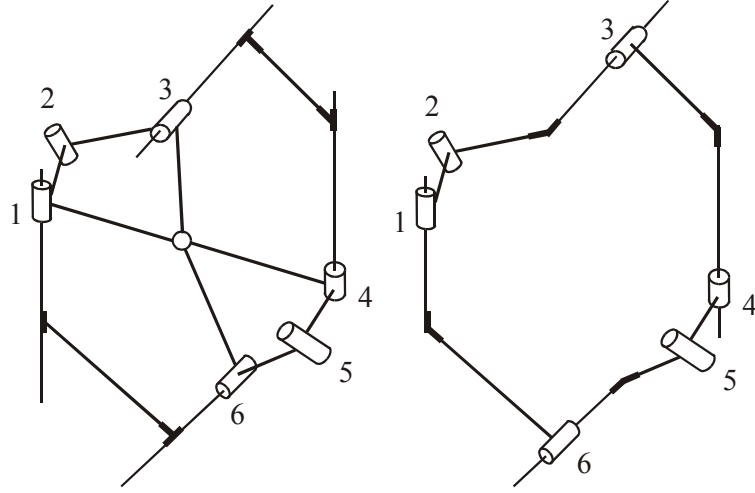


Fig. 2.2.13 Waldron hybrid linkage from two Bennett linkages.

- Schatz linkage

The Schatz linkage discovered and patented by Schatz was derived from a special trihedral Bricard linkage (Phillips, 1990). First, set this trihedral Bricard linkage in a configuration such that angles between the adjacent links are all $\pi/2$, see Fig. 2.2.14. Then replace links 61, 12, and 56 with a new link 61 of zero twist and a new pair of parallel shafts 12 and 56 (two links of zero length). By now, a new asymmetrical 6R linkage has been obtained with single degree of mobility. The dimension constraints of the linkage are as follows.

$$a_{12} = a_{56} = 0, \quad a_{23} = a_{34} = a_{45} = a, \quad a_{61} = \sqrt{3}a$$

$$\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{45} = \alpha_{56} = \frac{\pi}{2}, \quad \alpha_{61} = 0 \quad (2.2.13)$$

$$R_1 = -R_6, \quad R_2 = R_3 = R_4 = R_5 = 0$$

Brát (1969) studied the kinematic description of the Schatz linkage using the matrix method. The drum (link 34 in Fig. 2.2.14) was investigated in detail. The motion and trajectory of the centre of gravity of the drum were solved. Baker, Duclong, and Khoo (1982) carried out an exhaustive kinematic analysis of this linkage, comprising the determination of the linear and angular velocity and acceleration components for all moving links. The joint forces and driving torque required for dynamic equilibrium of the linkage were found.

This linkage is also known by the name Turbula because it constitutes the essential mechanism of a machine by that name, see Fig. 2.2.15. It is used for mixing fluids and powders.

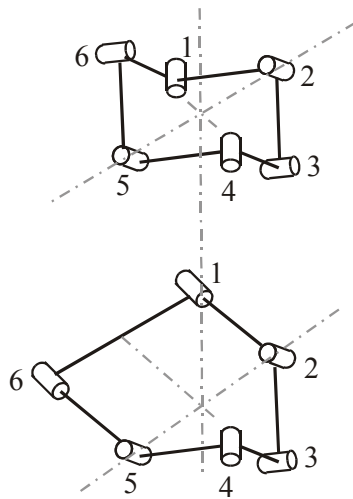


Fig. 2.2.14 Schatz linkage.

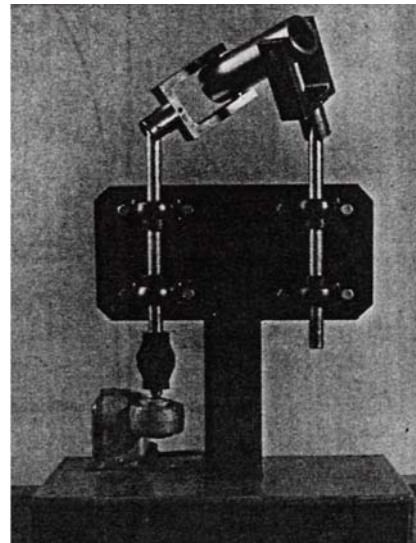


Fig. 2.2.15 Turbula machine.

- Wohlhart 6R linkage

Wohlhart (1987) presented a new 6R linkage, which can be regarded as a generalisation of the Bricard trihedral 6R linkage. In this linkage, the axes of all joints intersect a straight line, called the transversal line, see Fig. 2.2.16.

This linkage shows three partial plane symmetries, that is to say, three groups of two binary links with a symmetry plane are assembled in such a way that they form a loop. The conditions of its geometric parameters are as follows.

$$\begin{aligned}
 a_{12} &= a_{23}, \quad a_{34} = a_{45}, \quad a_{56} = a_{61} \\
 \alpha_{12} &= 2\pi - \alpha_{23}, \quad \alpha_{34} = 2\pi - \alpha_{45}, \quad \alpha_{56} = 2\pi - \alpha_{61} \\
 R_6 &= -R_2 - R_4, \quad R_1 = R_3 = R_5 = 0
 \end{aligned} \tag{2.2.14}$$

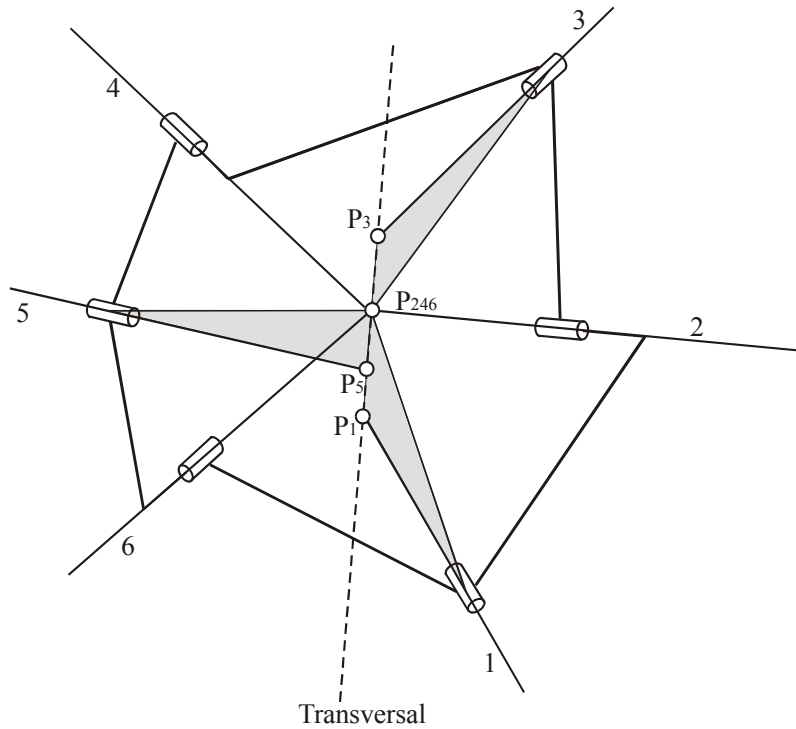


Fig. 2.2.16 Wohlhart 6R linkage.

- Wohlhart double-Goldberg linkage

Wohlhart (1991) also described another $6R$ overconstrained linkage, the synthesis of which is achieved by coalescing two appropriate generalised Goldberg $5R$ linkages and removing the two common links, see Fig. 2.2.17.

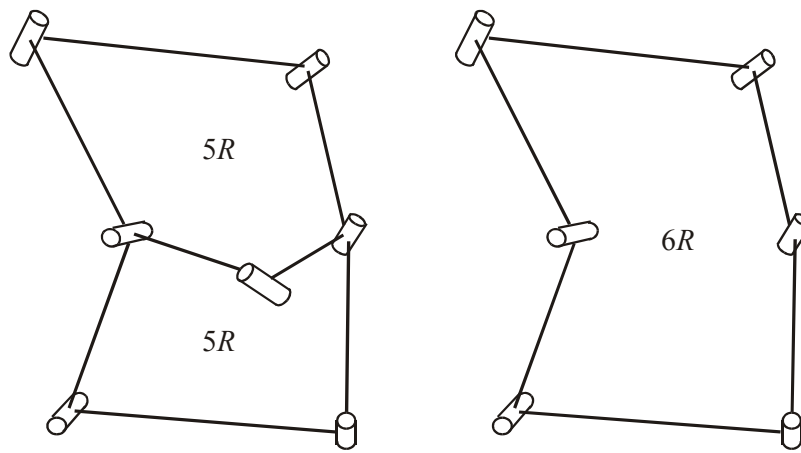


Fig. 2.2.17 Wohlhart double-Goldberg linkage.

- Bennett-joint $6R$ linkage

This linkage was discovered by Mavroidis and Roth (1994) as a by-product of their effort to develop a systematic method to deal with overconstrained linkages. They found that the manipulator inverse-kinematics problem, i.e. the problem of finding the values of the manipulator's joint variables that correspond to an end-effector position and orientation, is the same as that of finding the assembly configurations for the corresponding six-link closed-loop linkage. Hence the methods for solution of the manipulator inverse-kinematics problem could be used to prove overconstraint and to

calculate the input-output equations. Using this new method, they discovered a new overconstrained 6R linkage with the following parameter conditions.

$$\begin{aligned}
 a_{34} &= a_{12}, \quad a_{56} = a_{23}, \quad a_{61} = a_{45} \\
 \alpha_{34} &= \alpha_{12}, \quad \alpha_{56} = \alpha_{23}, \quad \alpha_{61} = \alpha_{45} \\
 R_1 &= R_4 = 0, \quad R_5 = R_2 \text{ (or } R_3), \quad R_6 = R_3 \text{ (or } R_2) \\
 \frac{\sin \alpha_i}{a_i} &= k_B \quad (i = 1, 2, \dots, 6)
 \end{aligned} \tag{2.2.15}$$

Mavroidis and Roth defined the term *Bennett joint* as a set of Bennett axes which are three adjacent revolute joints of a Bennett linkage. The constant k_B in the above equation is a parameter of the Bennett joint, and is called *Bennett ratio*, and the mobile linkages which contain at least one Bennett joint are defined as *Bennett based linkages*. So for this new 6R linkages in Fig. 2.2.18, joint axes 6, 1, 2 and 3, 4, 5 form Bennett joints A and B respectively with the same Bennett ratios and with no common axis.

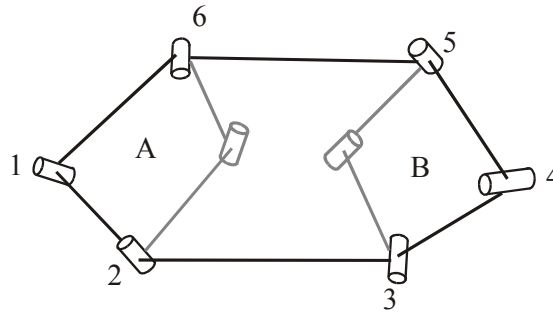


Fig. 2.2.18 Bennett-joint 6R linkage.

- Dietmaier 6R linkage

Dietmaier (1995) discovered a new family of overconstrained 6R linkages with the aid of the numerical method. The necessary conditions for a 6R linkage to be mobile were set up and solved numerically. The relationships of the geometrical parameters can be stated as follows.

$$a_{45} = a_{12}$$

$$\alpha_{45} = \alpha_{12}$$

$$R_3 = R_1, R_6 = R_4, R_2 = R_5 = 0 \quad (2.2.16)$$

$$\frac{a_3}{\sin \alpha_3} = \frac{a_2}{\sin \alpha_2}, \frac{a_6}{\sin \alpha_6} = \frac{a_5}{\sin \alpha_5}$$

$$\frac{a_2(\cos \alpha_2 + \cos \alpha_3)}{\sin \alpha_2} = \frac{a_5(\cos \alpha_5 + \cos \alpha_6)}{\sin \alpha_5}$$

Note that if the geometric parameters are restricted further by requiring $a_5 = a_3$, $\alpha_5 = \alpha_3$, $a_6 = a_2$, and $\alpha_6 = \alpha_2$, the Bennett-joint 6R linkage by Mavroidis and Roth will be obtained.

2.2.4 Summary

3D overconstrained linkages reviewed in this chapter are made of 4, 5 or 6 links connected by revolute (hinge) joints. There are fifteen types in total. Only two of these linkages, the Bennett linkage and the Bricard linkage, can be regarded as basic linkages, while others are combinations or derivatives of the basic linkages. A summary is given

in Table 2.2.1. Our research is concentrated on the construction of structural mechanisms based on the two basic linkages.

Table 2.2.1 3D overconstrained linkages with only revolute joints
and their dependent linkages.

Number of links	Linkages	Dependent linkages
4	Bennett linkage	—
5	Goldberg 5R linkage	Bennett linkage
5	Myard linkage	Bennett linkage
6	Altmann linkage	Bricard linkage
6	Bennett 6R hybrid linkage	Bennett linkage
6	Bennett-joint 6R linkage	Bennett linkage
6	Bricard linkages	—
6	Dietmaier 6R linkage	Bennett linkage
6	Double-Hooke's-joint linkage	Bennett linkage
6	Goldberg 6R linkage	Bennett linkage
6	Sarrus linkage	Bennett linkage
6	Schatz linkage	Bricard linkage
6	Waldron hybrid linkages	Four-bar linkage with lower joints
6	Wohlhart 6R linkage	Bricard linkage
6	Wohlhart double-Goldberg linkage	Bennett linkage

2.3 TILINGS AND PATTERNS

2.3.1 General Tilings and Patterns

A *plane tiling* is a countable family of closed sets which cover the plane without gaps or overlaps. The closed sets are called *tiles* of the tiling. Tiling is also called *tessellation*. A *pattern* is a design which repeats some motif in a more or less systematic manner.

The art of tiling must have originated very early in the history of civilisation. As soon as man began to build, he would use stones to cover the floors and the walls of his houses, and as soon as he started to select the shapes and colours of his stones to make a pleasing design, he could be said to have begun tiling. Patterns must have originated in a similar manner, and their history is probably as old as, if not older than, that of tilings (Grünbaum and Shephard, 1986).

The tiles could be either regular geometric shapes or irregular shapes such as animals or flowers. Figure 2.3.1 shows a honeycomb of bees, which is a typical example of tiling with regular tiles. Irregular tiling can be illustrated by Echer's work, see Fig. 2.3.2.

The art of designing tilings and patterns is clearly extremely old and well developed (Beverley, 1999; Evans, 1931; Rossi, 1970). By contrast, the science of tilings and patterns, which means the study of their mathematical properties, is comparatively recent and many parts of the subject have yet to be explored in depth.



Fig. 2.3.1 A honeycomb of bees.



Fig. 2.3.2 Escher's Woodcut 'Sky and Water' in 1938.

Magnus (1974) dealt with the triangle tessellations in planar Euclidean, Hyperbolic, and Elliptic geometry and also the triangle tessellations of the sphere. Cundy and Rollett (1961) found a convenient link between plane diagrams and solid configuration by the study of plane tessellations. They formed a suitable introduction to the polyhedra and their nets. In fact a plane tessellation is the special case of an infinite polyhedron. Critchlow (1969) and Steinhaus (1983) pointed out that there are *seventeen* different tiles consisting of polyhedrons, but only *eleven* of them can be extended over a whole plane without overlapping. The most systematic study of tilings and patterns can be found in Grünbaum and Shephard's book (1986).

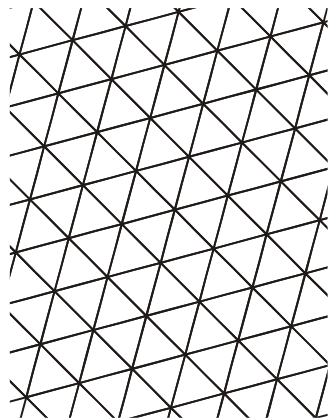
2.3.2 Tilings by Regular Polygons

Tilings are usually represented by the number of sides of the polygons around any cross point in the clockwise or anti-clockwise order. For instance, (3^6) represents a tiling in which each of the points is surrounded by six triangles, 3 is the number of the sides of a triangle and superscript 6 is the number of triangles. Similarly, $(3^3.4^2)$ means three triangles and two squares around a cross point. $(3^6;3^2.6^2)$ represents a 2-uniform tiling in which there are two types of points, one type is surrounded by six triangles while the other type is surrounded by two triangles and two hexagons.

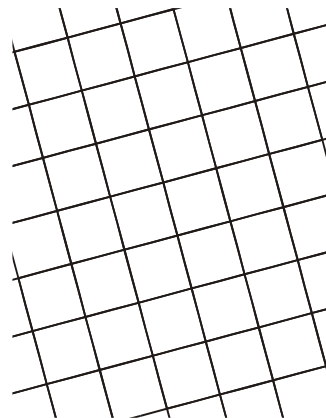
Grünbaum and Shephard (1986) reported that the tilings accommodating regular polygons can be classified into four types: regular and uniform tilings, k -uniform tilings, equitransitive and edge-transitive tilings, and tilings that are not edge-to-edge.

- *Regular and uniform tilings*

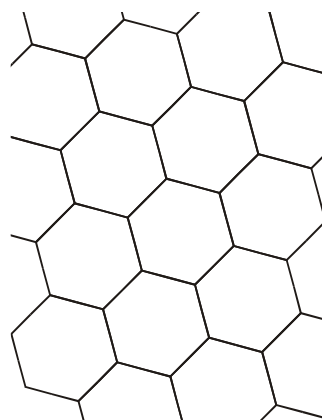
The only edge-to-edge monohedral tilings by regular polygons are the three regular tilings shown in Fig. 2.3.3. The basic tiles are identical equilateral triangles, squares and regular hexagons, respectively. There exist precisely eleven distinct edge-to-edge tilings by regular polygons such that all vertices are of the same type. They are (3^6) , $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2) , (4^4) , $(4.6.12)$, (4.8^2) and (6^3) , see Figs. 2.3.3 and 2.3.4.



(3^6)

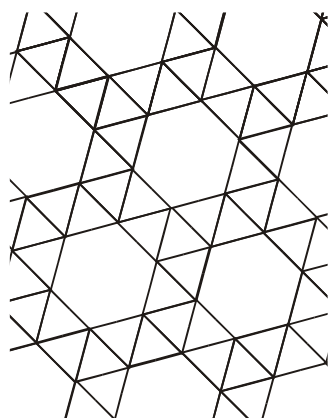


(4^4)

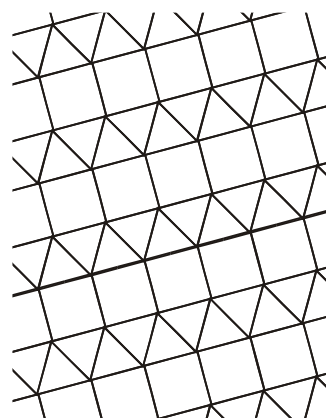


(6^3)

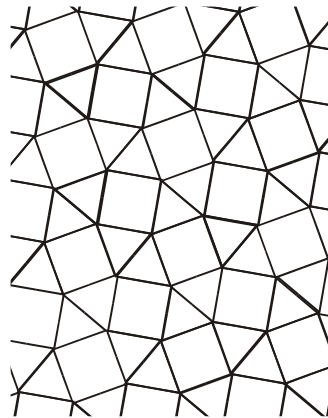
Fig. 2.3.3 The edge-to-edge monohedral tilings by regular polygons.



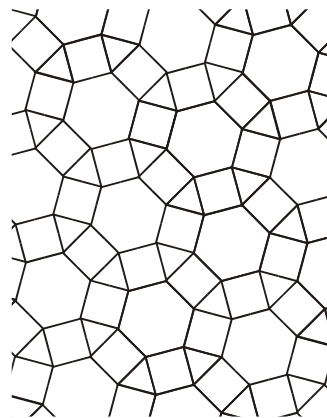
$(3^4.6)$



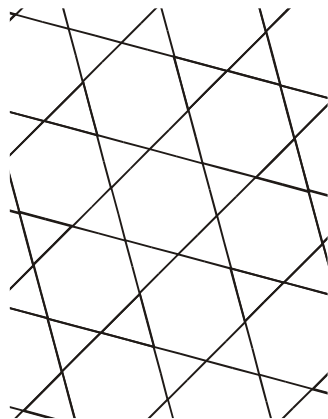
$(3^3.4^2)$



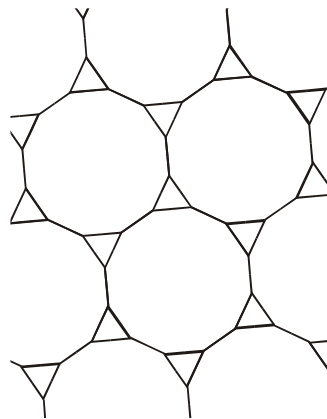
$(3^2.4.3.4)$



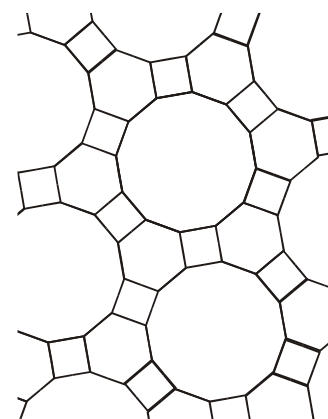
$(3.4.6.4)$



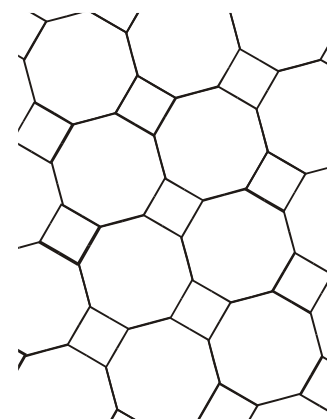
$(3.6.3.6)$



(3.12^2)



$(4.6.12)$



(4.8^2)

Fig. 2.3.4 Eight distinct edge-to-edge tilings by different regular polygons.

- *k-uniform tilings*

An edge-to-edge tiling by regular polygons is called k -uniform if its vertices form precisely k transitivity classes with respect to the group of symmetries of the tilings. In other words, the tiling is k -uniform if and only if it is k -isogonal and its tiles are regular polygons. There exist twenty distinct types of 2-uniform edge-to-edge tilings by regular polygons, namely: $(3^6;3^4.6)_1$, $(3^6;3^4.6)_2$, $(3^6;3^3.4^2)_1$, $(3^6;3^3.4^2)_2$, $(3^6;3^2.4.12)$, $(3^6;3^2.4.3.4)$, $(3^6;3^2.6^2)$, $(3^4.6;3^2.6^2)$, $(3^3.4^2; 3^2.4.3.4)_1$, $(3^3.4^2; 3^2.4.3.4)_2$, $(3^3.4^2; 3.4.6.4)$, $(3^3.4^2; 4^4)_1$, $(3^3.4^2;4^4)_2$, $(3^2.4.3.4;3.4.6.4)$, $(3^2.6^2;3.6.3.6)$, $(3.4.3.12;3.12^2)$, $(3.4^2.6;3.4.6.4)$, $(3.4^2.6;3.6.3.6)_1$, $(3.4^2.6;3.6.3.6)_2$, and $(3.4.6.4;4.6.12)$, where $()_1$ and $()_2$ denote two different arrangements of 2-uniform edge-to-edge tilings. Some of these tilings are shown in Fig. 2.3.5.

Denote $K(k)$ as the number of distinct k -uniform tilings. $K(1)=11$, $K(2)=20$, $K(3)=39$, $K(4)=33$, $K(5)=15$, $K(6)=10$, $K(7)=7$, and $K(k)=0$ for each $k \geq 8$.

- *Equitransitive and edge-transitive tilings*

A tiling by regular polygons is equitransitive if each set of mutually congruent tiles forms one transitivity class. One such tiling is shown in Fig. 2.3.6.

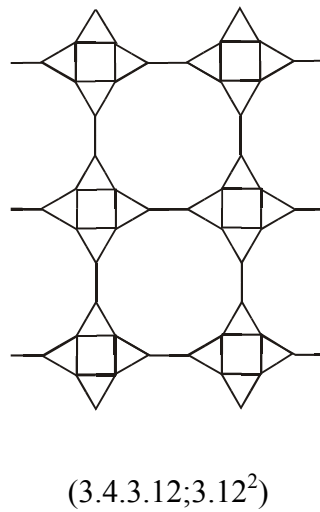
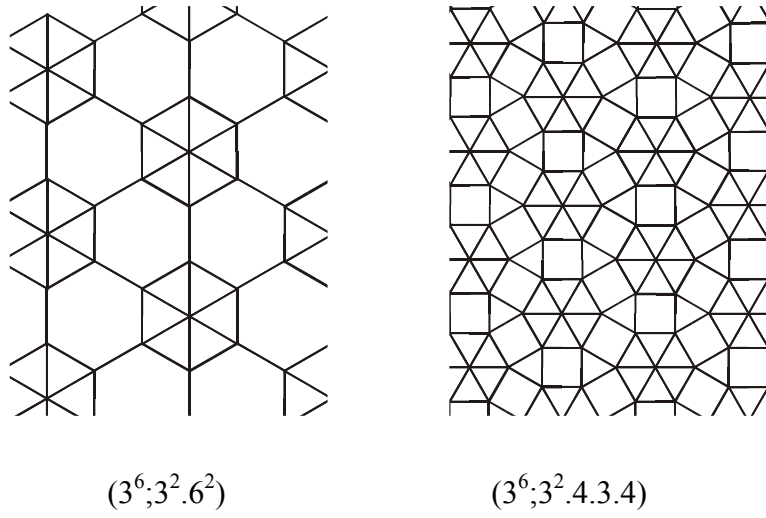


Fig. 2.3.5 Examples of 2-uniform tilings.

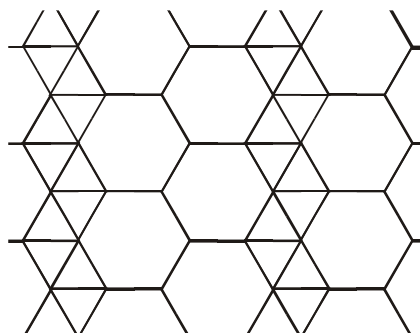


Fig. 2.3.6 An example of equitransitive tilings.

- *Tilings that are not edge-to-edge*

The tilings by regular polygons without the requirement that the tiling is edge-to-edge are considered here. Some examples are shown in Fig. 2.3.7.

Further consideration should be given to patterns with overlapping motifs. One such example is given in Fig. 2.3.8. The variety of such patterns is restricted by the number of basic tilings.

2.3.3 Summary - 3 Types of Simplified Tilings

According to the regular and uniform tilings, there are only three ways to cover the plane with an identical unit. Tiling (3^6) makes the unit spread in three directions; tiling (4^4) makes the unit spread in four directions; tiling (6^3) makes the unit spread in six directions. So all the tilings and patterns can be considered as a unit tessellates in one of these three ways, see Fig. 2.3.9, though within each unit, a repeatable pattern can be used. This idea is applied in our research to develop a systematic approach in determining the layout for mobile assemblies.

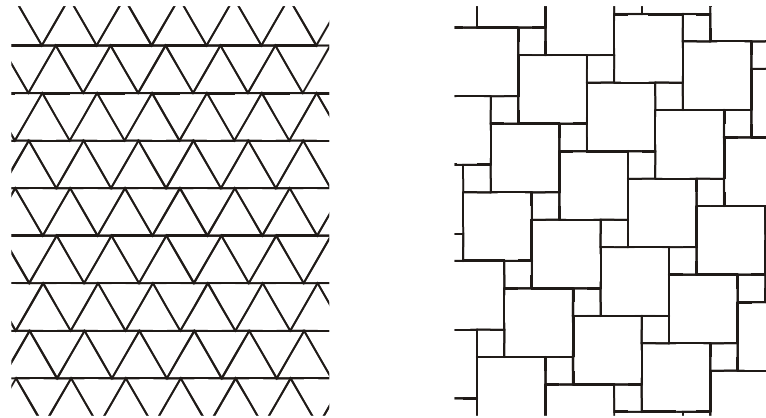


Fig. 2.3.7 Tilings that are not edge-to-edge.

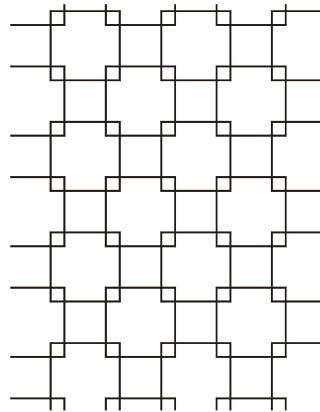
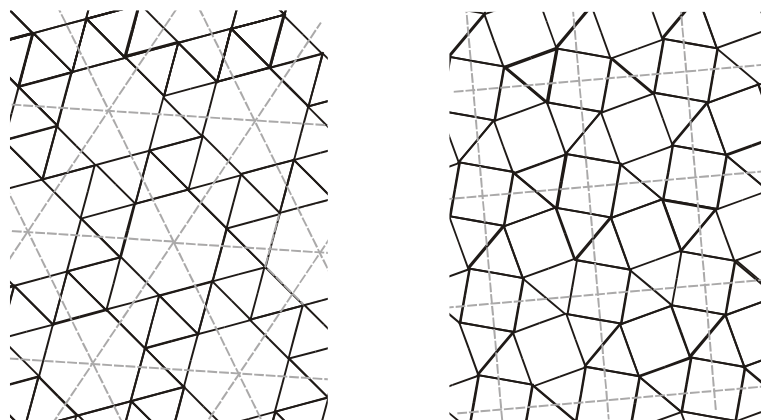
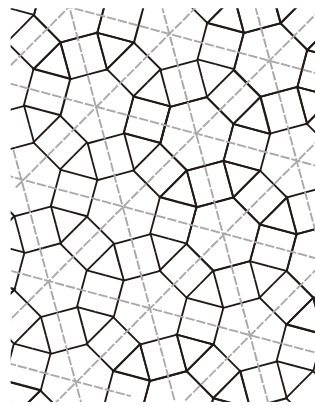


Fig. 2.3.8 Pattern with overlapping motifs.

