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Opinion Formation in Dynamic Social Networks

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This is my own work (except where otherwise indicated).

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Abstract

Opinion dynamics in a society of interacting agents may lead to consensus or to the coexistence of different opinions. The interplay between social network change and opinion formation is complex, because the agents, their social interactions and the changing social structure over time, are themselves complex.

DeGroot proposed a prescriptive model for achieving consensus, where agents revise their opinions at each time step by taking a weighted average of the opinions of neighbours. This thesis contains three main contributions. First, we introduce a generalisation of the DeGroot model and examine the long-time behaviour of the model, with and without insistent agents. Second, we consider opinion formation on networks which are themselves dynamic, where the dynamics may be completely random or based on homophily and triadic closure. The weights that agents place on the opinions of neighbours are also dynamic, based on a rule where weights decrease with increased difference in opinions. Third, we examine the effect of a sudden, temporary or permanent shift in the opinions of some agents.

Two dynamics are considered for the network change over time; random switching (RS) network dynamics, and homophily and triadic closure (HT) network dynamics. We prove that the RS network dynamics enhances consensus formation and network connectivity, compared to the HT network dynamics where we show by simulation that different opinions can persist.

We investigate the influence of the presence of a minority of insistent agents and prove that for a connected static network, insistent agents with the same opinion influence the final opinions to converge to their own opinion, thus leading to consensus. In contrast, lack of consensus persists when insistent agents have different opinions. This conclusion also holds for the RS network dynamics model. However, for the HT network dynamics model, coexistence of different opinions can persist even when insistent agents have the same opinion. This finding regarding the HT dynamics is of particular interest as it relates to observations in the real-world.

We also investigate the influence of a sudden shift in the opinions of some agents on the outcome of final opinions. The case of either a temporary shift in opinions or a permanent shift in opinions is examined. Additionally, the influence of the time of the introduction of a shift, the number and the network positions of initial recipients of the shift in opinions is investigated. The overall effect of an opinion shift is measured by its influence on the stabilisation time of the final opinions.

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Chapter 1

Introduction

The dynamics of opinion formation, leading to agreement or disagreement, is a complex phenomenon, because the individuals, their social interactions and their evolving social structure are themselves complex (Castellano et al., 2009; Iñiguez et al., 2012). In recent years, much effort has been invested in uncovering the complexities that underpin social opinion formation. Mathematical models have been proposed of opinion dynamics, many of which are based on models from physics studied using computer simulations (Castellano et al., 2009; Sobkowicz, 2009a; Stauffer, 2009; Sîrbu et al., 2017).

To understand the complexities that underpin opinion formation, the developing field of complex networks has attracted the use of social network tools to model opinions. The social groups, opinions, and interactions are defined on networks by representing agents as nodes and the interactions between them as edges. The flow and exchange of information, social capital, culture, norms and values are mediated and channelled through the structure of the network which governs their interactions (White et al., 1976; Iñiguez et al., 2009). This network approach provides a simplified representation of the interconnectedness and the interaction patterns of a system which lays the foundations for understanding the structures and the corresponding behaviour of the system.

A number of opinion formation models that incorporate social network structures have been proposed (Krause, 2000; Hegselmann and Krause, 2002; DeMarzo et al., 2003; Jackson, 2010; Golub and Jackson, 2010; Pan, 2010; Friedkin and Johnsen, 1990, 1997, 2011; Mirtabatabaei and Bullo, 2012). In these models, however, the network structure is considered to be static. In forming opinions in these models, agents are assumed to place a weight on the opinions of others, which can be seen as a measure of trust on the opinions of others. These weights are considered to static. The social network and the weights being static imply that agents interact with the same agents and place the same weight on opinions at each time step, irrespective of changes in opinions.

Furthermore, models of opinion formation mostly focus on internal dynamics (for a survey see Castellano et al., 2009, chap. 3) where agents only continue to revise their existing opinions over time as they interact, without accounting for exposure to external influence that can cause a sudden shift in the opinions of interacting agents. Models that look at the influence of external shifts in opinions do not consider the effect of timing and the characteristics of the agents initially exposed to the external influence (Holyst et al., 2001; Kuperman and Zanette, 2002; Sobkowicz, 2009a; Iñiguez et al., 2012). It is mostly assumed in these models that all agents are exposed to external influence at all time steps. Hence, the influence of the time of the occurrence of a sudden shift in opinions and the characteristics of initial recipients is not accounted for (Holyst et al., 2001; Kuperman and Zanette, 2002; Sobkowicz, 2009a; Iñiguez et al., 2012).

Motivated by these considerations, this thesis seeks to investigate the complexities that underpin the interdependencies of the network evolution, the dynamics of opinion formation, and the effect of sudden shifts in opinions. In particular, how consensus or lack of consensus emerges is investigated, depending on the evolution of the network, the dynamics of opinion change and shifts in opinions. Given a shift in opinions, we are interested in investigating the influence of the time of the introduction of a shift, the influence of the number of initial recipients and the influence of the difference in population variance just before and just after the introduction of a sudden shift on the stabilisation time of final opinions.

1.1 Background

This section provides a brief overview of existing work in the area of opinion formation, network dynamics and the influence of an opinion shift. There are numerous publications in these areas, for surveys see Castellano et al. (2009), Sobkowicz (2009a) and Sîrbu et al. (2017). Here, we focus mainly on literature that contributes to the construction of this thesis.

1.1.1 Opinion Dynamics

A pioneering work on opinion formation is proposed by DeGroot (1974) which is a formalization of earlier work by French Jr (1956).

In the DeGroot model, a set of n agents start with initial opinions $\boldsymbol{p}(0) = (p_1(0), p_2(0), \ldots p_n(0))$, a vector in $\mathbb{R}^n$. Agents interact and assign weights on the opinions of others through an $n \times n$ stochastic matrix $\boldsymbol{T}$ with elements $\boldsymbol{T}_{ij} \in [0, 1]$ such that $\sum_{j=1}^{n} \boldsymbol{T}_{ij} = 1$. The weight $\boldsymbol{T}_{ij}$ is viewed as the level of trust that agent i places on the opinion of agent j.

Opinions are updated at discrete times $t = 1, 2, \ldots$ by taking a weighted average of the opinions of neighbours. The opinion updating process can be formalised as:

$$\boldsymbol{p}(t+1) = \boldsymbol{T}\boldsymbol{p}(t) = \boldsymbol{T}^t\boldsymbol{p}(0), \tag{1.1}$$

where $\boldsymbol{p}(t)$ is the opinion vector at time t and $\boldsymbol{T}^t$ is the t-th power of $\boldsymbol{T}$. DeGroot (1974) adopts the standard theory of Markov chains in examining convergence to consensus or the lack of consensus in opinions. Consensus is reached if all the recurrent states in $\boldsymbol{T}$ communicate with each other and are aperiodic. The model and definition of terms are discussed in detail in Section 2.1.

The DeGroot model has been extended by many researchers. The first group of extensions considers the propensity to change opinions. Notable examples include Friedkin and Johnsen (1990, 1997, 2011), DeMarzo et al. (2003) and Pan (2010). DeMarzo et al. (2003) considered a variation of the DeGroot model with empirical findings in a framework which is justified as a reflection of *persuasion bias*. Persuasion bias implies that individuals fail to fully or properly account for possible repetitions of information they receive. The model is given by

$$\boldsymbol{p}(t+1) = \big[(1-\lambda_t)\boldsymbol{I} + \lambda_t\boldsymbol{T}\big]\boldsymbol{p}(t),$$

where $\lambda_t \in (0,1]$ is a parameter that allows agents to vary the weight they give to their own current opinions relative to the opinions of neighbours. Friedkin and Johnsen (1990, 1997, 2011) considered an extension of the DeGroot model of the form

$$\boldsymbol{p}(t+1) = \boldsymbol{A}\boldsymbol{T}\boldsymbol{p}(t) + (\boldsymbol{I}-\boldsymbol{A})\boldsymbol{p}(0),$$

where $\boldsymbol{A}$ is an $n \times n$ matrix with positive elements $a_{ii}(0 \leq a_{ii} \leq 1)$ on the diagonal and zero elements elsewhere. The parameter a_{ii} is interpreted as the susceptibility of an agent to interpersonal influence, and it allows agents to mix their starting opinions with some of their neighbours' recent opinions. They found that the final opinions of agents can converge to consensus or the coexistence of different opinions depending on the specification of a_{ii}.

The second group of extensions of the DeGroot model considers the weight on differing opinions. In the so called *bounded confidence* models (Krause, 2000; Deffuant et al., 2000; Hegselmann and Krause, 2002; Weisbuch et al., 2002; Lorenz, 2005; Mirtabatabaei and Bullo, 2012), each agent i has a confidence or threshold limit $\epsilon_i > 0$ used in deciding who they interact with. The bounded confidence model is of the form

$$p_i(t+1) = |I_i(p_i(t))|^{-1} \sum_{j \in I_i(p_i(t))} p_j(t),$$

where $I_i(p_i) = \{1 \leq j \leq n, |p_i - p_j| \leq \epsilon_i\}$.

The weighted averaging is applied to only those agents whose opinions are within their confidence limit $|p - \epsilon, p + \epsilon|$. Reaching consensus or not depends on the specification of ϵ. Increasing values of ϵ imply that agents can easily communicate, leading to consensus formation, while decreasing values of ϵ can lead to the coexistence of different opinions (Deffuant et al., 2000; Hegselmann and Krause, 2002; Lorenz, 2005; Mirtabatabaei and Bullo, 2012). Krause (2000) and Hegselmann and Krause (2002) include other characteristics of convergence to consensus including asymmetric values of ϵ. Lorenz (2005) gives an analytical characterization of consensus for bounded confidence dynamics.

Regan et al. (2006) also considered a variation of the DeGroot with a dynamic weight formalised as:

$$\boldsymbol{T}_{ij}(\boldsymbol{p}(t)) = \frac{1 - |p_i(t) - p_j(t)|}{\sum\limits_{k=1}^{n} 1 - |p_i(t) - p_k(t)|}.$$

They found that the opinions converge to a consensus. The influence of network structures are not considered in the extensions in Regan et al. (2006).

In all these models, the network structure if considered is assumed to be static over time. Additionally, the influence weight matrix is considered to be constant over time (DeGroot, 1974; DeMarzo et al., 2003; Friedkin and Johnsen, 1990, 1997, 2011).

This thesis is mainly motived by the extensions in Pan (2010, 2012), that considered a variation of the DeGroot with a dynamic weight matrix $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{p}(t))$

with elements inversely proportional to the closeness in opinions. For a positive constant $d > 0$, the dynamics is defined by:

$$\boldsymbol{p}(t+1) = \boldsymbol{T}(\boldsymbol{p}(t))\boldsymbol{p}(t) \qquad \text{for } t \geq 0,$$

where

$$\boldsymbol{T}_{ij}(\boldsymbol{p}(t)) \propto \begin{cases} 1/|p_i(t) - p_j(t)| & \text{for } \; p_i(t) \neq p_j(t), \\ \\ 1/d & \text{for } \; p_i(t) = p_j(t), \;\; i \neq j. \end{cases} \tag{1.2}$$

This is a consensus model in which the opinions of all agents converge to a consensus. Pan also studies the effect of agents who insist on their own opinions on the outcome of final opinions, and finds that when insisting agents have the same opinion, the opinions of all agents converge to the opinion of the insisting agents. The extension in Pan (2012) accounts for the influence of static network structures on opinions.

It is worth noting here that the weight function considered in Pan's models is not continuous over the differences in opinions. Additionally, the network is assumed to be static and the weights do not account for dynamic changes in the network. The variation considered here differs from Pan (2010, 2012) and the above mentioned models in the sense that we propose a dynamic network model and a dynamic weight update rule which is continuous and which depends on changes in the current opinions as well as changes in the network.

1.1.2 Network Dynamics

Social networks may evolve by removing or adding edges as well as vertices. The altering of the network can happen at random or by interactions among the agents. The current work investigates two different mechanisms for network dynamics with respect to their effect on opinion formation in reaching a consensus or a lack of consensus.

When the dynamics of the network is random, agents break and/or add edges at random. In our models, it is assumed that interaction among agents is independent of the changes in the opinions of the agents. Agents will randomly break edges to interact with others even when there is agreement in opinions. Our first random network dynamics model *termed the RS model* is based on the switching algorithm of Gkantsidis et al. (2003). The RS algorithm selects two edges connecting four different agents at random, and the edges are switched if the switching keeps the network connected. The algorithm of Gkantsidis et al. (2003) naturally preserves the degree and connectivity of the network. We extend the algorithm to consider a more general case of not imposing network connectivity. We construct an analytical proof of the convergence to a consensus under these random network dynamics. The algorithm is discussed in detail in Section 3.1.

In the absence of randomness, triadic closure and homophily are two strong principles considered to influence the dynamics of social networks (Lazarsfeld and Merton, 1954; McPherson et al., 2001; Kossinets and Watts, 2006; Crimaldi et al., 2015). Triadic closure is the tendency that an edge will be established between two agents who share common neighbours (or friends). If agent i interacts with agent j and agent k, then triadic closure asserts that there is a high tendency that an edge will be established between agent k and agent j. Homophily on the other hand, is the tendency for individuals to associate with others similar to themselves. Compared to triadic closure, homophily depends on the features (here: opinions) of the agents, whereas triadic closure establishes an edge solely due to the existence of common neighbours (friends).

Studies from different fields of research support empirically the prevalence of homophily and triadic closure in friendship networks (Baron et al., 1996; Kossinets and Watts, 2006). Although both homophily and triadic closure are fundamental mechanisms of edge formation, especially in social networks, models on network dynamics often either focus on triadic closure (Bianconi et al., 2014) or homophily (Currarini and Vega-Redondo, 2013).

This thesis proposes a model that combines homophily and triadic closure in the dynamics of the network. The combined approach is termed *the HT network dynamics model*. It has long been believed that social interaction that is based on triadic closure and homophily enhances consensus. The principles of triadic closure and homophily are evident in statements such as *birds of a feather flock together* and *similarity breeds connection* (Lazarsfeld and Merton, 1954; McPherson et al., 2001).

It is argued here that although individuals become more similar in opinions as they interact with alike others leading to consensus, the opposite phenomenon can occur. Homophily coupled with triadic interaction can result in the coexistence of diverse opinions and network clusters. On the other hand, random switching (RS) interaction enhances consensus in opinions and network connectivity.

In particular, if the network is disconnected, then convergence to a consensus may not occur. The HT dynamics can result in weakly connected clusters and the opinions within the different clusters can be different. This can result in the coexistence of different opinions for the HT dynamics. The RS dynamics in contrast tends to more homogeneous networks and thus convergence to a consensus is easier to achieve. Hence the network dynamics can strongly influence the opinion dynamics.

1.1.3 Opinion Shift

Over time, agents might experience a sudden shift in their opinions, and this might be caused for example by the influence of external information from the media.

Information coming from the media could be influential in shaping people's minds, ideas, opinions and attitudes over time (Shibanai et al., 2001). On the other hand, earlier studies have concluded that information originating from the media comes to strengthen previous opinions and that interpersonal interactions tend to be more effective in changing opinions (Katz and Lazarsfeld, 1970). In any case, external sources such as the media are good sources of information for building up knowledge. The acquired knowledge shapes perceptions and could eventually change opinions.

In the context of opinion dynamics, the ultimate goal of external influence may be to shift the opinions of people in order that they might, for example, prefer a specific product over others or vote for particular politicians or policies (Petty et al., 2009). For example, Bond et al. (2012) uses a randomised controlled trial of political mobilization of messages to 61 million Facebook users during the 2010 US congressional elections. They observe that the messages increased the voting turnout by shifting the opinions of people who might not have voted to actually vote. A further increase in voting turnout is observed for messages that showed the faces of friends who have already voted.

Motivated by these studies, we investigate the effect of a sudden shift in opinions, the time of its introduction and the number of initial recipients on the stabilisation time of opinions.

Mathematically, there has not been much research on the influence of a sudden shift in opinions, the time of introduction, the number of initial recipients and the network positions of initial recipients on the outcome of final opinions. Particularly, the time of the introduction of a shift in opinions, the number of initial recipients and the network positions of initial recipients, as well as the interaction effect of time and number initial recipients of a shift in opinions are not explicitly studied. Models that examine the influence of media information for example are centred mostly on the impact of the strength of the information with the assumption that information globally influences the opinions of all agents in the population at all times (Holyst et al., 2001; Sobkowicz, 2009b; Iñiguez et al., 2012). The influence of the time and the number of initial exposures to the information are not considered as each agent is assumed to experience a shift in opinion.

At the occurrence of a sudden shift in opinions, agents may experience a permanent shift in opinions, or a temporary shift in opinions (Alpert and Anderson, 1973; Liu and Johnson, 2011). In this thesis, these two processes of the influence of a shift in opinions are modelled separately. It is assumed that not all agents experience a shift in opinions and also, a shift in opinions does not affect all agents at all time steps.

We investigate how the outcome of final opinions is influenced by (1) a shift in opinions, (2) the time at which the shift in opinion is introduced and (3) the number and the network positions of initial recipients of the shift in opinions and (4) the interaction effect between the time and the number of initial recipients of a shift in opinions. The thesis seeks to investigate these questions by looking at the influence of both a temporary and a permanent shift in opinions.

1.2 Research Questions and Objectives

The DeGroot model (DeGroot, 1974) is introduced as a prescriptive proposal for opinion formation in a homogeneous population. The prescriptive DeGroot model is introduced with the view to provide a mechanism by which a group of agents can achieve consensus, but we shall see that consensus is not always achieved when the assumptions of the model are relaxed.

The aim of this thesis is to construct a generalisation of the DeGroot model of opinion formation on dynamic social networks and investigate variants of the models which yield consensus, or which yield coexistence of different opinions. Three mechanisms are considered to influence consensus or the coexistence of different opinions in the DeGroot model. These are:

- The mechanisms of opinion dynamics via the introduction of dynamic weight updating and the influence of a minority of insistent agents.

- The mechanisms underpinning the dynamics of the social network.
- The mechanisms of external influence via the introduction of a shift in opinions.

Specific questions of interest include:

1. How does a dynamic social network influence consensus or the coexistence of different opinions?
2. How does the presence of a minority of insistent agents influence the outcome of the final opinions and the network?
3. How does a shift in opinions influence the final opinions and the network? Given the introduction of a shift in opinions:
   (a) How does the time of introduction of a shift in opinions influence the outcome of the final opinions?
   (b) How does the number of initial recipients of a shift in opinions influence the outcome of the final opinions?
   (c) How do the network positions of initial recipients influence the final opinions?
   (d) How does the time scale for the system to stabilise change? And how is the stabilisation time related to the difference in the population variance of opinions, or to the maximal difference in opinions just before and just after the introduction of a shift in opinions?
4. How does all this depend on the size of the network?

1.3 Organisation and Contribution

The contributions of each chapter, and the organisation of the thesis are as follows:

Chapter 2 - Modelling Opinion Formation on Static Networks:

This chapter focuses on the DeGroot model. We first introduce the model, establish conditions for convergence of opinions and investigate the value opinions converge to. We establish conditions for convergence of opinions for the model with the presence of a group of agents who insist on their own opinions. The DeGroot model is extended to establish the convergence properties for the case of dynamic weights on opinions.

As mentioned in Section 1.1.1, the DeGroot model and other models of opinion formation (Bala and Goyal, 1998; DeMarzo et al., 2003; Jackson, 2010; Golub and Jackson, 2010; Friedkin and Johnsen, 2011), assume that the weights are static. Dynamic weight updating has been considered in Hegselmann and Krause (2002) and in Weisbuch et al. (2002). However, agents in their models only place positive weights on those with opinions within a defined confidence interval, whereas agents here place positive weights on all other agents, with weights decreasing, as a function of the difference in opinions.

Regan et al. (2006) and Pan (2010) have also considered a dynamic weight updating model close to our proposed modelling framework. However in the framework presented here, agents attach weights directly to their own opinions whereas agents in Pan's model do not attach weights directly to their own opinions. Additionally, in our model, the proposed dynamics of the weights is based on the rule that the weight between two agents is a decreasing exponential function of the difference in opinions standardised by the sum of exponential differences in opinions, which is not the case in the model by Regan et al. or Pan.

The choice of the proposed weight function is motived by some conceptual notions grounded in psychophysical judgement theory (Davis, 1996). Davis (1996, pg. 44-45) and the references therein argue that the interpersonal weights of agents is a decreasing exponential function of the difference in opinions. One plausible argument for the decreasing exponential function is based on the assumption that as the difference in opinions (preferences) between two agents increases, a constant fraction of the weight is lost with each unit of distance. As a consequence, weight decays exponentially (Davis, 1996, pg.45).

Under dynamic weight updating, we prove that the opinions of agents converge to a consensus for a connected static network. We characterise the conditions for convergence in opinions and establish an upper bound on the maximum difference between the opinions of any two agents over time.

The case of inhomogeneous updating of opinions is analysed by introducing a group of agents who insist on their own opinions throughout the updating period (termed *insistent agents*). We establish the convergence characteristics of final opinions in the case of inhomogeneous updating of opinions.

For static weight on opinions, we prove that in the special case where insistent agents have the same opinion, the final opinions converge to the opinion of the insistent agents. However, when insistent agents have different opinions, lack of consensus emerges and we prove that the opinions of the non-insistent agents converge to a weighted average of the opinions of the insistent agents alone. Using simulations, we show that the results for the case of a static weight on opinions hold when the weight on opinions is dynamic.

Chapter 3 - Generalised Modelling on Dynamic Networks:

This chapter considers opinion formation with dynamic social network structure. Social networks evolve over time either by adding new edges or by deleting old edges (Wasserman and Faust, 1994; Doreian and Stokman, 1997; Kossinets and Watts, 2006). The addition or removal of edges can occur at random or by the deeds of agents.

A novel feature of this chapter is that two models of network dynamics are proposed and investigated; the random switching network dynamics model (the RS Model) and the homophily and triadic closure network dynamics model (the HT Model). The RS Model assumes that edges are created and broken at random, while the HT Model assumes that agents create or break edges by means of homophily and triadic closure. The influence of the RS and the HT network dynamics on the final opinions is investigated separately. For the RS dynamics, we prove analytical results, whereas the HT dynamics is investigated using simulations.

At each time step, the random switching algorithm selects two edges with distinct agents at random and switches the edges, if the network remains connected after edge switch. This algorithm is of interest here because it preserves the degree and connectivity of the network. We establish conditions for convergence in opinions and prove that for the random switching network dynamics (RS) model, the opinions of agents will converge to a consensus.

We also extend the RS model to consider a more general case where the network connectivity is not preserved. We prove that the final opinions will converge to a consensus even when clusters emerge in the network.

The homophily and triadic closure network dynamics algorithm creates new edges by means of triadic closure and removes edges by means of homophily. Here, the degree and connectivity of the network are not preserved. The network can segregate into clusters with single isolated vertices. Using simulations, we find that the opinions can converge to a state of lack of consensus, where different opinions coexist.

Comparing results form the RS dynamics to the HT dynamics, we illustrate that the dynamics of the network could be central in determining whether or not final opinions will converge to a consensus.

Chapter 4 - The Influence of a Shift in Opinions:

In this chapter, we study the influence of a sudden shift in the opinions of some of the agents, which might be caused by external sources of information. Models of opinion formation on networks mostly do not account for the influence of shifts in opinions which can influence the outcomes of final opinions (see survey reviews in Castellano et al., 2009; Sobkowicz, 2009a; Sîrbu et al., 2017).

The occurrence of a shift in a system of opinions and network dynamics could cause a temporal shift in the opinions of agents, or a permanent shift in their opinions. Here, we investigate the influence of a shift in opinions for the case of a temporal shift in opinions and a permanent shift in opinions.

A novel feature of the present work is that for both a temporal and a permanent shift in opinions, we investigate the influence of the time of the occurrence of a shift in opinions and the influence of the number and the network positions of initial recipients on the time to stabilisation of final opinions. Thus, we draw relationships between the time of the occurrence of a shift, the number of initial recipients and the stabilisation time of final opinions.

Additionally, we investigate how the stabilisation time of opinions is influenced by the difference in the population variance of opinions just before and just after the introduction of a shift in opinions, and by the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift in opinions.

Chapter 5 - Conclusion:

The conclusion chapter summarises the work, discusses the research findings and ideas that can be drawn from the study. Additionally, it highlights and describes some ideas and directions for future research which are outside the scope of the present work.

Chapter 2

Modelling Opinion Formation on Static Networks

2.1 The DeGroot Model

Let $N = \{1, 2, \ldots, n\}$ be a finite set of n interacting agents who share and exchange opinions. The opinions of the agents are represented by an $n \times 1$ vector $\boldsymbol{p}(t) = \{p_1(t), p_2(t), \ldots, p_n(t)\}$ where $p_i(t) \in \mathbb{R}$ is the opinion of agent i at time $t = \{0, 1, 2, \ldots\}$.

Let $\boldsymbol{T}$ be an $n \times n$ stochastic weight matrix with $\sum_{j=1}^{n} \boldsymbol{T}_{ij} = 1$ and $\boldsymbol{T}_{ij} \geq 0 \; \forall i, j$. We think of $\boldsymbol{T}$ as a social influence matrix.

The DeGroot dynamical model (DeGroot, 1974) is given by

$$p_i(t+1) = \sum_{k=1}^{n} \boldsymbol{T}_{ik} p_k(t), \quad \forall i \in N.$$

In matrix form, we have

$$\boldsymbol{p}(t+1) = \boldsymbol{T}\boldsymbol{p}(t) = \boldsymbol{T}^t \boldsymbol{p}(0), \tag{2.1}$$

where $\boldsymbol{p}(0)$ is a an $n \times 1$ vector of initial opinions.

The model assumes that agents revise their opinions homogeneously for the next time step by taking a weighted average of the current opinions of their neighbours.

2.1.1 Convergence in the DeGroot Model

The DeGroot model is conceived as a prescriptive model for achieving consensus. In this section we recall some definitions and give some results by DeGroot (1974) for convergence to consensus in opinions.

Definition 2.1 *A sequence of opinion vectors* $\boldsymbol{p}(t)|_{t=1}^{\infty} \in \mathbb{R}^n$ *is* convergent *if* $\lim_{t\to\infty} \boldsymbol{p}(t)$ *exists, in the sense that there exists* $\boldsymbol{p}^* \in \mathbb{R}^n$*, such that* $\forall \varepsilon > 0$*, there is a* $t^* > 0$ *such that* $||\boldsymbol{p}(t) - \boldsymbol{p}^*|| \leq \varepsilon \ \forall t > t^*$*, where* $||\boldsymbol{p}|| = \sqrt{\sum_{i=1}^{n} \boldsymbol{p}_i^2}$ *is the Euclidean norm for an* $n \times 1$ *vector* $\boldsymbol{p}$*.*

Definition 2.2 *The sequence of opinion vectors* $\boldsymbol{p}(t)|_{t=1}^{\infty} \in \mathbb{R}^n$ converges to a consensus *if and only if there exists* $p^* \in \mathbb{R}$ *such that* $\lim_{t\to\infty} p_i(t) = p^*$ *for all* $i \in N$*. Thus, the opinions of agents converge to a specific value* $p^* \in \mathbb{R}$ *common to all* *agents. In this case we say that consensus is reached.*

It is worth noting here that consensus may not be achieved in finite time. For example, if $p_1(t) = \dfrac{a}{t}$ and $p_2(t) = \dfrac{b}{t}$, where $a, b > 0$, then as $t \to \infty$, $p_1(t) \to 0$ and $p_2(t) \to 0$, but for all t, $p_1(t) \neq p_2(t)$.

Definition 2.3 *A sequence of matrices $\boldsymbol{T}(t)|_{t=1}^{\infty}$ is* convergent *if $\lim_{t\to\infty} \boldsymbol{T}(t)$ exists, in the sense that there exists $\boldsymbol{T}^* \in \mathbb{R}^{n\times n}$, such that $\forall \varepsilon > 0$, there is a $t^* > 0$ such that $||\boldsymbol{T}(t) - \boldsymbol{T}^*|| \leq \varepsilon \ \forall t > t^*$, where $||\boldsymbol{T}|| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(\boldsymbol{T}_{ij})^2}$ for a $n \times n$ matrix $\boldsymbol{T}$.*

Next, we recall some basic notions of Markov Chains; see Norris (1998).

Definition 2.4 *(Section 1.1 of Norris (1998)) Let I be a finite state space. Let $\boldsymbol{X}_0$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which takes value $i \in I$ with probability λ_i, so that $\sum_{i\in I} \lambda_i = 1$. Let $\boldsymbol{T} = (\boldsymbol{T}_{ij} : i, j \in I)$ be a matrix with elements $\boldsymbol{T}_{ij} \in [0, 1]$ such that for all $i \in I$, $\sum_{j\in I} \boldsymbol{T}_{ij} = 1$.*

We say that $(\boldsymbol{X}_n)_{n\geq 0}$ is a Markov chain *starting from $\boldsymbol{X}_0$ if*

(i) $\mathbb{P}(\boldsymbol{X}_0 = i) = \lambda_i$ *for* $i \in I$,

(ii) for $n \geq 0$, conditional on $\boldsymbol{X}_n = i$, $\boldsymbol{X}_{n+1}$ has a distribution $(\boldsymbol{T}_{ij} : j \in \boldsymbol{I})$ and is independent of $\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n-1}$, so that $\mathbb{P}(\boldsymbol{X}_{n+1} = i_{n+1}|\boldsymbol{X}_0 = i_0, \ldots, \boldsymbol{X}_n = i_n) = \boldsymbol{T}_{i_n i_{n+1}}$.

Definition 2.5 *A Markov chain is* aperiodic *if the chain starting from a state can return to the state only at integer multiples of 1.*

Definition 2.6 *State i* communicates *with state j ($i\leftrightarrow j$) if there exists $n, m \geq 0$*

such that

$$\mathbb{P}(\boldsymbol{X}_n = i|\boldsymbol{X}_0 = j) > 0 \quad and \quad \mathbb{P}(\boldsymbol{X}_n = j|\boldsymbol{X}_0 = i) > 0.$$

Definition 2.7 *A Markov chain $\{X_n\}$ with transition matrix $\boldsymbol{T}$ is* irreducible *if all states $i \in \boldsymbol{I}$ form only one communicating class, that is, $i \leftrightarrow j$ for all $i, j \in \boldsymbol{I}$.*

Equation (2.1) describes a Markov chain on $\mathbb{R}^n$ with transition matrix $\boldsymbol{T}$. The convergence of opinions, $\boldsymbol{p}(t)$ as $t \to \infty$, in the DeGroot model can be inferred from the convergence of $\boldsymbol{T}^t$ as $t \to \infty$, using the standard theorems of Markov chains, see (Seneta, 1981, chap. 4).

Theorem 2.1 *(Theorem 1 of DeGroot (1974)) If there exists a positive integer t such that every element in at least one column of the matrix $\boldsymbol{T}^t$ is positive, then a consensus is reached.*

Theorem 2.2 *(Theorem 2 of DeGroot (1974)) If the Markov chain $\boldsymbol{T}$ is irreducible and aperiodic, then a consensus is reached.*

It is worth noting here that reaching consensus is not necessarily dependent on $\boldsymbol{T}$ alone but can also depend on the initial opinions $\boldsymbol{p}(0)$. For example, if agents start with the same initial opinions, $p_1(0) = p_2(0) = \ldots = p_n(0)$, then irrespective of the entries of $\boldsymbol{T}$, agents will reach consensus, since $\boldsymbol{p}(t+1) = \boldsymbol{T}^t\boldsymbol{p}(0) = \boldsymbol{p}(0)$; see Berger (1981) for the dependence of consensus on $\boldsymbol{p}(0)$.

2.1.2 The Consensus Value

Lemma 2.1 *Let $\boldsymbol{T}$ be an irreducible and aperiodic row stochastic matrix and $\boldsymbol{s} = \{s_1, , \ldots, s_n\}$ be a positive eigenvector for $\boldsymbol{T}$, for the eigenvalue 1. Then* *the consensus value* $\boldsymbol{p}^* = \boldsymbol{p}^*_{|\boldsymbol{p}(0)}$ *satisfies:*

$$p^* = \frac{1}{\sum_{i=1}^{n} s_i} \sum_{j=1}^{n} s_j p_j(0).$$

Proof.

