Moving Boundary Problems for Reflected SPDEs



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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2019

Dedicated to my parents.
Without their love and support, this would not have been possible.

Acknowledgements

I would like to thank my supervisor Professor Ben Hambly for his support, guidance and patience. I am very grateful for having had the opportunity to work with him and hope that we will stay in touch in the future.

This work would not have been possible without the financial support provided by the Engineering and Physical Sciences Research Council.

Thank you to all of the friends that I have made in the Maths department and Keble college. I've had a wonderful time getting to know you all and I hope that our friendships will last long after I leave Oxford.

Finally, I would like to thank my family and Rosie for helping and encouraging me throughout my time in Oxford.

Abstract

The work in this thesis is based on the study of reflected SPDE (stochastic partial differential equation) moving boundary problems. These are systems consisting of two competing profiles which each evolve according to a reflected stochastic heat equation in one spatial dimension, and share a common boundary point. The reflection here minimally pushes the profiles upwards in order to maintain positivity. The evolution of the shared boundary depends on the state of the profiles, and so is coupled with the dynamics of the two competing sides. Such equations are suited to modelling competition between two types. An example of this is the limit order book. We can think of the competing profiles as being order volumes to buy/sell an asset at different prices, with positivity of these ensured by the reflection terms. The shared boundary then represents the current midprice.

Key results include scaling laws, which show how we can connect particle systems to particular reflected SPDE moving boundary problems, and existence and uniqueness results for different classes of reflected SPDE moving boundary models. In the latter case, we first study the situation where the boundary mechanism is driven by the competing profiles considered as continuous functions. Following this we impose stronger assumptions and examine the case where the boundary mechanism is driven by the derivatives of the competing profiles at the shared interface, as in the classical Stefan problem. We also study other properties of our systems, including their Hölder regularity and the existence of invariant measures.

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Chapter 1

Introduction

Moving interfaces arise naturally in many contexts across different scientific disciplines including Physics, Biology and Economics. Often, there are two or more regions/types, and these can be associated with different behaviour/regimes. A classic example is the scenario where a large body of ice is in contact with a body of water. One can ask the question as to how the point of contact between the ice and water evolves as the ice melts or the water freezes. Another interesting problem of this type is the modelling of organisms of two types competing for territory. We can also consider the Limit Order Book (L.O.B.). In mathematical finance, the L.O.B. is the record of unexecuted, uncancelled orders to buy/sell an asset in a Limit Order Market. The bid is the best (highest) offer to buy which is currently on record on the book, whilst the ask is the best (lowest) offer to sell. In high frequency markets the spread (difference between the bid and ask) is typically very small, and is therefore assumed to be zero in many models. The price can then be thought of as a moving interface which sits between the offers to buy the asset and the offers to sell the asset.

Common amongst these different scenarios is the natural complexity which arises from the coupling of the dynamics of the shared interface and the dynamics of the rest of the system. For example, in the case of the ice melting problem, the dynamics of the boundary between the ice and the water depends on the evolution of the heat distribution throughout the body of water, which itself is affected by the current position of the interface. This makes the problem more interesting and challenging, with the difficulty amplified by the presence of noise.

1.1 Main Aims

The focus of this thesis is to study one-dimensional models made up of two SPDEs which share a common moving boundary. We will work in a general setting, and aim to provide broad frameworks which can be used to describe many scenarios similar to those discussed above. The models discussed in this thesis fall into three distinct categories, with each of these handling the boundary motion in different ways. We will examine these analytically, and will also provide numerical illustrations of some of the possible behaviours which can arise from such equations.

Our equations will be driven by space-time white noise. By comparison with more classical deterministic models, the presence of this noise leads to more complex behaviours in many situations where high frequency random components are important in the dynamics. For example, in typical one-dimensional biological models in the literature, space-time white noise terms often arise as scaling limits of birth and death in a population (see, for example, [20]). When thinking of the equations as a model for the Order Book, space-time white noise can naturally be thought of as representing high frequency trading in the system.

In all of the models which we consider, the dynamics of the shared boundary are influenced by the profiles on either side i.e. the volume and location of the two types on either side of the interface. For example, in a competition model the function will typically be the difference of the mass of each type close to the boundary. In the biological model discussed, this corresponds to the relative numbers of the two species at the interface between the types, whilst in the LOB example this represents the levels of supply and demand at the best prices.

1.2 An Overview of Reflected SPDEs in the Literature

In the absence of boundary movements, the equations which we will be studying in this thesis take the form

$$\frac{\partial u}{\partial t} = \Delta u + f(t, x, u(t, \cdot)) + \sigma(t, x, u(t, \cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta.$$

In this equation W represents a Brownian sheet, whilst the term η is a measure whose presence ensures positivity of the solutions. This makes the system a more natural choice for modelling situations where negative values are not expected. We note for completeness that all of the results in this thesis can also be shown in the absence of this term. Reflected SPDEs of the form (1.2) were originally studied by Nualart and Pardoux in [40], in which existence and uniqueness was proved in the case where σ is constant. Donati-Martin and Pardoux then proved existence for a wide class of σ in [16], with the function only needing to satisfy standard Lipschitz and linear growth conditions. Finally, Xu and Zhang proved existence *and* uniqueness for coefficients satisfying Lipschitz and linear growth conditions with a much shorter proof based on a two-step Picard argument in [52]. All of these papers focused on the case where the spatial domain is a finite interval, $[0, 1]$, with Dirichlet conditions imposed on the endpoints. Otobe extended the existence and uniqueness theory to the case when the spatial domain is $\mathbb{R}$ in [42], although uniqueness in the case of non-constant volatility remained an open question. Versions of the problem with smoother driving noises have also been considered. In [35], the problem is studied in $[0, T] \times \mathbb{R}^d$, with finite dimensional driving noise. Following this, in [15], infinite dimensional noise of certain types is permitted e.g. spatially coloured noise. However, space-time white noise is not allowed in this framework. In both of these cases, the coefficients can depend on the divergence of the solution, which lives in the Sobolev space H_0^1.

As well as advances in existence and uniqueness theory for the equations, various properties of the solutions have been proved. In [12], the contact sets for the solutions are studied in the case where the drift, f, is zero and the volatility, σ, is constant. In particular, it is shown that at all positive times, the solution is equal to zero at at most four points almost surely. Zambotti proved in [56] that, under the same conditions on the drift and volatility, the law of the 3D-Bessel bridge is in fact an invariant measure for the equation. Otobe then extended this result to the case where the spatial domain is $\mathbb{R}$ in [41], proving that the invariant measure is such that the conditional law in an interval is a 3D Bessel bridge, with suitable distributions for the endpoints. In [57], Zambotti examines the behaviour of the reflection measure in more detail, showing that it is absolutely continuous with respect to Lebesgue measure in the space variable, and also that for each point x in space these densities can be viewed as renormalised local time processes for $(u(t,x))_{t\geq 0}$. Zhang proved the strong Feller property of solutions in [60], and together with Xu proved a large deviation result for sequences of solutions to such equations with vanishingly small noise in [52].

The case with two reflecting walls has also received attention. Here, rather than having the equation simply be reflected at zero, the profile u is forced to lie in between two given functions, $h_1(t,x)$ and $h_2(t,x)$. We therefore have the presence of two separate reflection measures in the equation. In [43], existence and uniqueness is proved for such equations in the case of constant volatility, where the spatial domain for the equation is $[0,1]$. In [53], Yang and Zhang extended the existence and uniqueness theory to the case of non-constant volatility, before proving existence and uniqueness of invariant measures in [54] for the case where the drift coefficient, volatility coefficient and the obstacles do not depend on time. Xie proved a log-Harnack inequality for the equations in [51], and applied this to establish uniqueness of invariant measures. Zhang then studied the large deviations of these invariant measures in [58]. In [55], Yue shows that for $(t,x) \in (0,\infty)\times(0,1)$, provided that the volatility is non-zero, the law of the solution at time t and spatial position x, $u(t,x)$, is absolutely continuous with respect to Lebesgue measure.

1.3 The Classical Stefan Problem and SPDE Moving Boundary Problems in the Literature

Stefan problems have been extensively studied since the original work by Josef Stefan in 1889 (see [45], [46], [47] and [48]). The problem is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{for } x \in (0, p(t)) \\ \frac{\partial u}{\partial x} = 0 & \text{on } x = 0 \\ u = 0 & \text{on } x = p(t) \\ p'(t) = -\gamma \frac{\partial u}{\partial x}(t, p(t)) & \\ u(0,x) = u_0(x) > 0 & \\ p(0) = 0. & \end{cases} \tag{1.3.1}$$

We typically think of the function $u(t,x)$ as representing the temperature of a liquid which occupies the domain $(0,p(t))$ at time t. The region $[p(t),\infty)$ is thought of as being occupied by a solid mass whose temperature is assumed to be fixed at the freezing/melting temperature. The point $p(t)$ then evolves as the liquid freezes/the solid melts, with the boundary condition $p'(t) = -\gamma\frac{\partial u}{\partial x}(t,p(t))$ accounting for the amount of thermal energy lost/gained in the phase transition. For $\gamma < 0$, the equation models a supercooling scenario. We think of the temperature of the water as being measured in negative units by the equation, so that the water has been cooled to below its freezing temperature, and the water then freezes as the equation evolves i.e. $p(t)$ is decreasing. If the initial undercooling is too great, then it is possible to see an explosion of the boundary speed in finite time. If $\gamma > 0$, the equation represents a more stable regime where the ice is melting. The temperature of the water is now measured in positive units by the equation, and the heat energy is lost as the boundary moves outwards, converting ice to water. This classical version of the problem is very well understood, and useful references include [18], [19], [28] and [44], although there are certainly many more.

More recently, there has been increasing interest in stochastic versions of this problem, and moving boundary problems for SPDEs. In [32], existence and uniqueness for solutions to a Stefan problem for an SPDE driven by spatially coloured noise is proved. The corresponding problem in the case when the SPDE is driven by space-time white noise was then studied in [61] under the condition that the volatility vanishes quickly enough at the moving interface. More recent work on such problems include the models in [31] and in [36]. In these papers the focus is on essentially the same equations as (1.1) but without reflection and with coloured noise. Different boundary conditions are imposed at the interface in the two papers and in particular they are able to include a Brownian motion in the dynamics for the motion of the boundary. A system of two SPDEs with discrete boundary movements and constant volatilities is proposed by Newbury in [38] as a model for the limit order book. This is connected to a particle model by suitable scaling limits. Stochastic Stefan problems have also arisen as models for contagion and common exposure in large financial systems (e.g. in [17], [26], [33], [34] and [37]). Here, the solution $u(t,x)$ represents the density of banks at distance x to default at time t, with the position of the boundary representing default. Banks are assumed to have common exposures, which manifests itself as a transport term driven by Brownian motion in the SPDE for u. The Stefan condition at the boundary reflects that, as banks default, they move the default barrier closer to the other banks. As in the supercooled regime for the classical Stefan problem, this can lead to feedback loops which cause blow-up of the system.

1.4 Limit Order Book Models in the Literature

The equations studied in this thesis can be thought of as natural models for the Limit Order Book, and the work here fits well in the growing literature on such models. Limit Order Markets are the most common trading mechanism employed by exchanges globally, and they facilitate

trade of a given asset by allowing participants to place orders of three main types. Limit orders allow traders to indicate a price at which they wish to trade, together with a volume of the order they wish to trade and a direction i.e. buy/sell. If possible, such orders are matched with a corresponding offer of the opposite type. Otherwise, they are recorded on the order book, staying there until they are cancelled or they are matched with a new incoming order. Market orders are *impatient* orders, where the trader indicates a volume and the wish to buy/sell only, with these resulting in immediate transactions with the best offers of the opposite type currently on the book at that time. By placing such an order, the trader is indicating a willingness to sacrifice in terms of the price that they pay, in order to have their order fulfilled immediately. Finally, as alluded to earlier, unexecuted limit orders can be cancelled at any time. The Limit Order Book at a given time is then the record of unexecuted, uncancelled orders at that time. Figure 1.1 shows a typical order book profile, with prices on the x-axis and order volumes on the y-axis.

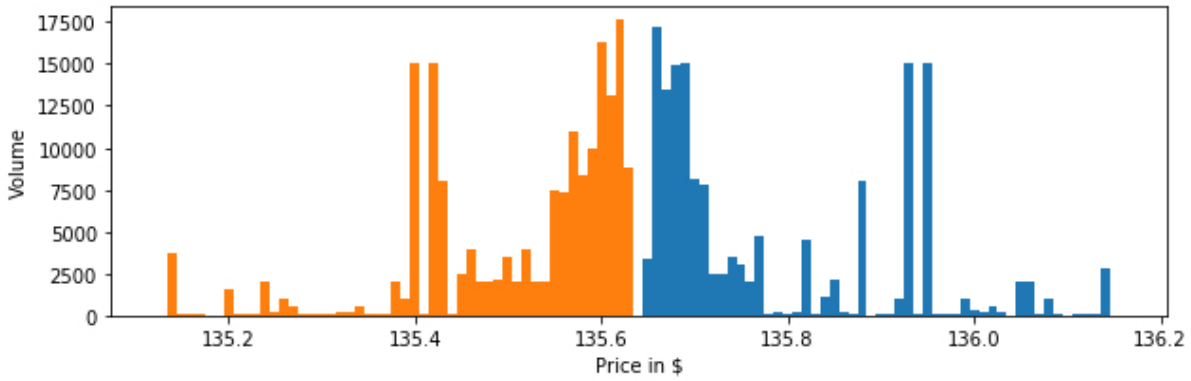


Figure 1.1: Order Volumes for the 50 best levels on both sides of the Limit Order Book for the SPDR Trust Series I on June 21 2012 at 09:30:00.000.

Our equations can be put into the context of the order book by once again thinking of the x-axis as representing the different price points at which one can trade and the y-axis as the density of the number of unexecuted and uncancelled orders currently on the book. In Section 3 of this thesis, we connect a particle queuing system model driven by Poisson processes to an SPDE equation, obtaining a system of SDEs as an intermediate model along the way. This can be interpreted as obtaining approximate dynamics for the evolution of the order book in the limit where order sizes are sent to zero, tick sizes (the smallest price change possible on the book) are sent to zero and order frequency is sent to infinity. Such a scaling is natural in certain high frequency markets, where the presence of automated trading algorithms mean that the number of orders and cancellations can be of the order of 10^6 per hour.

Papers which model the order book as a queuing system include [2], [5], [8], [39], [29]. Cont and Larrard considered a diffusion limit of the order book process from a queueing theory perspective in [7]. Here, only the number of orders at the highest offer to buy (bid) and the lowest offer to sell (ask) is kept track of at any given time. Once the bid or ask queue is depleted,

the depths of the new bid and ask queues are drawn from some stationary distribution. There have also been many other attempts to establish scaling limits for order book models. In [6], the authors start with a particle model and take a hydrodynamic scaling limit, obtaining a non-linear parabolic PDE in the limit. A similar approach is taken in [14], with the authors here arriving at an ODE limit. In [3] and [27] Horst et al. show how one can arrive at PDE or SPDE limits for an order book model, depending on the initial scaling procedure. The SPDE moving boundary equations studied in [61], [38], [31] and [36] were also motivated by order book dynamics.

1.5 Thesis Outline, Main Results and Contributions

The work in this thesis is based on the papers [23], [24], [25] and [30]. Section 2 is dedicated to a brief introduction to martingale measures and reflected SPDEs. As we will work within the SPDE framework introduced by Walsh throughout this thesis, an understanding of the construction of stochastic integrals against martingale measures is essential in order to grasp the work which follows. A key result here is the Burkhölder-Davis-Gundy inequality, which provides us with an L^p bound for integrals against space-time white noise. We also present our notion of solution to reflected SPDEs, and provide a description of the parabolic obstacle problem which corresponds to these equations.

In Section 3 we follow [23], adapting and extending the work of Newbury in [38]. A novel SPDE moving boundary model is considered, where the shared boundary moves in discrete jumps. The jumps occur at Poisson rates, the intensity of which is given by the state of the system at a given time. As in [38], the model is obtained as a scaling limit of a competition model of interacting SDEs, which in turn is connected to a particle model via another scaling limit. We call these three scales the macroscopic, mesoscopic and microscopic scales respectively. Convergence of the particle model to a system of reflecting, interacting SDEs is proved by classical results relating the convergence of Markov processes to convergence of the corresponding generators, together with some analysis related to the movement of the boundary. Moving from the mesoscopic to the macroscopic model uses the work by T.Zhang in [59], where static reflected SDE systems are shown to converge to static reflected SPDEs. The differences with the model in [38] include that we allow for drift and volatility coefficients which are functions of space and the solution itself. In [38] a constant volatility and space dependent drift was assumed. The mechanism for boundary movements also differs here. In [38], boundary movements were directly triggered by the SPDEs satisfying certain conditions at some time, whilst here boundary movements are driven by a Poisson process whose speed is dictated by the state of the SPDEs. We also generalise the reinitialisation mechanism for the new profiles of the SPDEs following boundary movements, with new profiles having their law given by a function of the state of the system just before the boundary movement.

Following the analytical work in this section, we demonstrate the applicability of the model by fitting it to Order Book data and simulating the system numerically. We illustrate that realistic behaviours can be captured by relatively simple systems which fall within the framework.

Section 4 largely follows [24] and is devoted to the study of moving boundary problems for reflected SPDEs, where the shared interface now evolves continuously with time. That is, we examine systems of the form:

$$\begin{aligned}
\frac{\partial u^1}{\partial t} &= \Delta u^1 + f_1(p(t) - x, u^1(t, p(t) - \cdot)) + \sigma_1(p(t) - x, u^1(t, p(t) - \cdot))\dot{W} + \eta^1 \\
\frac{\partial u^2}{\partial t} &= \Delta u^2 + f_2(x - p(t), u^2(t, p(t) + \cdot)) + \sigma_2(x - p(t), u^2(t, p(t) + \cdot))\dot{W} + \eta^2, \\
p'(t) &= h(u^1(t, p(t) - \cdot), u^2(t, p(t) + \cdot)).
\end{aligned} \tag{1.5.1}$$

We first establish our notion of solution, motivating this by multiplication with a test function and integration by parts. The problem is examined in two cases- firstly when the spatial domain is restricted to a fixed distance from the moving boundary and secondly when the spatial domain is infinite in both directions. Existence and uniqueness for the system is proved in both of these cases. Our approach is similar to that of [52], where existence and unqiueness is proved for static reflected SPDEs. As in [52], we decouple the problem into studying a deterministic obstacle problem and applying SPDE estimates. The non-Lipschitz term created by the moving boundary is controlled by a suitably truncated version of the problem, for which existence and uniqueness is proved by a Picard argument. Consistency among the solutions of the truncated problems allows us to piece these together to obtain a solution to our original problem which exists until some blow-up time.

In the case when the spatial domain is infinite, our analysis extends the known uniqueness theory for reflected SPDEs on infinite spatial domains. In [42], uniqueness for the reflected stochastic heat equation on $\mathbb{R}$ is proved only for the case when $\sigma \equiv 1$. We obtain uniqueness for a class of volatility coefficients which are allowed to depend on space and the solution itself at that particular point in space-time. The main condition here is that the coefficient σ is Lipschitz in its second argument, with a Lipschitz constant which decays exponentially fast in the spatial variable.

The local Hölder continuity of the solutions to our equations is also established, in both the case when the spatial domain is finite and when it is infinite. As one might expect, solutions can be shown to be up to $\frac{1}{4}$-Hölder continuous in time and up to $\frac{1}{2}$- Hölder continuous in space. In the case of the infinite spatial domain, this is a new result even for static reflected SPDEs. We argue by suitably adapting the proof in [13], in which the corresponding result was proved for solutions to static reflected SPDEs on compact spatial domains. As a corollary, we can also show that the derivative of the boundary is up to $\frac{1}{4}$-Hölder continuous in time, with this regularity inherited from the solution and a Lipschitz condition on the function h in (1.5.1).

Invariant measures are also studied. We establish the existence of these for the *profiles* of our solutions, $(u^1(t, p(t) - \cdot), u^2(t, p(t) + \cdot))_{t \geq 0}$, when the functions f_i, σ_i and h are all globally

bounded, with a sufficiently small boundary speed assigned to h. We note here that this, of course, establishes the existence of invariant measures for static reflected SPDEs with bounded drift and volatility coefficients. The content here is taken mainly from [30], in which the result for the static problem is proved, with some additions made to account for the boundary term. Since we only have an obstacle below our SPDE, we must first prove an a-priori L^p bound on our solutions. This distinguishes the argument from that in [54], where the existence of invariant measures is proved in the two barrier case. In [54] an a-priori bound is not necessary as we know that the SPDE solutions remain bounded at all times due to obstacles above and below.

Finally, in Section 5, we aim to take a first step towards studying reflected stochastic Stefan problems by proving an existence and uniqueness result. This section follows [25]. The problem is similar to that in Section 4, with the crucial difference that the boundary movement is driven by a local Stefan condition. That is, the system is of the form:

$$\begin{aligned}
\frac{\partial u^1}{\partial t} &= \Delta u^1 + f_1(p(t)-x,t,u^1(t,p(t)-\cdot)) + \sigma_1(p(t)-x,t,u^1(t,p(t)-\cdot))\dot{W} + \eta^1 \\
\frac{\partial u^2}{\partial t} &= \Delta u^2 + f_2(x-p(t),t,u^2(t,\cdot-p(t))) + \sigma_2(x-p(t),t,u^2(t,\cdot-p(t)))\dot{W} + \eta^2, \\
p'(t) &= h\left(\frac{\partial u^1}{\partial x}(t,p(t)^-), \frac{\partial u^2}{\partial x}(t,p(t)^+)\right).
\end{aligned} \tag{1.5.2}$$

The main additional requirement is that the volatility decays at least linearly as the boundary is approached, and we present a general condition on σ which ensures that this is the case. The results here extend the work in [61] by incorporating reflection and allowing for drift and volatility coefficients which depend on the spatial variable as well as the solution itself, and are Lipschitz in the solution. The condition on the decay of the volatility at the boundary is also less restrictive, as the authors in [61] assumed space dependent volatility which vanished faster than $x^{3/2}$ at the boundary. In addition, we are able to choose a general Lipschitz function h to determine the boundary evolution. We present a statement of our main existence and uniqueness theorem, before adapting the work on the deterministic parabolic obstacle problem to fit this problem. This is important, as it ensures that the reflection component will not prevent our SPDE solutions from having derivatives at the boundary. We then establish some key estimates for the main proof before proving the main result of this section, existence and uniqueness for (1.5.2).

The section is concluded with simple simulations of the equations. We first illustrate the appearance of derivatives when the coefficients fall within our framework and explore scenarios where this is not the case. Following this, we briefly investigate possible blow-ups for the system. Motivated by the classical Stefan problem, we study stability in two scenarios - one which resembles the problem of ice melting when in contact with warm water, and another which resembles the typically unstable set-up where a supercooled liquid is freezing. Finally, we numerically illustrate the possibility of a rough scaling limit for the boundary motion in the case where the boundary function is symmetric in the two profiles and has the correct sign.

Chapter 2

Preliminaries

In this section we introduce some of the key mathematical background required for the thesis. This is split into two main sections. The first of these is a brief introduction to martingale measures and the theory of SPDEs which was pioneered by Walsh. This will allow us to make sense of the white noise terms in our equations, and we will also present some of the associated estimates which we will make use of. The second part will be dedicated to reflected SPDEs. We define a parabolic obstacle problem which is essential both for constructing solutions and for gaining an intuition for the equation. We then define a notion of solution for such equations, before presenting an existence and uniqueness theorem.

2.1 Martingale Measures and Stochastic Integration

We are interested in adding space-time white noise to PDEs in order to obtain equations which account for high frequency random perturbations. Analogously to how SDEs are made sense of in classical stochastic analysis, we first try to understand how to itegrate against the noise process. We follow the approach of Walsh, and build an SPDE theory starting with an understanding of martingale measures. We give a brief introduction to this here, following the book [9], to which we refer the reader for further information on the topic.

We work in a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ throughout this section. Since it is the main integrator which we will be considering in this thesis, we first define white noise.

Definition 2.1.1. *Fix some $N \in \mathbb{N}$, $D \in \mathscr{B}(\mathbb{R}^N)$ and let $I := \mathscr{B}(D)$. We define N-parameter white noise on D to be the unique Gaussian process $\{\dot{W}(A) \mid A \in \mathscr{B}(D)\}$ such that*

1. *$\mu(A) := \mathbb{E}[\dot{W}(A)] = 0$ for every $A \in \mathscr{B}(D)$.*
2. *$C(A, B) := cov(\dot{W}(A), \dot{W}(B)) = \mu_{leb}(A \cap B)$ for every $A, B \in \mathscr{B}(D)$.*

Definition 2.1.2. *Let $N \in \mathbb{N}$. We define the N-parameter Brownian sheet to be the unique Gaussian process $\{W((t_1, ...t_N)) \mid (t_1, ...t_N) \in [0, \infty)^N\}$ such that*

1. *$\mu((t_1, ...t_N)) := \mathbb{E}[W((t_1, ...t_N))] = 0$ for every $(t_1, ...t_N) \in [0, \infty)^N$.*

2. $C((s_1,...s_N),(t_1,...t_N)) := cov(W((s_1,...s_N)),W((t_1,...t_N))) = \prod_{i=1}^{N} \min(s_i,t_i)$ *for every* $(s_1,...s_N),(t_1,...t_N) \in [0,\infty)^N$.

Remark 2.1.3. *We note that if* $\dot{W}$ *is an two-parameter white noise defined on* $[0,\infty)^2$*, we can define a Brownian sheet,* W *by setting*

$$W((t,x)) = \dot{W}([0,t]\times[0,x]).$$

For this reason, we interchangeably use the notation $\dot{W}$ *and* $\frac{\partial^2 W}{\partial t \partial x}$ *for white noise Gaussian processes.*

We now define a more general class of integrators, martingale measures.

Definition 2.1.4. *We say that the collection* $\{M_t(A) \mid t \geq 0, A \in \mathscr{B}(\mathbb{R}^{N-1})\}$ *is a martingale measure w.r.t.* $\mathscr{F}_t$ *if*

1. $\forall A \in \mathscr{B}(\mathbb{R}^{N-1})$, $M_0(A) = 0$ *almost surely.*

2. $\forall t > 0$, M_t *is a* σ*-finite* $L^2(\Omega)$*-valued measure. That is,*

   (a) $M_t(\emptyset) = 0$.

   (b) $\forall$ *disjoint (non-random) sets* $(A_n)_{n\geq 1}$ *in* $\mathscr{B}(\mathbb{R}^N)$,

   $$\mathbb{P}\left(M_t\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M_t(A_n)\right) = 1,$$

   where $\sum_{n=1}^{\infty} M_t(A_n)$ *is a limit in* $L^2(\Omega)$.

3. $\forall A \in \mathscr{B}(\mathbb{R}^{N-1})$, $\{M_t(A) \mid t \geq 0\}$ *is a mean-zero martingale w.r.t.* $\mathscr{F}_t$.

The martingale measure associated with white noise is the white noise process, and is defined as follows.

Definition 2.1.5. *Let* $\{\dot{W}(A) \mid A \in \mathscr{B}(\mathbb{R}^+ \times \mathbb{R})\}$ *be a white noise. The white noise process associated with* $\dot{W}$ *is then given by* $\{W_t(A) \mid t \geq 0, A \in \mathscr{B}(\mathbb{R})\}$*, where*

$$W_t(A) := \dot{W}([0,t]\times A)$$

for every $t \geq 0$ *and every* $A \in \mathscr{B}(\mathbb{R})$*. This has a natural filtration given by*

$$\mathscr{F}_t := \sigma(W_s(A) \mid s \in [0,t], A \in \mathscr{B}(\mathbb{R}^{N-1})).$$

It can then be shown that $\{W_t(A) \mid t \geq 0, A \in \mathscr{B}(\mathbb{R})\}$ *is a martingale measure.*

Now that we have established a candidate class of integrators, we may begin to construct stochastic integrals against these processes. Similarly to in the construction of the Itô integral, we begin by defining the integral for a simple class of integrands.

Definition 2.1.6. *We say f is an elementary function if $f : \mathbb{R}^+ \times \mathbb{R}^{N-1} \times \Omega \to \mathbb{R}$ such that*

$$f(t, \mathbf{x}, \omega) = X(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_A(\mathbf{x}),$$

where X is bounded and $\mathscr{F}_a$-measurable, $A \in \mathscr{B}(\mathbb{R}^{N-1})$.

Definition 2.1.7. *We say f is a simple function if it is a linear combination of elementary functions. The class of simple functions is denoted by $\mathscr{L}$.*

It is clear how one should define the stochastic integral for elementary/simple integrands, so we start by doing this.

Definition 2.1.8. *For an elementary function $f : \mathbb{R}^+ \times \mathbb{R}^{N-1} \times \Omega \to \mathbb{R}$ such that*

$$f(t, \mathbf{x}, \omega) = X(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_A(\mathbf{x}),$$

we define the integral of f against the martingale measure M to be the martingale measure $f.M$ where, for $B \in \mathscr{B}(\mathbb{R}^{N-1})$,

$$(f.M)_t(B)(\omega) := X(\omega)\left[M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B)\right](\omega).$$

We extend this definition to the space of simple functions, $\mathscr{L}$, by linearity.

Definition 2.1.9. *The predictable sigma algebra $\mathscr{P}$ is the sigma algebra on $\mathbb{R}^+ \times \mathbb{R}^{N-1} \times \Omega$ generated by the functions in $\mathscr{L}$.*

The following is a generalisation of quadratic variation for martingale measures.

Definition 2.1.10. *Let M be a martingale measure.*

1. *The covariance functional is given by*

$$\bar{Q}_t^M(A, B) := \langle M.(A), M.(B)\rangle_t$$

   for $t \geq 0$, $A, B \in \mathscr{B}(\mathbb{R}^{N-1})$.‘

2. *The covariance set function is*

$$Q_M(A, B, (s, t]) := \bar{Q}_t^M(A, B) - \bar{Q}_s^M(A, B)$$

   for $s, t \geq 0$, $A, B \in \mathscr{B}(\mathbb{R}^{N-1})$.

We extend Q_M to act on disjoint unions $\bigcup_{i=1}^{n}\left(A_i \times B_i \times (s_i, t_i]\right)$ by linearity.

We aim to further enlarge the domain of Q_M so that it is a random measure on $\mathscr{B}(\mathbb{R}^{N-1}) \times \mathscr{B}(\mathbb{R}^{N-1}) \times \mathscr{B}(\mathbb{R}^+)$. This will allow us to extend our definition of integrals against a martingale measures to a wider class of integrands. We will also be able to recover Itô's isometry and Burkholder's inequality in this context. In order to do this, we need conditions on Q_M.

Definition 2.1.11. *A martingale measure M is called* worthy *if there exists a random σ-finite measure K on $\mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \times \mathbb{R}^+$ such that*

(i) For any $C \in \mathscr{B}(\mathbb{R}^+)$, $\omega \in \Omega$, the map

$$A \times B \mapsto K(A \times B \times C)(\omega)$$

from $\mathscr{B}(\mathbb{R}^{N-1}) \times \mathscr{B}(\mathbb{R}^{N-1}) \to \mathbb{R}^+$ is symmetric.

(ii) $\forall A, B \in \mathscr{B}(\mathbb{R}^{N-1})$, $(K(A \times B \times (0,t]))_{t \geq 0}$ is a predictable process.

(iii) $\forall A, B \in \mathscr{B}(\mathbb{R}^{N-1})$ compact, $\forall t \geq 0$,

$$\mathbb{E}\left[K(A \times B \times (0,t])\right] < \infty.$$

(iv) $\forall A, B \in \mathscr{B}(\mathbb{R}^{N-1})$, $\forall t \geq 0$,

$$|Q_M(A \times B \times (0,t])| \leq K(A \times B \times (0,t])$$

almost surely.

We then call K a dominating measure for M.

Remark 2.1.12. *If M is a worthy martingale measure there exists a unique extension of Q_M to a random measure on $\mathscr{B}(\mathbb{R}^{N-1}) \times \mathscr{B}(\mathbb{R}^{N-1}) \times \mathscr{B}(\mathbb{R}^+)$.*

Proposition 2.1.13. *Suppose M is a worthy martingale measure with dominating measure K, and $f \in \mathscr{L}$. Then $(f.M)$ is a worthy martingale measure and*

1. *$Q_{f.M}(dx,dy,dt) = f(x,t)f(y,t)Q_M(dx,dy,dt)$.*
2. *$K_{f.M}(dx,dy,dt) = |f(x,t)f(y,t)|K_M(dx,dy,dt)$ is a dominating measure for $(f.M)$.*

The following is a rough outline of how we can extend our stochastic integral when the integrator is a worthy martingale measure. We once again refer to [9] and [50] for further details. Let M be a worthy martingale measure with a dominating measure K_M. Fix $T \geq 0$ and define for $f : (0,T] \times \mathbb{R}^{N-1} \times \Omega \to \mathbb{R}$

$$\|f\|_M^2 := \mathbb{E}\left[\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \int_{(0,T]} |f(x,t)f(y,t)| K_M(\mathrm{dx},\mathrm{dy},\mathrm{dt})\right].$$

Define $\mathscr{P}_M$ to be the set of predictable functions $f : (0,T] \times \mathbb{R}^{N-1} \times \Omega \to \mathbb{R}$ with $\|f\|_M < \infty$. $\mathscr{L}$ is then dense in $\mathscr{P}_M$. We can show that, for $f \in \mathscr{L}$, $t \in [0,T]$, $B \in \mathscr{B}(\mathbb{R}^{N-1})$

$$\mathbb{E}\left[\left((f.M)_t(B)\right)^2\right] \leq \|f\|_M^2.$$

We can therefore define $(f.M)$ for a general $f \in \mathscr{P}_M$, by setting $(f.M)_t(B)$ to be the $L^2(\Omega)$ limit of $(f_n.M)_t(B)$ for a sequence of simple f_n approximating f in $\mathscr{P}_M$. The previous proposition now holds for all $f \in \mathscr{P}_M$.

The following estimates will be very useful in proving some of the main results in this thesis.

Theorem 2.1.14 (Theorem 5.26, [9]). *Let M be a worthy martingale measure, $f \in \mathscr{P}_M$. Then for $t \in [0,T]$, $A, B \in \mathscr{B}(\mathbb{R}^{N-1})$*

1. $\left\langle (f.M)_\cdot(A), (f.M)_\cdot(B) \right\rangle_t = \int_A \int_B \int_{(0,t]} f(x,s) f(y,s) Q_M(dx,dy,ds)$.
2. $\mathbb{E}\left[\left((f.M)_t(B) \right)^2 \right] \leq \|f\|_M^2$.

Theorem 2.1.15. Burkholder's Inequality, Theorem 5.27, [9]
For every $p \geq 2$, there exists $c_p \in (0,\infty)$ such that for every predictable f and every $t \geq 0$,

$$\mathbb{E}\left[|(f.M)_t(B)|^p \right] \leq C_p \mathbb{E}\left[\left(\int_B \int_B \int_{(0,t]} |f(x,s) f(y,s)| K_M(dx,dy,ds) \right)^{\frac{p}{2}} \right].$$

2.2 Obstacle Problems and Reflected SPDEs

In this section, we give a brief overview of *reflected* SPDEs. These are equations of the form

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x,u(t,\cdot)) + \sigma(t,x,u(t,\cdot)) \frac{\partial^2 W}{\partial x \partial t} + \eta. \tag{2.2.1}$$

W here is a Brownian sheet, whilst η is a source term which maintains positivity of the solution. The measure η is analogous to the local time of a diffusion process.

2.2.1 The Deterministic Obstacle Problem

In order to both gain a strong intuition for reflected SPDEs and to develop some of the key techniques in proving some of our later results, we must first understand the deterministic parabolic obstacle problem. Intuitively, the problem is as follows. Given a suitable obstacle function, v, we look for a function z such that:

1. $z \geq v$.
2. z evolves according to the heat equation on the set $\{z > v\}$.
3. z is *minimally* pushed upwards on the set $\{z = v\}$, so that it remains above v.

In essence, the problem is a functional analogue of the Skorohod problem, where the aim is to minimally push a real-valued process upwards so that it remains above a given threshold.

We now define our notion of solution for such problems.

Definition 2.2.1. *Let $v \in C([0,T]; C_0((0,1)))$ with $v(0,x) \leq 0$. We say that the pair (z,η) satisfies the heat equation with obstacle v if:*

(i) $z \in C([0,T]; C_0((0,1)))$, $z(0,x) \equiv 0$ *and* $z \geq v$.

(ii) η *is a measure on* $(0,1) \times [0,T]$.

(iii) z weakly solves the PDE

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \eta \tag{2.2.2}$$

That is, for every $t \in [0,T]$ and every $\phi \in C^2((0,1)) \cap C_0((0,1))$,

$$\int_0^1 z(t,x)\phi(x)\,dx = \int_0^t \int_0^1 z(s,x)\phi''(x)\,dxds + \int_0^t \int_0^1 \phi(x)\ \eta(dx,ds).$$

(iv) $\int_0^t \int_0^1 (z(s,x) - v(s,x))\ \eta(dx,\,ds) = 0.$

An important property of this problem is that the map from obstacles to solutions is in fact a contraction. This was proved in [40] and used in the Picard argument in [52], allowing the authors to control reflected SPDEs by corresponding unreflected equations.

Theorem 2.2.2 ([40], Theorem 1.4)**.** *For every $v \in C([0,T]; C_0((0,1)))$ with $v(0,x) \leq 0$, there exists a unique solution to the above obstacle problem. In addition, if (z_1, η_1) and (z_2, η_2) solve the obstacle problems with obstacles v_1 and v_2 respectively, then for $t \in [0,T]$, we have the estimate*

$$\|z_1 - z_2\|_{\infty,t} \leq \|v_1 - v_2\|_{\infty,t} \tag{2.2.3}$$

where $\|.\|_{\infty,t}$ is defined for $u \in C([0,T] \times [0,1])$ by

$$\|u\|_{\infty,t} := \sup_{s\in[0,t]} \sup_{x\in[0,1]} |u(s,x)|.$$

The proof of this result is based on a penalisation method, where the limit is taken as $\epsilon \downarrow 0$ for solutions to the equations

$$\frac{\partial z_\epsilon}{\partial t} = \Delta z_\epsilon + \frac{(z_\epsilon - v)^-}{\epsilon}. \tag{2.2.4}$$

We refer the reader to [40] for details.

2.2.2 Reflected SPDEs

Definition 2.2.3. *Let $f, \sigma : [0,\infty) \times [0,1] \times C_0((0,1)) \to \mathbb{R}$ be measurable functions. We say that the pair (u,η) is a solution the SPDE with reflection*

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x,u(t,\cdot)) + \sigma(t,x,u(t,\cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta \tag{2.2.5}$$

with Dirichlet conditions $u(t,0) = u(t,1) = 0$ and initial data $u(0,x) = u_0 \in C_0((0,1))^+$ if

(i) u is a continuous adapted random field on $\mathbb{R}^+ \times [0,1]$ such that $u \geq 0$ almost surely.

(ii) η is a random measure on $\mathbb{R}^+ \times (0,1)$ such that:

(a) For every $t \geq 0$, $\eta(\{t\} \times (0,1)) = 0$,

(b) For every $t \geq 0$, $\int_0^t \int_0^1 x(1-x)\eta(ds,dx) < \infty$,

(c) η is adapted in the sense that for any measurable mapping ψ:

$$\int_0^t \int_0^1 \psi(s,x)\ \eta(ds,dx) \ \text{ is } \mathscr{F}_t - \text{measurable.}$$

(iii) For every $t \geq 0$ and every $\phi \in C^2([0,1])$ with $\phi(0) = \phi(1) = 0$,

$$\begin{aligned}\int_0^1 u(t,x)\phi(x)dx = &\int_0^1 u(0,x)\phi(x)dx + \int_0^t \int_0^1 u(s,x)\phi''(x)dxds \\ &+ \int_0^t \int_0^1 f(s,x,u(s,\cdot))\phi(x)dxds \\ &+ \int_0^t \int_0^1 \sigma(s,x,u(s,\cdot))\phi(x)W(ds,dx) \\ &+ \int_0^t \int_0^1 \phi(x)\ \eta(ds,dx)\end{aligned}$$

almost surely.

(iv) $\int_0^\infty \int_0^1 u(t,x)\ \eta(dt,dx) = 0$.

Theorem 2.2.4 ([52], Theorem 2.1)**.** *Suppose that f and σ are such that for every $T, M > 0$, there exists $C_{T,M}$ such that for every $u, v \in C_0((0,1))$ with $\|u\|_\infty, \|v\|_\infty \leq M$, every $t \leq T$ and every $x \in [0,1]$,*

$$|f(t,x,u) - f(t,x,v)| + |\sigma(t,x,u) - \sigma(t,x,v)| \leq C_{T,M}\|u - v\|_\infty.$$

Then for every $u_0 \in C_0((0,1))^+$, there exists a unique solution to (2.2.5) with initial data u_0.

Chapter 3

Model with Discrete Boundary Motion

In this section, we work towards the first of our SPDE moving boundary models, where boundary movements occur as jumps. The work here is an adaptation and extension of the work in [38]. We imagine that the boundary separates particles of two types, type a and type b. We keep track of the volumes of these two types at fixed, finite distances from the shared interface, whose own motion depends on the volumes of the two particles at different spatial locations at any given time. Given existence and uniqueness for reflected SPDEs one can, of course, write down such models immediately. We could define the volume dynamics of each of the two types so that they evolve according to two separate SPDEs. We could then define suitable stopping times determining when the boundary should move, and also specify how to reinitialise the equations following a boundary movement. Uniqueness would follow simply from uniqueness for the underlying SPDE models. The main contribution in this section, however, is that we are able to connect such models to a microscopic particle picture by taking scaling limits. We begin by introducing the microscopic model.

3.1 Microscopic Model in a Static Setting

We first describe the dynamics of our microscopic model in the absence of boundary movements, and assume that the position of the boundary is fixed at $m \in \mathbb{R}$. The index n here will be used later in order to take a diffusive scaling of the model. For every $n \in \mathbb{N}$, we consider two $N-1$- dimensional processes, $Z_n^a = (Z_n^{a,1}, ...Z_n^{a,N-1})$ and $Z_n^b = (Z_n^{b,1}, ...Z_n^{b,N-1})$, each taking values in $\mathbb{Z}^{N-1}$. The spatial grid is discrete, with each type occupying positions indexed by $\{0, 1, 2, ...N\}$, representing locations on either side of the shared boundary, m. The values of $Z_n^{a,0}$, $Z_n^{a,N}$, $Z_n^{b,0}$ and $Z_n^{b,N}$ are all fixed at zero. $Z_n^{a,i}$ and $Z_n^{b,i}$ represent the number of type a/type b particles, currently positioned at the i^{th} spatial location to the left/right of the boundary respectively. Particles are assumed to move, arrive and disappear from the system at different locations according to a combination of the following three components:

1. At each location, there is a common high frequency rate for arrival and disappearance of particles. We allow for these rates to be dependent on the particle type, relative location (with respect to the boundary position), current position of the boundary, and the number of particles currently at that position.

2. Residual imbalance between arrivals and disappearances at different locations gives lower frequency terms. These are once again allowed to be dependent on position, the current boundary location and the number of particles at that position.

3. All particles undergo a random walk. This has a smoothing effect - if there are more particles on average at the two neighbouring positions, then the net flow of the random walking particles will be positive at this position. Similarly, the net flow of the random walking particles will be negative there if a given location has more particles than the average of it's neighbours.

Altogether, this motivates the following description for the dynamics of the type a particles in our model. To simplify the notation here, we define e_i to be the usual basis functions for $\mathbb{R}^{N-1}$ for $i = 1, 2, ...N-1$, and we use the convention $e_0 = e_N = 0$. We also note that in the following, for $k \in \{a, b\}$, every n and every $m \in \mathbb{R}$, $\sigma_{k,m,n}$, $f_{k,m,n}$ and $g_{k,m,n}$ are maps from $\{1, 2, ...N-1\} \times \mathbb{N} \to \mathbb{R}^+$.

(i) For $i \in \{1, ..., N-1\}$, $Z_n^a \to Z_n^a + e_i$ at exponential rate

$$\frac{1}{2}\sigma^2_{a,m,n}(i, Z_n^{a,i}) \left(1 + \mathbb{1}_{\left\{Z_n^{a,i}=0\right\}}\right) + f_{a,m,n}\left(i, Z_n^{a,i}\right).$$

(ii) For $i \in \{1, ..., N-1\}$, $Z_n^a \to Z_n^a - e_i$ at exponential rate

$$\frac{1}{2}\sigma^2_{a,m,n}(i, Z_n^{a,i}) \mathbb{1}_{\left\{Z_n^{a,i}\geq 1\right\}} + g_{a,m,n}\left(i, Z_n^{a,i}\right) \mathbb{1}_{\left\{Z_n^{a,i}\geq 1\right\}}.$$

(iii) For $i \in \{1, ..., N-1\}$, $Z_n^a \to Z_n^a + e_{i-1} - e_i$ at exponential rate

$$\alpha_{a,n} Z_n^{a,i}$$

(iv) For $i \in \{1, ..., N-1\}$, $Z_n^a \to Z_n^a + e_{i+1} - e_i$ at exponential rate

$$\alpha_{a,n} Z_n^{a,i}$$

Similarly, the dynamics for the type b particles are given by:

(i) For $i \in \{1, ..., N-1\}$, $Z_n^b \to Z_n^b + e_i$ at exponential rate

$$\frac{1}{2}\sigma^2_{b,m,n}(i, Z_n^{b,i}) \left(1 + \mathbb{1}_{\left\{Z_n^{b,i}=0\right\}}\right) + f_{b,m,n}\left(i, Z_n^{b,i}\right).$$

(ii) For $i \in \{1, ..., N-1\}$, $Z_n^b \to Z_n^b - e_i$ at exponential rate

$$\frac{1}{2}\sigma^2_{b,m,n}(i, Z_n^{b,i})\mathbb{1}_{\left\{Z_n^{b,i}\geq 1\right\}} + g_{b,m,n}\left(i, Z_n^{b,i}\right)\mathbb{1}_{\left\{Z_n^{b,i}\geq 1\right\}}.$$

(iii) For $i \in \{1, ..., N-1\}$, $Z_n^b \to Z_n^b + e_{i-1} - e_i$ at exponential rate

$$\alpha_{b,n} Z_n^{b,i}$$

(iv) For $i \in \{1, ..., N-1\}$, $Z_n^b \to Z_n^b + e_{i+1} - e_i$ at exponential rate

$$\alpha_{b,n} Z_n^{b,i}$$

3.2 Heavy Traffic Diffusion Approximation in a Static Setting

Using classical theory on the convergence of Markov processes, we are able to obtain a diffusion limit for our microscopic static model. Recall that the word static here refers to the fact that the boundary position is fixed. We will call the limit the static mesoscopic model. The convergence result for the static models will be the basis for proving convergence of the corresponding dynamic models, in which the boundary is able to move.

The processes are scaled with the parameter n by accelerating time by a factor of n and dividing volumes by $\sqrt{n}$. That is, we look for limits of

$$\tilde{Z}_n^a(t) := \frac{Z_n^a(nt)}{\sqrt{n}} \quad \text{and} \quad \tilde{Z}_n^b(t) := \frac{Z_n^b(nt)}{\sqrt{n}}.$$

Therefore, these processes take values in $\frac{1}{\sqrt{n}}\mathbb{N}^{N-1}$ and the limiting processes will take values in $(\mathbb{R}^+)^{N-1}$. In order to obtain convergence, we need that various quantities in our microscopic model converge suitably. We will assume that

(i) For $k \in \{a, b\}$, every $m \in \mathbb{R}$ and every $n \geq 1$

$$\sigma_{k,m,n}(i, u) = \sigma_{k,m}\left(i, \frac{u}{\sqrt{n}}\right)$$

for every $i \in \{1, 2, ...N-1\}$ and every $u \in \mathbb{N}$.

(ii) For $k \in \{a, b\}$, every $m \in \mathbb{R}$ and every $n \geq 1$

$$f_{k,m,n}(i, u) = \frac{1}{\sqrt{n}} f_{k,m}\left(i, \frac{u}{\sqrt{n}}\right),$$

$$g_{k,m,n}(i, u) = \frac{1}{\sqrt{n}} g_{k,m}\left(i, \frac{u}{\sqrt{n}}\right),$$

for every $i \in \{1, 2, ...N-1\}$ and every $u \in \mathbb{N}$.

(iii) For $k \in \{a, b\}$, $\alpha_{k,n} = \frac{1}{n}\alpha_k > 0$.

(iv) For $k \in \{a, b\}$, $\frac{Z_n^k(0)}{\sqrt{n}} \implies X^k(0)$ in law in $(\mathbb{R}^+)^{N-1}$.

Here, the functions $\sigma_{k,m}$, $f_{k,m}, g_{k,m}$ and $h_{k,m} := f_{k,m} - g_{k,m}$ are all measurable from $\{1, 2, ...N-1\} \times \mathbb{R}^+ \to \mathbb{R}^+$. For technical reasons, we further assume that these functions are Lipschitz continuous in the second argument. We note here that this implies that they are also bounded on compact sets, with linear growth in the second argument.

Theorem 3.2.1. *The $\frac{1}{\sqrt{n}}\mathbb{N}^{N-1} \times \frac{1}{\sqrt{n}}\mathbb{N}^{N-1}$- valued process $(\tilde{Z}_n^a, \tilde{Z}_n^b)$ converges weakly in $\mathscr{M}\left(\mathbb{D}([0,\infty);\mathbb{R}^{N-1}) \times \mathbb{D}([0,\infty);\mathbb{R}^{N-1})\right)$ as $n \to \infty$ to the unique $[0,\infty)^{N-1} \times [0,\infty)^{N-1}$-valued strong Markov diffusion process (X^a, X^b) which satisfies the following system of reflected SDEs:*

$$dX_t^{a,i} = \alpha_a(X_t^{a,i+1} + X_t^{a,i-1} - 2X_t^{a,i})dt + h_{a,m}(i, X_t^{a,i})dt + \sigma_{a,m}(i, X_t^{a,i})dW_t^{a,i} + d\eta_t^{a,i},$$
$$dX_t^{b,i} = \alpha_b(X_t^{b,i+1} + X_t^{b,i-1} - 2X_t^{b,i})dt + h_{b,m}(i, X_t^{b,i})dt + \sigma_{b,m}(i, X_t^{b,i})dW_t^{b,i} + d\eta_t^{b,i},$$

for $i = 1, ..., N-1$ with the pinning conditions that $X^{k,0} = X^{k,N} = 0$, where $W^{k,i}$ are independent Brownian motions. The $\eta^{k,i}$ are reflection measures which maintain positivity of the $X^{k,i}$.

Proof. Since the particle/SDE systems decouple into two independent systems, it is sufficient to prove convergence of $\tilde{Z}_n^b$ to X^b in $\mathscr{M}(\mathbb{D}([0,\infty);\mathbb{R}^{N-1}))$. It follows from the dynamics of Z_n^b that the rescaled process, $\tilde{Z}_n^b(t)$, has dynamics given by:

(i) For $i \in \{1, ..., N-1\}$, $\tilde{Z}_n^b \to \tilde{Z}_n^b + \frac{e_i}{\sqrt{n}}$ at exponential rate

$$\frac{n}{2}\sigma_{b,m}^2(i, \tilde{Z}_n^{b,i})\left(1 + \mathbb{1}_{\left\{\tilde{Z}_n^{b,i}=0\right\}}\right) + \sqrt{n}f_{b,m}\left(i, \tilde{Z}_n^{b,i}\right).$$

(ii) For $i \in \{1, ..., N-1\}$, $\tilde{Z}_n^b \to \tilde{Z}_n^b - \frac{e_i}{\sqrt{n}}$ at exponential rate

$$\frac{n}{2}\sigma_{b,m}^2(i, \tilde{Z}_n^{b,i})\mathbb{1}_{\left\{\tilde{Z}_n^{b,i} \geq \frac{1}{\sqrt{n}}\right\}} + \sqrt{n}g_{b,m}\left(i, \tilde{Z}_n^{b,i}\right)\mathbb{1}_{\left\{\tilde{Z}_n^{b,i} \geq \frac{1}{\sqrt{n}}\right\}}.$$

(iii) For $i \in \{1, ..., N-1\}$, $\tilde{Z}_n^b \to \tilde{Z}_n^b + \frac{e_{i-1}}{\sqrt{n}} - \frac{e_i}{\sqrt{n}}$ at exponential rate

$$\alpha_b\sqrt{n}\tilde{Z}_n^{b,i}$$

(iv) For $i \in \{1, ..., N-1\}$, $\tilde{Z}_n^b \to \tilde{Z}_n^b + \frac{e_{i+1}}{\sqrt{n}} - \frac{e_i}{\sqrt{n}}$ at exponential rate

$$\alpha_b\sqrt{n}\tilde{Z}_n^{b,i}$$

We note that the indicator functions above will ensure that our processes converge to *reflected* SDEs. In order to prove convergence of these processes, we will argue that their infinitesimal generators converge (see Corollary 8.7, [21]). We start by calculating the infinitesimal generators for the processes $\tilde{Z}_n^b$. Define:

$$\Delta_{n,k}^l F(y) := \sqrt{n}\left[F(y) - F\left(y - \frac{e_k}{\sqrt{n}}\right)\right],$$
$$\Delta_{n,k}^r F(y) := \sqrt{n}\left[F\left(y + \frac{e_k}{\sqrt{n}}\right) - F(y)\right],$$
$$\Delta_{n,k}^2 := n\left[F\left(y + \frac{e_k}{\sqrt{n}}\right) + F\left(y - \frac{e_k}{\sqrt{n}}\right) - 2F(y)\right].$$

For ease of notation, we use the convention that terms are zero whenever they refer to the 0^{th} or N^{th} queues. Then we have that, for $F \in C_0([0,\infty)^{N-1}))$, the continuous functions on $[0,\infty)^{N-1}$ which vanish at ∞,

$$\begin{aligned}\frac{1}{t}\left(\mathbb{E}\left[F(\tilde{Z}_n(t))|\tilde{Z}_n(0)=y\right]-F(y)\right) &= \sum_{k=1}^{N-1}\Delta^r_{n,k}F(y)\left(\frac{\sqrt{n}}{2}\sigma^2_{b,m}\left(k,y_k\right)\left(1+\mathbb{1}_{\{y_k=0\}}\right)\right)\\ &+\sum_{k=1}^{N-1}\Delta^r_{n,k}F(y)f_{b,m}\left(k,y_k\right)\\ &+\sum_{k=1}^{N-1}\left(\Delta^r_{n,k-1}F(y-\frac{e_k}{\sqrt{n}})-\Delta^l_{n,k}F(y)\right)y_k\\ &+\sum_{k=1}^{N-1}\Delta^l_{n,k}F(y)\left(-\frac{\sqrt{n}}{2}\sigma^2_{b,m}\left(k,y_k\right)\mathbb{1}_{\left\{y_k\geq\frac{1}{\sqrt{n}}\right\}}\right)\\ &+\sum_{k=1}^{N-1}\Delta^l_{n,k}F(y)\left(-g_{b,m}\left(k,y_k\right)\mathbb{1}_{\left\{y_k\geq\frac{1}{\sqrt{n}}\right\}}\right)\\ &+\sum_{k=1}^{N-1}\left(\Delta^r_{n,k+1}F(y-\frac{e_k}{\sqrt{n}})-\Delta^l_{n,k}F(y)\right)y_k\\ &+R_{y,t},\end{aligned}$$

where $R_{y,t}\to 0$ as $t\to 0$ uniformly in compact sets for y. Note that this control on $R_{y,t}$ is a consequence of the local boundedness of σ, f and g, since these conditions ensure that the jump rate of the process after the first jump from y can be bounded. If we further assumed that F were smooth with compact support, so $F\in C^\infty_c([0,\infty)^{N-1})$, we obtain that the remainder term $R_{y,t}$ converges uniformly to zero for $y\in(\frac{1}{\sqrt{n}}\mathbb{N})^{N-1}$. Therefore, writing A_n for the generator of $\tilde{Z}_n$ and rearranging gives that for $F\in C^\infty_c([0,\infty)^{N-1})$,

$$\begin{aligned}A_nF(y) &= \sum_{k=1}^{N-1}\frac{1}{2}\Delta^2_{n,k}F(y)\mathbb{1}_{\left\{y_k\geq\frac{1}{\sqrt{n}}\right\}}\sigma^2_{b,m}(k,y_k)+\sum_{k=1}^{N-1}\Delta^r_{n,k}F(y)\sqrt{n}\sigma^2_{b,m}(k,y_k)\mathbb{1}_{\{y_k=0\}}\\ &+\sum_{k=1}^{N-1}\Delta^r_{n,k}F(y)f_{b,m}(k,y_k)-\sum_{k=1}^{N-1}\Delta^l_{n,k}F(y)g_{b,m}(k,y_k)\mathbb{1}_{\left\{y_k\geq\frac{1}{\sqrt{n}}\right\}}\\ &+\sum_{k=1}^{N-1}\left(\Delta^r_{n,k-1}F(y-\frac{e_k}{\sqrt{n}})+\Delta^r_{n,k+1}F(y-\frac{e_k}{\sqrt{n}})-2\Delta^l_{n,k}F(y)\right)y_k.\end{aligned}$$

Recall that our candidate limiting process, X^b, satisfies the system of reflected SDEs

$$\mathrm{d}X^{b,i}_t=\alpha_b(X^{b,i+1}_t+X^{b,i-1}_t-2X^{b,i}_t)\mathrm{d}t+h_{b,m}(i,X^{b,i}_t)\mathrm{d}t+\sigma_{b,m}(i,X^{b,i}_t)\mathrm{d}W^{b,i}_t+\mathrm{d}\eta^{b,i}_t,$$

for $i=1,2,...N-1$, with $X^0=X^N=0$ and $h_{b,m}:=f_{b,m}-g_{b,m}$. Note that by Theorem 4.1 in [49], we have existence of a strong solution and pathwise uniqueness for this system of SDEs. Inspection of the proof also reveals that the solution here, where the diffusion and drift coefficients do not have any explicit time-dependence, is continuous. We can calculate the corresponding generator to be

$$AF(x)=\frac{1}{2}\sum_{k=1}^{N-1}\sigma^2_{b,m}(k,x_k)\frac{\partial^2F}{\partial x_k^2}\ +\ \sum_{k=1}^{N-1}\left[h_{b,m}(k,x_k)+\alpha_b(x_{k+1}+x_{k-1}-2x_k)\right]\frac{\partial F}{\partial x_k},$$

acting on the domain

$$\mathscr{D}(A) = \left\{ F \in C_0^2([0,\infty)^{N-1}) \text{ s.t. } \forall k, \ \left.\frac{\partial F}{\partial x_k}\right|_{x_k=0} = 0 \right\}.$$

This has a core given by

$$C(A) = \left\{ F \in C_c^\infty([0,\infty)^{N-1}) \text{ s.t. } \forall k, \ \left.\frac{\partial F}{\partial x_k}\right|_{x_k=0} = 0 \right\}.$$

It is enough to prove (see Corollary 8.7 in [21]) that for $F \in C(A)$

$$\sup_{y \in \frac{1}{\sqrt{n}}\mathbb{N}^{N-1}} |A_n F(y) - AF(y)| \to 0.$$

First suppose that $y_k \geq \frac{1}{\sqrt{n}}$. Then, by Taylor's Theorem we have that

$$\frac{1}{2}\sigma_{b,m}^2(k,y_k)\left|\Delta_{n,k}^2 F(y) - \frac{\partial^2 F}{\partial y_k^2}(y)\right| \leq \frac{1}{6\sqrt{n}}\left\|\frac{\partial^3 F}{\partial y_k^3}\right\|_\infty \|\sigma_{b,m}^2\|_{F,\infty},$$

where $\|\sigma_{b,m}^2\|_{F,\infty}$ is the supremum of $\sigma_{b,m}^2$ over the support of F. Similarly

$$\left|\Delta_{n,k}^r F(y) - \frac{\partial F}{\partial y_k}(y)\right| f_{b,m}(k,y_k) \leq \frac{1}{2\sqrt{n}}\left\|\frac{\partial^2 F}{\partial y_k^2}\right\|_\infty \|f_{b,m}\|_{F,\infty},$$

and

$$\left|\Delta_{n,k}^l F(y) - \frac{\partial F}{\partial y_k}(y)\right| g_{b,m}(k,y_k) \leq \frac{1}{2\sqrt{n}}\left\|\frac{\partial^2 F}{\partial y_k^2}\right\|_\infty \|g_{b,m}\|_{F,\infty}.$$

Now suppose that $y_k = 0$. Then, again by Taylor's theorem (extending F by reflection about the axes and using that $\frac{\partial F}{\partial y_k}$ is zero when $y_k = 0$) we have that

$$\sigma_{b,m}^2(k,y_k)\left|\sqrt{n}\Delta_{n,k}^r F(y) - \frac{1}{2}\frac{\partial^2 F}{\partial y_k^2}(y)\right| \leq \frac{1}{6\sqrt{n}}\left\|\frac{\partial^3 F}{\partial y_k^3}\right\|_\infty \|\sigma_{b,m}^2\|_{F,\infty},$$

$$f_{b,m}(k,y_k)\left|\Delta_{n,k}^r F(y) - \frac{\partial F}{\partial y_k}(y)\right| = f_{b,m}(k,y_k)\left|\Delta_{n,k}^r F(y)\right| \leq \frac{1}{2\sqrt{n}}\left\|\frac{\partial^2 F}{\partial y_k^2}\right\|_\infty \|f_{b,m}\|_{F,\infty},$$

$$g_{b,m}(k,y_k)\left|\Delta_{n,k}^l F(y)\mathbb{1}_{\left\{y_k \geq \frac{1}{\sqrt{n}}\right\}} - \frac{\partial F}{\partial y_k}(y)\right| = 0.$$

We can argue similarly for the sum

$$\sum_{k=1}^{N-1}\left(\Delta_{n,k-1}^r F(y - \frac{e_k}{\sqrt{n}}) + \Delta_{n,k+1}^r F(y - \frac{e_k}{\sqrt{n}}) - 2\Delta_{n,k}^l F(y)\right) y_k,$$

and find that

$$\sup_{y \in \frac{1}{\sqrt{n}}\mathbb{N}^{N-1}}\left|BF(y) - \sum_{k=1}^{N-1}\left[\Delta_{n,k-1}^r F(y - \frac{e_k}{\sqrt{n}}) + \Delta_{n,k+1}^r F(y - \frac{e_k}{\sqrt{n}}) - 2\Delta_{n,k}^l F(y)\right] y_k\right| \to 0,$$

where

$$BF(y) := \sum_{k=1}^{N-1} \left[\frac{\partial F}{\partial y_{k-1}} + \frac{\partial F}{\partial y_{k+1}} - 2\frac{\partial F}{\partial y_k} \right] y_k.$$

Recalling our notational convention that terms referring to the 0^{th} and N^{th} queues are zero, we have that

$$BF(y) = \sum_{k=1}^{N-1} \frac{\partial F}{\partial y_k} \left(y_{k+1} + y_{k-1} - 2y_k \right).$$

Putting these cases together gives that Theorem 3.2.1 holds. □

3.3 Microscopic Model in a Dynamic Setting

We now describe the mechanism for boundary movements in the model to give our microscopic dynamic model. Boundary movements in both directions will be assumed to occur at positive rates which depend on the state of the processes Z_n^a, Z_n^b and the current position of the boundary at any given time. Our motivating example is the case where these rates are dependent on the imbalance of the number of particles which are currently at the locations directly on either side of the boundary. Therefore, the driving factor for the boundary moving in either direction is the local dominance of one type over the other. We think of the boundary movement as one of the two types conquering territory from the other, pushing the interface into the region of the opposite type.

To formalise this, we introduce functions $\theta_{u,m}^n$ and $\theta_{d,m}^n$ for $n \geq 1$ and $m \in \mathbb{R}$, which map $\mathbb{N}^{N-1} \times \mathbb{N}^{N-1} \to \mathbb{R}_{>0}$. These will determine the rate of upward and downward boundary movements respectively as a function of the profiles Z_n^a and Z_n^b, when the boundary is currently at position m. We also define for $n \geq 1$ functions R^n. These map $\mathbb{N}^{N-1} \times \mathbb{N}^{N-1} \times \{u, d\} \to \mathscr{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1})$ and will determine the distribution of the new profiles for Z_n^a and Z_n^b following boundary movements as functions of the profiles at the time of the boundary movement and the direction of the boundary movement. We fix some $\epsilon > 0$, which determines the size of boundary movements. We typically assume that the size of boundary movements correspond to the profiles Z_n^a and Z_n^b being shifted by some integer number of queues, so that the boundary motion and the profiles are scaled with one another in a natural way. We additionally introduce i.i.d. rate one exponential random variables $(Y_{n,u}^i)_{i=1}^\infty$ and $(Y_{n,d}^i)_{i=1}^\infty$ which will be used in the construction. These are independent of each other and of the driving Poisson processes in the model. With this is place, we can start to construct our dynamic process. Let $Z_{n,1}^a(0), Z_{n,1}^b(0) \in \mathbb{N}^{N-1}$ be the initial profiles, and let m_n^1 be the initial boundary position for the n^{th} dynamic microscopic model. We denote by $Z_{n,1}^a, Z_{n,1}^b$ the processes evolving according to our dynamics for the static model with boundary position m_n^1 and initial profiles $Z_{n,1}^a(0), Z_{n,1}^b(0)$. The stopping times $\tau_{n,u}^1$ and $\tau_{n,d}^1$ are defined as follows:

$$\tau_{n,u}^1 := \inf \left\{ t \geq 0 \;\middle|\; \int_0^t \theta_{u,m_n^1}^n (Z_{n,1}^a(s), Z_{n,1}^b(s)) \, \mathrm{d}s \geq Y_{n,u}^1 \right\},$$

$$\tau^1_{n,d} := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \theta^n_{d,m^1_n}(Z^a_{n,1}(s), Z^b_{n,1}(s))\, \mathrm{d}s \geq Y^1_{n,d} \right\}.$$

We also define $\tau^1_n := \tau^1_{n,u} \wedge \tau^1_{n,d}$. The time τ^1_n triggers a jump in the boundary position. If $\tau^1_n = \tau^1_{n,u}$, the boundary position will increase to the point $m^2_n = m^1_n + \epsilon$. Similarly, if $\tau^1_n = \tau^1_{n,d}$, the boundary position will decrease to $m^2_n = m^1_n - \epsilon$. We then let $Z^a_{n,2}, Z^b_{n,2}$ be the new processes which follow the dynamics of the static microscopic model with boundary position m^2_n. The initial profile has the law of $R^n(Z^a_{n,1}(\tau^1_n), Z^b_{n,1}(\tau^1_n), u)$ if the boundary moved upwards and $R^n(Z^a_{n,1}(\tau^1_n), Z^b_{n,1}(\tau^1_n), d)$ if the boundary moved downwards, and is conditionally independent of the past behaviour of the process given $(Z^a_{n,1}(\tau^1_n), Z^b_{n,1}(\tau^1_n))$ and the direction of the boundary movement. The processes $Z^a_{n,2}$, $Z^b_{n,2}$ are taken to be conditionally independent of the past of the dynamic microscopic model given the initial profile and m^2_n. We can iterate this procedure to define further stopping times, boundary positions and processes $Z^a_{n,i}, Z^b_{n,i}$, describing the dynamics after the i^{th} jump in the boundary position. Having defined $(Z^a_{n,i})_{i=1}^M$, $(Z^b_{n,i})_{i=1}^M$, $(\tau^i_n)_{i=1}^{M-1}$ and $(m^i_n)_{i=1}^M$, we set

$$\tau^M_{n,u} := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \theta^n_{u,m^M_n}(Z^a_{n,M}(s), Z^b_{n,M}(s))\, \mathrm{d}s \geq Y^M_{n,u} \right\},$$

$$\tau^M_{n,d} := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \theta^n_{d,m^M_n}(Z^a_{n,M}(s), Z^b_{n,M}(s))\, \mathrm{d}s \geq Y^M_{n,d} \right\}.$$

As before, we define $\tau^M_n := \tau^M_{n,u} \wedge \tau^M_{n,d}$, with the time τ^M_n triggering a change in the boundary position. If $\tau^M_n = \tau^M_{n,u}$, we set $m^{M+1}_n = m^M_n + \epsilon$ and if $\tau^M_n = \tau^M_{n,d}$, we set $m^{M+1}_n = m^M_n - \epsilon$. We then let $Z^a_{n,M+1}, Z^b_{n,M+1}$ be new processes which follow the dynamics of the static microscopic model with the boundary position now given by m^{M+1}_n, and the initial profile having the law of $R^n(Z^a_{n,M}(\tau^M_n), Z^b_{n,M}(\tau^M_n), k)$, where k is the direction of the boundary movement at time τ^M_n. Once again, $Z^a_{n,M+1}$, $Z^b_{n,M+1}$ are taken to be conditionally independent of the past of the dynamic microscopic model given the initial profile and the new boundary position. Our dynamic microscopic model is then described by the processes $(\hat{Z}^a_n(t), \hat{Z}^b_n(t), m_n(t))$, representing the evolution of the volumes of the two particle types in positions relative to the boundary through time, and the current boundary position, where

$$\hat{Z}^a_n(t) := \sum_{i=1}^{\infty} Z^a_{n,i}\left(t - \sum_{j=1}^{i-1} \tau^j_n \right) \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau^j_n \leq t < \sum_{j=1}^{i} \tau^j_n \right\}},$$

$$\hat{Z}^b_n(t) := \sum_{i=1}^{\infty} Z^b_{n,i}\left(t - \sum_{j=1}^{i-1} \tau^j_n \right) \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau^j_n \leq t < \sum_{j=1}^{i} \tau^j_n \right\}},$$

$$m_n(t) := \sum_{i=1}^{\infty} m^i_n \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau^j_n \leq t < \sum_{j=1}^{i} \tau^j_n \right\}}.$$

3.4 Heavy Traffic Diffusion Approximation in a Dynamic Setting

We now present the convergence of our dynamic microscopic model to a dynamic mesoscopic model. We should first, of course, define the dynamic mesoscopic model. This is done in essentially the same way as in the microscopic case. We therefore only give an overview here and refer to the previous section for the precise details. The static mesoscopic model determines the behaviour of the dynamic model in between boundary movements. The functions determining rates of upward and downward boundary movements as functions of the profiles are now denoted by $\theta_{u,m}$ and $\theta_{d,m}$ respectively. They are now bounded maps from $(\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1} \to \mathbb{R}_{>0}$ which are uniformly bounded over m and such that there exists a constant $c > 0$ with $\theta_{u,m}, \theta_{d,m} \geq c$. We further assume that these maps are continuous. Our regenerative function which determines the distribution of the new profiles following a boundary movement is denoted by R and now maps $(\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1} \times \{u, d\} \to \mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$. This function is also assumed to be continuous, where $\mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$ is equipped with the topology of weak convergence. For ease of notation, we introduce the maps $P_n : \mathbb{N}^{N-1} \times \mathbb{N}^{N-1} \mapsto (\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}$, which simply divide each coordinate by $\sqrt{n}$. The functions here are then approximated by the corresponding functions in the microscopic model in the following ways.

(i) For $k \in \{u, d\}$, $v_1, v_2 \in \mathbb{N}^{N-1}$ and $u_1, u_2 \in (\mathbb{R}^+)^{N-1}$,

$$\left| n\theta^n_{k,m}(v_1, v_2) - \theta_{k,m}(u_1, u_2) \right| \leq r(\|P_n((v_1, v_2)) - (u_1, u_2)\|),$$

where $\lim_{x \to 0} r(x) = 0$.

(ii) For $k \in \{u, d\}$, if (v^n_1, v^n_2) is a sequence in $\mathbb{N}^{N-1} \times \mathbb{N}^{N-1}$ such that $P_n((v^n_1, v^n_2)) \to (u_1, u_2)$, then

$$R^n(v^n_1, v^2_n, k) \circ P_n^{-1} \implies R(u_1, u_2, k)$$

in law in $\mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$.

With this in place, we define the processes X^a_i, X^b_i, stopping times τ^i_u, τ^i_d, τ^i and boundary sequence m^i analogously to the previous section, with boundary movements once again of size ϵ. The underlying collections of exponential random variables which are used in the construction of the stopping times τ^i_u and τ^i_d are now denoted Y^i_u and Y^i_d. Hence, X_i is the sequence of static mesoscopic models, (τ^i_u, τ^i_d) are stopping times determining the times in between boundary movements as well as the direction of these movements, and m^i is the sequence of boundary positions. Our dynamic mesoscopic model is then given by $(\hat{X}^a(t), \hat{X}^b(t), m(t))$, where

$$\hat{X}^a(t) := \sum_{i=1}^{\infty} X^a_i \left(t - \sum_{j=1}^{i-1} \tau^j \right) \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau^j \leq t < \sum_{j=1}^{i} \tau^j \right\}},$$

$$\hat{X}^b(t) := \sum_{i=1}^{\infty} X_i^b \left(t - \sum_{j=1}^{i-1} \tau^j \right) \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau^j \le t < \sum_{j=1}^{i} \tau^j \right\}},$$

$$m(t) := \sum_{i=1}^{\infty} m^i \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau^j \le t < \sum_{j=1}^{i} \tau^j \right\}}.$$

The following is our convergence result for approximating the dynamic mesoscopic model with our sequence of rescaled dynamic microscopic models.

