CHAPTER 5

Voting

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Voting is not as simple as it may seem. For example, in a tournament with at least three candidates *A*, *B*, and *C*, and at least three voters, *A* can win a majority against *B*, who in turn wins a majority against *C*, who wins a majority against *A*. This situation was pointed out by the Marquis de Condorcet in 1785,[1] but his work was lost for almost a century. In 1876 it was given its modern name of a 'cycle' by C. L. Dodgson.[2] Dodgson's work on the mathematics of voting then fell into neglect for a century until it was rediscovered by Duncan Black and other writers.

Voting theory before Dodgson

Dodgson is one of the great figures in the axiomatic theory of voting, which is the study of cycling and other paradoxes and the properties of choice systems. His only peers are Ramon Llull (*c.*1235–1315), Nicholas of Cusa (1401–64), the Marquis de Condorcet (1743–94), Jean-Charles de Borda (1733–99), and (more recently) Duncan Black (1908–91) and Kenneth J. Arrow (1920–2017).[3]

Several of these thinkers were, like Dodgson, strikingly eccentric. All, like Dodgson, saw that the simplest version of majority rule is seriously flawed. This version is the one used in parliamentary elections in the UK, the USA, India, and Canada, but hardly anywhere outside the former British Empire, and known in the UK as 'first-past-the-post'. To elect a single person where there are two or more candidates, you simply choose the

The Mathematical World of Charles L. Dodgson (Lewis Carroll). Robin Wilson and Amirouche Moktefi.
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one with the most votes, even when they add up to fewer than half of the votes cast. As Dodgson observed, it is easy to demonstrate the 'extraordinary injustice' of this method: a candidate may win a plurality of votes – that is, more than any of the others – while being ranked last by more than half of the voters. Here is Dodgson's example (Table 1):[4]

> Let us suppose that there are eleven electors, and four candidates, *a, b, c, d*; and that each elector has arranged in a column the names of the candidates, in the order of his preference; and that the eleven columns stand thus:–
>
> | | | | | | | | | | | |
> |---|---|---|---|---|---|---|---|---|---|---|
> | *a* | *a* | *a* | *b* | *b* | *b* | *b* | *c* | *c* | *c* | *d* |
> | *c* | *c* | *c* | *a* | *a* | *a* | *a* | *a* | *a* | *a* | *a* |
> | *d* | *d* | *d* | *c* | *c* | *c* | *c* | *d* | *d* | *d* | *c* |
> | *b* | *b* | *b* | *d* | *d* | *d* | *d* | *b* | *b* | *b* | *b* |
>
> Here *a* is considered best by *three* of the electors, and second by all the rest. It seems clear that he ought to be elected; and yet, by the above method, *b* would be the winner – a candidate who is considered *worst* by *seven* of the electors!

Ramon Llull was the first person to see this difficulty, and also to propose solutions. He considered the case of a monastic house where the monks or nuns must elect their own leader. As we shall see, as the first Westerner to apply algebra to voting, he invented a matrix notation for tournaments in which each candidate is compared with each other in turn.[5]

The only known copy of one of Llull's papers was made by the next great figure, Nicolaus Cusanus (Nicholas of Cusa), whose scheme differed from Llull's. Cusanus proposed that Electors to the position of Holy Roman Emperor (who were then mostly German princes) should hand in anonymous ballot papers, marking 1 for the candidate they least wanted, up to 10 (if there were ten candidates) for their favourite. We now know this as the *Borda count*, from its second proposer over three centuries later.

During the Revolutionary years Borda and Condorcet were contemporaries, but enemies, in the French Academy of Science: Borda survived, while Condorcet died in the Terror of 1794. While Borda again proposed the rank-order count, Condorcet observed that, although simple, it could produce perverse results. Borda himself found out one of his scheme's perversities: when used for electing new members of the Academy of Sciences, his scheme was manipulated by voters placing last the most dangerous rival to their favourite: 'My scheme is intended only for honest men', he said plaintively.[6]

Instead of the Borda rank-order count, Condorcet proposed a scheme to look for what is now called the *Condorcet winner*: the candidate who beats each of the others in the exhaustive pairwise tournament first proposed by Llull. Unfortunately, as he went on to discover, there is no Condorcet winner when there is a cycle at the top, because each


Ramon Llull and Nicolaus Cusanus


The Marquis de Condorcet and Jean-Charles de Borda

candidate loses to at least one other. Condorcet had a tie-break procedure, but it was not understood for a further two centuries.[7]

Dodgson and voting theory

Rather unusually in intellectual history, we can be sure that none of this prior work was known to Dodgson in December 1873 when he wrote his first seminal paper, *A Discussion of the Various Methods of Procedure in Conducting Elections*.[8] Llull and Cusanus were not rediscovered in this context until the 1980s.[9] As for Borda and Condorcet, Christ Church Library in Oxford has a run of the proceedings of the *Académie Royale des Sciences*

dating back to its 17th-century foundation, but the pages in the 1781 volume that contain Borda's method are uncut: nobody has ever read them in their copy.[10] But whereas the Christ Church Library evidence is still there, another piece of evidence, discovered in the 1940s by Duncan Black, has disappeared. Christ Church has no copy of Condorcet's *Essai* of 1785, but Oxford's main university library, the Bodleian Library, has, and in Black's time some of its pages, too, were not cut.[11] (They have been now.)

The occasion for Dodgson's 1873 paper was the election of two new Students of Christ Church. Dodgson reviewed methods used for such elections and proved that all those in common use were faulty. His summary dismissal of 'first-past-the-post' was presented in Table 1.

One important point that Dodgson made was that 'no election' should also be treated as if it were the name of a candidate. The processes of starting with the question 'Should we elect anyone?', and of finishing with 'Should we elect the candidate who has won the most votes?', were both unfair to any voter who preferred one set of candidates to 'no election' but who preferred 'no election' to another set of candidates. This simple insight of Dodgson's is still too rarely understood by appointing committees.

