Derived Complex Analytic Geometry Via Bornological Methods

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October 2025

Abstract

We present an exposition of several results on sheaves on analytic spaces and stacks. A simplification of the Verdier duality for bornological sheaves of [45] on complex manifolds is presented in the second chapter, using duality arguments of [13]. A thorough account of the abstract theory of six-functor formalisms of [28], [34] , and the foundations of derived analytic geometry via bornological methods, are given in the following two chapters. The results of [54] are adapted to the setting of overconvergent geometry in the fifth chapter. The final chapter applies six-functor formalism to present the analytic Beilinson-Bernstein localization of [51].

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1 Introduction

For the last seventy years, sheaf theory has been an indispensable tool for algebraic and analytic geometers. It elegantly unifies and simplifies the work of geometers from before modern categorical methods, and facilitated the translation of cohomological techniques to modern geometric contexts.

Sheaf theory evolved parallel to homological algebra, both serving as theories that operate under a common maxim of homotopical algebra: theories should be built on simple objects, and complicated objects should be considered relative to, or *resolved in terms of* simple objects. Algebras should be resolved by free objects, sheaves by injective or flat sheaves, algebras by polynomial algebras, and so on. The techniques of homological algebra and sheaf theory in algebraic geometry are compatible, and as they evolved, and their scope extended modules and quasicoherent sheaves, the notions of *abelian category, triangulated category and t-structure* were crystallized. However, the methods of complex analysis and sheaves on complex spaces had long made use of the topological structure on the sheaves, and so these methods were not immediately tractable via these abstract notions.

This problem can be resolved using the formalism of exact categories of Quillen [46], vastly extending the scope of the techniques of homological algebra. However, while the topological vector spaces arising in complex geometry don't define abelian categories, they define *quasi-abelian categories* in the sense of Schneiders [50]. While in an abelian category all morphisms are *strict* (images and coimages coincide), we cannot expect this for topological vector spaces, as the resulting quotient topology need only map *continuously* to the image, *not* homeomorphically. However, those morphisms which are strict satisfy some preservation properties under pullbacks and pushforwards, and this is enough to develop a theory of homological algebra remarkably similar to that of abelian categories. In this way, for a complex space X, $\mathcal{O}(X)$-modules endowed with the canonical Fréchet structure on local sections can be studied using the same homological techniques of algebraic geometry, with only minor technical adjustments.

Fundamental to sheaf theory is their functoriality with respect to base spaces. These functorial properties, along with their compatibilities in the form of base change, the projection formula, and the Künneth formula, lead Grothendieck to his eponymous "six functors". For decades mathematicians have worked within a "yoga" of six functors, emulating Grothendieck's six functors, and used their pleasant functorial properties to great effect. Recently various authors [12] [28] [34] [52] have crystallized the common features of these formalisms, providing a robust categorical framework to formalize and construct six-functor formalisms. The benefits of this characterization are hard to overstate:

1. The characterization is of great pedagogical value. While the working

mathematician will have to learn the intricacies of any individual formalism in which they work, it provides a universal framework in which to understand them, and a direct means of comparison with more familiar formalisms.

2. Eluded to above, it provides a robust method of *comparison* of six-functor formalisms, meaning they can be studied extrinsically using categorical techniques, and useful notions from one formalism can be extended to others.
3. The flexibility of six-functor formalism allows one to vary the choice of formalism in a particular context, providing a broader range of techniques for the study of geometric objects in a particular context. This sort of idea can already be seen in foundational work in algebraic geometry, where great progress was made in extending the scope of open immersion from open embedding to étale map.

In this paper, the main chapter adapts the six-functor formalism of quasicoherent sheaves on derived rigid analytic spaces [54] to derived overconvergent geometry (and in particular derived complex geometry).

Let us now outline the contents of the paper.

Chapter 2 details the content of the author's confirmation thesis. Using the quasi-abelian homological algebra of Schneiders, a simplified proof of the Verdier duality theorem for the structure sheaves of complex manifolds of [45] is presented, adapting duality arguments of [16]. Thereby, the endomorphisms of the structure sheaf are connected to infinite-order differential operators:

$$\mathbb{R}\mathcal{H}\mathrm{om}(\mathrm{IB}(\mathcal{O}_X), \mathrm{IB}(\mathcal{O}_X)) \cong \mathcal{D}_X^\infty \tag{1}$$

and the reconstruction theorem for perfect complexes of [45] is reproduced.

In chapter 3, the foundations of derived analytic geometry via bornological methods are recalled from [7]. Homotopical algebra and Lawvere theories controlling the overconvergent disc algebras are recalled, and the resulting theory of stacks relative to these algebras is developed. A comparison to other familiar theories of derived analytic spaces is briefly discussed at the end.

In chapter 4, the abstract theory of six-functor formalisms is discussed. The main reference for this chapter is [52]. Their definition, methods of construction and extension are first discussed, and familiar sheaf-theoretic notions of duality, finiteness and smoothness are cast in this formal language. The last section details pertinent examples of abelian sheaves on topological spaces, coherent sheaves, and algebraic $\mathcal{D}$-modules.

Chapter 5 adapts the theory of quasicoherent sheaves on derived rigid analytic spaces of [54] to derived analytic spaces built from derived dagger Stein spaces, and defined with respect to a weak topology. The formalism is defined on derived dagger Stein spaces, then extended formally to sheaves with respect to the weak topology, and restricted to these spaces. The formalism is demonstrated to have good descent properties with respect to particular infinite covers, making it tractable for the study of a broad class of derived analytic spaces. In the final section, Betti stacks, germs, crystals and $\mathcal{D}^{\infty}$-modules are defined and compared. A particular advantage of the definition of derived analytic spaces in terms of overconvergent discs is the expression of germs as limits of analytic neighbourhoods is very similar to the definition of overconvergent discs.

In the final chapter, we review an application of six functor formalisms to the representation theory of reductive groups, following the presentation of [51]. All results are found in loc. cit., but it is an excellent demonstration of the benefits of working with six-functor formalisms, and of the utility of the constructions of the previous chapter.

Acknowledgements

Tremendous thanks go to my supervisors, Professor Kobi Kremnizer and Professor Kevin McGerty, who have offered me spectacular support and tutelage throughout my DPhil programme. I would like to thank Professor Konstantin Ardakov and Professor Dan Ciubotaru for their support of my previous submissions, and Jack Kelly and Federico Bambozzi for their feedback on this work.

2 The Bornological Verdier Dual of the Structure Sheaf of a Complex Manifold

2.1 Quasi-Abelian Categories and Sheaves

Let $\mathcal{C}$ be an additive category with kernels and cokernels.

Definition 2.1. A morphism $f : X \to Y$ is *strict* if it induces an isomorphism $\tilde{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$.

The category $\mathcal{C}$ is abelian if all maps are strict. Remark also that kernels and cokernels are always strict, and so any strict morphism admits a decomposition as a strict epimorphism, an isomorphism (which can simply be ignored by absorbing into on of the factors), and a strict monomorphism. Quasi-abelian categories instead satisfy a stability condition on strict morphisms, and so they generalise abelian categories:

Definition 2.2. - An additive category $\mathcal{C}$ with kernels and cokernels is quasi abelian, if:

1. The pullback of a strict epimorphism is strict: in the following pullback diagram, f' being a strict epimorphism implies that f is a strict epimorphism:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

2. Dually, the pushforward of a strict monomorphism is a strict monomorphism - so in the diagram above, instead take it to be a pushout square. Then f a strict monomorphism implies f' a strict monomorphism.

Examples - Functional Analysis is a rich source of quasi-abelian categories, such as topological abelian groups, locally convex topological vector spaces and, in particular, Fréchet spaces. Of particular interest to us are complete convex bornological spaces - vector spaces with a distinguished class of *bounded* subsets. This category will be denoted CBorn_k.

Quasi-abelian categories possess familiar homological properties, which in turn also grants sheaves valued in them such familiar properties, while also being much more rigid in structure than, say, Quillen exact categories. We summarise those properties proven in [50] - let $\mathcal{C}$ be a quasi-abelian category:

1. We may place t-structures on the homotopy category $\mathcal{K}(\mathcal{E})$, but because the image and coimage of a map are generally distinct, there is no distinguished candidate to describe left truncation. This leads to two t-structures, the *left and right* t-structures. All concepts dependent on the

choice of t-structure, such as the hearts, will be called left or right accordingly.

2. The *derived category* $\mathcal{D}(\mathcal{E})$ is the localisation of $\mathcal{K}(\mathcal{E})$ with respect to the saturated null system of *strictly exact complexes*. A complex is strictly exact in degree n, if $\ker(d^n) \cong \mathrm{im}(d^{n-1})$, and d^n is strict, and a complex is strict if it's strict in all degrees.

3. Because homotopic complexes are strictly exact in the same degrees, and strict exactness in a given degree is inherited by mapping cones, it follows that the t-structures on $\mathcal{K}(\mathcal{E})$ localise to define t-structures on $\mathcal{D}(\mathcal{E})$.

4. The left heart of $\mathcal{D}(\mathcal{E})$ is denoted $\mathcal{LH}(\mathcal{E})$, and is equivalent to the full subcategory of $\mathcal{K}(\mathcal{E})$ of complexes with a single nonzero monomorphic map between the only two nonzero terms. Taking the cokernels of such maps, and noting that the cokernels of parallel arrows of a cocartesian diagram are isomorphic, this enables us to embed C in $\mathcal{LH}(\mathcal{C})$, and this embedding has a left adjoint. This embedding induces a derived isomorphism

$$\mathcal{D}(\mathcal{E}) \cong \mathcal{D}(\mathcal{LH}(\mathcal{E})) \tag{2}$$

   Therefore, at the derived level, quasi-abelian categories can be understood in terms of the conceptually easier abelian heart (recall that the heart of a t-structure is always an abelian category - see [20]).

5. Derived functors can be defined similarly to the classical case, with only minor technical adjustments.

6. The category of sheaves on a topological space X valued in an elementary quasi-abelian category C, $\mathrm{Sh}(X;\mathcal{C})$, is quasi-abelian. Strict exactness can be measured stalkwise, and the category admits strong properties with respect to limits, generators and exactness, so that the operations used later in the category of complex convex bornological sheaves will be valid.

7. $\mathcal{LH}(\mathrm{Sh}(X;\mathcal{C})) \cong \mathrm{Sh}(X;\mathcal{LH}(\mathcal{C}))$, and combining this with the previous derived equivalence, we see that sheaves valued in elementary quasi-abelian categories may also be understood in purely abelian terms at the derived level:

$$\mathcal{D}(\mathrm{Sh}(X;\mathcal{LH}(\mathcal{C}))) \simeq \mathcal{D}(\mathrm{Sh}(X;\mathcal{C})) \tag{3}$$

   From this one can derive theorems for derived categories of sheaves valued in a quasi abelian category from analogous results for abelian categories.

2.2 Ind(Ban) and Bornological Sheaves

We are interested in convex bornological sheaves for two reasons. Firstly, most of the sheaves of interest in algebraic analysis are valued in some functional-analytic category which embeds into CBorn_k. Secondly, CBorn_k is a particularly well behaved quasi-abelian category, with favourable closed structure. In [45], results are proven for the category $\mathrm{Ind}(\mathrm{Ban_k})$, but in fact the subcategory of reduced inductive systems is equivalent to CBorn_k [37].

Definition 2.3. The functor $\mathrm{IB} : \mathrm{Tc} \to \mathrm{Ind}(\mathrm{Ban}_k)$ is defined on objects by

$$\mathrm{IB}(V) = \text{“}\varinjlim_{B \in \mathcal{B}_V}\text{”} \hat{V}_B \tag{4}$$

where $\mathcal{B}_V$ is the category of absolutely convex bounded subsets $B \subseteq V$, and $\hat{V}_B$ is the completion of the linear span of B. A continuous map of locally convex topological vector spaces will induce maps on all these linear spans, and hence on their completions. These maps extend to the formal inductive limits, thereby defining a functor $\mathrm{IB} : \mathrm{Tc} \to \mathrm{Ind}(\mathrm{Ban})$. This may then be applied to the structure sheaf of a complex manifold, viewed as a sheaf valued in Tc.

Let us now discuss how we can adapt this functor into one which maps to the conceptually simpler category of complete bornological vector spaces:

Definition 2.4. [37] [2] - Let V be a real or complex vector space. A *disc*, or a *balanced or circled set* , is a subset $B \subseteq V$ which is *balanced* and *convex*: it is closed under multiplication by scalars with norm less than one, and any line segment L between two points of B is contained in B.

To any disc, we may associate the gauge seminorm and the associated seminormed space V_B on its linear span. The disc is *complete* if V_B is complete.

A bornological space is then complete if every bounded set is contained in a complete bounded disc.

Define an order on the discs of V: $B \leq B'$ if B' *absorbs* B - that is, $B \subseteq rB'$ for some positive real number r. If $B \leq B'$, there is an injective bounded linear map $V_B \to V_{B'}$. Given a bounded linear map $V \to W$, by definition this induces a map of the directed sets of bounded discs, which defines a morphism of inductive systems. This functor is called *dissection* and denoted diss.

On an ordinary bornological space, the target of diss is Ind(Sns), the Ind completion of the category of seminormed spaces. If V is complete then, by assumption, the spaces V_B are complete, and so diss may be regarded as a functor $\mathrm{CBorn}_k \to \mathrm{Ind}(\mathrm{Ban}_k)$. The main theorem relevance to us is the following:

Theorem 2.1. *[37] - The functor* diss : $\mathrm{CBorn}_k \to \mathrm{Ind}(\mathrm{Ban}_k)$ *is fully faithful, with essential image the reduced systems - those with injective transition functions. We therefore have the fundamental equivalence:*

$$\mathrm{CBorn}_k \simeq \mathrm{Ind}(\mathrm{Ban}_k)_{\mathrm{red}} \tag{5}$$

The functor IB associates to a locally convex topological vector space, a *reduced* inductive system of Banach spaces. We may therefore restrict the codomain of IB and regard it as a functor valued in complete bornological vector spaces. By abuse of notation, we shall denote the associated functor by $\mathrm{IB} : \mathrm{Tc} \to \mathrm{CBorn}_k$.