Theorem 3.4.1. *Suppose that*

$$\left(\frac{Z_{n,1}^a(0)}{\sqrt{n}}, \frac{Z_{n,1}^b(0)}{\sqrt{n}} \right) \implies (X_1^a(0), X_1^b(0))$$

weakly in $\mathscr{M}\left((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1} \right)$. *Let* $(\hat{Z}_n^a(t), \hat{Z}_n^b(t), m_n(t))$ *be dynamic microscopic models with initial data*

$$\left(\frac{Z_{n,1}^a(0)}{\sqrt{n}}, \frac{Z_{n,1}^b(0)}{\sqrt{n}}, m^1 \right),$$

and let $(\hat{X}^a(t), \hat{X}^b(t), m(t))$ *be the dynamic mesoscopic model with initial data* $(X_1^a(0), X_1^b(0), m^1)$. *Then*

$$\left(\frac{\hat{Z}_n^a(nt)}{\sqrt{n}}, \frac{\hat{Z}_n^b(nt)}{\sqrt{n}}, m_n(nt) \right) \implies (\hat{X}^a(t), \hat{X}^b(t), m(t))$$

weakly in $\mathscr{M}\left(\mathbb{D}([0,\infty); \mathbb{R}^{N-1}) \times \mathbb{D}([0,\infty); \mathbb{R}^{N-1}) \times \mathbb{D}([0,\infty); \mathbb{R}) \right)$.

The proof of Theorem 3.2.1 will be used as a basis for the proof of Theorem 3.4.1. We start by proving some continuity type results for certain maps, connecting some of the features of the microscopic models to their mesoscopic counterparts. These will be used in an inductive argument which will allow us to prove that

$$\left(\left(Z_{n,i}^a(nt)/\sqrt{n} \right)_{i=1}^{\infty}, \left(Z_{n,i}^b(nt)/\sqrt{n} \right)_{i=1}^{\infty}, (\tau_n^i/n)_{i=1}^{\infty}, (m_n^i)_{i=1}^{\infty} \right) \implies ((X_i^a)_{i=1}^{\infty}, (X_i^b)_{i=1}^{\infty}, (\tau^i)_{i=1}^{\infty}, (m^i)_{i=1}^{\infty})$$

in law in $\mathbb{D}([0,\infty); \mathbb{R}^{N-1})^{\mathbb{N}} \times \mathbb{D}([0,\infty); \mathbb{R}^{N-1})^{\mathbb{N}} \times [0,\infty]^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Note that, for a metric space M, we use the topology of pointwise convergence for $M^{\mathbb{N}}$, which is itself metrizable. We then prove a continuity-type result for the map which sends these processes to their associated dynamic models. Finally, we conclude with an application of the Skorohod representation theorem.

Proposition 3.4.2. *Fix some* $m \in \mathbb{R}$. *Let* $w_n : [0,\infty) \to \mathbb{Z}^{N-1} \times \mathbb{Z}^{N-1}$ *and* $w : [0,\infty) \to \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ *be such that, for every* $T > 0$,

$$\sup_{t \in [0,T]} \left| \frac{1}{\sqrt{n}} (w_n^1(nt), w_n^2(nt)) - (w^1(t), w^2(t)) \right| \to 0.$$

Then we have that, for every $T > 0$,

$$\sup_{t\in[0,T]} \left| n\theta^n_{u,m}(w_n(nt)) - \theta_{u,m}(w(t)) \right| \to 0,$$

and

$$\sup_{t\in[0,T]} \left| n\theta^n_{d,m}(w_n(nt)) - \theta_{d,m}(w(t)) \right| \to 0.$$

Proof. This is a direct consequence of assumption (i) in Section 3.4. □

Proposition 3.4.3. *Suppose (Z^1_n, Z^2_n) is a sequence in $\mathscr{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1})$ such that $P_n((Z^1_n, Z^2_n)) \implies (X^1, X^2)$ in $\mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$. Define the maps $\tilde{R}_n : \mathscr{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1}) \times \{u, d\} \to \mathscr{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1})$ and $\tilde{R} : \mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}) \times \{u, d\} \to \mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$ such that, for $(\mu, k) \in \mathscr{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1}) \times \{u, d\}$ and $(\nu, k) \in \mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}) \times \{u, d\}$,*

(i) $\tilde{R}_n(\mu, k)(A) := \int_{\mathbb{N}^{N-1}\times\mathbb{N}^{N-1}} R_n(x_1, x_2, k)(A)\ \mu(dx_1, dx_2)$.

(ii) $\tilde{R}(\nu, k)(B) := \int_{(\mathbb{R}^+)^{N-1}\times(\mathbb{R}^+)^{N-1}} R(x_1, x_2, k)(A)\ \nu(dx_1, dx_2)$.

Then, for $k \in \{u, d\}$,

$$\tilde{R}_n(Z^1_n, Z^2_n, k) \circ P_n^{-1} \implies \tilde{R}(X^1, X^2, k)$$

in $\mathscr{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$.

Proof. We apply the Skorohod representation theorem to the weak convergence of $P_n((Z^1_n, Z^2_n))$ to (X^1, X^2), so we assume that $P_n((Z^1_n, Z^2_n)) \to (X^1, X^2)$ almost surely on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We then have that

$$R_n(Z^1_n, Z^2_n, k) \circ P_n^{-1} \to R(X^1, X^2, k)$$

in $\mathscr{M}((\mathbb{R}^+)^{N-1}) \times (\mathbb{R}^+)^{N-1})$ $\mathbb{P}$- almost surely. Let $A \in \mathscr{B}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})$ be a continuity set for the measure $\tilde{R}(X^1, X^2, k)$. We then have that A is a continuity set for $R(X^1, X^2, k)$ $\mathbb{P}$-almost surely, from which it follows that

$$R_n(Z^1_n, Z^2_n, k) \circ P_n^{-1}(A) \to R(X^1, X^2, k)(A)$$

$\mathbb{P}$ - almost surely. Taking expectations then gives the result. □

Proposition 3.4.4. *Suppose that $f_n : [0, \infty) \to \mathbb{R}_{>0}$ and $f : [0, \infty) \to \mathbb{R}_{>0}$ are such that f is continuous, there exists a constant $c > 0$ such that $f \geq c$, and $f_n \to f$ uniformly on compact sets. Suppose also that $x_n \in \mathbb{R}$, $x \in \mathbb{R}$ such that $x_n \to x$. Define*

$$\tau_n := \inf \left\{ t \geq 0 \;\middle|\; x_n \leq \int_0^t f_n(s)ds \right\}$$

and

$$\tau := \inf \left\{ t \geq 0 \;\middle|\; x \leq \int_0^t f(s)ds \right\}.$$

Then $\tau_n \to \tau$.

Proof. Fix $\delta > 0$. We argue that, for n large enough, $\tau_n \leq \tau + \delta$. Note that

$$\left| \left(x_n - \int_0^\tau f_n(s)\mathrm{ds} \right) - \left(x - \int_0^\tau f(s)\mathrm{ds} \right) \right| \leq |x_n - x| + \tau \times \sup_{t\in[0,\tau]} |f_n(t) - f(t)|.$$

Since $f \geq c > 0$, we have that for n large enough $\inf_{t\in[0,\tau]} f_n(t) > \frac{c}{2}$. Also, for n large enough,

$$|x_n - x| + \tau \times \sup_{t\in[0,\tau]} |f_n(t) - f(t)| < \frac{\delta c}{2}.$$

Therefore,

$$x_n - \int_0^\tau f_n(s)\mathrm{ds} < x - \int_0^\tau f(s)\mathrm{ds} + \frac{\delta c}{2} = \frac{\delta c}{2}$$

for large enough n. So

$$x_n - \int_0^{\tau+\delta} f_n(s)\mathrm{ds} \leq x_n - \int_0^\tau f_n(s)\mathrm{ds} - \frac{\delta c}{2} < 0.$$

Therefore $\tau_n \leq \tau + \delta$ for large enough n. Similarly, we have $\tau_n \geq \tau - \delta$ for large enough n, concluding the proof. □

We can now prove the following result, whose proof constitutes the main step in proving Theorem 3.4.1.

Proposition 3.4.5. *Suppose that*

$$\left(\frac{Z^a_{n,1}(0)}{\sqrt{n}}, \frac{Z^b_{n,1}(0)}{\sqrt{n}} \right) \implies (X^a_1(0), X^b_1(0))$$

in law in $(\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}$. *Let* $(\hat{Z}^a_n(t), \hat{Z}^b_n(t), m_n(t))$ *be dynamic microscopic models with initial data* $(Z^a_{n,1}(0), Z^b_{n,1}(0), m^1)$, *and let* $(\hat{X}^a(t), \hat{X}^b(t), m(t))$ *be the dynamic mesoscopic model with initial data* $(X^a_1(0), X^b_1(0), m^1)$. *Then*

$$\left(\left(Z^a_{n,i}(nt)/\sqrt{n} \right)_{i=1}^\infty, \left(Z^b_{n,i}(nt)/\sqrt{n} \right)_{i=1}^\infty, (\tau^i_n/n)_{i=1}^\infty, (m^i_n)_{i=1}^\infty \right) \implies ((X^a_i)_{i=1}^\infty, (X^b_i)_{i=1}^\infty, (\tau^i)_{i=1}^\infty, (m^i)_{i=1}^\infty)$$

in law in $\mathbb{D}([0,\infty), \mathbb{R}^{N-1})^{\mathbb{N}} \times \mathbb{D}([0,\infty), \mathbb{R}^{N-1})^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$.

Proof. We begin the proof by introducing the following notation.

(i) $\mathscr{U}^{a,M}_n(t) := (Z^a_{n,i}(nt)/\sqrt{n})_{i=1}^M, \qquad \mathscr{U}^{a,M}(t) := (X^a_i(t))_{i=1}^M.$

(ii) $\mathscr{U}^{b,M}_n(t) := (Z^b_{n,i}(nt)/\sqrt{n})_{i=1}^M, \qquad \mathscr{U}^{b,M}(t) := (X^b_i(t))_{i=1}^M.$

(iii) $\tilde{\tau}^M_{n,u} := (\tau^i_{n,u}/n)_{i=1}^M, \qquad \tilde{\tau}^M_u := (\tau^i_u)_{i=1}^M.$

(iv) $\tilde{\tau}^M_{n,d} := (\tau^i_{n,d}/n)_{i=1}^M, \qquad \tilde{\tau}^M_d := (\tau^i_d)_{i=1}^M.$

(v) $\tilde{m}^M_n := (m^i_n)_{i=1}^M, \qquad \tilde{m}^M := (m^i)_{i=1}^M.$

We will prove by induction that for every $M \geq 1$,

$$(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,a}^{M-1}, \tilde{\tau}_{n,b}^{M-1}, \tilde{m}_n^M) \implies (\mathscr{U}^{a,M}, \mathscr{U}^{b,M}, \tilde{\tau}_a^{M-1}, \tilde{\tau}_b^{M-1}, \tilde{m}^M) \tag{3.4.1}$$

in law in $\mathbb{D}([0,\infty), \mathbb{R}^{N-1})^M \times \mathbb{D}([0,\infty), \mathbb{R}^{N-1})^M \times [0,\infty)^{M-1} \times [0,\infty)^{M-1} \times \mathbb{R}^M$, from which the result follows. Note that by applying Theorem 3.2.1, we have

$$(\mathscr{U}_n^{a,1}, \mathscr{U}_n^{b,1}, \tilde{m}_n^1) \implies (\mathscr{U}^{a,1}, \mathscr{U}^{b,1}, \tilde{m}^1)$$

in $\mathbb{D}([0,\infty), \mathbb{R}^{N-1}) \times \mathbb{D}([0,\infty), \mathbb{R}^{N-1}) \times \mathbb{R}$. Suppose we had for some $M \geq 1$ that (3.4.1) holds. Then we have that

$$(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,a}^{M-1}, \tilde{\tau}_{n,b}^{M-1}, \tilde{m}_n^M, Y_{n,a}^M, Y_{n,b}^M) \implies (\mathscr{U}^{a,M}, \mathscr{U}^{b,M}, \tilde{\tau}_a^{M-1}, \tilde{\tau}_b^{M-1}, \tilde{m}^M, Y_a^M, Y_b^M).$$

By the Skorohod convergence theorem, we can assume that this convergence holds almost surely. Recall that

$$\tau_{n,u}^M := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \theta_{u,m_n^M}^n (Z_{n,M}^a(s), Z_{n,M}^b(s))\, \mathrm{d}s \geq Y_{n,u}^M \right\},$$

$$\tau_{n,d}^M := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \theta_{d,m_n^M}^n (Z_{n,M}^a(s), Z_{n,M}^b(s))\, \mathrm{d}s \geq Y_{n,d}^M \right\}.$$

By a change of variables,

$$\int_0^t \theta_{u,m_n^M}^n (Z_{n,M}^a(s), Z_{n,M}^b(s)) \mathrm{d}s = \int_0^{t/n} n\theta_{u,m_n^M}^n (Z_{n,M}^a(nr), Z_{n,M}^b(nr)) \mathrm{d}r.$$

Therefore,

$$\begin{aligned}\frac{1}{n}\tau_{n,u}^M &= \frac{1}{n} \inf\left\{ t \geq 0 \;\middle|\; \int_0^{t/n} n\theta_{u,m_n^M}^n (Z_{n,M}^a(nr), Z_{n,M}^b(nr)) \mathrm{d}r \geq Y_{n,u}^M \right\} \\ &= \inf\left\{ t \geq 0 \;\middle|\; \int_0^t n\theta_{u,m_n^M}^n (Z_{n,M}^a(nr), Z_{n,M}^b(nr)) \mathrm{d}r \geq Y_{n,u}^M \right\}.\end{aligned}$$

As m_n^M and m^M both take values in $m^1 + \epsilon\mathbb{Z}$ and $m_n^M \to m^M$ almost surely, we have that $m_n^M = m^M$ for large enough n almost surely. Note that since the limits X_M^a, X_M^b are continuous, convergence of $Z_{n,M}^a(nt)/\sqrt{n}$ to $X_M^a(t)$ and $Z_{n,M}^b(nt)/\sqrt{n}$ to $X_M^b(t)$ in the Skorohod topology implies that, for every $T > 0$,

$$\sup_{t \in [0,T]} \left| \frac{1}{\sqrt{n}} (Z_{n,M}^a(nt), Z_{n,M}^b(nt)) - (X_M^a(t), X_M^b(t)) \right| \to 0.$$

Since $\theta_{u,m}$ was assumed to be continuous and X_M^a, X_M^b are continuous, $\theta_{u,m}(X_M^a(t), X_M^b(t))$ is continuous almost surely. We can therefore apply Propositions 3.4.2 and 3.4.4 to deduce that

$$\frac{1}{n}\tau_{n,u}^M \to \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \theta_{u,m}(X_M^a(s), X_M^b(s))\, \mathrm{d}s \geq Y_u^M \right\} = \tau_u^M$$

almost surely. Similarly, $\frac{1}{n}\tau_{n,d}^M \to \tau_d^M$ almost surely. Note that $\tau_u^M \neq \tau_d^M$ almost surely. This implies that $m_n^{M+1} \to m^{M+1}$ almost surely. Therefore, we have deduced that

$$(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \implies (\mathscr{U}^{a,M}, \mathscr{U}^{b,M}, \tilde{\tau}_u^M, \tilde{\tau}_d^M, \tilde{m}^{M+1}). \tag{3.4.2}$$

For $i = 1, ...M+1$, let $A_i \in \mathscr{B}(\mathbb{D}([0,\infty); \mathbb{R}^{N-1}))$ be a continuity set for the $\mathbb{D}([0,\infty); \mathbb{R}^{N-1})$-valued random variable X_i^a. Similarly, let B_i be continuity sets for the random variables X_i^b, and let $p_i \in (m^1 + \epsilon\mathbb{Z})$. For $i = 1, ...M$, let C_i, D_i be continuity sets for the random variables τ_u^i and τ_d^i respectively. Define:

$$\tilde{A}_K := \prod_{i=1}^K A_i, \qquad \tilde{B}_K := \prod_{i=1}^K B_i, \qquad \tilde{C}_K := \prod_{i=1}^K C_i, \qquad \tilde{D}_K := \prod_{i=1}^K D_i, \qquad P_K := \prod_{i=1}^K \{p_i\}.$$

Suppose that

$$\mathbb{P}\left[(\mathscr{U}^{a,M}, \mathscr{U}^{b,M}, \tilde{\tau}_u^M, \tilde{\tau}_d^M, \tilde{m}^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\right] > 0.$$

We then have that, for large enough n,

$$\mathbb{P}\left[(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\right] > 0,$$

from which it follows that

$$\begin{aligned}
&\mathbb{P}\left[(\mathscr{U}_n^{a,M+1}, \mathscr{U}_n^{b,M+1}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \in \tilde{A}_{M+1} \times \tilde{B}_{M+1} \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\right] \\
=&\mathbb{P}\left[(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\right] \\
&\times \mathbb{P}_1\left[\frac{1}{\sqrt{n}}(Z_{n,M+1}^a(nt), Z_{n,M+1}^b(nt)) \in A_{M+1} \times B_{M+1}\right],
\end{aligned}$$

where $\mathbb{P}_1$ denotes the conditional probability law given the event

$$(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}.$$

By (3.4.2), we know that

$$\begin{aligned}
&\mathbb{P}\left[(\mathscr{U}_n^{a,M}, \mathscr{U}_n^{b,M}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\right] \\
&\qquad \to \mathbb{P}\left[(\mathscr{U}^{a,M}, \mathscr{U}^{b,M}, \tilde{\tau}_u^M, \tilde{\tau}_d^M, \tilde{m}^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\right].
\end{aligned}$$

Let $\mathbb{Q}_n$ be the law on $(\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}$ induced by $R^n(Z_{n,M}^a(\tau_n^M), Z_{n,M}^b(\tau_n^M), k) \circ P_n^{-1}$ and the probability measure $\mathbb{P}_1$, where k is the direction of the last price change. That is

$$\mathbb{Q}_n(A) = \mathbb{E}^{\mathbb{P}_1}\left[(R^n(Z_{n,M}^a(\tau_n^M), Z_{n,M}^b(\tau_n^M), k) \circ P_n^{-1})(A)\right]$$

Similarly, we define the measure $\mathbb{Q}$ on $(\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}$ by setting

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}_2}\left[R(X_M^a(\tau^M), X_M^b(\tau^M), k)(A)\right],$$

where $\mathbb{P}_2$ here is the conditional probability law given the event that

$$(\mathscr{U}^{a,M}, \mathscr{U}^{b,M}, \tilde{\tau}_u^M, \tilde{\tau}_d^M, \tilde{m}^{M+1}) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}.$$

It follows from (3.4.2), and the fact that (X_M^a, X_M^b) is continuous almost surely, that the law of $\frac{1}{\sqrt{n}}(Z_{n,M}^a(\tau_n^M), Z_{n,M}^b(\tau_n^M), k)$ under the measure $\mathbb{P}_1$ converges to the law of $(X_M^a(\tau^M), X_M^b(\tau^M), k)$ under $\mathbb{P}_2$. It then follows by an application of Proposition 3.4.3 that $\mathbb{Q}_n \implies \mathbb{Q}$. We can now apply our result for convergence in a static setting, Theorem 3.2.1, to deduce that

$$\mathbb{P}_1\left[\frac{1}{\sqrt{n}}(Z_{n,M+1}^a(n\cdot), Z_{n,M+1}^b(n\cdot)) \in A_{M+1} \times B_{M+1}\right] \to \mathbb{P}_2\left[(X_{M+1}^a, X_{M+1}^b) \in A_{M+1} \times B_{M+1}\right].$$

We have therefore deduced that

$$\begin{aligned}\mathbb{P}\Big[(\mathscr{U}_n^{a,M+1}, \mathscr{U}_n^{b,M+1}, \tilde{\tau}_{n,u}^M, \tilde{\tau}_{n,d}^M, \tilde{m}_n^{M+1}) \in \tilde{A}_{M+1} \times \tilde{B}_{M+1} \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\Big] \\ \to \mathbb{P}\Big[(\mathscr{U}^{a,M+1}, \mathscr{U}^{b,M+1}, \tilde{\tau}_u^M, \tilde{\tau}_d^M, \tilde{m}^{M+1}) \in \tilde{A}_{M+1} \times \tilde{B}_{M+1} \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\Big].\end{aligned}$$

So we have that $(\mathscr{U}_n^{a,M+1}, \mathscr{U}_u^{b,M+1}, \tilde{\tau}_{n,a}^M, \tilde{\tau}_{n,b}^M, \tilde{m}_n^{M+1}) \implies (\mathscr{U}^{a,M+1}, \mathscr{U}^{b,M+1}, \tilde{\tau}_a^M, \tilde{\tau}_b^M, \tilde{m}^{M+1})$ which concludes our inductive argument. □

Proposition 3.4.6. *Let $g : \mathbb{D}([0,\infty), \mathbb{R}^{N-1})^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \to \mathbb{D}([0,\infty), \mathbb{R}^{N-1})$ such that*

$$g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)(t) := \sum_{j=1}^\infty u_j\left(t - \sum_{i=1}^{j-1} t_i\right) \mathbb{1}_{\left\{\sum_{i=1}^{j-1} t_i \le t < \sum_{i=1}^{j} t_i\right\}}.$$

Suppose that

(i) $t_i > 0$ *for every* i.

(ii) $\sum_{i=1}^\infty t_i = \infty$.

(iii) u_i *is continuous for every* i

Then g is continuous at the point $((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)$.

Proof. Suppose that the sequence $((u_i^n)_{i=1}^\infty, (t_i^n)_{i=1}^\infty)$ converges to $((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)$ in $\mathbb{D}([0,\infty); \mathbb{R}^{N-1})^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}}$. It is sufficient to prove that

$$g((u_i^n)_{i=1}^\infty, (t_i^n)_{i=1}^\infty) \to g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)$$

in $\mathbb{D}([0,T]; \mathbb{R}^{N-1})$ for every $T \ge 0$ such that $g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)$ is continuous at T. That is, it is enough to prove convergence in $\mathbb{D}([0,T]; \mathbb{R}^{N-1})$ for every $T \ge 0$ such that

$$\sum_{i=1}^L t_i \ne T$$

for every L. Let $T \ge 0$ be such a point. By condition (ii), there exists $K \ge 1$ such that

$$\sum_{i=1}^K t_i > T. \tag{3.4.3}$$

Since the functions u_i are continuous, we have by convergence of u_i^n to u_i in $\mathbb{D}([0,\infty),\mathbb{R}^{N-1})$ that

$$\sup_{t\in[0,T]}|u_i^n(t)-u_i(t)|\to 0$$

for each i. Recall that $f_n\to f$ in $\mathscr{D}_{[0,T]}(M)$ for a metric space M iff $\exists\lambda_n:[0,T]\to[0,T]$, strictly increasing continuous bijections such that

(a) $\sup_{t\in[0,T]}|\lambda_n(t)-t|\to 0$ and

(b) $\sup_{t\in[0,T]}|f_n(\lambda_n(t))-f(t)|\to 0$.

For our case, we define the continuous bijections $\lambda_n:[0,T]\to[0,T]$ such that for $m\geq 1$ we set $\lambda_n\left(\sum_{i=1}^m t_i\right):=\sum_{i=1}^m t_i^n$, and linearly interpolate between these points, together with the endpoints $\lambda_n(0)=0$ and $\lambda_n(T)=T$ (this may define the function for values greater than T as well but we only care about the restriction). Then, for large enough n so that the t_i^n are all strictly positive, λ_n is a strictly increasing continuous bijection from $[0,T]$ to $[0,T]$, and the λ_n satisfy (a) above. Note that this can only be done if T is a continuity point for g. We have that, for $t\in[0,T]$,

$$\begin{aligned}g((u_i^n)_{i=1}^\infty,(t_i^n)_{i=1}^\infty)(\lambda_n(t))&=\sum_{j=1}^\infty u_j^n\left(\lambda_n(t)-\sum_{i=1}^{j-1}t_i^n\right)\mathbf{1}_{\left\{\sum_{i=0}^{j-1}t_i^n\leq\lambda_n(t)<\sum_{i=1}^{j}t_i^n\right\}}\\&=\sum_{j=1}^\infty u_{j-1}^n\left(\lambda_n(t)-\sum_{i=0}^{j-1}t_i^n\right)\mathbf{1}_{\left\{\sum_{i=0}^{j-1}t_i\leq t<\sum_{i=0}^{j}t_i\right\}}.\end{aligned}$$

It follows from (3.4.3) that

$$\begin{aligned}&\sup_{t\in[0,T]}|g((u_i^n)_{i=1}^\infty,(t_i^n)_{i=1}^\infty)(\lambda_n(t))-g((u_i)_{i=1}^\infty,(t_i)_{i=1}^\infty)(t)|\\&\leq\sup_{j\leq K}\sup_{t\in[0,T]}|u_{j-1}^n(t)-u_{j-1}(t)|\to 0.\end{aligned}$$

Therefore (b) holds and we have the result. □

We are now in position to conclude the proof of Theorem 3.4.1.

Proposition 3.4.7. *Let $h:[0,\infty)^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}\to\mathbb{D}([0,\infty),\mathbb{R})$ such that*

$$h((t_i)_{i=1}^\infty,(m_i)_{i=1}^\infty):=\sum_{i=1}^\infty m_i\mathbb{1}_{\left\{\sum_{j=1}^{i-1}t_j\leq t<\sum_{j=1}^{i}t_j\right\}}.$$

Suppose that

(i) $t_i>0$ *for every* i.

(ii) $\sum_{i=1}^\infty t_i=\infty$.

Then h is continuous at the point $((t_i)_{i=1}^\infty,(m_i)_{i=1}^\infty)$.

Proof. This is essentially the same as the proof of Proposition 3.4.6. □

We are now in a position to conclude the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. We begin by applying the Skorohod representation theorem to the weak convergence statement in Proposition 3.4.5. This gives that

$$\left(\left(Z^a_{n,i}(nt)/\sqrt{n}\right)_{i=1}^{\infty}, \left(Z^b_{n,i}(nt)/\sqrt{n}\right)_{i=1}^{\infty}, (\tau^i_n/n)_{i=1}^{\infty}, (m^i_n)_{i=1}^{\infty}\right) \\ \to ((X^a_i)_{i=1}^{\infty}, (X^b_i)_{i=1}^{\infty}, (\tau^i)_{i=1}^{\infty}, (m^i)_{i=1}^{\infty})$$

in $\mathbb{D}([0,\infty), \mathbb{R}^{N-1})^{\mathbb{N}} \times \mathbb{D}([0,\infty), \mathbb{R}^{N-1})^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ almost surely, where the vectors represent the key objects for the dynamic microscopic and mesoscopic models respectively. By Proposition 3.4.6, we clearly have that $((X^a_i)_{i=1}^{\infty}, (\tau^i)_{i=1}^{\infty})$ and $((X^b_i)_{i=1}^{\infty}, (\tau^i)_{i=1}^{\infty})$ are continuity points of the map g almost surely. Similarly, by Proposition 3.4.7, $((\tau^i)_{i=1}^{\infty}, (m^i)_{i=1}^{\infty})$ is a continuity point for the map h almost surely. The result follows. □

3.5 SPDE limit

In this section we will rescale our previously obtained dynamic mesoscopic system and to obtain a macroscopic SPDE limit. We do so by mapping the profiles in the mesoscopic model onto functions on $[0,1]$ and rescaling suitably, taking the limit as the number of queues tends to infinity. As with the dynamic result in the previous section, the proof of this relies heavily on convergence of the static models which we present first. We are able to obtain convergence in the static setting by applying Theorem 2.1 in T. Zhang [59].

Remark 3.5.1. Note that, although we take the limit as the number of queues tends to infinity, the boundary process moves according to macroscopic jumps. This both simplifies the analysis, avoiding the need to consider a model with a continuously evolving boundary in this case, and allows us to maintain a separation of time-scales between the evolution of the SPDEs and the boundary events. This is better suited to certain modelling situations e.g. modelling the limit order book, where order book events often take place on a faster timescale compared to price movements.

The sequence of rescaled static mesoscopic models which we will consider are as follows. For every $N \geq 1$ we let $X^a_N(t), X^b_N(t)$ satisfy the dynamics of the two sides of the static mesoscopic model on the price grid $\{0,1,2,...N\}$ with boundary $m \in \mathbb{R}$. These models are now indexed by N, which was suppressed in the previous section. We wish to emphasise this here since we will be taking N to infinity. We will rescale space and time appropriately and map the coordinates of $X^a_N(t)$ and $X^b_N(t)$ to equally spaced points on $[0,1]$. Define here the function $Q_N : \mathbb{R}^{N-1} \to C_0((0,1))$, such that

(i) For $i = 1,2,...N-1$, $Q_N(x)(\frac{i}{N}) = \frac{x_i}{\sqrt{N}}$.

(ii) For $i = 0, 1, 2, ...N-1$, $Q_N(x)$ is linear between the points $[\frac{i}{N}, \frac{i+1}{N}]$.

We will take the limit of $(Q_N(X_N^a(N^2t)), Q_N(X_N^b(N^2t)))$. For this to be sensible, the parameters for our SDE systems should be consistent in some way. Given the rescaling, they are chosen such that for $k \in \{a, b\}$, $m \in \mathbb{R}$, $N \geq 1$, $i = 0, 1, ...N-1$ and $u \in (\mathbb{R}^+)^{N-1}$:

(i) $h_{k,m}^N(i, u) := N^{-\frac{3}{2}} h_{k,m}(\frac{i}{N}, \frac{u}{\sqrt{N}})$,

(ii) $\sigma_{k,m}^N(i, u) := \sigma_{k,m}(\frac{i}{N}, \frac{u}{\sqrt{N}})$,

where $h_{k,m}$, $\sigma_{k,m}$ are measurable maps from $[0, 1] \times [0, \infty) \to \mathbb{R}$. We further assume the following Lipschitz and linear growth conditions. These are required in order to prove convergence.

1. $|h_{k,m}(x, u) - h_{k,m}(y, v)| + |\sigma_{k,m}(x, u) - \sigma_{k,m}(y, v)| \leq C(|x - y| + |u - v|)$ and
2. $|h_{k,m}(x, u)| + |\sigma_{k,m}(x, u)| \leq C(1 + |u|)$.

We now pass to the macroscopic limit in the static setting. The proof relies on an adaptation of Theorem 2.1 in T.Zhang [59].

Theorem 3.5.2 (Theorem 2.1, [59]). *Let (u, η) be a solution of the following reflected stochastic heat equation*

$$\frac{\partial u}{\partial t} = \alpha \Delta u + h(x, u(t, x)) + \sigma(x, u(t, x)) \frac{\partial^2 W}{\partial t \partial} + \eta, \tag{3.5.1}$$

with initial data $u_0 \in C_0((0, 1))^+$. Suppose that σ and h are Lipschitz in both variables. For $n \geq 1$, let $(W^{n,k})_{k=1}^{n-1}$ be the independent family of Brownian motions given by

$$W_t^{n,k} := \sqrt{n} \left[W\left(t, \frac{k+1}{n}\right) - W\left(t, \frac{k}{n}\right) \right],$$

and let u^n be the solution of the system of reflected SDEs

$$du_t^{n,k} = \alpha n^2 \left(u_t^{n,k+1} + u_t^{n,k-1} - 2u_t^{n,k} \right) dt + h\left(k/n, u_t^{n,k}\right) dt + \sqrt{n}\sigma\left(k/n, u_t^{k,n}\right) dW_t^{n,k} + d\eta_t^{n,k},$$

for $k = 1, ...n-1$ with $u^{n,0} = u^{n,n} = 0$ and initial data $u_0^n \in (\mathbb{R}^+)^{n-1}$. For each $t \geq 0$, define the function $u^n(t, x)$ by setting $u^n(t, \frac{k}{n}) := u_t^{n,k}$ and linearly interpolating between these points. Suppose that

$$\sup_{x \in [0,1]} |u^n(0, x) - u_0(x)| \to 0$$

almost surely. Then for every $p \geq 1$ and every $t \in [0, T]$

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{t \in [0,T]} \sup_{x \in [0,1]} |u^n(t, x) - u(t, x)|^p \right] = 0.$$

The following corollary states that, as a consequence of Theorem 3.5.2, we have convergence in law of the SDE system to the reflected SPDE in $C([0, \infty); C_0((0, 1)))$ in the case where we have random initial data converging in law. The space $C([0, \infty); C_0((0, 1)))$ is equipped with the topology of uniform convergence on compact sets in the time variable, so $f_n \to f$ in $C([0, \infty); C_0((0, 1)))$ if and only if $f_n \to f$ in $C([0, T]; C_0((0, 1)))$ for every $T > 0$.

Corollary 3.5.3. *Let u be the solution of the reflected stochastic heat equation (3.5.1) with initial data given by the law $\mu \in \mathscr{M}(C_0((0,1))^+)$. Suppose that σ and h are Lipschitz in both variables. For $n \geq 1$, let $(W^{n,k})_{k=1}^{n-1}$ be an independent family of Brownian motions and let u^n be the solution of the system of reflected SDEs*

$$du_t^{n,k} = \alpha n^2 \left(u_t^{n,k+1} + u_t^{n,k-1} - 2u_t^{n,k}\right) dt + h\left(k/n, u_t^{n,k}\right) dt + \sqrt{n}\sigma\left(k/n, u_t^{k,n}\right) dW_t^{n,k} + d\eta_t^{n,k},$$

for $k = 1, ...n-1$ with $u^{n,0} = u^{n,n} = 0$ and initial data given by the law $\nu_n \in \mathscr{M}((\mathbb{R}^+)^{n-1})$. For each $t \geq 0$, define the function $u^n(t,x)$ by setting $u^n(t,0) = 0$, $u^n(t,1) = 0$ and $u^n(t, \frac{k}{n}) := u_t^{n,k}$ for $k = 1, ...n-1$, and linearly interpolating between these points. Suppose that $(\nu_n \circ (\sqrt{n}Q_n)^{-1}) \implies \mu$ in law in $\mathscr{M}(C_0((0,1)))$. Then $u^n \implies u$ in law in $\mathscr{M}(C([0,\infty); C_0((0,1))))$.

Proof. In the case where the initial data are deterministic, convergence in $\mathscr{M}(C([0,\infty); C_0((0,1)))$ follows from convergence in $L^p(\Omega; C([0,T] \times [0,1]))$ for every $T > 0$. Turning to the case of random initial data, let $f \in C_b(C([0,\infty); C_0((0,1))))$. Define $\mu_n = \nu_n \circ (\sqrt{n}Q_n)^{-1}$. We want to prove that

$$\mathbb{E}^{\mu^n}\left[f(u^n)\right] \to \mathbb{E}^{\mu}\left[f(u)\right].$$

Let $F_n : C_0((0,1)) \to C_0((0,1))$ such that $F_n(u)(0) = F_n(u)(1) = 0$, $F_n(u)(i/n) = u(i/n)$ for $i = 1, ...n-1$, and $F_n(u)$ is linear in the intervals $[i/n, (i+1)/n]$ for $i = 0, 1, ...n-1$. For $n \geq 1$ and $v \in C_0((0,1))$, define

$$g_n(v) := \mathbb{E}\left[f(u^n) \mid u^n(0,\cdot) = F_n(v)\right].$$

Similarly, for $v \in C_0((0,1))$, define

$$g(v) := \mathbb{E}\left[f(u) \mid u(0,\cdot) = v\right].$$

Then we have that

$$\mathbb{E}^{\mu_n}\left[f(u^n)\right] = \int_{C_0((0,1))} g_n(v)\, \mu_n(\mathrm{d}v),$$

and

$$\mathbb{E}^{\mu}\left[f(u)\right] = \int_{C_0((0,1))} g(v)\, \mu(\mathrm{d}v),$$

By the Skorohod representation theorem, we can realise the convergence of μ_n to μ with the random variables u_0^n and u_0 on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$, so that $u_0^n \to u_0$ in $C_0((0,1))$ $\mathbb{P}$-almost surely. By the deterministic initial data case, we then have

$$g_n(u_0^n) \to g(u_0)$$

$\mathbb{P}$- almost surely. The result then follows by the DCT. □

Theorem 3.5.4. *Suppose that $(Q_N(X_N^a(0)), Q_N(X_N^b(0))) \implies (v_0^a, v_0^b)$ in law in $C_0((0,1))^+ \times C_0((0,1))^+$. Then $(Q_N(X_N^a(N^2t)), Q_N(X_N^b(N^2t)) \implies (v^a, v^b)$ in law in $C([0,\infty); C_0((0,1)))$, where (v^a, v^b) is the unique solution to the pair of reflected stochastic heat equations*

$$\frac{\partial v^a}{\partial t} = \alpha_a \Delta v^a + h_{a,m}(x, v^a) + \sigma_{a,m}(x, v^a)\frac{\partial^2 W^a}{\partial x \partial t} + \eta^a(dt, dx),$$

$$\frac{\partial v^b}{\partial t} = \alpha_b \Delta v^b + h_{b,m}(x, v^b) + \sigma_{b,m}(x, v^b)\frac{\partial^2 W^b}{\partial x \partial t} + \eta^b(dt, dx),$$

with pinning conditions $v^a(t,0) = v^a(t,1) = v^b(t,0) = v^b(t,1) = 0$ *and initial data* $v^a(0,x) = v_0^a(x)$, $v^b(0,x) = v_0^b(x)$, *where space-time white noises* W^a *and* W^b *are independent.*

Proof. As in the case of the proof of Theorem 3.2.1, we notice that the static mesoscopic systems decouple into two independent problems on either side of the boundary. We therefore focus once more on proving the convergence on one side of the boundary. Recall that the dynamics for the type a side of the N^{th} static mesoscopic model are given by

$$\begin{aligned}\mathrm{d}X_N^{a,i}(t) =& \alpha_a(X_N^{a,i+1}(t) + X_N^{a,i-1}(t) - 2X_N^{a,i}(t))\mathrm{d}t + h_{a,m}^N(i, X_N^{a,i}(t))\mathrm{d}t \\ &+ \sigma_{a,m}^N(i, X_N^{a,i}(t))\mathrm{d}W_t^{N,a,i} + \mathrm{d}\eta_t^{N,a,i}\end{aligned}$$

for $i = 1, ...N-1$ with $X^{a,0} = X^{a,N} = 0$. It follows that

$$\begin{aligned}\mathrm{d}v_t^{N,a,i} =& \alpha_a N^2 \left(v_t^{N,a,i+1} + v_t^{N,a,i-1} - 2v_t^{N,a,i}\right)\mathrm{d}t + h_{a,m}\left(k/N, v_t^{N,,a,i}\right)\mathrm{d}t \\ &+ \sqrt{N}\sigma_{a,m}\left(i/N, v_t^{N,a,i}\right)\mathrm{d}\tilde{W}_t^{N,a,i} + \mathrm{d}\eta_t^{N,a,i},\end{aligned}$$

where we define $v_t^{N,a,i} := \frac{1}{\sqrt{N}} X_N^{a,i}(N^2 t)$, and the Brownian motions $\tilde{W}^{N,a,i}$ are given by

$$\tilde{W}_t^{N,a,i} = \frac{1}{N} W_{N^2 t}^{N,a,i}.$$

The result then follows by an application of Corollary 3.5.3. □

3.6 Dynamic SPDE Limit

Let $(\hat{X}_N^a(t), \hat{X}_N^b(t), m_N(t))$ be our N^{th} dynamic mesoscopic model, with the parameters defined as in the previous section. The price change rates are now denoted by $\theta_{a,m}^N$ and $\theta_{b,m}^N$, and the regenerative functions by R^N. We aim to prove convergence to a dynamic macroscopic model by taking the limit of the sequence $(Q_N(\hat{X}_N^a(N^2 t)), Q_N(\hat{X}_N^b(N^2 t)), m_N(N^2 t))$.
For the dynamic macroscopic model, we once again assume the dynamics follow the static model in between boundary movements, with these occurring at state driven rates. As in previous sections, the stopping times determining the time in between the $(i-1)^{\text{th}}$ and i^{th} boundary movement, as well as the direction of this jump, are denoted τ_u^i and τ_d^i. When an upward/downward boundary movement takes place, the position of the interface $m(t)$ increases/decreases by ϵ. The functions which give the volumes of the type a and type b particles at the different relative distances from the boundary in between the $(i-1)^{\text{th}}$ and i^{th} price change are denoted by u_i^a and u_i^b respectively. A regeneration function then determines the law of these profiles following a boundary movement, with the dynamics then following a new static model. We introduce $\theta_{u,m}$ and $\theta_{d,m}$ for $m \in \mathbb{R}$, which are uniformly bounded over m and continuous from $C_0((0,1)) \to \mathbb{R}_{>0}$, with there existing a constant $c > 0$ such that $\theta_{u,m}, \theta_{d,m} \geq c$. These functions determine the rates of upward and downward boundary movements respectively, as functions of the profiles of the two sides, when the shared boundary is at m. We also introduce the regenerative function for our

macroscopic model, R, which maps $C_0((0,1)) \times C_0((0,1)) \times \{u,d\} \to \mathscr{M}(C_0((0,1) \times C_0((0,1)))$. This tells us the new profiles following a boundary jump, as a function of the profiles at the time of the jump and the direction of the jump. For ease of notation, we introduce here the map $\tilde{Q}_N : \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \to C_0((0,1)) \times C_0((0,1))$, which simply applies Q_N to each coordinate. We connect the dynamic aspects of the mesoscopic and macroscopic models by assuming that

(i) For $k \in \{u,d\}$, every $m \in \mathbb{R}$, $(X^1, X^2) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ and $(u_1, u_2) \in C_0((0,1)) \times C_0((0,1))$

$$\left| N^2 \theta^N_{k,m}((X^1, X^2)) - \theta_{k,m}((u_1, u_2)) \right| \leq r(\|\tilde{Q}_N((X_1, X_2)) - (u_1, u_2)\|),$$

where $\lim_{x \to 0} r(x) = 0$.

(ii) For every $m \in \mathbb{R}$, if $(X^1, X^2) \in \mathbb{N}^{N-1} \times \mathbb{N}^{N-1}$ and $(u_1, u_2) \in C_0((0,1)) \times C_0((0,1))$ are such that

$$\|\tilde{Q}_N((X_1, X_2)) - (u_1, u_2)\| \to 0,$$

then, for $k \in \{u,d\}$,

$$R^N(X^1, X^2, k) \circ \tilde{Q}_N^{-1} \implies R(u_1, u_2, k)$$

in law in $C_0((0,1)) \times C_0((0,1))$.

Remark 3.6.1. For $X \in \mathbb{N}^{N-1}$ and $u \in C_0((0,1))$,

$$\|Q_N(X) - u\| \to 0$$

is equivalent to the condition that

$$\sup_{i=1,2,\ldots N-1} \left| X^i - u\left(i/N\right) \right| \to 0.$$

Remark 3.6.2. *Typically, we think of $N = kM$ for some fixed $k \in \mathbb{N}$, and we take the limit as $M \to \infty$. The jump size for the boundary is then taken to be $\epsilon = \frac{1}{k}$, equivalent to M queues. Essentially, we think of there being regions of space of size $\frac{1}{k}$ which are either captured by type a or type b particles. The boundary is the interface between the type a and type b regions.*

Theorem 3.6.3. *Suppose that $(Q_N(X^a_N(0)), Q_N(X^b_N(0))) \implies (u^a(0), u^b(0))$ in law in $C_0((0,1))^+ \times C_0((0,1))^+$. Let $(\hat{X}^a_N(t), \hat{X}^b_N(t), m_N(t))$ be our dynamic mesoscopic model with initial data*

$$(X^a_N(0), X^b_N(0), m(0)),$$

and let $(u^a(t), u^b(t), m(t))$ be our dynamic macroscopic model with initial data

$$(u^a(0), u^b(0), m(0)).$$

Then we have that

$$(Q_N(\hat{X}^a_N(N^2t)), Q_N(\hat{X}^b_N(N^2t)), m_N(N^2t)) \implies (u^a(t), u^b(t), m(t))$$

in law in the space $\mathbb{D}([0,\infty); C_0((0,1))) \times \mathbb{D}([0,\infty); C_0((0,1))) \times \mathbb{D}\left([0,\infty); \mathbb{R}\right)$.

As in the proof of Theorem 3.4.1, we build towards a proof by first presenting a series of smaller results. The arguments here reflect those made when upgrading the proof of Theorem 3.2.1 to a proof of Theorem 3.4.1. We therefore refer to the proof of Theorem 3.4.1 to illustrate how to prove the key results in this section. Similarly to the proof of Theorem 3.4.1, the main part of the proof is showing that

$$((Q_N((X^a_{N,i}(N^2t,\cdot)))_{i=1}^\infty,(Q_N((X^b_{N,i}(N^2t,\cdot)))_{i=1}^\infty,(\tau^i_{N,u}/N^2)_{i=1}^\infty,(\tau^i_{N,d}/N^2)_{i=1}^\infty,(m^i_N)_{i=1}^\infty) \\ \implies ((u^a_i(t,\cdot))_{i=1}^\infty,(u^b_i(t,\cdot))_{i=1}^\infty,(\tau^i_u)_{i=1}^\infty,(\tau^i_d)_{i=1}^\infty,(m^i)_{i=1}^\infty)$$

in law in $\mathbb{D}([0,\infty);C_0((0,1)))^{\mathbb{N}}\times\mathbb{D}([0,\infty);C_0((0,1)))^{\mathbb{N}}\times[0,\infty)^{\mathbb{N}}\times[0,\infty)^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$. Once again, for a metric space M we equip $M^{\mathbb{N}}$ with the topology of pointwise convergence, which is metrizable.

Proposition 3.6.4. *Fix some $m\in\mathbb{R}$. Let $w_N:[0,\infty)\to\mathbb{R}^{N-1}$ and $w:[0,\infty)\to C_0((0,1))$ be such that, for every $T>0$*

$$\sup_{t\in[0,T]}|Q_N(w_N(t))-w(t)|\to 0.$$

Then we have that, for every $T>0$,

$$\sup_{t\in[0,T]}\left|N^2\theta^N_{b,Nm}(w_N(t))-\theta_{b,m}(w(t))\right|\to 0,$$

and

$$\sup_{t\in[0,T]}\left|N^2\theta^N_{a,Nm}(w_N(t))-\theta_{a,m}(w(t))\right|\to 0.$$

Proof. This is a direct application of assumption (i) in Section 3.6. □

Proposition 3.6.5. *Suppose (X^1_N,X^2_N) is a sequence in $\mathscr{M}(\mathbb{R}^{N-1}\times\mathbb{R}^{N-1})$ such that $(Q_N(X^1_N),Q_N(X^2_N))\implies(u^1,u^2)$ in $\mathscr{M}(C_0((0,1))\times C_0((0,1)))$. Define the maps $\tilde{R}^N:\mathscr{M}(\mathbb{R}^{N-1}\times\mathbb{R}^{N-1})\times\{u,d\}\to\mathscr{M}(\mathbb{R}^{N-1}\times\mathbb{R}^{N-1})$ and $\tilde{R}:\mathscr{M}(C_0((0,1))\times C_0((0,1))\times\{u,d\}\to\mathscr{M}(C_0((0,1))\times C_0((0,1)))$ such that, for $(\mu,k)\in\mathscr{M}(\mathbb{R}^{N-1}\times\mathbb{R}^{N-1})\times\{u,d\}$ and for (ν,k) $\mathscr{M}(C_0((0,1))\times C_0((0,1)))\times\{u,d\}$*

(i) $\tilde{R}^N(\mu,k)(A):=\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}R^N(x_1,x_2,k)(A)\,\mu(dx_1,dx_2)$

(ii) $\tilde{R}(\nu,k)(B):=\int_{C_0((0,1))\times C_0((0,1))}R(u^1,u^2,k)(B)\,\nu(dx_1,dx_2)$.

Fix some $k\in\{u,d\}$. Then

$$\tilde{R}^N((X^1_N,X^2_N),k)\circ Q_N^{-1}\implies\tilde{R}((u^1,u^2),k)$$

in $\mathscr{M}(C_0((0,1))\times C_0((0,1)))$.

Proof. This is essentially the same as the proof of Proposition 3.4.3. □

Theorem 3.6.6. *Suppose that*

$$(Q_N(X^a_{N,1}(0)), Q_N(X^b_{N,1}(0))) \implies (u^a(0), u^b(0))$$

in law in $C_0((0,1))^+ \times C_0((0,1))^+$. *Let* $(X^a_N(t), X^b_N(t), m_N(t))$ *be dynamic microscopic models with initial data* $(X^a_{N,1}(0), X^b_{N,1}(0), m^1)$, *and let* $(u^a(t), u^b(t), m(t))$ *be the dynamic macroscopic model with initial data* $(u^a(0), u^b(0), m^1)$. *Then*

$$\begin{aligned}((Q_N(X^a_{N,i}(N^2t)))_{i=1}^\infty, (Q_N(X^b_{N,i}(N^2t)))_{i=1}^\infty, (\tau^i_{N,a}/N^2)_{i=1}^\infty, (\tau^i_{N,b}/N^2)_{i=1}^\infty, (m^i_N)_{i=1}^\infty)\\ \implies ((u^a_i)_{i=1}^\infty, (u^b_i)_{i=1}^\infty, (\tau^i_a)_{i=1}^\infty, (\tau^i_b)_{i=1}^\infty, (m^i)_{i=1}^\infty) \quad (3.6.1)\end{aligned}$$

in law in $\mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times \mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$.

Proof. This follows the proof of Theorem 3.4.5 and we refer to this for the details. We once again use an inductive argument, and show that for every $M \geq 1$,

$$\begin{aligned}((Q_N(X^a_{N,i}(N^2t)))_{i=1}^M, (Q_N(X^b_{N,i}(N^2t)))_{i=1}^M, (\tau^i_{N,a}/N^2)_{i=1}^{M-1}, (\tau^i_{N,b}/N^2)_{i=1}^{M-1}, (m^i_N)_{i=1}^M)\\ \implies ((u^a_i)_{i=1}^M, (u^b_i)_{i=1}^M, (\tau^i_a)_{i=1}^{M-1}, (\tau^i_b)_{i=1}^{M-1}, (m^i)_{i=1}^M) \quad (3.6.2)\end{aligned}$$

in law in $\mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times \mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, from which the result follows. Given the inductive hypothesis, we use the Skorohod representation theorem and Propositions 3.6.4 and 3.4.4 to obtain convergence of the next rescaled stopping times in the sequence. It then follows that the next boundary position in the sequence also converges. By conditioning as in the proof of Theorem 3.4.2 and making use of Proposition 3.6.5, we can obtain convergence of the process up to the next stopping time, concluding the inductive argument. □

Proposition 3.6.7. *Let* $g : \mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \to \mathbb{D}([0,\infty); C_0((0,1)))$ *such that*

$$g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)(t) := \sum_{j=1}^\infty u_j \left(t - \sum_{i=1}^{j-1} t_i\right) \mathbb{1}_{\left\{\sum_{i=1}^{j-1} t_i \leq t < \sum_{i=1}^{j} t_i\right\}}.$$

Suppose that

(i) $t_i > 0$ *for every* i.

(ii) $\sum_{i=1}^\infty t_i = \infty$.

(iii) u_i *is continuous for every* i

Then g *is continuous at the point* $((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)$.

Proof. This is essentially the same as the proof of Theorem 3.4.6. □

Proof of Theorem 3.6.3. We can now conclude the proof of Theorem 3.6.3. We begin by applying the Skorohod representation theorem to the convergence in Theorem 3.6.6. So we have that

$$((Q_N(X^a_{N,i}(N^2t)))_{i=1}^\infty, (Q_N(X^b_{N,i}(N^2t)))_{i=1}^\infty, (\tau^i_{N,a}/N^2)_{i=1}^\infty, (\tau^i_{N,b}/N^2)_{i=1}^\infty, (m^i_N)_{i=1}^\infty) \\ \to ((u^a_i)_{i=1}^\infty, (u^b_i)_{i=1}^\infty, (\tau^i_a)_{i=1}^\infty, (\tau^i_b)_{i=1}^\infty, (m^i)_{i=1}^\infty) \quad (3.6.3)$$

in $\mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times \mathbb{D}([0,\infty); C_0((0,1)))^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ almost surely. By Proposition 3.6.7, we clearly have that $((u^a_i)_{i=1}^\infty, (\tau^i)_{i=1}^\infty)$ and $((u^b_i)_{i=1}^\infty, (\tau^i)_{i=1}^\infty)$ are continuity points of the map g almost surely. Similarly, by Proposition 3.4.7, $((\tau^i)_{i=1}^\infty, (m^i)_{i=1}^\infty)$ is a continuity point for the map h almost surely. The result follows. □

3.7 Numerical Application: Modelling the Limit Order Book

A natural application of these equations is the modelling of the Limit Order Book, a record of offers to buy/sell an asset. In a Limit Order Market, all traders are able to place orders of three types. Limit Orders are offers to buy/sell which do not lead to an immediate transaction. For this reason, they are often referred to as patient offers, and only result in a transaction when they are matched with incoming market orders. By placing a limit order, a trader is indicating a willingness to wait for the transaction to take place, in order to achieve the price which they have quoted. Market Orders, on the other hand, are offers to buy/sell the asset which match with an existing limit order and so result in immediate transaction at the best available prices. They are therefore referred to as impatient orders. Finally, traders are able to cancel limit orders which they previously placed. The Order Book can consequently be thought of as a mechanism through which liquidity can be provided to impatient traders by patient traders. We refer to the best offer to buy on the book at a given time as the bid. The best offer to sell is the ask. The mid (which we will also refer to as the price) is the midpoint between the bid and the ask, whilst the difference between the bid and the ask is called the bid-ask spread.

When thinking of the dynamics in this context, the two particle types can be thought of offers to buy/sell the asset respectively, with the shared boundary representing the mid price. Specifically, in the case of the dynamic macroscopic model, $u^b(t,x)$ and $u^a(t,x)$ represent the density of offers currently on the book at prices which are at a distance x below/above the mid respectively (where a suitable scaling is chosen to convert the interval $[0,1]$ into units of currency). Note that we swap the indexes 'a' and 'b' in this section, as 'b' is a natural choice for referring to the *bid* side of the book i.e. the offers to buy the asset, which will be at lower prices, and 'a' is a natural choice for referring to the *ask* side, the offers to sell, which will be at higher prices. The arrival and disappearance of particles in the system is naturally interpreted as the arrival and cancellation of limit orders at the different price levels (as well as market orders when considering the queues at the best prices, the bid/ask). One can think of the high

frequency nature of the arrivals and cancellations of orders as a representation of high frequency trading in the system.

In the following numerical work, we demonstrate that simple versions of the model can reproduce some of the features of the price series and order books seen in financial data. We describe the particular parameter choices for the model which will be used in this section, before giving an overview of the numerical scheme and introducing the LOBSTER dataset. Following this, we briefly explain the parameter estimation procedure used, before presenting numerical illustrations and results by plugging in the estimated parameters into the simulation algorithms. We would like to emphasise here that the intention behind this section is to demonstrate that sensible order book simulations can readily be obtained from the model - we do not claim that the numerical scheme implemented, or the methods used to fit the parameters, are optimal.

3.7.1 The Model

We will work with a special case of the models which fall within the framework described in the earlier sections. The two sides of the book, u^b and u^a, will evolve in between price changes according to the reflected SPDEs

$$\frac{\partial u^i}{\partial t} = \alpha \Delta u^i + f(x) + \sigma(x) \frac{\partial^2 W^i}{\partial x \partial t} + \eta^i$$

for $i = b, a$. The implicit assumption here is that the order arrival rates, which give rise to the drift and volatility terms f and σ in our SPDE limit, depend on the distance from the mid only, and there is no dependence on the number of orders which are currently on the book at that price. We have also imposed that the coefficients f and σ here do not depend on i. This is due to the symmetry in the order arrival rates for the bid and ask sides of the book, which can be seen in the data.

We will now describe the form of the rate functions which will be used in this section, $\theta_u(u)$ and $\theta_d(u)$, which determine the rates at which the price moves up and down respectively. The rate at time t for an upward price jump will be given by

$$\theta_u(u^a(t,\cdot), u^b(t,\cdot)) = \gamma \max\left(\int_0^\epsilon \Big(u^b(t,x) - u^a(t,x) \Big) \, \mathrm{d}x, 0 \right) + \delta \tag{3.7.1}$$

and the rate of a downward price movement will similarly be given by

$$\theta_d(u^a(t,\cdot), u^b(t,\cdot)) = \gamma \max\left(\int_0^\epsilon \Big(u^a(t,x) - u^b(t,x) \Big) \, \mathrm{d}x, 0 \right) + \delta. \tag{3.7.2}$$

The first terms in these rates represents the contribution of order imbalance close to the mid as a driving factor for price movement, whilst the second terms here represent price movements due to additional exogenous factors.

Our regeneration functions which determine the profiles of the two sides of the book following price changes, will simply shift the profiles of the two sides of the book by the size of a price

jump, in the relevant direction. That is, R is given by

$$R\left(u^b, u^a, u\right) = \begin{cases} (0, u^a(x+\epsilon)) & \text{for } x \in [0, \epsilon], \\ (u^b(x-\epsilon), u^a(x+\epsilon)) & \text{for } x \in [\epsilon, 1-\epsilon], \\ (u^b(x-\epsilon), 0) & \text{for } x \in [1-\epsilon, 1], \end{cases}$$
$$R\left(u^b, u^a, d\right) = \begin{cases} (u^b(x+\epsilon), 0) & \text{for } x \in [0, \epsilon], \\ (u^b(x+\epsilon), u^a(x-\epsilon)) & \text{for } x \in [\epsilon, 1-\epsilon], \\ (0, u^a(x-\epsilon)) & \text{for } x \in [1-\epsilon, 1], \end{cases} \tag{3.7.3}$$

We note here that, strictly speaking, some of these new profiles do not satisfy the Dirichlet conditions imposed earlier in the analysis. However, the equations can be shown to have solutions when started from a general positive continuous initial profile, with the solution in $C_0(0,1)$ at all positive times. In addition, the regeneration function as described here has a natural interpretation simply as the best bid/ask queue having been depleted to trigger the price change.

3.7.2 The Numerical Scheme

We use a forward time-stepping scheme in order to simulate our equations in between price movements. The equation is discretised into M time steps and N space steps, and we define $t_i := iT/M$, $x_i := i/N$. We denote the simulated height of the bid/ask sides of the book at time t_j and position x_i by $u^b(t_j, x_i)$ and $u^a(t_j, x_i)$ respectively. The simulated price process at time t_j is denoted by $p(t_j)$. Given our simulated solution up to the j^{th} time step, we begin our approximation of the following time-step by first determining whether there should be a price movement at this time-step. Note that we should take the price jump size ϵ to be equal to a multiple of the space step i.e. $\epsilon = kL/N$ for an integer k. In our simulations, we will take $k = 1$. We start by defining $\pi^+[t_j]$ and $\pi^-[t_j]$ which give the probabilities of upward of downward price jumps in that time-step. These are approximations of the probabilities of price jumps in that time period given by the rates $\theta_b(u^a(t,\cdot), u^b(t,\cdot))$ and $\theta_b(u^a(t,\cdot), u^b(t,\cdot))$ as in (3.7.1) and (3.7.2). Writing these explicitly, we have that

$$\pi^+[t_j] = \max(\frac{\gamma}{2N}\left(u^b(t_j, x_1) - u^a(t_j, x_1)\right) \times T/M, 0) + \delta T/M$$

and

$$\pi^-[t_j] = \max(\frac{\gamma}{2N}\left(u^a(t_j, x_1) - u^b(t_j, x_1)\right) \times T/M, 0) + \delta T/M.$$

We simulate a uniform random variable on $[0,1]$, which we denote by $Y[t_j]$. If $Y[t_j] < \pi^+[t_j]$ we take this as an indication that the price has moved up in that time-step. Similarly, if $\pi^+[t_j] \leq Y[t_j] < \pi^+[t_j] + \pi^-[t_j]$, we take this as an indication that there has been a downward price change at this time-step. If we are in neither of these cases, the simulated order book does not change price during this time-step. If there has been a price increase, we update

our price process by setting $p[t_{j+1}] = p[t_j] + \epsilon$, and adjust the profiles u^b and u^a by discrete approximations to (3.7.3). That is, we simply set

$$u^b(t_j, x_0) = u^b(t_j, x_N) = u^b(t_j, x_0) = u^b(t_j, x_N) = 0,$$

and define

$$u^b(t_j, x_i) = u^b(t_j, x_{i-1})$$

and

$$u^a(t_j, x_i) = u^a(t_j, x_{i+1})$$

for $i \in \{1, 2, ...N-1\}$. We analogously update u^b, u^a and p in the event of a downward price movement. If there is no price change at this time-step, we do not update u^b, u^a at this point, and set $p(t_{j+1}) = p(t_j)$. At the end of this price updating procedure, we then simulate the profiles of u^b and u^a at the next time-step by setting, for $i \in \{1, 2, ...N-1\}$,

$$\begin{aligned} u^b(t_{j+1}, x_i) := \max \Bigg\{ & u^b(t_j, x_i) + \frac{TN^2}{M} \left(u^b(t_j, x_{i+1}) + u^b(t_j, x_{i-1}) - 2u^b(t_j, x_i) \right) \\ & + \frac{T}{M} f(x_i) + \frac{\sqrt{TN}}{\sqrt{M}} \sigma(x_i) Z^b_{i,j}, 0 \Bigg\}, \end{aligned}$$

and similarly

$$\begin{aligned} u^a(t_{j+1}, x_i) := \max \Bigg\{ & u^a(t_j, x_i) + \frac{TN^2}{M} \left(u^a(t_j, x_{i+1}) + u^a(t_j, x_{i-1}) - 2u^a(t_j, x_i) \right) \\ & + \frac{T}{M} f(x_i) + \frac{\sqrt{TN}}{\sqrt{M}} \sigma(x_i) Z^a_{i,j}, 0 \Bigg\}. \end{aligned}$$

The $Z^k_{i,j}$ here are simply simulated unit normal random variables, and appear due to the discretisation of the space-time white noise component of our equations. We take the maximum with zero at each time-step in order to capture the influence of the reflection measures in the equations. This completes our forward time-step, and we return the profiles $u^b(t_{j+1}, \cdot)$, $u^a(t_{j+1}, \cdot)$ and the price $p(t_{j+1})$ as our simulated order book at time t_{j+1}.

3.7.3 Dataset Description

We will use a data sample which covers the 50 best levels of the SPDR Trust Series 1 over a one hour period, taken from 21$^{\text{st}}$ June 2012 between 09:30 and 10:30 EST. This dataset has been obtained from the LOBSTER (Limit Order Book System, The Efficient Reconstructor) database, which gives access to order book data for all NASDAQ traded stocks from June 2007 onwards. The data comes in the form of a two files - a message file and an orderbook file. The message file contains a table of timestamped order book events. That is, it is a log of the order arrivals, cancellations and executions over that time period, together with the time at which these events occurred. The orderbook file contains the order volumes recorded on the limit order book at the times of order book events, with information for the 50 best levels on the bid and ask sides of the book.

3.7.4 Parameter Fitting

In this section, we aim to obtain input parameters for our model based on the data described in the previous section. We would like to emphasise here that we do not claim that our fitting methods in this section are particularly robust - we simply aim to fit the parameters in a reasonable way so that we can demonstrate that our model can provide sensible simulations of the evolution of the order book. In addition, we do not fit the smoothing parameter, α and instead choose this value such that the profiles obtained are on the correct scale. In order to account for the reflection measure, we will only use the data of limit orders placed in price levels which are currently occupied. This is done since we should only take into consideration the order arrivals when the order volumes are away from zero, so as not to bias our drift estimates with components which could be attributed to the reflection measure.

Throughout this section, we will denote by $X^b_{j,i}$ the order volume in the i^{th} price point below the best bid at the j^{th} time-step for our dataset. We similarly denote by $X^a_{j,i}$ the order volume in the i^{th} price point above the best ask at the j^{th} time-step for our dataset.

The scaling used will measure order volumes in units of 10^4 orders, and the 50 queues of the order book will map to a spatial interval of length 1. This simply results in the SPDEs for each side of the book each being on a spatial interval of length 1. Since each queue of our dataset represents a price change of size 1 cent, a spatial interval of size 0.02 corresponds to 1 cent under our scaling. Time units for the SPDEs will be measured in minutes. In this scale, our choice of smoothing parameter, α will be given by $\alpha = 0.01$.

3.7.5 Volatility and Drift Estimates

We begin this section by fitting the volatility and drift estimates for our SPDE. Let $\hat{\sigma}(i/51)$ denote our estimate for $\sigma(i/51)$, the volatility at spatial position $i/51$, corresponding to i price points away from the best bid/ask. We calculate $\hat{\sigma}$ by equating

$$51 \times 60 \times \hat{\sigma}^2(i/51) = \frac{1}{2} \sum \left[\text{Order Size} \times 10^{-4} \right]^2,$$

where the sum is taken over all orders on the bid/ask side of the book of all types which are i price points away from the best bid/ask price respectively, with the omission of limit orders placed in unoccupied queues. This simply matches the quadratic variation of different spatial intervals in the model with the corresponding value from the dataset. The factors of 51 and 3600 appear here due to our spatial and time scaling respectively, whilst the $1/2$ is present simply as we are summing data from both sides of the book, rather than just one side. The factor of 10^{-4} appears as we are working with units of 10^4 orders.

We now fit the drift term. Let d^b_i be the total number of bid limit orders placed in the hour long period in queues which are i price points away from the best bid, once again disregarding those orders placed in unoccupied queues. We similarly define d^a_i, with c^b_i and c^a_i the corresponding values for market/cancellation orders at the different relative price points for the bid

and ask sides of the book respectively. The net order flow per unit time at a price point which is i ticks away from the best bid/ask queue is then obtained from the dataset as

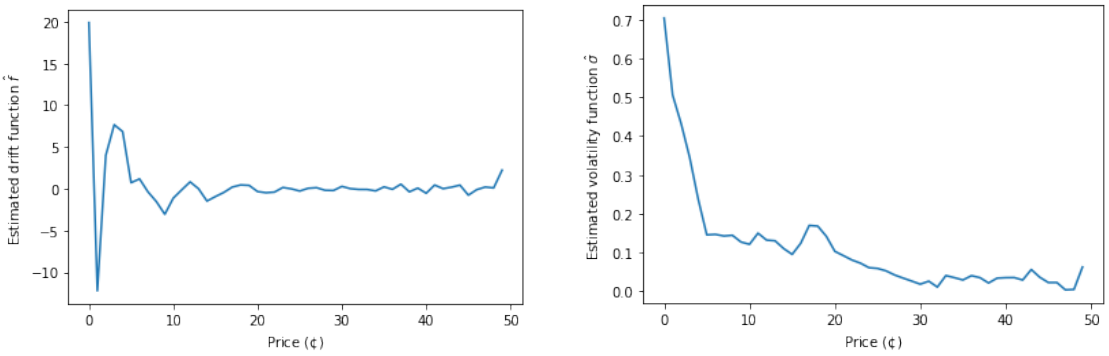
$$\frac{1}{2}\left(d_i^a + d_i^b - c_i^a - c_i^b\right) \times 10^{-4}.$$

Denote by $\tilde{f}(i/51)$ our estimate for the drift term at spatial position i. We estimate the drift parameter by equating

$$60 \times (\hat{f}(i/51) + 0.01\hat{\Delta}_i u) = \frac{1}{2}\left(d_i^a + d_i^b - c_i^a - c_i^b\right) \times 10^{-4}. \tag{3.7.4}$$

The term $0.01\hat{\Delta}_i u$ here represents the contribution of the laplacian term in our equation to the net order flow at the i^{th} level. $\hat{\Delta}_i u$ is obtained by calculating the average *Dirichlet* laplacian at the bid/ask sides of the book at price points which are i ticks away from the best bid/ask prices. By Dirichlet here, we mean that the laplacian is calculated at the first and last queues using the convention that the 0^{th} and 51^{th} queues are empty.

Figure 1 displays the estimated drift and volatility functions obtained from the data by the techniques described above. We note that the volatility is at its largest close to the mid, representing that there was significantly more activity at these price points over the trading period, as one would expect.



(a) (b)

Figure 3.1: Estimated Drift (a), and volatility (b).

3.7.6 Estimates for Rates of Price Changes

It is only left to produce estimates for γ, the rate at which the model changes price due to imbalance of the bid and ask queues, and δ, the rate at which the model changes price due to exogenous factors. Let $P(t)$ denote the price process of our dataset, measured in cents (recall that the tick size for that data is one cent). Our estimate of γ, $\hat{\gamma}$, is chosen such that it satisfies the equation

$$P(1) - P(0) = \hat{\gamma} \times 60 \times I. \tag{3.7.5}$$

The value I here is the average local imbalance of the data over the entire period, given by

$$I = \frac{1}{J}\sum_{j=1}^{J}\left[\int_0^{1/51}\left(\tilde{X}_j^b(x) - \tilde{X}_j^a(x)\right)\,\mathrm{d}x\right],$$

where $\tilde{X}^b_j(x)$ and $\tilde{X}^a_j(x)$ are obtained from $X^b_{j,i}$ and $X^a_{j,i}$ by setting $\tilde{X}^b_j(i/51) = 10^{-4}X^b_{j,i}$ and $\tilde{X}^a_j(i/51) = 10^{-4}X^a_{j,i}$, and then linearly interpolating in between these points. Writing I in terms of X^b and X^a, we have

$$I = \frac{1}{2 \times 51 \times 10^4 \times J} \sum_{j=1}^{J} \left(X^b_{j,1} - X^a_{j,1} \right).$$

The factor of 60 appears in (3.7.5) since we are working in units of minutes, and we have used an hour of data. The rate for price movements in either direction due to exogenous factors, $\hat{\delta}$, is then fitted so that the quadratic variation of the price process from the dataset matches with the expected quadratic variation had the price moved due to our price changing mechanism with parameters $\hat{\gamma}$ and $\hat{\delta}$. The expected number of price changes due to the local imbalance component is given by $60\hat{\gamma}\tilde{I}$, where

$$\tilde{I} = \frac{1}{2 \times 51 \times 10^4 \times J} \sum_{j=1}^{J} \left| X^b_{j,1} - X^a_{j,1} \right|.$$

For a given δ we would expect an extra $2 \times 60 \times \delta$ price changes over the time period. Our parameter $\hat{\delta}$ is therefore chosen such that it solves

$$120\hat{\delta} = \left[\sum_{j=1}^{J} \left(P(j/J) - P((j-1)/J) \right)^2 \right] - 60\hat{\gamma}\tilde{I}.$$

Implementing the procedures described, we obtain the following values for $\hat{\delta}$ and $\hat{\gamma}$.

$\hat{\gamma}$	$\hat{\delta}$
2181	28.80

Table 3.1: Estimated Price Change Rates.

3.7.7 Simulation Results

In this section, we will present the results of our numerical simulation of the order book under our model. In addition to the parameters which were fitted in the previous section, we fix the following additional parameters for the simulation.

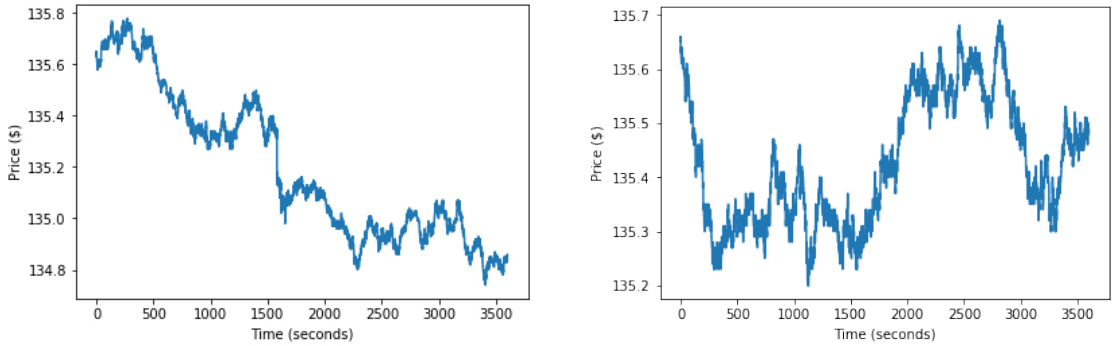
α	L	T	N	M
0.01	1	60	51	1500000

Table 3.2: Additional Model Parameters.

Recall that α refers to the smoothing parameter, L to the length of the spatial intervals on which each side of the book is modelled, T the time period in seconds for which we run the simulation and N, M the number of space and time steps used respectively.

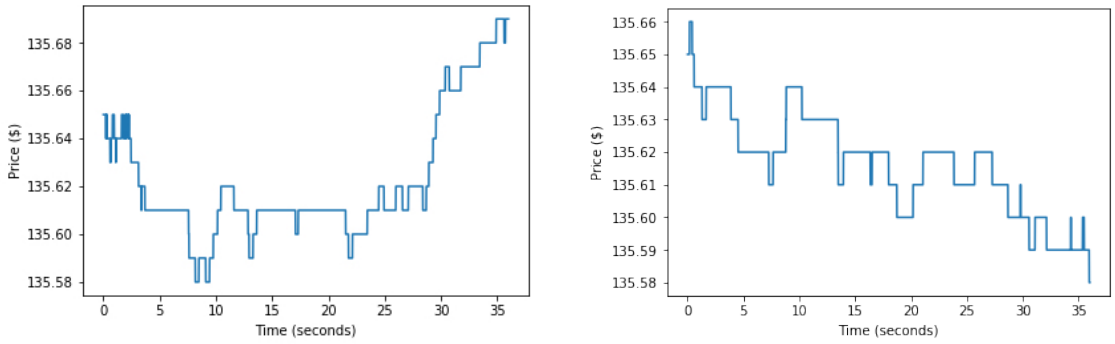
We will now present graphs illustrating the outcome of three simulations of the model, each representing a time period of one hour. In our simulations, both sides of the order book were

initialised with the time-averaged order book profiles from our dataset (see Figure 1.4(a)). In order to emphasise the strength of the fit, we present graphs of our simulations next to the corresponding graphs from the dataset.