Dodgson went on to discuss a knockout tournament, which he called the 'method of elimination'. He proved that it may lead to the 'preposterous' conclusion that the result of an election can depend on the order in which pairs of candidates are voted on, and gave a four-candidate example in which the chance of different pairs being drawn first could lead to any of the four winning.

He next discussed the 'method of marks', in which the voters have a fixed number of points to distribute among the candidates as they wish. Using reasoning that would later be considered as *game theory*, Dodgson showed that this would very quickly degenerate into simple plurality voting, which (as he had shown earlier) can elect an absolute majority loser:[12]

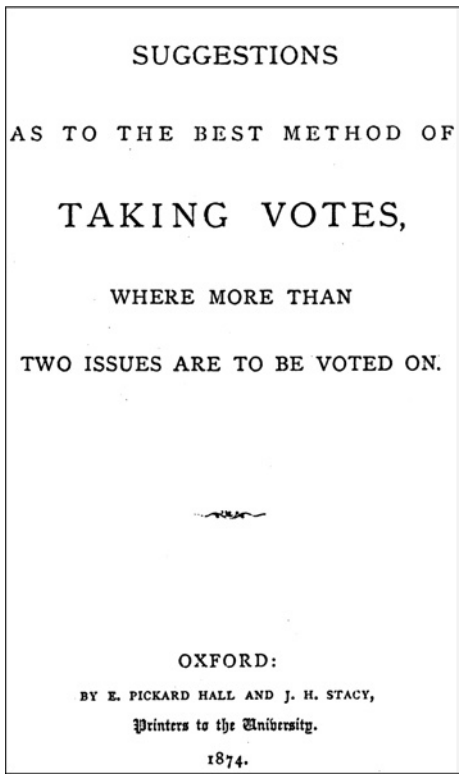
> Each elector would feel that it was *possible* for each other elector to assign the entire number of marks to his favourite candidate, giving to all the other candidates zero: and he would conclude that, in order to give his *own* favourite candidate any chance of success, he must do the same for him.

This was an unknowing echo of Borda's 'My scheme is intended only for honest men', which is ironic because Dodgson went on to recommend precisely the Borda count, with 'no election' treated as if it were a candidate.

The college did indeed use Dodgson's modified Borda procedure, but a rather disturbing thing happened. The voting records, discovered by Duncan Black, show that the Borda winner (Becker) received 48 points, but lost a direct pairwise vote against the

Borda runner-up (Baynes), who received 47 points.[13] The college elected Baynes, and Dodgson found that his Borda count would have elected a Condorcet non-winner.

As well as electing candidates to fellowships, the college had to decide on a new belfry for the south-east corner of its main quadrangle, in a sensitive site between the cathedral and the Great Hall. Henry Liddell, Dean of Christ Church, was an architectural enthusiast who liked designing belfries. Opinion was sharply divided, in more than one direction (materials?/look?/size?), and Dodgson was openly scornful of Liddell's ideas. Immediately before a meeting of the Governing Body to discuss belfries, which (as Black discovered) was to last five hours, Dodgson rushed out his second pamphlet, *Suggestions As to the Best Method of Taking Votes, Where More Than Two Issues Are to Be Voted On*.[14] He rescinded his support for the Borda count, and instead proposed an exhaustive pairwise vote – in other words, a Condorcet comparison – reserving the Borda count for the case where the Condorcet comparison failed to produce a clear answer.


SUGGESTIONS

AS TO THE BEST METHOD OF

TAKING VOTES,

WHERE MORE THAN

TWO ISSUES ARE TO BE VOTED ON.

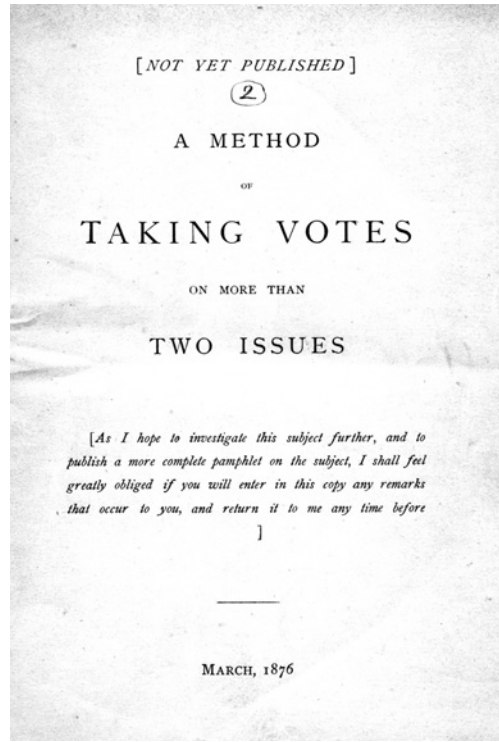
OXFORD:

BY E. PICKARD HALL AND J. H. STACY,

Printers to the University.

1874.

Title page of *Suggestions As to the Best Method of Taking Votes*, 1874


[*NOT YET PUBLISHED*]

(2)

A METHOD

OF

TAKING VOTES

ON MORE THAN

TWO ISSUES

[*As I hope to investigate this subject further, and to publish a more complete pamphlet on the subject, I shall feel greatly obliged if you will enter in this copy any remarks that occur to you, and return it to me any time before*

]

MARCH, 1876

Preprint version of *A Method of Taking Votes*, 1876

[Courtesy of Edward Wakeling]

Dodgson went on thinking about these matters. His third pamphlet, *A Method of Taking Votes on More Than Two Issues*, appeared in March 1876. It was not produced in the shadow of an urgent vote, and is all the better for that. It starts with a set of Dodgsonian rules – very precise and intricate, and mystifying to anyone not as steeped in the subject as the author. An *absolute majority winner* is a candidate who (or an option which) wins the first preference of more than half of the voters. As before, Dodgson first proposed that if one outcome is an absolute majority winner then it should be selected immediately.