By Equation (2.1),

$$\boldsymbol{p}(t+1) = \boldsymbol{T}^t \boldsymbol{p}(0).$$

As $\boldsymbol{s}$ is a positive eigenvector for $\boldsymbol{T}$, for the eigenvalue 1, it holds that

$$\boldsymbol{s} = \boldsymbol{s}\boldsymbol{T} = \boldsymbol{s}\boldsymbol{T}^t \quad \text{for all } t,$$

and hence

$$\boldsymbol{s}\boldsymbol{p}(t+1) = \boldsymbol{s}\boldsymbol{T}^t \boldsymbol{p}(0),$$

letting $t \to \infty$, we have

$$\boldsymbol{s}\boldsymbol{p}^* = \boldsymbol{s}\boldsymbol{p}(0).$$

As $\boldsymbol{p}^* = p^*(1, \ldots, 1)$,

$$\begin{aligned} p^* &= \frac{1}{\sum_{i=1}^{n} s_i} \sum_{j=1}^{n} s_j p_j(0) \\ &= \sum_{j=1}^{n} c_j p_j(0), \end{aligned}$$

where $c_j = s_j / \sum_{i=1}^{n} s_i$. □

For each agent, c_k measures the relative weight of influence of agent k on the final opinion (Jackson, 2010).

2.1.3 The DeGroot Model with Insistent Agents

In this subsection, the DeGroot model and the analysis are generalised to include insistent agents; agents who insist on their own opinions over time. Insistent agents do not change their opinions, irrespective of the changes in the opinions of their neighbours. The introduction of insistent agents relaxes the assumption that all agents update their opinions by taking a weighted average of the opinions of their neighbours. The idea of introducing insistent agents is motived by Pan's work on the influence of a group of agents who perform limited interactions (Pan, 2010).

In the next two sections, we review some results from linear algebra, and analyse the DeGroot model with the presence of insistent agents.

2.1.3.1 Preliminary Definitions and Theorems

The following definitions and theorems about matrices are found, for example, in Varga (2000).

Definition 2.8 *Let* $\lambda_i,\ i = 1, \ldots, n$*, be the eigenvalues of an* $n \times n$ *matrix* $\boldsymbol{U}$*. Then the* spectral radius $\rho(U)$ *of the matrix* $\boldsymbol{U}$ *is defined as* $\rho(U) = \max\limits_{1 \leq i \leq n} |\lambda_i|$*, where* $|\lambda_i|$ *is the modulus of* λ_i*.*

Definition 2.9 *A* permutation matrix *is a square matrix in which every row and column contains only one entry of unity with all other entries being zero.*

Definition 2.10 *An $n \times n$ matrix $\boldsymbol{T}$ is* reducible *if there exists a permutation matrix $\boldsymbol{P}$ such that*

$$PTP^T = \begin{bmatrix} \boldsymbol{T}^{11} & \boldsymbol{T}^{12} \\ \boldsymbol{0} & \boldsymbol{T}^{22} \end{bmatrix} \tag{2.2}$$

where $\boldsymbol{P}^T$ is the transpose of the matrix $\boldsymbol{P}$, $\boldsymbol{0}$ is a zero matrix, $\boldsymbol{T}^{11}$ is a $k \times k$ matrix, $\boldsymbol{T}^{12}$ is a $k \times (n-k)$ submatrix and $\boldsymbol{T}^{22}$ is an $(n-k) \times (n-k)$ submatrix where $1 \leq k < n$. If no such permutation matrix exists, then the matrix is irreducible.

Lemma 2.2 *(Lemma 2.8 of Varga (2000)) If $\boldsymbol{U}$ is an $n \times n$ irreducible matrix, then either*

$$\sum_{j=1}^{n} \boldsymbol{U}_{ij} = \rho(U) \quad \forall i \in N,$$

or

$$\min_{1 \leq i \leq n} \left(\sum_{j=1}^{n} \boldsymbol{U}_{ij} \right) < \rho(U) < \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} \boldsymbol{U}_{ij} \right).$$

Definition 2.11 *Let $\boldsymbol{T}$ and $\boldsymbol{U}$ be two real $n \times m$ matrices. Then $\boldsymbol{T} \leq \boldsymbol{U}$ $(> \boldsymbol{U})$ if $\boldsymbol{T}_{ij} \leq \boldsymbol{U}_{ij}$ $(> \boldsymbol{U}_{ij})$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. If $\boldsymbol{0}$ is a zero matrix and $\boldsymbol{T} \geq \boldsymbol{0}$ $(> \boldsymbol{0})$, then we say that $\boldsymbol{T}$ is a* nonnegative (positive) matrix.

Definition 2.12 *Let $\boldsymbol{T}$ be an $n \times n$ matrix. Then $\boldsymbol{T}$* converges to zero *if and only if the sequence of matrices $\boldsymbol{T}, \boldsymbol{T}^2, \boldsymbol{T}^3, \ldots$ converges to the zero matrix $\boldsymbol{0}$, and $\boldsymbol{T}$ is said to be* divergent *otherwise.*

Theorem 2.3 *(Theorem 1.10 of Varga (2000)) If $\boldsymbol{T}$ is an $n \times n$ matrix, then $\boldsymbol{T}$ converges to zero if and only if $\rho(T) < 1$.*

Theorem 2.4 *(Theorem 3.15 of Varga (2000)) If $\boldsymbol{T}$ is an arbitrary $n \times n$ matrix* *with $\rho(T) < 1$, then $\boldsymbol{I} - \boldsymbol{T}$ is nonsingular and the series*

$$(\boldsymbol{I} - \boldsymbol{T})^{-1} = \boldsymbol{I} + \boldsymbol{T} + \boldsymbol{T}^2 + \dots,$$

converges. Conversely, if the series converges, then $\rho(T) < 1$.

Definition 2.13 *A* simple eigenvalue *is an eigenvalue that has a multiplicity of one as a root of the characteristic polynomial.*

Theorem 2.5 *(Perron-Frobenius) Let $\boldsymbol{T} \geq \boldsymbol{0}$ be an irreducible $n \times n$ matrix. Then*

1. *$\boldsymbol{T}$ has a positive real eigenvalue equal to its spectral radius.*
2. *To $\rho(T)$ there corresponds an eigenvector $\boldsymbol{x} > \boldsymbol{0}$.*
3. *$\rho(T)$ increases when an entry of $\boldsymbol{T}$ increases.*
4. *$\rho(T)$ is a simple eigenvalue of $\boldsymbol{T}$.*

(Theorem 2.7 of Varga (2000))

2.1.3.2 Analysis of the DeGroot Model with Insistent Agents

Let $Z = \{1, 2, \dots, z\}$ be the set of insistent agents and $\overline{Z} = N \setminus Z$ be the set of non-insistent agents.

Order the n agents so that the first $\overline{z}$ are non-insistent agents and the last $z = n - \overline{z}$ are insistent agents. Introduce two matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ where $\boldsymbol{Q}$ is a $\overline{z} \times \overline{z}$ matrix and $\boldsymbol{R}$ is a $\overline{z} \times z$ matrix such that $\boldsymbol{Q}_{ij} = \boldsymbol{T}_{ij}$ for $i, j = 1, \dots, \overline{z}$ and $\boldsymbol{R}_{ik} = \boldsymbol{T}_{i,\overline{z}+k}$ for $k = 1, \dots, z$. Let $\boldsymbol{p}_z(t)$ be the opinion vector of insistent agents and $\boldsymbol{p}_{\overline{z}}(t)$ be the opinion vector of non-insistent agents at time t. Then

$$\boldsymbol{p}(t) = (\boldsymbol{p}_{\overline{z}}(t), \boldsymbol{p}_z(t)),$$

and the matrix $\boldsymbol{T}$ has a block structure of the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \tag{2.3}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{0}$ is a $z \times \overline{z}$ zero matrix.

Therefore, in the presence of insistent agents, the dynamics can be summarised as

$$\begin{aligned} \boldsymbol{p}_{\overline{z}}(t+1) &= \boldsymbol{Q}\boldsymbol{p}_{\overline{z}}(t) + \boldsymbol{R}\boldsymbol{p}_z(0) \\ \boldsymbol{p}_z(t+1) &= \ \boldsymbol{p}_z(0) \end{aligned} \tag{2.4}$$

where $\boldsymbol{p}_z(t) = \boldsymbol{p}_z(t-1) = \boldsymbol{p}_z(0)\ \forall t, z \in Z$.

Assume that there is at least one insistent agent who interacts with a non-insistent agent. Then, $\sum_{j=1}^{\overline{z}} \boldsymbol{Q}_{ij} \leq 1$ for non-insistent agents who are not connected to insistent agents, and $\sum_{j=1}^{\overline{z}} \boldsymbol{Q}_{ij} < 1$ (strictly less than one) for non-insistent agents who are connected to insistent agents. Therefore

$$\min_{1 \leq i \leq \overline{z}} \left(\sum_{j=1}^{\overline{z}} \boldsymbol{Q}_{ij} \right) < 1,$$

and

$$\max_{1 \leq i \leq \overline{z}} \left(\sum_{j=1}^{\overline{z}} \boldsymbol{Q}_{ij} \right) \leq 1.$$

Assume that the matrix $\boldsymbol{Q}$ is irreducible, then by Lemma 2.2 the spectral radius

$$\rho(\boldsymbol{Q}) < \max_{1 \leq i \leq \overline{z}} \left(\sum_{j=1}^{\overline{z}} \boldsymbol{Q}_{ij} \right),$$

which implies that

$$\rho(\boldsymbol{Q}) < 1. \tag{2.5}$$

Thus, the spectral radius is strictly less than one when at least one non-insistent agent is connected to an insistent agent and $\boldsymbol{Q}$ is irreducible.

Theorem 2.6 *Let the initial opinion vector of non-insistent agents be* $\boldsymbol{p}_{\overline{z}}(0)$ *and the opinion vector of insistent agents be* $\boldsymbol{p}_z(0)$*. Suppose that at each time* t*, the vector of opinions of non-insistent agents evolves according to* (2.4)*. Then at time* t *the opinion vector of non-insistent agents is*

$$\boldsymbol{p}_{\overline{z}}(t) = \boldsymbol{Q}^t \boldsymbol{p}_{\overline{z}}(0) + \sum_{\tau=0}^{t-1} \boldsymbol{Q}^{\tau} \boldsymbol{R} \boldsymbol{p}_z(0), \tag{2.6}$$

where $\boldsymbol{Q}^t$ *is the* t*-th power of* $\boldsymbol{Q}$*.*

Proof.

The proof is by mathematical induction. By substituting $t = 1$ into Equation (2.6), it follows that

$$\boldsymbol{p}_{\overline{z}}(1) = \boldsymbol{Q} \boldsymbol{p}_{\overline{z}}(0) + \boldsymbol{R} \boldsymbol{p}_z(0),$$

which is the first step in the DeGroot model, given by (2.4). If we know the equation is true for $t-1$, given by

$$\boldsymbol{p}_{\bar{z}}(t-1) = \boldsymbol{Q}^{t-1}\boldsymbol{p}_{\bar{z}}(0) + \sum_{\tau=0}^{t-2} \boldsymbol{Q}^{\tau}\boldsymbol{R}\boldsymbol{p}_{z}(0),$$

then it follows that

$$\begin{aligned}
\boldsymbol{p}_{\bar{z}}(t) &= \boldsymbol{Q}\boldsymbol{p}_{\bar{z}}(t-1) + \boldsymbol{R}\boldsymbol{p}_{z}(0), \\
&= \boldsymbol{Q}^{t}\boldsymbol{p}_{\bar{z}}(0) + \sum_{\tau=1}^{t-1} \boldsymbol{Q}^{\tau}\boldsymbol{R}\boldsymbol{p}_{z}(0) + \boldsymbol{R}\boldsymbol{p}_{z}(0), \\
&= \boldsymbol{Q}^{t}\boldsymbol{p}_{\bar{z}}(0) + \sum_{\tau=0}^{t-1} \boldsymbol{Q}^{\tau}\boldsymbol{R}\boldsymbol{p}_{z}(0), \qquad (2.7)
\end{aligned}$$

as required. □

The next lemma gives conditions to ensure that the limiting opinions of non-insistent agents is a weighted average of the initial opinion vector of only the insistent agents.

Lemma 2.3 *Assume that $\boldsymbol{Q}$ is irreducible and that there is at least one insistent agent in the network and at least one insistent agent interacts with a non-insistent agent, so that the dynamics of opinion change is given by Equation* (2.4)*. Then*

$$\lim_{t\to\infty} \boldsymbol{p}_{\bar{z}}(t) = \left[(\boldsymbol{I}-\boldsymbol{Q})^{-1}\boldsymbol{R}\right]\boldsymbol{p}_{z}(0).$$

Proof.

We can rewrite Equation (2.6) as:

$$\boldsymbol{p}_{\bar{z}}(t) = \boldsymbol{Q}^{t}\boldsymbol{p}_{\bar{z}}(0) + (\boldsymbol{I}-\boldsymbol{Q})^{-1}(\boldsymbol{I}-\boldsymbol{Q}^{t})\boldsymbol{R}\boldsymbol{p}_{z}(0). \qquad (2.8)$$

To see this, let $\boldsymbol{W}^t = \sum_{\tau=0}^{t-1} \boldsymbol{Q}^\tau$, then it follows that

$$\begin{aligned}
\boldsymbol{Q}\boldsymbol{W}^t &= \sum_{\tau=1}^{t} \boldsymbol{Q}^\tau, \\
&= \boldsymbol{Q}^t + \sum_{\tau=0}^{t-1} \boldsymbol{Q}^\tau - \boldsymbol{I}, \\
&= \boldsymbol{Q}^t + \boldsymbol{W}^t - \boldsymbol{I}. \qquad (2.9)
\end{aligned}$$

By rearranging Equation (2.9), one finds that $(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{W}^t = \boldsymbol{I}-\boldsymbol{Q}^t$. The inverse of $\boldsymbol{I}-\boldsymbol{Q}$ exists by Theorem 2.4. Therefore $\boldsymbol{W}^t = (\boldsymbol{I}-\boldsymbol{Q})^{-1}(\boldsymbol{I}-\boldsymbol{Q}^t)$, and Equation (2.8) follows by substituting $\boldsymbol{W}^t$ into Equation (2.6).

Now, taking the limit of Equation (2.8) as $t \to \infty$, we have:

$$\lim_{t\to\infty} \boldsymbol{p}_{\bar{z}}(t) = \left[(\boldsymbol{I}-\boldsymbol{Q})^{-1}\boldsymbol{R}\right] \boldsymbol{p}_z(0), \qquad (2.10)$$

since by Equation (2.5) and Theorem 2.3, it follows that $\lim_{t\to\infty} \boldsymbol{Q}^t = \boldsymbol{0}$. $\square$

Equation (2.10) implies that in the presence of insistent agents in the model, the limiting opinions of non-insistent agents are weighted dependent only upon the opinion vector of the insistent agents.

Corollary 2.1 *Assume that there is only one insistent agent in the model given by Equation* (2.4)*, or that all insistent agents have the same initial opinion. Then the limiting opinions of the non-insistent agents will converge to the opinion of the insistent agents given a connected network.*

Proof.

We use the notation from the proof of Lemma 2.3. From Equation (2.10), let $\boldsymbol{U} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R}$, then $\boldsymbol{U}$ is a $\overline{z} \times z$ matrix and $\boldsymbol{U}$ is row stochastic so that

$$\sum_{j=1}^{z} \boldsymbol{U}_{ij} = \sum_{j=1}^{z} \left[(\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R}\right]_{ij} = 1 \quad \forall i \in \overline{Z}.$$

Hence, in the special case that all insistent agents have the same initial opinion $p_j(0) = \overline{p}_z \ \forall j \in Z$, or if there is only one insistent agent, then the solution to Equation (2.10) is, for each $i \in \overline{Z}$:

$$p_i(\infty) = \overline{p}_z, \quad \forall i \notin Z.$$

□

Thus, the opinions of all the non-insistent agents converge to the opinion of the insistent agents if there is only one insistent agent in the model, or if all the insistent agents have the same initial opinion. In general, the opinions of the non-insistent agents will converge to a weighted average of the opinions of the insistent agents, if insistent agents have different opinions as indicated in Lemma 2.3.

Note that the weight matrix $\boldsymbol{T} = \{\boldsymbol{T}_{ij}\}, \ i, j \in N$, could be seen as a weighted network with $\boldsymbol{T}_{ij}$ being the weight of edge (i, j). In the next section, we disentangle the weights from the network structure, as a foundation for allowing the network structure themselves to change.

2.2 Introduction of Dynamic Weight Updates

We introduce an underlying network structure by an $n \times n$ adjacency matrix $\boldsymbol{G} = \{\mathscr{G}_{ij}\},\ i, j \in N$, via which agents interact, such that, $\mathscr{G}_{ij} = 1$ if i and j are neighbours ($i \sim j$) and $\mathscr{G}_{ij} = 0$ if i and j are not neighbours ($i \nsim j$).

The weight matrix $\boldsymbol{T}$ is such that $\boldsymbol{T}_{ij} > 0$ if $\mathscr{G}_{ij} = 1$ and $\boldsymbol{T}_{ij} = 0$ if $\mathscr{G}_{ij} = 0$. It is assumed that $\mathscr{G}_{ii} = 1\ \forall i \in N$, so that $\boldsymbol{T}_{ii} > 0\ \forall i \in N$ and that $\boldsymbol{T}$ is row stochastic ($\sum_j \boldsymbol{T}_{ij} = 1$) and might be non-symmetric ($\boldsymbol{T}_{ij} \neq \boldsymbol{T}_{ji}$). The assumption that $\mathscr{G}_{ii} = 1$ and $\boldsymbol{T}_{ii} > 0$ for all $i \in N$ means that all agents self-communicate and place a non-zero positive weight on their own opinions. We assume that $\boldsymbol{G}$ is symmetric so that the network is undirected.

We now introduce a new generalisation of the DeGroot model which deviates from the simple averaging of opinions using static weights. Instead, the dynamics of the weight matrix $\boldsymbol{T}$ is based on the rule that weights decrease with increasing difference in opinions.

The dynamics of the weight is formalised as:

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t)) = \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i(t) - p_j(t)|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i(t) - p_k(t)|\}}, \tag{2.11}$$

where $\lambda_{ij} \geq 0\ \forall i \in N$. The matrix $\boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))$ is row stochastic and may be non-symmetric so that for some $i, j \in N$, $\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t)) \neq \boldsymbol{T}_{ji}(\boldsymbol{G}, \boldsymbol{p}(t))$. A special case, which reduces to a subclass of the DeGroot Model, is when $\lambda_{ij} = 0,\ \forall i, j \in N$. Simulation studies suggest that $\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t))$ is robust for small changes in the parameter values of λ. As explained in Section 1.3, the choice of the weight function in Equation (2.11) is based on conceptual notions grounded in psychophysical judgement theory (Davis, 1996, pg. 44-45) and the references therein. Following the opinion updating rule in Equation (2.1), the updating of opinions is modelled as:

$$\boldsymbol{p}(t+1) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))\boldsymbol{p}(t) \quad \text{for } t > 0. \tag{2.12}$$

As the weight $\boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))$ changes over time, Equation (2.12) can be rewritten as

$$\boldsymbol{p}(t+1) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t)) \times \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t-1)) \times, \ldots, \times \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(0))\boldsymbol{p}(0). \tag{2.13}$$

Here, the standard theorems of homogeneous Markov chains considered in Section 2.1 no longer holds for Equation (2.13). We therefore rely on the averaging contraction properties of row stochastic matrices established in Seneta (1981) in the following analysis.

2.2.1 Convergence with Dynamic Weight Updates

The main result in this section is Theorem 2.8, stating that the model given by Equation (2.11) and (2.12) converge to consensus. The proof uses results from Seneta (1981).

Definition 2.14 *(Definition 3.1 of Seneta (1981)) An $n \times n$ matrix $\boldsymbol{T} \geq \boldsymbol{0}$ is said to be* row-allowable *if it has at least one positive entry in each row.*

Definition 2.15 *(Definition 3.2 of Seneta (1981)) A row-allowable matrix $\boldsymbol{T} \geq \boldsymbol{0}$ is called* scrambling *if any two rows have at least one positive element in a* *coincident position.*

That is, $\boldsymbol{T} \geq \mathbf{0}$ is *scrambling* if for all pairs of rows i, j there exists a column k such that $\boldsymbol{T}_{ik}, \boldsymbol{T}_{jk} > 0$.

Theorem 2.7 *(Theorem 3.1 of Seneta (1981, p. 81)) Let $\boldsymbol{u} = \{u_k\}$ be an arbitrary vector and $\boldsymbol{T}$ an $n \times n$ stochastic matrix. If $\boldsymbol{q} = \boldsymbol{T}\boldsymbol{u}$, then for any two indices i, j,*

$$q_i - q_j \;\leq\; \frac{1}{2}\sum_{k=1}^{n} |\boldsymbol{T}_{ik} - \boldsymbol{T}_{jk}|\{\max_{k,l} |u_k - u_l|\}, \tag{2.14}$$

and

$$\max_{i,j} |q_i - q_j| \;\leq\; \mu(\boldsymbol{T})\{\max_{k,l} |u_k - u_l|\}, \tag{2.15}$$

where

$$\mu(\boldsymbol{T}) = \frac{1}{2}\max_{i,j}\sum_{k=1}^{n} |\boldsymbol{T}_{ik} - \boldsymbol{T}_{jk}| = 1 - \min_{i,j}\sum_{k=1}^{n} \min(\boldsymbol{T}_{ik}, \boldsymbol{T}_{jk}),$$

and $\mu(\boldsymbol{T}) < 1$ if and only if $\boldsymbol{T}$ is scrambling.

Definition 2.16 *There is a* path *between two vertices i, j in a graph if there exists a sequence of edges and vertices that connects i to j.*

For our main result, we assume that $\boldsymbol{G}$ is connected in the following sense.

Definition 2.17 *The graph $\boldsymbol{G}$ is* connected *if for each pair $i, j \in N$, there is a path between i and j.*

Definition 2.18 *Let $\boldsymbol{G}$ be a connected graph. The* shortest path length *between two vertices $i, j \in \boldsymbol{G}$ is the minimum path length between i and j. The diameter of a graph $\boldsymbol{G}$ is the maximum shortest path length between any two vertices $i, j \in \boldsymbol{G}$.*

We now state our main result for this section.

Theorem 2.8 *Assume that $\boldsymbol{G}$ is connected and self-communicating. Then the sequence $\boldsymbol{p}(t)|_{t=1}^{\infty}$ converges to a consensus as $t \to \infty$ for all initial $\boldsymbol{p}(0)$ and $\boldsymbol{G}$ given the updating rule in Equation (2.12). In particular, for all $t \geq L$,*

$$\max_{i,j} |p_i(t+1) - p_j(t+1)| \leq (1 - \underline{B}^L)^{\lfloor \frac{t}{L} \rfloor} \breve{p}, \tag{2.16}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x, L is the diameter of the graph, $\breve{p} = \max_i p_i(0) - \min_i p_i(0)$, and $\underline{B} = \dfrac{\exp\{-\lambda \breve{p}\}}{\max_i \deg(i)}$ for $\lambda = \max_{ij \in N} \lambda_{ij}$ and $\deg$*(i) is the degree of agent i.*

Proof.

For initial opinions $\boldsymbol{p}(0) = \{p_1(0), p_2(0), \ldots, p_n(0)\}$, let $\underline{p} = \min\{\boldsymbol{p}(0)\}$, $\overline{p} = \max\{\boldsymbol{p}(0)\}$ and $\breve{p} = \overline{p} - \underline{p}$. Since $p_i(t+1)$ is a convex combination of the $p_i(t)$'s for $t > 0$, we have that $p_i(t+1) \in [\underline{p}, \overline{p}]$ $\forall i$ and

$$0 \leq \lambda_{ij} |p_i(t) - p_j(t)| \leq \lambda_{ij} \breve{p},$$

which implies that

$$\exp\{-\lambda_{ij} \breve{p}\} \leq \exp\{-\lambda_{ij} |p_i(t) - p_k(t)|\} \leq 1. \tag{2.17}$$

For $\boldsymbol{p}(t) \in [\underline{p}, \overline{p}]^n$, let $\lambda = \max_{ij} \lambda_{ij} \geq 0 \;\; \forall i, j$, and let

$$w_{ij}(\boldsymbol{p}(t)) = \frac{\exp\{-\lambda_{ij} |p_i(t) - p_j(t)|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik} |p_i(t) - p_k(t)|\}},$$

then from (2.17)

$$\frac{\mathscr{G}_{ij}\exp\{-\lambda\breve{p}\}}{\sum_{k=1}^{n}\mathscr{G}_{ik}\exp\{-\lambda_{ik}|p_i(t)-p_k(t)|\}} \;\leq\; \mathscr{G}_{ij}w_{ij}(\boldsymbol{p}(t)) \;\leq\; \frac{\mathscr{G}_{ij}}{\sum_{k=1}^{n}\mathscr{G}_{ik}\exp\{-\lambda_{ik}|p_i(t)-p_k(t)|\}}.$$

Hence

$$\mathscr{G}_{ij}\frac{\exp\{-\lambda\breve{p}\}}{\sum_{k=1}^{n}\mathscr{G}_{ik}} \;\leq\; \mathscr{G}_{ij}w_{ij}(\boldsymbol{p}(t)) \;\leq\; \mathscr{G}_{ij}\frac{1}{1+\sum_{k\neq i}\mathscr{G}_{ik}\exp\{-\lambda_{ik}\breve{p}\}},$$

so that

$$\mathscr{G}_{ij}\frac{\exp\{-\lambda\breve{p}\}}{\deg(i)} \;\leq\; \mathscr{G}_{ij}w_{ij}(\boldsymbol{p}(t)) \;\leq\; \mathscr{G}_{ij}\frac{1}{1+(\deg(i)-1)\exp\{-\lambda\breve{p}\}}.$$

Let $\underline{B}=\dfrac{\exp\{-\lambda\breve{p}\}}{\max_i\deg(i)}$. It holds that $0<\underline{B}<1$, and

$$\mathscr{G}_{ij}w_{ij}(\boldsymbol{p}) \geq \mathscr{G}_{ij}\underline{B}. \tag{2.18}$$

Let L be the diameter of $\boldsymbol{G}$. Define an operator $\boldsymbol{A}^{(L)}:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ by

$$\boldsymbol{A}^{(L)}(\boldsymbol{q})=\boldsymbol{T}(\boldsymbol{G},\boldsymbol{q}^{(L)})\times\boldsymbol{T}(\boldsymbol{G},\boldsymbol{q}^{(L-1)})\times\ldots\times\boldsymbol{T}(\boldsymbol{G},\boldsymbol{q}^{(0)}) \tag{2.19}$$

where for $r>0$

$$q^{(r)}=\boldsymbol{T}\left(\boldsymbol{G},q^{(r-1)}\right)q^{(r-1)}.$$

Then by Equation (2.13)

$$\boldsymbol{p}(t+1) = \boldsymbol{A}^{(L)}\big(\boldsymbol{p}(t-L)\big)\boldsymbol{p}(t-L),$$

where $(\boldsymbol{p}(t))^{(L)} = \boldsymbol{p}(t+L)$.

Next we show that the matrix $\boldsymbol{A}^{(L)}(\boldsymbol{q})$ is scrambling for all $\boldsymbol{q}$, that is, for $t \geq L$, there exists $k \in N$ such that $\forall i,j \in N$, $\big(\boldsymbol{A}^{(L)}(\boldsymbol{q})\big)_{ik}\big(\boldsymbol{A}^{(L)}(\boldsymbol{q})\big)_{jk} > 0$, where $\big(\boldsymbol{A}^{(L)}(\boldsymbol{q})\big)_{ij}$ is the (i,j)th element of the matrix $\boldsymbol{A}^{(L)}(\boldsymbol{q})$.

To see this, note that for a single agent $i \in N$,

$$p_i(t+1) = \sum_k \Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{p}(t-L)\big)\Big)_{ik} p_k(t-L) \quad \text{for} \quad t \geq L,$$

where

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{p}(t-L)\big)\Big)_{ij} = \sum_{k_1,\ldots,k_L=1}^{n} \prod_{\tau=1}^{L+1} \mathscr{G}_{k_{r-1},k_r}\ w_{k_{r-1},k_r}(\boldsymbol{p}(t-\tau+1)),$$

with $k_0 = i$ and $k_L = j$.

Hence by Equation (2.18), for $t \geq L$

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{p}(t-L)\big)\Big)_{ij} \ \geq\ \underline{B}^L \sum_{k_1,\ldots,k_L=1}^{n} \prod_{\tau=1}^{L+1} \mathscr{G}_{k_{r-1},k_r}.$$

As L is the diameter of a connected graph $\boldsymbol{G}$ and $i \sim i\ \forall i$, it is the case that for all $i, j \in N$ there is a path of length L between i and j. Hence

$$\sum_{k_1,\ldots,k_L=1}^{n} \prod_{\tau=1}^{L+1} \mathscr{G}_{k_{r-1},k_r} \geq 1,$$

and so

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{p}(t-L)\big)\Big)_{ij} \ \geq\ \underline{B}^{L}. \tag{2.20}$$

In particular all elements $\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{p}(t-L)\big)\Big)_{ij} > 0\ \forall i, j$. Hence for $\boldsymbol{p} \in [\underline{p}, \overline{p}]^n,\ \boldsymbol{A}^{(L)}(\boldsymbol{p})$ is scrambling for all $\boldsymbol{p}$.

It follows from Equation (2.15) of Theorem 2.7 that for $t > L$,

$$\max_{i,j} |p_i(t+1) - p_j(t+1)| \leq \mu(\boldsymbol{A}^{(L)})\{\max_{i,j} |p_i(t-L) - p_j(t-L)|\}, \tag{2.21}$$

where

$$\mu(\boldsymbol{A}^{(L)}) = 1 - \min_{i,j} \sum_{k=1}^{n} \min\left[\boldsymbol{A}_{ik}^{(L)}, \boldsymbol{A}_{jk}^{(L)}\right]. \tag{2.22}$$

Hence by substituting Equation (2.20) into Equation (2.22) we have

$$\mu(\boldsymbol{A}^{(L)}) \leq 1 - \underline{B}^{L}. \tag{2.23}$$

We can rewrite Equation (2.21) by substituting Equation (2.23) as

$$\max_{i,j} |p_i(t+1) - p_j(t+1)| \leq (1 - \underline{B}^{L})\{\max_{i,j} |p_i(t-L) - p_j(t-L)|\} \quad \text{for} \quad t \geq L.$$

For $t \geq mL$, $m \leq \lfloor \frac{t}{L} \rfloor$ for a fixed L, and we can iterate as follows:

$$\begin{aligned}
\max_{i,j} |p_i(t+1) - p_j(t+1)| &\leq (1 - \underline{B}^L) \max_{i,j} |p_i(t-L) - p_j(t-L)|, \\
&\leq (1 - \underline{B}^L)^2 \max_{i,j} |p_i(t-2L) - p_j(t-2L)|, \\
&\vdots \\
&\leq (1 - \underline{B}^L)^{\lfloor \frac{t}{L} \rfloor} \max_{i,j} |p_i(t-mL) - p_j(t-mL)|.
\end{aligned}$$

Recall that $\breve{p} = \overline{p} - \underline{p}$ and $\underline{p} \leq \max_{i,j} |p_i(t-mL) - p_j(t-mL)| \leq \overline{p}$ is bounded. Hence

$$\max_{i,j} |p_i(t+1) - p_j(t+1)| \leq (1 - \underline{B}^L)^{\lfloor \frac{t}{L} \rfloor} \breve{p}, \tag{2.24}$$

and Equation (2.16) follows. Hence as $t \to \infty$, $|p_i(t+1) - p_j(t+1)| \to 0 \;\; \forall \boldsymbol{p}(0)$.

This final part follows directly by using an argument from Pan (2010) to show that the sequence $\boldsymbol{p}(t)|_{t=0}^{\infty}$ converges to a consensus. Recall that $\boldsymbol{p}(t)|_{t=0}^{\infty}$ is a sequence in a compact set $[\underline{p}, \overline{p}]^n$ where $\underline{p} = \min\{\boldsymbol{p}(0)\}$ and $\overline{p} = \max\{\boldsymbol{p}(0)\}$. Therefore $\boldsymbol{p}(t)|_{t=0}^{\infty}$ has a subsequence that is convergent. Suppose that the subsequence converges to $\boldsymbol{p}^*$. If $\boldsymbol{p}(t)$ does not converge to $\boldsymbol{p}^*$, then there exists another subsequence of $\boldsymbol{p}(t)|_{t=0}^{\infty}$ that converges to $\boldsymbol{p}^{**} \neq \boldsymbol{p}^*$.