(a) (b)

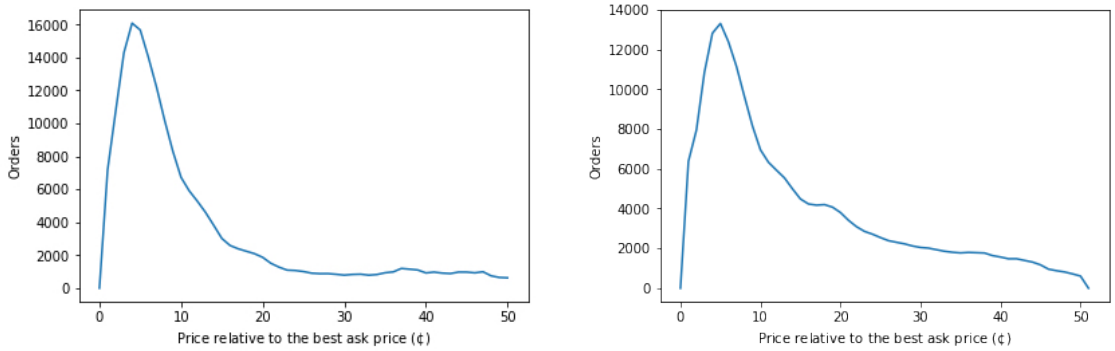
Figure 3.2: Price process of SPDR1 (a), simulated price process from fitted model (b).



(a) (b)

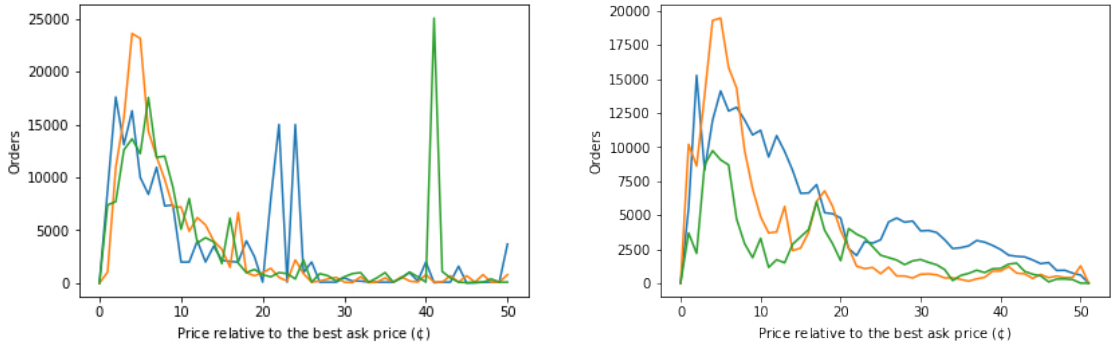
Figure 3.3: Price process of SPDR1 (a), simulated price process from fitted model (b) over the first 36 seconds of the trading period.

We note that the quadratic variation of the first path over the hour long time period is 0.421 (where the y-axis is measured in dollars), and the quadratic variation of the simulated path is 0.4143. When fitting, we attributed 0.3455 of the quadratic variation of the price from our dataset to exogenous price movements, with the remaining considered to be due to local imbalance. Based on this fitting, 0.3455 of the quadratic variation for our simulated price process can also be attributed to exogenous movements, and once again the remainder here is due to local imbalance. Therefore, price movements due to local imbalance were produced at a realistic rate by the simulation. This is compatible with the fact that the average order book profile over the hour long period appears to match well with that which was observed in the original data, which would mean that in particular we might expect the local imbalance for the real data and the simulated process to be of similar magnitude on average. The following figures show the average profiles of the ask side of the book for the real and simulated processes, together with snapshots of these profiles.



(a) (b)

Figure 3.4: Time averaged order book profiles for the ask side of SPDR1 (a) and our simulation (b).



(a) (b)

Figure 3.5: Static snapshots of order book profiles at 20 minute intervals for the ask side of SPDR1 (a) and our simulation (b).

We note that the average profile from our simulation is on the correct scale, and, like the average profile from the data, has most mass concentrated close to the mid. The snapshots from the simulation also demonstrate this. The nature of the decay away from the mid, however, is not quite as good a fit, with slightly higher average order volumes being apparent further into the book. Another feature which doesn't appear to have been captured by the simulation is the infrequent appearance of very high order volume at prices deep into the book as seen in Figure 5(a).

Chapter 4

Model with Continuous Boundary Motion

The aim of this section is to establish existence and uniqueness for systems of the form

$$\begin{aligned}
\frac{\partial u^1}{\partial t} &= \Delta u^1 + f_1(t, p(t) - x, u^1(t, p(t) - \cdot)) + \sigma_1(t, p(t) - x, u^1(t, p(t) - \cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta^1 \\
\frac{\partial u^2}{\partial t} &= \Delta u^2 + f_2(t, x - p(t), u^2(t, p(t) + \cdot)) + \sigma_2(t, x - p(t), u^2(t, p(t) + \cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta^2 \\
p'(t) &= h(u^1(t, p(t) - \cdot), u^2(t, p(t) + \cdot)),
\end{aligned} \tag{4.0.1}$$

where u^1 and u^2 have spatial domains on either side of the point $p(t)$ at any given time, and η^1, η^2 are reflection measures which keep the profiles positive. The key difference to the model from the previous section is that the boundary now moves continuously in time, constantly evolving in $\mathbb{R}$ rather than jumping from location to location on a discrete lattice. We impose Dirichlet boundary conditions so that $u^1(t, p(t)) = u^2(t, p(t)) = 0$, and consider the system in two cases - when the profiles are supported on finite intervals of fixed length on either side of the shared boundary and when the profiles are supported on semi-infinite regions on either side of the boundary. In short, the equation is made up of two reflected SPDEs sharing a moving boundary, whose derivative at a given time is a function of the two profiles u^1 and u^2 at that time. Within the class of systems produced by this model are approximations of the Stefan problem, where the motion of the boundary would be given by

$$p'(t) = \gamma \left(\frac{\partial u^1}{\partial x}(t, p(t)) + \frac{\partial u^2}{\partial x}(t, p(t)) \right),$$

where $\gamma \in \mathbb{R}$. The combination of the space-time white noise and the reflection measure mean that, in general, we do not have differentiability of u^1 and u^2 at the boundary, making it difficult to make sense of this Stefan condition. We show in Section 5 that, at the cost of more restrictive conditions on the decay of the volatility at the shared boundary, we can prove existence and uniqueness for this particular problem as well.

4.1 The Moving Boundary Problem on Finite Intervals in the Relative Frame

We are interested in the system (4.0.1), where at any time $t \geq 0$, the functions $u^1(t,\cdot)$ and $u^2(t,\cdot)$ are compactly supported in the sets $[p(t)-1, p(t)]$ and $[p(t), p(t)+1]$ respectively. Note that $\frac{\partial^2 W}{\partial x \partial t}$ is a space-time white noise, $f_i, \sigma_i : [0,\infty) \times [0,1] \times C_0((0,1)) \mapsto \mathbb{R}$ and $h : C_0((0,1)) \times C_0((0,1)) \mapsto \mathbb{R}$, with these functions satisfying certain conditions which we will introduce later. (η^1, η^2) are reflection measures for the functions u^1 and u^2 respectively, keeping the profiles positive and satifying the conditions

(i) $\text{supp}(\eta^1) \subset \big\{(t,x) \mid x \in (p(t)-1, p(t))\big\}$,

(ii) $\text{supp}(\eta^2) \subset \big\{(t,x) \mid x \in (p(t), p(t)+1)\big\}$,

(iii) $\int_0^\infty \int_{\mathbb{R}} u^1(t,x)\, \eta^1(\mathrm{d}t, \mathrm{d}x) = 0$,

(iv) $\int_0^\infty \int_{\mathbb{R}} u^2(t,x)\, \eta^2(\mathrm{d}t, \mathrm{d}x) = 0$.

4.1.1 Formulation of the Problem

We would like to formalise what we mean by (4.0.1) in the finite interval case. Before doing so, we define what we mean for a space-time white noise to respect a given filtration. This will be useful in some of the measurability arguments which follow.

Definition 4.1.1. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Suppose that $\dot{W}$ is a space-time white noise defined on this space. Let W be the white noise process associated with $\dot{W}$ so that for $t \geq 0$ and $A \in \mathscr{B}(\mathbb{R})$,*

$$W_t(A) := \dot{W}([0,t] \times A).$$

We say that $\dot{W}$ respects the filtration $\mathscr{F}_t$ if $(W_t(A))_{t\geq 0, A \in \mathscr{B}(\mathbb{R})}$ is an $\mathscr{F}_t$- martingale measure i.e. if for every $A \in \mathscr{B}(\mathbb{R})$, $(W_t(A))_{t\geq 0}$ is an $\mathscr{F}_t$-martingale.

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete probability space, and $\dot{W}$ space-time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $(u^1, \eta^1, u^2, \eta^2, p)$ is an $\mathscr{F}_t$-adapted process solving (4.0.1). Then $p : \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}$ is an $\mathscr{F}_t$-adapted process such that the paths of $p(t)$ are almost surely C^1 (note that, in particular, p is $\mathscr{F}_t$-predictable). Let $\varphi \in C_c^\infty([0,\infty) \times (0,1))$, and define the function ϕ by setting $\phi(t,x) = \varphi(t, p(t) - x)$. By multiplying the equation for u^1 in (4.0.1) by such a ϕ and integrating over space and time, interpreting the derivatives in the

usual weak sense, we obtain the expression

$$\begin{aligned}\int_{\mathbb{R}} u^1(t,x)\phi(t,x)\mathrm{d}x = &\int_{\mathbb{R}} u^1(0,x)\phi(0,x)\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u^1(s,x)\frac{\partial\phi}{\partial t}(s,x)\mathrm{d}x\mathrm{d}s\\ &+ \int_{\mathbb{R}}\int_0^t u^1(s,x)\frac{\partial^2\phi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s\\ &+ \int_{\mathbb{R}}\int_0^t f_1(s,p(s)-x,u^1(t,p(s)-\cdot))\phi(s,x)\mathrm{d}x\mathrm{d}s\\ &+ \int_{\mathbb{R}}\int_0^t \sigma_1(s,p(s)-x,u^1(t,p(s)-\cdot))\phi(s,x)W(\mathrm{d}x,\mathrm{d}s)\\ &+ \int_{\mathbb{R}}\int_0^t \phi(s,x)\eta^1(ds,dx).\end{aligned}$$

We now introduce a change in the spatial variable in order to associate our problem with a fixed boundary problem. Setting $v^1(t,x) = u^1(t,p(t)-x)$, the above equation becomes

$$\begin{aligned}\int_0^1 v^1(t,x)\phi(t,p(t)-x)\mathrm{d}x = &\int_0^1 v^1(0,x)\phi(0,p(0)-x)\mathrm{d}x + \int_0^T\int_0^1 v^1(s,x)\frac{\partial\phi}{\partial t}(s,p(s)-x)\mathrm{d}x\mathrm{d}s\\ &+ \int_0^1\int_0^t v^1(s,x)\frac{\partial^2\phi}{\partial x^2}(s,p(s)-x)\mathrm{d}x\mathrm{d}s\\ &+ \int_0^1\int_0^t f_1(s,x,v^1(s,\cdot))\phi(s,p(s)-x)\mathrm{d}x\mathrm{d}s\\ &+ \int_0^1\int_0^t \sigma_1(s,x,v^1(s,\cdot))\phi(s,p(s)-x)W_p(\mathrm{d}x,\mathrm{d}s)\\ &+ \int_0^1\int_0^t \phi(s,p(s)-x)\eta_p^1(ds,dx).\end{aligned}$$

Here, $\dot{W}_p$ and η_p^1 are obtained from W and η by shifting by $p(t)$. That is, for $t \in \mathbb{R}^+$ and $A \in \mathscr{B}(\mathbb{R})$,

$$\dot{W}_p([0,t]\times A) = \int_0^t \int_{p(s)-A} W(\mathrm{d}s,\mathrm{d}y), \qquad \eta_p^1([0,t]\times A) = \int_0^t\int_{p(s)-A}\eta^1(\mathrm{d}y,\mathrm{d}s),$$

Note that, since the process $p(t)$ is $\mathscr{F}_t$-predictable, $\dot{W}_p$ is then also a space time white noise which respects the filtration $\mathscr{F}_t$. Also, η_p^1 is a reflection measure for v, so that

$$\int_0^T\int_0^1 v^1(t,x)\ \eta_p^1(\mathrm{d}t,\mathrm{d}x) = 0.$$

We can calculate

$$\frac{\partial\phi}{\partial t}(s,x) = \frac{\partial\varphi}{\partial t}(t,p(t)-x) + p'(t)\frac{\partial\varphi}{\partial x}(t,p(t)-x). \tag{4.1.1}$$

It therefore follows that

$$\begin{aligned}\int_0^1 v^1(t,x)\varphi(t,x)\mathrm{d}x = &\int_0^1 v^1(0,x)\varphi(0,x)\mathrm{d}x + \int_0^t\int_0^1 v^1(s,x)\frac{\partial\varphi}{\partial t}(s,x)\mathrm{d}x\mathrm{d}s\\ &+\int_0^t\int_0^1 v^1(s,x)p'(s)\frac{\partial\varphi}{\partial x}(s,x)\mathrm{d}x\mathrm{d}s + \int_0^1\int_0^t v^1(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s\\ &+\int_0^1\int_0^t f_1(s,x,v^1(s,\cdot))\varphi(s,x)\mathrm{d}x\mathrm{d}s\\ &+\int_0^1\int_0^t \sigma_1(s,x,v^1(s,\cdot))\varphi(s,x)W_p(\mathrm{d}x,\mathrm{d}s)\\ &+\int_0^1\int_0^t \varphi(s,x)\ \eta_p^1(ds,dx).\end{aligned}$$

We can perform similar manipulations to obtain a weak form for $v^2(t,x) := u^2(t,p(t)+x)$. This yields that for test functions $\varphi \in C_c^\infty([0,\infty)\times(0,1))$, we should have that

$$\begin{aligned}\int_0^1 v^2(t,x)\varphi(t,x)\mathrm{d}x = &\int_0^1 v^2(0,x)\varphi(0,x)\mathrm{d}x + \int_0^t\int_0^1 v^2(s,x)\frac{\partial\varphi}{\partial t}(s,x)\mathrm{d}x\mathrm{d}s\\ &-\int_0^t\int_0^1 v^2(s,x)p'(s)\frac{\partial\varphi}{\partial x}(s,x)\mathrm{d}x\mathrm{d}s\\ &+\int_0^1\int_0^t v^2(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s + \int_0^1\int_0^t f_2(s,x,v^2(s,\cdot))\varphi(s,x)\mathrm{d}x\mathrm{d}s\\ &+\int_0^1\int_0^t \sigma_2(s,x,v^2(s,\cdot))\varphi(s,x)W_p^-(\mathrm{d}x,\mathrm{d}s) + \int_0^1\int_0^t \varphi(s,x)\ \eta_p^2(ds,dx),\end{aligned}$$

where $\dot{W}_p^-$ is given by

$$\dot{W}_p^-([0,t]\times A) = \dot{W}_p([0,t]\times(-A)).$$

Note also that, since $(u^1,\eta^1,u^2,\eta^2,p)$ is $\mathscr{F}_t$-adapted, we know that $(v^1,\eta_p^1,v^2,\eta_p^1,p)$ is also $\mathscr{F}_t$-adapted.

Remark 4.1.2. *The above formulation would need to be adjusted if we were anticipating rough paths for $p(t)$. For example, if $p(t)$ were a semimartingale, we would have to apply Itô's formula for the change of variables (4.1.1), which would change our weak form.*

We now define what we mean by a solution to a particular class of reflected SPDEs. The preceding calculation will allow us to connect the solutions to these SPDEs to our moving boundary problem.

Definition 4.1.3. *Let $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $\tilde{v}$ is a continuous $\mathscr{F}_t$-adapted process taking values in $C_0(0,1)$. Let $f,\sigma : [0,\infty)\times[0,1]\times C_0((0,1))\mapsto\mathbb{R}$, $h : C_0(0,1)\times C_0(0,1)\to\mathbb{R}$ and $F : C_0(0,1)\to C_0(0,1)$. For the $\mathscr{F}_t$-stopping time τ, we say that the pair (v,η) is a local solution to the reflected SPDE*

$$\frac{\partial v}{\partial t} = \Delta v + h(v(t,\cdot),\tilde{v}(t,\cdot))\frac{\partial F(v)}{\partial x} + f(t,x,v) + \sigma(t,x,v)\dot{W} + \eta$$

with Dirichlet boundary conditions $v(t,0) = v(t,1) = 0$ and initial data $v_0 \in C_0(0,1)^+$, until time τ, if

(i) *For every $t \geq 0$ and every $x \in [0,1]$, $v(t,x)$ is $\mathscr{F}_t$-measurable.*

(ii) *$v \geq 0$ almost surely.*

(iii) *$v\big|_{[0,\tau)\times[0,1]} \in C([0,\tau) \times [0,1])$ almost surely.*

(iv) *$v(t,x) = \infty$ for every $t \geq \tau$ and every $x \in [0,1]$ almost surely.*

(v) *η is a measure on $(0,1) \times [0,\infty)$ such that*

(a) *For every measurable map $\psi : [0,1] \times [0,\infty) \to \mathbb{R}$,*

$$\int_0^t \int_0^1 \psi(x,s)\ \eta(dx,ds)$$

is $\mathscr{F}_t$-measurable.

(b) $\int_0^\infty \int_0^1 v(t,x)\ \eta(dx,dt) = 0$.

(vi) *There exists a localising sequence of stopping times $\tau_n \uparrow \tau$ almost surely, such that for every $\varphi \in C_c^{1,2}([0,\infty) \times [0,1])$ such that $\varphi(\cdot,0) = \varphi(\cdot,1) = 0$,*

$$\begin{aligned}\int_0^1 v(t\wedge\tau_n,x)\varphi(t\wedge\tau_n,x)\,dx = &\int_0^1 v(0,x)\varphi(0,x)\,dx + \int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial\varphi}{\partial t}(s,x)\,dx\,ds\\ &+ \int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\,dx\,ds\\ &- \int_0^{t\wedge\tau_n}\int_0^1 F(v(s,\cdot))(x)h(v(s,\cdot),\tilde{v}(s,\cdot))\frac{\partial\varphi}{\partial x}(s,x)\,dx\,ds\\ &+ \int_0^{t\wedge\tau_n}\int_0^1 f(s,x,v(s,\cdot))\varphi(s,x)\,dx\,ds\\ &+ \int_0^{t\wedge\tau_n}\int_0^1 \sigma(s,x,v(s,\cdot))\varphi(s,x)W(dx,ds)\\ &+ \int_0^{t\wedge\tau_n}\int_0^1 \varphi(s,x)\ \eta(ds,dx).\end{aligned} \tag{4.1.2}$$

for every $t \geq 0$ almost surely.

We say that a local solution is *maximal* if it cannot be extended to a solution on a larger stochastic interval. We say that a local solution is *global* if we can take $\tau_n = \infty$ in (4.1.2).

Remark 4.1.4. *Condition (iv) above is included for the purposes of discussing uniqueness only.*

Before stating the formal definition for our moving boundary problem we introduce some notation which will allow us to easily write down the profiles in suitable relativised coordinates.

Definition 4.1.5. *For $p_0 \in \mathbb{R}$, we define $\Theta^1_{p_0} : \mathbb{R} \to \mathbb{R}$ such that*

$$\Theta^1_{p_0}(x) = p_0 - x.$$

For a function $p : [0,\infty) \to \mathbb{R}$ we then define $\theta_p^1 : [0,\infty) \times \mathbb{R} \to [0,\infty) \times \mathbb{R}$ such that

$$\theta_p^1(t,x) := (t, \Theta^1_{p(t)}(x)).$$

We similarly define $\Theta^2_{p_0} : \mathbb{R} \to \mathbb{R}$ such that

$$\Theta^2_{p_0}(x) = x - p_0,$$

and $\theta_p^2 : [0,\infty) \times \mathbb{R} \to [0,\infty) \times \mathbb{R}$ such that

$$\theta_p^2(t,x) := (t, \Theta^2_{p(t)}(x)).$$

Definition 4.1.6. *For a space time white noise $\dot{W}$, we denote by $\dot{W}^-$ the space time white noise such that $\dot{W}^-([0,t] \times A) = \dot{W}([0,t] \times (-A))$.*

Definition 4.1.7. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. For $i = 1, 2$, let u_0^i be such that $u_0^i \circ (\Theta^i_{p_0})^{-1} \in C_0((0,1))^+$, and let $p_0 \in \mathbb{R}$. We say that the quintuple $(u^1, \eta^1, u^2, \eta^2, p)$ is a local solution to the moving boundary problem with initial data (u_0^1, u_0^2, p_0), up to the $\mathscr{F}_t$-stopping time τ if*

(i) $(v^1, \tilde{\eta}^1) := (u^1 \circ (\theta_p^1)^{-1}, \eta^1 \circ (\theta_p^1)^{-1})$ solves the reflected SPDE

$$\frac{\partial v^1}{\partial t} = \Delta v^1 - h(v^1(t,\cdot), v^2(t,\cdot))\frac{\partial v^1}{\partial x} + f_1(t,x,v^1) + \sigma_1(t,x,v^1)\dot{W} + \tilde{\eta}^1$$

with Dirichlet boundary conditions $v^1(0) = v^1(1) = 0$ and initial data $v^1(0,\cdot) = u_0^1 \circ (\Theta^1_{p_0})^{-1}$ until time τ.

(ii) $(v^2, \tilde{\eta}^2) := (u^2 \circ (\theta_p^2)^{-1}, \eta^2 \circ (\theta_p^2)^{-1})$ solves the reflected SPDE

$$\frac{\partial v^2}{\partial t} = \Delta v^2 + h(v^1(t,\cdot), v^2(t,\cdot))\frac{\partial v^2}{\partial x} + f_2(t,x,v^2) + \sigma_2(t,x,v^2)\dot{W}^- + \tilde{\eta}^2$$

with Dirichlet boundary conditions $v^2(0) = v^2(1) = 0$ and initial data $v^2(0,\cdot) = u_0^2 \circ (\Theta^2_{p_0})^{-1}$ until time τ.

(iii) $p'(t)$ is $\mathscr{F}_t$-adapted, $p(0) = p_0$, and $p'(t) = h(v^1(t,\cdot), v^2(t,\cdot))$.

We refer to $(v^1, \tilde{\eta}^1, v^2, \tilde{\eta}^2)$ as the solution to the moving boundary problem in the relative frame.

We now introduce the precise conditions on the coefficients. We suppose that for $i = 1, 2$, $f_i, \sigma_i : [0,\infty) \times [0,1] \times C_0((0,1)) \to \mathbb{R}$ are measurable mappings, and that $h : C_0((0,1)) \times C_0((0,1)) \to \mathbb{R}$ is a measurable function such that

(I) For every $t \geq 0$, $x \in [0,1]$, $u, v \in C_0((0,1))$,

$$|f_i(t,x,u) - f_i(t,x,v)| + |\sigma_i(t,x,u) - \sigma_i(t,x,v)| \leq C\|u - v\|_\infty$$

for some constant C.

(II) For every $t \geq 0$, $x \in [0,1]$, $u, v \in C_0((0,1))$,

$$|f_i(t,x,u)| + |\sigma_i(t,x,u)| \leq R(1 + \|u\|_\infty)$$

for some constant R.

(III) h is Lipschitz, so there exists a constant K such that for every $u_1, u_2, v_1, v_2 \in C_0((0,1))$, $|h(u_1,v_1) - h(u_2,v_2)| \leq K(\|u_1 - u_2\|_\infty + \|v_1 - v_2\|_\infty)$.

4.1.2 Existence and Uniqueness

We now prove existence and uniqueness for our problems. In order to do this, we will first consider corresponding equations where the non-linear terms arising from the moving boundary have been suitably truncated. Following this, we use a Picard argument, with the approach here similar to that in the work by T. Xu and T. Zhang [52], in which existence and uniqueness is established for static reflected SPDEs.

Theorem 4.1.8. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that f_i, σ_i and h satisfy the conditions (I)-(IV). Then there exists a unique maximal solution $(u^1, \eta^1, u^2, \eta^2, p)$ to the moving boundary problem. The blow-up time, τ, is given by*

$$\tau := \sup_{M>0} \left[\inf \left\{ t \geq 0 \mid \|u^1\|_{\infty,t} + \|u^2\|_{\infty,t} \geq M \right\} \right],$$

with $\tau > 0$ almost surely.

The following notation for the Dirichlet heat kernel will be used throughout the rest of this section.

Definition 4.1.9. *We define $H(t,x,y)$ to be the Dirichlet heat kernel on $[0,1]$, so that*

$$H(t,x,y) := \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-y+2n)^2}{4t} \right) - \exp\left(-\frac{(x+y+2n)^2}{4t} \right) \right].$$

We will prove that we have global existence to the problem where the non-linear term is truncated. Before doing so, we present the following result which will be applied in the argument.

Proposition 4.1.10. *Fix $T > 0$ and let $v \in L^p(\Omega; L^\infty([0,T] \times [0,1]))$, where $p > 10$. Define, for $t \in [0,T]$,*

$$w(t,x) := \int_0^t \int_0^1 H(t-s,x,y) v(s,y)\, W(dy, ds).$$

Then we have that w is continuous, and for $\tau \in [0,T]$,

$$\mathbb{E}\left[\|w\|^p_{\infty,\tau} \right] \leq C_{p,T} \mathbb{E}\left[\int_0^\tau \|v\|^p_{\infty,s} ds \right]$$

Proof. Let $\tau \in [0,T]$ and suppose that $t,s \in [0,\tau]$. We have that

$$\begin{aligned}
&\mathbb{E}\left[|w(t,x)-w(s,y)|^p\right] \\
\leq& C_p\mathbb{E}\left[\left|\int_s^t\int_0^1 H(t-r,x,z)v(r,z)\mathrm{W}(\mathrm{d}z,\mathrm{d}r)\right|^p\right] \\
&+ C_p\mathbb{E}\left[\left|\int_0^s\int_0^1 \left[H(t-r,x,z)-H(t-r,y,z)\right]v(r,z)\mathrm{W}(\mathrm{d}z,\mathrm{d}r)\right|^p\right] \\
&+ C_p\mathbb{E}\left[\left|\int_0^s\int_0^1 \left[H(t-r,y,z)-H(s-r,y,z)\right]v(r,z)\mathrm{W}(\mathrm{d}z,\mathrm{d}r)\right|^p\right].
\end{aligned}$$

For the first term, we have by Burkholder's inequality that it is at most

$$C_p\mathbb{E}\left[\left|\int_s^t\int_0^1 H(t-r,x,z)^2v(r,z)^2\mathrm{d}z\mathrm{d}r\right|^{p/2}\right] \leq \mathbb{E}\left[\left|\int_s^t\left(\int_0^1 H(t-r,x,z)^2\mathrm{d}z\right)\|v\|_{\infty,r}^2\,\mathrm{d}r\right|^{p/2}\right].$$

An application of Hölder's inequality then gives that this is at most

$$\left(\int_s^t\left[\int_0^1 H(t-r,x,z)^2\mathrm{d}r\right]^{p/(p-2)}\mathrm{d}z\right)^{(p-2)/2}\times\int_s^t\mathbb{E}\left[\|v\|_{\infty,r}^p\right]\mathrm{d}r.$$

By the estimate (2) of Proposition A.1.5, we have that this is at most

$$C_{p,T}|t-s|^{(p-4)/4}\int_s^t\mathbb{E}\left[\|v\|_{\infty,r}^p\right]\mathrm{d}r.$$

By arguing similarly and making use of estimates (1) and (3) from Proposition A.1.5, we obtain that for every $t,s\in[0,\tau]$,

$$\begin{aligned}
\mathbb{E}\left[|w(t,x)-w(s,y)|^p\right] \leq& C_{p,T}\left(|t-s|^{1/2}+|x-y|\right)^{(p-4)/2}\times\int_0^\tau\mathbb{E}\left[\|v\|_{\infty,r}^p\right]\mathrm{d}r \\
=& C_{p,T}\left(|t-s|^{1/2}+|x-y|\right)^{3+\frac{p-10}{2}}\times\int_0^\tau\mathbb{E}\left[\|v\|_{\infty,r}^p\right]\mathrm{d}r.
\end{aligned}$$

Applying Corollary A.3 in [11] then gives that there exists a positive random variable $X\in L^p$ such that

$$\mathbb{E}\left[X^p\right] \leq C_{p,T}\int_0^\tau\mathbb{E}\left[\|v\|_{\infty,r}^p\right]\mathrm{d}r,$$

and, for every $t,s\in[0,\tau]$,

$$|w(t,x)-w(s,y)| \leq X(|t-s|^{1/2}+|x-y|)^{\frac{p-10}{4p}}.$$

It follows that

$$\sup_{t\in[0,\tau]}\sup_{x\in[0,1]}|w(t,x)| = \sup_{t\in[0,\tau]}\sup_{x\in[0,1]}|w(t,x)-w(t,0)| \leq C_{p,T}X. \tag{4.1.3}$$

Taking the L^p norm of both sides of (4.1.3) then gives the result. □

Proposition 4.1.11. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Let f_i, σ_i and h satisfy the conditions (I)-(IV) and suppose that v_0^1, $v_0^2 \in C_0((0,1))^+$. Define for $M \geq 0$ the function $h_M : C_0((0,1)) \times C_0((0,1)) \to \mathbb{R}$ such that*

$$h_M(v_1, v_2) := h(v_1 \wedge M, v_2 \wedge M).$$

Then for every $M \geq 0$, there exists a unique pair of $C_0((0,1))$-valued processes v^1, v^2 together with their reflection measures η^1, η^2, such that

1. *(v^1, η^1) is a global solution to the reflected SPDE*

$$\frac{\partial v^1}{\partial t} = \Delta v^1 - h_M(v^1, v^2)\frac{\partial}{\partial x}(v^1 \wedge M) + f_1(t, x, v^1) + \sigma_1(t, x, v^1)\dot{W} + \eta^1 \tag{4.1.4}$$

   with Dirichlet boundary conditions $v^1(t,0) = v^1(t,1) = 0$ and initial data v_0^1.

2. *(v^2, η^2) is a global solution to the reflected SPDE*

$$\frac{\partial v^2}{\partial t} = \Delta v^2 + h_M(v^1, v^2)\frac{\partial}{\partial x}(v^2 \wedge M) + f_2(t, x, v^2) + \sigma_2(t, x, v^2)\dot{W}^- + \eta^2 \tag{4.1.5}$$

   with Dirichlet boundary conditions $v^2(t,0) = v^2(t,1) = 0$ and initial data v_0^2.

Furthermore, for every $p \geq 1$ and every $T > 0$, we have that $(v_1, v_2) \in L^p(\Omega; C([0,T] \times [0,1])) \times L^p(\Omega; C([0,T] \times [0,1]))$. We call $(v^1, \eta^1, v^2, \eta^2)$ the solution to the M-truncated problem in the relative frame.

Proof. Note that by a concatenation argument, it is sufficient to prove existence and uniqueness on the time interval $[0,T]$ for any $T > 0$. Fix $T > 0$. We perform a Picard iteration in order to obtain existence. The first approximations are given by setting $v_0^1(t,x) = v_0^1(x)$ and $v_0^2(t,x) = v_0^2(x)$ for every $x \in [0,1]$ and every $t \in [0,T]$. For $n \geq 1$, we let w_{n+1}^1 solve the SPDE

$$\frac{\partial w_{n+1}^1}{\partial t} = \Delta w_{n+1}^1 - h_M(v_n^1, v_n^2)\frac{\partial}{\partial x}(v_n^1 \wedge M) + f_1(t, x, v_n^1) + \sigma_1(t, x, v_n^1)\dot{W}$$

with Dirichlet boundary conditions $w_{n+1}^1(t,0) = w_{n+1}^1(t,1) = 0$ and initial data v_0^1. We then set $v_{n+1}^1 := w_{n+1}^1 + z_{n+1}^1$, where z_{n+1}^1 solves the obstacle problem with obstacle $-w_{n+1}^1$. We similarly define v_{n+1}^2 in terms of v_n^1 and v_n^2, via functions w_{n+1}^2 and z_{n+1}^2. Writing the equation for w_{n+1}^1 in mild form gives the expression

$$\begin{aligned} w_{n+1}^1(t,x) = &\int_0^1 H(t,x,y)v_0^1(y)\mathrm{d}y \\ &+ \int_0^t \int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)h_M(v_n^1(s,\cdot), v_n^2(s,\cdot))\left[v_n^1(s,y) \wedge M\right]\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 H(t-s,x,y)f_1(s,y,v_n^1(s,\cdot))\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_0^1 H(t-s,x,y)\sigma_1(s,y,v_n^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s), \end{aligned} \tag{4.1.6}$$

where H is the Dirichlet heat kernel as in Definition 4.1.9. Recall that by Theorem 2.2.2, we have that $\|v^1_{n+1} - v^1_n\|_{\infty,t} \leq 2\|w^1_{n+1} - w^1_n\|_{\infty,t}$ almost surely. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\|v^1_{n+1} - v^1_n\|^p_{\infty,t}\right] \leq 2^p \mathbb{E}\left[\|w^1_{n+1} - w^1_n\|^p_{\infty,t}\right] \\
&\leq C_p \mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 H(s-r,x,y)\left[f_1(s,y,v^1_n(r,\cdot)) - f_1(s,y,v^1_{n-1}(r,\cdot)\right]\mathrm{d}y\mathrm{d}r\right|^p\right] \\
&+ C_p \mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 H(s-r,x,y)\left[\sigma_1(s,y,v^1_n(r,\cdot)) - \sigma_1(s,y,v^1_{n-1}(r,\cdot))\right]\mathrm{W}(\mathrm{dy},\mathrm{dr})\right|^p\right] \\
&+ C_p \mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 \frac{\partial H}{\partial y}(s-r,x,y)\left[h_M(v^1_n(r,\cdot),v^2_n(r,\cdot))(v^1_n(r,y)\wedge M)\right.\right. \\
&\qquad \left.\left. - h_M(v^1_{n-1}(r,\cdot),v^2_{n-1}(r,\cdot))(v^1_{n-1}(r,y)\wedge M)\right]\mathrm{d}y\mathrm{d}r\right|^p\right]
\end{aligned}$$

We deal with the three terms separately. For the first, we apply Hölder's inequality with respect to the measures $\mu_{s,x}$ on $[0,s]\times[0,1]$ given by

$$\mu_{s,x}(A) = \int_A H(s-r,x,y)\,\mathrm{d}y\mathrm{d}r$$

to see that it is at most

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 H(s-r,x,y)\left[f_1(s,y,v^1_n(r,\cdot)) - f_1(s,y,v^1_{n-1}(r,\cdot)\right]^p\mathrm{d}y\mathrm{d}r\right|\right. \\
&\quad\left.\times\left|\int_0^s\int_0^1 H(s-r,x,y)\mathrm{d}y\mathrm{d}r\right|^{p/q}\right] \\
&\leq C_T\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 H(s-r,x,y)\left[f_1(s,y,v^1_n(r,\cdot)) - f_1(s,y,v^1_{n-1}(r,\cdot)\right]^p\mathrm{d}y\mathrm{d}r\right|\right].
\end{aligned}$$

Making use of the Lipschitz property of the function f_1, this is at most

$$\begin{aligned}
&C_T\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 H(s-r,x,y)\|v^1_n(r,\cdot) - v^1_{n-1}(r,\cdot)\|^p_\infty\mathrm{d}y\mathrm{d}r\right|\right] \\
&\leq C_T\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\left(\int_0^1 H(s-r,x,y)\mathrm{d}y\right)\|v^1_n - v^1_{n-1}\|^p_{\infty,r}\mathrm{d}s\right|\right] \\
&\leq C_T\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\int_0^s\|v^1_n - v^1_{n-1}\|^p_{\infty,r}\mathrm{d}r\right]. \\
&\leq C_T\mathbb{E}\left[\int_0^t\|v^1_n - v^1_{n-1}\|^p_{\infty,r}\mathrm{d}r\right].
\end{aligned}$$

For the second term, we apply Proposition 4.1.10 and the Lipschitz property of σ_1 to deduce

that, for $p > 10$ and $t \in [0,T]$,

$$\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1 H(s-r,x,y)\left[\sigma_1(s,y,v_n^1(r,\cdot))-\sigma_1(s,y,v_{n-1}^1(r,\cdot))\right]\mathrm{W}(\mathrm{dy},\mathrm{dr})\right|^p\right]$$
$$\leq C_{p,T}\int_0^t \mathbb{E}\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r}^p\right]\mathrm{d}r.$$

Finally, we deal with the third term. Using that h is Lipschitz and bounded on bounded sets, we see that this is at most

$$C_M\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\int_0^1\left|\frac{\partial H}{\partial y}(s-r,x,y)\right|\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r} + \|v_n^2 - v_{n-1}^2\|_{\infty,r}\right]\mathrm{d}y\mathrm{d}r\right|^p\right]$$
$$= C_M\mathbb{E}\left[\sup_{s\in[0,t]}\sup_{x\in[0,1]}\left|\int_0^s\left(\int_0^1\left|\frac{\partial H}{\partial y}(s-r,x,y)\right|\mathrm{d}y\right)\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r} + \|v_n^2 - v_{n-1}^2\|_{\infty,r}\right]\mathrm{d}r\right|^p\right].$$

Applying Hölder's inequality and Proposition A.1.6 gives that this is at most

$$C_M\left(\sup_{x\in[0,1]}\int_0^t\left[\int_0^1\left|\frac{\partial H}{\partial y}(s-r,x,y)\right|\mathrm{d}y\right]^q\mathrm{d}r\right)^{p/q}$$
$$\times\int_0^t\mathbb{E}\left[\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r} + \|v_n^2 - v_{n-1}^2\|_{\infty,r}\right]^p\right]\mathrm{d}r$$
$$\leq C_M\times\left(\int_0^t\frac{1}{(t-r)^{q/2}}\mathrm{d}r\right)^{p/q}\times\int_0^t\mathbb{E}\left[\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r} + \|v_n^2 - v_{n-1}^2\|_{\infty,r}\right]^p\right]\mathrm{d}r$$
$$= C_M\times\left(\int_0^t\frac{1}{r^{q/2}}\mathrm{d}r\right)^{p/q}\times\int_0^t\mathbb{E}\left[\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r} + \|v_n^2 - v_{n-1}^2\|_{\infty,r}\right]^p\right]\mathrm{d}r,$$

where $q = \frac{p}{p-1}$. For $p > 10$ and corresponding $q \in (1, \frac{10}{9})$, this is at most

$$C_{M,p,T}\times\int_0^t\mathbb{E}\left[\|v_n^1 - v_{n-1}^1\|_{\infty,r} + \|v_n^2 - v_{n-1}^2\|_{\infty,r}\right]^p\mathrm{d}r.$$

Putting this all together, we have shown that for any $t \in [0,T]$,

$$\mathbb{E}\left[\|v_{n+1}^1 - v_n^1\|_{\infty,t}^p\right] \leq C_{M,p,T}\int_0^t\mathbb{E}\left[\|v_n^1 - v_{n-1}^1\|_{\infty,s}^p + \|v_n^2 - v_{n-1}^2\|_{\infty,s}^p\right]\mathrm{d}s.$$

We can repeat these arguments to obtain similar bounds for v^2. Together, this gives

$$\mathbb{E}\left[\|v_{n+1}^1 - v_n^1\|_{\infty,T}^p + \|v_{n+1}^2 - v_n^2\|_{\infty,T}^p\right] \leq C_{M,p,T}\int_0^T\mathbb{E}\left[\|v_n^1 - v_{n-1}^1\|_{\infty,s}^p + \|v_n^2 - v_{n-1}^2\|_{\infty,s}^p\right]\mathrm{d}s.$$

We can then argue that

$$\begin{aligned}
&\mathbb{E}\left[\|v^1_{n+1}-v^1_n\|^p_{\infty,T}+\|v^2_{n+1}-v^2_n\|^p_{\infty,T}\right]\\
\leq& C_{M,p,T}\int_0^T \mathbb{E}\left[\|v^1_{n+1}-v^1_n\|^p_{\infty,s}+\|v^2_{n+1}-v^2_n\|^p_{\infty,s}\right]\mathrm{d}s\\
\leq& C^2_{M,p,T}\int_0^T\int_0^s \mathbb{E}\left[\|v^1_{n-1}-v^1_{n-2}\|^p_{\infty,u}+\|v^2_{n-1}-v^2_{n-2}\|^p_{\infty,u}\right]\mathrm{d}u\,\mathrm{d}s\\
=& C^2_{M,p,T}\int_0^T\int_u^T \mathbb{E}\left[\|v^1_{n-1}-v^1_{n-2}\|^p_{\infty,u}+\|v^2_{n-1}-v^2_{n-2}\|^p_{\infty,u}\right]\mathrm{d}s\,\mathrm{d}u\\
=& C^2_{M,p,T}\int_0^T \mathbb{E}\left[\|v^1_{n-1}-v^1_{n-2}\|^p_{\infty,u}+\|v^2_{n-1}-v^2_{n-2}\|^p_{\infty,u}\right](T-u)\,\mathrm{d}u.
\end{aligned}$$

Iterating this, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|v^1_{n+1}-v^1_n\|^p_{\infty,T}+\|v^2_{n+1}-v^2_n\|^p_{\infty,T}\right]\\
&\leq C^n_{M,p,T}\int_0^T \mathbb{E}\left[\|v^1_1-v^1_0\|^p_{\infty,s}+\|v^2_1-v^2_0\|^p_{\infty,s}\right]\frac{(T-s)^{n-1}}{(n-1)!}\mathrm{d}s\\
&\leq C^n_{M,p,T}\times\mathbb{E}\left[\|v^1_1-v^1_0\|^p_{\infty,T}+\|v^2_1-v^2_0\|^p_{\infty,T}\right]\frac{T^n}{n!}.
\end{aligned}$$

Therefore, for $m>n\geq 1$ we have that

$$\begin{aligned}
&\mathbb{E}\left[\|v^1_m-v^1_n\|^p_{\infty,T}+\|v^2_m-v^2_n\|^p_{\infty,T}\right]^{1/p}\\
&\leq\sum_{k=n}^{m-1}\left[\frac{\tilde{C}^k_{M,p,T}T^k}{k!}\right]^{1/p}\mathbb{E}\left[\|v^1_1-v^1_0\|^p_{\infty,T}+\|v^2_1-v^2_0\|^p_{\infty,T}\right]^{1/p}\to 0.
\end{aligned}$$

as $m,n\to\infty$. Hence, the sequence (v^1_n,v^2_n) is Cauchy in the space $L^p(\Omega;C([0,T]\times[0,1]))^2$ and so converges to some pair (v^1,v^2). We now verify that this is indeed a solution to our evolution equation. Let $\tilde{w}^1$ be given by

$$\begin{aligned}
\tilde{w}^1(t,x)=&\int_0^1 H(t,x,y)v^1_0(y)\mathrm{d}y\\
&+\int_0^t\int_0^1\frac{\partial H}{\partial y}(t-s,x,y)h_M(v^1(s,\cdot),v^2(s,\cdot))\left[v^1(s,y)\wedge M\right]\mathrm{d}y\mathrm{d}s\\
&+\int_0^t\int_0^1 H(t-s,x,y)f_1(s,y,v^1(s,\cdot))\mathrm{d}x\mathrm{d}s\\
&+\int_0^t\int_0^1 H(t-s,x,y)\sigma_1(s,y,v^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s).
\end{aligned}$$

Then $\tilde{w}^1$ solves the SPDE

$$\frac{\partial\tilde{w}^1}{\partial t}=\Delta\tilde{w}^1-h_M(v^1,v^2)\frac{\partial}{\partial x}(v^1\wedge M)+f_1(t,x,v^1)+\sigma(t,x,v^1)\dot{W}.$$

with Dirichlet boundary conditions $\tilde{w}^1(t,0)=\tilde{w}^1(t,1)=0$ and initial data v^1_0. Define $\tilde{v}^1=\tilde{w}^1+\tilde{z}^1$, where $\tilde{z}^1$, together with a measure $\tilde{\eta}^1$, solves our obstacle problem with obstacle $-\tilde{w}^1$. We then have that $(\tilde{v}^1,\tilde{\eta}^1)$ solves the reflected SPDE

$$\frac{\partial\tilde{v}^1}{\partial t}=\Delta\tilde{v}^1-h_M(v^1,v^2)\frac{\partial}{\partial x}(v^1\wedge M)+f_1(t,x,v^1)+\sigma(t,x,v^1)\dot{W}+\tilde{\eta}^1.$$

By arguing as in the Picard argument, we see that

$$\begin{aligned}\mathbb{E}\left[\|\tilde{v}^1 - v_n^1\|_{\infty,T}^p\right] \leq & C_{M,p,T}\int_0^T \mathbb{E}\left[\|v^1 - v_{n-1}^1\|_{\infty,t}^p + \|v^2 - v_{n-1}^2\|_{\infty,t}^p\right] \mathrm{d}t \\ \leq & \tilde{C}_{M,p,T} \times \mathbb{E}\left[\|v^1 - v_{n-1}^1\|_{\infty,T}^p + \|v^2 - v_{n-1}^2\|_{\infty,T}^p\right] \to 0.\end{aligned}$$

It follows that $\tilde{v}^1 = v^1$ in $L^p(\Omega; C([0,T]\times[0,1]))$. The same applies to v^2, so it follows that the pair (v^1, v^2), together with the reflection measures $(\tilde{\eta}^1, \tilde{\eta}^2)$, do indeed satisfy our problem.

Uniqueness follows by essentially the same argument. Given two solutions with the same initial data, (v_1^1, v_1^2) and (v_2^1, v_2^2) (together with their reflection measures), we define the stopping time

$$\tau_N = \inf\left\{t \geq 0 \mid \|(v_j^i)\|_{\infty,t} \geq N \text{ for some } i,j = 1,2\right\}.$$

This stopping time is introduced to ensure that the our weak form solutions can be written in mild form. We can then argue as in our Picard argument to obtain that, for $t \in [0,T]$

$$\mathbb{E}\left[\|v_1^1 - v_2^1\|_{\infty,t\wedge\tau_N}^p + \|v_1^2 - v_2^2\|_{\infty,t\wedge\tau_N}^p\right] \leq \int_0^t \mathbb{E}\left[\|v_1^1 - v_2^1\|_{\infty,s\wedge\tau_N}^p + \|v_1^2 - v_2^2\|_{\infty,s\wedge\tau_N}^p\right] \mathrm{d}s.$$

It then follows from Gronwall's inequality that

$$\mathbb{E}\left[\|v_1^1 - v_2^1\|_{\infty,t\wedge\tau_N}^p + \|v_1^2 - v_2^2\|_{\infty,t\wedge\tau_N}^p\right] = 0.$$

Letting $N \uparrow \infty$ and applying the MCT then gives the result. □

We are now in position to prove Theorem 4.1.8. This essentially amounts to showing that the solutions to our truncated problems coincide for different M. We use this to define a candidate function, and then check the conditions for this candidate.

Proof of Theorem 4.1.8. For every $M > 0$, let $(v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ be the solution to the M-truncated problem. Suppose $M_1 \leq M_2$. Then we clearly have that $(v_{M_2}^1, \eta_{M_2}^1, v_{M_2}^2, \eta_{M_2}^2)$ solves the M_1-truncated problem in the relative frame, until the random time

$$\tilde{\tau} = \inf\left\{t \geq 0 \mid \|(v_{M_2}^1, v_{M_2}^2)\|_{\infty,t} \geq M_1\right\}.$$

We can then argue as in the proof of Proposition 4.1.11 to deduce that

$$\mathbb{E}\left[\|(v_{M_1}^1, v_{M_1}^2) - (v_{M_2}^1, v_{M_2}^2)\|_{\infty,\tilde{\tau}}^p\right] = 0,$$

This consistency allows us to define our candidate solution in the relative frame, $(v^1, \eta^1, v^2, \eta^2)$, by setting $(v^1, \eta^1, v^2, \eta^2) = (v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ on $[0, \tau_M]$, where

$$\tau_M = \inf\left\{t \geq 0 \mid \|(v_M^1, v_M^2)\|_{\infty,t} \geq M\right\}.$$

This defines $(v^1, \eta^1, v^2, \eta^2)$ on the interval $[0, \tau)$, where $\tau = \sup_{M>0} \tau_M$. It is clear from the definition that in fact

$$\tau = \sup_{M>0}\left[\inf\left\{t \geq 0 \mid \|(v^1, v^2)\|_{\infty,t} \geq M\right\}\right].$$

Clearly, $(v^1, \eta^1, v^2, \eta^2)$ is then a maximal solution to the moving boundary problem in the relative frame until the explosion time τ, with localising sequence τ_M. We now prove uniqueness. If $(v_1^1, \eta_1^1, v_1^2, \eta_1^2)$ and $(v_2^1, \eta_2^1, v_2^2, \eta_2^2)$ are both maximal solutions, they both satisfy the M-truncated problem until they exceed M in the infinity norm, and so we can once again argue as in Proposition 4.1.11 to obtain that they both agree with the unique solution of the M-truncated problem until these times. Since this holds for every M, it follows that they agree until a common explosion time. We therefore have the result. □

Proposition 4.1.12. *Suppose that h is a bounded function. Then the solution to the moving boundary problem is global.*

Proof. Fix $T > 0$. Let $(v^1, \eta^1, v^2, \eta^2)$ be the unique maximal solution to the moving boundary problem in the relative frame, and let τ be the blow-up time for this solution. We consider the solutions to the corresponding truncated solutions, (v_M^1, v_M^2), for $M > 0$, with the same initial data i.e. $(v_M^1(0,x), v_M^2(0,x)) = (v_0^1(x), v_0^2(x))$. Let w_M^1 solve the SPDE

$$\frac{\partial w_M^1}{\partial t} = \Delta w_M^1 - h_M(v_M^1, v_M^2)\frac{\partial}{\partial x}(v_M^1 \wedge M) + f_1(t,x,v_M^1) + \sigma_1(t,x,v_M^1)\dot{W}$$

with Dirichlet boundary conditions $w_M^1(t,0) = w_M^1(t,1) = 0$ and initial data $w_M^1(0,x) = v_0^1(x)$. Define w_M^2 similarly. Noting that $\|v_M^1\|_{\infty,T} \leq 2\|w_M^1\|_{\infty,T}$ and making use of the mild form for w_M^1, we have that

$$\begin{aligned}
\mathbb{E}\left[\|v_M^1\|_{\infty,T}^p\right] \leq\ & C_p \sup_{t\in[0,T]}\sup_{x\in[0,1]}\left|\int_0^1 H(t,x,y)v_0^1(y)\mathrm{d}y\right|^p \\
& + C_p\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in[0,1]}\left|\int_0^t\int_0^1 H(t-s,x,y)f_1(s,y,v_M^1(s,\cdot))\mathrm{d}y\mathrm{d}s\right|^p\right] \\
& + C_p\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in[0,1]}\left|\int_0^t\int_0^1 H(t-s,x,y)\sigma_1(s,y,v_M^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right|^p\right] \\
& + C_p\|h\|_\infty^p\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in[0,1]}\left(\int_0^t\int_0^1\left|\frac{\partial H}{\partial y}(t-s,x,y)\right|\|v_M^1\|_{\infty,s}\mathrm{d}y\mathrm{d}s\right)^p\right]
\end{aligned}$$

By arguing as in Proposition 4.1.11, we obtain that for $t \in [0,T]$

$$\mathbb{E}\left[\|v_M^1\|_{\infty,t}^p\right] \leq C_{p,T,\|h\|_\infty}\left(\|v_0^1\|_\infty + \int_0^t \mathbb{E}\left[\|v_M^1\|_{\infty,s}^p\right]\mathrm{d}s\right).$$

By noting that $C_{p,T,\|h\|_\infty}$ and $\|v_0^1\|$ do not depend on M here, we can apply Gronwall's Lemma (as stated in Lemma 3.4, [22]) to obtain that

$$\sup_{M>0}\mathbb{E}[\|v_M^1\|_{\infty,T}^p] < \infty.$$

It follows that

$$\mathbb{E}[\|v^1\|_{\infty,\tau\wedge T}^p] \leq \mathbb{E}\left[\liminf_{M\to\infty}\|v_M^1\|_{\infty,T}^p\right] \leq \liminf_{M\to\infty}\mathbb{E}\left[\|v_M^1\|_{\infty,T}^p\right] < \infty.$$

Similarly,

$$\mathbb{E}\left[\|v^2\|^p_{\infty,\tau\wedge T}\right] < \infty.$$

This can only hold if there is almost surely no blow-up before time T i.e. $\tau > T$ almost surely. Since this holds for every $T > 0$, we must have that $\tau = \infty$ almost surely. □

4.1.3 Hölder Continuity

We now prove that, as in the case of the static reflected SPDE, our equations enjoy the expected Hölder continuity- up to 1/4-Hölder in time and up to 1/2-Hölder in space. The details of the proof here are a simplification of those used in [13], where Hölder continuity is proved for the equations when there is no moving boundary.

The following is Lemma 3.1 in [13].

Lemma 4.1.13. *Let $V \in C_b^{1,2}([0,T]\times[0,1])$ and $\psi, F \in C([0,T]\times[0,1])$ with $\psi \leq 0$. Suppose that*

$$\frac{\partial V}{\partial t} = \frac{1}{2}V'' + \psi V + \psi F \tag{4.1.7}$$

with Dirichlet or Neumann boundary conditions at zero, and zero initial data. Then

$$\|V\|_{T,\infty} \leq \|F\|_{T,\infty}.$$

We now present an adaptation of Lemma 3.2 in [13]. This will allow us to smooth solutions to certain SPDEs, which will be crucial in our later arguments in this section.

Lemma 4.1.14. *Let $f : [0,T]\times[0,1] \to \mathbb{R}$ be such that $f(\cdot,0) = f(\cdot,1) = 0$, and for every $t,s \in [0,T]$ and every $x,y \in [0,1]$*

$$|f(t,x) - f(s,y)| \leq C_f\left(\sum_{i=1}^n |t-s|^{\alpha_i} + \sum_{i=1}^m |x-y|^{\beta_i}\right).$$

Then, for every $p,q > 0$ there exists a smooth function $f_{p,q} : [0,T]\times[0,1] \to \mathbb{R}$ such that $f^{p,q}(\cdot,0) = f^{p,q}(\cdot,1) = 0$, and

(i) $\|f_{p,q}\|_\infty \leq \|f\|_\infty$.

(ii) $\|f_{p,q} - f\|_\infty \leq C_{\alpha_i,\beta_i} C_f \left(\sum_{i=1}^n p^{\alpha_i} + \sum_{i=1}^m q^{\beta_i}\right)$.

(iii) $\left\|\frac{\partial f_{p,q}}{\partial t}\right\|_\infty \leq C_{\alpha_i,\beta_i} C_f \sum_{i=1}^n p^{\alpha_i - 1}$.

(iv) $\left\|\frac{\partial f_{p,q}}{\partial x}\right\|_\infty \leq C_{\alpha_i\beta_i} C_f \sum_{i=1}^m q^{\beta_i - 1}$.

Proof. The argument mainly follows the steps of the proof of Lemma 3.2 in [13]. For $u > 0$ and $x, y \in \mathbb{R}$, we define the function

$$P_u(x,y) := \frac{1}{\sqrt{2\pi u}} \exp\left(-\frac{(x-y)^2}{2u}\right).$$

Define $f^{p,q}$ to be given by

$$f^{p,q}(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} P_{p^2}(t,s) P_{q^2}(x,y) \tilde{f}(s,y) \, \mathrm{d}s\mathrm{d}y,$$

where the function $\tilde{f}$ is an extension of f given by

$$\tilde{f}(t,x) = \sum_{n=-\infty}^{\infty} \left[f(t, x-2n) 1_{\{x \in [2n, 2n+1]\}} - f(t, 2n+2-x) 1_{\{x \in [2n+1, 2n+2]\}} \right].$$

Note that $\tilde{f}$ then satisfies

$$|\tilde{f}(t,x) - \tilde{f}(s,y)| \leq 2C_f \left(\sum_{i=1}^{n} |t-s|^{\alpha_i} + \sum_{i=1}^{m} |x-y|^{\beta_i} \right)$$

for every $x, y \in \mathbb{R}$ and every $t, s \in [0,T]$. We now verify that $f^{p,q}$ satisfies the statements claimed. First note that we do have $f^{p,q}(t,0) = f^{p,q}(t,1) = 0$, since

$$\begin{aligned}
f^{p,q}(t,0) = &\sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \int_{2n}^{2n+1} P_{p^2}(t,s) P_{q^2}(0,y) f(s, y-2n) \, \mathrm{d}y\mathrm{d}s \\
&- \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \int_{2n+1}^{2n+2} P_{p^2}(t,s) P_{q^2}(0,y) f(s, 2n+2-y) \, \mathrm{d}y\mathrm{d}s \\
= &\sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \int_{0}^{1} P_{p^2}(t,s) P_{q^2}(0,y+2n) f(s,y) \, \mathrm{d}y\mathrm{d}s \\
&- \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \int_{0}^{1} P_{p^2}(t,s) P_{q^2}(0,2n+2-y) f(s,y) \, \mathrm{d}y\mathrm{d}s = 0.
\end{aligned}$$

The calculation for $f^{p,q}(t,1)$ is similar. We now prove inequality (i). Note that

$$\begin{aligned}
|f^{p,q}(t,x)| = &\left| \int_{\mathbb{R}} \int_{\mathbb{R}} P_{p^2}(t,s) P_{q^2}(x,y) \tilde{f}(s,y) \, \mathrm{d}s\mathrm{d}y \right| \\
\leq &\|f\|_{\infty,T} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} P_{p^2}(t,s) P_{q^2}(x,y) \, \mathrm{d}s\mathrm{d}y \right| = \|f\|_{\infty,T}.
\end{aligned}$$

For inequality (ii), we note that

$$\begin{aligned}
|f^{p,q}(t,x) - f(t,x)| = &\int_{\mathbb{R}} \int_{\mathbb{R}} P_{p^2}(t,s) P_{q^2}(x,y) \left[f(s,y) - f(t,x) \right] \, \mathrm{d}s\mathrm{d}y \\
\leq &C_f \left(\int_{\mathbb{R}} \int_{\mathbb{R}} P_{p^2}(t,x) P_{q^2}(x,y) \left[\sum_{i=1}^{n} |t-s|^{\alpha_i} + \sum_{i=1}^{m} |x-y|^{\beta_i} \right] \, \mathrm{d}s\mathrm{d}y \right) \\
= &C_f \left(\int_{\mathbb{R}} P_{p^2}(t,s) \sum_{i=1}^{n} |t-s|^{\alpha_i} \, \mathrm{d}s + \int_{\mathbb{R}} P_{q^2}(t,s) \sum_{i=1}^{m} |x-y|^{\beta_i} \, \mathrm{d}s \right) \\
\leq &C_{\alpha_i, \beta_i} C_f \left(\sum_{i=1}^{n} p^{\alpha_i} + \sum_{i=1}^{m} q^{\beta_i} \right).
\end{aligned} \tag{4.1.8}$$

We now bound the time derivative of $f^{p,q}$. We have that

$$\begin{aligned}\left|\frac{\partial f^{p,q}}{\partial t}(t,x)\right| =& \left|\frac{1}{p^2}\int_{\mathbb{R}}\int_{\mathbb{R}} P_{p^2}(t,s)(t-s)P_{q^2}(x,y)\tilde{f}(s,y)\,\mathrm{d}s\mathrm{d}y\right| \\ =& \left|\frac{1}{p^2}\int_{\mathbb{R}}\int_{\mathbb{R}} P_{p^2}(t,s)(t-s)P_{q^2}(x,y)\left[\tilde{f}(s,y)-\tilde{f}(t,y)\right]\,\mathrm{d}s\mathrm{d}y\right| \\ \leq& C_f\frac{1}{p^2}\int_{\mathbb{R}}\int_{\mathbb{R}} P_{p^2}(t,s)P_{q^2}(x,y)\sum_{i=1}^{n}|t-s|^{\alpha_i+1}\,\mathrm{d}s\mathrm{d}y \\ =& C_f\frac{1}{p^2}\int_{\mathbb{R}} P_{p^2}(t,s)\sum_{i=1}^{n}|t-s|^{\alpha_i+1}\,\mathrm{d}s \\ \leq& C_f C_{\alpha_i,\beta_i}\frac{1}{p^2}\sum_{i=1}^{n}p^{1+\alpha_i} = C_f C_{\alpha_i,\beta_i}\sum_{i=1}^{n}p^{\alpha_i-1}.\end{aligned}$$

The argument for bounding the spatial derivative of $f^{p,q}$ is similar. □

Before proceeding to the main arguments for Hölder continuity of the solutions to our moving boundary problem, we first present a direct proof of a classical result which controls the Hölder norm of the solution to the heat equation with the Hölder norm of the initial data.

Proposition 4.1.15. *Let u solve the heat equation $\frac{\partial u}{\partial t} = \Delta u$ with Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$ and initial data $u_0 \in C_0((0,1)) \cap C^{\alpha}([0,1])$, where $\alpha > 0$. Then we have that*

$$\sup_{t,s\geq 0, t\neq s}\sup_{x,y\in[0,1],x\neq y}\frac{|u(t,x)-u(s,y)|}{|t-s|^{\alpha/2}+|x-y|^{\alpha}} \leq C_{\alpha}\sup_{x,y\in[0,1],x\neq y}\frac{|u_0(x)-u_0(y)|}{|x-y|^{\alpha}}.$$

Proof. Define the extension of u_0, u_0^e given by

$$u_0^e(x) = \sum_{n=-\infty}^{\infty}\left[u_0(x-2n)1_{\{x\in[2n,2n+1]\}} - u_0(2n+2-x)1_{\{x\in[2n+1,2n+2]\}}\right].$$

Then

$$\sup_{x,y\in\mathbb{R},x\neq y}\frac{|u_0^e(x)-u_0^e(y)|}{|x-y|^{\alpha}} \leq 2\sup_{x,y\in[0,1],x\neq y}\frac{|u_0(x)-u_0(y)|}{|x-y|^{\alpha}}.$$

We then have that

$$u(t,x) = \int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-(x-y)^2/4t}u_0^e(y)\,\mathrm{d}y.$$

for $(t,x)\in[0,\infty)\times[0,1]$. Therefore, for $t\geq 0$ and $x,y\in[0,1]$

$$\begin{aligned}|u(t,x)-u(t,y)| =& \left|\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-(x-z)^2/4t}u_0^e(z)\,\mathrm{d}z - \int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-(y-z)^2/4t}u_0^e(z)\,\mathrm{d}z\right| \\ \leq& \int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-z^2/4t}|u_0^e(z+x)-u_0^e(z+y)|\,\mathrm{d}y \\ \leq& 2|x-y|^{\alpha}\sup_{x,y\in[0,1],x\neq y}\frac{|u_0(x)-u_0(y)|}{|x-y|^{\alpha}}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-z^2/4t}\,\mathrm{d}y \\ =& 2|x-y|^{\alpha}\sup_{x,y\in[0,1],x\neq y}\frac{|u_0(x)-u_0(y)|}{|x-y|^{\alpha}}.\end{aligned}$$

This proves the required Hölder bound in space. For the Hölder continuity in time, we note that for $t \geq 0$ and $x \in [0,1]$

$$\begin{aligned}
|u(t,x) - u_0(x)| =& \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-z)^2/4t} \left[u_0^e(z) - u_0^e(x) \right] \, \mathrm{d}z \right| \\
\leq& 2 \sup_{x,y \in [0,1], x \neq y} \frac{|u_0(x) - u_0(y)|}{|x-y|^\alpha} \left(\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-z)^2/4t} |z-x|^\alpha \, \mathrm{d}z \right) \qquad (4.1.9) \\
=& C_\alpha \sup_{x,y \in [0,1], x \neq y} \frac{|u_0(x) - u_0(y)|}{|x-y|^\alpha} \times t^{\alpha/2}.
\end{aligned}$$

For a general $t > s \geq 0$, we note that $u(t,\cdot)$ is the solution to the heat equation with initial data $u(s,\cdot)$ at time $t-s$. We also have from our earlier calculation that the α- Hölder norm of $u(s,\cdot)$ in space is bounded by twice that of u_0. Therefore, it follows from the calculation (4.2.13) that

$$|u(t,x) - u(s,x)| \leq 2C_\alpha \sup_{x,y \in [0,1], x \neq y} \frac{|u_0(x) - u_0(y)|}{|x-y|^\alpha} \times |t-s|^{\alpha/2}.$$

This concludes the proof. □

We now present the main result of this section, regarding the Hölder continuity of solutions to our moving boundary problem. In addition to allowing for the extra term in the equation, corresponding to the moving boundary, our proof here slightly differs from the approach used in [13] in another way. In [13], the solution to the obstacle problem

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x,u) + \sigma(t,x,u)\dot{W} + \eta \qquad (4.1.10)$$

is approximated by the solutions of the penalised SPDEs

$$\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon + f(t,x,u_\epsilon) + \sigma(t,x,u_\epsilon)\dot{W} + g_\epsilon(u_\epsilon), \qquad (4.1.11)$$

where $g_\epsilon(x) = \frac{1}{\epsilon} \arctan([x \wedge 0]^2)$. Hölder continuity of the solution to (4.1.10) is then shown by uniformly controlling the Hölder continuity of the equations (4.1.11). Here, we instead approximate u by the solutions to the equations

$$\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon + f(t,x,u) + \sigma(t,x,u)\dot{W} + g_\epsilon(u_\epsilon).$$

By using u in the coefficients of our approximating SPDEs f and σ here, we limit the problem of uniformly controlling the Hölder coefficients to studying the deterministic obstacle problem.

Theorem 4.1.16. *For $i = 1,2$, let $p_0 \in \mathbb{R}$ and u_0^i be such that, for every $\xi \in (0,1/2)$, $u_0^i \circ (\Theta_{p_0}^i)^{-1} \in C_0((0,1)) \cap C^\xi([0,1])$. For every $T > 0$, $M > 0$ and every $\gamma \in (0,1)$ the solution (v_M^1, v_M^2) to the M-truncated problem in the relative frame, described by equations (4.1.4) and (4.1.5) with initial data $(u_0^1 \circ (\Theta_{p_0}^1)^{-1}, u_0^2 \circ (\Theta_{p_0}^2)^{-1})$, is $\gamma/4$-Hölder in time and $\gamma/2$-Hölder in space on $[0,T] \times [0,1]$. In particular, if $(u^1, \eta^1, u^2, \eta^2, p)$ is the solution to our moving boundary problem with initial data (u_0^1, u_0^2, p_0), then (u^1, u^2) enjoys the same Hölder regularity locally until the blow-up time, τ.*

Proof. We consider v_M^1 only, since the argument for v_M^2 is identical. Define $v_0^i := u_0^i \circ (\Theta_{p_0}^i)^{-1}$ and let w_M^1 be the $C_0((0,1))$-valued process given by

$$\begin{aligned} w_M^1(t,x) = &\int_0^1 H(t,x,y)v_0^1(y)\,\mathrm{d}y \\ &+ \int_0^t \int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)h_M(v_M^1(s,.),v_M^2(s,.))(v_M^1(s,y)\wedge M)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 H(t-s,x,y)f_1(s,y,v_M^1(s,\cdot))\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 H(t-s,x,y)\sigma_1(s,y,v_M^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s). \end{aligned}$$

Let a be such that $\frac{a}{a+5} > \gamma$, and set $r = 2a+10$, so that $a = r/2-5$. By applying Proposition 4.1.15 to the first term, and using the inequalities from Propositions A.1.5 and A.1.7 together with Burkholder's inequality for the additional terms, we see that

$$\mathbb{E}\left[|w_M^1(t,x) - w_M^1(s,y)|^r\right] \leq C(|t-s|^{1/2} + |x-y|)^{\frac{r}{2}-2} = C(|t-s|^{1/2} + |x-y|)^{a+3}.$$

We note that it is Proposition A.1.7 which allows us to control the extra term arising due to the moving boundary. Note that $\frac{a}{r} = \frac{a}{2a+10} > \frac{\gamma}{2}$. It then follows by Corollary A.3 in [11] that there exists a random variable $X \in L^r(\Omega)$ such that, for every $t,s \in [0,T]$ and every $x,y \in [0,1]$,

$$|w_M^1(t,x) - w_M^1(s,y)| \leq X(|t-s|^{1/2} + |x-y|)^{\frac{\gamma}{2}}$$

almost surely. In order to obtain Hölder continuity for v_M^1, it is only left to prove Hölder continuity of z, where z solves the obstacle problem with obstacle $-w_M^1$. From here, the argument follows the steps from [13], so we give an outline only and refer the reader to the proof of Theorem 3.3 in [13] for further details. For each $\epsilon > 0$, let z^ϵ solve the PDE

$$\frac{\partial z^\epsilon}{\partial t} = \Delta z^\epsilon + g_\epsilon(z^\epsilon + w_M^1)$$

on $[0,1]$ with Dirichlet boundary conditions and zero initial data, where we once again define

$$g_\epsilon(x) := \frac{1}{\epsilon}\arctan([x\wedge 0]^2).$$

We then have (see [40]) that $z^\epsilon + w_M^1$ increases to v_M^1. Let $(w_M^1)_{p,q}$ be a smoothing of w_M^1 as in Proposition 4.1.14, with respect to the random variable X, the Hölder coefficients $\gamma/2$ and $\gamma/4$, and the constants p, q which are yet to be determined. Define $z_{p,q}^\epsilon$ to be the solution of the PDE

$$\frac{\partial z_{p,q}^\epsilon}{\partial t} = \Delta z_{p,q}^\epsilon + g_\epsilon(z_{p,q}^\epsilon + (w_M^1)_{p,q}) \tag{4.1.12}$$

with Dirichlet boundary condition at zero and and zero initial data. We then have that (see the proof of Theorem 1.4 in [40] for details)

$$\|z^\epsilon - z_{p,q}^\epsilon\|_{T,\infty} \leq \|w_M^1 - (w_M^1)_{p,q}\|_{T,\infty},$$

Define $\alpha^\epsilon_{p,q} := \frac{\partial z^\epsilon_{p,q}}{\partial t}$ and $\beta^\epsilon_{p,q} := \frac{\partial z^\epsilon_{p,q}}{\partial x}$. By differentiating the equation (4.1.12) in time we obtain

$$\frac{\partial \alpha^\epsilon_{p,q}}{\partial t} = \Delta \alpha^\epsilon_{p,q} + g'_\epsilon(z^\epsilon_{p,q} + (w^1_M)_{p,q}) \left[\alpha^\epsilon_{p,q} + \frac{\partial (w^1_M)_{p,q}}{\partial t} \right], \tag{4.1.13}$$

with zero initial data and Dirichlet boundary conditions $\alpha^\epsilon_{p,q}(t,0) = \alpha^\epsilon_{p,q}(t,1) = 0$. Similarly, if we differentiate (4.1.12) in space

$$\frac{\partial \beta^\epsilon_{p,q}}{\partial t} = \Delta \beta^\epsilon_{p,q} + g'_\epsilon(z^\epsilon_{p,q} + (w^1_M)_{p,q}) \left[\beta^\epsilon_{p,q} + \frac{\partial (w^1_M)_{p,q}}{\partial x} \right], \tag{4.1.14}$$

with initial data $z'_0 = 0$ and Neumann boundary conditions $\frac{\partial \beta^\epsilon_{p,q}}{\partial x}(t,0) = \frac{\partial \beta^\epsilon_{p,q}}{\partial x}(t,1) = 0$. Applying Lemma 4.1.13 to equations (4.1.13) and (4.1.14), we obtain that

$$\|\alpha^\epsilon_{p,q}\|_{T,\infty} \leq \left\| \frac{\partial (w^1_M)_{p,q}}{\partial t} \right\|_{T,\infty},$$

and

$$\|\beta^\epsilon_{p,q}\|_{T,\infty} \leq \left\| \frac{\partial (w^1_M)_{p,q}}{\partial x} \right\|_{T,\infty}.$$

Note that this control is uniform over ϵ. Fix $t, s \in [0,T]$ and $x, y \in [0,1]$. By using the bounds from Lemma 4.1.14 and choosing $p = |t-s|$, $q = |x-y|$, we obtain

$$\sup_{\epsilon>0} \|\alpha^\epsilon_{p,q}\|_{T,\infty} \leq C_\gamma X |t-s|^{(\gamma-4)/4}$$

and

$$\sup_{\epsilon>0} \|\beta^\epsilon_{p,q}\|_{T,\infty} \leq C_\gamma X |x-y|^{(\gamma-2)/2}.$$

Therefore, for $\epsilon > 0$,

$$\begin{aligned}
|z^\epsilon(t,x) - z^\epsilon(s,y)| &\leq |z^\epsilon(t,x) - z^\epsilon_{p,q}(t,x)| + |z^\epsilon_{p,q}(t,x) - z^\epsilon_{p,q}(s,y)| + |z^\epsilon_{p,q}(s,y) - z^\epsilon(s,y)| \\
&\leq 2\|z^\epsilon - z^\epsilon_{p,q}\|_{T,\infty} + \|\alpha^\epsilon_{p,q}\|_{T,\infty}|t-s| + \|\beta^\epsilon_{p,q}\|_{T,\infty}|x-y| \\
&\leq 2\|w^1_M - (w^1_M)_{p,q}\|_{T\infty} + C_\gamma X(|t-s|^{\gamma/4} + |x-y|^{\gamma/2}) \\
&= C_\gamma X(|t-s|^{\gamma/4} + |x-y|^{\gamma/2}).
\end{aligned}$$

Letting $\epsilon \downarrow 0$, we obtain the desired Hölder bound for z, from which the Hölder bound for v^1_M follows. We also have Hölder continuity of $u^1_M(t,x) := v^1_M(t, p_M(t) - x)$, where

$$p_M(t) = p_0 + \int_0^t h_M(v^1_M(s,\cdot), v^2_M(s,\cdot))\, \mathrm{d}s,$$

since

$$\begin{aligned}
|u^1_M(t,x) - u^1_M(s,y)| &= |v^1_M(t, p(t) - x) - v^1_M(s, p(s) - y)| \\
&\leq C_{\gamma,M} X(|t-s|^{\gamma/4} + |p(t) - p(s)|^{\gamma/2} + |x-y|^{\gamma/2}) \\
&\leq C_{\gamma,M,T} X(|t-s|^{\gamma/4} + |x-y|^{\gamma/2}).
\end{aligned}$$

□

Corollary 4.1.17. *For every $\gamma \in (0,1)$, the derivative of the boundary is locally $\gamma/4$-Hölder continuous on $[0,\tau)$, where τ is the blow-up time.*

Proof. Fix $\gamma \in (0,1)$. Recall that $p'(t) = h(v^1(t,\cdot), v^2(t,\cdot))$, where $v^1(t,x) = u^1(t, p(t) - \cdot)$ and $v^2(t,x) = u^2(t, p(t) + \cdot)$. For $M > 0$, define

$$\tau_M = \inf\left\{ t \geq 0 \mid \|(v^1, v^2)\|_{\infty,t} \geq M \right\}.$$

Note that $\tau_M \uparrow \tau$ as $M \to \infty$. Let $t, s \in [0, \tau_M]$. We have, by the Lipschitz property of h, that

$$\begin{aligned} |p'(t) - p'(s)| &\leq K\left(\|v^1(t,\cdot) - v^1(s,\cdot)\|_\infty + \|v^2(t,\cdot) - v^2(s,\cdot)\|_\infty \right) \\ &= K\left(\|v^1_M(t,\cdot) - v^1_M(s,\cdot)\|_\infty + \|v^2_M(t,\cdot) - v^2_M(s,\cdot)\|_\infty \right). \end{aligned}$$

The result then follows by Theorem 4.1.16. □

4.1.4 Existence of Invariant Measures

The aim in this section is to argue for the existence of invariant measures for the *profiles* of our moving boundary problem, in the case where the drift and volatility coefficients have no time dependence. That is, there exists a measure μ on $C_0((0,1))^2$ such that, if $(v^1, \eta^1, v^2, \eta^2)$ solves the moving boundary problem in the *relative* frame,

$$\begin{aligned} \frac{\partial v^1}{\partial t} =& \Delta v^1 - h(v^1, v^2)\frac{\partial v^1}{\partial x} + f_1(x, v^1) + \sigma_1(x, v^1)\dot{W} + \eta^1 \\ \frac{\partial v^2}{\partial t} =& \Delta v^2 + h(v^1, v^2)\frac{\partial v^2}{\partial x} + f_2(x, v^2) + \sigma_2(x, v^2)\dot{W}^- + \eta^2 \end{aligned} \tag{4.1.15}$$

with initial data given by the law μ, then for every $t > 0$ we have that $(v^1(t,\cdot), v^2(t,\cdot))$ also has law μ. The additional assumptions made in this section, besides f_i and σ_i not depending on time, are that the volatility coefficients, σ_i, and the function determining the boundary dynamic, h, are globally bounded (indeed, we have only proved the existence of global solutions in the case when h is globally bounded). We will prove that invariant measures exist for our system provided that the linear growth coefficient for f and the global bound for h are sufficiently small. Therefore, we now assume that for $i = 1, 2$, f_i, σ_i are measurable mappings $f_i, \sigma_i : [0,1] \times C_0((0,1)) \to \mathbb{R}$ and that $h : C_0((0,1))^2 \to \mathbb{R}$ is a measurable function, with there existing constants C, C_f, C_σ and C_h such that

(I) For every $x \in [0,1]$, $u, v \in C_0((0,1))$,

$$|f_i(x,u) - f_i(x,v)| \leq C\|u - v\|_\infty.$$

(II) For every $x \in [0,1]$, $u, v \in C_0((0,1))$,

$$|f_i(x,u)| \leq C_f(1 + \|u\|_\infty).$$

(III) For every $x \in [0,1]$, $u, v \in C_0((0,1))$,

$$|\sigma_i(x,u) - \sigma_i(x,v)| \leq C\|u - v\|_\infty.$$

(IV) For every $x \in [0,1]$, $u, v \in C_0((0,1))$,

$$|\sigma_i(x,u)| \leq C_\sigma.$$

(V) For every $u_1, u_2, v_1, v_2 \in C_0((0,1))$,

$$|h(u_1,v_1) - h(u_2,v_2)| \leq C(\|u_1 - u_2\|_\infty + \|v_1 - v_2\|_\infty).$$

(VI) For every $u, v \in C_0((0,1))$,

$$|h(u,v)| \leq C_h.$$

The following is a formal statement of the main theorem of this section.