The next step is to check whether a Condorcet winner exists, by conducting a full pairwise tournament. If there is a winner, then the process stops. If not, then there must be a 'top cycle', which might embrace all the options. Any options that are not in the top cycle may be discarded. To break the cycle, Dodgson had a new proposal:[15]

> When the issues to be further debated consist of, or have been reduced to, a single cycle, the Chairman shall inform the meeting how many alterations of votes each issue requires to give it a majority over every other separately.

Armed with this information the voters vote again, to see whether that breaks the cycle. If not, then there is no election.

To some people the existence of cycles is obvious; to others it comes as a great surprise. Here is an example of Dodgson's (Table 2):[16]

> Let us suppose that there are 11 electors, and 4 candidates, *a*, *b*, *c*, *d*; and that each elector has arranged in a column the names of the candidates in the order of his preference; and that the 11 columns stand thus:–
>
> | | | | | | | | | | | |
> |---|---|---|---|---|---|---|---|---|---|---|
> | *a* | *a* | *a* | *a* | *b* | *b* | *b* | *c* | *c* | *c* | *d* |
> | *d* | *d* | *b* | *b* | *c* | *c* | *d* | *b* | *b* | *b* | *c* |
> | *c* | *c* | *d* | *d* | *a* | *a* | *c* | *d* | *d* | *d* | *b* |
> | *b* | *b* | *c* | *c* | *d* | *d* | *a* | *a* | *a* | *a* | *a* |
>
> Here the majorities are cyclical, in the order *a d c b a*, each beating the one next following.

To take this more slowly than Dodgson did, we observe that *a* beats *d*, *d* beats *c*, and *c* beats *b* (each by 6 votes to 5), but *b* beats *a* (by 7 votes to 4). So we can write $a > d > c > b > a$: here the symbol '>' is to be read as 'beats'. It may be easier to read this off from a vote matrix, as we show later.

'Majority rule' has a clear meaning when only two candidates or issues are in contention. If one option wins more than half of the votes cast, then it is unambiguously the majority's choice. Simple majority rule satisfies some classically desirable properties of fairness; furthermore, it was proved by Kenneth O. May in 1952 that it is the *only* voting procedure that does so.[17]

But as soon as there are more than two candidates, simple majority voting may perform very badly. For instance, a candidate may be the plurality winner (getting the largest number of votes of any candidate, but fewer than half of the votes cast), while being the absolute majority loser (being ranked last by more than half of the voters). Dodgson's example in Table 1 was designed to show this.

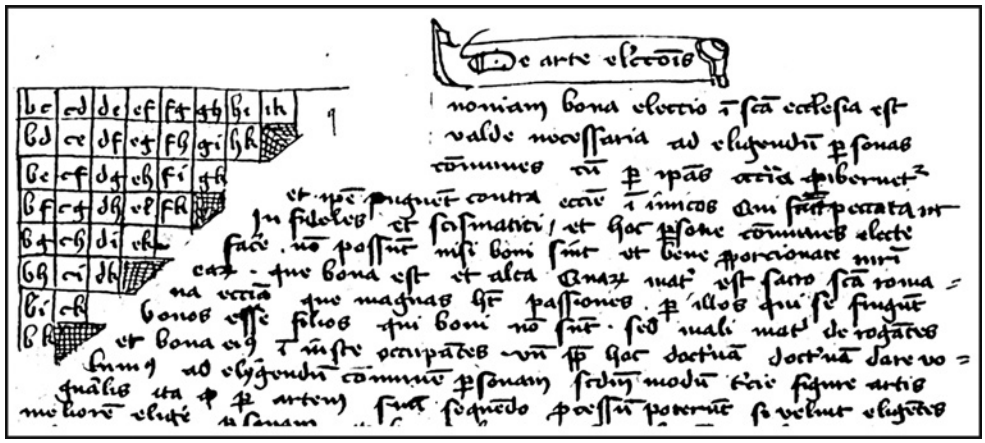
An obvious way to avoid this difficulty is to conduct exhaustive pairwise voting, where each candidate is compared directly with each of the others. If the voters are asked to rank the candidates, then this may be done at the second stage in the procedure, the 'aggregation stage' – there is no need for the voters actually to vote on each pair. Exhaustive pairwise voting was first proposed by Ramon Llull in 1283. If it yields a clear winner – a candidate who has won a majority against every other candidate, taken one at a time – then that candidate (the Condorcet winner) has an obvious claim to be considered the best.

But, as Condorcet had found out ninety years earlier, Dodgson discovered that the Condorcet winner may not exist (see Table 2). When a Condorcet top cycle exists, 'majority rule' seems to have no meaning: whatever society ends up with, a majority of voters would have preferred somebody (or something) else. This has deep implications for democratic theory.

The Cusanus (or Borda) rank-order count obviates this difficulty. It was described in Borda's paper, with Condorcet's commentary, neither of which Dodgson had read. The Borda count, of which the Cusanus count is a special case, works as follows. Each voter ranks the candidates from best to worst. Each last place is scored as a and each interval between places is scored as b ($b > 0$). Where there are n candidates, this is most easily done by giving n votes to one's favourite, $n - 1$ to the next, and so on down to 1 vote for one's least-liked, as proposed by Nicolaus Cusanus (so $a = 1$ and $b = 1$), or (better) from $n - 1$ at the top to 0 at the bottom, as proposed by Borda (so $a = 0$ and $b = 1$); there are rules for equal places that need not concern us. These scores are then simply added up and the candidate with the highest aggregate score is elected.

This is all beautifully simple and (unlike the Condorcet procedure) always gives a clear result. But it may fail to select the Condorcet winner, even when one exists. It may also lead to peculiar outcomes when the set of candidates is expanded or contracted, and it can patently be manipulated by voters placing the most dangerous rival to their favourite at the bottom of their lists.