We have shown that $p_i(t) - p_j(t) \to 0$ for all i, j as $t \to \infty$, and so $\boldsymbol{p}^* = (p^*, \ldots, p^*)$ and $\boldsymbol{p}^{**} = (p^{**}, \ldots, p^{**})$. Also, $\overline{p}(t) - \underline{p}(t) \to 0$. For $\varepsilon = |p^* - p^{**}|$, $\varepsilon > 0$, by definition, there exists t^* such that for $t > t^*$, we have $\overline{p}(t) - \underline{p}(t) < \varepsilon$, which means that $|p^* - p^{**}| < \varepsilon$ since $p^*, p^{**} \in [\underline{p}(t), \overline{p}(t)]$. This is a contradiction as $\varepsilon = |p^* - p^{**}|$. Therefore, $p^* = p^{**}$ and $\lim_{t \to \infty} \boldsymbol{p}(t) = (p^*, \ldots, p^*)^T$. $\square$

Theorem 2.9 *Assume that $\boldsymbol{G}$ is a connected network. Then, the sequence*

$\boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))|_{t=1}^{\infty}$ *converges to a non-negative matrix* $\boldsymbol{T}^*$*, for all initial* $\boldsymbol{p}(0)$ *and* $\boldsymbol{G}$ *given the dynamic weight updating rules in Equation* (2.11) *and* (2.12)*. Moreover* $\boldsymbol{T}^*_{i,j} = \mathscr{G}_{ij}\dfrac{1}{\deg(i)}$ $\forall i, j \in N$*, where* $\deg$*(i) is the degree of agent i.*

Proof.

Recall from Equation (2.11) that for two agents $i, j \in N$,

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t)) = \frac{\mathscr{G}_{ij}\exp\{-\lambda_{ij}|p_i(t) - p_j(t)|\}}{\sum\limits_{k=1}^{n}\mathscr{G}_{ik}\exp\{-\lambda_{ik}|p_i(t) - p_k(t)|\}}.$$

By Theorem 2.8, $|p_i(t) - p_j(t)| \to 0$ as $t \to \infty$. By continuity, if $\lambda > 0$, then

$$\lim_{t\to\infty}\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t)) \to \mathscr{G}_{ij}\frac{1}{\sum\limits_{k=1}^{n}\mathscr{G}_{ik}} = \mathscr{G}_{ij}\frac{1}{\deg(i)} \quad \forall i, k \in N.$$

If $\lambda = 0$ so that $\lambda_{ij} = 0$ $\forall i, j \in N$, then $\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t)) = \mathscr{G}_{ij}\dfrac{1}{\deg(i)}$. $\square$

2.2.2 Dynamic Weight Updates with Insistent Agents

We now investigate the behaviour of the model given by Equation (2.11) and (2.12) when insistent agents are present.

Recall from Section 2.1.3 that the set of insistent agents is represented by $Z = \{1, 2, \ldots, z\}$, and the set of non-insistent agents is represented by $\overline{Z} = N \setminus Z$.

As insistent agents keep to their own opinions throughout the updating period, the updating rule can be formulated as

$$\begin{aligned} \boldsymbol{p}_z(t+1) &= \boldsymbol{p}_z(t) = \boldsymbol{p}_z(0) \quad \forall z \in Z, \\ \boldsymbol{p}_{\overline{z}}(t+1) &= \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))\boldsymbol{p}(t) \quad \forall \overline{z} \in \overline{Z}, \end{aligned} \tag{2.25}$$

where $\boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))$ is given by Equation (2.11),

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}(t)) = \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i(t) - p_j(t)|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i(t) - p_k(t)|\}}, \qquad i \in \overline{Z}, j \in N.$$

Given the complicated nature of the nonlinear dynamics considered here, we use computer simulation to illustrate and investigate the influence of insistent agents.

2.2.2.1 Simulation Parameters

Recall that while the opinions may converge to consensus, consensus may not be achieved in finite time steps. Also, while we are ultimately interested in convergence, in simulations we can often not be certain that the system has converged. Instead the concept of stabilisation is used, defined as follows:

Definition 2.19 *Let $\varepsilon > 0$. The vector of opinions $\boldsymbol{p}(t)$ at time t is* stabilised *at level ε if*

$$\sum_{i=1}^{n} |p_i(t) - p_i(t-1)| < \varepsilon. \tag{2.26}$$

Definition 2.20 *The* stabilisation time *of opinions is the number of time steps taken for opinions to stabilise.*

It is worth noting here that after the stabilisation time, opinions might become non-stabilised again. We focus only on the behaviour of opinions before stabilisation. The simulation is driven until the opinions are stabilised. Also, note that one can observe stabilisation without convergence, and the time to convergence is generally different from the stabilisation time.

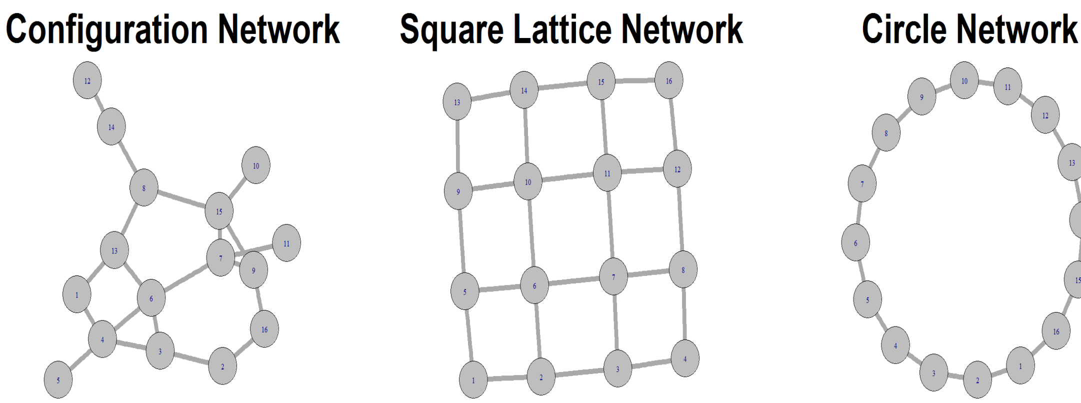


Figure 2.1. *An example of a random configuration network with a maximum degree of four, a square lattice network and a circle network. Each network has 16 vertices.*

Three initial network structures are considered for the simulations: a configuration network, a square lattice network and a circle network. An example of a random configuration network of size 10 with a maximum degree of four, a square lattice network of size 16 and a circle network of size 10 is given in Figure 2.1. For all the networks considered in this report, it is assumed that all agents self-communicate.

The square lattice network and the circle network are self-explanatory. The configuration random network can be obtained as follows:

1. Choose a non-negative integer vector $\boldsymbol{k} = \{k_1, k_2, \ldots, k_n\}$ such that $\sum_{i=1}^{n} k_i$ is even and $k_1, k_2, \ldots, k_n$ is a feasible degree sequence (i.e. can occur as a degree sequence of a simple graph).

2. Create an empty network that has n vertices with no edges.

3. Create a vector v (termed a *stub list*) such that for each $1 \leq i \leq n$, the vector v contains k_i copies of i, and take a random permutation of the entries of the list v.

4. Sample uniformly at random from v, two stubs i, j and remove i, j from the stub list v. Add an edge between vertex i and vertex j in the network.

5. Repeat Step 4 until the stub list v is empty.

The resulting network is a configuration random network. In the present work, the network is simplified to avoid multiple edges but self-loops are maintained.

The *igraph* package in *R* (Csardi and Nepusz, 2006) is used to generate the networks. For the configuration network, it allows one to generate a degree sequence (for Step 1) by specifying a single number which represents the maximum expected degree. The maximum expected degree is then used to generate a feasible degree sequence for the construction of the network. Here, a maximum degree of four is used to generate the degree sequence (Step 1), and the network realisation is kept fixed for all the simulations using the *set.seed* command in R. Considering that the circle network has a maximum degree of two and the square lattice network has a maximum degree of four, we choose a maximum degree of four for the configuration network for comparison.

At the initial time step, the size of the configuration network, the square lattice network and the circle network is set to $n = 225$ for Figure 2.2 and $n = 1024$ for Figure 2.3, $\varepsilon = 10^{-6}$ and the number of insistent agents to $z = 22$ for Figure 2.2 and $z = 102$ for Figure 2.3.

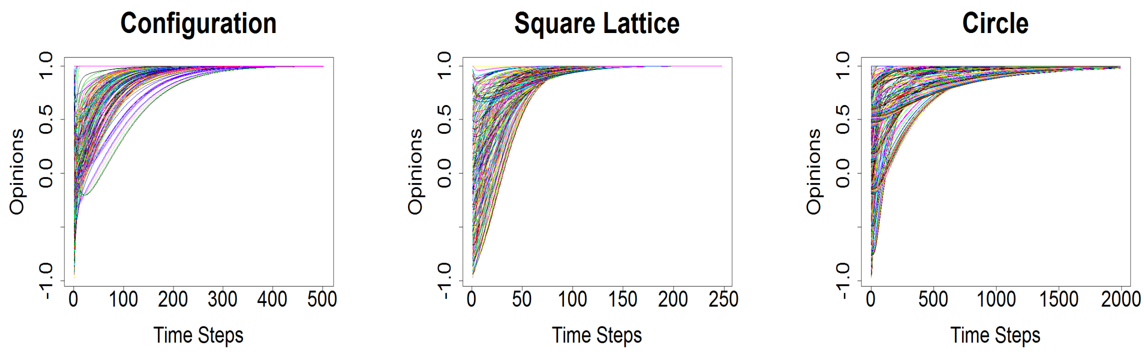
For the purpose of illustration, two scenarios are considered; where insistent agents have the same initial opinion (Figure 2.2a and Figure 2.3a), and where insistent agents have two different initial opinions (Figure 2.2b and Figure 2.3b).

First, the initial opinions $\boldsymbol{p}(0)$ are drawn from $\boldsymbol{p}(0) \overset{\text{iid}}{\sim} N(0, 1)$. Then the initial opinions of insistent agents, when they are assumed to have the same opinion, are assigned to be the maximum of $\boldsymbol{p}(0)$, Figure 2.2a and Figure 2.3a. When insistent agents are assumed to have two different opinions, Figure 2.2b and Figure 2.3b, half of the insistent agents are randomly chosen and assigned initial opinions to be the maximum of $\boldsymbol{p}(0)$ and the remaining half have initial opinions to be the minimum of $\boldsymbol{p}(0)$. If opinions do not stabilise, then the first 1000 iterations are reported.

Figures 2.2-2.3 illustrate the influence of insistent agents for the dynamic weight model introduced in Equation (2.25). Figure 2.2a and Figure 2.3a show that the opinions of all agents stabilise to the opinion of insistent agents when insistent agents have the same opinion. However, coexistence of different opinions persists when insistent agents have different opinions, as shown in Figure 2.2b and Figure 2.3b. It takes a longer time for opinions to stabilise in the circle network compared to the configuration network and the square lattice network. This can be explained by the high average shortest path length between agents in the circle network compared to the configuration network and the square lattice network.

Additionally, the outcome of opinions for $n = 225$ Figure 2.2 is consistent with Figure 2.3 when the network size is increased from $n = 225$ to $n = 1024$ and when the number of insistent agents is increased from $z = 22$ to $z = 102$.

(a) Insistent agents having the same opinion



(b) Insistent agents having two different opinions

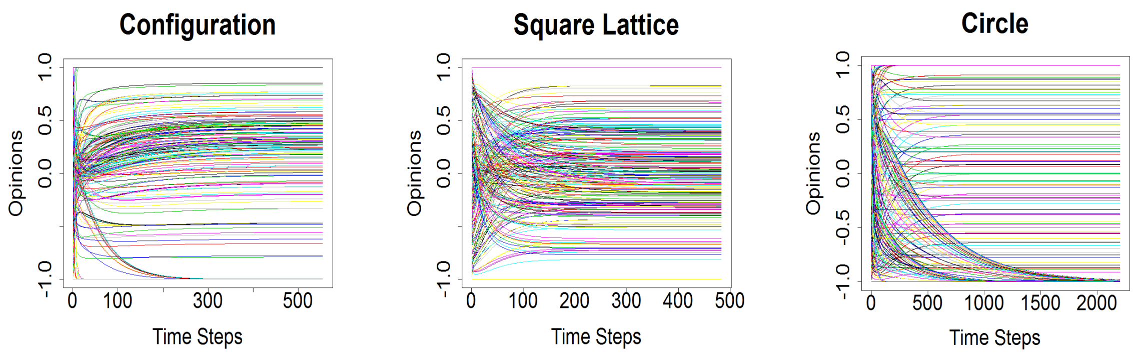


Figure 2.2. *Stabilisation of opinions for insistent agents in a configuration network, a square lattice network and a circle network. The size of the network is* $n = 225$*, the number of insistent agents is* $z = 22$ *and* $\varepsilon = 10^{-6}$*. In (a), insistent agents have the same initial opinions chosen to be the maximum of* $\boldsymbol{p}(0)$ *and in (b), insistent agents have two different initial opinions chosen to be the minimum and the maximum of* $\boldsymbol{p}(0)$*. It is observed from (a) that opinions stabilise to the opinion of insistent agents when insistent agents have the same opinion. For insistent agents having different opinions, coexistence of different opinions persists.*

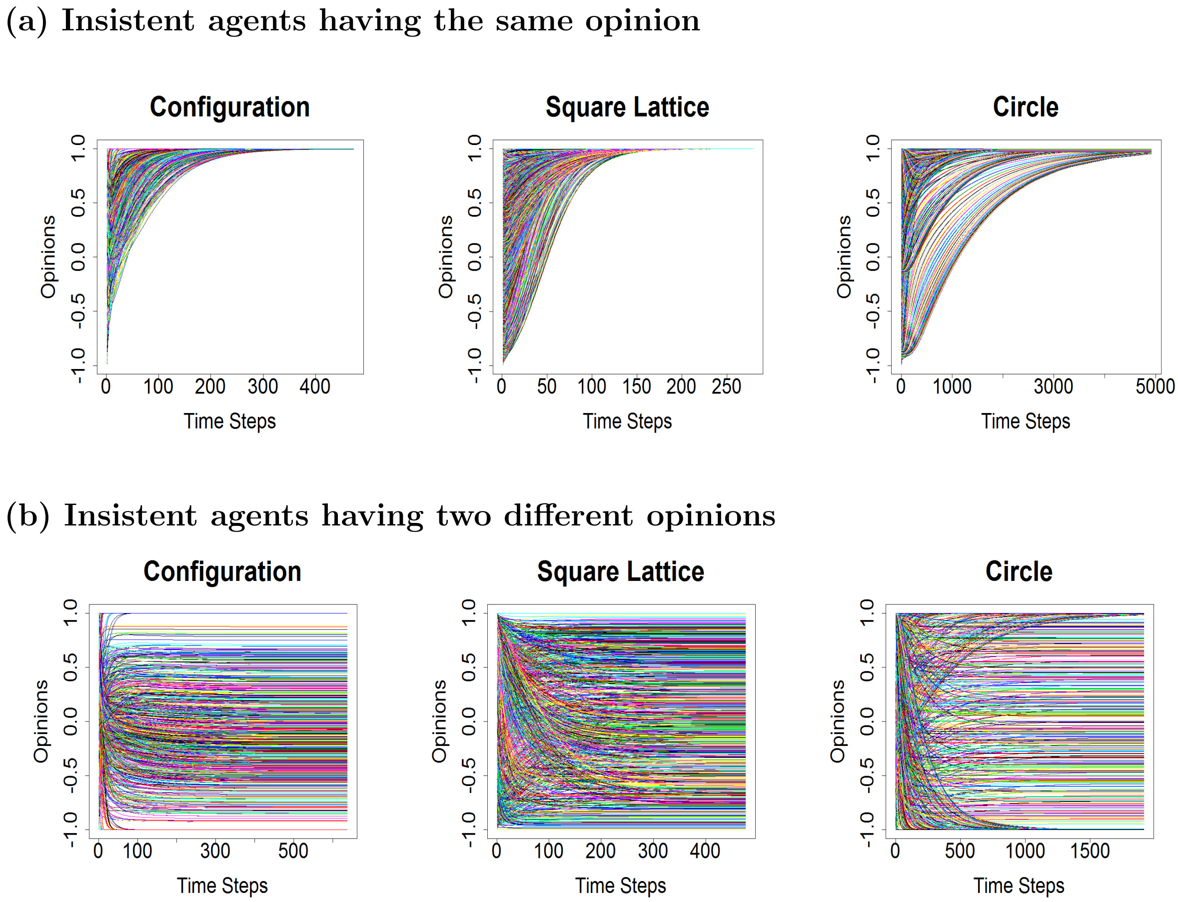


Figure 2.3. *Stabilisation of opinions for insistent agents in a configuration network, a square lattice network and a circle network. The size of the network is* $n = 1024$*, the number of insistent agents is* $z = 102$ *and* $\varepsilon = 10^{-6}$*. In (a), insistent agents have the same initial opinions chosen to be the maximum of* $\boldsymbol{p}(0)$ *and in (b), insistent agents have two different initial opinions chosen to be the minimum and the maximum of* $\boldsymbol{p}(0)$*. It is observed from (a) that opinions stabilise to the opinion of insistent agents when insistent agents have the same opinion. For insistent agents having different opinions, coexistence of different opinions persists.*

2.3 Conclusion

We have introduced the DeGroot model and extended the model to include dynamic weight updates and inhomogeneous updating of opinions. New theorems regarding the long-time behaviour of both the DeGroot model and the extended models have been proved.

In the DeGroot model, each agent has a real number which describes its opinion. Agents update their opinions at each time step by taking a weighted average of the opinions of their neighbours. The DeGroot model provides a prescriptive proposal for achieving consensus on the assumptions that:

1. Agents place the same weights on the opinions of their neighbours at each time step.

2. All agents homogeneously update their opinions by taking a weighted average of the opinions of their neighbours at each time step.

3. All agents interact with one another, resulting in a static pattern of interaction.

These assumptions are relaxed in this chapter to examine convergence to consensus in opinions. The third assumption will further be relaxed in the next chapter to allow for dynamical networks.

A dynamic weight based on the changes in the current opinion of agents is proposed. Agents update their weights based on the rule that weight decreases with increased difference in opinions.

We have proved in Theorem 2.9 that the final weight of an agent converges to a state inversely proportional to the degree of the agent. Additionally, we have proved in Theorem 2.8 that the opinions of agents converge to a consensus under a dynamic weight updates for a connected network. We derived an upper bound for the maximum difference between the opinions of any two agents in Equation (2.24) of the proof of Theorem 2.8.

A minority group of agents who insist on their opinions throughout the updating period is introduced to relax the assumption that all agents homogeneously update their opinions. These agents are termed *insistent agents*.

We have proved in Corollary 2.1 that the DeGroot model with insistent agents converge to a consensus when all insistent agents have the same opinion. However different opinions coexist when insistent agents hold different opinions. We have proved in Lemma 2.3 that when insistent agents have different opinions, the opinions of the non-insistent agents converge to a weighted average of the opinions of insistent agents.

The results on insistent agents for the DeGroot model also hold for the extended model with dynamic weight updating. The dynamic weight update model with insistent agents is studied by simulation with illustrations given in Figure 2.2 and Figure 2.3.

Chapter 3

Generalised Modelling on Dynamic Networks

In this chapter, we introduce a further generalisation of the proposed model in Section 2.2. We relax the assumption that the network is static over time and instead introduce two models for the network dynamics. The dynamics of the network is assumed to change on the same time scale as the opinion dynamics.

In Iñiguez et al. (2009) a model is considered in which the network change and the opinion change are coupled, so that agents prefer to form edges with agents of similar opinions. Here we consider two models; one in which the network change is independent of the opinion change, and one in which the network change is depends on the differences in opinions as well as changes in the network topology.

For a model in which the network change is independent of opinion change, an edge switch algorithm proposed by Gkantsidis et al. (2003) is used, which we term here as the *random switching* network dynamics model (the RS model), and it is explained in detail in Section 3.1. When the opinion change and the network change are coupled, we propose a network dynamics model based on homophily and triadic closure, called the *homophily and triadic closure* network dynamics model (the HT model), and it is explained in detail in Section 3.2. In both models, it is assumed that agents are always connected to themselves through a self-edge.

Starting with a network $\boldsymbol{G}(0)$ at time 0, let $\boldsymbol{G}(t) = \{\mathscr{G}_{ij}(t)\}$ be the network at time t. Then the updating of opinions with a dynamic network can be formalised as:

$$\boldsymbol{p}(t+1) = \boldsymbol{T}(\boldsymbol{G}(t), \boldsymbol{p}(t))\boldsymbol{p}(t), \quad \text{for } t > 0, \tag{3.1}$$

where

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}) = \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}, \quad \text{with} \quad \lambda_{ij} \geq 0 \; \forall i, j \in N, \tag{3.2}$$

is both network and opinion dependent. Thus, agents update their opinions by taking the current changes in their network and the current changes in the opinions of their neighbours into account.

For both the RS model and the HT model, agents apply the same updating rule, Equation (3.1), in updating their opinions and the weights placed on the opinions of their neighbours after the network change.

For the RS model, we give some theoretical result on the convergence of opinions. However, given the complex nature of the nonlinear dynamics considered in this Chapter, computer simulation is used as the main methodological tool for obtaining the results of the HT model. All simulations are carried out in R using the igraph package (Csardi and Nepusz, 2006).

3.1 Random Switching (RS) Network Dynamics

Here, we use the random switching (RS) algorithm proposed by Gkantsidis et al. (2003). The RS Model assumes that agents update the network by randomly switching edges to form new edges, independently of opinions and the network structure. This provides a degree preserving dynamics on the network, such that agents at each time step rewire edges without changing the degree and connectivity of the initial network. The RS dynamics is summarised as:

1. Select two edges (i, j) and (x, y) at random such that i, j, x and y are distinct agents.

2. If agents i, j are not neighbours of agents x or y, switch edges by removing the edges (i, j) and (x, y) and inserting the edges (i, y) and (j, x) to form $\boldsymbol{G}'$. Else return to Step 1.

3. If $\boldsymbol{G}'$ is connected, set $\boldsymbol{G}(t + 1) = \boldsymbol{G}'$. Else return to Step 1.

A variation of the RS model will be considered in section 3.1.2 where Step 3 is omitted so that the network $\boldsymbol{G}'$ may be disconnected. Note that $\boldsymbol{G}'$ will never have any isolated vertices.

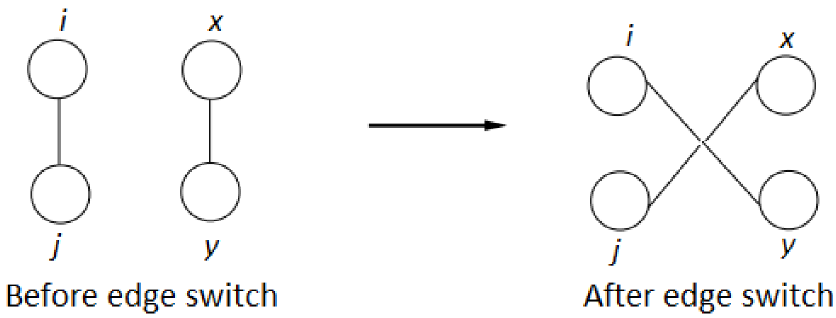


Figure 3.1. *An illustration for the random switching algorithm for agents before and after an edge switch.*

An illustration of the random switching algorithm is given in Figure 3.1. Before an edge switch, two edges are selected uniformly at random, say (i, j) and (x, y) such that i, j, x and y are distinct agents. In Step 2, agents switch edges by removing edges (i, j) and (x, y) and inserting edges (i, y) and (j, x) as shown in Figure 3.1 after the edge switch. Hence, the network changes randomly with agents switching edges at each time step.

3.1.1 Convergence for the RS Model without Insisting Agents

Given the updating rule in Equation (3.1), one might conjecture that coexistence of different opinions could arise for the RS network dynamics, since agents adopt opinions through random interaction in the network, and also put less weight on distant opinions (Pan, 2010). In contrast, we prove convergence of opinions to a consensus.

Theorem 3.1 *Assume that* $\boldsymbol{G}(0)$ *is connected and self-communicating. Then the* *sequence* $\boldsymbol{p}(t)|_{t=1}^{\infty}$ *converges to a consensus for all initial* $\boldsymbol{p}(0)$ *and* $\boldsymbol{G}(0)$ *given the RS dynamics and the opinion updating rule in* (3.1).

Theorem 3.1 states that when the network dynamics is such that agents switch edges as described in Section 3.1, and update their opinions using Equation (3.1), then the final opinions will converge to a consensus.

The proof of Theorem 3.1 is adapted from the proof of Theorem 2.8. The assumptions required for Theorem 2.8 to hold are that (1) the network is connected, (2) all agents self-communicate so that $\mathscr{G}_{ii}(t) = 1$ and $\boldsymbol{T}_{ii}(t) > 0\ \forall t$, and (3) agents assign non-negative weights on the opinions of their neighbours, $\boldsymbol{T}_{ij} \geq 0$ and $\sum_{j=1}^{n} \boldsymbol{T}_{ij} = 1\ \forall i, j \in N$.

The random switching network dynamics preserves the connectivity and degree of the network at each time step. In particular, $\deg_{\boldsymbol{G}(t)}(i) = \deg_{\boldsymbol{G}(0)}(i)\ \forall t$. Also as agents do not rewire self edges (by assumption), $\mathscr{G}_{ii}(t) = 1\ \forall t, i \in N$, and so the weight $\boldsymbol{T}_{ii}(\boldsymbol{G}(t), \boldsymbol{p}(t)) > 0\ \forall t$. Therefore, if the connectivity of the network and the vertex degrees are preserved, then the proof of Theorem 2.8 can be adapted to prove Theorem 3.1 as follows.

Proof of Theorem 3.1.

For a dynamic network $\boldsymbol{G}(t) = \{\mathscr{G}_{ij}(t)\}$, we use the notation

$$w_{ij}(\boldsymbol{G}, \boldsymbol{p}) = \frac{\exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}},$$

so that $\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}) = \mathscr{G}_{ij} w_{ij}(\boldsymbol{G}, \boldsymbol{p})$. Recall from the proof of Theorem 2.8 that $\breve{p} = \overline{p} - \underline{p}$, where $\underline{p} = \min_i p_i(0)$ and $\overline{p} = \max_i p_i(0)$. Let $\lambda = \max_{ij} \lambda_{ij} > 0\ \forall i, j$.

Then

$$\frac{\mathscr{G}_{ij}(t)\exp\{-\lambda\breve{p}\}}{\sum_{k=1}^{n}\mathscr{G}_{ik}(t)\exp\{-\lambda_{ik}|p_i(t)-p_k(t)|\}} \leq \mathscr{G}_{ij}(t)w_{ij}\big(\boldsymbol{G}(t),\boldsymbol{p}(t)\big)$$

$$\leq \frac{\mathscr{G}_{ij}(t)}{\sum_{k=1}^{n}\mathscr{G}_{ik}(t)\exp\{-\lambda_{ik}|p_i(t)-p_k(t)|\}}.$$

Let $\deg_{\boldsymbol{G}(t)}(i)$ be the degree of agent i in the network $\boldsymbol{G}(t)$. Then

$$\mathscr{G}_{ij}(t)\frac{\exp\{-\lambda\breve{p}\}}{\deg_{\boldsymbol{G}(t)}(i)} \leq \mathscr{G}_{ij}(t)w_{ij}\big(\boldsymbol{G}(t),\boldsymbol{p}(t)\big) \leq \mathscr{G}_{ij}(t)\frac{1}{1+(\deg_{\boldsymbol{G}(t)}(i)-1)\exp\{-\lambda\breve{p}\}}.$$

Note that the RS dynamics preserves the connectivity and degree of the network at each time step, and so $\deg_{\boldsymbol{G}(t)}(i) = \deg_{\boldsymbol{G}(0)}(i) = \deg(i)\ \forall t$.

Therefore, if $\underline{B} = \dfrac{\exp\{-\lambda\breve{p}\}}{\max_i \deg(i)}$, then it holds that $0 < \underline{B} < 1$, and

$$\mathscr{G}_{ij}(t)w_{ij}(\boldsymbol{G},\boldsymbol{p}) \geq \mathscr{G}_{ij}(t)\underline{B}. \tag{3.3}$$

Let L be the maximum shortest path length between any two vertices $i,j \in \boldsymbol{G}(t)$. Then L is bounded by $n-1$ as long as the network is connected.

The operator $\boldsymbol{A}^{(L)} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ from Equation (2.19) can be adapted as:

$$\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\ldots,\boldsymbol{G}(t),\boldsymbol{q}\big) = \boldsymbol{T}(\boldsymbol{G}t),\boldsymbol{q}^{(L)})\times\boldsymbol{T}(\boldsymbol{G}(t-1),\boldsymbol{q}^{(L-1)})\times\ldots\times\boldsymbol{T}(\boldsymbol{G}(t-L),\boldsymbol{q}^{(0)}),$$

where for $r > 0$

$$q^{(r)} = \boldsymbol{T}\left(\boldsymbol{G}^{(r-1)},q^{(r-1)}\right)q^{(r-1)}.$$

Hence, for a network evolution $\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t)$, Equation (3.1) is equivalent to

$$\boldsymbol{p}(t+1) = \boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)\boldsymbol{p}(t-L),$$

where $(\boldsymbol{p}(t))^{(L)} = \boldsymbol{p}(t+L)$.

Next we show that $\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t),\boldsymbol{q}\big)$ is scrambling using arguments similar to the proof of Theorem 2.8.

To see this, note that for a single agent $i \in N$,

$$p_i(t+1) = \sum_k \Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)\Big)_{ik} p_k(t-L), \quad \text{for} \ \ t \geq L,$$

where

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)\Big)_{ij} = \sum_{k_1,\dots,k_L=1}^{n}\prod_{\tau=1}^{L+1}\mathscr{G}_{k_{r-1},k_r}(t-\tau+1)\, w_{k_{r-1},k_r}\big(\boldsymbol{p}(t-\tau+1)\big),$$

with $k_0 = i$ and $k_L = j$.

Hence by Equation (3.3), for $t \geq L$

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)\Big)_{ij} \ \geq \ \underline{B}^L \sum_{k_1,\dots,k_L=1}^{n}\prod_{\tau=1}^{L+1}\mathscr{G}_{k_{r-1},k_r}(t-\tau+1).$$

Now, we show that there exists an $L > 0$ such that for all $i, j \in N$, there is a path of length L between i and j, that is, for any RS history $\boldsymbol{G}(t-L),\dots,\boldsymbol{G}(t)$,

$$\sum_{k_1,\dots,k_L=1}^{n}\prod_{\tau=1}^{L+1}\mathscr{G}_{k_{r-1},k_r}(t-\tau+1) \geq 1,$$

To this end, let

$$\begin{aligned} M = M(\boldsymbol{G}(0)) = \{\boldsymbol{G} : \boldsymbol{G} &\text{ is a graph on } n \text{ vertices with } \mathscr{G}_{ii} = 1 \ \forall i \in N. \\ &\boldsymbol{G} \text{ is connected and } \deg_{\boldsymbol{G}(t)}(i) = \deg_{\boldsymbol{G}(0)}(i) = \deg(i) \ \forall t\}. \end{aligned}$$

Then starting from $\boldsymbol{G}(0)$, the set M contains the history of all possible networks $\boldsymbol{G}$ that can arise from randomly switching $\boldsymbol{G}(0)$. It is worth noting here that for a given degree sequence, the chain is irreducible on connected graphs (see Taylor, 1982).

As the network is assumed to be always connected with self-loops, after

$$L = |M| + 1,$$

steps of the algorithm, at least one network will have been visited twice. After

$$L = |M|^{n-1} + 1,$$

steps at least one network will have been visited n times, and due to the self-loops, after $|M|^{n-1} + 1$ there will be a path between any two vertices in the network. Therefore, after $|M|^{n-1} + 1$ time steps, we can be sure that there is a path of length L between i and j.

Hence

$$\sum_{k_1,\ldots,k_L=1}^{n} \prod_{\tau=1}^{L+1} \mathscr{G}_{k_{r-1},k_r}(t - \tau + 1) \geq 1,$$

and so

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\ldots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)\Big)_{ij} \;\geq\; \underline{B}^{L}.$$

In particular, all elements

$$\Big(\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\ldots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)\Big)_{ij} > 0 \;\; \forall i,j \in N.$$

Hence for $\boldsymbol{p} \in [\underline{p},\overline{p}]^n$, the matrix $\boldsymbol{A}^{(L)}\big(\boldsymbol{G}(t-L),\ldots,\boldsymbol{G}(t),\boldsymbol{p}(t-L)\big)$ is scrambling $\forall \boldsymbol{p}, \boldsymbol{G}$.