Theorem 4.1.18. *Suppose that f_i, σ_i and h satisfy the conditions (I)- (VI). Then, for C_f and C_h sufficiently small, there exists a measure $\mu \in \mathscr{M}(C_0((0,1)) \times C_0((0,1))$ which is invariant for the process $((u^1 \circ (\theta_p^1)^{-1})(t,\cdot), (u^2 \circ (\theta_p^2)^{-1})(t,\cdot))_{t\geq 0}$.*

To prove the result, we will argue via the Krylov-Bogolyubov theorem, (Theorem 3.1.1 in [10]). This gives that it is sufficient for us to prove that the solutions to our moving boundary problem in the relative frame are Feller Markov processes, and that the family of probability distributions on $\mathscr{M}(C_0((0,1)) \times C_0((0,1)))$ induced by the solution to the moving boundary problem starting from zero initial data at times $t \geq 0$ is tight.

4.1.4.1 An L^p bound for solutions to reflected SPDEs

The aim of this section is to prove Theorem 4.1.19. The key feature of the estimate is the uniformity in T, which will be essential in arguing tightness of the sequence $(v^1(t,\cdot), v^2(t,\cdot))_{t\geq 0}$.

Theorem 4.1.19. *Suppose that $(v^1, \eta^1, v^2, \eta^2)$ solves (4.1.15) and that f_i, σ_i and h satisfy conditions (I)-(VI). Then we have that, for C_h and C_f small enough, and for any $\alpha > 0$, $p \geq 1$, and $i = 1, 2$*

$$\sup_{T>0} \mathbb{E}\left[\sup_{t\leq T} \sup_{x\in[0,1]} |v^i(t,x)e^{-\alpha(T-t)}|^p\right] < \infty. \tag{4.1.16}$$

Such a bound will later enable us to obtain uniform Hölder-type estimates for the functions $v^i(t,\cdot)$. The first step towards obtaining this bound is understanding the equations satisfied by $\tilde{v}^i(t,x) := e^{-\alpha(T-t)}v^i(t,x)$.

Proposition 4.1.20. *Suppose that $(v^1, \eta^1, v^2, \eta^2)$ solves (4.1.15). Fix $\alpha, T > 0$. Let $\tilde{v}^i(t,x) := e^{-\alpha(T-t)}v^i(t,x)$, and define the measures $\tilde{\eta}_T^i$ such that, for $A \in \mathscr{B}([0,\infty) \times [0,1])$,*

$$\tilde{\eta}_T^i(A) = \iint_A e^{-\alpha(T-t)}\, \eta^i(dy, ds).$$

Then $(\tilde{v}^1, \tilde{\eta}_T^1, \tilde{v}^2, \tilde{\eta}_T^2)$ solves the system of non-linear reflected SPDEs

$$\begin{aligned}
\frac{\partial \tilde{v}^1}{\partial t} =& \Delta \tilde{v}^1 - \tilde{h}_T(t,\tilde{v}^1,\tilde{v}^2)\frac{\partial \tilde{v}^1}{\partial x} + \tilde{f}_{1,T}(t,x,\tilde{v}^1) + \tilde{\sigma}_{1,T}(t,x,\tilde{v}^1)\dot{\tilde{W}} + \tilde{\eta}_T^1 \\
\frac{\partial \tilde{v}^2}{\partial t} =& \Delta \tilde{v}^2 + \tilde{h}_T(t,\tilde{v}^1,\tilde{v}^2)\frac{\partial \tilde{v}^2}{\partial x} + \tilde{f}_{2,T}(t,x,\tilde{v}^2) + \tilde{\sigma}_{2,T}(t,x,\tilde{v}^2)\dot{\tilde{W}}^- + \tilde{\eta}_T^2
\end{aligned} \tag{4.1.17}$$

on the domain $[0,T] \times [0,1]$, where

1. *For $i = 1, 2$ and $(t, x, z) \in [0, \infty) \times [0, 1] \times C_0((0, 1))$,*

$$\tilde{f}_{i,T}(t, x, z) = e^{-\alpha(T-t)} f_i(x, e^{\alpha(T-t)} z) + \alpha z.$$

2. *For $i = 1, 2$ and $(t, x, z) \in [0, \infty) \times [0, 1] \times C_0((0, 1))$,*

$$\tilde{\sigma}_{i,T}(t, x, z) = e^{-\alpha(T-t)} \sigma_i(x, e^{\alpha(T-t)} z).$$

3. *For $(z_1, z_2) \in C_0((0, 1))$,*

$$\tilde{h}_T(t, z_1, z_2) = h(e^{\alpha(T-t)} z_1, e^{\alpha(T-t)} z_2)$$

Proof. We prove the result for $\tilde{v}^1$, and note that the arguments for $\tilde{v}^2$ are essentially the same. Let $\varphi \in C_c^{1,2}([0, \infty) \times [0, 1])$ with $\varphi(t, 0) = \varphi(t, 1) = 0$ for every $t \geq 0$. We then have that, for $t \in [0, T]$,

$$\int_0^1 \tilde{v}^1(t, x)\varphi(t, x)\mathrm{d}x = \int_0^1 v^1(t, x) e^{-\alpha(T-t)} \varphi(t, x)\mathrm{d}x.$$

Define $\phi(t, x) := \varphi(t, x)e^{-\alpha(T-t)}$. Then we have that $\phi \in C_c^{1,2}([0, \infty) \times [0, 1])$, with $\phi(t, 0) = \phi(t, 1) = 0$. Therefore, ϕ is a valid test function for the equation for v^1. It follows that

$$\begin{aligned}
\int_0^1 v^1(t, x)\phi(t, x)\mathrm{d}x = &\int_0^1 v^1(0, x)\phi(0, x)\ \mathrm{d}x + \int_0^t \int_0^1 v^1(s, y) \frac{\partial \phi}{\partial t}(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 v^1(s, y) h(v^1(s, \cdot), v^2(s, \cdot)) \frac{\partial \phi}{\partial x}(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 v^1(s, y) \frac{\partial^2 \phi}{\partial x^2}(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 f_1(y, v^1(s, \cdot))\phi(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 \sigma_1(s, v^1(s, \cdot))\phi(s, y)\ W(\mathrm{d}y, \mathrm{d}s) \\
&+ \int_0^t \int_0^1 \phi(s, y)\ \eta^1(\mathrm{d}y, \mathrm{d}s).
\end{aligned}$$

Note that $\frac{\partial \phi}{\partial t} = e^{-\alpha(T-t)} \left(\alpha\varphi + \frac{\partial \varphi}{\partial t} \right)$. Therefore, we have that

$$\begin{aligned}
\int_0^1 \tilde{v}^1(t, x)\varphi(t, x)\mathrm{d}x = &\int_0^1 \tilde{v}^1(0, x)\phi(0, x)\ \mathrm{d}x + \int_0^t \int_0^1 \tilde{v}^1(s, y) \left[\alpha\varphi(s, y) + \frac{\partial \varphi}{\partial t}(s, y) \right]\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 \tilde{v}^1(s, y) h_T(s, \tilde{v}^1(s, \cdot), \tilde{v}^2(s, \cdot)) \frac{\partial \varphi}{\partial x}(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 \tilde{v}^1(s, y) \frac{\partial^2 \varphi}{\partial x^2}(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 f_1(y, e^{-\alpha(T-s)} \tilde{v}^1(s, \cdot)) e^{-\alpha(T-s)} \varphi(s, y)\ \mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 \sigma_1(s, e^{-\alpha(T-s)} \tilde{v}^1(s, \cdot)) e^{-\alpha(T-s)} \varphi(s, y)\ W(\mathrm{d}y, \mathrm{d}s) \\
&+ \int_0^t \int_0^1 \varphi(s, y)\ \tilde{\eta}_T^1(\mathrm{d}y, \mathrm{d}s).
\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^1 \tilde{v}^1(t,x)\varphi(t,x)\mathrm{d}x = &\int_0^1 \tilde{v}^1(0,x)\phi(0,x)\ \mathrm{d}x + \int_0^t \int_0^1 \tilde{v}^1(s,y)\frac{\partial\varphi}{\partial t}(s,y)\ \mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 \tilde{v}^1(s,y)h_T(s,\tilde{v}^1(s,\cdot),\tilde{v}^2(s,\cdot))\frac{\partial\varphi}{\partial x}(s,y)\ \mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 \tilde{v}^1(s,y)\frac{\partial^2\varphi}{\partial x^2}(s,y)\ \mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 \tilde{f}_{1,T}(y,\tilde{v}^1(s,\cdot))\varphi(s,y)\ \mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^1 \tilde{\sigma}_{1,T}(s,\tilde{v}^1(s,\cdot))\varphi(s,y)\ W(\mathrm{d}y,\mathrm{d}s) \\ &+ \int_0^t \int_0^1 \varphi(s,y)\ \tilde{\eta}_T^1(\mathrm{d}y,\mathrm{d}s).\end{aligned}$$

Also note that, since v^1 is equal to zero on the support of η^1, and we have that $supp(\eta^1) = supp(\tilde{\eta}_T^1)$, with $v^1 = 0$ precisely when $\tilde{v}^1 = 0$, we have that $\tilde{v}^1 = 0$ on the support of $\tilde{\eta}_T^1$. Therefore, $(\tilde{v}^1, \tilde{\eta}_T^1)$ solves the reflected SPDE as claimed. □

We now present some estimates for the Dirichlet heat kernel on $[0,1]$, which we denote by H. We will then be able to bound the solutions to our SPDEs by first writing them in mild form and then applying these estimates, together with Burkholder's inequality and Hölder's inequality.

Proposition 4.1.21. *The following estimates hold:*

(i)

$$\sup_{x\in[0,1]} \int_0^\infty \int_0^1 H(s,x,y)\ dyds < \infty.$$

(ii)

$$\sup_{x\in[0,1]} \int_0^\infty \int_0^1 \left|\frac{\partial H}{\partial y}(s,x,y)\right|\ dyds < \infty.$$

Proof. Using the spectral decomposition for H, we have that

$$H(s,x,y) = 2\sum_{k=1}^\infty e^{-k^2\pi^2 s}\sin(k\pi x)\sin(k\pi y).$$

Calculating, we have that

$$\begin{aligned}\int_0^\infty \int_0^1 H(t,x,y)\ \mathrm{d}y\mathrm{d}t &= 2\left|\int_0^\infty \int_0^1 \sum_{k=1}^\infty e^{-k^2\pi^2 t}\sin(k\pi x)\sin(k\pi y)\ \mathrm{d}y\mathrm{d}t\right| \\ &\leq 2\int_0^\infty \int_0^1 \sum_{k=1}^\infty e^{-k^2\pi^2 t}|\sin(k\pi y)|\ \mathrm{d}y\mathrm{d}t = 2\sum_{k=1}^\infty \int_0^\infty e^{-k^2\pi^2 t}\left(\int_0^1 |\sin(k\pi y)|\mathrm{d}y\right)\mathrm{d}t \\ &\leq 2\sum_{k=1}^\infty \int_0^\infty e^{-k^2\pi^2 t}\mathrm{d}t = \sum_{k=1}^\infty \frac{2}{k^2\pi^2} = \frac{1}{3}.\end{aligned}$$

For the second estimate, we first note that by Proposition A.1.6,

$$\sup_{x\in[0,1]}\int_0^1\int_0^1\left|\frac{\partial H}{\partial y}(s,x,y)\right|\,\mathrm{d}y\mathrm{d}s<\infty.$$

For $(s,x,y)\in[1,\infty)\times[0,1]\times[0,1]$, we have that

$$\begin{aligned}\left|\frac{\partial H}{\partial y}(s,x,y)\right| =&2\left|\sum_{k=1}^{\infty}e^{-k^2\pi^2 s}k\pi\sin(k\pi x)\cos(k\pi y)\right|\\ \leq&2\sum_{k=1}^{\infty}k\pi e^{-k^2\pi^2 s}\\ =&2e^{-\pi^2 s}\sum_{k=1}^{\infty}k\pi e^{-(k^2-1)\pi^2 s}\\ \leq&2e^{-\pi^2 s}\sum_{k=1}^{\infty}k\pi e^{-(k^2-1)\pi^2}\\ =&Ce^{-\pi^2 s}.\end{aligned}$$

Therefore,

$$\int_1^{\infty}\int_0^1\left|\frac{\partial H}{\partial y}(s,x,y)\right|\,\mathrm{d}y\mathrm{d}s<\infty.$$

So we have the result. □

We present some additional estimates in the following Proposition. These will allow us to control the tail of the heat kernel.

Proposition 4.1.22. *There exists $C>0$ such that for every $t\in[1,\infty)$*

1. $\sup_{x\in[0,1]}\sup_{y\in[0,1]}\left|\frac{\partial H}{\partial x}(t,x,y)\right|<Ce^{-\pi^2 t}$.
2. $\sup_{x\in[0,1]}\sup_{y\in[0,1]}\left|\frac{\partial H}{\partial t}(t,x,y)\right|<Ce^{-\pi^2 t}$.
3. $\sup_{x\in[0,1]}\sup_{y\in[0,1]}\left|\frac{\partial H}{\partial y\partial x}(t,x,y)\right|<Ce^{-\pi^2 t}$.
4. $\sup_{x\in[0,1]}\sup_{y\in[0,1]}\left|\frac{\partial H}{\partial y\partial t}(t,x,y)\right|<Ce^{-\pi^2 t}$.

Proof. We once again make use of the expression

$$H(s,x,y)=2\sum_{k=1}^{\infty}e^{-k^2\pi^2 s}\sin(k\pi x)\sin(k\pi y).$$

Differentiating with respect to x, we obtain that, for $t \geq 1$,

$$\begin{aligned}\left|\frac{\partial H}{\partial x}(t,x,y)\right| =& 2\left|\sum_{k=1}^{\infty} e^{-k^2\pi^2 t} k\pi \cos(k\pi x)\sin(k\pi y)\right| \\ \leq& 2\sum_{k=1}^{\infty} k\pi e^{-k^2\pi^2 t} \\ \leq& 2e^{-\pi^2 t}\sum_{k=1}^{\infty} k\pi e^{-(k^2-1)\pi^2} \\ =& Ce^{-\pi^2 t}.\end{aligned}$$

As the manipulations for inequalities (2), (3) and (4) are similar, we omit these proofs. □

Proposition 4.1.23. *The following estimates hold for $p > 4$*

1. *For every $s, t > 0$ such that $|t-s| \leq 1$*

$$\sup_{x\geq 0}\left[\int_s^t \left(\int_0^1 H(t-r,x,z)^2 dz\right)^{p/(p-2)} dr\right]^{(p-2)/2} \leq C_p|t-s|^{(p-4)/4}.$$

2. *For every $0 \leq s \leq t$ such that $|t-s| \leq 1$*

$$\sup_{x\geq 0}\left[\int_0^s \left(\int_0^1 (H(t-r,x,z)-H(s-r,x,z)^2 dz\right)^{p/(p-2)} dr\right]^{(p-2)/2} \leq C_p|t-s|^{(p-4)/4}.$$

3. *For every $x, y \in [0,1]$*

$$\sup_{t\geq 0}\left[\int_0^t \left(\int_0^1 (H(t-r,x,z)-H(t-r,y,z)^2 dz\right)^{p/(p-2)} dr\right]^{(p-2)/2} \leq C_p|x-y|^{(p-4)/2}.$$

Proof. We already have that these bounds hold for $t, s \in [0,2]$ (see Proposition A.1.5). We therefore need only to extend the estimates to the case where $t, s \geq 0$. The proof of (1) is the same as previously, as performing the change of variables $u = t - r$ we see that it depends only on $t - s$. For the second estimate, we have that

$$\begin{aligned}&\int_0^s \left(\int_0^1 (H(t-r,x,z)-H(s-r,x,z)^2 \mathrm{d}z\right)^{p/(p-2)} \mathrm{d}r \\ &= \int_{s-1}^s \left(\int_0^1 (H(t-r,x,z)-H(s-r,x,z)^2 \mathrm{d}z\right)^{p/(p-2)} \mathrm{d}r \\ &\quad + \int_0^{s-1} \left(\int_0^1 (H(t-r,x,z)-H(s-r,x,z)^2 \mathrm{d}z\right)^{p/(p-2)} \mathrm{d}r.\end{aligned}$$

We have that

$$\begin{aligned}&\int_{s-1}^s \left(\int_0^1 (H(t-r,x,z)-H(s-r,x,z)^2 \mathrm{d}z\right)^{p/(p-2)} \mathrm{d}r \\ &= \int_0^1 \left(\int_0^1 (H(1+(t-s)-r,x,z)-H(1-r,x,z)^2 \mathrm{d}z\right)^{p/(p-2)} \mathrm{d}r.\end{aligned} \tag{4.1.18}$$

Applying our estimate for $t, s \in [0,2]$ from Proposition A.1.5, with $1+(t-s)$ in place of t and 1 in place of s, we have that this is at most $|t-s|^{\frac{p-4}{4}\times\frac{2}{p-2}}$. By applying estimate (2) of Proposition 4.1.22, we also have that

$$\int_0^{s-1}\left(\int_0^1 (H(t-r,x,z)-H(s-r,x,z)^2 \mathrm{d}z\right)^{p/(p-2)} \mathrm{d}r \leq C_p \int_0^\infty \left(|t-s|^2 e^{-2\pi^2 r}\right)^{p/(p-2)} \mathrm{d}r$$
$$= C_p |t-s|^{2p/(p-2)}. \tag{4.1.19}$$

Putting (4.1.18) and (4.1.19) together, we obtain estimate (2). The argument for estimate (3) is similar. □

Proposition 4.1.24. *The following estimates hold for $p > 4$*

1. *For every $0 \leq s \leq t$ such that $|t-s| \leq 1$*

$$\sup_{x\in[0,1]}\left[\int_0^s \left(\int_0^1 \left|\frac{\partial H}{\partial z}(t-r,x,z) - \frac{\partial H}{\partial z}(s-r,x,z)\right| dz\right)^{p/(p-2)} dr\right]^{(p-2)/p} \leq C|t-s|^{(p-4)/4p}.$$

2. *For every $x, y \in [0,1]$,*

$$\sup_{s\geq 0}\left[\int_0^s \left(\int_0^1 \left|\frac{\partial H}{\partial z}(s-r,x,z) - \frac{\partial H}{\partial z}(s-r,y,z)\right| dz\right)^{p/(p-2)} dr\right]^{(p-2)/p} \leq C|x-y|^{(p-4)/2p}.$$

3. *For every $0 \leq s \leq t$ such that $|t-s| \leq 1$*

$$\sup_{x\in[0,1]}\left[\int_s^t \left(\int_0^1 \left|\frac{\partial H}{\partial z}(t-r,x,z)\right| dz\right)^{p/(p-2)} dr\right]^{(p-2)/p} \leq C|t-s|^{(p-4)/4p}.$$

Proof. We can argue as in the proof of estimate (2) in Proposition 4.1.23. We split the integrals and use Proposition A.1.7 to bound where the time parameter in the heat kernels is close to zero, and use the bounds from Proposition 4.1.22 to bound the components from the tail of the heat kernel. □

Equipped with these heat kernel estimates, we can now prove the following bound on the white noise term which will appear in the mild forms for $\tilde{v}^i$, where $\tilde{v}^i(t,x) := e^{-\alpha(T-t)}v^i(t,x)$.

Proposition 4.1.25. *Suppose that σ is bounded and $\alpha > 0$. Define for $t \leq T$, $x \in [0,1]$*

$$I_3^T(t,x) := \int_0^t \int_0^1 e^{-\alpha(T-s)} H(t-s,x,y)\sigma(y, v^i(s,\cdot))W(dy,ds).$$

Then for $p \geq 1$ and $\gamma \in (0,1)$,

$$\sup_{T>0}\mathbb{E}\left[\sup_{s,t\in[0,T],s\neq T}\ \sup_{x,y\in[0,1],x\neq y}\left(\frac{|I_3^T(t,x)-I_3^T(s,y)|}{|t-s|^{\gamma/4}+|x-y|^{\gamma/2}}\right)^p\right] < \infty.$$

Proof. Fix $n \in \mathbb{N}$. Let $t, s \in [n, n+1]$ and $x, y \in [0,1]$. Assume wlog that $s \leq t$. We have that

$$\begin{aligned}
\mathbb{E}\left[|I_3^T(t,x) - I_3^T(s,y)|^p\right] &\leq C_p\mathbb{E}\left[\left|\int_s^t \int_0^1 e^{-\alpha(T-r)} H(t-r,x,z)\sigma(z,v^i(r,\cdot))W(\mathrm{d}z,\mathrm{d}r)\right|^p\right] \\
&+ C_p\mathbb{E}\left[\left|\int_0^s \int_0^1 e^{-\alpha(T-r)} \sigma(z,v^i(r,\cdot))(H(t-r,x,z) - H(s-r,x,z))W(\mathrm{d}z,\mathrm{d}r)\right|^p\right] \\
&+ C_p\mathbb{E}\left[\left|\int_0^s \int_0^1 e^{-\alpha(T-r)} \sigma(z,v^i(r,\cdot))(H(s-r,x,z) - H(s-r,y,z))W(\mathrm{d}z,\mathrm{d}r)\right|^p\right].
\end{aligned}$$

Applying Burkholder's inequality to each of these terms allows us to bound the right hand side by

$$\begin{aligned}
&C_p\mathbb{E}\left[\left|\int_s^t \int_0^1 e^{-2\alpha(T-r)} H(t-r,x,z)^2\sigma^2(z,v^i(r,\cdot))\mathrm{d}z\mathrm{d}r\right|^{p/2}\right] \\
&+ C_p\mathbb{E}\left[\left|\int_0^s \int_0^1 e^{-2\alpha(T-r)} (H(t-r,x,z) - H(s-r,x,z))^2\sigma^2(z,v^i(r,\cdot))\mathrm{d}z\mathrm{d}r\right|^{p/2}\right] \\
&+ C_p\mathbb{E}\left[\left|\int_0^s \int_0^1 e^{-2\alpha(T-r)} (H(s-r,x,z) - H(s-r,y,z))^2\sigma^2(z,v^i(r,\cdot))\mathrm{d}z\mathrm{d}r\right|^{p/2}\right].
\end{aligned}$$

We focus on the first of these terms and note that the arguments for the other two are essentially the same, the difference being in which inequality from Proposition 4.1.23 we apply. Since σ is bounded and $s, t \in [n, n+1]$, we have that $e^{-\alpha(T-r)}\sigma^2(z, v^i(r,z)) \leq \|\sigma\|_\infty^2 e^{-\alpha(T-(n+1))}$ for $(r,z) \in [0,t] \times [0,1]$. This gives that:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t \int_0^1 e^{-2\alpha(T-r)} G^2(t-r,x,z)\sigma^2(z,v^i(r,\cdot))\mathrm{d}z\mathrm{d}r\right|^{p/2}\right] \\
&\qquad \leq C_\sigma e^{-\alpha p(T-(n+1))/2}\left|\int_s^t \int_0^1 e^{-\alpha(T-r)} G^2(t-r,x,z)\mathrm{d}z\mathrm{d}r\right|^{p/2}.
\end{aligned}$$

By Hölder's inequality we have that

$$\begin{aligned}
&\left|\int_s^t \int_0^1 e^{-\alpha(T-r)} H(t-r,x,z)^2\mathrm{d}z\mathrm{d}r\right|^{p/2} \\
&= \left|\int_s^t e^{-\alpha(T-r)}\left(\int_0^1 H(t-r,x,z)^2\mathrm{d}z\right)\mathrm{d}r\right|^{p/2} \\
&\leq \int_s^t e^{-\alpha p(T-r)/2}\mathrm{d}r \times \left[\int_s^t \left(\int_0^1 H(t-r,x,z)^2\mathrm{d}r\right)^{p/(p-2)}\mathrm{d}z\right]^{(p-2)/2} \\
&\leq \int_0^\infty e^{-\alpha p t/2}\mathrm{d}t \times \left[\int_s^t \left(\int_0^1 H(t-r,x,z)^2\mathrm{d}r\right)^{p/(p-2)}\mathrm{d}z\right]^{(p-2)/2}.
\end{aligned}$$

By inequality (1) from Proposition 4.1.23, this is at most $C_{p,\alpha}|t-s|^{(p-4)/4}$. Arguing similarly for the other terms, we obtain that for $t, s \in [n, n+1]$ and $x, y \in [0,1]$

$$\mathbb{E}\left[|I_3^T(t,x) - I_3^T(s,y)|^p\right] \leq C_{p,\alpha,\sigma} e^{-\alpha p(T-(n+1))/2} \left(|t-s|^{1/2} + |x-y|\right)^{(p-4)/2}.$$

Let p be large enough so that $\gamma < (p-10)/p$. We can then apply Corollary A.3 from [11] to obtain that for $t, s \in [n, n+1]$ and $x, y \in [0,1]$

$$|I_3^T(t,x) - I_3^T(s,y)| \leq X_n(|t-s|^{1/2} + |x-y|)^{\gamma/2}.$$

almost surely, where

$$\mathbb{E}\left[X_n^p\right] \leq C_{\gamma,p,\sigma} e^{-\alpha p(T-(n+1))/2}. \tag{4.1.20}$$

Therefore, for every $t, s \in [0,T]$, $x, y \in [0,1]$,

$$|I_3^T(t,x) - I_3^T(s,y)| \leq \left(\sum_{n=0}^{\lfloor T \rfloor} X_n\right) \times (|t-s|^{1/2} + |x-y|)^{\gamma/2} \tag{4.1.21}$$

Let $X := \sum_{n=0}^{\lfloor T \rfloor} X_n$. Calculating and applying Hölder's inequality gives that

$$\begin{aligned}
\mathbb{E}\left[X^P\right] &= \mathbb{E}\left[\left(\sum_{n=0}^{\lfloor T \rfloor} X_n\right)^p\right] \\
&\leq \mathbb{E}\left[\left(\sum_{n=0}^{\lfloor T \rfloor} e^{\alpha p(T-(n+1))/4} X_n^p\right)\right] \times \left(\sum_{n=0}^{\lfloor T \rfloor} e^{-\alpha q(T-(n+1))/4}\right)^{p/q} \\
&= \left(\sum_{n=0}^{\lfloor T \rfloor} e^{\alpha p(T-(n+1))/4} \times \mathbb{E}\left[X_n^p\right]\right) \times \left(\sum_{n=0}^{\lfloor T \rfloor} e^{-\alpha q(T-(n+1))/4}\right)^{p/q},
\end{aligned}$$

where $q = p/(p-1)$. By (4.1.20), we obtain that this is at most

$$C_{\gamma,p,\sigma}\left(\sum_{n=0}^{\infty} e^{-\alpha p(n+1)/4}\right) \times \left(\sum_{n=0}^{\infty} e^{-\alpha q(n+1)/4}\right)^{p/q} = C_{\gamma,p,\sigma,\alpha} < \infty.$$

Importantly, this is independent of T. By (4.1.21), we then have that

$$\mathbb{E}\left[\sup_{s,t\in[0,T],s\neq T} \sup_{x,y\in[0,1],x\neq y} \left(\frac{|I_3^T(t,x) - I_3^T(s,y)|}{|t-s|^{\gamma/4} + |x-y|^{\gamma/2}}\right)^p\right] \leq C_{\gamma,p,\sigma,\alpha}.$$

Taking the supremum over $T > 0$ concludes the proof. □

Corollary 4.1.26. *Let I_3^T be as in Proposition 4.1.25. For $p \geq 1$, we have*

$$\sup_{T>0} \mathbb{E}\left[\|I_3^T\|_{\infty,T}^p\right] \leq C_{p,\sigma,\alpha}.$$

Proof. Note that $I_3^T(t,0) = 0$ for every $t \geq 0$ almost surely. Let $\gamma = 1/2$. Then, for $x \in (0,1]$,

$$|I_3^T(t,x)| = |I_3^T(t,x) - I_3^T(t,0)| \leq \frac{|I_3^T(t,x) - I_3^T(t,0)|}{x^{\gamma/2}} \leq \sup_{((t,x),(s,y))\in A_T} \frac{|I_3^T(t,x) - I_3^T(s,y)|}{|t-s|^{\gamma/4} + |x-y|^{\gamma/2}}. \tag{4.1.22}$$

By taking the supremum on the left hand side and then taking the $L^p(\Omega)$-norm, we see that the result follows from Proposition 4.1.25. □

We are now in position to prove the main Theorem for this section.

Proof of Theorem 4.1.19. Fix $T > 0$. Let $(\tilde{v}^1, \tilde{\eta}^1, \tilde{v}^2, \tilde{\eta}^2)$ be as in Proposition 4.1.20. Then $(\tilde{v}^1, \tilde{\eta}^1, \tilde{v}^2, \tilde{\eta}^2)$ solves the reflected SPDE (4.1.17). Let $\tilde{z}^1(t,x)$ solve the SPDE

$$\frac{\partial \tilde{z}^1}{\partial t} = \Delta \tilde{z}^1 - \tilde{h}_T(t, \tilde{v}^1, \tilde{v}^2)\frac{\partial \tilde{v}^1}{\partial x} + \tilde{f}_1(t,x,\tilde{v}^1(t,\cdot)) + \tilde{\sigma}_1(t,x,\tilde{v}^1(t,\cdot))\frac{\partial^2 W}{\partial x \partial t}.$$

We then have, by Theorem 1.4 in [40], that $\|\tilde{v}^1\|_{\infty,T} \leq 2\|\tilde{z}^1\|_{\infty,T}$ almost surely. Writing $\tilde{z}^1$ in mild form gives

$$\begin{aligned}
\tilde{z}^1(t,x) =& \int_0^t \int_0^1 H(t-s,x,y)\tilde{f}_1(s,y,\tilde{v}^1(s,\cdot))\,\mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)\tilde{h}_T(s,\tilde{v}^1(s,\cdot),\tilde{v}^2(s,\cdot))\tilde{v}^1(s,y)\,\mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_0^1 H(t-s,x,y)\tilde{\sigma}(s,y,\tilde{v}^1(s,\cdot))\,W(\mathrm{d}y\mathrm{d}s) =: I_1^T(t,x) + I_2^T(t,x) + I_3^T(t,x).
\end{aligned}$$

It follows that

$$\mathbb{E}\left[\|\tilde{z}^1\|_{\infty,T}^p\right] \leq C_p\left(\mathbb{E}\left[\|I_1^T\|_{\infty,T}^p\right] + \mathbb{E}\left[\|I_2^T\|_{\infty,T}^p\right] + \mathbb{E}\left[\|I_3^T\|_{\infty,T}^p\right]\right). \tag{4.1.23}$$

Bounding the I_1^T term, we obtain by applying Proposition 4.1.21:

$$\begin{aligned}
\left|I_1^T(t,x)\right| =& \left|\int_0^t \int_0^1 H(t-s,x,y)\tilde{f}(s,y,\tilde{v}^1)\,\mathrm{d}y\mathrm{d}s\right| \\
&\leq (C_f+\alpha)\left|\int_0^t \int_0^1 H(t-s,x,y)\tilde{v}^1(s,y)\,\mathrm{d}y\mathrm{d}s\right| + C_f\left|\int_0^t \int_0^1 H(t-s,x,y)\,\mathrm{d}y\mathrm{d}s\right| \\
&\leq \frac{(C_f+\alpha)\|\tilde{v}^1\|_{\infty,T} + C_f}{3}.
\end{aligned}$$

This gives that

$$\mathbb{E}\left[\|I_1^T\|_{\infty,T}^p\right] \leq C_p\frac{(C_f+\alpha)^p\|\tilde{v}^1\|_{\infty,T}^p}{3^p} + C_p\frac{C_f^p}{3^p}.$$

For the term I_2^T, we note that:

$$\begin{aligned}
|I_T^2(t,x)| =& \left|\int_0^t \int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)\tilde{h}(\tilde{v}^1(s,\cdot),\tilde{v}^2(s,\cdot))\tilde{v}^1(s,y)\,\mathrm{d}y\mathrm{d}s\right| \\
\leq& C_h\|\tilde{v}^1\|_{\infty,T} \times \int_0^T \int_0^1 \left|\frac{\partial H}{\partial y}(T-s,x,y)\right|\,\mathrm{d}y\mathrm{d}s \\
=& C_h\|\tilde{v}^1\|_{\infty,T} \times \int_0^T \int_0^1 \left|\frac{\partial H}{\partial y}(s,x,y)\right|\,\mathrm{d}y\mathrm{d}s \\
=& C_h\|\tilde{v}^1\|_{\infty,T}.
\end{aligned}$$

Applying Corollary 4.1.26 and using the inequality (4.1.23), we then obtain that

$$\mathbb{E}\left[\|\tilde{v}^1\|^p_{\infty,T}\right] \leq C_{p,\sigma,\alpha,f} + \left[\frac{1}{3^p}C_p(C_f+\alpha)^p + C_pC_h^p\right]\ \mathbb{E}\left[\|\tilde{v}^1\|^p_{\infty,T}\right]. \tag{4.1.24}$$

Therefore, for C_f, C_h and α small enough such that

$$\left[\frac{1}{3^p}C_p(C_f+\alpha)^p + C_pC_h^p\right] < 1,$$

it follows by simple rearrangement of (4.1.24) that

$$\mathbb{E}\left[\|\tilde{v}^1\|^p_{\infty,T}\right] \leq \tilde{C}_{p,\sigma,\alpha,f,h}.$$

Note that we are able to choose α small since the result for larger α follows from the result for small α. Since this bound is independent of T, we have the result. □

4.1.4.2 Tightness of the Sequence $(v^1(t,\cdot), v^2(t,\cdot))_{t\geq 0}$ and Proof of Theorem 5.1.9

We recall that, by the Arzela-Ascoli Theorem, the relatively compact sets in $C_0((0,1))$ are those which are equicontinuous. It follows that collections of functions for which we can uniformly bound some Hölder norm are relatively compact. Therefore, in order to prove tightness of $(v^1(t,\cdot), v^2(t,\cdot))_{t\geq 0}$, it is enough to show that for $i = 1, 2$

$$\sup_{T\geq 0}\mathbb{E}\left[\sup_{x,y\in[0,1],x\neq y}\frac{|v^i(T,x)-v^i(T,y)|^p}{|x-y|^{\alpha p}}\right] < \infty \tag{4.1.25}$$

for some $\alpha > 0$, $p \geq 1$. To show this, we use estimate (4.1.16) and follow the work of Dalang and Zhang in [13], in which the authors prove Hölder continuity for reflected SDPEs. Since the supremum over T appears outside the expectation, we can apply the reasoning from Zhang [13] to $\tilde{v}^i(t,x) = e^{-\alpha(T-t)}v^i(t,x)$ for each $T > 0$, and then take the supremum over T. That is, in order to prove (4.1.25), we will prove that

$$\sup_{T\geq 0}\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x,y\in[0,1],x\neq y}\frac{|\tilde{v}^i(t,x)-\tilde{v}^i(t,y)|^p}{|x-y|^{\alpha p}}\right] < \infty.$$

Proof of Theorem 4.1.18. Let $T > 0$ and define $\tilde{v}^i(t,x) := e^{-\alpha(T-t)}v^i(t,x)$ for $i = 1, 2$. By Proposition 4.1.20, we have that $(\tilde{v}^1, \tilde{\eta}^1, \tilde{v}^2, \tilde{\eta}^2)$ solves the system of reflected SPDEs (4.1.17). Define $\tilde{z}^1$ as the solution to the SPDE

$$\frac{\partial \tilde{z}^1}{\partial t} = \Delta\tilde{z}^1 - \tilde{h}_T(t,\tilde{v}^1(t,\cdot),\tilde{v}^2(t,\cdot))\frac{\partial\tilde{v}^1}{\partial x} + \tilde{f}(t,x,\tilde{v}^1(t,\cdot)) + \tilde{\sigma}(t,x,\tilde{v}^1(t,\cdot))\frac{\partial^2 W}{\partial x\partial t},$$

with Dirichlet boundary conditions $\tilde{z}^1(t,0) = \tilde{z}^1(t,1) = 0$ and zero initial data. We now examine the Hölder continuity of $\tilde{z}^1$. Writing $\tilde{z}^1$ in mild form, we have that

$$\begin{aligned}\tilde{z}^1(t,x) &= \int_0^t\int_0^1 H(t-s,x,y)\tilde{f}(s,y,\tilde{v}^1(s,\cdot))\,\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t\int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)\tilde{h}_T(s,\tilde{v}^1(s,\cdot),\tilde{v}^2(s,\cdot))\tilde{v}^1(s,y)\,\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t\int_0^1 H(t-s,x,y)\tilde{\sigma}(s,y,\tilde{v}^1(s,\cdot))\,W(\mathrm{d}y,\mathrm{d}s) =: I_1^T(t,x) + I_2^T(t,x) + I_3^T(t,x).\end{aligned}$$

Fix $\gamma \in (0,1)$. By Proposition 4.1.25, we have that for $p \geq 1$ there exists $X^T \in L^p(\Omega)$ such that, for every $s,t \in [0,T]$ and every $x,y \in [0,1]$

$$|I_3^T(t,x) - I_3^T(s,y)| \leq X^T(|t-s|^{\gamma/4} + |x-y|^{\gamma/2}) \tag{4.1.26}$$

almost surely, with the following uniform bound on the random variables X^T:

$$\sup_{T>0} \mathbb{E}\left[|X^T|^p\right] < \infty.$$

We now control the Hölder norm of I_1^T. We have that

$$\begin{aligned}
|I_1^T(t,x) - I_1^T(s,y)| \leq &\left| \int_s^t \int_0^1 H(t-r,x,z)\tilde{f}(r,z,\tilde{v}^1(r,z))\mathrm{d}z\mathrm{d}r \right| \\
&+ \left| \int_0^s \int_0^1 (H(t-r,x,z) - H(s-r,x,z))\tilde{f}(r,z,\tilde{v}^1(r,z))\mathrm{d}z\mathrm{d}r \right| \\
&+ \left| \int_0^s \int_0^1 (H(s-r,x,z) - H(s-r,y,z))\tilde{f}(r,z,\tilde{v}^1(r,z))\mathrm{d}z\mathrm{d}r \right|.
\end{aligned}$$

For the first of these terms, we note that for $n \in \mathbb{N}$ and $\tilde{s},\tilde{t} \in [n,n+1]$, we have that

$$\begin{aligned}
&\left| \int_{\tilde{s}}^{\tilde{t}} \int_0^1 H(\tilde{t}-r,x,z)\tilde{f}(r,z,\tilde{v}^1(r,z))\mathrm{d}z\mathrm{d}r \right| \\
&\leq (C_f + \alpha) \left| \int_{\tilde{s}}^{\tilde{t}} \int_0^1 H(\tilde{t}-r,x,z)\mathrm{d}z\mathrm{d}r \right| e^{-\alpha/2(T-(n+1))} \sup_{r\in[0,T]} \sup_{z\in[0,1]} \left[1 + e^{\alpha/2(T-r)}\tilde{v}^1(r,z)\right],
\end{aligned} \tag{4.1.27}$$

where we make use of the bound

$$\begin{aligned}
|\tilde{f}(r,z,\tilde{v}^1(r,x)| &\leq C_f e^{-\alpha(T-r)} + (C_f+\alpha)\tilde{v}^1(r,x) \\
&\leq (C_f+\alpha)e^{-\alpha/2(T-r)}(1 + e^{\alpha/2(T-r)}\tilde{v}^1(r,z)).
\end{aligned}$$

Let

$$Y^T := \sup_{t\in[0,T]} \sup_{x\in[0,1]} \left[1 + e^{\alpha/2(T-r)}\tilde{v}^1(t,x)\right].$$

By Theorem 4.1.19, we have that

$$\sup_{t\in[0,T]} \mathbb{E}\left[(Y^T)^p\right] < \infty.$$

We then have that, for p large enough so that $(p-4)/p > \gamma$,

$$\begin{aligned}
&\left| \int_{\tilde{s}}^{\tilde{t}} \int_0^1 H(\tilde{t}-r,x,z)\tilde{f}(r,z,\tilde{v}^1(r,z))\mathrm{d}z\mathrm{d}r \right| \leq C_{f,\alpha} Y^T e^{-\alpha/2(T-(n+1))} \left| \int_{\tilde{s}}^{\tilde{t}} \int_0^1 H(\tilde{t}-r,x,z)\mathrm{d}z\mathrm{d}r \right| \\
&\leq C_{f,\alpha} Y^T e^{-\alpha/2(T-(n+1))} \left(\int_{\tilde{s}}^{\tilde{t}} \left[\int_0^1 H(\tilde{t}-r,x,z)\mathrm{d}z \right]^{2p/(p-2)} \mathrm{d}r \right)^{(p-2)/2p} \\
&\leq C_{f,\alpha} Y^T e^{-\alpha/2(T-(n+1))} \left(\int_{\tilde{s}}^{\tilde{t}} \left[\int_0^1 H(\tilde{t}-r,x,z)^2\mathrm{d}z \right]^{p/(p-2)} \mathrm{d}r \right)^{(p-2)/2p} \\
&\leq C_{f,\alpha,p} Y^T e^{-\alpha/2(T-(n+1))} |\tilde{t}-\tilde{s}|^{p-4/4p} \\
&\leq C_{f,\alpha,p} Y^T e^{-\alpha/2(T-(n+1))} |\tilde{t}-\tilde{s}|^{\gamma/4},
\end{aligned}$$

Hence, for a general $s,t\in[0,T]$

$$\begin{aligned}\left|\int_s^t\int_0^1 H(t-r,x,z)\tilde{f}(r,z,\tilde{v}^1(r,z))\mathrm{d}z\mathrm{d}r\right| \leq& C_{f,\alpha,p}Y^T\left[\sum_{n=0}^{\lfloor T\rfloor}e^{-\alpha/2(T-(n+1))}\right]|t-s|^{\gamma/4}\\ \leq& C_{f,\alpha,p}Y^T\left[\sum_{n=0}^{\infty}e^{-\alpha/2(n+1)}\right]|t-s|^{\gamma/4}\\ =& C_{f,\alpha,p}Y^T|t-s|^{\gamma/4}.\end{aligned}$$

Arguing in the same way and applying the other estimates from Proposition 4.1.23, we obtain that for every $t,s\in[0,T]$ and every $x,y\in[0,1]$

$$|I_1^T(t,x)-I_1^T(s,y)|\leq\tilde{Y}^T(|t-s|^{\gamma/4}+|x-y|^{\gamma/2}) \tag{4.1.28}$$

almost surely, where $\sup_{T>0}\mathbb{E}\left[|\tilde{Y}^T|^p\right]<\infty$. For the I_2^T term, we have that

$$\begin{aligned}&|I_2^T(t,x)-I_2^T(s,y)|\\ &\leq\left|\int_s^t\int_0^1\frac{\partial H}{\partial y}(t-r,x,z)\tilde{h}_T(r,\tilde{v}^1(r,\cdot),\tilde{v}^2(r,\cdot))\tilde{v}^1(r,z)\,\mathrm{d}z\mathrm{d}r\right|\\ &\quad+\left|\int_0^s\int_0^1\left(\frac{\partial H}{\partial y}(t-r,x,z)-\frac{\partial H}{\partial y}(s-r,x,z)\right)\tilde{h}_T(r,\tilde{v}^1(r,\cdot),\tilde{v}^2(r,\cdot))\tilde{v}^1(r,z)\,\mathrm{d}z\mathrm{d}r\right|\\ &\quad+\left|\int_0^s\int_0^1\left(\frac{\partial H}{\partial y}(s-r,x,z)-\frac{\partial H}{\partial y}(s-r,y,z)\right)\tilde{h}_T(r,\tilde{v}^1(r,\cdot),\tilde{v}^2(r,\cdot))\tilde{v}^1(r,z)\,\mathrm{d}z\mathrm{d}r\right|.\end{aligned}$$

We have that, for $n\in\mathbb{N}$ and $\tilde{s},\tilde{t}\in[n,n+1]$

$$\begin{aligned}&\left|\int_{\tilde{s}}^{\tilde{t}}\int_0^1\frac{\partial H}{\partial y}(\tilde{t}-r,x,z)\tilde{h}_T(r,\tilde{v}^1(r,\cdot),\tilde{v}^2(r,\cdot))\tilde{v}^1(r,z)\,\mathrm{d}z\mathrm{d}r\right|\\ &\leq C_h\int_{\tilde{s}}^{\tilde{t}}\int_0^1\left|\frac{\partial H}{\partial y}(\tilde{t}-r,x,z)\right|\tilde{v}^1(r,z)\,\mathrm{d}z\mathrm{d}r\\ &\leq C_h\left[\sup_{r\in[0,T]}\sup_{z\in[0,1]}e^{\alpha/2(T-r)}\tilde{v}^1(r,z)\right]e^{-\alpha/2(T-(n+1))}\int_{\tilde{s}}^{\tilde{t}}\int_0^1\left|\frac{\partial H}{\partial y}(\tilde{t}-r,x,z)\right|\,\mathrm{d}z\mathrm{d}r\\ &\leq C_h\left[\sup_{r\in[0,T]}\sup_{z\in[0,1]}e^{\alpha/2(T-r)}\tilde{v}^1(r,z)\right]e^{-\alpha/2(T-(n+1))}\\ &\quad\times\left(\int_{\tilde{s}}^{\tilde{t}}\left[\int_0^1\left|\frac{\partial H}{\partial y}(\tilde{t}-r,x,z)\right|\,\mathrm{d}z\right]^{p/(p-2)}\mathrm{d}r\right)^{(p-2)/p}.\\ &=C_hZ^Te^{-\alpha/2(T-(n+1))}\left(\int_{\tilde{s}}^{\tilde{t}}\left[\int_0^1\left|\frac{\partial H}{\partial y}(\tilde{t}-r,x,z)\right|\,\mathrm{d}z\right]^{p/(p-2)}\mathrm{d}r\right)^{(p-2)/p}.\end{aligned}$$

where $\sup_{T\geq0}\mathbb{E}\left[|Z^T|^p\right]<\infty$. By estimate (3) of Proposition 4.1.24, we have that this is at most

$$C_{h,p}Z^Te^{-\alpha/2(T-(n+1))}|\tilde{t}-\tilde{s}|^{(p-4)/4p},$$

We then argue as we did for the I_1^T term to obtain that

$$|I_2^T(t,x) - I_2^T(s,y)| \leq \tilde{Z}^T(|t-s|^{\gamma/4} + |x-y|^{\gamma/2}), \tag{4.1.29}$$

almost surely, where $\sup_{T>0} \mathbb{E}\left[|\tilde{Z}^T|^p\right] < \infty$. Setting $R^T = X^T + \tilde{Y}^T + \tilde{Z}^T$, we have from (4.1.28) and (4.1.29) that R^T bounds the $(\gamma/4, \gamma/2)$-Hölder norm of $\tilde{z}^1$. By arguing as in the proof of Theorem 4.1.16, which is based on the arguments in Theorem 3.3 of [13], it is shown that the Hölder norm of $\tilde{z}^1$ controls the Hölder norm of $\tilde{v}^1$. More precisely, we have that

$$|\tilde{v}^1(t,x) - \tilde{v}^1(s,y)| \leq C_\gamma R^T(|t-s|^{\gamma/4} + |x-y|^{\gamma/2})$$

for every $t, s \in [0,T]$ and every $x, y \in [0,1]$ almost surely. In particular, we obtain

$$\mathbb{E}\left[\sup_{x,y\in[0,1],x\neq y} \frac{|v^1(T,x) - v^1(T,y)|^p}{|x-y|^{\gamma p/2}}\right] = \mathbb{E}\left[\sup_{x,y\in[0,1],x\neq y} \frac{|\tilde{v}^1(T,x) - \tilde{v}^1(T,y)|^p}{|x-y|^{\gamma p/2}}\right] \leq C_\gamma \mathbb{E}\left[|Z^T|^p\right].$$

Noting that $\sup_{T>0} \mathbb{E}\left[|Z^T|^p\right] < \infty$ concludes the proof. □

Remark 4.1.27. *A clear corollary of the result in this section is the existence of invariant measures for static reflected SPDEs*

$$\frac{\partial u}{\partial t} = \Delta u + f(x, u(t,\cdot)) + \sigma(x, u(t,\cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta,$$

on the domain $[0,\infty) \times [0,1]$, when the coefficients f and σ are globally bounded. This is a new result of interest in the study of static reflected SPDEs.

4.1.5 Extension to General Elliptic Operators

In order to simplify our calculations, we have focused throughout the last subsection on the case where the elliptic operator associated with our SPDEs has been the Laplacian. In this subsection, we illustrate how the analysis can easily be adapted to the case where we replace the Laplacian in our equations with elliptic operators of the form

$$(\mathscr{L}f)(x) = a(x)\frac{\partial^2 f}{\partial x^2}(x) + b(x)\frac{\partial f}{\partial x}(x),$$

where $a \in C^2([0,1])$ with $a(x) > 0$ for every $x \in [0,1]$, and $b \in C^1([0,1])$. We introduce here the the adjoint of $\mathscr{L}$, $\mathscr{L}^\star$, given by

$$(\mathscr{L}^\star f)(x) = \frac{\partial^2}{\partial x^2}(a(x)f(x)) - \frac{\partial}{\partial x}(b(x)f(x)).$$

Our moving boundary problem then takes the form

$$\begin{aligned}
\frac{\partial u^1}{\partial t} &= \mathscr{L}_1 u^1 + f_1(t, p(t) - x, u^1(t, p(t) - \cdot)) + \sigma_1(t, p(t) - x, u^1(t, p(t) - \cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta^1 \\
\frac{\partial u^2}{\partial t} &= \mathscr{L}_2 u^2 + f_2(t, x - p(t), u^2(t, p(t) + \cdot)) + \sigma_2(t, x - p(t), u^2(t, p(t) + \cdot))\frac{\partial^2 W}{\partial x \partial t} + \eta^2 \\
p'(t) &= h(u^1(t, p(t) - \cdot), u^2(t, p(t) + \cdot)).,
\end{aligned} \tag{4.1.30}$$

We begin by defining our notion of solution for these equations, via a definition for solutions to a class of non-linear SPDEs. Our definition will once again be motivated by changing variables to the relative frame, and integrating by parts.

Definition 4.1.28. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $\tilde{v}$ is a continuous $\mathscr{F}_t$-adapted process taking values in $C_0(0,1)$. Let $f_i, \sigma_i : [0,\infty) \times [0,1] \times C_0((0,1)) \to \mathbb{R}$, $h : C_0(0,1) \times C_0(0,1) \to \mathbb{R}$, $F : C_0(0,1) \to C_0(0,1)$ and $c \in C^1([0,1])$. Finally, suppose that $\mathscr{L}$ is an elliptic operator given by*

$$(\mathscr{L}f)(x) := a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x},$$

where $a \in C^2([0,1])$ with $a(x) > 0$ for all $x \in [0,1]$, and $b \in C^1([0,1])$. For the $\mathscr{F}_t$-stopping time τ, we say that the pair (v, η) is a local solution to the reflected SPDE

$$\frac{\partial v}{\partial t} = \mathscr{L}v + c(x)h(v,\tilde{v})\frac{\partial F(v)}{\partial x} + f(t,x,v) + \sigma(t,x,v)\dot{W} + \eta \tag{4.1.31}$$

with Dirichlet boundary conditions $v(t,0) = v(t,1) = 0$ and initial data $v_0 \in C_0(0,1)^+$, until time τ, if

(i) For every $x \in [0,1]$ and every $t \in [0,\infty)$, $v(t,x)$ is $\mathscr{F}_t$-measurable.

(ii) $v \geq 0$ almost surely.

(iii) $v\big|_{[0,\tau)\times[0,1]} \in C([0,\tau) \times [0,1])$ almost surely.

(iv) $v(t,x) = \infty$ for every $t \geq \tau$ almost surely.

(v) η is a measure on $(0,1) \times [0,\infty)$ such that

(a) For every measurable map $\psi : [0,1] \times [0,\infty) \to \mathbb{R}$,

$$\int_0^t \int_0^1 \psi(x,s)\, \eta(dx, ds)$$

is $\mathscr{F}_t$-measurable.

(b) $\int_0^\infty \int_0^1 v(t,x)\, \eta(dx,dt) = 0$.

(vi) There exists a localising sequence of stopping times $\tau_n \uparrow \tau$ almost surely, such that for

every $\varphi \in C_c^{1,2}([0,\infty)\times[0,1])$ *such that* $\varphi(s,0)=\varphi(s,1)=0$ *for every* $s\geq 0$,

$$\begin{aligned}\int_0^1 v(t\wedge\tau_n,x)\varphi(t\wedge\tau_n,x)\,dx = &\int_0^1 v(0,x)\varphi(0,x)\,dx + \int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial\varphi}{\partial t}(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 v(s,x)(\mathscr{L}^\star\varphi(s,\cdot))(x)\,dx\,ds\\ &-\int_0^{t\wedge\tau_n}\int_0^1 F(v(s,\cdot))(x)h(v(s,\cdot),\tilde{v}(s,\cdot))c(x)\frac{\partial\varphi}{\partial x}(s,x)\,dx\,ds\\ &-\int_0^{t\wedge\tau_n}\int_0^1 F(v(s,\cdot))(x)h(v(s,\cdot),\tilde{v}(s,\cdot))c'(x)\varphi(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 f(s,x,v(s,\cdot))\varphi(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 \sigma(s,x,v(s,\cdot))\varphi(s,x)W(dx,ds)\\ &+\int_0^{t\wedge\tau_n}\int_0^1 \varphi(s,x)\,\tilde{\eta}(ds,dx).\end{aligned} \tag{4.1.32}$$

for every $t\geq 0$ *almost surely.*

Definition 4.1.29. *Let* $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ *be a complete filtered probability space. Let* $\dot{W}$ *be a space time white noise on this space which respect the filtration* $\mathscr{F}_t$*. Finally, suppose that* $\mathscr{L}_i$ *are the elliptic operators given by*

$$(\mathscr{L}_i f)(x) := a_i(x)\frac{\partial^2 f}{\partial x^2} + b_i(x)\frac{\partial f}{\partial x}$$

for $i=1,2$*, where* $a\in C^2([0,1])$ *with* $a(x)>0$ *for all* $x\in[0,1]$*, and* $b\in C^1([0,1])$*. We say that the quintuple* $(u^1,\eta^1,u^2,\eta^2,p)$ *is a local solution to the moving boundary problem with initial data* (u_0^1,u_0^2,p_0)*, where* $p_0\in\mathbb{R}$ *and* $(u_0^1\circ(\Theta_{p_0}^1)^{-1},u_0^2\circ(\Theta_{p_0}^2)^{-1})\in C_0((0,1))$ *up to the* $\mathscr{F}_t$*-stopping time* τ *if*

(i) $p'(t)$ *is* $\mathscr{F}_t$*-adapted and* $p'(t)=h(v^1(t,\cdot),v^2(t,\cdot))$.

(ii) $(v^1,\tilde{\eta}^1):=(u^1\circ(\theta_p^1)^{-1},\eta^1\circ(\theta_p^1)^{-1})$ *solves the reflected SPDE*

$$\frac{\partial v^1}{\partial t} = \mathscr{L}_1 v^1 - p'(t)\frac{\partial v^1}{\partial x} + f_1(t,x,v^1) + \sigma_1(t,x,v^1)\dot{W} + \tilde{\eta}^1$$

with Dirichlet boundary conditions $v^1(0)=v^1(1)=0$ *and initial data* $v_0^1=u_0^1\circ(\Theta_{p_0}^1)^{-1}$ *until time* τ.

(iii) $(v^2,\tilde{\eta}^2):=(u^2\circ(\theta_p^2)^{-1},\eta^2\circ(\theta_p^2)^{-1})$ *solves the reflected SPDE*

$$\frac{\partial v^2}{\partial t} = \mathscr{L}_2 v^2 + p'(t)\frac{\partial v^2}{\partial x} + f_2(t,x,v^2) + \sigma_2(t,x,v^2)\dot{W}^- + \tilde{\eta}^2$$

with Dirichlet boundary conditions $v^2(0)=v^2(1)=0$ *and initial data* $v_0^2=u_0^2\circ(\Theta_{p_0}^2)^{-1}$ *until time* τ.

At this point, we note that there are two potential ways in which we could proceed in order to prove the results of this section in the setting where the Laplacian has been replaced by $\mathscr{L}_i$. One approach would be to try to reproduce analogous estimates to those for the heat kernel for the parabolic operators $(\frac{\partial}{\partial t} - \mathscr{L}_i)$. We note, however, that many of our estimates were obtained by explicit calculation, and that we will not, in general, have explicit expressions for the kernels associated with $(\frac{\partial}{\partial t} - \mathscr{L}_i)$. We therefore present an alternative approach here, where a spatial change of variables is used in order to convert our equations to equations in which the leading term is once again given by the Laplacian. The following proposition indicates how we can do this for non-linear reflected SPDEs of the form (4.1.31), and Corollary 4.1.32 then connects this back to our original moving boundary problem (5.4.10).

Proposition 4.1.30. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space, and $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Let $\tilde{v}$ be a continuous $\mathscr{F}_t$-adapted process taking values in $C_0(0,1)$, with $f_i, \sigma_i : [0,\infty) \times [0,1] \times C_0((0,1)) \to \mathbb{R}$, $h : C_0(0,1) \times C_0(0,1) \to \mathbb{R}$ and $F : C_0(0,1) \to C_0(0,1)$. Suppose that $a \in C^2([0,1])$ and $b \in C^1([0,1])$, with $a(x) > 0$ for every $x \in [0,1]$. Let $\mathscr{L}$ be the elliptic operator defined by*

$$(\mathscr{L} f)(x) := a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x}. \tag{4.1.33}$$

Assume that

$$\int_0^1 \frac{1}{\sqrt{a(y)}}\, dy = 1,$$

and define $\Gamma : [0,1] \mapsto [0,1]$ to be the unique function whose inverse is given by

$$\Gamma^{-1}(x) := \int_0^x \frac{1}{\sqrt{a(y)}}\, dy.$$

Then we have that the pair (u, η) is a local solution to the reflected SPDE

$$\frac{\partial u}{\partial t} = \mathscr{L} u + h(u, \tilde{v})\frac{\partial F(u)}{\partial x} + f(t,x,u) + \sigma(t,x,u)\dot{W} + \eta \tag{4.1.34}$$

with Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$ and initial data $u_0 \in C_0((0,1))^+$ until time τ if and only if the pair (v, η_Γ) is a local solution to the reflected SPDE

$$\begin{aligned}\frac{\partial v}{\partial t} =& \Delta v + \frac{\partial v}{\partial x}\frac{1}{\sqrt{a(\Gamma(x))}}\left[b(\Gamma(x)) - \frac{1}{2}a'(\Gamma(x))\right] + \frac{1}{\sqrt{a(\Gamma(x))}}h(v(s,\Gamma^{-1}(\cdot)), \tilde{v}(s,\cdot))\frac{\partial G(v)}{\partial x} \\ &+ \tilde{f}(t,x,v) + \tilde{\sigma}(t,x,v)\dot{W}_\Gamma + \eta_\Gamma\end{aligned} \tag{4.1.35}$$

with Dirichlet boundary conditions $v(t,0) = v(t,1) = 0$ and initial data $v_0 \in C_0((0,1))^+$ until time τ, where

(i) $v(t,x) := u(t,\Gamma(x))$.

(ii) $v_0(x) = u_0(\Gamma(x))$.

(iii) $G : C_0((0,1)) \to C_0((0,1))$ *such that, for* $v \in C_0((0,1))$ *and* $x \in [0,1]$

$$G(v)(x) = F(v(\Gamma^{-1}(\cdot)))(\Gamma(x)).$$

(iv) For $t \geq 0$, $x \in [0,1]$ *and* $u \in C_0((0,1))^+$

$$\tilde{f}(t,x,u(t,\cdot)) = f(t,\Gamma(x),u(t,\Gamma^{-1}(\cdot))).$$

(v) For $t \geq 0$, $x \in [0,1]$ *and* $u \in C_0((0,1))^+$

$$\tilde{\sigma}(t,x,u(t,\cdot)) = \frac{1}{\sqrt{\Gamma'(x)}}\sigma(t,\Gamma(x),u(t,\Gamma^{-1}(\cdot))).$$

(vi) $\dot{W}_\Gamma$ *is the space-time white noise defined by*

$$\dot{W}_\Gamma([0,t]\times[0,x]) = \int_0^t \int_0^{\Gamma(x)} \sqrt{(\Gamma^{-1})'(y)}\, W(dy,ds).$$

(vii) η_Γ *is the measure defined by*

$$\eta_\Gamma([0,t]\times[0,x]) := \int_0^t \int_0^{\Gamma(x)} \frac{1}{\Gamma'(\Gamma^{-1}(y))}\, \eta(dy,ds).$$

Remark 4.1.31. *We note that we do not lose any generality by assuming that* $\int_0^1 \frac{1}{\sqrt{a_i(x)}}\, dx = 1$. *We can ensure that this is the case by rescaling e.g. in the time variable.*

Proof of Proposition 4.1.30. We will prove the result for global solutions, noting that the argument for local solutions is essentially the same. Suppose that (u,η) solves (4.1.34). Let $\phi \in C^2((0,1)) \cap C_0((0,1))$. By a change of variables, we have that

$$\int_0^1 v(t,x)\phi(x)\,\mathrm{d}x = \int_0^1 u(t,x)\phi(\Gamma^{-1}(x))\frac{1}{\sqrt{a(x)}}\,\mathrm{d}x.$$

Define $\varphi(x) := \phi(\Gamma^{-1}(x))\frac{1}{\sqrt{a(x)}}$. We note that $\varphi \in C^2((0,1)) \cap C_0((0,1))$, and so we can test the equation for u with φ. This gives that

$$\begin{aligned}
\int_0^1 u(t,x)\phi(\Gamma^{-1}(x))\frac{1}{\sqrt{a(x)}}\,\mathrm{d}x = &\int_0^1 u(0,x)\varphi(x)\mathrm{d}x \\
&- \int_0^t \int_0^1 F(u(s,\cdot))(x)h(u(s,\cdot),\tilde{v}(s,\cdot))\frac{\partial\varphi}{\partial x}(x)\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_0^1 u(s,x)(\mathscr{L}^\star\varphi(\cdot))(x)\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_0^1 f(s,x,u(s,\cdot))\varphi(x)\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_0^1 \sigma(s,x,u(s,\cdot))\varphi(x)W(\mathrm{d}x,\mathrm{d}s) \\
&+ \int_0^t \int_0^1 \varphi(x)\,\eta(\mathrm{d}s,\mathrm{d}x).
\end{aligned} \tag{4.1.36}$$

By changing variables, we have that

- $\int_0^1 u_0(x)\varphi(x)\,\mathrm{d}x = \int_0^1 v_0(x)\phi(x)\,\mathrm{d}x$.
- $\int_0^t \int_0^1 f(s,x,u(s,\cdot))\varphi(x)\,\mathrm{d}x\mathrm{d}s = \int_0^1 \int_0^t f(s,\Gamma(x),v(s,\Gamma^{-1}(\cdot)))\phi(x)\,\mathrm{d}x\mathrm{d}s$.

By differentiating $\varphi(x) = \phi(\Gamma^{-1}(x))\frac{1}{\sqrt{a(x)}}$ with respect to x and then changing variables, we are able to obtain (via some lengthy calculations) that

$$\begin{aligned}
&\int_0^t \int_0^1 F(u(s,\cdot))(x)h(u(s,\cdot),\tilde{v}(s,\cdot))\frac{\partial\varphi}{\partial x}(x)\,\mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_0^1 G(v(s,\cdot))(x)h(v(s,\Gamma^{-1}(\cdot)),\tilde{v}(s,\cdot))\left[\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{a(\Gamma(x))}}\phi(x)\right)\right]\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Similarly, we have that

$$\int_0^t \int_0^1 u(s,x)(\mathscr{L}^\star\varphi(\cdot))(x)\,\mathrm{d}x\mathrm{d}s = \int_0^t \int_0^1 v(s,x)(\mathscr{G}^\star\phi(\cdot))(x)\,\mathrm{d}x\mathrm{d}s,$$

where $\mathscr{G}$ is the differential operator given by

$$(\mathscr{G}f)(x) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{1}{\sqrt{a(\Gamma(x))}}\left[b(\Gamma(x)) - \frac{1}{2}a'(\Gamma(x))\right].$$

For the stochastic term, we have that

$$\begin{aligned}
&\int_0^t \int_0^1 \sigma(s,x,v(s,\Gamma^{-1}(\cdot)))\varphi(x)\,W(\mathrm{d}y,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \sigma(s,x,v(s,\Gamma^{-1}(\cdot)))\varphi(x)\frac{1}{\sqrt{(\Gamma^{-1})'(x)}}\sqrt{(\Gamma^{-1})'(x)}\,W(\mathrm{d}x,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \sigma(s,\Gamma(x),v(s,\Gamma^{-1}(\cdot)))\varphi(\Gamma(x))\frac{1}{\sqrt{(\Gamma^{-1})'(\Gamma(x))}}W_\Gamma(\mathrm{d}x,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \sigma(s,\Gamma(x),v(s,\Gamma^{-1}(\cdot)))\phi(x)\frac{1}{\sqrt{a(\Gamma(x))}}\frac{1}{\sqrt{(\Gamma^{-1})'(\Gamma(x))}}W_\Gamma(\mathrm{d}x,\mathrm{d}s).
\end{aligned}$$

Since $\frac{1}{\sqrt{a(x)}} = (\Gamma^{-1})'(x)$, we have that this is equal to

$$\begin{aligned}
&\int_0^t \int_0^1 \sigma(s,\Gamma(x),v(s,\Gamma^{-1}(\cdot)))\phi(x)\sqrt{(\Gamma^{-1})'(\Gamma(x))}W_\Gamma(\mathrm{d}x,\mathrm{d}s) \\
&\quad = \int_0^t \int_0^1 \sigma(s,\Gamma(x),v(s,\Gamma^{-1}(\cdot)))\phi(x)\frac{1}{\sqrt{\Gamma'(x)}}W_\Gamma(\mathrm{d}x,\mathrm{d}s) \\
&\quad = \int_0^t \int_0^1 \tilde{\sigma}(s,x,v(s,\cdot))\phi(x)W_\Gamma(\mathrm{d}x,\mathrm{d}s).
\end{aligned}$$

Finally, we perform a change of variables on the measure term. This gives

$$\begin{aligned}
\int_0^t \int_0^1 \varphi(x)\,\eta(\mathrm{d}x,\mathrm{d}s) &= \int_0^t \int_0^1 \phi(\Gamma^{-1}(x))\frac{1}{\sqrt{a(x)}}\,\eta(\mathrm{d}x,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \phi(\Gamma^{-1}(x))\frac{1}{\sqrt{a(x)}}\Gamma'(\Gamma^{-1}(x))\frac{1}{\Gamma'(\Gamma^{-1}(x))}\,\eta(\mathrm{d}x,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \phi(x)\frac{1}{\sqrt{a(\Gamma(x))}}\Gamma'(x)\,\eta_\Gamma(\mathrm{d}x,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \phi(x)\left[(\Gamma^{-1})'(\Gamma(x))\right]\Gamma'(x)\,\eta_\Gamma(\mathrm{d}x,\mathrm{d}s) \\
&= \int_0^t \int_0^1 \phi(x)\,\eta_\Gamma(\mathrm{d}x,\mathrm{d}s).
\end{aligned}$$

Putting these different components together, we see that equation (4.1.36) becomes

$$\begin{aligned}\int_0^1 v(t,x)\phi(x)\,\mathrm{d}x = &\int_0^1 v_0(x)\phi(x)\,\mathrm{d}x \\ &- \int_0^t\int_0^1 G(v(s,\cdot))(x)h(v(s,\Gamma^{-1}(\cdot)),\tilde{v}(s,\cdot))\left[\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{a(\Gamma(x))}}\phi(x)\right)\right]\,\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t\int_0^1 v(s,x)(\mathscr{G}^\star\phi(\cdot))(x)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_0^1 \tilde{f}(s,x,v(s,\cdot))\phi(x)\,\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_0^1 \tilde{\sigma}(s,x,v(s,\cdot))\phi(x)\,W_\Gamma(\mathrm{d}y,\mathrm{d}s) + \int_0^t\int_0^1 \phi(x)\,\eta_\Gamma(\mathrm{d}x,\mathrm{d}s).\end{aligned}$$

We also note that

$$\int_0^\infty\int_0^1 v(s,x)\,\eta_\Gamma(\mathrm{d}x,\mathrm{d}s) = \int_0^\infty\int_0^1 u(s,\Gamma(x))\,\eta_\Gamma(\mathrm{d}x,\mathrm{d}s) = \int_0^t\int_0^1 u(s,x)\frac{1}{\Gamma'(\Gamma^{-1}(x))}\,\eta(\mathrm{d}x,\mathrm{d}s). \tag{4.1.37}$$

Since we know that $\mathrm{supp}(\eta)\subset\{(t,x)\mid u(t,x)=0\}$, we have that (4.1.37) is equal to zero. This concludes the proof that, if (u,η) solves (4.1.34) then (v,η_Γ) solves (4.1.35). The argument for the converse is similar. □

Corollary 4.1.32. *For $i=1,2$, suppose that $\int_0^1 \frac{1}{\sqrt{a_i(y)}}\,dy = 1$, and define the function $\Gamma_i : [0,1]\mapsto[0,1]$ such that their inverses are given by*

$$\Gamma_i^{-1}(x) := \int_0^x \frac{1}{\sqrt{a_i(y)}}\,dy$$

for $i=1,2$. The quintuple $(u^1,\eta^1,u^2,\eta^2,p)$ is a local solution to the moving boundary problem, with initial data (u_0^1,u_0^2,p_0) with $u_0^i\circ(\Theta^i_{p_0})^{-1}\in C_0((0,1))$ for $i=1,2$, until time τ if and only if $(v^1,\tilde{\eta}^1,v^2,\tilde{\eta}^2,p)$ is a local solution to the system of non-linear reflected SPDEs

$$\frac{\partial v_1}{\partial t} = \mathscr{G}_1 v_1 - \frac{1}{\sqrt{a_1(\Gamma_1(x))}}p'(t)\frac{\partial v_1}{\partial x} + \tilde{f}_1(t,x,v_1(t,\cdot)) + \tilde{\sigma}_1(t,x,v_1(t,\cdot))\dot{W}_{\Gamma_1} + \tilde{\eta}^1 \tag{4.1.38}$$

$$\frac{\partial v_2}{\partial t} = \mathscr{G}_2 v_2 + \frac{1}{\sqrt{a_2(\Gamma_2(x))}}p'(t)\frac{\partial v_2}{\partial x} + \tilde{f}_2(t,x,v_2(t,\cdot)) + \tilde{\sigma}_2(t,x,v_2(t,\cdot))\dot{W}^-_{\Gamma_2} + \tilde{\eta}^2 \tag{4.1.39}$$

$$p(t) = p_0 + \int_0^t h(v^1(s,\Gamma_1^{-1}(\cdot)),v^2(s,\Gamma_2^{-1}(\cdot)))\,ds,$$

with initial data $(u_0^1\circ(\Theta^1_{p_0})^{-1},u_0^2\circ(\Theta^2_{p_0})^{-1})$, until time τ, where

(i) For $i=1,2$, $\mathscr{G}_i$ is the differential operator

$$(\mathscr{G}_i f)(x) := \frac{\partial^2 f}{\partial x^2} + \frac{1}{\sqrt{a(\Gamma(x))}}\left[b(\Gamma(x)) - \frac{1}{2}a'(\Gamma(x))\right]\frac{\partial f}{\partial x}.$$

(ii) For $i=1,2$, $v^i(t,x) := u^i(t,\Gamma_i((\theta^i_p)^{-1}(x)))$.