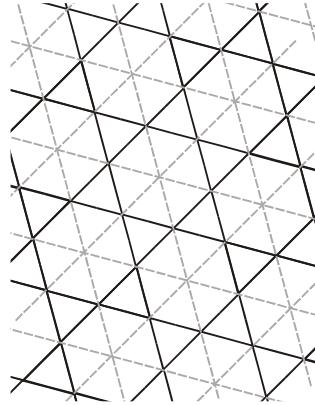


(a) $(3^4.6)$

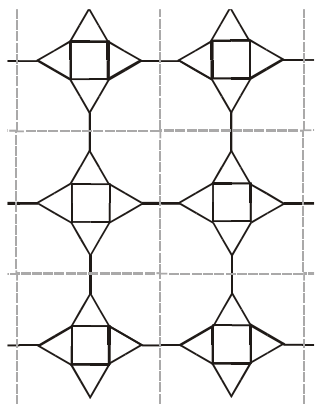
(b) $(3^2.4.3.4)$



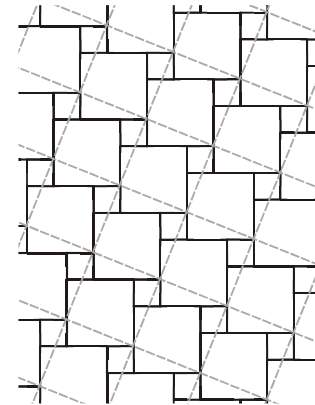
(c) (3.4.6.4)



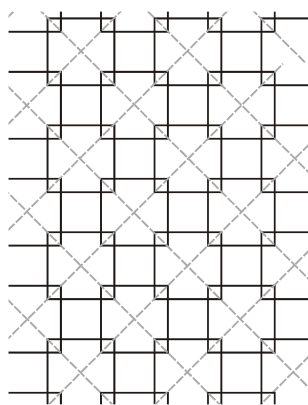
(d) (3.6.3.6)



(e) (3.4.3.12;3.12²)



(f) Tiling that are not edge-to-edge



(g) Pattern with overlapping motifs

Fig. 2.3.9 Units, represented by grey dash lines, in tilings and patterns.

Bennett Linkage and its Networks

3.1 INTRODUCTION

In this chapter, the possibility of building large space deployable structures using the Bennett linkage as the basic element is investigated. We examine whether the Bennett linkage can be used to form large structural mechanisms, and if so, whether it is possible to achieve the highest expansion/packaging ratio.

Before we proceed, a definition is given here. Consider two Bennett linkages, of which one has lengths and twists a_1 , b_1 , α_1 and β_1 , while the other has a_2 , b_2 , α_2 and β_2 . If

$$\alpha_1 = \alpha_2, \beta_1 = \beta_2 \text{ (or } \alpha_1 = -\alpha_2, \beta_1 = -\beta_2 \text{)}$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \tag{3.1.1}$$

these two Bennett linkages are considered as *similar* Bennett linkages.

The layout of the chapter is as follows. Section 3.2 presents the method to form a network of Bennett linkages. The mathematical derivation of the compact folding and maximum expansion conditions is given in Section 3.3, resulting in an alternative form of the Bennett linkage. A simple method to build this alternative form is introduced and

verified. The alternative form is then extended to the network of the Bennett linkages, leading to large structural mechanisms which can be compactly folded up. Further discussion in Section 3.4 concludes the chapter.

3.2 NETWORK OF BENNETT LINKAGES

3.2.1 Single-layer Network of Bennett Linkages

For convenience, we adopt a simple schematic diagram, shown in Fig. 3.2.1, to illustrate a Bennett linkage. Four links are represented by thick lines, and four revolute joints are represented by cross points of the lines. The twists of the joint axes are denoted as α and β , respectively, which are marked alongside the link connecting the joints. The lengths of links $a_{AB} = a_{CD} = a$, $a_{BC} = a_{DA} = b$ due to (2.2.1a) and (2.2.1b). They are not marked in the diagram, but it will not affect the generality as long as we bear in mind that opposite sides always have equal lengths. For the sake of clarity, this diagram is adopted in the derivation.

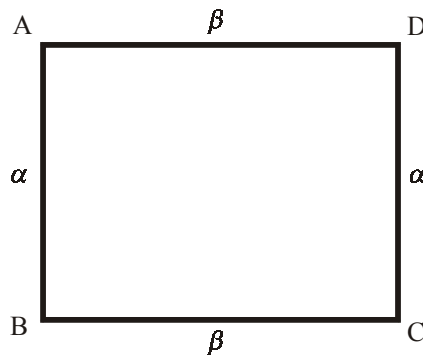


Fig. 3.2.1 A schematic diagram of the Bennett linkage.

Many such identical Bennett linkages with lengths a , b and twists α , β can be connected forming a network as shown in Fig. 3.2.2(a). In order to retain the single degree of mobility, it is necessary to assume that every small $4R$ linkage in the network should also be a Bennett linkage. While large Bennett linkage ABCD has lengths a , b and twists α , β , the smaller Bennett linkages around it, which are marked with numbers 1 to 8, will have lengths a_i , b_i and twists α_i , β_i ($i = 1, 2, \dots, 8$), see Fig. 3.2.2(b).

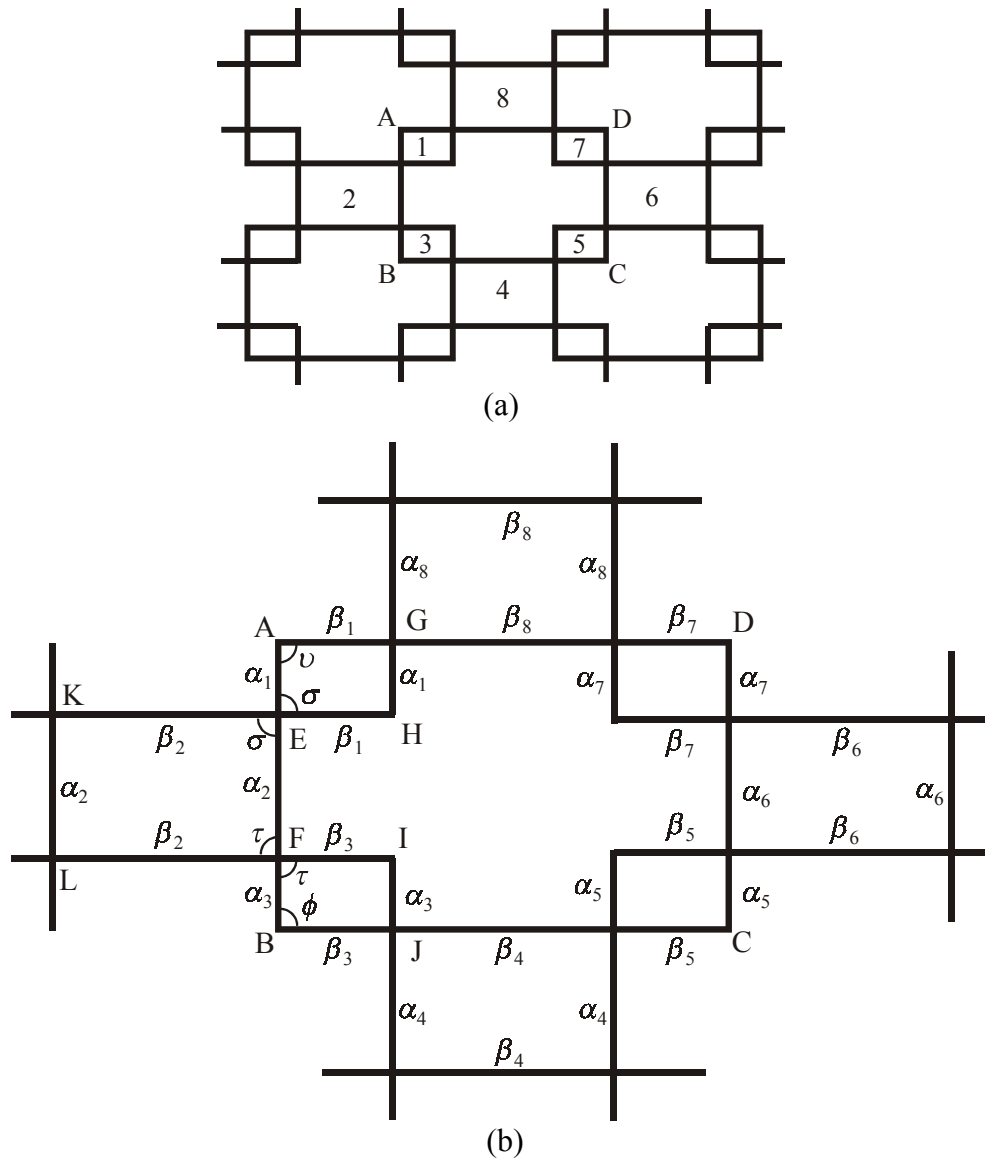


Fig. 3.2.2 Single-layer network of Bennett linkages.

(a) A portion of the network; (b) enlarged connection details.

Considering links AB, BC, CD, and DA, we have

$$\begin{aligned}
 a_1 + a_2 + a_3 &= a, \quad \alpha_1 + \alpha_2 + \alpha_3 = \alpha \\
 b_3 + b_4 + b_5 &= b, \quad \beta_3 + \beta_4 + \beta_5 = \beta \\
 a_5 + a_6 + a_7 &= a, \quad \alpha_5 + \alpha_6 + \alpha_7 = \alpha \\
 b_7 + b_8 + b_1 &= b, \quad \beta_7 + \beta_8 + \beta_1 = \beta
 \end{aligned} \tag{3.2.1}$$

Define angles σ , τ , ν and ϕ as revolute variables shown in Fig. 3.2.2(b). For Bennett linkage 1, there is

$$\tan \frac{\pi - \nu}{2} \tan \frac{\pi - \sigma}{2} = \frac{\sin \frac{1}{2}(\beta_1 + \alpha_1)}{\sin \frac{1}{2}(\beta_1 - \alpha_1)} \tag{3.2.2}$$

For Bennett linkage 2,

$$\tan \frac{\pi - \sigma}{2} \tan \frac{\pi - \tau}{2} = \frac{\sin \frac{1}{2}(\beta_2 + \alpha_2)}{\sin \frac{1}{2}(\beta_2 - \alpha_2)} \tag{3.2.3}$$

For Bennett linkage 3,

$$\tan \frac{\pi - \tau}{2} \tan \frac{\pi - \phi}{2} = \frac{\sin \frac{1}{2}(\beta_3 + \alpha_3)}{\sin \frac{1}{2}(\beta_3 - \alpha_3)} \tag{3.2.4}$$

Finally, for Bennett linkage ABCD,

$$\tan \frac{\pi - \nu}{2} \tan \frac{\pi - \phi}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \tag{3.2.5}$$

Combining (3.2.2) – (3.2.5) gives

$$\frac{\sin \frac{1}{2}(\beta_1 + \alpha_1)}{\sin \frac{1}{2}(\beta_1 - \alpha_1)} \cdot \frac{\sin \frac{1}{2}(\beta_3 + \alpha_3)}{\sin \frac{1}{2}(\beta_3 - \alpha_3)} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \cdot \frac{\sin \frac{1}{2}(\beta_2 + \alpha_2)}{\sin \frac{1}{2}(\beta_2 - \alpha_2)} \quad (3.2.6)$$

This is a non-linear equation and many solutions may exist. By observation, two solutions can be immediately determined, which are

$$\begin{aligned} \alpha_3 &= \alpha & \alpha_2 &= -\alpha_1 \\ \beta_3 &= \beta & \beta_2 &= -\beta_1 \end{aligned} \quad (3.2.7a)$$

and

$$\begin{aligned} \alpha_1 &= \alpha & \alpha_2 &= -\alpha_3 \\ \beta_1 &= \beta & \beta_2 &= -\beta_3 \end{aligned} \quad (3.2.7b)$$

Similar analysis can be applied to Bennett linkages around links BC, CD and DA, see in Fig. 3.2.2(a). The twists of Bennett linkages 3, 4 and 5 should therefore satisfy

$$\begin{aligned} \alpha_3 &= \alpha & \alpha_4 &= -\alpha_5 \\ \beta_3 &= \beta & \beta_4 &= -\beta_5 \end{aligned} \quad (3.2.8a)$$

or

$$\begin{aligned} \alpha_5 &= \alpha & \alpha_4 &= -\alpha_3 \\ \beta_5 &= \beta & \beta_4 &= -\beta_3 \end{aligned} \quad (3.2.8b)$$

The twists of Bennett linkages 5, 6 and 7 should satisfy

$$\begin{aligned} \alpha_7 &= \alpha & \alpha_6 &= -\alpha_5 \\ \beta_7 &= \beta & \beta_6 &= -\beta_5 \end{aligned} \quad (3.2.9a)$$

or

$$\begin{aligned} \alpha_5 &= \alpha & \alpha_6 &= -\alpha_7 \\ \beta_5 &= \beta & \beta_6 &= -\beta_7 \end{aligned} \quad (3.2.9b)$$

The twists of Bennett linkages 7, 8 and 1 should satisfy

$$\begin{aligned}\alpha_7 &= \alpha & \alpha_8 &= -\alpha_1 \\ \beta_7 &= \beta & \beta_8 &= -\beta_1\end{aligned}\tag{3.2.10a}$$

or

$$\begin{aligned}\alpha_1 &= \alpha & \alpha_8 &= -\alpha_7 \\ \beta_1 &= \beta & \beta_8 &= -\beta_7\end{aligned}\tag{3.2.10b}$$

Combining four sets of solutions (3.2.7) – (3.2.10), two common solutions, which enable the mobility of the network in Fig. 3.2.2(a), are obtained:

$$\begin{aligned}\alpha_1 &= \alpha_5 = \alpha & \alpha_2 &= \alpha_4 = -\alpha_3 & \alpha_6 &= \alpha_8 = -\alpha_7 \\ \beta_1 &= \beta_5 = \beta & \beta_2 &= \beta_4 = -\beta_3 & \beta_6 &= \beta_8 = -\beta_7\end{aligned}\tag{3.2.11a}$$

and

$$\begin{aligned}\alpha_3 &= \alpha_7 = \alpha & \alpha_2 &= \alpha_8 = -\alpha_1 & \alpha_6 &= \alpha_4 = -\alpha_5 \\ \beta_3 &= \beta_7 = \beta & \beta_2 &= \beta_8 = -\beta_1 & \beta_6 &= \beta_4 = -\beta_5\end{aligned}\tag{3.2.11b}$$

Because of symmetry, (3.2.11a) and (3.2.11b) are essentially same. Thus, we take only (3.2.11a) in the derivation next.

Note that, due to (2.2.2), the corresponding lengths of each Bennett linkage should satisfy

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin \alpha_i}{\sin \beta_i} = \frac{a}{b} = \frac{a_i}{b_i} \quad (i = 1, 2, \dots, 8)\tag{3.2.12}$$

A close examination of (3.2.11a) and (3.2.12) reveals the order of distribution of Bennett linkages within a mobile network. Diagonally from top left corner to bottom

right, there are two types of rows of Bennett linkages, see Fig. 3.2.3. The first type consists of Bennett linkages with lengths proportional to a and b , and twists being α and β , which are denoted as ‘I’ in Fig. 3.2.3. The second type, next to the first one, are made of Bennett linkages with lengths proportional to a and b , and twists being α_i and β_i or $-\alpha_i$ and $-\beta_i$, which are denoted as ‘II_i’ or ‘-II_i’, respectively, and

$$\frac{\sin \alpha_i}{\sin \beta_i} = \frac{a}{b}$$

A further study has shown that the first row of Bennett linkages in Fig. 3.2.3 can actually have twists α_j and β_j , which may be different from α and β , as long as

$$\frac{\sin \alpha_j}{\sin \beta_j} = \frac{a}{b}$$

If we denote this type of Bennett linkages as ‘I_j’, the arrangement of the mobile network can be redrawn as that in Fig. 3.2.4, in which a very interesting pattern emerges: diagonally from top left to bottom right, each row of Bennett linkages belong to the same type. This leads to an important feature: the joints, which form the diagonal direction from top left to bottom right, remain along a straight line during deployment. Therefore, if we draw a set of guidelines in that direction, each of which passes through a number of joints, these lines remain straight and parallel to each other during deployment though the distances between the guidelines may vary.

Generally, the network shown in Fig. 3.2.4 deploys into a cylindrical profile. The joints forming those parallel guidelines remain straight. They expand along the longitudinal

direction of the cylinder. In the other diagonal direction, joints generally deploy spirally on the surface of the cylinder, see Fig. 3.2.5.

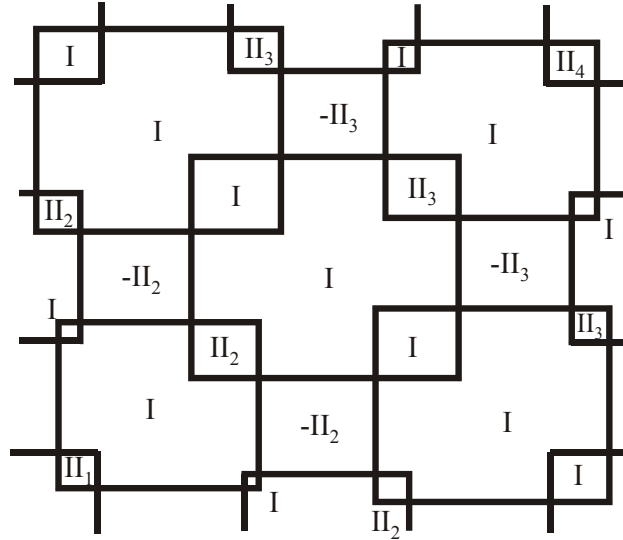


Fig. 3.2.3 Network of Bennett linkages with the same twists.

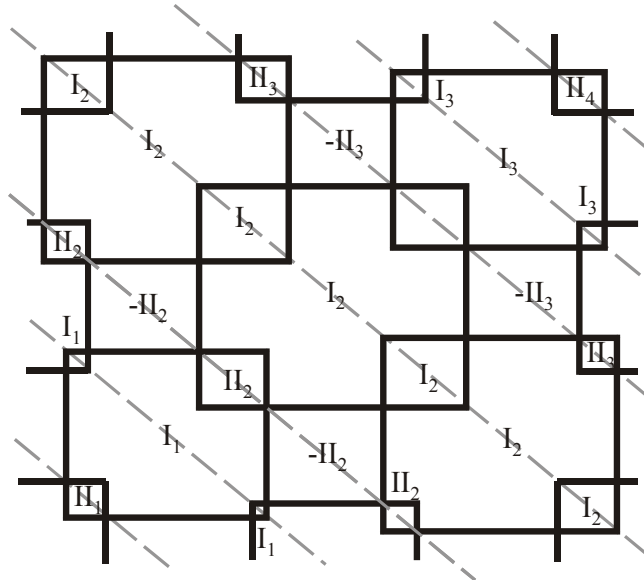


Fig. 3.2.4 Network of similar Bennett linkages with guidelines.

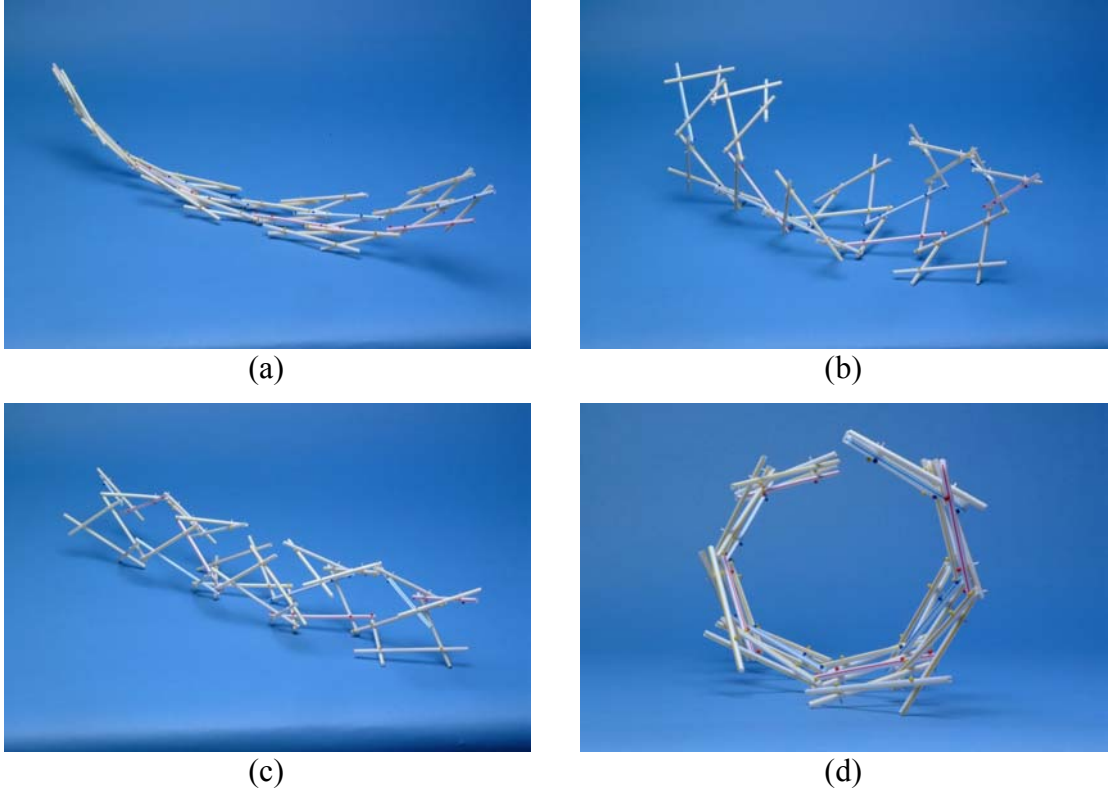


Fig. 3.2.5 A special case of single-layer network of Bennett linkages.

(a) – (c) Deployment sequence; (d) view of cross section of network.

The curvature of the cylinder inscribed the network depends on the values of the lengths and twists of links of each Bennett linkage and the deployment angle of the network.

For instance, for the model shown in Fig. 3.2.5, where the lengths and twists of the links are

$$a_i = \frac{a}{3}, b_i = \frac{b}{3} \quad (i = 1, 2, \dots, 8)$$

$$\alpha_1 = \alpha_2 = -\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = -\alpha_7 = \alpha_8 = \alpha$$

$$\beta_1 = \beta_2 = -\beta_3 = \beta_4 = \beta_5 = \beta_6 = -\beta_7 = \beta_8 = \beta$$

the radius of its deployed cylinder, which is inscribed the network, is

$$R = \frac{t \sin \theta}{1 + t^2 + 2t \cos \theta} \sqrt{-\left(1 + t^2 + 2t \frac{1 + \cos \theta \cos \varphi}{\cos \theta + \cos \varphi}\right)} \cdot a$$

where $t = \frac{b}{a}$, θ and φ are deployment angles related by (2.2.6).

When $\alpha = \frac{\pi}{4}$, the relationship between deployment angle θ and unit radius R/a is shown in Fig. 3.2.6.

During deployment, the network expands in the helical direction on the surface of the cylindrical profile. In the special case where the link lengths are the same, i.e. $a = b$, the network expands only in the circumferential direction, forming a circular arch as shown in Fig. 3.2.7. In another special case where the twists are $\alpha_1 = -\alpha_2 = \alpha_3 = \alpha$ and $\beta_1 = -\beta_2 = \beta_3 = \beta$, the network remains completely flat during the deployment, see Fig. 3.2.8.

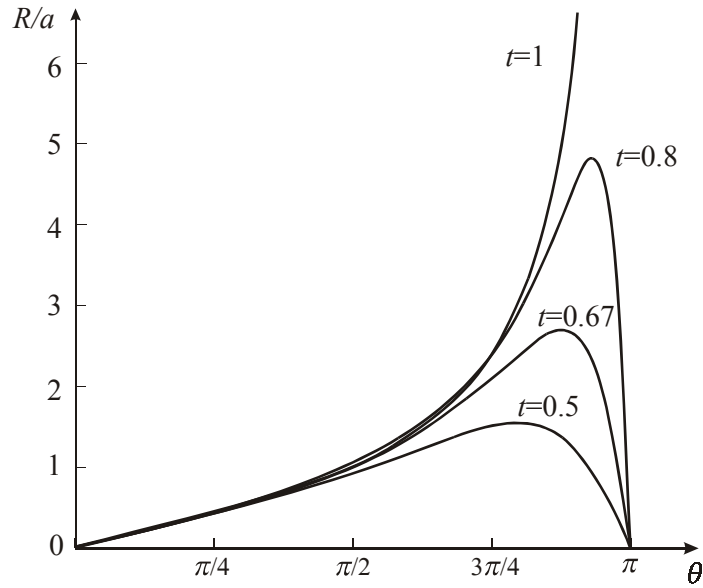
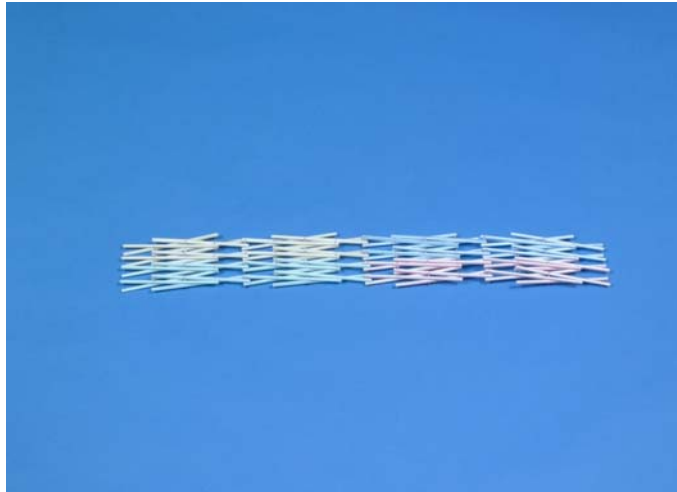
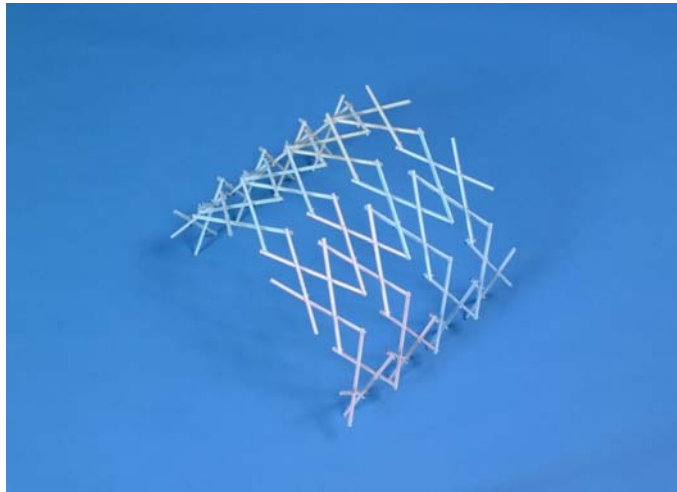


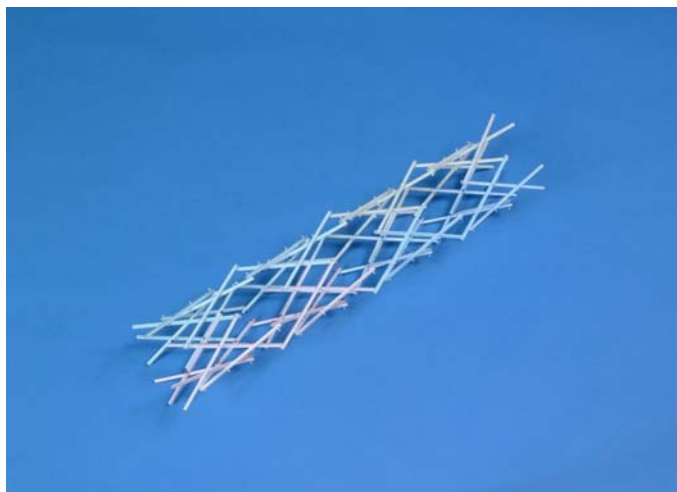
Fig. 3.2.6 R/a vs θ for different t ($t = b/a$).



(a)



(b)



(c)

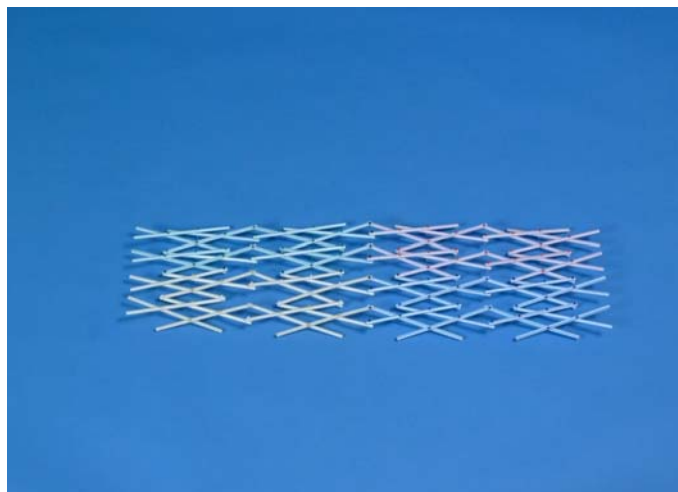
Fig. 3.2.7 (a) – (c) Deployment sequence of a deployable arch.