Now, it follows from the argument in the proof of Theorem 2.8, from Equation (2.21) to Equation (2.24), that the sequence $\boldsymbol{p}(t)|_{t=1}^{\infty}$ converges to a consensus for all initial $\boldsymbol{p}$ and $\boldsymbol{G}$ given the RS updating rule. $\square$

3.1.2 Convergence for the RS Model without Imposing Network Connectivity

In this section, we show that convergence to a consensus is attained without imposing network connectivity (Step 3) in the RS dynamics as long as there are no isolated vertices in the initial network.

Let $\overline{RS}$ be the model for the RS dynamics without imposing the network connectivity rule of Step 3 at each time step. Then, the $\overline{RS}$ dynamics is given by

1. Select two edges (i, j) and (x, y) at random such that i, j, x and y are distinct agents.

2. If agents i, j are not neighbours to any of agents x, y, switch edges by remov-

ing edges (i, j) and (x, y) and insert edges (i, y) and (j, x) to form $\boldsymbol{G}(t+1)$, else return to Step 1.

Intuitively, the opinions contract slightly after each updating, as there are no isolated vertices and as all opinions will be different with probability 1. We formally state and prove this in Theorem 3.3 using the result of Krause (2009) and Lorenz and Lorenz (2010). In Lorenz and Lorenz (2010), a d-dimensional opinion space S is considered but here, we consider the case that $d = 1$. First, consider the following definitions from Lorenz and Lorenz (2010).

Definition 3.1 *Let $S \subset \mathbb{R}$ be the opinion space and $x(t) \in S^n$ be the opinion profile at time t with coordinate $x_i(t) \in S$. A* generalised barycentric coordinate map *y is a continuous function $y : S^n \to S^m$ which maps an opinion profile x to a set of m vectors $y(x) = (y_1(x), \ldots, y_m(x))$ such that for $x \in S^n$ and $i \in N$, it holds that*

$$x_i \in \left[\min_{j\in m} y_j(x),\ \max_{j\in m} y_j(x)\right].$$

In particular, taking y as the identity ($y(x) = x$) with $m = n$, it holds that

$$x_i \in \left[\min_{j\in n} x_j,\ \max_{j\in n} x_j\right].$$

Definition 3.2 *Let $S \subset \mathbb{R}$ and let $y : S^n \to S^m$ be a generalized barycentric coordinate map. A mapping $f : S^n \to S^n$ is called a y-*averaging map *if for every $x \in S^n$ it holds that*

$$\left[\min_{j\in m} y_j(f(x)),\ \max_{j\in m} y_j(f(x))\right] \subset \left[\min_{j\in m} y_j(x),\ \max_{j\in m} y_j(x)\right]. \tag{3.4}$$

In particular, taking y as the identity with $m = n$, it holds that

$$\left[\min_{j\in n} (f(x))_j,\ \max_{j\in n} (f(x))_j\right] \subset \left[\min_{j\in n} x_j,\ \max_{j\in n} x_j\right].$$

Definition 3.3 *A* proper y-averaging map *is a y-averaging map, such that for $x \in S^n$ which has all components taking on different values, the inclusion in Equation* (3.4) *is strict.*

We now state a special case of Lorenz and Lorenz Theorem 2.4 for a 1-dimensional opinions space and a finite F.

Theorem 3.2 *(Theorem 2.4 of Lorenz and Lorenz (2010)) Let $S \subset \mathbb{R}$, and let y be a generalized barycentric coordinate map. Let F be a finite family of proper y-averaging maps on S^n. Then it holds for any sequence $(f_t)_{t\in\mathbb{N}}$ with $f_t \in F$ and any $x(0) \in S^n$ that the dynamical system of the form*

$$x(t+1) = f_t(x(t)),$$

converges to a consensus. That is, there exists $\gamma = \gamma(x(0)) \in S$ such that for all $i \in N$ it holds that

$$\lim_{t\to\infty} x_i(t) = \gamma.$$

In Lorenz and Lorenz (2010), the result is derived for a family of functions F which can be uncountable but has to satisfy additional assumptions. We only need the case of a finite F, defined here as follows. Let

$$M_0 = \{\boldsymbol{G} = \{\mathscr{G}_{ij}\} \in \{0,1\}^{\binom{n}{2}} : \mathscr{G}_{ii}(t) = 1, \deg_{\boldsymbol{G}}(i) = \deg_{\boldsymbol{G}(0)}(i)\ \forall i \in \mathbb{N},$$

and $\boldsymbol{G}$ has no isolated vertices$\}$.

Then M_0 contains the history of all possible $\overline{RS}$ networks from randomly switching $\boldsymbol{G}(0)$. Again, the chain is irreducible for all graph with a given degree sequence, connected or unconnected, see Taylor (1982) for further theoretical results.

For $\boldsymbol{G} \in M_0$ and $\boldsymbol{p} \in \mathbb{R}^n$, we set $f_{\boldsymbol{G}} : \mathbb{R}^n \to \mathbb{R}^n$,

$$f_{\boldsymbol{G}}(\boldsymbol{p}) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p})\boldsymbol{p}, \text{ and } F = \{f_{\boldsymbol{G}} : \boldsymbol{G} \in M_0\}.$$

Then F is a finite set because M_0 is finite.

Next, we first proof some useful lemmas before stating our theorem.

Lemma 3.1 *If the initial network $\boldsymbol{G}(0)$ has no isolated vertices, and $\boldsymbol{G}(t)$ for $t > 0$ arises from randomly switching $\boldsymbol{G}(0)$, then $\boldsymbol{G}(t)$ has no isolated vertices.*

Proof

Let $\boldsymbol{G}(0)$ be a network with no isolated vertices. Let (u, v) and (x, y) be edges with distinct vertices in $\boldsymbol{G}(0)$, then for any $\boldsymbol{G}(t)$ arising from $\boldsymbol{G}(0)$, randomly switching edges replaces edges (u, v) and (x, y) with edges (u, x) and (v, y). The switching preserves the degree of vertices u, v, x and y. Hence, it can not create isolated vertices.

Lemma 3.2 *Let $\boldsymbol{p}(0) = \{p_1(0), \ldots, p_n(0)\}$ be independently drawn from a continuous distribution D. Let $\boldsymbol{p}(t)$ be defined by (3.1) and (3.2) with $\lambda = \max_{ij} \lambda_{ij} > 0$, for a $\overline{RS}$ history $\boldsymbol{G}(0), \boldsymbol{G}(1), \ldots, \boldsymbol{G}(t)$. Then for all $t \in \mathbb{N}$, with probability 1, all entries of $\boldsymbol{p}(t)$ are different.*

Proof

We proceed by induction. First, note that due to the independence assumption, with probability 1, all entries of $\boldsymbol{p}(0) = \{p_1(0), \ldots, p_n(0)\}$ are different.

Let $f_{\boldsymbol{G}}(\boldsymbol{p}) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p})\boldsymbol{p}$ with $\boldsymbol{G} \in \{\boldsymbol{G}(0), \ldots, \boldsymbol{G}(t)\}$. Then the dynamics of opinion change can be rewritten from Equation (3.1) as:

$$\boldsymbol{p}(t+1) = f_{\boldsymbol{G}(t)}(\boldsymbol{p}(t)).$$

For an agent $i \in N$,

$$(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = \sum_{j=1}^{n} p_j \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}. \tag{3.5}$$

We want to show that for all $l \neq i$

$$\mathbb{P}[(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = (f_{\boldsymbol{G}}(\boldsymbol{p}))_l \mid \boldsymbol{p}(0) \overset{\text{iid}}{\sim} D] = 0.$$

To this end, let $\Delta_i(\boldsymbol{p})$ be the vector space spanned by the vectors $\{(p_i - p_j)e_j,\ j = 1, \ldots, n\}$ where e_j is the jth canonical unit vector. Note that $p_i - p_i = 0$ and hence, $\dim(\Delta_i(\boldsymbol{p})) \leq n - 1$. From (3.5)

$$\begin{aligned}(f_{\boldsymbol{G}}(\boldsymbol{p}))_i &= p_i + \sum_{j=1}^{n} (p_j - p_i) \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}, \\ &= p_i + h(\Delta_i(\boldsymbol{p})) \quad \text{for a function } h.\end{aligned}$$

Now, we proceed by induction. If we assume that $(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = (f_{\boldsymbol{G}}(\boldsymbol{p}))_l$ for some $l \neq i$, then it implies that

$$\begin{aligned} p_i + h(\Delta_i(\boldsymbol{p})) &= p_l + h(\Delta_l(\boldsymbol{p})), \\ &= p_l + \tilde{h}(\Delta_i(\boldsymbol{p})), \end{aligned}$$

for a function $\tilde{h}$.

Note that $p_l - p_j = p_l - p_i + p_i - p_j$, and so $\{(p_i - p_j)e_j,\ j = 1, \ldots, n\}$ and $\{(p_l - p_j)e_j,\ j = 1, \ldots, n\}$ span the same vector space, hence $\operatorname{span}(\Delta_l(\boldsymbol{p})) = \operatorname{span}(\Delta_i(\boldsymbol{p}))$. Therefore

$$p_i = p_l + \tilde{h}(\Delta_i(\boldsymbol{p})) - h(\Delta_i(\boldsymbol{p})). \tag{3.6}$$

Note that the right hand side does not contain p_i. Hence, if $\wp_i^* = \{\boldsymbol{p} \in \mathbb{R}^n :$ at least two components of $\boldsymbol{p}$ are same, or $\boldsymbol{p}$ satisfies Equation (3.6)$\}$. Then the dimension of $\wp_i^* \leq n-1$ as p_i is fixed through Equation (3.6). Therefore

$$\mathbb{P}[(f_G(\boldsymbol{p}))_i \in \wp_i^* \mid \boldsymbol{p}(0) \overset{\text{iid}}{\sim} D] = 0.$$

Hence, we have shown that if $\boldsymbol{p}$ has all components different, then so does $f_G(\boldsymbol{p})$, with probability 1. Iterating the argument gives that $\boldsymbol{p}(t)$ has all components different with probability 1. $\square$

Theorem 3.3 *Assume that $\boldsymbol{G}(0)$ is connected and self-communicating so that $\mathscr{G}_{ii}(0) = 1\ \forall i$ and that $\boldsymbol{p}(0) \overset{\text{iid}}{\sim} D$. Then, with probability 1, the sequence $\boldsymbol{p}(t)|_{t=1}^{\infty}$*

converges to a consensus for all initial $\boldsymbol{p}(0)$ *and* $\boldsymbol{G}(0)$ *given the* $\overline{RS}$ *dynamics and the opinion updating rule in* (3.1)*.*

Proof

Theorem 3.3 can be considered as a special case of Theorem 3.2 with y as the identity and $f_t = f_{\boldsymbol{G}}$ given by $f_{\boldsymbol{G}}(\boldsymbol{p}) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p})\boldsymbol{p}$, where $\boldsymbol{G}$ is the network defined by the $\overline{RS}$ network dynamics and $\boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p})$ is defined by Equation (3.2).

If we can show that $f_{\boldsymbol{G}}$ is a proper averaging map, then by Theorem 3.2 the dynamics of Equation (3.1) given by

$$\boldsymbol{p}(t+1) = f_{\boldsymbol{G}(t)}(\boldsymbol{p}(t))$$

converges to a consensus.

By Definition 3.3, an averaging map in a one dimensional space is said to be proper if for all $\boldsymbol{p} = \{p_1, p_2, \ldots, p_n\}$ which components are not all equal, it holds that

$$\begin{aligned}
&\min_j p_j < (f_{\boldsymbol{G}}(\boldsymbol{p}))_i \leq \max_j p_j \quad \forall i \in N \text{ or} \\
&\min_j p_j \leq (f_{\boldsymbol{G}}(\boldsymbol{p}))_i < \max_j p_j \quad \forall i \in N.
\end{aligned}$$

We show that $(f_{\boldsymbol{G}}(\boldsymbol{p}))_i < \max_j p_j \quad \forall i \in N$. For an agent i,

$$(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = \sum_j p_j \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}. \tag{3.7}$$

Recall that for each agent $i \in N$, $\mathscr{G}_{ii} = 1$. By Lemma 3.1, for all t and $i, j \in N$,

$\exists i, j,\ j \neq i$ such that $\mathscr{G}_{ij} = 1$. Additionally, by Lemma 3.2, $\forall i \in N,\ p_i \leq \max\limits_j p_j$ and for $i \neq \max\limits_j p_j$, $p_i < \max\limits_j p_j$.

Let $p_{max} = \max_j p_j$ be the maximum opinion. Then

$$\begin{aligned}
(f_{\boldsymbol{G}}(\boldsymbol{p}))_i &= \sum_j p_j \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}, \\
&< p_{max} \sum_j \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}} \\
&= p_{max}.
\end{aligned}$$

Hence

$$(f_{\boldsymbol{G}}(\boldsymbol{p}))_i < \max_j p_j, \tag{3.8}$$

which implies that $f_{\boldsymbol{G}}$ is a proper averaging map. Hence, Theorem 3.2 can be applied and therefore Theorem 3.3 holds. □

Note that the assumption that $\boldsymbol{p}(0)$ be drawn from a continuous distribution is sufficient but not generally necessary.

3.1.3 Illustration by Simulation for the RS Model

In this section, we explore the RS model further by simulation.

At the initial time step, the size of the network is set to $n = 225$ for Figure 3.2 and $n = 1024$ for Figure 3.3. For both Figure 3.2 and Figure 3.3, $\varepsilon = 10^{-6}$ and

$\boldsymbol{p}(0) \overset{\text{iid}}{\sim} N(0,1)$. For $t > 0$ agents rewire their connections using the RS dynamics given in Section 3.1. The weight $\boldsymbol{T}(\boldsymbol{G}(t), \boldsymbol{p}(t))$ is then computed and used to update opinions for the next time step.

Figures 3.2-3.3 give an illustration of the simulation outcome for the configuration network, the square lattice network and the circle network for $n = 225$ and $n = 1024$ respectively. The horizontal axis of the the plot of opinions against time steps is set to $x = t/n$ to scale for population size. The vertical axis and the colours show the different opinions of agents. Here we record the stabilisation times.

In agreement with Theorem 3.1, opinions stabilise to a consensus for the different initial networks as shown in Figure 3.2. Similarly, opinions stabilise to a consensus in Figure 3.3 when the size of the network is increased from $n = 225$ to $n = 1024$.

The time to stabilisation in the opinions is observed to depend on the initial structure of the network and the size of the network. Notice that opinions stabilise faster for the square lattice network compared to the configuration network and the circle network in both Figure 3.2 and Figure 3.3. Additionally the stabilisation time increases when the size of the network is increased from $n = 225$ to $n = 1024$.

We note that opinions stabilise to a consensus in finite time due to rounding errors, whereas in the theoretical model with probability 1, consensus will not be achieved in finite time.

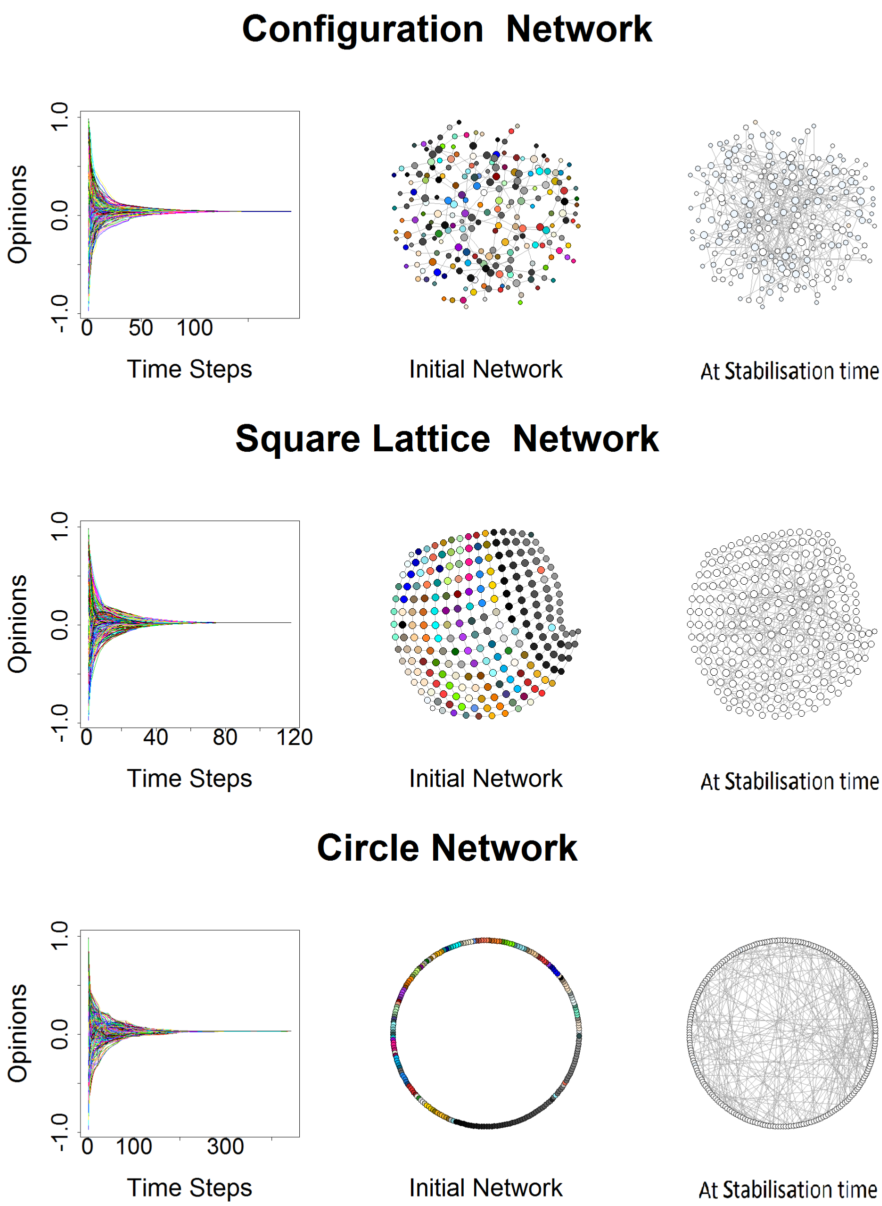


Figure 3.2. *Stabilisation of opinions, initial network and final network with* $n = 225$ *agents for a configuration network, a square lattice network and a circle network. The initial network and the network at stabilisation time are coloured by the different opinions in the network. The single colour in the network at opinion stabilisation is an indication that opinions stabilise to a consensus for the different initial networks.*

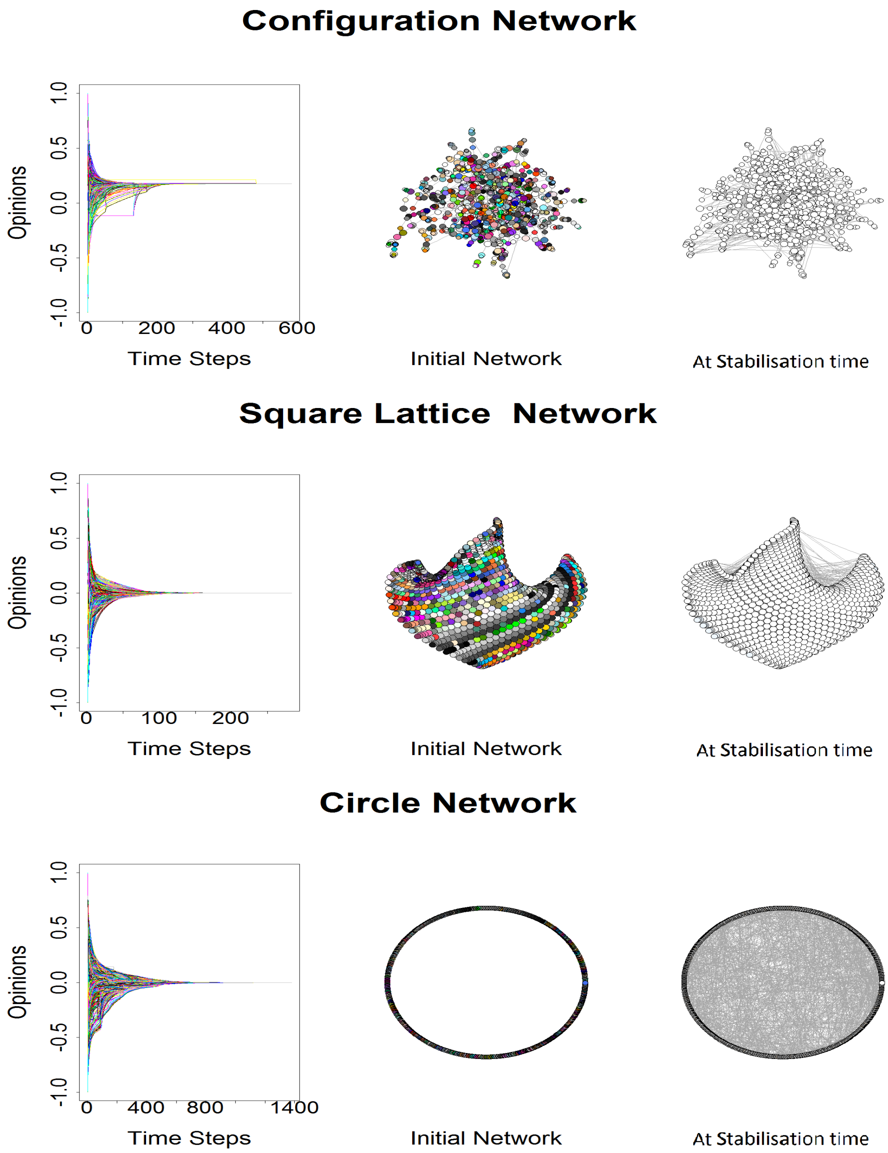


Figure 3.3. *Stabilisation of opinions, initial network and final network with* $n = 1024$ *agents for a configuration network, a square lattice network and a circle network. The initial network and the network at stabilisation time are coloured by the different opinions in the network. The single colour in the network at opinion stabilisation is an indication that opinions stabilise to a consensus for the different networks even when the network size is increased.*

3.2 Homophily and Triadic Closure (HT) Network Dynamics

We now propose an alternative network updating rule where agents update the network by means of homophily and triadic closure (the HT model). In the RS model, agents are assumed to randomly switch edges over time. Here, new edges are created between neighbours of neighbours (triadic closure) and existing edges are deleted based on differences in opinion (homophily).

The HT dynamics is as follows:

1. Select at random an agent i, one of its neighbours j, and all neighbours k of j which are not connected to i.
2. Add an edge (i, k) between i and all such k. Suppose that a total of m_t edges are added (*triadic closure*).
3. Compute the quantity $o_{il}(t) = |p_i(t) - p_l(t)|$ for agent i and all its neighbours l including new neighbours from Step 2.
4. Delete m_t edges from the neighbours of agent i, starting with agent j with the highest value $o_{ij}(t)$ (*homophily*).

As explained in Section 1.1.2, the choice of homophily and triadic closure dynamics is motived by the empirical support for the prevalence of homophily and triadic closure in social networks (Baron et al., 1996; Kossinets and Watts, 2006).

An illustration for the homophily and triadic closure dynamics is given in Figure 3.4, where at each time step t, an agent i and one of its neighbours j are selected at random together with all the neighbours of j. The neighbours of agent j are agent x and agent y as shown in Figure 3.4a. Agent i first creates new edges with agent x and agent y so that edges (i, x) and (i, y) are added, giving a total $m_t = 2$ added edges as shown in Figure 3.4b (triadic closure).

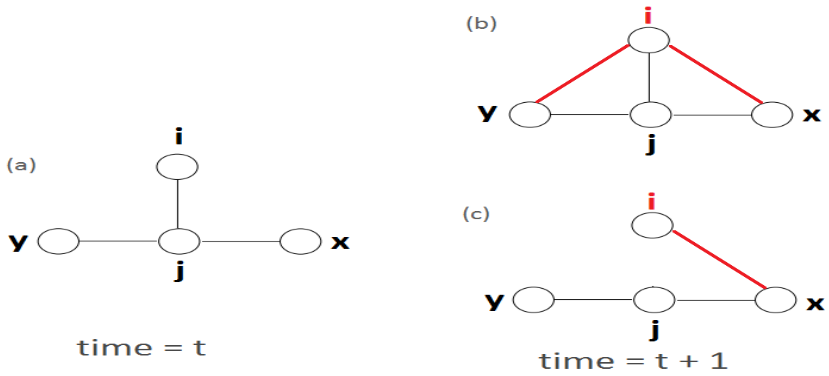


Figure 3.4. *An illustration for the homophily and triadic closure algorithm for updating the network. It is assumed that agent i is first selected followed by agent j and all agents x, y, the neighbours of j. If agent x has the closest opinion to agent i, then the final network at time $t+1$ is as shown in (c), where $m_t = 2$, the two new edges are the solid red lines in (b).*

The set of neighbours of agent i now consists of edges to its originally connected neighbours and m_t of edges to the neighbours of agent j. The degree of a selected vertex is kept fixed by deleting the same number of added edges.

To that purpose agent i computes the quantity $o^t_{ij} = |p^t_i - p^t_j|$ for all its neighbours with new neighbours included and deletes the same number of m_t added edges in decreasing order of o^t_{ij} as shown in Figure 3.4(c). In Figure 3.4(c), it is assumed that agent x has the closest opinion to agent i (homophily). By deleting the same number of added edges of a selected vertex, the degree of other agents might change but the total number of edges and hence the average degree of the network is kept constant over time.

The deletion process is such that if all the edges an agent deletes are part of the m_t added edges, then the network remains unchanged. If however the edges an agent deletes are all originally connected edges, such that all the neighbours of the agent are made up of edges in m_t, then the agent takes on a new set of neighbours and breaks communication with all originally connected neighbours to associate with others of a similar opinion (homophily).

After rewiring of the network, agents update opinions and weights using Equation (3.1).

3.2.1 Convergence for the HT Model without Insisting Agents

Given the updating rule in Equation (3.1), one might conjecture that by the assumption of homophily and triadic closure, agents' opinions will become more similar and the network will remain connected over time, leading to consensus in the final opinions (Lazarsfeld and Merton, 1954; McPherson et al., 2001). In contrast, we found that homophily coupled with triadic closure can lead to the coexistence of different opinions and cluster formation in the network.

Proposition 3.1 *Assume that* $\boldsymbol{G}(0)$ *is connected and self-communicating. Then the sequence* $\boldsymbol{p}(t)|_{t=1}^{\infty}$ *can converge to a state of lack of consensus for all initial* $\boldsymbol{p}(0)$ *and* $\boldsymbol{G}(0)$ *given the HT dynamics and the opinion updating rule in* (3.1).

Proof.

We proceed by counter example to show that $f_{\boldsymbol{G}}(\boldsymbol{p})$ is not a proper averaging map for all $\boldsymbol{p}$ when there are isolated vertices. Recall that by Definition 3.3, an averaging map is proper if for all $\boldsymbol{p} = \{p_1, p_2, \ldots, p_n\}$ in which the components are not all equal, it holds that

$$\begin{aligned} \min_j p_j &< (f_{\boldsymbol{G}}(\boldsymbol{p}))_i \quad \forall i \in N \quad \text{or} \\ \max_j p_j &> (f_{\boldsymbol{G}}(\boldsymbol{p}))_i \quad \forall i \in N. \end{aligned}$$

We show that for some $i \in N$, $(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = \max_j p_j \quad \forall i \in N$.

Recall from the proof of Theorem 3.3 that $f_{\boldsymbol{G}}(\boldsymbol{p}) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p})\boldsymbol{p}$, and therefore, $\boldsymbol{p}(t+1) = f_{\boldsymbol{G}(t)}(\boldsymbol{p}(t))$. For an agent $i \in N$,

$$(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = \sum_j p_j \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}.$$

Note that in the HT model, it is possible that two agents with different opinions each become isolated. These agents will then stay isolated and hence will never change their opinions. As a consequence, their opinions will not converge to a consensus.

Assume that i is an isolated vertex and $p_i = p_{max}$, where $p_{max} = \max_j p_j$. Then, $\mathscr{G}_{ii} = 1$ and $\mathscr{G}_{ij} = 0 \quad \forall j \neq i$, hence

$$\begin{aligned} (f_{\boldsymbol{G}}(\boldsymbol{p}))_i &= \sum_j p_j \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_i|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ii} \exp\{-\lambda_{i}|p_i - p_k|\}}, \\ &= p_i \frac{\mathscr{G}_{ii} \exp\{-\lambda_{ii}|p_i - p_i|\}}{\mathscr{G}_{ii} \exp\{-\lambda_{ii}|p_i - p_i|\}}, \end{aligned}$$

$$= p_i,$$

$$= p_{max}.$$

Therefore, $(f_{\boldsymbol{G}}(\boldsymbol{p}))_i = \max_j p_j$, which implies that $f_{\boldsymbol{G}}(\boldsymbol{p})$ is not a proper averaging map for all $\boldsymbol{p}$. Indeed, every isolated vertex will never change its opinion and will stay isolated. Hence, lack of consensus can emerge. $\square$

Compared to the RS and the $\overline{RS}$ model from Section 3.1 where the opinions of agents always converge to a consensus, the HT model can converge to a state of lack of consensus.

Under the HT model, the network stabilises to a fixed network. The stabilisation in the network occurs after the opinions have stabilised. Although agents start with an initially connected network, the final network can segregate into groups of connected agents with few connections across groups. Individuals with similar opinions begin to form groups in the network, leading to consensus within groups but different opinions across groups.

These findings are in agreement with the findings by Bianconi et al. (2014) and Iñiguez et al. (2009, 2012). Bianconi et al. (2014) found community structures to emerge when triadic closure is used as a generating mechanism for complex networks. Similarly, Iñiguez et al. (2009, 2012) found that the incorporation of homophily and triadic closure in modelling opinion formation can result in the segregation of the final network into communities.

As $t \to \infty$, the network stabilises and remains unchanged for the HT model. Intuitively, the network stabilises because after some time steps, an agent's opinion will be closer to the opinion of its direct neighbours than to the opinions of the neighbours of neighbours (friends of friends). When the opinion of an agent is similar to the opinions of its neighbours, newly added edges will be deleted in order to keep the degree fixed, leaving the network unchanged.

3.2.2 Illustration by Simulation for the HT Model

In this section, we explore the HT model further by simulation.

The system is initialised with $n = 225$ for Figure 3.5 and $n = 1024$ for Figure 3.6. For both Figure 3.5 and Figure 3.6, $\boldsymbol{p}(0) \overset{\text{iid}}{\sim} N(0, 1)$ and $\varepsilon = 10^{-6}$. For $t > 0$ agents update their network using the HT dynamics specified in Section 3.2. They then update opinions taking into account the current changes in the network and the currents changes in the opinions of neighbours using Equation (3.1).

An illustration of the simulation outcome for a configuration network with maximum degree four, square lattice network and a circle network is given in Figure 3.5 for $n = 225$ and Figure 3.6 for $n = 1024$. The plot of opinions over time in Figure 3.5 shows that there is lack of consensus leading to the coexistence of different opinions. The initial network compared to the network at opinion stabilisation shows that although the network is initially connected, groups with few edges connecting groups can emerge. A similar result is observed in Figure 3.6 when the size of the network is increased from $n = 225$ to $n = 1024$. Figure 3.6 shows that different opinions will persist even when the network size is increased.

Notice that as the size of the network is increased from $n = 225$ in Figure 3.5 to $n = 1024$ in Figure 3.6, the stabilisation time of opinions increases for the different initial networks. Additionally, the circle network takes more time for opinions to stabilise and it has many different coexisting opinions compared to the configuration network and the square lattice network in both Figure 3.5 and Figure 3.6. A plausible explanation for this is the high average shortest path length between agents in the circle network compared to the configuration network and the square lattice network (see also Chapter 4).