(iii) For $i=1,2$, $\tilde{\eta}^i$ is the measure defined by

$$\tilde{\eta}^i([0,t]\times[0,x]) := \int_0^t\int_0^{\Gamma_i(x)} \frac{1}{\Gamma_i'(\Gamma_i^{-1}(y))}\,(\eta^i\circ(\theta^i_p)^{-1})(dy,ds).$$

(iv) *For* $i = 1, 2$, $v_0^i(x) = u_0^i(\Gamma_i(x))$.

(v) *For* $i = 1, 2$, $t \geq 0$, $x \in [0, 1]$ *and* $u \in C_0((0, 1))^+$

$$\tilde{f}_i(t, x, u(t, \cdot)) = f_i(t, \Gamma(x), u(t, \Gamma^{-1}(\cdot))).$$

(vi) *For* $i = 1, 2$, $t \geq 0$, $x \in [0, 1]$ *and* $u \in C_0((0, 1))^+$

$$\tilde{\sigma}_i(t, x, u(t, \cdot)) = \frac{1}{\sqrt{\Gamma'(x)}} \sigma_i(t, \Gamma(x), u(t, \Gamma^{-1}(\cdot))).$$

(vii) $\dot{W_{\Gamma_1}}$ *is the space-time white noise defined by*

$$\dot{W_{\Gamma_1}}([0, t] \times [0, x]) = \int_0^t \int_0^{\Gamma_1(x)} \sqrt{(\Gamma_1^{-1})'(y)}\, W(dy, ds).$$

(viii) $\dot{W}^-_{\Gamma_2}$ *is the space-time white noise defined by*

$$\dot{W}^-_{\Gamma_2}([0, t] \times [0, x]) = \int_0^t \int_{-\Gamma_2(x)}^0 \sqrt{(\Gamma_2^{-1})'(y)}\, W(dy, ds).$$

Equipped with Corollary 4.1.32, we see how our moving boundary problem with elliptic operators can be associated with a system of coupled non-linear SPDEs where the leading term is the Laplacian. From here, we are able to argue in essentially the same way as before to obtain the results on existence, uniqueness, Hölder continuity and invariant measures in this context.

4.2 The Moving Boundary Problem on Semi-Infinite Intervals

We now consider the analogous problem, where the two sides of the equation satisfy SPDEs on the infinite halflines $(-\infty, p(t)]$ and $[p(t), \infty)$ respectively. That is

$$\frac{\partial u^1}{\partial t} = \Delta u^1 + f_1(t, p(t) - x, u^1(t, p(t) - \cdot)) + \sigma_1(t, p(t) - x, u^1(t, p(t) - \cdot))\dot{W} + \eta^1$$

on $[0, \infty) \times (-\infty, p(t)]$, and

$$\frac{\partial u^2}{\partial t} = \Delta u^2 + f_2(t, x - p(t), u^2(t, p(t) + \cdot)) + \sigma_2(t, x - p(t), u^2(t, p(t) + \cdot))\dot{W} + \eta^2,$$

on $[0, \infty) \times [p(t), \infty)$. We once again have Dirichlet conditions at the boundary, $p(t)$, so that $u^1(t, p(t)) = u^2(t, p(t)) = 0$, with the point $p(t)$ evolving according to the equation

$$p'(t) = h(u^1(t, p(t) - \cdot), u^2(t, p(t) + \cdot)).$$

Here, W is a space-time white noise and h is a function of the two profiles of the equation on either side of the shared boundary. As before, η^1 and η^2 are reflection measures for the functions u^1 and u^2 respectively, keeping the profiles positive and satisfying the conditions

(i) $\text{supp}(\eta^1) \subset \big\{(t, x) \mid x \in (\infty, p(t)]\big\}$,

(ii) $\operatorname{supp}(\eta^2) \subset \big\{(t,x) \mid x \in [p(t),\infty)\big\}$,

(iii) $\int_0^\infty \int_{\mathbb{R}} u^1(t,x)\, \eta^1(\mathrm{d}t, \mathrm{d}x) = 0$,

(iv) $\int_0^\infty \int_{\mathbb{R}} u^2(t,x)\, \eta^2(\mathrm{d}t, \mathrm{d}x) = 0$.

Remark 4.2.1. *As in the previous section, the results throughout this section can be adapted for the case where the Laplacian terms appearing in our equations are replaced with elliptic operators $\mathscr{L}_i$ of the form*

$$(\mathscr{L}_i f)(x) = a_i(x)\frac{\partial^2 f}{\partial x^2} + b_i(x)\frac{\partial f}{\partial x},$$

where $a_i(x) \in C_b^2([0,\infty))$, $b_i \in C^1([0,\infty))$ and there exists $c > 0$ such that $a_i(x) \geq c$ for every $x \in [0,\infty)$.

4.2.1 Formulation of the Problem

We introduce the spaces $\mathscr{C}_r$ and $\mathscr{C}_r^T$, which we will be working with throughout this section.

Definition 4.2.2. *For $r \in \mathbb{R}$, we say that $u : [0,\infty) \to \mathbb{R}$ is in the space $\mathscr{L}_r$ if u is measurable and*

$$\|u\|_{\mathscr{L}_r} := \sup_{x \geq 0} e^{-rx}|u(x)| < \infty. \tag{4.2.1}$$

We say that $u \in \mathscr{C}_r$ if $u \in \mathscr{L}_r$ and u is continuous. $\mathscr{C}_r$ is equipped with the same norm as $\mathscr{L}_r$.

Definition 4.2.3. *We say that $u : [0,T] \times [0,\infty) \to \mathbb{R}$ is in the space $\mathscr{C}_r^T$ if u is continuous and*

$$\sup_{t \in [0,T]} \sup_{x \geq 0} e^{-rx}|u(t,x)| < \infty. \tag{4.2.2}$$

This space presents issues when handling both the non-Lipschitz term arising due to the moving boundary and the stochastic term. Truncating the non-linear term requires more care, as we are now trying to control the $\mathscr{C}_r^T$-norm of the process. We are also unable to suitably control the supremum of the stochastic terms using our previous arguments, as they are not well suited to unbounded domains. For this reason, we introduce extra decay for the growth of the volatility relative to the growth of the drift term. Fixing $r \in \mathbb{R}$ we take, for $i = 1, 2$, f_i and σ_i to be measurable mappings from $[0,\infty) \times [0,\infty) \times \mathscr{C}_r \to \mathbb{R}$ and $h : \mathscr{C}_r \times \mathscr{C}_r \to \mathbb{R}$ to be a measurable function such that, for some $C, \delta > 0$,

(I) For every $t \geq 0$, $x \in [0,\infty)$, $u, v \in \mathscr{C}_r$,

$$|f_i(t,x,u) - f_i(t,x,v)| \leq Ce^{rx}\|u - v\|_{\mathscr{C}_r}.$$

(II) For every $t \geq 0$, $x \in [0,\infty)$, $u \in \mathscr{C}_r$,

$$|f_i(t,x,u)| \leq Ce^{rx}(1 + \|u\|_{\mathscr{C}_r}).$$

(III) For every $t \geq 0$, $x \in [0, \infty)$, $u, v \in \mathscr{C}_r$,

$$|\sigma_i(t, x, u) - \sigma_i(t, x, v)| \leq Ce^{(r-\delta)x}\|u - v\|_{\mathscr{C}_r}.$$

(IV) For every $t \geq 0$, $x \in [0, \infty)$, $u \in \mathscr{C}_r$,

$$|\sigma_i(t, x, u)| \leq Re^{(r-\delta)x}(1 + \|u\|_{\mathscr{C}_r}).$$

(V) For every $u_1, u_2, v_1, v_2 \in \mathscr{C}_r$,

$$|h(u_1, v_1) - h(u_2, v_2)| \leq K(\|u_1 - u_2\|_{\mathscr{C}_r} + \|v_1 - v_2\|_{\mathscr{C}_r}).$$

Since our notion of solution here is motivated by the same ideas as in the compact case, we move straight to the definitions for solutions to non-linear SPDEs and moving boundary problems on $\mathbb{R}$.

Definition 4.2.4. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $\tilde{v}$ is a continuous $\mathscr{F}_t$-adapted process taking values in $\mathscr{C}_r$. Let $f, \sigma : [0, \infty) \times [0, \infty) \times \mathscr{C}_r \to \mathbb{R}$, $h : \mathscr{C}_r \times \mathscr{C}_r \to \mathbb{R}$ and $F : \mathscr{C}_r \to \mathscr{C}_r$. For the $\mathscr{F}_t$-stopping time τ, we say that the pair (v, η) is a local $\mathscr{C}_r$-valued solution to the reflected SPDE*

$$\frac{\partial v}{\partial t} = \Delta v + h(v, \tilde{v})\frac{\partial F(v)}{\partial x} + f(t, x, v) + \sigma(t, x, v)\dot{W} + \eta$$

with Dirichlet boundary condition $v(t, 0) = 0$ and initial data $v_0 \in \mathscr{C}_r^+$ with $v_0(0) = 0$, until time τ, if

(i) For every $x \geq 0$ and every $t \geq 0$, $v(x, t)$ is $\mathscr{F}_t$- measurable.

(ii) $v \geq 0$ almost surely.

(iii) $v\big|_{[0,t]\times[0,\infty)} \in \mathscr{C}_r^t$ for every $t < \tau$ almost surely.

(iv) $v(t, x) = \infty$ for every $t \geq \tau$ almost surely.

(v) η is a measure on $[0, \infty) \times [0, \infty)$ such that

(a) For every measurable map $\psi : [0, \infty) \times [0, \infty) \to \mathbb{R}$,

$$\int_0^\infty \int_0^\infty \psi(x, s)\, \eta(dx, ds)$$

is $\mathscr{F}_t$-measurable.

(b) $\int_0^\infty \int_0^\infty v(t, x)\, \eta(dx, dt) = 0$.

(vi) There exists a localising sequence of stopping times $\tau_n \uparrow \tau$ almost surely, such that for every $\varphi \in C_c^{1,2}([0,\infty)\times[0,\infty))$ with $\varphi(\cdot,0)=0$, and for every $t \geq 0$,

$$\begin{aligned}\int_0^\infty v(t\wedge\tau_n,x)\varphi(t\wedge\tau_n,x)\,dx = &\int_0^\infty v(0,x)\varphi(0,x)\,dx \\ &+ \int_0^{t\wedge\tau_n}\int_0^\infty v(s,x)\frac{\partial\varphi}{\partial t}(s,x)\,dx\,ds \\ &+ \int_0^{t\wedge\tau_n}\int_0^\infty v(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\,dx\,ds \\ &- \int_0^{t\wedge\tau_n}\int_0^\infty F(v(s,\cdot))(x)h(v(s,\cdot),\tilde{v}(s,\cdot))\frac{\partial\varphi}{\partial x}(s,x)\,dx\,ds \\ &+ \int_0^{t\wedge\tau_n}\int_0^\infty f(s,x,v(s,\cdot))\varphi(s,x)\,dx\,ds \\ &+ \int_0^{t\wedge\tau_n}\int_0^\infty \sigma(s,x,v(s,\cdot))\varphi(s,x)W(dx,ds) \\ &+ \int_0^{t\wedge\tau_n}\int_0^\infty \varphi(s,x)\,\tilde{\eta}(ds,dx).\end{aligned} \tag{4.2.3}$$

almost surely.

Similarly to as in Section 4.1, we say that a local $\mathscr{C}_r$-valued solution is *maximal* if it cannot be extended to a $\mathscr{C}_r$-valued solution on a larger stochastic interval, and we say that a local solution is *global* if we can take $\tau_n = \infty$ in (4.2.3).

Definition 4.2.5. *Let $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. We say that the quintuple $(u^1,\eta^1,u^2,\eta^2,p)$ is a local solution to the moving boundary problem on $\mathbb{R}$ with exponential growth r and initial data (u_0^1,u_0^2,p_0), where $p_0\in\mathbb{R}$ and $(u_0^1\circ(\Theta_{p_0}^1)^{-1},u_0^2\circ(\Theta_{p_0}^2)^{-1})\in\mathscr{C}_r^2$, up to the $\mathscr{F}_t$-stopping time τ if*

(i) $(v^1,\tilde{\eta}^1) := (u^1\circ(\theta_p^1)^{-1},\eta^1\circ(\theta_p^1)^{-1})$ is a $\mathscr{C}_r$-valued solution to the non-linear reflected SPDE

$$\frac{\partial v^1}{\partial t} = \Delta v^1 - h(v^1(t,\cdot),v^2(t,\cdot))\frac{\partial v^1}{\partial x} + f_1(t,x,v^1) + \sigma_1(t,x,v^1)\dot{W} + \tilde{\eta}^1$$

with Dirichlet boundary condition $v^1(t,0)=0$ and initial data $v_0^1 = u_0^1\circ(\Theta_{p_0}^1)^{-1}\in\mathscr{C}_r$, until time τ.

(ii) $(v^2,\tilde{\eta}^2) := (u^2\circ(\theta_p^2)^{-1},\eta^2\circ(\theta_p^2)^{-1})$ is a $\mathscr{C}_r$-valued solution to the non-linear reflected SPDE

$$\frac{\partial v^2}{\partial t} = \Delta v^2 + h(v^1(t,\cdot),v^2(t,\cdot))\frac{\partial v^2}{\partial x} + f_2(t,x,v^2) + \sigma_2(t,x,v^2)\dot{W}^- + \tilde{\eta}^2$$

with Dirichlet boundary condition $v^2(t,0)=0$ and initial data $v_0^2 = u_0^2\circ(\Theta_{p_0}^2)^{-1}\in\mathscr{C}_r$, until time τ.

(iii) $p(t) = p_0 + \int_0^t h(v^1(s,\cdot),v^2(s,\cdot))\,ds$.

We refer to $(v^1,\tilde{\eta}^1,v^2,\tilde{\eta}^2)$ as the solution to the moving boundary problem in the relative frame.

4.2.2 Deterministic Obstacle Problem on $[0, \infty)$

Before we continue with our analysis for this problem, we first look at the deterministic obstacle problem on the unbounded domain $[0, \infty)$. If we are to obtain an existence and uniqueness result by arguing in the same spirit as in the previous section, we will require an analogous result to Theorem 2.2.2. That is, we need a Lipschitz bound on the map from obstacles to solutions in a suitable function space. The obstacle problem in this setting is defined as follows.

Definition 4.2.6. *Fix some $r \in \mathbb{R}$. Let $v \in \mathscr{C}_r^T$ such that $v(t, 0) = 0$ for every $t \in [0, T]$. We say that the pair (z, η) satisfies the heat equation with obstacle v and exponential growth r on $[0, \infty)$ if:*

(i) $z \in \mathscr{C}_r^T$, $z(t, 0) = 0$, $z(0, x) = 0$ *and* $z \geq v$.

(ii) η *is a measure on* $(0, \infty) \times [0, T]$.

(iii) z *weakly solves the PDE*

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \eta. \tag{4.2.4}$$

That is, for every $t \in [0, T]$ and every $\phi \in C_c^\infty([0, \infty))$ with $\phi(0) = 0$,

$$\int_0^\infty z(t, x)\phi(x)\,dx = \int_0^t \int_0^\infty z(s, x)\phi''(x)\,dx\,ds + \int_0^t \int_0^\infty \phi(x)\ \eta(dx, ds).$$

(iv) $\int_0^t \int_0^\infty (z(s, x) - v(s, x))\ \eta(dx, ds) = 0$.

We note that the deterministic obstacle problem on the spatial domain $\mathbb{R}$ is considered in [42]. Here, we pose the problem in $\mathscr{C}_r^T$ for any $r \in \mathbb{R}$, and we work on the spatial domain $[0, \infty)$ rather than $\mathbb{R}$. We now present the following existence, uniqueness and Lipschitz continuity result for the problem.

Theorem 4.2.7. *For every $r \in \mathbb{R}$ and every $v \in \mathscr{C}_r^T$ such that $v(t, 0) = 0$, there exists a unique solution (z, η) to the heat equation on $[0, \infty)$ with Dirichlet condition and obstacle v. Furthermore, if $v_1, v_2 \in \mathscr{C}_r^T$, we have that*

$$\|z_1 - z_2\|_{\mathscr{C}_r^T} \leq C_{r,T}\|v_1 - v_2\|_{\mathscr{C}_r^T}, \tag{4.2.5}$$

where z_i is the solution to the obstacle problem corresponding to v_i.

Proof. See Section A.2 in the Appendix. □

4.2.3 Existence and Uniqueness

As in the proof of Theorem 4.1.8, we will use a Picard iteration in order to prove existence and uniqueness for a truncated version of this problem. There is some extra complexity introduced when trying to do this in the case of an infinite spatial domain. In particular, we should be more careful in how we truncate the problem.

Theorem 4.2.8. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $f_i, \sigma_i : [0,\infty) \times [0,\infty) \times \mathscr{C}_r \to \mathbb{R}$ and $h : \mathscr{C}_r \times \mathscr{C}_r \to \mathbb{R}$ are measurable, satisfying conditions (I)- (VI). Then there exists a unique maximal solution $(u^1, \eta^1, u^2, \eta^2, p)$ to the moving boundary problem on $\mathbb{R}$, with the blow-up time given by*

$$\tau := \sup_{M>0} \left[\inf \left\{ t \geq 0 \;\middle|\; \|u^1\|_{\mathscr{C}_r} + \|u^2\|_{\mathscr{C}_r} \geq M \right\} \right].$$

The following notation will be used throughout the rest of this section.

Definition 4.2.9. *We define $G(t,x,y)$ to be the Dirichlet heat kernel on $[0,\infty)$, that is*

$$G(t,x,y) := \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(x-y)^2}{4t} \right) - \exp\left(-\frac{(x+y)^2}{4t} \right) \right].$$

For $r \in \mathbb{R}$, we also define the notation

$$G_r(t,x,y) := e^{-r(x-y)} G(t,x,y).$$

Before proving Theorem 4.2.8, we present here some results which will be essential to the proof.

Lemma 4.2.10. *Let $r \in \mathbb{R}$. Suppose that $u \in L^1([0,T]; \mathscr{L}_r)$. Then we have that, for $\tau \in [0,T]$,*

$$\sup_{t\in[0,\tau]} \sup_{x\geq 0} \left| e^{-rx} \int_0^t \int_0^\infty G(t-s,x,y) u(s,y)\, dy\, ds \right| \leq C_{r,T} \int_0^\tau \|u\|_{t,\mathscr{L}_r}\, dt.$$

Proof. Let $\tau \in [0,T]$ and $t \in [0,\tau]$.

$$\begin{aligned} \left| e^{-rx} \int_0^t \int_0^\infty G(t-s,x,y) u(s,y) \mathrm{d}y \mathrm{d}s \right| &= \left| \int_0^t \int_0^\infty G_r(t-s,x,y) e^{-ry} u(s,y) \mathrm{d}y \mathrm{d}s \right| \\ &\leq \int_0^t \left(\int_0^\infty G_r(t-s,x,y) \mathrm{d}y \right) \|u\|_{s,\mathscr{L}_r} \mathrm{d}s. \end{aligned}$$

Applying Proposition A.1.1, we see that this is at most

$$C_{r,T} \int_0^\tau \|u\|_{s,\mathscr{L}_r} \mathrm{d}s.$$

This concludes the proof. □

We would like an analogous result which would allow us to control the noise term appearing in the mild formulation. The following lemmas will enable us to obtain such an estimate.

Proposition 4.2.11. *Suppose that $r \in \mathbb{R}$. Let $u \in L^p(\Omega; L^\infty([0,T]; \mathscr{L}_r))$ with $p > 10$. Define*

$$w(t,x) := e^{-rx} \int_0^t \int_0^\infty G(t-s,x,z) u(s,z)\, W(dz,ds).$$

Then w is continuous almost surely and for $x, y \in [0,\infty)$ and $0 \le s \le t \le \tau \le T$ we have that

$$\mathbb{E}\left[|w(t,x) - w(s,y)|^p\right] \le C_{p,T,r} \mathbb{E}\left[\int_0^t \|u\|^p_{\mathscr{L}_r,s} ds\right] \left(|t-s|^{1/4} + |x-y|^{1/2}\right)^{p-4},$$

Proof. We have that

$$\begin{aligned}
&\mathbb{E}\left[|w(t,x) - w(s,y)|^p\right] \\
\le& C_p \mathbb{E}\left[\left|\int_s^t \int_0^\infty G_r(t-q,x,z) e^{-rz} u(q,z) \mathrm{W}(\mathrm{d}z, \mathrm{d}q)\right|^p\right] \\
&+ C_p \mathbb{E}\left[\left|\int_0^s \int_0^\infty \left[G_r(t-q,x,z) - G_r(t-q,y,z)\right] e^{-rz} u(q,z) \mathrm{W}(\mathrm{d}z, \mathrm{d}q)\right|^p\right] \\
&+ C_p \mathbb{E}\left[\left|\int_0^t \int_0^\infty \left[G_r(t-q,y,z) - G_r(s-q,y,z)\right] e^{-rz} u(q,z) \mathrm{W}(\mathrm{d}z, \mathrm{d}q)\right|^p\right].
\end{aligned}$$

We bound the first term only, and note that the other terms follow similarly by the estimates from Proposition A.1.2. Burkholder's inequality gives

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t \int_0^\infty G_r(t-q,x,z) e^{-rz} u(q,z) \mathrm{W}(\mathrm{d}z, \mathrm{d}q)\right|^p\right] \\
&\le C_p \mathbb{E}\left[\left|\int_s^t \int_0^\infty G_r(t-q,x,z)^2 e^{-2rz} u(q,z)^2 \mathrm{d}z \mathrm{d}q\right|^{p/2}\right] \\
&\le C_p \mathbb{E}\left[\left|\int_s^t \left(\int_0^\infty G_r(t-q,x,z)^2 \mathrm{d}z\right) \|u\|^2_{q,\mathscr{L}_r} \mathrm{d}q\right|^{p/2}\right]
\end{aligned}$$

Hölder's inequality then gives

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t \left(\int_0^\infty G_r(t-q,x,z)^2 \mathrm{d}z\right) \|u\|^2_{\mathscr{L}_r,q} \mathrm{d}q\right|^{p/2}\right] \\
&\le C_{p,T} \left(\int_s^t \left[\int_0^\infty G_r(t-q,x,z)^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}q\right)^{(p-2)/2} \times \mathbb{E}\left[\int_s^t \|u\|^p_{\mathscr{L}_r,q} \mathrm{d}q\right].
\end{aligned}$$

Applying the first bound from Proposition A.1.2 then gives that this is at most

$$C_{r,p,T} |t-s|^{(p-4)/4} \times \mathbb{E}\left[\int_s^t \|u\|^p_{\mathscr{L}_r,q} \mathrm{d}q\right].$$

By making similar arguments, using the other bounds from Proposition A.1.2, we obtain that

$$\mathbb{E}\left[|w(t,x) - w(s,y)|^p\right] \le C_{r,p,T} \mathbb{E}\left[\int_0^\tau \|u\|^p_{\mathscr{L}_r,q} \mathrm{d}q\right] \times \left(|t-s|^{1/4} + |x-y|^{1/2}\right)^{p-4}.$$

Continuity of w then follows by Corollary A.3 in [11]. □

Proposition 4.2.12. *Let $p \geq 1$ and $q, K, \delta > 0$. Suppose that $w : [0,T] \times [0,\infty) \to \mathbb{R}$ is a random field such that for every $s,t \in [0,T]$, every n and every $x,y \in [n,n+1]$,*

$$\mathbb{E}\left[|w(t,x) - w(s,y)|^p\right] \leq K(|t-s|^{1/2} + |x-y|)^{3+q}.$$

Then $w \in C([0,T]; \mathscr{C}_\delta)$ almost surely, and there exists a finite constant C depending only on p, q, T and δ, and a non-negative random variable Y such that

$$\|w\|_{\mathscr{C}_\delta^T} \leq |w(0,0)| + Y,$$

almost surely, where

$$\mathbb{E}[Y^p] \leq C_{p,q,\delta,T} K.$$

Proof. Fix $n \in \mathbb{N}$. We have that for every $s,t \in [0,T]$ and every $x,y \in [n,n+1]$ that

$$\mathbb{E}\left[|w(t,x) - w(s,y)|^p\right] \leq K(|t-s|^{1/2} + |x-y|)^{q+3}.$$

Let $\gamma = q/2p$. Applying Corollary A.3 from [11], we find random variables ψ_n such that $\sup_{n\in\mathbb{N}} \mathbb{E}[(\psi_n)^p] \leq C_{p,q,T} K < \infty$ and

$$|w(t,x) - w(s,y)| \leq \psi_n(|t-s|^{\frac{\gamma}{2}} + |x-y|^\gamma).$$

for every $t,s \in [0,T]$ and $x,y \in [n,n+1]$ almost surely. It follows that, for $t \in [0,T]$ and $x \in [n,n+1]$,

$$\begin{aligned}
e^{-\delta x}|w(t,x)| &\leq e^{-\delta x}\left(|w(0,0)| + |w(t,0) - w(0,0)| + |w(t,x) - w(t,0)|\right) \\
&\leq e^{-\delta x}|w(0,0)| + e^{-\delta x}|w(t,0) - w(0,0)| + e^{-\delta x}|w(t,x) - w(t,n)| \\
&\quad + e^{-\delta x}\sum_{m=0}^{n-1} |w(t,m+1) - w(t,m)| \\
&\leq e^{-\delta x}\left(|w(0,0)| + T^{\gamma/2}\psi_0 + \sum_{m=0}^{n} \psi_m\right) \\
&\leq |w(0,0)| + T^{\gamma/2}\psi_0 + \sum_{m=0}^{n} e^{-\delta m}\psi_m \\
&\leq |w(0,0)| + T^{\gamma/2}\psi_0 + \sum_{m=0}^{\infty} e^{-\delta m}\psi_m.
\end{aligned}$$

Define $Y := T^{\gamma/2}\psi_0 + \sum_{m=0}^{\infty} e^{-\delta m}\psi_m$. Defining $\tilde{p} = p/(p-1)$, we have that

$$\begin{aligned}
\mathbb{E}\left[Y^p\right] &\leq C_p T^{p\gamma/2}\mathbb{E}\left[\psi_0^p\right] + C_p\mathbb{E}\left[\left(\sum_{m=0}^{\infty} e^{-\delta m}\psi_m\right)^p\right] \\
&\leq C_{p,q,T}K + C_p\left(\sum_{m=0}^{\infty} e^{-\delta m\tilde{p}/2}\right)^{p/\tilde{p}}\mathbb{E}\left[\sum_{m=0}^{\infty} e^{-\delta pm}\psi_m^p\right] \\
&\leq C_{p,q,T}K + C_p\left(\sum_{m=0}^{\infty} e^{-\delta m\tilde{p}/2}\right)^{p/\tilde{p}}\left(\sum_{m=0}^{\infty} e^{-\delta pm}\mathbb{E}\left[\psi_m^p\right]\right) \\
&\leq C_{p,q,\delta,T}K.
\end{aligned}$$

It is only left to show that the map $t \mapsto w(t,\cdot)$ is continuous from $[0,T]$ into $\mathscr{C}_\delta$ almost surely. Let $s,t \in [0,T]$. Then for $x \geq 0$,

$$\begin{aligned}
e^{-\delta x}|w(t,x) - w(s,x)| &\leq e^{-\delta x}\psi_{\lfloor x\rfloor}|t-s|^{\gamma/2} \\
&\leq |t-s|^{\gamma/2}\sum_{m=0}^{\infty} e^{-\delta m}\psi_m \\
&\leq Y|t-s|^{\gamma/2}.
\end{aligned}$$

In particular, we have that

$$\|w(t,\cdot) - w(s,\cdot)\|_{\mathscr{C}_\delta} \leq Y|t-s|^{\gamma/2},$$

concluding the proof. □

Proposition 4.2.13. *Let $r \in \mathbb{R}$. Suppose that $u \in L^p(\Omega; L^\infty([0,T]; \mathscr{L}_{r-\epsilon}))$. Define*

$$F(t,x) := \int_0^t \int_0^\infty G(t-s,x,y)u(s,y)\,W(dy,ds).$$

Then we have that, for $p > 10$ and $\tau \in [0,T]$,

$$\mathbb{E}\left[\|F\|^p_{\mathscr{C}_r^\tau}\right] \leq C_{p,r,T,\epsilon}\mathbb{E}\left[\int_0^\tau \|u\|^p_{s,\mathscr{L}_{r-\epsilon}} ds\right].$$

Proof. Define

$$w(t,x) := e^{-(r-\epsilon)x}\int_0^t \int_0^\infty G(t-s,x,z)u(s,z)\mathrm{W}(\mathrm{d}y,\mathrm{d}s).$$

Then Proposition 4.2.11 gives that, for $s,t \in [0,\tau]$ and $x,y \in [0,\infty)$

$$\mathbb{E}\left[|w(t,x) - w(s,y)|^p\right] \leq C_{p,T,r,\epsilon}\mathbb{E}\left[\int_0^\tau \|u\|^p_{q,\mathscr{L}_{(r-\epsilon)}}\mathrm{d}q\right]\left(|t-s|^{1/4} + |x-y|^{1/2}\right)^{p-4}.$$

It then follows by Proposition 4.2.12 that

$$\mathbb{E}\left[\|w\|^p_{\mathscr{C}_\epsilon^\tau}\right] = \mathbb{E}\left[\|F\|^p_{\mathscr{C}_r^\tau}\right] \leq C_{p,T,r,\epsilon}\mathbb{E}\left[\int_0^\tau \|u\|^p_{s,\mathscr{L}_{(r-\epsilon)}}\mathrm{d}s\right].$$

So we have the result. □

We are now in position to prove Theorem 4.2.8 with a Picard iteration. Since the ideas for the remaining arguments are similar to those in the proof of Theorem 4.1.8, we give an outline of the strategy only.

Proof of Theorem 4.2.8. Our strategy is as follows:

1. We note that, by the definition, it is sufficient to prove existence and uniqueness for maximal solutions to the coupled reflected SPDEs

$$\frac{\partial v^1}{\partial t} = \Delta v^1 - h(v^1(t,\cdot), v^2(t,\cdot))\frac{\partial v^1}{\partial x} + f_1(t,x,v^1(t,\cdot)) + \sigma_1(t,x,v^1(t,\cdot))\dot{W} + \eta^1$$

with Dirichlet boundary condition $v^1(0) = 0$ and initial data $v_0^1 = u_0^1 \circ (\Theta_{p_0}^1)^{-1} \in \mathscr{C}_r^+$, and

$$\frac{\partial v^2}{\partial t} = \Delta v^2 + h(v^1(t,\cdot), v^2(t,\cdot))\frac{\partial v^2}{\partial x} + f_2(t,x,v^2(t,\cdot)) + \sigma_2(t,x,v^2(t,\cdot))\dot{W}^- + \eta^2$$

with Dirichlet boundary condition $v^2(0) = 0$ and initial data $v_0^2 = u_0^2 \circ (\Theta_{p_0}^2)^{-1} \in \mathscr{C}_r^+$.

2. We once again consider a truncated version of the problem. That is, we find $(v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ such that (v_M^1, η_M^1) solves the reflected SPDE

$$\frac{\partial v_M^1}{\partial t} = \Delta v_M^1 - h_{M,r}(v_M^1, v_M^2)\frac{\partial}{\partial x}(F_{M,r}(v_M^1)) + f_1(t,x,v_M^1) + \sigma_1(t,x,v_M^1)\dot{W} + \eta_M^1 \quad (4.2.6)$$

with Dirichlet boundary condition $v_M^1(0) = 0$ and initial data $v_0^1 = u_0^1 \circ (\Theta_{p_0}^1)^{-1} \in \mathscr{C}_r^+$, and (v_M^2, η_M^2) solves the reflected SPDE

$$\frac{\partial v_M^2}{\partial t} = \Delta v_M^2 + h_{M,r}(v_M^1, v_M^2)\frac{\partial}{\partial x}(F_{M,r}(v_M^2)) + f_2(t,x,v_M^2) + \sigma_2(t,x,v_M^2)\dot{W}^- + \eta_M^2 \quad (4.2.7)$$

with Dirichlet boundary condition $v_M^2(0) = 0$ and initial data $v_0^2 = u_0^2 \circ (\Theta_{p_0}^2)^{-1} \in \mathscr{C}_r^+$.

As in Proposition 4.1.11, $h_{M,r}$ is defined by applying h to suitably truncated inputs, with the truncation function here being $F_{M,r}$. Formalising this, we define $h_{M,r} : \mathscr{C}_r \times \mathscr{C}_r \to \mathbb{R}$ by

$$h_{M,r}(v_1, v_2) := h(F_{M,r}(v_1), F_{M,r}(v_2)),$$

with $F_{M,r} : \mathscr{C}_r \to \mathscr{C}_r$ given by

$$F_{M,r}(u)(x) := e^{rx}\min(e^{-rx}u(x), M).$$

3. Use a Picard argument to prove global existence and uniqueness for the solution to the truncated problem on the finite time interval $[0,T]$. The first approximations are given by

$$v_{M,n}^1(t,x) = \left(u_0^1 \circ (\Theta_{p_0}^1)^{-1}\right)(x)$$

and

$$v_{M,n}^2(t,x) = \left(u_0^2 \circ (\Theta_{p_0}^2)^{-1}\right)(x).$$

For $n \geq 1$, we let $w^1_{M,n+1}$ solve the SPDE

$$\frac{\partial w^1_{M,n+1}}{\partial t} = \Delta w^1_{M,n+1} - h_{M,r}(v^1_{M,n}, v^2_{M,n}) \frac{\partial}{\partial x}(F_{M,r}(v^1_{M,n})) + f_1(t,x,v^1_{M,n}) + \sigma_1(t,x,v^1_{M,n})\dot{W}$$

with Dirichlet boundary condition $w^1_{M,n+1}(t,0) = 0$ and initial data $v^1_0 := u^1_0 \circ (\Theta^1_{p_0})^{-1}$. Writing this in mild form gives the expression

$$\begin{aligned} w^1_{M,n+1}(t,x) = &\int_0^\infty G(t,x,y)v^1_0(y)\mathrm{d}y \\ &+ \int_0^t \int_0^\infty \frac{\partial G}{\partial y}(t-s,x,y) h_{M,r}(v^1_{M,n}(s,\cdot), v^2_{M,n}(s,\cdot)) F_{M,r}(v^1_{M,n}(s,\cdot))(y)\ \mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^\infty G(t-s,x,y) f_1(s,y,v^1_{M,n}(s,\cdot))\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_0^\infty G(t-s,x,y)\sigma_1(s,y,v^1_{M,n}(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s). \end{aligned}$$

We then set $v^1_{M,n+1} := w^1_{M,n+1} + z^1_{M,n+1}$, where $z^1_{M,n+1}$ solves the obstacle problem with obstacle $-w^1_{M,n+1}$. We similarly define $w^2_{M,n+1}$ and $v^2_{M,n+1}$. Our Lipschitz conditions on h, f and σ, together with the estimates from Lemma 4.2.10, Proposition 4.2.13 and Proposition A.1.3 allow us to argue as in the proof of Propsition 4.1.11 to obtain that, for $t \in [0,T]$,

$$\begin{aligned} &\mathbb{E}\left[\|w^1_{M,n+1} - w^1_{M,n}\|^p_{\mathscr{C}^T_r} + \|w^2_{M,n+1} - w^2_{M,n}\|^p_{\mathscr{C}^T_r}\right] \\ &\qquad \leq C_{M,p,T,r,\delta} \int_0^t \mathbb{E}\left[\|v^1_{M,n} - v^1_{M,n-1}\|^p_{\infty,s} + \|v^2_{M,n} - v^2_{M,n-1}\|^p_{\infty,s}\right] \mathrm{d}s. \quad (4.2.8) \end{aligned}$$

Theorem 4.2.7 gives that, for $t \in [0,T]$ and $i = 1,2$,

$$\|v^i_{M,n+1} - v^i_{M,n}\|_{\mathscr{C}^T_r} \leq C_{r,T}\|w^i_{M,n+1} - w^i_{M,n}\|_{\mathscr{C}^T_r}.$$

Plugging this into (4.2.8) then gives that, for $t \in [0,T]$,

$$\begin{aligned} &\mathbb{E}\left[\|v^1_{M,n+1} - v^1_{M,n}\|^p_{\mathscr{C}^T_r} + \|v^2_{M,n+1} - v^2_{M,n}\|^p_{\mathscr{C}^T_r}\right] \\ &\qquad \leq C_{M,p,T,r,\delta} \int_0^t \mathbb{E}\left[\|v^1_{M,n} - v^1_{M,n-1}\|^p_{\infty,s} + \|v^2_{M,n} - v^2_{M,n-1}\|^p_{\infty,s}\right] \mathrm{d}s. \end{aligned}$$

Arguing as in Proposition 4.1.11, we see that $(v^1_{M,n}, v^2_{M,n})_{n\geq 1}$ is Cauchy in $L^p(\Omega; \mathscr{C}^T_r)^2$ for large enough p, and the limit solves the truncated problem given by equations (4.2.6) and (4.2.7).

4. As in Proposition 4.1.11, uniqueness for the truncated problem can be shown by applying the same estimates as in the proof of existence and concluding with a Gronwall argument.

5. We note the consistency of the truncated problems with different truncation values M and use this to define a solution to the problem until the $\|.\|_{\mathscr{C}_r}$ norm blows up.

6. We observe that uniqueness of the truncated problems implies uniqueness for the original moving boundary problem.

7. We finally note that $\tau > 0$ almost surely. We have that

$$\tau_M = \inf\{t \geq 0 \mid \|v_M^1\|_{\mathscr{C}_r^t} + \|v_M^2\|_{\mathscr{C}_r^t} \geq M\} = \inf\{t \geq 0 \mid \|v^1\|_{\mathscr{C}_r^t} + \|v^2\|_{\mathscr{C}_r^t} \geq M\},$$

and $\tau_M \uparrow \tau$ almost surely. It is therefore sufficient to show that there exists an M such that $\tau_M > 0$ almost surely. Let w_M^1 solve the SPDE

$$\frac{\partial w_M^1}{\partial t} = \Delta w_M^1 - h_{M,r}(v_M^1, v_M^2)\frac{\partial}{\partial x}(F_{M,r}(v_M^1)) + f_1(t,x,v_M^1) + \sigma_1(t,x,v_M^1)\dot{W}$$

with Dirichlet boundary condition $w(t,0) = 0$ and initial data $w(0,x) = v_0^1(x)$. By Theorem 4.2.7, we have that for every $t \in [0,T]$, $\|v_M^1\|_{\mathscr{C}_r^t} \leq C_{r,T}\|w_M^1\|_{\mathscr{C}_r^t}$. It follows that $\tau_M \leq \rho_M$, where

$$\rho_M := \inf\{t \geq 0 \mid \|w_M^1\|_{\mathscr{C}_r^t} + \|w_M^2\|_{\mathscr{C}_r^t} \geq M/C_{r,T}\}.$$

We have that

$$\begin{aligned} w_M^1(t,x) = &\int_0^\infty G(t,x,y)v_0^1(y)\mathrm{d}y \\ &+ \int_0^t \int_0^\infty \frac{\partial G}{\partial y}(t-s,x,y)h_{M,r}(v_M^1(s,\cdot), v_M^2(s,\cdot))F_{M,r}(v_M^1(s,\cdot))(y)\ \mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^\infty G(t-s,x,y)f_1(s,y,v_M^1(s,\cdot))\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_0^\infty G(t-s,x,y)\sigma_1(s,y,v_M^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s). \end{aligned} \tag{4.2.9}$$

We consider each of these terms separately. We have that

$$\sup_{t\in[0,T]} \int_0^\infty G(t,x,y)v_0^1(y)\mathrm{d}y \leq \|v_0^1\|_{\mathscr{C}_r} \sup_{t\in[0,T]} \int_0^\infty G_r(t,x,y)\mathrm{d}y = K_{r,T}\|v_0^1\|_{\mathscr{C}_r^T}.$$

It follows from Proposition A.1.3, and the fact that h is bounded on bounded sets, that for $t \in [0,T]$

$$\begin{aligned} &\left| \int_0^t \int_0^\infty \frac{\partial G}{\partial y}(t-s,x,y)h_{M,r}(v_M^1(s,\cdot), v_M^2(s,\cdot))F_{M,r}(v_M^1(s,\cdot))(y)\ \mathrm{d}y\mathrm{d}s \right| \\ &\leq \int_0^t \int_0^\infty \left| e^{-r(x-y)}\frac{\partial G}{\partial y}(t-s,x,y)h_{M,r}(v_M^1(s,\cdot), v_M^2(s,\cdot)) \right| \|F_{M,r}(v_M^1(s,\cdot))\|_{\mathscr{C}_r}\ \mathrm{d}y\mathrm{d}s \\ &\leq K_{M,h,T} \int_0^t \int_0^\infty \left| e^{-r(x-y)}\frac{\partial G}{\partial y}(t-s,x,y) \right| \ \mathrm{d}y\mathrm{d}s \\ &\leq K_{M,h,T} \int_0^t \frac{1}{\sqrt{t-s}}\ \mathrm{d}s. \end{aligned}$$

By Lemma 4.2.10, we have that for $t \in [0,T]$

$$\begin{aligned} \int_0^t \int_0^\infty G(t-s,x,y)f_1(y,v_M^1(s,\cdot))\mathrm{d}x\mathrm{d}s &\leq K_{r,T}\int_0^t \|f_1(\cdot,v_M^1(\cdot,\cdot)\|_{\mathscr{C}_r^s}\ \mathrm{d}s \\ &\leq K_{r,T,f}\int_0^t 1 + \|v_M^1\|_{\mathscr{C}_r^s}\ \mathrm{d}s. \end{aligned}$$

Finally, Proposition 4.2.13 gives that the stochastic term in (4.2.9) lies in the space $C([0,T];\mathscr{C}_r)$. Putting this together, we see that for M large enough there almost surely exists $t > 0$ such that for every $s \in [0,t]$,

$$\|w_M^1\|_{\mathscr{C}_r^s} \leq M/C_{r,T}.$$

Therefore $\rho_M \geq t > 0$. □

Proposition 4.2.14. *Suppose that h is bounded. Then the solution to the moving boundary problem on $\mathbb{R}$ is global.*

Proof. We argue as in Proposition 4.1.12, replacing bounds on H with the corresponding bounds on $G_r(t,x,y)$ and $G_{r+\delta}(t,x,y)$. □

Remark 4.2.15. *We note that our uniqueness result here extends the existing theory for uniqueness for reflected SPDEs on infinite spatial domains. Prior to this thesis and the paper [24] on which this section is based, uniqueness had only been shown for equations*

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x,u) + \sigma(t,x,u)\dot{W} + \eta$$

in the case when σ is constant. This was proved in [42], where the spatial domain was $\mathbb{R}$ (this makes no difference to the arguments here). Choosing $h = 0$ in our equations i.e. a static boundary, we obtain uniqueness for solutions to these equations in the spaces $\mathscr{C}_r$, provided that the dependence of the volatility on the solution itself decays exponentially, as in conditions (iii) and (iv) in the formulation of the problem.

4.2.4 Hölder Continuity

Theorem 4.2.16. *For $i = 1,2$, let u_0^i be such that $v_0^i := u_0^i \circ (\Theta_{p_0}^i)^{-1} \in \mathscr{C}_r^+$. Suppose that for every compact interval $I \subset [0,\infty)$ and every $\gamma \in (0,1)$, there exists $C_{\gamma,|I|}$ such that for every $x, y \in I$*

$$|e^{-rx}v_0^i(x) - e^{-ry}v_0^i(y)| \leq C_{\gamma,|I|}|x-y|^{\gamma/2}$$

Then, for every $M > 0$ and every $\gamma \in (0,1)$ the solution (v_M^1, v_M^2) to the M-truncated problem with initial data (v_0^1, v_0^2), described by equations (4.2.6) and (4.2.7), is locally $\gamma/4$-Hölder in time and $\gamma/2$-Hölder in space. In particular, if $(u^1, \eta^1, u^2, \eta^2, p)$ is the solution to our moving boundary problem with initial data (u_0^1, u_0^2, p_0), then (u^1, u^2) enjoys the same Hölder regularity locally until the blow-up time, τ.

Remark 4.2.17. *Our condition on the initial data in Theorem 4.2.16 implies that, for $\delta > 0$ and $\gamma \in (0,1)$, there exists $C_{\delta,\gamma}$ such that for every $x, y \in [0,\infty)$,*

$$|e^{-(\delta+r)x}v_0^i(x) - e^{-(\delta+r)y}v_0^i(y)| \leq C_{\delta,\gamma}|x-y|^{\gamma/2}.$$

The argument is as follows. We have that, for $n \in \mathbb{N}$, $\gamma \in (0,1)$ and $x, y \in [n, n+1]$

$$|e^{-rx}v_0^i(x) - e^{-ry}v_0^i(y)| \leq C_\gamma|x-y|^{\gamma/2}$$

Therefore, for $n \in \mathbb{N}$, $\gamma \in (0,1)$ *and* $x, y \in [n, n+1]$

$$|e^{-(\delta+r)x}v_0^i(x) - e^{-(\delta+r)y}v_0^i(y)| \leq e^{-\delta x}|e^{-rx}v_0^i(x) - e^{-ry}v_0^i(y)| + |e^{-\delta x} - e^{-\delta y}||e^{-ry}v_0^i(y)|$$
$$\leq C_\gamma e^{-\delta n}|x-y|^{\gamma/2}.$$

It follows that, for $\delta > 0$ *and* $x, y \in [0, \infty)$

$$|e^{-(\delta+r)x}v_0^i(x) - e^{-(\delta+r)y}v_0^i(y)| \leq C_\gamma \left(\sum_{n=0}^{\infty} e^{-\delta n}\right)|x-y|^{\gamma/2} = C_{\gamma,\delta}|x-y|^{\gamma/2}.$$

The proof of Theorem 4.2.16 is similar in spirit to that of Theorem 4.1.16. There are, however, some intricate differences which arise due to the infinite spatial domain. We first introduce here a modified version of Lemma 4.1.13 which is suitable for this context.

Proposition 4.2.18. *Let* $r \in \mathbb{R}$. *Let* $V \in C_b^{1,2}([0,T] \times [0,\infty))$ *and* $\psi, F \in C([0,T] \times [0,\infty))$ *with* $\psi \leq 0$ *bounded and* $|F(t,x)| \leq Ke^{Rx}$ *for some* $K, R > 0$. *Suppose that*

$$\frac{\partial V}{\partial t} = \frac{1}{2}V'' + \psi V + \psi F$$

with Dirichlet or Neumann boundary conditions at zero, and zero initial data. Then there exists a constant $C_{r,T}$ *such that*

$$\|V\|_{\mathscr{C}_r^T} \leq C_{r,T}\|F\|_{\mathscr{C}_r^T}.$$

Proof. We will prove the result for the Neumann boundary condition- the argument for the Dirichlet condition is essentially the same. Let $(B_t^x)_{t\geq 0}$ be a Brownian motion on $[0,\infty)$ with reflection at 0, started at x. Then by arguing as in Lemma 3.6 in [12], we have that

$$V_t(x) = \mathbb{E}\left[\int_0^t \exp\left(\int_0^s \psi_{t-r}(B_r^x)\mathrm{d}r\right)\psi_{t-s}(B_s^x)F_{t-s}(B_s^x)\mathrm{d}s\right].$$

Therefore

$$e^{-rx}V_t(x) = \mathbb{E}\left[\int_0^t \exp\left(\int_0^s \psi_{t-r}(B_r^x)\mathrm{d}r\right)\psi_{t-s}(B_s^x)F_{t-s}(B_s^x)e^{-rB_s^x}(e^{rB_s^x - rx})\mathrm{d}s\right].$$

Hence

$$\begin{aligned}
|e^{-rx}V_t(x)| \leq& \|F\|_{\mathscr{C}_r^T} \times \mathbb{E}\left[-\int_0^t \exp\left(\int_0^s \psi_{t-r}(B_r^x)\mathrm{d}r\right)\psi_{t-s}(B_s^x)(e^{rB_s^x - rx})\mathrm{d}s\right] \\
\leq& \|F\|_{\mathscr{C}_r^T} \times \mathbb{E}\left[-\int_0^t \exp\left(\int_0^s \psi_{t-r}(B_r^x)\mathrm{d}r\right)\psi_{t-s}(B_s^x)\mathrm{d}s\left(\sup_{s\in[0,t]}(e^{rB_s^x - rx})\right)\right] \\
\leq& \|F\|_{\mathscr{C}_r^T} \times \mathbb{E}\left[\left(1 - \exp\left(\int_0^t \psi_{t-r}(B_r^x)\mathrm{d}r\right)\right)\left(\sup_{s\in[0,t]}(e^{rB_s^x - rx})\right)\right] \\
\leq& \|F\|_{\mathscr{C}_r^T} \times \mathbb{E}\left[\sup_{s\in[0,t]}(e^{rB_s^x - rx})\right] \leq \|F\|_{\mathscr{C}_r^T} \times \mathbb{E}\left[\sup_{s\in[0,T]}(e^{rB_s^x - rx})\right] \\
=& \|F\|_{\mathscr{C}_r^T} \times e^{-rx} \times \mathbb{E}\left[\sup_{s\in[0,T]} e^{rB_s^x}\right].
\end{aligned} \tag{4.2.10}$$

We note that the law of B^x is simply the law of $|W^x|$, where W is a standard Brownian motion (no reflection) started from x. Therefore

$$e^{-rx}\mathbb{E}\left[\sup_{s\in[0,T]} e^{rB_s^x}\right] = e^{-rx}\mathbb{E}\left[\sup_{s\in[0,T]}\left(e^{rW_s^x}\mathbb{1}_{\{W_s^x\geq 0\}} + e^{-rW_s^x}\mathbb{1}_{\{-W_s^x\geq 0\}}\right)\right].$$

By the symmetry of Brownian motion, this is at most

$$e^{-rx}\mathbb{E}\left[\sup_{s\in[0,T]} e^{rW_s^x}\right] + e^{-rx}\mathbb{E}\left[\sup_{s\in[0,T]} e^{rW_s^{-x}}\right] = (1+e^{-2rx})\times\mathbb{E}\left[\sup_{s\in[0,T]} e^{rW_s^0}\right]$$
$$\leq 2\mathbb{E}\left[\sup_{s\in[0,T]} e^{rW_s^0}\right] \leq 2e^{Tr^2/2}\times\mathbb{E}\left[\sup_{s\in[0,T]} e^{rW_s^0 - sr^2/2}\right].$$

Since $e^{rW_t^0 - tr^2/2}$ is a square integrable martingale, we apply Cauchy-Schwarz and Doob's L^2 inequality to get

$$2e^{Tr^2/2}\mathbb{E}\left[\sup_{s\in[0,T]} e^{rW_s^0 - sr^2/2}\right] \leq 2e^{Tr^2/2}\mathbb{E}\left[\sup_{s\in[0,T]} e^{2rW_s^0 - sr^2}\right]^{\frac{1}{2}} \leq 4e^{Tr^2/2}\mathbb{E}\left[e^{2rW_T^0 - Tr^2}\right]^{\frac{1}{2}}$$

We now note that $e^{2rW_t^0 - 2tr^2}$ is also a square integrable martingale, and therefore

$$\mathbb{E}\left[e^{2rW_T^0 - 2Tr^2}\right] = 1.$$

It follows that

$$4e^{Tr^2/2}\mathbb{E}\left[e^{2rW_T^0 - Tr^2}\right]^{\frac{1}{2}} = 4e^{Tr^2}. \tag{4.2.11}$$

Therefore, we have shown that

$$e^{-rx}\mathbb{E}\left[\sup_{s\in[0,T]} e^{rB_s^x}\right] \leq 4e^{Tr^2}.$$

Plugging this into (4.2.10), we obtain that

$$\|V\|_{\mathscr{C}_r^T} \leq 4e^{Tr^2}\|F\|_{\mathscr{C}_r^T}.$$

So we have the result. □

The following results will be important in the main arguments for local-Hölder continuity of our solutions. They are analogues of Lemma 4.1.14 and Proposition 4.1.15, adapted to the case where the spatial domain is $[0,\infty)$. Consequently, the following Lemma is also an adaptation of Lemma 3.2 in [13].

Lemma 4.2.19. *Let $f : [0,T]\times[0,\infty)\to\mathbb{R}$ be such that $f(\cdot,0) = 0$, and for every $t,s\in[0,T]$ and every $x,y\in[0,\infty)$*

$$|f(t,x) - f(s,y)| \leq C_f\left(\sum_{i=1}^n |t-s|^{\alpha_i} + \sum_{i=1}^m |x-y|^{\beta_i}\right).$$

Then, for every $p,q>0$ there exists a smooth function $f_{p,q} : [0,T]\times[0,\infty)\to\mathbb{R}$ such that $f^{p,q}(\cdot,0) = 0$, and

(i) $\|f_{p,q}\|_\infty \leq \|f\|_\infty$.

(ii) $\|f_{p,q} - f\|_\infty \leq C_{\alpha_i,\beta_i} C_f \left(\sum_{i=1}^n p^{\alpha_i} + \sum_{i=1}^m q^{\beta_i} \right)$.

(iii) $\left\| \frac{\partial f_{p,q}}{\partial t} \right\|_\infty \leq C_{\alpha_i,\beta_i} C_f \sum_{i=1}^n p^{\alpha_i - 1}$.

(iv) $\left\| \frac{\partial f_{p,q}}{\partial x} \right\|_\infty \leq C_{\alpha_i \beta_i} C_f \sum_{i=1}^m q^{\beta_i - 1}$.

Proof. The proof is essentially the same as the proof of Lemma 4.1.14 and so we omit this. □

Proposition 4.2.20. *Let $u \in \mathscr{C}_r^T$ solve the heat equation $\frac{\partial u}{\partial t} = \Delta u$ on $[0,T] \times [0,\infty)$ with Dirichlet boundary condition $u(t,0) = 0$ and initial data $u_0 \in \mathscr{C}_r$. Suppose also that there exists $\delta \geq r$ such that, for every $x, y \in [0,\infty)$,*

$$|e^{-\delta x} u_0(x) - e^{-\delta y} u_0(y)| \leq C_{\delta,\alpha,u_0} |x-y|^\alpha, \tag{4.2.12}$$

where $\alpha \in (0,1]$. Then we have that for $\tilde{\delta} > \delta$, $s,t \in [0,T]$ and $x, y \in [0,\infty)$

$$|e^{-\tilde{\delta x}} u(t,x) - e^{-\tilde{\delta} y} u(s,y)| \leq C_{\tilde{\delta},\delta,\alpha,u_0,T} \left(|x-y|^\alpha + |t-s|^{\alpha/2} \right).$$

Proof. The proof is an adaptation of the argument in the proof of Proposition 4.1.15. Define the extension of u_0, u_0^e given by

$$u_0^e(x) = u_0(x) 1_{\{x \in [0,\infty)\}} - u_0(-x) 1_{\{x \in (-\infty, 0)\}}.$$

Then

$$\sup_{x,y \in \mathbb{R}, x \neq y} \frac{|e^{-\delta|x|} u_0^e(x) - e^{-\delta|y|} u_0^e(y)|}{|x-y|^\alpha} \leq 2 \sup_{x,y \in [0,\infty), x \neq y} \frac{|e^{-\delta x} u_0(x) - e^{-\delta y} u_0(y)|}{|x-y|^\alpha}.$$

We then have that

$$u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} u_0^e(y) \, \mathrm{d}y.$$

for $(t,x) \in [0,T] \times [0,\infty)$. We note that

$$\begin{aligned}
|e^{-rx} u(t,x)| =& e^{-rx} \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} u_0^e(y) \, \mathrm{d}y \right| \\
\leq& e^{-rx} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} |u_0^e(y)| \, \mathrm{d}y \\
=& \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} e^{-r(x-|y|)} |e^{-r|y|} u_0^e(y)| \, \mathrm{d}y \\
\leq& C_{\delta,T} \|u_0\|_{\mathscr{C}_r}
\end{aligned}$$

We now prove the Hölder bound in space. By Remark 4.2.17, it is enough to show that for $s,t \in [0,T]$ and $0 \leq x \leq y$ with $|x-y| \leq 1$,

$$|e^{-\delta x} u(t,x) - e^{-\delta y} u(s,y)| \leq C_{\delta,\alpha,u_0,T} \left(|x-y|^\alpha + |t-s|^{\alpha/2} \right).$$

Let $t \in [0,T]$ and $0 \le x \le y$ with $|x-y| \le 1$.

$$
\begin{aligned}
&|e^{-\delta x}u(t,x) - e^{-\delta y}u(t,y)| \\
&= \left| e^{-\delta x} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-z)^2/4t} u_0^e(z)\, \mathrm{d}z - e^{-\delta y} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(y-z)^2/4t} u_0^e(z)\, \mathrm{d}z \right| \\
&= \left| e^{-\delta x} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} u_0^e(z+x)\, \mathrm{d}z - e^{-\delta y} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} u_0^e(z+y)\, \mathrm{d}z \right| \\
&= \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} \left[e^{-\delta(x-|x+z|)} e^{-\delta|z+x|} u_0^e(z+x) - e^{-\delta(y-|z+y|)} e^{-\delta|z+y|} u_0^e(z+y) \right] \mathrm{d}z \right| \\
&\le \int_{-x}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} e^{\delta z} |e^{-\delta|z+x|} u_0^e(z+x) - e^{-\delta|z+y|} u_0^e(z+y)|\, \mathrm{d}z \\
&\quad + \int_{-y}^{-x} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} |e^{-\delta(2x+z)} e^{-\delta|z+x|} u_0^e(z+x) - e^{\delta z} e^{-\delta|z+y|} u_0^e(z+y)|\, \mathrm{d}z \\
&\quad + \int_{-\infty}^{-y} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} |e^{-\delta(2x+z)} e^{-\delta|z+x|} u_0^e(z+x) - e^{-\delta(2y+z)} e^{-\delta|z+y|} u_0^e(z+y)|\, \mathrm{d}z
\end{aligned}
$$

For the first of these terms, we have that

$$
\begin{aligned}
&\int_{-x}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} e^{\delta z} |e^{-\delta|z+x|} u_0^e(z+x) - e^{-\delta|z+y|} u_0^e(z+y)|\, \mathrm{d}z \\
&\qquad \le \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} e^{\delta z}\, \mathrm{d}z \right| \times 2C_\delta |x-y|^\alpha = C_{\delta,\alpha,u_0,T} |x-y|^\alpha.
\end{aligned}
$$

The second term can be bounded by noting that

$$
\begin{aligned}
&\int_{-y}^{-x} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} |e^{-\delta(2x+z)} e^{-\delta|z+x|} u_0^e(z+x) - e^{\delta z} e^{-\delta|z+y|} u_0^e(z+y)|\, \mathrm{d}z \\
&\le \int_{-y}^{-x} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} e^{\delta z} |e^{-\delta|z+x|} u_0^e(z+x) - e^{-\delta|z+y|} u_0^e(z+y)|\, \mathrm{d}z \\
&\quad + \int_{-y}^{-x} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} |e^{-\delta|z+x|} u_0^e(z+x)| |e^{\delta z} - e^{-\delta(2x+z)}|\, \mathrm{d}z \\
&\le C_{\delta,\alpha,u_0,T} |x-y|^\alpha + C_{\delta,\alpha,u_0,T} \sup_{z \in [-y,-x]} |z + (2x+z)| \|u_0\|_{\mathscr{C}_\delta} \\
&\le C_{\delta,\alpha,u_0,T} |x-y|^\alpha.
\end{aligned}
$$

We note that the manipulations for the third term are similar, so we omit these here. For the Hölder continuity in time, we note that for $t \in [0,T]$ and $x \in [0,\infty)$

$$
\begin{aligned}
|e^{-\delta x}u(t,x) - e^{-\delta x}u_0(x)| =& e^{-\delta x} \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-z)^2/4t} \left[u_0^e(z) - u_0^e(x) \right] \mathrm{d}z \right| \\
=& e^{-\delta x} \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} \left[u_0^e(z+x) - u_0^e(x) \right] \mathrm{d}z \right|
\end{aligned}
\tag{4.2.13}
$$

We note that it follows from (4.2.12) that, for $x \ge y \ge 0$,

$$
\begin{aligned}
|u_0(x) - u_0(y)| \le& e^{\delta x} \left(|e^{-\delta x}u_0(x) - e^{-\delta y}u_0(y)| + |e^{-\delta x} - e^{-\delta y}| |u_0(y)| \right) \\
\le& e^{\delta x} \left(|e^{-\delta x}u_0(x) - e^{-\delta y}u_0(y)| + |1 - e^{-\delta(x-y)}| \|u_0\|_{\mathscr{C}_\delta} \right) \\
\le& e^{\delta x} C_{\delta,\alpha,u_0} \left(1 + \|u_0\|_{\mathscr{C}_\delta} \right) |x-y|^\alpha.
\end{aligned}
$$

It follows that, for $x, y \in \mathbb{R}$,

$$
|u_0^e(x) - u_0^e(y)| \le 2C_{\delta,\alpha,u_0} \left(1 + \|u_0\|_{\mathscr{C}_\delta} \right) e^{\delta \max(|x|,|y|)} |x-y|^\alpha.
$$

Therefore, for $z \in \mathbb{R}$ and $x \in [0, \infty)$, we have that

$$|u_0^e(z+x) - u_0^e(z)| \leq 2C_{\delta,\alpha,u_0} \left(1 + \|u_0\|_{\mathscr{C}_\delta}\right) e^{\delta(x+|z|)} |z|^\alpha.$$

Recalling (4.2.13), we then have that

$$\begin{aligned} |e^{-\delta x} u(t,x) - e^{-\delta x} u_0(x)| \leq & 2C_{\delta,\alpha,u_0} \left(1 + \|u_0\|_{\mathscr{C}_\delta}\right) \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} e^{\delta|z|} |z|^\alpha \, \mathrm{d}z \right| \\ = & C_{\delta,\alpha,u_0} \left| \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t} e^{\delta z} z^\alpha \, \mathrm{d}z \right|. \end{aligned} \tag{4.2.14}$$

Applying the change of variables $y = z/\sqrt{t}$, we see that this is equal to

$$C_{\delta,\alpha,u_0} \int_0^\infty e^{-y^2/4} e^{\delta\sqrt{t}y} y^\alpha t^{\alpha/2} \, \mathrm{d}y \leq C_{\delta,\alpha,u_0} t^{\alpha/2} \int_0^\infty e^{-y^2/4} e^{\delta\sqrt{T}y} y^\alpha \, \mathrm{d}y \leq C_{\delta,T,\alpha,u_0} t^{\alpha/2}.$$

For a general $t > s \geq 0$, we note that $u(t,\cdot)$ is the solution to the heat equation with initial data $v(s,\cdot)$ at time $t-s$. We also have from our earlier calculations that, for $\tilde{\delta} > \delta$, the Hölder continuity of $e^{-\tilde{\delta}(\cdot)} u(s,\cdot)$ and the $\mathscr{C}_{\tilde{\delta}}$ norm of $u(s,\cdot)$ are controlled by those of u_0. Hölder continuity in time therefore follows by applying (4.2.14) to the heat equation at time $t-s$ with initial data $u(s,\cdot)$. This concludes the proof. □

We are now in position to prove the main result of this section, Theorem 4.2.16.