We summarise the main facts about IB proven in [45]:

1. IB commutes with *reduced* inductive limits of systems of Fréchet spaces over $\mathbb{N}$ - those with all transition maps injective:

$$\varinjlim_{n\in\mathbb{N}} \mathrm{IB}(F_n) \cong \mathrm{IB}(\varinjlim_{n\in\mathbb{N}}(F_n)) \tag{6}$$

2. For $V, W \in \mathrm{Tc}$ we can bestow the space $L_b(V,W) = \mathrm{Hom}_{\mathrm{Tc}}(V,W)$ with the structure of a locally convex topological vector space, with seminorms

$$\{p_B : B \subseteq V \,\mathrm{bounded}, p\,\mathrm{a\,continuous\,seminorm\,on\,W}\}$$

and $p_B(f) = \sup_{v\in B} p(f(e))$. So, p_B evaluates on f the supreme value that $p \circ f$ takes on B.

The tensor product of two locally convex topological vector spaces V, W, inherits a canonical family of seminorms $\{p \otimes q\}$ for p, q seminorms on V, W respectively. For x an element of $V \otimes W$

$$(p \otimes q)(x) = \inf_{x=\sum\limits_{i\in I} v_i \otimes w_i} \Sigma p(v_i) q(w_j) \tag{7}$$

where the infimum runs over all possible representations of x as an element of the tensor product. We record the isomorphisms of note - firstly for V bornological and W complete:

$$\mathrm{IB}(\mathrm{L}_b(V,W)) \cong \mathrm{Hom}_{\mathrm{Ind}(\mathrm{Ban}_k)}(\mathrm{IB}(V), \mathrm{IB}(W)) \tag{8}$$

Here $\mathrm{L}_b(E,F)$ denotes $\mathrm{Hom}_{\mathcal{T}_c}(E,F)$, bestowed with the locally convex structure via the system of seminorms p_B, parameterized by seminorms p on F and bounded subsets $B \subseteq E$, defined by the following for $h \in \mathrm{L}_b(E,F)$:

$$p_B(h) = \sup_{e\in B} p(h(e)). \tag{9}$$

For arbitrary locally convex V, W:

$$\mathrm{IB}(V)\hat{\otimes}\mathrm{IB}(W) \cong \mathrm{IB}(E \otimes F) \tag{10}$$

where $E \otimes F$ has the locally convex structure from the system of seminorms defined above, and the internal tensor product on Ind(Ban) objects is obtained by the unique extension of the closed structure to Ind objects.

3. Whenever E is a DFN space and F is an FN space:

$$\mathrm{Hom}(\mathrm{IB}(E), \mathrm{IB}(F)) \cong \mathbb{R}\mathrm{Hom}(\mathrm{IB}(E), \mathrm{IB}(F)) \tag{11}$$

$$L(\mathrm{IB}(E), \mathrm{IB}(F)) \cong \mathbb{R}L(\mathrm{IB}(E), \mathrm{IB}(F)), \tag{12}$$

where L denotes the internal Hom. Whenever X, Y objects of Ind(Ban) with X nuclear (meaning that for any index i, there is a $j \geq i$ with $X(i) \to X(j)$ nuclear), then

$$X \hat{\otimes}^{\mathbb{L}} Y \cong X \hat{\otimes} Y \tag{13}$$

4. IB respects sheaves in the following sense: if F is a presheaf of Fréchet spaces, which is a sheaf when viewed as a presheaf of vector spaces on second countable space X (such as a complex manifold), then the presheaf characterised by

$$\mathrm{IB}(F)(U) = \mathrm{IB}(F(U)) \tag{14}$$

is a sheaf.

5. Cartan's Theorem B holds for $\mathrm{IB}(\mathcal{O}_X)$, which by (14) is a sheaf:

$$\mathbb{R}\Gamma(U, \mathrm{IB}(\mathcal{O}_X)) \cong \mathrm{IB}(\mathcal{O}_X(U)) \tag{15}$$

whenever U has vanishing algebraic nonzero cohomology cohomology, $H^k(U, \mathcal{O}_X) = 0, k > 0$.

2.3 The Verdier Dual of $\mathrm{IB}(\mathcal{O}_X)$

In this chapter we present an alternative calculation of the Verdier dual of $\mathrm{IB}(\mathcal{O}_X)$ and $\mathrm{IB}(\Omega_X^{n-p})$, inspired by the argument from [16]. By the equivalence between reduced inductive systems of Banach spaces and complete bornological vector spaces, we shall henceforth consider complete bornological sheaves on an n-dimensional complex manifold X. All derived functors are in the sense of Schneiders [50].

Definition 2.5. The *Verdier dual* of a bornological sheaf F on X, $D(F)$, is characterised by the formula, for all open $U \subseteq X$:

$$D(F)(U) = \mathbb{R}L(\mathbb{R}\Gamma_c(U; F), \mathrm{IB}(\mathbb{C})) \tag{16}$$

Here $\mathbb{R}\Gamma_c$ denotes the derived compactly supported global sections functor.

Theorem 2.2. *(Verdier Dual of* $\mathrm{IB}(\mathcal{O}_X)$*) - There are isomorphisms for all positive integers* $p \leq n$*:*

$$\mathrm{IB}(\Omega_X^{n-p}) \simeq D(\mathrm{IB}(\Omega_X^p)) \tag{17}$$

In [45] this result is proven by constructing a perfect pairing between $\mathrm{IB}(\Omega_X^{n-p})$ and $\mathrm{IB}(\Omega_X^p)$ by generalising the cup product and fibre integration to sheaves valued in $\mathrm{Ind}(\mathrm{Ban}_k)$. Their argument thereafter is a double induction on the dimensions of the spaces considered and the number of irreducible components involved. After a long series of reductions, it is eventually shown to be a corollary of Stokes' theorem.

We propose a different proof, using an approach by descent to the calculation of the Verdier dual in the non-Archimedean context, detailed in [16]. Rather than making the long series of reductions as in the above proof, the idea of this proof is as follows:

1. Carry out the proof in the special case of complex affine space, $\mathbb{C}^n$. This is done by the construction of a *residue map* on compactly supported cohomology, $H_c^n(X, \omega) \to \mathbb{C}$, which composed with a natural pairing on $H^0(\mathcal{O}_X) \times H_c^n(X, \omega)$ establishes duality.
2. Given a closed immersion $i : X \to \mathbb{C}^n$ of a complex submanifold, we can express cohomology on X in terms of cohomology of $\mathbb{C}^n$, using the argument given in [13]. Moreover, a residue map on X can be defined in terms of the residue on the ambient affine space, independent of the choice of Stein neighbourhood and of the choice closed immersion $i : X \to \mathbb{C}^n$.
3. Any *Stein manifold* admits a closed immersion into some complex affine space. For details see [22] or [27]. Therefore the duality theorem can be deduced for Stein domains from the affine case.
4. Any point on a complex manifold admits a Stein neighbourhood, which may be assumed to be irreducible (and of course of constant dimension), and such that the intersections of these spaces are also irreducible Stein spaces. This is because holomorphic separability and convexity are preserved by the intersection of such domains, so one only has to take irreducible components [21].
5. We therefore have unique canonical local duality isomorphisms, induced by unique canonical residue maps on these Stein neighbourhoods, and also on the overlaps of these domains. We conclude that all the duality isomorphisms glue to a global duality isomorphism for any complex manifold.

(4) and (5) require no further explanation, so below we give a proof of (1), and briefly explain the pulling back of cohomology in (2) for closed immersed Stein domains. Before delving into the proof, we recall the necessary facts about Stein spaces:

Theorem 2.3. *[48] - A Stein manifold X of dimension n admits a proper embedding into $\mathbb{C}^{2n+1}$.*

Theorem 2.4. *[27] - The topology of every complex space has a basis of open Stein neighbourhoods. In particular, any point of a complex manifold possesses a neighbourhood basis of Stein manifolds.*

Therefore to locally study a complex manifold, we may take a Stein neighbourhood, which has nice cohomological properties, and exploit the cohomological relationship between the Stein manifold and a complex space into which it embeds. This is the approach we take in our alternative calculation of the Verdier dual. We need only a few additional facts:

Theorem 2.5. *(Extension Principle [21] - For a closed complex subspace $X \subseteq Y$, an analytic sheaf F on X is coherent if and only if its pushforward i_*F is coherent. This will apply in particular to the embeddings of Stein manifolds into affine space.*

Theorem 2.6. *(Conormal Exact Sequence, [23]) - For an embedding of smooth complex manifolds $Y \hookrightarrow X$, if $\mathcal{I}$ is the associated ideal sheaf, there is a canonical exact sequence*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_X|_Y{}^1 \longrightarrow \Omega^1_Y \longrightarrow 0 \tag{18}$$

Moreover, this is an exact sequence of Fréchet sheaves, and therefore an exact sequence of bornological sheaves for the canonically induced bornologies.

Through the use of Koszul resolutions, we can understand the cohomology of an embedded Stein manifold, or indeed any closed complex subspace, in terms of its local equations. First we need a definition:

Definition 2.6. [27] [56] - Let R be a ring. A finite sequence of elements $(f_1, \cdots, f_n)$ of R is a *regular sequence* if each f_i is a nonzerodivisor modulo $f_1, \cdots, f_{i-1}$.

We apply this definition to the local ring of a point (without loss of generality the origin) in complex affine space in the image of an embedded smooth Stein manifold X. We remark that the local equations f_i for X can be chosen such that the associated stalk elements $(f_i)_0$ form a regular sequence by smoothness - the Stein arises locally as an embedded subspace, and thus as the zero locus of complex variables. This could fail for other embeddings, such as the embedding of a union of two spaces which meet tangentially at a point.

Theorem 2.7. *(Koszul Resolution, [27]) - For any open region $U \subseteq \mathbb{C}^n$, and $i : X \hookrightarrow U$, $i(x) = 0$ the embedding of an analytic subset defined by equations $f_1, \cdots, f_p$, such that $(f_1)_0, \cdots, (f_p)_0$ defines a regular $(\mathcal{O}_U)_0$-sequence, the push-forward $i_*\mathcal{O}_X$ admits a free resolution locally at x, of length at most p. That is, on some open neighbourhood $0 \in V \subseteq U$, there is an exact sequence:*

$$0 \longrightarrow \mathcal{O}_V^{m_p} \longrightarrow \mathcal{O}_V^{m_{p-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_V^{m_0} \longrightarrow (i_*\mathcal{O}_X)|_V \longrightarrow 0 \tag{19}$$

By translation, the assertion that $i(x) = 0$ is redundant, and so we may conclude that for any embedded Stein manifold into affine space, the structure sheaf admits a free resolution locally.

Proof of Verdier Duality in the Affine Case, $U = \mathbb{C}^n$

1. As in the proof in [45], all the cases for various p are equivalent, so we shall consider only the case of top differential forms.

2. $\mathbb{C}^n$ possesses a filter by compact polydiscs P_m of polyradius $(m, \cdots, m)$, for m running over all positive real numbers (or integers). These polydiscs are cofinal in the system of compact subsets under inclusions, and so a compactly supported section is supported on one of these balls.

3. There is a long exact sequence

$$0 \to H^0_{P_m}(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) \to H^0(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) \to H^0(\mathbb{C}^n \backslash P_m, \Omega_{\mathbb{C}^n}) \to H^1_{P_m}(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) \to \cdots \tag{20}$$

   where $H^0_{P_m}(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) = \mathrm{Ker}(H^0(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) \to H^0(\mathbb{C}^n \setminus P_m, \Omega_{\mathbb{C}^n}))$, and $H^i_{P_m}$ denote the derived functors. By Cartan's Theorem B for coherent sheaves, $H^i(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) = 0$ for $i > 0$, and so the long exact sequence yields isomorphisms

$$H^n_{P_m}(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) \cong H^{n-1}(\mathbb{C}^n \backslash P_m, \Omega_{\mathbb{C}^n}) \tag{21}$$

4. $U \backslash P_m$ comes equipped with an acyclic cover by those subspaces, for varying i, with i^{th} coordinate greater than m:

$$V_i = \{(z_1, ..., z_n) : |z_i| > m\} \tag{22}$$

   These are Stein domains, and therefore defines an acyclic cover by Cartan's theorem B. Therefore the associated Čech complex is adequate to calculate cohomology [23].

5. Čech cohomology with respect to this cover yields acyclicity in all degrees below n.

   k-cocycles consist of the data of tuples of top degree differential forms with coefficients holomorphic functions on the intersection of k of the subspaces V_i. For $\mathcal{I} = (i_1, \cdots, i_k) \subseteq \{1, \cdots, n\}$ with distinct elements, the $(i_1, \cdots, i_n)$-component of the cocycle has holomorphic coefficient which admits a Laurent expansion

   $$\sum_{j_r \in \mathbb{Z}} a_{j_1, \cdots, j_n} z_1^{j_1} \cdots z_n^{j_n}. \tag{23}$$

   If $r \notin \mathcal{I}$, the coefficients $a_{j_1, \cdots, j_n}$ vanish whenever $j_r < 0$. The cocycle condition can then be used to demonstrate that the components of any cocycle can be modified by a coboundary to admit even stronger conditions on these series, and this can be repeated until the components are annihilated, and thus the cohomology vanishes.

   We illustrate this with the simple example of 2-cocycles in the case $n = 3$, the other cases are similar but more intricate. A 1-cocycle consists of three top differential forms with coefficient functions (f, g, h), where

   (a) f admits an expansion with vanishing coefficients for those terms with negative z_3 exponent. Similarly, the coefficients of g vanish for terms with negative z_1 exponent, and those of h for negative z_2-exponents.
   (b) These functions satisfy the cocycle condition $f - g + h = 0$, or equivalently, $g = f + h$.

   This expresses g, a series which does not permit terms with negative z_2-exponents, as a sum of f and h which both do. This shows that f and h have the same principal z_2-component, with opposite signs. This principal component defines a holomorphic function on V_2, in the notation above, and so there is a corresponding coboundary. Modifying by this coboundary, we obtain a new tuple (f', g, h'), where now the coefficients of f' vanish for those terms with negative exponents for either z_2 or z_3, and the coefficients of h' vanish for terms with negative exponents for either z_1 or z_2.

   We can repeat this argument with for both the other pairs to yield a triple of entire functions, which is then easily expressed as a coboundary. Thus all cocycles have been demonstrated to be coboundaries, and the cohomology vanishes.