In his exposition Dodgson proposed a matrix in which each candidate is scored against each other. This revived a similar notation that, unknown to him, had been proposed by Llull in 1283: one of Llull's half-matrices is shown below. Such a half-matrix is all that is needed for entering data, if two numbers are inserted into each cell.



One of Llull's half-matrices

But for computation it is easier to construct a square matrix, in which the votes for *a* against *b* are entered in cell *ab* and the votes for *b* against *a* are entered in cell *ba;* the matrix corresponding to Table 2 is:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 4 & 4 & 6 \\ 7 & 0 & 5 & 8 \\ 7 & 6 & 0 & 5 \\ 5 & 3 & 6 & 0 \end{pmatrix} \end{array}$$

It is then easy to carry out arithmetic checks to confirm that the numbers have been entered correctly. In particular, when there are m candidates and n voters the cells in the matrix should add up to $(m-1)!\,n$.

If the votes for each candidate are scored against each other in such a square matrix, then the Borda and Condorcet winners may both be read directly from the matrix. If Borda scores are assigned with $a = 0$ and $b = 1$, with 0 for the least-liked of n candidates and $n-1$ for the voter's favourite, then they also record the number of victories that each candidate has over the others in an exhaustive tournament; Borda noticed this equivalence himself. The Condorcet winner is computed by comparing each cell with its mirror image across the main diagonal; this main diagonal is empty because an option's score against itself has no meaning. So the Borda scores for each candidate are horizontal sums of their pairwise votes. In these small elections the Condorcet comparison is done by eye, with each cell compared with its mirror image. In larger elections it is easily programmable.

The Dodgson matrix is a handy device that may be used in any context where we want to decide the Condorcet and Borda winners, provided that we have a rule that tells us what to do when they are different. Dodgson proposed two different tie-breaks, one in his 1874 pamphlet and a different one in 1876.

Lawn tennis tournaments

Exhaustive pairwise voting, as discussed by Llull and Dodgson, is essentially a tournament in which every player plays each other once. It is therefore not surprising that Dodgson next turned his attention to the new sport of lawn tennis, after watching a game in the Oxford University Parks in 1880.

For a large number of players, the number of pairs may be unmanageably large. If there are n candidates, then the number of pairs is $n(n-1)/2$, and in the 1880s a lawn tennis competition (such as at Wimbledon, where the All-England competition began in 1877) typically had 32 players, requiring an unmanageable 496 matches.[18] Before Dodgson, the authorities running the evolving sport had favoured a knockout tournament, in which each player plays one other in the first round, after which half of the players are eliminated. In each subsequent round, the number of players still in competition halves again as losers drop out, until the final is held between the last two survivors.

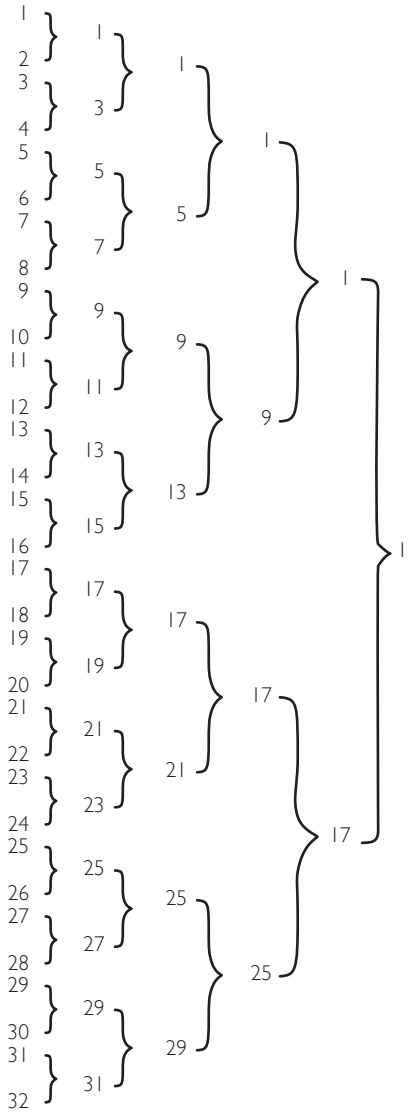
But a knockout system has a fatal flaw which Dodgson had already identified in 1873 in the context of voting among candidates. In his pamphlet *Lawn Tennis Tournaments*, published in 1883, he opened with the following words:[19]

> At a Lawn Tennis tournament, where I chanced, some while ago, to be a spectator, the present method of assigning prizes was brought to my notice by the lamentations of one of the Players, who had been beaten (and had thus lost all chance of a prize) early in the contest, and who had the mortification of seeing the 2nd prize carried off by a Player whom he knew to be quite inferior to himself.

For, suppose that the 32 players in the tournament had been paired initially in descending order of ability (see overleaf). Then, in the first round, all even-numbered players are eliminated. In the second round, player 1 eliminates player 3, player 5 eliminates player 7,…, and player 29 eliminates player 31. So the process proceeds until the final match takes place between the best player and the 17th-best. Because numbers 2 to 8 will all have been eliminated, the third prize will be awarded to the 9th-best, and the fourth prize to the 25th-best.

In consequence, Dodgson asserted that 'the present method of assigning prizes is, except in the case of the first prize, entirely unmeaning', and proposed to lay before the reader 'A system of rules for conducting Tournaments, which, while requiring even less time than the present system, shall secure equitable results'.

Nowadays, this problem is dealt with by 'seeding' players according to their performance in earlier tournaments, and arranging that players with a similar seeding play one another: this mitigates the problem, but does not eliminate it. Dodgson's solution is more interesting: his core idea was this:[20]



A tennis tournament with 32 players

> A list is kept, and against each name is entered, at the end of each contest, the name of any one who has been proved superior to him – whether by actually beating him, or by beating some one who has done so (thus, if A beats B, and B beats C, A and B are both "superiors" of C). So soon as any name has 3 "superiors" entered against it, it is struck out of the list.