Proof of Theorem 4.2.16. The argument broadly follows the steps in the proof of Theorem 4.1.16 and consequently those in [11]. Fix some $T > 0$. Let (v_M^1, v_M^2) solve the M-truncated problem. Define w_M^1 so that

$$\begin{aligned} w_M^1(t,x) = & \int_0^\infty G(t,x,y) v_0^1(y) \, \mathrm{d}y \\ & - \int_0^t \int_0^\infty \frac{\partial G}{\partial y}(t-s,x,y) h_{M,r}(v_M^1(s,\cdot), v_M^2(s,\cdot)) F_{M,r}(v_M^1(s,\cdot))(y) \mathrm{d}y \mathrm{d}s \\ & + \int_0^t \int_0^\infty G(t-s,x,y) f_1(s,y,v_M^1(s,\cdot)) \mathrm{d}x \mathrm{d}s \\ & + \int_0^t \int_0^\infty G(t-s,x,y) \sigma_1(s,y,v_M^1(s,\cdot)) \mathrm{W}(\mathrm{d}y, \mathrm{d}s). \end{aligned} \tag{4.2.15}$$

where we define $h_{M,r} : \mathscr{C}_r \times \mathscr{C}_r \to \mathbb{R}$ by

$$h_{M,r}(v_1, v_2) := h(F_{M,r}(v_1), F_{M,r}(v_2)),$$

and $F_{M,r} : \mathscr{C}_r \to \mathscr{C}_r$ by

$$F_{M,r}(u)(x) := e^{rx} \min(e^{-rx} u(x), M).$$

Then w_M^1 solves the SPDE

$$\frac{\partial w_M^1}{\partial t} = \Delta w_M^1 + h_{M,r}(v_M^1, v_m^2) \frac{\partial}{\partial x}(F_{M,r}(v_M^1)) + f_1(t,x,v_M^1) + \sigma(t,x,v_M^1)\dot{W},$$

with Dirichlet boundary condition $w_m^1(\cdot,0) = 0$ and initial data $w_M^1(0,\cdot) = v_0^1$. We examine the Hölder continuity in space of the third term on the right hand side of (4.2.15). Let a be such

that $\frac{a}{a+5} > \gamma$, and set $p = 2a + 10$, so that $p = a/2 - 5$. Fix $\delta > 0$ and define $\tilde{r} := r + \delta$. We then have that, for $|x - z| \leq 1$,

$$\begin{aligned}
&\left| \int_0^t \int_0^\infty \left(e^{-\tilde{r}x} G(t-s,x,y) f_1(s,y,v_M^1(s,\cdot)) - e^{-\tilde{r}z} G(t-s,z,y) f_1(s,y,v_M^1(s,\cdot)) \right) \mathrm{d}y\mathrm{d}s \right|^p \\
&= \left| \int_0^t \int_0^\infty e^{-\tilde{r}y} f_1(s,y,v_M^1(s,\cdot)) \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right) \mathrm{d}y\mathrm{d}s \right|^p \\
&\leq \left| \int_0^t \int_0^\infty e^{-\delta y} \| f_1(\cdot,\cdot,v_M^1(\cdot,\cdot)) \|_{\mathscr{C}_{r,s}} \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right) \mathrm{d}y\mathrm{d}s \right|^p .
\end{aligned}$$

By Hölder's inequality, this is at most

$$\begin{aligned}
&\left[\int_0^t \int_0^\infty e^{-2\delta y} \| f_1(\cdot,\cdot,v_M^1(\cdot,\cdot)) \|_{\mathscr{C}_{r,s}}^2 \mathrm{d}y\mathrm{d}s \right]^{p/2} \times \left[\int_0^t \int_0^\infty \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right)^2 \mathrm{d}y\mathrm{d}s \right]^{\frac{p}{2}} \\
&= C_{p,\delta} \left[\int_0^t \| f_1(\cdot,\cdot,v_M^1(\cdot,\cdot)) \|_{\mathscr{C}_{r,s}}^2 \mathrm{d}s \right]^{p/2} \times \left[\int_0^t \int_0^\infty \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right)^2 \mathrm{d}y\mathrm{d}s \right]^{\frac{p}{2}} \\
&\leq C_{p,\delta,T} \left[\int_0^t \| f_1(\cdot,\cdot,v_M^1(\cdot,\cdot)) \|_{\mathscr{C}_{r,s}}^p \mathrm{d}s \right] \times \left[\int_0^t \int_0^\infty \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right)^2 \mathrm{d}y\mathrm{d}s \right]^{\frac{p}{2}}
\end{aligned}$$

By the linear growth property of f, and since we have from our construction of v_M that $v_M \in L^p(\Omega; \mathscr{C}_r^T)$, we have that

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_0^t \int_0^\infty \left(e^{-\tilde{r}x} G(t-s,x,y) f_1(s,y,v_M^1(s,\cdot)) - e^{-\tilde{r}z} G(t-s,z,y) f_1(s,y,v_M^1(s,\cdot)) \right) \mathrm{d}y\mathrm{d}s \right|^p \right] \\
&\leq C \left[\int_0^t \int_0^\infty \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right)^2 \mathrm{d}y\mathrm{d}s \right]^{\frac{p}{2}} \\
&\leq C \left[\int_0^t \left[\int_0^\infty \left(G_{\tilde{r}}(t-s,x,y) - G_{\tilde{r}}(t-s,z,y) \right)^2 \mathrm{d}y \right]^{\frac{p}{p-2}} \mathrm{d}s \right]^{\frac{p-2}{2}} \\
&\leq C|x-z|^{(p-4)/2},
\end{aligned}$$

Note that we have made use of inequality (2) from Proposition A.1.2 in the above calculation. Proposition 4.2.20 and similar applications of the inequalities from Propositions A.1.2 and A.1.4, together with Hölder's and Burkholder's inequality give that, for $|x - y| \leq 1$,

$$\begin{aligned}
\mathbb{E}\left[|e^{-\tilde{r}x} w_M^1(t,x) - e^{-\tilde{r}y} w_M^1(s,y)|^p \right] &\leq C(|t-s|^{1/2} + |x-y|)^{\frac{p}{2}-2} \\
&= C(|t-s|^{1/2} + |x-y|)^{a+3}.
\end{aligned}$$

Since $\gamma/2 \in (0, a/p)$ by our choice of a and p, we can apply Corollary A.3 from [11] to find random variables ψ_m such that $\sup_{m \in \mathbb{N}} \mathbb{E}[(\psi_m)^p] < \infty$, and for $t, s \in [0,T]$, $x, y \in [m, m+1]$

$$|e^{-\tilde{r}x} w_M^1(t,x) - e^{-\tilde{r}y} w_M^1(s,y)| \leq \psi_m (|t-s|^{\frac{\gamma}{2}} + |x-y|^\gamma).$$

Let $\mu > 0$. We note that, for $x, y \in [m, m+1]$ and $s, t \in [0, T]$,

$$\begin{aligned}|e^{-(\tilde{r}+\mu)x} w_M^1(t,x) - e^{-(\tilde{r}+\mu)y} w_M^1(s,y)| \leq & e^{-\mu y} |e^{-\tilde{r}x} w_M^1(t,x) - e^{-\tilde{r}y} w_M^1(s,y)| \\ & + e^{-\tilde{r}x} |w_M^1(t,x)||e^{-\mu x} - e^{-\mu y}| \\ \leq & e^{-m\mu} |e^{-\tilde{r}x} w_M^1(t,x) - e^{-\tilde{r}y} w_M^1(s,y)| \\ & + e^{-\tilde{r}x} |w_M^1(t,x)||e^{-y\mu} - e^{-x\mu}|.\end{aligned}$$

By considering the derivative of $e^{-\mu x}$ we see that

$$|e^{-y\mu} - e^{-x\mu}| \leq \mu |x - y| e^{-m\mu}.$$

Therefore

$$\begin{aligned}|e^{-(\tilde{r}+\mu)x} w_M^1(t,x) - e^{-(\tilde{r}+\mu)y} w_M^1(s,y)| \leq & e^{-m\mu} |e^{-\tilde{r}x} w_M^1(t,x) - e^{-\tilde{r}y} w_M^1(s,y)| \\ & + \mu |x - y| e^{-m\mu} |e^{-\tilde{r}x} w_M^1(t,x)|.\end{aligned}$$

Since $|x - y| \leq 1$, this is at most

$$e^{-m\mu} |e^{-\tilde{r}x} w_M^1(t,x) - e^{-\tilde{r}y} w_M^1(s,y)| + \mu |x - y|^\gamma e^{-m\mu} |e^{-\tilde{r}x} w_M^1(t,x)|$$

Define the random variable

$$Z := \mu \|w_M^1\|_{\mathscr{C}_{\tilde{r}}^T}.$$

Recalling the construction of v_M^1 and w_M^1 by a Picard argument in $L^p(\Omega; \mathscr{C}_r^T)$, we know that $\mathbb{E}[Z^p] < \infty$. Setting $Y_m := \psi_m + Z$ we have that $\sup_{m \geq 0} \mathbb{E}[Y_m^p] < \infty$, and that for $m \in \mathbb{N}$, $x, y \in [m, m+1]$ and $s, t \in [0, T]$

$$|e^{-(\tilde{r}+\mu)x} w_M^1(t,x) - e^{-(\tilde{r}+\mu)y} w_M^1(s,y)| \leq Y_m e^{-m\mu} (|t - s|^{\gamma/2} + |x - y|^\gamma).$$

Therefore, for any $t, s \in [0, T]$ and $x, y \geq 0$ we have that

$$|e^{-(\tilde{r}+\mu)x} w_M^1(t,x) - e^{-(\tilde{r}+\mu)y} w_M^1(s,y)| \leq \left(\sum_{m=0}^{\infty} Y_m e^{-m\mu} \right) \times (|t - s|^{\frac{\gamma}{2}} + |x - y|^\gamma). \tag{4.2.16}$$

Define

$$X := \sum_{m=0}^{\infty} Y_m e^{-m\mu}.$$

Then

$$\begin{aligned}\mathbb{E}[X^p] = \mathbb{E}\left[\left(\sum_{m=0}^{\infty} Y_m e^{-m\mu} \right)^p \right] \leq & \mathbb{E}\left[\sum_{m=0}^{\infty} Y_m^p e^{-m\mu p/2} \right] \times \left(\sum_{m=0}^{\infty} e^{-m\mu q/2} \right)^{\frac{p}{q}} \\ = & C_{r,p,\mu} \sum_{m=0}^{\infty} \mathbb{E}[Y_m^p] e^{-mp\mu/2} = C_{r,p,\mu,R} < \infty.\end{aligned}$$

Now let z^ϵ solve the PDE

$$\frac{\partial z^\epsilon}{\partial t} = \Delta z^\epsilon + g_\epsilon(z^\epsilon + w_M^1)$$

on $[0,\infty)$ with Dirichlet boundary condition at zero, and zero initial data, where we define

$$g_\epsilon(x) := \frac{1}{\epsilon}\arctan([x \wedge 0]^2).$$

Then by Proposition A.2.3, $z^\epsilon + w_M^1$ increases to v_M^1, the solution of the reflected SPDE on $[0,\infty)$. Let $(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}$ be a smoothing of $e^{-(\tilde{r}+\mu)x}w_M^1$ as in Lemma 4.2.19, with respect to the Hölder coefficients $\gamma/2$ and $\gamma/4$, with the parameters $\tilde{p}$ and $\tilde{q}$ to be determined later. Define $z^{\epsilon,\alpha,\beta}$ to be the solution of the PDE

$$\frac{\partial z^{\epsilon,\tilde{p},\tilde{q}}}{\partial t} = \Delta z^{\epsilon,\tilde{p},\tilde{q}} + g_\epsilon(z^{\epsilon,\tilde{p},\tilde{q}} + e^{(\tilde{r}+\mu)x}(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}) \tag{4.2.17}$$

with Dirichlet boundary condition at zero and and zero initial data. By Proposition A.2.2, we obtain that

$$\|z^{\epsilon,\tilde{p},\tilde{q}}\|_{\mathscr{C}^T_{\tilde{r}+\mu}} \le \|(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\|_{T,\infty}.$$

Differentiating (4.2.17) with respect to t, we see that $q^{\epsilon,\tilde{p},\tilde{q}} := \frac{\partial z^{\epsilon,\tilde{p},\tilde{q}}}{\partial t}$ solves

$$\frac{\partial q^{\epsilon,\tilde{p},\tilde{q}}}{\partial t} = \Delta q^{\epsilon,\tilde{p},\tilde{q}} + g'_\epsilon(z^{\epsilon,\tilde{p},\tilde{q}} + e^{(\tilde{r}+\mu)x}(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}) \left[q^{\epsilon,\tilde{p},\tilde{q}} + e^{(\tilde{r}+\mu)x}\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial t}\right].$$

The boundary condition is Dirichlet, since z does not change at 0, which means the time derivative is zero there. The initial data is $z_0'' = 0$, since z_0 is identically zero. Note that g'_ϵ is negative, so we can use Proposition 4.2.18 to deduce that

$$\|q^{\epsilon,\tilde{p},\tilde{q}}\|_{\mathscr{C}^T_{\tilde{r}+\mu}} \le C_{\tilde{r}+\mu,T}\left\|\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial t}\right\|_{\infty,T}.$$

Differentiating (4.2.17) with respect to x, we obtain, that $y^{\epsilon,\tilde{p},\tilde{q}} := \frac{\partial z^{\epsilon,\tilde{p},\tilde{q}}}{\partial x}$ satisfies

$$\begin{aligned}\frac{\partial y^{\epsilon,\tilde{p},\tilde{q}}}{\partial x} =& \Delta y^{\epsilon,\tilde{p},\tilde{q}} + g'_\epsilon(z^{\epsilon,\tilde{p},\tilde{q}} + e^{(\tilde{r}+\mu)x}(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}})\Big[y^{\epsilon,\tilde{p},\tilde{q}} \\ &+ e^{(\tilde{r}+\mu)x}\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial x} + (\tilde{r}+\mu)e^{(\tilde{r}+\mu)x}(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\Big].\end{aligned}$$

with initial data $z_0' = 0$ and Neumann boundary condition at zero. Proposition 4.2.18 then gives

$$\|y^{\epsilon,\tilde{p},\tilde{q}}\|_{\mathscr{C}^T_{\tilde{r}+\mu}} \le C_{\tilde{r}+\mu,T}\left(\left\|\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial x}\right\|_{\infty,T} + \|(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\|_{\infty,T}\right).$$

Another application of Proposition A.2.2 gives

$$\|z^\epsilon - z^{\epsilon,\tilde{p},\tilde{q}}\|_{\mathscr{C}^T_{\tilde{r}+\mu}} \le \|e^{-(\tilde{r}+\mu)x}w_M^1 - (e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\|_{\infty,T}.$$

For $t,s \in [0,T]$, $x,y \ge 0$, we have that

$$\begin{aligned}|e^{-(\tilde{r}+\mu)x}z^\epsilon(t,x) - e^{-(\tilde{r}+\mu)y}z^\epsilon(s,y)| \le& 2\|z^\epsilon - z^{\epsilon,\tilde{p},\tilde{q}}\|_{\mathscr{C}^T_{\tilde{r}+\mu}} \\ &+ |e^{-(\tilde{r}+\mu)x}z^{\epsilon,\tilde{p},\tilde{q}}(t,x) - e^{-(\tilde{r}+\mu)x}z^{\epsilon,\tilde{p},\tilde{q}}(s,x)| \\ &+ |e^{-(\tilde{r}+\mu)x}z^{\epsilon,\tilde{p},\tilde{q}}(s,x) - e^{-(\tilde{r}+\mu)y}z^{\epsilon,\tilde{p},\tilde{q}}(s,y)|.\end{aligned}$$

By the estimates deduced above, we can bound this by

$$2C_{\tilde{r}+\mu,T}\|e^{-(\tilde{r}+\mu)x}w_M^1-(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\|_{\infty,T}+|t-s|\left\|\partial z^{\epsilon,\tilde{p},\tilde{q}}/\partial t\right\|_{\mathscr{C}^T_{\tilde{r}+\mu}}\\ +|x-y|\left\|\partial(e^{-(\tilde{r}+\mu)x}z^{\epsilon,\tilde{p},\tilde{q}})/\partial x\right\|_{\infty,T}.$$

This is at most

$$2\|e^{-(\tilde{r}+\mu)x}w_M^1-(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\|_{\infty,T}+C_{\tilde{r}+\mu,T}|t-s|\left\|\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial t}\right\|_{\infty,T}\\ +|x-y|\left(\left\|\frac{\partial z^{\epsilon,\tilde{p},\tilde{q}}}{\partial x}\right\|_{\mathscr{C}^T_{\tilde{r}+\mu}}+\|(\tilde{r}+\mu)z^{\epsilon,\tilde{p},\tilde{q}}\|_{\mathscr{C}^T_{\tilde{r}+\mu}}\right)$$

$$\leq 2\|e^{-(\tilde{r}+\mu)x}w_M^1-(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}\|_{\infty}+C_{\tilde{r}+\mu,T}|t-s|\left\|\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial t}\right\|_{\infty,T}\\ +C_{\tilde{r}+\mu,T}|x-y|\left(\left\|\frac{\partial(e^{-(\tilde{r}+\mu)x}w_M^1)^{\tilde{p},\tilde{q}}}{\partial x}\right\|_{\infty,T}+\|w_M^1\|_{\mathscr{C}^T_{\tilde{r}+\mu}}\right).$$

Making the choices $\tilde{p}=|t-s|$, $\tilde{q}=|x-y|$ and recalling Lemma 4.1.14, this is at most

$$C_{\gamma,T,\tilde{r}+\mu}\left(X|t-s|^{\gamma/2}+X|x-y|^{\gamma}+|x-y|\|w_M^1\|_{\mathscr{C}^T_{\tilde{r}+\mu}}\right).$$

Since X and $\|w_M^1\|_{\mathscr{C}^T_{\tilde{r}+\mu}}$ are in L^p, they are finite almost surely. Letting $\epsilon\downarrow 0$ and noting the inequality (4.2.16) gives that, for every $x,y\geq 0$ and every $s,t\in[0,T]$

$$|e^{-(\tilde{r}+\mu)x}v_M^1(t,x)-e^{-(\tilde{r}+\mu)y}v_M^1(s,y)|\leq C_{\gamma,T,\tilde{r}+\mu}\left(X|t-s|^{\gamma/2}+X|x-y|^{\gamma}+|x-y|\|w_M^1\|_{\mathscr{C}^T_{\tilde{r}+\mu}}\right) \tag{4.2.18}$$

almost surely. It follows that, for $x,y\leq K$, $s,t\in[0,T]$

$$\begin{aligned}|v_M^1(t,x)-v_M^1(s,y)|\leq& e^{(\tilde{r}+\mu)x}|e^{-(\tilde{r}+\mu)x}v_M^1(t,x)-e^{-(\tilde{r}+\mu)y}v_M^1(s,y)|+(1-e^{(\tilde{r}+\mu)(x-y)})|v_M^1(s,y)|\\ \leq& e^{(\tilde{r}+\mu)K}|e^{-(\tilde{r}+\mu)x}v_M^1(t,x)-e^{-(\tilde{r}+\mu)y}v_M^1(s,y)|\\ &+(\tilde{r}+\mu)e^{(\tilde{r}+\mu)K}|v_M^1(s,y)||x-y|\\ \leq& e^{(\tilde{r}+\mu)K}|e^{-(\tilde{r}+\mu)x}v_M^1(t,x)-e^{-(\tilde{r}+\mu)y}v_M^1(s,y)|\\ &+(\tilde{r}+\mu)e^{(2\tilde{r}+\mu)K}|e^{-\tilde{r}y}v_M^1(s,y)||x-y|.\end{aligned}$$

Since we know that $v_M^1\in L^p(\Omega;\mathscr{C}_r^T)$, we have that $\|v_M^1\|_{\mathscr{C}^T_{\tilde{r}+\mu}}<\infty$ almost surely. Therefore, v_M^1 is locally $\gamma/4$ Hölder in time and $\gamma/2$ Hölder in space almost surely. □

Corollary 4.2.21. *Suppose that there exists $\tilde{\mu}>0$ such that for every $u_1,u_2,v_1,v_2\in\mathscr{C}_r$,*

$$|h(u_1,v_1)-h(u_2,v_2)|\leq K(\|u_1-u_2\|_{\mathscr{C}_{r+\tilde{\mu}}}+\|v_1-v_2\|_{\mathscr{C}_{r+\tilde{\mu}}}).$$

Then, for every $\gamma\in(0,1)$, the derivative of the boundary is locally $\gamma/4$-Hölder continuous on $[0,\tau)$, where

$$\tau:=\inf\left\{t\geq 0\;\middle|\;\|u^1\|_{\mathscr{C}_r}+\|u^2\|_{\mathscr{C}_r}=\infty\right\}.$$

Proof. Fix $\gamma \in (0,1)$. Recall that $p'(t) = h(v^1(t,\cdot), v^2(t,\cdot))$, where $v^1(t,x) = u^1(t, p(t) - \cdot)$ and $v^2(t,x) = u^2(t, p(t) + \cdot)$. For $M > 0$, define

$$\tau_M := \inf \left\{ t \geq 0 \;\middle|\; \|u^1\|_{\mathscr{C}_r} + \|u^2\|_{\mathscr{C}_r} > M \right\}$$

Note that $\tau_M \uparrow \tau$ as $M \to \infty$. Let $t, s \in [0, \tau_M]$. We have, by the Lipschitz property of h, that

$$|p'(t) - p'(s)| \leq K \left(\|v^1(t,\cdot) - v^1(s,\cdot)\|_{\mathscr{C}_{r+\tilde{\mu}}} + \|v^2(t,\cdot) - v^2(s,\cdot)\|_{\mathscr{C}_{r+\tilde{\mu}}} \right).$$

By equation (4.2.18) in the proof of Theorem 4.2.16, choosing δ and μ such that $\delta + \mu = \tilde{\mu}$, there exists an almost surely finite random variable C_M such that this is at most $KC_M|t-s|^{\gamma/4}$, which gives the result. □

Remark 4.2.22. *Choosing $h = 0$, we see that in particular the solutions to the reflected SPDEs*

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x,u) + \sigma(t,x,u)\dot{W} + \eta$$

on $[0,\infty) \times [0,\infty)$ (or $[0,\infty) \times \mathbb{R}$ by the same arguments) are locally up to $1/4$-Hölder continuous in time and $1/2$-Hölder continuous in space.

Chapter 5

The Stefan Problem for Reflected SPDEs

In this section we consider systems of the form

$$\begin{aligned}\frac{\partial u^1}{\partial t} &= \Delta u^1 + f_1(p(t)-x,t,u^1(t,p(t)-\cdot)) + \sigma_1(p(t)-x,t,u^1(t,p(t)-\cdot))\dot{W} + \eta^1 \\ \frac{\partial u^2}{\partial t} &= \Delta u^2 + f_2(x-p(t),t,u^2(t,p(t)+\cdot)) + \sigma_2(x-p(t),t,u^2(t,p(t)+\cdot))\dot{W} + \eta^2,\end{aligned} \tag{5.0.1}$$

where, at any time $t \geq 0$, u^1 is supported on $[p(t)-1, p(t)]$, u^2 is supported on $[p(t), p(t)+1]$ and $u^1(t,p(t)-1) = u^1(t,p(t)) = u^2(t,p(t)) = u^2(t,p(t)+1) = 0$. The point $p(t)$ evolves according to the equation

$$p'(t) = h\left(\frac{\partial u^1}{\partial x}(t,p(t)^-), \frac{\partial u^2}{\partial x}(t,p(t)^+)\right).$$

Here, $\dot{W}$ is a space-time white noise, f_i and σ_i, for $i = 1, 2$, are the drift and volatility functions, which are determined by the distance to the moving boundary $p(t)$ and (η^1, η^2) are reflection measures for the functions u^1 and u^2 respectively, keeping the profiles positive and satisfying the conditions

(i) $\int_0^\infty \int_{\mathbb{R}} u^1(t,x)\, \eta^1(\mathrm{d}t, \mathrm{d}x) = 0$, and

(ii) $\int_0^\infty \int_{\mathbb{R}} u^2(t,x)\, \eta^2(\mathrm{d}t, \mathrm{d}x) = 0$.

Therefore, the model here is similar to that of the previous section, with the crucial difference that the boundary dynamics are now driven by a Stefan condition i.e. a *local* condition in the SPDEs near the shared interface. This makes such equations more natural in contexts where it isn't sensible to think that the instantaneous motion of the boundary is affected by non-local factors. It also, however, introduces extra complexities into the analysis.

5.1 Notion of Solution and Statement of Main Theorem

We begin this section by defining the space $\mathscr{H}$. The solutions to our Stefan problem will be $\mathscr{H}$-valued processes.

Definition 5.1.1.

$$\mathscr{H} := \left\{ f \in C([0,1]) \mid f(0) = f(1) = 0, f'(0) \text{ exists} \right\}.$$

We equip $\mathscr{H}$ with the norm

$$\|f\|_{\mathscr{H}} := \sup_{x \in (0,1]} \left| \frac{f(x)}{x} \right|.$$

Remark 5.1.2. *$(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$ is a Banach space. In addition, we can characterise $\mathscr{H}$ as follows*

$$\mathscr{H} = \left\{ f \in C_0((0,1)) \mid \exists g \in C([0,1]) \text{ s.t. } f(x) = xg(x) \right\}$$

Clearly, for any $f \in \mathscr{H}$, the function $g \in C([0,1])$ such that $f(x) = xg(x)$ is unique, and we have that

$$\|f\|_{\mathscr{H}} = \|g\|_{\infty}.$$

Definition 5.1.3. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $\tilde{v}$ is a continuous $\mathscr{F}_t$-adapted process taking values in $\mathscr{H}$. Let $F : \mathscr{H} \to \mathscr{H}$ and $h : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$. For the $\mathscr{F}_t$-stopping time τ, we say that the pair (v, η) is a local solution to the reflected SPDE*

$$\frac{\partial v}{\partial t} = \Delta v + h(v(t,\cdot), \tilde{v}(t,\cdot)) \frac{\partial F(v)}{\partial x} + f(x,t,v(t,\cdot)) + \sigma(x,t,v(t,\cdot))\dot{W} + \eta$$

on $[0,\infty) \times [0,1]$ with Dirichlet boundary conditions $v(t,0) = v(t,1) = 0$ and initial data $v_0 \in \mathscr{H}$, with $v_0 \geq 0$, until time τ, if

(i) For every $x \in [0,1]$ and every $t \geq 0$, $v(t,x)$ is an $\mathscr{F}_t$-measurable random variable.

(ii) $v \geq 0$ almost surely.

(iii) $v\big|_{[0,\tau) \times [0,1]} \in C([0,\tau); \mathscr{H})$ almost surely.

(iv) $v(t,x) = \infty$ for every $t \geq \tau$ and $x \in [0,1]$ almost surely.

(v) η is a measure on $(0,1) \times [0,\infty)$ such that

(a) For every measurable map $\psi : [0,1] \times [0,\infty) \to \mathbb{R}$,

$$\int_0^t \int_0^1 \psi(x,s)\, \eta(dx, ds)$$

is $\mathscr{F}_t$-measurable.

(b) $\int_0^\infty \int_0^1 v(t,x)\, \eta(dx,dt) = 0$.

(vi) There exists a sequence of stopping times $\tau_n \uparrow \tau$ almost surely, such that for every $\varphi \in C_c^{1,2}([0,\infty)\times[0,1])$ such that $\varphi(s,0)=\varphi(s,1)=0$ for every $s\in[0,\infty)$,

$$\begin{aligned}\int_0^1 v(t\wedge\tau_n,x)\varphi(t,x)\,dx = &\int_0^1 v(0,x)\varphi(0,x)\,dx + \int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial\varphi}{\partial t}(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\,dx\,ds\\ &-\int_0^{t\wedge\tau_n}\int_0^1 h(v(s,\cdot),\tilde{v}(s,\cdot))F(v(s,\cdot))(x)\frac{\partial\varphi}{\partial x}(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 f(x,s,v(s,\cdot))\varphi(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 \sigma(x,s,v(s,\cdot))\varphi(s,x)W(dx,ds)\\ &+\int_0^{t\wedge\tau_n}\int_0^1 \varphi(s,x)\;\eta(ds,dx).\end{aligned} \tag{5.1.1}$$

for every $t\geq 0$ almost surely.

We say that a local solution is *maximal* if there does not exist a solution to the equation on a larger stochastic interval, and we say that a local solution is *global* if we can take $\tau_n=\infty$ in (5.1.1).

Recall Definitions 4.1.1 and 4.1.5, which provide us with notation allowing us to easily move between the relative frame (measured with respect to the current position of the boundary) and the fixed frame when discussing solutions to our equations.

Before finally stating our definition for solutions to (5.0.1), we first outline the conditions on our coefficients f_i, σ_i and h.

(i) For $i=1,2$, $f_i,\sigma_i : [0,1]\times[0,\infty)\times\mathscr{H}\to\mathbb{R}$ are measurable mappings.

(ii) $h:\mathscr{H}\times\mathscr{H}\to\mathbb{R}$ is a measurable map.

(iii) For $i=1,2$, f_i is such that for every $T>0$ there exists C_T such that for every $x\in[0,1]$ and every $t\in[0,T]$,

$$|f_i(x,t,0)|\leq C_T.$$

(iv) For $i=1,2$, f_i satisfies the local Lipschitz condition that for every $M,T>0$, there exists $C_{T,M}$ such that for every $x\in[0,1]$, every $t\in[0,T]$ and every $u,v\in\mathscr{H}$ with $\|u\|_{\mathscr{H}},\|v\|_{\mathscr{H}}\leq M$,

$$|f_i(x,t,u)-f_i(x,t,v)|\leq C_{T,M}\|u-v\|_{\mathscr{H}}.$$

(v) For $i=1,2$, σ_i is such that for every $T>0$ there exists C_T such that for every $x\in[0,1]$ and every $t\in[0,T]$

$$|\sigma_i(x,t,0)|\leq C_T x$$

(vi) For $i = 1, 2$, σ_i satisfies the local Lipschitz condition that for every $M, T > 0$, there exists $C_{T,M}$ such that for every $x \in [0,1]$, every $t \in [0,T]$ and every $u, v \in \mathscr{H}$ with $\|u\|_{\mathscr{H}}, \|v\|_{\mathscr{H}} \leq M$,

$$|\sigma_i(x,t,u) - \sigma_i(x,t,v)| \leq C_{T,M} x \|u - v\|_{\mathscr{H}}.$$

(vii) h satisfies the local Lipschitz condition that for every $M > 0$, there exists C_M such that for every $u_i, v_i \in \mathscr{H}$ with $\|u_i\|_{\mathscr{H}}, \|v_i\|_{\mathscr{H}} \leq M$,

$$|h(u_1, v_1) - h(u_2, v_2)| \leq C_M(\|u_1 - u_2\|_{\mathscr{H}} + \|v_1 - v_2\|_{\mathscr{H}}).$$

Remark 5.1.4. *A consequence of the conditions on f_i and σ_i is that they satisfy local linear growth conditions. That is, for every $T, M > 0$ there exists a constant $C_{T,M}$ such that for every $x \in [0,1]$, every $t \in [0,T]$ and every $u \in \mathscr{H}$ with $\|u\|_{\mathscr{H}} \leq M$,*

$$f_i(x,t,u) \leq C_{T,M}\left(1 + \|u\|_{\mathscr{H}}\right),$$

$$\sigma_i(x,t,u) \leq C_{T,M}\left(1 + \|u\|_{\mathscr{H}}\right).$$

Remark 5.1.5. *As a consequence of condition (vii) on h, we have that h is bounded on bounded sets. If $u, v \in \mathscr{H}$ with $\|u\|_{\mathscr{H}}, \|v\|_{\mathscr{H}} \leq M$, then*

$$\begin{aligned} |h(u,v)| \leq & |h(0,0)| + |h(u,v) - h(0,0)| \\ \leq & |h(0,0)| + C_M(\|u\|_{\mathscr{H}} + \|v\|_{\mathscr{H}}) \\ \leq & |h(0,0)| + 2MC_M. \end{aligned}$$

Remark 5.1.6. *The class of permitted functions for h includes as a subclass functions of the form*

$$h(u,v) = \tilde{h}\left(\frac{\partial u}{\partial x}(0), \frac{\partial v}{\partial x}(0)\right),$$

where $\tilde{h}$ is a Lipschitz function from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Remark 5.1.7. *The x term in the linear growth and Lipschitz conditions for σ ensure that the volatility will decay at least linearly as we approach the boundary i.e. as $x \downarrow 0$. We note that, in the case when $\sigma(x,t,u) = \tilde{\sigma}(x,t,u(x))$, the typical Lipschitz condition*

$$|\tilde{\sigma}(x,t,u) - \tilde{\sigma}(x,t,v)| \leq C|u - v|$$

implies the Lipschitz condition (vi). In particular, volatility functions of the form

$$\sigma(x,t,u) = \sigma_1(x) + \sigma_2(x)u(x),$$

where $\frac{\sigma_1}{x}, \sigma_2 \in L^\infty([0,1])$, are permitted.

We are now in position to state our definition for a solution to problem (5.0.1). Note that this definition is motivated by the calculations made in Section 4.1.1. Following this, we state the main result of this chapter, namely existence and uniqueness for maximal solutions to these equations.

Definition 5.1.8. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space, and $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $p_0 \in \mathbb{R}$, and for $i = 1, 2$, u_0^i is such that $u_0^i \circ (\Theta_{p_0}^i)^{-1} \in \mathscr{H}^+$. We say that the quintuple $(u^1, \eta^1, u^2, \eta^2, p)$ satisfies the reflected stochastic Stefan problem with initial data (u_0^1, u_0^2, p_0), up to the $\mathscr{F}_t$-stopping time τ if*

(i) $(v^1, \tilde{\eta}^1) := (u^1 \circ (\theta_p^1)^{-1}, \eta^1 \circ (\theta_p^1)^{-1})$ *solves, in the sense of Definition 5.1.3, the reflected SPDE*

$$\frac{\partial v^1}{\partial t} = \Delta v^1 - h(v^1(s,\cdot), v^2(s,\cdot))\frac{\partial v^1}{\partial x} + f_1(x,t,v^1(t,\cdot)) + \sigma_1(x,t,v^1(t,\cdot))\dot{W} + \tilde{\eta}^1$$

with Dirichlet boundary conditions $v^1(0) = v^1(1) = 0$ and initial data $u_0^1 \circ (\Theta_{p_0}^1)^{-1}$, until time τ.

(ii) $(v^2, \tilde{\eta}^2) := (u^2 \circ (\theta_p^2)^{-1}, \eta^2 \circ (\theta_p^2)^{-1})$ *solves, in the sense of Definition 5.1.3, the reflected SPDE*

$$\frac{\partial v^2}{\partial t} = \Delta v^2 + h(v^1(s,\cdot), v^2(s,\cdot))\frac{\partial v^2}{\partial x} + f_2(x,t,v^2(t,\cdot)) + \sigma_2(x,t,v^2(t,\cdot))\dot{W}^- + \tilde{\eta}^2$$

with Dirichlet boundary conditions $v^2(0) = v^2(1) = 0$ and initial data $u_0^2 \circ (\Theta_{p_0}^2)^{-1}$, until time τ.

(iii) $p(t) = p_0 + \int_0^t h(v^1(s,\cdot), v^2(s,\cdot))\, ds$.

We refer to $(v^1, \tilde{\eta}^1, v^2, \tilde{\eta}^2)$ as the solution to the moving boundary problem in the relative frame.

Theorem 5.1.9. *There exists a unique maximal solution to the reflected stochastic Stefan problem.*

Remark 5.1.10. *It is immediate from the definition that existence of a unique maximal solution to the reflected stochastic Stefan problem is equivalent to the existence of a unique maximal solution to the system of coupled reflected SPDEs*

$$\begin{aligned}
\frac{\partial v^1}{\partial t} &= \Delta v^1 - h(v^1(s,\cdot), v^2(s,\cdot))\frac{\partial v^1}{\partial x} + f_1(x,t,v^1(t,\cdot)) + \sigma_1(x,t,v^1(t,\cdot))\dot{W} + \eta^1, \\
\frac{\partial v^2}{\partial t} &= \Delta v^2 + h(v^1(s,\cdot), v^2(s,\cdot))\frac{\partial v^2}{\partial x} + f_2(x,t,v^2(t,\cdot)) + \sigma_2(x,t,v^2(t,\cdot))\dot{W}^- + \eta^2.
\end{aligned} \tag{5.1.2}$$

5.2 The Deterministic Obstacle Problem in $\mathscr{H}$

Recall the deterministic obstacle problem and Theorem 2.2.2. In this section we adapt this result, and show that solutions to the obstacle problem in the case when the obstacle lies in $C([0,T];\mathscr{H})$ are themselves in the space $C([0,T];\mathscr{H})$. We also prove the corresponding estimate, that we can control the $C([0,T];\mathscr{H})$-norm of the difference in the solutions by the $C([0,T];\mathscr{H})$-norm of the difference of the obstacle. This will be essential in proving existence for the system (5.0.1).

Theorem 5.2.1. *Suppose that $z_1, z_2 \in C([0,T];\mathscr{H})$. Let w_1, w_2 solve the corresponding parabolic obstacle problems with obstacles z_1 and z_2. Then we have that $w_1, w_2 \in C([0,T];\mathscr{H})$ and*

$$\|w_1 - w_2\|_{\mathscr{H},T} \leq \|z_1 - z_2\|_{\mathscr{H},T}.$$

Proof. Define $\delta := \|z_1 - z_2\|_{\mathscr{H},T}$. For $i = 1, 2$ and $\epsilon > 0$, let w_i^ϵ solve the following penalised equations

$$\frac{\partial w_i^\epsilon}{\partial t} = \Delta w_i^\epsilon + g_\epsilon(w_i^\epsilon - z),$$

with Dirichlet boundary conditions $w_i^\epsilon(t,0) = w_i^\epsilon(t,1) = 0$ and zero initial data, where we define

$$g_\epsilon(x) := \frac{1}{\epsilon}\arctan((x \wedge 0)^2).$$

Then we know that $w_i^\epsilon \uparrow w_i$ for $i = 1, 2$ (see, for example, the proof of Theorem 1.4 in [40]). We define $k^\epsilon(t,x) := w_1^\epsilon - w_2^\epsilon - x\delta$. Then k^ϵ solves the equation

$$\frac{\partial k^\epsilon}{\partial t} = \Delta k^\epsilon + g_\epsilon(w_1^\epsilon - z_1) - g_\epsilon(w_2^\epsilon - z_2),$$

with boundary conditions $k^\epsilon(t,0) = 0$, $k^\epsilon(t,1) = -\delta$ and negative initial data $k^\epsilon(0,x) = -x\delta$. Testing this equation with $(k^\epsilon)^+$ and integrating over space and time, we obtain that

$$\begin{aligned}\frac{1}{2}\|(k_T^\epsilon)^+\|_{L^2}^2 =& \frac{1}{2}\|(k_0^\epsilon)^+\|_{L^2}^2 - \int_0^T\int_0^1 \left|\frac{\partial (k_t^\epsilon)^+}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1 g_\epsilon(w_1^\epsilon(t,x) - z_1(t,x))(k^\epsilon(t,x))^+\mathrm{d}x\mathrm{d}t \\ &- \int_0^T\int_0^1 g_\epsilon(w_2^\epsilon(t,x) - z_2(t,x))(k^\epsilon(t,x))^+\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.2.1}$$

Note that $(k_0^\epsilon)^+ = 0$ and g_ϵ is a decreasing function. Also, on the set $k^\epsilon \geq 0$, $w_1^\epsilon - z_1 \geq w_2^\epsilon - z_2$. It follows that the right hand side of (5.2.1) is negative. Therefore, $(k^\epsilon)^+ = 0$, and so

$$w_1^\epsilon - w_2^\epsilon \leq x\delta = x\|z_1 - z_2\|_{\mathscr{H},T}.$$

Interchanging w_1^ϵ and w_2^ϵ, we obtain that, for every $\epsilon > 0$,

$$\sup_{t\in[0,T]}\sup_{x\in(0,1]}\left|\frac{w_1^\epsilon(t,x)}{x} - \frac{w_2^\epsilon(t,x)}{x}\right| \leq \|z_1 - z_2\|_{\mathscr{H},T}.$$

Letting $\epsilon \downarrow 0$, we deduce that

$$\sup_{t\in[0,T]}\sup_{x\in(0,1]}\left|\frac{w_1(t,x)}{x} - \frac{w_2(t,x)}{x}\right| \leq \|z_1 - z_2\|_{\mathscr{H},T}. \tag{5.2.2}$$

We now argue that the functions $w_i \in C([0,T];\mathscr{H})$. Note that we have $C([0,T];\mathscr{H})$ regularity for solutions to the obstacle problem provided that the obstacle is smooth (see, for example, Theorem 4.1 in [1]). Let z_n be a sequence of smooth obstacles such that $z_n \to z$ in $C([0,T];\mathscr{H})$, and denote by w_n the solution to the obstacle problem with obstacle z_n. Then, by (5.2.2), the sequence w_n is Cauchy in $C([0,T];\mathscr{H})$, and so converges in $C([0,T];\mathscr{H})$ to some $\tilde{w}$. We also have that

$$\sup_{t\in[0,T]}\sup_{x\in(0,1]}\left|\frac{w_n(t,x)}{x} - \frac{w(t,x)}{x}\right| \leq \|z_n - z\|_{\mathscr{H},T} \to 0.$$

Therefore, we have that $\tilde{w} = w$, and so $w \in C([0,T];\mathscr{H})$. This concludes the proof. □

5.3 Key Estimates

In order to prove existence for our reflected SPDE, we will perform a Picard iteration in the space $L^p(\Omega; C([0,T];\mathscr{H})) \times L^p(\Omega; C([0,T];\mathscr{H}))$. Having defined $(v^{1,n}, \eta^{1,n}, v^{2,n}, \eta^{2,n}, p^n)$, we will define our $(n+1)^{\text{th}}$ approximation of a solution to be given by the solutions of

$$\begin{aligned}
\frac{\partial v^{1,n+1}}{\partial t} &= \Delta v^{1,n+1} + f(x,t,v^{1,n}(t,\cdot)) + \frac{\partial v^{1,n}}{\partial x} h\left(v^{1,n}(t,\cdot), v^{2,n}(t,\cdot)\right) \\
&\quad + \sigma(x,t,v^{i,n}(t,\cdot))\dot{W} + \eta^{1,n+1}, \\
\frac{\partial v^{2,n+1}}{\partial t} &= \Delta v^{2,n+1} + f(x,t,v^{2,n}(t,\cdot)) - \frac{\partial v^{2,n}}{\partial x} h\left(v^{1,n}(t,\cdot), v^{2,n}(t,\cdot)\right) \\
&\quad + \sigma(x,t,v^{i,n}(t,\cdot))\dot{W}^- + \eta^{2,n+1}, \\
p^{n+1}(t) &= p_0 + \int_0^t h\left(v^{1,n}(s,\cdot), v^{2,n}(s,\cdot)\right) \mathrm{d}s.
\end{aligned}$$

In order to control the difference $\mathbb{E}\left[\|v^{i,n+1} - v^{i,n}\|^p_{\mathscr{H},T}\right]$, we will use Theorem 5.2.1. This gives

$$\mathbb{E}\left[\|v^{i,n+1} - v^{i,n}\|^p_{\mathscr{H},T}\right] \leq C_p \mathbb{E}\left[\|z^{i,n+1} - z^{i,n}\|^p_{\mathscr{H},T}\right],$$

where z_i^{n+1} solves

$$\frac{\partial z^{i,n+1}}{\partial t} = \Delta z^{i,n+1} \pm \frac{\partial u^{i,n}}{\partial x} p'(t) + f(t,x,u^{i,n}) + \sigma(t,x,u^{i,n}) \frac{\partial^2 W}{\partial x \partial t}.$$

By writing this in mild form, we see that we will require estimates on the terms

(i)
$$\int_0^t \int_0^1 \frac{1}{x} G(t-s,x,y) \left[f(s,x,v^{i,n}) - f(s,x,v^{i-1,n})\right] \mathrm{d}y \mathrm{d}s. \tag{5.3.1}$$

(ii)
$$\int_0^t \int_0^1 \frac{y}{x} G(t-s,x,y) \left[\frac{\sigma(s,x,v^{i,n}) - \sigma(s,x,v^{i-1,n})}{y}\right] W(\mathrm{d}y, \mathrm{d}s). \tag{5.3.2}$$

(iii)
$$\int_0^t \int_0^1 \frac{y}{x} \frac{\partial G}{\partial y}(t-s,x,y) \left[\frac{v^{i,n}(s,y)}{y}(p^n)'(s) - \frac{v^{i,n-1}(s,y)}{y}(p^{n-1})'(s)\right] \mathrm{d}y \mathrm{d}s. \tag{5.3.3}$$

Obtaining such estimates will be the focus in this section.

5.3.1 Heat Kernel Estimates

In this section we will state and prove the heat kernel estimates which will be crucial in proving existence and uniqueness for our equations. In particular, they allow us to show that terms appearing in the mild forms of our reflected SPDEs lie in the space $C([0,T];\mathscr{H})$, and also enable us to obtain estimates in $L^p(\Omega; C([0,T];\mathscr{H}))$ which form the basis of a Picard argument. As in Section 4, H is the Dirichlet heat kernel on $[0,1]$ and can be expressed as

$$H(t,x,y) := \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-y+2n)^2}{4t}\right) - \exp\left(-\frac{(x+y+2n)^2}{4t}\right)\right].$$

We then define for $t > 0$, $y \in [0,1]$ and $x \in (0,1]$

$$\tilde{H}(t,x,y) := \frac{y}{x} H(t,x,y).$$

For $t > 0$ and $y \in [0,1]$, we set

$$\tilde{H}(t,0,y) := y \frac{\partial H}{\partial x}(t,0,y).$$

We also define

$$\tilde{J}(t,x,y) := \frac{y}{x} \frac{\partial H}{\partial y}(t,x,y)$$

for $t \in [0,T]$, $x \in (0,1]$ and $y \in [0,1]$, and set

$$\tilde{J}(t,0,y) := y \frac{\partial^2 H}{\partial y \partial x}(t,0,y)$$

for $t \in [0,T]$ and $y \in [0,1]$.

Remark 5.3.1. *Note that*

$$\begin{aligned} H(t,x,y) =& \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(x-y)^2}{4t} \right) - \exp\left(-\frac{(x+y)^2}{4t} \right) \right] \\ &+ \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(x-y-2)^2}{4t} \right) - \exp\left(-\frac{(x+y-2)^2}{4t} \right) \right] + L(t,x,y), \end{aligned}$$

where L is a smooth function on $[0,T] \times [0,1] \times [0,1]$, vanishing on $t = 0$ and $x = 0, 1$. Since the arguments for bounding the second term on the right hand side are similar to those for bounding the first, and estimates for L are simple due to L being smooth, it is sufficient to prove the estimates for the first term,

$$G(t,x,y) := \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(x-y)^2}{4t} \right) - \exp\left(-\frac{(x+y)^2}{4t} \right) \right].$$

Proposition 5.3.2. *The following estimates hold for H, $\tilde{H}$ and $\tilde{J}$.*

1. *$\exists\, C_T > 0$ such that for $t \in [0,T]$,*

$$\sup_{x \in (0,1]} \int_0^1 \left| \frac{1}{x} H(t,x,y) \right| dy \leq \frac{C_T}{\sqrt{t}}.$$

2. *$\exists\, C_T > 0$ such that for $t \in [0,T]$,*

$$\sup_{x \in [0,1]} \int_0^1 \tilde{H}(s,x,y)^2 dy \leq \frac{C_T}{\sqrt{t}}.$$

3. *For $T > 0$ and $q \in (1,2)$, $\exists\, C_{T,q} > 0$ such that, for every $x, y \in [0,1]$*

$$\sup_{t \in [0,T]} \int_0^t \left[\int_0^1 \left(\tilde{H}(s,x,z) - \tilde{H}(t,y,z) \right)^2 dy \right]^q ds \leq C_{T,q} |x-y|^{(2-q)/3}.$$

4. For $T > 0$, $0 \leq s \leq t \leq T$ and $q \in (1,2)$, $\exists\, C_{T,q} > 0$ such that

$$\sup_{x\in[0,1]} \int_0^s \left[\int_0^1 (\tilde{H}(t-r,x,y) - \tilde{H}(s-r,x,y))^2 dy \right]^q dr \leq C_{T,q}|t-s|^{(2-q)/2}.$$

5. $\exists C_T > 0$ such that, for $t \in [0,T]$,

$$\sup_{x\in[0,1]} \int_0^1 \left| \tilde{J}(t,x,y) \right| dy \leq \frac{C}{\sqrt{t}}.$$

Proof. The proof for inequality (2) can be found in [61]. The other inequalities can be shown by adapting some of the arguments from [61], and their proofs are deferred to the appendix. □

5.3.2 Mild form Estimates and Continuity

We would now like to translate the estimates on the heat kernel into estimates on the terms (5.3.1), (5.3.2) and (5.3.3) appearing in the mild form for the problem. We would also like to ensure continuity of these terms. In the case of the drift and moving boundary terms in the mild form, this is essentially an immediate consequence of the heat kernel estimates. For the stochastic term, more work is required, as we need to control the L^p-norm of the supremum of the term over the space-time interval $[0,T]\times[0,1]$. We can achieve this by a standard application of the Garsia-Rodemich-Rumsey Lemma.

Proposition 5.3.3. *Let $p > 2$ and suppose that $f \in L^p(\Omega; L^\infty([0,T]\times[0,1]))$. Define $F(t,x)$ such that, for $t \in [0,T]$ and $x \in [0,1]$,*

$$F(t,x) := \int_0^t \int_0^1 H(t-s,x,y) f(s,y)\, dy\, dy.$$

Then we have that

1. *$F \in C([0,T]; \mathscr{H})$ almost surely.*
2. *For every $t \in [0,T]$,*

$$\mathbb{E}\left[\|F\|^p_{\mathscr{H},t} \right] \leq C_{p,T} \int_0^t \mathbb{E}\left[\|f\|^p_{\infty,s} \right] ds.$$

Proof. $F \in C([0,T]; \mathscr{H})$ is clear, since F is simply the solution to the heat equation with Dirichlet boundary conditions at $x = 0, 1$, initial data 0 and source term f. For the second part, we note that for $0 \leq s \leq t \leq T$ and $x \in (0,1]$,

$$\begin{aligned}
\left| \int_0^s \int_0^1 \frac{1}{x} H(s-r,x,y) f(r,y)\, \mathrm{d}y \mathrm{d}r \right|^p &\leq \left| \int_0^s \left(\int_0^1 \frac{1}{x} H(s-r,x,y) \mathrm{d}y \right) \|f\|_{\infty,r}\, \mathrm{d}r \right|^p \\
&\leq \left[\int_0^s \left(\int_0^1 \frac{1}{x} H(s-r,x,y) \mathrm{d}y \right)^q \mathrm{d}r \right]^{p/q} \times \int_0^s \|f\|^p_{\infty,r}\, \mathrm{d}r,
\end{aligned} \tag{5.3.4}$$

where $q = p/(p-1)$. By estimate (1) in Proposition 5.3.2, we have that

$$\left[\int_0^s \left(\int_0^1 \frac{1}{x} H(s-r,x,y)\mathrm{d}y\right)^q \mathrm{d}r\right]^{p/q} \leq C_{T,p}\left[\int_0^s \left(\frac{1}{\sqrt{s-r}}\right)^q \mathrm{d}r\right]^{p/q} \leq C_{T,p}.$$

Taking the supremum over $s \in [0,t]$ and $x \in (0,1]$ in the inequality (5.3.4) then gives that

$$\|F\|^p_{\mathscr{H},t} \leq C_{T,p}\int_0^t \|f\|^p_{\infty,r}\,\mathrm{d}r.$$

We can then take expectations of both sides to conclude the argument. □

Proposition 5.3.4. *Let $p > 3$ and suppose that $f \in L^p(\Omega; C([0,T];\mathscr{H}))$. Define $F(t,x)$ so that*

$$F(t,x) := \int_0^t \int_0^1 \frac{\partial H}{\partial y}(t-s,x,y) f(s,y)\,dy ds.$$

Then we have that

1. *$F \in C([0,T];\mathscr{H})$ almost surely.*
2. *For $t \in [0,T]$,*

$$\mathbb{E}\left[\|F\|^p_{\mathscr{H},t}\right] \leq C_{T,p}\int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},s}\right] ds.$$

Proof. We have that, for $x \in (0,1]$ and $t \in [0,T]$,

$$\begin{aligned}\frac{F(t,x)}{x} :=& \int_0^t \int_0^1 \frac{y}{x}\frac{\partial H}{\partial y}(t-s,x,y)\frac{f(s,y)}{y}\mathrm{d}y\mathrm{d}s \\ =& \int_0^t \int_0^1 \frac{y}{x}\frac{\partial H}{\partial y}(s,x,y)\frac{f(t-s,y)}{y}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Note that, by the estimates in Proposition A.1.5, we know that $\frac{F(t,x)}{x}$ is well defined by our integral expression and continuous on $[0,T]\times(0,1]$. So it is left to show that $\frac{F(t,x)}{x}$ can be extended to a continuous function on $[0,T]\times[0,1]$. In the following arguments, we focus on the G component of H (see Remark 5.3.1). Suppose that (t_n,x_n) is a sequence in $[0,T]\times(0,1]$ converging to $(t,0)$. Note that

$$\begin{aligned}&\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\frac{y}{x_n}\left|\frac{\partial G}{\partial y}(s,x_n,y)\frac{f(t_n-s,y)}{y}\right|\mathrm{d}y\mathrm{d}s \\ \leq& \|f\|_{\mathscr{H},T}\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\frac{y}{x_n}\left|\frac{\partial G}{\partial y}(s,x_n,y)\right|\mathrm{d}y\mathrm{d}s \\ \leq& 2\|f\|_{\mathscr{H},T}\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\left|\frac{\partial G}{\partial y}(s,x_n,y)\right|\mathrm{d}y\mathrm{d}s.\end{aligned}$$

For $p > 3$, we have that this is at most

$$2\|f\|_{\mathscr{H},T}\left(\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\mathrm{d}y\mathrm{d}s\right)^{1/p} \times \left(\int_0^{t_n}\int_0^1 \left|\frac{\partial G}{\partial y}(s,x_n,y)\right|^q \mathrm{d}y\mathrm{d}s\right)^{1/q}, \tag{5.3.5}$$

where $q = p/(p-1)$. Note that $q \in (1, 3/2)$, and so

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} \int_0^t \int_0^1 \left| \frac{\partial G}{\partial y}(s, x, y) \right|^q \mathrm{d}y\mathrm{d}s < \infty.$$

We therefore have that (5.3.5) converges to zero as $n \to \infty$. Now note that

$$\mathbb{1}_{\{y\geq 2x_n, s\in[0,t_n]\}} \frac{y}{x_n} \frac{\partial G}{\partial y}(s, x_n, y) \frac{f(t_n - s, y)}{y} \to \mathbb{1}_{\{s\in[0,t]\}} y \frac{\partial^2 G}{\partial y \partial x}(s, 0, y) \frac{f(t-s, y)}{y}$$

almost everywhere in $[0, T] \times [0, 1]$. In addition we have that for $s \in [0, T]$, $y \in [0, 1]$ and $n \geq 1$,

$$\begin{aligned}
&\left| \frac{y}{x_n} \frac{\partial G}{\partial y}(s, x_n, y) \right| \\
&= C \left| \frac{y}{x_n s\sqrt{s}} \left((x_n - y) e^{-(x_n-y)^2/4s} + (x_n + y) e^{-(x_n+y)^2/4s} \right) \right| \\
&\leq C \left| \frac{y}{s\sqrt{s}} \left(e^{-(x_n-y)^2/4s} + e^{-(x_n+y)^2/4s} \right) \right| + C \left| \frac{y^2}{x_n s\sqrt{s}} \left(e^{-(x_n-y)^2/4s} - e^{-(x_n+y)/4s} \right) \right| \\
&\leq C \left| \frac{y}{s\sqrt{s}} \left(e^{-(x_n-y)^2/4s} + e^{-(x_n+y)^2/4s} \right) \right| + C \left| \frac{y^2}{x_n s\sqrt{s}} e^{-(x_n-y)^2/4s} \left(1 - e^{-x_n y/s} \right) \right|.
\end{aligned} \tag{5.3.6}$$

On the set $\{y \geq 2x_n\}$, we have that

$$e^{-(x_n-y)^2/4s} \leq e^{-y^2/16s}.$$

Note also that, for $x \geq 0$, $1 - e^{-x} \leq x$. Therefore, we can bound (5.3.6) on the set $\{y \geq 2x_n\}$ by

$$\left| \frac{y}{s\sqrt{s}} \left(e^{-y^2/16s} + e^{-y^2/4s} \right) \right| + \left| \frac{y^2}{x_n s\sqrt{s}} e^{-y^2/16s} \times \frac{x_n y}{s} \right| \leq C \left(\frac{y}{s\sqrt{s}} e^{-y^2/16s} + \frac{y^3}{s^2\sqrt{s}} e^{-y^2/16s} \right).$$

It follows that, for every n,

$$\left| \mathbb{1}_{\{y\geq 2x_n, s\in[0,t_n]\}} \frac{y}{x_n} \frac{\partial G}{\partial y}(s, x_n, y) \frac{f(t_n - s, y)}{y} \right| \leq C \left(\frac{y}{s\sqrt{s}} e^{-y^2/16s} + \frac{y^3}{s^2\sqrt{s}} e^{-y^2/16s} \right) \|f\|_{\mathscr{H},T}$$

This function is integrable, and so we can apply the DCT to obtain that

$$\begin{aligned}
&\int_0^{t_n} \int_0^1 \mathbb{1}_{\{y\geq 2x_n\}} \frac{y}{x_n} \frac{\partial G}{\partial y}(t_n - s, x_n, y) \frac{f(s, y)}{y} \mathrm{d}y\mathrm{d}s \\
&= \int_0^{t_n} \int_0^1 \mathbb{1}_{\{y\geq 2x_n\}} \frac{y}{x_n} \frac{\partial G}{\partial y}(s, x_n, y) \frac{f(t_n - s, y)}{y} \mathrm{d}y\mathrm{d}s \to \int_0^t \int_0^1 \frac{\partial^2 G}{\partial y \partial x}(s, 0, y) f(t-s, y) \mathrm{d}y\mathrm{d}s \\
&= \int_0^t \int_0^1 \frac{\partial^2 G}{\partial y \partial x}(t - s, 0, y) f(s, y) \mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Another application of the DCT (note that we can use the same dominating function as in (**??**)) gives that, if t_n is a sequence in $[0, T]$ and $t_n \to t$, then

$$\int_0^{t_n} \int_0^1 \frac{\partial^2 G}{\partial y \partial x}(t_n - s, 0, y) f(s, y) \mathrm{d}y\mathrm{d}s \to \int_0^t \int_0^1 \frac{\partial^2 G}{\partial y \partial x}(t - s, 0, y) f(s, y) \mathrm{d}y\mathrm{d}s.$$

So we have shown that $F \in C([0,T];\mathscr{H})$ almost surely. We now prove the bound (2). Let $0 \le s \le t \le T$. For $p > 2$, we have that

$$\left|\frac{F(s,x)}{x}\right| \le \left(\int_0^s \left[\int_0^1 \left|\frac{y}{x}\frac{\partial H}{\partial y}(r,x,y)\right| \mathrm{d}y\right]^q \mathrm{d}r\right)^{1/q} \times \left(\int_0^s \|f\|^p_{\mathscr{H},r} \mathrm{d}r\right)^{1/p}.$$

Arguing as before, considering the cases $y < 2x$ and $y \ge 2x$ separately, we have that

$$\left|\frac{y}{x}\frac{\partial H}{\partial y}(r,x,y)\right| \le C\left(\left|\frac{\partial H}{\partial y}(r,x,y)\right| + \frac{y}{r\sqrt{r}}e^{-y^2/16r} + \frac{y^3}{r^2\sqrt{r}}e^{-y^2/16r}\right).$$

For $p > 2$ and corresponding $q = p/(p-1) \in (1,2)$, and we have that

$$\begin{aligned}\int_0^s \left[\int_0^1 \left(\left|\frac{\partial H}{\partial y}(r,x,y)\right| + \frac{y}{r\sqrt{r}}e^{-y^2/16r} + \frac{y^3}{r^2\sqrt{r}}e^{-y^2/16r}\right)\mathrm{d}y\right]^q \mathrm{d}r \\ \le C\int_0^s \left(\frac{1}{\sqrt{r}}\right)^q \mathrm{d}r \le C_{T,p} < \infty.\end{aligned}$$

It follows that, for $p > 2$,

$$\left|\frac{F(s,x)}{x}\right| \le C_{T,p}\left(\int_0^s \|f\|^p_{\mathscr{H},r}\mathrm{d}r\right)^{1/p}.$$

Therefore,

$$\sup_{s\in[0,t]}\sup_{x\in(0,1]}\left|\frac{F(s,x)}{x}\right|^p \le C_{T,p}\int_0^s \|f\|^p_{\mathscr{H},r}\mathrm{d}r.$$

The result then follows by taking expectations. □

Proposition 5.3.5. *Suppose that $p > 22$ and let $f \in L^p(\Omega; L^\infty([0,T];\mathscr{H}))$. Define $F(t,x)$ such that, for $t \in [0,T]$ and $x \in [0,1]$,*

$$F(t,x) := \int_0^t \int_0^1 H(t-s,x,y)f(s,y)W(dy,ds).$$

Then we have that

1. *$F \in C([0,T];\mathscr{H})$ almost surely.*
2. *For $t \in [0,T]$,*

$$\mathbb{E}\left[\|F\|^p_{\mathscr{H},t}\right] \le C_{T,p}\int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},s}\right] ds.$$

Proof. In this case, we will need to apply the Garsia-Rodemich-Rumsey Lemma (in the form of Corollary A.3 in [11]) to obtain continuity of $\frac{F(t,x)}{x}$, and to suitably control the supremum of this process. Define for $t \in [0,T]$ and $x \in [0,1]$

$$K(t,x) := \int_0^t \int_0^1 \tilde{H}(t-s,x,y)\frac{f(s,y)}{y}W(\mathrm{d}y,\mathrm{d}s).$$

Note that, for $t \in [0,T]$ and $x \in (0,1]$, we then have that $K(t,x) = F(t,x)/x$. For $x, z \in [0,1]$ and $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}\mathbb{E}\left[|K(t,x) - K(s,z)|^p\right] \leq & \mathbb{E}\left[\left|\int_s^t \int_0^1 \tilde{H}(t-r,x,y)\frac{f(r,y)}{y} W(\mathrm{d}y,\mathrm{d}r)\right|^p\right] \\ & + \mathbb{E}\left[\left|\int_0^s \int_0^1 \left[\tilde{H}(t-r,x,y) - \tilde{H}(s-r,z,y)\right]\frac{f(r,y)}{y} W(\mathrm{d}y,\mathrm{d}r)\right|^p\right].\end{aligned} \tag{5.3.7}$$

Applying Burkholder's inequality gives that the first term on the right hand side is at most

$$\mathbb{E}\left[\left|\int_s^t \int_0^1 \left(\tilde{H}(t-r,x,y)\frac{f(r,y)}{y}\right)^2 \mathrm{d}y\mathrm{d}r\right|^{p/2}\right] \leq \mathbb{E}\left[\left|\int_s^t \left(\int_0^1 \tilde{H}(t-r,x,y)^2 \mathrm{d}y\right)\|f\|^2_{\mathscr{H},r}\mathrm{d}r\right|^{p/2}\right].$$

An application of Hölder's inequality then bounds this by

$$\left[\int_s^t \left(\int_0^1 \tilde{H}(t-r,x,y)^2\mathrm{d}y\right)^{p/(p-2)}\mathrm{d}r\right]^{(p-2)/2} \times \int_s^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\mathrm{d}r\right].$$

By estimate (2) in Proposition 5.3.2, this can be bounded by

$$C|t-s|^{(p-4)/4} \times \int_s^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\mathrm{d}r\right].$$

Arguing similarly for the second term in (5.3.7), making use of the other estimates (3) and (4) in Proposition 5.3.2, we see that

$$\mathbb{E}\left[|K(t,x) - K(s,z)|^p\right] \leq C_{T,p}\left(|t-s|^{1/4} + |x-z|^{1/6}\right)^{p-4} \times \int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\mathrm{d}r.$$

In particular, for $x, z \in [0,1]$ and $0 \leq s \leq \tau \leq t \leq T$,

$$\begin{aligned}\mathbb{E}\left[|K(\tau,x) - K(s,z)|^p\right] &\leq C_{T,p}\left(|\tau-s|^{1/2} + |x-z|\right)^{(p-4)/6} \times \int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\mathrm{d}r \\ &= C_{T,p}\left(|\tau-s|^{1/2} + |x-z|\right)^{\frac{p-22}{6}+3} \times \int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\mathrm{d}r\end{aligned}$$

Applying the Garsia-Rodemich-Rumsey Lemma in the form of Corollary A.3 from [11] we see that there exists a random variable $X_t \geq 0$ such that

1. $\mathbb{E}\left[X_t^p\right] \leq C_{T,p}\int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\mathrm{d}r$.
2. For $s, \tau \in [0,t]$,

$$|K(\tau,x) - K(s,y)| \leq X_t\left(|\tau-s|^{1/4} + |x-y|^{1/6}\right)^{(p-22)/p}$$

In particular, setting $t = T$, we see that K is continuous on $[0,T]\times[0,1]$ almost surely, from which it follows that $F \in C([0,T];\mathscr{H})$ almost surely. In addition, we have that for $t \in [0,T]$, $\tau \in [0,t]$ and $x \in [0,1]$

$$|K(\tau,x)| = |K(\tau,x) - K(0,x)| \leq C_{T,p} \times X_t.$$

It follows that

$$\mathbb{E}\left[\|F\|^p_{\mathscr{H},t}\right] = \mathbb{E}\left[\|K\|^p_{\infty,t}\right] \leq C_{T,p}\int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\,\mathrm{d}r.$$

□

5.4 Existence and Uniqueness

We are now in position to prove existence and uniqueness for our Stefan problem. We once again do this via a Picard argument for a truncated version of the problem. Existence for the main problem is then deduced by concatenation. Before doing so, the following truncation map is introduced.

Definition 5.4.1. *For $M > 0$, we define the map $F_M : \mathscr{H} \to \mathscr{H}$ such that*

$$F_M(u)(x) := \begin{cases} x\left[\frac{u(x)}{x} \wedge M\right] & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

Remark 5.4.2. *$F_M(u)$ is a well-defined map from $\mathscr{H}$ to $\mathscr{H}$ i.e. $F_M(u) \in \mathscr{H}$ for $u \in \mathscr{H}$. Continuity on $(0,1]$ is clear, and we have defined $F_M(u)(0) = 0$. We therefore only need to argue the existence of a derivative for $F_M(u)$ at $x = 0$. Note that, for $x > 0$,*

$$\frac{F_M(u)(x)}{x} = \left[\frac{u(x)}{x} \wedge M\right].$$

This converges as $x \downarrow 0$, since $\frac{u(x)}{x}$ converges.

Theorem 5.4.3. *Fix some $M > 0$. Let $v_0^1, v_0^2 \in \mathscr{H}^+$. Then there exists a unique solution $(v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ to the system of coupled SPDEs*

$$\begin{aligned}
\frac{\partial v_M^1}{\partial t} =& \Delta v_M^1 - h_M\left(v_M^1(s,\cdot), v_M^2(s,\cdot)\right)\frac{\partial(F_M(v_M^1))}{\partial x} + f_{1,M}(x,t,v_M^1(t,\cdot)) \\
&+ \sigma_{1,M}(x,t,v_M^1(t,\cdot))\dot{W} + \eta_M^1, \\
\frac{\partial v_M^2}{\partial t} =& \Delta v_M^2 + h_M\left(v_M^1(s,\cdot), v_M^2(s,\cdot)\right)\frac{\partial(F_M(v_M^2))}{\partial x} + f_{2,M}(x,t,v_M^2(t,\cdot)) \\
&+ \sigma_{2,M}(x,t,v_M^2(t,\cdot))\dot{W}^- + \eta_M^2,
\end{aligned}$$

where the Dirichlet conditions $v_M^1(t,0) = v_M^1(t,1) = v_M^2(t,0) = v_M^2(t,1) = 0$ are satisfied, and the equation has initial data $(v_0^1, v_0^2) \in \mathscr{H}^+ \times \mathscr{H}^+$. The functions $f_{i,M}$, $\sigma_{i,M}$ and h_M here are given by

$$f_{i,M}(x,t,u) := f_i(x,t,F_M(u)),$$

$$\sigma_{i,M}(x,t,u) := \sigma_i(x,t,F_M(u)),$$

$$h_M(u,v) := h(F_M(u), F_M(v)).$$

Proof. Note that existence and uniqueness for the problem on the infinite time interval $[0,\infty)$ follow from existence and uniqueness on the finite time intervals $[0,T]$, for $T>0$. Fix some $T>0$ and some $p>22$. We perform a Picard iteration in the space $L^p(\Omega, C([0,T];\mathscr{H}))\times L^p(\Omega, C([0,T];\mathscr{H}))$. Let $v^{i,0}(t,x)=v_0^i(x)$ for $i=1,2$. Given $(v^{1,n},v^{2,n})\in L^p(\Omega;C([0,T];\mathscr{H}))\times L^p(\Omega;C([0,T];\mathscr{H}))$, we define $z^{1,n+1}$ and $z^{2,n+1}$ to solve the equations

$$\begin{aligned}\frac{\partial z^{1,n+1}}{\partial t} =&\Delta z^{1,n+1} - h_M\left(v^{1,n}(t,\cdot),v^{2,n}(t,\cdot)\right)\frac{\partial(F_M(v^{1,n}))}{\partial x}\\ &+f_{1,M}(x,t,v^{1,n}(t,\cdot))+\sigma_{1,M}(x,t,v^{1,n}(t,\cdot))\dot{W},\end{aligned}\tag{5.4.1}$$

and

$$\begin{aligned}\frac{\partial z^{2,n+1}}{\partial t} =&\Delta z^{2,n+1} + h_M\left(v^{1,n}(t,\cdot),v^{2,n}(t,\cdot)\right)\frac{\partial(F_M(v^{2,n}))}{\partial x}\\ &+f_{2,M}(x,t,v^{2,n}(t,\cdot))+\sigma_{2,M}(x,t,v^{2,n}(t,\cdot))\dot{W}^-,\end{aligned}\tag{5.4.2}$$

with initial data v_0^1 and v_0^2 respectively. We then define, for $i=1,2$, $w^{i,n+1}$ to be given by the solution to the obstacle problem

$$\begin{aligned}&\frac{\partial w_{n+1}^i}{\partial t}=\Delta w_{n+1}^i+\eta_{n+1}^i,\qquad w_{n+1}^i\geq -z_{n+1}^i,\\ &\int_0^T\int_0^1\left[w_{n+1}^i(t,x)+z_{n+1}^i(t,x)\right]\eta_{n+1}^i(\mathrm{d}x,\mathrm{d}t)=0,\end{aligned}$$

and set our $(n+1)^{\text{th}}$ approximate solution to be given by $(v^{1,n+1},v^{2,n+1}):=(z^{1,n+1}+w^{1,n+1},z^{2,n+1}+w^{2,n+1})$. For ease of notation, we define here

$$g_n(t,x):=h_M\left(v^{1,n}(t,\cdot),v^{2,n}(t,\cdot)\right)F_M(v^{1,n}(t,\cdot))(x).$$

Writing $z^{1,n+1}$ in mild form, we have that

$$\begin{aligned}z^{1,n+1}(t,x)=&\int_0^1 H(t,x,y)v_0^1(y)\mathrm{d}y\\ &+\int_0^t\int_0^1\frac{\partial H}{\partial y}(t-s,x,y)g_n(s,y)\mathrm{d}y\mathrm{d}s\\ &+\int_0^t\int_0^1 H(t-s,x,y)f_{1,M}(y,s,v^{1,n}(s,\cdot))\mathrm{d}x\mathrm{d}s\\ &+\int_0^t\int_0^1 H(t-s,x,y)\sigma_{1,M}(y,s,v^{1,n}(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s).\end{aligned}$$

Note that, by Propositions 5.3.3, 5.3.4 and 5.3.5, we can see from this expression that $z^{1,n+1}\in C([0,T];\mathscr{H})$ almost surely. Using the analogous expression for $z^{1,n}$ and taking the difference,

we see that for $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\|z^{1,n+1} - z^{1,n}\|^p_{\mathscr{H},t}\right] \\
\leq &\mathbb{E}\left[\left\|\int_0^t \int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)\left[g_n(s,y) - g_{n-1}(s,y)\right] \mathrm{d}y\mathrm{d}s\right\|^p_{\mathscr{H},t}\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t \int_0^1 H(t-s,x,y)\left[f_{1,M}(y,s,v^{1,n}(s,\cdot)) - f_{1,M}(y,s,v^{1,n-1}(s,\cdot))\right] \mathrm{d}x\mathrm{d}s\right\|^p_{\mathscr{H},t}\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t \int_0^1 H(t-s,x,y)\left[\sigma_{1,M}(y,s,v^{1,n}(s,\cdot)) - \sigma_{1,M}(y,s,v^{1,n-1}(s,\cdot))\right] \mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right\|^p_{\mathscr{H},t}\right].
\end{aligned} \tag{5.4.3}$$

Note that

$$\|g_n - g_{n-1}\|_{\mathscr{H},s} \leq C_{M,h}\left(\|v^{1,n} - v^{1,n-1}\|_{\mathscr{H},s} + \|v^{2,n} - v^{2,n-1}\|_{\mathscr{H},s}\right). \tag{5.4.4}$$

By making use of Propositions 5.3.3, 5.3.4 and 5.3.5, we are able to bound (5.4.3). We obtain that, for $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\left[\|z^{1,n+1} - z^{1,n}\|^p_{\mathscr{H},t}\right] \leq &C_{T,p}\int_0^t \mathbb{E}\left[\|f_{1,M}(\cdot,s,v^{1,n}(s,\cdot)) - f_{1,M}(\cdot,s,v^{1,n-1}(s,\cdot))\|^p_\infty\right] \mathrm{d}s \\
&+ \int_0^t \mathbb{E}\left[\|g_n - g_{n-1}\|^p_{\mathscr{H},s}\right] \mathrm{d}s \\
&+ \int_0^t \mathbb{E}\left[\|\sigma_{1,M}(\cdot,s,v^{1,n}(s,\cdot)) - \sigma_{1,M}(\cdot,s,v^{1,n-1}(s,\cdot))\|^p_{\mathscr{H}}\right] \mathrm{d}s.
\end{aligned}$$

By the Lipschitz-type conditions on f, σ and the inequality (5.4.4), we see that this is at most

$$C_{M,h,T,p}\int_0^t \mathbb{E}\left[\|v^{1,n} - v^{1,n-1}\|^p_{\mathscr{H},s} + \|v^{2,n} - v^{2,n-1}\|^p_{\mathscr{H},s}\right] \mathrm{d}s.$$

By arguing in the same way, we obtain the same bound for $\mathbb{E}\left[\|z^{2,n+1} - z^{2,n}\|^p_{\mathscr{H},t}\right]$. Adding these together gives that

$$\begin{aligned}
&\mathbb{E}\left[\|z^{1,n+1} - z^{1,n}\|^p_{\mathscr{H},t}\right] + \mathbb{E}\left[\|z^{2,n+1} - z^{2,n}\|^p_{\mathscr{H},t}\right] \\
&\leq C_{M,h,T,p}\int_0^t \mathbb{E}\left[\|v^{1,n} - v^{1,n-1}\|^p_{\mathscr{H},s}\right] + \mathbb{E}\left[\|v^{2,n} - v^{2,n-1}\|^p_{\mathscr{H},s}\right] \mathrm{d}s.
\end{aligned} \tag{5.4.5}$$

By Theorem 5.2.1, we have that for $i = 1, 2$,

$$\|w^{i,n+1} - w^{i,n}\|_{\mathscr{H},t} \leq \|z^{i,n+1} - z^{i,n}\|_{\mathscr{H},t}$$

almost surely. Therefore, for $i = 1, 2$,

$$\|v^{i,n+1} - v^{i,n}\|_{\mathscr{H},t} \leq 2\|z^{i,n+1} - z^{i,n}\|_{\mathscr{H},t} \tag{5.4.6}$$

almost surely. It follows from (5.4.5) and (5.4.6) that

$$\begin{aligned}
&\mathbb{E}\left[\|v^{1,n+1} - v^{1,n}\|^p_{\mathscr{H},t}\right] + \mathbb{E}\left[\|v^{2,n+1} - v^{2,n}\|^p_{\mathscr{H},t}\right] \\
&\leq C_{M,h,T,p}\int_0^t \mathbb{E}\left[\|v^{1,n} - v^{1,n-1}\|^p_{\mathscr{H},s}\right] + \mathbb{E}\left[\|v^{2,n} - v^{2,n-1}\|^p_{\mathscr{H},s}\right] \mathrm{d}s.
\end{aligned}$$

By iterating this inequality, we obtain that, for $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\|v^{1,n+1}-v^{1,n}\|^p_{\mathscr{H},t}+\|v^{2,n+1}-v^{2,n}\|^p_{\mathscr{H},t}\right]\\
\leq& C_{M,p,T}\int_0^t \mathbb{E}\left[\|v^{1,n}-v^{1,n-1}\|^p_{\mathscr{H},s}+\|v^{2,n}-v^{2,n-1}\|^p_{\mathscr{H},s}\right]\mathrm{d}s\\
\leq& C^2_{M,p,T}\int_0^t\int_0^s \mathbb{E}\left[\|v^{1,n-1}-v^{1,n-2}\|^p_{\mathscr{H},u}+\|v^{2,n-1}-v^{2,n-2}\|^p_{\mathscr{H},u}\right]\mathrm{d}u\,\mathrm{d}s\\
=& C^2_{M,p,T}\int_0^t\int_u^t \mathbb{E}\left[\|v^{1,n-1}-v^{1,n-2}\|^p_{\mathscr{H},u}+\|v^{2,n-1}-v^{2,n-2}\|^p_{\mathscr{H},u}\right]\mathrm{d}s\,\mathrm{d}u\\
=& C^2_{M,p,T}\int_0^t \mathbb{E}\left[\|v^{1,n-1}-v^{1,n-2}\|^p_{\mathscr{H},u}+\|v^{2,n-1}-v^{2,n-2}\|^p_{\mathscr{H},u}\right](t-u)\,\mathrm{d}u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\|v^{1,n+1}-v^{1,n}\|^p_{\mathscr{H},T}+\|v^{2,n+1}-v^{2,n}\|^p_{\mathscr{H},T}\right]\\
&\leq C^n_{M,p,T}\int_0^T \mathbb{E}\left[\|v^{1,1}-v^{1,0}\|^p_{\mathscr{H},s}+\|v^{2,1}-v^{2,0}\|^p_{\mathscr{H},s}\right]\frac{(T-s)^{n-1}}{(n-1)!}\mathrm{d}s\\
&\leq C^n_{M,p,T}\times\mathbb{E}\left[\|v^{1,1}-v^{1,0}\|^p_{\mathscr{H},T}+\|v^{2,1}-v^{2,0}\|^p_{\mathscr{H},T}\right]\frac{T^n}{n!}.
\end{aligned}$$

Hence, for $m>n\geq 1$, we have that

$$\begin{aligned}
&\mathbb{E}\left[\|v^{1,m}-v^{1,n}\|^p_{\mathscr{H},T}+\|v^{2,m}-v^{2,n}\|^p_{\mathscr{H},T}\right]^{1/p}\\
&\leq\sum_{k=n}^{m-1}\left[\frac{\tilde{C}^k_{M,p,T}T^k}{k!}\right]^{1/p}\mathbb{E}\left[\|v^1_1-v^1_0\|^p_{\mathscr{H},T}+\|v^2_1-v^2_0\|^p_{\mathscr{H},T}\right]^{1/p}\to 0.
\end{aligned}$$

as $m,n\to\infty$. Therefore, the sequence $(v^{1,n},v^{2,n})$ is Cauchy in the space $L^p(\Omega;C([0,T];\mathscr{H})))^2$ and so converges to some pair (v^1_M,v^2_M). We now verify that this is indeed a solution to our evolution equation. Let (z^1_M,z^2_M) solve the SPDEs

$$\begin{aligned}
\frac{\partial z^1_M}{\partial t}&=\Delta z^1_M-p'(t)\frac{\partial(F_M(v^1_M))}{\partial x}+f_{1,M}(x,t,v^1_M)+\sigma_{1,M}(x,t,v^1_M)\dot{W},\\
\frac{\partial z^2_M}{\partial t}&=\Delta z^2_M+p'(t)\frac{\partial(F_M(v^2_M))}{\partial x}+f_{2,M}(x,t,v^2_M)+\sigma_{2,M}(x,t,v^2_M)\dot{W}^-,
\end{aligned}$$

where

$$p(t):=p_0+\int_0^t h_M\left(v^1_M(s,\cdot),v^2_M(s,\cdot)\right)\,\mathrm{d}s.$$

We then define (w^i_M,η^i_M) as solutions to the obstacle problem, with obstacles $-z^i_M$, and set $\tilde{v}^i_M:=z^i_M+w^i_M$. Therefore, we have that

$$\begin{aligned}
\frac{\partial\tilde{v}^1_M}{\partial t}&=\Delta\tilde{v}^1_M-p'(t)\frac{\partial(F_M(v^1_M))}{\partial x}+f_{1,M}(x,t,v^1_M)+\sigma_{1,M}(x,t,v^1_M)\dot{W}+\eta^1_M,\\
\frac{\partial\tilde{v}^2_M}{\partial t}&=\Delta\tilde{v}^2_M+p'(t)\frac{\partial(F_M(v^2_M))}{\partial x}+f_{2,M}(x,t,v^2_M)+\sigma_{2,M}(x,t,v^2_M)\dot{W}+\eta^2_M.
\end{aligned}$$

By reproducing the same argument as that used to achieve the estimate (5.4.6), we are able to show that

$$\begin{aligned}
&\mathbb{E}\left[\|\tilde{v}^1_M-v^{1,n}\|^p_{\mathscr{H},T}+\|\tilde{v}^2_M-v^{2,n}\|^p_{\mathscr{H},T}\right]\\
&\leq C_{M,h,T,p}\int_0^T\mathbb{E}\left[\|v^1_M-v^{1,n}\|^p_{\mathscr{H},s}+\|v^2_M-v^{2,n}\|^p_{\mathscr{H},s}\right]\mathrm{d}s\to 0.
\end{aligned}$$

It follows that $\tilde{v}_M^1 = v_M^1$ in $L^p(\Omega; C([0,T];\mathscr{H}))$. The same applies to v_M^2, so it follows that the pair (v_M^1, v_M^2), together with the reflection measures (η_M^1, η_M^2), do indeed satisfy our problem.