6. The n^{th} cohomology $H^n_{P_m}(\mathbb{C}^n, \Omega_{\mathbb{C}^n})$ can be identified with the Fréchet space of Laurent series converging on the intersection of the annulus

$$V = \{(z_1, ..., z_n) : |z_i| \geq m \ \forall i\}, \tag{24}$$

Such Laurent series can be written in the form

$$\underset{m_i<0}{\Sigma} a_m z^m dz/z \tag{25}$$

where m is the multi-index $(m_1, \cdots, m_d)$, $z^m = \prod_i z_i^{m_i}$, and $dz/z = \bigwedge_i dz_i/z_i$. Note that in this notation, the coefficient of $dz/z = (z_1 \cdots z_n)^{-1} dz_1 \wedge \cdots \wedge dz_n$ is a_0.

7. As compactly supported cohomology is the colimit of cohomology supported on these balls, we find that $H^n_c(\mathbb{C}^n, \Omega_{\mathbb{C}^n})$ can be identified with the space of such Laurent series, convergent outside *some* ball B.

8. Bearing in mind the statement of Verdier duality, we would like to relate this cohomology space to the space of bounded linear maps from $H^0(\mathcal{O}_{\mathbb{C}^n})$ to $\mathbb{C}$. Since all of the spaces involved are Fréchet spaces, boundedness and continuity are equivalent. Therefore, following the argument of [16], we associate such a continuous linear map to the Laurent series (25) as follows: a global section in $H^0(\mathcal{O}_{\mathbb{C}^n})$ can be identified with a convergent power series, and we define the map by

   $$f = \Sigma c_m z^n \mapsto \Sigma a_m c_m \tag{26}$$

   and since the projections onto the values of the coefficients are continuous and linear, every continuous linear map has this form. This proves the affine case:

   $$H^n_c(\Omega_{\mathbb{C}^n}) \cong \mathrm{Ext}^n_c(\mathcal{O}_{\mathbb{C}^n}, \Omega_{\mathbb{C}^n}) \tag{27}$$

   and the other Exts are zero, so by the above the dual is the translated structure sheaf.

9. We may extend this result. Firstly, we have established a bilinear map, from which the two isomorphisms below can be deduced by dualization ([16] Lemma 2.5, 2-3).

   Secondly we can consider more general scalars than those in $\mathbb{C}$, and consider scalars in a finitely-generated $\mathbb{C}$-vector space V.

   We define the *residue* $\mathrm{Res} : H^n_c(\mathbb{C}^n, \Omega_{\mathbb{C}^n}) \to \mathbb{C}$ to be the map sending the class associated to a convergent sum (25) to its lowest degree coefficient a_0. The *trace* on $H^n_c(X, \omega \otimes V) \cong H^n_c(X, \omega) \otimes V$ is simply $\mathrm{Res} \otimes 1_V$. Then we deduce the duality results from the bilinear form

$$H_c^n(X, \omega \otimes V) \times H^0(\mathcal{O}_X) \to V \tag{28}$$

$$\left(\underset{m_i<0}{\Sigma} a_{m_i} z_i^{m_i} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_d, \Sigma c_m z^n \right) \mapsto \Sigma a_m c_m \tag{29}$$

We deduce the isomorphisms

$$H_c^n(\Omega_{\mathbb{C}^n} \otimes V) \cong \mathrm{Ext}_c^n(\mathcal{O}_{\mathbb{C}^n}, \Omega_{\mathbb{C}^n} \otimes V) \cong \mathrm{Hom}_{\mathrm{cont}}(H^0(\mathcal{O}_X), V) \tag{30}$$

$$H^0(\Omega_{\mathbb{C}^n} \otimes V) \cong \mathrm{Ext}^0(\mathcal{O}_{\mathbb{C}^n}, \Omega_{\mathbb{C}^n} \otimes V) \cong \mathrm{Hom}_{\mathrm{cont}}(H_c^n(\mathcal{O}_X), V) \tag{31}$$

where $\mathrm{Hom}_{\mathrm{cont}}$ denotes the space of continuous maps.

10. Finally, these arguments can be generalised to a coherent $\mathcal{O}_X$-module F in place of $\mathcal{O}_X$ by taking a local presentation. Therefore, we have the following isomorphisms for any coherent sheaf F on $\mathbb{C}^n$ and any finitely-generated $\mathbb{C}$-vector space ([16] Proposition 2.6):

$$\mathrm{Ext}_c^p(F, \Omega_{\mathbb{C}^n}) = 0, \ p \leq n \tag{32}$$

$$\mathrm{Ext}_c^n(F, \Omega_{\mathbb{C}^n}) \cong \mathrm{Hom}_{\mathrm{cont}}(H^0(F), \mathbb{C}) \tag{33}$$

$$\mathrm{Hom}(F, \Omega_{\mathbb{C}^n}) \cong \mathrm{Hom}_{\mathrm{cont}}(H_c^n(F), \mathbb{C}). \tag{34}$$

The proof of these statements are no different to the proof found in loc. cit. For the vanishing of lower exts, choose $R \in \mathbb{R}_{>0}$, and global sections $e_1, \cdots, e_{m_0}$ of F which generate the local sections of F on P_R, the polydisc of polyradius R^n. This defines a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\pi) \arrow[r] & \mathcal{O}_{\mathbb{C}^n}^{m_0} \arrow[r, "\pi"] & \mathrm{Im}(\pi) \arrow[r] & 0 \end{tikzcd}$$

Let $R' < R$. Then by assumption that π is surjective over $B(R)$, $\mathrm{Ext}^p_{P_{R'}}(\mathrm{Im}(\pi), \Omega_{\mathbb{C}^n} \otimes M) \cong \mathrm{Ext}^p_{P_{R'}}(F, \Omega_{\mathbb{C}^n} \otimes M)$, and so we deduce from the long exact cohomology sequence that, by induction on p:

$$\mathrm{Ext}^p_{P_{R'}}(F, \Omega_{\mathbb{C}^n} \otimes M) = 0 \ \forall p < n \tag{35}$$

Letting R vary yields the vanishing of lower exts, and the other results follow by similar reasoning.

Remark - Van der Put [16] instead considers affine space over a rigid space Y, and so V is replaced by a finitely generated $\mathcal{O}_Y$-module. For our purposes complex affine space is sufficient, and so Y is taken to be a point, and V is just a complex vector space.

Pulling back Cohomology to an embedded smooth Stein manifold

We follow the argument given in [13]. The argument is given in the rigid analytic context also, where Steins have a similar definition, and have similar properties to complex Stein domains:

1. Let $i : X \hookrightarrow \mathbb{C}^n$ be a closed embedding, and cover the image of X by open subsets U_i, such that on U_i, X has local equations $f_1^i, \cdots, f_{n-q}^i$. These open sets may be chosen such that the stalks of the equations f_j^i form a regular $\mathcal{O}_x$-sequence at any point $x \in X$. We can describe the structure sheaf on X explicitly:

$$\mathcal{O}_X(V_i) \cong \mathcal{O}_{\mathbb{C}^n}(U_i)/(f_1^i, \cdots, f_{n-q}^i) \tag{36}$$

2. Because i is a closed immersion there is an *underived* exceptional pullback $i^!$ - that is, a right adjoint to the pushforward i_* on the category of sheaves. One finds by the following series of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n}}(i_*\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{i^{-1}\mathcal{O}_{\mathbb{C}^n}}(\mathcal{F}, i^!\mathcal{G}) && (37)\\
&= \mathrm{Hom}_{i^{-1}\mathcal{O}_{\mathbb{C}^n}}(\mathcal{F}, \mathcal{H}\mathrm{om}_{i^{-1}\mathcal{O}_{\mathbb{C}^n}}(\mathcal{O}_X, i^!\mathcal{G})) && (38)\\
&= \mathrm{Hom}_{i^{-1}\mathcal{O}_{\mathbb{C}^n}}(\mathcal{F}, i^*\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{C}^n}}(i_*\mathcal{O}_X, \mathcal{G})) && (39)\\
&= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i^*\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{C}^n}}(i_*\mathcal{O}_X, \mathcal{G})) && (40)
\end{aligned}$$

Therefore by uniqueness of representatives we have a description of the functor $i^!$:

$$i^!\mathcal{G} \cong i^*\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{C}^n}}(i_*\mathcal{O}_X, \mathcal{G}) \tag{41}$$

3. By taking the n^{th} exterior power, we obtain an isomorphism

$$i^*\Omega^n_{\mathbb{C}^n} \otimes \mathcal{O}_X \cong \bigwedge^{n-q} \mathcal{I}/\mathcal{I}^2 \otimes \Omega^q_X \tag{42}$$

This isomorphism is defined in the obvious way, as described in [24]. That is, we can choose local generators $e_1, ..., e_n$ of $\Omega^1_{\mathbb{C}^n} \otimes \mathcal{O}_X$, such that $e_1, ..., e_{n-q}$ generate $\mathcal{I}/\mathcal{I}^2$, and the images of the remaining generators generate Ω^1_X. This choice determines the isomorphism.

The wedge product on the right is invertible, yielding the isomorphism:

$$\Omega^q_X \cong i^*\Omega^n_{\mathbb{C}^n} \otimes \left(\bigwedge^{n-q} \mathcal{I}/\mathcal{I}^2 \right)^* \tag{43}$$

4. To analyse $\mathbb{R}i^!$ one considers the restriction to open affine neighbourhoods as in (36). Then, one can analyse the local Ext sheaves $\mathcal{E}\mathrm{xt}^j_{\mathcal{O}_X|_{U_i}}(\mathcal{O}_X|_{U_i}, \Omega^n_{\mathbb{C}^n}|_{U_i})$ using Koszul resolutions.

   (a) Consider a neighbourhood U_i as in (36), so that X is locally defined by the equations $f^i_1 = \cdots = f^i_{n-q} = 0$.

   (b) Denote the Koszul resolution of $i_*\mathcal{O}_X|_{U_i}$ by $K_\bullet(f^i_1, \cdots, f^i_{n-q})$, and for any sheaf F of $\mathcal{O}_{U_i}$-modules we define the *dual complex*:

   $$K^\bullet(f^i_1, \cdots, f^i_{n-q}; F) = \mathcal{H}\mathrm{om}^\bullet_{\mathcal{O}_X}(K_\bullet(f^i_1, \cdots, f^i_{n-q}), F) \tag{44}$$

   We see that because the Koszul complex is locally free, this calculates the sheaf Ext complex:

   $$\begin{aligned} K^\bullet(f^i_1, \cdots, f^i_{n-q}; F) &\cong \mathcal{E}\mathrm{xt}^\bullet_{\mathcal{O}_{U_i}}(K_\bullet(f^i_1, \cdots, f^i_{n-q}), F) && (45) \\ &\cong \mathcal{E}\mathrm{xt}^\bullet_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}/(f^i_1, \cdots, f^i_{n-q}), F) && (46) \\ &\cong \mathcal{E}\mathrm{xt}^\bullet_{\mathcal{O}_{U_i}}(\mathcal{O}_X|_{V_i}, F) && (47) \end{aligned}$$

   We deduce an isomorphism on global cohomology

   $$\mathrm{Ext}^{n-q}(\mathcal{O}_X|_{V_i}, F) \to H^n K^\bullet(f^i_1, \cdots, f^i_{n-q}; F) \to F/(f^i_1, \cdots, f^i_{n-q})F \tag{48}$$

   The first map is the induced map on $(n-q)^{\mathrm{th}}$ cohomology. The second morphism $H^n(f^i_1, \cdots, f^i_{n-q}; F) \to F/(f^i_1, \cdots, f^i_{n-q})F$, sends $\alpha \in K^n(f^i_1, \cdots, f^i_{n-q}; F)$ to $\alpha_{1,\cdots,n}$.

   (c) The argument of Proposition 7.2 from [24] adapts to our context, and allows us to conclude that

   $$\mathcal{E}\mathrm{xt}^j_{\mathcal{O}_X|_{U_i}}(\mathcal{O}_X|_{U_i}, \Omega^n_{\mathbb{C}^n}|_{U_i}) = 0,\ j \neq n-q \tag{49}$$

   and therefore these Ext sheaves vanish globally also.

   (d) By (48) there are isomorphisms depending on the chosen equations for X

   $$\phi_i : \mathcal{E}\mathrm{xt}^{n-q}_{\mathcal{O}_X|_{U_i}}(\mathcal{O}_X|_{U_i}, \Omega^n_{\mathbb{C}^n}|_{U_i}) \cong \frac{\Omega^n_{\mathbb{C}^n}|_{U_i}}{\mathcal{I}|_{U_i}\Omega^n_{\mathbb{C}^n}|_{U_i}} \tag{50}$$

   and these isomorphisms glue via the maps relating the systems of equations on overlaps. That is, given two systems of equations X

on U_i and U_j, $(f_1^i, \cdots, f_{n-q}^i)$ and $(g_1^i, \cdots, g_{n-q}^i)$ respectively, then on the overlap $U_i \cap U_j$, these equations can be written in terms of each other. Equivalently, there is a matrix c_{kl} with $g_k = \Sigma c_{kl} f_l$. By lemma 7.1 of [24] these isomorphisms differ exactly by the canonical map $\det(c_{ij})$. These morphisms ϕ_i can therefore be glued along these maps, yielding a global isomorphism:

$$\phi : \mathcal{E}\mathrm{xt}^{n-q}_{\mathcal{O}_X|_{U_i}}(\mathcal{O}_X|_{U_i}, \Omega^n_{\mathbb{C}^n}|_{U_i}) \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\bigwedge^{n-q} \frac{\mathcal{I}}{\mathcal{I}^2}, \frac{\Omega^n_{\mathbb{C}^n}|_{U_i}}{\mathcal{I}|_{U_i}\Omega^n_{\mathbb{C}^n}|_{U_i}}\right) \tag{51}$$

as the gluing data $\{\det(c_{ij})\}$ is described by the automorphisms of $\bigwedge^{n-q} \frac{\mathcal{I}}{\mathcal{I}^2}$. To be precise, this map ϕ evaluated on $f_1^i, \cdots, f_{n-q}^i$ is exactly ϕ_i.