Over a century later, social choice-theorists rediscovered the concept, which is now called 'covering' and which helps break some Condorcet

Let the players be arranged alphabetically, and call them A, B, C, etc., and let their relative skill be represented by the following numbers :—

A	B	C	D	E	F	G	H	J	K	L	M	N	P	Q	R
6	10	13	5	15	9	7	12	1	14	3	2	8	16	11	4

These numbers will enable the reader to decide which will be the victor in any contest : but of course they are not supposed to be known to the Tournament Committee, who have nothing to guide them but the results of actual contests. In the following tables, "I. (e)" means "first day, evening," and so on : also a player, who is *virtually* proved superior to another, is entered thus "(A)." The victor in each contest is marked *.

TABLE I. (PAIRS.)

I. (e)	II. (m)	(e)	III. (m)	(e)	IV. (m)	(e)
A, B *	A, D *	D, G *	D, J	D, J (a)	D, M *	D, L
C, D (a)	F, G *	J, R *	A, M *	M *	L, R *	
E, F (a)	J, M *	A, F *	G, R *	R		
G, H *	N, R (a)	H, N *	L, N *	L		
J, K *	B, C *	M, Q *				
L, M (a)	E, H (a)	B, L (a)				
N, P	K, L (a)					
Q, R (a)	P, Q (a)					

Part of a letter by Dodgson, published on 1 August 1883, in the *St James's Gazette*

cycles.[21] Where there is no Condorcet winner, there may still be an 'uncovered set' in which the plausible winners may be found.

Proportional representation

Until 1883 all of Dodgson's work on choice procedures addressed the case of choosing a single winner (person or outcome), to be selected by a tournament method. The problem of proportional representation is different and emerges from the *microcosm theory*. This theory for representative bodies holds that the body should represent those who elect it, in such a way that each relevant characteristic of the electors (such as age, gender, ethnicity, or political opinion) is represented in the assembly in the same proportion as it appears in the electorate. However, this concept of proportionality has no meaning in (for instance) a presidential election: there is only one president, who cannot be white, black, gay, straight, female, male, left, and right in the same proportion as the electorate. So the study of proportional representation (PR) is a distinct branch of the mathematics of voting.[22]

Here, unlike in the single-winner case, Dodgson knew some of the prior and contemporary work. From his perspective, first-past-the-post was bad enough, but the actual UK parliamentary electoral system was worse. Many UK constituencies, including Oxford University, returned more than one member, and this only exacerbated the problem of first-past-the-post that Dodgson had identified in 1873. If an absolute majority loser could win one seat, then two absolute majority losers could win each of two seats.

Various enthusiasts, publicized by John Stuart Mill, had been proposing PR schemes in Britain for most of the 19th century, and even implementing them in some colonial assemblies.[23] In four letters to the *St James's Gazette*,[24] Dodgson showed that some of their ideas were mathematically unsound. In particular, the system boosted by Mill, and now known world-wide as the 'single transferable vote' (STV), may elect the wrong set of candidates.

STV depends on successive eliminations and transfers of surpluses, but the latter are a minor problem. The eliminations are a major problem, however, for the reason that Dodgson had already identified – that a top cycle member or Borda winner may be eliminated early on, like the unlucky knockout victim of a tennis tournament. Dodgson's own proposals were hard to follow but, as with his work on social choice, they are now recognized as unique.

In Dodgson's day the widening of the franchise in the United Kingdom was a heated issue. Conservatives knew that it was difficult to resist the principle, yet feared that they would lose out substantially. The franchise was reformed in 1832 and 1867–68 when qualifications to vote were slightly lowered, and a few 'rotten boroughs' with very small electorates were abolished

and their seats redistributed to growing industrial cities. The 1867 Reform Act included a provision that some of the large English cities should each form a three-member district, with voters having only two votes each: this was known as the 'limited vote', and it was added to the bill by Lord Cairns's hostile amendment in the House of Lords which the Commons Tory leader, Benjamin Disraeli, unexpectedly accepted.[25] The impact of the limited vote on Dodgson's thinking is discussed below.

Except in Ireland, the British General Election of 1880 saw the closest approach since 1841 to a straight two-party contest, between Liberals and Conservatives. It illustrated both the bias and the exaggerative effect of the first-past-the-post electoral system. The Liberals won roughly the same proportion of seats as votes, but the Conservatives won many fewer seats than their vote share, and the Irish Party many more.

So the Liberals formed a government, and in 1884 they proposed a further widening of the franchise. The Marquess of Salisbury, leader of the Conservatives, noted the bias and exaggeration in the 1880 result. He envisaged that, unless the extension of the franchise were accompanied by a redistribution of seats, the Conservatives could be decimated in Parliament, even if the franchise extension reduced their share of the vote only slightly (or not at all). In the most sophisticated piece of election analysis yet written by a UK parliamentary leader, he showed that if the electorate of a 17-seat legislature with single-member districts were split between imaginary parties (which he named 'Catholics' and 'Liberals') in the proportion 8 to 9, then the 'Liberals' would win all 17 seats in two cases:[26] either where the population was exactly evenly mixed, *or* where it was completely segregated (say, into a 'Liberal' city surrounded by 'Catholic' countryside), but where constituencies were drawn radially from the city centre in such a way that each constituency contained the same ratio of 'Catholics' to 'Liberals' as the population.