Uniqueness follows by essentially the same argument. Given two solutions with the same initial data, (v_1^1, v_1^2) and (v_2^1, v_2^2) (together with their reflection measures), we argue as before to obtain that, for $t \in [0,T]$,

$$\mathbb{E}\left[\|v_1^1 - v_2^1\|_{\mathscr{H},t}^p + \|v_1^2 - v_2^2\|_{\mathscr{H},t}^p\right] \leq \int_0^t \mathbb{E}\left[\|v_1^1 - v_2^1\|_{\mathscr{H},s}^p + \|v_1^2 - v_2^2\|_{\mathscr{H},s}^p\right] \mathrm{d}s.$$

The equivalence then follows by Gronwall's inequality. □

Before stating and proving our main theorem, we first prove the following technical result which gives us a universal representation for terminal times of maximal solutions, and a canonical localising sequence.

Proposition 5.4.4. *Let $(v^1, \eta^1, v^2, \eta^2)$ be a maximal solution to the moving boundary problem in the relative frame. Then we have that the terminal time, τ, is given by*

$$\tau := \sup_{M>0} \tau_M,$$

where

$$\tau_M := \inf\left\{t \geq 0 \mid \|v^1\|_{\mathscr{H},t} + \|v^2\|_{\mathscr{H},s} \geq M\right\}.$$

Furthermore, τ_M can be taken as a localising sequence for the solution.

Proof. Let σ_n be a localising sequence for the maximal solution $(v^1, \eta^1, v^2, \eta^2)$. Define $\sigma := \sup_{n\geq 1} \sigma_n$. Fix some $M > 0$ and consider the localising sequence $(\sigma_n \wedge \tau_M)_{n\geq 1}$ for the solution, with this solution on a potentially smaller stochastic interval. Taking the limit as $n \to \infty$ for each $M > 0$, we obtain a local solution which agrees with $(v^1, \eta^1, v^2, \eta^2)$ until $\tau \wedge \sigma$ (its own terminal time), and has the localising sequence $(\sigma \wedge \tau_M)_{M>0}$. Arguing as in Theorem 5.4.3, for every $M > 0$, this must agree with the M-truncated problem until the time $\sigma \wedge \tau_M$. Therefore, for every $M > 0$, we have that $\sigma \geq \tau_M$ almost surely, as otherwise we could use the solution to the M-truncated problem to propagate the solution to a later time. Hence, the localising sequence $\sigma \wedge \tau_M$ is simply just τ_M. In addition $\sigma \geq \tau_M$ for every M almost surely implies that $\sigma \geq \tau$ almost surely. Clearly, by agreement with the M-truncated problems until the times τ_M, we have that $\tau \geq \sigma$, since the solution cannot be propagated beyond a blow-up in the $\mathscr{H}$-norm. We therefore have the result. □

We are now in position to prove our main result, Theorem 5.1.9.

Proof of Theorem 5.1.9. We prove existence of a maximal solution by concatenating solutions to the truncated problems. For $M > 0$, let $\mathscr{V}_M := (v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ solve the M-truncated problem as in Theorem 5.4.3. Note that for $M_1 \geq M_2 > 0$, $\mathscr{V}_{M_1}$ solves (5.1.2) until the random stopping time

$$\tau := \inf\left\{t \in [0,\infty) \mid \max\left\{\|v_{M_1}^1\|_{\mathscr{H},t}, \|v_{M_1}^2\|_{\mathscr{H},t}\right\} \geq M_2\right\}.$$

Let $z^i_{M_j}$ solve the unreflected SPDEs associated with $v^i_{M_j}$ for $i,j=1,2$, as in the equation (5.4.2). By considering the difference between $z^i_{M_1}(t,x)\mathbb{1}_{\{t\leq\tau\}}$ and $z^i_{M_2}(t,x)\mathbb{1}_{\{t\leq\tau\}}$ for $i=1,2$, we are able to argue as in the proof of Theorem 5.4.3 to obtain that, for $T>0$, $p>22$ and $t\in[0,T]$

$$\begin{aligned}&\mathbb{E}\left[\|z^1_{M_1}-z^1_{M_2}\|^p_{\mathscr{H},\tau\wedge t}+\|z^2_{M_1}-z^2_{M_2}\|^p_{\mathscr{H},\tau\wedge t}\right]\\&\quad\leq C_{T,p}\int_0^t\mathbb{E}\left[\|v^1_{M_1}-v^1_{M_2}\|^p_{\mathscr{H},\tau\wedge s}+\|v^2_{M_1}-v^2_{M_2}\|^p_{\mathscr{H},\tau\wedge s}\right]\mathrm{d}s.\end{aligned}\tag{5.4.7}$$

Since $(v^i_{M_j}-z^i_{M_j},\eta^i_{M_j})$ are simply the solutions to the obstacle problem with obstacles $z^i_{M_j}$, it follows that

$$\begin{aligned}&\mathbb{E}\left[\|v^1_{M_1}-v^1_{M_2}\|^p_{\mathscr{H},\tau\wedge t}+\|v^2_{M_1}-v^2_{M_2}\|^p_{\mathscr{H},\tau\wedge t}\right]\\&\quad\leq C_{T,p}\int_0^t\mathbb{E}\left[\|v^1_{M_1}-v^1_{M_2}\|^p_{\mathscr{H},\tau\wedge s}+\|v^2_{M_1}-v^2_{M_2}\|^p_{\mathscr{H},\tau\wedge s}\right]\mathrm{d}s.\end{aligned}$$

Gronwall's lemma then applies to give that $v^i_{M_1}=v^i_{M_2}$ almost surely for $i=1,2$, until the random time $\tau\wedge T$. Equation (5.4.7) then gives that $z^i_{M_1}=z^i_{M_2}$ almost surely for $i=1,2$, until the random time $\tau\wedge T$, and so it follows from uniqueness for the obstacle problem that $(v^1_{M_1},\eta^1_{M_1},v^2_{M_1},\eta^2_{M_1})=(v^1_{M_2},\eta^1_{M_2},v^2_{M_2},\eta^2_{M_2})$ until time $\tau\wedge T$. This holds for every $T>0$, so we have that they almost surely agree until time τ. We have therefore shown that the solutions to our M-truncated problems are consistent. This allows us to concatenate them. We define the process (v^1,η^1,v^2,η^2) such that, for all $M>0$, it agrees with $(v^1_M,\eta^1_M,v^2_M,\eta^2_M)$ on $[0,\tau_M]$, where

$$\tau_M:=\inf\left\{t\in[0,\infty)\mid\max\left\{\|v^1_M\|_{\mathscr{H},t},\|v^2_M\|_{\mathscr{H},t}\right\}\geq M\right\}.$$

Note that

$$\tau_M=\inf\left\{t\in[0,\infty)\mid\max\left\{\|v^1\|_{\mathscr{H},t},\|v^2\|_{\mathscr{H},t}\right\}\geq M\right\}.$$

Since $\mathscr{V}_M$ solves (5.1.2) until time τ_M, we have that (v^1,η^1,v^2,η^2) solves (5.1.2) on the random interval $[0,\tilde{\tau})$, where

$$\tilde{\tau}=\sup_{M>0}\tau_M,$$

with localising sequence τ_M. This solution is maximal since, on the set $\{\tilde{\tau}<\infty\}$, we have that

$$\max\left\{\|v^1\|_{\mathscr{H},t},\|v^2\|_{\mathscr{H},t}\right\}\to\infty$$

as $t\uparrow\tilde{\tau}$ almost surely. Therefore, we know that a maximal solution exists. Uniqueness follows by the same arguments as those made for consistency among solutions to the truncated problems. Fix $T>0$ and let $(\tilde{v}^1,\tilde{\eta}^1,\tilde{v}^2,\tilde{\eta}^2)$ be a maximal solution to (5.1.2). Note that the localising sequence can be taken as in Proposition 5.4.4. We then have that for every $M>0$, it must agree with the solution to the M-truncated problem, and therefore with (v^1,η^1,v^2,η^2), until

$$\inf\left\{t\in[0,T]\mid\max\left\{\|\tilde{v}^1\|_{\mathscr{H},t},\|\tilde{v}^2\|_{\mathscr{H},t}\right\}\geq M\right\}.$$

Uniqueness then follows. □

We can now show that, as one might expect, blow up for the problem occurs precisely when the boundary speed blows up.

Proposition 5.4.5. *Let $(v^1, \eta^1, v^2, \eta^2)$ be a solution to the moving boundary problem in the relative frame on the maximal interval $[0, \tau)$. Then*

$$\tau = \sup_{M>0} \sigma_M,$$

where

$$\sigma_M = \inf\left\{t \in [0,\infty) \mid h(v^1(t,\cdot), v^2(t,\cdot)) \geq M\right\}.$$

Proof. By Proposition 5.4.4, it is sufficient to show that for every $M > 0$, $\sigma_M \leq \tau$ almost surely. Suppose that, for every $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,(\sigma_M\wedge\tau\wedge T))}\left[\|v^1(t,\cdot)\|^p_{\mathscr{H}} + \|v^2(t,\cdot)\|^p_{\mathscr{H}}\right]\right] < \infty. \tag{5.4.8}$$

Then, since

$$\sup_{t\in[0,\tau\wedge T)}\left[\|v^1(t,\cdot)\|_{\mathscr{H}} + \|v^2(t,\cdot)\|_{\mathscr{H}}\right] = \infty$$

almost surely on the set that $\{\tau \leq T\}$ we must have that $\sigma_M < \tau \wedge T$ almost surely on the set $\{\tau \leq T\}$. Therefore, for every $T > 0$, we would have that

$$\sigma_M \mathbb{1}_{\{\tau\leq T\}} < \tau \wedge T$$

almost surely. Letting $T \uparrow \infty$ (via a countable sequence), we obtain that

$$\sigma_M \mathbb{1}_{\{\tau<\infty\}} \leq \tau$$

almost surely, so that $\sigma_M \leq \tau$ almost surely, giving the result. So it is sufficient to prove that (5.4.8) holds for every $M, T > 0$. Fix some $T > 0$ and some $M > 0$. By Theorem 5.2.1,

$$\mathbb{E}\left[\sup_{t\in[0,(\sigma_M\wedge\tau\wedge T))}\|v^1(t,\cdot)\|^p_{\mathscr{H}}\right] \leq 2^p\mathbb{E}\left[\sup_{t\in[0,(\sigma_M\wedge\tau\wedge T))}\|z^1(t,\cdot)\|^p_{\mathscr{H}}\right],$$

where z^1 solves the unreflected problem corresponding to v^1 on the random interval $[0, \tau \wedge T)$. That is, z^1 solves

$$\frac{\partial z^1}{\partial t} = \Delta z^1 - h(v^1(t,\cdot), v^2(t,\cdot))\frac{\partial v^1}{\partial x} + f(x,t,v^1(t,\cdot)) + \sigma(x,t,v^1(t,\cdot))\dot{W}(\mathrm{d}x,\mathrm{d}t),$$

on the interval $[0, \tau \wedge T)$, where we take the localising sequence

$$\tau_N := \inf\left\{t \in [0,\infty) \mid \max\left\{\|v^1\|_{\mathscr{H},t} + \|v^2\|_{\mathscr{H},t}\right\} \geq N\right\}.$$

Writing z^1 in mild form, we have that for $t \in [0,T]$

$$\begin{aligned} z^1(t,x)\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}} = \\ &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^1 H(t,x,y)u_0^1(y)\mathrm{d}y\right] + \\ &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)h(v^1(s,\cdot),v^2(s,\cdot))v^1(s,y)\mathrm{d}y\mathrm{d}s\right] + \\ &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 H(t-s,x,y)f(y,s,v^1(s,\cdot))\mathrm{d}x\mathrm{d}s\right] + \\ &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 H(t-s,x,y)\sigma(y,s,v^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right], \end{aligned} \tag{5.4.9}$$

We aim to bound the $L^p(\Omega;L^\infty([0,T];\mathscr{H}))$ norms of the four terms on the right hand side uniformly over N. The first term is simple, so we omit the proof for this. For the second term, note that

$$\begin{aligned} &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)h(v^1(s,\cdot),v^2(s,\cdot))v^1(s,y)\mathrm{d}y\mathrm{d}s\right] \\ &\qquad\leq \left[\int_0^t\int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)h(v^1(s,\cdot),v^2(s,\cdot))v^1(s,y)\mathbb{1}_{\{s<\sigma_M\wedge\tau_N\wedge T\}}\mathrm{d}y\mathrm{d}s\right]. \end{aligned}$$

By the definition of σ_M, this is at most

$$J(t,x) := C_M\left[\int_0^t\int_0^1 \frac{\partial H}{\partial y}(t-s,x,y)v^1(s,y)\mathbb{1}_{\{s<\sigma_M\wedge\tau_N\wedge T\}}\mathrm{d}y\mathrm{d}s\right].$$

An application of Proposition 5.3.4 then gives that

$$\mathbb{E}\left[\|J\|^p_{\mathscr{H},t}\right] \leq C_{M,T,p}\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]}\|v^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right]\mathrm{d}s.$$

Arguing in the same way, applying Proposition 5.3.3 in place of Proposition 5.3.4, we are able to control the third term of (5.4.9). For the fourth term of (5.4.9), we note that

$$\begin{aligned} &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left|\int_0^t\int_0^1 H(t-s,x,y)\sigma(y,s,v^1(s,\cdot))\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right| \\ &\qquad\leq \left|\int_0^t\int_0^1 H(t-s,x,y)\sigma(y,s,v^1(s,\cdot))\mathbb{1}_{\{s<\sigma_M\wedge\tau_N\wedge T\}}\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right| =: K(t,x). \end{aligned}$$

Applying Proposition 5.3.5 and noting the linear growth condition on σ then gives that, for $t \in [0,T]$

$$\mathbb{E}\left[\|K\|^p_{\mathscr{H},t}\right] \leq C_{T,p}\left[1+\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]}\|v^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right]\mathrm{d}s\right].$$

Putting the four terms together, we have that, for $t \in [0,T]$,

$$\mathbb{E}\left[\sup_{r\in[0,t]}\left\|z^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\right\|^p_{\mathscr{H}}\right] \leq C_{T,p,M}\left[1+\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]}\|v^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right]\mathrm{d}s\right].$$

Therefore,

$$\mathbb{E}\left[\sup_{r\in[0,t]}\left\|v^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\right\|_{\mathscr{H}}^p\right]\leq C_{T,p,M}\left[1+\int_0^t\mathbb{E}\left[\sup_{r\in[0,s]}\|v^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|_{\mathscr{H}}^p\right]\mathrm{d}s\right].$$

It follows by Gronwall's inequality that

$$\sup_{N>0}\mathbb{E}\left[\sup_{t\in[0,\sigma_M\wedge\tau_N\wedge T)}\left\|v^1(t,\cdot)\right\|_{\mathscr{H}}^p\right]=\sup_{N>0}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|v^1(t,\cdot)\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\right\|_{\mathscr{H}}^p\right]\leq K_{T,p,M}<\infty.$$

Note that, by the MCT

$$\sup_{N>0}\mathbb{E}\left[\sup_{t\in[0,\sigma_M\wedge\tau_N\wedge T)}\left\|v^1(t,\cdot)\right\|_{\mathscr{H}}^p\right]=\mathbb{E}\left[\sup_{t\in[0,\sigma_M\wedge\tau\wedge T)}\left\|v^1(t,\cdot)\right\|_{\mathscr{H}}^p\right]$$

The same estimates hold for v^2. This concludes the proof. □

Remark 5.4.6. *The previous result gives that solutions to the moving boundary problem can only blow-up in ways which cause the boundary speed to blow up. For example, if we have the classical boundary behaviour given by*

$$h(u,v)=\gamma(u'(0)-v'(0)),$$

where γ is a constant, we know that blow-up can only occur when one of the spatial derivatives at the shared interface blow up.

Corollary 5.4.7. *Suppose that the function h is globally bounded. Then the unique maximal solution to (5.1.2) is global i.e. $\tau=\infty$ almost surely.*

Remark 5.4.8. *By arguing as in Section 4.1.5, we can extend our results in this Chapter to Stefan problems of the form*

$$\begin{aligned}
\frac{\partial u^1}{\partial t}&=\mathscr{L}_1u^1+f_1(t,p(t)-x,u^1(t,p(t)-\cdot))+\sigma_1(t,p(t)-x,u^1(t,p(t)-\cdot))\frac{\partial^2W}{\partial x\partial t}+\eta^1\\
\frac{\partial u^2}{\partial t}&=\mathscr{L}_2u^2+f_2(t,x-p(t),u^2(t,p(t)+\cdot))+\sigma_2(t,x-p(t),u^2(t,p(t)+\cdot))\frac{\partial^2W}{\partial x\partial t}+\eta^2\\
p'(t)&=h(u^1(t,p(t)-\cdot),u^2(t,p(t)+\cdot)),
\end{aligned}\tag{5.4.10}$$

under the same conditions (i) - (vii) on f, σ and h as stated in Section 5.1, and the assumption that, for $i=1,2$, $\mathscr{L}_i$ is of the form

$$(\mathscr{L}f)(x)=a_i(x)\frac{\partial^2f}{\partial x^2}(x)+b_i(x)\frac{\partial f}{\partial x}(x),$$

where $a_i\in C^2([0,1])$ with $a_i(x)>0$ for every $x\in[0,1]$, and $b_i\in C^1([0,1])$. We refer the reader to Section 4.1.5 for details.

5.5 Numerical Illustrations of Boundary Derivatives

In this section, we implement a numerical scheme in order to illustrate some typical profiles for solutions to our Stefan problems and point out some of the features which were highlighted in the previous analysis. In particular, we are able to see the presence of spatial derivatives at the shared boundary when we choose the drift and volatility parameters appropriately. We will also contrast this with the behaviour at the boundary when the volatility only decays there like $\sqrt{x}$, providing numerical evidence that a spatial derivative does not exist at the boundary in this case. This is of interest as we have not shown the linear decay condition which we imposed on the volatility to be optimal. To begin, we briefly describe the numerical scheme used. We will consider drift and volatility functions which only depend on the position in space (in the relative frame) and the value of the solution at that particular position at that particular time. In particular, these functions will be the same for both sides of the Stefan problem. In addition, our functions h which determine the boundary behaviour will take the classical form

$$h(u,v) = \gamma\left(u'(0) - v'(0)\right),$$

where γ is a positive constant. We simulate the process on the interval $[0,T]$ and space interval $[p(t)-1, p(t)+1]$ at a given $t \in [0,T]$. That is, we simulate the equation in the relative frame on $[0,T]\times[-1,1]$, with v^1, v^2 supported on $[0,T]\times[0,1]$. We discretise the equation into M time steps and N space steps, and define $t_i := iT/M$, $x_i := i/N$. Given our simulated solution up to the j^{th} time step, we define the values of the approximate solution at the $(j+1)^{\text{th}}$ time step by setting

$$\begin{aligned} v^1(t_{j+1},x_i) := \Bigg| & v^1(t_j,x_i) + \frac{TN^2}{M}\left(v^1(t_j,x_{i+1}) + v^1(t_j,x_{i-1}) - 2v^1(t_j,x_i)\right) \\ & -\gamma\frac{TN^2}{M}\left(v^1(t_j,x_1) - v^2(t_j,x_1)\right)\left(v^1(t_j,x_{i+1} - v^1(t_j,x_i)\right) \\ & +\frac{T}{M}f(x_i,v^1(t_j,x_i)) + \frac{\sqrt{TN}}{\sqrt{M}}\sigma(x_i,v^1(t_j,x_i))Z_{i,j}\Bigg|, \end{aligned}$$

where the $Z_{i,j}$ are independent unit normal random variables. $v^2(t_{j+1},x_i)$ is defined similarly. We note that taking the absolute value on the right hand side here is intended to capture the fact that the process is reflected at zero. Given our simulated processes v^1 and v^2, we are then able to reproduce the boundary process by noting that $p'(t) = h(v^1(t,\cdot), v^2(t,\cdot))$, and imposing the condition $p_0 = 0$. That is, we take

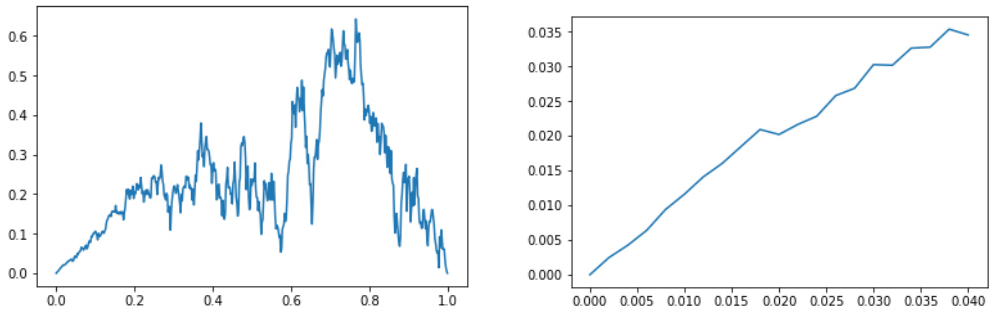
$$p(t_j) := \sum_{k=1}^{j} \frac{TN}{M}\left(v^1(t_k,x_1) - v^2(t_k,x_1)\right).$$

As we are mainly interested in the effects of different volatility functions on our equations, we will fix the other parameters here. Throughout all of our simulations, we will choose the drift function f and the parameter γ such that $f = 1$ and $\gamma = 10$. The initial data for both

v^1 and v^2 will be given by the function u_0, where $u_0(0) = u_0(1) = 0$, and $u_0(0.5) = 1$, with linear interpolation in between these points. The volatility functions for which we perform our simulations are given by $\sigma_a(x), \sigma_b(u)$ and $\sigma_c(x)$, where

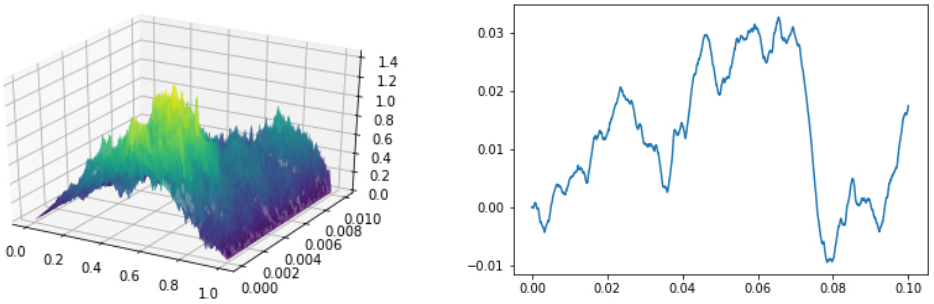
1. $\sigma_a(x)$ is linear between the points 0, 0.5 and 1, with $\sigma_a(0) = 0$, $\sigma_a(0.5) = \sigma_a(1) = 1$.
2. $\sigma_b(x, u) = u(x)$.
3. $\sigma_c(x) = \sqrt{x}$.

The following figures depict the results of our simulations.



(a) (b)

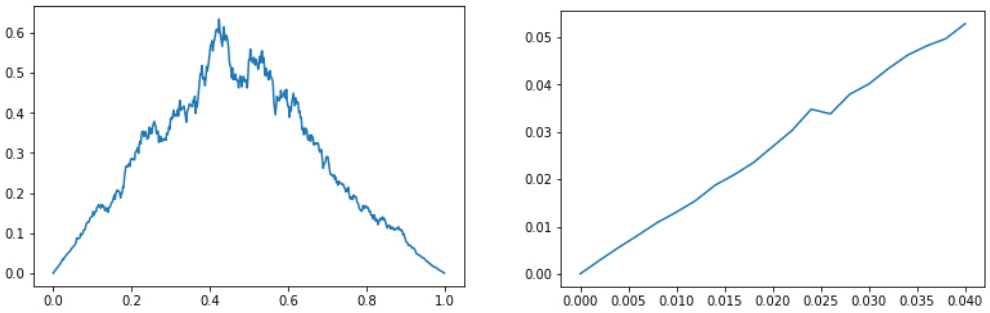
Figure 5.1: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_a(x)$, so that σ decays linearly at the boundary.



(a) (b)

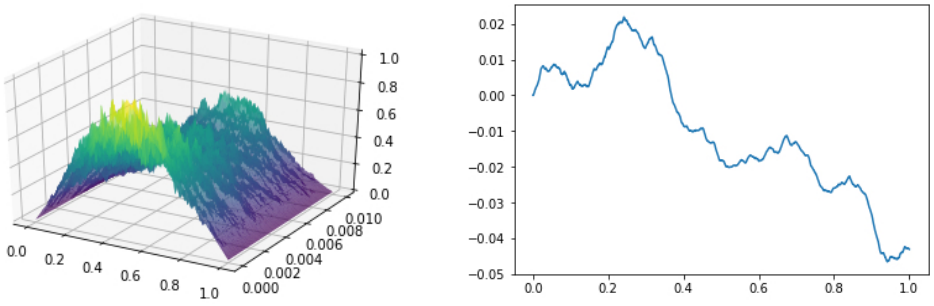
Figure 5.2: 3D plot of v^1 and plot of the boundary motion, $p(t)$ in the case where $\sigma = \sigma_a(x)$, so that σ decays linearly at the boundary.

We see in Figure 5.1 that the profile is less volatile as the boundary $x = 0$ is approached, and the derivative at the boundary is visible. The boundary function appears to be smooth to some degree in Figure 5.2 (b), as expected. Our second set of simulations deals with the case when the volatility is multiplicative, which also falls within the framework of our earlier analysis. Figures 5.3 and 5.4 show the results of our simulation in this case.



(a) (b)

Figure 5.3: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_b(u) = u$.

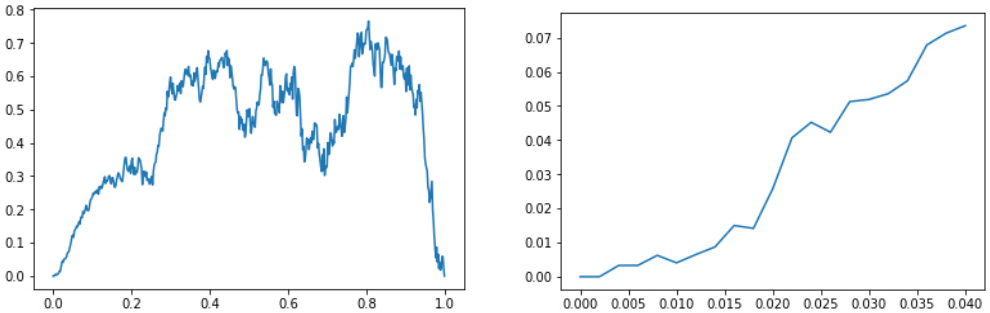


(a) (b)

Figure 5.4: 3D plot of v^1 and plot of the boundary motion, $p(t)$ in the case where $\sigma = \sigma_b(u) = u$.

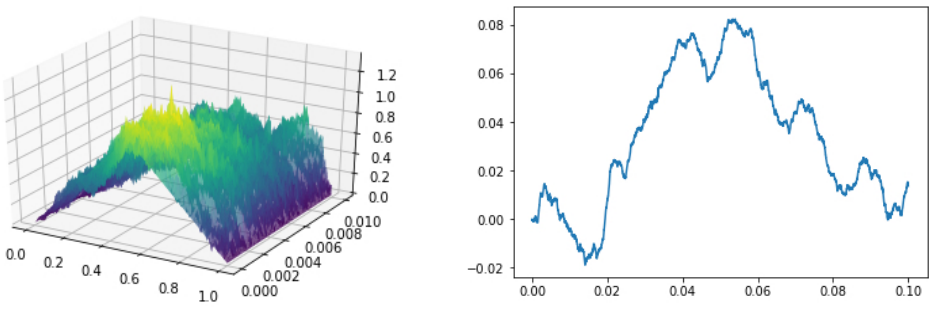
The presence of spatial derivatives at both ends of the profile is visible in Figure 5.3 (a). This is simply because the equation is symmetric. The Dirichlet condition is imposed at both 0 and 1, so that the volatility decays sufficiently quickly at both ends.

We now present the case where the volatility decays like $\sqrt{x}$ at the shared boundary, which falls outside of our proof for existence and uniqueness.



(a) (b)

Figure 5.5: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_c(x) = \sqrt{x}$.



(a) (b)

Figure 5.6: 3D plot of v^1 and plot of the boundary motion, $p(t)$, in the case where $\sigma = \sigma_c(x) = \sqrt{x}$.

In this case, the profiles can be seen to be significantly rougher close to the boundary. They do not appear to have spatial derivatives there (although this is inconclusive of course). Interestingly, the boundary motion appears to be rough to some degree. We note, however, that the scheme did not blow up- we were able to obtain sensible profiles and sensible processes for the movement of the boundary. This suggests that, perhaps by considering the derivative at the boundary in a suitable weak sense, equations of this form may be solvable without the need for linear decay of the volatility, and could produce rough paths for the boundary motion.

We perform a simple numerical test in order to give some (albeit naive) quantification of how linear the profiles are close to the boundary in the cases presented above. For a profile at a given time, we fit the values in the spatial interval $[0, 0.04]$ to a line passing through the origin by the method of least squares. Calculating the sum of the squared residuals and dividing by the square of height of the fitted line at position 0.04, we obtain values indicating how well the profile can be fitted to a line, with these values now independent of scale and so comparable across different profiles. Time averaging these values for each of the three cases, we obtain the following values, which clearly indicate that there is significant deviation from being linear close to zero when $\sigma = \sigma_c(x) = \sqrt{x}$.

$\sigma_a(x)$	$\sigma_b(u)$	$\sigma_c(x)$
0.04274	0.01473	0.28378

Table 5.1: Time-Averaged measure of deviation of profile from linear fit on the spatial interval $[0, 0.04]$.

5.6 Numerical Illustrations of Blow-up Scenarios

In Section 5.4, we proved the existence of solutions to our stochastic Stefan problem until blow-up, with global solutions only guaranteed in the case where the function determining the boundary speed, h, is globally bounded. In particular, we were unable to say anything about potential blow-ups for typical Stefan boundary conditions of the form

$$h(u, v) = \gamma(u'(0) - v'(0)), \tag{5.6.1}$$

where $\gamma \in \mathbb{R}$. In this section, we will numerically investigate potential blow-ups of (5.6.1). We begin by considering the classical Stefan problem, for which the blow up scenarios are well understood, and will then use these results to help guide our investigations of the stochastic problem. The overview of the classical problem here has been informed by [18], [19], [28] and [44], to which we refer the reader for further details.

5.6.1 Blow ups in the Classical Stefan Problem

The classical Stefan problem takes the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{for } x \in (0, p(t)) \\ \frac{\partial u}{\partial x} = 0 & \text{on } x = 0 \\ u = 0 & \text{on } x = p(t) \\ p'(t) = -\gamma \frac{\partial u}{\partial x}(t, p(t)) & \\ u(0, x) = u_0(x) > 0 & \\ p(0) = 0. & \end{cases} \tag{5.6.2}$$

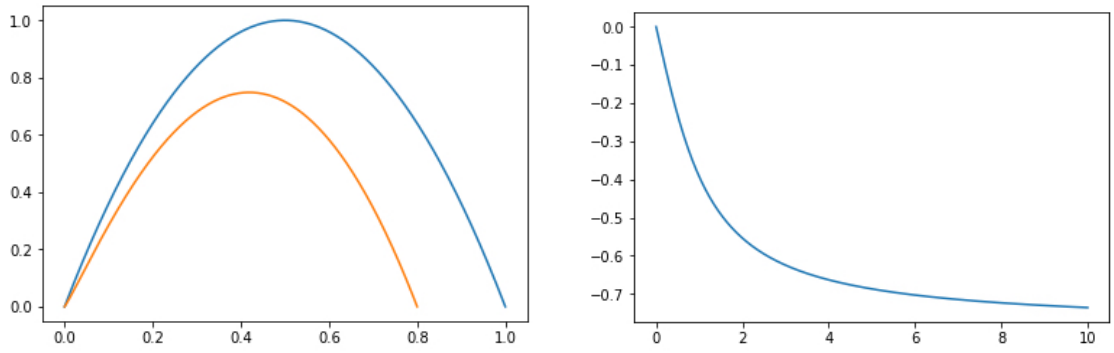
Similarly to before, we imagine that the region $(0, p(t))$ is occupied by a body of water, with the region $[p(t), \infty)$ occupied by ice. Typically, we have that $\gamma > 0$ and we imagine that the temperature of the water is positive. The boundary condition $p'(t) = -\gamma \frac{\partial u}{\partial x}$ arises from balancing the thermal energy lost due to the Dirichlet condition at the boundary, with the energy required to convert ice into water. In this case, solutions to the problem are global, with no possibility of blow up. The boundary recedes as the ice melts, with there being an infinite block of ice with which the thermal energy of the water can be expended in the melting process.

The case where $\gamma < 0$, however, is less stable, and it is in this case that blow-up for the problem is possible. Here, the boundary moves inwards, so that $p(t)$ decreases. If we imagine that u now represents the temperature of the water in *negative* units, we have that the equation describes a scenario of *supercooling*, where the water temperature is less than that of the freezing temperature. The water warms up as it freezes, as it gains thermal energy from being transformed into ice. There are three possible situations which can arise from this:

1. Incomplete freezing of the water. In this scenario, the water is gaining sufficient thermal energy from transforming to ice such that freezing at the boundary slows down, with the limiting state still containing a region of water. In terms of our equation, this corresponds to having a global solution, $(u(t, \cdot), p(t))_{t \geq 0}$, with $p(t) \to p > 0$ and $u(t, x) \to 0$ for $x \in (0, p)$ as $t \to \infty$. Figure 5.7 gives an illustration of this.

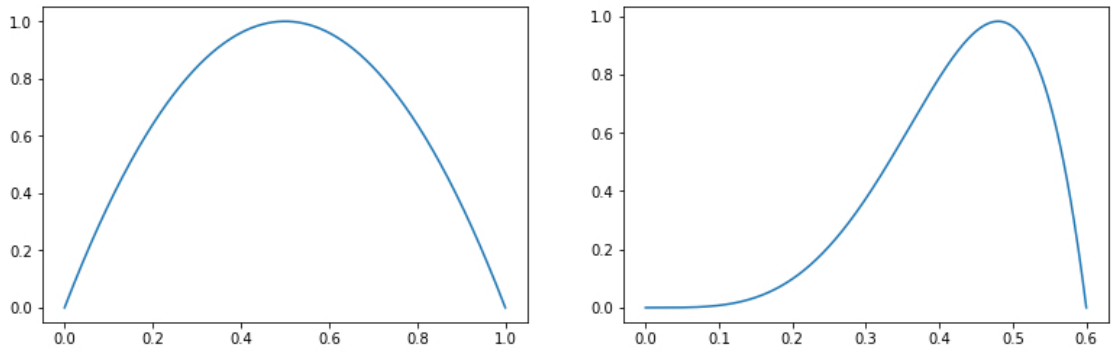
2. Complete freezing of the water, where at some finite time t_{terminal} we have that the region of water has completely vanished.

3. Finite time blow-up. Here, the water is sufficiently cold that the temperature gained by transforming it to ice isn't enough to increase the temperature to the freezing temperature. As the water freezes, the gradient of the water temperature at the interface becomes



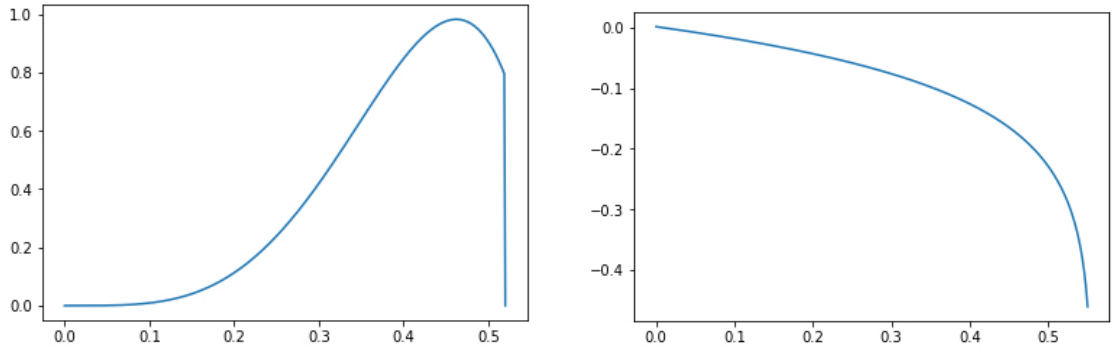
(a) (b)

Figure 5.7: Typical evolution of temperature distribution of supercooled liquid in negative units, in the case where blow-up does not occur. In (a), the blue line represents the initial temperature of the liquid in negative units, and the orange line represents the same after the equation has evolved for some time. We see that the liquid has heated up as it freezes, in such a way that the gradient at the interface has not increased. Graph (b) represents the evolution of the boundary against time. It is converging to some position $p \in (0, 1)$ as $t \to \infty$.



(a) (b)

Figure 5.8: Typical evolution of temperature distribution of supercooled liquid in negative units, in the case of blow-up. Initial profile is on the left, but as the liquid freezes at the right boundary, the boundary moves to the left, causing the gradient at the interface to become steeper. This then makes the boundary move even faster, and can lead to an infinite gradient at the interface i.e. blow-up.



(a) (b)

Figure 5.9: Evolution of temperature distribution of supercooled liquid in negative units, in the case of blow-up. In (a), we see that the boundary derivative has now exploded. In (b), we see a plot of the position of the boundary against time. We see that the boundary speed is approaching infinity.

infinitely large and negative, causing the boundary speed to become infinite. Figures 5.8 and 5.9 gives an illustration of how this can occur.

5.6.2 Investigation of Blow-ups for the Stochastic Stefan Problem

Our understanding of the classical Stefan problem gives us some clues as to when our stochastic problem might blow up. As in Section 5.5, we will consider systems of the form

$$\begin{aligned}
\frac{\partial u^1}{\partial t} &= \Delta u^1 + f_1(p(t)-x) + \sigma_1(p(t)-x)\frac{\partial^2 W}{\partial x \partial t} + \eta^1 \\
\frac{\partial u^2}{\partial t} &= \Delta u^2 + f_2(x-p(t)) + \sigma_2(x-p(t))\frac{\partial^2 W}{\partial x \partial t} + \eta^2, \\
p'(t) &= -\gamma\left(\frac{\partial u^1}{\partial x}(t,p(t)^-) + \frac{\partial u^2}{\partial x}(t,p(t)^+)\right).
\end{aligned} \tag{5.6.3}$$

Writing this in the relative frame, we obtain the system of coupled reflected SPDEs

$$\begin{aligned}
\frac{\partial v^1}{\partial t} &= \Delta v^1 - p'(t)\frac{\partial v^1}{\partial x} + f_1(x) + \sigma_1(x)\frac{\partial^2 W}{\partial x \partial t} + \eta^1 \\
\frac{\partial v^2}{\partial t} &= \Delta v^2 + p'(t)\frac{\partial v^2}{\partial x} + f_2(x) + \sigma_2(x)\frac{\partial^2 W}{\partial x \partial t} + \eta^2, \\
p'(t) &= \gamma\left(\frac{\partial v^1}{\partial x}(t,0^+) - \frac{\partial v^2}{\partial x}(t,0^+)\right).
\end{aligned} \tag{5.6.4}$$

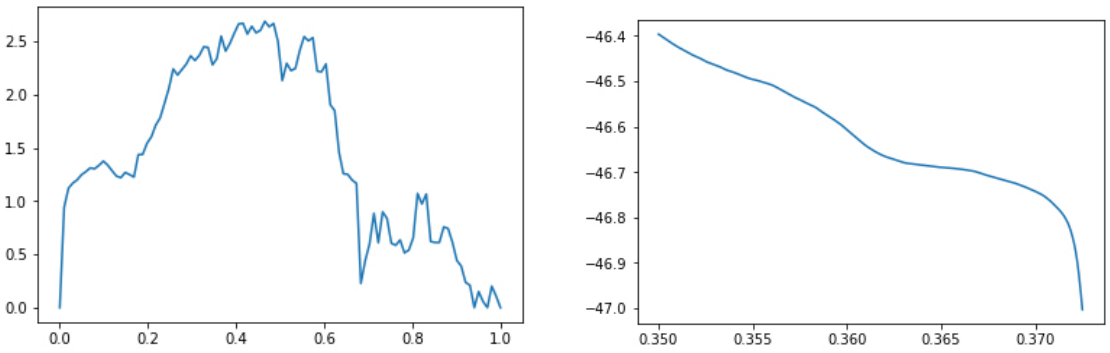
The simulation scheme used here is as in Section 5.5. As before, we expect to see more stable behaviour when $\gamma > 0$, where large gradients for a profile at the interface push the boundary *away* from that profile, and less stable behaviour when $\gamma < 0$, where large gradients pull the boundary *towards* the profile. In light of Proposition 5.4.5 we also expect to see that, when there is blow-up, it is indeed a derivative at the shared boundary which has exploded.

5.6.2.1 One-sided Stochastic Stefan Problem with $\gamma < 0$

A simple case where the deterministic problem clearly does not blow up is if we start the system with zero data, and set $f_i \equiv 0$. If we now introduce noise to one side, and impose that $\gamma < 0$ we would expect that if the noise is sufficiently large (or perhaps if we look over a long enough time period) we will see blow-up once more. The intuition is that the noise will populate one side, and at some point enough mass will collect near the boundary so as to trigger a blow-up due to the nature of the boundary condition there when γ is negative. We choose $\sigma_1(x) = 5x$, $\sigma_2 \equiv 0$ and $\gamma = -5$. Doing so produced a blow up at time $t = 0.3725$ (see Figures 5.10 and 5.11). We note that blow up of the derivative at zero in (a) of Figure 5.10 is consistent with Proposition 5.4.5.

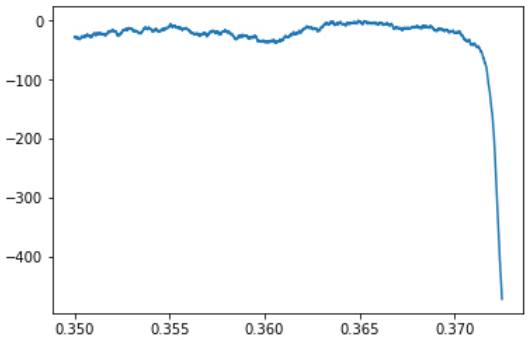
5.6.2.2 Two-sided Stochastic Stefan Problem with $\gamma < 0$

Figures 5.12 and 5.13 illustrate the results of simulating the two-sided problem with $f_i \equiv 0$, $\sigma_i(x) = 5x$ and $\gamma = -2$. We give both sides of the equation initial data $u_0(x) = 4x(1-x)$. Here, at time $t = 0.058$, we see that the system explodes. Examination of profiles shows that



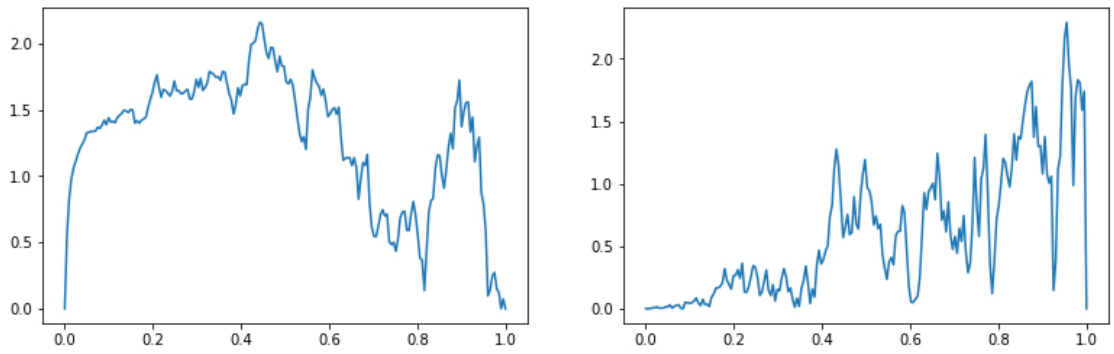
(a) (b)

Figure 5.10: Development of a blow-up in the Stochastic Stefan Problem, started with zero initial data. (a) represents the profile at time $t = 0.3725$, with (b) the corresponding boundary position over the time period $[0.35, 0.3725]$.



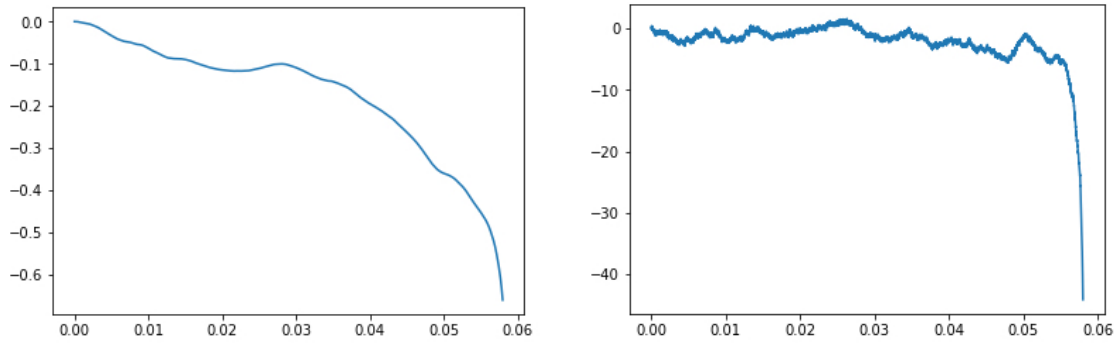
(a)

Figure 5.11: Derivative at the boundary for the Stochastic Stefan Problem, started with zero initial data, over the time period $[0.35, 0.3725]$.



(a) (b)

Figure 5.12: Development of a blow-up in the two-sided Stochastic Stefan Problem. (a) represents the profile v^1 at time $t = 0.058$, with (b) the profile v^2 at this time.



(a) (b)

Figure 5.13: Development of a blow-up in the two-sided Stochastic Stefan Problem. (a) is the position of the boundary against time, for $t \in [0, 0.058]$, and (b) depicts the evolution and blow-up of the boundary derivative over this same time period.

that u^1 profile has accumulated a lot of mass close to the boundary, causing the blow up, whilst the u^2 profile has very little mass close to the boundary, since the boundary has moved away from this profile as u^1 has become dominant at the shared interface.

5.6.2.3 Two-Sided Stochastic Stefan Problem with $\gamma > 0$.

We simulate the case where $\gamma > 0$. Here, we have that accumulation of mass for a profile close to the shared boundary pushes the boundary point *away* from that profile, which suggests stable behaviour as in the case of the classical deterministic problem. We also have the intuition that for the static reflected SPDE, we have *global* existence of $\mathscr{H}$-valued solutions, and that if blow-ups occur they must be due to the profile derivative at the boundary, as shown in Proposition 5.4.5. Moving the boundary away when mass accumulates there should make blow-up less likely for the problem than in the static case.

Our numerical results support this hypothesis, as over all of our simulations of this case we did not see blow-up of the system. Figure 5.14 shows the boundary paths obtained from our simulations, where we choose $\sigma_i(x) = x$, $\sigma_i(x) = 2x$ and $\sigma_i(x) = 5x$ respectively. In all cases, we set $f_i \equiv 0$ and $\gamma = 5$. We note that the moving boundary now tends to change

course rather than accelerating in a single direction as in the $\gamma < 0$ case. The reason for this is clear- if the gradient for one of the profiles becomes large, the boundary moves away from that side and therefore reduces the gradient there, rebalancing the gradients at the shared interface. When we set $\gamma < 0$, dominance of one side at the interface was reinforced by the boundary moving towards that side. Figure 5.15 gives an example of how the derivative of the boundary can evolve. The rough nature of this path is as expected- indeed it was shown in [61] that one should expect Hölder regularity of 1/4 for the boundary derivative for the corresponding equation in the absence of reflection. Most interestingly is that, when viewed over increasingly large timescales, the evolution of the boundary itself starts to look rough. We can see how this develops in Figures 5.16 and 5.17. Here we see the evolution of the boundary when the equation is run until times $t = 1$, $t = 10$, $t = 100$ and $t = 1000$ respectively, for three separate simulations in the case when $\sigma(x) = x$. We can estimate the Hurst parameter of our boundary processes by calculating

$$\tau(s)^2 := \langle |p(t+s) - p(t)|^2 \rangle_t$$

for different values of s. The $\langle \cdot \rangle_t$ notation indicates that we average over all possible values of t for the time series. Our estimate for the Hurst parameter, H, is then given by the slope of the graph $\log(s)$ against $\log(\tau(s))$. That is, the choice of H fitted according to the linear relationship

$$\log(\tau(s)) = H \log(s) + C.$$

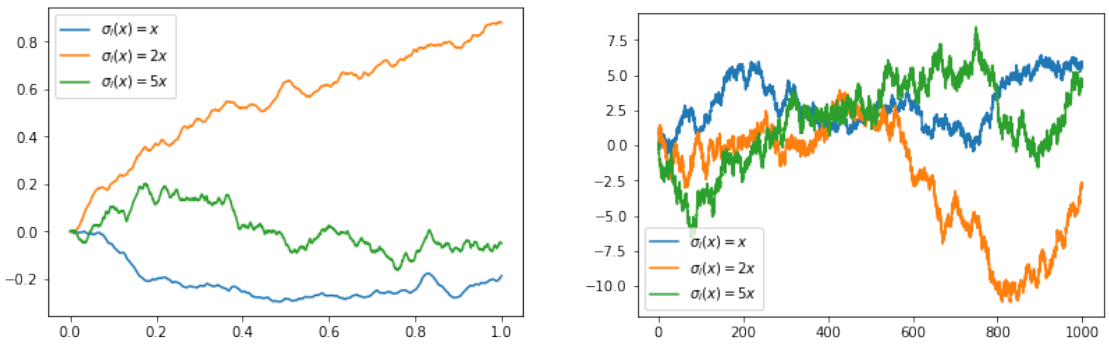
Figure 5.18 (a) depicts the graph of $\log(\tau(s))$ against $\log(s)$ for our three simulations run until time $t = 1000$, and Figure 5.18 (b) shows the derivative of this graph. We see that, for small lag sizes s, the derivative of $\log(\tau(s))$ with respect to $\log(s)$ is approximately 1, corresponding to the fact that the boundary process is C^1. As we increase s, the slope decreases and appears to be converging. This agrees with our intuition that there is a scaling law, with our boundary process converging to a rough process over large timescales. Approximating the limiting slope for large s by

$$H = 2 \int_{0.5}^{1} \frac{\mathrm{d}(\log(\tau(s)))}{\mathrm{d}(\log(s))} \, \mathrm{d}s,$$

we obtain the values $H = 0.505$, $H = 0.515$ and $H = 0.516$ for our simulations. Since these values are close to 0.5, the Hurst parameter for Brownian motion, they hint at a Brownian limiting process. In Figure 5.19 (b), we calculate the quadratic variation of four different time series obtained from sampling the boundary process of our first simulated boundary at regular intervals, where we define the quadratic variation of a time series $(X_{t_i})_{i=1}^n$ at time t_k to be given by

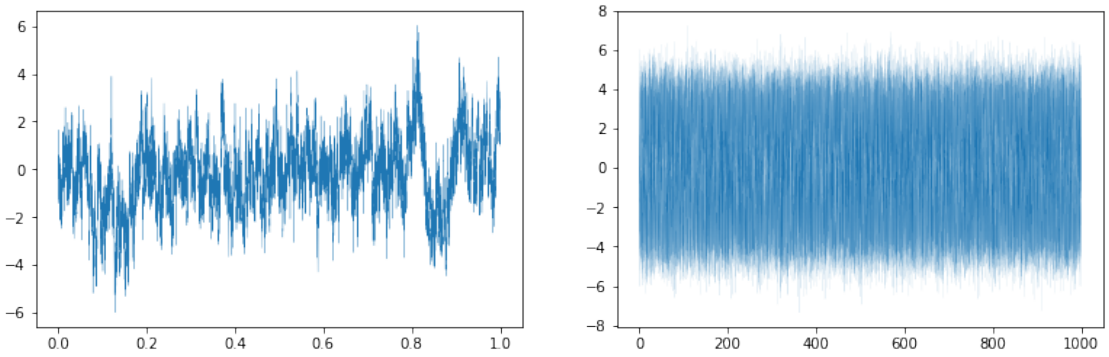
$$QV(X)_{t_k} := \sum_{i=1}^{k-1} (X_{t_{i+1}} - X_{t_i})^2.$$

We sample from the boundary process at time intervals of length $0.5, 0.75$, 1 and 1.5 respectively. Note that choosing a finer mesh would not be consistent with the scale on which our process starts to look Brownian. We see in Figure 5.19 (b) that the quadratic variation does grow



(a) (b)

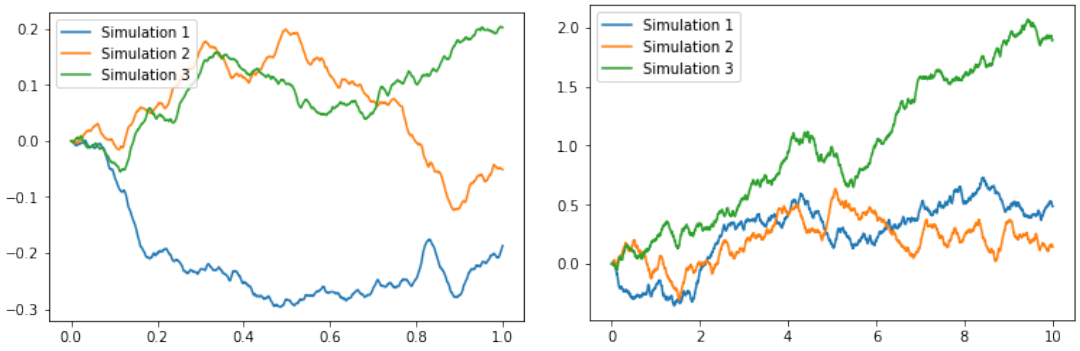
Figure 5.14: Boundary paths in the cases $\sigma_i(x) = x$, $\sigma_i(x) = 2x$ and $\sigma_i(x) = 5x$, over the time intervals $[0, 1]$ and $[0, 1000]$ respectively.



(a) (b)

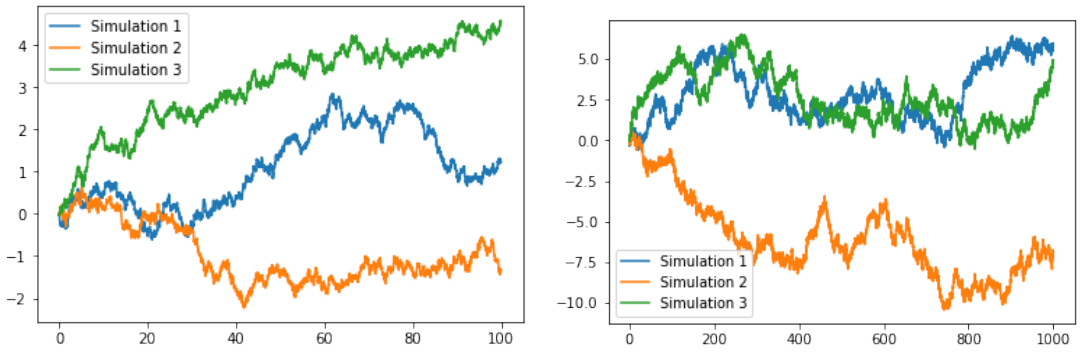
Figure 5.15: Derivative of the boundary in a simulation of the two-sided Stochastic Stefan Problem when $f_i \equiv 0$, $\sigma_i(x) = x$ and $\gamma = 5$, over time intervals $[0, 1]$ and $[0, 1000]$ respectively.

approximately linearly in time, and that the results obtained from the partitions chosen are consistent with one another. This further supports the conjecture that there is a Brownian scaling law for the boundary over long time periods. We note that we only expect this behaviour in the case where the parameters for the two sides of the Stefan problem are chosen to be symmetric, and we have a symmetric Stefan condition imposed at the shared interface with γ negative. In the case where the coefficients and/or boundary mechanism are not symmetric in the two competing profiles, we would expect the boundary to drift upwards/downwards.



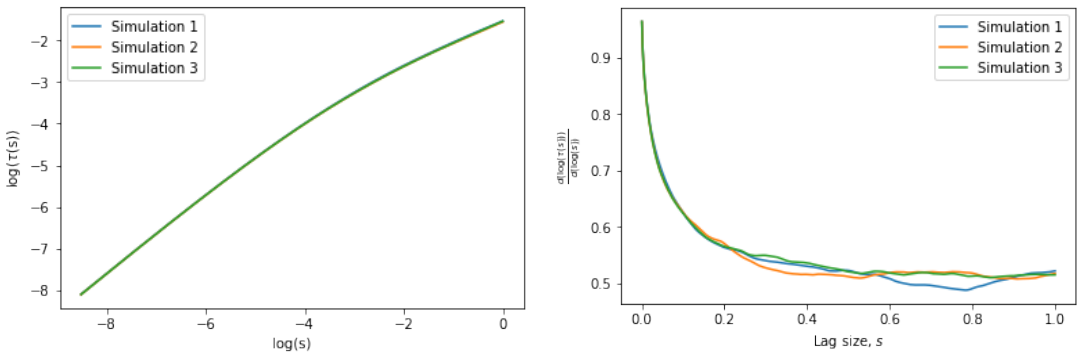
(a) (b)

Figure 5.16: Boundary evolution in the two-sided Stochastic Stefan Problem when $f_i \equiv 0$, $\sigma_i(x) = x$ and $\gamma = 5$, for $t \in [0, 1]$ and $t \in [0, 10]$ respectively.



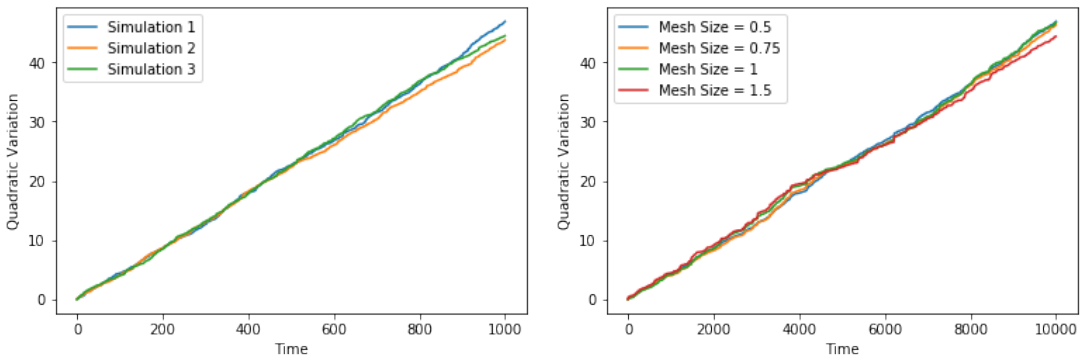
(a) (b)

Figure 5.17: Boundary evolution in the two-sided Stochastic Stefan Problem when $f_i \equiv 0$, $\sigma_i(x) = x$ and $\gamma = 5$, for $t \in [0, 100]$ and $t \in [0, 1000]$ respectively.



(a) (b)

Figure 5.18: Figure (a) illustrates $\log(\tau(s))$ against $\log(s)$ for our three simulations. Figure (b) shows how the derivative of (a) changes with s.



(a) (b)

Figure 5.19: Figures (a) shows the quadratic variation of our three simulated boundary processes, where the boundary processes are sampled at time intervals of length 0.5. Figure (b) shows the quadratic variation process for the first simulated path when it is sampled at time intervals of lengths $0.5, 0.75, 1$ and 1.5 respectively.

Chapter 6

Conclusions and Ideas for Future Work

In this thesis we have explored various reflected SPDE moving boundary problems. The models and results complement the existing literature on reflected SPDEs, extending known results and introducing interesting new features to the established models. Allowing for reflection measures in the dynamics make the models a more natural choice for scenarios where negative values aren't expected (e.g. order volume in a limit order book) without assuming a change of variables such as a logarithmic scale. In Chapter 3, we demonstrated how equations of this type can arise from natural microscopic dynamics of interacting particle systems and systems of reflected SDEs. Following this, we investigated systems of reflected SPDEs which share boundaries evolving continuously in time, with the boundaries driven by the profiles of the competing interfaces as functions in $C_0((0,1))$. Existence and uniqueness results have been established, together with certain properties of the solution, such as Hölder continuity and the existence of invariant measures (under suitable conditions). In addition, we have provided conditions under which the continuously evolving boundary can be driven by a Stefan condition - that is, the spatial derivatives of the two competing profiles at the shared interface. This is particularly interesting due to the presence of highly irregular space-time white noise in our equations.

There are many potential directions for future work in this area. With regards to the first of our models, in which boundary movements occur as discrete jumps, we can consider whether analogous results can be proved in the case where the Laplacian is replaced by a general elliptic operator. The analysis of T.Zhang in [59] made use of explicit and non-classical estimates on the heat kernel and discrete heat kernel, which are no longer available to us in the general case. A different approach may be required here.

In the case of our models with continuous boundary motion, there are many avenues to investigate further. We have shown that, in general, solutions exist until blow-up times. Our numerical work in Section 5 gives a strong indication that, in the case of the Stefan boundary condition, blow-up is possible for our systems when the boundary mechanism represents a "supercooling" regime i.e. when a profile having more mass locally near the shared boundary pulls the boundary *towards* this profile. It is unclear, however, whether blow-up is possible when

we are in a classically stable regime instead, where mass at the boundary pushes the boundary away. It is also interesting to consider whether blow up is possible in any circumstance when the boundary is driven by the profiles as functions in $C_0((0,1))$, as in Chapter 4. An open question specific to the Stefan problem is whether the linear decay condition of the volatility at the boundary may be relaxed. Although this condition is an improvement over the more strict condition in [61], we were unable to prove its optimality. Finally, it might also be possible to prove that the models from Sections 4 and 5 can be built from natural microscopic and/or mesoscopic models, as in Section 3. One might expect, for example, that adapting the model from Section 3 so that boundary jumps are on a smaller scale and occur on a faster time-scale would produce these equations in the limit.

Another one could investigate is whether we can incorporate boundary processes which are rough in our models. In [31] and [36], the authors include Brownian motion into the boundary dynamics in the case where the driving noise in the SPDE equations is coloured in space, with the extra spatial regularity that this affords important to the arguments. In Chapter 5, we also saw that reducing the linear decay condition on our volatility function seemed to produce paths for the boundary process which appear rough. This should, however, be treated with caution as it is unclear that the numerical scheme used accurately represents the moving boundary problem in this case. Finally, as mentioned in Chapter 5, it may be interesting to explore whether scaling laws exist for the boundary motion in certain versions of our existing models. For example, we might expect that a Stefan problem which is symmetric in the equations for u_1 and u_2, with a boundary mechanism where mass near the boundary pushes the profile away, could have a boundary process which converges to Brownian motion when rescaled appropriately.

Appendix A

Section 4 Appendix

A.1 Heat Kernel Estimates

We present here some of the simple estimates for the heat kernels on $[0,1]$ and $[0,\infty)$ which were used throughout, and details of their proofs.

A.1.1 Heat Kernel on $[0,\infty)$

Recall that the Dirichlet heat kernels on $[0,\infty)$ is given by

$$G(t,x,y) := \frac{1}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right],$$

and that $G_r(t,x,y) := e^{-r(x-y)}G(t,x,y)$. We define the functions

$$F_1(t,x,y) := \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x-y)^2}{4t}\right),$$

and

$$F_2(t,x,y) := \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x+y)^2}{4t}\right).$$

In this section, the proofs focus on the F_1 component of G. The arguments for the F_2 components are similar.