(e) Applying i^* and the isomorphism (43), we deduce

$$i^*\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^n_{\mathbb{C}^n}) \cong i^*\Omega^n_{\mathbb{C}^n} \otimes \left(\bigwedge^{n-q} \mathcal{I}/\mathcal{I}^2\right)^* \cong \Omega^q_X \tag{52}$$

We therefore finally arrive at a neat description of the derived exceptional pullback, according to (41)

$$\mathbb{R}i^!(\Omega^n_{\mathbb{C}^n}) \cong \Omega^q_X[q-n] \tag{53}$$

5. Recall that i_* preserves coherence. To prove coherent duality we take an injective resolution of top forms

$$0 \longrightarrow \Omega^n_{\mathbb{C}^n} \longrightarrow I^\bullet$$

and take cohomology of the adjunction isomorphism between i_* and $i^!$. This yields the duality isomorphisms, for any coherent $\mathcal{O}_X$-module

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{C}^n}}(i_*M, \Omega^n_{\mathbb{C}^n}) \cong \mathrm{Ext}^{i-(n-q)}_{\mathcal{O}_X}(M, \Omega^q_X) \tag{54}$$

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_{\mathbb{C}^n}}(i_*M, \Omega^n_{\mathbb{C}^n}) \cong i_*\mathcal{E}\mathrm{xt}^{i-(n-q)}_{\mathcal{O}_X}(M, \Omega^q_X) \tag{55}$$

6. Because i is proper, the correspondence between compact subsets yields the isomorphism on compactly supported cohomology, for any sheaf of $\mathcal{O}_X$-modules M:

$$H^*_c(\mathbb{C}^n, i_*M) \cong H^*_c(X, M) \tag{56}$$

These isomorphisms are already enough to pull back cohomology from affine space to the closed immersed Stein. From these we deduce the necessary isomorphisms. However, the isomorphisms need to be sufficiently canonical in order for us to deduce a global isomorphism. In particular, the isomorphism must not depend on the choice of embedding into affine space, and two different Stein covers should yield the same isomorphism. This is done by constructing an invariant trace map for Steins, based on the affine definition.

7. We can define for an embedding $\phi : X \to \mathbb{C}^n$, a residue map on X:

$$\begin{tikzcd}
H^n_c(X, \Omega_X) \arrow[r, "\sim"] & H^n_c(\mathbb{C}^n, \mathcal{E}\mathrm{xt}^{n-q}(\phi_*\mathcal{O}_X, \Omega_{\mathbb{C}^n})) \arrow[r, "\alpha"] & \mathrm{Ext}^N_c(\phi_*\mathcal{O}_X, \Omega_{\mathbb{C}^n}) \arrow[dl, "\beta"] \\
 & \mathrm{Hom\,cont}(H^0(\mathcal{O}_X), \mathbb{C}) \arrow[r, "\gamma"] & \mathbb{C}
\end{tikzcd} \tag{57}$$

The map α is an isomorphism, coming from the spectral sequence for Ext^N_c. The isomorphism β comes from the coherent duality isomorphism. The final map γ is the evaluation on the constant function 1 on X. The trace map is defined by taking the same sequence, tensoring the sheaves of differential forms (including the trivial one $\mathbb{C}$ on the point) with a complex vector space V.

8. It remains to show the following, which we explain below, following the argument of [16]:
   (a) Given two distinct immersions into distinct affine spaces, the residues associated both equal the residue associated to the sum of the maps $(i_1, i_2) : X \to \mathbb{C}^{(n_1+n_2)}$
   (b) Given an open immersion $X_1 \to X_2$ of Steins, the composition

$$H^n_c(X_1, \Omega_{X_1}) \to H^n_c(X_2, \Omega_{X_2}) \xrightarrow{\mathrm{res}_{X_2}} \mathbb{C} \tag{58}$$

   is Res_{X_1}

Invariance of Trace and Residue

Lemma 2.1. *The residue map on affine space can be written as a composition*

$$H^n_c(\mathbb{C}^n, \omega) \xrightarrow{\phi} H^n_c(\mathbb{P}^n, \omega) \xrightarrow{\sim} H^n(\mathbb{P}^n, \omega) \xrightarrow{\psi} \mathbb{C}$$

for some final isomorphism $\psi : H^n(\mathbb{P}^n, \omega) \to \mathbb{C}$.

Proof - by GAGA [53] the cohomology group of $\mathbb{P}^n$ can be calculated algebraically. To prove the statement we show that ϕ and Res_n have the same kernel. Then, ψ can be chosen such that $\psi \circ \phi = \mathrm{Res}_n$.

We consider the action of $(S^1)^n$ on $\mathbb{C}^n$. It respects the polydiscs P_m, as well as the open cover of $\mathbb{C}^n \setminus P_m$, and therefore defines an action on $H^n_c(X, \omega)$. The extension of this action to $H^n(\mathbb{P}^n)$ is trivial. Therefore for any cohomology class ξ, and any operator $\sigma = (\lambda_1, \cdots, \lambda_n) \in (S^1)^n$ we have $\phi(\sigma\xi - \xi) = 0$. Representing ξ as a convergent sum $\underset{m_i<0}{\Sigma} a_m z^m dz/z$, we see that

$$\sigma\xi - \xi = \underset{m_i<0}{\Sigma} a_m z^m (\lambda^m - 1) dz/z \tag{59}$$

With careful choices of λ such that the terms $\lambda^m - 1$ don't vanish for $m \neq 0$, we see that the terms that vanish under ϕ are exactly those terms with $a_0 = 0$ - that is, $\ker \mathrm{Res}_n = \ker \phi$. This proves the factorisation, and then because $H^n(\mathbb{P}^n, \omega)$ is invariant under the projective general linear group, so is the residue.

Lemma 2.2. *For any automorphism σ of $\mathbb{C}^n$, there exists a unique invertible $f \in H^0(\mathcal{O}_X)$ such that $\mathrm{Res}_n \circ \sigma = f \cdot \mathrm{Res}_n$ as maps $H^n_c(\mathbb{C}^n, \omega) \to \mathbb{C}$.*

Proof - The map $\mathrm{Res}_n \circ \sigma$ is a continuous linear map $H^n_c(\omega) \to \mathbb{C}$. Because ω is coherent, the duality statement implies that $\mathrm{Res}_n \circ \sigma(\xi) = \mathrm{Res}_n(f\xi)$. We may apply the same reasoning to σ^{-1}, and so we conclude that f is invertible.

With this, we may conclude invariance of Res_n with respect to certain automorphisms of $\mathbb{C}^n$:

Lemma 2.3. *Automorphisms of the form*

$$\sigma(z_1, \cdots, z_n) = (z_1, \cdots, z_m, z_{m+1} + k_{m+1}, \cdots, z_n + k_n) \tag{60}$$

where the k_j are holomorphic functions varying with $z_1, \cdots, z_m$, are Res_n-invariant: $\mathrm{Res}_n \circ \sigma = \mathrm{Res}_n$.

Proof - Such automorphisms can be written as a commutator composed with a translation. Commutators are Res_n-invariant because of the last lemma. Translations are Res_n-invariant from the power series formulation.

These lemmas are enough to prove the invariance properties of the generalised residue map (57).

Theorem 2.8. *The residue (57) does not depend on the choice of embedding ϕ.*

Proof - A closed immersion of steins $\phi : X_1 \to X_2$ of dimensions m_1, m_2 respectively, induces a map $\tilde{\phi}$ functorially on cohomology, by pulling back sheaves of differential forms:

$$\begin{tikzcd}
H_c^{m_1}(X_1, \Omega_{X_1}) \arrow[r, "\sim"] & H_c^{m_1}(X_2, \mathcal{E}\mathrm{xt}^{m_2-m_1}(\phi_*\mathcal{O}_X, \Omega_{X_2})) \arrow[r] & \mathrm{Ext}_c^{m_2}(\phi_*\mathcal{O}_{X_1}, \Omega_{X_2}) \arrow[d] \\
 & & H_c^{m_2}(X_2, \Omega_{X_2})
\end{tikzcd}$$

Now given two closed immersions $X \hookrightarrow \mathbb{C}^{n_1}$, $X \hookrightarrow \mathbb{C}^{n_2}$, we show that the residue associated to these embeddings is equal to the residue associated to the coproduct of the embeddings. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathbb{C}^{n_1} \arrow[rr, "l_1"] \arrow[dr, "k_1"] & & \mathbb{C}^{n_1+n_2} \arrow[dl, "\tau_1"] \\
X \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] \arrow[rr] & & \mathbb{C}^{n_1+n_2} & \\
 & \mathbb{C}^{n_2} \arrow[rr, "l_2"'] \arrow[ur, "k_2"] & & \mathbb{C}^{n_1+n_2} \arrow[ul, "\tau_2"']
\end{tikzcd}$$

where k_1 is the holomorphic extension of $\phi_2 \circ \phi_1^{-1}$ on $\phi_1(X)$, l_1 is the embedding $z \mapsto (z, 0)$, and this determines a unique map τ_1. The other maps are defined similarly.

For the map l_1, the induced map $\tilde{l_1}$ on cohomology induces the canonical map on power series rings. Therefore, it respects the lowest order coefficient, which is exactly the residue.

So, the residue associated with ϕ_1 is the same as the residue associated with $l_1 \circ \phi_1$. As τ_1 is an automorphism of the form considered before it respects residues. As this composition equals $k_1 \circ \phi_1$, the associated residues are equal. The same reasoning applies to the bottom half of the diagram, concluding the proof.

With this proven, we are entitled to unambiguously denote the residue on a Stein manifold X by Res_X, without reference to an ambient affine space into which it embeds. It remains only to prove the following:

Proposition 2.1. - *For a closed immersion of Stein spaces $\phi : X_1 \to X_2$, the composition*

$$H_c^n(X_1, \Omega_{X_1}) \xrightarrow{\tilde{\phi}} H_c^n(X_2, \Omega_{X_2}) \xrightarrow{\mathrm{Res}_{X_2}} \mathbb{C}$$

is Res_{X_1}*. Therefore, the residue map and the coherent duality isomorphisms do not depend on the choice of Stein neighbourhood chosen.*

Proof - $\mathrm{Res}_{X_2} \circ H^n_c(\phi)$ is continuous and linear, and so by coherent duality, it corresponds to a unique element $a(X_1, X_2) \in H^0(X_1, \mathcal{O}_{X_1})$:

$$\mathrm{Res}_{X_2} \circ H^n_c(\phi)(\xi) = \mathrm{Res}_{X_1}(a(X_1, X_2)\xi) \tag{61}$$

Our task is to show that $a(X_1, X_2) = 1$. First this is proven for the prototypical embedding of a polydisc into affine space: $P^n(R) \subseteq \mathbb{C}^n$. The action of $(S^1)^n$ respects both the polydisc and affine space, and the residue maps are invariant under the action of each. From this symmetry, we see that the element $a(X_1, X_2)$ must be a scalar, because it's holomorphic and constant on tori.

Next we reduce to the one-dimensional case. We consider the commutative diagram:

$$\begin{tikzcd}
P^n(R) \arrow[r] & \mathbb{C}^n \\
P^1(R) \arrow[r] \arrow[u] & \mathbb{C} \arrow[u]
\end{tikzcd}$$

where the vertical maps are the closed immersions, mapping to the first coordinate. Because these are closed immersions, the maps on cohomology are canonical, and the scalars $a(P^n(R), \mathbb{C}^n)$ and $a(P^1(R), \mathbb{C})$ must agree, reducing to dimension 1.

In [16], it is very elegantly proven that $a^2 = a$ by considering the following diagram:

$$\begin{tikzcd}
 & P^2(R) \arrow[rr] & & P^1(R) \times \mathbb{C} \arrow[rr] & & \mathbb{C}^2 & \\
0 \times P^1(R) \arrow[ur] \arrow[rr] & & 0 \times \mathbb{C} \arrow[ur] & & P^1(R) \times 0 \arrow[ul] \arrow[rr] & & \mathbb{C} \times 0 \arrow[ul]
\end{tikzcd}$$

Every map in this diagram is the evident embedding. Now on one hand, a is determined by the composition of the top two horizontal maps, since this is the embedding of a polydisc. On the other hand, a is also determined by both the horizontal lower maps, and since the upwards maps are all closed immersions, we conclude that $a^2 = a$, and $a \neq 0$ because $\mathrm{Res}_n \circ H^n_c(\phi) \neq 0$.

It remains to prove the result for any open immersion of Stein manifolds $\phi : X_1 \to X_2$ - that is, we'd like to show that for any point $p \in X_1$ that $a(X_1, X_2)(p) = 1$. Embed each X_i into affine space, $\phi_i : X_i \to \mathbb{C}^{n_i}$, such that $p \mapsto 0$ under both embeddings. Choose small polydiscs $B_i \subseteq \mathbb{C}^{n_i}$ whose preimages in X_1 and X_2 are the same open neighbourhood U of p in X_1: $U = \phi_1^{-1}(B_1) = \phi_2^{-1}(B_2)$. The commutative diagrams

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X_i \arrow[d] \\
B_i \arrow[r] & \mathbb{C}_i^n
\end{tikzcd}$$

demonstrate that $a(U, X_i) = 1$ as functions on U, and by functoriality of associated maps on cohomology we have $a(U, X_2) = a(U, X_1)a(X_1, X_2)$, meaning that the restriction of $a(X_1, X_2)$ to any open neighbourhood is identically 1, and so it is globally constant of value 1, as required. Q.E.D

2.4 The Reconstruction Theorem

We outline the motivation given in [45]. Recall the Riemann-Hilbert correspondence for the structure sheaf on a complex analytic manifold X. One classically defines the *holomorphic solutions* to a system of equations with associated $\mathcal{D}_X$-module $\mathcal{M}$ to be $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. We define the more sophisticated functor Sol by

$$\mathrm{Sol}(\mathcal{M}) = \mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \tag{62}$$

which also encodes all compatibility criteria on solutions. The Riemann-Hilbert correspondence, states that this induces an equivalence between the derived category of regular holonomic $\mathcal{D}_X$-modules, and the derived category of constructible $\mathbb{C}_X$-sheaves. As a consequence one can reconstruct the full complex of regular holonomic $\mathcal{D}_X$-modules from the complex $\mathrm{Sol}(\mathcal{M})$. The idea of Prosmans' & Schneiders' reconstruction theorem is to extend this reconstruction to perfect complexes of $\mathcal{D}_X^\infty$-modules. We outline the arguments of chapters 6 and 7 of [45]:

Definition 2.7. For complex manifolds X and Y, the *sheaf of differentials of bidegree* (r, s), $\Omega^{(r,s)}_{X\times Y}$ is the sheaf whose sections are locally finite sums of terms $a_{i,j}dx_{i_1} \wedge \cdots \wedge dx_{i_r} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_s}$.

