

BLOW-UP PHENOMENA IN NONLOCAL EIGENVALUE PROBLEMS: WHEN THEORIES OF L^1 AND L^2 MEET

HARDY CHAN, DAVID GÓMEZ-CASTRO, AND JUAN LUIS VÁZQUEZ

ABSTRACT. We develop a linear theory of very weak solutions for nonlocal eigenvalue problems $\mathcal{L}u = \lambda u + f$ involving integro-differential operators posed in bounded domains with homogeneous Dirichlet exterior condition, with and without singular boundary data. We consider mild hypotheses on the Green's function and the standard eigenbasis of the operator. The main examples in mind are the fractional Laplacian operators.

Without singular boundary datum and when λ is not an eigenvalue of the operator, we construct an L^2 -projected theory of solutions, which we extend to the optimal space of data for the operator \mathcal{L} . We present a Fredholm alternative as λ tends to the eigenspace and characterise the possible blow-up limit. The main new ingredient is the transfer of orthogonality to the test function.

We then extend the results to singular boundary data and study the so-called large solutions, which blow up at the boundary. For that problem we show that, for any regular value λ , there exist “large eigenfunctions” that are singular on the boundary and regular inside. We are also able to present a Fredholm alternative in this setting, as λ approaches the values of the spectrum.

We also obtain a maximum principle for weighted L^1 solutions when the operator is L^2 -positive. It yields a global blow-up phenomenon as the first eigenvalue is approached from below.

Finally, we recover the classical Dirichlet problem as the fractional exponent approaches one under mild assumptions on the Green's functions. Thus “large eigenfunctions” represent a purely nonlocal phenomenon.

Keywords and phrases. Integro-differential operators, eigenvalue problems, large solutions, fractional Laplacian.

2010 Mathematics Subject Classification. 35R09, 35R11, 35D30, 45C05.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded of class $C^{1,1}$. Let \mathcal{L} be an integro-differential operator of order $2s \in (0, 2)$ in Ω with exterior or boundary Dirichlet condition, whichever is applicable. Typical choices for \mathcal{L} are the restricted and spectral versions of the fractional Laplacian $(-\Delta)^s$ for all $s \in (0, 1)$, briefly RFL and SFL, but our theory includes a wider class of operators described in Section 2.1. Let $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$. From standard theory, it is known that non-trivial solutions for the problems

$$\begin{cases} (-\Delta)_{\text{RFL}}^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \cup \Omega^c, \end{cases} \quad \text{or} \quad \begin{cases} (-\Delta)_{\text{SFL}}^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

exist if and only if λ belongs to the standard spectrum, Σ , a discrete subset of $(0, +\infty)$. Moreover, the corresponding eigenfunctions are continuous up to the boundary, though with different regularity. Surprisingly, even the homogeneous Dirichlet problem

$$\begin{cases} (-\Delta)_{\text{RFL}}^s u = 0 & \text{in } B_1 \\ u = 0 & \text{in } B_1^c, \end{cases}$$

without condition *on* the boundary is ill-posed as shown by the non-trivial solution given by

$$u(x) = \begin{cases} (1 - |x|^2)^{s-1} & \text{in } B_1 \\ 0 & \text{in } B_1^c, \end{cases}$$

found in Hmissi [30] and later Bogdan–Byczkowski–Kulczycki–Ryznar–Song–Vondraček [8]. This initiated the study of *large solutions*, or solutions with boundary blow-up, which are the leitmotif of this paper. We need one further notation for the distance function to the boundary, $\delta : \Omega \rightarrow (0, +\infty)$:

$$\delta(x) = \text{dist}(x, \partial\Omega).$$

Among other things, Abatangelo [1] finds large RFL-harmonic solutions blowing up at the boundary of a general domain like δ^{s-1} , and Abatangelo–Dupaigne [3] establishes large solutions for SFL growing like δ^{2s-2} . Using two-sided estimates for Green’s function, Abatangelo–Gómez-Castro–Vázquez [4] provides a unified theory which also includes the censored fractional Laplacian (CFL), where no large solutions exist. The authors of [4] also show the boundary behavior for the large solution (or its candidate) is $\delta^{-(1-2s+\gamma)}$, where γ is the parameter that determines the optimal boundary behavior; thus $\gamma = s$ for RFL, $\gamma = 1$ for SFL, and $\gamma = 2s - 1 \in (0, 1)$ for CFL. Notice that for the classical Laplacian, $s = \gamma = 1$, so $1 - 2s + \gamma = 0$, consistent with the fact that no large harmonic functions exist.

Problem and type of results. In this paper we are interested in “large eigenfunctions”, or *large solutions* of the eigenvalue problem, namely

$$\begin{cases} \mathcal{L}u - \lambda u = 0 & \text{in } \Omega, \\ u(x) \rightarrow +\infty & \text{as } x \rightarrow \partial\Omega, \\ u = 0 & \text{in } \Omega^c \quad (\text{if applicable}) \end{cases}$$

with $\lambda \in \mathbb{R}$. The precise equations are introduced in (2.4), (2.5). The class of operators is precisely defined in Section 2.1, and a suitable notion of solution in Section 2.3. In contrast to the standard theory, we will establish the existence of a unique large solution for *any* $\lambda \in \mathbb{R} \setminus \Sigma$, and for all prescribed (non-zero) boundary data in a suitable class. Moreover, as λ approaches the standard spectrum Σ of \mathcal{L} , one expects a blow-up phenomenon, which we will precisely identify. See Theorem 2.6. It follows from the study that “large eigenfunctions” are different from the standard eigenfunctions in three striking ways:

- they exist when λ stays away from the standard spectrum;
- they are singular on the boundary;
- they blow up in the interior, when λ tends to the standard spectrum.

For these reasons, although “large eigenfunctions” solve an equation similar to standard eigenfunctions, they show drastically different behavior from the standard ones. We emphasize this fact by using the quotation marks.

Related work. For the homogeneous eigenvalue problem, to our best knowledge, only locally bounded large solutions are known for $\lambda \in (-\infty, 0)$ (absorption) or for $\lambda > 0$ (reaction) that is sufficiently small [1, 3].

Díaz–Gómez–Castro–Vázquez [22] study very weak solutions of the RFL-Schrödinger equation where the potential $-\lambda$ (when $\lambda \leq 0$) is replaced by a general non-negative function $V \geq 0$.

Interesting boundary blow-up solutions of nonlinear elliptic equations are also found in various settings, see for instance [1, 3, 5, 6, 9, 19, 25].

In the classical case $\mathcal{L} = -\Delta$, Keller [33] and Osserman [37] have found independently a necessary and sufficient condition for the existence of solutions of

$$(1.1) \quad -\Delta u + f(u) = 0 \quad \text{in } \Omega,$$

with explosive boundary behavior,¹ when f is non-negative and non-decreasing with $f(0) = 0$. In fact, they show the equivalence of

- *Keller–Osserman condition*, or superlinear growth of the nonlinearity as in $\int_0^{+\infty} (\int_0^t f)^{-1/2} dt < +\infty$ (which implies an *a priori* estimate in terms of the radial solution),
- existence of large solutions of (1.1) when Ω is a bounded smooth domain, and
- non-existence of entire solutions of (1.1) when $\Omega = \mathbb{R}^n$.

In particular, when $f(u) = -\lambda u$ (for $\lambda \leq 0$) the first condition is violated so no large solutions exist. See also Dumont–Dupaigne–Goubet–Rădulescu [23] for a study with oscillating nonlinearity. In the fractional case, a sufficient condition for the existence of solutions of $(-\Delta)_{\text{RFL}}^s + f(u) = 0$ with a blow-up rate faster than δ^{s-1} , or a partial fractional Keller–Osserman condition, is obtained by Abatangelo [2].

¹More precisely, unbounded behavior on $\partial\Omega$ for $\mathcal{L} = -\Delta$, and more generally, blowing up faster than $\delta^{s-1}(x)$ as $x \rightarrow \partial\Omega$ for $\mathcal{L} = (-\Delta)_{\text{RFL}}^s$, $s \in (0, 1)$.

2. PRELIMINARIES, DEFINITIONS AND PRECISE STATEMENT OF RESULTS

2.1. The operator. The operator \mathcal{L} that we consider includes the two most popular versions of the fractional Laplacian in a bounded domain. \mathcal{L} takes the sign convention which is the one of $-\Delta$, hence is a non-negative operator. We make the following assumptions on \mathcal{L} throughout the paper:

- (K1) \mathcal{L} has a Green's function² $\mathcal{G}_0 : (\Omega \times \Omega) \setminus \{(x, y) \in \Omega \times \Omega : x = y\} \rightarrow (0, +\infty)$, such that³

$$\begin{aligned} \mathcal{G}_0(x, y) = \mathcal{G}_0(y, x) &\asymp \frac{1}{|x - y|^{n-2s}} \left(\frac{\delta^\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x - y|^\gamma} \wedge 1 \right) \\ &\asymp \frac{1}{|x - y|^{n-2s}} \left(\frac{\delta^\gamma(x)\delta^\gamma(y)}{|x - y|^{2\gamma}} \wedge 1 \right), \end{aligned}$$

for $\gamma \in (0, 1]$, $n \neq 2s$.

- (K2) \mathcal{G}_0 has an (inner) γ -normal derivative $D_\gamma \mathcal{G}_0 : \partial\Omega \times \Omega \rightarrow (0, +\infty)$,

$$D_\gamma \mathcal{G}_0(z, x) := \lim_{\Omega \ni y \rightarrow z} \frac{\mathcal{G}_0(y, x)}{\delta^\gamma(y)} \asymp \frac{\delta^\gamma(x)}{|x - z|^{n-2s+2\gamma}}, \quad x \in \Omega, z \in \partial\Omega,$$

which is referred to as the *Martin's kernel* in this setting.

Remark 2.1. In similar settings, some authors included the following additional assumptions: the linear operator $\mathcal{L} : \text{dom}(\mathcal{L}) \subset L^1(\Omega) \rightarrow L^1(\Omega)$ is assume to be densely defined and sub-Markovian,

- (A1) \mathcal{L} is m -accretive on $L^1(\Omega)$,
(A2) If $0 \leq f \leq 1$, then $0 \leq e^{-t\mathcal{L}} f \leq 1$.

We do not need them in this elliptic context.

Remark 2.2. Note that when $n = 2s = 1$, \mathcal{G}_0 has a logarithmic singularity.

Remark 2.3. By [9, Proposition 5.1], (K1) (in fact $0 \leq \mathcal{G}_0 \leq C|x - y|^{-(n-2s)}$) are sufficient for \mathbb{G}_0 to be a compact operator $L^2(\Omega) \rightarrow L^2(\Omega)$. They apply the RieszFrchetKolmogorov Theorem. Hence \mathcal{L} has a discrete standard spectrum $\Sigma = (\lambda_i)_{i \geq 1}$, containing a non-decreasing, divergent sequence of positive Dirichlet eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty,$$

repeated according to multiplicity, and the corresponding standard eigenfunctions, $(\varphi_j)_{j \geq 1} \subset C(\overline{\Omega})$, form an orthonormal basis of $L^2(\Omega)$, with $\varphi_1 \geq 0$.

The parameter γ is responsible for the optimal boundary behavior in the sense of Hopf, and the first eigenfunction satisfies $\varphi_1 \asymp \delta^\gamma$ (see [9]).

Remark 2.4. Considering the range of $\gamma \in (0, 1]$, we will at times look at the extra condition $\gamma < 2s$, which is not present in [3, 4]. The advantage in such regime is that L^1 weak-dual solutions can be directly defined, see Remark 2.7. On the other hand, from [4] or the discussion of Section 2.2, one obtains a large solution if and only if

²The zero subscript in \mathcal{G}_0 suggests $\lambda = 0$.

³ $f \asymp g$ means $C^{-1}f \leq g \leq Cf$ for a constant C . $a \wedge b$ denotes the minimum of a and b .

$\gamma > 2s - 1$. If $\gamma \leq 2s - 1$, then an L^2 -theory suffices and Theorem 2.6 becomes simply an exercise.

We discuss two concrete examples. They take the usual form

$$\mathcal{L}u(x) = \int_{\Omega} (u(x) - u(y))\mathcal{J}(x, y)dy + \kappa(x)u(x).$$

In [9], the authors check that the following two operators are m -accretive and sub-Markovian.

- The *restricted fractional Laplacian* (RFL) is defined by imposing the exterior Dirichlet condition $u = 0$ in Ω^c in the singular integral formula in the whole space, i.e. for $u \in C_{\text{loc}}^{2s+\epsilon}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; (1 + |x|)^{-n-2s})$,

$$(-\Delta)_{\text{RFL}}^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where $C_{n,s} = \frac{2^{2s}\Gamma(\frac{n+2s}{2})}{|\Gamma(-s)|\pi^{n/2}}$. It leads to

$$\mathcal{J}(x, y) = \frac{C_{n,s}}{|x - y|^{n+2s}}, \quad \kappa(x) = \int_{\Omega^c} \frac{C_{n,s}}{|x - y|^{n+2s}} dy \asymp \delta^{-2s}(x).$$

By [39], $\gamma = s$ in (K1). Note that $\gamma < 2s$ is satisfied for all $s \in (0, 1)$. Considering the exterior Dirichlet condition as a part the definition of the operator, we will not repeat it in the equations.

- The *spectral fractional Laplacian* (SFL) is defined by taking a fractional power of the Dirichlet Laplacian, typically done by an eigenfunction expansion. This operator is constructed so that φ_j are the basis of $L^2(\Omega)$ -eigenfunctions of $-\Delta$ associated to eigenvalues (say) μ_j , and $\lambda_j = \mu_j^s$. In terms of the heat kernel $\mathcal{K}(t, x, y)$ of $-\Delta$ in Ω , one has [44, 3],

$$\begin{aligned} \mathcal{J}(x, y) &= \frac{s}{\Gamma(1-s)} \int_0^{+\infty} \mathcal{K}(t, x, y) \frac{dt}{t^{1+s}} \asymp \frac{1}{|x - y|^{n+2s}} \left(\frac{\delta(x)\delta(y)}{|x - y|^2} \wedge 1 \right) \\ \kappa(x) &= \frac{s}{\Gamma(1-s)} \int_0^{+\infty} \left(1 - \int_{\Omega} \mathcal{K}(t, x, y) dy \right) \frac{dt}{t^{1+s}} \asymp \delta^{-2s}(x). \end{aligned}$$

By [16], $\gamma = 1$ in (K1). See also [14]. Note that $\gamma < 2s$ if and only if $s \in (\frac{1}{2}, 1)$.

Remark 2.5. In [10, 9, 4] the reader may find some other operators of fractional type satisfying (A1), (A2) and (K1). Whether hypothesis (K2) is satisfied is usually a difficult problem, which we will not discuss here. The study of other operators of Schrödinger relativistic type $\mathcal{L} = (-\Delta + m^2 I)^s - m^{2s} I$ have also been of significant recent interest. Conditions (A1)–(A2) are easy to check. We do not know if (K1) and (K2) hold.

We will make some further comments on the sharp structure of compact embeddings and functional spaces related to our operators in Appendix A.

2.2. Large \mathcal{L} -harmonic functions. This material corresponds to the case $\lambda = 0$ of our equation. The knowledge of the precise boundary blow-up rate of the large \mathcal{L} -harmonic functions, [4], leads us to introduce the important exponent

$$(2.1) \quad b = 1 - 2s + \gamma.$$

In particular,

$$b = \begin{cases} 1 - s & \text{for } \mathcal{L} = (-\Delta)_{\text{RFL}}^s, s \in (0, 1) \\ 2 - 2s & \text{for } \mathcal{L} = (-\Delta)_{\text{SFL}}^s, s \in (0, 1) \\ 0 & \text{for } \mathcal{L} = -\Delta \quad \text{or} \quad \mathcal{L} = (-\Delta)_{\text{CFL}}^s, s \in (\frac{1}{2}, 1). \end{cases}$$

We are interested in the case $b > 0$. Define the Martin's operator

$$\begin{aligned} \mathbb{M} : C(\partial\Omega) &\longrightarrow \delta^{-b} L^\infty(\Omega) \\ h &\longmapsto \mathbb{M}(h) \end{aligned}$$

by

$$(2.2) \quad \mathbb{M}(h)(x) = \int_{\partial\Omega} D_\gamma \mathcal{G}_0(z, x) h(z) d\mathcal{H}^{n-1}(z), \quad \text{for } x \in \partial\Omega,$$

where $D_\gamma \mathcal{G}_0(z, x)$ is the (inner) γ -normal derivative defined in (K2). (When $\gamma = 1$, it is reduced to the Poisson kernel.) For the reader's convenience, we include, in Proposition 3.8, a short proof showing that the Martin's operator is well-defined, in the hope of demystifying the naturally-occurring exponent $b = 1 - 2s + \gamma$. When $h \equiv 1$, in particular, [4, Corollary 4.3] asserts that

$$\mathbb{M}(1)(x) = \int_{\partial\Omega} D_\gamma \mathcal{G}_0(z, x) d\mathcal{H}^{n-1}(z) \asymp \delta^{-b}.$$

When normalized by $\mathbb{M}(1)$, the equation (2.2) actually leads to the representation formula for \mathcal{L} -harmonic functions with prescribed (large) boundary data, i.e. $u = \mathbb{M}(h)$ solves

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ \frac{u}{\mathbb{M}(1)} = h & \text{on } \partial\Omega, \\ u = 0 & \text{in } \Omega^c \text{ (if applicable).} \end{cases}$$

Since we will only focus on the case of null exterior condition, we indicate this in the future. Indeed, by [4, Theorem 4.13] (see also [1, 3]), for $h \in C(\partial\Omega)$, the limit

$$\frac{\mathbb{M}(h)}{\mathbb{M}(1)}(z) := \lim_{\Omega \ni x \rightarrow z} \frac{\mathbb{M}(h)(x)}{\mathbb{M}(1)(x)}$$

exists uniformly in $z \in \partial\Omega$ and equals $h(z)$. Even when $h \in L^1(\partial\Omega)$, such boundary trace can still be understood in a weaker sense, as an average over constricting tubular neighborhoods of $\partial\Omega$, see [4, Theorem 4.15].

On the other hand, the weighted trace operator

$$\begin{aligned} B : \delta^{-b} C(\overline{\Omega}) &\longrightarrow L^\infty(\partial\Omega) \\ u &\longmapsto Bu \end{aligned}$$

given by

$$Bu(z) = \lim_{\Omega \ni x \rightarrow z} \frac{u(x)}{\mathbb{M}(1)(x)} \quad \text{for } z \in \partial\Omega,$$

is well-defined. In the case $\mathcal{L} = (-\Delta)_{\text{RFL}}^s$, thanks to the connection pointed out to us by Ros-Oton, we even have the explicit formula

$$(2.3) \quad Bu(z) = \Gamma(1+s)^2 \lim_{\Omega \ni x \rightarrow z} \delta^{1-s}(x)u(x),$$

The Ros-Oton–Serra constant $\Gamma(1+s)^2$ appears already in the fractional Pohožaev identity [40]. Through its relation to the integration-by-parts formula of Abatangelo [1], the precise expression (2.3) can be obtained. See Lemma B.1 for the proof.

2.3. Main equation and notions of solution. Given $h \in C(\partial\Omega)$ and $\lambda \in \mathbb{R} \setminus \Sigma$, we consider L^1 -solutions of the eigenvalue problem

$$(2.4) \quad \begin{cases} \mathcal{L}u - \lambda u = 0 & \text{in } \Omega \\ Bu = h & \text{on } \partial\Omega, \end{cases}$$

understood in a certain weak sense that we describe below. One of our main results, Theorem 2.6, which is also the motivation of this work, is to determine the precise blow-up behavior of its solutions.

More generally, consider the equation

$$(2.5) \quad \begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = h & \text{on } \partial\Omega. \end{cases}$$

Let us recall the notion of L^1 or very weak solution used in the literature [1, 3, 4]. Denote the Green's function of \mathcal{L} in Ω by \mathcal{G}_0 , and the Green's operator by

$$(2.6) \quad \mathbb{G}_0(f)(x) = \int_{\Omega} \mathcal{G}_0(x, y) f(y) dy.$$

We stress two facts: it was shown in [4] that the largest class of functional data f admissible is $L^1(\Omega, \delta^\gamma)$ and that, if $f \in L_c^\infty(\Omega)$, then $|\mathbb{G}_0(f)| \leq C\delta^\gamma$.

The notion of *very weak formulation* first introduced by H. Brezis for local operators, was later extended to the notion of *weak-dual formulation* and extensively used for instance in [9, 4]. Its merit is that it avoids defining \mathcal{L} on the test function. As a matter of fact, the regularity needed to define \mathcal{L} classically calls for more (i.e. $2s + \epsilon$, $\epsilon > 0$) than what a Schauder estimate can recover (i.e. $2s$, or $2s - \epsilon = 1 - \epsilon$ when $2s = 1$). This issue will not be present when the Green's function is used instead. We will specify the duality precisely by naming the Green's function used in the definition.

Definition 2.1 ($\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solutions [1, 3]). Suppose $f \in L^1(\Omega, \delta^\gamma)$ and $h \in C(\partial\Omega)$. We say that $u \in L^1(\Omega, \delta^\gamma)$ is a $\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solution of (2.5) if

$$(2.7) \quad \int_{\Omega} u(\mathcal{L}\zeta - \lambda\zeta) dx = \int_{\Omega} f\zeta dx + \int_{\partial\Omega} D_\gamma\zeta h d\mathcal{H}^{n-1}, \quad \forall \zeta \in \mathbb{G}_0(C_c^\infty(\Omega)),$$

or, equivalently,

$$(2.8) \quad \int_{\Omega} u(\psi - \lambda \mathbb{G}_0(\psi)) dx = \int_{\Omega} f \mathbb{G}_0(\psi) dx + \int_{\partial\Omega} D_{\gamma} \mathbb{G}_0(\psi) h d\mathcal{H}^{n-1}, \quad \forall \psi \in C_c^{\infty}(\Omega).$$

Formulation (2.7) corresponds to the notion of very weak solution introduced by Brezis, where all “derivatives” are passed to the test function. Formulation (2.8) corresponds to the notion of weak-dual solution, in which derivatives are not present.

Remark 2.6. We point out that

- (1) The integrability $u \in L^1(\Omega, \delta^{\gamma})$ is sufficient in this setting. Notice that, since ψ is bounded and compactly supported, then $u\psi$ is integrable. Furthermore, since $|\mathcal{G}_0(\psi)| \leq C\delta^{\gamma}$, then $u\mathcal{G}_0(\psi)$ is also integrable.
- (2) Notice that the γ -normal derivative $D_{\gamma} \mathbb{G}_0(\psi)$ is well-defined on $\partial\Omega$ by the assumption (K2), see Proposition 3.10. To gain an intuition of the boundary integral in (2.8), suppose $\lambda = 0$ and $f = 0$. Testing the representation formula

$$u(x) = \int_{\partial\Omega} D_{\gamma} \mathcal{G}_0(z, x) h(z) d\mathcal{H}^{n-1}(z)$$

against $\psi \in C_c^{\infty}(\Omega)$, Proposition 3.10 implies

$$\begin{aligned} \int_{\Omega} u\psi dx &= \int_{\Omega} \left(\int_{\partial\Omega} D_{\gamma} \mathcal{G}_0(z, x) h(z) d\mathcal{H}^{n-1}(z) \right) \psi(x) dx \\ &= \int_{\partial\Omega} \left(\int_{\Omega} D_{\gamma} \mathcal{G}_0(z, x) \psi(x) dx \right) h(z) d\mathcal{H}^{n-1}(z) \\ &= \int_{\partial\Omega} D_{\gamma} \mathbb{G}_0(\psi)(z) h(z) d\mathcal{H}^{n-1}(z). \end{aligned}$$

For an L^1 -theory, it is convenient to enlarge the class of test functions to bounded functions with compact support.

Definition 2.2 ($\mathbb{G}_0(L_c^{\infty}(\Omega))$ -weak solutions [4]). Suppose $f \in L^1(\Omega, \delta^{\gamma})$ and $h \in C(\partial\Omega)$. We say that $u \in L^1(\Omega, \delta^{\gamma})$ is a $\mathbb{G}_0(L_c^{\infty}(\Omega))$ -weak solution of (2.5) if

$$\int_{\Omega} u(\psi - \lambda \mathbb{G}_0(\psi)) dx = \int_{\Omega} f \mathbb{G}_0(\psi) dx + \int_{\partial\Omega} D_{\gamma} \mathbb{G}_0(\psi) h d\mathcal{H}^{n-1}, \quad \forall \psi \in L_c^{\infty}(\Omega).$$

Let us now restrict ourselves to the case $h = 0$, which happens in most parts of the present article.

$$(2.9) \quad \begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that in this case, the boundary condition can be rewritten as $\delta^b u = 0$.