(a)



(b)



(c)

Fig. 3.2.8 (a) – (c) Deployment sequence of a flat deployable structure.

Now let us examine whether the top right small Bennett linkage PVND can be connected with the bottom left one JBKS while retaining mobility, see Fig. 3.2.9(b).

This can only be possible if and only if all the rectangular loops are Bennett linkages.

The deployment angles and twists for each of the loops are shown in Fig. 3.2.9(b). Note that $\tau + \phi = \pi$. Thus, the following relationships can be found.

For ABCD,

$$\tan \frac{\pi - \nu}{2} \tan \frac{\pi - \sigma}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \quad (3.2.13)$$

For JBKS,

$$\tan \frac{\pi - \nu}{2} \tan \frac{\pi - \tau}{2} = \frac{\sin \frac{1}{2}(\beta_3 + \alpha_3)}{\sin \frac{1}{2}(\beta_3 - \alpha_3)} \quad (3.2.14)$$

For PVND,

$$\tan \frac{\pi - \nu}{2} \tan \frac{\pi - \tau}{2} = \frac{\sin \frac{1}{2}(\beta_7 + \alpha_7)}{\sin \frac{1}{2}(\beta_7 - \alpha_7)} \quad (3.2.15)$$

For RSTV,

$$\tan \frac{\pi - \nu}{2} \tan \frac{\pi - \sigma}{2} = \frac{\sin \frac{1}{2}(\beta_0 + \alpha_0)}{\sin \frac{1}{2}(\beta_0 - \alpha_0)} \quad (3.2.16)$$

For AJTP with twists $\alpha_0 + \alpha_7$, $\beta_0 + \beta_3$, and variables $\pi - \phi, \pi - \sigma$,

$$\tan \frac{\pi - \phi}{2} \tan \frac{\pi - \sigma}{2} = \frac{\sin \frac{1}{2}((\beta_0 + \beta_3) + (\alpha_0 + \alpha_7))}{\sin \frac{1}{2}((\beta_0 + \beta_3) - (\alpha_0 + \alpha_7))} \quad (3.2.17)$$

For RKCNC with twists $\alpha_0 + \alpha_3$, $\beta_0 + \beta_7$, and variables $\pi - \phi, \pi - \sigma$,

$$\tan \frac{\pi - \phi}{2} \tan \frac{\pi - \sigma}{2} = \frac{\sin \frac{1}{2}((\beta_0 + \beta_7) + (\alpha_0 + \alpha_3))}{\sin \frac{1}{2}((\beta_0 + \beta_7) - (\alpha_0 + \alpha_3))} \quad (3.2.18)$$

From (3.2.13) – (3.2.18), the following relationships among the twists can be obtained,

$$\begin{aligned} \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} &= \frac{\sin \frac{1}{2}(\beta_0 + \alpha_0)}{\sin \frac{1}{2}(\beta_0 - \alpha_0)} \\ \frac{\sin \frac{1}{2}(\beta_3 + \alpha_3)}{\sin \frac{1}{2}(\beta_3 - \alpha_3)} &= \frac{\sin \frac{1}{2}(\beta_7 + \alpha_7)}{\sin \frac{1}{2}(\beta_7 - \alpha_7)} \\ \frac{\sin \frac{1}{2}((\beta_0 + \beta_3) + (\alpha_0 + \alpha_7))}{\sin \frac{1}{2}((\beta_0 + \beta_3) - (\alpha_0 + \alpha_7))} &= \frac{\sin \frac{1}{2}((\beta_0 + \beta_7) + (\alpha_0 + \alpha_3))}{\sin \frac{1}{2}((\beta_0 + \beta_7) - (\alpha_0 + \alpha_3))} \\ \frac{\sin \frac{1}{2}(\beta_3 + \alpha_3)}{\sin \frac{1}{2}(\beta_3 - \alpha_3)} \cdot \frac{\sin \frac{1}{2}((\beta_0 + \beta_7) + (\alpha_0 + \alpha_3))}{\sin \frac{1}{2}((\beta_0 + \beta_7) - (\alpha_0 + \alpha_3))} &= \frac{\sin \frac{1}{2}(\beta_0 + \alpha_0)}{\sin \frac{1}{2}(\beta_0 - \alpha_0)} \end{aligned} \quad (3.2.19)$$

Two sets of solution can be found if $\alpha, \beta \neq 0$ or π , which are

$$\begin{aligned} \alpha_3 &= \alpha_7 = \alpha, \alpha_0 = -\alpha \\ \beta_3 &= \beta_7 = \beta, \beta_0 = -\beta \end{aligned} \quad (3.2.20)$$

and

$$\alpha_3 = \alpha_7 = 0, \alpha_0 = \alpha \quad (3.2.21)$$

$$\beta_3 = \beta_7 = 0, \beta_0 = \beta$$

When $\alpha, \beta = 0$ or π , the network becomes a number of overlapped planar crossed isograms or planar parallelograms.

Denote the lengths of links for small Bennett linkages JBKS, PVND, and RSTV as a_3, b_3, a_7, b_7 and a_0, b_0 . Obviously,

$$\frac{a_3}{b_3} = \frac{a_7}{b_7} = \frac{a_0}{b_0} = \frac{a}{b} \quad (3.2.22a)$$

$$a_3 + a_0 + a_7 = a \quad (3.2.22b)$$

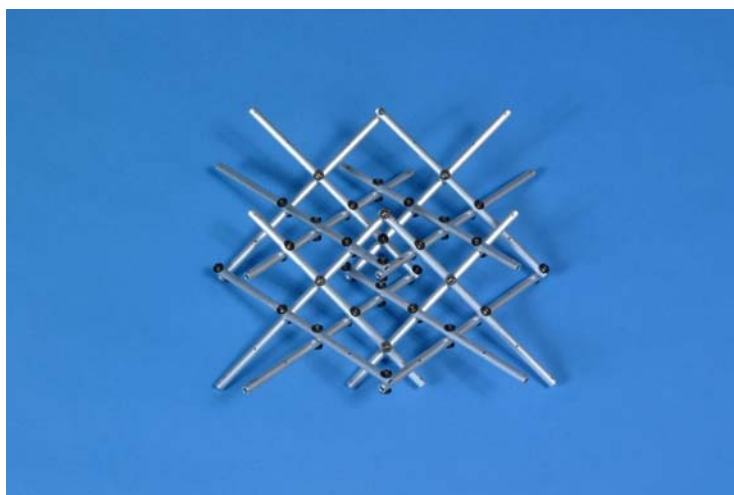
$$b_3 + b_0 + b_7 = b$$

The solutions given in (3.2.20) can be easily applied to connect the top left Bennett linkage AQEI and the bottom right one GMCL, see Fig. 3.2.9(c). Doing so, we are able to create a multi-layer connection as shown in Fig. 3.2.9(d). When the twists of both parts are (3.2.20) or (3.2.21), the profile of network is flat. When the twists of one part are (3.2.20) and those of the other are (3.2.21), the network forms a curved profile.

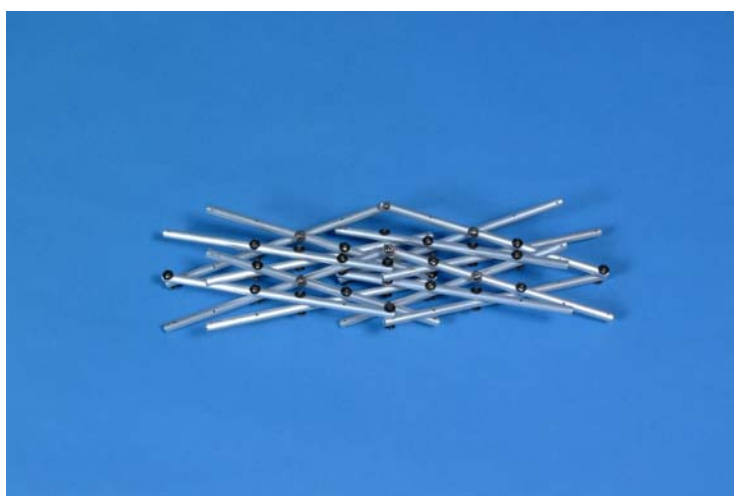
The current arrangement can be extended by repetition. For example, links in one of the central smaller Bennett linkages shown in Fig. 3.2.9(d) can be extended and connected again to another Bennett linkage similar to ABCD but on a different layer. This process will not affect the mobility of the entire assembly. A model of such multi-layer structure is shown in Fig. 3.2.10.



(a)



(b)



(c)

Fig. 3.2.10 (a) – (c) Deployment sequence of a multi-layer Bennett network.

3.2.3 Connectivity of Bennett Linkages

Baker and Hu (1986) attempted to connect two similar Bennett linkages together with only revolute joints, see Fig. 3.2.11. They found that the assembly lost mobility, i.e. it becomes a rigid structure.

In this section, we intend to show that, based on the work presented in previous sections, the mobility of the assembly can be retained if two Bennett linkages are connected by a type of Bennett connection. Note that the Bennett connection is different from the Bennett joint described in Section 2.2.3 (page 32).

Let us start with the structure shown in Fig. 3.2.12(a), which is based on the solution given in (3.2.21), in which loops AJSP and RKC� are Bennett linkages similar to Bennett linkage ABCD. If the lengths for ABCD are a and b , then loop AJSP has lengths of k_1a and k_1b , while loop RKC� has lengths of k_2a and k_2b .

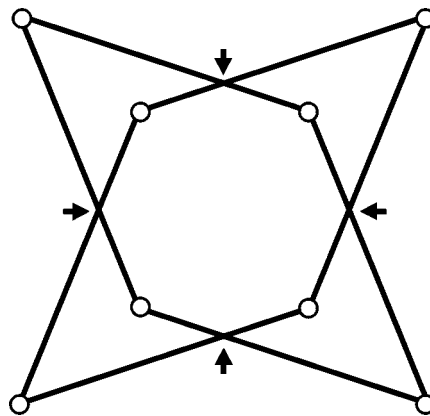


Fig. 3.2.11 Connection of two similar Bennett linkages by four revolute joints at locations marked by arrows.

We can also create a complementary set of extensions as sketched in dash lines in Fig. 3.2.12(b). Again loops IBLH and QFMD are Bennett linkages similar to ABCD, whose lengths are proportional to those of ABCD with ratios k_3 and k_4 , respectively.

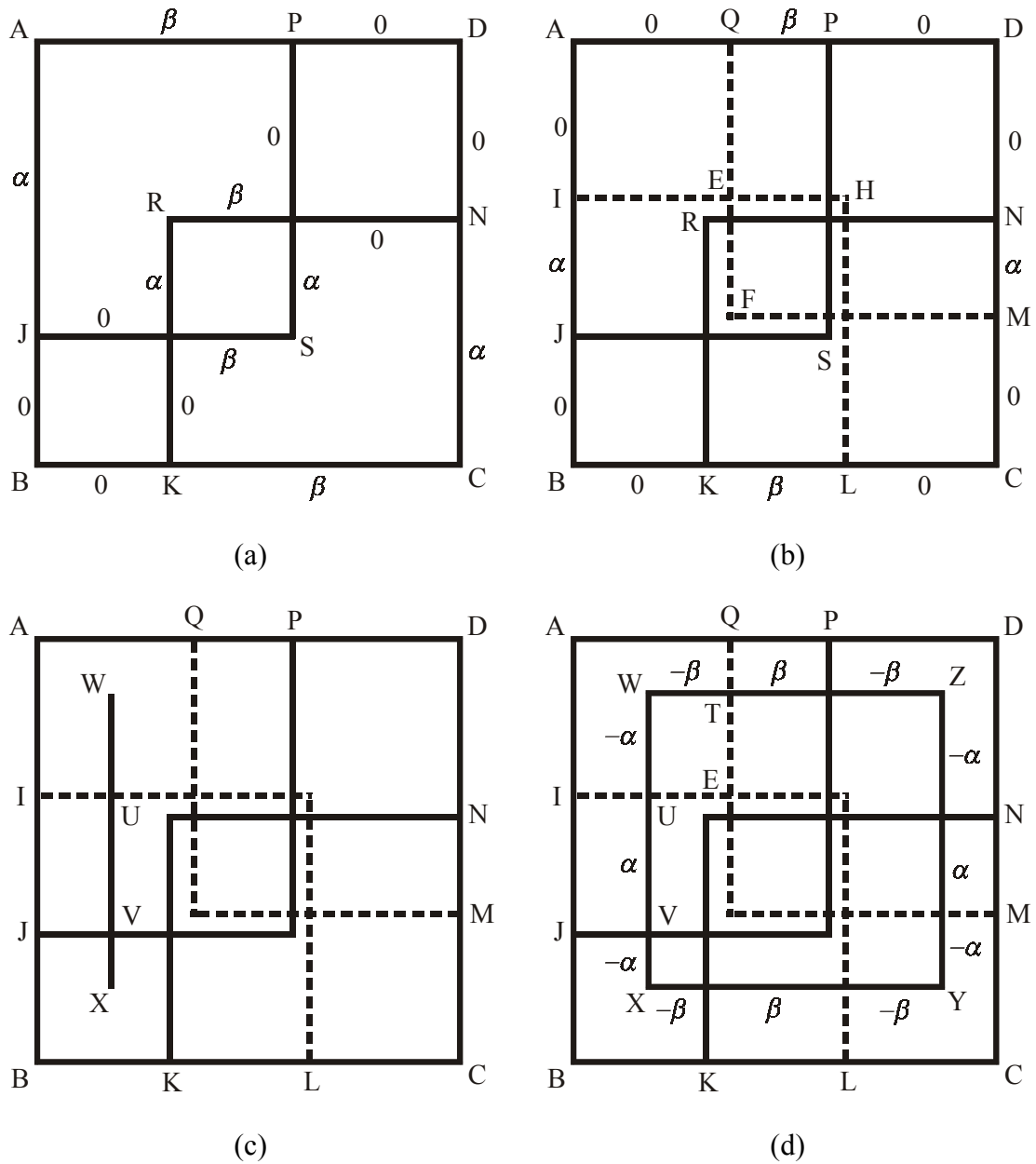


Fig. 3.2.12 Connection of two Bennett linkages ABCD and WXYZ.

(a) Addition of four bars; (b) a complementary set;

(c) further extension; (d) formation of the inner Bennett linkage.

Then the lengths of section IJ, MN, KL and PQ are

$$\begin{aligned}
 a_{IJ} &= (k_1 + k_3 - 1)a \\
 a_{MN} &= (k_2 + k_4 - 1)a \\
 a_{KL} &= (k_2 + k_3 - 1)b \\
 a_{PQ} &= (k_1 + k_4 - 1)b
 \end{aligned}
 \tag{3.2.23}$$

Now introduce bar WX by connecting it to the existing structure at U and V using revolute joints, see Fig. 3.2.12(c). To ensure connectivity without the loss of mobility, loop IJVU must be a Bennett linkage similar to ABCD. Thus, there must be

$$\begin{aligned}
 a_{UV} &= a_{IJ} = (k_1 + k_3 - 1)a \\
 a_{IU} &= a_{JV} = (k_1 + k_3 - 1)b
 \end{aligned}
 \tag{3.2.24}$$

while the twists are α and β , respectively.

The same can be done to the remaining sides of the structure, see Fig. 3.2.12(d). Again, to retain mobility, all of the rectangular loops, except those with zero twists such as AIEQ, must be Bennett linkages similar to ABCD. Based on this requirement, the twists marked alongside each link in Fig. 3.2.12(d) should be adopted. Loop WXYZ also becomes a Bennett linkage similar to ABCD with

$$\begin{aligned}
 a_{WX} &= a_{YZ} = (3 - k_1 - k_2 - k_3 - k_4)a \\
 a_{WZ} &= a_{XY} = (3 - k_1 - k_2 - k_3 - k_4)b
 \end{aligned}
 \tag{3.2.25}$$

and the twists $-\alpha$ and $-\beta$, respectively.

The mechanism shown in Fig. 3.2.12(d) is a complex one. If the central part of the assembly is removed, it is clear that two Bennett linkages, ABCD and WXYZ, are connected by four smaller Bennett linkages whose dimension is (3.2.23), see Fig. 3.2.13. All of the Bennett linkages, large or small, are similar ones.

A model of such a structure has been constructed, see Fig. 3.2.14, in which $k_1 = 0.25$, $k_2 = 0.65$, $k_3 = 0.6$ and $k_4 = 0.65$. Then the ratios between the lengths of the four smaller Bennett linkages and those of Bennett linkage ABCD are -0.15 , 0.3 , 0.25 and -0.1 . The negative ratios mean that the corresponding smaller Bennett linkages are in fact located outside of the larger Bennett linkage ABCD.

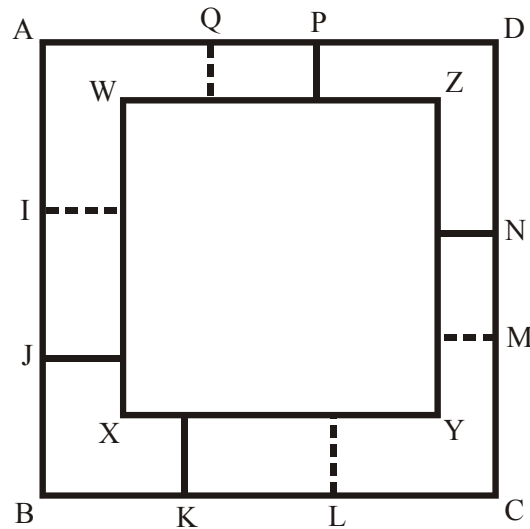
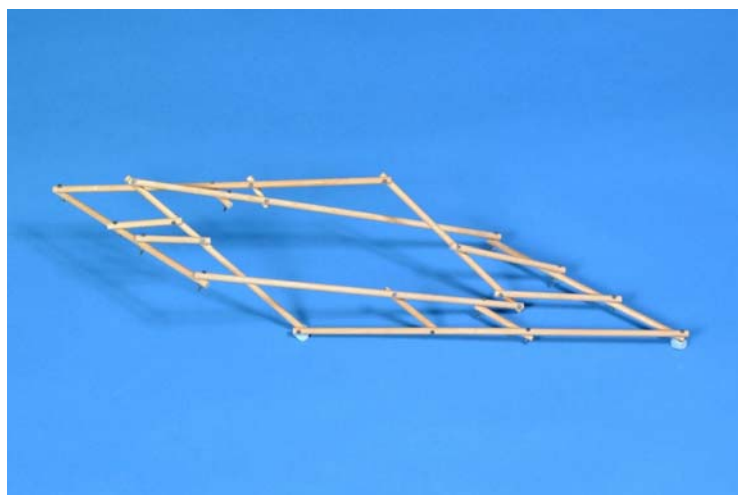


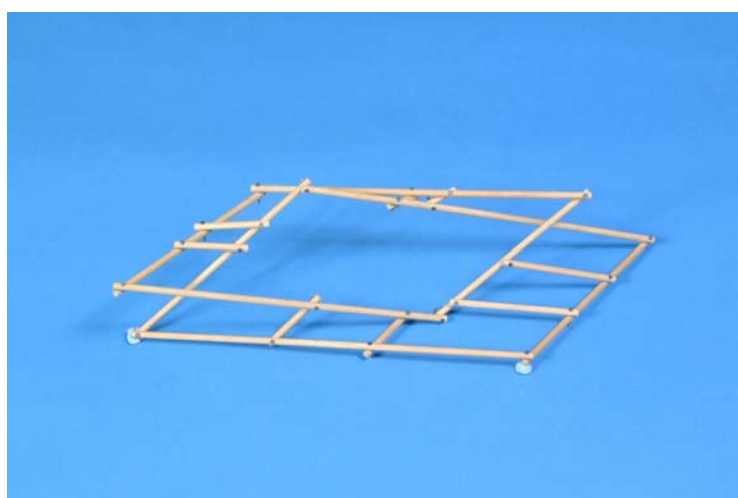
Fig. 3.2.13 Two similar Bennett linkages connected by four smaller ones.



(a)



(b)



(c)

Fig. 3.2.14 (a) – (c) Three configurations of connection of two similar Bennett linkages.

(3.2.25) also determines the size of the inner Bennett linkage. Obviously, when $k_1 + k_2 + k_3 + k_4 = 2$, the inner Bennett linkage becomes identical in size to the outer one. From (3.2.23), when $k_1 = k_2$, $k_3 = k_4$ and $k_1 + k_3 = 1$, the size of the four smaller connection Bennett linkages reduces to zero. We define this type of zero-sized Bennett linkages as the *Bennett connection*. The key difference between a Bennett connection and a conventional revolute joint is the relative motions of the pair of links connected. The links remain co-planar when connected with a conventional revolute joint, which is not true when connected by a Bennett connection. The discovery presented here actually indirectly confirms that Baker and Hu's assembly is rigid.

3.3 ALTERNATIVE FORM OF BENNETT LINKAGE

3.3.1 Alternative Form of Bennett Linkage

For the Bennett linkage shown in Fig. 3.3.1, (2.2.6) indicates that φ must be close to π while θ approaches 0, or *vice versa*, which means that the distance between B and D become smallest while that between A and C is largest. As a result, the Bennett linkage cannot be folded completely in both directions simultaneously. However, we are to demonstrate in this section that modification can be carried out to make compact folding possible.

For the purpose of geometric derivation, the joints of Bennett linkage are marked with letters, i.e. A, B, C, etc. So the lengths and twists of linkage are also marked with letters in the subscripts. For instance, α_{AB} is the twist between joint A and joint B.

Figure 3.3.2(a) shows an equilateral special case of Bennett linkage, where

$$a_{AB} = a_{BC} = a_{CD} = a_{DA} = l \quad (3.3.1)$$

$$\alpha_{AB} = \alpha_{CD} = \alpha \quad (3.3.2)$$

$$\alpha_{BC} = \alpha_{DA} = \pi - \alpha$$

This linkage is symmetric about both the plane through AC and perpendicular BD and the plane through BD and perpendicular AC even though lines AC and BD may not cross each other. The axes of revolute joints are marked as dash-dot lines at A, B, C and D. The positive directions of the axes are also shown.

Denote M and N as the respective middle points of BD and AC, see Fig. 3.3.2(b). Obviously, $\triangle ABD$ and $\triangle CDB$ are isosceles and identical triangles due to (3.3.1). So are $\triangle BCA$ and $\triangle DAC$. These lead to the conclusion that $\triangle AMC$ and $\triangle BND$ are both isosceles triangles. Hence MN is perpendicular to both AC and BD. Furthermore, extensions of the axes of revolute joints must meet with the extension of MN at P and Q, respectively, due to symmetry.

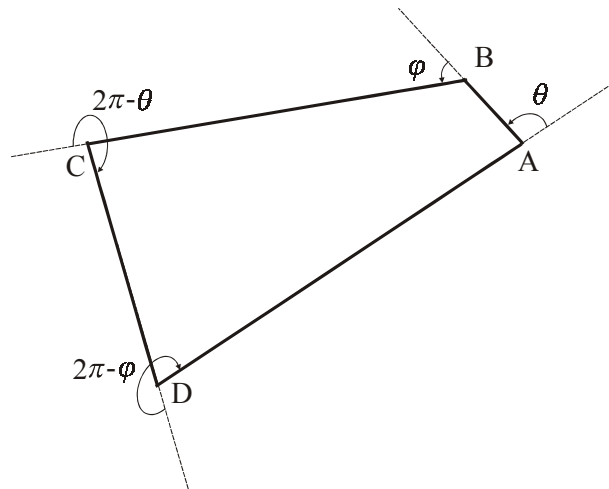


Fig. 3.3.1 A Bennett linkage.

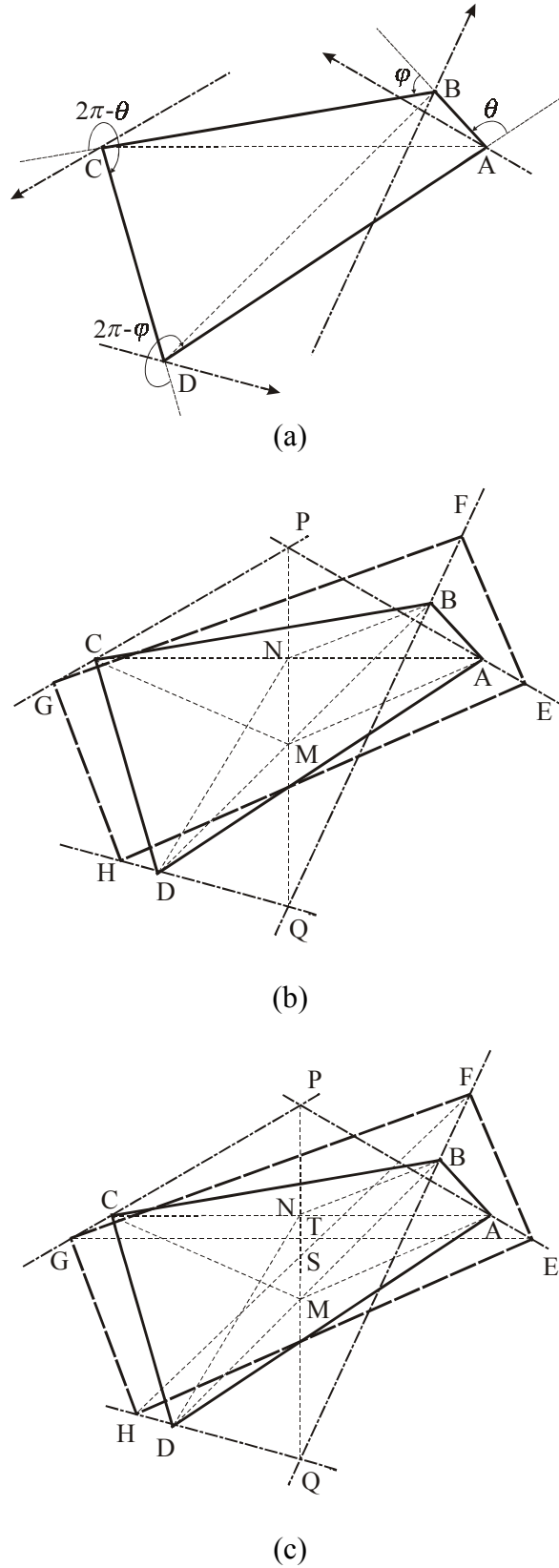


Fig. 3.3.2 Equilateral Bennett linkage. Certain new lines are introduced in (a), (b) and (c) for derivation of compact folding and maximum expansion conditions.

Consider now four alternative connection points E, F, G and H along the extensions of the revolute axes AP, BQ, CP and DQ, respectively. To preserve symmetry, define

$$\overline{GC} = \overline{AE} = c$$

$$\overline{BF} = \overline{DH} = d$$

Hence,

$$\begin{aligned}\overline{EF}^2 &= l^2 + c^2 + d^2 - 2cd \cos(\pi - \alpha_{AB}) \\ \overline{FG}^2 &= l^2 + c^2 + d^2 - 2cd \cos \alpha_{BC}\end{aligned}\tag{3.3.3}$$

Substituting (3.3.2) into (3.3.3) gives

$$\overline{EF} = \overline{FG} = \sqrt{l^2 + c^2 + d^2 + 2cd \cos \alpha}\tag{3.3.4}$$

Similarly,

$$\overline{EF} = \overline{FG} = \overline{GH} = \overline{HE}\tag{3.3.5}$$

which means EFGH is also equilateral.

For any given Bennett linkage ABCD, (3.3.4) and (3.3.5) show that EF, FG, GH and HE have constant length provided that both c and d are given. They do not vary with the revolute variables, θ and φ . Thus, it is possible to replace EF, FG, GH and HE with bars connected by the revolute joints whose axes are along BF, CG, DH and AE, respectively. EFGH is therefore an *alternative form* of the Bennett linkage ABCD.

For each given set of c and d , an alternative form for the Bennett linkage can be obtained. The next step is to examine if a particular form would provide the most compact folding.

When the linkage in the alternative form displaces, the distance between E and G varies. So does the distance between F and H. Assume that when the structure is fully folded, deployment angles θ and φ become θ_f and φ_f , respectively. The condition for the most compact folding is

$$\overline{EG} = \overline{FH} = 0 \quad (3.3.6)$$

This means that physically the mechanism becomes a bundle.

(3.3.6) can be written in term of c , d and the deployment angles, which is done next.