As this is a sheaf of nuclear Fréchet spaces, we may apply IB to obtain a sheaf valued in Ind(Ban). Moreover by ([45], Chapter 4) there is a box product isomorphism

$$\mathrm{IB}(\Omega^{(r,s)}_{X\times Y}) \cong \mathrm{IB}(\Omega^r_X)\hat{\boxtimes}^{\mathbb{L}}\mathrm{IB}(\Omega^s_Y) \tag{63}$$

Theorem 2.9. *([45] Theorem 6.3) - For X, Y complex manifolds of dimensions d_X and d_Y, there is a canonical isomorphism*

$$\mathrm{IB}(\Omega^{(d_X-r,s)}_{X\times Y})[d_X] \cong \mathbb{R}L(q_X^{-1}\mathrm{IB}(\Omega^r_X), q_Y^!\mathrm{IB}(\Omega^s_Y)) \tag{64}$$

Proof - The proof is a series of isomorphisms arising from Verdier duality, where $\omega_{X\times Y}$ is the dualising complex:

$$\begin{aligned}
\mathbb{R}L(q_X^{-1}\mathrm{IB}(\Omega_X^r), q_Y^! \mathrm{IB}(\Omega_Y^s)[d_Y]) &\simeq \mathbb{R}L(q_X^{-1}\mathrm{IB}(\Omega_X^r), q_Y^! D\mathrm{IB}(\Omega_Y^{d_Y - s})) \quad (65)\\
&\simeq \mathbb{R}L(q_X^{-1}\mathrm{IB}(\Omega_X^r), D(\mathrm{IB}(q_Y^{-1}\Omega_Y^{d_Y - s}))) \quad (66)\\
&\simeq \mathbb{R}L(q_X^{-1}\mathrm{IB}(\Omega_X^r), \mathbb{R}L(\mathrm{IB}(q_Y^{-1}\Omega_Y^{d_Y - s}), \omega_{X\times Y})) \quad (67)\\
&\simeq \mathbb{R}L(\mathrm{IB}(\Omega_X^r)\hat{\boxtimes}^{\mathbb{L}}\mathrm{IB}(\Omega_Y^{d_Y - s}), \omega_{X\times Y}) \quad (68)\\
&\simeq \mathbb{R}L(\mathrm{IB}(\Omega_{X\times Y}^{(r, d_Y - s)}), \omega_{X\times Y}) \quad (69)\\
&\simeq D(\mathrm{IB}(\Omega_{X\times Y}^{(r, d_Y - s)}) \quad (70)\\
&\simeq \mathrm{IB}(\Omega_{X\times Y}^{(d_X - r, s)})[d_X + d_Y] \quad (71)
\end{aligned}$$

We have successively used our calculation of the Verdier dual, made a series of standard manipulations, used the box product isomorphism (63), and concluded using the Verdier dual calculation once again.

We now state some useful corollaries for what follows:

1. The following quasi-isomorphism from the theory of abelian sheaves generalises immediately to the quasi-abelian context, for sheaves $\mathcal{F}$ and $\mathcal{G}$ on X and Y respectively ([45] Corollary 6.4):

   $$\mathbb{R}\Gamma_{K\times Y}(X \times Y, \mathbb{R}L(q_X^{-1}\mathcal{F}, q_Y^! \mathcal{G})) \simeq \mathbb{R}L(\mathbb{R}\Gamma(K, \mathcal{F}), \mathbb{R}\Gamma(Y, \mathcal{G})) \quad (72)$$

   Combined with the previous isomorphism we obtain the quasi-isomorphism ([45] Corollary 6.5):

   $$\mathbb{R}(X \times Y, \mathrm{IB}(\Omega_{X\times Y}^{(d_x - r, s)})[d_X]) \simeq \mathbb{R}L(\mathbb{R}\Gamma(K, \mathrm{IB}(\Omega_X^r)), \mathbb{R}\Gamma(Y, \mathrm{IB}(\Omega_Y^s))) \quad (73)$$

   Furthermore, if X and Y are Stein manifolds, and K is holomorphically convex in X, by Cartan's theorem B, we may rewrite the arguments of the right side of this quasi-isomorphism

   $$\begin{aligned}
   \mathbb{R}L(\mathbb{R}\Gamma(K, \mathrm{IB}(\Omega_X^r)), \mathbb{R}\Gamma(Y, \mathrm{IB}(\Omega_Y^s))) &\simeq \mathbb{R}L(\mathrm{IB}(\Omega_X^r(K)), \mathrm{IB}(\Omega_Y^s(Y))) \quad (74)\\
   &\simeq L(\mathrm{IB}(\Omega_X^r(K)), \mathrm{IB}(\Omega_Y^s(Y))) \quad (75)\\
   &\simeq \mathrm{IB}(L_b(\Omega_X^r(K), \Omega_Y^s(Y))) \quad (76)
   \end{aligned}$$

   The second quasi-isomorphism follows from the fact that the arguments are Fréchet spaces, and so it is underived ([45]), and the third follows from (8).

2. For an embedding of complex manifolds $Y \subseteq X$ of dimensions d_Y, d_X, the complex

   $$\mathbb{R}\Gamma_Y(\mathrm{IB}(\mathcal{O}_X)) \tag{77}$$

   is concentrated in degree $d_X - d_Y$. This may be verified locally, and so open Stein neighbourhoods in Y and X of $y \in Y$ may be identified with open Stein neighbourhoods of affine space of the origin. We are therefore reduced to the local statement:

   $$LH^k(\mathbb{R}\Gamma_{0\times V}(U \times V, \mathrm{IB}(\mathcal{O}_{U\times V}))) \simeq 0, \qquad k \neq d_X - d_Y \tag{78}$$

   This then follows from the previous item.

Theorem 2.10. *([45] Theorem 6.7) - A morphism of complex manifolds $f : X \to Y$, let Δ_f denote the graph of the morphism, and let $\delta_f : X \to X \times Y$ be the embedding into the graph. Then there is a canonical quasi-isomorphism*

$$\mathbb{R}L(f^{-1}\mathrm{IB}(\mathcal{O}_Y), \mathrm{IB}(\mathcal{O}_X)) \simeq \delta_f^{-1}\mathbb{R}\Gamma_{\Delta_f}\mathrm{IB}(\Omega^{(0,d_Y)}_{X\times Y})[d_Y] \tag{79}$$

These complexes are therefore concentrated in degree 0 and the 0-term is isomorphic to $\mathcal{D}^\infty_{X\to Y}$. In particular, applying the theorem to the identify $f : X \to X$ yields

$$\mathbb{R}L(\mathrm{IB}(\mathcal{O}_X), \mathrm{IB}(\mathcal{O}_X)) \simeq \delta^{-1}\mathbb{R}\Gamma_\Delta \mathrm{IB}(\Omega_{X\times X^{(0,d_X)}[d_X]}) \tag{80}$$

and the only nonzero cohomology object is the 0^{th}, isomorphic to $\mathcal{D}^\infty_X$.

Proof - We apply $\delta_f^!$ to the quasi-isomorphism (64), choosing $r = d_X$ and $s = d_Y$ (note the roles of X and Y are reversed)

$$\begin{aligned}
\delta_f^!\mathrm{IB}(\Omega^{(0,d_Y)}_{X\times Y})[d_Y] &\simeq \delta_f^!\mathbb{R}L(q_Y^{-1}\mathrm{IB}(\mathcal{O}_Y), q_X^!\mathrm{IB}(\mathcal{O}_X)) && (81)\\
&\simeq \mathbb{R}L(\delta_f^{-1}q_Y^{-1}\mathrm{IB}(\mathcal{O}_Y), \delta_f^! q_X^!\mathrm{IB}(\mathcal{O}_X)) && (82)\\
&\simeq \mathbb{R}L((q_Y \circ \delta_f)^{-1}\mathrm{IB}(\mathcal{O}_Y), (q_X \circ \delta_f)^{-1}\mathrm{IB}(\mathcal{O}_X)) && (83)\\
&\simeq \mathbb{R}L(f^{-1}\mathrm{IB}(\mathcal{O}_Y), \mathrm{IB}(\mathcal{O}_X)) && (84)
\end{aligned}$$

where the last isomorphism follows from $q_Y \circ \delta_f = f$ and $q_X \circ \delta_f = \mathrm{id}_X$. The result now follows from the quasi-isomorphism [49]:

$$\mathcal{D}^\infty_{X\to Y} \simeq \delta_f^{-1}\mathbb{R}\Gamma_{\Delta_f}\Omega^{(0,d_Y)}_{X\times Y}p[d_Y] \tag{85}$$

Remark - If $f - \mathrm{id}_X$, we conclude that

$$\mathbb{R}\mathcal{H}\mathrm{om}(\mathrm{IB}(\mathcal{O}_X), \mathrm{IB}(\mathcal{O}_X)) \cong \mathcal{D}^\infty_X \tag{86}$$

We now outline the reconstruction theorem. Let $\mathcal{R}$ be a ring in $\mathrm{Sh}(X)$ and let $\mathrm{Mod}(\mathcal{R})$ be the category of $\mathcal{R}$-modules. Given two $\mathcal{R}$-modules $\mathcal{M}, \mathcal{N}$, the internal Hom $L(\mathcal{M}, \mathcal{N})$ has a left and right $\mathcal{R}$-module structure. The object of $\mathcal{R}$-module homomorphisms $L_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ is defined to be the kernel of the associated pair of structure maps $L(\mathcal{M}, \mathcal{N}) \to L(\mathcal{R}, L(\mathcal{M}, \mathcal{N}))$.

As $\mathrm{Mod}(\mathcal{R})$ has enough injectives, the bifunctor $L_{\mathcal{R}}(\cdot, \cdot)$ may be derived.

Similarly, for any sheaf $\mathcal{F}$ valued in $\mathrm{Ind(Ban)}$ and $\mathcal{R}$-module $\mathcal{N}$, $L(\mathcal{F}, \mathcal{N})$ is also an $\mathcal{R}$-module. Then this bifunctor can also be derived, by resolving a sheaf valued in $\mathrm{Ind(Ban)}$ by a complex of sheaves of the form $\bigoplus (P_i)_U$ for P_i projective objects of $\mathrm{Ind(Ban)}$ and U arbitrary open sets. The $\mathcal{R}$-module component is resolved by a complex of flabby sheaves.

Definition 2.8. A complex $\mathcal{M} \in \mathcal{D}^b(\mathrm{Mod}(\mathcal{R}))$ is *perfect* if there are integers $p \leq q$ such that, for each $x \in X$ has an open neighbourhood U on which the complex is isomorphic to a complex of the form

$$0 \longrightarrow \mathcal{R}|_U^{k_p} \longrightarrow \cdots \longrightarrow \mathcal{R}|_U^{k_q} \longrightarrow 0$$

with $\mathcal{R}_U^{k_i}$ in the i^{th} position. That is, a perfect complex has a local *generalised free presentation* locally to any point, the length of which is uniformly bounded over X. We denote the triangulated subcategory of perfect complexes of $\mathcal{R}$-modules by $\mathcal{D}^b_{\mathrm{pf}}(\mathrm{Mod}(\mathcal{R})$

Proposition 2.2. *(Abstract Reconstruction Theorem, [45] Proposition 7.1) - Any sheaf $\mathcal{N}$ valued in* $\mathrm{Ind(Ban)}$ *is an $L(\mathcal{N}, \mathcal{N})$-module. If $\mathcal{N}$ has no derived automorphisms - that is, for $k \neq 0$:*

$$LH^k(\mathbb{R}L(\mathcal{N}, \mathcal{N})) = 0 \tag{87}$$

Then the functor $\mathbb{R}L_{\mathcal{R}}(\cdot, \mathcal{N})$ restricts to a functor on $\mathcal{D}^b_{\mathrm{pf}}(\mathrm{Mod}(\mathcal{R})$, and we can recover $\mathcal{M}$ from $\mathbb{R}L(\mathcal{N}, \mathcal{N})$:

$$\mathbb{R}L(\mathbb{R}L_{\mathcal{R}}(\mathcal{M}, \mathcal{N}), \mathcal{N}) \simeq \mathcal{M} \tag{88}$$

Therefore, the functor $\mathbb{R}L_{\mathcal{R}}(\cdot, \mathcal{N})$ embeds the category of perfect complexes $\mathcal{R}$-modules by $\mathcal{D}^b_{\mathrm{pf}}(\mathrm{Mod}(\mathcal{R}))$ as a full subcategory of $\mathcal{D}^b(\mathrm{Sh}(X, \mathrm{Ind(Ban)}))$.

Proof - The first statement is clear. Since $\mathbb{R}L_{\mathcal{R}}(\mathcal{R}, N) \cong \mathcal{N}$, it follows that for any perfect complex $\mathcal{M}$, $\mathbb{R}L_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ is a bounded complex of sheaves. Therefore the functor does restrict to $\mathcal{D}^b_{\mathrm{pf}}(\mathrm{Mod}(\mathcal{R}))$.

For the last statement we argue similarly. We first show demonstrate the reconstruction quasi-isomorphism (88) for $\mathcal{M} = \mathcal{R}$:

$$\begin{aligned} \mathbb{R}L(\mathbb{R}L_{\mathcal{R}}(\mathcal{M},\mathcal{N}),\mathcal{N}) &\simeq \mathbb{R}L(\mathcal{N},\mathcal{N}) && (89)\\ &\simeq L(\mathcal{N},\mathcal{N}) && (90)\\ &\simeq \mathcal{R} && (91) \end{aligned}$$

The second quasi-isomorphism follows from the acyclicity assumption (87), and the third is simply true by definition. Therefore it is an isomorphism for any free $\mathcal{R}$-module. and therefore any perfect complex of $\mathcal{R}$-modules, which generically matches the local structure of $\mathcal{M}$. Q.E.D

To apply this to $\mathcal{D}_X^\infty$, we consider a minor modification to IB - an adjunction between $\mathrm{Vect}_\mathbb{C}$ and Ind(Ban). The left adjoint is given by

$$\begin{aligned} I_{\mathrm{Vect}_\mathbb{C}} : \mathrm{Vect}_\mathbb{C} &\to \mathrm{Ind(Ban)} && (92)\\ V &\mapsto \text{"}\varinjlim_{W\subseteq V}\text{"}W && (93) \end{aligned}$$

where the inductive limit runs over all finite dimensional subspaces. The right adjoint is given by

$$\begin{aligned} L_{\mathrm{Vect}_\mathbb{C}} : \mathrm{Ind(Ban)} &\to \mathrm{Vect}_\mathbb{C} && (94)\\ \text{“}\varinjlim\text{”}W &\mapsto \varinjlim W && (95) \end{aligned}$$

Both are exact functors, and since every (complex) vector space is the direct limit of its finite dimensional subspaces, $L_{\mathrm{Vect}_\mathbb{C}} \circ I_{\mathrm{Vect}_\mathbb{C}} = \mathrm{id}$. In a similar fashion to IB, we extend their definitions to sheaves, denoted $\tilde{I}_{\mathrm{Vect}_\mathbb{C}}$ and $\tilde{I}_{\mathrm{Vect}_\mathbb{C}}$.