For later purposes, it is more convenient to use the test function space $\mathbb{G}_{\lambda}(L^{\infty}(\Omega))$, where \mathbb{G}_{λ} is the solution operator of the full Helmholtz problem (2.9), when λ is not an eigenvalue of \mathcal{L} . By spectral decomposition, it can be defined simply by (4.2) below. In Theorem 2.2 we will show this operator has a unique extension to $L^1(\Omega, \delta^{\gamma})$ data. We note that, to our best knowledge, the exact behavior of its Green’s function \mathcal{G}_{λ} has not yet been studied.

Remark 2.7. Further along, we will want to perform projections onto the basis of eigenfunctions, which do not preserve the compact support. Hence $\psi \in L_c^\infty(\Omega)$ seems not to be convenient for us. Fortunately, we know that the eigenfunction $\varphi_j \in \delta^\gamma L^\infty(\Omega)$. We need sharp information on the boundary behaviour. In [4] the authors prove that, for $\alpha > -1 - \gamma$,

$$(2.10) \quad \mathbb{G}_0(\delta^\alpha) \asymp \begin{cases} \delta^{\alpha+2s} & \text{for } \alpha + 2s < \gamma, \\ \delta^\gamma(1 + |\ln \delta|) & \text{for } \alpha + 2s = \gamma, \\ \delta^\gamma & \text{for } \alpha + 2s > \gamma. \end{cases}$$

In particular,

$$\mathbb{G}_0(\delta^\gamma) \asymp \delta^\gamma, \quad \mathbb{G}_0(1) \asymp \begin{cases} \delta^\gamma & \text{for } \gamma < 2s, \\ \delta^{2s}(1 + |\ln \delta|) & \text{for } \gamma = 2s, \\ \delta^{2s} & \text{for } \gamma > 2s. \end{cases}$$

It is easy to see that

$$(2.11) \quad \mathbb{G}_0(\delta^\alpha) \geq c_\alpha \delta^\gamma.$$

The case of equality involves a logarithm. Notice that, for the case of the SFL, we have $\gamma = 1$ so the second range is achieved for $s < \frac{1}{2}$. We must adapt our data for this setting. This relation can be seen in the singular solution $\mathbb{M}(1) \asymp \delta^{2s-\gamma-1}$. If $2s > \gamma$ then $\mathbb{M}(1) \in L^1(\Omega)$. This ranges simplifies the theory immensely. Nevertheless, $\mathbb{M}(1) \in L^1(\Omega, \delta^\gamma)$ for all s and γ .

Definition 2.3 ($\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solutions). Suppose $f \in L^1(\Omega, \delta^\gamma)$. We say that $u \in L^1(\Omega, \delta^\gamma)$ is a $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (2.9) if

$$\int_\Omega u \psi \, dx = \int_\Omega f \mathbb{G}_\lambda(\psi) \, dx, \quad \forall \psi \in \delta^\gamma L^\infty(\Omega).$$

Note that by Proposition 4.3, $\mathbb{G}_\lambda(\psi)$ is well-defined and controlled in $L^\infty(\Omega)$ by δ^γ , which is crucial for the right hand integral to be defined.

After establishing the linear theory, one may talk about solutions of (2.9) in the sense of Green's operator, or briefly, *Green solution*.⁴

Definition 2.4 (Green solutions). Let $u, f \in L^1(\Omega; \delta^\gamma)$. We say that u is a \mathbb{G}_0 -Green solution of (2.9) if

$$u = \lambda \mathbb{G}_0(u) + \mathbb{G}_0(f) \quad \text{a.e. in } \Omega,$$

for \mathbb{G}_0 defined by the integral formula in (2.6). We say that u is a \mathbb{G}_λ -Green solution of (2.9) if

$$u = \mathbb{G}_\lambda(f) \quad \text{a.e. in } \Omega,$$

where $\mathbb{G}_\lambda : L^1(\Omega; \delta^\gamma) \rightarrow L^1(\Omega; \delta^\gamma)$ is defined in Theorem 2.2.

Remark 2.8. Notice that, in order to define \mathbb{G}_0 -Green solutions, u must be in the admissible class of data for \mathbb{G}_0 .

⁴This is in fact a convenient abuse of nomenclature because it is defined by solving the PDE and not by studying the Green's function for the operator $\mathcal{L} - \lambda$.

When $f \in L^2(\Omega)$, solutions are in

$$\mathbb{H}_{\mathcal{L}}^2(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j \geq 1} \lambda_j^2 \langle v, \varphi_j \rangle^2 < +\infty \right\},$$

and are naturally understood in the *spectral* sense.

Definition 2.5 (Spectral solutions). Suppose $f \in L^2(\Omega)$. We say that $u \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$ is a *spectral solution* of (2.9) if

$$(\lambda_j - \lambda) \int_{\Omega} u \varphi_j dx = \int_{\Omega} f \varphi_j dx, \quad \forall j \geq 1.$$

Remark 2.9. We will show below that the eigenfunctions enjoy the optimal boundary regularity $\varphi_j \in \delta^\gamma L^\infty(\Omega)$, therefore these equations are defined for $f \in L^1(\Omega, \delta^\gamma)$. However, when $f \notin L^2(\Omega)$, u is not in $\mathbb{H}_{\mathcal{L}}^2(\Omega)$, so the resulting series may not converge to u .

It is a simple exercise of approximation of the test functions to show that

Theorem 2.1 (Equivalence of notions of solution). Assume (K1). Suppose $f, u \in L^1(\Omega, \delta^\gamma)$. Then the following are equivalent:

- (1) u is a \mathbb{G}_λ -Green solution of (2.9).
- (2) u is a $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (2.9).
- (3) u is a $\mathbb{G}_0(L_c^\infty(\Omega))$ -weak solution of (2.9).
- (4) u is a $\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solution of (2.9).
- (5) u is a \mathbb{G}_0 -Green solution of (2.9).

Suppose, in addition, that $f \in L^2(\Omega)$ and $u \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$. Then any of the above is equivalent to

- (6) u is a spectral solution of (2.9).

The proofs is given in Section 4.3, using the linear theory Theorem 2.2 (see below) and the integration-by-parts formulae in Section 4.2.

2.4. General linear theory. Secondly, we prove that \mathbb{G}_λ is well-defined in various Lebesgue spaces and establish its mapping properties. Denote the eigenvalue of \mathcal{L} just larger than λ by

$$(2.12) \quad \bar{\lambda} = \lambda_{I+1}, \quad \text{where} \quad I = \min \{j : \lambda_j \geq \lambda \text{ and } \lambda_{j+1} > \lambda_j\}.$$

and the distance of λ to the nearest eigenvalue by

$$(2.13) \quad d_\Sigma(\lambda) = \min_j |\lambda_j - \lambda|.$$

Notice that, with this choice, $\lambda_{I+1} > \lambda_I \geq \lambda$.

Theorem 2.2 (The solution operator \mathbb{G}_λ). Assume (K1). Let $\lambda \in \mathbb{R} \setminus \Sigma$. Let $\bar{\lambda}, d_\Sigma(\lambda)$ be defined in (2.12), (2.13) respectively. Then

(1) \mathbb{G}_λ is well-defined on $L^1(\Omega; \delta^\gamma)$ and maps continuously from

$$\begin{aligned} L^1(\Omega; \mathbb{G}_0(\delta^\alpha)) &\longrightarrow L^1(\Omega, \delta^\alpha), & \text{for } \alpha > -1 - \gamma, \\ L^1(\Omega) &\longrightarrow L^p(\Omega), & \text{for } p \in [1, \frac{n}{n-2s}), \\ L^2(\Omega) &\longrightarrow \mathbb{H}_{\mathcal{L}}^2(\Omega), \\ L^{p_0}(\Omega) &\longrightarrow L^{p_1}(\Omega) & \text{for } p_0 \in (1, \frac{n}{2s}) \text{ and } \frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}, \\ L^q(\Omega) &\longrightarrow L^\infty(\Omega), & \text{for } q \in (\frac{n}{2s}, +\infty), \\ \delta^\alpha L^\infty(\Omega) &\longrightarrow \mathbb{G}_0(\delta^\alpha) L^\infty(\Omega), & \text{for } \alpha > -1 - \gamma. \end{aligned}$$

with corresponding operator norms depending only on $\bar{\lambda}, d_\Sigma(\lambda)$ and, where appropriate, p or q .

(2) \mathbb{G}_λ maps continuously from

$$\begin{aligned} L^1(\Omega; \delta^\gamma) &\longrightarrow L_{\text{loc}}^p(\Omega), & \text{for } p \in (1, \frac{n}{n-2s}), \\ L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^{p_0}(\Omega) &\longrightarrow L_{\text{loc}}^{p_1}(\Omega), & \text{for } p_0 \in (1, \frac{n}{2s}) \text{ and } \frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}, \\ L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^q(\Omega) &\longrightarrow L_{\text{loc}}^\infty(\Omega), & \text{for } q \in (\frac{n}{2s}, +\infty). \end{aligned}$$

Furthermore, for $f \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$, $K \Subset K_0 \Subset \Omega$ and $u = \mathbb{G}_\lambda(f)$ we have that

$$\|u\|_{L^\infty(K)} \leq C(\bar{\lambda}, d_\Sigma(\lambda), K, K_0) \left(\|f\|_{L^\infty(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right).$$

The L^2 theory is classical and the regularity with high integrability data follows essentially from the behavior of the Green's function. In fact, under stronger assumptions on the Green's function, one may even obtain $u \in C^\alpha(\bar{\Omega})$, for $\alpha > \gamma$. Indeed, for RFL and SFL, it follows from [24, Theorem 1.1] and [16, Theorem 1.5 (2)] respectively.⁵ The case of $L^1(\Omega; \delta^\gamma)$ data is an elaboration of the classical paper of Brezis–Cazenave–Martel–Ramiandrisoa [11]. Note that we do not use the maximum principle, in order to cover the case of large λ (i.e. $\lambda \geq \lambda_1$). Also, we obtain directly L^p -estimates for some $p > 1$. The proof is provided in Section 4.1 for (1) global estimates and Section 4.4 for (2) local boundedness.

Remark 2.10. Since the fundamental notions and results concerning \mathbb{G}_λ are grouped for easier presentation and are not ordered linearly, we make an explicit note on the logical line of thought.

- (1) For $\psi \in L^2(\Omega)$, $\zeta = \mathbb{G}_\lambda(\psi)$ is defined in Lemma 4.1 as the explicit spectral solution of $\mathcal{L}\zeta - \lambda\zeta = \psi$.
- (2) For ψ such that $\psi/\delta^\gamma \in L^\infty(\Omega)$, $\mathbb{G}_\lambda(\psi)$ also enjoys the boundary regularity $\mathbb{G}_\lambda(\psi)/\delta^\gamma \in L^\infty(\Omega)$, and so do the eigenfunctions, as shown in Proposition 3.7.
- (3) The $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak formulation in Definition 2.3 makes sense.
- (4) The definition of \mathbb{G}_λ is extended to $L^1(\Omega; \delta^\gamma) \rightarrow L^1(\Omega; \delta^\gamma)$ in Proposition 4.4.
- (5) An integration-by-parts formula is obtained for \mathbb{G}_λ in Lemma 4.7.
- (6) All notions of solutions are shown to be equivalent in Section 4.3.

⁵In [24] the Morrey space \mathcal{M}_β is used. Note that by Lemma C.1, $L^q(\Omega)$ is continuously embedded into $\mathcal{M}_{n/q}$.

- (7) The remaining theory, including the local bounds, the projection into E^\perp and the maximum principle, is developed.

Remark 2.11. When $\lambda = \lambda_i$ is an eigenvalue, the classical Fredholm alternative applies: \mathbb{G}_λ is well-defined on the E_i^\perp , the space of functions orthogonal to all eigenfunctions associated to λ_i , and in which case the solution space has the same dimension as E_i . Results can be obtained as a limiting case of Theorem 2.3 and Theorem 2.4.

2.5. Projected linear theory. Our third result concerns the solution operator \mathbb{G}_λ on the subspace $E^\perp \subset L^2(\Omega)$, whose elements are orthogonal to any eigenfunction with eigenvalue at most $\bar{\lambda}$. For further clarity, let us write

$$E = \text{span}\{\varphi_1, \dots, \varphi_I\}, \quad E^\perp = \text{span}\{\varphi_{I+1}, \varphi_{I+2}, \dots\}$$

where $\lambda \leq \lambda_I < \lambda_{I+1} = \bar{\lambda}$ and span is taken topologically in $L^2(\Omega)$. We prove uniform estimates that are independent of $d_\Sigma(\lambda)$. Furthermore, let us define

$$\widetilde{E}^\perp = \left\{ f \in L^1(\Omega, \delta^\gamma) : \int_\Omega f \varphi_j = 0, \text{ for } j = 1, \dots, I \right\}.$$

It is easy to see that

$$\widetilde{E}^\perp = \overline{E^\perp}^{L^1(\Omega; \delta^\gamma)}.$$

Indeed, the backward inclusion follows by taking the $L^1(\Omega; \delta^\gamma)$ -closure in $E^\perp \subset \widetilde{E}^\perp$ (note that the pairing with eigenfunction is continuous in $L^1(\Omega; \delta^\gamma)$), and the converse follows by considering the projection in E^\perp of the sequence $f_k = (|f| \wedge k) \text{sign}(f) \in L^2(\Omega)$, which converges to f in $L^1(\Omega; \delta^\gamma)$.

Theorem 2.3 (The solution operator \mathbb{G}_λ on E^\perp). *Assume (K1). Let $\lambda \in \mathbb{R} \setminus \Sigma$. Let $\bar{\lambda}$ be defined in (2.12). Then \mathbb{G}_λ maps continuously from*

$$\begin{aligned} L^1(\Omega; \mathbb{G}_0(\delta^\alpha)) \cap \widetilde{E}^\perp &\longrightarrow L^1(\Omega, \delta^\alpha) \cap \widetilde{E}^\perp, & \text{for } \alpha > -1 - \gamma, \\ L^1(\Omega) \cap \widetilde{E}^\perp &\longrightarrow L^p(\Omega) \cap \widetilde{E}^\perp, & \text{for } p \in [1, \frac{n}{n-2s}), \\ L^2(\Omega) \cap E^\perp &\longrightarrow \mathbb{H}_\mathcal{L}^2(\Omega) \cap E^\perp, \\ L^{p_0}(\Omega) \cap \widetilde{E}^\perp &\longrightarrow L^{p_1}(\Omega) \cap \widetilde{E}^\perp, & \text{for } p_0 \in (1, \frac{n}{2s}) \text{ and } \frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}, \\ L^q(\Omega) \cap E^\perp &\longrightarrow L^\infty(\Omega) \cap E^\perp, & \text{for } q \in (\frac{n}{2s}, +\infty), \\ \delta^\alpha L^\infty(\Omega) \cap E^\perp &\longrightarrow \mathbb{G}_0(\delta^\alpha) L^\infty(\Omega) \cap E^\perp, & \text{for } \alpha > -1 - \gamma. \end{aligned}$$

with corresponding operator norms depending only on $\bar{\lambda}$ and, where appropriate, p or q .

By the orthogonal decomposition $L^2(\Omega) = E \oplus E^\perp$, one writes down the solution of the eigenvalue problem as an explicit part in E and a uniformly controlled part in E^\perp . When $f \in L^2(\Omega)$ we can write

$$f = \sum_{j=1}^I \langle f, \varphi_j \rangle \varphi_j + \sum_{j=I+1}^{+\infty} \langle f, \varphi_j \rangle \varphi_j = P_E(f) + P_{E^\perp}(f).$$

If $f \in L^1(\Omega, \delta^\gamma)$ this projection is not possible in general. Nevertheless, since $\varphi_j \in \delta^\gamma L^\infty(\Omega)$ for any eigenspace we have the well defined projection

$$\begin{aligned} P_{E_i} : L^1(\Omega, \delta^\gamma) &\longrightarrow E_i \subset \delta^\gamma L^\infty(\Omega) \\ f &\longmapsto \sum_{j: \lambda_j = \lambda_i} \langle f, \varphi_j \rangle \varphi_j. \end{aligned}$$

and

$$P_E(f) = \sum_{j=1}^I \langle f, \varphi_j \rangle \varphi_j.$$

We can define

$$(2.14) \quad f^\perp = f - P_E(f) = f - \sum_{j=1}^I \langle f, \varphi_j \rangle \varphi_j.$$

It is clear that $f^\perp \in L^1(\Omega, \delta^\gamma)$. In fact, $f^\perp \in \overline{E^\perp}^{L^1(\Omega; \delta^\gamma)}$ and, for $f \in L^2(\Omega)$, $f^\perp = P_{E^\perp}(f)$.

Theorem 2.4 (Projection and uniform estimates). *Assume (K1). Let $\lambda \in \mathbb{R} \setminus \Sigma$. Let $b, \bar{\lambda}$ be defined in (2.1), (2.12) respectively. For $f \in L^1(\Omega; \delta^\gamma)$ let $u \in L^1(\Omega, \delta^\gamma)$ be the unique $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of*

$$\begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for f^\perp given by (2.14) and $u^\perp = \mathbb{G}_\lambda(f^\perp)$, i.e. the unique solution of

$$\begin{cases} \mathcal{L}u^\perp - \lambda u^\perp = f^\perp & \text{in } \Omega \\ Bu^\perp = 0 & \text{on } \partial\Omega, \end{cases}$$

we have that

$$u = \sum_{j=1}^I \frac{\langle f, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j + u^\perp.$$

We have the uniform estimate

$$\|u^\perp \delta^\gamma\|_{L^1(\Omega)} \leq C(\bar{\lambda}) \|f \delta^\gamma\|_{L^1(\Omega)}.$$

If $f \in L^1(\Omega)$ then, for any $p \in [1, \frac{n}{n-2s})$ we have that

$$\|u^\perp\|_{L^1(\Omega)} \leq C(p, \bar{\lambda}) \|f\|_{L^p(\Omega)},$$

If $f \in L^{p_0}(\Omega)$ then, for any $\frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}$ we have that

$$\|u^\perp\|_{L^{p_1}(\Omega)} \leq C(p_0, p_1, \bar{\lambda}) \|f\|_{L^{p_0}(\Omega)},$$

Moreover, if $f \in L^\infty_{\text{loc}}(\Omega)$, then for any $K \Subset K_0 \Subset \Omega$,

$$\|u^\perp\|_{L^\infty(K)} \leq C(\bar{\lambda}, K, K_0) \left(\|f\|_{L^\infty(K_0)} + \|f \delta^\gamma\|_{L^1(\Omega)} \right).$$

Remark 2.12. The uniform dependence on $\bar{\lambda} = \lambda_{I+1}$ is actually a dependence on $\lambda_{I+1} - \lambda_I$.

These closely related main theorems are proved in Section 5. While the bootstrap regularity arguments remain the same as in Theorem 2.2, the L^2 -norm of u^\perp can now be bounded independently of $d_\Sigma(\lambda)$. In the (weighted) L^1 -theory, it is impossible to write down the full eigenfunction expansion. Nonetheless, it still makes sense to project an $L^1(\Omega; \delta^\gamma)$ -function into a finite dimensional subspace. Such interplay of L^1 - and L^2 -theory is demonstrated in the proof of Theorem 5.5, which is the cornerstone of this article. As the uniform estimate is available only in the orthogonal complement E^\perp , it is indispensable to transfer the orthogonality to the test function. This simple while decisive step is seen in (5.4).

The basic idea behind, i.e. to single out “bad” directions, runs in parallel with the finite dimensional Lyapunov–Schmidt reduction in the construction of solutions of PDEs. A nonlinear analogue can be seen in the recent paper [21], where the authors construct helical vortex filaments in the three-dimensional Ginzburg–Landau equation, and the orthogonality of the larger part of the nonlinear perturbation is crucially used.

Remark 2.13. A natural question is what happens as $\lambda \rightarrow \lambda_i$. Since we are going to move λ , let us be precise about notation. The value $\bar{\lambda}$ was defined for λ fixed. We want to be able approach λ_i from above or below, let $\bar{\lambda} = \lambda_{I+1} > \lambda_i$. Looking at the decomposition $f = P_E(f) + f^\perp$ giving a solution $u = \mathbb{G}_\lambda(P_E(f)) + \mathbb{G}_\lambda(f^\perp)$, the estimates above tell us that $\mathbb{G}_\lambda(f^\perp)$ is uniformly bounded in the corresponding spaces. On the contrary, looking at

$$\mathbb{G}_\lambda(P_E(f)) = \sum_{j=1}^I \frac{\langle f, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j = \sum_{\substack{j=1 \\ \lambda_j \neq \lambda_i}}^I \frac{\langle f, \varphi_j \rangle}{\lambda_j - \lambda} + \frac{1}{\lambda_i - \lambda} \sum_{j: \lambda_j = \lambda_i} \langle f, \varphi_j \rangle \varphi_j,$$

there is a blow-up at any point such that $P_{E_i}(f)(x) = \sum_{j: \lambda_j = \lambda_i} \langle f, \varphi_j \rangle \varphi_j(x) \neq 0$. We recover the classical Fredholm alternative:

As $\lambda \rightarrow \lambda_i$,

- (1) If $P_{E_i}(f) = 0$, then $\mathbb{G}_\lambda(f) \rightarrow \mathbb{G}_{\lambda_i}(f^\perp) = \mathbb{G}_{\lambda_i}(f)$ in $L^1(\Omega, \delta^\gamma)$
- (2) If $P_{E_i}(f) \neq 0$, then $|\mathbb{G}_\lambda(f)| \rightarrow +\infty$ on a set of positive measure.

Notice that, if $\lambda \rightarrow \lambda_i^\pm$, we can estimate the sign of the blow-up at a point x , by combining the sign of $P_{E_i}(f)(x) \neq 0$. At the end we point out that, since the eigenfunctions are linearly independent $P_{E_i}(f) = 0$ if and only if $\langle f, \varphi_j \rangle = 0$ for all j such that $\lambda_i = \lambda_j$.

2.6. Maximum principle. When $\lambda < \lambda_1$, one obtains a maximum principle for weighted L^1 -solutions, even though the positivity via Poincaré inequality makes sense only in the L^2 -setting. This is our fourth result.

Theorem 2.5 (Maximum principle for L^1 -solutions). *Assume (K1). Let $\lambda \in (-\infty, \lambda_1)$. Let b be defined in (2.1). Suppose $f \in L^1(\Omega; \delta^\gamma)$, and $u \in L^1(\Omega; \delta^\gamma)$ is a $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of*

$$\begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, if $f \geq 0$ a.e. in Ω , we have $u \geq 0$ a.e. in Ω .

The proof is given in Section 6. A related maximum principle can be found in [17]. As customary, for L^1 -solution one applies an L^2 maximum principle to the test function; the negative part of an L^2 -solution is a valid test function. Classically one uses the fact that $\nabla u_- \cdot \nabla u_+ = 0$ when $u \in H^1$. In the nonlocal case, one choice of dealing with the cross term is by Kato's inequality, tested against the solution itself, as can be seen in the singular integral formulation. Such cross term appears for example in the recent breakthrough related to the De Giorgi conjecture [27]. For the full pointwise inequality see [20, 15, 22] for RFL and [3] for SFL. In our approach where the equation is expressed in terms of the Green's function which is non-negative, however, the cross term clearly has a sign. The burden of proof shifts to the Poincaré inequality, which has already been established in [9].

The maximum principle for the fractional Laplacian ($\lambda = 0$) or associated coercive operators (including $\lambda < 0$) are well-known. In the case of the fractional Laplacian it follows automatically from the condition $\mathcal{G}_0(x, y) \geq 0$. See for example [43, 3, 22].

2.7. Inhomogeneous eigenvalue problem. We are finally addressing the problem of singular boundary data, i.e., large solutions. Let $g \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$ and $h \in C(\partial\Omega)$. Consider $\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solutions of

$$(2.15) \quad \begin{cases} \mathcal{L}v - \lambda v = g & \text{in } \Omega \\ Bv = h & \text{in } \partial\Omega. \end{cases}$$

Let us consider the large \mathcal{L} -harmonic function $v_h \in \delta^{-b}L^\infty(\Omega) \subset L^1(\Omega, \delta^\gamma)$ solving

$$(2.16) \quad \begin{cases} \mathcal{L}v_h = 0 & \text{in } \Omega \\ Bv_h = h & \text{on } \partial\Omega. \end{cases}$$

Existence, uniqueness and kernel representation has been extensively studied in [1] (RFL), [3] (SFL) and [4] (unified approach). The solution is obtained by either justifying the Martin's kernel \mathbb{M} representation or as a limit of the interior theory. We present the corresponding results in Section 3.3. The difference $u = v - v_h$ solves

$$\begin{cases} \mathcal{L}u - \lambda u = g + \lambda v_h & \text{in } \Omega \\ Bu = 0 & \text{in } \partial\Omega. \end{cases}$$

Since $g + \lambda v_h \in L^1(\Omega, \delta^\gamma)$, when $\lambda \notin \Sigma$ this problem has a unique solution in theory above, given by $u = \mathbb{G}_\lambda(g + \lambda v_h)$. Then

$$(2.17) \quad v = v_h + \mathbb{G}_\lambda(g + \lambda v_h).$$

Using Theorem 2.4, we obtain, as our third main result, the structure of solutions and Fredholm alternative as $\lambda \rightarrow \Sigma$.