Consider $\triangle ADC$ in Fig. 3.3.2(b). It can be found that

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 - 2\overline{AD} \cdot \overline{CD} \cos(\pi - \varphi) = 2l^2(1 + \cos \varphi) \quad (3.3.7)$$

Similarly, in $\triangle ABD$, there is

$$\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 - 2\overline{AB} \cdot \overline{AD} \cos(\pi - \theta) = 2l^2(1 + \cos \theta) \quad (3.3.8)$$

while in $\triangle BCM$,

$$\overline{CM}^2 = \overline{BC}^2 - \overline{BM}^2 = \frac{l^2}{2}(1 - \cos \theta) \quad (3.3.9)$$

Thus, from $\triangle AMC$,

$$\cos \angle AMC = 1 - 2 \frac{1 + \cos \varphi}{1 - \cos \theta} \quad (3.3.10)$$

From quadrilateral PAMC,

$$\cos \angle APC = -\cos \angle AMC = 2 \frac{1 + \cos \varphi}{1 - \cos \theta} - 1 \quad (3.3.11)$$

Because in $\triangle APC$,

$$\overline{AC}^2 = 2\overline{PC}^2(1 - \cos \angle APC) = 4\overline{PC}^2 \frac{-\cos \theta - \cos \varphi}{1 - \cos \theta} \quad (3.3.12)$$

Comparing (3.3.7) with (3.3.12) yields

$$\overline{PC}^2 = l^2 \frac{(1 + \cos \varphi)(1 - \cos \theta)}{-2(\cos \varphi + \cos \theta)} \quad (3.3.13)$$

Similarly it can be obtained that

$$\overline{QB}^2 = l^2 \frac{(1 + \cos \theta)(1 - \cos \varphi)}{-2(\cos \varphi + \cos \theta)} \quad (3.3.14)$$

In ΔEPG , there is

$$\overline{EG}^2 = 2(c + \overline{PC})^2 (1 - \cos \angle APC) = 4(c + \overline{PC})^2 \frac{-\cos \theta - \cos \varphi}{1 - \cos \theta} \quad (3.3.15)$$

and similarly

$$\overline{FH}^2 = 2(d + \overline{QB})^2 (1 - \cos \angle BQD) = 4(d + \overline{QB})^2 \frac{-\cos \theta - \cos \varphi}{1 - \cos \varphi} \quad (3.3.16)$$

In general, $\angle APC$ and $\angle BQD$ cannot reach zero at the same time. Substituting (3.3.15) and (3.3.16) into (3.3.6) yields

$$\begin{aligned} c = -\overline{PC} &= -l \sqrt{\frac{(1 + \cos \varphi_f)(1 - \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} \\ d = -\overline{QB} &= -l \sqrt{\frac{(1 - \cos \varphi_f)(1 + \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} \end{aligned} \quad (3.3.17)$$

The above equations show how the values of c and d are related to the fully folded deployment angles θ_f and φ_f . In fact, c and d can be determined graphically as solutions (3.3.17) simply mean that E and G should move to a single point P, and F and H go to Q if the configuration shown Fig. 3.3.2(b) represents the fully folded

configuration of linkage EFGH. In this case, positive values of c and d will be used in the following derivation. Thus, (3.3.17) becomes

$$\begin{aligned} c &= l \sqrt{\frac{(1 + \cos \varphi_f)(1 - \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} \\ d &= l \sqrt{\frac{(1 - \cos \varphi_f)(1 + \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} \end{aligned} \quad (3.3.18)$$

Having obtained the linkage corresponding to the most efficient folding configuration, what is the form of Bennett linkage that covers the largest area? To answer this question, it is necessary to find out the geometrical condition relating to the maximum coverage.

Figure 3.3.2(c) shows the alternative form of the Bennett linkage EFGH. Due to symmetry, a line between E with G will intersect MN at T, and that between F and H will intersect MN at S. The projection of EFGH will cover a maximum area if

$$\overline{ST} = 0 \quad (3.3.19)$$

when deployment angles reach θ_d and φ_d . This implies that EFGH is completely flattened to a rhombus.

Again, $\overline{ST}$ can be expressed in term of c , d and deployment angles.

Based on (3.3.7) and (3.3.9),

$$\overline{MN}^2 = \overline{CM}^2 - \overline{CN}^2 = \overline{CM}^2 - \frac{\overline{AC}^2}{4} = -\frac{l^2}{2}(\cos \theta + \cos \varphi) \quad (3.3.20)$$

Considering (3.3.11) gives

$$\sin \angle \text{PGE} = \sin\left(\frac{\pi}{2} - \frac{1}{2} \angle \text{APC}\right) = \frac{\cos \frac{\varphi}{2}}{\sin \frac{\theta}{2}}$$

So,

$$\overline{\text{NT}} = c \cdot \sin \angle \text{PGE} = c \cdot \frac{\cos \frac{\varphi}{2}}{\sin \frac{\theta}{2}} \quad (3.3.21)$$

Similarly,

$$\overline{\text{MS}} = d \cdot \sin \angle \text{QFH} = d \cdot \frac{\cos \frac{\theta}{2}}{\sin \frac{\varphi}{2}} \quad (3.3.22)$$

Hence, considering (3.3.20), (3.3.21) and (3.3.22), $\overline{\text{ST}}$ can be written as

$$\overline{\text{ST}} = \overline{\text{MN}} - \overline{\text{MS}} - \overline{\text{NT}} = l \sqrt{-\frac{(\cos \theta + \cos \varphi)}{2}} - d \frac{\cos \frac{\theta}{2}}{\sin \frac{\varphi}{2}} - c \frac{\cos \frac{\varphi}{2}}{\sin \frac{\theta}{2}} \quad (3.3.23)$$

When $\theta = \theta_d$ and $\varphi = \varphi_d$,

$$\overline{\text{ST}} = l \sqrt{-\frac{(\cos \theta_d + \cos \varphi_d)}{2}} - d \frac{\cos \frac{\theta_d}{2}}{\sin \frac{\varphi_d}{2}} - c \frac{\cos \frac{\varphi_d}{2}}{\sin \frac{\theta_d}{2}} = 0 \quad (3.3.24)$$

due to (3.3.19).

Substituting c and d obtained from (3.3.18) into (3.3.24) and considering

$$\tan \frac{\theta_d}{2} \tan \frac{\varphi_d}{2} = \tan \frac{\theta_f}{2} \tan \frac{\varphi_f}{2} = \frac{1}{\cos \alpha} \quad (3.3.25)$$

we have

$$\tan^4 \alpha \tan^2 \frac{\theta_d}{2} \tan^2 \frac{\theta_f}{2} = \left(\sec^2 \frac{\theta_d}{2} \sec^2 \frac{\theta_f}{2} + \tan^2 \alpha \right)^2 \quad (3.3.26)$$

If $0 \leq \theta_f \leq \pi$, then $\pi \leq \theta_d \leq 2\pi$. (3.3.26) becomes

$$-\tan^2 \alpha \tan \frac{\theta_d}{2} \tan \frac{\theta_f}{2} = \sec^2 \frac{\theta_d}{2} \sec^2 \frac{\theta_f}{2} + \tan^2 \alpha \quad (3.3.27)$$

Solutions to (3.3.27) only exist when the value of α is in the range between $\arccos(1/3)$ and $\pi - \arccos(1/3)$, i.e., 70.53° to 109.47° . Within this range, the relationship between θ_d and θ_f for a set of given α is shown in Fig. 3.3.3. Note that in most circumstances, each θ_f corresponds to two values of θ_d . This means that there are two possible deployed configurations in which the structure built using the alternative forms of the Bennett linkage can be flattened. For $\alpha < \arccos(1/3)$ or $\alpha > \pi - \arccos(1/3)$, there is no pair of θ_f and θ_d satisfying (3.3.27). Therefore the structure is incapable of being flattened despite that it can be folded up compactly, or *vice versa*.

From (3.3.4), the actual side length of the alternative form of the Bennett linkage, L , can be obtained as

$$L = \sqrt{l^2 + c^2 + d^2 + 2cd \cos \alpha} \quad (3.3.28)$$

Substituting (3.3.18) into (3.3.28), we have

$$\frac{L}{l} = \sqrt{-\frac{2}{\cos \theta_f + \cos \varphi_f}} \quad (3.3.29)$$

Figure 3.3.4 shows the relationship between θ_f and L/l , c/l , d/l when $\alpha = 7\pi/12$.

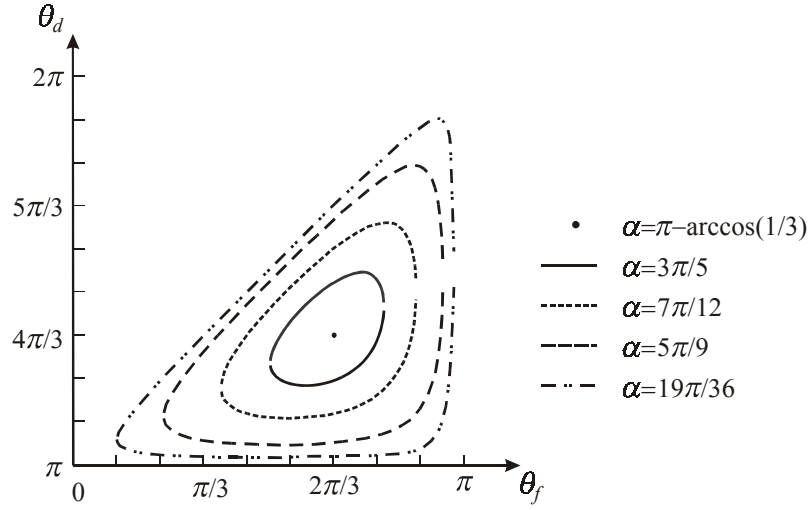


Fig. 3.3.3 θ_d vs θ_f for a set of given α .

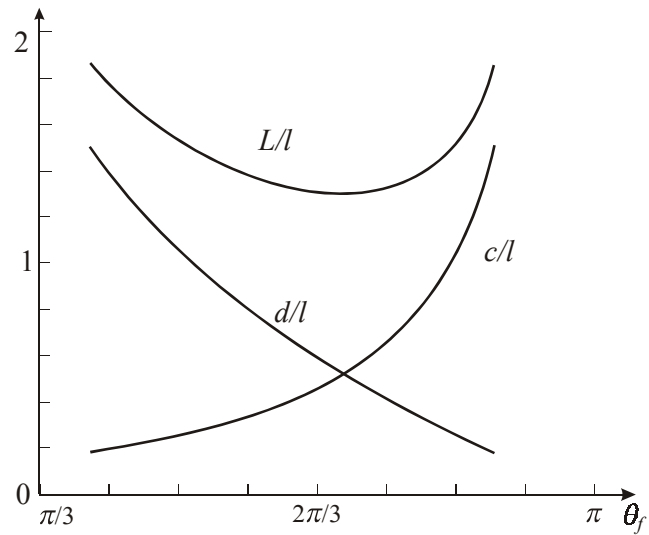


Fig. 3.3.4 L/l , c/l and d/l vs θ_f when $\alpha = 7\pi/12$.

Denote δ as the angle between two adjacent sides of the alternative form of the Bennett linkage in its flattened configuration when $\theta = \theta_d$, $\varphi = \varphi_d$. Thus, $\delta = \angle FGH$ when S and T in Fig. 3.3.2(c) become one point. We have

$$\tan \frac{\delta}{2} = \frac{\frac{1}{2} \overline{FH}}{\frac{1}{2} \overline{EG}} = \frac{\overline{FH}}{\overline{EG}}$$

Expressing $\overline{FH}$ and $\overline{EG}$ in terms of angles gives

$$\delta = 2 \arctan \left(\frac{\sqrt{\frac{(1 - \cos \varphi_f)(1 + \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} + \sqrt{\frac{(1 - \cos \varphi_d)(1 + \cos \theta_d)}{-2(\cos \varphi_d + \cos \theta_d)}}}{\sqrt{\frac{(1 + \cos \varphi_f)(1 - \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} + \sqrt{\frac{(1 + \cos \varphi_d)(1 - \cos \theta_d)}{-2(\cos \varphi_d + \cos \theta_d)}}} \cdot \sqrt{\frac{1 - \cos \theta_d}{1 - \cos \varphi_d}} \right) \quad (3.3.30)$$

This relationship is plotted in Fig. 3.3.5 when $\alpha = 7\pi/12$. Similar to relationship between θ_d and θ_f , there are two values of δ for each θ_f .

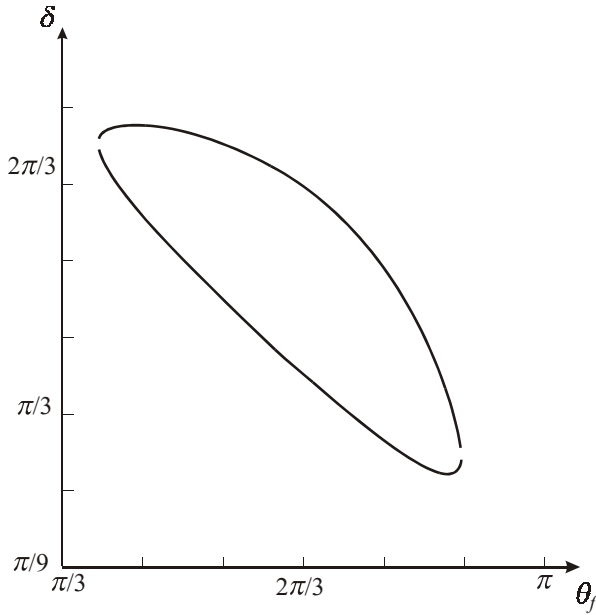


Fig. 3.3.5 δ vs θ_f when $\alpha = 7\pi/12$.

It is interesting to note that among the rhombuses with the same side-length, the square has the largest area, i.e.,

$$\delta = \frac{\pi}{2} \quad (3.3.31)$$

Considering (3.3.30), (3.3.31) becomes

$$\begin{aligned} & \left(\sqrt{\frac{(1 + \cos \varphi_f)(1 - \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} + \sqrt{\frac{(1 + \cos \varphi_d)(1 - \cos \theta_d)}{-2(\cos \varphi_d + \cos \theta_d)}} \right)^2 \frac{-(\cos \varphi_d + \cos \theta_d)}{(1 - \cos \theta_d)} = \\ & \left(\sqrt{\frac{(1 - \cos \varphi_f)(1 + \cos \theta_f)}{-2(\cos \varphi_f + \cos \theta_f)}} + \sqrt{\frac{(1 - \cos \varphi_d)(1 + \cos \theta_d)}{-2(\cos \varphi_d + \cos \theta_d)}} \right)^2 \frac{-(\cos \varphi_d + \cos \theta_d)}{(1 - \cos \varphi_d)} \end{aligned} \quad (3.3.32)$$

Considering (3.3.25), (3.3.32) can be simplified as

$$\tan^2 \frac{\theta_f}{2} \sec^2 \frac{\theta_d}{2} = \tan^2 \frac{\theta_d}{2} + \sec^2 \alpha \quad (3.3.33)$$

So when α , θ_d and θ_f satisfy both of (3.3.27) and (3.3.33), the fully deployed configuration of alternative form of the Bennett linkage is a square.

Solving (3.3.27) and (3.3.33), we obtain,

$$\theta_d = 2\theta_f$$

$$\tan^2 \alpha = \sec^2 \theta_f \left(\tan^2 \frac{\theta_f}{2} - 1 \right)$$

This relationship is shown in Fig. 3.3.6, which is in fact the projection of the intersected curve between the surface of (3.3.27) and that of (3.3.33).

Also, very interestingly, for any square fully deployed configuration, we always have

$$L = \sqrt{2}l \quad (3.3.34)$$

due to (3.3.27), (3.3.29) and (3.3.33).

Finally, we like to point out that normally, for a given α , there are two sets of θ_d and θ_f , in which configuration of the corresponding alternative form of Bennett linkage is square. However, when

$$\alpha = \arccos \frac{1}{3} \text{ or } \pi - \arccos \frac{1}{3},$$

there is only one solution,

$$\theta_f = \frac{2}{3}\pi \text{ and } \theta_d = \frac{4}{3}\pi.$$

In this case,

$$c = d = \frac{\sqrt{6}}{4}l, \quad L = \sqrt{2}l.$$

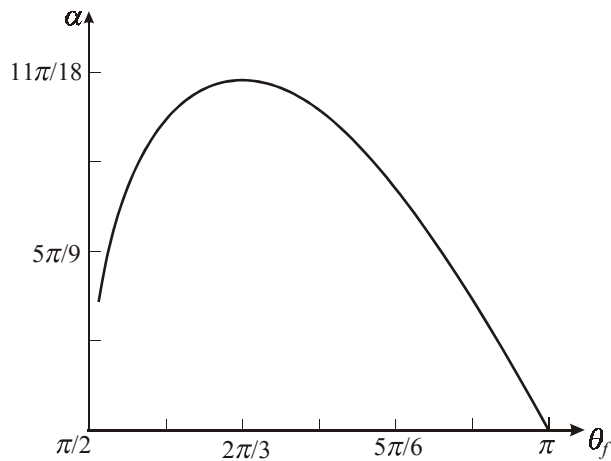


Fig. 3.3.6 α vs θ_f when the fully deployed structure based on the alternative form of the Bennett linkage forms a square.

3.3.2 Manufacture of Alternative Form of Bennett linkage

According to Crawford et al. (1973), for a close loop consisting n bars with identical cross-section, the cross-section of the whole structure in the folded configuration should be n regular polygon in order to achieve the most compact folding. So for the alternative form of Bennett linkage, each link could be made of bars with square cross-section. Pellegrino et al. (2000) revealed a model of $4R$ linkage which is in fact an alternative form of the Bennett linkage. Figure 3.3.7 shows such a model though angle ω in the reported model was $\pi/4$ in the fully deployed configuration. This particular example shows that construction of a Bennett linkage with compact folding and maximum expansion is not only mathematically feasible, as we proved in the previous section, but also practically possible.

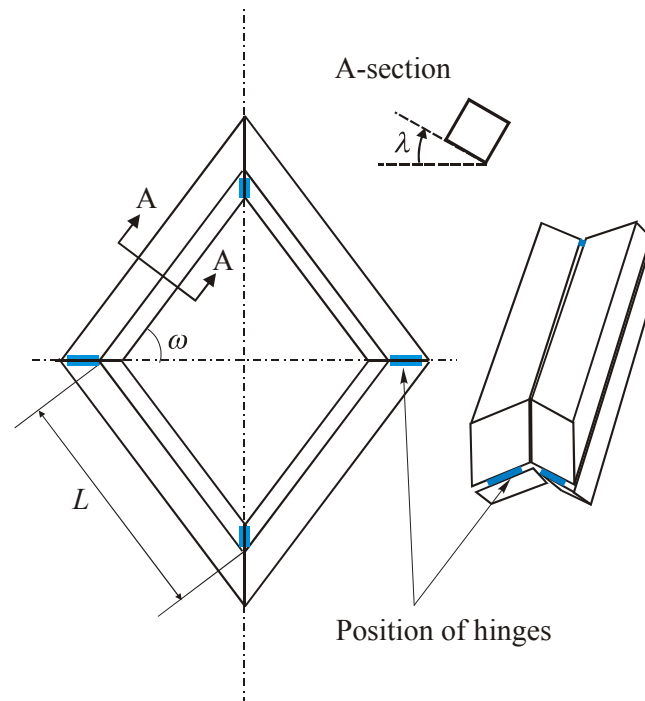


Fig. 3.3.7 The alternative form of Bennett linkage made from square cross-section bars in the deployed and folded configurations.

In general, the model shown in Fig. 3.3.7 has three design parameters: L , λ and ω . The number matches that presented in the previous section in which the geometric parameters are l , α , c (or d). In this section, we will establish the relationships among two sets of parameters.

Figure 3.3.8 shows a fully deployed linkage EFGH which is the alternative form of the Bennett linkage. The linkage has reached its maximum expansion and thus, EFGH becomes a plane rhombus. For simplification in description, let us define a coordinate system where axis x passes through FH and axis y through EG. The linkage EFGH is therefore symmetric about both xoz and yo z planes where o is the centre of EFGH and axis z is perpendicular to plane xoy . Now introduce a square bar, shown in grey colour in Fig. 3.3.8 to replace link FG in such a way that one of the edges of the square bar lies along FG. The bar is terminated by planes GIJK and FUVW, created by slicing the bar by the plane yo z and xoz , respectively.

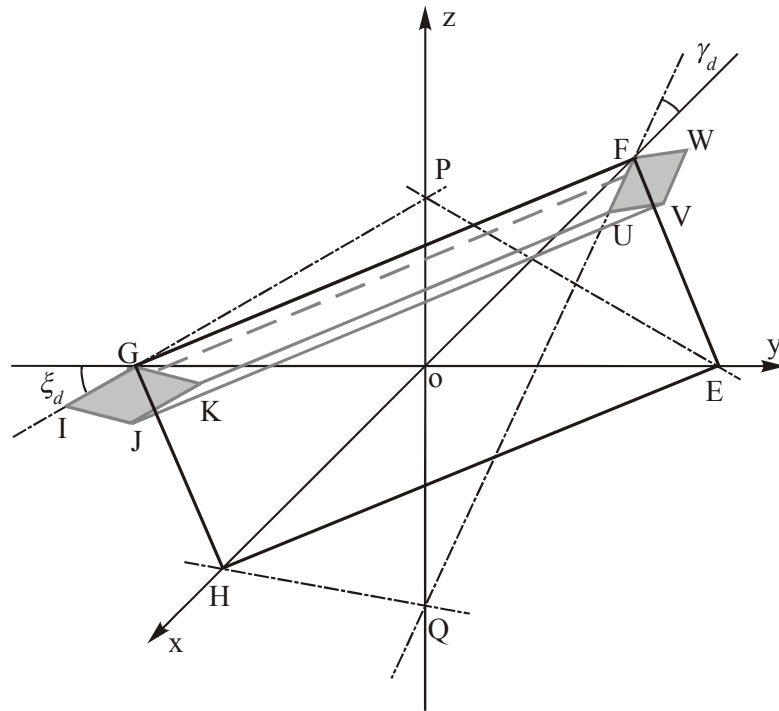


Fig. 3.3.8 The alternative form of Bennett linkage with square cross-section bars in deployed configuration.

An enlarged diagram of bar FG is shown in Fig. 3.3.9(a), in which GI and FU are the axes of the revolute joints. Plane $x'o'y'$ is through JV and parallel to plane xoy . A typical square cross-section is marked as RXYZ. The projection of the bar is given in Fig. 3.3.9(b), in which X' , R' and Z' , etc. represent the projection of X, R and Z on the plane $x'o'y'$, etc.

Assume

$$\overline{RX} = \overline{XY} = \overline{YZ} = \overline{ZR} = 1$$

The following relationships can be obtained from Fig. 3.3.9(b).

$$\overline{X'Y} = \overline{R'Z'} = 1 \cdot \cos \lambda$$

$$\overline{X'R'} = \overline{YZ'} = 1 \cdot \sin \lambda$$

$$\overline{X'Z'} = 1 \cdot (\sin \lambda + \cos \lambda)$$

$$\overline{II'} = \overline{XX'} = \overline{WW'} = 1 \cdot \sin \lambda$$

$$\overline{KK'} = \overline{ZZ'} = \overline{UU'} = 1 \cdot \cos \lambda$$

$$\overline{F'U'} = \overline{VW'} = \frac{\overline{R'Z'}}{\cos \omega} = \frac{\cos \lambda}{\cos \omega}$$

$$\overline{F'W'} = \overline{U'V} = \frac{\overline{X'R'}}{\cos \omega} = \frac{\sin \lambda}{\cos \omega}$$

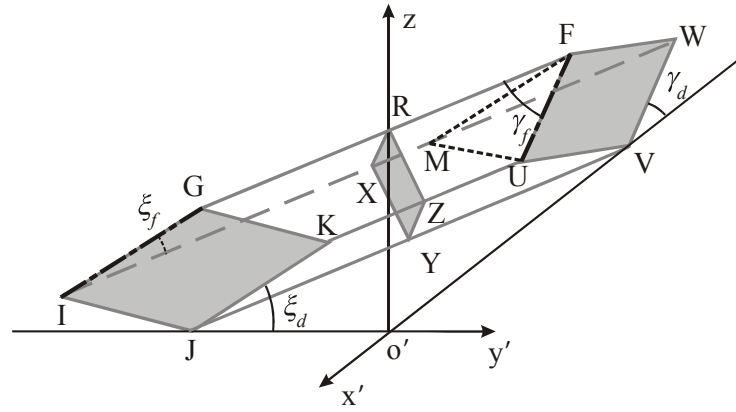
$$\overline{G'K'} = \overline{I'J} = \frac{\overline{R'Z'}}{\sin \omega} = \frac{\cos \lambda}{\sin \omega}$$

$$\overline{G'I'} = \overline{JK'} = \frac{\overline{X'R'}}{\sin \omega} = \frac{\sin \lambda}{\sin \omega}$$

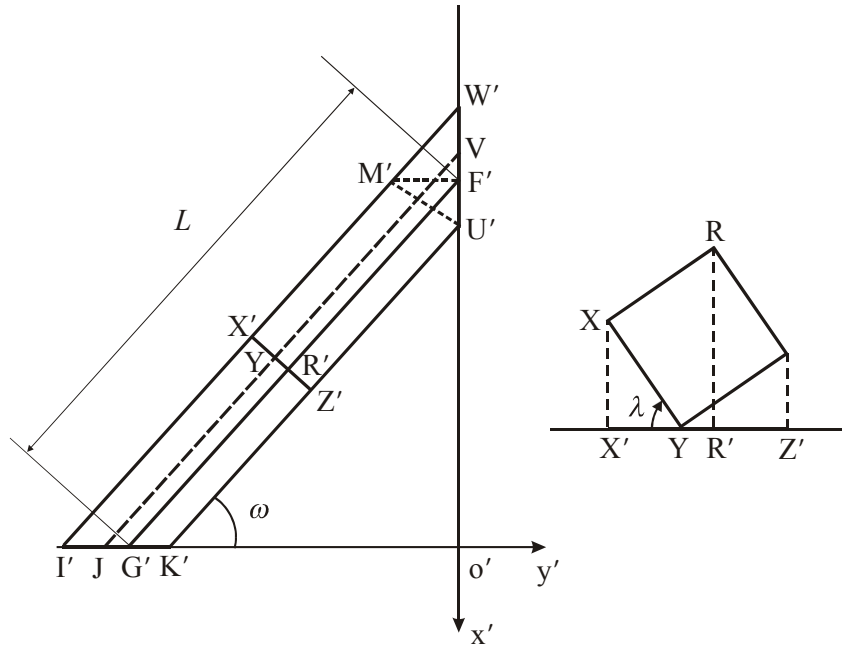
So we have

$$\overline{GI} = \overline{JK} = \sqrt{JK'^2 + KK'^2} = \sqrt{\frac{\sin^2 \lambda}{\sin^2 \omega} + \cos^2 \lambda} \quad (3.3.35)$$

$$\overline{FU} = \overline{VW} = \sqrt{VW'^2 + WW'^2} = \sqrt{\frac{\cos^2 \lambda}{\cos^2 \omega} + \sin^2 \lambda} \quad (3.3.36)$$



(a)



(b)

Fig. 3.3.9 The geometry of the square cross-section bar.

(a) In 3D; (b) projection on the plane $x'o'y'$ and the cross section RXYZ.

Let us now draw line FM parallel to GI and cross IW at M. Thus,

$$\angle MFU = \alpha$$

and

$$\cos \alpha = \frac{\overline{FM}^2 + \overline{FU}^2 - \overline{UM}^2}{2 \cdot \overline{FM} \cdot \overline{FU}} \quad (3.3.37)$$

Because of FM // GI and FG // MI, we have

$$\overline{FM} = \overline{GI} = \sqrt{\frac{\sin^2 \lambda}{\sin^2 \omega} + \cos^2 \lambda} \quad (3.3.38)$$

$$\overline{F'M'} = \overline{G'I'} = \frac{\sin \lambda}{\sin \omega} \quad (3.3.39)$$

and $F'M' \perp W'U'$.

Then,

$$\overline{U'M'}^2 = \overline{F'M'}^2 + \overline{F'U'}^2 \quad (3.3.40)$$

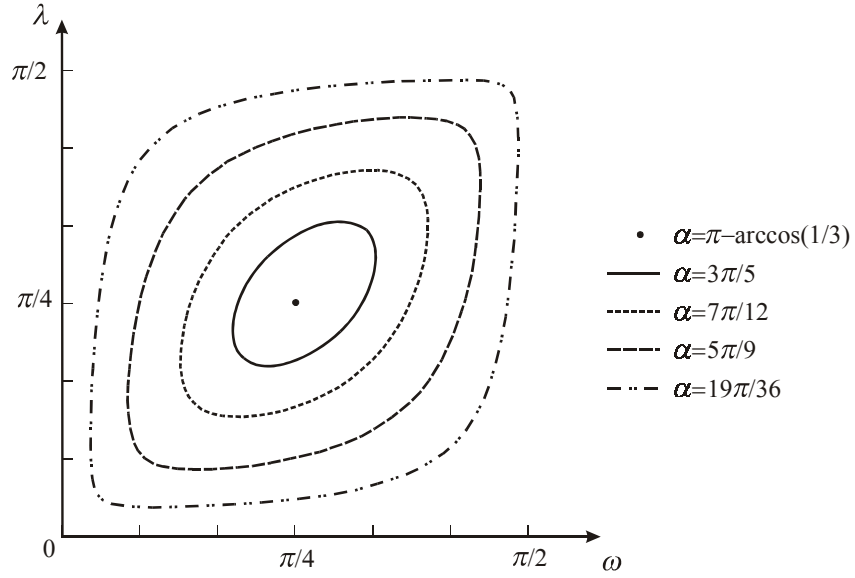
and

$$\overline{UM}^2 = \overline{U'M'}^2 + (\overline{MM'} - \overline{UU'})^2 \quad (3.3.41)$$

Substituting (3.3.36), (3.3.38) and (3.3.41) into (3.3.37), we obtain

$$\cos \alpha = \pm \frac{\sin 2\lambda \sin 2\omega}{\sqrt{\sin^2 2\lambda \sin^2 2\omega + 8(1 - \cos 2\lambda \cos 2\omega)}} \quad (3.3.42)$$

(3.3.42) is the relationship between design parameters ω , λ and twist α of the original Bennett linkage, which is plotted in Fig. 3.3.10 for a set of given α . It is interesting to note that, for $0 \leq \lambda \leq \pi/2$ and $0 \leq \omega \leq \pi/2$, the range of α is between $\arccos(1/3)$ and $\pi - \arccos(1/3)$, which is the same as that obtained from (3.3.27).


 Fig. 3.3.10 λ vs ω for a set of given α .