Ireland was even more threatening to Salisbury's interests. The whole of Ireland was incorporated into the United Kingdom in 1801, but the union was never legitimate in the eyes of Irish Catholics, who comprised some seven-eighths of the population.[27] Since 1874, seats in Catholic Ireland had been falling to supporters of Home Rule, who used every procedural means open to them to disrupt Parliament. The franchise reform of 1884 proposed to give the vote in Ireland, as in the rest of the country, to rural male householders. But would this not mean a great boost to Charles Stuart Parnell, the Home Rule leader, with consequent threats to public order and the unity of the UK? As Sir John Lubbock, one of the leading advocates of STV, wrote:[28]

> At the general election of 1880, 86 seats were contested [in Ireland]. Of these the Home-rulers secured 52, the Liberal and Conservatives together only 34. Yet the Home-rule electors were only 48,000, while the Liberals and Conservatives together were no less than 105,000…we are

Lord Salisbury, Lord Cairns, and Sir John Lubbock

told…that under the new Redistribution Act the Home-Rulers will secure 90 seats out of 100, leaving only a dozen to the Liberals and Conservatives together…The result of this system, then, will be that Ireland will be entirely misrepresented, and that we shall have gratuitously created serious and unnecessary difficulties for ourselves. To adopt, indeed, a system of representation by which we shall exclude from the representation of Ireland one-third of the electors, and give almost the whole power to two-thirds, would, under any circumstances, be unjust; but to do so when the one-third comprise those who are moderate and loyal, while the two-thirds are led by men not only opposed to the Union, but in many cases animated by a bitter and extraordinary hatred of this country, seems to be an act of political madness.

He then went on to draw attention to the US Presidential election of 1860, in which Abraham Lincoln won an absolute majority of the Electoral College on less than 40 per cent of the vote.

Lubbock's prediction was spot on. The Home Rulers won 85 seats in the 1885 General Election, and continued to do so at every election until 1910. Anglo-Irish war was about to break out in 1914, but was delayed for the First World War to take place. The Irish wars lasted from 1919 to 1923. Electoral systems have important consequences.

This was the context for Dodgson's *Principles of Parliamentary Representation* of 1884. Dodgson was a political Conservative as well as a temperamental conservative. He met Salisbury and his family in 1870, uncharacteristically using his fame as the author of *Alice* to obtain an introduction to Salisbury's wife and daughters. Despite the gulfs of class and temperament, he was welcomed by the Salisbury family and spent the New Year at their great house, Hatfield, several times during the 1870s and 1880s.

Dodgson seems to have thought about PR for the first time in 1882, in connection with college politics, but it was the reform crisis of 1884 that brought him into print. His letters to the *St James's Gazette* show his continually evolving ideas. In June he hit on the most

distinctive feature of his scheme, 'the giving to each candidate the power of transferring to any other candidate the votes given for him', and in the following month he sent it to Salisbury, saying:[29]

> How I wish the enclosed could have appeared as *your* scheme…That *some* such scheme is needed, and much more needed than *any* scheme for mere redistribution of electoral districts, I feel sure.

Salisbury replied immediately, acknowledging the need for electoral reform but stressing the difficulty of getting a hearing for 'anything…absolutely new…however Conservative'. Dodgson responded the next day. After congratulating Salisbury for insisting that the Conservatives (who controlled the House of Lords) would not accept franchise reform unless it were linked with redistribution, Dodgson continued:

> *please* don't call my scheme for Proportionate Representation a 'Conservative' one…all I aim at is to secure that, *whatever* be the proportions of opinion among the Electors, the same shall exist among the Members.

The House of Lords did indeed return the franchise bill to the Commons with an added clause insisting that it must be 'accompanied by provisions for so apportioning the right to return members of parliament as to insure a true and fair representation of the people'. This amendment was moved by Lord Cairns, the same peer who had inserted the limited vote provision into the 1867 Reform Act. No doubt the motives of the majority Conservative peers mingled self-interest with a desire for fair representation. In Britain, fair representation was expected to mean protecting the Conservatives from being wiped out in terms of seats in a General Election where they narrowly came second in terms of votes. Salisbury may have wished to preserve the limited vote, but in the end he went down a different road.

Like Lubbock, Dodgson saw that Salisbury had failed to accept the implications of his own argument. No redistribution based on single-member districts with the plurality voting rule could be guaranteed to save the Conservatives in Britain, nor either British party in Ireland. As Salisbury had pointed out in the *National Review*, and as Dodgson repeated in his *Principles of Parliamentary Representation*, single-member districts when combined with an even distribution of supporters of two parties around the country could lead to the larger of the two wiping out the smaller in terms of seats. This happened in Ireland in every election from 1885 to 1910, and in Scotland in 2015.

Salisbury could not shift his perspective from majoritarian to proportional in order to see the true implications of his own argument. The concepts of political and physical proportionality interact with one another. Conservative supporters were fairly evenly spread around Great Britain; Irish Nationalists were to be found only in Ireland (and in Liverpool, where they held a seat from 1885 to 1929). So he turned to a scheme that achieved physical

proportionality in Britain, although not in Ireland, and masterminded the scheme of single-member districts 'according to the occupation of the people' that was embodied in the Redistribution of Seats Act of 1885. This scheme, the outcome of his negotiations with the Liberals' Sir Charles Dilke in the autumn of 1884, has frequently been hailed as Salisbury's stroke of genius. It constructed suburban seats where the new concentration of Conservative voters was to be found in what contemporaries called the 'villa vote',[30] but it is unclear whether it would have saved the Conservatives if the debacle that Salisbury feared had come about. For in 1886 the Liberals were torn apart over Irish Home Rule, leading to a twenty-year Conservative hegemony not foreseen by Salisbury or anybody else in 1884. Thus, in the end, no Conservative had to take Dodgson's arguments seriously out of self-interest.

It is unfortunate that certain points that Dodgson quickly passed over in his *Principles of Parliamentary Representation* were exactly the ones that mainstream politicians could not accept, even when it was in their own interest. Dodgson took for granted that guaranteeing the survival of minorities in parliament requires multi-member districts and minority representation (which the politicians should have accepted, but did not), and that the number of electors per Member of Parliament should be equal (which almost no parliamentarian in the 1884 debates did). These assumptions were probably enough on their own to blind contemporaries to the more striking features of the *Principles*.

The *Principles* is the earliest known work to discuss both the assignment of seats to each of a number of multi-member districts (the *apportionment problem*) and the assignment of seats within each district to the parties (the *PR problem*). Not until 1982 was the formal congruence of the two problems fully understood.