Theorem 2.6. *Assume (K1) and (K2). Let $\lambda \in \mathbb{R} \setminus \Sigma$. Let $b, \bar{\lambda}$ be defined in (2.1), (2.12) respectively. Let*

$$(2.18) \quad g \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega).$$

Given $h \in C(\partial\Omega)$, let

$$(2.19) \quad v_h \in \delta^{-b} L^\infty(\Omega)$$

be the large \mathcal{L} -harmonic function solving (2.16). Then the following hold.

- (1) (Existence-uniqueness) For any $\lambda \in \mathbb{R} \setminus \Sigma$, (2.15) has a unique $\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solution

$$v_\lambda \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega).$$

- (2) (Representation formula) The solution v_λ is given by

$$v_\lambda = v_h + \sum_{\varphi_j \in E} \frac{\langle g + \lambda v_h, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j + u^\perp,$$

where $u^\perp \in L^p(\Omega) \cap L_{\text{loc}}^\infty(\Omega) \cap \widetilde{E}^\perp$, for all $p \in [1, \frac{n}{n-2s})$, is the unique solution (in any of the sense (1)–(5) in Theorem 2.1) of

$$(2.20) \quad \begin{cases} \mathcal{L}u^\perp - \lambda u^\perp = g^\perp + \lambda v_h^\perp & \text{in } \Omega \\ Bu^\perp = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$v_h^\perp = v_h - \sum_{\varphi_j \in E} \langle v_h, \varphi_j \rangle \varphi_j, \quad g^\perp = g - \sum_{\varphi_j \in E} \langle g, \varphi_j \rangle \varphi_j.$$

- (3) (Uniform estimates) We have that

$$\|u^\perp \delta^\gamma\|_{L^1(\Omega)} \leq C \left(\|g\|_{L^1(\Omega)} + \|h\|_{L^\infty(\partial\Omega)} \right),$$

where $C = C(\bar{\lambda})$, and for any $K \Subset K_0 \Subset \Omega$ we have that

$$(2.21) \quad \|v_h\|_{L^\infty(K)} + \|u^\perp\|_{L^\infty(K)} \leq C \left(\|g\|_{L^\infty(K_0)} + \|g\delta^\gamma\|_{L^1(\Omega)} + \|h\|_{L^\infty(\partial\Omega)} \right),$$

for $C = (\bar{\lambda}, K, K_0)$.

- (4) (Fredholm alternative) For $i = 1, 2, \dots$, let E_i be the eigenspace associated to the $\lambda_i \in \Sigma$. Let $\lambda \rightarrow \lambda_i$. Then exactly one of the following holds.

- (a) If $P_{E_i}(g + \lambda_i v_h) = 0$, then

$$v_\lambda \longrightarrow v_h - P_{E_i}(v_h) + \mathbb{G}_{\lambda_i}(g + \lambda_i v_h).$$

in $L^1(\Omega, \delta^\gamma)$.

- (b) If $P_{E_i}(g + \lambda_i v_h) \neq 0$ then $|v_\lambda| \rightarrow +\infty$ uniformly in a set of positive measure. Furthermore, let, for $\varepsilon > 0$

$$A_i^+ := \{P_{E_i}(g + \lambda_i v_h) > \varepsilon\}, \quad A_i^- := \{P_{E_i}(g + \lambda_i v_h) < -\varepsilon\}.$$

Then for any $K \Subset \Omega$,

$$\begin{cases} \lim_{\lambda \nearrow \lambda_i} \inf_{K \cap A_i^+} v_\lambda = \lim_{\lambda \searrow \lambda_i} \inf_{K \cap A_i^-} v_\lambda = +\infty, \\ \lim_{\lambda \nearrow \lambda_i} \sup_{K \cap A_i^-} v_\lambda = \lim_{\lambda \searrow \lambda_i} \sup_{K \cap A_i^+} v_\lambda = -\infty. \end{cases}$$

(5) (*Global blow-up*) If $g \geq 0$ in Ω and $h > 0$ on $\partial\Omega$, and $i = 1$, then

$$\lim_{\lambda \nearrow \lambda_1} \inf_{\Omega} v_{\lambda} = +\infty.$$

We give the proof in Section 7. The main idea is straightforward: by subtracting the large \mathcal{L} -harmonic function we obtain an equation with zero weighted (by δ^b) trace, and a right hand side in $\delta^{-b}L^\infty(\Omega) \subset L^1(\Omega; \delta^\gamma)$. Then the previously developed linear theory in Theorem 2.4 applies. The interior blow-up behaviors as $\lambda \rightarrow \Sigma$ can be seen from the explicit part with the low eigenfunctions, because the orthogonal part is shown to be bounded in compact subsets.

Remark 2.14. Note that by Fubini's Theorem, one may rewrite

$$\langle \lambda_i v_h, \varphi_j \rangle = \int_{\partial\Omega} D_\gamma \varphi_j(z) h(z) d\mathcal{H}^{n-1}(z), \quad \text{for } \varphi_j \in E_i.$$

Remark 2.15. It is possible that the solution does not blow up in the interior, i.e. $A_+ = A_- = \emptyset$. For example, this happens when $n = 3$, $\Omega = B_1$, $g = 0$, $h = 1$ and $i = 2$, in which case $U = (1 - |x|^2)_+^{s-1}$ [8] and all eigenfunctions corresponding to λ_2 are anti-symmetric [26]. The anti-symmetry of higher eigenfunctions for the restricted fractional Laplacian appears to be a challenging problem.

2.8. Convergence and compactness as $s \nearrow 1$. As the parameter s tends to 1, we also have $\gamma \rightarrow 1$. Thus the exponent $b = 1 - 2s + \gamma$ tends to 0. Therefore the limit no longer blows up at the boundary. In fact, an L^2 -theory is sufficient due to the slow blow-up rate for s close to 1.

Under additional assumptions on the Green's function, we show various convergence results and prove that the solution of the inhomogeneous eigenvalue problem converges to that of the corresponding Dirichlet problem. Since the limiting solution is bounded, "large eigenfunctions" are purely nonlocal objects.

These results are presented in Section 8. One main ingredient for the strong convergence is the compactness of the sequence of Green's operator on bounded L^2 functions, in the spirit of the Riesz–Fréchet–Kolmogorov theorem.

2.9. Notations and organization. For convenience, we list some notations used throughout the paper.

- $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$ is the complement of Ω .
- $K_1 \Subset K_0$ means that the closure of K_1 is contained in the interior of K_0 .
- s is half the order of \mathcal{L} , and ranges over $(0, 1)$ for RFL and over $(\frac{1}{2}, 1)$ for SFL.
- γ is the parameter for the boundary behavior; $\gamma = s$ for RFL and $\gamma = 1$ for SFL.
- $\bar{\lambda}$ is the eigenvalue just above λ , defined in (2.12).
- (λ_j, φ_j) are the eigenpairs, with $\lambda_j \in \Sigma$ repeated according to multiplicity.
- $d_\Sigma(\lambda)$ is the distance of λ to the spectrum, defined in (2.13).
- E_i is the eigenspace associated to λ_i .
- E is the span of all eigenfunctions with eigenvalue at most $\bar{\lambda}$.
- E^\perp is the orthogonal complement of E in $L^2(\Omega)$.
- $\langle u, \varphi \rangle = \int_\Omega u \varphi dx$ is the $L^2(\Omega)$ inner product.

- $\mathcal{G}_0(x, y)$ is the Green's function for \mathcal{L} .
- $\mathbb{G}_0(u)(x) = \int_{\Omega} \mathcal{G}_0(x, y)u(y) dy$ is the Green's operator for \mathcal{L} .
- $\mathbb{G}_{\lambda}(u)$ is the solution operator for $\mathcal{L} - \lambda$.
- $a \wedge b$ is the minimum of the real numbers a and b .
- $a_+ = \max\{a, 0\}$ is the positive part of a .
- $f \asymp g$ means $C^{-1}f \leq g \leq Cf$, for positive functions f and g .
- $\eta_{\varepsilon} = \varepsilon^{-n}\eta(\varepsilon^{-1}\cdot)$ is the standard mollifier, where $\eta \in C_c^{\infty}(B_1)$ is an L^1 -normalized bump function.
- Constants depending only on n, s, Ω, γ are considered universal and denoted generically by C . Any additional dependence is indicated.

The paper is organized as follows. We derive regularity estimates for the Green's function in Section 3, and apply them to establish a linear theory for L^1 -solutions in Section 4. We show the equivalence of various notions of solution, and obtain local boundedness of L^1 -solutions. Then we derive the L^1 -linear theory in the L^2 -projected space in Section 5. In Section 6 we prove a maximum principle for L^1 -solutions. All these are applied in Section 7 to prove the target theorem, classifying blow-up phenomena of the nonlocal eigenvalue problem. We recover the classical eigenvalue problem as $s \nearrow 1$ in Section 8. In Appendix A, we discuss the natural Hilbert spaces and some compactness properties. We give an explicit expression for the weighted trace for the RFL in Appendix B and collect elementary embedding results into Morrey spaces in Appendix C.

3. ESTIMATES FOR $\lambda = 0$

3.1. Regularisation between weighted L^p spaces.

3.1.1. *Regularisation between L^1 weight spaces.* The first item to notice is that, as shown in [4], we have that

Proposition 3.1. *The Green's operator \mathbb{G}_0 for \mathcal{L} is continuous from*

$$\mathbb{G}_0 : L^1(\Omega, \mathbb{G}_0(\delta^{\alpha})) \longrightarrow L^1(\Omega, \delta^{\alpha}).$$

for $\alpha > -1 - \gamma$, with its operator norm bounded by 1.

Proof. If $u = \mathbb{G}_0(f)$, then one can take $\psi = \text{sign}(u)\delta^{\alpha}$ as a test function to see

$$\int_{\Omega} |u|\delta^{\alpha} = \int_{\Omega} f\mathbb{G}_0(\text{sign}(u)\delta^{\alpha}) \leq \int_{\Omega} |f|\mathbb{G}_0(\delta^{\alpha}). \quad \square$$

Remark 3.1. For $\alpha = \gamma$ we have that

$$\mathbb{G}_0 : L^1(\Omega, \delta^{\gamma}) \rightarrow L^1(\Omega, \delta^{\gamma}).$$

Since, for $\alpha > -1 - \gamma$, $\mathbb{G}_0(\delta^{\alpha}) \geq c_{\alpha}\delta^{\gamma}$, we have that $L^1(\Omega, \delta^{\alpha}) \hookrightarrow L^1(\Omega, \delta^{\gamma})$. Hence, solutions u is always in $L^1(\Omega, \delta^{\gamma})$.

Remark 3.2. Since, by (2.10) we reduce the power of the weight needed, we always improve the integrability. In particular, if $\gamma < 2s$ then

$$\mathbb{G}_0 : L^1(\Omega, \delta^{\gamma}) \longrightarrow L^1(\Omega)$$

and, if $\gamma > 2s$ then

$$\begin{aligned}\mathbb{G}_0 : L^1(\Omega, \delta^{2s}) &\longrightarrow L^1(\Omega), \\ \mathbb{G}_0 : L^1(\Omega, \delta^\beta) &\longrightarrow L^1(\Omega, \delta^{\beta-2s}), & \text{for any } \beta \in (\gamma, 2s), \\ \mathbb{G}_0 : L^1(\Omega, \delta^\gamma) &\longrightarrow L^1(\Omega, \delta^\alpha), & \text{for any } \alpha > \gamma - 2s.\end{aligned}$$

Again, $\gamma < 2s$, all admissible data give L^1 solutions.

3.1.2. Regularisation from L^p to L^q and Marcinkiewicz spaces. We start with a definition

Definition 3.1. Let u be a measurable function in Ω . Let $p \in (1, +\infty)$ and let p' be its conjugate exponent. We define the Marcinkiewicz norm $M^p(\Omega)$ as

$$\|u\|_{M^p(\Omega)} = \sup_{\substack{K \subset \Omega \\ \text{measurable}}} \frac{\int_K |u(x)| dx}{|K|^{\frac{1}{p'}}}.$$

We define the Marcinkiewicz space $M^p(\Omega)$ as the space of functions with bounded $M^p(\Omega)$ norm.

Notice that the extension by zero produces an isometric embedding $M^p(\Omega) \hookrightarrow M^p(\mathbb{R}^n)$. The embedding $M^p(\Omega) \hookrightarrow L^q(\Omega)$ for $q < p$ is well known (see, e.g. [7, Lemma A.2]).

The aim of this subsection is to prove

Proposition 3.2. *The Green's operator \mathbb{G}_0 for \mathcal{L} is continuous from*

$$(3.1) \quad \mathbb{G}_0 : L^1(\Omega) \longrightarrow M^{\frac{n}{n-2s}}(\Omega).$$

In particular $\mathbb{G}_0 : L^1(\Omega) \longrightarrow L^p(\Omega)$ and

$$(3.2) \quad \mathbb{G}_0 : L^{p_0}(\Omega) \longrightarrow L^{p_1}(\Omega) \text{ for } p_0 \in (1, \frac{n}{2s}) \text{ and } \frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}.$$

Remark 3.3. Notice that $p_1 = p_0 \frac{n}{n-2sp_0}$. As $p_0 \rightarrow \frac{n}{2s}$ then $p_1 \rightarrow +\infty$. Hence, for every $q \geq 1$, there exists k finite such that $\mathbb{G}_0^k : L^1(\Omega) \rightarrow L^q(\Omega)$.

In [28], the authors give a direct computation of the second result. We will give better results using stronger machinery. For $p = 1$, we will give a different proof applying of the convolution estimates to Marcinkiewicz spaces given in [7, Appendix]. For $p > 1$ we apply Hardy–Littlewood–Sobolev estimates for Riesz potentials.

Proposition 3.3 (Chapter V, §1, Theorem 1 in [46]). *Let $\alpha \in (0, n)$ and*

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\alpha} dy.$$

Then, for $p_0 \in (1, \frac{n}{2s})$,

$$\|I_\alpha(f)\|_{M^{\frac{n}{n-2s}}(\mathbb{R}^n)} \leq C \|f\|_{L^1(\mathbb{R}^n)}, \quad \|I_\alpha(f)\|_{L^{p_1}} \leq C(p_0) \|f\|_{L^{p_0}(\mathbb{R}^n)}$$

where $\frac{1}{p_1} = \frac{1}{p_0} + 1 - \frac{\alpha}{n}$.

Proof of Proposition 3.2. We have that

$$|u(x)| \leq C \int_{\Omega} \frac{|f(y)|}{|x-y|^{n-2s}} dy = C \int_{\mathbb{R}^n} \frac{|f(y)|\chi_{\Omega}}{|x-y|^{n-2s}} dy = CI_{n-2s}(|f|\chi_{\Omega}).$$

Therefore

$$\|u\|_{M^{\frac{n}{n-2s}}(\Omega)} \leq C\|f\chi_{\Omega}\|_{L^1(\mathbb{R}^n)} = C\|f\|_{L^1(\mathbb{R}^n)}, \quad \|u\|_{L^{p_1}(\Omega)} \leq C(p_0)\|f\|_{L^{p_0}(\Omega)}.$$

This completes the proof. \square

3.1.3. *Regularisation from L^p to L^∞ .* By duality from Proposition 3.2, we have that

Proposition 3.4. *The Green's operator \mathbb{G}_0 for \mathcal{L} is continuous from*

$$(3.3) \quad \mathbb{G}_0 : L^q(\Omega) \longrightarrow L^\infty(\Omega), \quad q \in (\frac{n}{2s}, +\infty).$$

In particular

$$(3.4) \quad \|\mathbb{G}_0(f)\|_{L^\infty(\Omega)} \leq C(q_0) \|f\|_{L^{q_0}(\Omega)}.$$

3.1.4. *Improvement of weighted integrability.* It is immediate to prove that

Proposition 3.5. *The Green's operator \mathbb{G}_0 for \mathcal{L} is continuous from*

$$\mathbb{G}_0 : \delta^\alpha L^\infty(\Omega) \longrightarrow \mathbb{G}_0(\delta^\alpha) L^\infty(\Omega),$$

with operator norm bounded by 1.

We recall that $\mathbb{G}_0(\delta^\alpha) \asymp \delta^{(\alpha+2s)\wedge\gamma}$ if $\alpha + 2s \neq \gamma$.

Proof. Assume that $|f| \leq C\delta^\alpha$. Then

$$\left| \frac{\mathbb{G}_0(f)}{\mathbb{G}_0(\delta^\alpha)} \right| \leq \frac{\mathbb{G}_0(|f|)}{\mathbb{G}_0(\delta^\alpha)} \leq \left\| \frac{f}{\delta^\alpha} \right\|_{L^\infty(\Omega)} \frac{\mathbb{G}_0(\delta^\alpha)}{\mathbb{G}_0(\delta^\alpha)} \leq \left\| \frac{f}{\delta^\alpha} \right\|_{L^\infty(\Omega)}. \quad \square$$

In particular, for $\gamma < 2s$ we have that

$$\mathbb{G}_0 : L^\infty(\Omega) \longrightarrow \delta^\gamma L^\infty(\Omega)$$

and for $\gamma > 2s$ we have that

$$\begin{aligned} \mathbb{G}_0 : L^\infty(\Omega) &\longrightarrow \delta^{2s} L^\infty(\Omega), \\ \mathbb{G}_0 : \delta^\alpha L^\infty(\Omega) &\longrightarrow \delta^{\alpha+2s} L^\infty(\Omega), & \text{for any } \alpha \in (0, \gamma - 2s), \\ \mathbb{G}_0 : \delta^\alpha L^\infty(\Omega) &\longrightarrow \delta^\gamma L^\infty(\Omega), & \text{for any } \alpha > \gamma - 2s. \end{aligned}$$

3.1.5. *Local integrability.*

Proposition 3.6. *The operator \mathbb{G}_0 is continuous from*

$$\begin{aligned} L^1(\Omega; \delta^\gamma) &\longrightarrow L_{\text{loc}}^p(\Omega), & \text{for } p \in (1, \frac{n}{n-2s}), \\ L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^{p_0}(\Omega) &\longrightarrow L_{\text{loc}}^{p_1}(\Omega), & \text{for } p_0 \in (1, \frac{n}{2s}), \frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}, \\ L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^{q_0}(\Omega) &\longrightarrow L_{\text{loc}}^\infty(\Omega), & \text{for } q_0 \in (\frac{n}{2s}, +\infty), \end{aligned}$$

For $K_1 \Subset K_0 \Subset \Omega$ we have estimates

$$(3.5) \quad \|\mathbb{G}_0(f)\|_{L^p(K_1)} \leq C(p, K_1) \|f\delta^\gamma\|_{L^1(\Omega)},$$

$$(3.6) \quad \|\mathbb{G}_0(f)\|_{L^{p_1}(K_1)} \leq C(p_0, K_0, K_1) \left(\|f\|_{L^{p_0}(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right),$$

$$(3.7) \quad \|\mathbb{G}_0(f)\|_{L^\infty(K_1)} \leq C(q_0, K_0, K_1) \left(\|f\|_{L^{q_0}(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right).$$

Proof. We write

$$f = f\chi_{K_0} + f\chi_{\Omega \setminus K_0}.$$

for $x \in K_1$ we have that

$$\begin{aligned} |\mathbb{G}_0(f\chi_{\Omega \setminus K_0})(x)| &\leq C \int_{\Omega \setminus K_0} \frac{|f(y)|\delta(y)^\gamma}{|x-y|^{n-2s+\gamma}} dy \\ &\leq C \text{dist}(K_1, \Omega \setminus K_0)^{2s-n-\gamma} \|f\delta^\gamma\|_{L^1}. \end{aligned}$$

where C depends only on the kernel estimate. Therefore

$$|u(x)| \leq |\mathbb{G}_0(f\chi_{K_0})| + C \text{dist}(K_1, \Omega \setminus K_0)^{2s-n-\gamma} \|f\delta^\gamma\|_{L^1(\Omega)}.$$

(1) If $f \in L^1(\Omega, \delta^\gamma)$ then

$$\begin{aligned} \|f\chi_{K_0}\|_{L^1(\Omega)} &= \|f\|_{L^1(K_0)} \leq \|\delta^{-\gamma}\|_{L^\infty(K_0)} \|f\delta^\gamma\|_{L^1(\Omega)} \\ &\leq \text{dist}(K_0, \partial\Omega)^{-\gamma} \|f\delta^\gamma\|_{L^1(\Omega)}. \end{aligned}$$

Hence

$$\|\mathbb{G}_0(f\chi_{K_0})\|_{L^p(\Omega)} \leq C(p) \|f\chi_{K_0}\|_{L^1(\Omega)} \leq C(p) \text{dist}(K_0, \partial\Omega)^{-\gamma} \|f\delta^\gamma\|_{L^1(\Omega)}.$$

Note that in (3.5) the constant can be chosen to depend only on K_1 , for example by fixing $K_0 = \{x \in \Omega : \delta(x) > \text{dist}(K_1, \Omega)/2\}$.

(2) If $f\chi_{K_0} \in L^{p_0}(K_0)$ with $p_0 \in (1, \frac{n}{2s})$ we have, analogously

$$\|\mathbb{G}_0(f\chi_{K_0})\|_{L^{p_1}(\Omega)} \leq C(p_0) \|f\chi_{K_0}\|_{L^{p_0}(\Omega)} = C(p_0) \|f\|_{L^{p_0}(K_0)}.$$

(3) If $f\chi_{K_0} \in L^{q_0}(K_0)$ with $q_0 > \frac{n}{2s}$ we have

$$\|\mathbb{G}_0(f\chi_{K_0})\|_{L^\infty(\Omega)} \leq C(q_0) \|f\|_{L^{q_0}(K_0)}. \quad \square$$

3.2. Eigenfunction estimates. In [9, Proposition 5.3 and Proposition 5.4] the authors prove that the eigenfunctions satisfy $|\varphi_j| \leq \kappa_j \delta^\gamma$ and that $\varphi_1 \asymp \delta^\gamma$. For the sake of completeness, we give a short proof of this fact.

Proposition 3.7. *Let $\varphi \in L^1(\Omega, \delta^\gamma)$ be an eigenfunction of \mathbb{G}_0 , i.e. $\varphi = \lambda \mathbb{G}_0(\varphi)$ for some λ . Then $\varphi \in \delta^\gamma L^\infty(\Omega)$.*

Proof. We apply a bootstrap argument. Clearly,

$$\varphi = \lambda \mathbb{G}_0(\varphi) = \lambda \mathbb{G}_0(\lambda \mathbb{G}_0(\varphi)) = \lambda^2 \mathbb{G}_0^2(\varphi) = \dots = \lambda^k \mathbb{G}_0^k(\varphi).$$

Since $\varphi \in L^1(\Omega, \delta^\gamma)$, by Proposition 3.1 for k large enough $\mathbb{G}_0^k(\varphi) \in L^1(\Omega)$. By Remark 3.3 we have that, for k large enough $\mathbb{G}_0^k(\varphi) \in L^p(\Omega)$ for all $p > \frac{n}{2s}$. Therefore, by Proposition 3.4, $\mathbb{G}_0^{k+1}(\varphi) \in L^\infty(\Omega)$. Finally, by Proposition 3.5, for k large enough $\varphi = \lambda^k \mathbb{G}_0^k(\varphi) \in \delta^\gamma L^\infty(\Omega)$. \square

3.3. Martin kernel. We recall the following result

Theorem ([4, Corollary 4.3, Theorem 4.6, Theorem 4.13]). *Let $b = 1 - 2s + \gamma$. For any $h \in C(\partial\Omega)$, there exists a unique $\mathbb{G}_0(L_c^\infty(\Omega))$ -weak solution $v_h \in \delta^{-b}L^\infty(\Omega)$ given by*

$$(3.8) \quad v_h(x) := M(h)(x) = \int_{\partial\Omega} D_\gamma \mathcal{G}_0(z, x) h(z) d\mathcal{H}^{n-1}(z), \quad \text{for } x \in \Omega,$$

which satisfies

$$(3.9) \quad \lim_{\Omega \ni x \rightarrow z} \frac{v_h(x)}{v_1(x)} = h(z), \quad \text{for } z \in \partial\Omega,$$

uniformly in $z \in \partial\Omega$, where $v_1 := M(1) \asymp \delta^{-b}$. In particular,

$$\|v_h \delta^b\|_{L^\infty(\Omega)} \leq C \|h\|_{L^\infty(\partial\Omega)},$$

and so for any $K \Subset \Omega$,

$$(3.10) \quad \|v_h\|_{L^\infty(K)} \leq C(K) \|h\|_{L^\infty(\partial\Omega)}.$$

Furthermore, we can show the following

Proposition 3.8 (Martin's operator). *The Martin's operator*

$$\begin{aligned} \mathbb{M} : L^\infty(\partial\Omega) &\longrightarrow \delta^{-b}L^\infty(\Omega) \\ h &\longmapsto \mathbb{M}(h) \end{aligned}$$

defined by

$$\mathbb{M}(h)(x) = \int_{\partial\Omega} D_\gamma \mathcal{G}_0(z, x) h(z) d\mathcal{H}^{n-1}(z)$$

is a continuous operator.