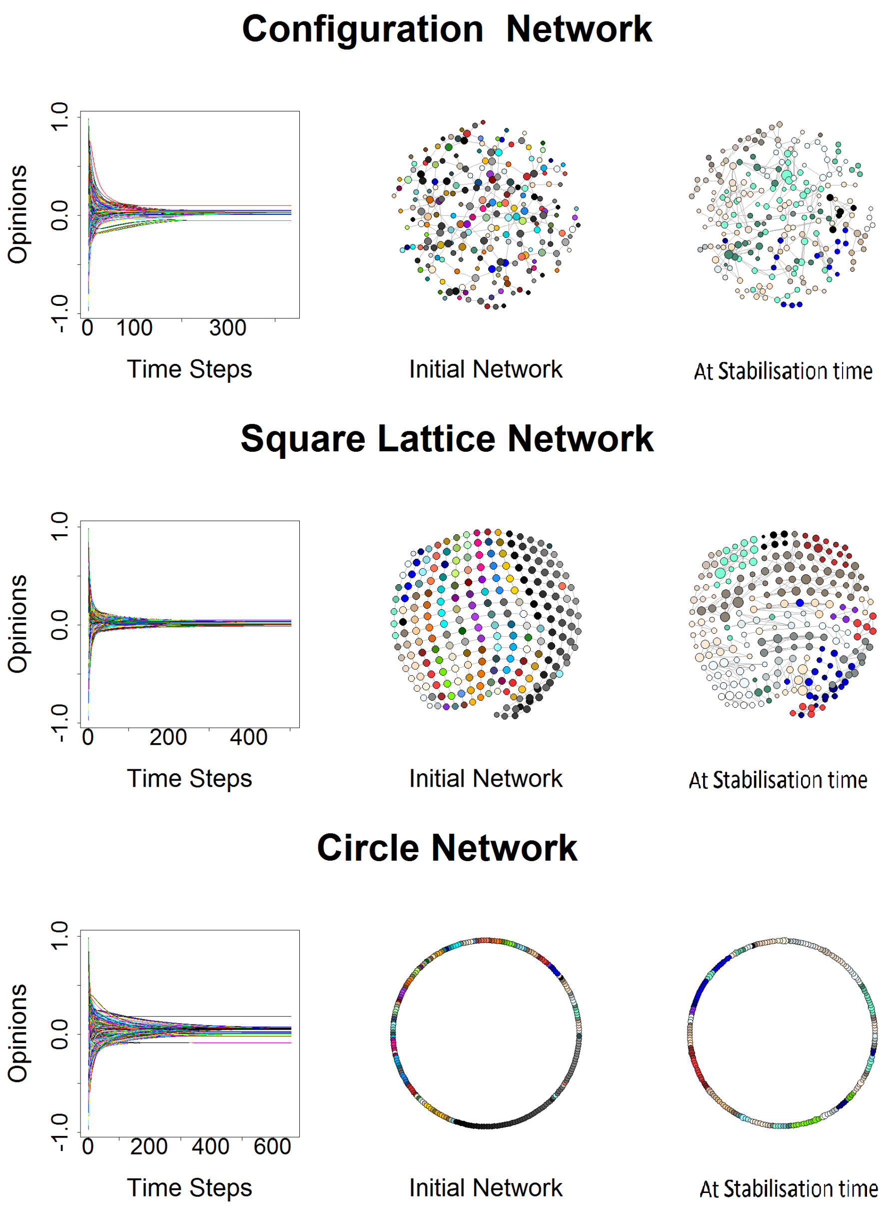


Figure 3.5. *Stabilisation of opinions, initial network and final network with* $n = 225$ *and* $\varepsilon = 10^{-6}$ *for a configuration network, a square lattice network and a circle network. The initial network and the network at stabilisation time are coloured by the different opinions of agents. The different opinions at stabilisation time implies that the opinions stabilise to a state of lack of consensus.*

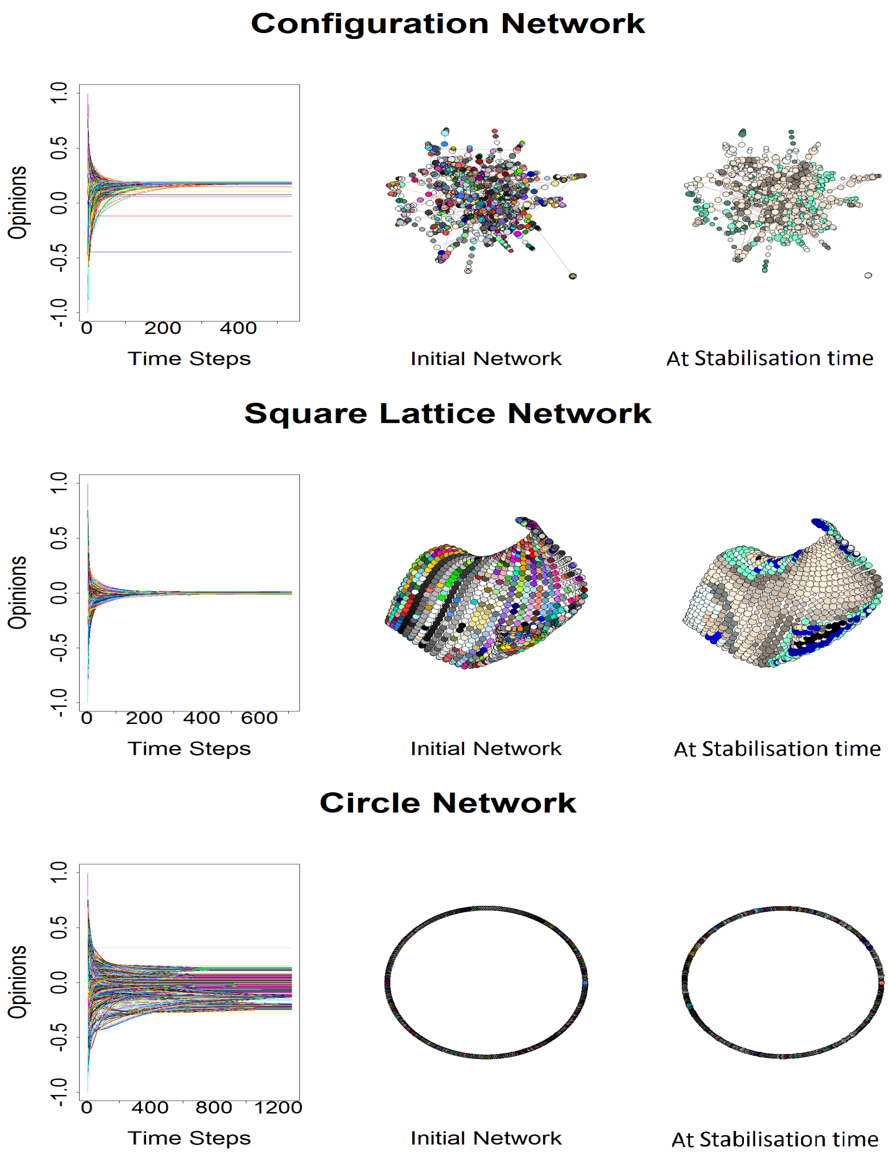


Figure 3.6. *Stabilisation of opinions, initial network and final network with $n = 1024$ and $\varepsilon = 10^{-6}$ for a configuration network, a square lattice network and a circle network. The initial network and the network at stabilisation time are coloured by the different opinions of agents. The different opinions at stabilisation time implies that the opinions stabilise to a state of lack of consensus when the network size is increased from $n = 225$ to $n = 1024$.*

3.3 Dynamic Networks with Insistent Agents

Recall from Section 2.1.3 that insistent agents are agents who do not change their opinions over time. The set of insistent agents is represented by $Z = \{1, 2, \ldots, z\}$, and the set of non-insistent agents is represented by $\overline{Z} = N \setminus Z$.

The network is updated at each time step using the RS network dynamics for the RS model and the HT network dynamics for the HT model described in Section 3.1 and Section 3.2 respectively.

After updating the network, insistent agents keep to their own opinions, while non-insistent agents revise their opinions using Equation (3.1), so that

$$\begin{aligned}
\boldsymbol{p}_z(t+1) &= \boldsymbol{p}_z(t) = \boldsymbol{p}_z(0) \quad \forall z \in Z, \\
\boldsymbol{p}_{\overline{z}}(t+1) &= \boldsymbol{T}(\boldsymbol{G}(t), \boldsymbol{p}(t))\boldsymbol{p}_{\overline{z}}(t) \quad \forall \overline{z} \in \overline{Z},
\end{aligned} \tag{3.9}$$

where $\boldsymbol{T}(\boldsymbol{G}(t), \boldsymbol{p}(t))$ is given by

$$\boldsymbol{T}_{ij}(\boldsymbol{G}(t), \boldsymbol{p}(t)) = \frac{\mathscr{G}_{ij}(t) \exp\{-\lambda_{ij}|p_i(t) - p_j(t)|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik}(t) \exp\{-\lambda_{ik}|p_i(t) - p_k(t)|\}}, \qquad i \in \overline{Z}, j \in N.$$

Note that the averaging of opinions by non-insistent agents is performed over the current neighbours of the agents in the network $\boldsymbol{G}(t)$. Simulation results for the RS model and the HT model are given in Figure 3.7 to Figure 3.10.

Figure 3.7 and Figure 3.8 show the influence of network dynamics on final opinions in the presence of insistent agents for the RS model. Results are displayed for the configuration network, the square lattice network and the circle network as in

Figure 2.1. The size of the network is set to $n = 225$ for Figure 3.7 and $n = 1024$ for Figure 3.8, the number of insistent agents to $z = 22$ for Figure 3.7 and $z = 102$ for Figure 3.8. For both Figure 3.7 and Figure 3.8, $\varepsilon = 10^{-6}$ and $\boldsymbol{p}(0) \overset{\text{iid}}{\sim} N(0, 1)$.

For Figure 3.7a and Figure 3.8a, the opinions of insistent agents are the maximum of $\boldsymbol{p}(0)$ while for Figure 3.7b and Figure 3.8b, half of the insistent agents are randomly chosen and assigned initial opinions to be the maximum of $\boldsymbol{p}(0)$ and the remaining half have initial opinions to be the minimum of $\boldsymbol{p}(0)$. The first 1000 iterations are reported when opinions do not stabilise.

When insistent agents have the same opinion, the RS network dynamics model stabilises to a consensus which is the opinions of the insistent agents, Figure 3.7a and Figure 3.8a. This is similar to the static network case, where we proved in Corollary 2.1 that when insistent agents have the same opinion, opinions will converge to a consensus which is the opinions of the insistent agents. On the other hand, the HT network dynamics model can result in the persistence of different opinions even when all insistent agents have the same opinion.

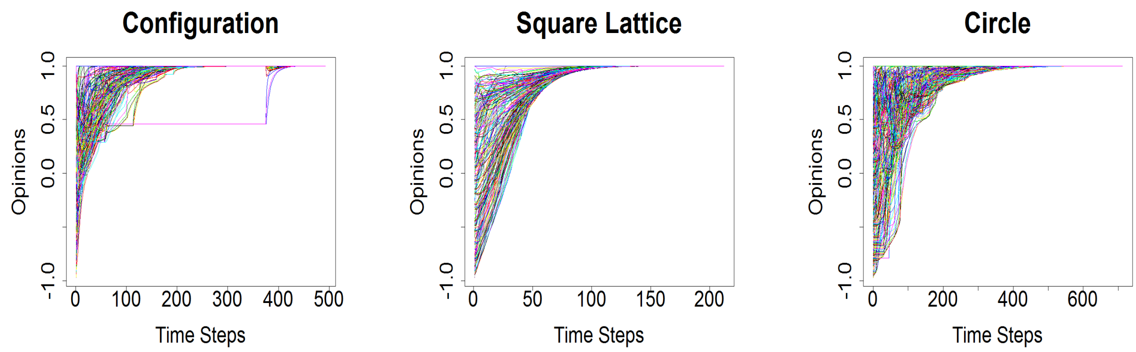
When insistent agents have two different opinions, coexistence of different opinions persists, Figure 3.7b and Figure 3.8b. Additionally, notice in Figure 3.7b and Figure 3.8b that non-insistent agents continue to switch opinions between the two groups of insistent agents. Therefore we do not observe stabilisation in the opinions of the non-insistent agents, and the first 1000 iterations are reported for Figures 3.7b and 3.8b.

The findings for $n = 225$ (Figure 3.7) are consistent with the findings for $n = 1024$ (Figure 3.8) when the size of the network is increased from $n = 225$ to

$n = 1024$ and when the number of insistent agents is increased from $z = 22$ to $z = 102$. Furthermore, the stabilisation time of opinions increases across the different networks when the size of the network is increased from $n = 225$ to $n = 1024$.

The RS Model with Insistent Agents ($n = 225$, $z = 22$)

(a) Insistent agents having the same opinion



(b) Insistent agents having two different opinions

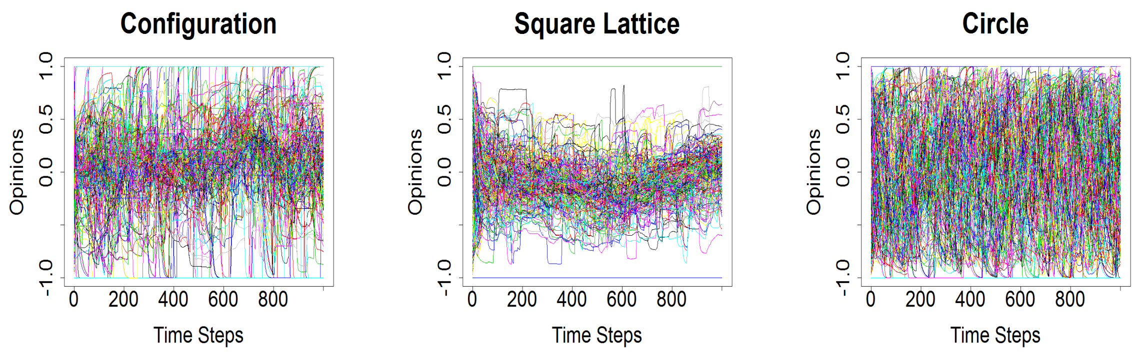
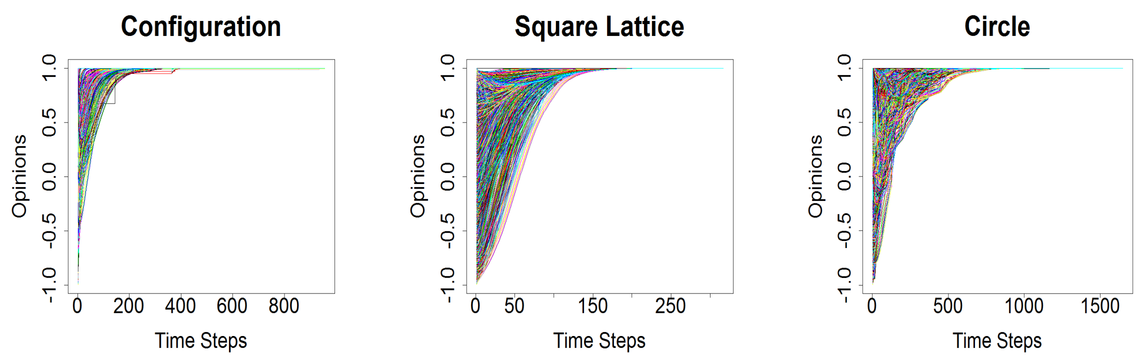


Figure 3.7. *Stabilisation of opinions for insistent agents in a configuration network, a square lattice network and a circle network. The size of the network is $n = 225$, the number of insistent agents is $z = 22$ and $\varepsilon = 10^{-6}$. In (a), insistent agents have the same initial opinions chosen to be the maximum of $\boldsymbol{p}(0)$ and in (b), insistent agents have two different initial opinions chosen to be the minimum and the maximum of $\boldsymbol{p}(0)$. (a) indicates that the opinions will stabilise to the initial opinion of insistent agents when all insistent agents have the same initial opinions. However when insistent agents have two different initial opinions (b), the coexistence of different opinions persist. The first 1000 iterations are reported for (b) as we do not observe stabilisation due to the opinions of non-insistent agents randomly switching between the opinions of the insistent agents.*

The RS Model with Insistent Agents ($n = 1024$, $z = 102$)

(a) Insistent agents having the same opinion



(b) Insistent agents having two different opinions

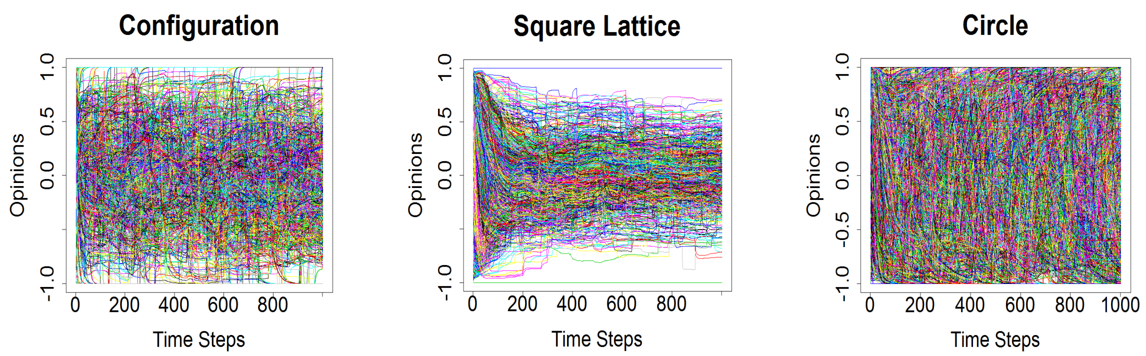


Figure 3.8. *Stabilisation of opinions for insistent agents in a configuration network, a square lattice network and a circle network. The size of the network is* $n = 1024$*, the number of insistent agents is* $z = 102$ *and* $\varepsilon = 10^{-6}$*. In (a), insistent agents have the same initial opinions chosen to be the maximum of* $\boldsymbol{p}(0)$ *and in (b), insistent agents have two different initial opinions chosen to be the minimum and the maximum of* $\boldsymbol{p}(0)$*. (a) indicates that the opinions will stabilise to the initial opinion of insistent agents when all insistent agents have the same initial opinions (a). However when insistent agents have two different initial opinions (b), the coexistence of different opinions persist. The first 1000 iterations are reported for (b) as we do not observe stabilisation due to the opinions of non-insistent agents randomly switching between the opinions of the insistent agents.*

Figure 3.9 and Figure 3.10 show the influence of insistent agents in the HT network dynamics model for the configuration network, the square lattice network and the circle network using the same network realisations. The size of the network is set to $n = 225$ for Figure 3.9 and $n = 1024$ for Figure 3.10, the number of insistent agents to $z = 22$ for Figure 3.9 and $z = 102$ for Figure 3.10. For both Figure 3.9 and Figure 3.10, $\varepsilon = 10^{-6}$ and $\boldsymbol{p}(0) \overset{\text{iid}}{\sim} N(0, 1)$.

For Figure 3.9a and Figure 3.10a, the opinions of insistent agents are the maximum of $\boldsymbol{p}(0)$ while for Figure 3.9b and Figure 3.10b, half of the insistent agents are randomly chosen and assigned initial opinions to be the maximum of $\boldsymbol{p}(0)$ and the remaining half have initial opinions to be the minimum of $\boldsymbol{p}(0)$. The first 1000 iterations are reported when opinions do not stabilise.

Figure 3.9a and Figure 3.10a show that when insistent agents have the same opinion, the final opinions might not necessarily stabilise to the initial opinion of the insistent agents. Thus, insistent agents having the same opinion might not guarantee stabilisation in opinions to their own opinion. When insistent agents have two different opinions, Figure 3.9b and Figure 3.10b, obviously, coexistence of different opinions persists.

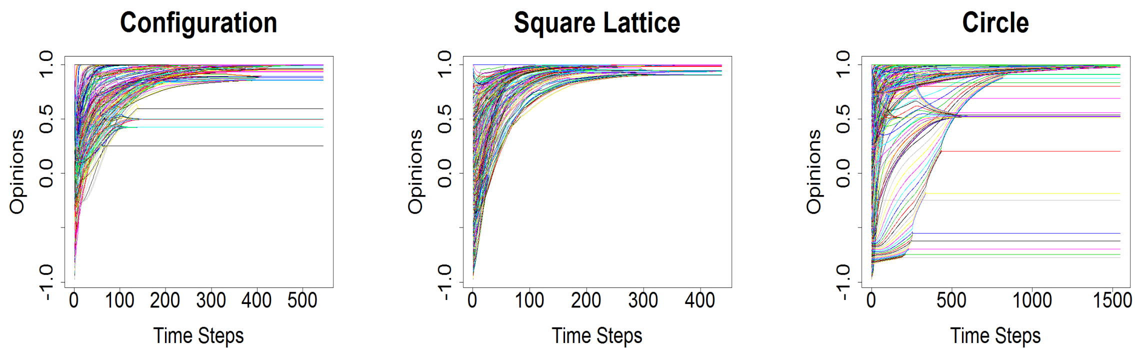
Comparing the simulation results of the RS network dynamics model in Figure 3.7b and Figure 3.8b to the HT network dynamics model in Figure 3.9b and Figure 3.10b, the different opinions coexisting in the HT model stabilise compared to the different opinions coexisting in the RS model where opinions do not stabilise as the non-insistent agents continue to switch opinions between the opinions of the insistent agents.

These findings are consistent when the size of the network is increased from $n = 225$

to $n = 1024$ and when the number of insistent agents is increased from $z = 22$ to $z = 102$. Moreover, the time to stabilisation of opinions increases for the different initial networks when the size of the network is increased from $n = 225$ (Figure 3.7a) to $n = 1024$ (Figure 3.8a).

The HT Model with Insistent Agents ($n = 225$, $z = 22$)

(a) Insistent agents having the same opinion



(b) Insistent agents having two different opinions

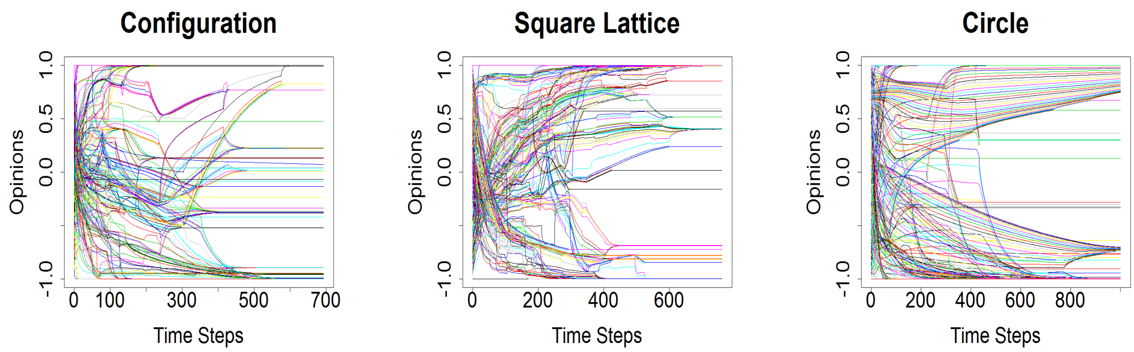
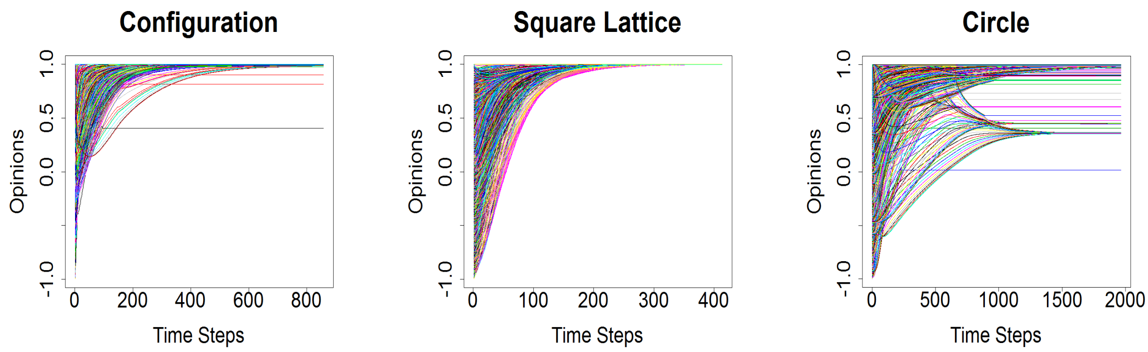


Figure 3.9. *Stabilisation of opinions for insistent agents in a configuration network, a square lattice network and a circle network. The size of the network is* $n = 225$*, the number of insistent agents is* $z = 22$ *and* $\varepsilon = 10^{-6}$*. In (a), insistent agents have the same initial opinion chosen to be the maximum of* $\boldsymbol{p}(0)$ *and in (b), insistent agents have two different initial opinions chosen to be the minimum and the maximum of* $\boldsymbol{p}(0)$*. It is indicated in (a) that insistent agents having the same opinion does not imply convergence to a consensus. For insistent agents having different opinions (b), coexistence of different opinions persists.*

The HT Model with Insistent Agents ($n = 1024$, $z = 102$)

(a) Insistent agents having the same opinion



(b) Insistent agents having two different opinions

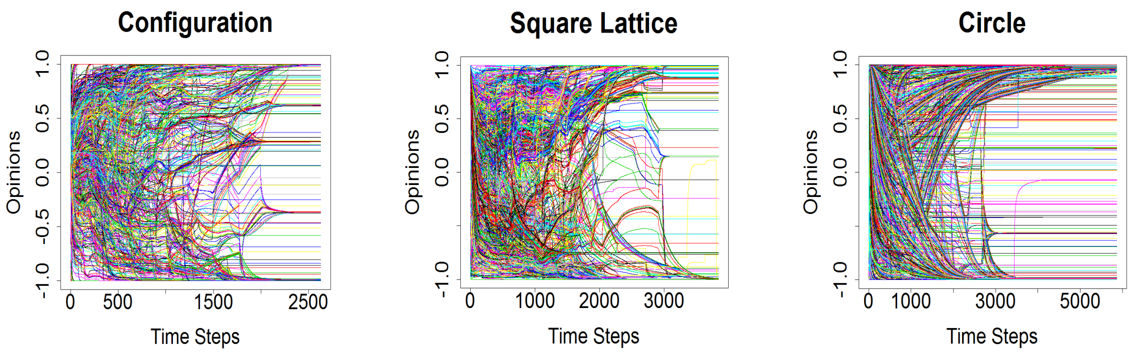


Figure 3.10. *Stabilisation of opinions for insistent agents in a configuration network, a square lattice network and a circle network. The size of the network is* $n = 1024$*, the number of insistent agents is* $z = 102$ *and* $\varepsilon = 10^{-6}$*. In (a), insistent agents have the same initial opinion chosen to be the maximum of* $\boldsymbol{p}(0)$ *and in (b), insistent agents have two different initial opinions chosen to be the minimum and the maximum of* $\boldsymbol{p}(0)$*. It is indicated in (a) that insistent agents having the same opinion does not imply convergence to a consensus. For insistent agents having different opinions (b), coexistence of different opinions persists.*

3.4 Conclusion

This chapter extends Chapter 2 to account for the influence of social network dynamics. We have investigated how the dynamics of the network structure can influence consensus or the lack of consensus of opinions. Two mechanisms of network dynamics have been considered. First, we consider the case where the network changes randomly independent of opinions. The term *random switching* network dynamics model (the RS model) is used. The RS model is based on the edge switch algorithm proposed by Gkantsidis et al. (2003). Second, we consider the case where changes in the network depends on changes in opinions. Here, we propose a network dynamics model based on homophily and triadic closure, which we term the *homophily and triadic closure* network dynamics model (the HT model).

In the RS model, two edges with distinct vertices are selected at random and the edges are switched if the switching keeps the network connected. The RS dynamics naturally preserves the degree and connectivity of the network. As a result, we have proved analytically in Theorem 3.1 that the opinions will converge to a consensus, by adopting the proof of convergence in the case of a static network. We then relaxed the condition of connectedness of the network in the RS dynamics and termed the dynamics the $\overline{RS}$ *model*. Using a different proof technique, we have proved analytically that convergence to consensus will hold for the $\overline{RS}$ dynamics if the initial opinions are drawn from a continuous distribution.

While still artificial, the HT model might be more realistic of social network dynamics compared to the RS dynamics. In the HT model, agents create new edges with neighbours of neighbours (triadic closure), and delete edges of neighbours whose opinions are far from their own opinions (homophily). We have shown that the HT dynamic can result in lack of consensus even for a connected initial network. Additionally, the network might segregate into groups with similar opinions within groups and different opinions across groups.

The RS and the $\overline{RS}$ dynamics compared to the HT dynamics lead to convergence to a consensus, while the HT dynamics enhances the lack of consensus formation. We conclude that the dynamics of the network influences the outcome opinions converge to.

Intuitively, if the network disconnects into isolated agents with low probability of reconnection, faster than the dynamics of opinions in reaching consensus, then coexistence of different opinions might emerge (as in the case of the HT model). However, if the network remains mostly connected (as in the case of the RS model), or if the network disconnects into groups without isolated agents and with high probability of reconnection (as in the case of the $\overline{RS}$ model), then opinions will converge to a consensus.

With the introduction of insistent agents, it is observed that generally insistent agents could alter the outcome of the final opinions. In particular, it is found that when insistent agents have the same opinion in the RS model, they influence the opinions of the entire population to converge to theirs. A similar result is observed for the case of a static network. However, in the case where insistent agents have different opinions, they influence the opinions of the population to a state where different opinions coexist. The opinions of agents switch between the opinions of insistent agents leading to unstable opinions. As a result, we do not observe stabilisation of opinions in the simulation.

On the other hand, for the HT model, insistent agents having the same opinion do not necessarily enhance their ability to influence the opinions of the entire population to converge to theirs. This finding is of particular interest as it relates to observations in the real-world. Here, the influential power of insistent agents of the same opinion to sway the opinions of others to converge to theirs is diminished for the HT network dynamics. Therefore whether insistent agents have the same or different opinions, the opinions can converge to a state of lack of consensus. In the simulation, unlike the RS model in Figure 3.7b and Figure 3.8b, the HT model leads to more stable opinions as shown in Figure 3.9b and Figure 3.10b even in the case where insistent agents have different opinions.

Chapter 4

The Influence of a Shift in Opinions

This chapter investigates the influence of a sudden shift in the opinions of agents over time. A sudden shift in opinions may ensue from exposure to external information other than interpersonal interactions among agents.

The influence of a sudden shift in opinions could have interesting implications for areas such as political decision making, marketing new products, the introduction of new technology and the spread of new ideas (Rogers, 2003; Valente, 2005; Light et al., 2013; Greenan, 2015). In these areas, timing and target population can be crucial in achieving intended goals and objectives. There is therefore the need to understand when to introduce a shift in opinions, and what fraction of agents should first be exposed to the shift. This chapter focuses on the influence of the time of the introduction of a sudden shift in opinions and the influence of the number of initial recipients of a shift in opinions on the average stabilisation time of opinions. Additionally, the influence of network structures and network positions of initial recipients are investigated.

Recall that the initial opinions are represented by a vector $\boldsymbol{p}(0) = \{p_1(0), p_2(0), \ldots, p_n(0)\}$, with $\underline{p} = \min\{\boldsymbol{p}(0)\}$ and $\overline{p} = \max\{\boldsymbol{p}(0)\}$. Let $p^S \in [\underline{p}, \overline{p}]$ be a shift parameter stemming from an external information source.

As agents interact and update their network and opinions, a shift in opinion p^S is introduced once, at a specified time and to a specified number of initial recipients. The time taken for opinions to stabilise after the introduction of an opinion shift is recorded. This process is repeated for different times of introducing a shift in opinions and for different numbers of initial recipients.

Let $N = \{1, 2, \ldots, n\}$ be a set of n agents, $N_{\text{shift}} = \{1, \ldots, n_{\text{shift}}\} \subset N$ be a set of n_{shift} randomly selected initial recipients who experience a shift in their opinions, and let t_{shift} be the time at which a shift is introduced. Let t_{stab} be the average stabilisation time of the model without the introduction of a shift. Let $t_{\text{stab}}^{\text{shift}}$ be the average stabilisation time of the model with the introduction of a shift in opinions.

When a shift is introduced, we assume that the n_{shift} agents experience either a temporary shift in opinions or a permanent shift in opinions (Alpert and Anderson, 1973; Liu and Johnson, 2011). These two scenarios are investigated separately.

For a *temporary shift* in opinions, the current opinions of the n_{shift} agents at time t_{shift}, change from p_i to $p_i + p^S$ for all $i \in N_{\text{shift}}$. For a *permanent shift* in opinions, the opinions of the n_{shift} agents at time t_{shift}, change from p_i to p^S for all $i \in N_{\text{shift}}$, and they keep to this opinion for the rest of the updating period, thus turning into insistent agents. In both scenarios, the opinions of the rest of the population at time t_{shift}, remain the same, $p_j \to p_j$ for $j \notin N_{\text{shift}}$.

The introduction of a shift p^S in opinions at time t_{shift} can be formalised for $i \in N_{\text{shift}}$ as:

$$p_i(t_{\text{shift}}) = \begin{cases} p_i(t_{\text{shift}}) + p^S & \text{if the shift is temporary,} \\ p^S & \text{if the shift is permanent,} \end{cases}$$

and for any time $t > t_{\text{shift}}$,

$$p_i(t+1) = \begin{cases} \sum_k \boldsymbol{T}_{ik}(\boldsymbol{G}(t), \boldsymbol{p}(t)) p_k(t) & \text{if the shift is temporary,} \\ p^S & \text{if the shift is permanent.} \end{cases}$$

Thus, agents who experience a temporary shift in opinions continue to update their opinions as usual using Equation (3.1). The opinions of agents permanently influenced by the shift remain the same at p^S for the rest of the updating period, thus such agents become insistent agents. In both a temporary shift and a permanent shift, the agents in $N \setminus N_{\text{shift}}$ updated their opinions according to the usual updating rule given in Equation (3.1).