We remark that, once again, the essential image of $I_{\mathrm{Vect}_\mathbb{C}}$ consists of reduced inductive systems, and therefore this may be adjusted to furnish an adjunction between $\mathrm{Vect}_\mathbb{C}$ and $\mathrm{CBorn}_\mathbb{C}$. All the theorems below remain valid in this context.

We may then prove the following statement in an entirely similar way to the previous proposition:

Proposition 2.3. *([45] Proposition 7.2) - For $\mathcal{N}$ a sheaf on X valued in* $\mathrm{Ind}(\mathrm{Ban}_\mathbb{C})$*, such that*

$$LH^k(\mathbb{R}\mathcal{H}\mathrm{om}(\mathcal{N},\mathcal{N})) = 0 \qquad (96)$$

Let $\mathcal{R}_{\mathrm{Vect}_\mathbb{C}}$ be the ring $\mathcal{H}\mathrm{om}(\mathcal{N},\mathcal{N})$. Then $\mathcal{N}$ is an $\tilde{I}_{\mathrm{Vect}_\mathbb{C}}(\mathcal{R}_{\mathrm{Vect}_\mathbb{C}})$-module, and the functor $\mathbb{R}L_{\tilde{I}_{\mathrm{Vect}_\mathbb{C}}(\mathcal{R}_{\mathrm{Vect}_\mathbb{C}})}(\tilde{I}_{\mathrm{Vect}_\mathbb{C}}(\cdot),\mathcal{N})$ restricts to the category of perfect complexes of $\mathcal{R}_{\mathrm{Vect}_\mathbb{C}}$-modules, and there is a similar canonical isomorphism:

$$M \cong \mathbb{R}\mathcal{H}\mathrm{om}(\mathbb{R}L_{\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{R}_{\mathrm{Vect}_{\mathbb{C}}})}(\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{M}), \mathcal{N}), \mathcal{N}) \tag{97}$$

Therefore once again, this functor fully and faithfully embeds the category of perfect complexes into the category of bounded complex of sheaves valued in Ind(Ban)*.*

Proof - Since $\tilde{L}_{\mathrm{Vect}_{\mathbb{C}}} \circ L \simeq \mathcal{H}\mathrm{om}$, $\tilde{L}_{\mathrm{Vect}_{\mathbb{C}}} \circ \mathbb{R}L \simeq \mathbb{R}\mathcal{H}\mathrm{om}$. We apply $\tilde{L}_{\mathrm{Vect}_{\mathbb{C}}}$ to the evaluation morphism

$$I_{\mathrm{Vect}_{\mathbb{C}}}(M) \to \mathbb{R}L(\mathbb{R}L_{\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{R}_{\mathrm{Vect}_{\mathbb{C}}})}(\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{M}), \mathcal{N}), \mathcal{N}) \tag{98}$$

This yields a morphism

$$M \to \mathbb{R}\mathcal{H}\mathrm{om}(\mathbb{R}L_{\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{R}_{\mathrm{Vect}_{\mathbb{C}}})}(\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{M}), \mathcal{N}), \mathcal{N}) \tag{99}$$

The proof is then entirely equivalent - show that this is an isomorphism for $M = R$ by a similar series of quasi-isomorphisms, and therefore for arbitrary powers of $\mathcal{R}$ and free complexes.

The reconstruction theorem now follows by the substitution $\mathcal{N} = \mathrm{IB}(\mathcal{O}_X)$ and the representation of infinite order differential operators, $\mathcal{D}_X^{\infty} \cong \mathbb{R}\mathcal{H}\mathrm{om}(\mathrm{IB}(\mathcal{O}_X), \mathrm{IB}(\mathcal{O}_X))$.

Theorem 2.11. *(Reconstruction Theorem, [45] Theorem 7.3) - For X a d_X-dimensional complex manifold, the sheaf* $\mathrm{IB}(\mathcal{O}_X)$ *is an $\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{D}_X^{\infty})$-module, and the functor $\mathbb{R}L_{\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{R}_{\mathrm{Vect}_{\mathbb{C}}})}(\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\cdot), \mathcal{N})$ restricts to the category of perfect complexes of $\mathcal{D}_X^{\infty}$-modules, and the abstract reconstruction isomorphism (100) specialises to an isomorphism:*

$$M \cong \mathbb{R}\mathcal{H}\mathrm{om}(\mathbb{R}L_{\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{D}_X^{\infty})}(\tilde{I}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{M}), \mathrm{IB}(\mathcal{O}_X)), \mathrm{IB}(\mathcal{O}_X)) \tag{100}$$

Therefore, again, the functor fully faithfully embeds perfect complexes of $\mathcal{D}_X^{\infty}$-modules into the category of bounded sheaves valued in $\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})$ *or* $\mathrm{CBorn}_{\mathbb{C}}$*.*

3 Foundations of Derived Analytic Geometry

3.1 Motivation

This chapter summarizes the theory of derived analytic geometry via bornological methods developed in [7]. The objectives of the chapter are to review the motivation behind this theory and establish conventions for the following chapters. The presentation in loc. cit. is very technical and thorough, but the presentation found below is expository, intended to provide the minimum details to appreciate what follows, with a focus on examples to provide adequate intuition.

We begin, much as in the cited work, with the motivation to study analytic geometry as a form of relative algebraic geometry. We then say a few words about how this is done in practice.

Let us briefly recall Grothendieck's relative algebraic geometry. The functor of global sections

$$\mathcal{O} : \text{Locally Ringed Spaces} \to \text{Rings} \tag{101}$$

restricts to an equivalence on the category of affine schemes. This therefore determines the fully-faithful left adjoint spectrum functor Spec : $\text{Rings}^{\text{op}} \to$ Spaces.

More general spaces are then defined to be *prestacks*: contravariant functors $\chi : \text{Aff}^{\text{op}} \to \text{Grpd}$. These encapsulate affine schemes via their functor of points, but also captures formal moduli spaces. Moreover, through the use of Grothendieck topologies on the category of affine schemes, we can then define various kinds of *geometric stack*, such as Deligne-Mumford and Artin stacks.

As a first generalization, we may replace rings with internal algebra objects to some symmetric monoidal category $\mathcal{C}$. Grothendieck's theory is then the case of $\mathcal{C} = \mathbb{Z}$. Prestacks and geometric stacks can then be defined in an entirely analogous manner. Furthermore, this generalization can be extended homotopically, taking $\mathcal{C}$ to be a symmetric monoidal *model category.* With the appropriate adjustments (prestacks replaced with simplicial prestacks, Grothendieck topologies on $\mathcal{C}$ replaced with Grothendieck topologies on $\text{Ho}(\mathcal{C})$), a higher relative algebraic geometry suitable for the study of derived algebraic geometry and brave new geometry is developed in [55].

We now consider an analogous theory suitable for the study of derived analytic geometry. We will consider derived complex analytic geometry. We begin with a class of algebras/affines for which the global sections functor will define an equivalence. The natural candidates are the categories of *Stein spaces and algebras*, by Stein duality and Kiehl & Cartan's theorems A & B. Regarding the Stein algebras as complete locally convex topological vector spaces, we have another fully faithful functor

$$\mathcal{O} : \text{Stein Spaces} \to \text{Comm}(\widehat{\mathcal{T}_c}). \tag{102}$$

Therefore, one might attempt to cast analytic geometry as geometry relative to complete locally convex topological vector spaces. This has a number of deficiencies:

1. $\widehat{\mathcal{T}_c}$ is not an abelian category. It does however have an exact structure, so that homological arguments can be reproduced. We may furthermore embed in the category of $\mathcal{T}_c$ of all locally-convex topological vector spaces, which is *quasi-abelian* in the sense of Schneiders. These are particularly nice exact categories, whose homological algebra closely mirrors that of abelian categories. Therefore this deficiency can be easily resolved.

2. The category $\widehat{\mathcal{T}_c}$ is not a closed symmetric monoidal category, is not presentable, and does not have enough projectives. This means that many of the fundamental constructions central to homological algebra and geometry are not available.

The problem is therefore to replace complete locally convex topological vector spaces with a better-behaved exact category. The author knows of two established methods:

1. In Scholze & Clausen's theory of condensed mathematics, condensed sets are considered. These are sheaves of sets on the pro-étale site of a point, or equivalently, sheaves on the site of profinite sets with jointly-surjective families of maps as covers. To avoid set-theoretic difficulties, a strong limit cardinal κ is fixed, and only sheaves on the site of profinite sets of cardinality less than κ are considered. Then topological spaces/groups/rings/vector spaces embed faithfully into κ-condensed sets/groups/rings/vector spaces, and this embedding is fully faithful on the subcategory of κ-compactly generated objects: those objects whose topology is generated by the collection of all $S \to X$, where S is compact Hausdorff and $|S| < \kappa$ ([15] Proposition 1.7).

2. In the theory developed in [7], we work within the quasi-abelian categories $\text{Ind}(\text{Ban}_R)$ of formal inductive limits of Banach spaces, or in CBorn_R, the quasi-abelian category of complete convex bornological vector spaces.

As quasi-abelian categories, they admit particularly well-behaved homological algebra, and they are both presentable and closed symmetric monoidal. They are in fact equivalent, and will even be monoidally equivalent in most cases of interest (when the value ring is *proper*, [3]).

Since the target category is closed symmetric monoidal and presentable, we can proceed as before to obtain a global sections functor on analytic spaces

$$\mathcal{O} : \text{Analytic Spaces} \to \text{Ind}(\text{Ban}_R) \tag{103}$$

where Stein spaces have been taken as the basic affine objects. Analytic geometry can then be realized as geometry relative to $\mathrm{Ind}(\mathrm{Ban}_R)$.

We will discuss derived enhancements of this theory in what follows. In algebraic geometry this has been done to enrich the theory to great effect. However, in our theory *it is essential that we adopt the derived viewpoint from the outset.* The distinction comes from the topologies we consider: in algebraic geometry, localizations and topologies of interest are flat, and so one can proceed without considering derived enhancements. Analytic geometers are not so fortunate: there are quite simple examples of localizations that are not flat.

Moreover, we will require tailored concepts of familiar classes of maps (smooth, étale, finite presentation) in analytic geometric contexts, as the abstract definition in terms of internal monoids do not apply to even the most basic localizations. To resolve this, we encode the algebraic structures of interest through the use of Lawvere theories, and redefine these classes of maps in terms of Lawvere theories.

Analytic geometry is then developed relative to a monoidal exact category $\mathcal{E}$, a Lawvere theory T, and a faithful, finite coproduct-preserving functor $T \to \mathrm{Comm}(\mathcal{E})$: a "realization" of T-algebras as internal monoids.

In a series of short chapters, we outline the theory from [7] to construct the context controlling dagger Stein geometry.

3.2 Homotopical Algebra

Homotopical algebra is developed in chapter 2 of [7] in an analogous manner to chapter 1.2 of [55]. The analogue of the HA contexts of Toën & Vezzosi are $(\infty, 1)$-algebra contexts, which have some additional structure for the simultaneous treatment of simplicial commutative algebras and $\mathbb{E}_\infty$-algebras. Many of the results of chapter 7 of [31] are reproduced in this context, and several variants of $(\infty, 1)$-algebra contexts are defined to exhibit good behaviour. Generalizations of Raksit's derived algebraic contexts [47] are developed, permitting the reconstruction of derived commutative rings from loc. cit. . Because of the existence and familiar properties of the cotangent complex, notions of formal smoothness and étaleness can be developed as in chapter 1.2 of [55].

3.3 Lawvere Theories

Lawvere theories are developed to pick out particular geometries within the broader homotopical algebraic context in which we work. Recall that Lawvere theories functorially determine algebraic structure of their algebras. We first generalize to *multisorted Lawvere theories* - generalizations which permit free T-algebras over a broader class of parameters. For our purposes, we require polyradii with possibly infinite components as coefficients for algebras of holomorphic functions on arbitrary complex polydiscs. We also consider functorial parameters for these algebras, which are useful for considering systems of polydisc algebras and for the consideration of limiting objects, such as overconvergent algebras.

Chapters 4.1 and 4.2 develop the basic homotopical algebra of Lawvere and Fermat theories - Lawvere theories that admit a formal differentiation, and thereby a formal infinitesimal theory. Chapter 4.3 explains how to realize T-algebras as monoids internal to a symmetric monoidal model category: given a closed symmetric monoidal category $\mathcal{C}$ and a collection of compact objects Z, such that the model structure they define on s$\mathcal{C}$ transfers to internal monoids sComm$(\mathcal{C})$, the realization is given by a finite-coproduct-preserving functor

$$F : T^{\mathrm{op}} \to \mathrm{sComm}(\mathcal{C}). \tag{104}$$

The right hand side presents $L^H(\mathrm{sComm}(\mathcal{C}))$, and this functor extends along sifted colimits to a functor $P_\Sigma(T^{\mathrm{op}}) \cong \mathbf{sAlg}_T \cong L^H(\mathrm{sAlg}_T) \to L^H(\mathrm{sComm}(\mathcal{C})$.

Amongst such Lawvere theories, those which control algebras which are completions of polynomial algebras are of particular importance to analytic geometry, as local models of convergent power series are of this form. Such Lawvere theories are axiomatized as those of *homotopy polynomial type*, and this completeness condition is imposed by imposing that, for all $\underline{\lambda} \in \Lambda^n$, there is a homotopy epimorphism

$$\mathrm{Sym}(\mathbb{I}^n) \to F(T(\underline{\lambda})). \tag{105}$$

The Lawvere theory of interest to us is $\mathrm{Disc}_{\mathbb{C}}$, the Lawvere theory controlling complex Stein geometry. We now recall the theory from [4] [5] [7]. First, classical affinoids are considered, and classical Steins are built from them. Then derived Steins are defined and shown to admit a homotopical version of the geometric presentations of classical Steins.