Proof. Let $h \in L^\infty(\partial\Omega)$. Using the behavior of $D_\gamma \mathcal{G}_0$ stated in (K2), write

$$\begin{aligned} \delta^b(x) \mathbb{M}(h)(x) &= \int_{\partial\Omega} \delta^{1-2s+\gamma}(x) D_\gamma \mathcal{G}_0(z, x) h(z) d\mathcal{H}^{n-1}z \\ &\leq C \|h\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \frac{\delta^{1-2s+2\gamma}(x)}{|z-x|^{n-2s+2\gamma}} d\mathcal{H}^{n-1}(z) \end{aligned}$$

Using a system of coordinates centered at $z_x = \arg \min \{|z-x| : z \in \partial\Omega\}$ and aligning e_n with the inward normal at z_x , the last integral converges as $\delta(x) \searrow 0$ to

$$\int_{\mathbb{R}^{n-1}} \frac{1}{|z-e_n|^{n-2s+2\gamma}} dz$$

after considering a dilation of factor $1/\delta(x)$, and hence is bounded independently of $\delta(x)$. \square

3.4. The range $2s > \gamma$. For $\lambda = 0$, we prove regularity estimates in various Lebesgue spaces. With the assumption $\gamma \in (0, 2s)$, $|\cdot|^{-(n-2s+\gamma)}$ is integrable, so that either factor $\delta^\gamma(x)$ or $\delta^\gamma(y)$ becomes available. The results on the whole domain is classical, and we include the proof for the reader's convenience. We are particularly interested in the local regularity for $L^1(\Omega; \delta^\gamma)$ data, where we use a localized version of the classical Hardy–Littlewood–Sobolev inequality. Compare with [4, Theorem 2.1, Theorem 2.10, Theorem 2.11], where the full range $\gamma \in (0, 1]$ is covered.

Theorem 3.9 (Local mapping properties of \mathbb{G}_0). *The Green's operator \mathbb{G}_0 for \mathcal{L} is continuous from*

$$\begin{aligned} L^1(\Omega; \delta^\gamma) &\longrightarrow L^p(\Omega), & \text{for } p \in [1, \frac{n}{n-2s+\gamma}), \\ L^q(\Omega) &\longrightarrow \delta^\gamma L^\infty(\Omega) & \text{for } q \in (\frac{n}{2s-\gamma}, +\infty). \end{aligned}$$

Moreover,

$$(3.11) \quad \|\mathbb{G}_0(f)\|_{L^p(\Omega)} \leq C(p) \|f\delta^\gamma\|_{L^1(\Omega)},$$

$$(3.12) \quad \|\mathbb{G}_0(f)/\delta^\gamma\|_{L^\infty(\Omega)} \leq C(q) \|f\|_{L^q(\Omega)}.$$

Remark 3.4. Under additional assumptions on the Green's function, high integrability data lead to Hölder continuous solutions. However, we will not need this for the rest of the paper.

Proof. (1) For any $f \in L^1(\Omega; \delta^\gamma)$, and $x \in \Omega$, we have

$$|\mathbb{G}_0(f)(x)| \leq C \int_{\Omega} \frac{1}{|x-y|^{n-2s}} \frac{\delta^\gamma(y)}{|x-y|^\gamma} |f(y)| dy.$$

Since $|\cdot - y|^{-(n-2s+\gamma)}$ is uniformly integrable in $y \in \Omega$, we have

$$\|\mathbb{G}_0(f)\|_{L^1(\Omega)} \leq C \|f\delta^\gamma\|_{L^1(\Omega)}.$$

This proves the case $p = 1$. When $p \in (1, \frac{n}{n-2s+\gamma})$, we apply Jensen's inequality with the probability measure

$$d\mu(y) = \frac{|f(y)|\delta^\gamma(y) dy}{\|f\delta^\gamma\|_{L^1(\Omega)}},$$

to obtain, from

$$|\mathbb{G}_0(f)(x)| \leq C \|f\delta^\gamma\|_{L^1(\Omega)} \int_{\Omega} \frac{1}{|x-y|^{n-2s+\gamma}} d\mu(y),$$

that

$$\begin{aligned} \|\mathbb{G}_0(f)\|_{L^p(\Omega)}^p &\leq C \|f\delta^\gamma\|_{L^1(\Omega)}^p \int_{\Omega} \left(\int_{\Omega} \frac{1}{|x-y|^{n-2s+\gamma}} d\mu(y) \right)^p dx \\ &\leq C \|f\delta^\gamma\|_{L^1(\Omega)}^p \int_{\Omega} \int_{\Omega} \frac{1}{|x-y|^{(n-2s+\gamma)p}} dx d\mu(y) \\ &\leq C(p) \|f\delta^\gamma\|_{L^1(\Omega)}^p. \end{aligned}$$

(2) For any $f \in L^q(\Omega)$ with $q > \frac{n}{2s-\gamma}$, by Hölder's inequality, we have for any $x \in \Omega$,

$$\begin{aligned} \frac{|\mathbb{G}_0(f)(x)|}{\delta^\gamma(x)} &\leq \int_{\Omega} \frac{|f(y)|}{|x-y|^{n-2s+\gamma}} dy \\ &\leq \|f\|_{L^q(\Omega)} \left(\int_{\Omega} \frac{1}{|x-y|^{(n-2s+\gamma)q'}} dy \right)^{\frac{1}{q'}} \\ &\leq C(q) \|f\|_{L^q(\Omega)}, \end{aligned}$$

since the conjugate exponent $q' = \frac{q}{q-1}$ satisfies $(n-2s+\gamma)q' < n$. \square

Then we prove that for bounded data the \mathbb{G}_0 -Green solution has well-defined γ -normal derivative. Here we use again the fact that $\gamma < 2s$. Compare with [4, Theorem 3.15].

Proposition 3.10 (γ -normal derivative of Green's operator).

$$D_\gamma \mathbb{G}_0 : L^\infty(\Omega) \longrightarrow L^\infty(\partial\Omega)$$

is a continuous operator. More precisely, For any $f \in L^\infty(\Omega)$, $D_\gamma \mathbb{G}_0(f) : \partial\Omega \rightarrow \mathbb{R}$ is well-defined and equal to

$$D_\gamma \mathbb{G}_0(f)(z) = \int_{\Omega} D_\gamma \mathcal{G}_0(z, y) f(y) dy, \quad \forall z \in \partial\Omega.$$

Proof. For any $x \in \Omega$,

$$\frac{\mathbb{G}_0(f)(x)}{\delta^\gamma(x)} = \int_{\Omega} \frac{\mathcal{G}_0(x, y)}{\delta^\gamma(x)} f(y) dy,$$

and the right hand side is uniformly bounded as $x \rightarrow z \in \partial\Omega$ because

$$\begin{aligned} \left| \int_{\Omega} \frac{\mathcal{G}_0(x, y)}{\delta^\gamma(x)} f(y) dy \right| &\leq \int_{\Omega} \frac{1}{|x-y|^{n-2s+\gamma}} \|f\|_{L^\infty(\Omega)} dy \\ &\leq C \|f\|_{L^\infty(\Omega)}. \end{aligned}$$

By Dominated Convergence Theorem, one may pass to the limit $x \rightarrow z$. \square

4. GENERAL LINEAR THEORY

4.1. Global estimates. Let $\lambda \in \mathbb{R} \setminus \Sigma$, and $\bar{\lambda}$ and $d_\Sigma(\lambda)$ be as in (2.12) and (2.13). Let $b = 1 - 2s + \gamma$. Consider the equation

$$(4.1) \quad \begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

The definition of \mathbb{G}_λ relies on the well-posedness of (4.1), which we will show on $L^2(\Omega)$, $L^q(\Omega)$ for large q , and finally $L^1(\Omega; \delta^\gamma)$.

We start with an L^2 -theory.

Lemma 4.1 (Solvability in L^2). *If $f \in L^2(\Omega)$, then there exists a unique spectral solution $u \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$ of (4.1), given by*

$$u(x) = \sum_{j \geq 1} \frac{\langle f, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j(x).$$

As a consequence,

$$\|u\|_{L^2(\Omega)} \leq d_{\Sigma}(\lambda)^{-1} \|f\|_{L^2(\Omega)}.$$

Proof. The result follows directly from eigenfunction decomposition. \square

Hence for any $\lambda \in \mathbb{R} \setminus \Sigma$, one may define $\mathbb{G}_{\lambda} : L^2(\Omega) \rightarrow \mathbb{H}_{\mathcal{L}}^2(\Omega)$ by

$$(4.2) \quad \mathbb{G}_{\lambda}(f) = \sum_{j \geq 1} \frac{\langle f, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j, \quad \forall f \in L^2(\Omega).$$

This includes the particular case $\lambda = 0$, where \mathbb{G}_0 is alternatively given by the integral formula (2.6).

Lemma 4.2 (Spectral solutions are Green solutions). *Suppose $f \in L^2(\Omega)$ and $u \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$. Let \mathbb{G}_{λ} and \mathbb{G}_0 be as defined in (4.2). Then the following are equivalent.*

- (1) $u = \mathbb{G}_{\lambda}(f)$ a.e.
- (2) $(\lambda_j - \lambda) \langle u, \varphi_j \rangle = \langle f, \varphi_j \rangle$ for all $j \geq 1$.
- (3) $u - \lambda \mathbb{G}_0(u) = \mathbb{G}_0(f)$ a.e.

Proof. (1) \Leftrightarrow (2) is by projection. To see (2) \Leftrightarrow (3), one may rearrange

$$(\lambda_j - \lambda) \langle u, \varphi_j \rangle = \langle f, \varphi_j \rangle$$

to

$$\langle u, \varphi_j \rangle - \lambda \frac{\langle u, \varphi_j \rangle}{\lambda_j} = \frac{\langle f, \varphi_j \rangle}{\lambda_j}.$$

\square

Remark 4.1. Notice that

$$\begin{aligned} u &= \lambda \mathbb{G}_0(\lambda \mathbb{G}_0(u) + \mathbb{G}_0(f)) + \mathbb{G}_0(f) = \lambda^2 \mathbb{G}_0^2(u) + \lambda \mathbb{G}_0^2(f) + \mathbb{G}_0(f) = \cdots = \\ &= \lambda^k \mathbb{G}_0^k(u) + \sum_{m=1}^k \lambda^{m-1} \mathbb{G}_0^m(f). \end{aligned}$$

Since the regularity of the second term of the sum is always that of $\mathbb{G}_0(f)$, we can iterate the bootstrap in k up to its regularity as in the proof of Proposition 3.7. This shows us that the regularisation properties of \mathbb{G}_{λ} are precisely those of \mathbb{G}_0 .

Proposition 4.3 (Solvability for high integrability data). *Let $q \in (\frac{n}{2s}, +\infty]$. For any $f \in L^q(\Omega)$ there is a unique spectral solution $u \in L^{\infty}(\Omega)$ of (4.1), with*

$$\|u\|_{L^{\infty}(\Omega)} \leq C(q, \bar{\lambda}) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} \right),$$

and

$$\|u\|_{L^{\infty}(\Omega)} \leq C(q, \bar{\lambda}, d_{\Sigma}(\lambda)) \|f\|_{L^q(\Omega)}.$$

If $f \in \delta^\alpha L^\infty(\Omega)$ then $u \in \mathbb{G}_0(\delta^\alpha) L^\infty(\Omega)$ we have that

$$\left\| \frac{u}{\mathbb{G}_0(\delta^\alpha)} \right\|_{L^\infty(\Omega)} \leq C(\alpha, \bar{\lambda}) \left(\|u\|_{L^2(\Omega)} + \left\| \frac{f}{\delta^\alpha} \right\|_{L^\infty(\Omega)} \right).$$

Furthermore, if $\gamma < 2s$, then for $q \in (\frac{n}{2s-\gamma}, +\infty]$,

$$\left\| \frac{u}{\delta^\gamma} \right\|_{L^\infty(\Omega)} \leq C(q, \bar{\lambda}) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} \right).$$

Proof. By Lemma 4.1 and Lemma 4.2, (4.1) has a unique spectral solution which satisfies

$$u = \lambda \mathbb{G}_0(u) + \mathbb{G}_0(f).$$

We can now bootstrap the regularity using the results from Section 3. We proceed as follows:

- (1) Let $p_0 = 2$ and define p_k by

$$\frac{1}{p_k} = \frac{1}{p_0} - k \frac{2s}{n}$$

Let k_0 be the smallest integer that $p_{k_0} > \frac{n}{2s}$. (We can avoid $p_{k_0} = \frac{n}{2s}$ by decreasing p_0 .)

- (2) For $k = 0, \dots, k_0 - 1$, since $\lambda u + f \in L^{p_k}(\Omega)$, we have

$$\begin{aligned} \|u\|_{L^{p_{k+1}}(\Omega)} &\leq C(\bar{\lambda}) \left(\|u\|_{L^{p_0}(\Omega)} + \|f\|_{L^{p_0}(\Omega)} \right) \\ &\leq C(\bar{\lambda}) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} \right). \end{aligned}$$

- (3) Since $\lambda u + f \in L^{p_{k_0}}(\Omega)$ for $p_{k_0} > \frac{n}{2s}$, by (3.4) we have

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq C(\bar{\lambda}) \left(\|u\|_{L^{p_{k_0}}(\Omega)} + \|f\|_{L^{p_{k_0}}(\Omega)} \right) \\ &\leq C(\bar{\lambda}) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} \right). \end{aligned}$$

- (4) If $f \in \delta^\alpha L^\infty(\Omega)$ then, applying the previous steps $u \in L^\infty(\Omega)$. Applying Proposition 3.5 a finite number of times, the result follows.

- (5) If $\gamma < 2s$, since $\lambda u + f \in L^q(\Omega)$ for $q > \frac{n}{2s-\gamma}$, by (3.12) we have

$$\begin{aligned} \|u/\delta^\gamma\|_{L^\infty(\Omega)} &\leq C(q, \bar{\lambda}) \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)} \right) \\ &\leq C(q, \bar{\lambda}) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} \right). \end{aligned}$$

The last estimate without $\|u\|_{L^2(\Omega)}$ follows from (4.1). \square

Proposition 4.4 (Solvability for low integrability data). *Let $f \in L^1(\Omega, \mathbb{G}_0(\delta^\alpha))$ for $\alpha \geq 0$. Then, there exists a unique $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -solution $u \in L^1(\Omega, \delta^\alpha)$ of (4.1).*

$$(4.3) \quad \|u\delta^\alpha\|_{L^1(\Omega)} \leq C \|f\mathbb{G}_0(\delta^\alpha)\|_{L^1(\Omega)},$$

$C = C(\alpha, \bar{\lambda}, d_\Sigma(\lambda))$. Furthermore, if $f \in L^1(\Omega)$, then $u \in L^p(\Omega)$ for any $p \in [1, \frac{n}{n-2s})$ and

$$(4.4) \quad \|u\|_{L^p(\Omega)} \leq C \|f\|_{L^1(\Omega)},$$

for $C = C(p, \bar{\lambda}, d_\Sigma(\lambda))$.

Proof. Consider the cut-off data $f_k = (f \wedge k) \vee (-k) \in L^\infty(\Omega)$, which tends to f in $L^1(\Omega; \mathbb{G}_0(\delta^\alpha))$ by the Dominated Convergence Theorem, and hence in $L^1(\Omega, \delta^\gamma)$ by (2.11). By Proposition 4.3, there is a unique spectral solution $u_k \in L^\infty(\Omega)$, satisfying

$$(4.5) \quad \int_{\Omega} u_k \psi dx = \int_{\Omega} f_k \mathbb{G}_\lambda(\psi) dx, \quad \forall \psi \in \delta^\gamma L^\infty(\Omega)$$

If $\psi \in L^\infty(\Omega)$, we can approximate it in $\delta^\gamma L^\infty(\Omega)$ by

$$\psi_m = \psi (\delta^{-\gamma} \wedge m) \delta^\gamma \in \delta^\gamma L^\infty(\Omega).$$

Therefore $\mathbb{G}_0(\psi_m) \rightarrow \mathbb{G}_0(\psi)$ in $L^\infty(\Omega)$. Since $u_k, f_k \in L^\infty(\Omega)$, passing to the limit under the integral

$$(4.6) \quad \int_{\Omega} u_k \psi dx = \int_{\Omega} f_k \mathbb{G}_\lambda(\psi) dx, \quad \forall \psi \in L^\infty(\Omega).$$

For any $k, \ell \geq 0$,

$$\int_{\Omega} (u_k - u_\ell) \psi dx = \int_{\Omega} (f_k - f_\ell) \mathbb{G}_\lambda(\psi) dx, \quad \forall \psi \in L^\infty(\Omega).$$

Taking $\psi = \text{sign}(u_k - u_\ell) \delta^\gamma \in \delta^\gamma L^\infty(\Omega)$, and then using the fact that $|\mathbb{G}_\lambda(\delta^\gamma)| \leq C(\bar{\lambda}, d_\Sigma(\lambda)) \delta^\gamma$, we have that

$$\int_{\Omega} |u_k - u_\ell| \delta^\gamma dx \leq C(\bar{\lambda}, d_\Sigma(\lambda)) \int_{\Omega} |f_k - f_\ell| \delta^\gamma dx.$$

Therefore, u is a Cauchy sequence in $L^1(\Omega, \delta^\gamma)$. Passing to the limit under the integral sign, we conclude that u is the $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (4.1).

Taking $\psi = \text{sign}(u_k - u_\ell) \delta^\alpha \in \delta^\alpha L^\infty(\Omega)$, and using that $|\mathbb{G}_\lambda(\delta^\alpha)| \leq C(\alpha, \bar{\lambda}, d_\Sigma(\lambda)) \delta^\alpha$ we recover analogously that u_k is a Cauchy sequence in $L^1(\Omega, \delta^\alpha)$ and has a limit u . To see (4.3), we put $\psi = \text{sign}(u_k) \delta^\alpha \in \delta^\alpha L^\infty(\Omega)$ in (4.6) and use Proposition 4.3 to obtain

$$\int_{\Omega} |u_k| \delta^\alpha dx \leq C(\alpha, \bar{\lambda}, d_\Sigma(\lambda)) \int_{\Omega} |f_k| \mathbb{G}_0(\delta^\alpha) dx.$$

Passing to the limit we recover (4.3).

If \tilde{u} is any other solution, then

$$\int_{\Omega} (u - \tilde{u}) \psi dx = 0, \quad \forall \psi \in \delta^\gamma L^\infty(\Omega),$$

and by taking $\psi = \text{sign}(u - \tilde{u}) \delta^\gamma$ we must have $u = \tilde{u}$ a.e. in Ω .

For $p \in (1, \frac{n}{n-2s})$, let $k = 1, 2, \dots$, and take instead $\psi = (|u| \wedge k)^{p-1} \text{sign}(u) \in L^\infty(\Omega)$ which is moreover uniformly bounded in $L^{\frac{p}{p-1}}(\Omega)$. Since $p < \frac{n}{n-2s}$, we have $p' = \frac{p}{p-1} > \frac{n}{2s}$

and so Proposition 4.3 again applies to yield

$$\begin{aligned}
\int_{\Omega} (|u| \wedge k)^p dx &\leq \int_{\Omega} |u| (|u| \wedge k)^{p-1} dx = \|f\|_{L^1} \left\| \mathbb{G}_{\lambda}(|u| \wedge k)^{p-1} \text{sign}(u) \right\|_{L^\infty} \\
&\leq C(p, \bar{\lambda}, d_{\Sigma}(\lambda)) \|f\|_{L^1} \left\| (|u| \wedge k)^{p-1} \text{sign}(u) \right\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\leq C(p, \bar{\lambda}, d_{\Sigma}(\lambda)) \left(\int_{\Omega} (|u| \wedge k)^p dx \right)^{\frac{p-1}{p}} \|f\|_{L^1(\Omega)}.
\end{aligned}$$

Hence

$$\| |u| \wedge k \|_{L^p(\Omega)} \leq C \|f\|_{L^1(\Omega)},$$

and the (4.3) follows by taking $k \rightarrow +\infty$. \square

Proof of Theorem 2.2 (1). It follows from Lemma 4.1, Proposition 4.3 and Proposition 4.4. \square

4.2. Integration-by-parts formulae. We prove an integration by parts formula for \mathbb{G}_0 and then for \mathbb{G}_{λ} . This confirms the duality seen in their mapping properties.

Lemma 4.5 (Integration by parts with \mathbb{G}_0). *Suppose $f \in L^1(\Omega; \delta^\gamma)$ and $g \in \delta^\gamma L^\infty(\Omega)$. Then we have*

$$\int_{\Omega} f \mathbb{G}_0(g) dx = \int_{\Omega} g \mathbb{G}_0(f) dx.$$

Proof. Both sides are equal to

$$\int_{\Omega} \int_{\Omega} \mathcal{G}_0(x, y) f(x) g(y) dx dy. \quad \square$$

The linear theory Theorem 2.2 shows in particular that \mathbb{G}_{λ} is also a bounded operator from

$$\begin{aligned}
\mathbb{G}_{\lambda} : L^1(\Omega; \delta^\gamma) &\longrightarrow L^1(\Omega; \delta^\gamma) \\
\mathbb{G}_{\lambda} : \delta^\gamma L^\infty(\Omega) &\longrightarrow \delta^\gamma L^\infty(\Omega).
\end{aligned}$$

Therefore, an analogous integration by parts formula holds for \mathbb{G}_{λ} because of that for the differential operator $\mathcal{L} - \lambda$, even without knowledge of the its kernel.

Lemma 4.6 (Integration by parts with \mathcal{L}). *Suppose $u, v \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$. Then*

$$\int_{\Omega} u \mathcal{L} v dx = \int_{\Omega} v \mathcal{L} u dx.$$

Proof. In the eigenbasis,

$$u(x) = \sum_{j \geq 1} \langle u, \varphi_j \rangle \varphi_j(x), \quad v(x) = \sum_{j \geq 1} \langle v, \varphi_j \rangle \varphi_j(x),$$

both integrals are equal to

$$\sum_{j \geq 1} \lambda_j \langle u, \varphi_j \rangle \langle v, \varphi_j \rangle. \quad \square$$

Lemma 4.7 (Integration by parts with \mathbb{G}_λ). *Suppose $f \in L^1(\Omega; \delta^\gamma)$ and $g \in \delta^\gamma L^\infty(\Omega)$. Then we have*

$$\int_{\Omega} f \mathbb{G}_\lambda(g) dx = \int_{\Omega} g \mathbb{G}_\lambda(f) dx.$$

Proof. Suppose first $f, g \in L^2(\Omega)$. Write $u = \mathbb{G}_\lambda(f)$ and $v = \mathbb{G}_\lambda(g)$, so that $u, v \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$. Then⁶

$$\int_{\Omega} f \mathbb{G}_\lambda(g) dx = \int_{\Omega} (\mathcal{L}u - \lambda u)v dx = \int_{\Omega} (\mathcal{L}v - \lambda v)u dx = \int_{\Omega} g \mathbb{G}_\lambda(f) dx.$$

In general, let $f_k, g_k \in C_c^\infty(\Omega) \subset L^2(\Omega)$ such that $f_k \rightarrow f$ in $L^1(\Omega; \delta^\gamma)$ and $g_k \rightarrow g$ in $\delta^\gamma L^\infty(\Omega)$. Since g_k are uniformly bounded in $L^\infty(\Omega)$, $\mathbb{G}_\lambda(g_k)$ is uniformly bounded in $\delta^\gamma L^\infty(\Omega)$. On the other hand, f_k are uniformly bounded in $L^1(\Omega; \delta^\gamma)$, and so $\mathbb{G}_\lambda(f_k)$ are uniformly bounded in $L^1(\Omega, \delta^\gamma)$. By Theorem 2.2,

$$\begin{aligned} \|(\mathbb{G}_\lambda(f_k) - \mathbb{G}_\lambda(f))\delta^\gamma\|_{L^1(\Omega)} &= \|\mathbb{G}_\lambda(f_k - f)\delta^\gamma\|_{L^1(\Omega)} \leq C \|(f_k - f)\delta^\gamma\|_{L^1(\Omega)}, \\ \left\| \frac{\mathbb{G}_\lambda(g_k) - \mathbb{G}_\lambda(g)}{\delta^\gamma} \right\|_{L^\infty(\Omega)} &= \left\| \frac{\mathbb{G}_\lambda(g_k - g)}{\delta^\gamma} \right\|_{L^\infty(\Omega)} \leq C \left\| \frac{g_k - g}{\delta^\gamma} \right\|_{L^\infty(\Omega)}. \end{aligned}$$

This implies

$$\int_{\Omega} f_k \mathbb{G}_\lambda(g_k) dx = \int_{\Omega} f_k \delta^\gamma \frac{\mathbb{G}_\lambda(g_k)}{\delta^\gamma} dx \longrightarrow \int_{\Omega} f \delta^\gamma \frac{\mathbb{G}_\lambda(g)}{\delta^\gamma} dx = \int_{\Omega} f \mathbb{G}_\lambda(g) dx$$

and, analogously,

$$\int_{\Omega} g_k \mathbb{G}_\lambda(f_k) dx = \int_{\Omega} \frac{g_k}{\delta^\gamma} \mathbb{G}_\lambda(f_k) \delta^\gamma dx \longrightarrow \int_{\Omega} \frac{g}{\delta^\gamma} \mathbb{G}_\lambda(f) \delta^\gamma dx = \int_{\Omega} g \mathbb{G}_\lambda(f) dx$$

as $k \rightarrow +\infty$. This proves the result. \square

4.3. Equivalent notions of solution. Suppose $f \in L^1(\Omega; \delta^\gamma)$ and $u \in L^1(\Omega; \delta^\gamma)$. We show that the notions of L^1 -solution are equivalent, for the equation

$$(4.7) \quad \begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

For convenience we restate the statements in Theorem 2.1.

$$(1) \quad u = \mathbb{G}_\lambda(f) \quad \text{in } L^1(\Omega, \delta^\gamma).$$

$$(2) \quad \int_{\Omega} u \psi dx = \int_{\Omega} f \mathbb{G}_\lambda(\psi) dx, \quad \forall \psi \in \delta^\gamma L^\infty(\Omega).$$

$$(3) \quad \int_{\Omega} u(\phi - \lambda \mathbb{G}_0(\phi)) dx = \int_{\Omega} f \mathbb{G}_0(\phi) dx, \quad \forall \phi \in L_c^\infty(\Omega).$$

$$(4) \quad \int_{\Omega} u(\varphi - \lambda \mathbb{G}_0(\varphi)) dx = \int_{\Omega} f \mathbb{G}_0(\varphi) dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

⁶Alternatively, the same computations can be done using the spectral decomposition using (4.2), as in [10].

$$(5) \quad u - \lambda \mathbb{G}_0(u) = \mathbb{G}_0(f) \quad \text{in } L^1(\Omega; \delta^\gamma).$$

$$(6) \quad (\lambda_j - \lambda) \langle u, \varphi_j \rangle = \langle f, \varphi_j \rangle \quad \forall j \geq 1.$$

We remark that when $f \in L^2(\Omega)$ and $u \in \mathbb{H}_\mathcal{L}^2(\Omega)$, (1) \Leftrightarrow (5) \Leftrightarrow (6) is already proved in Lemma 4.2.