For each type of shift in opinions, the influence of the time of the introduction of a shift, and the influence of the number of initial recipients are examined. The overall influence of a shift in opinions is measured by assessing the impact of the shift on the average stabilisation time of opinions using simulation.

4.1 The Immediate Effect of a Shift in Opinions

We conjecture that a shift in opinions has an immediate effect on the population variance of opinions which can influence the average stabilisation time of opinions. In this Section, we establish the relationship between the population variance of opinions just before and just after a shift is introduced. In Section 4.1.3, we investigate the relationship between the stabilisation time of opinions and the difference in the population variance of opinions just before and just after the introduction of a shift.

Consider a society of n agents each with an opinion p_i, $i = 1, \ldots, n$. The mean and variance of these opinions are:

$$\bar{p}_0 = \frac{1}{n} \sum_i p_i,$$

and

$$\sigma_0^2 = \frac{1}{n} \sum_i (p_i - \bar{p}_0)^2.$$

4.1.1 The Effect of a Temporary Shift in Opinions

Now consider the effect on the mean and variance of a shift of opinion amongst a subset of the society. Recall that p^S is an opinion shift parameter and n_{shift} is the number of agents who shift opinions. Introduce an indicator variable u_i such that $u_i = 1$ indicates an agent that will shift its opinion, and $u_i = 0$ indicates an agent that does not shift opinion. Then

$$n_{\text{shift}} = \sum_i u_i.$$

Suppose that a change of opinion occurs that shift each opinion according to the model:

$$p_i \to p_i + u_i p^S,$$

for some opinion increment p^S.

Then, the new mean and variance are thus functions of p^S and are given by the expressions:

$$\bar{p}(p^S) = \frac{1}{n}\sum_i (p_i + u_i p^S),$$

and

$$\sigma^2(p^S) = \frac{1}{n}\sum_i (p_i + u_i p^S - \bar{p}(p^S))^2.$$

The equation for $\bar{p}(p^S)$ can be reduced to the expression:

$$\bar{p}(p^S) = \bar{p}_0 + \frac{n_{\text{shift}}}{n} p^S.$$

The equation for $\sigma^2(p^S)$ can be written in the form,

$$\sigma^2(p^S) = \frac{1}{n}\sum_i \Big(p_i - \bar{p}_0 + \big(u_i - \frac{n_{\text{shift}}}{n}\big)p^S\Big)^2.$$

Observe that $\sigma^2(p^S) \geq 0$ and that $\sigma^2(p^S)$ is a quadratic function of p^S.

We now calculate the value, $p^{\tilde{S}}$, of p^S, that minimises $\sigma^2(p^S)$. Differentiating with respect to p^S gives the derivative,

$$\frac{\partial \sigma^2(p^S)}{\partial p^S} = \frac{2}{n} \sum_i \{ \Big(p_i - \bar{p}_0 + \big(u_i - \frac{n_{\text{shift}}}{n} \big) p^S \Big) \big(u_i - \frac{n_{\text{shift}}}{n} \big) \}.$$

This equation reduces to,

$$\frac{\partial \sigma^2(p^S)}{\partial p^S} = \frac{2}{n} \sum_i \{ \big(p_i - \bar{p}_0 \big) \big(u_i - \frac{n_{\text{shift}}}{n} \big) \} + \frac{2p^S}{n} \sum_i (u_i - \frac{n_{\text{shift}}}{n})^2.$$

This leads to the value of $p^{\tilde{S}}$,

$$p^{\tilde{S}} = -\frac{\sum_i \{ \big(p_i - \bar{p}_0 \big) \big(u_i - \frac{n_{\text{shift}}}{n} \big) \}}{\sum_i (u_i - \frac{n_{\text{shift}}}{n})^2}.$$

Now note that $u_i^2 = u_i$ to evaluate the denominator in the previous equation as:

$$\sum_i (u_i - \frac{n_{\text{shift}}}{n})^2 = \sum_i \left\{ u_i^2 - 2u_i \frac{n_{\text{shift}}}{n} + \frac{(n_{\text{shift}})^2}{n^2} \right\} = n_{\text{shift}} \big(1 - \frac{n_{\text{shift}}}{n} \big).$$

The second derivative of $\sigma^2(p^S)$ is thus given by,

$$\frac{\partial^2 \sigma^2(p^S)}{\partial {p^S}^2} = \frac{2}{n} n_{\text{shift}} \big(1 - \frac{n_{\text{shift}}}{n} \big).$$

We can now find the quadratic function for $\sigma^2(p^S)$ by expanding in a Taylor series around the minimiser $p^{\tilde{S}}$. Note that because $\sigma^2(p^S)$ is quadratic, then the Taylor series terminates at the second order. Integrate the second derivative with respect to p^S, eliminating the constant of integration by using the fact that the derivative is zero at the minimiser. Then integrate with respect to p^S a second time, and eliminate the constant of integration using the value of $\sigma^2(p^S)$ at the minimiser.

Thus we find that,

$$\sigma^2(p^S) = \sigma^2(p^{\tilde{S}}) + \frac{n_{\text{shift}}}{n}\big(1 - \frac{n_{\text{shift}}}{n}\big)(p^S - p^{\tilde{S}})^2.$$

Now consider the effect of changing p^S from $p^S = 0$, corresponding to undisturbed opinions, where the variance is the undisturbed variance $\sigma_0^2 = \sigma^2(0)$, one sees that if p^S moves from zero a small distance towards the minimiser $p^{\tilde{S}}$, then the population variance must decrease. However, if p^S moves from zero a small distance away from the minimiser $p^{\tilde{S}}$, then the population variance must increase.

4.1.2 The Effect of a Permanent Shift in Opinions

Now consider the case where there are two populations, one of size n_a and the other of size n_b with opinions a_i, with $i = 1, ..., n_a$, and b_j, with $j = 1, ..., n_b$.

The two populations have means and variances given by,

$$\bar{a} = \frac{1}{n_a}\sum_i a_i,$$

$$\sigma_a^2 = \overline{a^2} - \bar{a}^2,$$

where

$$\overline{a^2} = \frac{1}{n_a}\sum_i a_i^2.$$

Similarly, for the second population,

$$\bar{b} = \frac{1}{n_b}\sum_j b_j,$$

$$\sigma_b^2 = \overline{b^2} - \bar{b}^2,$$

where

$$\overline{b^2} = \frac{1}{n_b} \sum_j b_j^2.$$

Now consider the combined population of size $n = n_a + n_b$, then the mean and pooled variance of opinions is given by,

$$\bar{c} = \frac{1}{n}(n_a \bar{a} + n_b \bar{b}),$$

and

$$\sigma_c^2 = \frac{1}{n}\big(n_a \overline{a^2} + n_b \overline{b^2} - \frac{1}{n}(n_a \bar{a} + n_b \bar{b})^2\big).$$

By differentiating σ_c^2 with respect to b_j for $j = 1, ..., n_b$ and setting the derivative to zero, one finds that the values of b_j for a minimum total variance are that,

$$b_j = \bar{a}.$$

for each value of j.

Now consider the situation of a permanent shift in opinion in the second population from some starting opinions b_j to some permanently changed opinion, p^S. The combined population variance is then given by,

$$\sigma^2(p^S) = \frac{1}{n}\big(n_a \overline{a^2} + n_b (p^S)^2 - \frac{1}{n}(n_a \bar{a} + n_b p^S)^2\big).$$

By computing the derivative of $\sigma^2(p^S)$ and setting the derivative to zero one finds that the value of p^S that minimises the population variance after a shift of opinion amongst the second population to p^S is $p^S = \bar{a}$.

Thus it follows that a permanent shift of $b_j \to \bar{a}$ gives the smallest population variance (for all the a_i's fixed). It also follows that if the b_j's are different from $\bar{a}$, then a shift to $b_j \to \bar{a}$ will reduce the total variance. It also follows that a shift to a large enough absolute value of p^S will increase the variance from some starting value.

Thus for both temporary and permanent shifts in opinion it is, in general, possible for the variance to decrease or to increase.

4.1.3 The Influence of Variance on Stabilisation Time without Network Dynamics

In this section, we investigate how the stabilisation time of opinions is influenced by an increase or a decrease in the population variance of opinions introduced by a shift in opinions. We also investigate the relationship between the stabilisation time of opinions and the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift.

Intuitively, if the introduction of a shift increases the population variance, then we would expect that it will take more time steps on average for opinions to stabilise. However, if the introduction of a shift decreases the population variance, then we would expect that it will take a fewer time steps on average for opinions to stabilise.

Similarly, we would expect that if the introduction of a shift increases the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift, then it should on average take more time steps for opinions to stabilise. If however, the introduction of a shift in opinions decreases the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift, then it should on average take fewer time steps for opinions to stabilise.

Illustrations from simulation are given in Figure 4.1 for the difference in population variance of opinions just before and just after the introduction of a shift, and in Figure 4.2 for the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the shift introduction, and the stabilisation time of opinions. Figure 4.1 and Figure 4.2 addresses both a temporary shift and a permanent shift.

We consider the dynamic weight model from Equation (2.11) & (2.12) with a fixed network structure given as

$$\boldsymbol{p}(t+1) = \boldsymbol{T}(\boldsymbol{G}, \boldsymbol{p}(t))\boldsymbol{p}(t) \quad \text{for } t > 0,$$

where

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}) = \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}.$$

At the initial time step, the size of the network is set to $n = 225$, $p_i(0) \overset{\text{iid}}{\sim} N(0,1)$ and $\varepsilon = 10^{-6}$. We choose $t_{\text{shift}} \approx \frac{1}{2} t_{\text{stab}}$, (midway through the updating process) to enhance comparison, so that $t_{\text{shift}} = 200$ for the configuration network, $t_{\text{shift}} = 400$

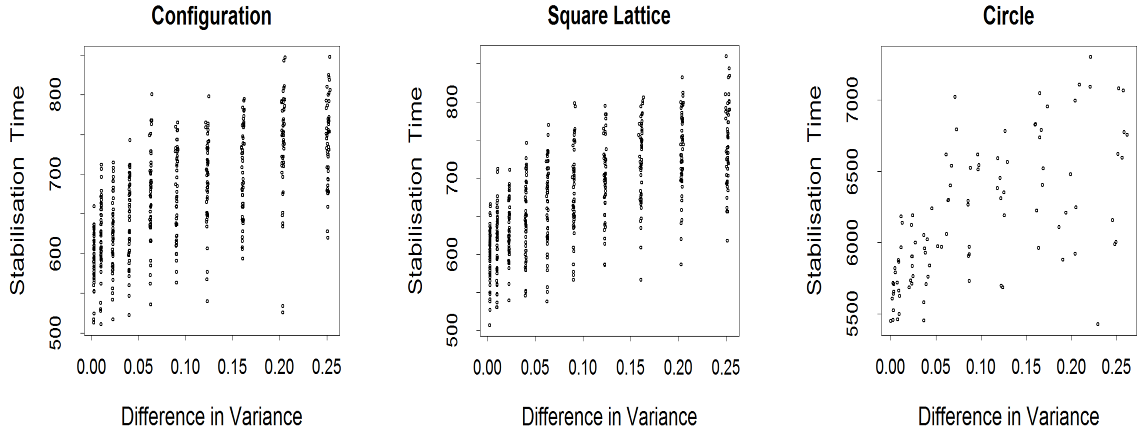
for the square lattice network and $t_{\text{shift}} = 4500$ for the circle network. The number of initial recipients is set to $n_{\text{shift}} = 112$ ($\approx 0.5 \times n$) for all the networks, so that we expect to see a larger effect in the shift of opinions. The shift p^S in opinions takes the values $p^S = 0.1, 0.2, \ldots, 1$.

For each value of p^S, the simulation is repeated 50 times, randomising the set of the n_{shift} initial recipients. At each introduction of a shift p^S in opinions, we record (1) the population variance just before and just after the shift introduction, (2) the difference between the maximum opinion and the minimum opinion just before and just after the shift introduction, and (3) the stabilisation time of opinions.

For both Figure 4.1 and Figure 4.2, a positive trend is observed for the influence of the difference in population variance and the maximal difference in opinions just before and just after the introduction of a shift. This supports the intuition that an increase (decrease) in the population variance of opinions or in the maximal difference of opinions just before and just after the introduction of a shift tends to increase (decrease) the stabilisation time of opinions.

It is worth noting here that although in theory, the introduction of a shift can increase or decrease the population variance, in the simulations however, a decrease in variance rarely occurs.

Temporary Shift in Opinions



Permanent Shift in Opinions

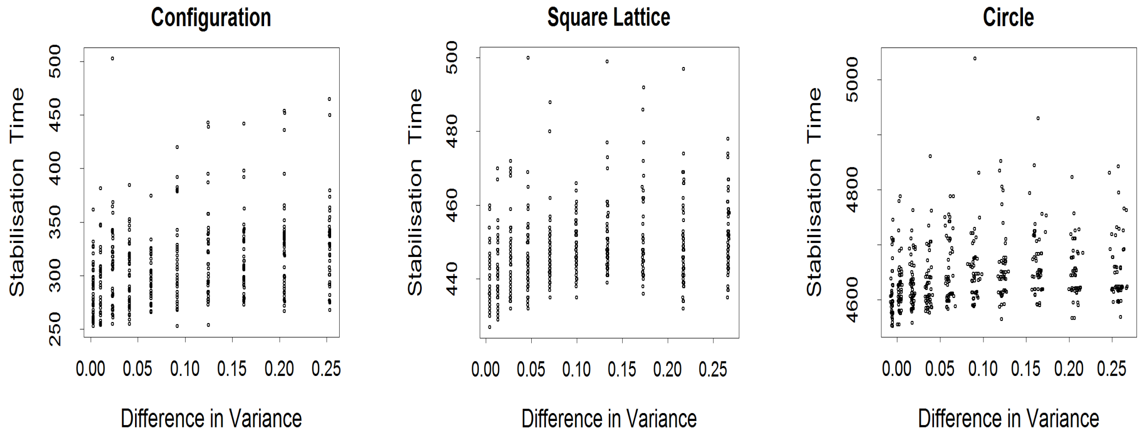
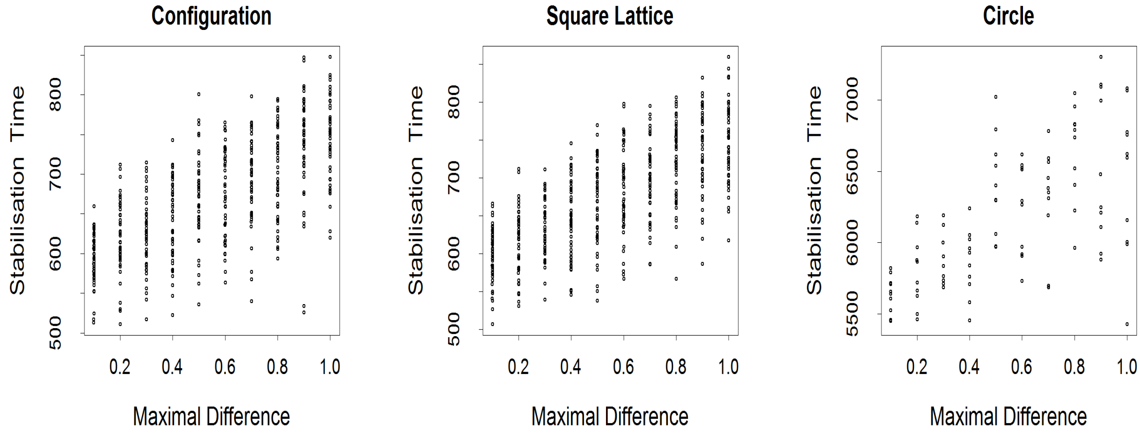


Figure 4.1. *An illustration of the difference in population variance of opinions just before and just after the introduction of a shift on the stabilisation time of opinions. The size of the network is set to* $n = 225$, $p_i(0) \overset{\text{iid}}{\sim} N(0, 1)$ *and* $\varepsilon = 10^{-6}$. *The time of a shift in opinions is set to* $t_{\text{shift}} = 200$ *for the configuration network, to* $t_{\text{shift}} = 400$ *for the square lattice network and to* $t_{\text{shift}} = 4500$ *for the circle network. The number of initial recipients is set to* $n_{\text{shift}} = 112$ $(\approx 0.5 \times n)$ *for all the networks. The shift* p^S *in opinions takes the values* $p^S = 0.1, 0.2, \ldots, 1$. *For each value of* p^S, *the simulation is repeated 50 times, randomising the set of the* n_{shift} *initial recipients.*

Temporary Shift in Opinions



Permanent Shift in Opinions

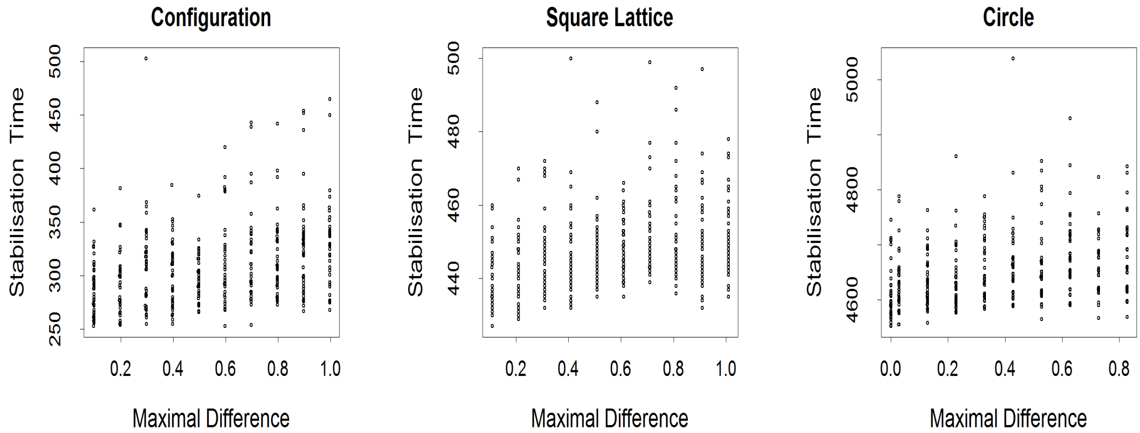


Figure 4.2. *An illustration of the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the shift introduction on the stabilisation time of opinions. The size of the network is set to* $n = 225$, $p_i(0) \overset{\text{iid}}{\sim} N(0,1)$ *and* $\varepsilon = 10^{-6}$. *The time of a shift in opinions is set to* $t_{\text{shift}} = 200$ *for the configuration network, to* $t_{\text{shift}} = 400$ *for the square lattice network and to* $t_{\text{shift}} = 4500$ *for the circle network. The number of initial recipients is set to* $n_{\text{shift}} = 112$ *(*$\approx 0.5 \times n$*) for all the networks. The shift* p^S *in opinions takes the values* $p^S = 0.1, 0.2, \ldots, 1$. *For each value of* p^S*, the simulation is repeated 50 times, randomising the set of the* n_{shift} *initial recipients.*

4.2 The Influence of Opinion Shift in Static Networks

In this section, we investigate the influence of opinion shift in static networks. In particular, we investigate the effect of the time t_{shift} and the effect of the number of initial recipients n_{shift} on the stabilisation time of opinions.

Again, consider the dynamic weight model with a static network structure introduced in Section 2.2. The updating rule for opinions is given by

$$p_i(t+1) = \sum_k \boldsymbol{T}_{ik}(\boldsymbol{G}, \boldsymbol{p}(t))p_k(t) \quad \text{for } t > 0, \tag{4.1}$$

where

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}) = \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum\limits_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}, \quad \lambda_{ij} \geq 0.$$

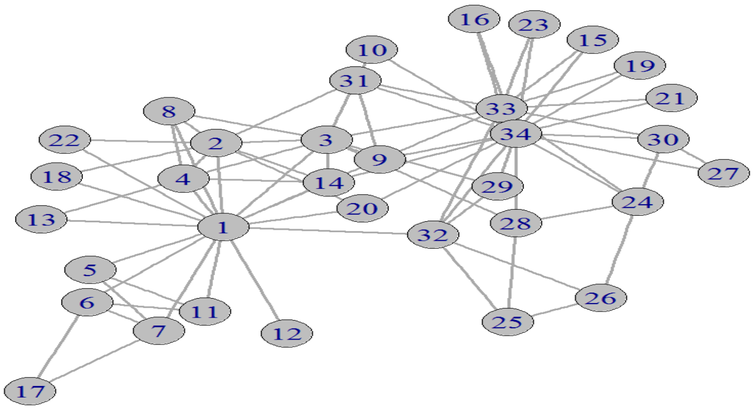
The social structures considered here are the same network realisations for the configuration network, the square lattice network and the circle network introduced in Section 2.2.2.1. Additional to these three networks, we also consider two real world networks; the Zachary karate club network (Zachary, 1977) and the Krackhardt network (Krackhardt, 1987). The karate club network consists of 34 members of a karate club, with 78 undirected edges between club members. During the study, there was a conflict between the manager (agent 34) and the coach (agent 1) which lead to a split of the club into two groups. In this section, the network structure is assumed to be static over time. The case of a dynamic network structure is considered in Section 4.3.

The Krackhardt (1987) network data set has as agents, 21 managers from a high-tech equipment manufacturing company with approximately 100 employees located on the West coast of the United States. The data set consist of an advice network, a friendship network and a report network. These three data sets are obtained by asking each manager,

1. To whom do you go for advice? (the advice network)
2. Who is your friend? (the friendship network)
3. To whom do you report? (the report network)

Figure 4.3a shows the Zachary karate club network and Figure 4.3b shows the Krackhardt friendship network. Both networks are connected. The Krackhardt friendship network is symmetrised to form an undirected network. By symmetrising the network, it fits into the framework of this thesis. Disregarding the direction of the edges clearly loses some information. In this thesis, the purpose of using these networks is solely to provide a more realistic underlying networks to study opinion formation than the simulated networks. The small size of the networks allows for a more detailed analysis than larger networks.

(a)



(b)

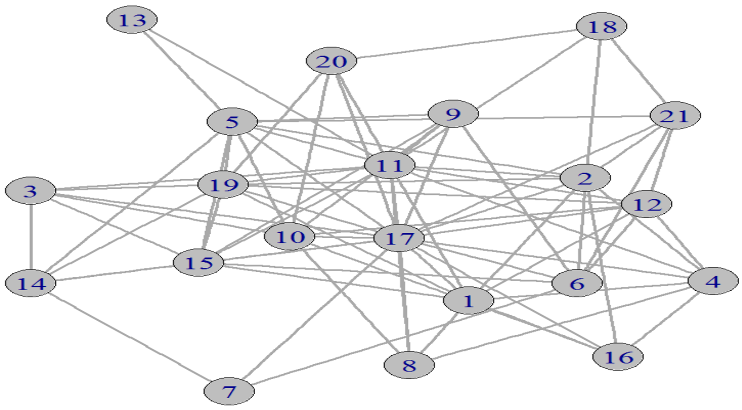


Figure 4.3. *(a) shows the Zachary karate club network and (b) shows the Krackhardt friendship network. It is assumed that every agent in both networks self communicates*

4.2.1 Simulation Parameters

The initial parameters are fixed for all simulations, $\varepsilon = 10^{-6}$, $\lambda_{ij} = 1\ \forall ij$, $p_i(0) \overset{\text{iid}}{\sim} N(0,1)$ and $p^S \in [\underline{p}, \overline{p}]$ is randomly selected from the range of initial opinions. For the configuration, the square lattice and the circle network $n = 225$.

Without the introduction of a shift in opinions, as shown in Figure 4.4 for example, it takes 13 time steps for opinions to stabilise for the Krackhardt friendship network and 56 time steps for the karate club network. It takes about 255, 566 and 5495 time steps respectively for the configuration, the square lattice and the circle network, for opinions to stabilise without the introduction of opinion shift.

To study the effect of the time of an opinion shift, n_{shift} is fixed at $n_{\text{shift}} = 0.5 \times n$ rounded to the nearest whole number for all the networks. The values of t_{shift} are set to $t_{\text{shift}} = 50, 100, \ldots, 400$ (in steps of 50) for the configuration network, $t_{\text{shift}} = 50, 100, \ldots, 700$ (in steps of 50) for the square lattice network, $t_{\text{shift}} = 100, 600, 1100, \ldots, 6500$ (in steps of 500) for the circle network, $t_{\text{shift}} = 10, 20, \ldots, 80$ (in steps of 10) for the karate club network and $t_{\text{shift}} = 2, 4, \ldots, 20$ (in steps of 2) for the Krackhardt friendship network.

By extending the time of the introduction of a shift in opinions beyond the stabilisation time t_{stab}, the behaviour of the stabilisation time $t_{\text{stab}}^{\text{shift}}$ with opinion shift can be examined before and after the stabilisation time t_{stab}.

For the effect of the number of initial recipients, the values of n_{shift} are set to $n_{\text{shift}} = 0.1 \times n, 0.2 \times n, \ldots, 1 \times n$. For the purpose of illustration, $t_{\text{shift}} \approx \frac{1}{2} t_{\text{stab}}$, to enhance comparison, so that $t_{\text{shift}} = 100$ for the configuration network, $t_{\text{shift}} = 200$

for the square lattice network, $t_{\text{shift}} = 2000$ for the circle network, $t_{\text{shift}} = 20$ for the karate club network and $t_{\text{shift}} = 5$ for the Krackhardt friendship network.

For each value of t_{shift} and n_{shift} the simulation is repeated 100 times, randomly changing the sample of agents who form the initial recipients. The results are displayed using boxplots.

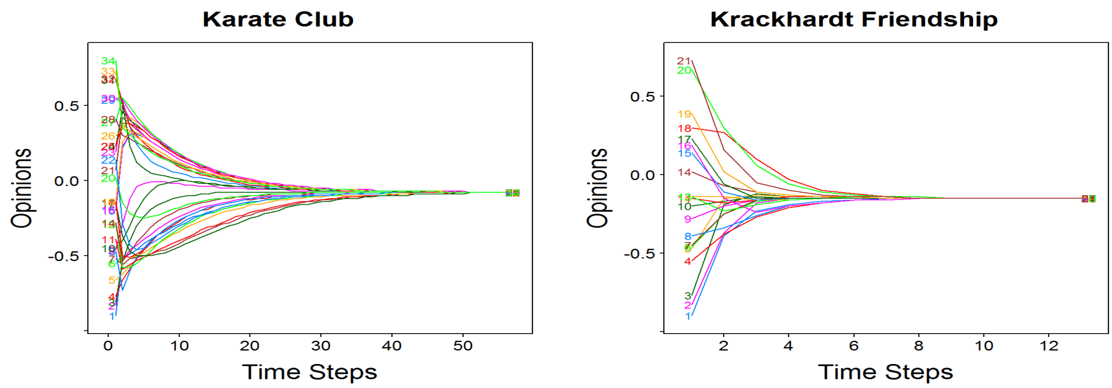


Figure 4.4. *Stabilisation path in opinions for a model without the introduction of a shift in opinions. The opinions of agents stabilises to a consensus for both the karate club network and the Krackhardt friendship network. It takes 56 time steps for the karate club network and 13 time steps for opinions to stabilise for the Krackhardt friendship network.*

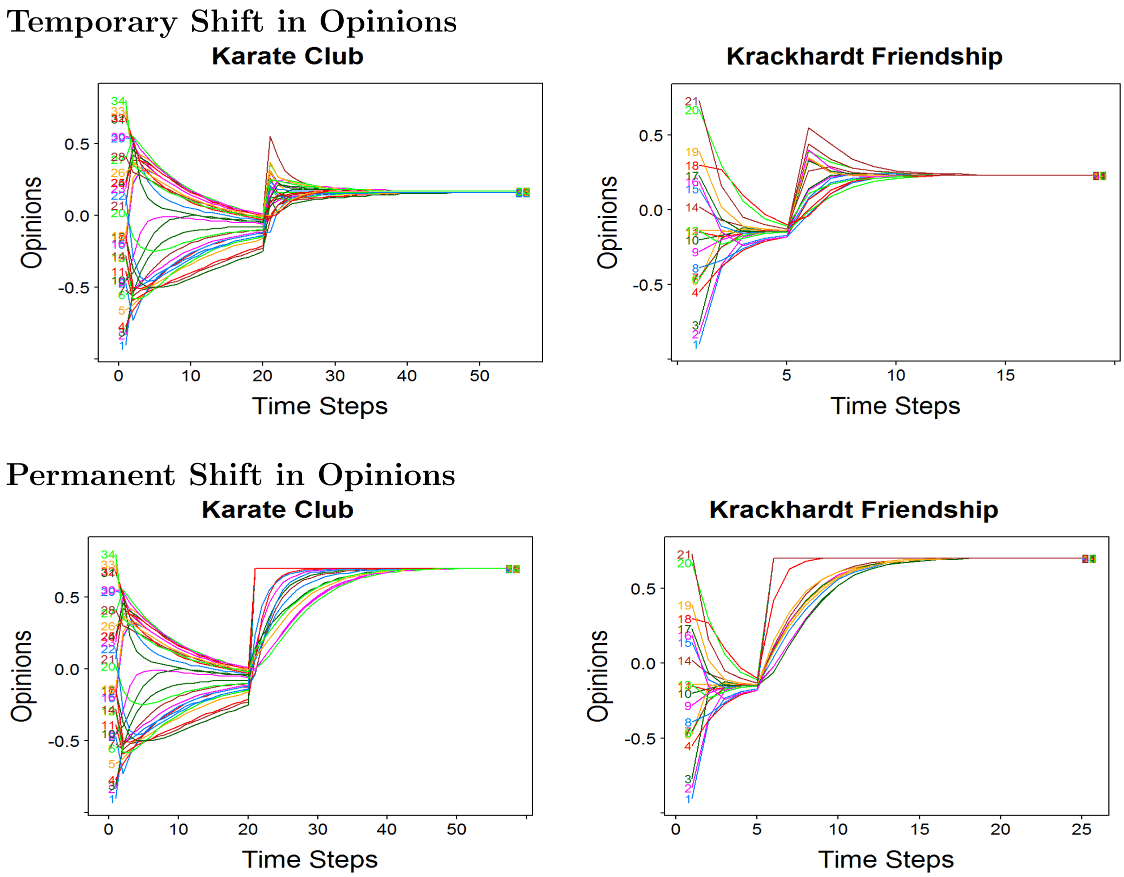


Figure 4.5. *An illustration of the influence of a temporary shift and a permanent shift in opinions. The value of t_{shift} is fixed at $t_{\text{shift}} = 5$ for the Krackhardt friendship network and $t_{\text{shift}} = 20$ for the karate club network. The value of n_{shift} randomly selected agents is set to $n_{\text{shift}} = 17$ for the karate club network and $n_{\text{shift}} = 10$ for the Krackhardt friendship network. The colours show the initial opinions of agents in the karate club and the Krackhardt friendship network. The opinions stabilise to a consensus for both a temporary shift and a permanent shift in opinions.*

Figure 4.5 shows a single run illustrating the influence of a temporary shift in opinions and a permanent shift in opinions, for the karate club network and the Krackhardt friendship network. The value of t_{shift} is fixed at $t_{\text{shift}} = 5$ for the Krackhardt friendship network and $t_{\text{shift}} = 20$ for the karate club network. The value of n_{shift} randomly selected agents is set to $n_{\text{shift}} = 10$ for the Krackhardt friendship network and $n_{\text{shift}} = 17$ for the karate club network.

For both a temporary and a permanent shift in opinions, the opinions stabilise to a consensus given the introduction of a shift. The observed consensus in opinions can be explained using Theorem 2.8. It is shown in Theorem 2.8 that opinions will converge to a consensus for all initial opinions $\boldsymbol{p}(0)$. To see this, imagine that the updating process only started when the shift in opinions is introduced. Then Theorem 2.8 applies to the new starting opinions.

Therefore, given the proposed updating rule in Equation (4.1), convergence to a consensus will be observed given that the network is connected. Also, the opinions will converge to a consensus for the introduction of a shift at any time t_{shift} and to any number of initial recipients n_{shift}.

Again, we note that in the simulation, opinions stabilise to a consensus in finite time steps due to digit truncation or rounding errors. Theoretically, convergence to consensus might not be achieved in finite time steps.