Dodgson's own preference was for electoral districts with two to five members, in which electors were given just one vote. For such multi-member constituencies Dodgson calculated the percentages of the votes required for a political party to return a specified number of members, and presented his results in the following table:

number of members allocated to district	*number of seats to fill* 1	2	3	4	5	6
1	**51**					
2	34	**67**				
3	26	51	76			
4	21	**41**	61	**81**		
5	17	34	51	67	84	
6	15	29	43	58	72	86

For example, in a single-member district a party requires over half of the votes in order to fill the seat – that is, at least 51 per cent (line 1, column 1), if we stick to whole numbers. In order to return two members, a party requires at least 67 per cent of the vote (line 2, column 2) in a two-member district, and at least 41 per cent (line 4, column 2) in a four-member one. In general, as Dodgson discovered, a political party wishing to return k members in an n-member district requires more than $k / (n + 1)$ of the votes; for example, in order to fill two seats ($k = 2$) in a four-member district ($n = 4$), a party needs more than two-fifths of the votes – that is, at least 41 per cent. The last number in each line gives the percentage of the electorate represented by its members; for example, in a four-member district, 81 per cent of the electorate is so represented.

Thus, for the assignment of seats to parties, Dodgson essentially recommended what came to be known as the standard 'Droop quota' $V / (S + 1)$, rounded up to the next integer, where V is the total number of votes cast and S is the number of seats to be filled. In 1881 the mathematician Henry R. Droop had pointed out that a higher quota proposed by Thomas Hare and endorsed by Mill could lead to too few candidates being elected. Dodgson showed that the rules of the Proportional Representation Society (now the Electoral Reform Society) to run a single transferable vote could lead to the defeat of a candidate who had obtained a Droop quota. This cannot happen at the first stages of an STV election, but Dodgson showed that it can happen at subsequent stages. He briskly concluded:[31]

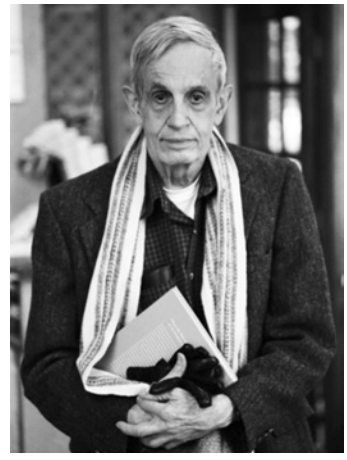
I think I have sufficiently proved the fallacy of its method for disposing of surplus votes…Clearly *somebody* must have authority to dispose of them: it cannot be the Elector (as we have proved); it will never do to refer it to a Committee. There remains *the Candidate himself, for whom the votes have been given*.

Henry R. Droop
[The Master and Fellows of Trinity College, Cambridge]

Dodgson's surprising formula does not get to the root problem of STV and all other elimination systems – namely, that they use information about voters' preferences after their first in an arbitrary way. Preference orderings are not treated equally: the $(n + 1)$th preference of a voter whose nth preference has been eliminated is counted, whereas the $(n + 1)$th preference of a voter whose nth preference has been elected with a surplus is counted with reduced weight, and the $(n + 1)$th preference of a voter whose nth preference is elected with nothing to spare is not counted.

Dodgson presented his alternative in a compressed and elliptical argument. He identified what game theorists now call the *Nash equilibrium strategy* for two parties, after the Nobel Prizewinner John Nash. A *Nash equilibrium*, not so defined until 1951, is a point in a game from which, once reached, it benefits neither player to depart. Dodgson considered the class of methods in which voters may each cast v unranked ballots in an m-member district, where $v < m$. The limited vote introduced by Lord Cairns operated in Manchester, Birmingham, and Liverpool with $v = 2$ and $m = 3$, and in the City of London with $v = 3$ and $m = 4$. In Birmingham the Liberals had manipulated the limited vote by dividing the city into three zones and asking their supporters in each zone to vote for a different pair from the Liberal candidates: they thereby won all three seats in each of the general elections in the period. In 1874 the Conservatives did not run at all. In 1868 the Liberals controlled 73% of the votes cast, and in 1880 they controlled 67%.

Were the Birmingham parties rational strategists? This may be answered directly from Dodgson's pamphlet, once his reasoning is understood. Given perfect information and common knowledge about party strengths, Dodgson knew, but failed to make explicit, that the optimum strategy for each party is to put up exactly as many candidates as it can



John Nash
[Wikimedia Commons]

fill seats, and to instruct its supporters to divide their ballots among its candidates as evenly as possible.

Dodgson's aim was to find the voting procedure that leaves the fewest voters 'unrepresented' by votes that do not contribute to the outcome. He concluded that, within the class of limited voting procedures, the fairest method – in the sense that it can be predicted to leave the fewest voters unrepresented – is that where $v = 1$ for any m. The fairness of the system increases with m, so the fairest of this class of systems is that which divides the country into multi-member constituencies in which each voter has only one vote. This was the system used in Japan for national elections from the end of the Second World War until 1993. It was generally labelled the *single non-transferable vote* (SNTV), and has also been used in Taiwan and Korea. In these countries, m has typically had a value of around 5.

From Dodgson's calculations we find that, for $v = 2$ and $m = 3$, a party with at least 61% of the vote can guarantee to win all three seats, so in Birmingham the Liberal strategy was optimal. The Conservative strategy was optimal in 1874, and was sub-optimal in 1868 and 1880, at any rate from the perspective of Dodgson's game. If it was common knowledge that the Liberals controlled over 60% of the probable voters, then there was no point in the Conservatives' running candidates, unless for the purpose of forcing an election and putting the Liberals to some expense.

Dodgson's result on SNTV has been independently rediscovered by Gary Cox and Emerson Niou.[32] It can be obtained by postulating either that the parties are rational (in which case they do what the Birmingham Liberals did) or that the voters are rational (in which case, given perfect and free information, they would clump their ballots in the same way as the Birmingham Liberals directed them to). The outcome of a procedure in which the parties seek to minimize waste of 'their' votes is the same as that of a procedure in which the voters seek to avoid wasting their votes.