Proof of Theorem 2.1. We prove the equivalences pair by pair.

(1) \Leftrightarrow (2): Given (1), we multiply both sides by $\psi \in \delta^\gamma L^\infty(\Omega)$ and integrate by parts using (4.7) to get (2). Given (2), integrate by parts using (4.7) to yield

$$\int_{\Omega} (u - \mathbb{G}_\lambda(f)) \psi \, dx = 0, \quad \forall \psi \in \delta^\gamma L^\infty(\Omega).$$

and take $\psi = \text{sign}(u - \mathbb{G}_\lambda(f)) \delta^\gamma$ to get (1).

(2) \Leftrightarrow (3): Given (2) and $\phi \in L_c^\infty(\Omega)$, choose $\zeta = \mathbb{G}_0(\phi)$ and take $\psi = \phi - \lambda \zeta \in \delta^\gamma L^\infty(\Omega)$ so that by uniqueness $\zeta = \mathbb{G}_\lambda(\psi)$. Then $\psi = \phi - \lambda \mathbb{G}_0(\phi)$ and $\mathbb{G}_\lambda(\psi) = \mathbb{G}_0(\phi)$, so (2) implies (3). Given (3) and $\psi \in \delta^\gamma L^\infty(\Omega)$, write $\zeta = \mathbb{G}_\lambda(\psi)$ and choose $\phi = \psi + \lambda \zeta \in \delta^\gamma L^\infty(\Omega)$, so that $\phi - \lambda \mathbb{G}_0(\phi) = \psi$ and $\mathbb{G}_0(\phi) = \mathbb{G}_\lambda(\psi)$. While ϕ is not a valid test function for (3), we choose $\phi_K = \phi \mathbf{1}_K \in L_c^\infty(\Omega)$ for any $K \Subset \Omega$. Integrating (3) with ϕ_K by parts using Lemma 4.5 yields

$$\int_{\Omega} (u - \lambda \mathbb{G}_0(u)) \phi_K \, dx = \int_{\Omega} \mathbb{G}_0(f) \phi_K \, dx,$$

which, as a $L^1(\Omega, \delta^\gamma)$ – $\delta^\gamma L^\infty(\Omega)$ pairing, can be passed to the limit $K \nearrow \Omega$ using Dominated Convergence Theorem. The resulting identity becomes (2).

(3) \Leftrightarrow (4): Since $C_c^\infty(\Omega) \subset L_c^\infty(\Omega)$, “ \Rightarrow ” is trivial. Assume now (4). Given $\phi \in L_c^\infty(\Omega)$, take $\varphi_\varepsilon = \phi * \eta_\varepsilon \in C_c^\infty(\Omega)$, where $\varepsilon > 0$ and η_ε is the standard mollifier. Integrating (4) with φ_ε by parts using Lemma 4.5, we have

$$\int_{\Omega} (u - \lambda \mathbb{G}_0(u)) \varphi_\varepsilon \, dx = \int_{\Omega} \mathbb{G}_0(f) \varphi_\varepsilon \, dx,$$

which, as a $L^1(\Omega, \delta^\gamma)$ – $\delta^\gamma L^\infty(\Omega)$ pairing, can be passed to the limit $\varepsilon \searrow 0$ using Dominated Convergence Theorem. The resulting identity becomes (3).

(3) \Leftrightarrow (5): Integrate (3) by parts, we have

$$\int_{\Omega} (u - \lambda \mathbb{G}_0(u) - \mathbb{G}_0(f)) \phi \, dx = 0 \quad \forall \phi \in L_c^\infty(\Omega).$$

For any $K \Subset \Omega$, choosing $\phi = \text{sign}(u - \lambda \mathbb{G}_0(u) - \mathbb{G}_0(f)) \mathbf{1}_K$ yields (5). The converse is trivial, indeed by just testing (5) against $\phi \in L_c^\infty(\Omega)$ one obtains (3). \square

4.4. Local boundedness. In this section we obtain a local boundedness result for \mathbb{G}_0 -Green solutions, which are also $\mathbb{G}_\lambda(L^\infty(\Omega))$ -weak solutions according to Theorem 2.1, of the equation

$$\begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus we study the equivalent integral equation

$$(4.8) \quad u = \lambda \mathbb{G}_0(u) + \mathbb{G}_0(f) \quad \text{in } L^1(\Omega; \delta^\gamma).$$

The main estimates come from Theorem 3.9, which are in turn based on a localized version of the classical Hardy–Littlewood–Sobolev inequality.

Proposition 4.8. *The operator $f \mapsto u = \mathbb{G}_\lambda(f)$ is continuous*

$$\begin{aligned} L^1(\Omega; \delta^\gamma) &\longrightarrow L^p_{\text{loc}}(\Omega), & \text{for } p \in [1, \frac{n}{n-2s}), \\ L^1(\Omega; \delta^\gamma) \cap L^{p_0}_{\text{loc}}(\Omega) &\longrightarrow L^{p_1}_{\text{loc}}(\Omega), & \text{for } p_0 \in (1, \frac{n}{2s}), \frac{1}{p_1} = \frac{1}{p_0} - \frac{2s}{n}, \\ L^1(\Omega; \delta^\gamma) \cap L^{q_0}_{\text{loc}}(\Omega) &\longrightarrow L^\infty_{\text{loc}}(\Omega), & \text{for } q_0 \in (\frac{n}{2s}, +\infty), \end{aligned}$$

Moreover, for $K \Subset K_0 \Subset \Omega$,

$$\begin{aligned} \|u\|_{L^p(K)} &\leq C(\bar{\lambda}, p, K) \left(\|u\delta^\gamma\|_{L^1(\Omega)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right), \\ \|u\|_{L^{p_1}(K)} &\leq C(\bar{\lambda}, p_0, K_0, K) \left(\|u\delta^\gamma\|_{L^1(\Omega)} + \|f\|_{L^{p_0}(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right), \\ \|u\|_{L^\infty(K)} &\leq C(\bar{\lambda}, q_0, K_0, K) \left(\|u\delta^\gamma\|_{L^1(\Omega)} + \|f\|_{L^{q_0}(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right). \end{aligned}$$

Proof. We use a standard bootstrap argument using Proposition 3.6.

- (1) From $\lambda u + f \in L^1(\Omega; \delta^\gamma)$, we see from (4.8) that

$$\begin{aligned} \|u\|_{L^p(K)} &\leq C(p, K) \|(\lambda u + f)\delta^\gamma\|_{L^1(\Omega)} \\ &\leq C(p, K) \|(\lambda u + f)\delta^\gamma\|_{L^1(\Omega)} \\ &\leq C(p, K)(1 + |\lambda|) \left(\|u\delta^\gamma\|_{L^1(\Omega)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right), \end{aligned}$$

for some fixed $p \in (1, \frac{n}{n-2s})$.

- (2) Fix any $p_0 \in (1, \frac{n}{n-2s} \wedge \frac{n}{2s})$. Define p_k by

$$\frac{1}{p_k} = \frac{1}{p_0} - k \frac{2s}{n}.$$

Let k_0 be smallest integer such that $p_{k_0} \in (\frac{2s}{n}, +\infty)$. (One may avoid the critical situation $p_{k_0} = \frac{2s}{n}$ by increasing p_0 if necessary.) For any $K \Subset K_0 \Subset \Omega$, we choose a sequence of compact subsets

$$K \Subset K_{k_0} \Subset K_{k_0-1} \Subset \dots \Subset K_1 \Subset K_0 \Subset \Omega.$$

- (3) For $k = 0, 1, \dots, k_0 - 1$, from $u \in L^1(\Omega; \delta^\gamma) \cap L^{p_k}(K_k)$ and $f \in L^1(\Omega; \delta^\gamma) \cap L^\infty(K_0)$, we deduce from (4.8) and (3.6) that

$$\begin{aligned} \|u\|_{L^{p_{k+1}}(K_{k+1})} &\leq C(\bar{\lambda}, K, K_0) \left(\|u\|_{L^{p_k}(K_k)} + \|u\delta^\gamma\|_{L^1(\Omega)} + \|f\|_{L^\infty(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right) \\ &\leq C(\bar{\lambda}, K, K_0) \left(\|u\delta^\gamma\|_{L^1(\Omega)} + \|f\|_{L^\infty(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right). \end{aligned}$$

- (4) Since $u \in L^1(\Omega; \delta^\gamma) \cap L^{p_{k_0}}(K_{k_0})$ with $p_{k_0} > \frac{2s}{n}$, and $f \in L^1(\Omega; \delta^\gamma) \cap L^\infty(K_0)$, (4.8) and (3.7) implies

$$\begin{aligned} \|u\|_{L^\infty(K)} &\leq C(\bar{\lambda}, K, K_0) \left(\|u\|_{L^{p_{k_0}}(K_{k_0})} + \|u\delta^\gamma\|_{L^1(\Omega)} + \|f\|_{L^\infty(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right) \\ &\leq C(\bar{\lambda}, K, K_0) \left(\|u\delta^\gamma\|_{L^1(\Omega)} + \|f\|_{L^\infty(K_0)} + \|f\delta^\gamma\|_{L^1(\Omega)} \right). \quad \square \end{aligned}$$

Proof of Theorem 2.2 (2). It follows from Proposition 4.8, and Theorem 2.2 (1) which bounds $\|u\delta^\gamma\|_{L^1(\Omega)}$ in terms of f and $d_\Sigma(\lambda)$. \square

5. PROJECTED LINEAR THEORY

Given $\lambda \in \mathbb{R} \setminus \Sigma$, we let $\bar{\lambda}$, $d_\Sigma(\lambda)$ be as in (2.12), (2.13). We decompose $L^2(\Omega) = E \oplus E^\perp$, where E is the span of eigenfunctions associated to eigenvalues from λ_1 up to $\bar{\lambda}$, and E^\perp is its orthogonal complement. For the orthogonal component (i.e. in E^\perp) of the solution, we derive estimates independent of $d_\Sigma(\lambda)$.

For any datum $f \in L^1(\Omega; \delta^\gamma)$, we project the equation

$$(5.1) \quad \begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

into \widetilde{E}^\perp , according to

$$(5.2) \quad \begin{aligned} f^\perp(x) &= f(x) - \sum_{\varphi_j \in E_i} \langle f, \varphi_j \rangle \varphi_j(x) \\ u^\perp(x) &= u(x) - \sum_{\varphi_j \in E_i} \frac{\langle f, \varphi_j \rangle}{\lambda_j - \bar{\lambda}} \varphi_j(x), \end{aligned}$$

to arrive at

$$(5.3) \quad \begin{cases} \mathcal{L}u^\perp - \lambda u^\perp = f^\perp & \text{in } \Omega \\ Bu^\perp = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 5.1 (Projection). *Let $p \in [1, \frac{n-2s}{n})$ and $q \in (\frac{n}{2s}, +\infty)$. Under the projection (5.2),*

- (1) *$u \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$ is a spectral solution of (5.1) for $f \in L^2(\Omega)$ if and only if $u^\perp \in \mathbb{H}_{\mathcal{L}}^2(\Omega) \cap E^\perp$ is a spectral solution of (5.3) for $f^\perp \in L^2(\Omega) \cap E^\perp$.*
- (2) *$u \in L^\infty(\Omega)$ is a spectral solution of (5.1) for $f \in L^q(\Omega)$ if and only if $u^\perp \in L^\infty(\Omega) \cap E^\perp$ is a spectral solution of (5.3) for $f^\perp \in L^q(\Omega) \cap E^\perp$.*
- (3) *$u \in L^1(\Omega; \delta^\gamma)$ is a $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (5.1) for $f \in L^1(\Omega; \delta^\gamma)$ if and only if $u^\perp \in \widetilde{E}^\perp$ is a $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (5.3) for $f^\perp \in \widetilde{E}^\perp$.*

Proof. Since there are finitely many smooth eigenfunctions involved in (5.2), the projected and unprojected functions lie in the same space, and their equations are directly verified. \square

Corollary 5.2 (Well-posedness). *Let $p \in [1, \frac{n-2s}{n})$ and $q \in (\frac{n}{2s}, +\infty)$. Under the projection (5.2):*

- (1) *For any $f^\perp \in L^2(\Omega) \cap E^\perp$, there exists a unique spectral solution $u^\perp \in \mathbb{H}_{\mathcal{L}}^2(\Omega) \cap E^\perp$ of (5.3).*
- (2) *For any $f^\perp \in L^q(\Omega) \cap E^\perp$, there exists a unique spectral solution $u^\perp \in L^\infty(\Omega) \cap E^\perp$ of (5.3).*
- (3) *For any $f^\perp \in \widetilde{E}^\perp$, there exists a unique $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution $u^\perp \in \widetilde{E}^\perp$ of (5.3).*

Proof. The result follows directly from Theorem 2.2 and Lemma 5.1. \square

The underlying reason for the uniform estimate is seen in the L^2 -theory.

Lemma 5.3 (Uniform estimate for L^2 data). *Suppose $f^\perp \in L^2(\Omega) \cap E^\perp$ and $u^\perp \in \mathbb{H}_L^2(\Omega) \cap E^\perp$ is a spectral solution of (5.3). Then*

$$u^\perp(x) = \sum_{j=I+1}^{+\infty} \frac{\langle f, \varphi_j \rangle}{\lambda_j - \bar{\lambda}} \varphi_j(x).$$

In particular,

$$\|u^\perp\|_{L^2(\Omega)} \leq C(\bar{\lambda}) \|f^\perp\|_{L^2(\Omega)}.$$

Proof. This is an immediate consequence of the eigenfunction expansion. \square

Then we obtain boundary regularity with uniform estimates.

Proposition 5.4 (Uniform estimate for high integrability data). *Let $f^\perp \in L^q(\Omega) \cap E^\perp$ for $q \in (\frac{n}{2s}, +\infty)$, and $u^\perp \in L^\infty(\Omega) \cap E^\perp$ be a spectral solution of (5.3). Then*

$$\|u^\perp\|_{L^\infty(\Omega)} \leq C(q, \bar{\lambda}) \|f^\perp\|_{L^q(\Omega)}.$$

Moreover, if $f^\perp \in \delta^\alpha L^\infty(\Omega) \cap E^\perp$, then $u \in \mathbb{G}_0(\delta^\alpha) L^\infty(\Omega) \cap E^\perp$ satisfies

$$\left\| \frac{u^\perp}{\mathbb{G}_0(\delta^\alpha)} \right\|_{L^\infty(\Omega)} \leq C(\alpha, \bar{\lambda}) \left\| \frac{f^\perp}{\delta^\alpha} \right\|_{L^\infty(\Omega)}.$$

Proof. By Proposition 4.3 applied to (5.3),

$$\|u^\perp\|_{L^\infty(\Omega)} \leq C(q, \bar{\lambda}) \left(\|u^\perp\|_{L^2(\Omega)} + \|f^\perp\|_{L^q(\Omega)} \right).$$

Then we use Lemma 5.3 to bound

$$\|u^\perp\|_{L^2(\Omega)} \leq C(\bar{\lambda}) \|f^\perp\|_{L^2(\Omega)} \leq C(\bar{\lambda}) \|f^\perp\|_{L^\infty(\Omega)}.$$

Analogously with weights. \square

The next result on the uniform estimate in the projected space is crucial and we call it a theorem due to its importance.

Theorem 5.5 (Uniform estimate for low integrability data). *Let $f^\perp \in L^1(\Omega; \mathbb{G}_0(\delta^\alpha)) \cap \widetilde{E}^\perp$, and $u^\perp \in L^1(\Omega, \delta^\alpha) \cap \widetilde{E}^\perp$ be the $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (5.3). Then*

$$\|u^\perp \delta^\alpha\|_{L^1(\Omega)} \leq C \|f^\perp \mathbb{G}_0(\delta^\alpha)\|_{L^1(\Omega)},$$

for $C = C(\bar{\lambda})$. The corresponding integrabilities with weight and $L^{p_0} - L^{p_1}$ estimates also hold, with a uniform constant depending on $\bar{\lambda}$.

Remark 5.1. The dependence on $\bar{\lambda} = \lambda_{I+1}$ is actually on $\lambda_{I+1} - \lambda_I$.

Proof. Recall that $\mathbb{G}_\lambda(\psi)$ is the spectral solution of

$$\begin{cases} \mathcal{L}\mathbb{G}_\lambda(\psi) - \lambda\mathbb{G}_\lambda(\psi) = \psi & \text{in } \Omega \\ B\mathbb{G}_\lambda(\psi) = 0 & \text{on } \partial\Omega. \end{cases}$$

Projecting this equation into \widetilde{E}^\perp according to (5.2), we have

$$\begin{cases} \mathcal{L}\mathbb{G}_\lambda(\psi)^\perp - \lambda\mathbb{G}_\lambda(\psi)^\perp = \psi^\perp & \text{in } \Omega \\ B\mathbb{G}_\lambda(\psi)^\perp = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} \psi^\perp(x) &= \psi(x) - \sum_{j=1}^I \langle \psi, \varphi_j \rangle \varphi_j(x) \\ \mathbb{G}_\lambda(\psi)^\perp(x) &= \mathbb{G}_\lambda(\psi)(x) - \sum_{j=1}^I \frac{\langle \psi, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j(x) = \mathbb{G}_\lambda(\psi^\perp). \end{aligned}$$

By Proposition 5.4 for $\psi \in \delta^\alpha L^\infty(\Omega)$

$$|\mathbb{G}_\lambda(\psi)^\perp| \leq C(\alpha, \bar{\lambda}) \left\| \frac{\psi^\perp}{\delta^\alpha} \right\|_{L^\infty(\Omega)} \mathbb{G}_0(\delta^\alpha).$$

Since for any $f^\perp \in E^\perp$,

$$\int_{\Omega} f^\perp \mathbb{G}_\lambda(\psi) dx = \int_{\Omega} f^\perp \left(\sum_{j=1}^I \frac{\langle \psi, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j + \mathbb{G}_\lambda(\psi)^\perp \right) dx = \int_{\Omega} f^\perp \mathbb{G}_\lambda(\psi)^\perp dx.$$

Therefore, if $f \in L^1(\Omega, \mathbb{G}_0(\delta^\alpha))$ we have that

$$(5.4) \quad \int_{\Omega} u^\perp \psi dx = \int_{\Omega} f^\perp \mathbb{G}_\lambda(\psi)^\perp dx, \quad \forall \psi \in \delta^\alpha L^\infty(\Omega).$$

We take $\psi = \text{sign}(u^\perp) \delta^\alpha \in \delta^\alpha L^\infty(\Omega)$ such that

$$\int_{\Omega} |u^\perp| \delta^\alpha dx = \int_{\Omega} f^\perp \mathbb{G}_\lambda(\psi)^\perp dx \leq C(\alpha, \bar{\lambda}) \int_{\Omega} |f^\perp| \mathbb{G}_0(\delta^\alpha) dx.$$

When $p \in (1, \frac{n}{2s})$, for $k = 1, 2, \dots$ we take $\psi = (|u^\perp| \wedge k)^{p-1} \text{sign}(u^\perp) \in L^\infty(\Omega)$, which has uniformly bounded $L^{\frac{p}{p-1}}(\Omega)$ -norm, such that

$$\int_{\Omega} (|u^\perp| \wedge k)^p dx \leq \int_{\Omega} |u^\perp| (|u^\perp| \wedge k)^{p-1} dx \leq C(q, \bar{\lambda}) \|\psi^\perp\|_{L^{\frac{p}{p-1}}(\Omega)} \int_{\Omega} |f^\perp| \delta^\gamma dx.$$

Since, by Hölder's inequality,

$$\begin{aligned} \|\psi^\perp\|_{L^{\frac{p}{p-1}}(\Omega)} &\leq \|\psi\|_{L^{\frac{p}{p-1}}(\Omega)} + \sum_{j=1}^I \langle \psi, \varphi_j \rangle \|\varphi_j\|_{L^{\frac{p}{p-1}}(\Omega)} \\ &\leq \|\psi\|_{L^{\frac{p}{p-1}}(\Omega)} \left(1 + \sum_{j=1}^I \|\varphi_j\|_{L^p(\Omega)} \|\varphi_j\|_{L^{\frac{p}{p-1}}(\Omega)} \right) \\ &\leq C(p, \bar{\lambda}) \|\psi\|_{L^{\frac{p}{p-1}}(\Omega)} \end{aligned}$$

and

$$\|\psi\|_{L^{\frac{p}{p-1}}(\Omega)} = \left(\int_{\Omega} (|u^\perp| \wedge k) dx \right)^{p-1} \leq \left(\int_{\Omega} (|u^\perp| \wedge k)^p dx \right)^{\frac{p-1}{p}},$$

We conclude that

$$\| |u^\perp| \wedge k \|_{L^p(\Omega)} \leq C(p, \bar{\lambda}) \int_{\Omega} |f^\perp| \delta^\gamma dx.$$

The desired estimate then follows by taking $k \rightarrow +\infty$.

A similar argument applies for $p = 1$. \square

Proof of Theorem 2.3. The existence-uniqueness of solutions of the projected equation follows from Lemma 5.1 and Theorem 2.2. Thus \mathbb{G}_λ is well-defined in corresponding subspaces of E^\perp . The uniform estimates follow from Lemma 5.3, Proposition 5.4 and Theorem 5.5. \square

We turn to the uniform L^∞ -bound in compact subsets.

Proposition 5.6. *Suppose $f^\perp \in L_{\text{loc}}^\infty(\Omega) \cap \widetilde{E}^\perp$, and $u^\perp \in \widetilde{E}^\perp$ is the $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (5.3). Then for any $K \Subset K_0 \Subset \Omega$,*

$$\|u^\perp\|_{L^\infty(K)} \leq C(\bar{\lambda}, K, K_0) \left(\|f^\perp\|_{L^\infty(K_0)} + \|f^\perp \delta^\gamma\|_{L^1(\Omega)} \right).$$

Proof. By Proposition 4.8 applied to (5.3),

$$\|u^\perp\|_{L^\infty(K)} \leq C(\bar{\lambda}, K, K_0) \left(\|u^\perp \delta^\gamma\|_{L^1(\Omega)} + \|f^\perp\|_{L^\infty(K_0)} + \|f^\perp \delta^\gamma\|_{L^1(\Omega)} \right).$$

Now Theorem 5.5 with $\alpha = \gamma$ allows $\|u^\perp \delta^\gamma\|_{L^1(\Omega)}$ to be absorbed by $\|f^\perp \delta^\gamma\|_{L^1(\Omega)}$ independently of $d_\Sigma(\lambda)$. \square

We conclude this section by controlling norms of f^\perp in terms of those of f .