Proposition A.1.1. *For every $r \in \mathbb{R}$ and every T there exists a constant $C_{r,T} > 0$ such that*

$$\sup_{t\in[0,T]}\sup_{x\geq 0}\int_0^\infty G_r(t,x,y)\,dy \leq C_{r,T}.$$

Proof. We have that, for $x, y \geq 0$, $r \in \mathbb{R}$ and $t \in [0,T]$

$$e^{-r(x-y)}e^{-(x-y)^2/4t} = e^{-(x-y+2rt)^2/4t}e^{r^2t} \leq C_{r,T}e^{-(x-y+2rt)^2/4t}.$$

Therefore, for $t \in [0,T]$ and $x \geq 0$, we have

$$\int_0^\infty e^{-r(x-y)}F_1(t,x,y)\mathrm{d}y \leq C_{r,T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-(x-y+2rt)^2/4t}\mathrm{d}y = C_{r,T}.$$

□

Proposition A.1.2. *Fix $r \in \mathbb{R}$, $T > 0$. Then, for $p > 4$, we have that*

1. *For every $t, s \in [0, T]$,*

$$\sup_{x \geq 0} \left(\int_s^t \left[\int_0^\infty G_r(t-u, x, z)^2 dz \right]^{p/(p-2)} du \right)^{\frac{p-2}{2}} \leq C_{p,r,T} |t-s|^{(p-4)/4}.$$

2. *For every $x, y \in [0, \infty)$*

$$\sup_{s \in [0,T]} \left(\int_0^s \left[\int_0^\infty \left| G_r(s-u, x, z) - G_r(s-u, y, z) \right|^2 dz \right]^{p/(p-2)} du \right)^{\frac{p-2}{2}} \leq C_{p,r,T} |x-y|^{(p-4)/2}.$$

3. *For every $t, s \in [0, T]$,*

$$\sup_{x \geq 0} \left(\int_0^s \left[\int_0^\infty \left| G_r(t-u, x, z) - G_r(s-u, x, z) \right|^2 dz \right]^{p/(p-2)} du \right)^{\frac{p-2}{2}} \leq C_{p,r,T} |t-s|^{(p-4)/4}.$$

Proof of 1. Note that

$$e^{-r(x-z)} F_1(t, x, z) = e^{r^2 t} F_1(t, x + 2rt, z) = e^{r^2 t} F_1(t, x, z - 2rt). \tag{A.1.1}$$

Therefore,

$$\begin{aligned}
&\int_s^t \left[\int_0^\infty \left| e^{-r(x-z)} F_1(t-u, x, z) \right|^2 \mathrm{d}z \right]^{\frac{p}{p-2}} \mathrm{d}u \\
&\quad \leq C_{p,r,T} \int_0^{t-s} \left[\int_{\mathbb{R}} \left| F_1(u, x, z) \right|^2 \mathrm{d}z \right]^{p/(p-2)} \mathrm{d}u \\
&\quad = C_{p,r,T} \int_0^{t-s} \left[\int_{\mathbb{R}} \frac{1}{4\pi u} \exp\left(-\frac{(x-z)^2}{2u} \right) \mathrm{d}z \right]^{p/(p-2)} \mathrm{d}u \\
&\quad \leq C_{p,r,T} \int_0^{t-s} u^{-\frac{p}{2(p-2)}} \mathrm{d}u.
\end{aligned}$$

If $p > 4$, we have that $-\frac{p}{2(p-2)} > -1$, and this is equal to

$$C_{p,r,T} |t-s|^{\frac{p-4}{2(p-2)}}.$$

The result follows. □

Proof of 2. By (A.1.1), we have that

$$\begin{aligned}
&\int_0^s \left[\int_0^\infty |e^{-r(x-z)}F_1(s-u,x,z) - e^{-r(y-z)}F_1(s-u,y,z)|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u \\
&\leq C_{p,r,T}\int_0^s \left[\int_0^\infty |F_1(s-u,x,z-2rs) - F_1(s-u,y,z-2rs)|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u \\
&\leq C_{p,r,T}\int_0^s \left[\int_{\mathbb{R}} |F_1(s-u,x,z) - F_1(s-u,y,z)|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u \\
&= C_{p,r,T}\int_0^s \left[\int_{\mathbb{R}} \frac{1}{u}\left[\exp\left(-\frac{((x-y)+(y-z))^2}{4u}\right) - \exp\left(-\frac{(y-z)^2}{4u}\right)\right]^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u \\
&= C_{p,r,T}\int_0^\infty \left[\int_{\mathbb{R}} \frac{1}{u}\left[\exp\left(-\frac{((x-y)+h)^2}{4u}\right) - \exp\left(-\frac{h^2}{4u}\right)\right]^2 \mathrm{d}h\right]^{p/(p-2)} \mathrm{d}u.
\end{aligned}$$

Making the change of variables $w = \frac{h}{|x-y|}$ and $v = \frac{u}{(x-y)^2}$, we see that this is equal to

$$C_{p.r,T}|x-y|^{\frac{p-4}{p-2}} \int_0^\infty \left[\int_{\mathbb{R}} \frac{1}{v}\left|e^{-(1+w)^2/4v} - e^{-w^2/4v}\right|^2 \mathrm{d}w\right]^{p/(p-2)} \mathrm{d}v.$$

The integral here converges provided that $p > 4$, and only depends on p. We therefore have the result. □

Proof of 3. We again make use of equation (A.1.1). This gives that

$$\begin{aligned}
\left|e^{-r(x-z)}F_1(t,x,z) - e^{-r(x-z)}F_1(s,x,z)\right| \leq& e^{Tr^2}|F_1(t,x,z-2rt) - F_1(s,x,z-2rs)| \\
&+ F_1(s,x,z-2rs)|e^{r^2t} - e^{r^2s}| \\
\leq& e^{Tr^2}|F_1(t,x,z-2rt) - F_1(s,x,z-2rt)| \\
&+ e^{Tr^2}|F_1(s,x,z-2rt) - F_1(s,x,z-2rs)| \\
&+ r^2e^{Tr^2}F_1(s,x,z-2rs)|t-s|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\int_0^s \left[\int_0^\infty e^{-r(x-z)}|F_1(t-u,x,z) - F_1(s-u,x,z)|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u\right)^{\frac{p-2}{2}} \\
&\leq C_{p,r,T}\left(\int_0^s \left[\int_{\mathbb{R}} |F_1(t-u,x,z) - F_1(s-u,x,z)|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u\right)^{\frac{p-2}{2}} \\
&\quad + C_{p,r,T}\left(\int_0^s \left[\int_{\mathbb{R}} |F_1(s-u,x,z) - F_1(s-u,x,z-2r(t-s))|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u\right)^{\frac{p-2}{2}} \\
&\quad + C_{p,r,T}|t-s|^p\left(\int_0^s \left[\int_{\mathbb{R}} |F_1(s-u,x,z)|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u\right)^{\frac{p-2}{2}}.
\end{aligned}$$

The second term can be bounded by inequality (2) of this Proposition. This gives that it is at most

$$C_{p,r,T}|t-s|^{(p-4)/2}.$$

The integral in the third term is integrable if $p > 4$. Therefore, the third term is equal to $C_{p,r,T}|t-s|^p$ for $p > 4$. For the first term, we have that it is equal to

$$C_{p,r,T}\int_0^s \left[\int_{\mathbb{R}} \left|\frac{1}{\sqrt{(t-u)}}e^{-(x-z)^2/4((t-s)+(s-u))} - \frac{1}{\sqrt{s-u}}e^{-(x-z)^2/4(s-u)}\right|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u$$
$$= C_{p,r,T}\int_0^s \left[\int_{\mathbb{R}} \left|\frac{1}{\sqrt{(t-s)+u}}e^{-(x-z)^2/4((t-s)+u)} - \frac{1}{\sqrt{u}}e^{-(x-z)^2/4u}\right|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}u.$$

By making the substitution $v = \frac{u}{|t-s|}$, we see that this integral is at most

$$|t-s|\int_0^\infty \left[\int_{\mathbb{R}} \left|\frac{1}{\sqrt{(t-s)(1+v)}}e^{-z^2/4((t-s)(1+v))} - \frac{1}{\sqrt{(t-s)v}}e^{-z^2/4(t-s)v}\right|^2 \mathrm{d}z\right]^{p/(p-2)} \mathrm{d}v.$$

The substitution $\tilde{z} = \frac{z}{\sqrt{t-s}}$ then gives that this is equal to

$$|t-s|^{\frac{p-4}{2(p-2)}}\int_0^\infty \left[\int_{\mathbb{R}} \left|\frac{1}{\sqrt{1+v}}e^{-\tilde{z}^2/4(1+v)} - \frac{1}{\sqrt{v}}e^{-\tilde{z}^2/4v}\right|^2 \mathrm{d}\tilde{z}\right]^{p/(p-2)} \mathrm{d}v. \tag{A.1.2}$$

The integral in (A.1.2) converges provided that $p > 4$, giving the result. □

Proposition A.1.3. *Let $r \in \mathbb{R}$. Then for $t \in [0,T]$,*

$$\sup_{x\geq 0}\left(\int_0^\infty e^{-r(x-y)}\left|\frac{\partial G}{\partial y}(t,x,y)\right| dy\right) \leq \frac{C_{r,T}}{\sqrt{t}}.$$

Proof. It is sufficient to prove the result for F_1 and F_2 separately. Calculating gives that

$$e^{-r(x-y)}\frac{\partial F_1}{\partial y}(t,x,y) = \frac{(x-y)}{4\sqrt{\pi}t^{3/2}}\exp\left(-\frac{(x-y+2rt)^2}{4t}\right)\exp\left(r^2t\right)$$
$$\leq \frac{(x-y)}{4\sqrt{\pi}t^{3/2}}\exp\left(-\frac{(x-y+2rt)^2}{4t}\right)\exp\left(r^2T\right).$$

Therefore

$$\int_0^\infty e^{-r(x-y)}\left|\frac{\partial F_1}{\partial y}(t,x,y)\right| \mathrm{d}y \leq C_{r,T}\int_0^\infty \frac{(x-y)}{t^{3/2}}\exp\left(-\frac{(x-y+2rt)^2}{4t}\right)\mathrm{d}y$$
$$\leq C_{r,T}\int_0^\infty \frac{(x-y+2rt)}{t^{3/2}}\exp\left(-\frac{(x-y+2rt)^2}{4t}\right)\mathrm{d}y$$
$$\leq C_{r,T}\int_{\mathbb{R}} \frac{y}{t^{3/2}}\exp\left(-\frac{y^2}{4t}\right)\mathrm{d}y = \frac{C_{r,T}}{\sqrt{t}}.$$

The proof for the F_2 part of the heat kernel is similar. By noting that

$$e^{-r(x-y)}\frac{\partial F_2}{\partial y}(t,x,y) = \frac{x+y}{8t^{3/2}}\exp\left(-\frac{(x+y-2rt)^2}{4t}\right)\exp\left(r^2t-2rx\right)$$

and arguing as before, we obtain the result. □

Proposition A.1.4. *Fix $r \in \mathbb{R}$, $T > 0$. Then for $p > 4$, we have that*

1. *For every $x, y \in [0, \infty)$ with $|x - y| \leq 1$*

$$\sup_{s\in[0,T]} \left[\int_0^s \left(\int_0^\infty \left| e^{-r(x-z)} \frac{\partial G}{\partial z}(s-u, x, z) - e^{-r(y-z)} \frac{\partial G}{\partial z}(s-u, y, z) \right| dz \right)^{p/(p-2)} du \right]^{(p-2)/p} \leq C_{p,T,r} |x-y|^{(p-4)/2p},$$

2. *For $s, t \in [0, T]$,*

$$\sup_{x\in[0,\infty)} \left[\int_0^s \left(\int_0^\infty \left| e^{-r(x-z)} \frac{\partial G}{\partial z}(t-u, x, z) - e^{-r(x-z)} \frac{\partial G}{\partial z}(s-u, x, z) \right| dz \right)^{p/(p-2)} du \right]^{(p-2)/p} \leq C_{p,T,r} |t-s|^{(p-4)/4p},$$

3. *For $s, t \in [0, T]$,*

$$\sup_{x\in[0,\infty)} \left[\int_s^t \left(\int_0^\infty \left| e^{-r(x-z)} \frac{\partial G}{\partial z}(t-u, x, z) \right| dz \right)^{p/(p-2)} du \right]^{(p-2)/p} \leq C_{p,T,r} |t-s|^{(p-4)/4p}.$$

Proof of 1. First note that

$$\begin{aligned} e^{-r(x-z)} \frac{\partial F_1}{\partial z}(u, x, z) =& \frac{(x-z+2ru)}{4\sqrt{\pi} u^{3/2}} \exp\left(-\frac{(x-z+2ru)^2}{4u} \right) \exp\left(r^2 u \right) \\ & - \frac{r}{2\sqrt{\pi}\sqrt{u}} \exp\left(-\frac{(x-z+2ru)^2}{4u} \right) \exp\left(r^2 u \right). \end{aligned} \tag{A.1.3}$$

We bound the terms corresponding to these two components separately. Let $q = p/(p-2)$. Then, by applying Hölder's inequality, we have that

$$\begin{aligned} & \int_0^s \left(\int_0^\infty \left| \frac{(x-z+2ru)}{u^{3/2}} \exp\left(-\frac{(x-z+2ru)^2}{4u} \right) - \frac{(y-z+2ru)}{u^{3/2}} \exp\left(-\frac{(y-z+2ru)^2}{4u} \right) \right| \mathrm{d}z \right)^q \mathrm{d}u \\ & \leq \int_0^s \left(\int_{\mathbb{R}} \left| \frac{(x-y+h)}{u^{3/2}} \exp\left(-\frac{(x-y+h)^2}{4u} \right) - \frac{h}{u^{3/2}} \exp\left(-\frac{h^2}{4u} \right) \right| \mathrm{d}h \right)^q \mathrm{d}u \\ & \leq \int_0^\infty \left(\int_{\mathbb{R}} \left| \frac{(x-y+h)}{u^{3/2}} \exp\left(-\frac{(x-y+h)^2}{4u} \right) - \frac{h}{u^{3/2}} \exp\left(-\frac{h^2}{4u} \right) \right| \mathrm{d}h \right)^q \mathrm{d}u. \end{aligned}$$

By letting $h = |x-y|v$, we see that this is equal to

$$\int_0^\infty \left(\int_{\mathbb{R}} |x-y|^2 \left| \frac{(v+1)}{u^{3/2}} \exp\left(-\frac{(v+1)^2 (x-y)^2}{4u} \right) - \frac{v}{u^{3/2}} \exp\left(-\frac{v^2 (x-y)^2}{4u} \right) \right| \mathrm{d}v \right)^q \mathrm{d}u$$

We split the time integral into two parts- the integral on $[0, |x-y|]$ and the integral on $(|x-y|, \infty)$. For the first of these domains, we have

$$\begin{aligned}
&\int_0^{|x-y|} \left(\int_{\mathbb{R}} |x-y|^2 \left| \frac{(v+1)}{u^{3/2}} \exp\left(-\frac{(v+1)^2(x-y)^2}{4u} \right) - \frac{v}{u^{3/2}} \exp\left(-\frac{v^2(x-y)^2}{4u} \right) \right| \mathrm{d}v \right)^q \mathrm{d}u \\
&\quad \leq C_p |x-y|^{2p/(p-2)} \int_0^{|x-y|} \left(\frac{1}{\sqrt{u}(x-y)^2} \right)^q \mathrm{d}u \\
&\quad = C_p \int_0^{|x-y|} u^{-p/(2p-4)} \mathrm{d}u = C_p |x-y|^{(p-4)/(2p-4)}.
\end{aligned}$$

To bound on $(|x-y|, \infty)$, we note that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xe^{-x^2/4u}) = e^{-x^2/4u} - \frac{x^2}{2u} e^{-x^2/4u} \leq Ce^{-x^2/8u}.$$

Therefore, for $v \in (0, \infty)$, we have that

$$\begin{aligned}
&\left| \frac{(v+1)}{u^{3/2}} \exp\left(-\frac{(v+1)^2(x-y)^2}{4u} \right) - \frac{v}{u^{3/2}} \exp\left(-\frac{v^2(x-y)^2}{4u} \right) \right| \\
&\leq \frac{C}{u^{3/2}} e^{-v^2(x-y)^2/8u}.
\end{aligned}$$

Similarly, for $v \in (-\infty, -1)$, we have that

$$\begin{aligned}
&\left| \frac{(v+1)}{u^{3/2}} \exp\left(-\frac{(v+1)^2(x-y)^2}{4u} \right) - \frac{v}{u^{3/2}} \exp\left(-\frac{v^2(x-y)^2}{4u} \right) \right| \\
&\leq \frac{C}{u^{3/2}} e^{-(v+1)^2(x-y)^2/8u}.
\end{aligned}$$

In the region $v \in [-1, 0]$, we have that

$$\left| \frac{(v+1)}{u^{3/2}} \exp\left(-\frac{(v+1)^2(x-y)^2}{4u} \right) - \frac{v}{u^{3/2}} \exp\left(-\frac{v^2(x-y)^2}{4u} \right) \right| \leq C \frac{1}{u^{3/2}}.$$

It follows that

$$\begin{aligned}
&\int_{|x-y|}^{\infty} \left(\int_{\mathbb{R}} |x-y|^2 \left| \frac{(v+1)}{u^{3/2}} \exp\left(-\frac{(v+1)^2(x-y)^2}{4u} \right) - \frac{v}{u^{3/2}} \exp\left(-\frac{v^2(x-y)^2}{4u} \right) \right| \mathrm{d}v \right)^q \mathrm{d}u \\
&\quad \leq C_p \int_{|x-y|}^{\infty} \left[\frac{|x-y|^2}{u^{3/2}} \left(1 + \int_{\mathbb{R}} e^{-v^2(x-y)^2/8u} \mathrm{d}v \right) \right]^q \mathrm{d}u \\
&\quad \leq C_p |x-y|^{2q} \int_{|x-y|}^{\infty} u^{-3q/2} + (u|x-y|)^{-q} \mathrm{d}u \\
&\quad = C_p |x-y|^{2q} (|x-y|^{-(3q-2)/2} + |x-y|^{(1-2q)})
\end{aligned}$$

Since $|x-y| \leq 1$, this is at most $C_p |x-y|^{(p-4)/(2p-4)}$. We have therefore deduced inequality (1) for the first component on the right hand side of expression (A.1.3). For the second component

of (A.1.3), we have that that

$$\begin{aligned}
&\int_0^\infty \left| \frac{1}{\sqrt{r}} \exp\left(-\frac{(x-z+2ru)^2}{4u}\right) \exp\left(r^2 u\right) - \frac{1}{\sqrt{u}} \exp\left(-\frac{(y-z+2ru)^2}{4u}\right) \exp\left(r^2 u\right) \right| \mathrm{d}z \\
&\quad \leq \int_{\mathbb{R}} \left| \frac{1}{\sqrt{u}} \exp\left(-\frac{(x-y+h)^2}{4u}\right) \exp\left(r^2 u\right) - \frac{1}{\sqrt{u}} \exp\left(-\frac{h^2}{4u}\right) \exp\left(r^2 u\right) \right| \mathrm{d}h \\
&\quad \leq C_{r,T} \int_{\mathbb{R}} \left| \frac{1}{\sqrt{u}} \exp\left(-\frac{(x-y+h)^2}{4u}\right) - \frac{1}{\sqrt{u}} \exp\left(-\frac{h^2}{4u}\right) \right| \mathrm{d}h.
\end{aligned}$$

Since

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} (e^{-x^2/4u}) \right| \leq \frac{C}{\sqrt{u}} e^{-x^2/8u},$$

we have that, for $h \notin [-|x-y|, 0]$,

$$\left| \exp\left(-\frac{(x-y+h)^2}{4u}\right) - \exp\left(-\frac{h^2}{4u}\right) \right| \leq \frac{C}{\sqrt{u}} |x-y| \max\{e^{-h^2/8u}, e^{-(x-y+h)^2/8u}\}.$$

For $h \in [-|x-y|, 0]$, we use the simple bound

$$\left| \exp\left(-\frac{(x-y+h)^2}{4u}\right) - \exp\left(-\frac{h^2}{4u}\right) \right| \leq 2.$$

Putting this together, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \frac{1}{\sqrt{u}} \exp\left(-\frac{(x-y+h)^2}{4u}\right) \exp\left(r^2 u\right) - \frac{1}{\sqrt{u}} \exp\left(-\frac{h^2}{4u}\right) \exp\left(r^2 u\right) \right| \mathrm{d}h \\
&\quad \leq C_{\delta,T} \left(\frac{|x-y|}{\sqrt{u}} + |x-y| \int_{\mathbb{R}} \frac{1}{u} e^{-h^2/8u} \mathrm{d}u \right) = C_{\delta,T} \frac{|x-y|}{\sqrt{u}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^s \left[\int_0^\infty \frac{1}{\sqrt{u}} \exp\left(r^2 u\right) \left| \exp\left(-\frac{(x-z+2ru)^2}{4u}\right) - \exp\left(-\frac{(y-z+2ru)^2}{4u}\right) \right| \mathrm{d}z \right]^{p/(p-2)} \mathrm{d}u \\
&\quad \leq C_{T,r,p} |x-y|^{p/(p-2)}.
\end{aligned}$$

Inequality (1) for the second component of (A.1.3) is then a simple consequence of this. The manipulations required to prove inequalities (2) and (3) are similar, and we therefore omit these lengthy calculations. □

A.1.2 Heat Kernel on $[0, 1]$

Recall that

$$H(t, x, y) := \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-y+2n)^2}{4t}\right) - \exp\left(-\frac{(x+y+2n)^2}{4t}\right) \right].$$

We make the observation here that this expression can be written as

$$H(t,x,y) = G(t,x,y) - \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x+y-2)^2}{4t}\right) + L(t,x,y),$$

where L is a smooth function of t, x, y which vanishes at $t = 0$. Consequently, we are able to prove the estimates for H in this section in analogous ways to how we proved the corresponding results for G. We therefore omit the proofs here.

Proposition A.1.5. *Fix $T > 0$. Then, for $p > 4$, we have that*

1. *For every $t, s \in [0,T]$,*

$$\sup_{x\in[0,1]} \left(\int_s^t \left[\int_0^1 H(t-r,x,z)^2 dr \right]^{\frac{p}{p-2}} dz \right)^{\frac{p-2}{2}} \leq C_p |t-s|^{(p-4)/4}.$$

2. *For every $t, s \in [0,T]$,*

$$\sup_{x\in[0,1]} \left(\int_0^s \left[\int_0^1 \big(H(t-r,x,z) - H(s-r,x,z)\big)^2 ds \right]^{\frac{p}{p-2}} dz \right)^{\frac{p-2}{2}} \leq C_p |t-s|^{(p-4)/4}.$$

3. *For every $x, y \in [0,1]$,*

$$\sup_{s\in[0,T]} \left(\int_0^s \left[\int_0^1 \big(H(s-r,x,z) - H(s-r,y,z)\big)^2 dz \right]^{p/(p-2)} dr \right)^{\frac{p-2}{2}} \leq C_p |x-y|^{(p-4)/2}.$$

Proposition A.1.6. *For $t \in [0,T]$,*

$$\sup_{x\geq 0} \left(\int_0^\infty \left| \frac{\partial H}{\partial y}(t,x,y) \right| dy \right) \leq \frac{C_T}{\sqrt{t}}.$$

Proposition A.1.7. *Fix $T > 0$ and $p > 4$. We have that*

1. *For every $t, s \in [0,T]$,*

$$\sup_{x\in[0,1]} \left[\int_0^s \left(\int_0^1 \left| \frac{\partial H}{\partial z}(t-r,x,z) - \frac{\partial H}{\partial z}(s-r,x,z) \right| dz \right)^{p/(p-2)} dr \right]^{(p-2)/p} \leq C|t-s|^{(p-4)/4p},$$

2. *For every $x, y \in [0,1]$,*

$$\sup_{s\in[0,T]} \left[\int_0^s \left(\int_0^1 \left| \frac{\partial H}{\partial z}(s-r,x,z) - \frac{\partial H}{\partial z}(s-r,y,z) \right| dz \right)^{p/(p-2)} dr \right]^{(p-2)/p} \leq C|x-y|^{(p-4)/2p},$$

3. *For every $t, s \in [0,T]$,*

$$\sup_{x\in[0,1]} \left[\int_s^t \left(\int_0^1 \left| \frac{\partial H}{\partial z}(t-r,x,z) \right| dz \right)^{p/(p-2)} dr \right]^{(p-2)/p} \leq C|t-s|^{(p-4)/4p}.$$

A.2 Proof of Theorem 4.2.7

In this section, we will prove a series of Propositions which together constitute a proof of Theorem 4.2.7.

Proposition A.2.1. *Let $r \in \mathbb{R}$ and $v \in \mathscr{C}_r^T$ such that $v(t,0) = 0$. For $\epsilon > 0$, let z^ϵ be the solution to the PDE*

$$\frac{\partial z^\epsilon}{\partial t} = \Delta z^\epsilon + \frac{1}{\epsilon}\arctan(((z^\epsilon - v) \wedge 0)^2). \tag{A.2.1}$$

Then z^ϵ increases as $\epsilon \downarrow 0$.

Proof. Let $0 < \epsilon_2 \leq \epsilon_1$. Fix some $\delta > 0$ and set $y(t,x) := e^{-\delta x}(z^{\epsilon_1}(t,x) - z^{\epsilon_2}(t,x))$. We have that y satisfies

$$\frac{\partial y}{\partial t} = \Delta y + 2\delta\frac{\partial y}{\partial x} + \delta^2 y + \frac{e^{-\delta x}}{\epsilon_1}\arctan(((z^{\epsilon_1} - v) \wedge 0)^2) - \frac{e^{-\delta x}}{\epsilon_2}\arctan(((z^{\epsilon_2} - v) \wedge 0)^2).$$

We know that

$$\|y\|_{T,\mathscr{C}_\delta} = \|z^{\epsilon_1} - z^{\epsilon_2}\|_{T,\infty} < \infty.$$

Testing our equation for y with the positive part of y, we obtain that

$$\begin{aligned}\|y_T^+\|_{L^2}^2 = &-\int_0^T \left\|\frac{\partial y_t^+}{\partial x}\right\|_{L^2}^2 \mathrm{d}t + \delta\int_0^T\int_0^\infty \frac{\partial((y^+)2)}{\partial x}(s,x)\mathrm{d}x\mathrm{d}s + \delta^2\int_0^T \|y_t^+\|_{L^2}^2\mathrm{d}t \\ &+ \text{Negative Part.}\end{aligned}$$

We note that testing against the last two terms gives a negative contribution, since when $y \geq 0$, we have that $z_{\epsilon_1} \geq z_{\epsilon_2}$ and so $((z^{\epsilon_1} - v) \wedge 0)^2 \leq ((z^{\epsilon_2} - v) \wedge 0)^2$, from which it follows that

$$\frac{e^{-\delta x}}{\epsilon_1}\arctan(((z^{\epsilon_1} - v) \wedge 0)^2) - \frac{e^{-\delta x}}{\epsilon_2}\arctan(((z^{\epsilon_2} - v) \wedge 0)^2 \leq 0.$$

Putting this together, we see that

$$\|y_T^+\|_{L^2}^2 \leq \delta^2\int_0^T \|y_t^+\|_{L^2}^2\mathrm{d}t.$$

Gronwall's inequality then gives that $y_T^+ = 0$ i.e. that $z_{\epsilon_1} \leq z_{\epsilon_2}$. □

The following bound will allow us to control solutions of our obstacle problems by the obstacles themselves.

Proposition A.2.2. *Let $r \in \mathbb{R}$ and v_1,$v_2 \in \mathscr{C}_r^T$ such that $v_1(t,0) = v_2(t,0) = 0$. Fix $\epsilon > 0$. For $i = 1,2$, let z_i^ϵ be the solution to the PDE*

$$\frac{\partial z_i^\epsilon}{\partial t} = \Delta z_i^\epsilon + \frac{1}{\epsilon}\arctan(((z_i^\epsilon - v_i) \wedge 0)^2) \tag{A.2.2}$$

with boundary condition $z_i^\epsilon(t,0) = 0$ and zero initial data. Then there exists a constant $C_{r,T}$ such that

$$\|z_1^\epsilon - z_2^\epsilon\|_{\mathscr{C}_r^T} \leq C_{r,T}\|v_1 - v_2\|_{\mathscr{C}_r^T}.$$

Proof. Let $\delta := \|v_1 - v_2\|_{\mathscr{C}_r^T}$, and define $w : [0,T] \times [0,\infty) \to \mathbb{R}$ such that

$$w(t,x) := \delta e^{rx+r^2t}.$$

Then we have that w solves the heat equation

$$\frac{\partial w}{\partial t} = \Delta w,$$

with initial data

$$w(0,x) = \delta e^{rx}$$

and boundary condition $w(t,0) = \delta e^{r^2t}$. Define

$$\tilde{z}(t,x) := e^{-(r+1)x}(z_1^\epsilon(t,x) - z_2^\epsilon(t,x) - w(t,x)).$$

Then $\tilde{z}$ solves the equation

$$\begin{aligned}\frac{\partial \tilde{z}}{\partial t} =& \Delta \tilde{z} + 2(r+1)\frac{\partial \tilde{z}}{\partial x} + (r+1)^2\tilde{z} + \frac{e^{-(r+1)x}}{\epsilon}\arctan(((z_1^\epsilon - v_1) \wedge 0)^2) \\ &- \frac{e^{-(r+1)x}}{\epsilon}\arctan(((z_2^\epsilon - v_2) \wedge 0)^2),\end{aligned} \tag{A.2.3}$$

with initial data $\tilde{z}(0,x) = -\delta e^{-x}$ and boundary condition $\tilde{z}(t,0) = -\delta e^{r^2t}$. Note that when $\tilde{z} \geq 0$, we have

$$\begin{aligned}z_1^\epsilon(t,x) - z_2^\epsilon(t,x) \geq e^{rx+r^2t}\|v_1 - v_2\|_{\mathscr{C}_r^T} &\geq e^{rx+r^2t}|e^{-rx}v_1(t,x) - e^{-rx}v_2(t,x)| \\ &\geq v_1(t,x) - v_2(t,x).\end{aligned}$$

Hence, when $\tilde{z} \geq 0$,

$$\frac{e^{-(r+1)x}}{\epsilon}\arctan(((z_1^\epsilon - v_1) \wedge 0)^2) - \frac{e^{-(r+1)x}}{\epsilon}\arctan(((z_2^\epsilon - v_2) \wedge 0)^2) \leq 0.$$

Therefore, testing the equation with $\tilde{z}^+$ we obtain that for $t \in [0,T]$

$$\begin{aligned}\int_0^t\int_0^\infty \frac{\partial \tilde{z}}{\partial t}(s,x)\tilde{z}^+(s,x)\mathrm{d}x\mathrm{d}s \leq& \int_0^t\int_0^\infty \Delta\tilde{z}(s,x)\tilde{z}^+(s,x)\mathrm{d}x\mathrm{d}s \\ &+ 2(r+1)\int_0^t\int_0^\infty \frac{\partial \tilde{z}}{\partial x}(s,x)\tilde{z}^+(s,x)\mathrm{d}x\mathrm{d}s \\ &+ (r+1)^2\int_0^\infty \tilde{z}(s,x)\tilde{z}^+(s,x)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

By integrating by parts and noting that $\tilde{z}^+$ is zero at time $t = 0$ and vanishes at $x = 0$ and $x = \infty$, we obtain that

$$\frac{1}{2}\|\tilde{z}^+(t,\cdot)\|_{L^2}^2 \leq (r+1)^2\int_0^t \|\tilde{z}^+(s,\cdot)\|_{L^2}^2\mathrm{d}s.$$

It follows by an application of Gronwall's inequality that $z^+(t,\cdot) = 0$. Therefore, we have that

$$z_1^\epsilon(t,x) - z_2^\epsilon(t,x) \leq w(t,x).$$

Interchanging z_1^ϵ and z_2^ϵ, we also have that

$$z_2^\epsilon(t,x) - z_1^\epsilon(t,x) \leq w(t,x).$$

It follows that

$$\|z_1^\epsilon - z_2^\epsilon\|_{\mathscr{C}_r^T} \leq \|w\|_{\mathscr{C}_r^T} = e^{r^2 T}\|v_1 - v_2\|_{\mathscr{C}_r^T}.$$

□

We are now in position to argue existence for the obstacle problem on $[0,\infty)$.

Proposition A.2.3. *Let $r \in \mathbb{R}$ and $v \in \mathscr{C}_r^T$, with $v(0,\cdot) \leq 0$. Then there exists (z,η) solving the heat equation with obstacle v and exponential growth r.*

Proof. Proposition A.2.1 gives that the solutions z^ϵ to the equations (A.2.2) are increasing as $\epsilon \downarrow 0$. For $x \geq 0$ and $t \in [0,T]$, let

$$z(t,x) := \lim_{\epsilon \downarrow 0} z^\epsilon(t,x).$$

By Proposition A.2.2, we have that

$$\sup_{\epsilon > 0} \|z^\epsilon\|_{\mathscr{C}_r^T} \leq C_{r,T}\|v\|_{\mathscr{C}_r^T}.$$

Letting $\epsilon \downarrow 0$, it follows that

$$\sup_{t\in[0,T]} \sup_{x \in [0,1]} |e^{-rx} z(t,x)| \leq C_{r,T}\|v\|_{\mathscr{C}_r^T}. \tag{A.2.4}$$

We now argue that z is continuous. Let $w_n \in C_c^\infty([0,T]\times(0,\infty))$ be such that $w_n(0,\cdot) \leq 0$ and

$$\sup_{t\in[0,T]} \sup_{x \geq 0} |w_n(t,x) - e^{-(r+\delta)x} v(t,x)| \to 0,$$

where $\delta > 0$. Let $v_n(t,x) := e^{(r+\delta)x} w_n(t,x)$, so we have that

$$\|v - v_n\|_{\mathscr{C}_{r+\delta}^T} \to 0.$$

As we did in order to construct z, we define the functions z_n^ϵ to be the solutions to the equations

$$\frac{\partial z_n^\epsilon}{\partial t} = \Delta z_n^\epsilon + \frac{1}{\epsilon}\arctan(((z_n^\epsilon - v_n) \wedge 0)^2). \tag{A.2.5}$$

We can then argue as in the proof of Theorem 4.2.16, differentiating the equation in space and time respectively, and applying Proposition 4.2.18 to see that

$$\left\| \frac{\partial z_n^\epsilon}{\partial x} \right\|_{\mathscr{C}_{r+\delta}^T} \leq C_{r+\delta,T} \left\| \frac{\partial v_n}{\partial x} \right\|_{\mathscr{C}_{r+\delta}^T},$$

and

$$\left\| \frac{\partial z_n^\epsilon}{\partial t} \right\|_{\mathscr{C}_{r+\delta}^T} \leq C_{r+\delta,T} \left\| \frac{\partial v_n}{\partial t} \right\|_{\mathscr{C}_{r+\delta}^T}$$

Note that we use the condition that $v_n(0,\cdot) \leq 0$ for this step, in order to ensure that the initial data for $\frac{\partial z_n^\epsilon}{\partial t}$ and $\frac{\partial z_n^\epsilon}{\partial x}$ is zero. Therefore, $\frac{\partial z_n^\epsilon}{\partial x}$ and $\frac{\partial z_n^\epsilon}{\partial t}$ are uniformly bounded over ϵ on compact subsets of $[0,T] \times [0,\infty)$. We can now argue that z is continuous. We have that, for $M > 0$, $(t,x),(s,y) \in [0,T] \times [0,M]$ and $n \geq 1$,

$$\begin{aligned}|z(t,x) - z(s,y)| &= \lim_{\epsilon \downarrow 0} |z^\epsilon(t,x) - z^\epsilon(s,y)| \\ &\leq \liminf_{\epsilon \downarrow 0} \left[|z_n^\epsilon(t,x) - z_n^\epsilon(s,y)| + 2 \sup_{t \in [0,T]} \sup_{x \in [0,M]} \left| z^\epsilon(t,x) - z_n^\epsilon(t,x) \right| \right].\end{aligned}$$

We know that

$$\sup_{\epsilon > 0} \sup_{t \in [0,T]} \sup_{x \in [0,M]} \left| z^\epsilon(t,x) - z_n^\epsilon(t,x) \right| \leq e^{(r+\delta)M} \sup_{\epsilon > 0} \|z^\epsilon - z_n^\epsilon\|_{\mathscr{C}^T_{r+\delta}} \leq C_{r+\delta,T,M} \|v - v_n\|_{\mathscr{C}^T_{r+\delta}} \to 0$$

as $n \to \infty$, using the bound from Proposition A.2.2. We also have that, for $n \geq 1$,

$$\begin{aligned}\sup_{\epsilon > 0} |z_n^\epsilon(t,x) - z_n^\epsilon(s,y)| &\leq e^{(r+\delta)M} \sup_{\epsilon > 0} \left[\left\| \frac{\partial z_n^\epsilon}{\partial x} \right\|_{\mathscr{C}^T_{r+\delta}} + \left\| \frac{\partial z_n^\epsilon}{\partial t} \right\|_{\mathscr{C}^T_{r+\delta}} \right] \big(|t-s| + |x-y| \big) \\ &\leq C_{M,r+\delta,T} \left[\left\| \frac{\partial v_n}{\partial x} \right\|_{\mathscr{C}^T_{r+\delta}} + \left\| \frac{\partial v_n}{\partial t} \right\|_{\mathscr{C}^T_{r+\delta}} \right] \big(|t-s| + |x-y| \big) \to 0\end{aligned}$$

as $(s,y) \to (t,x)$. It follows that

$$|z(t,x) - z(s,y)| \to 0$$

as $(s,y) \to (t,x)$, and so z is continuous. Therefore, $z \in \mathscr{C}^T_r$. We now want to verify that z solves the obstacle problem. Clearly, $z(t,0) = 0$ and $z(0,x) = 0$ for all x. Let $\varphi \in C_c^\infty([0,T] \times [0,\infty))$ with $\varphi(t,0) = 0$ for every t. Testing the equation for z^ϵ with φ, we see that

$$\begin{aligned}\int_0^\infty z^\epsilon(T,x)\varphi(T,x)\mathrm{d}x = &\int_0^T \int_0^\infty z^\epsilon(t,x) \frac{\partial \varphi}{\partial t}(t,x)\mathrm{d}x\mathrm{d}t + \int_0^T \int_0^\infty z^\epsilon(t,x) \frac{\partial^2 \varphi}{\partial x^2}(t,x)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_0^\infty \varphi(t,x) \frac{1}{\epsilon} \arctan(((z^\epsilon(t,x) - v(t,x)) \wedge 0)^2)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{A.2.6}$$

Define

$$\eta_\epsilon(\mathrm{d}t,\mathrm{d}x) := \frac{1}{\epsilon} \arctan(((z^\epsilon - v) \wedge 0)^2)\mathrm{d}x\mathrm{d}t.$$

Letting $\epsilon \to 0$, we see that $\eta_\epsilon \to \eta$ in distribution on $[0,T] \times (0,\infty)$, to some positive distribution η. It follows from it's positivity that η is a measure, and we have that for every $\varphi \in C_c^\infty([0,T] \times [0,\infty))$ with $\varphi(t,0) = 0$ for every t,

$$\begin{aligned}\int_0^\infty z(T,x)\varphi(T,x)\mathrm{d}x = &\int_0^T \int_0^\infty z(t,x) \frac{\partial \varphi}{\partial t}(t,x)\mathrm{d}x\mathrm{d}t + \int_0^T \int_0^\infty z(t,x) \frac{\partial^2 \varphi}{\partial x^2}(t,x)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_0^\infty \varphi(t,x)\eta(\mathrm{d}t,\mathrm{d}x).\end{aligned}$$

It is left to check that $z - v \geq 0$ and that the integral of $z - v$ against the measure η is zero (i.e. the reflection measure is supported on the set where z hits the obstacle, v). Multiplying (A.2.6) by ϵ, we obtain that

$$\int_0^T \int_0^\infty \varphi(s,x) \arctan(((z^\epsilon - v) \wedge 0)^2) \mathrm{d}x\mathrm{d}s = o(\epsilon).$$

Letting $\epsilon \to 0$, we see that

$$\int_0^T \int_0^\infty \varphi(s,x) \arctan(((z - v) \wedge 0)^2) \mathrm{d}x\mathrm{d}s = 0.$$

It follows that we must have $z - v \geq 0$. Finally, we want to have that

$$\int_0^T \int_0^\infty (z(t,x) - v(t,x)) \, \eta(\mathrm{dx},\mathrm{dt}) = 0.$$

Since z^ϵ is increasing as ϵ decreases, we see that $\mathrm{supp}(\eta) \subset \mathrm{supp}(\eta_\epsilon)$ for every $\epsilon > 0$. Also, we have that for every $\epsilon > 0$, $z^\epsilon - v \leq 0$ on the support of η_ϵ, and so on the support of η. Therefore, for $(t,x) \in \mathrm{supp}(\eta)$, we have that

$$z(t,x) - v(t,x) = \lim_{\epsilon \downarrow 0} (z^\epsilon(t,x) - v(t,x)) \leq 0.$$

Since $z - v \geq 0$, we have that $z - v = 0$ on $\mathrm{supp}(\eta)$. So we have the result, and (z,η) is a solution to the obstacle problem. □

We now turn to the problem of uniqueness. The following lemma is an adaptation of the result from Section 2.3 in [40].

Lemma A.2.4. *Let (z_1,η_1) and (z_2,η_2) be two solutions to the obstacle problem with obstacle $v \in \mathscr{C}_r^T$. Set $\psi(t,x) := z_1(t,x) - z_2(t,x)$. Then, for $\phi \in C_c^\infty([0,\infty))$ with $\phi(0) = 0$ and $t \in [0,T]$, we have that*

$$\int_0^\infty \psi^2(t,x)\phi^2(x)dx \leq \int_0^t \int_0^\infty \psi^2(s,x)(\phi^2)''(x)dx\,ds.$$

Proof. Fix some $\phi \in C_c^\infty([0,\infty))$ with $\phi(0) = 0$. The result would follow if we could test the equation for ψ with the function $\psi(t,x)\phi^2(x)$. Since this isn't possible, as ψ isn't regular enough, we must test with a smooth approximation of this function and take a limit. Let ϵ be a non-negative function supported on $[-1,1]$ which is symmetric, smooth and so that

$$\int_{-1}^1 \epsilon(x)\mathrm{d}x = 1.$$

Assume further that ϵ is non-negative definite, so that for any $n \geq 1$ and any $(x_1,...x_n),(y_1,...y_n) \in \mathbb{R}^n$

$$\sum_{i=1}^n \sum_{j=1}^n \epsilon(x_i - x_j) y_i y_j \geq 0.$$

We then obtain approximations of the identity, given by $\epsilon_n(x) := n\epsilon(nx)$. We now define the function of two variables, $\epsilon_{n,m}$ to be

$$\epsilon_{n,m}(t,x) := \epsilon_n(t)\epsilon_m(x).$$

We continuously extend ψ to a function on $\mathbb{R}\times\mathbb{R}$ by setting $\psi\equiv 0$ on $\big(\mathbb{R}\times(-\infty,0)\big)\cup\big((-\infty,0)\times\mathbb{R}\big)$, and setting $\psi(t,x)=\psi(T,x)$ for $t>T$ and $x\in\mathbb{R}$. Define the function $d_{n,m}$ to be given by

$$d_{n,m}:=((\psi\phi)*\epsilon_{n,m})\phi,$$

where $*$ here denotes the convolution on $\mathbb{R}^2$. That is, we define

$$\begin{aligned}d_{n,m}(t,x)&=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(s,y)\phi(y)\epsilon_n(t-s)\epsilon_m(x-y)\mathrm{d}y\mathrm{d}s\right)\phi(x)\\&=\left(\int_{(t-1/n)^+}^{(t+1/n)}\int_0^\infty\psi(s,y)\phi(y)\epsilon_n(t-s)\epsilon_m(x-y)\mathrm{d}y\mathrm{d}s\right)\phi(x).\end{aligned}$$

The function $d_{n,m}$ is now a smooth compactly supported approximation of $\psi(t,x)\phi(x)^2$. Also, since $\phi(0)=0$, we have that $d_{n,m}(\cdot,0)=0$. We can therefore test the equation for ψ against this function. Doing so gives that for $t\in[0,T]$

$$\begin{aligned}\int_0^\infty d_{n,m}(t,x)\psi(t,x)\,\mathrm{d}x=&\int_0^t\int_0^\infty\frac{\partial(d_{n,m})}{\partial t}(s,x)\psi(s,x)\,\mathrm{d}x\mathrm{d}s+\int_0^t\int_0^\infty\psi(s,x)\frac{\partial^2 d_{n,m}}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s\\&+\int_0^t\int_0^\infty d_{n,m}(s,x)\,\eta_1(\mathrm{dx},\mathrm{dt})-\int_0^t\int_0^\infty d_{n,m}(s,x)\,\eta_2(\mathrm{dx},\mathrm{dt}).\end{aligned}$$

We take the limit for each term separately. The first term is

$$\int_0^\infty d_{n,m}(t,x)\psi(t,x)\,\mathrm{d}x.$$

As $n,m\to\infty$, we can apply the DCT to see that this converges to

$$\int_0^\infty d_{n,m}(t,x)\psi(t,x)\,\mathrm{d}x\to\int_0^\infty\psi^2(t,x)\phi^2(x)\mathrm{d}x.$$

For the second term, we have that

$$\begin{aligned}&\int_0^t\int_0^\infty\frac{\partial(d_{n,m})}{\partial t}(s,x)\psi(s,x)\,\mathrm{d}x\mathrm{d}s\\&\quad=\int_0^t\int_{s-1/n}^{s+1/n}\epsilon_n'(s-r)\left(\int_0^\infty\int_0^\infty\psi(r,y)\phi(y)\epsilon_m(x-y)\phi(x)\psi(s,x)\mathrm{d}x\mathrm{d}y\right)\mathrm{d}r\mathrm{d}s.\end{aligned}\tag{A.2.7}$$

Define

$$\Gamma_m(r,s):=\int_0^\infty\int_0^\infty\psi(r,y)\phi(y)\epsilon_m(x-y)\phi(x)\psi(s,x)\mathrm{d}x\mathrm{d}y.$$

Then, since Γ is symmetric in r,s and that $\epsilon'(r)=-\epsilon'(-r)$, we see that by symmetry, (A.2.7) is equal to

$$\int_{t-\frac{1}{n}}^t\int_t^{s+\frac{1}{n}}\epsilon_n'(s-r)\Gamma_m(r,s)\mathrm{d}r\mathrm{d}s.\tag{A.2.8}$$

Let $A_n:=\{(r,s)\mid s\in[t-1/n,t],\ r\in[t,s+1/n]\}$. We can choose ϵ so that it's derivative is approximately equal to 1 on the interval $[-1,0]$, and therefore the derivative of ϵ_n is approximately n^2 on $[-1/n,0]$. We also have that $\Gamma_m(r,s)$ is equal to

$$\int_0^\infty\psi^2(t,x)\phi^2(x)\mathrm{d}x+R_{m,n}$$

on A_n, where $R_{n,m} \to 0$ as $n, m \to \infty$. It follows from these calculations, and the fact that $\mu_{\text{Leb}}(A) = 1/2n^2$ that (A.2.8) converges to

$$\frac{1}{2}\int_0^\infty \psi^2(t,x)\phi^2(x)\mathrm{d}x$$

as $n, m \to \infty$. The limit of the combination of the reflection terms is at most zero, since

$$\begin{aligned}&\lim_{n,m\to\infty}\left[\int_0^t\int_0^\infty d_{n,m}(t,x)\left(\eta_1(\mathrm{d}t,\mathrm{d}x)-\eta_2(\mathrm{d}t,\mathrm{d}x)\right)\right]\\&=\int_0^\infty \psi(t,x)\phi(x)^2\left(\eta_1(\mathrm{d}t,\mathrm{d}x)-\eta_2(\mathrm{d}t,\mathrm{d}x)\right).\end{aligned}$$

Since $z_i - v$ is zero on the support of η_i and $z_i - v \geq 0$, we have that

$$\begin{aligned}&\int_0^\infty \psi(t,x)\phi(x)^2\left(\eta_1(\mathrm{d}t,\mathrm{d}x)-\eta_2(\mathrm{d}t,\mathrm{d}x)\right)\\&=\int_0^\infty \left(z_1(t,x)-z_2(t,x)\right)\phi(x)^2\,\eta_1(\mathrm{d}t,\mathrm{d}x)+\int_0^\infty \left(z_2(t,x)-z_1(t,x)\right)\phi(x)^2\,\eta_2(\mathrm{d}t,\mathrm{d}x)\\&=\int_0^\infty \left(v(t,x)-z_2(t,x)\right)\phi(x)^2\,\eta_1(\mathrm{d}t,\mathrm{d}x)+\int_0^\infty \left(v(t,x)-z_1(t,x)\right)\phi(x)^2\,\eta_2(\mathrm{d}t,\mathrm{d}x)\leq 0.\end{aligned}$$

Finally, we deal with the second derivative term,

$$\int_0^t\int_0^\infty \psi(s,x)\frac{\partial^2 d_{n,m}}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s. \tag{A.2.9}$$

Letting $n \to \infty$, we have that, for each $m \geq 1$, this converges to

$$\int_0^t\int_0^\infty \psi(s,x)\frac{\partial^2 d_m}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s \tag{A.2.10}$$

where

$$d_m(t,x) = \left(\int_0^\infty \psi(t,y)\phi(y)\epsilon_m(x-y)\mathrm{d}y\right)\phi(x).$$

In order to bound (A.2.10), we first suppose that ψ is smooth. Integrating by parts we obtain that (using angle bracket notation to indicate integration over $[0,\infty)$ and indexing d_m by ψ to highlight it's dependence on the function ψ)

$$\begin{aligned}\left\langle \frac{\partial^2}{\partial x^2}d_m^\psi(t,\cdot),\psi(t,\cdot)\right\rangle &= \left\langle \frac{\partial^2}{\partial x^2}(((\psi(t,\cdot)\phi(\cdot))*\epsilon_m)\phi(\cdot)),\psi(t,\cdot)\right\rangle\\&=-\left\langle \frac{\partial}{\partial x}(((\psi(t,\cdot)\phi(\cdot))*\epsilon_m)\phi(\cdot)),\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle\\&=-\left\langle (\frac{\partial}{\partial x}(\psi(t,\cdot)\phi(\cdot))*\epsilon_m)\phi(\cdot),\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle\\&\quad-\left\langle ((\psi(t,\cdot)\phi(\cdot))*\epsilon_m)\frac{\partial}{\partial x}\phi(\cdot),\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle\\&=-\left\langle (\frac{\partial}{\partial x}\psi(t,\cdot)\phi(\cdot))*\epsilon_m,\phi(\cdot)\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle\\&\quad-\left\langle (\psi(t,\cdot)\frac{\partial}{\partial x}\phi(\cdot))*\epsilon_m,\phi(\cdot)\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle\\&\quad-\left\langle ((\psi(t,\cdot)\phi(\cdot))*\epsilon_m),\frac{\partial}{\partial x}\phi(\cdot)\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle.\end{aligned} \tag{A.2.11}$$

We bound this last expression. Positive definiteness of ϵ ensures that the first term is negative. We split the last term into two, using that

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\psi\right) - \frac{\partial^2\phi}{\partial x^2}\psi,$$

Together, this gives that (A.2.11) is at most

$$\begin{aligned}
&-\left\langle (\psi(t,\cdot)\frac{\partial}{\partial x}\phi(\cdot)) * \epsilon_m, \phi(\cdot)\frac{\partial}{\partial x}\psi(t,\cdot)\right\rangle - \left\langle ((\psi(t,\cdot)\phi(\cdot)) * \epsilon_m), \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\psi\right) - \frac{\partial^2\phi}{\partial x^2}\psi\right\rangle.\\
&= -\left\langle ((\psi(t,\cdot)\frac{\partial\phi}{\partial x}(\cdot)) * \epsilon_m), \phi(\cdot)\frac{\partial\psi}{\partial x}(t,\cdot)\right\rangle + \left\langle ((\frac{\partial\psi}{\partial x}(t,\cdot)\phi(\cdot)) * \epsilon_m), \frac{\partial\phi}{\partial x}(\cdot)\psi(t,\cdot)\right\rangle\\
&\quad + \left\langle ((\psi(t,\cdot)\frac{\partial\phi}{\partial x}(\cdot)) * \epsilon_m), \frac{\partial\phi}{\partial x}(\cdot)\psi(t,\cdot)\right\rangle + \left\langle ((\psi(t,\cdot)\phi(\cdot)) * \epsilon_m), \frac{\partial^2\phi}{\partial x^2}(\cdot)\psi(t,\cdot)\right\rangle\\
&= \left\langle ((\psi(t,\cdot)\frac{\partial\phi}{\partial x}(\cdot)) * \epsilon_m), \frac{\partial\phi}{\partial x}(\cdot)\psi(t,\cdot)\right\rangle + \left\langle ((\psi(t,\cdot)\phi(\cdot)) * \epsilon_m), \frac{\partial^2\phi}{\partial x^2}(\cdot)\psi(t,\cdot)\right\rangle
\end{aligned}$$

So we have shown that, if ψ were smooth, then

$$\left\langle \frac{\partial^2 d_m^\psi}{\partial x}(t,\cdot), \psi(t,\cdot)\right\rangle \leq \left\langle (\psi(t,\cdot)\phi(\cdot)) * \epsilon_m, \frac{\partial^2\phi}{\partial x^2}(\cdot)\psi(t,\cdot)\right\rangle + \left\langle (\psi(t,\cdot)\frac{\partial\phi}{\partial x}(\cdot)) * \epsilon_m, \frac{\partial\phi}{\partial x}(\cdot)\psi(t,\cdot)\right\rangle.$$

By taking a sequence of smooth functions which approximate ψ in the infinity norm on the support of ϕ, we obtain (making use of the compact support of ϕ in order to pass to the limit for the integrals) that this formula holds for our original choice of ψ, $\psi = z_1 - z_2$. Taking the lim sup of both sides gives that

$$\begin{aligned}
\limsup_{m\to\infty}\left\langle \frac{\partial^2 d_m}{\partial x}(t,\cdot), \psi(t,\cdot)\right\rangle &\leq \left\langle \psi(t,\cdot)\phi(\cdot), \frac{\partial^2\phi}{\partial x^2}(\cdot)\psi(t,\cdot)\right\rangle + \left\langle \psi(t,\cdot)\frac{\partial\phi}{\partial x}(\cdot), \frac{\partial\phi}{\partial x}(\cdot)\psi(t,\cdot)\right\rangle\\
&\leq \left\langle \psi^2(t,\cdot), \frac{\partial}{\partial x}(\phi\frac{\partial\phi}{\partial x})\right\rangle = \frac{1}{2}\left\langle \psi^2(t,\cdot), \frac{\partial^2(\phi^2)}{\partial x^2}\right\rangle.
\end{aligned}$$

By the reverse Fatou Lemma, we then obtain that

$$\begin{aligned}
\limsup_{m\to\infty}\int_0^t\int_0^\infty \psi(s,x)\frac{\partial^2 d_m}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s &\leq \int_0^t \limsup_{m\to\infty}\left\langle \frac{\partial^2 d_m}{\partial x}(s,\cdot), \psi(s,\cdot)\right\rangle \mathrm{d}s\\
&\leq \frac{1}{2}\int_0^t\int_0^\infty \psi^2(s,x)(\phi^2)''(x)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Altogether, we have shown that

$$\int_0^\infty \psi^2(t,x)\phi^2(x)\mathrm{d}x \leq \frac{1}{2}\int_0^\infty \psi^2(t,x)\phi^2(x)\mathrm{d}x + \frac{1}{2}\int_0^t\int_0^\infty \psi^2(s,x)(\phi^2)''(x)\mathrm{d}x.$$

A simple rearrangement then gives the result. □

Proposition A.2.5. *Let $r \in \mathbb{R}$. The parabolic obstacle problem with exponential growth r has a unique solution.*

Proof. The ideas for this proof borrow from those in Theorem 5.3 of [42]. Once again, $\psi = z_1 - z_2$ where (z_1, η_1) and (z_2, η_2) are two solutions to the obstacle problem with obstacle $v \in \mathscr{C}_r^T$. Let $\phi^2(x) = e^{-2x}\sinh(x)$, and let ϕ_n be a sequence of functions in $C_c^\infty([0,\infty))$ such that $\phi_n \to \phi$ in $H^{2,2}((0,\infty))$. Let $\tilde{r} := \max(r, 0)$. We then have that

$$\left|\int_0^\infty \psi^2(t,x)e^{-2\tilde{r}x}\left(\phi^2(x) - \phi_n^2(x)\right)\mathrm{d}x\right| \leq \sup_{t\in[0,T]}\sup_{x\geq 0}\left[\psi^2(t,x)e^{-2\tilde{r}x}\right] \times \int_0^\infty |(\phi^2(x) - \phi_n^2(x))|\mathrm{d}x.$$

This converges to zero as $n \to \infty$. Note that for a C^2 function φ, we have that

$$((e^{-\tilde{r}y}\varphi)^2)''(y) = e^{-2\tilde{r}y}\left(4\tilde{r}^2\varphi^2(y) - 8\tilde{r}\varphi(y)\varphi'(y) + 2\varphi''(y)\varphi(y) + 2(\varphi'(y))^2\right)$$

Since

$$\sup_{t\in[0,T]}\sup_{x\geq 0}\left[\psi^2(t,x)e^{-2\tilde{r}x}\right] < \infty$$

almost surely and $\phi_n \to \phi$ in $H^{2,2}((0,\infty))$, we have that

$$\begin{aligned}
&\int_0^T\int_0^\infty \psi^2(s,y)e^{-2\tilde{r}y}\left(4\tilde{r}^2\phi_n^2(y) - 8\tilde{r}\phi_n(y)\phi_n'(y) + 2\phi_n''(y)\phi_n(y) + 2(\phi_n'(y))^2\right)\mathrm{d}y\mathrm{d}s \\
&\to \int_0^T\int_0^\infty \psi^2(s,y)e^{-2\tilde{r}y}\left(4\tilde{r}^2\phi^2(y) - 8\tilde{r}\phi(y)\phi'(y) + 2\phi''(y)\phi(y) + 2(\phi'(y))^2\right)\mathrm{d}y\mathrm{d}s. \quad \text{(A.2.12)}
\end{aligned}$$

Hence, the inequality from Lemma 2 still holds with the function $e^{-(\tilde{r}+1)x}\sqrt{\sinh(x)}$. Note that

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{-2(\tilde{r}+1)x}\sinh(x)) &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(e^{-(2\tilde{r}+1)x} - e^{-(2\tilde{r}+3)x}\right) \\
&= \frac{1}{2}\left((2\tilde{r}+1)^2e^{-(2\tilde{r}+1)x} - (2\tilde{r}+3)^2e^{-(2\tilde{r}+3)x}\right) \\
&\leq \frac{1}{2}\left((2\tilde{r}+3)^2e^{-(2\tilde{r}+1)x} - (2\tilde{r}+3)^2e^{-(2\tilde{r}+3)x}\right) \\
&= C_r e^{-2(\tilde{r}+1)x}\sinh(x).
\end{aligned}$$

Applying the result with this function, we obtain that, for $t \in [0,T]$,

$$\int_0^\infty \psi^2(t,x)e^{-(2\tilde{r}+1)x}\sinh(x)\mathrm{d}x \leq C_r\int_0^t\int_0^\infty \psi^2(s,x)e^{-2(\tilde{r}+1)x}\sinh(x)\mathrm{d}x\mathrm{d}s.$$

By Gronwall, we see that $\int_0^\infty \psi^2(t,x)e^{-2(\tilde{r}+1)x}\sinh(x)\,\mathrm{d}x = 0$ for all $t \in [0,T]$. Therefore, $\psi = 0$, so we have uniqueness for our problem. □

Proof of Theorem 4.2.7. By Proposition A.2.3 and Proposition A.2.5, we have existence and uniqueness of a solution. Suppose now that v_1, $v_2 \in \mathscr{C}_r^T$ and z_1, z_2 are solutions to the associated obstacle problems. Then, as in the proof of Proposition A.2.3 and applying the estimate from Proposition A.2.2, we have for $\epsilon > 0$ functions z_1^ϵ and z_2^ϵ such that for $i = 1, 2$

$$z_i(t,x) = \lim_{\epsilon\downarrow 0} z_i^\epsilon(t,x)$$

and

$$\|z_1^\epsilon - z_2^\epsilon\|_{\mathscr{C}_r^T} \leq C_{r,T}\|v_1 - v_2\|_{\mathscr{C}_r^T}.$$

Letting $\epsilon \downarrow 0$, we obtain that

$$\|z_1 - z_2\|_{\mathscr{C}_r^T} \leq C_{r,T} \|v_1 - v_2\|_{\mathscr{C}_r^T}.$$

This concludes the proof. □

Appendix B

Section 5 Appendix

We provide the main details for the proofs of the inequalities stated in Proposition 4.2. Recalling Remark 5.3.1, we focus on proving the estimates for the component G of H, where G is given by

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right].$$

We also define $\tilde{G}$ such that

$$\tilde{G}(t,x,y) := \frac{y}{x}G(t,x,y). \tag{B.0.1}$$

The proofs of the estimates in Lemma 3.2 of [61] were very helpful for producing the following arguments.

B.1 Proof of Inequality 1

Proof. The first inequality in the Proposition states that, for every $T > 0$, there exists a constant C_T such that, for every $t \in (0,T]$,

$$\sup_{x\in(0,1]}\int_0^1 \frac{1}{x}H(t,x,y)\mathrm{d}y \leq \frac{C_T}{\sqrt{t}}.$$

We prove the corresponding bound for G.

$$\begin{aligned}\frac{1}{x}G(t,x,y) =& \frac{1}{x\sqrt{4\pi t}}\left[\exp\left(-(x-y)^2/4t\right) - \exp\left(-(x+y)^2/4t\right)\right]\\ =& \frac{1}{x\sqrt{4\pi t}}\exp\left(-(x-y)^2/4t\right)\left[1-\exp\left(-xy/t\right)\right].\end{aligned}$$

Note that, since $x \geq 0$

$$xy = x((y-x)+x) \leq 2x^2\mathbb{1}_{\{y-x\leq x\}} + 2x(y-x)\mathbb{1}_{\{y-x>x\}}.$$

Therefore

$$\left(1-\exp(-xy/t)\right) \leq \left(1-\exp(-2x^2/t)\right)\mathbb{1}_{\{y-x\leq x\}} + \left(1-\exp(-2x(y-x)/t)\right)\mathbb{1}_{\{y-x>x\}}.$$

Since, for $x \geq 0$

$$(1-e^{-x}) \leq \min(x,\sqrt{x}),$$

we have that

$$\big(1-\exp(-xy/t)\big) \le 2\left[\frac{x}{\sqrt{t}} + \frac{x|y-x|}{t}\right].$$

Putting these bounds together, we see that

$$\begin{aligned}\int_0^1 \frac{1}{x} G(t,x,y)\,\mathrm{d}y &\le C\int_0^1 \frac{1}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{4t}\right)\left[\frac{1}{\sqrt{t}} + \frac{|y-x|}{t}\right]\mathrm{d}y \\ &\le C\int_{\mathbb{R}}\left[\frac{1}{t}e^{-y^2/4t} + \frac{|y|}{t^{3/2}}e^{-y^2/4t}\right]\mathrm{d}y = \frac{C}{\sqrt{t}}.\end{aligned}$$

□

B.2 Proof of Inequality 3

Proof. We begin by recalling the inequality. For every $T > 0$ and $q \in (1,2)$, we claim that $\exists C_{T,q}$ such that for every $x, y \in [0,1]$,

$$\sup_{t\in[0,T]}\int_0^t \left[\int_0^1 \left(\tilde{H}(s,x,z) - \tilde{H}(s,y,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}s \le C_{T,q}|x-y|^{(2-q)/3}.$$

We will prove the corresponding bound for $\tilde{G}$. The arguments made in the proof of Lemma 3.2.1 of [61] are followed here, and we make adjustments where necessary. Assume wlog that $x \le y$, and define $h = y - x$. By following the proofs of inequalities (1) and (2) in Lemma 3.2.1 of [61], we arrive at the the inequality

$$\begin{aligned}0 \le \int_0^1 \left(\tilde{G}(s,x,z) - \tilde{G}(s,y,z)\right)^2 \mathrm{d}y \le &\frac{C}{\sqrt{s}}\left[1 + s\left(\frac{1-e^{-x^2/4s}}{x^2} + \frac{1-e^{-(x+h)^2/4s}}{(x+h)^2}\right)\right. \\ &\left. - e^{-h^2/16s}\left(1 + 2s\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right] =: CR(s,x,h).\end{aligned}$$

So it is enough to prove that

$$\int_0^T R(s,x,h)^q\,\mathrm{d}s \le C_{T,q}h^{(2-q)/3}.$$

In the spirit of [61] once again, we control this integral by proving two separate estimates for it which we will then combine. These estimates are obtained by splitting R up into different components. Note that

$$R(s,x,h) = R_1(s,x,h) + R_2(s,x,h) \ge 0,$$

where we define

$$R_1(s,x,h) := \frac{1}{\sqrt{s}}\left(1 - e^{-h^2/16s}\right),$$

and

$$R_2(s,x,h) := \sqrt{s}\left[\frac{1-e^{-x^2/4s}}{x^2} + \frac{1-e^{-(x+h)^2/4s}}{(x+h)^2} - 2e^{-h^2/16s}\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right].$$

Therefore, if $I(s,x,h) \geq 0$ such that $R_2(s,x,h) \leq I(s,x,h)$ we have that

$$R(s,x,h)^q \leq (R_1(s,x,h) + I(s,x,h))^q \leq C_q \left(R_1(s,x,h)^q + I(s,x,h)^q\right).$$

Our approach will therefore be to bound the integral of the R_1 and I terms separately, and we do so for two different functions I. Starting with the R_1 term, we first note that

$$R_1(s,x,h) \leq \frac{1}{\sqrt{s}}, \quad \text{and that} \quad R_1(s,x,h) \leq \frac{h^2}{16s^{3/2}},$$

where we once again make use of the inequality $(1 - e^{-x}) \leq x$. It then follows that

$$\int_0^T R_1(s,x,h)^q \mathrm{d}s \leq \int_0^{h^2} \left(\frac{1}{\sqrt{s}}\right)^q \mathrm{d}s + \int_{h^2}^{\infty} \frac{h^{2q}}{16^q s^{3q/2}} \mathrm{d}s,$$

Calculating, we see that this is equal to Ch^{2-q}. For the R_2 term, we have the bound

$$R_2(s,x,h) \leq \sqrt{s}\left[\frac{1}{2s} - 2e^{-h^2/16s}\left(\frac{1}{4s} - \frac{x(x+h)}{32s^2}\right)\right],$$

where we have made use of the fact that

$$x - \frac{x^2}{2} \leq 1 - e^{-x} \leq x.$$

This is equal to

$$\frac{1}{2\sqrt{s}}\left[1 - e^{-h^2/16s}\right] + \frac{x(x+h)}{16s^{3/2}} e^{-h^2/16s}.$$

The first of these terms is in the same form as R_1, and so we can control it in the same way. For the second term, we have that

$$\begin{aligned}
\left(\frac{x(x+h)}{16s^{3/2}} e^{-h^2/16s}\right)^q &= C_q x^q (x+h)^q \left(\frac{e^{-h^2/16s}}{s^{3/2}}\right)^q \\
&\leq C_q x^q (x+h)^q \left(\frac{s}{s^{3/2}h^2}\right)^q \\
&= C_q x^q (x+h)^q s^{-q/2} h^{-2q},
\end{aligned}$$

since $e^{-x} \leq \frac{C}{x}$. This bound then gives that

$$\int_0^T \left(\frac{x(x+h)}{16s^{3/2}} e^{-h^2/16s}\right)^q \mathrm{d}s \leq C_q x^q (x+h)^q \left(\int_0^{h^2} s^{-q/2} h^{-2q} \, \mathrm{d}s + \int_{h^2}^{\infty} \frac{1}{s^{3q/2}} \mathrm{d}s\right).$$

Calculating gives that the right hand side is equal to

$$C_q x^q (x+h)^q h^{2-3q}.$$

So we obtain our first bound

$$\int_0^T R(s,x,h)^q \, \mathrm{d}s \leq C_q \left[h^{2-q} + x^q (x+h)^q h^{2-3q}\right]. \tag{B.2.1}$$

We prove our second estimate for the integral by bounding R_2 differently. We have that

$$\begin{aligned} R_2(s,x,h) \leq & \sqrt{s}\left|\frac{1-e^{-x^2/4s}}{x^2}+\frac{1-e^{-(x+h)^2/4s}}{(x+h)^2}-2\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right| \\ & +\sqrt{s}\left|2\left(1-e^{-h^2/16s}\right)\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right|. \end{aligned} \tag{B.2.2}$$

The second term on the right hand side is at most

$$\frac{C}{\sqrt{s}}\left(1-e^{-h^2/16s}\right).$$

This is once again in the form of R_1, and so can be controlled in the same way. In order to bound the first term on the right hand side of (B.2.2), we define here the function

$$\phi(t,x) := \frac{1-e^{-x/4t}}{x}.$$

Then we can calculate that, for $x \in (0,1]$

$$\left|\frac{\partial\phi}{\partial x}(s,x)\right| = \left|-\frac{1}{x^2}(1-e^{-x/4s})+\frac{1}{4xs}e^{-x/4s}\right| \leq \frac{C}{xs}. \tag{B.2.3}$$

Therefore,

$$\begin{aligned} & \sqrt{s}\left|\frac{1-e^{-x^2/4s}}{x^2}+\frac{1-e^{-(x+h)^2/4s}}{(x+h)^2}-2\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right| \\ & \quad = \sqrt{s}\left|\phi(s,x^2)+\phi(s,(x+h)^2)-2\phi(s,x(x+h))\right|. \end{aligned}$$

Using (B.2.3), we see that this is at most

$$\frac{C}{x^2\sqrt{s}} \times (xh+h^2) = C\left(\frac{h}{x\sqrt{s}}+\frac{h^2}{x^2\sqrt{s}}\right).$$

We have that

$$C\int_0^T\left[\left(\frac{h}{x\sqrt{s}}+\frac{h^2}{x^2\sqrt{s}}\right)\right]^q \mathrm{d}s = C_{T,q}\left(\frac{h^q}{x^q}+\frac{h^{2q}}{x^{2q}}\right).$$

Putting these parts together, we have deduced our second estimate,

$$\int_0^T R(s,x,h)^q \mathrm{d}s \leq C_{T,q}\left(h^{2-q}+\frac{h^q}{x^q}+\frac{h^{2q}}{x^{2q}}\right).$$

We can now conclude our proof by splitting into cases, and using the two inequalities in the different scenarios. For $x \leq h^{\frac{4q-2}{3q}}$, we use our first bound, (B.2.1), and obtain that

$$\begin{aligned} \int_0^T R(s,x,h)^q \, \mathrm{d}s & \leq C_q\left[h^{2-q}+x^q(x+h)^q h^{2-3q}\right] \\ & \leq C_q\left[h^{2-q}+h^{\frac{8q-4+6-9q}{3}}\right] \leq C_q h^{\frac{2-q}{3}}, \end{aligned}$$

where we have used that, since $q \in (1,2)$, we have $(4q-2)/3q \in (2/3,1)$ and so $h \leq h^{(4q-2)/3q}$. If $x > h^{\frac{4q-2}{3q}}$, we have by our second bound that

$$\begin{aligned}\int_0^T R(s,x,h)^q \mathrm{d}s &\leq C_{T,q}\left(h^{2-q} + \frac{h^q}{x^q} + \frac{h^{2q}}{x^{2q}}\right) \\ &\leq C_{T,q}\left(h^{2-q} + h^{q-\frac{4q-2}{3}} + h^{2q-\frac{8q-4}{3}}\right) \\ &= C_{T,q}\left(h^{2-q} + h^{(2-q)/3} + h^{2(2-q)/3}\right) \\ &\leq C_{T,q} h^{(2-q)/3},\end{aligned}$$

recalling that $h \in [0,1]$ and $q \in (1,2)$. This concludes the proof. □

B.3 Proof of Inequality 4

Proof. We begin by recalling the inequality. For $T > 0$, $0 \leq s \leq t \leq T$ and $q \in (1,2)$, $\exists\, C_{T,q} > 0$ such that

$$\sup_{x\in[0,1]} \int_0^s \left[\int_0^1 (\tilde{H}(t-r,x,y) - \tilde{H}(s-r,x,y))^2 \mathrm{d}y\right]^q \mathrm{d}r \leq C_{T,q}|t-s|^{(2-q)/2}.$$

Once again, we will prove the corresponding bound for $\tilde{G}$, given by (B.0.1). The proof will have two steps. We will first perform a change of variables, which will allow us to bound the integral by $|t-s|^{(2-q)/2}$ multiplied by some integral which does not depend on t, s, which we denote by $I(x)$. We will then show that

$$\sup_{x\geq 0} I(x) < \infty.$$

Let $k := t - s$. We have that

$$\begin{aligned}&\int_0^s \left[\int_0^1 (\tilde{G}(s-r+k,x,y) - \tilde{G}(s-r,x,y))^2 \mathrm{d}y\right]^q \mathrm{d}r \\ &\quad= \int_0^s \left[\int_0^1 (\tilde{G}(r+k,x,y) - \tilde{G}(r,x,y))^2 \mathrm{d}y\right]^q \mathrm{d}r \\ &\quad= \int_0^s \left[\int_0^1 \frac{1}{k}\left(\tilde{G}\left(\frac{r}{k}+1, \frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}\right) - \tilde{G}\left(\frac{r}{k}, \frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}\right)\right)^2 \mathrm{d}y\right]^q \mathrm{d}r.\end{aligned}$$

Performing the change of variables $z := y/\sqrt{k}$ and $u = r/k$, we obtain that this is at most $k^{(2-q)/2} \times I(\frac{x}{\sqrt{k}})$, where

$$I(x) = \int_0^\infty \left[\int_0^\infty \left(\tilde{G}(u+1,x,z) - \tilde{G}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u.$$

By inequality (1) of Lemma 3.2.1 in [61], we have that there exists $C > 0$ such that for $u \geq 0$

$$\sup_{x\geq 0} \int_0^\infty \tilde{G}(u,x,z)^2 \mathrm{d}z \leq \frac{C}{\sqrt{u}}.$$

Therefore, for $u \geq 0$ we have

$$\sup_{x\geq 0}\int_0^\infty \left(\tilde{G}(u+1,x,z)-\tilde{G}(u,x,z)\right)^2 \mathrm{d}z \leq \frac{C}{\sqrt{u}}.$$

It follows that, for $q \in (1,2)$,

$$\int_0^1 \left[\int_0^\infty \left(\tilde{G}(u+1,x,z)-\tilde{G}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u = C_q < \infty.$$

So it is only left to control

$$\int_1^\infty \left[\int_0^\infty \left(\tilde{G}(u+1,x,z)-\tilde{G}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u. \tag{B.3.1}$$

We split the space integral here into two cases, when $\{z \geq 2x\}$, and when $\{z < 2x\}$. For the integral over the spatial region $\{z < 2x\}$, we have that,

$$\begin{aligned}
&\int_1^\infty \left[\int_0^{2x} \left(\tilde{G}(u+1,x,z)-\tilde{G}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq& C_q \int_1^\infty \left[\int_0^\infty \left(G(u+1,x,z)-G(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq& C_q \int_1^\infty \left[\int_0^\infty \frac{1}{u}\left[e^{-(x-z)^2/4(u+1)}-e^{-(x-z)^2/4u}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&+ C_q \int_1^\infty \left[\int_0^\infty \frac{1}{u}\left[e^{-(x+z)^2/4(u+1)}-e^{-(x+z)^2/4u}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&+ C_q \int_1^\infty \left[\int_0^\infty \left(\frac{1}{\sqrt{u}}-\frac{1}{\sqrt{u+1}}\right)^2 \left(e^{-(x-z)^2/4(u+1)}-e^{-(x+z)^2/4(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq& C_q \int_1^\infty \left[\int_{\mathbb{R}} \frac{1}{u}\left(e^{-z^2/4(u+1)}-e^{-z^2/4u}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&+ C_q \int_1^\infty \frac{1}{u^{3q}}\left[\int_0^\infty \left(e^{-(x-z)^2/4(u+1)}-e^{-(x+z)^2/4(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq& C_q \int_1^\infty \left[\int_{\mathbb{R}} \frac{1}{u} e^{-z^2/2(u+1)}\left(1-e^{-z^2(\frac{1}{4u}-\frac{1}{4(u+1)})}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u + C_q \int_1^\infty \frac{1}{u^{5q/2}}\mathrm{d}u \\
\leq& C_q \int_1^\infty \left[\int_{\mathbb{R}} \frac{z^4}{u^5} e^{-z^2/2(u+1)}\mathrm{d}z\right]^q \mathrm{d}u + C_q \int_1^\infty \frac{1}{u^{5q/2}}\mathrm{d}u = C_q \int_1^\infty \frac{1}{u^{5q/2}}\mathrm{d}u.
\end{aligned}$$

This last term is finite and doesn't depend on x, so we have successfully bounded (B.3.1) in the case when $\{z < 2x\}$. On the set $\{z \geq 2x\}$, we have that

$$
\begin{aligned}
&\int_1^\infty \left[\int_{2x}^\infty \left(\tilde{G}(u+1,x,z) - \tilde{G}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&\leq C_q \int_0^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\frac{1}{u}\left(\left(e^{-(x-z)^2/4(u+1)} - e^{-(x+z)^2/4(u+1)}\right) - \left(e^{-(x-z)^2/4u} - e^{-(x+z)^2/4u}\right)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&\qquad + C_q \int_0^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\left(\frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u+1}}\right)^2 \left(e^{-(x-z)^2/4(u+1)} - e^{-(x+z)^2/4(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&= C_q \int_0^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\frac{1}{u}\left(e^{-(x-z)^2/4(u+1)}\left(1 - e^{-xz/(u+1)}\right) - e^{-(x-z)^2/4u}\left(1 - e^{-xz/u}\right)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&\qquad + C_q \int_0^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\left(\frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u+1}}\right)^2 \left(e^{-(x-z)^2/4(u+1)} - e^{-(x+z)^2/4(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&\leq C_q \int_1^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\frac{1}{u}\left(e^{-xz/(u+1)} - e^{-xz/u}\right)^2 e^{-(x-z)^2/2(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \\
&\qquad + C_q \int_1^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\frac{1}{u}\left(1 - e^{-xz/u}\right)^2 \left(e^{-(x-z)^2/4(u+1)} - e^{-(x-z)^2/4u}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&\qquad + C_q \int_1^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\frac{1}{u^3} e^{-(x-z)^2/2(u+1)}\left(1 - e^{-xz/(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u
\end{aligned}
$$

This is at most

$$
\begin{aligned}
&C_q \int_1^\infty \left[\int_{2x}^\infty \frac{z^2}{x^2}\frac{1}{u} e^{-2xz/(u+1)}\left(1 - e^{-xz/u(u+1)}\right)^2 e^{-(x-z)^2/2(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \\
&\quad + C_q \int_1^\infty \left[\int_{2x}^\infty \frac{z^4}{u^3} e^{-(x-z)^2/2(u+1)}\left(1 - e^{-(x-z)^2/4u(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
&\quad + C_q \int_1^\infty \left[\int_{2x}^\infty \frac{z^4}{u^3(u+1)^2} e^{-(z-x)^2/2(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \\
&\leq C_q \int_1^\infty \left[\int_0^\infty \frac{z^4}{u^3(u+1)^2} e^{-z^2/8(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \\
&\quad + C_q \int_1^\infty \left[\int_0^\infty \frac{z^8}{u^5(u+1)^2} e^{-z^2/8(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \\
&\quad + C_q \int_1^\infty \left[\int_0^\infty \frac{z^4}{u^3(u+1)^2} e^{-z^2/8(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \\
&\leq C_q \int_1^\infty \frac{1}{u^{5q/2}} \mathrm{d}u = C_q < \infty.
\end{aligned}
$$

Therefore we have that $\sup_{x \geq 0} I(x) < \infty$, concluding the proof. □

B.4 Proof of Inequality 5

Proof. We want to prove that, for $t \in (0, T]$

$$\sup_{x\in[0,1]} \int_0^1 \left|\tilde{J}(t,x,y)\right| \mathrm{d}y \leq \frac{C_T}{\sqrt{t}}.$$

We prove the corresponding bound for K, where we define, for $x \in (0,1]$, $y \in [0,1]$ and $t \geq 0$,

$$K(t,x,y) := \frac{y}{x}\frac{\partial G}{\partial y}(t,x,y),$$

and for $y \in [0,1]$, $t \geq 0$ we define

$$K(t,0,y) := y\frac{\partial^2 G}{\partial x \partial t}(t,0,y).$$

Note that, for $t > 0$, $x \in (0,1]$, $y \in [0,1]$ and $y \geq 2x$,

$$\begin{aligned}
\left|K(t,x,y)\right| \leq & \frac{Cy}{t^{3/2}}\left[e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}\right] + \frac{Cy^2}{t^{3/2}x}\left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \\
\leq & \frac{Cy}{t^{3/2}}e^{-y^2/16t} + \frac{Cy^2}{t^{3/2}x}e^{-(x-y)^2/4t}\left[1 - e^{-xy/t}\right] \\
\leq & e^{-y^2/16t}\left[\frac{Cy}{t^{3/2}} + \frac{Cy^3}{t^{5/2}}\right].
\end{aligned}$$

By letting $x \downarrow 0$, we also have that the bound holds at $x = 0$ for $t > 0$ and $y \in [0,1]$. For $x \in (0,1]$, $y \in [0,1]$ and $y < 2x$, we have that

$$\begin{aligned}
|K(t,x,y)| \leq & 2\left|\frac{\partial G}{\partial y}(t,x,y)\right| \\
\leq & \frac{C}{t^{3/2}}\left[(x-y)e^{-(x-y)^2/4t} + (x+y)e^{-(x+y)^2/4t}\right] \\
\leq & \frac{C}{t^{3/2}}\left[(x-y)e^{-(x-y)^2/16t} + (x+y)e^{-(x+y)^2/16t}\right]
\end{aligned}$$

It follows that

$$\sup_{x\in[0,1]} \int_0^1 \left|K(t,x,y)\right| \mathrm{d}y \leq C\int_{\mathbb{R}} e^{-y^2/16t}\left[\frac{y}{t^{3/2}} + \frac{y^3}{t^{5/2}}\right] \mathrm{d}y \leq \frac{C}{\sqrt{t}}.$$

□

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