3.4 Stein Geometry

Following [4] [5] and passages from [7], we introduce the analytic algebras and Lawvere theories controlling Stein geometry. We begin with a brief recollection of dagger affinoid geometry from [4]:

Definition 3.1. - Let k be a complete nontrivially-valued field:

1. An algebra $A \in \mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_k)$ is *multiplicatively convex* (or an *m-algebra*) if it is isomorphic to an inductive limit of Banach algebras: $A \in \mathrm{Ind}(\mathrm{Comm}(\mathrm{Ban}_k))$.
2. The *Berkovich spectrum* of an m-algebra A is the limit of Berkovich spectra of its component algebras:

$$\mathcal{M}(A) = \mathcal{M}(\text{“}\varinjlim\text{”} A_i) = \varprojlim \mathcal{M}(A_i) \tag{106}$$

It is a compact Hausdorff space, by the argument of [3] used for the bornological spectrum.

3. For any polyradius $\rho = (r_1, \cdots, r_n)$, define the *ring of overconvergent functions on the polydisc of polyradius* ρ as follows:

$$\mathcal{W}_k^n(\rho) = \text{“}\varinjlim_{r>\rho}\text{”} \mathcal{T}_k^n(r) \tag{107}$$

   k-Dagger affinoid algebras are those algebras that can be expressed as a quotient $A \cong \mathcal{W}_k^n(\rho)/I$ of a ring of overconvergent functions. The category of k-dagger affinoids is denoted $\mathrm{Afnd_k}^{\dagger}$.

4. A *localization of k-dagger affinoids* is a morphism $A \to D$, which induces an injection of spectra, and if $A \to B$ is any map of affinoid algebras such that $\mathcal{M}(B) \to \mathcal{M}(A)$ factors through $\mathcal{M}(D)$, then $A \to B$ factors through D.

5. A *Weierstrass localization* of a k-dagger affinoid algebra is a morphism of the form

$$A \to \frac{A\left\langle r_1^{-1}X_1, \cdots, r_n^{-1}X_n\right\rangle}{(X_1 - f_1, \cdots, X_n - f_n)}, \tag{108}$$

   a *Laurent localization* is a map of the form

$$A \to \frac{A\left\langle r_1^{-1}X_1, \cdots, r_n^{-1}X_n, s_1^{-1}Y_1, \cdots, s_n^{-1}Y_n\right\rangle}{(X_1 - f_1, \cdots, X_n - f_n, g_1Y_1 - 1, \cdots, g_nY_n - 1)}, \tag{109}$$

   and a *rational localization* is a map of the form

$$A \to \frac{A\left\langle r_1^{-1}X_1, \cdots, r_n^{-1}X_n\right\rangle}{(hX_1 - f_1, \cdots, hX_n - f_n)}, \tag{110}$$

   where $f_i, \cdots, f_n, h \in A$ generated the unit ideal of A.

6. k-dagger affinoid spaces are defined to be Grothendieck topological spaces with underlying set the spectrum of a k-dagger affinoid algebra, and covering families finite covers by k-dagger affinoid subdomains - maps which are injective on underlying sets and isomorphisms on stalks (which are defined in the usual way). This topology is called the *weak dagger G-topology.*

   Dagger analytic spaces are defined as locally ringed Grothendieck topological spaces with a Berkovich net τ and atlas of dagger affinoid subdomains.

In chapter 5 of [4], it is proven that all dagger affinoid localizations define homotopy epimorphisms, and a homotopy epimorphism $f : A \to B$ of k-dagger affinoid algebras defines a k-dagger affinoid domain immersion $\mathcal{M}(B) \to \mathcal{M}(A)$.

We now move on to Stein algebras and domains, following [5]. Dagger Stein spaces admit filtrations by dagger affinoids $X = \bigcup_{i\in\mathbb{N}} U_i$ where $\mathcal{O}(U_{i+1}) \to \mathcal{O}_X(U_i)$ is a Weierstrass localizations. Using the methods of [4], [8], homotopy epimorphisms of Stein algebras are demonstrated to correspond to Stein localizations. This is not straightforward, as the projective and injective tensor product of bornological vector spaces don't commute. Section 3 of [5] is dedicated to resolving these issues in remarkable cases applicable to Stein domains.

Again, we assume k is a complete nontrivially valued field:

1. A *pro-multiplicatively convex algebra* (or *pro m-algebra*) is a small cofiltered projective limit of complete bornological m-algebras:

$$A \cong \varprojlim_{i\in I} A_i. \tag{111}$$

   Such an algebra is *densely defined* if each projection $\pi : A \to A_i$ has dense image.

2. A *dagger Stein algebra* over k is a complete bornological k-algebra A isomorphic to an inverse limit of k-dagger affinoid algebras in CBorn_k, $\varprojlim_{i\in\mathbb{Z}_{\geq 0}} A_i$, such that each map $A_{i+1} \to A_i$ is a Weierstrass localization, and $\mathcal{M}(A_i)$ is contained in the interior of $\mathcal{M}(A_{i+1})$. The category of dagger Stein algebras will be denoted $\mathrm{Stn}_k^{\dagger}$.

   Dagger Stein algebras are densely-defined, pro-multiplicatively convex, bornological Fréchet algebras ([5] Lemma 4.15). In the presentation $A \cong \varprojlim A_i$ it may be assumed that the A_i are strictly dagger affinoid (Lemma 4.19), and any morphism of dagger Stein algebras arises via a projective system of dagger affinoids (Lemma 4.24).

3. Just as in the dagger affinoid setting, homotopy epimorphisms of Stein algebras are characterized as localizations ([5], Theorems 5.5, 5.7).

We now consider derived Stein algebras and spaces, following the relevant sections of [7] - particularly sections 5.2-4. Fix a nontrivially valued Banach field k:

Definition 3.2. 1. A *bornological Fréchet-Stein algebra* A is a complete bornological algebra equipped with a system:

$$A \to \cdots \to \overline{A}_{n+1} \to A_{n+1} \to \cdots \to \overline{A}_0 \to A_0 \tag{112}$$

where

    (a) A is a bornological nuclear Fréchet space.
    (b) The $\overline{A}_n$ are strongly left Noetherian algebras, and their bornologies have countable bases.
    (c) Finitely-presented left $\overline{A}_n$-algebras are proper and nuclear.
    (d) The A_n are Fréchet algebras which is submultiplicative as a bornological algebra.
    (e) The projections $A \to A_n$ and $A \to \overline{A}_n$ are epimorphisms, and the images $A_{n+1} \to \overline{A}_n$ are strongly dense.
    (f) $\overline{A}_n$ is transverse over $\overline{A}_{n+1}$ to left finitely-generated $\overline{A}_{n+1}$-modules.

2. A *pre-quasicoherent sheaf* on a bornological Fréchet Stein algebra is a system of modules M_n on the A_n and $\overline{M}_n$ on the $\overline{A}_n$, and isomorphisms

$$\alpha_n : A_n \widehat{\otimes}_{\overline{A}_n} \overline{M}_n \to M_n$$
$$\beta_n : \overline{A}_n \widehat{\otimes}_{A_{n+1}} M_{n+1} \to \overline{M}_n,$$

   such that the evident compositions induce equivalences:

$$\overline{A}_n \otimes^{\mathbb{L}}_{\overline{A}_{n+1}} \overline{M}_{n+1} \cong \overline{M}_{n+1} \tag{113}$$

   Denote the global sections functor by $\Gamma : (M_n, \overline{M}_n, \alpha_n, \beta_n) \mapsto \varprojlim M_n$. The pre-quasicoherent sheaf is *quasicoherent* if the global sections defines a bornological nuclear Fréchet space, and the maps $\overline{M}_{n+1} \to \overline{M}_n$ are strongly dense.

3. A pre-quasicoherent sheaf $(M_n, \overline{M}_n, \alpha_n, \beta_n)$ is *coadmissible* if the M_n are finitely-generated left A_n-modules.

   Coadmissible modules form an abelian category, and $\Gamma : \mathrm{Coad}_{(A, A_n, \overline{A}_n)} \to \mathrm{Mod}_A(\mathrm{CBorn}_k)$ is exact ([7] Proposition 5.2.58).

4. An object $A \in \mathrm{DAlg}(\mathrm{Ind}(\mathrm{Ban}_k))$ is a *derived bornological Fréchet-Stein algebra* if $\pi_0(A)$ is a bornological Fréchet-Stein algebra, and each $\pi_n(A)$ is a coadmissible left $\pi_0(A)$-module.

Lemma 3.1. *([7] Lemma 5.2.68) - A connective derived algebra* $A \in \mathrm{DAlg}^{\mathrm{cn}}(\mathrm{Ind}(\mathrm{Ban}_k))$ *is a derived bornological Fréchet-Stein algebra, if and only if it admits a presentation*

$$A \to \cdots \to \overline{A}_{n+1} \to A_{n+1} \to \cdots \to \overline{A}_0 \to A_0 \tag{114}$$

where

1. *A is the inverse limit of each of the A_n and $\overline{A}_n$:*

$$A \cong \mathbb{R} \varprojlim A_n \cong \mathbb{R} \varprojlim \overline{A}_n \tag{115}$$

2. *$(\pi_0(A), \pi_0(A_n), \pi_0(\overline{A}_n))$ is a presentation of $\pi_0(A)$ as a bornological Fréchet-Stein algebra.*

3. *The system morphisms $\overline{A}_{n+1} \to \overline{A}_n$ are derived strong.*

Proof Sketch - For given a bornological Fréchet-Stein algebra, sequences $A_{\leq n}$ may be constructed inductively, starting from the base case $A_{\leq 0} = \pi_0(A)$ with geometric presentation by virtue of it being a bornological Fréchet-Stein algebra. The sequences $A_{\leq n}$ are constructed by square zero-extensions:

$$(\overline{A}_i)_{\leq n+1} = (A_i)_{\leq n} \oplus_{d_{i,n}} ((\overline{A}_i)_{\leq 0} \otimes_{\pi_0(A)} \pi_{n+2}(A)[n+1]) \tag{116}$$

$$(A_i)_{\leq n+1} = (A_i)_{\leq n} \oplus_{d_{i,n}} ((A_i)_{\leq 0} \otimes_{\pi_0(A)} \pi_{n+2}(A)[n+1]) \tag{117}$$

Conversely, if such a presentation exists as in (114), then the $\pi_m(\overline{A}_n)$ define coadmissible modules over $(\pi_0(A), \pi_0(A_n), \pi_0(\overline{A_n}))$. By a spectral sequence argument we see that $\pi_n(A) = \pi_n(\varprojlim A_i)$ is also coadmissible. Q.E.D.

Example - Let Disc_k be the Lawvere theory controlling disc algebras: for a polyradius $\rho \in \{\mathbb{R} \cup \infty\}$, the associated disc algebra is

$$\mathcal{O}(D^n_{\leq \rho, k}) = \varprojlim_{r \leq \rho} R\{r_1^{-1} X_1, \cdots, r_n^{-1} X_n\}. \tag{118}$$

These algebras generate the Lawvere theory Disc_k. It is an algebra concretely of homotopy $\text{Ind}(\text{Ban}_k)$-polynomial type, which roughly means that these generating algebras are completions of polynomial algebras, and the realization functor for T-algebras in internal monoids of $\text{Ind}(\text{Ban}_k)$ is fully faithful (see [7] Definitions 4.3.38, 4.3.44).

Lemma 3.2. *([7] Lemma 5.3.86) - Any discrete finitely Disc_k-presented algebra is a bornological Fréchet-Stein algebra.*

Proof - Quotients of bornological Fréchet-Stein algebras are bornological Fréchet-Stein algebras, and so it's sufficient to prove the claim for $\mathcal{O}(D^n_{k,\lambda})$. Then the geometric presentation is given as follows: let ρ^m be an increasing sequence converging to λ:

$$\begin{aligned}\mathcal{O}(D^n_{k,\lambda}) &\cong \varprojlim_m k \left\langle \rho_1^{-m} x_1, \cdots, \rho_n^{-m} x_n \right\rangle^{\dagger} \\ &\cong \varprojlim_m k \left\langle \rho_1^{-m} x_1, \cdots, \rho_n^{-m} x_n \right\rangle\end{aligned}$$

The canonical maps between the algebras of this system are dense, and the dagger algebras are strongly Noetherian, therefore furnishing a presentation of $\mathcal{O}(D^n_{k,\lambda})$ as a bornological Fréchet-Stein algebra. Q.E.D

3.5 Derived Stein Geometry

We conclude with the construction of a context for derived Stein geometry from chapter 9.2 of [7]. We first introduce the *Cartan contexts* of ([7] Definition 8.1.17). These are tuples

$$(\mathrm{Aff}^{\mathrm{cn}}_{\mathcal{C}}, \tau, \mathbb{P}, \mathcal{A}^{\heartsuit}, \mathbb{Q}^{\heartsuit}) \tag{119}$$

of a *strong relative* $(\infty,1)$*-geometry tuple* $(\mathrm{Aff}^{\mathrm{cn}}_{\mathcal{C}}, \tau, \mathbb{P}, \mathcal{A}^{\heartsuit})$, and a subpresheaf $\mathbb{Q}^{\heartsuit} \subseteq \mathrm{QCoh}^{\heartsuit}|_{\mathcal{A}^{\heartsuit}}$ subject to the various conditions as laid out in ([7] Definition 8.1.17) that ensure that the underlying $(\infty,1)$-geometry tuples satisfy strong descent conditions. $(\infty,1)$-pregeometry tuples are analogues of the HAG contexts of Toën & Vezzosi, consisting of an $(\infty,1)$-Grothendieck site and class of suitably "smooth" maps $\mathbb{P}$: all maps in covers of τ are in $\mathbb{P}$, $\mathbb{P}$ is τ-local, contains all isomorphisms and is closed under compositions and pullbacks.