Lemma 5.7 (Norms of f^\perp). *For any $f \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$,*

$$\|f^\perp \delta^\gamma\|_{L^1(\Omega)} \leq C(\bar{\lambda}) \|f \delta^\gamma\|_{L^1(\Omega)},$$

and for any $K_0 \Subset \Omega$,

$$\|f^\perp\|_{L^\infty(K_0)} \leq C(\bar{\lambda}, K_0) \left(\|f\|_{L^\infty(K_0)} + \|f \delta^\gamma\|_{L^1(\Omega)} \right).$$

Proof. Using Proposition 3.7, we compute

$$\begin{aligned} \int_{\Omega} |f^{\perp}| \delta^{\gamma} dx &\leq \int_{\Omega} |f| \delta^{\gamma} dx + \sum_{j=1}^I \int_{\Omega} |f| |\varphi_j| dx \int_{\Omega} |\varphi_j| \delta^{\gamma} dx \\ &\leq \int_{\Omega} |f| \delta^{\gamma} dx \left(1 + \sum_{j=1}^I \|\varphi_j / \delta^{\gamma}\|_{L^{\infty}(\Omega)} \|\varphi_j \delta^{\gamma}\|_{L^1(\Omega)} \right), \end{aligned}$$

and for any $K_0 \Subset \Omega$,

$$\begin{aligned} \|f^{\perp}\|_{L^{\infty}(K_0)} &\leq \|f\|_{L^{\infty}(K_0)} + \sum_{j=1}^I \int_{\Omega} |f| |\varphi_j| dx \cdot \|\varphi_j\|_{L^{\infty}(K_0)} \\ &\leq \|f\|_{L^{\infty}(K_0)} + \left(\sum_{j=1}^I \|\varphi_j / \delta^{\gamma}\|_{L^{\infty}(\Omega)} \|\varphi_j\|_{L^{\infty}(\Omega)} \right) \|f \delta^{\gamma}\|_{L^1(\Omega)}. \quad \square \end{aligned}$$

Proof of Theorem 2.4. The existence-uniqueness follows from Lemma 5.1 and Theorem 2.2. The uniform global $L^1(\Omega; \delta^{\gamma})$ estimate follows from Theorem 5.5 as well as Lemma 5.7, while the uniform local L^{∞} bound follows from Proposition 5.6 and Lemma 5.7. \square

6. MAXIMUM PRINCIPLE

In this section we prove a maximum principle for weighted L^1 functions when $\lambda < \lambda_1$, where the operator is known to be positive in the L^2 -sense. Suppose $f \in L^1(\Omega; \delta^{\gamma})$ and consider the $\mathbb{G}_{\lambda}(\delta^{\gamma} L^{\infty}(\Omega))$ -weak solution $u \in L^1(\Omega; \delta^{\gamma})$ of

$$(6.1) \quad \begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

We will use the Poincaré inequality [9, Proposition 5.1]

$$(6.2) \quad \lambda_1 \int_{\Omega} \phi \mathbb{G}_0(\phi) dx \leq \int_{\Omega} \phi^2 dx, \quad \forall \phi \in L^2(\Omega).$$

Lemma 6.1 (Maximum principle for spectral solutions). *Suppose $f \in L^2(\Omega)$ and $u \in \mathbb{H}_{\mathcal{L}}^2(\Omega)$ is a spectral solution of (6.1). If $f \geq 0$ and $\lambda < \lambda_1$, then $u \geq 0$.*

Proof. We write

$$\int_{\Omega} u \psi dx = \lambda \int_{\Omega} u \mathbb{G}_0(\psi) dx + \int_{\Omega} f \mathbb{G}_0(\psi) dx.$$

Taking $\psi = -u_- \leq 0$ we have that

$$\int_{\Omega} u_-^2 dx = -\lambda \int_{\Omega} u_+ \mathbb{G}_0(u_-) dx + \lambda \int_{\Omega} u_- \mathbb{G}_0(u_-) dx - \int_{\Omega} f \mathbb{G}_0(u_-) dx.$$

Since $u_- \geq 0$, it is clear that $\mathbb{G}_0(u_-) \geq 0$. Then we have

$$\int_{\Omega} u_-^2 dx \leq \lambda \int_{\Omega} u_- \mathbb{G}_0(u_-) dx.$$

Applying the Poincaré inequality , (6.2)

$$\lambda_1 \int_{\Omega} u_- \mathbb{G}_0(u_-) dx \leq \lambda \int_{\Omega} u_- \mathbb{G}_0(u_-) dx.$$

Since $\lambda < \lambda_1$ we have that

$$\int_{\Omega} u_- \mathbb{G}_0(u_-) dx = 0.$$

If $u_- \not\equiv 0$, then $\mathbb{G}_0(u_-) > 0$ in Ω and we arrive at a contradiction. Hence, we deduce that $u_- = 0$. \square

Now Theorem 2.5 is restated and proved in the following

Lemma 6.2 (Maximum principle for $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solutions). *Let $f \in L^1(\Omega; \delta^\gamma)$ and $u \in L^1(\Omega; \delta^\gamma)$ be a $\mathbb{G}_\lambda(\delta^\gamma L^\infty(\Omega))$ -weak solution of (6.1). If $f \geq 0$ and $\lambda < \lambda_1$, then $u \geq 0$.*

Proof. Split $u = u_+ - u_-$, with $u_+, u_- \in L^1(\Omega; \delta^\gamma)$ non-negative. Let $K \Subset \Omega$. Testing (6.1) with $\psi = \mathbf{1}_{\{u < 0\} \cap K} \geq 0$, we have

$$- \int_{\{u < 0\} \cap K} u_- dx = \int_{\Omega} u \psi dx = \int_{\Omega} f \mathbb{G}_\lambda(\psi) dx \geq 0,$$

in view of Lemma 6.1. This forces $u_- = 0$ a.e. in any $K \Subset \Omega$, i.e. $u \geq 0$ a.e. in Ω . \square

7. INHOMOGENEOUS EIGENVALUE PROBLEM

In this section we complete the proof of Theorem 2.6.

Proof of Theorem 2.6. We split the proof into a few steps.

- (1) Taking the difference $u_\lambda = v_\lambda - v_h$, where v_λ and v_h are $\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solutions of (2.15) and (2.16) respectively, we see that u_λ is a $\mathbb{G}_0(C_c^\infty(\Omega))$ -weak solution (hence in any of the sense (1)–(5) in Theorem 2.1) of

$$(7.1) \quad \begin{cases} \mathcal{L}u_\lambda - \lambda u_\lambda = g + \lambda v_h & \text{in } \Omega \\ Bu_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

We have that $v_h \in \delta^{-b} L^\infty(\Omega) \subset L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$; indeed,

$$(7.2) \quad \|v_h \delta^\gamma\|_{L^1(\Omega)} \leq \|v_h \delta^b \cdot \delta^{2s-1}\|_{L^1(\Omega)} \leq \|v_h \delta^b\|_{L^\infty(\Omega)} \|\delta^{2s-1}\|_{L^1(\Omega)} \leq C \|v_h \delta^b\|_{L^\infty(\Omega)}.$$

Hence, Theorem 2.2 implies the existence of a unique solution u_λ of (7.1) in $L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$. In particular, we obtain the existence of a unique solution

$$v_\lambda = v_h + u_\lambda \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$$

of (2.15), for any $\lambda \in \mathbb{R} \setminus \Sigma$.

- (2) The representation formula follows directly from Theorem 2.4 applied to $f = g + \lambda v_h \in L^1(\Omega; \delta^\gamma) \cap L_{\text{loc}}^\infty(\Omega)$, under the projection (5.2).

(3) By Theorem 2.4, we have the estimates

$$\|u_\lambda^\perp \delta^\gamma\|_{L^1(\Omega)} \leq C(\bar{\lambda}) \|(g + \lambda v_h) \delta^\gamma\|_{L^1(\Omega)}$$

and, for $K \Subset K_0 \Subset \Omega$,

$$\|u_\lambda^\perp\|_{L^\infty(K)} \leq C(\bar{\lambda}, K, K_0) \left(\|g + \lambda v_h\|_{L^\infty(K_0)} + \|(g + \lambda v_h) \delta^\gamma\|_{L^1(\Omega)} \right).$$

But then the norms of v_h can be controlled in terms of $\|h\|_{L^\infty(\partial\Omega)}$ by (3.10) and (7.2).

(4) Let us write

$$\begin{aligned} v_\lambda &= v_h + \sum_{\substack{j=1 \\ \lambda_j \neq \lambda_i}}^I \frac{\langle g + \lambda v_h, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j + \frac{1}{\lambda_i - \lambda} \sum_{j: \lambda_j = \lambda_i} \langle g + \lambda v_h, \varphi_j \rangle \varphi_j + \mathbb{G}_\lambda((g + \lambda v_h)^\perp) \\ &= v_h + \frac{P_{E_i}(g + \lambda v_h)}{\lambda_i - \lambda} + \mathbb{G}_\lambda(g + \lambda v_h - P_{E_i}(g + \lambda v_h)). \end{aligned}$$

By our previous construction $v_h + \mathbb{G}_\lambda((g + \lambda v_h)^\perp)$ is uniformly bounded in $L^\infty(K)$ for any $K \Subset \Omega$. It is easy to check that

$$v_h + \mathbb{G}_\lambda((g + \lambda v_h)^\perp) \rightarrow v_h + \mathbb{G}_{\lambda_i}((g + \lambda_i v_h)^\perp)$$

over compacts. The term

$$\sum_{\substack{j=1 \\ \lambda_j \neq \lambda_i}}^I \frac{\langle g + \lambda v_h, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j \longrightarrow \sum_{\substack{j=1 \\ \lambda_j \neq \lambda_i}}^I \frac{\langle g + \lambda_i v_h, \varphi_j \rangle}{\lambda_j - \lambda_i} \varphi_j$$

is in $\delta^\gamma L^\infty(\Omega)$. Hence,

$$v_h + \mathbb{G}_\lambda(g + \lambda v_h - P_{E_i}(g + \lambda v_h)) \rightarrow v_h + \mathbb{G}_{\lambda_i}(g + \lambda v_h - P_{E_i}(g + \lambda_i v_h))$$

in $L^1(\Omega, \delta^\gamma)$.

The last term is given by the projection

$$P_{E_i}(g + \lambda v_h) = \sum_{j: \lambda_j = \lambda_i} \langle g + \lambda v_h, \varphi_j \rangle \varphi_j.$$

By continuity of the projection,

$$P_{E_i}(g + \lambda v_h) \longrightarrow P_{E_i}(g + \lambda_i v_h)$$

as $\lambda \rightarrow \lambda_i$. Hence, either $P_{E_i}(g + \lambda_i v_h) = 0$ or there exists $K \Subset \Omega$ and ε of positive measure such that $|P_{E_i}(g + \lambda_i v_h)| > \varepsilon$ on K and therefore

$$\left| \frac{P_{E_i}(g + \lambda v_h)}{\lambda_i - \lambda} \right| \rightarrow +\infty \text{ on } K.$$

Since the rest of the terms of v_h are uniformly locally essentially bounded, this $\|v_h\|_{L^\infty(K)}$ blows up.

If $P_{E_i}(g + \lambda_i v_h) = 0$, since all the considered eigenfunctions are linearly independent, we have that $\langle g + \lambda_i v_h, \varphi_j \rangle = 0$. Then, we apply L'Hôpital's rule to deduce that

$$\lim_{\lambda \rightarrow \lambda_i} \frac{\langle g + \lambda v_h, \varphi_j \rangle}{\lambda_i - \lambda} = \lim_{\lambda \rightarrow \lambda_i} \frac{\langle g, \varphi_j \rangle + \lambda \langle v_h, \varphi_j \rangle}{\lambda_i - \lambda} = -\langle v_h, \varphi_j \rangle.$$

Therefore

$$\frac{P_{E_i}(g + \lambda v_h)}{\lambda_i - \lambda} = \frac{1}{\lambda_i - \lambda} \sum_{j: \lambda_j = \lambda_i} \langle g + \lambda v_h, \varphi_j \rangle \varphi_j \rightarrow - \sum_{j: \lambda_j = \lambda_i} \langle v_h, \varphi_j \rangle = -P_{E_i}(v_h)$$

in $\delta^\gamma L^\infty(\Omega)$, and the limit is characterised.

- (5) To prove the global blow-up behavior, suppose $g \geq 0$ and $h > 0$. By (3.8), $v_h > 0$. In fact, (3.9) and (3.8) imply that

$$(7.3) \quad v_h(x) \geq C \left(\inf_{\partial\Omega} h \right) \delta^{-b}(x) \quad \text{for } x \in \Omega.$$

Also, since $g + \lambda v_h > 0$, by the maximum principle Theorem 2.5, $u_\lambda = v - v_h \geq 0$.

The solution v_λ can be expressed in two ways, namely

$$(7.4) \quad v_\lambda = v_h + u_\lambda, \quad u_\lambda \geq 0,$$

and

$$(7.5) \quad v_\lambda = \frac{1}{\lambda_1 - \lambda} \langle g + \lambda v_h, \varphi_1 \rangle \varphi_1 + v_h + u_\lambda^\perp, \quad u_\lambda^\perp \in L_{\text{loc}}^\infty(\Omega).$$

Given any $\bar{C} > 0$, there exists $K = K(\bar{C}) \Subset \Omega$ such that $v_h \geq \bar{C}$ in $\Omega \setminus K$ by (7.3). Then (7.4) yields

$$\inf_{\Omega \setminus K} v_\lambda \geq \bar{C}.$$

On the other hand, (2.21) implies that

$$\|u^\perp\|_{L^\infty(K)} \leq C(\lambda_1, K) \left(\|g\|_{L^\infty(\{\delta > \text{dist}(K, \partial\Omega)/2\})} + \|g\delta^\gamma\|_{L^1(\Omega)} + \|h\|_{L^\infty(\partial\Omega)} \right).$$

Then (7.5) implies

$$\begin{aligned} \inf_K v_\lambda &\geq \frac{\lambda}{\lambda_1 - \lambda} \cdot C \left(\inf_{\partial\Omega} h \right) \langle \delta^{-b}, \varphi_1 \rangle \inf_K \varphi_1 \\ &\quad - C(K) \left(\|g\|_{L^\infty(\{\delta > \text{dist}(K, \partial\Omega)/2\})} + \|g\delta^\gamma\|_{L^1(\Omega)} + \|h\|_{L^\infty(\partial\Omega)} \right) \\ &\geq \frac{C(K, h)}{\lambda_1 - \lambda} - C(K, g, h) \\ &\geq \bar{C}, \end{aligned}$$

for all $\lambda \in (\lambda_{\bar{C}}(g, h), \lambda_1)$. This completes the proof. \square

8. THE LIMIT AS s TENDS TO 1

In this section we address the issue of convergence of our problems in the limit when s approaches 1. For convenience, we will index here our operators as \mathcal{L}_s . Originally, our interest was treating the RFL and SFL operators, where the passage to the limit is relatively simple and produces in the limit a classical problem involving the Laplacian. The interesting feature is that we may thus observe how the boundary blow-up disappears in the limit, since the corresponding blow-up exponent goes to zero as $s \rightarrow 1$.

We propose a different writing, since we found interesting to state a list of hypotheses on the general class \mathcal{L}_s for s near 1 that allow to show convergence of the original problems involving \mathcal{L}_s to a limit problem as $s \rightarrow 1$. Naturally, we assume that

$$(8.1) \quad \mathcal{L}_s \text{ satisfies (K1)-(K2) for all } s \in (0, 1].$$

and we will denote the limit operator by \mathcal{L}_1 . In our main case of interest $\mathcal{L}_1 = -\Delta$. At the end, we show that these assumptions hold for the RFL and SFL, see Subsection 8.6.

Let us proceed. On a first stage, let us simply assume that \mathcal{L}_1 is indeed the limit of \mathcal{L}_s , in the sense of resolvents:

$$(8.2) \quad \|\mathbb{G}_{\mathcal{L}_s}(f) - \mathbb{G}_{\mathcal{L}_1}(f)\|_{L^2(\Omega)} \leq \omega(1-s)\|f\|_{L^2}, \quad \text{where } \omega(1-s) \searrow 0 \text{ as } s \rightarrow 1.$$

We will show that it suffices to have

$$\mathbb{G}_{\mathcal{L}_s - \lambda}(f) \longrightarrow \mathbb{G}_{\mathcal{L}_1 - \lambda}(f) \quad \text{in } L^2(\Omega), \text{ for all } f \in L^2(\Omega),$$

at least when $\lambda \notin \Sigma(\mathcal{L}_1)$, so the nonsingular eigenvalue problem is studied.

To study the large eigenvalue problem, we will also assume that \mathcal{L}_s satisfies (K2). Since we are mostly interest in approximating the Laplacian's behaviour, let us assume that

$$(8.3) \quad 2s > \gamma(s).$$

for s sufficiently close to 1. This is satisfied for the RFL and the SFL with $s > \frac{1}{2}$. In this setting $\mathbb{M}_{\mathcal{L}_s}(h) \in L^1(\Omega)$ for $h \in L^\infty(\partial\Omega)$. We need to assume some convergence of the Martin operator. In particular, we will assume that

$$(8.4) \quad \mathbb{M}_{\mathcal{L}_s}(h) \longrightarrow \mathbb{M}_{\mathcal{L}_1}(h), \quad \text{in } L^1(\Omega) \text{ for all } h \in L^\infty(\partial\Omega).$$

Remark 8.1. Notice that $\mathbb{M}_{-\Delta}$ is the usual Poisson integral operator, i.e. $u = \mathbb{M}_{-\Delta}(h)$ is the unique solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

The last condition we will ask is that there are uniform $L^p \rightarrow L^q$ bounds in two special cases

$$(8.5) \quad \|\mathbb{G}_{\mathcal{L}_s - \lambda}(f)\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C(\lambda)\|f\|_{L^1(\Omega)}$$

$$(8.6) \quad \|\mathbb{G}_{\mathcal{L}_s - \lambda}(f)\|_{L^\infty(\Omega)} \leq C(\lambda)\|f\|_{L^\infty(\Omega)}$$

where the constant $C(\lambda)$ does not depend on s .

We will check that these two conditions are implied by

$$(8.7) \quad \mathcal{G}_{\mathcal{L}_s}(x, y) \leq C_\Omega |x - y|^{-(n-2s)} \quad \text{where } C_\Omega \text{ does not depend on } s \text{ close to } 1.$$

This condition is reasonable since all operators in the family satisfy (K1). This will be the condition that we check in the examples.

Under these hypothesis we will be able to prove the convergence of large solutions as $s \rightarrow 1^-$. Furthermore, we will have that, if f_s are bounded in $L^2(\Omega)$, then $\mathbb{G}_s(f_s)$ has a strongly convergent subsequence.

8.1. Compactness theorem.

Theorem 8.1. *Assume (8.1), (8.2) and let f_s be such that $\|f_s\|_{L^2(\Omega)}$ is bounded. Then, there exists a sequence $s_m \rightarrow 1$ such that $f_{s_m} \rightharpoonup f$ in weakly in $L^2(\Omega)$ and $\mathbb{G}_{\mathcal{L}_{s_m}}(f_{s_m}) \rightarrow \mathbb{G}_{\mathcal{L}_1}(f)$ strongly in $L^2(\Omega)$.*

For each s , the compactness of $\mathbb{G}_{\mathcal{L}_s}$ proved in [9, Proposition 5.1] uses the Riesz–Fréchet–Kolmogorov theorem and the fact that, since⁷ $\mathbb{G}_{\mathcal{L}_s}^{(2)}(x, y) = \mathbb{G}_{\mathcal{L}_s}(x, y)\chi_{|x-y|>\varepsilon} \in L^2(\Omega)$, then $\|\mathbb{G}_{\mathcal{L}_s}^{(2)}(\cdot + h, \cdot) - \mathbb{G}_{\mathcal{L}_s}^{(2)}\|_{L^2(\Omega \times \Omega)} \rightarrow 0$ as $h \rightarrow 0$. The authors prove that under (K1), for each $s \in (0, 1]$,

$$\lim_{|h| \rightarrow 0} \sup_{\|f\|_{L^2(\Omega)} \leq 1} \|\tau_h \mathbb{G}_{\mathcal{L}_s}(f_s) - \mathbb{G}_{\mathcal{L}_s}(f_s)\|_{L^2(\Omega)} = 0.$$

Let us denote the modulus of continuity by ω_s ,

$$\omega_s(\varepsilon) = \sup_{|h| \leq \varepsilon} \sup_{\|f\|_{L^2(\Omega)} \leq 1} \|\tau_h \mathbb{G}_{\mathcal{L}_s}(f_s) - \mathbb{G}_{\mathcal{L}_s}(f_s)\|_{L^2(\Omega)}.$$

The condition that this happens uniformly would be too severe. In order to prove our result we need a very specific form of the Riesz–Fréchet–Kolmogorov theorem. For $h \in \mathbb{R}^n$ we define

$$\tau_h u(x) = \begin{cases} u(x+h) & x+h \in \Omega, \\ 0 & x+h \notin \Omega. \end{cases}$$

Lemma 8.2 (Remark 6 (II) in [31]). *Let $u_m \in L^p(\Omega)$ ($1 \leq p < +\infty$) be a sequence such that*

$$\|\tau_h u_m - u_m\|_{L^p(\Omega)} \leq \omega_1(|h|) + \omega_2\left(\frac{1}{m}\right),$$

where $\omega_1(\varepsilon), \omega_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, u_m has a subsequence that converges strongly in $L^p(\Omega)$.

Proof of Theorem 8.1. We write

$$\begin{aligned} \|\tau_h \mathbb{G}_{\mathcal{L}_s}(f_s) - \mathbb{G}_{\mathcal{L}_s}(f_s)\|_{L^2(\Omega)} &\leq \|\tau_h \mathbb{G}_{\mathcal{L}_s}(f_s) - \tau_h \mathbb{G}_{\mathcal{L}_1}(f_s)\|_{L^2(\Omega)} \\ &\quad + \|\tau_h \mathbb{G}_{\mathcal{L}_1}(f_s) - \mathbb{G}_{\mathcal{L}_1}(f_s)\|_{L^2(\Omega)} \\ &\quad + \|\mathbb{G}_{\mathcal{L}_1}(f_s) - \mathbb{G}_{\mathcal{L}_s}(f_s)\|_{L^2(\Omega)} \\ &\leq 2\omega(1-s) + \omega_1(|h|)\|f_s\|_{L^2(\Omega)}, \end{aligned}$$

⁷Note that in $\mathbb{G}_{\mathcal{L}_s}^{(2)}$ the superscript denotes a second part, a notation borrowed from [9], as opposed to a power used in the rest of this paper.

where $\omega_1(|h|)$ follows from the continuity of $\mathbb{G}_{\mathcal{L}_1}$. Take any sequence $s_m \rightarrow 1^-$. Since f_{s_m} is bounded, up to a subsequence there exists $f \in L^2(\Omega)$ such that $f_{s_m} \rightharpoonup f$ weakly in $L^2(\Omega)$. Applying Lemma 8.2, we have that $u_m = \mathbb{G}_{\mathcal{L}_{s_m}}(f_{s_m})$ has a strongly $L^2(\Omega)$ convergent subsequence and its limit be v . Using the duality formula and (8.2) we have that $v = \mathbb{G}_{\mathcal{L}_1}(f)$. \square

8.2. Spectral convergence. It is well known that

$$\sup_j \text{dist}(\lambda_j(\mathbb{G}_{\mathcal{L}_s}), \Sigma(\mathbb{G}_{\mathcal{L}_1})) \leq \|\mathbb{G}_{\mathcal{L}_s} - \mathbb{G}_{\mathcal{L}_1}\|_{\mathcal{L}(L^2, L^2)} \leq \omega(1-s).$$

(see [32, Theorem 4.10 in Chapter 5]). Therefore

$$\sup_j \text{dist}(\lambda_j(\mathcal{L}_s)^{-1}, \Sigma(\mathcal{L}_1)^{-1}) \leq \omega(1-s).$$

Remark 8.2. The first eigenvalues are uniformly bounded below. Assuming (8.2) we have that

$$\lambda_1(\mathcal{L}_s)^{-1} \leq \lambda_1(\mathcal{L}_1)^{-1} + \text{dist}(\lambda_1(\mathcal{L}_s)^{-1}, \Sigma(\mathcal{L}_1)^{-1}) \leq \lambda_1(\mathcal{L}_1)^{-1} + \omega(1-s).$$

Hence

$$\lambda_1(\mathcal{L}_s) \geq \frac{1}{\lambda_1(\mathcal{L}_1)^{-1} + \omega(1-s)}.$$

Remark 8.3. Assuming only (8.7) also gives some information. Applying the Rayleigh–Faber–Krahn theorem guaranties and Proposition 3.3 we have that

$$\frac{1}{\lambda_1(\mathcal{L}_s)} = \inf_{f \neq 0} \frac{\|\mathbb{G}_{\mathcal{L}_s}(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq C_\Omega \inf_{f \neq 0} \frac{\|I_{n-2s}(f\chi_\Omega)\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} = C_\Omega \lambda_1(I_{n-2s}).$$

Furthermore, we have strong convergence of the eigenspaces.