Our next step is to obtain the relationship among λ , ω and θ_d , φ_d , θ_f , φ_f .

As described in (3.3.11), in deployed configuration,

$$\begin{aligned}\cos \angle EPG &= 2 \frac{1 + \cos \varphi_d}{1 - \cos \theta_d} - 1 \\ \cos \angle FQH &= 2 \frac{1 + \cos \theta_d}{1 - \cos \varphi_d} - 1\end{aligned}\tag{3.3.43}$$

From Fig. 3.3.8,

$$\begin{aligned}\angle EPG &= \pi - 2\xi_d \\ \angle FQH &= \pi - 2\gamma_d\end{aligned}\tag{3.3.44}$$

From Fig. 3.3.9(a),

$$\tan \xi_d = \frac{\overline{KK'}}{\overline{JK'}} = \frac{\cos \lambda}{\sin \lambda / \sin \omega} \quad (3.3.45)$$

$$\tan \gamma_d = \frac{\overline{WW'}}{\overline{VW'}} = \frac{\sin \lambda}{\cos \lambda / \cos \omega}$$

From (3.3.43) – (3.3.45), we can obtain that

$$\cos \theta_d = \frac{(\cos 4\lambda - 1) + (\cos 4\lambda \cos 2\omega - 4 \cos 2\lambda + 3 \cos 2\omega)}{4(1 - \cos 2\omega \cos 2\lambda)} \quad (3.3.46a)$$

$$\cos \varphi_d = \frac{(\cos 4\lambda - 1) - (\cos 4\lambda \cos 2\omega - 4 \cos 2\lambda + 3 \cos 2\omega)}{4(1 - \cos 2\omega \cos 2\lambda)} \quad (3.3.46b)$$

When the linkage is folded up, EFGH becomes a bundle ST, so does the square cross-section bars. Bar EF is shown Fig. 3.3.11 in grey colour.

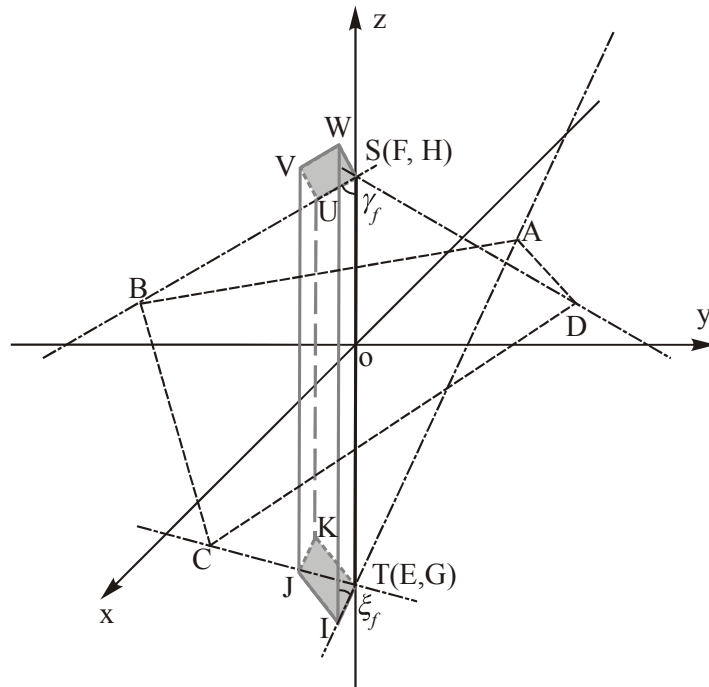


Fig. 3.3.11 The alternative form of Bennett linkage with square cross-section bars in folded configuration.

Similarly to (3.3.43), in folded configuration, we have

$$\cos \angle ATC = 2 \frac{1 + \cos \varphi_f}{1 - \cos \theta_f} - 1 \quad (3.3.47)$$

$$\cos \angle BSD = 2 \frac{1 + \cos \theta_f}{1 - \cos \varphi_f} - 1$$

From Fig. 3.3.11,

$$\angle ATC = 2\xi_f \quad (3.3.48)$$

$$\angle BSD = 2\gamma_f$$

From Fig. 3.3.9,

$$\begin{aligned} \tan \xi_f &= \frac{\overline{RX}}{\overline{XI} - \overline{RG}} = \frac{\overline{RX}}{\overline{R'X'} / \tan \omega} = \frac{\tan \omega}{\sin \lambda} \\ \tan \gamma_f &= \frac{\overline{RZ}}{\overline{RF} - \overline{UZ}} = \frac{\overline{RZ}}{\overline{R'Z'} \tan \omega} = \frac{1}{\cos \lambda \tan \omega} \end{aligned} \quad (3.3.49)$$

From (3.3.47) – (3.3.49), we can obtain that

$$\cos \theta_f = \frac{(\cos 4\omega - 1) + (\cos 4\omega \cos 2\lambda - 4 \cos 2\omega + 3 \cos 2\lambda)}{4(1 - \cos 2\omega \cos 2\lambda)} \quad (3.3.50a)$$

$$\cos \varphi_f = \frac{(\cos 4\omega - 1) - (\cos 4\omega \cos 2\lambda - 4 \cos 2\omega + 3 \cos 2\lambda)}{4(1 - \cos 2\omega \cos 2\lambda)} \quad (3.3.50b)$$

Now we can get the relationship among λ , ω and L/l , c/l , d/l .

Substituting (3.3.50) into (3.3.29) yields

$$\frac{L}{l} = 2 \sqrt{\frac{1 - \cos 2\omega \cos 2\lambda}{1 - \cos 4\omega}} \quad (3.3.51)$$

When the deployed configuration of the alternative form is square, that is $\omega = \frac{\pi}{4}$,

(3.3.51) gives

$$\frac{L}{l} = \sqrt{2}$$

which is the same as (3.3.34).

Substituting (3.3.50) into (3.3.18) yields

$$\begin{aligned} \frac{c}{l} &= \frac{\cos \lambda \sin^2 \omega}{\cos \omega} \sqrt{\frac{1 - \sin^2 \omega \sin^2 \lambda}{\sin^2 \lambda \cos^2 \omega + \sin^2 \omega \cos^2 \lambda}} \\ \frac{d}{l} &= \frac{\sin \lambda \cos^2 \omega}{\sin \omega} \sqrt{\frac{1 - \cos^2 \omega \cos^2 \lambda}{\sin^2 \lambda \cos^2 \omega + \sin^2 \omega \cos^2 \lambda}} \end{aligned} \quad (3.3.52)$$

Obviously, α , θ_d , and θ_f from (3.3.42), (3.3.46a), and (3.3.50a) satisfy (3.3.27).

So a set of equations have been obtained relating geometric parameters of an alternative form of Bennett linkage to the design parameters, based on which a model consisting of four square cross-section bars can be installed together with hinges as described in Fig. 3.3.7. The linkage has only one degree of mobility and can be folded into a compact bundle and deployed into a rhombus. In practice, it is not necessary to know the characteristics of the original Bennett linkage l , α , and extended distance on the axes of joints c (or d). The design work can be done based on the values of L , λ and ω , though it is much more convenient to use l , α and c (or d) when analysing the kinematic behaviour of the linkage such as θ_d , φ_d , θ_f and φ_f . Several models have been made which are shown in Figs. 3.3.12 – 3.3.15.

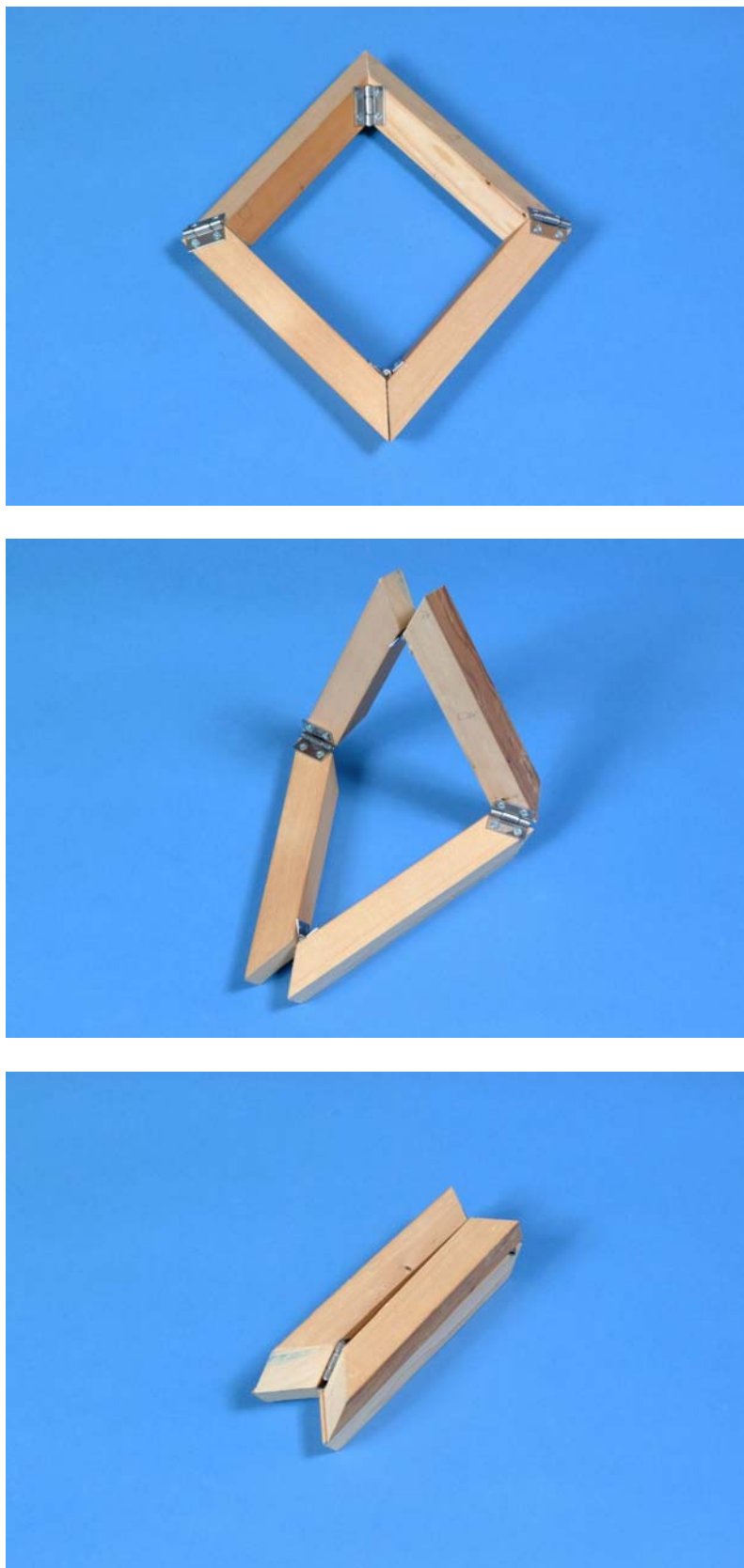


Fig. 3.3.12 Model that $\lambda = \pi / 6$ and $\omega = 53\pi / 180$.

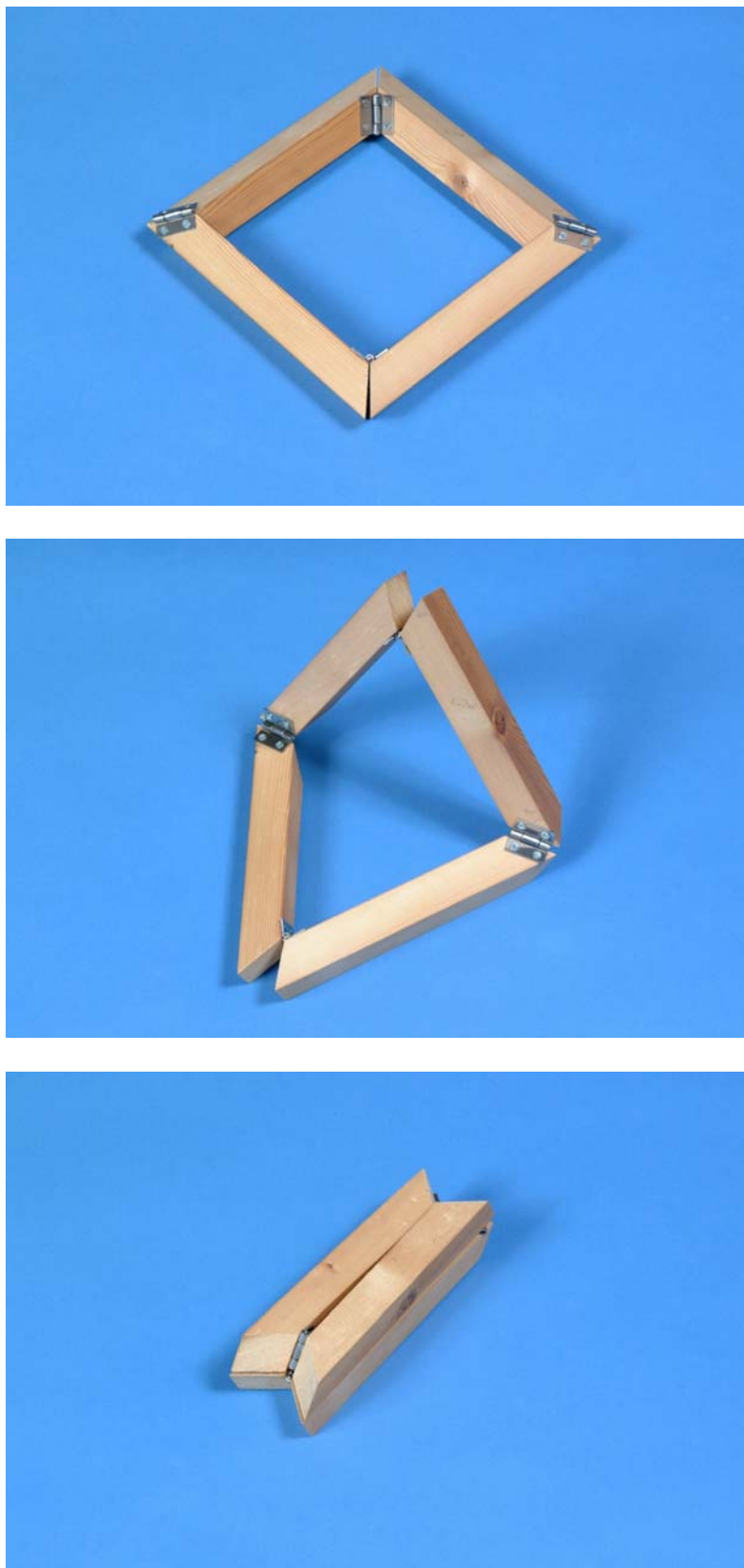


Fig. 3.3.13 Model that $\lambda = \pi / 6$ and $\omega = \pi / 4$.

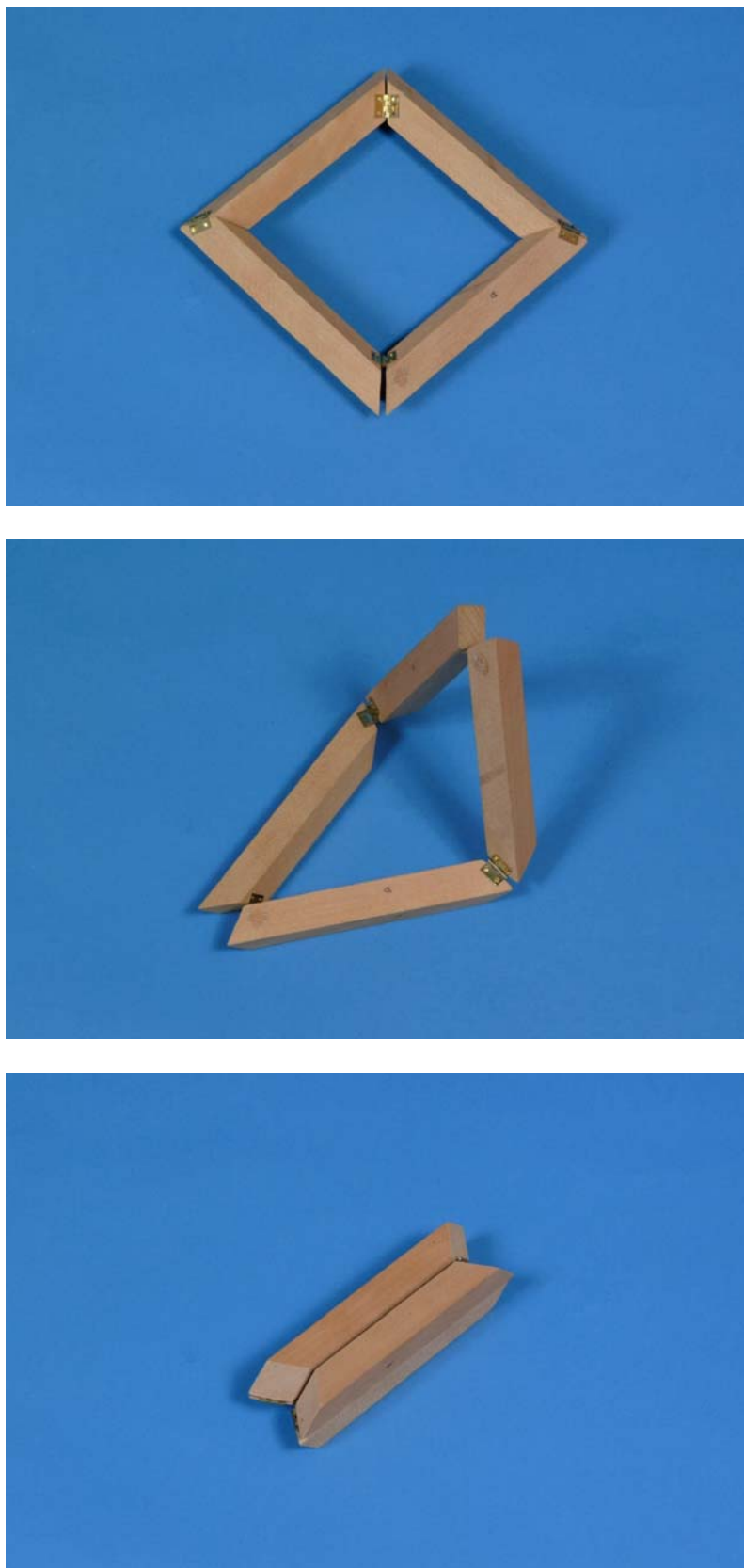


Fig. 3.3.14 Model that $\lambda = \pi / 4$ and $\omega = \pi / 4$.

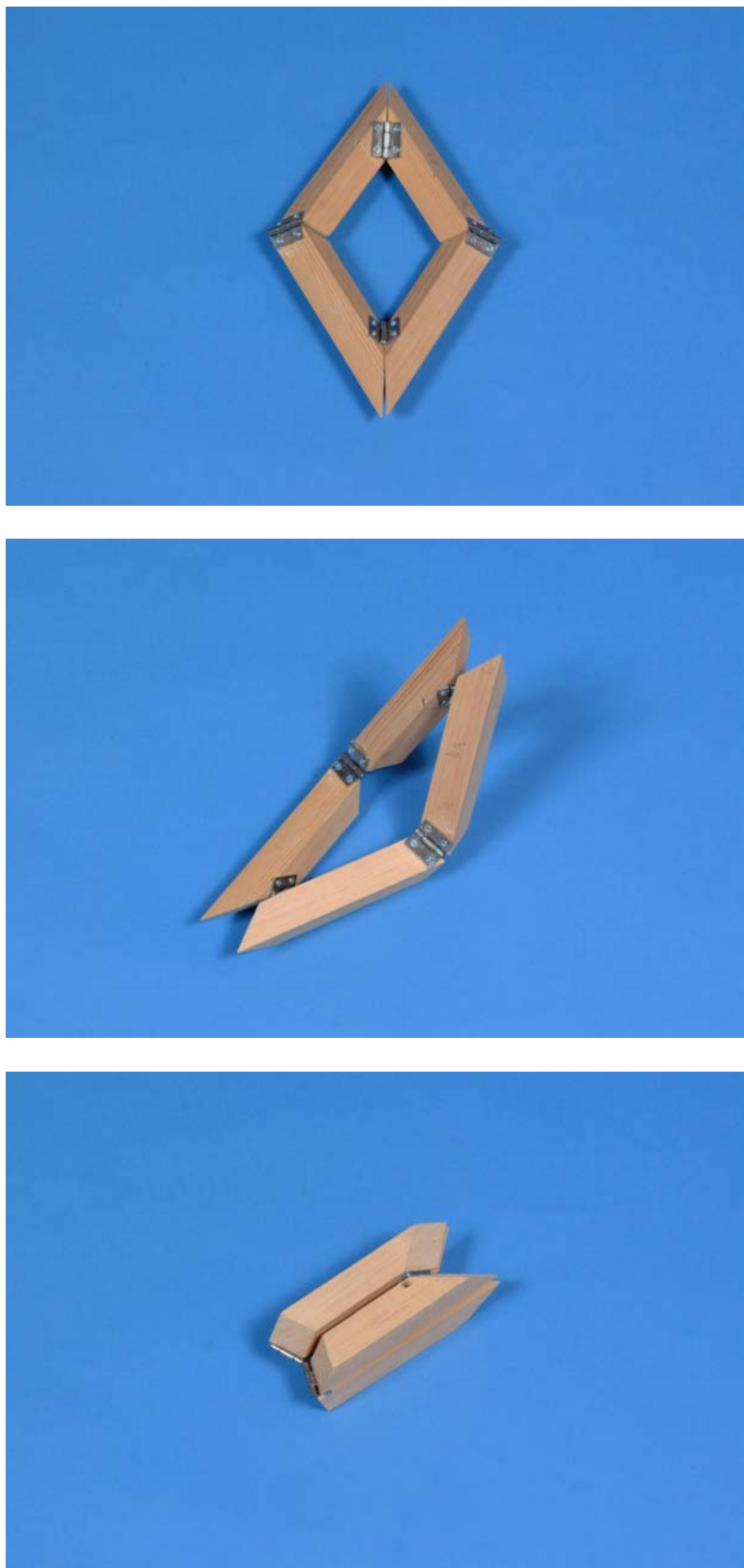


Fig. 3.3.15 Model that $\lambda = \pi / 4$ and $\omega = \pi / 3$.

3.3.3 Network of Alternative Form of Bennett Linkage

Similar to the original Bennett linkage, the alternative form of Bennett linkage can also form a network with the same layout. The model shown in Fig. 3.3.16 is the alternative form of the model in Fig. 3.2.8.

3.4 CONCLUSION AND DISCUSSION

This chapter has presented the first attempt to construct a large overconstrained structural mechanism using Bennett linkages as basic elements. The proposed mechanisms have the following features.

1. The mechanisms have single degree of mobility and therefore can be controlled with ease.
2. Only simple revolute joints are required, which means that the mechanisms can be made without much difficulty and furthermore it increases the reliability of the mechanisms.
3. They are very rigid as the Bennett linkage itself is geometrically overconstrained while the method to constructing the networks of Bennett linkages proposed here add further constraints.

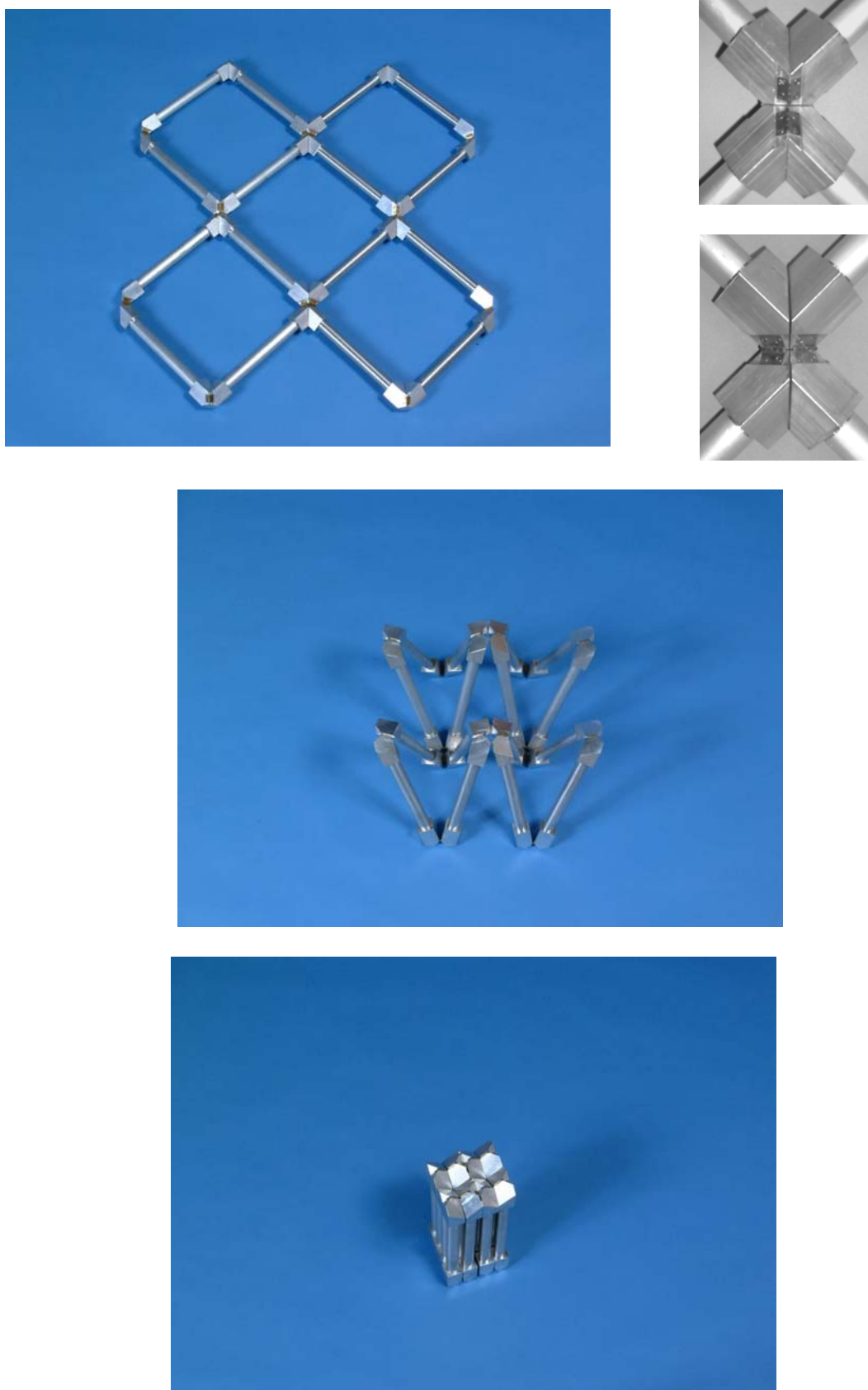


Fig. 3.3.16 Network of alternative form of Bennett linkage.

Firstly, the method to construct networks from original Bennett linkage has been derived. The basic layout can be considered as criteria to extend the network unlimitedly. There is a family of networks of Bennett linkages to form single-layer deployable structures, which in general deploy into profiles of cylindrical surface. Meanwhile, two special cases of the single-layer structures can be extended to construct multi-layer structural mechanism. One of them forms a planar profile, whereas the other forms a curved profile. According to this derivation, a Bennett connection is introduced to solve the problem to connect two similar Bennett linkages into a mobile structure.

Secondly, a mathematical proof that special forms of the Bennett linkage exist which enable the linkage to be folded completely has been presented. Under certain circumstances, the said forms also allow the linkage to be flattened into a rhombus. Hence, effective deployable elements can be created from the Bennett linkages. An easy method to make this alternative form of Bennett linkage has been introduced and verified, and a corresponding alternative network has been obtained following the similar layout of the original Bennett linkage with a flat profile. For any equilateral original Bennett linkages, its network also can be made into the alternative form with the same method.

The research presented here opens many possibilities of constructing deployable structure using Bennett linkage.

Hybrid Bricard Linkage and its Networks

4.1 INTRODUCTION

Of the $6R$ overconstrained linkages reviewed in Section 2.2.3 the most remarkable are the six types of Bricard linkages which he discovered as a result of his investigation into mobile octahedra. Bricard linkages are the only $6R$ overconstrained linkages that are not derived from $4R$, $5R$ or $6R$ ones, while other $6R$ linkages can be obtained by combining or generalising $4R$, $5R$ or $6R$ linkages. Due to this fact, in this chapter we choose to explore the possibility of forming large structural mechanisms using a particular type of Bricard linkage.

The layout of the chapter is as follows. Section 4.2 describes the construction process of a $6R$ hybrid linkage based on the Bricard linkage and its basic characteristics. Section 4.3 discusses several possibilities of forming networks of hybrid Bricard linkages. The bifurcation of this linkage is dealt with in Section 4.4. Section 4.5 analyses the alternative forms of the hybrid Bricard linkage. Finally, a conclusion is given in Section 4.6.

4.2 HYBRID BRICARD LINKAGES

Section 2.2.3 lists six types of Bricard linkages. By combining the general plane-symmetric and trihedral Bricard linkages, a new type of hybrid Bricard linkage can be obtained, whose geometric parameters satisfy the following conditions.

$$\begin{aligned}
 a_{12} &= a_{23} = a_{34} = a_{45} = a_{56} = a_{61} = l \\
 \alpha_{12} &= \alpha_{34} = \alpha_{56} = \alpha, \quad \alpha_{23} = \alpha_{45} = \alpha_{61} = 2\pi - \alpha \\
 R_i &= 0 \quad (i = 1, 2, \dots, 6)
 \end{aligned} \tag{4.2.1}$$

The configuration of this linkage is shown in Fig. 4.2.1. Next we shall prove that this linkage is mobile and has a single degree of mobility.