Neglect and rediscovery

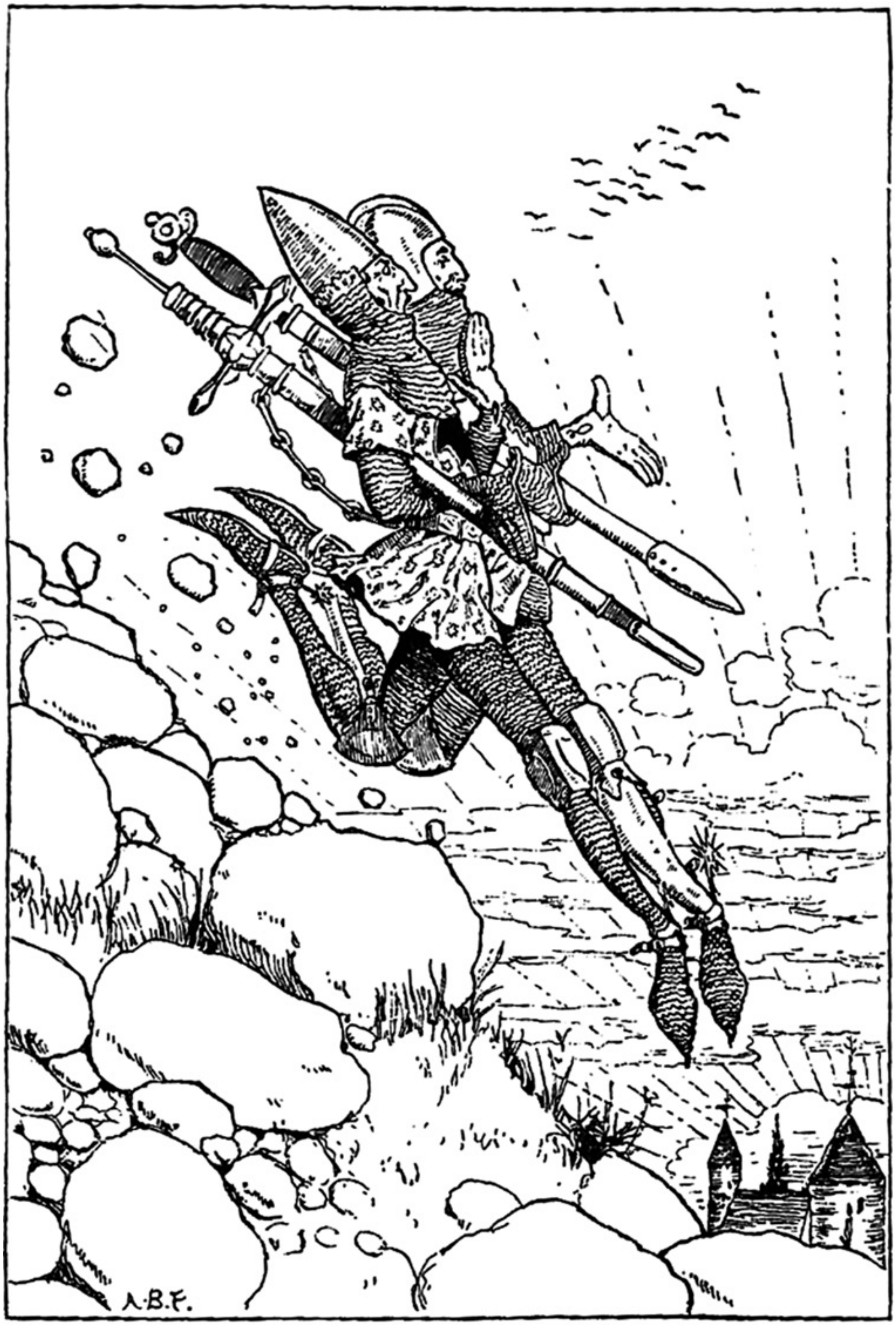
Many writers have constructed elaborate psychological theories to account for Dodgson's lack of recognition during his lifetime, in anything except *Alice* and *Snark*. However, nothing elaborate is required to explain Dodgson's lack of immediate influence in college, university, or world affairs. He could be pedantic, quarrelsome, and (when not quarrelsome) obscure.

By the time of the belfry controversy Dodgson had fallen out acrimoniously with Dean Liddell, publishing sarcastic pamphlets about him which, to a modern reader, have none of the lightness of the *Alice* books.[33] His correspondence with Lord Salisbury (which we reviewed above) shows that Dodgson was not always quarrelsome and that he could engage with a senior politician – indeed, one of the most influential politicians in the UK. It is a pity that he could not do so in the smaller world of Christ Church.

Even there, it has been argued that Dodgson had agenda-setting, or rule-setting, power. In the light of his behaviour towards Liddell it seems surprising that his colleagues ever listened to his advice, but we know that they did – for instance, by using the Borda count in the election of Baynes in 1873. Although that was not an unmitigated triumph (because Baynes was not the Borda winner), it spurred Dodgson on to his most fundamental work on voting.

Alice Adventures in Wonderland and *Through the Looking-Glass* are based on games, the latter being explicitly a game of chess. Dodgson's writings about voting were also about games, as Duncan Black was the first to see. Indeed, Dodgson's life was devoted to being serious about games and game-like about serious things. Occasionally he actually referred to a voting 'game', as when proving that there was what would now be called a *defective equilibrium* when rational voters used his 'method of marks'. But his more remarkable achievement was to have written about voting in game-theoretic terms, before game theory had been invented.

Alice is deep; *The Hunting of the Snark* is deep and dark; Dodgson's writings on voting and proportional representation were also deep, and were not understood for almost a century.



At a pace of six miles in the hour: from *Excelsior* (Knot I of *A Tangled Tale*)