Proposition 8.3. *Assume (8.2) and $\lambda_j(\mathcal{L}_s) \rightarrow \lambda_i(\mathcal{L}_1)$ for some j, i . Then $E_j(\mathcal{L}_s) \rightarrow E_i(\mathcal{L}_1)$ in $L^2(\Omega)$ for some i , in the sense that if $\phi_s \in E_j(\mathcal{L}_s)$ are such that $\|\phi_s\|_{L^2} = 1$, then there exists $\phi_1 \in E_i(\mathcal{L}_1)$ such that, up to a subsequence, $\phi_s \rightarrow \phi_1$ strongly in $L^2(\Omega)$.*

Proof. Since (ϕ_s) satisfies $\|\phi_s\|_{L^2(\Omega)} = 1$ and $\phi_s = \lambda_j(\mathcal{L}_s)\mathbb{G}_{\mathcal{L}_s}(\phi_s)$, we can apply Theorem 8.1 to show that there exists $\phi_{s_m} \rightharpoonup \phi_1$ in $L^2(\Omega)$ with $\mathbb{G}_{\mathcal{L}_{s_m}}(\phi_{s_m}) \rightarrow \mathbb{G}_{\mathcal{L}_1}(\phi_1)$ in $L^2(\Omega)$. But then, using $\lambda_j(\mathcal{L}_s) \rightarrow \lambda_i(\mathcal{L}_1)$,

$$\phi_{s_m} = \lambda_j(\mathcal{L}_{s_m})\mathbb{G}_{\mathcal{L}_{s_m}}(\phi_{s_m}) \rightarrow \lambda_i(\mathcal{L}_1)\mathbb{G}_{\mathcal{L}_1}(\phi_1)$$

strongly in $L^2(\Omega)$, so the convergence $\phi_{s_m} \rightarrow \phi_1$ is strong, $\|\phi_1\|_{L^2(\Omega)} = 1$ and

$$\phi_1 = \lambda_i(\mathcal{L}_1)\mathbb{G}_{\mathcal{L}_1}(\phi_1) \quad \text{in } L^2(\Omega),$$

i.e. $\phi_1 \in E_i(\mathcal{L}_1)$. \square

8.3. Convergence of $\mathcal{L}_s - \lambda$. Let $\lambda \notin \Sigma(\mathcal{L}_1)$, so $\lambda_i(\mathcal{L}_1) < \lambda < \lambda_{i+1}(\mathcal{L}_1)$ for some i . Let $s_0 = s_0(\lambda)$ be sufficiently close to 1 such that

$$\omega(1-s) \leq \frac{1}{2} \text{dist}(\lambda^{-1}, \Sigma(\mathcal{L}_1)^{-1}), \quad \text{for } s \in (s_0, 1).$$

Then, for $s \in (s_0, 1)$, we have that

$$\text{dist}(\lambda^{-1}, \Sigma(\mathcal{L}_s)^{-1}) \geq \text{dist}(\lambda^{-1}, \Sigma(\mathcal{L}_1)^{-1}) - \omega(1-s) \geq \frac{1}{2} \text{dist}(\lambda^{-1}, \Sigma(\mathcal{L}_1)^{-1}).$$

Therefore

$$\text{dist}(\lambda, \Sigma(\mathcal{L}_s)) \geq c > 0.$$

for $s \in (s_0, 1)$.

Proposition 8.4. *Let $\lambda \notin \Sigma(\mathcal{L}_1)$ and assume (8.2). Then*

$$\mathbb{G}_{\mathcal{L}_s - \lambda}(f) \rightarrow \mathbb{G}_{\mathcal{L}_1 - \lambda}(f) \quad \text{in } L^2(\Omega), \text{ for all } f \in L^2(\Omega).$$

Proof. Let $f \in L^2(\Omega)$ be fixed. Let $u_s = \mathbb{G}_{\mathcal{L}_s - \lambda}(f)$. We know that

$$\|u_s\|_{L^2(\Omega)} \leq \frac{1}{\text{dist}(\lambda, \Sigma(\mathcal{L}_s))} \|f\|_{L^2(\Omega)}.$$

Let us write the alternative formulation $u_s = \mathbb{G}_{\mathcal{L}_s}(\lambda u_s + f)$. Since $\lambda u_s + f$ is a bounded sequence in $L^2(\Omega)$, applying Theorem 8.1 there exists a subsequence $\lambda u_{s_m} + f \rightharpoonup g$ such that $u_{s_m} \rightarrow v = \mathbb{G}_{\mathcal{L}_1}(g)$ in $L^2(\Omega)$. Hence $g = f + \lambda v$ and $v = \mathbb{G}_{\mathcal{L}_1}(\lambda v + f)$. By uniqueness $v = \mathbb{G}_{\mathcal{L}_1 - \lambda}(f)$. Since every sequence of u_s have a convergent subsequence converging to the same v , the whole sequence converges. \square

8.4. Inhomogeneous eigenvalue value problem.

Proposition 8.5. *Assume (8.1)–(8.6), let $\lambda \notin \Sigma(\mathcal{L}_1)$, $g \in L^2(\Omega)$ and $h \in L^\infty(\partial\Omega)$. Then the solution of (2.15) weakly converges in $L^1(\Omega)$ to*

$$v_1 = \mathbb{M}_{\mathcal{L}_1}(h) + \mathbb{G}_{\mathcal{L}_1 - \lambda}(g + \lambda \mathbb{M}_{\mathcal{L}_1}(h)).$$

In some sense, this limit is the solution of

$$\begin{cases} \mathcal{L}_1 u - \lambda u = g & \Omega \\ B_1 u = h & \partial\Omega \end{cases}$$

Proof. Let consider the explicit form of the solution

$$v_s = \mathbb{M}_{\mathcal{L}_s}(h) + \mathbb{G}_{\mathcal{L}_s - \lambda}(g + \lambda \mathbb{M}_{\mathcal{L}_s}(h)).$$

We know that $\mathbb{M}_{\mathcal{L}_s}(h)$ and $\mathbb{G}_{\mathcal{L}_s - \lambda}(g)$ converge in $L^1(\Omega)$ to $\mathbb{M}_{\mathcal{L}_1}(h)$ and $\mathbb{G}_{\mathcal{L}_1 - \lambda}(g)$. Now we focus on $w_s = \mathbb{G}_{\mathcal{L}_s - \lambda}(\mathbb{M}_{\mathcal{L}_s}(h))$. We have that

$$(8.8) \quad \|w_s\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C(\lambda) \|\mathbb{M}_{\mathcal{L}_s}(h)\|_{L^1(\Omega)} \leq C.$$

Therefore, w_s as a weakly convergent subsequence in $L^{\frac{n}{n-1}}$ to some w_1 . Writing the weak formulation

$$\int_{\Omega} w_s \psi = \int_{\Omega} \mathbb{M}_{\mathcal{L}_s}(h) \mathbb{G}_{\mathcal{L}_s - \lambda}(\psi), \quad \forall \psi \in L_c^\infty(\Omega).$$

Since $\mathbb{G}_{\mathcal{L}_s-\lambda}(\psi)$ is bounded in L^∞ , (w_s) has a weak- \star - L^∞ convergent subsequence. Since it converges weakly in L^2 to $\mathbb{G}_{\mathcal{L}_1-\lambda}(\psi)$, this is its weak- \star - L^∞ limit. Passing to the limit

$$\int_{\Omega} w_1 \psi = \int_{\Omega} \mathbb{M}_{\mathcal{L}_1}(h) \mathbb{G}_{\mathcal{L}_1-\lambda}(\psi), \quad \forall \psi \in L_c^\infty(\Omega).$$

There $w_1 = \mathbb{G}_{\mathcal{L}_1-\lambda}(\mathbb{M}_{\mathcal{L}_1}(h))$. □

8.5. Uniform embeddings (8.5)–(8.6).

Remark 8.4. Notice that (8.5)–(8.6) are only used in Proposition 8.5. Here, they are used to show there exists a convergent subsequence. They can be replaced by $\|\mathbb{G}_{\mathcal{L}_s}(f) - \mathbb{G}_{\mathcal{L}_1}(f)\|_{L^1} \leq \omega(1-s)\|f\|_{L^1}$. If one assume that $\mathbb{M}_{\mathcal{L}_s}(h)$ converge in $L^2(\Omega)$, these hypothesis can be removed. This would not be abusive since $\mathbb{M}_{-\Delta}(h)$ is bounded.

Proposition 8.6. *Assume that (8.7). Then (8.5) and (8.6) hold.*

Proof. We already know that

$$\|\mathbb{G}_{\mathcal{L}_s-\lambda}(f)\|_{L^2} \leq \frac{\|f\|_{L^2(\Omega)}}{\text{dist}(\lambda, \Sigma(\mathcal{L}_s))} \leq 2 \frac{\|f\|_{L^2(\Omega)}}{\text{dist}(\lambda, \Sigma(\mathcal{L}_1))}$$

for s close to 1.

Let us go over the ideas in Section 4.1, being quite careful on the dependence on s . Let $f \in L^q$ for $q > \frac{n}{2s}$. The idea for the L^∞ estimate is to bootstrap via

$$u = \lambda^k \mathbb{G}_0^k(u) + \sum_{m=1}^k \lambda^{m-1} \mathbb{G}_0^m(f).$$

so that the sequence $p_0 = 2 - \varepsilon$,

$$\frac{1}{p_{m+1}} = \frac{1}{p_m} - \frac{2s}{n}$$

and pick ε_s and k_s so that $p_{k_s-1} < \frac{n}{2s} < p_{k_s}$. In order to this uniformly as $s \rightarrow 1$, for $s > \frac{3}{4}$ we take so that the sequence $p_0 = 2 - \varepsilon$,

$$\frac{1}{p_{m+1}} = \frac{1}{p_m} - \frac{3}{2n}$$

and pick ε and k_s so that $p_{k-1} < \frac{n}{2s} < \frac{2n}{3} < p_k$.

Taking into account the embedding $L^2 \hookrightarrow L^{2-\varepsilon}$, we have that

$$\begin{aligned} \|\mathbb{G}_{\mathcal{L}_s-\lambda}(f)\|_{L^\infty(\Omega)} &\leq 2|\Omega|^{\frac{2}{\varepsilon}} |\lambda|^k C_s(p_0, p_1) \cdots C_s(p_k, p_{k+1}) C_s(p_{k+1}, \infty) \frac{\|f\|_{L^2}}{\text{dist}(\lambda, \Sigma(\mathcal{L}_1))} \\ &\quad + \|f\|_{L^q(\Omega)} \sum_{m=1}^k |\lambda|^{m-1} C_s(q, \infty) C_s(\infty, \infty)^{m-1} \end{aligned}$$

where $C_s(p, q) = \|\mathbb{G}_{\mathcal{L}_s}\|_{\mathcal{L}(L^p, L^q)}$ is the corresponding continuity modulus. As seen in Section 4.1, this embedding constant is computed in two parts: the first, has to do with the constant

$$\mathcal{G}_{\mathcal{L}_s}(x, y) \leq C_\Omega |x - y|^{-(n+2s)}.$$

Then one, uses the fact that

$$|\mathbb{G}_{\mathcal{L}_s}(f)| \leq C_\Omega I_{n-2s}(|f|\chi_\Omega).$$

where I_α is the convolution with the Riesz potential. Hence

$$\|\mathbb{G}_{\mathcal{L}_s}(f)\|_{L^q} \leq C_\Omega \|I_{n-2s}(|f|\chi_\Omega)\|_{L^q}.$$

Thus

$$C_s(p, q) \leq C_\Omega \|I_{n-2s}(\cdot\chi_\Omega)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))}.$$

Notice that the right hand side can be bounded uniformly as $s \rightarrow 1$, even though it depends on the order $n - 2s$. On the other hand, following the proof of Proposition 4.4, we have that

$$\|\mathbb{G}_{\mathcal{L}_s-\lambda}\|_{\mathcal{L}(L^1, L^p)} \leq \|\mathbb{G}_{\mathcal{L}_s-\lambda}\|_{\mathcal{L}(L^{p'}, L^{+\infty})}. \quad \square$$

8.6. The two main examples. We are ready to get down into practical examples

1. *Restricted Fractional Laplacian.* For this operator we have $\gamma(s) = s$. Hence, (8.3) holds for every s .

When $\Omega = B_r$, the explicit form of the operator is known (see, e.g., [12])

$$\mathcal{G}_{(-\Delta)_{\text{RFL}}^s}(x, y) = \frac{\Gamma(\frac{n}{2})}{2^{2s}\Gamma^2(s)\pi^{\frac{n}{2}}} \frac{1}{|x - y|^{n-2s}} \int_0^{\frac{(r^2-|x|^2)(r^2-|y|^2)}{r^2|x-y|^2}} \frac{t^{s-1}}{(t+1)^{\frac{n}{2}}} dt.$$

It satisfies

$$0 \leq \mathcal{G}_{(-\Delta)_{\text{RFL}}^s}(x, y) \leq C(n, r)|x - y|^{-(n-2s)},$$

since the normalization constant is uniform as $s \rightarrow 1$ and an upper bound for the last integral is the one on $(0, +\infty)$, therefore, we have (8.7).

This formula is also valid when $s = 1$ and there is uniform convergence away from $x \neq y$. Hence, it is easy to see that (8.2) holds. Furthermore, we can compute

$$D_s \mathcal{G}_{(-\Delta)_{\text{RFL}}^s}(z, y) = \lim_{x \rightarrow z} \frac{\mathcal{G}_{(-\Delta)_{\text{RFL}}^s}(x, y)}{\delta(x)^s} = \lim_{x \rightarrow z} \frac{\mathcal{G}_{(-\Delta)_{\text{RFL}}^s}(x, y)}{(r - |x|)^s}$$

We can see that

$$\begin{aligned} \frac{1}{(r - |x|)^s} \int_0^{\frac{(r^2-|x|^2)(r^2-|y|^2)}{r^2|x-y|^2}} \frac{t^{s-1}}{(t+1)^{\frac{n}{2}}} dt &= \frac{(r^2 - |x|^2)^s}{(r - |x|)^s} \int_0^{\frac{r^2-|y|^2}{r^2|x-y|^2}} \frac{t^{s-1}}{((r^2 - |x|^2)t + 1)^{\frac{n}{2}}} dt \\ &\longrightarrow (2r)^s \int_0^{\frac{r^2-|y|^2}{r^2|z-y|^2}} t^{s-1} dt \end{aligned}$$

as $x \rightarrow z \in \partial B_r$. Hence

$$\begin{aligned} D_s \mathcal{G}_{(-\Delta)_{\text{RFL}}^s}(z, y) &= \frac{\Gamma(\frac{n}{2})}{2^s \Gamma^2(s) \pi^{\frac{n}{2}}} \frac{r^s}{|z - y|^{n-2s}} \int_0^{\frac{r^2-|y|^2}{r^2|z-y|^2}} t^{s-1} dt. \\ (8.9) \quad &= \frac{\Gamma(\frac{n}{2})}{2^s \Gamma^2(s) \pi^{\frac{n}{2}}} \frac{(r^2 - |y|^2)^s}{r^s |z - y|^n}. \end{aligned}$$

This formula for the s -normal derivative, which seems new in the literature, is also valid for $s = 1$. Indeed, as $s \rightarrow 1^-$, we have the uniform convergence in every compact set in B_r to the classical Poisson kernel (recall $|\mathbb{S}^{n-1}| = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$)

$$D_1 \mathcal{G}_{-\Delta}(z, y) = \frac{1}{|\mathbb{S}^{n-1}|} \frac{r^2 - |y|^2}{r|z - y|^n}, \quad y \in B_r, z \in \partial B_r.$$

Applying the Dominated Convergence Theorem we see that (8.4) holds. Therefore, we are in the correct setting at least when $\Omega = B_r$.

Comments on the general setting. In the general setting, some additional work is needed. Let us give some hints on a possible approach. To check (8.2) a possible scheme is as follows. First, use the Rayleigh quotient and Γ -convergence to check the convergence of the first eigenvalue

$$\lambda_1((-\Delta)_{\text{RFL}}^s) = \min_{0 \neq u \in H_0^s(\Omega)} \frac{\|(-\Delta)_{\text{RFL}}^{\frac{s}{2}} u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}$$

This shows strong L^2 convergence of subsequence of $\mathbb{G}_{(-\Delta)_{\text{RFL}}^s}(f)$. Since $(-\Delta)_{\text{RFL}}^s \varphi \rightarrow -\Delta \varphi$ for adequate test functions, it is easy to characterise the limit as the $\mathbb{G}_{-\Delta}(f)$. By uniqueness of the limit the whole sequence converges. Via the weak formulation and the uniform bounds, a rate of convergence can be recovered from that of $(-\Delta)_{\text{RFL}}^s \varphi$.

In [39], the authors prove that $\mathbb{G}_{(-\Delta)_{\text{RFL}}^s}(f)/\delta^s$ in $C^\alpha(\bar{\Omega})$. It is to be expected that the constants for this embedding are uniform for s close to 1, and hence have a uniformly convergent subsequence. By uniqueness of the limit one shows the whole sequence converges. This would imply that $D_s \mathbb{G}_{(-\Delta)_{\text{RFL}}^s}(\psi)$ uniformly converges to $D_1 \mathbb{G}_{(-\Delta)_{\text{RFL}}^s}(\psi)$, where D_1 is the standard normal derivative (as shown in Lemma B.1).

In order to check (8.4) (which is not studied in the main reference [1]), we can use the weak formulation

$$\int_{\Omega} \mathbb{M}_{(-\Delta)_{\text{RFL}}^s}(h) \psi = \int_{\partial \Omega} h D_s \mathbb{G}_{(-\Delta)_{\text{RFL}}^s}(\psi), \quad \forall \psi \in L_c^\infty(\Omega),$$

and some compactness to show that (8.4) holds. The proof of (8.5)–(8.6) can probably be done directly.

2. *Spectral Fractional Laplacian.* For this operator $\gamma(s) = 1$. Hence assumption (8.2) follows directly from the eigen-decomposition.

By [3], for instance,

$$\mathbb{G}_{(-\Delta)_{\text{SFL}}^s}(f)(x) = \int_{\Omega} \int_0^{+\infty} \mathcal{K}(t, x, y) f(y) \frac{t^{s-1}}{\Gamma(s)} dt dy,$$

where \mathcal{K} is the heat kernel for $u_t - \Delta$, which has the known estimates (see [34, Lemma 1.3])

$$\mathcal{K}(t, x, y) \leq \frac{C}{t^{\frac{n}{2}}} \exp\left(-\frac{|x - y|^2}{Ct}\right) \left(\frac{\delta(x)}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta(y)}{\sqrt{t}} \wedge 1\right).$$

Hence

$$\mathcal{G}_{(-\Delta)_{\text{SFL}}^s}(x, y) = \int_0^{+\infty} \mathcal{K}(t, x, y) \frac{t^{s-1}}{\Gamma(s)} dt$$

so hypothesis (8.7) holds.

This also holds for $s = 1$. Thus

$$\|\mathbb{G}_{(-\Delta)_{\text{SFL}}^s}(f) - \mathbb{G}_{-\Delta}(f)\|_{L^2(\Omega)}^2 = \int_{\Omega} \left(\int_{\Omega} \int_0^{+\infty} \mathcal{K}(t, x, y) f(y) \left(\frac{t^{s-1}}{\Gamma(s)} - 1 \right) dt dy \right)^2 dx.$$

From here (8.2) follows. Also, [3] shows that

$$\mathbb{M}_{(-\Delta)_{\text{SFL}}^s}(h)(x) = \int_{\partial\Omega} \int_0^{+\infty} -\frac{\partial \mathcal{K}(t, x, z)}{\partial \nu_z} h(z) \frac{t^{s-1}}{\Gamma(s)} dt dz.$$

Then

$$\begin{aligned} & \|\mathbb{M}_{(-\Delta)_{\text{SFL}}^s}(h) - \mathbb{M}_{-\Delta}(h)\|_{L^1(\Omega)} \\ & \leq \|h\|_{L^\infty(\partial\Omega)} \int_{\Omega} \int_{\partial\Omega} \int_0^{+\infty} \left| \frac{\partial \mathcal{K}(t, x, z)}{\partial \nu_z} \right| \left(\frac{t^{s-1}}{\Gamma(s)} - 1 \right) dt dz dx. \end{aligned}$$

From here it is not hard to deduce (8.4).

8.7. Some exotic examples. It is known for any non-negative bounded potential V , $\mathcal{L}_s = (-\Delta)_{\text{RFL}}^s + V(x)$ satisfies (A1), (A2) and (K1) (furthermore $\mathcal{G}_{\mathcal{L}_s} \leq \mathcal{G}_{(-\Delta)_{\text{RFL}}^s}$). If V is smooth, then (K2) also holds. However, moving the parameter s could lead to some strange behaviours. We could think about the family of operators

$$\mathcal{L}_s = (-\Delta)_{\text{RFL}}^s + \delta(x)^{-3} \wedge \frac{1}{1-s}.$$

These operators satisfy (K1), but not uniformly from below. The $\mathcal{L}_1 = -\Delta + \delta^{-3}$ where the solutions are flat $|u| \leq C \exp(-\frac{1}{\delta})$. The properties of the Green function of $\mathcal{L}_1 = -\Delta + \delta^{-3}$ are still not well understood.

COMMENTS, EXTENSIONS AND OPEN PROBLEMS

We collect here further issues that we would like to comment or propose.

- It could be interesting to find better properties of the Green's function for $\mathcal{L} - \lambda$.
- The theory could be applied to other examples of operators, e.g. the relativistic version $\sqrt{-\Delta + m^2}$.
- Schauder estimates should be found if \mathcal{G}_0 is smooth enough.
- Find versions of the main results when g, h are measures under certain conditions.
- Consider problems of the form

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \Omega, \\ u(x) \rightarrow +\infty & \text{as } x \rightarrow \partial\Omega, \\ u = 0 & \text{in } \Omega^c \text{ (if applicable)} \end{cases}$$

with suitable growth conditions on the nonlinearity f .

APPENDIX A. SOME COMMENTS ON COMPACTNESS AND FUNCTIONAL SPACES

Denoting by E_i the eigenspace corresponding to λ_i ($i \geq 1$), we have $\forall \varphi_j \in E_i$,

$$\begin{cases} \mathcal{L}\varphi_j = \lambda_i \varphi_j, & \text{in } \Omega \\ \varphi_j = 0 & \text{in } \overline{\Omega^c} \quad \text{or} \quad \text{on } \partial\Omega, \end{cases}$$

and for any $f \in L^2(\Omega)$,

$$f = \sum_{j \geq 1} \langle f, \varphi_j \rangle \varphi_j.$$

The solution operator in spectral form is given by

$$\mathbb{G}_0(f) = \sum_{j=1}^{+\infty} \frac{\langle f, \varphi_j \rangle}{\lambda_j} \varphi_j.$$

which is a well defined sum in $L^2(\Omega)$ since $\lambda_i \rightarrow +\infty$ and φ_j are orthonormal in L^2 so

$$\|\mathbb{G}_0(f)\|_{L^2(\Omega)}^2 = \sum_{j=1}^{+\infty} \frac{\langle f, \varphi_j \rangle^2}{\lambda_j^2}.$$

Therefore, defining $\mathbb{H}_{\mathcal{L}}^2(\Omega) = \mathbb{G}_0(L^2(\Omega))$ we easily see that

$$\mathbb{H}_{\mathcal{L}}^2(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j \geq 1} \lambda_j^2 \langle v, \varphi_j \rangle^2 < +\infty \right\}.$$

In fact, we can define *push-forward* norms for $k \geq 0$,

$$\|u\|_{\mathbb{H}_{\mathcal{L}}^k(\Omega)} = \sqrt{\sum_{j \geq 1} \lambda_j^k \langle u, \varphi_j \rangle^2},$$

so that $\|\mathbb{G}_0(f)\|_{\mathbb{H}_{\mathcal{L}}^2(\Omega)} = \|f\|_{L^2(\Omega)}$. Furthermore, $\mathcal{L} : \mathbb{H}_{\mathcal{L}}^2(\Omega) \rightarrow L^2(\Omega)$ is also an isometry. As in the theory of $-\Delta$, a natural energy space is

$$\mathbb{H}_{\mathcal{L}}^1(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j \geq 1} \lambda_j \langle v, \varphi_j \rangle^2 < +\infty \right\}.$$

This space was studied in [10]. Notice that

$$\int_{\Omega} \mathbb{G}_0(f) f = \sum_{j \geq 1} \frac{\langle f, \varphi_j \rangle^2}{\lambda_j} = \|\mathbb{G}_0(f)\|_{\mathbb{H}_{\mathcal{L}}^1}^2.$$

Therefore, for weak solutions

$$\|\mathbb{G}_0(f)\|_{\mathbb{H}_{\mathcal{L}}^1(\Omega)} \leq \|f\|_{L^2} \|\mathbb{G}_0(f)\|_{L^2}.$$

We always have that

$$\lambda_1 \|u\|_{L^2(\Omega)} \leq \|u\|_{\mathbb{H}_{\mathcal{L}}^1(\Omega)}$$

hence, since $\lambda_1 > 0$ then we can call this Poincar inequality and deduce that

$$\|\mathbb{G}_0(f)\|_{L^2(\Omega)} \leq \frac{1}{\lambda_1} \|f\|_{L^2(\Omega)}.$$

In [9] the authors prove that this operator \mathbb{G}_0 is compact in $L^2(\Omega)$, by a sharp application of the Riesz–Fréchet–Kolmogorov theorem. By definition, $\mathbb{H}_{\mathcal{L}}^2(\Omega) = \mathbb{G}_0(L^2(\Omega))$ with its norm is compactly embedded in $L^2(\Omega)$. However, since the authors of [9] estimate the translations $\|\tau_h \mathbb{G}_0(f) - \mathbb{G}_0(f)\|_{L^2}$ without rates, there is no estimate of $\mathbb{H}_{\mathcal{L}}^2(\Omega)$ in any of the Sobolev spaces $W^{t,2}(\Omega)$.