Because of symmetry, the six revolute variables must satisfy the following conditions.

$$\begin{aligned}
 \theta_1 &= \theta_3 = \theta_5 = \theta \\
 \theta_2 &= \theta_4 = \theta_6 = \varphi
 \end{aligned} \tag{4.2.2}$$

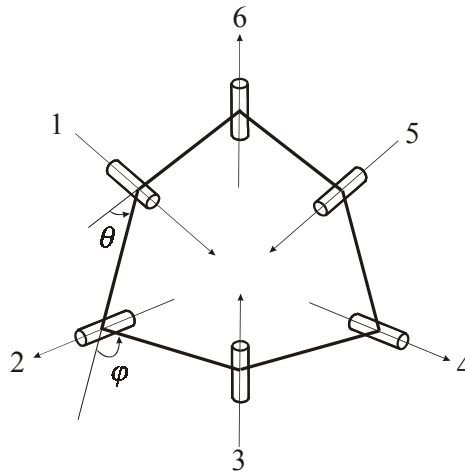


Fig. 4.2.1 Hybrid Bricard linkage.

The mobility condition (2.1.3) requires

$$[T_{61}][T_{56}][T_{45}][T_{34}][T_{23}][T_{12}] = [I] \quad (4.2.3)$$

or

$$[T_{34}][T_{23}][T_{12}] = [T_{45}]^{-1} [T_{56}]^{-1} [T_{61}]^{-1} = [T_{54}][T_{65}][T_{16}] \quad (4.2.4)$$

Hence, using the parameters in (4.2.1) and (4.2.2), (4.2.4) yields

$$\begin{aligned} \cos^2 \alpha \cos \theta - \cos^2 \alpha \cos \theta \cos \varphi - \cos^2 \alpha + \cos^2 \alpha \cos \varphi \\ + 2 \cos \alpha \sin \theta \sin \varphi - \cos \theta - \cos \theta \cos \varphi - \cos \varphi = 0 \end{aligned} \quad (4.2.5)$$

(4.2.2) and (4.2.5) form a set of independent closure equations for this 6R linkage. For any given α ($0 \leq \alpha \leq \pi$), (4.2.5) represents the compatibility condition of the mechanism. It is noted that the compatibility paths are periodic. The periods for θ and φ are both 2π . Hence, we can consider only the compatibility paths in the period of $-\pi \leq \theta \leq \pi$ and $-\pi \leq \varphi \leq \pi$ shown in Fig. 4.2.2.

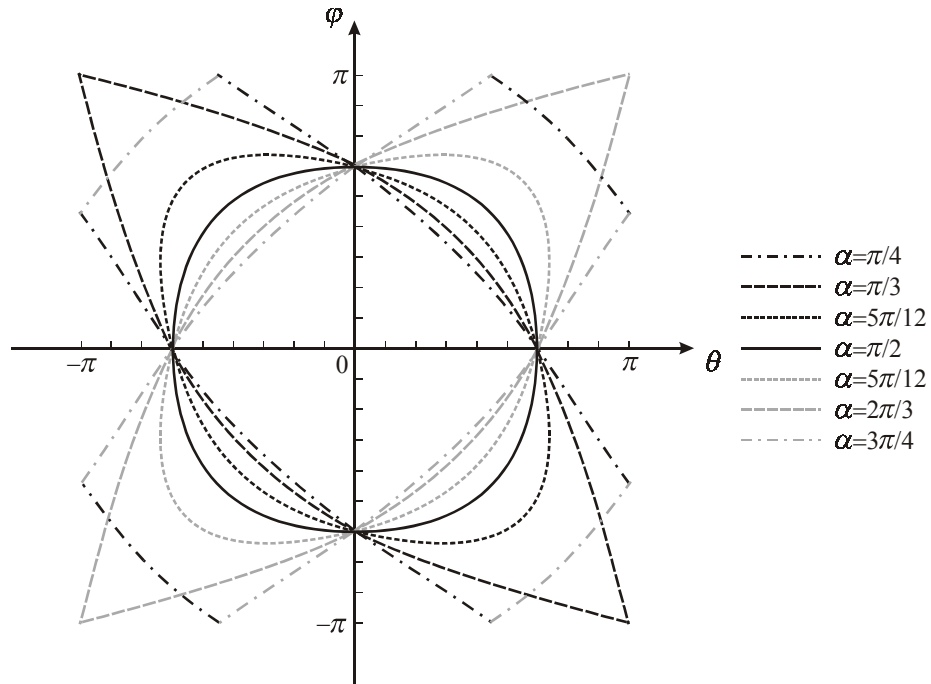
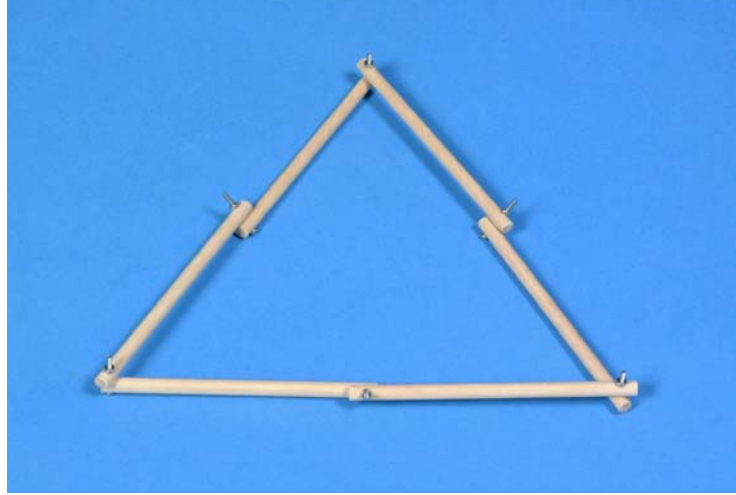


Fig. 4.2.2 φ vs θ for the hybrid Bricard linkage for a set of α in a period.

Note that Fig. 4.2.2 shows the compatibility paths for only a set of α , though $0 \leq \alpha \leq \pi$. A number of distinctive features of the hybrid Bricard linkage with any twist α can be summarised from Fig. 4.2.2. First of all, it shows that only one of six revolute variables can be free. Thus, in general, this hybrid Bricard linkage has one degree of mobility. Secondly, the linkage with twist α behaves the same as that whose twist is $\pi - \alpha$. Thirdly, all the compatibility paths pass through the points $(0, -2\pi/3)$, $(0, 2\pi/3)$, $(-2\pi/3, 0)$ and $(2\pi/3, 0)$, regardless the value of α . This means that all of the hybrid Bricard linkages can be flattened to form a planar equilateral triangle whose side length is $2l$. Fourthly, when $0 \leq \alpha < \pi/3$ or $2\pi/3 < \alpha \leq \pi$, the movement of the linkages is not continuous. It has been found by experiment that the linkage is physically blocked in the positions when all the links are crossed in the centre when either θ or φ reaches π or $-\pi$. Figure 4.2.3 shows a model for $\alpha = \pi/4$. The compatibility paths form a closed loop when $\pi/3 \leq \alpha \leq 2\pi/3$. So the linkage keeps moving continuously, see Fig. 4.2.4, which shows a model for $\alpha = 5\pi/12$.

Now let us focus on a particular set of compatibility paths in Fig. 4.2.2 when $\alpha = \pi/3$ or $2\pi/3$. Both θ and φ reach π or $-\pi$ simultaneously, which correspond to the configurations of the most compact folding, see Fig. 4.2.5(a). On the other hand, when $\theta = 0$, $\varphi = 2\pi/3$ (or $-2\pi/3$), or *vice versa*, the linkage forms a plane equilateral triangle as mentioned before, in accordance to the configuration of maximum expansion, see Fig. 4.2.5(c). Therefore, this particular hybrid Bricard linkage with twist $\pi/3$ or $2\pi/3$ shows characteristics similar to the alternative form of the Bennett linkages given in Section 3.3. Thus, it can be utilised to construct structural mechanisms, which will be discussed next.

Note that at points (π, π) , $(\pi, -\pi)$, $(-\pi, -\pi)$ and $(-\pi, \pi)$, the compatibility paths cross each other. The diagram indicates the existence of bifurcation. The detailed discussion is given in Section 4.4.



(a)

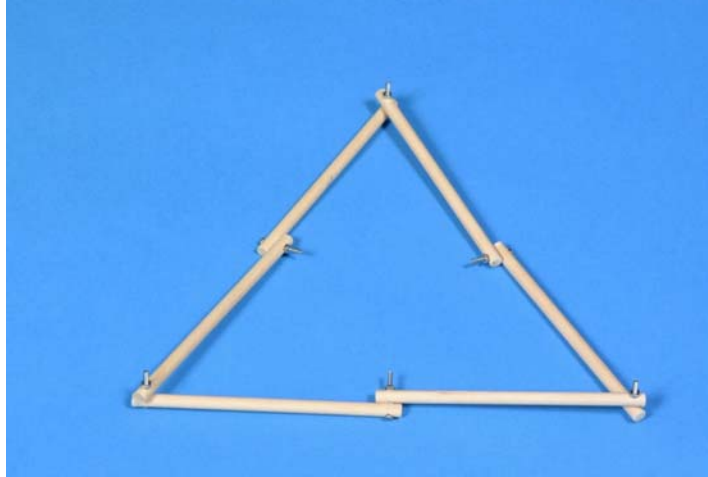


(b)

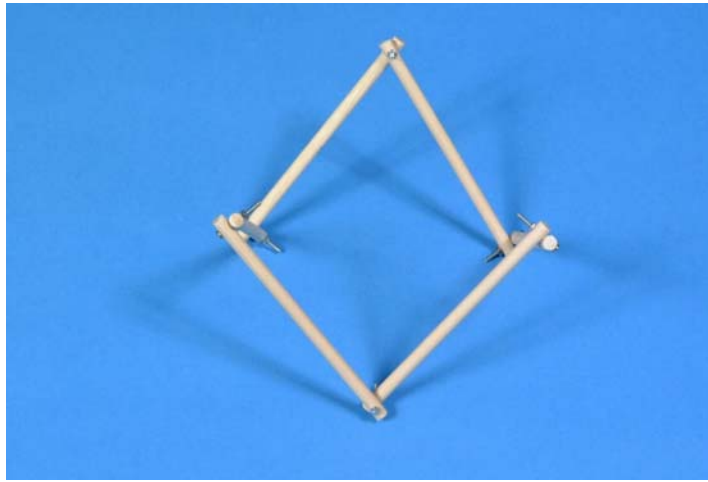
Fig. 4.2.3 Deployment sequence of a hybrid Bricard linkage with $\alpha = \pi / 4$.

(a) The configuration of planar equilateral triangle;

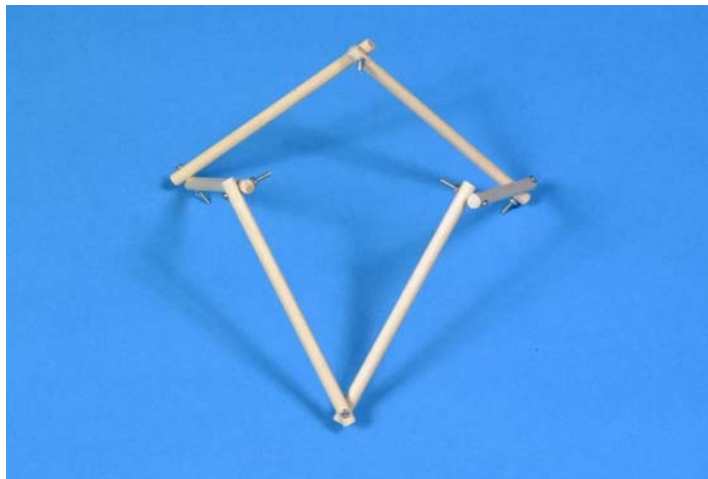
(b) the configuration in which the movement of linkage is physically blocked.



(a)



(b)



(c)

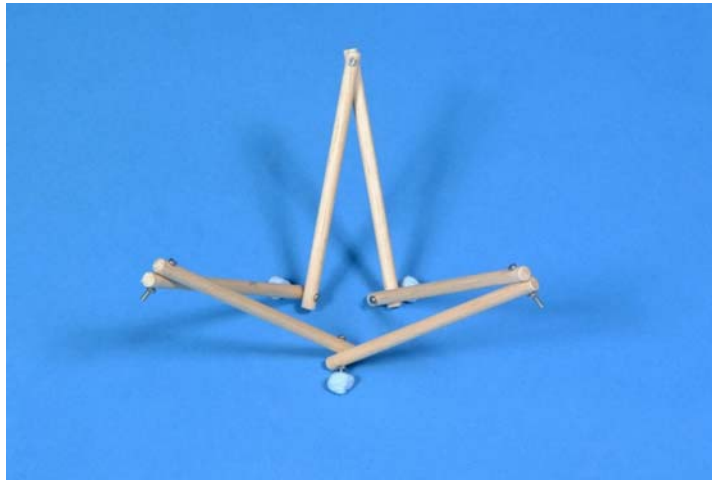
Fig. 4.2.4 Deployment sequence of a hybrid Bricard linkage with $\alpha = 5\pi / 12$.

(a) The configuration of planar equilateral triangle;

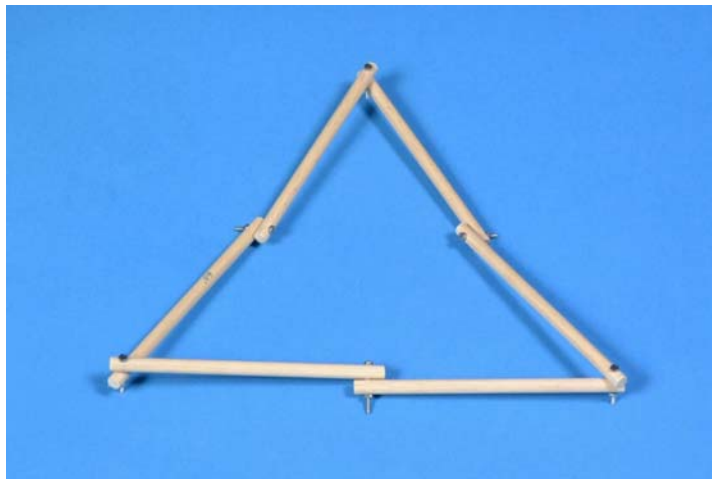
(b) and (c) the configurations during the process of movement.



(a)



(b)



(c)

Fig. 4.2.5 Deployment sequence of a hybrid Bricard linkage with $\alpha = \pi / 3$.

(a) The compact folded configuration; (b) the configuration during the process of deployment; (c) the maximum expanded configuration.

4.3 NETWORK OF HYBRID BRICARD LINKAGES

Because the hybrid Bricard linkages with twist $\pi/3$ or $2\pi/3$ have the same behaviour, we just take $\alpha = \pi/3$ here. This particular linkage, presented in the previous section, can be represented by the schematic diagram shown in Fig. 4.3.1(a), in which ‘○’ represents the hinge connecting the ends of the links and ‘●’ represents the hinge connecting the middle points of two links. Hence, a pair of links connected by a ‘●’ in the middle, see Fig. 4.3.1(b), behaves like a pair of scissors.

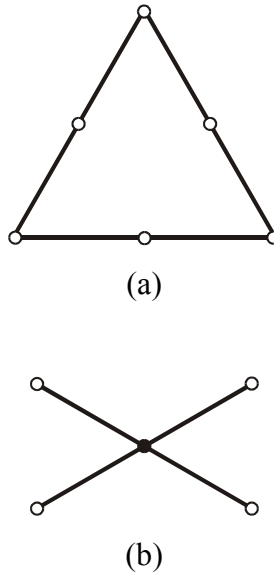
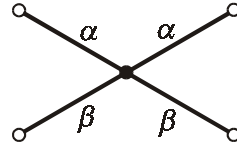


Fig. 4.3.1 (a) Schematic diagram of the hybrid Bricard linkage;

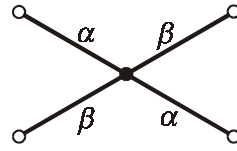
(b) one pair of the cross bars connected by a hinge in the middle.

The basic element consists of three pairs of cross bars with identical length. For each pair, there exist two possible arrangements of twists for a pair of cross bars, namely type A and B as shown in Fig. 4.3.2(a), in which $\alpha = \pi/3$ and $\beta = 2\pi - \pi/3$. The deployable element is formed by having three pairs in a loop with a hybrid Bricard

linkage with $\alpha = \pi/3$ at the centre. There are in total four possible combinations of three pairs of cross bars, A-A-A, A-A-B, A-B-B and B-B-B, one of which is shown in Fig. 4.3.2(b). Next, we will examine by experiments whether a network composed of these elements will retain the deployable character of the hybrid Bricard linkage.

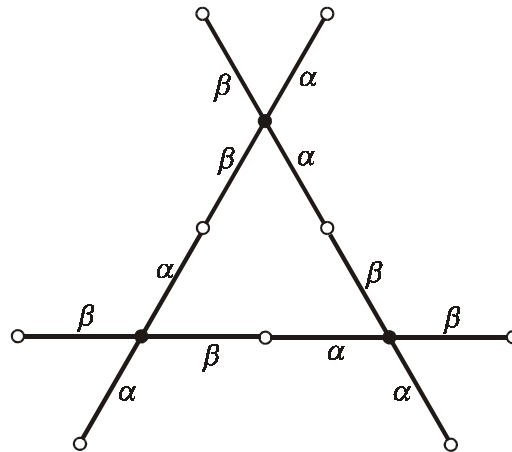


Type A



Type B

(a)



(b)

Fig. 4.3.2 Construction of the deployable element. (a) Two possible pairs of cross bars:

Type A and B; (b) one of the arrangements: A-A-B.

Consider a simple connection of two deployable elements as shown in Fig. 4.3.3. The marked pair of cross bars at the centre could be either type A or B. Figure 4.3.4 is a model where the intermediate pair of cross bars is a type A pair. It is obvious that two elements work well together. So the arrangement of type A can form a deployable network. In Fig. 4.3.5, two elements are connected with a type B pair. It cannot be fully folded. The links of each side interface with each other in the folding process, see Fig. 4.3.5(b). As the result, we conclude that a deployable network cannot be connected with type B pair. Thus, for the layout of element as Fig. 4.3.3, only element A-A-A can be used to form a deployable network. A typical portion of such a network is shown in Fig. 4.3.6. Figure 4.3.7 is a model of such a network.

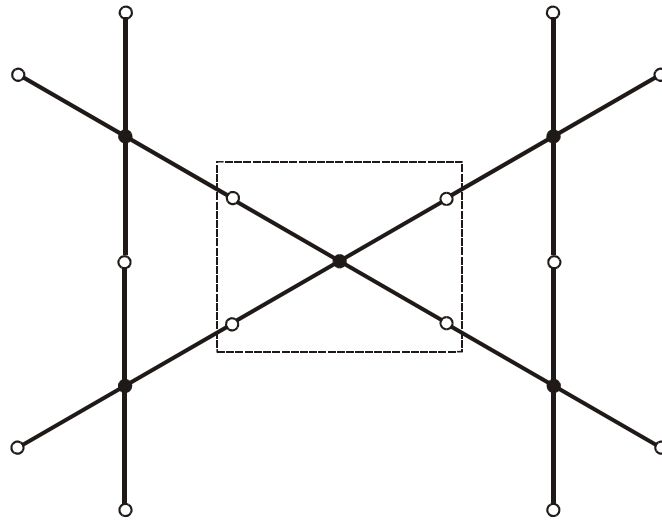
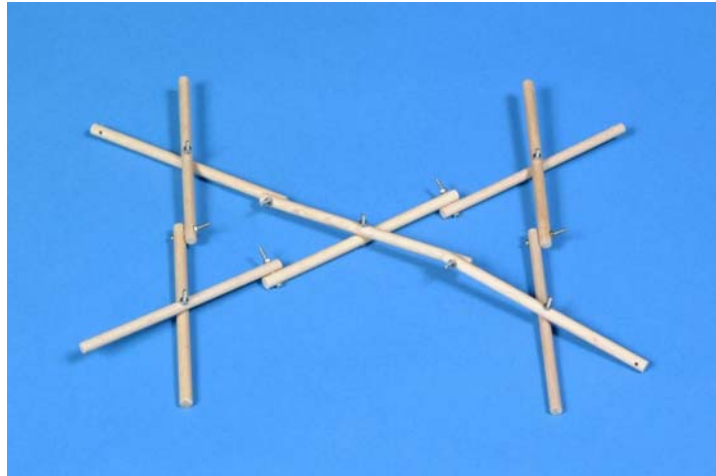


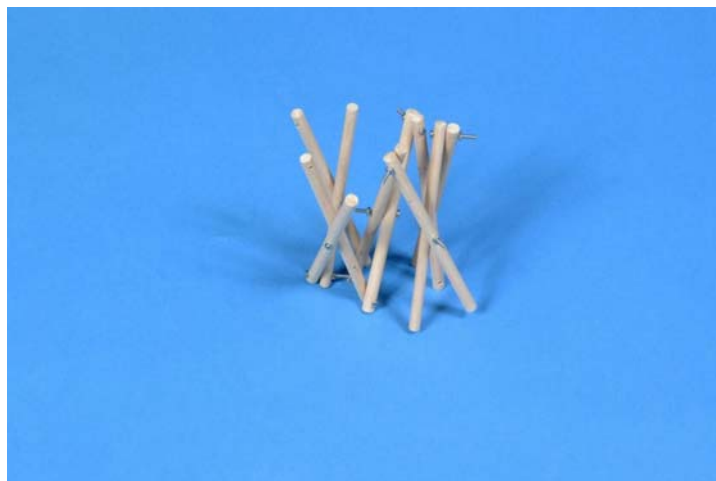
Fig. 4.3.3 Connectivity of deployable elements.



(a)



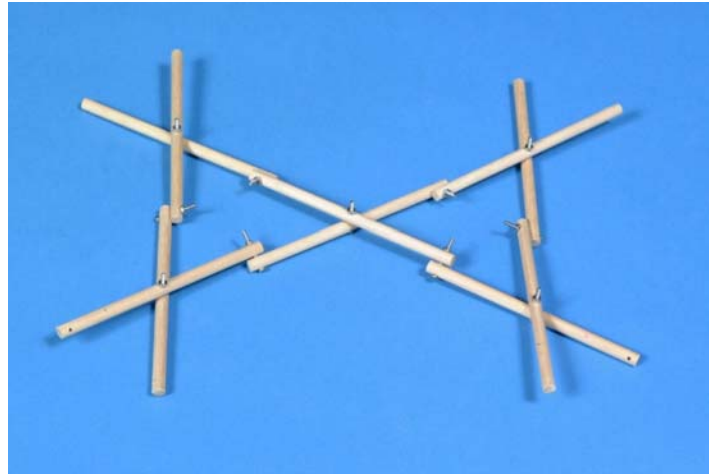
(b)



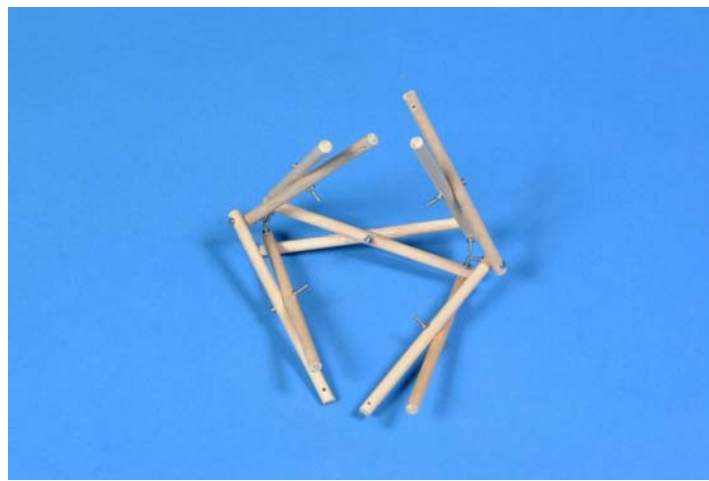
(c)

Fig. 4.3.4 Model of connectivity of deployable elements with a type A pair.

(a) Fully deployed; (b) during deployment; and (c) close to being folded.



(a)



(b)

Fig. 4.3.5 Model of connectivity of deployable elements with a type B pair.

(a) Fully deployed; (b) during deployment.

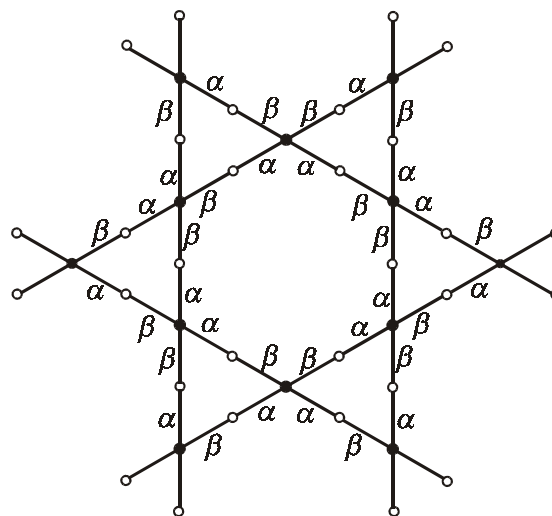


Fig. 4.3.6 Portion of a network of hybrid Bricard linkages.

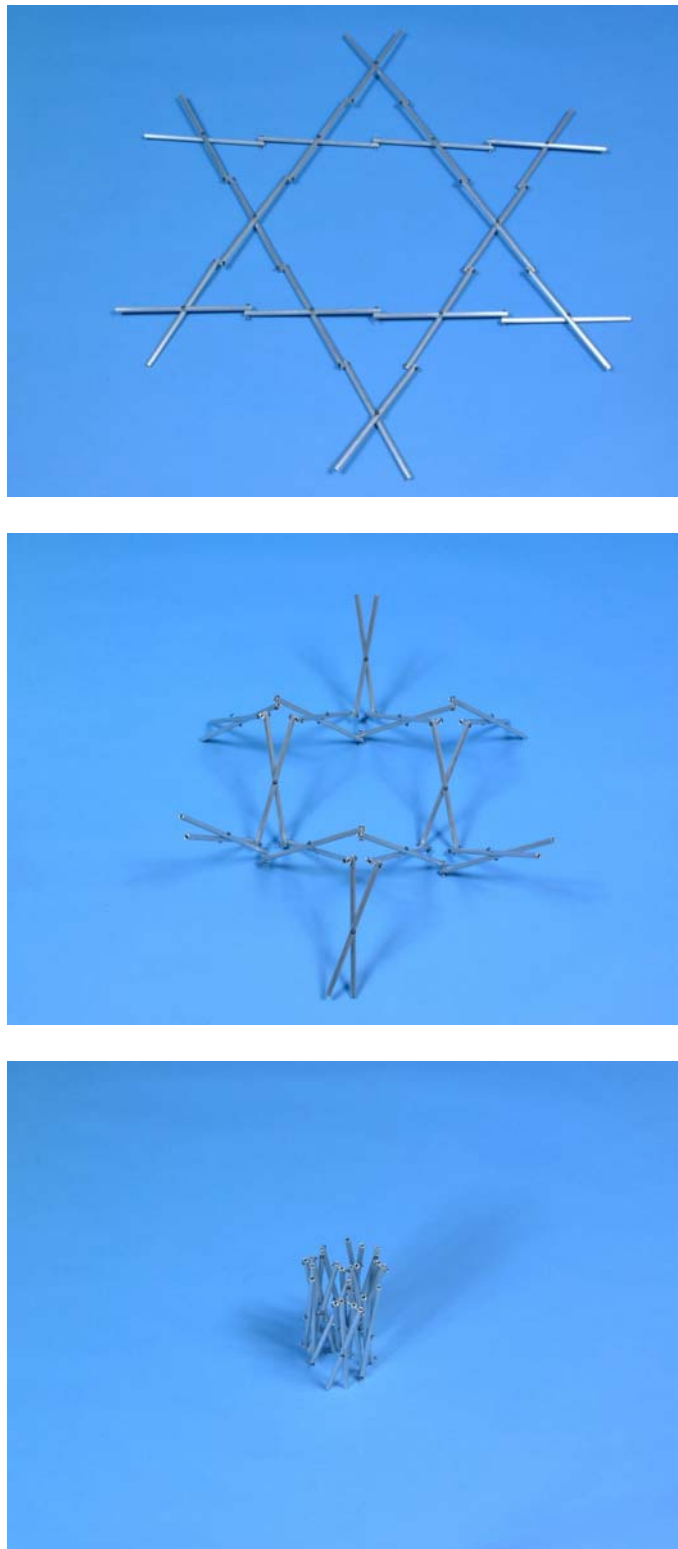


Fig. 4.3.7 Model of a network of hybrid Bricard linkages.

4.4 BIFURCATION OF HYBRID BRICARD LINKAGE

Figure 4.2.2 shows that, when $\alpha = \pi/3$ or $2\pi/3$, two compatibility paths cross each other at points (π, π) , $(-\pi, \pi)$, $(\pi, -\pi)$ and $(-\pi, -\pi)$. This raises the possibility of having a kinematic bifurcation at these four points.

Because of similarity, we consider the hybrid Bricard linkage with $\alpha = 2\pi/3$ at point (π, π) . The compatibility paths of this linkage in the period of $0 \leq \theta, \varphi \leq 2\pi$ is reproduced in Fig. 4.4.1. For a small imperfection ε ($\varepsilon > 0$) to twist α , the compatibility paths will alter slightly, see Fig. 4.4.1(a). If the compatibility condition (4.2.5) is plotted in 3D, see Fig. 4.4.1(b), the surface of compatibility will have a saddle point at $(\theta, \varphi, \alpha) = (\pi, \pi, 2\pi/3)$. Next, we will prove that this is a point of bifurcation.