The next two subsections are focused on the effect of the time at which a shift is introduced and the effect of the number of initial recipients on the stabilisation time of opinions. In particular, we are interested in the time taken for opinions to stabilise after the introduction of a shift, measured by $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$, for a given t_{shift} and n_{shift}. We investigate how different values of t_{shift} and n_{shift} influence the average stabilisation time $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$.

Additionally, the relationship between the stabilisation time t_{stab} without the introduction of a shift and the stabilisation time $t_{\text{stab}}^{\text{shift}}$ with the introduction of a shift is investigated. Thus, the relationship among t_{shift}, n_{shift} t_{stab}, $t_{\text{stab}}^{\text{shift}}$ and $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ is examined.

4.2.2 The Effect of the Time t_{shift} of Opinion Shift

Figures 4.6 - 4.7 illustrate the influence of the time t_{shift} of a shift in opinions on the stabilisation time for a temporary and a permanent shift in opinions. The green line (with pluses) shows the average stabilisation time t_{stab} without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time $t_{\text{stab}}^{\text{shift}}$ with the introduction of a shift in opinions, and the red line (with crosses) shows the effect of t_{shift} measured by the time difference between $t_{\text{stab}}^{\text{shift}}$ and t_{shift}.

For both a temporary and a permanent shift in opinions, the time effect (red line with crosses) of a shift in opinions initially decreases with later introduction of a shift, after which it approaches a constant behaviour. A constant behaviour is observed when a shift is introduced after the opinions of agents have stabilised.

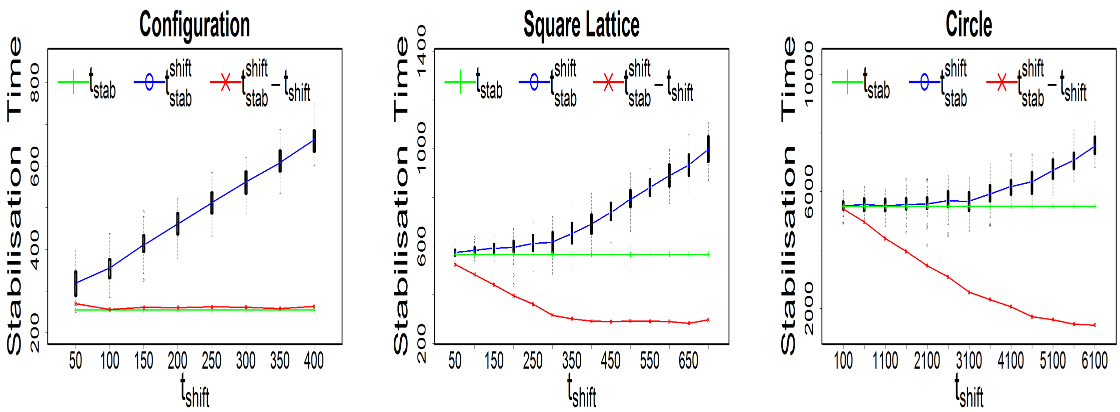
The observed time effect, indicated by the association between $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ and t_{shift}, implies that for both a temporary and a permanent shift, the later the introduction of a shift, the shorter the average time required for opinions to stabilise, however, introducing a shift after opinions have stabilise might not influence the time to stabilisation of opinions.

The time effect $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ is almost constant for the configuration network and the Krackhardt friendship network compared to the rest of the networks. This suggests that the topology of the network can influence the effect of the time of opinion shift. The observed constant behaviour suggests that for the the configuration network and the Krackhardt friendship network, the average stabilisation time of opinions might not be affected by the time at which a shift is introduced.

Intuitively, the observed differences across the different networks can be explained by the so called *small world effect* or the average shortest path length between agents. The average shortest path length is 7.42 for the configuration network, 10 for the square lattice network and 56.5 for the circle network. The configuration network has the smallest average distance between agents. Hence, it takes fewer time steps for a shift in opinions to diffuse in the network, leading to a faster stabilisation in opinions compared to the square lattice and the circle network. The circle network having the highest average shortest path length requires more time steps for opinions to stabilise compared to the other networks.

Similarly, the average shortest path length is 1.65 for the Krackhardt network and 2.41 for the karate club network. Hence, a shift in opinions diffuses faster in the Krackhardt network compared to the karate club network as shown in Figure 4.7.

Temporary Shift in Opinions



Permanent Shift in Opinions

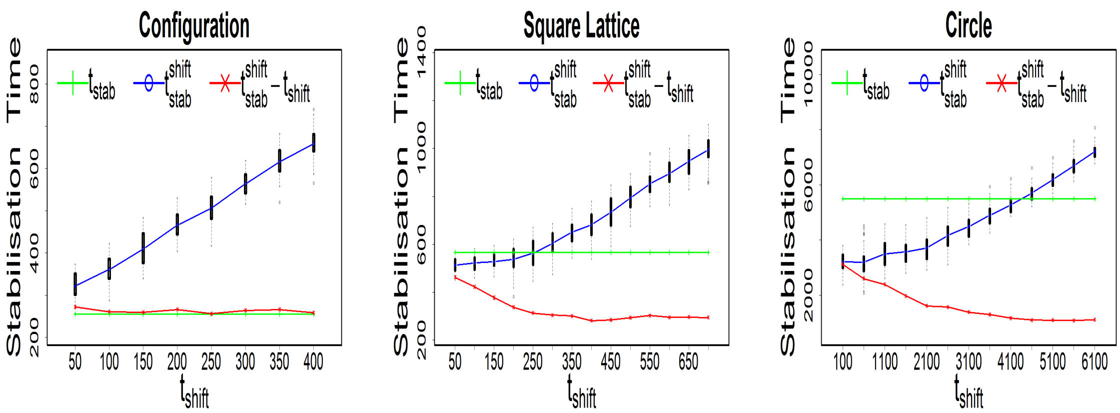
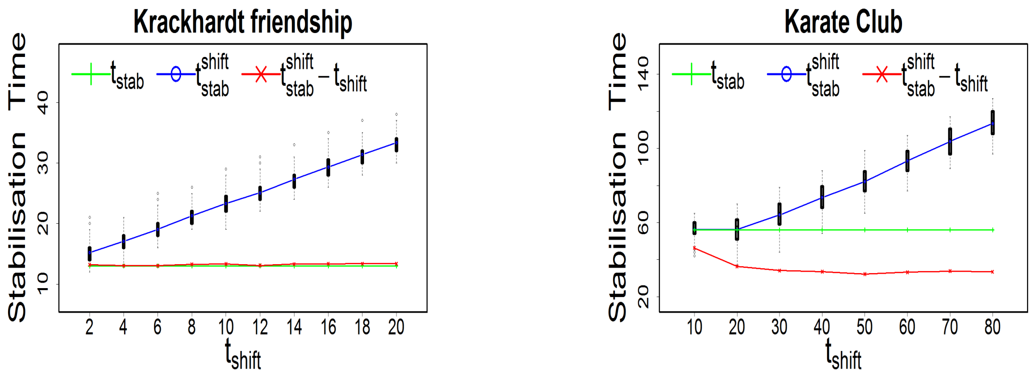


Figure 4.6. *A boxplot illustration of the influence of* t_{shift} *for a temporary and a permanent shift in opinions with* $p^S = 0.7$ *and* $n = 225$ *for the configuration, square lattice and the circle network. The green line (with pluses) shows the average stabilisation time* t_{stab} *without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time* t_{stab}^{shift} *with the introduction of opinion shift, and the red line (with crosses) shows the time difference between* t_{stab}^{shift} *and* t_{shift}*. A nonlinear decreasing effect of time (red line with crosses) is observed for the square lattice and the circle network, whereas an almost constant effect is observed for the configuration network.*

Temporary Shift in Opinions



Permanent Shift in Opinions

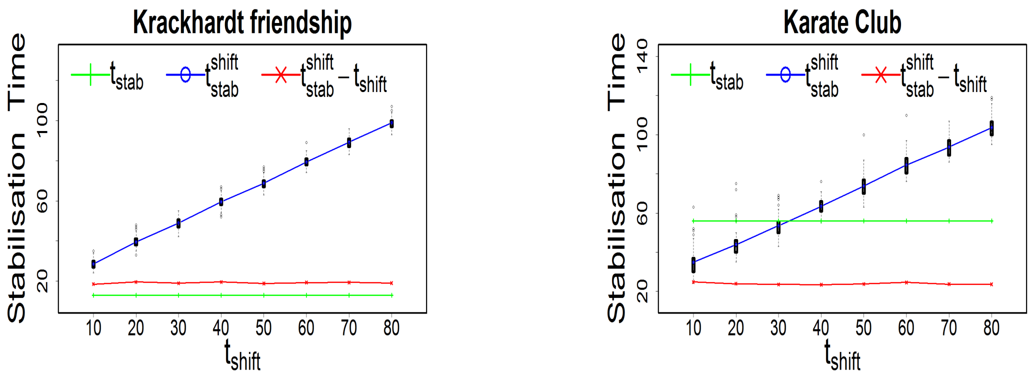


Figure 4.7. *A boxplot illustration of the influence of* t_{shift} *for a temporary and a permanent shift in opinions with* $p^S = 0.7$*. The value of* n_{shift} *is fixed at* $n_{\text{shift}} = 17$ *for the karate club network and* $n_{\text{shift}} = 11$ *for the Krackhardt friendship network. The simulation is carried out for different values of* p^S*, and all give similar trend. The green line (with pluses) shows the average stabilisation time* t_{stab} *without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time* t_{stab}^{shift} *with the introduction of opinion shift, and the red line (with crosses) shows the time difference between* t_{stab}^{shift} *and* t_{shift}*. A nonlinear decreasing effect of time (red line with crosses) is observed for the karate club network, whereas an almost constant effect is observed for the Krackhardt friendship network.*

4.2.3 The Effect of the Number of Initial Recipients n_{shift} of Opinion Shift

Figures 4.8 - 4.9 illustrate the influence of the number of initial recipients n_{shift} of a shift in opinions on the stabilisation time of opinions, for both a temporary and a permanent shift in opinions. The green line (with pluses) shows the average stabilisation time t_{stab} without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time $t_{\text{stab}}^{\text{shift}}$ with the introduction of a shift in opinions, and the red line (with crosses) shows the effect of n_{shift} measured by the time difference between $t_{\text{stab}}^{\text{shift}}$ and t_{shift}.

For a temporary shift, as the number of initial recipients n_{shift} increases, the effect on the average stabilisation time $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$, is almost quadratic especially for the configuration network (Figure 4.8) and the Krackhardt friendship network (Figure 4.9). The observed quadratic association for a temporary shift in opinions, suggests that the average stabilisation time first increases and then decreases as the numbers of initial recipients n_{shift} increases.

The observed quadratic effect is an indication that when only few agents experience a shift in opinions, then they will soon be re-converted to be close to the old opinions, and when many agents experience a shift in opinions, then the few left with other opinions will soon be converted to the new opinion. However when a shift is experienced by an average number of agents, then it is not clear whether the old opinions or the new opinion will dominate, and hence, it might take more time steps for opinions to stabilise.

On the other hand, when the shift is permanent, the average time to stabilisation

$t^{\text{shift}}_{\text{stab}} - t_{\text{shift}}$ decreases nonlinearly as the number of initial recipients n_{shift} increases. This suggests that for a permanent shift in opinion, the more agents are exposed to opinion shift, the shorter the time taken for opinions to stabilise. Compared to a temporary shift in opinions where the average stabilisation time is quadratic in nature, the average stabilisation time for a permanent shift in opinions decreases as n_{shift} is increased.

Intuitively, a permanent shift converts the opinions of agents to a fixed opinion. Therefore, if more opinions are converted by the shift, then fewer time steps are required for the remaining opinions to be converted. Hence, the observed decreasing time effect as the number of initial recipients increases is plausible.

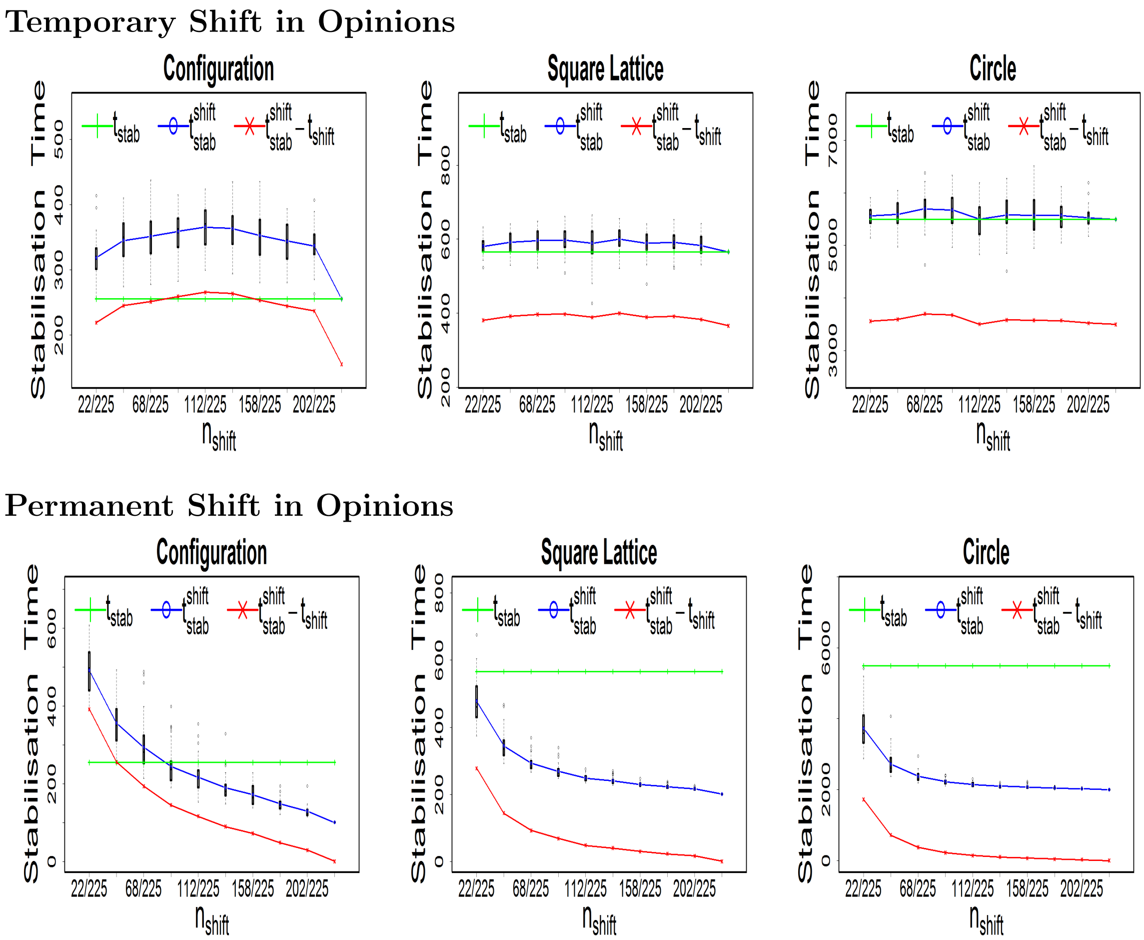
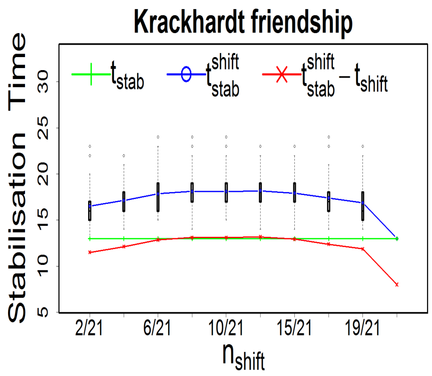
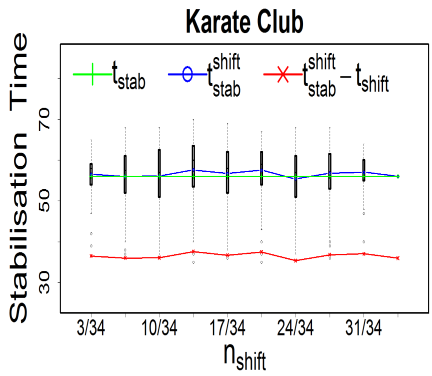


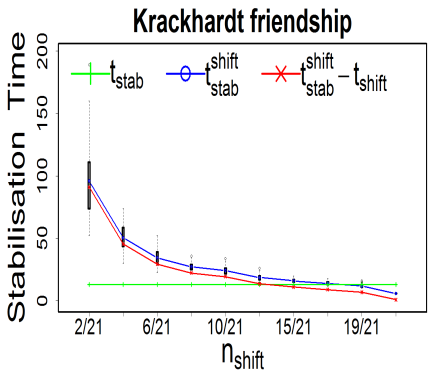
Figure 4.8. *A boxplot illustration of the influence of n_{shift} for a temporary and a permanent shift in opinions with $p^S = 0.7$ and $n = 225$ for the configuration, square lattice and the circle network. The green line (with pluses) shows the average stabilisation time t_{stab} without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time t^{shift}_{stab} with the introduction of opinion shift, and the red line (with crosses) shows the time difference between t^{shift}_{stab} and t_{shift}. For a temporary shift, the average stabilisation time (red line with crosses) is quadratic in nature especially for the configuration network, whereas for a permanent shift, the average stabilisation time decreases nonlinearly as n_{shift} is increased.*

Temporary Shift in Opinions





Permanent Shift in Opinions



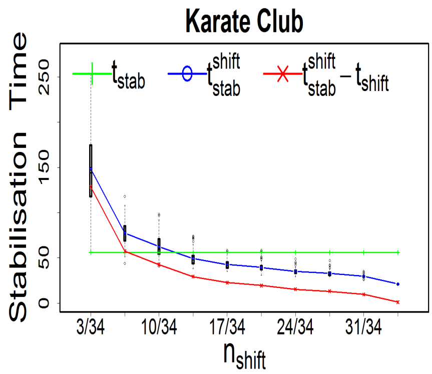


Figure 4.9. *A boxplot illustration of the influence of n_{shift} for a temporary shift in opinions. The value of $p^S = 0.7$, t_{shift} is fixed at $t_{\text{shift}} = 5$ for the Krackhardt network and $t_{\text{shift}} = 20$ for the karate club network. The green line (with pluses) shows the average average stabilisation time t_{stab} without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time t_{stab}^{shift} with the introduction of opinion shift, and the red line (with crosses) shows the time difference between t_{stab}^{shift} and t_{shift}. For a temporary shift in opinions, a quadratic behaviour is observed for t_{stab}^{shift}. However for a permanent shift in opinions, the average stabilisation time t_{stab}^{shift} decreases as n_{shift} increases.*

4.3 The Influence of Opinion Shift in Dynamic Networks

Here, we consider a shift in opinions with a dynamic social network structure. The dynamics of the network are based on the random switching (RS) model and the homophily and triadic closure (HT) model discussed in Chapter 3. The same initial parameters and the same network realisation for the configuration network, the square lattice network and the circle network as in Section 4.2.1 are used.

The opinions and the the weight matrix are updated using the dynamic network model from Equation (3.1) & (3.2) as

$$\boldsymbol{p}(t+1) = \boldsymbol{T}(\boldsymbol{G}(t), \boldsymbol{p}(t))\boldsymbol{p}(t), \quad \text{for } t > 0,$$

where

$$\boldsymbol{T}_{ij}(\boldsymbol{G}, \boldsymbol{p}) = \frac{\mathscr{G}_{ij} \exp\{-\lambda_{ij}|p_i - p_j|\}}{\sum_{k=1}^{n} \mathscr{G}_{ik} \exp\{-\lambda_{ik}|p_i - p_k|\}}, \quad \lambda_{ij} \geq 0.$$

4.3.1 The Effect of the Time t_{shift} of Opinion Shift

Figures 4.10 - 4.11 show the influence of the time t_{shift} of a shift on the stabilisation time of opinions for dynamic social networks. Figure 4.10 shows the effect of a temporary shift in opinions and Figure 4.11 shows the effect of a permanent shift in opinions. Plots are presented for the RS model and the HT model, for the configuration, the square lattice and the circle network.

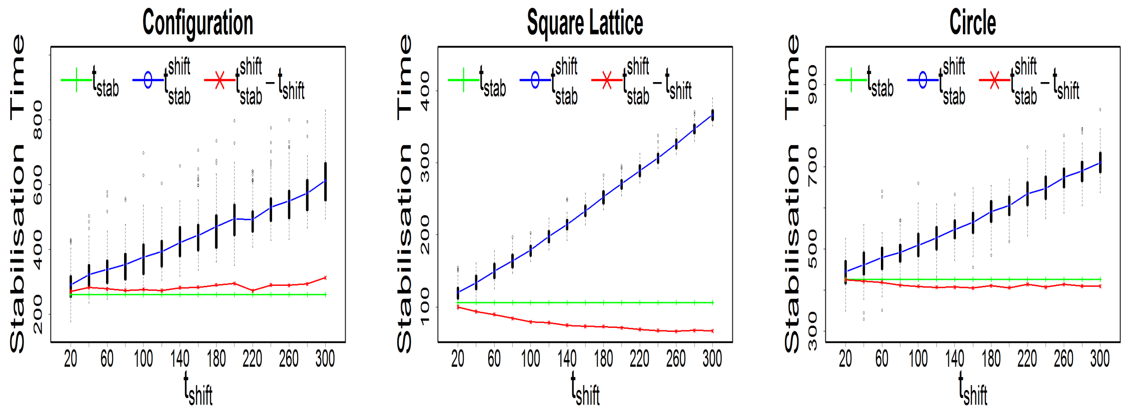
For a temporary shift in opinions, Figure 4.10, the time effect $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$, for the RS model is almost constant for the configuration and the circle network but decreases nonlinearly for the square lattice network. For the HT model, the time effect $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ decreases nonlinearly for the configuration network and the circle network but the effect is quadratic for the square lattice network. The observed differences among the different networks suggest that for a temporary shift, the initial network topology can be influential even for dynamic networks.

For a permanent shift in opinions, Figure 4.11, the average stabilisation time $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ is almost constant for the different values of t_{shift} for both the RS and the HT model. The constant effect for t_{shift} suggests that for a permanent shift, the average stabilisation time of opinions might not be influenced by the time at which a shift is introduced when the network is dynamic.

As mentioned in Section 4.2.2, the effect of the time of a shift can be explained by the average shortest path length between agents. When the network is dynamic, more shortcuts (edges) are created between agents. Hence, the average shortest path length between agents decreases, causing the opinions to stabilise faster for dynamic networks compared to when the network is static.

Temporary Shift in Opinions

The RS Model



The HT Model

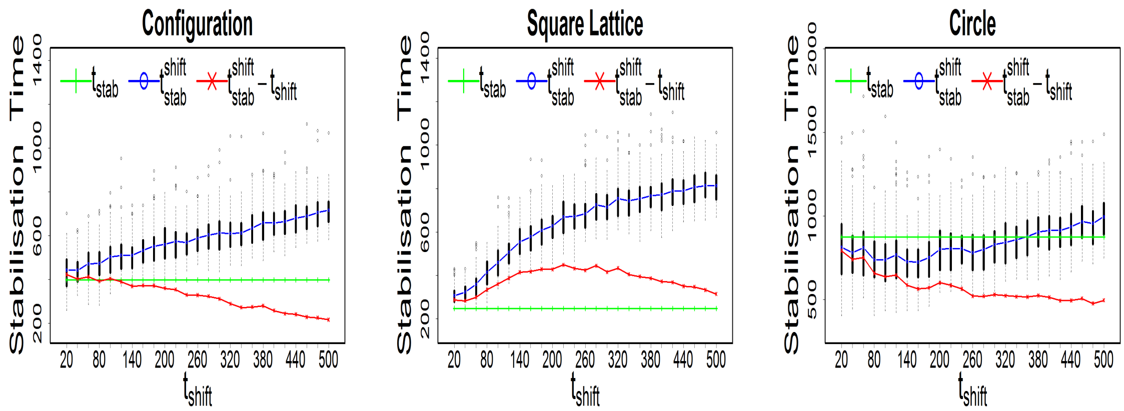
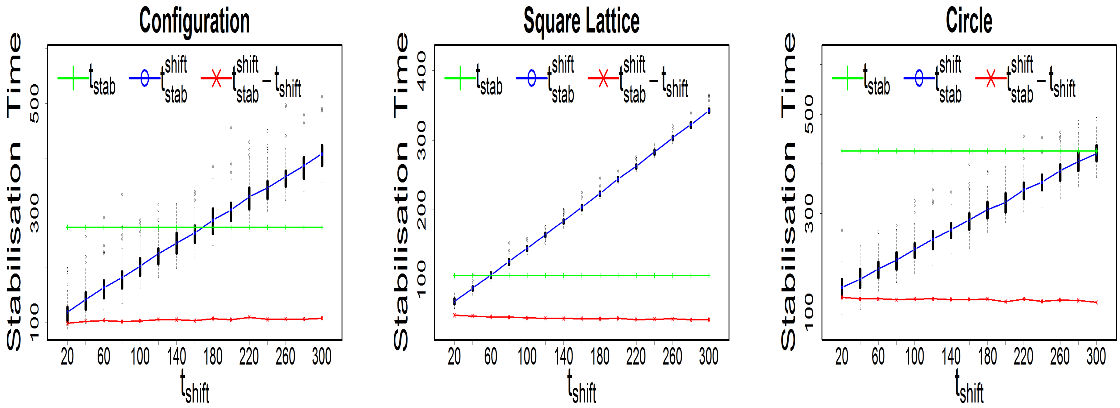


Figure 4.10. *A boxplot illustration of the influence of* t_{shift} *for a temporary shift in opinions, for the RS model and the HT model. The network size is* $n = 225$ *for the Configuration network, the square lattice network and the circle network. The green line (with pluses) shows the average stabilisation time* t_{stab} *without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time* t_{stab}^{shift} *with the introduction of opinion shift, and the red line (with crosses) shows the time difference between* t_{stab}^{shift} *and* t_{shift}*. For the RS model, the time effect* $t_{\text{stab}}^{shift} - t_{\text{shift}}$ *is almost constant for the configuration and circle network, but decreases for square lattice network. For the HT model, the time effect* $t_{\text{stab}}^{shift} - t_{\text{shift}}$ *decreases as* t_{shift} *increases for the configuration and circle network, but it is quadratic the square lattice network.*

Permanent Shift in Opinions

The RS Model



The HT Model

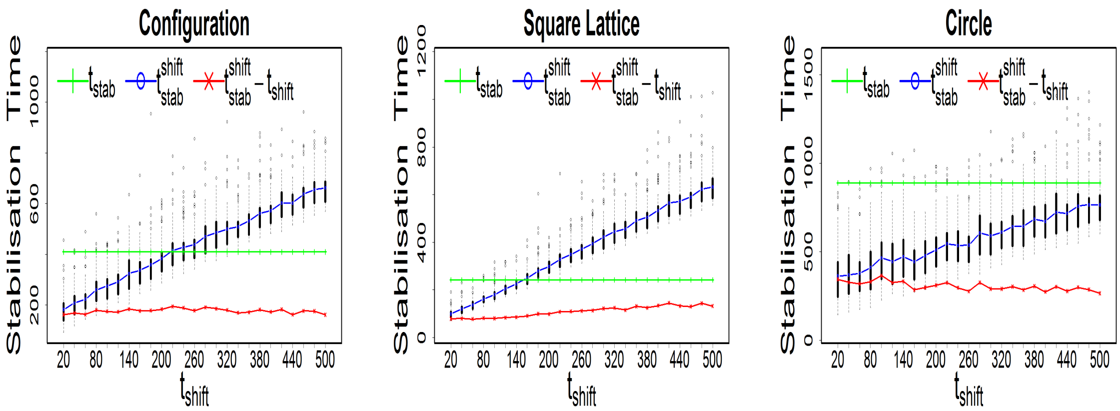


Figure 4.11. *A boxplot illustration of the influence of* t_{shift} *for a permanent shift in opinions, for the RS model and the HT model. The network size is* $n = 225$ *for the Configuration network, the square lattice network and the circle network. The green line (with pluses) shows the average stabilisation time* t_{stab} *without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time* t_{stab}^{shift} *with the introduction of opinion shift, and the red line (with crosses) shows the time difference between* t_{stab}^{shift} *and* t_{shift}*. A constant time effect* $t_{\mathrm{stab}}^{shift} - t_{\mathrm{shift}}$ *is observed for the different values of* t_{shift} *for both the RS and the HT model.*

4.3.2 The Effect of the Number of Initial Recipients n_{shift} of Opinion Shift

Figures 4.12 - 4.13 show the influence of the number of initial recipients n_{shift} on the stabilisation time of opinions for dynamic social networks. Figure 4.12 shows the effect of a temporary shift in opinions and Figure 4.13 shows the effect of a permanent shift in opinions. Plots are presented for the RS model and the HT model, for the configuration, the square lattice and the circle network.

For a temporary shift in opinions, Figure 4.12, the average stabilisation time $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ is nonlinear for the different numbers of initial recipients in both the RS and the HT model. For the RS model, the nonlinear effect is almost constant compared to the HT model. The constant effect for the RS model suggests that the stabilisation time of opinions might not be influenced by the number of initial recipients when the network is dynamic.

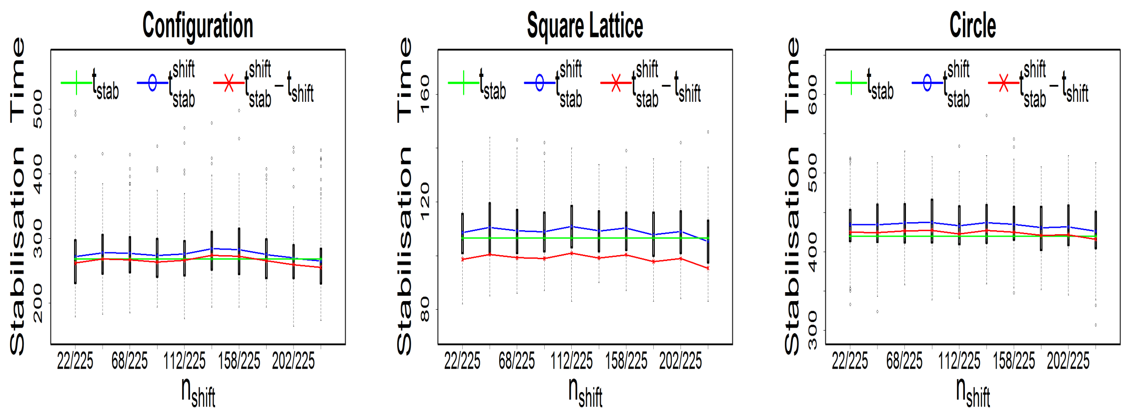
A plausible explanation for the observed behaviour in Figure 4.12 involves the average shortest path length between agents. Intuitively, when the network is dynamic, shortcuts (edges) are easily created between agents, especially for the random switching dynamics. Hence, the number of initial recipients might not have a strong influence on the average stabilisation time due to the creation of shortcuts between agents. The nonlinear effect observed for the HT dynamics especially for the square lattice network might be due to network segregation or cluster formation in the HT dynamics, giving rise to unstable behaviour in the average stabilisation time.

For the case of a permanent shift in opinions, Figure 4.13, there is a nonlinear decreasing effect on the average stabilisation time $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ when the number of initial recipients n_{shift} is increased. Thus, for a permanent shift in opinions, increasing the number of initial recipients decreases the time taken for opinions to stabilise. This observation is consistent across the different initial networks, and is similar to that observed in the static network case in Section 4.2.3.

Additionally as mentioned in Section 4.2.3, the observed decreasing time effect as the number of initial recipients increases, may stem from the fact that when the opinions of an increasing number of agents are converted by a permanent shift, it takes fewer time steps for the remaining opinions to stabilise.