The question of whether $\mathbb{H}_{\mathcal{L}}^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ is left open.

The readers are directed to [10] for a comprehensive introduction to both RFL and SFL operators in Sobolev spaces. Note that in both cases, the natural domain is identified as⁸

$$\mathbb{H}_{(-\Delta)_{\text{SFL}}^s}^1(\Omega) = \mathbb{H}_{(-\Delta)_{\text{RFL}}^s}^1(\Omega) = \begin{cases} H^s(\Omega) & \text{for } s \in (0, \frac{1}{2}), \\ H_{00}^{\frac{1}{2}}(\Omega) & \text{for } s = \frac{1}{2}, \\ H_0^s(\Omega) & \text{for } s \in (\frac{1}{2}, 1). \end{cases}$$

Note that exponent 1 in left-hand side becomes s in the right-hand side. Similar, the $\mathbb{H}_{(-\Delta)_{\text{SFL}}^s}^2(\Omega)$ are related to spaces of type H^{2s} .

APPENDIX B. THE WEIGHTED TRACE FOR THE RESTRICTED FRACTIONAL LAPLACIAN

Let $\mathcal{L} = (-\Delta)_{\text{RFL}}^s$. Recall the definition of the weighted trace operator

$$Bu(z) = \lim_{\Omega \ni x \rightarrow z} \frac{u(x)}{\mathbb{M}(1)(x)} \quad \text{for } z \in \partial\Omega.$$

Recall from [1, 4] that $\delta^{1-s}\mathbb{M}(1)$ is a positive, continuous function, bounded away from 0 and $+\infty$. Thanks to the connection pointed out by Ros-Oton, we show that it is actually a constant, independent of the domain.

Lemma B.1. *Let $v \in \delta^{s-1}C(\overline{\Omega})$. For any $z \in \partial\Omega$,*

$$Bv(z) = \Gamma(1+s)^2 \lim_{\Omega \ni x \rightarrow z} \delta^{1-s}(x)v(x).$$

We remark that as $s \nearrow 1$, we recover the usual trace.

Proof. Let $u \in (-\Delta)^{-s}C_c^\infty(\Omega)$. Recall that the integration-by-parts formula [1] applied to the functions u and $x \cdot \nabla u$ reads

$$\begin{aligned} \int_{\partial\Omega} E(x \cdot \nabla u)(z) \frac{u}{\delta^s} d\mathcal{H}^{n-1}(z) \\ = \int_{\Omega} (x \cdot \nabla u)(-\Delta)_{\text{RFL}}^s u \, dx - \int_{\Omega} u(-\Delta)_{\text{RFL}}^s (x \cdot \nabla u) \, dx. \end{aligned}$$

⁸Here $H_{00}^{\frac{1}{2}}(\Omega)$ is the Lions–Magènes spaces [36].

On the other hand, by the Pohožaev identity [40],

$$\begin{aligned} & \int_{\Omega} (x \cdot \nabla u) (-\Delta)_{\text{RFL}}^s u \, dx \\ &= -\frac{n-2s}{2} \int_{\Omega} u (-\Delta)_{\text{RFL}}^s u \, dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (z \cdot \nu) \, d\mathcal{H}^{n-1}(z). \end{aligned}$$

Using this identity together with the pointwise relation

$$(-\Delta)_{\text{RFL}}^s (x \cdot \nabla u) = x \cdot \nabla (-\Delta)_{\text{RFL}}^s u + 2s (-\Delta)_{\text{RFL}}^s u,$$

we have

$$\begin{aligned} & \int_{\Omega} u (-\Delta)_{\text{RFL}}^s (x \cdot \nabla u) \, dx \\ &= -\frac{n-2s}{2} \int_{\Omega} u (-\Delta)_{\text{RFL}}^s u \, dx + \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (z \cdot \nu) \, d\mathcal{H}^{n-1}(z). \end{aligned}$$

Combining,

$$(B.1) \quad \int_{\partial\Omega} E(x \cdot \nabla u)(z) \frac{u}{\delta^s} \, d\mathcal{H}^{n-1}(z) = -\Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (z \cdot \nu) \, d\mathcal{H}^{n-1}(z).$$

In a tubular neighborhood of $\partial\Omega$, choose an orthonormal frame $\nu(x), e_1(x), \dots, e_{n-1}(x)$, so that $\nu(x)$ agrees with the outward normal. We have for $z \in \partial\Omega$,

$$\begin{aligned} E(x \cdot \nabla u)(z) &= \lim_{\Omega \ni x \rightarrow z} \frac{x \cdot \nabla u(x)}{\mathbb{M}(1)(x)} \\ &= \lim_{\Omega \ni x \rightarrow z} \frac{(x \cdot \nu(x))(\nabla u(x) \cdot \nu(x)) + (x \cdot e_k(x))(\nabla u(x) \cdot e_k(x))}{\mathbb{M}(1)(x)}. \end{aligned}$$

Note that

$$\lim_{\Omega \ni x \rightarrow z} (x \cdot \nu(x)) = z \cdot \nu(z), \quad \limsup_{\Omega \ni x \rightarrow z} |x \cdot e_k(x)| \leq C,$$

where $\nu(z)$ is the outward normal at $z \in \partial\Omega$. Then, since

$$\nabla u(x) \cdot \nu(x) = \lim_{h \rightarrow 0} \frac{u(x + h\nu(x)) - u(x)}{h}, \quad \nabla u(x) \cdot e_k(x) = \lim_{h \rightarrow 0} \frac{u(x + he_k(x)) - u(x)}{h},$$

exist, one can take the particular sequence $h = \delta(x)$, since $\delta(x) \rightarrow 0$ as $x \rightarrow z \in \partial\Omega$. Recall that by [1, Lemma 3.4], $\delta^{1-s}\mathbb{M}(1)$ is bounded between positive constants. Also, by [40] u/δ^s can be extended to a Hölder continuous function in $\overline{\Omega}$. We estimate the

tangential terms by

$$\begin{aligned}
& \limsup_{\Omega \ni x \rightarrow z} \left| \frac{(x \cdot e_k(x))(\nabla u(x) \cdot e_k(x))}{\mathbb{M}(1)(x)} \right| \\
& \leq C \limsup_{\Omega \ni x \rightarrow z} \left| \delta(x)^{1-s} \frac{u(x + \delta(x)e_k(x)) - u(x)}{\delta(x)} \right| \\
& \leq C \limsup_{\Omega \ni x \rightarrow z} \left| \frac{u(x + \delta(x)e_k(x))}{\delta^s(x + \delta(x)e_k(x))} \frac{\delta^s(x + \delta(x)e_k(x))}{\delta^s(x)} - \frac{u(x)}{\delta^s(x)} \right| \\
& \leq C \left| \frac{u}{\delta^s}(z) \cdot 1 - \frac{u}{\delta^s}(z) \right| \\
& = 0.
\end{aligned}$$

On the other hand, we notice that

$$x + \delta(x)\nu(x) \in \partial\Omega \implies u(x + \delta(x)\nu(x)) = 0.$$

Therefore

$$\begin{aligned}
E(x \cdot \nabla u)(z) &= (z \cdot \nu(z)) \lim_{\Omega \ni x \rightarrow z} \frac{0 - u(x)}{\delta(x)\mathbb{M}(1)(x)} \\
&= -(z \cdot \nu(z)) \frac{u}{\delta^s}(z) \lim_{\Omega \ni x \rightarrow z} \frac{1}{\delta^{1-s}(x)\mathbb{M}(1)(x)}
\end{aligned}$$

Plugging this back into (B.1), we have

$$\begin{aligned}
& \int_{\partial\Omega} \left(\lim_{\Omega \ni x \rightarrow z} \frac{1}{\delta^{1-s}(x)\mathbb{M}(1)(x)} \right) \left(\frac{u}{\delta^s} \right)^2 (z \cdot \nu) d\mathcal{H}^{n-1}(z) \\
&= \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (z \cdot \nu) d\mathcal{H}^{n-1}(z).
\end{aligned}$$

By [1, Proposition 2], the limit

$$a(z) := \lim_{\Omega \ni x \rightarrow z} \frac{1}{\delta^{1-s}(x)\mathbb{M}(1)(x)}$$

is well-defined and continuous. Since the Pohožaev identity can be applied with any center $x_0 \in \mathbb{R}^n$, we can write

$$(B.2) \quad \int_{\partial\Omega} (a(z) - \Gamma(1+s)^2) \left(\frac{u}{\delta^s} \right)^2 ((z - x_0) \cdot \nu) d\mathcal{H}^{n-1}(z) = 0.$$

We will show that $a(z) \equiv \Gamma(1+s)^2$ by a contradiction argument choosing appropriate u . For any $x_1 \in \Omega$, we choose $u_\varepsilon(x) = (-\Delta)^{-s} \eta_\varepsilon(x - x_1)$ (where $\varepsilon < \delta(x_1)$) where $\eta_\varepsilon \in C_c^\infty(\Omega)$ is the standard mollifier centered at x_1 . Then by Proposition 3.10, we have

$$\frac{u_\varepsilon}{\delta^s}(z) = D_s u_\varepsilon(z) = \int_{\Omega} D_s \mathcal{G}_0(z, y) \eta_\varepsilon(y - x_1) dy.$$

Using (B.2) with $u = u_\varepsilon$,

$$\int_{\partial\Omega} (a(z) - \Gamma(1+s)^2) \left(\int_{\Omega} D_s \mathcal{G}_0(z, y) \eta_\varepsilon(y - x_1) dy \right)^2 ((z - x_0) \cdot \nu) d\mathcal{H}^{n-1}(z) = 0.$$

Taking $\varepsilon \searrow 0$,

$$(B.3) \quad \int_{\partial\Omega} (a(z) - \Gamma(1+s)^2) (D_s \mathcal{G}_0(z, x_1))^2 ((z - x_0) \cdot \nu) d\mathcal{H}^{n-1}(z) = 0,$$

for any $z \in \Omega$. Recall that

$$D_s \mathcal{G}_0(z, x_1) \asymp \frac{\delta^s(x_1)}{|x_1 - z|^n}.$$

Suppose $a(z_0) \neq \Gamma(1+s)^2$. Then there exists a neighborhood $z_0 \in \omega \subset \partial\Omega$ such that $a \neq \Gamma(1+s)^2$ on ω . Moreover, one can choose x_0 such that $(z - x_0) \cdot \nu \neq 0$ for $z \in \omega$. Dividing (B.3) by $\delta^{2s}(x_1)$, we have

$$\int_{\omega} \frac{|a(z) - \Gamma(1+s)^2| |(z - x_0) \cdot \nu|}{|x_1 - z|^{2n}} d\mathcal{H}^{n-1}(z) \leq C(\omega),$$

a contradiction as $x_1 \rightarrow z_0$. Therefore $a(z) \equiv \Gamma(1+s)^2$ and the proof is complete. \square

APPENDIX C. EMBEDDINGS INTO MORREY SPACES

Since we mention a special case of the regularity results of Fall [24], we indicate the corresponding embedding results into Morrey spaces. Recall that the Morrey space \mathcal{M}_β , $\beta \in [0, n]$, is defined by

$$(C.1) \quad \mathcal{M}_\beta(\Omega) = \left\{ f \in L^1(\Omega) : \|f\|_{\mathcal{M}_\beta(\Omega)} := \sup_{r \in (0,1), x \in \Omega} r^{\beta-n} \int_{B_r(x) \cap \Omega} |f(y)| dy < \infty \right\}$$

Lemma C.1 (High integrability data). *For any $p \in (\frac{n}{s}, \infty]$,*

$$L^p(\Omega) \hookrightarrow \mathcal{M}_\beta(\Omega)$$

for $\beta = \frac{n}{p} \in [0, s)$.

Proof. This is a direct consequence of Hölder inequality. Indeed, for any $B_r(x)$ with $x \in \Omega$,

$$\int_{B_r(x) \cap \Omega} |f| dx \leq \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} |B_r(x)|^{\frac{p-1}{p}} \leq C \|f\|_{L^p(\Omega)} r^{n-\frac{n}{p}},$$

for finite p , and the same is true when $p = \infty$. \square

It is also instructive to observe how the weighted L^∞ functions are embedded into Morrey spaces. This leads directly to regularity properties for the inhomogeneous eigenvalue problem for RFL, with the large RFL-harmonic function as right hand side.

Lemma C.2 (Weighted L^∞ data). *For $\beta \in [0, 1)$, we have the continuous inclusion*

$$\delta^{-\beta} L^\infty(\Omega) \hookrightarrow \mathcal{M}_\beta(\Omega)$$

Moreover, there holds

$$\|f\|_{\mathcal{M}_\beta(\Omega)} \leq C \left(\|f \delta^\beta\|_{L^\infty(\Omega)} + \|f\|_{L^1(\Omega)} \right),$$

whenever the right hand side is finite.

Proof. Let $C_0 = \|f\delta^\beta\|_{L^\infty(\Omega)} + \|f\|_{L^1(\Omega)}$. We want to show that

$$\int_{B_r(x) \cap \Omega} |f(y)| dy \leq CC_0 r^{n-\beta},$$

for any $0 < r < \text{diam}(\Omega)$ and $x \in \Omega$. Indeed, we distinguish between three cases as follows.

- (1) If $r < \delta(x)/2$, then $B_r(x) \subset B_{2r}(x) \subset \Omega$ and for any $y \in B_r(x)$, $\delta(y) \geq \delta(x)/2 > r$, so that

$$\int_{B_r(x) \cap \Omega} |f(y)| dy \leq CC_0 \int_{B_r(x)} \delta(y)^{-\beta} dy \leq CC_0 r^{-\beta} |B_r(x)|.$$

- (2) In a small enough neighborhood of $\partial\Omega$, the Fermi coordinates are well-defined and used to flatten the boundary. More precisely, we write a subset of $\partial\Omega$ as the image of a $C^{1,1}$ function

$$\psi : B'_{2r} \subset \mathbb{R}^{n-1} \rightarrow \partial\Omega,$$

with $\psi(0) = \arg \min_{x_0 \in \partial\Omega} \text{dist}(x, x_0)$. Then we define

$$\begin{aligned} \Psi : B'_{2r} \times (0, 4r) &\rightarrow \{\delta < 4r\} \subset \Omega \\ \Psi(z', z_n) &= \psi(z') - z_n \nu(\psi(z')), \end{aligned}$$

where $\nu(\psi(z'))$ is the outward normal of Ω at $\psi(z') \in \partial\Omega$.

If $r < r_1$ and r_1 is fixed sufficiently small, then $|\det \Psi(z', z_n)| \leq C$ for all $(z', z_n) \in B'_{2r} \times (0, 4r)$. By Area Formula,

$$\begin{aligned} \int_{B_r(x) \cap \Omega} |f(y)| dy &\leq \int_{\Psi(B'_{2r} \times (0, 4r))} C_0 \delta(y)^{-\beta} dy \\ &\leq CC_0 \int_{B'_{2r}} \int_0^{4r} z_n^{-\beta} dz_n dz' \\ &\leq CC_0 r^{n-\beta}. \end{aligned}$$

- (3) Since $f \in L^1(\Omega)$, if $r \geq r_1$, then

$$\int_{B_r(x) \cap \Omega} |f(y)| dy \leq Cr_1^{n-\beta} \int_{\Omega} |f(y)| dy \leq CC_0 r^{n-\beta}.$$

By taking supremum over r and x , we conclude that

$$\|f\|_{\mathcal{M}_\beta(\Omega)} \leq CC_0. \quad \square$$

ACKNOWLEDGEMENTS

H.C. has received funding from the European Research Council under the Grant Agreement No 721675. He acknowledges the kind hospitality received in the Universidad Aut3noma de Madrid during his visit in January 2020. The work of D.G-C. and JLV was funded by grant PGC2018-098440-B-I00 from the Spanish Government. J. L. V3zquez is also an Honorary Professor at Univ. Complutense de Madrid. H.C. is grateful to Xavier Ros-Oton for pointing out the connection between the weighted trace operator

and the Pohožaev identity, and to Yannick Sire for a helpful comment on interpolation spaces. H.C. also indebted to Alessio Figalli for an enlightening comment concerning regularity and for motivating encouragements.

REFERENCES

- [1] N. Abatangelo. Large s -harmonic functions and boundary blow-up solutions for the fractional Laplacian. *Discrete & Continuous Dynamical Systems - A*, 2015, 35 (12) : 5555–5607.
- [2] N. Abatangelo. Very large solutions for the fractional Laplacian: towards a fractional Keller-Osserman condition. *Adv. Nonlinear Anal.* 6 (2017), no. 4, 383–405.
- [3] N. Abatangelo, L. Dupaigne. Nonhomogeneous boundary conditions for the spectral fractional Laplacian. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34 (2017), no. 2, 439–467.
- [4] Nicola Abatangelo, David Gómez-Castro, Juan Luis Vázquez, Singular boundary behaviour and large solutions for fractional elliptic equations, arXiv 1910.00366.
- [5] W. Ao, H. Chan, A. DelaTorre, M.A. Fontelos, M.d.M. González, J. Wei. On higher-dimensional singularities for the fractional Yamabe problem: a nonlocal Mazzeo-Pacard program. *Duke Math. J.* 168 (2019), no. 17, 3297–3411.
- [6] Ben Chrouda, Mohamed; Ben Fredj, Mahmoud. Blow up boundary solutions of some semilinear fractional equations in the unit ball. *Nonlinear Anal.* 140 (2016), 236–253.
- [7] Bnilan, P., Brezis, H., Crandall, M. G. A semilinear equation in $L^1(R^N)$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 2 (1975), no.4, 523555.
- [8] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, and Z. Vondraček. *Potential analysis of stable processes and its extensions*. Springer, Berlin, 2009.
- [9] M. Bonforte, A. Figalli, J.L. Vázquez. Sharp boundary behaviour of solutions to semilinear non-local elliptic equations. *Calc. Var. Partial Differential Equations* 57 (2018), no. 2, Art. 57, 34 pp.
- [10] M. Bonforte, Y. Sire, J.L. Vázquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, *Discrete Contin. Dyn. Syst. Ser. A* 35 (12) (2015) 5725–5767.
- [11] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa. Blow up for $u_t - \Delta u = g(u)$ revisited. *Adv. Differential Equations* 1 (1996), no. 1, 73–90.
- [12] C. Bucur. Some observations on the Green function for the ball in the fractional Laplace framework. *Communications on Pure and Applied Analysis*, 15 (2) (2016), 657–699.
<https://doi.org/10.3934/cpaa.2016.15.657>
- [13] C. Bucur, A.L. Karakhanian. Potential theoretic approach to Schauder estimates for the fractional Laplacian. *Proc. Amer. Math. Soc.* 145 (2017), no. 2, 637–651.
- [14] X. Cabré, J. Tan. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.* 224 (2010), no. 5, 2052–2093.
- [15] L.A. Caffarelli, Y. Sire. *On some pointwise inequalities involving nonlocal operators*. Harmonic analysis, partial differential equations and applications, 118, *Appl. Numer. Harmon. Anal.*, Birkhuser/Springer, Cham, 2017.
- [16] L. Caffarelli, P. R. Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33 (2016), no. 3, 767–807.
- [17] H. Chan, A. DelaTorre. An analytic construction of singular solutions related to a critical Yamabe problem. arXiv:1912.10352.
- [18] Z.-Q. Chen, Multidimensional symmetric stable processes. *Korean J. Comput. Appl. Math.* 6 (1999), no. 2, 227–266.
- [19] H. Chen, P. Felmer, A. Quaas. Large solutions to elliptic equations involving fractional Laplacian. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32 (2015), no. 6, 1199–1228.
- [20] H. Chen, L. Véron. Semilinear fractional elliptic equations involving measures. *J. Differential Equations* 257 (2014), no. 5, 1457–1486.

- [21] J. Dávila, M. del Pino, M. Medina, R. Rodiac. Interacting helical vortex filaments in the 3-dimensional Ginzburg-Landau equation. arXiv:1901.02807.
- [22] J. I. Díaz, D. Gómez-Castro, J. L. Vázquez. The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach. *Nonlinear Anal.* 177 (2018), part A, 325–360.
- [23] S. Dumont, L. Dupaigne, O. Goubet, V. Rădulescu. Back to the Keller-Osserman condition for boundary blow-up solutions. *Adv. Nonlinear Stud.* 7 (2007), no. 2, 271–298.
- [24] M.M. Fall. Regularity estimates for nonlocal Schrödinger equations. *Discrete Contin. Dyn. Syst.* 39 (2019), no. 3, 1405–1456.
- [25] P. Felmer, A. Quaas. Boundary blow up solutions for fractional elliptic equations. *Asymptot. Anal.* 78 (2012), no. 3, 123–144.
- [26] Rui A. C. Ferreira. Anti-symmetry of the second eigenfunction of the fractional Laplace operator in a 3-D ball. arXiv:1706.04960.
- [27] A. Figalli, J. Serra. On stable solutions for boundary reactions: a De Giorgi-type result in dimension $4 + 1$. *Invent. Math.* 219 (2020), no. 1, 153–177.
- [28] Gmez-Castro, D., Vzquez, J. L. The fractional Schrödinger equation with singular potential and measure data. *Discrete & Continuous Dynamical Systems - A* 39 (2019) no. 12, 71137139.
- [29] A. Greco, R. Servadei, Hopf’s lemma and constrained radial symmetry for the fractional Laplacian. *Math. Res. Lett.* 23 (2016), no. 3, 863–885.
- [30] Hmissi, F. Fonctions harmoniques pour les potentiels de Riesz sur la boule unit. *Expositiones Mathematicae*, 12 (1994) no. 3, 281288.
- [31] H. Hanche-Olsen, H. Holden. The Kolmogorov-Riesz compactness theorem. *Expo. Math.* 2010; 28:385-394.
- [32] T. Kato, *Perturbation Theory for Linear Operators*, Springer (1980).
- [33] J. Keller. On solutions of $\Delta u = f(u)$. *Comm. Pure Appl. Math.* 10 (1957), 503–510.
- [34] Hui KM. A Fatou Theorem for the Solution of the Heat Equation at the Corner Points of a Cylinder. *Trans. Am. Math. Soc.* (1992);333:607.
- [35] E. Lieb, M. Loss, *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001. xxii+346 pp. ISBN: 0-8218-2783-9.
- [36] J.-L. Lions, E. Magenes. *Problèmes aux limites non homogènes et applications*. Vol. 1. (French) *Travaux et Recherches Mathématiques*, No. 17 Dunod, Paris 1968 xx+372 pp.
- [37] R. Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific J. Math.* 7 (1957), 1641–1647.
- [38] X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.* 60 (2016), no. 1, 3–26.
- [39] X. Ros-Oton, J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.* (9) 101 (2014), no. 3, 275–302.
- [40] X. Ros-Oton, J. Serra. The Pohozaev identity for the fractional Laplacian, *Arch. Rat. Mech. Anal.* 213 (2014), 587–628.
- [41] X. Ros-Oton, J. Serra. Regularity theory for general stable operators. *J. Differential Equations* 260 (2016), no. 12, 8675–8715.
- [42] R. Servadei, E. Valdinoci. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst.* 33 (2013), no. 5, 2105–2137.
- [43] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* 60 (2007), no. 1, 67–112.
- [44] R. Song, Z. Vondraček. Potential theory of subordinate killed Brownian motion in a domain. *Probab. Theory Related Fields* 125 (2003), no. 4, 578–592.
- [45] Ph. Souplet, Optimal regularity conditions for elliptic problems via L^p_δ -spaces. *Duke Math. J.* 127 (2005), no. 1, 175–192.
- [46] E.M. Stein, *Singular integrals and Differentiability Properties of Functions* (1970). Princeton: Princeton University Press.

(H. Chan) DEPT. OF MATHEMATICS, ETH ZÜRICH, RÄMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND

E-mail address: `hardy.chan@math.ethz.ch`

URL: `https://people.math.ethz.ch/~hchan/`

(D. Gómez-Castro) INSTITUTO DE MATEMÁTICA INTERDISCIPLINAR, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN

E-mail address: `dgcastro@ucm.es`

URL: `http://blogs.mat.ucm.es/dgcastro/`

(J. L. Vázquez) DEPTO. DE MATEMÁTICAS, UNIV. AUTÓNOMA DE MADRID (UAM), 28049 MADRID, SPAIN

E-mail address: `juanluis.vazquez@uam.es`

URL: `http://verso.mat.uam.es/~juanluis.vazquez/`