The bifurcation of a mechanism can be determined by examining the state of self-stress of the mechanism (Calladine, 1978; Calladine and Pellegrino, 1991). Normally in 3D space, seven links, connected by revolute joints, are required to form a linkage with a single degree of mobility. The hybrid Bricard linkage is a 6R linkage. Hence, it is overconstrained and must have a state of self-stress in general. By considering equilibrium of internal forces of the linkage, we are able to show the existence of a state of self-stress. Moreover, we can demonstrate that the number of independent states of self-stress becomes two at point (π, π) .

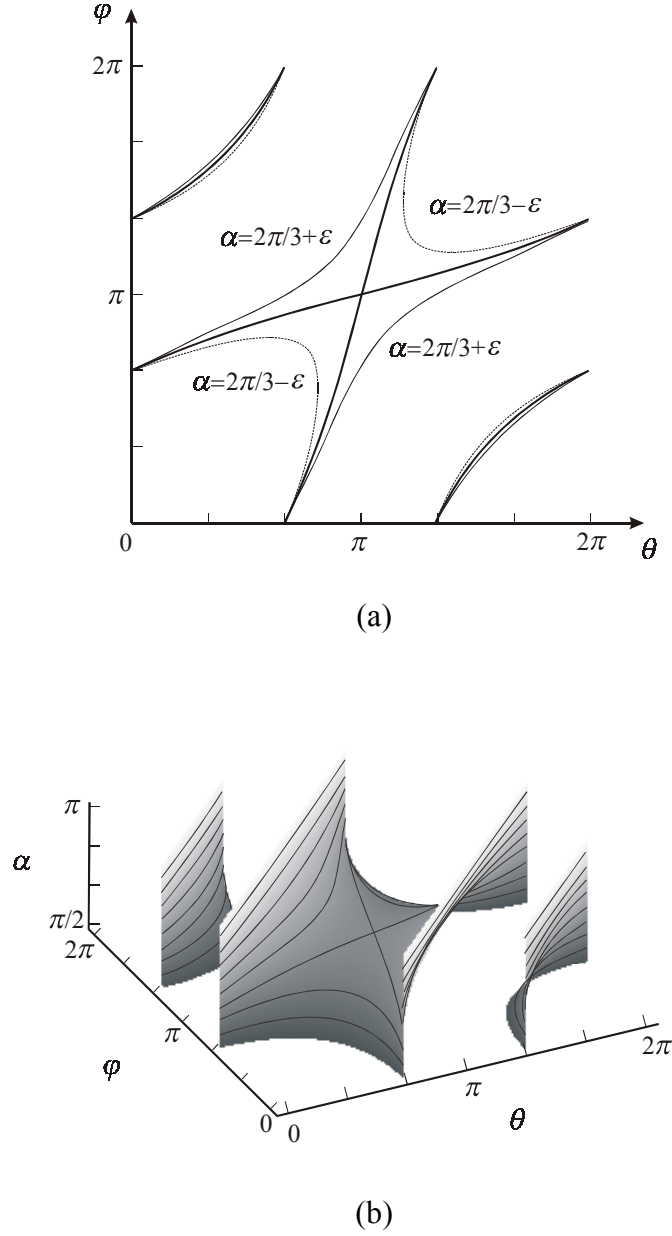


Fig. 4.4.1 The compatibility paths of the hybrid Bricard linkage.

(a) $\alpha = 2\pi/3$ and $\alpha = 2\pi/3 \pm \varepsilon$. (b) $\pi/2 \leq \alpha \leq \pi$.

Figure 4.4.2 shows the hybrid Bricard linkage with link length l and twist α . For the typical links 12 and 23, the internal forces and moments at each end are shown in Fig. 4.4.3(a). The forces and moments are joint specific. For example, at joint 2 of link 12, the forces and moments are defined in the local coordinate system $x_{21}y_{21}z_{21}$ in

which z_{21} is alongside the link while y_{21} is along the axis of joint 2. At joint 1, z_{12} is alongside the link while y_{12} is along the axis of joint 1. Similarly, we are able to define forces and moments for link 23.

Because of symmetry, the local forces and moments in link 34 should be the same as those in link 12.

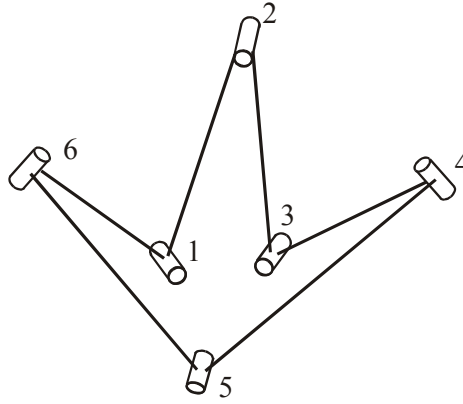


Fig. 4.4.2 A typical hybrid Bricard linkage.

Considering the equilibrium of link 12, we have

$$\begin{aligned}
 Nx_{21} + Nx_{12} \cos \alpha - Ny_{12} \sin \alpha &= 0 \\
 Ny_{21} + Nx_{12} \sin \alpha + Ny_{12} \cos \alpha &= 0 \\
 Nz_{21} + Nz_{12} &= 0 \\
 Mx_{21} + Mx_{12} \cos \alpha - My_{12} \sin \alpha - Ny_{21} \cdot l &= 0 \\
 My_{21} + Mx_{12} \sin \alpha + My_{12} \cos \alpha + Nx_{21} \cdot l &= 0 \\
 Mz_{21} + Mz_{12} &= 0
 \end{aligned} \tag{4.4.1}$$

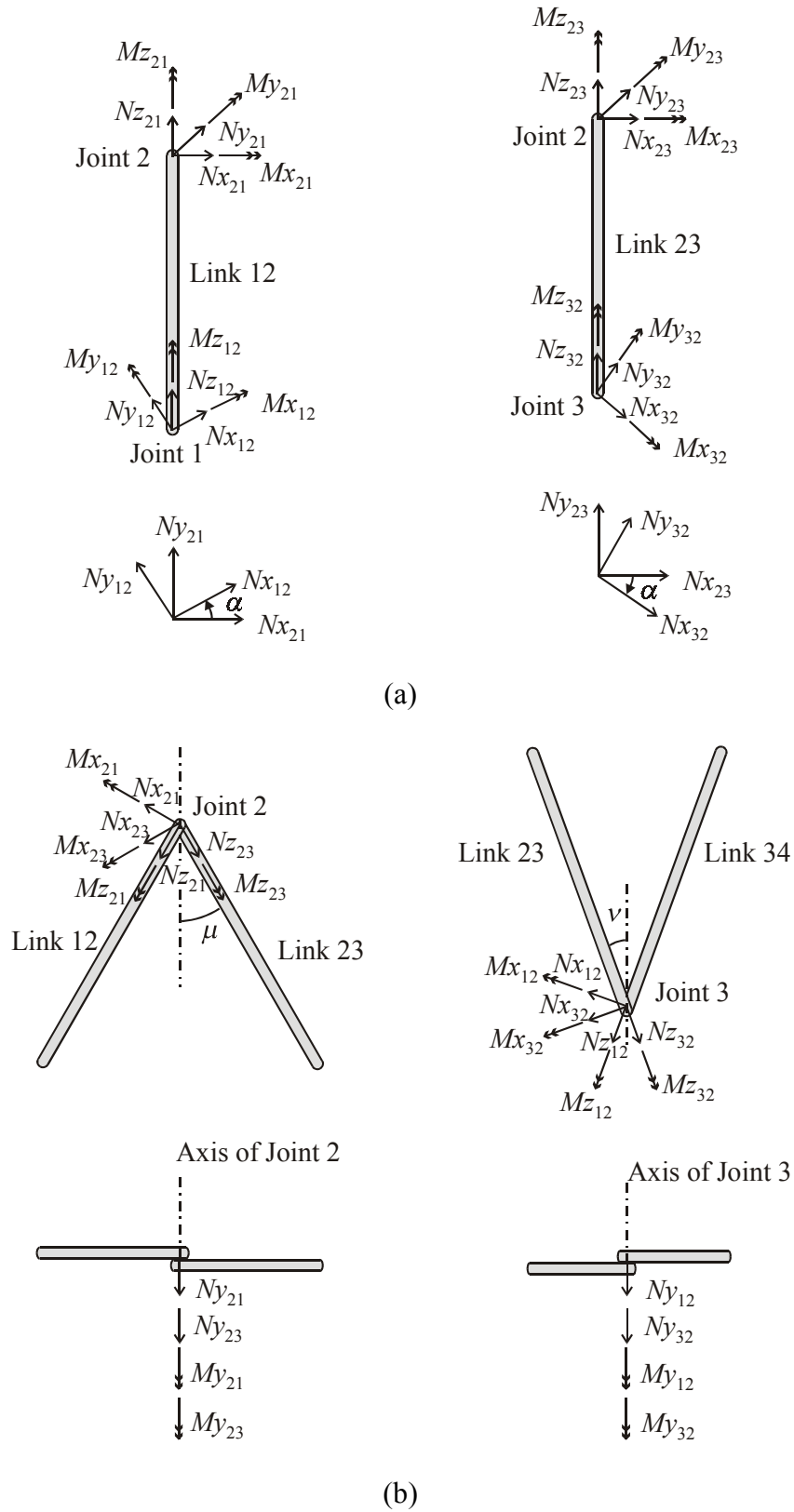


Fig. 4.4.3 Equilibrium of links and joints.

(a) Forces and moments in two typical links; (b) forces and moments at joints 2 and 3.

The equilibrium equations for link 23 are

$$\begin{aligned}
 Nx_{32} + Nx_{23} \cos \alpha - Ny_{23} \sin \alpha &= 0 \\
 Ny_{32} + Nx_{23} \sin \alpha + Ny_{23} \cos \alpha &= 0 \\
 Nz_{23} + Nz_{32} &= 0 \\
 Mx_{32} + Mx_{23} \cos \alpha - My_{23} \sin \alpha + Ny_{32} \cdot l &= 0 \\
 My_{32} + Mx_{23} \sin \alpha + My_{23} \cos \alpha - Nx_{32} \cdot l &= 0 \\
 Mz_{23} + Mz_{32} &= 0
 \end{aligned} \tag{4.4.2}$$

Links 12 and 23 are connected at joint 2 where the axis of the joint is in the direction of y_{21} and y_{23} , see Fig. 4.4.3(b). The equilibrium equations at joint 2 are

$$\begin{aligned}
 -Nx_{21} \cos \mu - Nx_{23} \cos \mu - Nz_{21} \sin \mu + Nz_{23} \sin \mu &= 0 \\
 -Ny_{21} - Ny_{23} &= 0 \\
 Nx_{21} \sin \mu - Nx_{23} \sin \mu - Nz_{21} \cos \mu - Nz_{23} \cos \mu &= 0 \\
 -Mx_{21} \cos \mu - Mx_{23} \cos \mu - Mz_{21} \sin \mu + Mz_{23} \sin \mu &= 0 \\
 -My_{21} - My_{23} &= 0 \\
 Mx_{21} \sin \mu - Mx_{23} \sin \mu - Mz_{21} \cos \mu - Mz_{23} \cos \mu &= 0
 \end{aligned} \tag{4.4.3}$$

where $\mu = \frac{\pi - \varphi}{2}$, which is half of the angle between links 12 and 23.

Links 23 and 34 are connected at joint 3 whose axis is in the same direction as y_{12} and y_{32} , see Fig. 4.4.3(b), because the local forces in link 34 are the same as those in link 12.

The equilibrium equations at joint 3 are

$$\begin{aligned}
 & -Nx_{12} \cos \nu - Nx_{32} \cos \nu - Nz_{12} \sin \nu + Nz_{32} \sin \nu = 0 \\
 & -Ny_{12} - Ny_{32} = 0 \\
 & Nx_{12} \sin \nu - Nx_{32} \sin \nu - Nz_{12} \cos \nu - Nz_{32} \cos \nu = 0 \\
 & -Mx_{12} \cos \nu - Mx_{32} \cos \nu - Mz_{12} \sin \nu + Mz_{32} \sin \nu = 0 \\
 & -My_{12} - My_{32} = 0 \\
 & Mx_{12} \sin \nu - Mx_{32} \sin \nu - Mz_{12} \cos \nu - Mz_{32} \cos \nu = 0
 \end{aligned} \tag{4.4.4}$$

where $\nu = \frac{\pi - \theta}{2}$, which is half of the angle between links 23 and 34.

Because the revolute joint cannot transform the moment in its axis direction,

$$\begin{aligned}
 My_{21} &= My_{23} = 0 \\
 My_{12} &= My_{32} = 0
 \end{aligned} \tag{4.4.5}$$

Substituting (4.4.5) into (4.4.1)-(4.4.4) gives

$$\begin{aligned}
 Nx_{21} &= Nz_{23} \cdot \tan \mu, \quad Ny_{21} = Nz_{23} \cdot \frac{\tan \nu - \tan \mu \cos \alpha}{\sin \alpha}, \quad Nz_{21} = -Nz_{23} \\
 Mx_{21} &= Nz_{23} \cdot \frac{\tan \nu}{\sin \alpha} \cdot l, \quad My_{21} = 0, \quad Mz_{21} = Nz_{23} \cdot \frac{\tan \mu \tan \nu}{\sin \alpha} \cdot l \\
 Nx_{12} &= -Nz_{23} \cdot \tan \nu, \quad Ny_{12} = Nz_{23} \cdot \frac{\tan \mu - \tan \nu \cos \alpha}{\sin \alpha}, \quad Nz_{12} = Nz_{23} \\
 Mx_{12} &= -Nz_{23} \cdot \frac{\tan \mu}{\sin \alpha} \cdot l, \quad My_{12} = 0, \quad Mz_{12} = -Nz_{23} \cdot \frac{\tan \mu \tan \nu}{\sin \alpha} \cdot l \\
 Nx_{23} &= Nz_{23} \cdot \tan \mu, \quad Ny_{23} = Nz_{23} \cdot \frac{-\tan \nu + \tan \mu \cos \alpha}{\sin \alpha}, \quad Nz_{23} = Nz_{23} \\
 Mx_{23} &= -Nz_{23} \cdot \frac{\tan \nu}{\sin \alpha} \cdot l, \quad My_{23} = 0, \quad Mz_{23} = Nz_{23} \cdot \frac{\tan \mu \tan \nu}{\sin \alpha} \cdot l
 \end{aligned} \tag{4.4.6}$$

$$N_{x_{32}} = -N_{z_{23}} \cdot \tan \nu, \quad N_{y_{32}} = N_{z_{23}} \cdot \frac{-\tan \mu + \tan \nu \cos \alpha}{\sin \alpha}, \quad N_{z_{32}} = -N_{z_{23}}$$

$$M_{x_{32}} = N_{z_{23}} \cdot \frac{\tan \mu}{\sin \alpha} \cdot l, \quad M_{y_{32}} = 0, \quad M_{z_{32}} = -N_{z_{23}} \cdot \frac{\tan \mu \tan \nu}{\sin \alpha} \cdot l$$

In general, (4.4.6) presents one state of self-stress because if one variable, i.e., $N_{z_{23}}$, is given, all the rest can be determined. However, when $\alpha = 2\pi/3$, this changes at point (π, π) , i.e. $\mu = \nu = 0$. Considering (4.4.5), the solution of (4.4.1)-(4.4.4) becomes

$$\begin{aligned} N_{z_{21}} &= -N_{z_{23}}, \quad N_{z_{12}} = N_{z_{23}} \\ N_{z_{23}} &= N_{z_{23}}, \quad N_{z_{32}} = -N_{z_{23}} \\ M_{z_{21}} &= -M_{z_{23}}, \quad M_{z_{12}} = M_{z_{23}} \\ M_{z_{23}} &= M_{z_{23}}, \quad M_{z_{32}} = -M_{z_{23}} \end{aligned} \tag{4.4.7}$$

The rest of forces and moments are zero.

Now, there are two independent variables, e.g., $N_{z_{23}}$ and $M_{z_{23}}$. Thus, two states of self-stress exist. Therefore, point (π, π) is a point of kinematic bifurcation.

So far, we have proven that the kinematic indeterminacy of the linkage increases by one at $\theta = \varphi = \pi$. The physical model shows that at $\theta = \varphi = \pi$, the degree of infinitesimal mobility indeed increases by 1 because in this configuration, the axes of the three joints at the top are coplanar and intersect at a single point, and so are the axes of the other three joints at the bottom, see Fig. 4.2.5(a). This leads to two degrees of infinitesimal mobility (Phillips, 1990). Although a bifurcation exists, it does not cause any problem in deployment of the linkage because the links would have to penetrate each other in order

to reach the bifurcated position, which is physically impossible. The movement of the linkage at $\theta = \varphi = \pi$ effectively changes from one path to another toward a direction such that the values of θ and φ decrease, instead of increasing. In such a situation, one of two of the infinitesimal mobilities becomes finite while the other disappears.

4.5 ALTERNATIVE FORMS OF HYBRID BRICARD LINKAGE

Although the original hybrid Bricard linkage has already produced a compact folded configuration, it is hard to realize in practice. The angle between six pairs of connected bars cannot become zero simultaneously without a complex design. Hence, it is important to investigate alternative forms of the hybrid Bricard linkages.

The method applied to find the alternative form of hybrid Bricard linkage is the same as that in Section 3.3. The axes of revolute joints are extended and joints are connected with bars not perpendicular to the axes, see Fig. 4.5.1, where the dashed lines present the links of the original linkage and the solid lines present the bars in the alternative form.

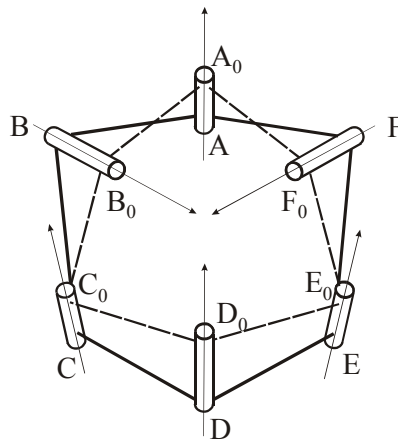


Fig. 4.5.1 The alternative form of hybrid Bricard linkage.

For simplicity, let us assume that symmetry is kept in the alternative form. Denote the link length and twist of the original hybrid Bricard linkage in Fig. 4.5.1 as l and α , respectively. We have

$$\begin{aligned}\overline{A_0A} = \overline{C_0C} = \overline{E_0E} &= c \\ \overline{B_0B} = \overline{D_0D} = \overline{F_0F} &= d\end{aligned}\tag{4.5.1}$$

So all the bars of the alternative form have the same length, L , which is

$$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{FA} = L = \sqrt{l^2 + c^2 + d^2 - 2cd \cos \alpha}\tag{4.5.2}$$

For each given set of c and d , an alternative form for the hybrid Bricard linkage can be obtained. The most compact folding can be achieved if simultaneously the points A, C and E meet at a point while the points B, D and F also meet at another point. This means that physically the linkage becomes a bundle whose length is L . Denote θ and φ in this fully folded configuration as θ_f and φ_f , respectively. On the other hand, when the linkage is full expanded, points A, B, C, D, E, and F are on the same plane, i.e. the linkage is completely flattened to form an equilateral hexagon. Take θ and φ in this configuration as θ_d and φ_d .

The relationship among the geometric parameters of the hybrid Bricard linkage and the design parameters of the linkage in the alternative forms can be found geometrically as we did in Section 3.3.2. Several alternative forms have been studied. Here only the most interesting ones are listed.

For the linkage found by Gan and Pellegrino (2003) shown in Fig. 4.5.2, the geometric parameters of its original hybrid Bricard linkage and its alternative form are

$$\begin{aligned}\alpha &= \pi - \arccos \frac{7}{13} \\ \theta_f &= \pi + \arccos \frac{19}{20}, \quad \varphi_f = \pi - \arccos \frac{19}{20} \\ \theta_d &= 2\pi + \arccos \frac{7}{20}, \quad \varphi_d = 2\pi - \arccos \frac{7}{20} \\ L &= \frac{3}{10} \sqrt{10} \cdot l \\ c = d &= \frac{\sqrt{130}}{60} \cdot l\end{aligned} \tag{4.5.3}$$

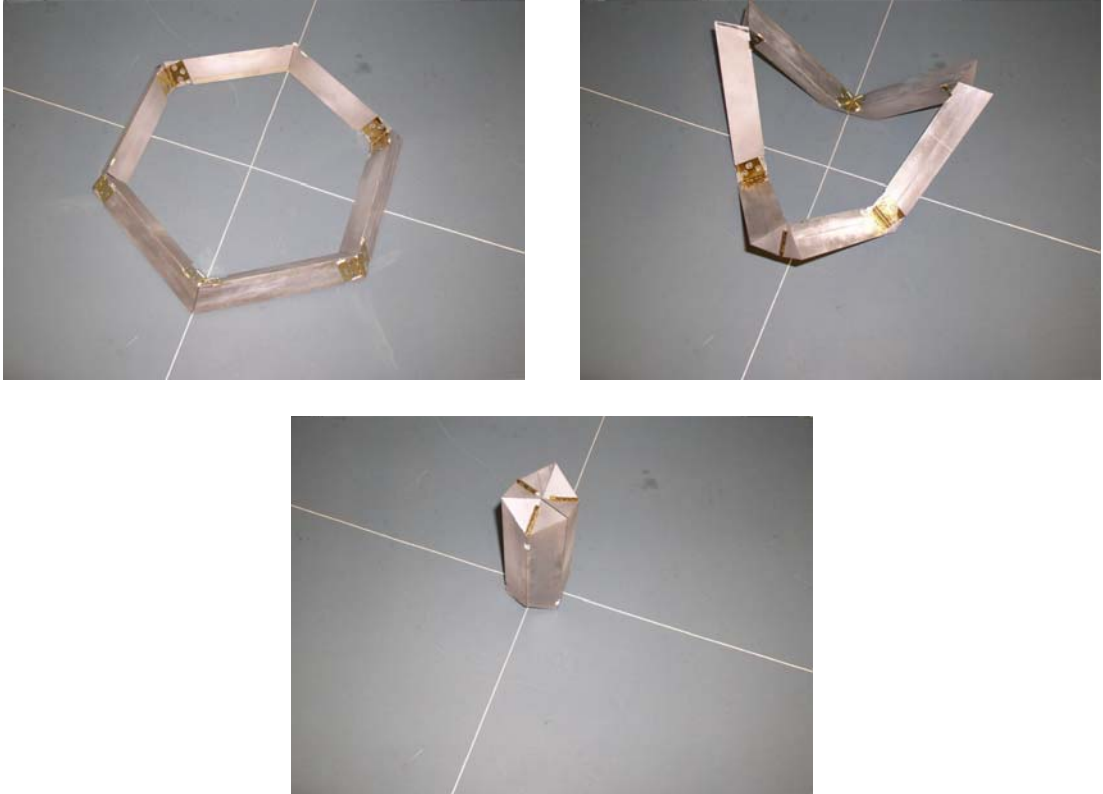


Fig. 4.5.2 A hybrid Bricard linkage in its alternative form.

(Courtesy of Professor Pellegrino of Cambridge University).

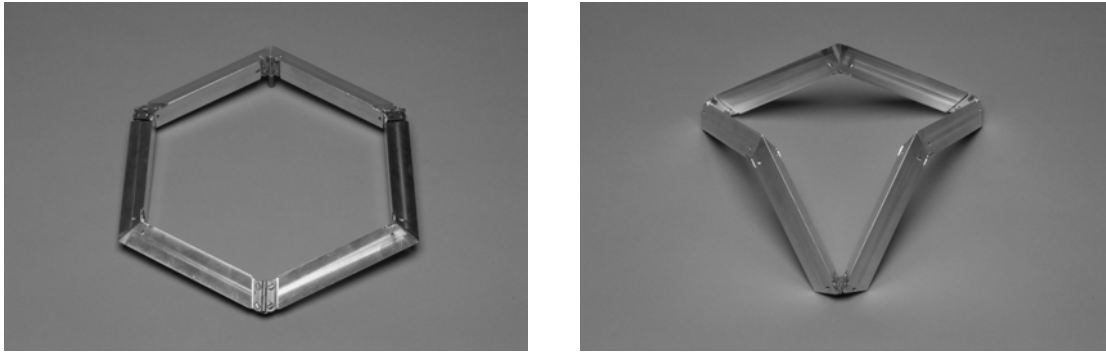
It is particularly interesting to mention two linkages, both of which are based on the *same* alternative form of the *same* hybrid Bricard linkage. During the folding process, one is blocked while the other is not.

The first linkage was proposed by Pellegrino (2002). For convenience, let us call it Linkage I. A reproduction of Linkage I is given in Fig. 4.5.3. The other, namely Linkage II, is shown in Fig. 4.5.4. The geometric parameters of the original hybrid Bricard linkage and its alternative form of both linkages are as follows.

$$\begin{aligned}
 \alpha &= \pi - \arctan 2 \\
 \theta_f &= -\frac{2}{3}\pi, \varphi_f = -\arctan \frac{\sqrt{15}}{7} \\
 \theta_d &= \frac{2}{3}\pi, \varphi_d = 0 \\
 L &= \frac{2}{3}\sqrt{3} \cdot l \\
 c &= \frac{\sqrt{3}}{6} \cdot l, d = \frac{\sqrt{15}}{6} \cdot l
 \end{aligned} \tag{4.5.4}$$

The compatibility path of both linkages is shown in Fig. 4.5.5. When the models are fully expanded, they are flat, corresponding to point D in the compatibility path. The completely folded position corresponds to point F in the path.

The similarity of these two linkages ends here.

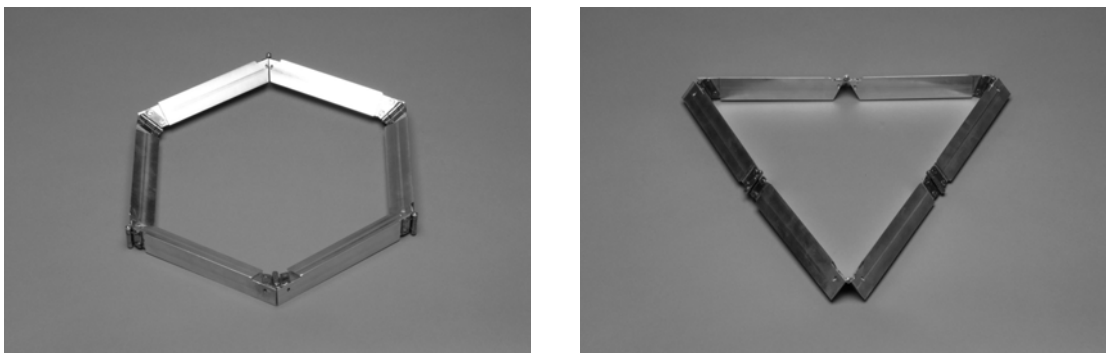


(a)

(b)

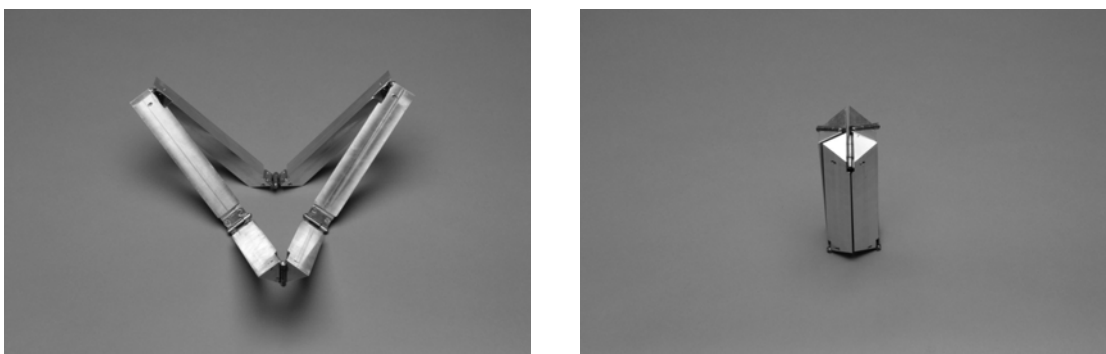
Fig. 4.5.3 Deployment of Linkage I.

(a) Fully expanded configuration; (b) blockage occurs during folding.



(a)

(b)



(c)

(d)

Fig. 4.5.4 Deployment of Linkage II.

(a) Fully expanded; (b) – (c) intermediate; (d) fully folded configurations.

For Linkage I, the folding process can be traced along the compatibility path from D to F via B, see Fig. 4.5.5. However, the movement of linkage is found to be physically blocked at point B because the ends of bars of hit each other, see Fig. 4.5.3(b). This model is made from solid steel bars and brass hinges, which are fairly rigid and allow almost no deformation. Hence, the deployment terminates at B.

Pellegrino (2002) also observed that the linkage did fold up if it is made from less rigid material such as card or weak hinges. While folding, a force has to be applied to linkage to enable joints deform slightly. A model shown in Fig. 4.5.6, which is made from card, demonstrates this process. After moving from the configuration shown in Fig. 4.5.6(a) to that in Fig. 4.5.6(b) during folding, a force is applied to make the model folded to that in Fig. 4.5.6(c) and then to that in Fig. 4.5.6(d).