Temporary Shift in Opinions

The RS Model



The HT Model

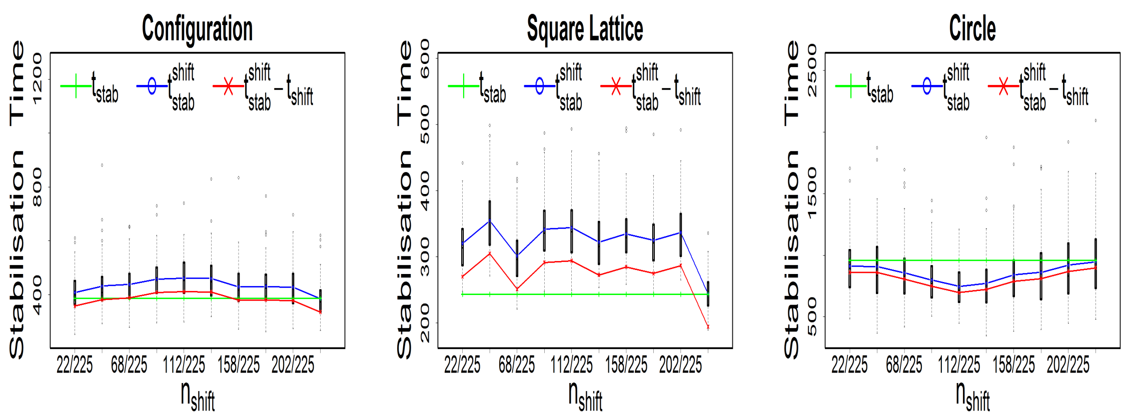
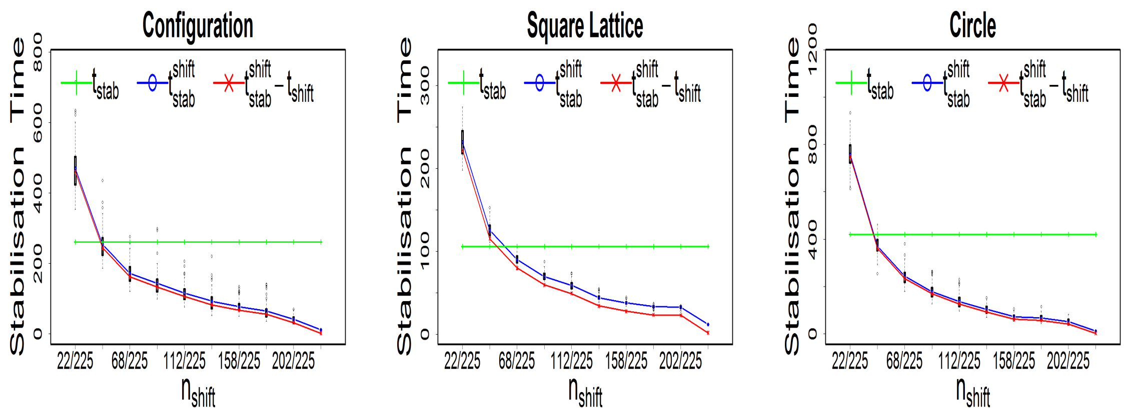


Figure 4.12. *A boxplot illustration of the influence of n_{shift} for a temporary shift in opinions, for the RS model and the HT model. The network size is $n = 225$ for the Configuration network, the square lattice network and the circle network. The green line (with pluses) shows the average stabilisation time t_{stab} without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time t_{stab}^{shift} with the introduction of opinion shift, and the red line (with crosses) shows the time difference between t_{stab}^{shift} and t_{shift}. An almost constant behaviour is observed for the effect of n_{shift} on the average stabilisation time $t_{stab}^{shift} - t_{shift}$ for the RS model. In the HT model however, the observed behaviour is nonlinear.*

Permanent Shift in Opinions

The RS Model



The HT Model

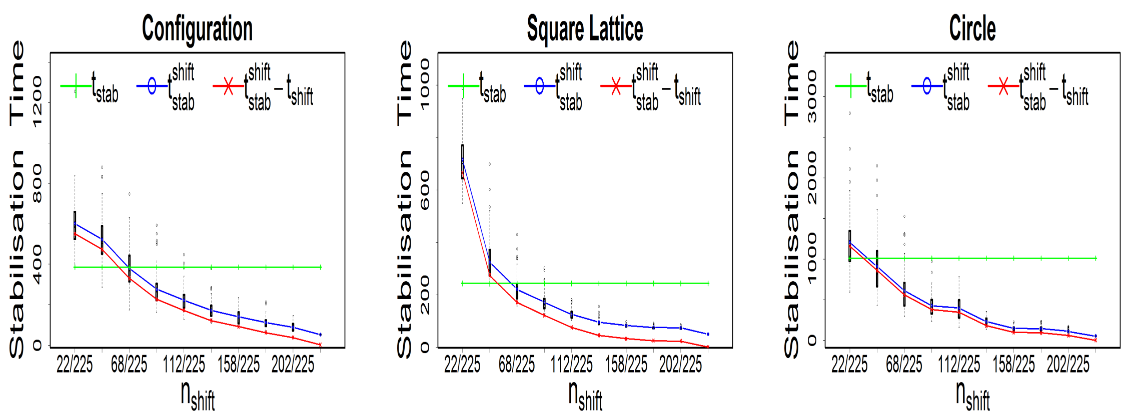


Figure 4.13. *A boxplot illustration of the influence of* n_{shift} *for a permanent shift in opinions, for the RS model and the HT model. The network size is* $n = 225$ *for the Configuration network, the square lattice network and the circle network. The green line (with pluses) shows the average stabilisation time* t_{stab} *without the introduction of a shift in opinions, the blue line (with circles) shows the average stabilisation time* t_{stab}^{shift} *with the introduction of opinion shift, and the red line (with crosses) shows the time difference between* t_{stab}^{shift} *and* t_{shift}*. There is a nonlinear decreasing effect on the average stabilisation time* $t_{stab}^{shift} - t_{shift}$ *as* n_{shift} *increases.*

4.4 Network Influence of a Shift in Opinions

In Section 4.2 and Section 4.3, we considered the influence of time and the number of initial recipients on the average stabilisation time of opinions. This section focuses on the influence of the network structure and the network positions of initial recipients of a permanent shift in opinions. Additionally, in Chapters 2 and 3, insistent agents are chosen at random. As a permanent shift in opinions converts agents into insistent agents, this section also extends Chapters 2 and 3 to consider the case where insistent agents are specifically chosen and not randomly chosen.

The structure of the network plays an important role in modelling the influence of a shift in opinions. The positions of initial recipients and how well connected they are in the network can influence the spread of a shift in the whole network. Especially, when the initial recipients are bridging agents, they can strongly influence opinions in their social clique as well as the outcome of opinions in other cliques depending on the structure of the network (Krackhardt, 1987). (Watts, 2002, pg. 5767) summarised this as,

> "Clearly, the more early adopters exist in the network, the more likely it is that an innovation will spread. But the extent of its growth and hence the susceptibility of the network as a whole depends not only on the number of early adopters, but on how connected they are to one another, and also to the much larger community consisting of the early and late majority, who do not tend to respond to the innovators directly, but who can be influenced indirectly if exposed to multiple

early adopters."

In the work of Kitamura and Namatame (2015), the authors used a dumbbell-like network to illustrate how the network positions of stubborn agents can influence the outcomes of team cooperation and negotiation. Using the setting of Kitamura and Namatame (2015), further insights are drawn on the influence of the network positions of initial recipients when a shift in opinions is introduced.

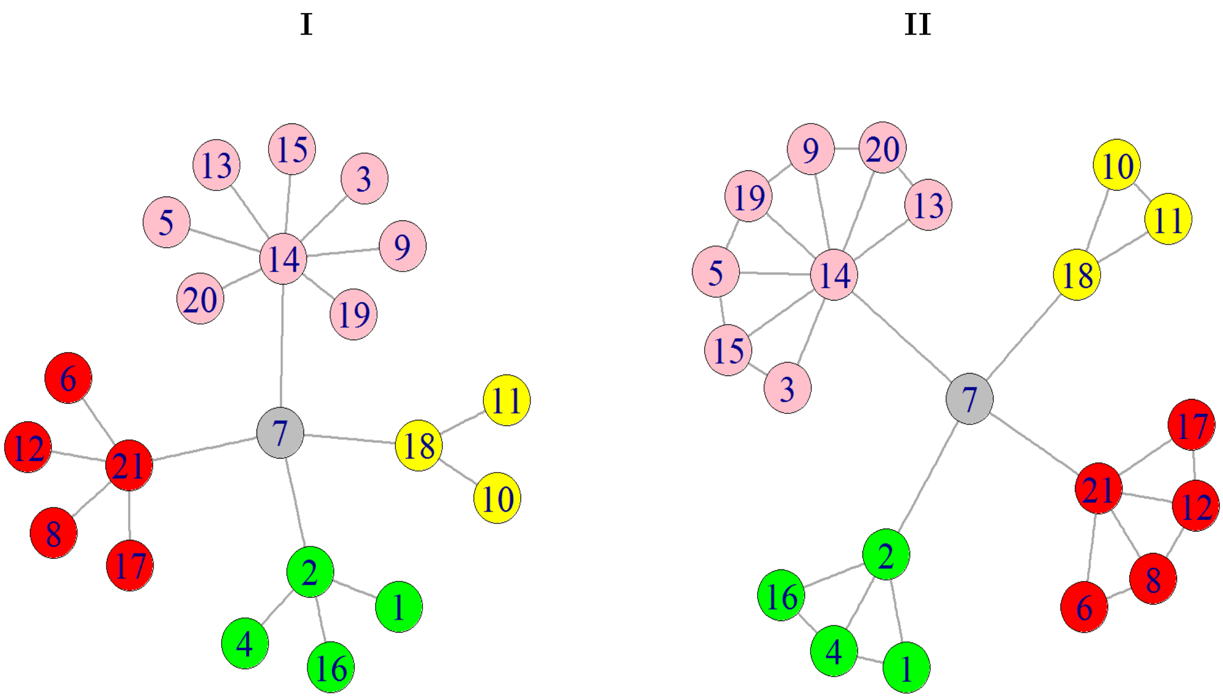


Figure 4.14. *Network I on the left hand side shows a symmetrised version of the Krackhardt report network and Network II is similar to Network I with interaction paths among agents in a group. The colours show the different groups of the agents.*

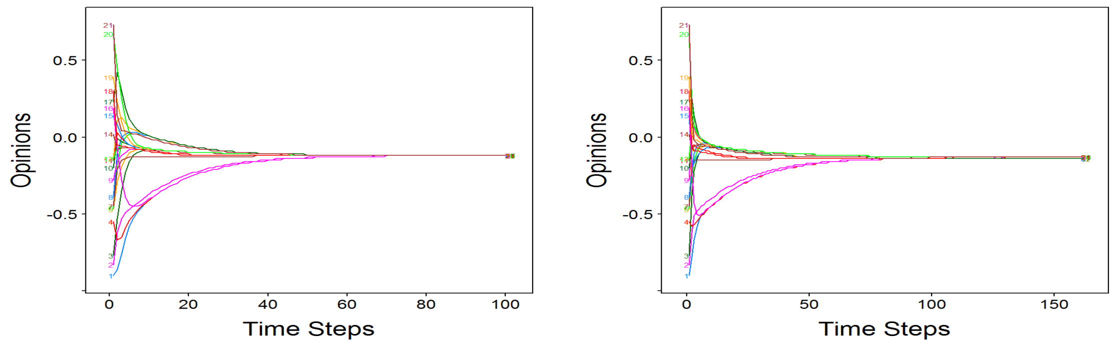
Consider the report network of Krackhardt (1987). Figure 4.14-I on the left hand side shows a symmetrised version of the Krackhardt report network. Figure 4.14-

II on the right hand side, is similar to Network I with added interactions among agents within the same group. The colours show the different groups in which agents operate, except for agent 7 who serves as a central bridge agent connecting all the groups.

We categorise agents {2, 14, 18 and 21} as bridge agents and agents {1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19 and 20} as non-bridge agents. The simulation results for Network I are shown on the left of each figure, and the results for Network II are shown on the right of each figure. The colours in figures represent the different initial opinions of agents.

Network I **Network II**

A: No Opinion Shift



B: Agent 7

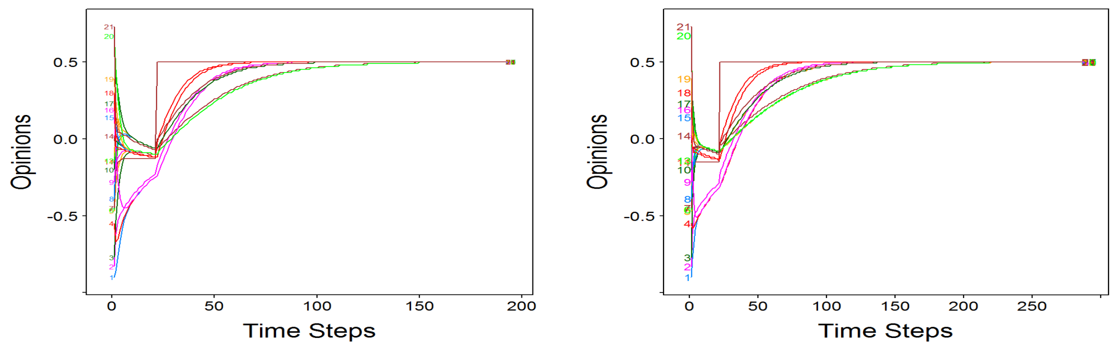


Figure A shows the outcome of opinions without the introduction of an opinion shift. All agents update their opinions using the basic model of Equation (4.1). Consensus in opinions is observed as shown in Figure A for both Network I and Network II. All agents first reach a local consensus within groups before an overall consensus across all groups. Recall from the proof of Theorem 2.8 that Equation (4.1) is a consensus model.

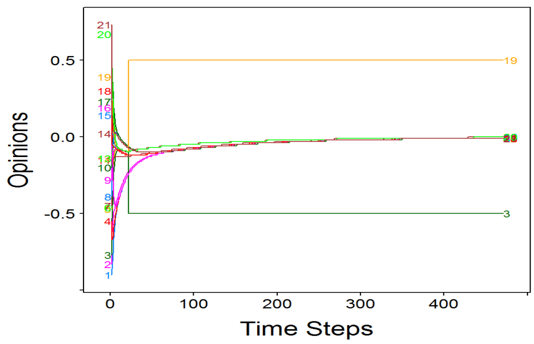
Figure B shows a shift in the opinion of agent 7. All agents reach consensus to the opinion of agent 7. Generally, a permanent shift p^S in the opinion of only one agent, or to a group of agents will lead to a consensus in opinions to p^S, irrespective of the network structure and the positions of the initial recipients. This is demonstrated in Corollary 2.1 for the DeGroot and Section 2.2.2 for the dynamic weight update model.

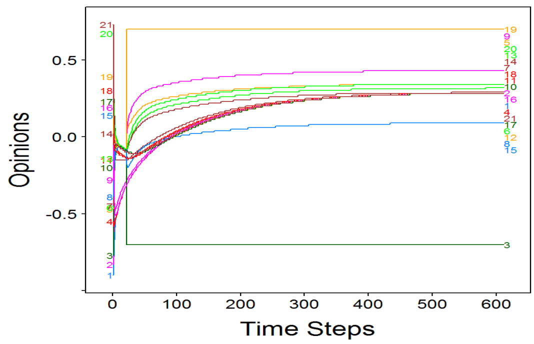
To clearly illustrate the influence of network structures and network positions of initial recipients, suppose that the introduction of a shift causes initial recipients to experience an opposite shift in their opinions. Two examples are considered here; one with opposite shift between non-bridge agents only (eg. Figure C), and one with opposite shift between bridge agents and non-bridge agents (eg. Figure D). The shift parameter p^S is considered to be -0.5 and 0.5 for the simulation.

Example 1

Network I **Network II**

C: Agent 3 & 19

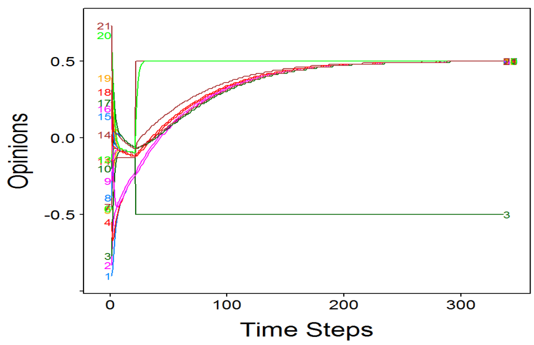


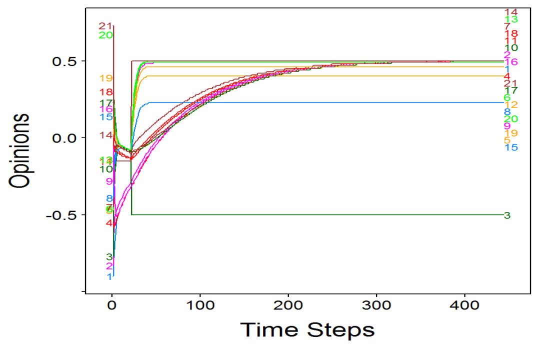


Example 2

Network I **Network II**

D: Agent 3 & 14





In Example 1, Figure C shows the introduction of opposite shift between agent 3 and agent 19 (both non-bridge agents). In Network I, all the other agents form an intermediate opinion. However, in Network II, where there exists interaction among non-bridge agents, the rest of the agents no longer stabilise to a common intermediate opinion. Furthermore agent 19 interacts directly with three other agents {5, 9 and 14}, compared to agent 3 who interacts with two agents {15 and

14}. Hence agent 19 draws the opinions of other agents closer to its own opinion in Network II.

In Example 2, Figure D demonstrates the introduction of opposite shift between non-bridge agent 3 and bridge agent 14. Agent 14 is the only bridge connecting both agents within its group and across to all agents outside of its group. Therefore, in Network I, the opinion of agent 14 is adopted by agents in its group and agents in all the other groups, whereas the opinion of agent 3 remain isolated. In Network 2, agent 14 draws opinions closer to its own opinion, however, not all agents fully adopt the opinion of agent 14 as in the case of Network 1, due to the existence of interaction between the non-bridge agents.

In conclusion, the observed differences in the examples highlight the importance of the structure of the network and the position of initial recipients on the influence of opinion shift. When agents are positioned as bridges, they influence opinions both within and across groups compared to non-bridge agents. However, the existence of interaction between non-bridge agents can alter the full adoption of the opinions of bridge agents by other agents. Additionally, the level of interactions within groups plays a key role in determining which values opinions will stabilise to and the number of coexisting opinions that will emerge at opinion stabilisation.

Furthermore, as initial recipients of a permanent shift can be viewed as insistent agents, the examples also illustrate that the network positions of insistent can have considerable effect on opinion formation (see Kitamura and Namatame (2015) for further studies on this effect).

4.5 Conclusion

In this chapter we have investigated the influence of a shift in opinions. The introduction of a shift is assumed to cause either a temporary or a permanent shift in the opinions of agents. As weights and opinions are updated in a network of interacting agents, a shift in opinions is introduced at a given time t_{shift} and to a given number of initial recipients n_{shift}. We have investigated the influence of time t_{shift} and number of initial recipients n_{shift} on the average stabilisation time $t_{\text{stab}}^{\text{shift}} - t_{\text{shift}}$ of opinions.

We have also investigated the relationship between the stabilisation time of opinions and:

1. The difference in population variance of opinions just before and just after the introduction of a shift in opinions.

2. The maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift in opinions.

The investigations are carried out for a temporary shift and a permanent shift in opinions and for both static networks and dynamic networks. For a temporary shift in opinions, the current opinions of initial recipients shift from p_i to $p_i + p^S$ for all $i \in N_{\text{shift}}$. After time t_{shift}, the agents continue to update their opinions as usual by taking a weighted average of the opinions of their neighbours. For a permanent shift in opinions, the current opinions of the n_{shift} agents change from p_i to p^S for all $i \in N_{\text{shift}}$, and the new opinion is maintained throughout the updating period. In this case the n_{shift} initial recipients become insistent agents. The rest of the agents continue to update their opinions by weighted averaging.

We have shown that the introduction of a shift in opinions can cause an immediate increase in the population variance or an immediate decrease in the population variance. Additionally, we found that the difference in population variance of opinions just before and just after the introduction of a shift, or the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift in opinions, can directly influence the average stabilisation time of opinions.

We found by simulation that there is a positive relationship between the average stabilisation time and the difference in population variance of opinions. Similarly, a positive relationship is found between the average stabilisation time and the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift in opinions.

The observed positive trend suggests that if a shift in opinions increases the variability in opinions or the maximal distance in opinions, then it will take more time steps for opinions to stabilise. Similarly, if the introduction of a shift in opinions decreases the variability in opinions or the maximal distance in opinions, then it will require fewer time steps for opinions to stabilise.

Considering the effect of the time t_{shift}, for a temporal shift in opinions, the average stabilisation time first decreases nonlinearly with later introduction of a shift, and then approaches a constant for a shift introduced after the opinions of agents have stabilised. The decreasing effect of time t_{shift} on the average stabilisation time suggests that the later the introduction of a shift, the fewer time steps required for opinions to stabilise. However, the introduction of a shift after opinions have stabilised might not influence the average stabilisation time. For a permanent shift in opinions, the average stabilisation time is almost constant for dynamic networks. But for static networks, the effect of time can be constant or nonlinearly decreasing depending on the structure of the network. The average stabilisation time is almost constant for the configuration network, the Krackhardt friendship network and the karate club network; whereas, for the square lattice network and the circle network, the average stabilisation time decreases nonlinearly with later introduction of a shift.

Intuitively, the average shortest path length between agents in the network can explain the observed behaviour in the stabilisation times of opinions. A network with a small average shortest path length between agents implies that a shift in opinions diffuses quicker in the network. This leads to a faster stabilisation time. The opposite is true for networks with high average shortest path length between agents, which leads to an increase in the average stabilisation time due to slow diffusion of an opinion shift.

Considering the effect of the number of initial recipients n_{shift}, when the shift is temporary, a nonlinear effect is observed for the influence of the number of initial recipients on the average stabilisation time. Depending on the structure of the initial network, this effect is found to be quadratic or almost constant. A constant effect suggests that the average stabilisation time of opinions is not influenced by the number of initial recipients. When the shift is permanent, a nonlinear decreasing effect is observed for the influence of n_{shift}. The decreasing effect suggests that on average, the more agents are exposed to opinion shift, the fewer time steps required for opinions to stabilise.

Intuitively, the decreasing effect observed for the permanent shift is because as the opinions of more and more agents are converted to a fixed opinion, the number of agents left to reach agreement decreases. Hence, it takes fewer time steps for the opinions of the remaining agents to stabilise.

We have also demonstrated by simulation, the importance of the structure of the network and the position of initial recipients on the influence of opinion shift. We have shown that initial recipients who are bridge agents influence opinions within and across groups compared to non-bridge agents. However, if the non-bridge agents interact, then they can alter the full adoption of the opinions of bridge agents by other agents. Additionally, we have illustrated that the values opinions stabilise to and the number of coexisting opinions at stabilisation is influenced by the structure of the network and how well agents are connected in the network.

Chapter 5

Conclusion

In this thesis, we have investigated the formation of opinions on social networks leading to consensus or a lack of consensus. The investigation is carried out on three main mechanisms considered to influence consensus or a lack of consensus:

1. The mechanisms of opinion dynamics via

   a. the introduction of dynamic weight updates and

   b. the influence of a minority of insistent agents.

2. The mechanisms underpinning the dynamics of the social network.

3. The mechanisms of external influence via the introduction of a shift in opinions.

In doing this, the DeGroot model is used as the foundation of our model development. The DeGroot model presents a pioneering work in the field of opinion dynamics. The model is introduced as a prescriptive approach for reaching consensus in opinions. The DeGroot model is developed on four assumptions which we relax in the three mechanisms above to investigate whether or not convergence to a consensus in opinions holds.

First, the DeGroot model assumes that at each time step, agents place the same weight on the opinions of others. This assumption is relaxed and consensus or a lack of consensus is investigated in Mechanism 1a. We have proposed a dynamic weight updating based on the rule that weights decrease with increased difference in opinions. The choice of the weight function proposed here is motived by conceptual notions grounded in psychophysical judgement theory (Davis, 1996). Davis (1996, pg. 44-45) and the references therein argue that the interpersonal weight between two agents is an exponential function of the difference in the opinions of the agents.

We have proved that opinions converge to a consensus for the proposed dynamic weight updates on opinions. Additionally, we have derived an upper bound for the maximum difference in opinions. Our finding of consensus for the proposed dynamic weight updates (Equation (2.11)) is consistent with the findings in (Pan, 2010) and Regan et al. (2006), where convergence to a consensus in opinions is observed for the different updating rules.

Second the DeGroot model assumes that all agents update their opinions homogeneously by taking a weighted average of the opinions of others. Following (Pan, 2010), we have relaxed this assumption as part of our investigations in Mechanism 1b, by introducing a minority group of agents who are insistent on their own opinions. These agents are termed *insistent agents*.

We have proved that the introduction of insistent agents will lead to consensus when all insistent agents have the same opinion. However, lack of consensus emerges when insistent agents have different opinions. We have shown that this conclusion also holds for dynamic weight updates using illustrations from simulations.

Third, the DeGroot model assumes that all agents interact with each other. Hence, there is no defined pattern of interaction among agents. Section 8.3 of Jackson (2010) introduces a social network defined on the weight matrix of the DeGroot model such that an edge exists between two agents i, j if a non-zero weight is placed on opinions ($\boldsymbol{T}_{ij} > 0$). Otherwise, if $\boldsymbol{T}_{ij} = 0$, then no direct edge exists between agents. With the introduction of a defined social network structure, the influence of different patterns of social interactions on opinions is investigated (Jackson, 2010). The social network is however considered to be static, which implies that agents interact with the same agents at each time step (Friedkin and Johnsen, 1990, 1997, 2011; DeMarzo et al., 2003; Jackson, 2010; Pan, 2010, 2012).

We have relaxed the assumption of a static network structure by introducing a dynamic network approach. We have investigated in Mechanism 2, the influence of two network dynamics models in reaching consensus or a lack of consensus in opinions. We first considered a dynamic network model where the network change is independent of the opinion change. The model is based on the random switching algorithm by Gkantsidis et al. (2003). Two versions of the random switching dynamics are considered; one that requires the network to be connected at all time steps, which we termed the *RS model* and a general case that does not require the network to be connected at all time steps, which we termed the

$\overline{RS}$ *model*. Second, we considered a dynamic network model where the network changes with changes in the opinions. Here, we proposed a dynamic network model based on homophily and triadic closure, which we termed the *HT model*.

We have shown that the dynamics of the network can be important in determining whether or not opinions will converge to a consensus. We have proved that opinions will converge to a consensus for network dynamics that preserves the degree and connectivity of the network (the RS model). We have also proved that opinions will converge to a consensus for the RS dynamics without necessarily imposing network connectivity (the $\overline{RS}$ model). On the other hand, we have illustrated through simulation that lack of consensus can emerge for the HT dynamics. In the HT dynamics, the network can segregate into single isolated agents and these agents will have their opinions unchanged, leading to the lack of consensus.

We have also investigated the influence of network dynamics with the presence of insistent agents using simulation. We have shown that for the RS model, when all the insistent agents have the same opinion, then the opinions of all agents will converge to the opinion of the insistent agents. This observation is similar to the static network case where we proved analytically that when all the insistent agents have the same opinion, the final opinions will converge to a consensus of the opinion of the insistent agents. However, for the HT model, we have illustrated through simulation that lack of consensus can emerge even when insistent agents have the same opinion. Thus, all insistent agents having the same opinion does not guarantee convergence to a consensus in the HT model. Clearly, if insistent agents have different opinions, then different opinions will persist in the network implying a lack of consensus.

Fourth, we have investigated in Mechanism 3, the influence of a sudden shift in opinions which might be caused by external influence other than interpersonal interactions among agents. The DeGroot model is based on interpersonal interactions among agents. Here, we have investigated the influence of a shift in two shift effect models; in the first model, the shift has a temporary effect on opinions, and in the second model, the shift has a permanent effect on opinions. Additionally, we have examined the influence of the time of the introduction of a shift and the influence of the number of initial recipients on the average stabilisation time of opinions.

We have shown by simulation that for both shift effect models, the time of a shift in opinions and the number of initial recipients can influence the average stabilisation time of opinions depending on the variability in opinions introduced by the shift. After establishing a relationship between the population variance of opinions prior to and just after the introduction of a shift in opinions, we find that the introduction of a shift in opinions can cause an immediate increase in the population variance or an immediate decrease in the population variance. Additionally, we find that the difference in population variance of opinions just before and just after the introduction of a shift can directly influence the average stabilisation time of opinions.

Our simulation illustrations shows that there is a positive relationship between the difference in population variance of opinions and the average stabilisation time, for both a temporal and a permanent shift in opinions. The observed positive trend suggest that if a shift in opinions increases the variability in opinions, then more time steps will be taken for opinions to stabilise. Similarly, if the introduction of a shift in opinions decreases the variability in opinions, then it will require fewer time steps for opinions to stabilise. This positive relationship also holds for the relationship between the average stabilisation time of opinions and the maximal difference in the difference between the maximum opinion and the minimum opinion just before and just after the introduction of a shift in opinions.

For the effect of the time t_{shift}, when the shift is temporary, the average stabilisation time first decreases nonlinearly with later introduction of a shift and approaches a constant when a shift is introduced after opinions have stabilised. This suggests that later introduction of a shift requires fewer time steps for opinions to stabilise compared to earlier introduction of a shift. However, after opinions have stabilise, the introduction of a shift might not influence the average stabilisation time. When the shift is permanent, the average stabilisation time can be constant or nonlinearly decreasing depending on the structure and dynamics of the network.

For the effect of the number of initial recipients n_{shift}, when the shift is temporary, depending on the structure and dynamics of the network, a quadratic effect or an almost constant effect is found. However, when the shift is permanent, the average stabilisation time decreases nonlinearly as the number of initial recipients n_{shift} increases. The decreasing effect suggests that increasing the number of initial recipients of a shift decrease the average stabilisation time.

We have also investigated the influence of network structure and network positions of initial recipients of opinion shift. We have shown by simulation that initial recipients who are positioned as bridge agents influence opinions both within and across groups compared to non-bridge agents. However, non-bridge agents can alter the full adoption of the opinions of bridge agents by other agents if the non-bridge agents interact. Additionally, we have illustrated that the structure of the network and how well initial recipients are connected in the network can influence the values opinions stabilise to and the number of coexisting opinions at stabilisation as well as the stabilisation time of opinions.

In conclusion, opinion formation is a complex phenomenon with no data available for how they are formed. We have shown that dynamic changes in the network can have profound effect. We are able to prove analytical results for convergence to consensus in random switching models. The original DeGroot model is prescriptive with the aim to device a procedure which converges to a consensus. When taking social network dynamics into account, unless one is prescriptive in these dynamics also (for example by imposing the RS model), convergence to a consensus may no longer be achieved.

Additionally, we have shown that the occurrence of sudden shifts in opinions can have profound effect on the average stabilisation time of opinions. The time of the introduction of a shift, the number and the network positions of initial recipients, as well as the structure and dynamics of the network can influence the average stabilisation time of opinions.

Next, we discuss possible ideas that can be explored in extending the current work.

5.1 Future Work

The current work can be extended by further investigations in the following directions:

5.1.1 Interaction Effect

In this thesis, the influence of the time and the number of initial recipients of the introduction of a shift in opinions is considered separately. However, is the introduction of a shift late, but to a large number of individuals related to the introduction of a shift early, but to fewer initial recipients? This interaction between the time and the number of initial recipients for the different networks could be investigated further.

5.1.2 Different Time Scales

The network dynamics and the opinion dynamics are assumed to occur on the same time scale. However, it is sometimes the case that opinions change faster than the network, or the network changes faster than opinions. The static network section already covers the case that network dynamics is much slower than the opinion dynamics, so that the network can be assumed to be static over the opinion formation period. If opinion formation is much slower than network evolution, then the main interest lies in the long-term behaviour of the network dynamics. Under the random switching models the network will never have single isolated agents and disconnected agents can reconnect to the network. Whereas under the HT model the network might break up into small disconnected components and single isolated agents can emerge with low probability of reconnection (Bianconi et al., 2014). This behaviour is also of interest and could further be investigated.

5.1.3 The Final Network Characteristics

Most of the results presented in this thesis are focused on the characteristics of the final opinions, giving less attention to the characteristics of the network structure at opinion stabilisation. Further investigations can be carried out for the characteristics of the network structure: the number of formed network clusters, changes in the degree distribution and changes in the size of the largest component. Also, investigations into how the introduction of sudden shifts in opinions and the introduction of insistent agents influences the characteristics of the network could be considered.

5.1.4 Multidimensional Opinion Space

The dimension of the opinion space of the current work could be extended from a one dimensional opinion space to a multidimensional opinion space. Here, the opinion space will comprise of possible different opinions an agent may have on issues of interest, and each agent updates across all opinions in order to reach a final opinion. The influence of the relationships among the different dimensions of opinions on the outcome of final opinions and the network can be investigated further. Additionally, the influence of insistent agents, network dynamics and sudden shifts in opinions can be studied. Also, the study of opinion formation under bounded confidence models with dynamical networks and sudden shifts in opinions could be interesting.

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