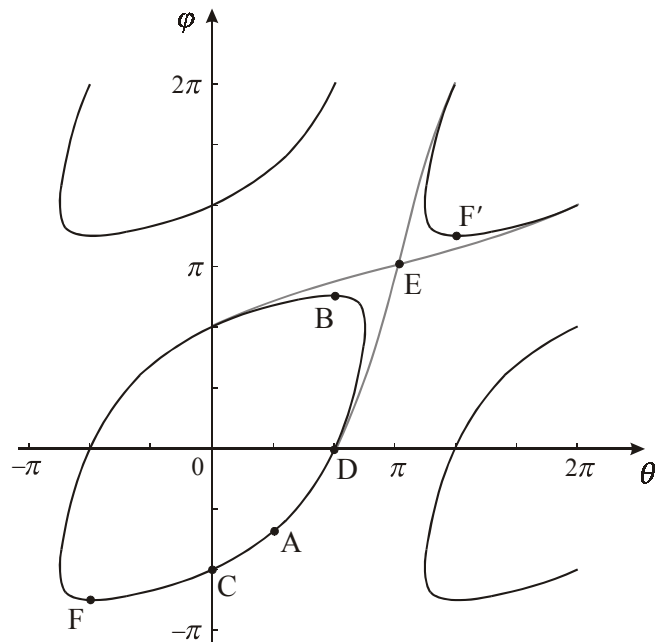


Fig. 4.5.5 The compatibility path of the 6R linkage with twist $\alpha = \pi - \arctan 2$.

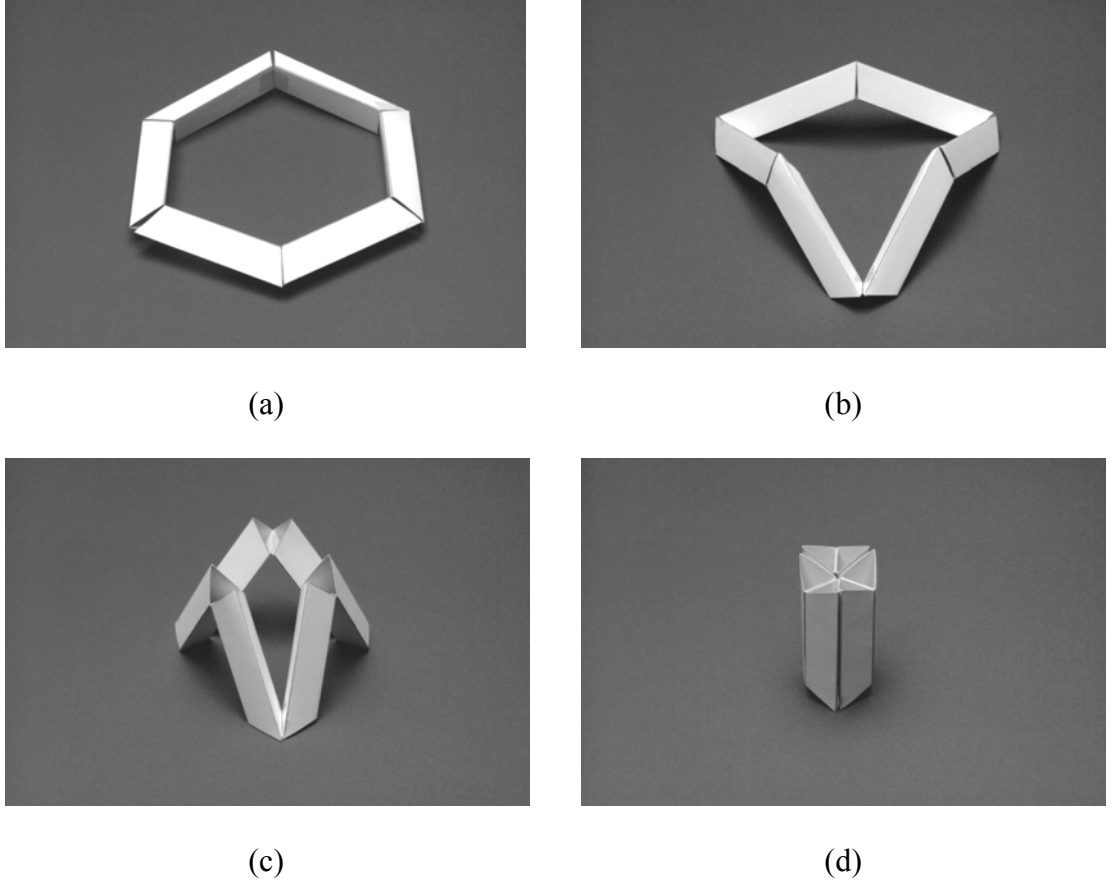


Fig. 4.5.6 Card model of Linkage I.

(a) At D, (b) B, (c) E and (d) F' of the compatibility path.

A close examination of the compatibility path reveals that, in the card model, the folding process corresponds to a movement from D to F', instead of F, due to the fact that compatibility path is periodic and $\theta_{F'} = 2\pi + \theta_F$, $\varphi_{F'} = 2\pi + \varphi_F$. The reason why this happens is because $\alpha = \pi - \arctan 2$, i.e. 116.57° , is sufficient close to $2\pi/3$, i.e. 120° . As we discussed in the previous section, $\theta = \varphi = \pi$ is a point of bifurcation for the 6R linkage with twist $2\pi/3$. Hence, when the force is applied to the linkage with $\alpha = \pi - \arctan 2$, an imperfection is introduced to the twist of the linkage which alters to $2\pi/3$. The folding process reaches bifurcation point E. When the force is released, the

twist of linkage changes back to $\alpha = \pi - \arctan 2$. Accordingly, (θ, φ) reaches point F'.

Linkage II behaves differently. The folding process takes route from D to F via A and C. There is no blockage during deployment and the structure can be folded up completely, as shown in Fig. 4.5.4.

4.6 CONCLUSION AND DISCUSSION

This chapter has presented a method to construct a large structural mechanism using the hybrid Bricard linkages as basic elements. It has the same features as the structural mechanism based on the Bennett linkage, e.g., it has a single degree of mobility, consists of simple revolute joints and possesses overconstrained geometry.

The chapter started with a derivation of the hybrid Bricard linkage from the general plane-symmetric and trihedral Bricard linkages. This hybrid linkage is overconstrained with a single degree of mobility. Its compatibility condition was presented.

The second part of this chapter was devoted to the study of the deployment characteristics of the hybrid Bricard linkage. We found all hybrid Bricard linkages can expand to a planar equilateral triangle, and in particular, when the twists are $\pi/3$ or $2\pi/3$, the hybrid Bricard linkages can be folded completely. This feature allows the construction of networks of hybrid Bricard linkages.

As a result, we proposed a scheme to build large deployable networks using the hybrid Bricard linkages. It has been proven to be successful.

It has also been noticed that branches exist at certain points along the compatibility paths. We have proved the existence of bifurcation points by identifying a change in the number of states of self-stresses in the linkage. We have also demonstrated that the bifurcation itself does not cause any problem in deployment.

Finally, alternative forms of hybrid Bricard linkage have been discussed with examples. Their deployed configuration is a hexagon and their folded configuration is a bundle. We have shown that the identical alternative form could lead to complete deployment behaviour: one with deployment blockage while the other without. Although the current method is not able to model deployment blockage of the structure, we have shown how a blocked configuration can be re-configured to achieve full deployment.

The work presented has shown that the approach is valid. It also points us to the fact that more work should be done to set up the relationship between the original hybrid Bricard linkage and its alternative forms, and to find a way to model deployment blockage. Networks of the hybrid Bricard linkage in the alternative forms could be developed as we did for the Bennett linkage.

Tilings for Construction of Structural Mechanisms

5.1 INTRODUCTION

In the previous two chapters, we presented the construction of networks of Bennett linkages and hybrid Bricard linkages. We did not, however, give any reason why a particular network pattern was chosen. Neither did we show the existence of other network patterns. In this chapter, we will explain how we arrived at those particular network patterns.

In general, a network could consist of an unlimited number of basic linkages. For practical reasons, it is sensible to consider that the number of the types of basic linkages is small and these linkages are regularly repeated in the network. Due to this simplification, the mathematic tiling technique can be adopted to find the possible arrangements of networks.

As reviewed in Section 2.3, the total number of distinct k -uniform tilings is 135. These tilings can be modified into many more tilings and patterns with methods such as

transformation of symmetry, transitivity and regularity, tilings that are not edge-to-edge, and patterns with overlap motifs, etc. In spite of their large number, all of the tilings can be considered as a tessellation unit in one of these three (3^6) , (4^4) and (6^3) tilings involving a single type of regular polygon. If, in the construction of structural mechanisms, we take the basic linkage or assembly of a set of linkages as a basic tessellation unit, the tiling technique can then be used to guide us towards suitable arrangements in which units are connected repeatedly following the three tilings mentioned above. Hence, the key questions are, first of all, how a tessellation unit in the form of a triangle, square or hexagon can be built and secondly whether the compatibility requirements can be met in a particular assembly.

The layout of the chapter is as follows. Sections 5.2 and 5.3 present tiling and its application to the networks of Bennett linkages and hybrid Bricard linkages, respectively. Some conclusions and discussion follow in Section 5.4.

5.2 NETWORKS OF BENNETT LINKAGES

The Bennett linkage consists of four bars and therefore tilings of squares can be used as possible arrangements of a network of Bennett linkages.

5.2.1 Case A

Figure 5.2.1(a) is a schematic diagram of a Bennett linkage, in which four links are

connected at A, B, C and D. (4^4) tiling presents the simplest way to form a network in which another similar Bennett linkage is connected on the right of CD, see Fig. 5.2.1(b). The process can be repeated for other sides.

Now let us examine the two Bennett linkages shown in Fig. 5.2.1(b). The structure has more than one degree of mobility because Bennett linkages ABCD and DCFE can move independently. Goldberg (1943) suggested that links BC and CF could be welded together, resulting in a Goldberg $5R$ linkage. However, the same process cannot be repeated for other adjacent Bennett linkages unlimitedly. Hence, this arrangement does not work.

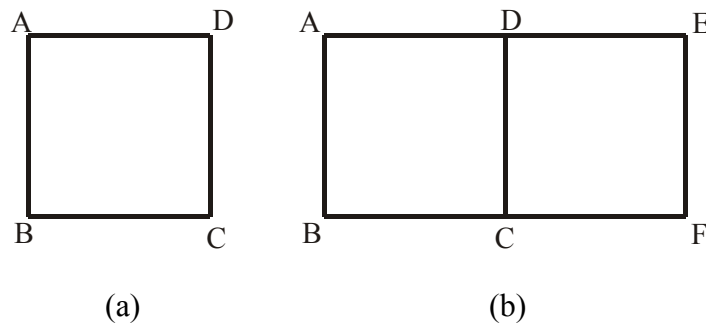


Fig. 5.2.1 (a) A Bennett linkage; (b) two Bennett linkages connected.

5.2.2 Case B

Let us now modify the unit to that shown in Fig. 5.2.2(a), in which the links are extended while the revolute joints remain at A, B, C and D. A network can then be formed, see Fig. 5.2.2(b). Next, we will examine whether such an arrangement will produce a structural mechanism.

Take

$$\begin{aligned}
 a_{AB} &= a_{CD} = a, \quad a_{BC} = a_{DA} = b \\
 \alpha_{AB} &= \alpha_{CD} = \alpha, \quad \alpha_{BC} = \alpha_{DA} = \beta \\
 a_{AE} &= a_{CG} = a_{BI} = a_{DK} = k_s a, \quad a_{AL} = a_{CJ} = a_{BF} = a_{DH} = k_s b \\
 \alpha_{AE} &= \alpha_{CG} = \alpha_{BI} = \alpha_{DK} = \alpha, \quad \alpha_{AL} = \alpha_{CJ} = \alpha_{BF} = \alpha_{DH} = \beta
 \end{aligned} \tag{5.2.1}$$

where $0 < k_s < 1$.

Hence, all of the Bennett linkages in Fig. 5.2.2(b) satisfy the geometric condition for Bennett linkages.

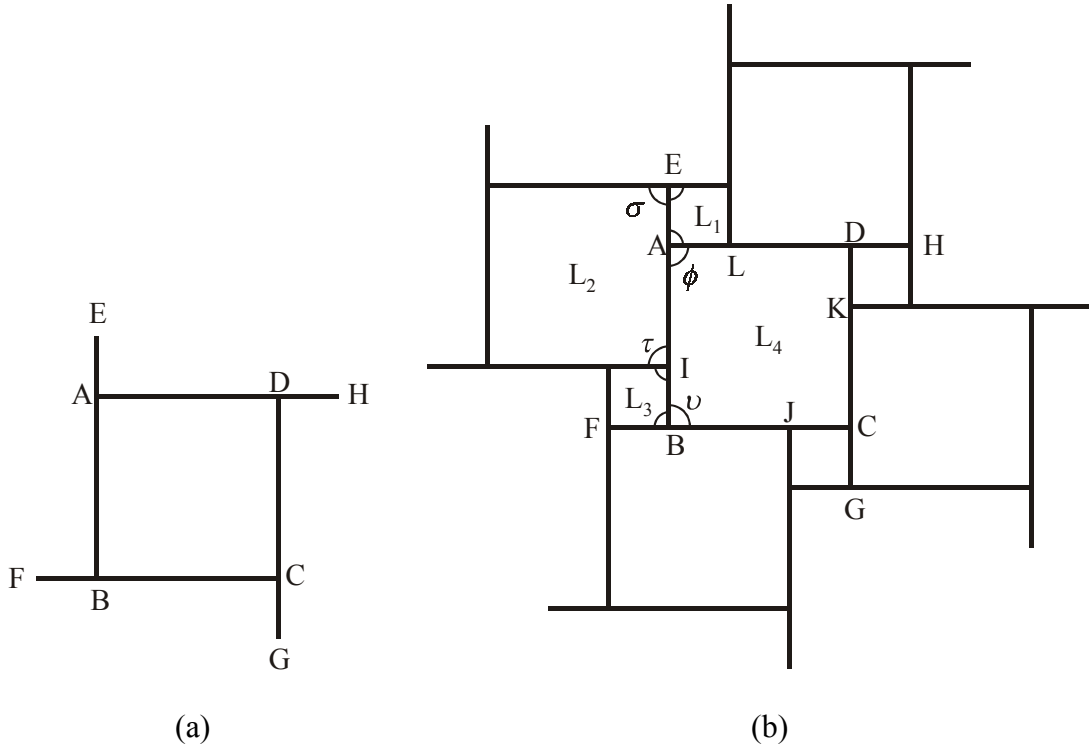


Fig. 5.2.2 (a) A unit based on the Bennett linkage; (b) network of units.

Considering Bennett linkages L_1 , L_2 , L_3 and L_4 , we have

$$\tan \frac{\sigma}{2} \tan \frac{\phi}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \quad (5.2.2a)$$

$$\tan \frac{\pi - \sigma}{2} \tan \frac{\pi - \tau}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \quad (5.2.2b)$$

$$\tan \frac{\tau}{2} \tan \frac{\nu}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \quad (5.2.2c)$$

$$\tan \frac{\pi - \phi}{2} \tan \frac{\pi - \nu}{2} = \frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \quad (5.2.2d)$$

Combining (5.2.2a) and (5.2.2c), as well as (5.2.2b) and (5.2.2d) gives

$$\tan \frac{\phi}{2} \tan \frac{\sigma}{2} \tan \frac{\tau}{2} \tan \frac{\nu}{2} = \left(\frac{\sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta - \alpha)} \right)^2 \quad (5.2.3)$$

$$\tan \frac{\phi}{2} \tan \frac{\sigma}{2} \tan \frac{\tau}{2} \tan \frac{\nu}{2} = \left(\frac{\sin \frac{1}{2}(\beta - \alpha)}{\sin \frac{1}{2}(\beta + \alpha)} \right)^2$$

From (5.2.3), we obtain that

$$\frac{\sin \frac{1}{2}(\beta - \alpha)}{\sin \frac{1}{2}(\beta + \alpha)} = \pm 1 \quad (5.2.4)$$

There are four possibilities for any $0 \leq \alpha, \beta \leq 2\pi$.

$$\begin{aligned}
 \beta - \alpha &= \beta + \alpha \\
 \beta - \alpha &= -(\beta + \alpha) \\
 \beta + \alpha &= (\beta - \alpha) + 2\pi \\
 \beta + \alpha &= -(\beta - \alpha) + 2\pi
 \end{aligned}
 \tag{5.2.5}$$

leading to

$$\alpha = 0, \text{ or } \alpha = \pi, \text{ and } \beta = 0, \text{ or } \beta = \pi
 \tag{5.2.6}$$

It can be concluded from (5.2.6) that only when α and β are 0 or π , can a mobile network based on the tiling shown in Fig. 5.2.2(b) be constructed. In each solution, the Bennett linkage is a 2D four-bar chain. For Bennett linkages with any other twists, it is impossible to construct a mobile network with the arrangement as shown in Fig. 5.2.2(b).

5.2.3 Case C

Now let us modify the unit further to that shown in Fig. 5.2.3, in which all of the links are extended at both ends.

Applying this unit to the (4^4) tiling produces the network shown in Fig. 3.2.4. From the discussion in Section 3.2, we know that the network is mobile with a single degree of mobility.

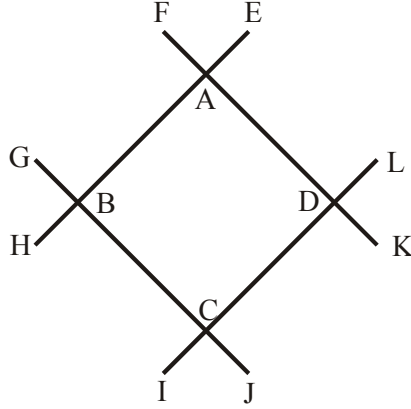


Fig. 5.2.3 A unit based on the Bennett linkage.

5.3 NETWORKS OF HYBRID BRICARD LINKAGES

5.3.1 Case A

The hybrid Bricard linkage has six equal links. Naturally, we think of (6^3) tiling to form a network.

Consider a portion of the network shown in Fig. 5.3.1. Loops L_1 , L_2 , and L_3 are all made of identical hybrid Bricard linkages with twists α and $2\pi - \alpha$, alternating from one link to the other. Take

$$\alpha_{AC} = \alpha \quad (5.3.1)$$

Then for loop L_2 ,

$$\alpha_{BA} = 2\pi - \alpha \quad (5.3.2)$$

Considering loop L_3 gives

$$\alpha_{AD} = 2\pi - \alpha \quad (5.3.3)$$

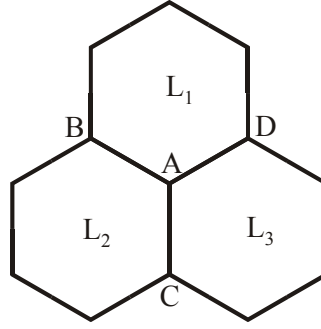


Fig. 5.3.1 A network of hybrid Bricard linkage.

Now a problem arises, as for loop L_1 ,

$$\alpha_{BA} = \alpha_{AD} = 2\pi - \alpha \quad (5.3.4)$$

which means that loop L_1 violates the condition of the hybrid Bricard linkage. Thus, the network consisting of more than one such deployable element will become immobile.

5.3.2 Case B

In Section 4.2, we showed that the hybrid Bricard linkage deploys a triangular shape shown in Fig. 5.3.2(a). Thus, it may be possible to use (3^6) tiling to construct a network.

A typical connection of hybrid Bricard linkages in the (3^6) tiling is as shown in Fig. 5.3.2(b), in which the central linkage shares two common links with each of the adjacent linkages. The projection of its probable deployment sequence is shown in Figs. 5.3.2(c) and 5.3.2(d). It becomes obvious that the central linkage behaves completely differently from those around it. Hence, such an arrangement cannot be used to construct a mobile network.

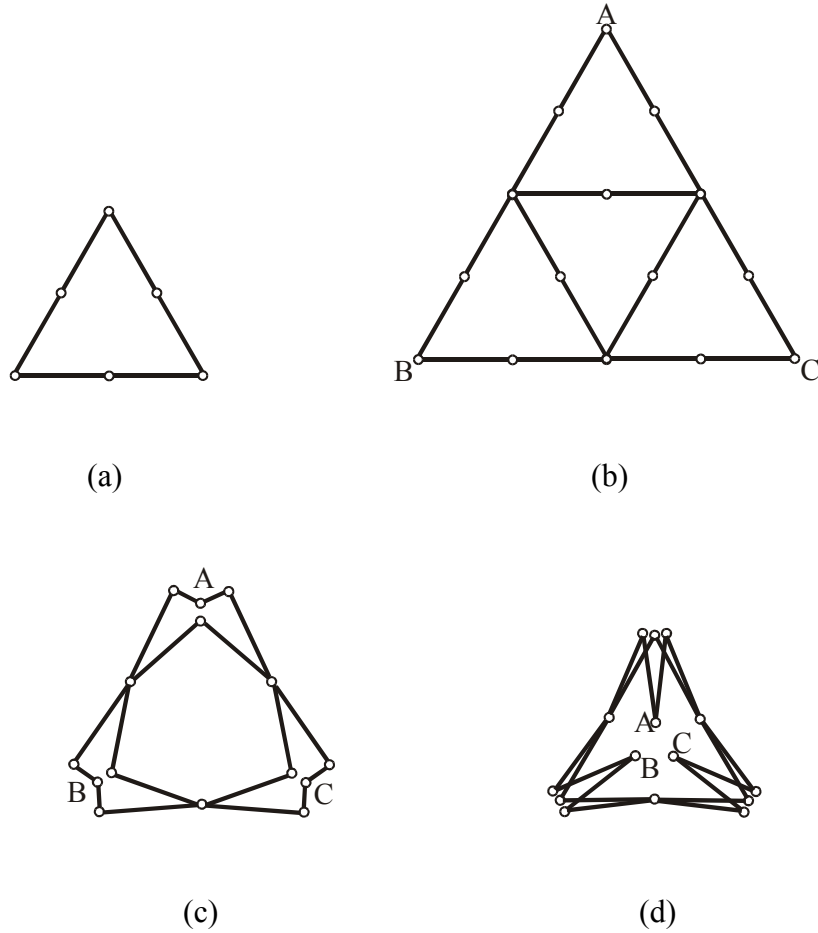


Fig. 5.3.2 (a) A hybrid Bricard linkage with twists of $\pi/3$ or $2\pi/3$;
 (b) possible connections; (c) and (d) projection of probable deployment sequence.

5.3.3 Case C

The unit can be modified further to that shown in Fig. 5.3.3. The unit can be tessellated by (3^6) tiling to make a network, which has been the subject of discussion in Section 4.3. The projection of the network during deployment is given in Fig. 5.3.4. It is shown that every hybrid Bricard linkage behaves in the same fashion all the time during the deployment process, which guarantees the compatibility of the network.

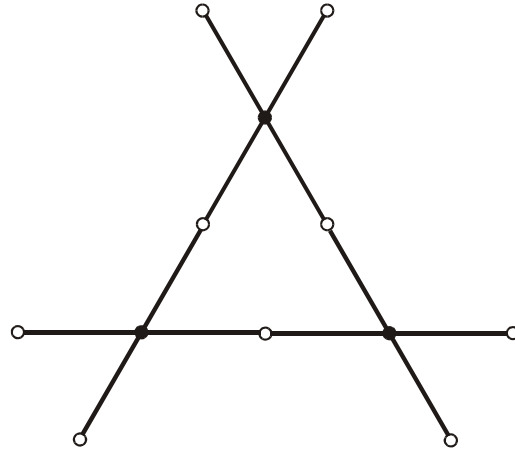


Fig. 5.3.3 A unit based on the hybrid Bricard linkage.

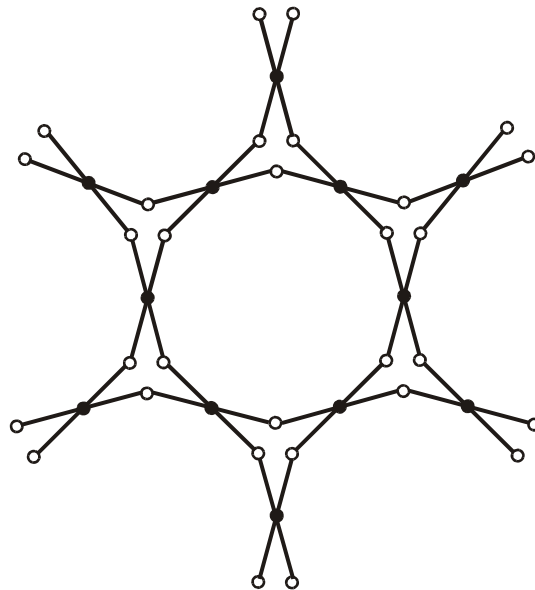


Fig. 5.3.4 Projection of a network of hybrid Bricard linkages during deployment.

5.4 CONCLUSION AND DISCUSSION

This chapter has presented a method to build structural mechanisms using deployable unit and tilings and patterns. It involves three steps: selection of suitable tilings, construction of units using overconstrained linkage and validation of compatibility.

This method has been applied to the networks consisting of Bennett linkages and hybrid Bricard linkages, respectively. (3^6) , (4^4) and (6^3) tilings have been adopted and several units have been discussed. The structural mechanisms based on Bennett linkages and hybrid Bricard linkages that described in previous two chapters were reconstructed by this tessellation method.

It should be pointed out that there are many ways of constructing deployable units including the units made from a combination of more than one type of 3D overconstrained linkages. Many examples concerning tilings and patterns can be found from references such as Grünbaum and Shephard (1986). We have not attempted all the possibilities. Instead a few simple examples have been used to illustrate that the approach is valid.

6

Final Remarks

The aim of this dissertation was to explore ways of constructing structural mechanisms using the existing 3D overconstrained linkages consisting of only hinged joints. In this chapter, we summarise the main achievements in the design of 3D structural mechanisms and highlight future work needed.

6.1 MAIN ACHIEVEMENTS

- Structural mechanisms based on Bennett linkages

The first effort of this dissertation was to construct large overconstrained structural mechanisms using Bennett linkages as basic elements.

We have found the basic layout of a structure allowing unlimited extension of the network. We were able to produce a family of networks of Bennett linkages to form single-layer structural mechanisms, which, in general, deploy into profiles of cylindrical surface. Meanwhile, we have discussed two special cases of the single-layer structures that can be extended to multi-layer structural mechanisms. In addition, according to our

derivation, we have found a solution to a problem which other researchers have been unable to solve, i.e., how to connect two similar Bennett linkages to form a mobile structure.

A mathematical proof of the existence of alternative forms of Bennett linkages has also been presented, which enables the linkage to be folded completely. Under certain circumstances, the said forms also allow the linkage to be flattened completely, giving maximum expansion. This work has resulted in the creation of the most effective deployable element based on Bennett linkage. A simple method to build the Bennett linkage in its alternative form has been introduced and verified. The corresponding networks have been obtained following the similar layouts of the original Bennett linkage.

All of the structural mechanisms contain only revolute joints, have a single degree of mobility and are geometrically overconstrained.

- Structural mechanisms based on hybrid Bricard linkages

A similar method has been developed to construct large overconstrained structural mechanisms using hybrid Bricard linkages as basic elements.

The hybrid Bricard linkage is a special case of the Bricard linkage. This linkage is overconstrained and with a single degree of mobility. We have derived its compatibility condition and studied its deployment behaviour. It has been found that for some

particular twists, the hybrid Bricard linkage can be folded completely into a bundle and deployed to a flat triangular profile. Based on this linkage, we have constructed a network of the hybrid Bricard linkages. Furthermore, we have studied the deployment characteristics including kinematic bifurcation and the alternative forms of the hybrid Bricard linkage.

The proposed structural mechanisms exhibit the same features as those based on the Bennett linkage.

- Tilings

We have applied the tilings and patterns to the construction of the networks based on the Bennett linkages and hybrid Bricard linkages. Several possible configurations have been discussed including those described previously.

A general approach to construct large structural mechanisms has been proposed, which can be divided into three steps: selection of suitable tilings, construction of overconstrained units and validation of compatibility.

6.2 FUTURE WORKS

The research reported in this dissertation opens up many opportunities for further study.

First of all, for assemblies of Bennett linkages our solution is largely based on the compatibility equations. There may be other solutions. Further study of the highly non-linear compatibility conditions is needed to either find or rule out other solutions.

Secondly, the work dealing with alternative forms of the hybrid Bricard linkage shows that the approach is valid. More work should be done to set up the general relationship between the original hybrid Bricard linkage and its alternative form. Networks of the hybrid Bricard linkage in the alternative forms could also be developed as we did for the Bennett linkage.

Thirdly, during the construction of networks of linkages, we have considered only one type of linkage at a time. With the application of tiling, units consisting of more than one type of linkage can be considered. For example, the deployable element based on the Bennett linkage and that based on the hybrid Bricard linkage maybe can be combined together following $(3^3.4^2)$ tiling.

Fourthly, the approach presented in this dissertation has only been applied to the Bennett and Bricard linkages. It can be extended to all the existing 3D overconstrained linkages, e.g. Myard linkage, Altmann linkage, Sarrus linkage, Waldron hybrid linkage, and Wohlhart linkage. More structural mechanisms could be found in this way.

Furthermore, in the mathematical derivation, the cross-sectional dimension of links has been ignored, so the current modelling is not capable of predicting conflict in the deployment process. Discussion on blockage of deployment has been mainly based on

experiments. Hence, a detailed modelling process, which is closer to the physical models, will be highly desirable.

Finally, although the approach presented in this dissertation has led to discovery of many new structures, it does not consider the deployed configuration in the design process. Instead, it is checked after a structural mechanism is obtained. In engineering practice, deployed configuration is the most important design consideration. How to integrate this consideration into the current approach will be a natural step forward to make this work more attractive to practical engineers.

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