Theorem 3.1. *([7] Theorem 9.2.14) - Let k be any nontrivially-valued Banach field. The tuples*

$$\begin{gathered}
(\mathrm{Aff}^{\mathrm{cn}}_{\mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k))}, \mathrm{G}^{\mathrm{pre},\aleph_1}_{\mathrm{Disc}_k}, \overline{\mathrm{sm}}^{\mathrm{Disc}_k,\aleph_1}_{\mathrm{o}}, (\mathrm{dStn}^{\dagger,f})^{\heartsuit}, \mathrm{Coad}^{\heartsuit}) \\
(\mathrm{Aff}^{\mathrm{cn}}_{\mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k))}, \mathrm{G}^{\mathrm{pre}}_{\mathrm{Disc}_k}, \overline{\mathrm{sm}}^{\mathrm{Disc}_k}_{\mathrm{o}}, (\mathrm{dStn}^{\dagger,f})^{\heartsuit}, \mathrm{Coad}^{\heartsuit}) \\
(\mathrm{Aff}^{\mathrm{cn}}_{\mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k))}, \mathrm{G}^{\mathrm{pre},\aleph_1}_{\mathrm{Disc}_k}, \overline{\mathrm{sm}}^{\mathrm{Disc}_k,\aleph_1}_{\mathrm{o}}, (\mathrm{dStn}^{\dagger,gf})^{\heartsuit}, \mathrm{Coad}^{\heartsuit}) \\
(\mathrm{Aff}^{\mathrm{cn}}_{\mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k))}, \mathrm{G}^{\mathrm{pre}}_{\mathrm{Disc}_k}, \overline{\mathrm{sm}}^{\mathrm{Disc}_k}_{\mathrm{o}}, (\mathrm{dStn}^{\dagger,gf})^{\heartsuit}, \mathrm{Coad}^{\heartsuit})
\end{gathered}$$

are Cartan contexts, where

1. $\mathrm{Aff}^{\mathrm{cn}}_{\mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k))}$ *is the category of connective affines relative to the left heart of* $\mathrm{Ind}(\mathrm{Ban}_R)$.

2. $\mathrm{G}^{\mathrm{pre},\aleph_1}_{\mathrm{Disc}_k}$ *and* $\mathrm{G}^{\mathrm{pre}}_{\mathrm{Disc}_k}$ *are the countable and finite G-T pretopologies: the pretopologies with covers collections of maps* $\{f_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$, *with all* f_i Disc_k*-rational localizations, such that all* A*-modules* M *satisfy descent along countable/finite subfamilies.* T*-rational localizations are the derived analogue of familiar rational localizations: for* $A \in \mathrm{DAlg}^{\mathrm{cn}}(\mathcal{C})$ *a derived algebra,* $f_0, f_1, \cdots, f_n : \mathbb{I} \to A$ *a collection of* Disc_k*-extendable maps (the associated map* $\mathrm{Sym}(\mathbb{I}^{n+1}) \to A$ *factorizes through a free* T*-algebra* $F(\mathrm{Free}_{\mathrm{Disc}_k}(\underline{\lambda}))$*) which generate the unit ideal, the associated* Disc_k*-rational localization is the derived quotient*

$$A \to A \otimes^{\mathbb{L}} F(\mathrm{Free}_T(\underline{\lambda}))//(f_0 y_{\lambda_1} - f_1, \cdots, f_0 y_{\lambda_n} - f_n). \tag{120}$$

3. $\overline{\mathrm{sm}}^{\mathrm{Disc}_k,\aleph_1}_{\mathrm{o}}$ *(respectively* $\overline{\mathrm{sm}}^{\mathrm{Disc}_k}_{\mathrm{o}}$*) are the classes of countable (respectively finite) strong* Disc_k*-open smooth maps relative to* Disc_k *([7] Definition 8.2.24, 8.2.29). These are maps which are locally standard* Disc_k*-smooth and* $\mathrm{Disc}_k - P$*-smooth with respect to the G-T topologies* G_T *generated by the pretopologies* $\mathrm{G}^{\mathrm{pre},\aleph_1}_{\mathrm{Disc}_k}$ *and* $\mathrm{G}^{\mathrm{pre}}_{\mathrm{Disc}_k}$.

4. *The classes of affine objects are taken to be the hearts of (globally) finitely embeddable dagger Steins.*
5. $\mathrm{Coad}^{\heartsuit}$ *is the heart of the category of coadmissible modules.*

We outline a few of the necessary verifications.

1. For any cover $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)\}$ of the pretopology $\mathrm{G}^{\mathrm{pre},\aleph_1}_{\mathrm{Disc}_k}$, the corresponding maps $A \to B_i$ must be in $\overline{\mathrm{sm}}_{\mathrm{o}}^{\mathrm{Disc}_k,\aleph_1}$.

   Let $A \to C$ be a map with C a discrete dagger Stein algebra. Theorem 5.5 of [5] applied to the map $B_i \to A$ implies that $C \to \pi_0(B_i \otimes_A C)$ is a homotopy monomorphism. Therefore $C \to B_i \otimes^{\mathbb{L}}_A C$ is derived strong, and so $B_i \otimes^{\mathbb{L}}_A C$ is a discrete dagger Stein algebra, as required of a smooth map (see section 4.5.3 of [7]).

2. $\mathrm{Coad}(\mathrm{Spec}(A))$ is thick and closed under finite limits and colimits for any finitely-embeddable dagger Stein algebra. For given a map of coadmissible modules on finitely-embeddable dagger Stein algebra $\mathrm{Stn}^{\dagger,f}_k$, we can find a geometric presentation $(A, A_i, \overline{A}_i)$ of the Stein upon which both coadmissible modules can be defined as systems of modules on the components of the system, and limits and colimits can be computed componentwise.

3. Coadmissible modules must satisfy descent for $G^{\aleph_1}_{\mathrm{Disc}_k}$. But by considering the geometric presentation of the dagger Stein by classical Steins, and the decomposition of the coadmissible module on its components, this follows from descent for coherent sheaves on classical Steins, Theorems A and B, and the open mapping theorem.

Lemma 3.3. *([7] Lemma 9.2.17) - For a nontrivially valued Banach field k, the countable* $\mathrm{G} - \mathrm{Disc}_k$ *topologies associated to* Disc_k*-rational and* Disc_k*-open maps, restricted to connective coherent affines are equivalent:*

$$\overline{\mathrm{G}}^{\aleph_1}_{\mathrm{Disc_k-rat}}|_{\mathrm{Aff}^{\mathrm{cn},\mathcal{T}-\mathrm{coh}}} \equiv \overline{\mathrm{G}}^{\aleph_1}_{\mathrm{Disc_k-open}}|_{\mathrm{Aff}^{\mathrm{cn},\mathcal{T}-\mathrm{coh}}} \tag{121}$$

Proof - Clearly, rational covers are open covers.

Conversely, let $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_{i \in \mathcal{I}}$ be a cover. Then the underlying cover $\{\pi_0(\mathrm{Spec}(A_i)) \to \pi_0(\mathrm{Spec}(A))\}$ is an open cover by Stein subdomains. By assumption there is a surjection $\mathcal{O}(k^n) \to \pi_0(A)$, and each open Stein subdomain $\mathrm{Spec}(A_i)$ can be expressed as the intersection of $\mathrm{Spec}(\pi_0(A))$ with an open subset $V \subseteq k^n$ (because of the finite embedding of $\pi_0(\mathrm{Spec}(A))$ in k^n). This V may be further covered by polydiscs $\{U_{j_i}\}_{j_i \in \mathcal{J}_i}$ in the ambient affine space k^n. The restrictions $\mathcal{O}(k^n) \to \mathcal{O}(U_j)$ are Disc_k-rational localizations, and by open coherence, so are the maps $\pi_0(A) \to \pi_0(A) \otimes^{\mathbb{L}}_{\mathcal{O}(k^n)} \mathcal{O}(U_{i_j})$. This is a rational refinement of the original cover. Conclude from the proposition above that this discrete refinement induces a derived rational refinement of the open cover.

Theorem 3.2. *([7] Theorem 9.2.18) - Let $A \in \mathrm{dStn}^{\dagger,f}$ be a finitely embeddable dagger Stein algebra, with each $\pi_n(A)$ transverse to open immersions. A collection of maps $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$ is a cover in $\overline{G}^{\aleph_1}_{\mathrm{Disc}_k}$ (respectively, in $\overline{G}^{\mathrm{pre},\aleph_1}_{\mathrm{Disc}_k}$) if and only if any of the equivalent conditions hold:*

1. *Each $A \to A_i$ is derived strong, and the underlying collection $\{\mathrm{Spec}(\pi_0(A_i)) \to \mathrm{Spec}(\pi_0(A))\}$ is a cover in the respective topology.*
2. *Each $A \to A_i$ is derived strong, and the associated cover $\{\mathrm{Max}(\pi_0(A_i)) \to \mathrm{Max}(\pi_0(A))\}$ is an open (respectively, rational) Stein cover.*
3. *Each $A \to A_i$ is derived strong, each $\pi_0(A) \to \pi_0(A_i)$ is a homotopy monomorphism (respectively, a Disc_k-rational localization), and there is a surjection $\bigsqcup \mathcal{M}(\pi_0(A_i)) \to \mathcal{M}(\pi_0(A))$ of spectra in the sense of ([5] Definition 4.1, [7] Definition 9.1.1).*

3.5.1 Comparison with Structured Topoi

In the theory of derived analytic geometry explored by Lurie in [30] and by Porta & Yue Yu in [42] [41] [39] [43], derived analytic spaces are defined as *structured spaces* : ∞-topoi $\mathcal{X}$ equipped with a $\mathcal{T}_{\mathrm{an}}(k)$-structure $\mathcal{O}_X$, where $\mathcal{T}_{\mathrm{an}}(k)$ is the *pregeometry* that controls smooth k-analytic spaces. Amongst such structured spaces, the derived k-analytic spaces $X = (\mathcal{X}, \mathcal{O}_X)$ are those whose truncation $t_{\leq 0}(X)$ corresponds to a usual k-analytic space, and each $\pi_n(\mathcal{O}_X)$ is a coherent $\pi_0(\mathcal{O}_X)$-module.

The full subcategory $\mathrm{dStn}_k \subseteq \mathrm{dAn}_k$ has objects whose underlying space $t_{\leq 0}(X)$ is a Stein space. All derived analytic spaces X admit effective epimorphisms $\bigsqcup U_i \to X$ where each $U_i \to X$ is an open immersion, and they admit pseudo-representable hypercovers by Stein spaces.

In [44], particular emphasis is placed on *globally finitely embeddable derived Stein spaces*: similar to the definition given in this chapter, this is the subcategory of derived Stein spaces that admit a closed immersion $t_{\leq 0} \to k^n$, such that the ideal of definition is finitely generated. Define $\mathrm{dAn}_k^{gf,<\infty}$ to be those analytic spaces that admit an n-coskeletal hypercover by finitely-embeddable derived Stein spaces. For descent arguments this class is suitable, as all Stein spaces are locally of this form, and they are much easier to work with - finite coskeletal covers permit inductive and spectral sequence arguments.

In [44], it is shown that global sections define a fully faithful embedding of finitely-embeddable derived Stein spaces into simplicial EFC-algebras:

$$\mathbb{R}\Gamma : (\mathrm{dStn}^f_{\mathbb{C}})^{\mathrm{op}} \to \mathrm{sAlg}_{\mathrm{EFC}_{\mathbb{C}}} \tag{122}$$

EFC stands for *entire functional calculus*: the 1-sorted Lawvere theory that controls entire functions. The essential image are those EFC algebras with $\pi_0(A)$

globally finitely embeddable and $\pi_n(A)$ coherent over $\pi_0(A)$. This embedding respects topologies, in the sense that covers $\{f_i : U_i \to U\}$ in dStn_k^f induce isomorphisms

$$\pi_n(\mathbb{R}\Gamma(U))\hat{\otimes}_{\pi_0(\mathbb{R}\Gamma(U))}\pi_0(\mathbb{R}\Gamma(U_i)) \to \pi_n(\mathbb{R}\Gamma(U_i)). \tag{123}$$

Now by Lemma 5.3.87 of [7]there is a further fully faithful functor

$$F : \mathrm{sAlg}_{\mathrm{EFC}_k} \to \mathrm{DAlg}^{\mathrm{cn}}(\mathrm{Ch}(\mathrm{Ind}(\mathrm{Ban}_k))) \tag{124}$$

and so composing this with the previous embedding produces an embedding $F \circ \mathbb{R}\Gamma$ of derived finitely embeddable Steins in the sense of structured topoi.

Theorem 3.3. *([7] Theorem 9.2.20) - Let $\{A \to B_i\}$ be a collection of maps in $\mathrm{sAlg}_{\mathrm{EFC}_k}$, such that $\pi_0(A)$ and each $\pi_0(B_i)$ corresponds to a globally finitely embeddable Stein space, and $\pi_n(A)$ and $\pi_n(B_i)$ are the global sections of some coherent sheaf. Then this corresponds to a cover of τ_{an} if and only if $\{F(A) \to F(B_i)\}$ corresponds to a cover of $\overline{\tau}_{\mathrm{disc}_k-\mathrm{open}}$. Therefore, there is a fully faithful functor*

$$\mathrm{dAn}^{f,<\infty} \to \mathrm{Stk}_{\mathrm{geom}}(\mathrm{Aff}^{\mathrm{cn}}_{\mathrm{Ind}(\mathrm{Ban}_k)}, \mathrm{dStn}_k^{\dagger,gf}, \overline{\mathrm{sm}}_{\mathrm{Disc}_k}, \overline{\tau}^{\aleph_1}_{\mathrm{disc}_k-\mathrm{open}}) \tag{125}$$

4 Abstract Six Functor Formalisms

Here we record the abstract theory of six functor formalisms. This theory is built upon the work of Liu-Zheng [28] and Mann [34], and an excellent exposition of the theory is given by Scholze [52], in which all the results below can be found.

4.1 Basic Definitions

The naive idea of a six-functor formalism is as follows. First, define a finitely-complete category C of "geometric objects", and a class of morphisms E for which the exceptional pushforward will be assumed to exist, stable under composition and pullback. A three-functor formalism is essentially a functor from C to symmetric monoidal ∞-categories, such that the basic formulae of sheaf theory are assumed to hold:

- The assignment of pullbacks is functorial and compatible with the monoidal structure.
- The assignment of pushforwards of morphisms of E is functorial, and pushforwards satisfy base change and projection formula.

The third functor is simply the monoidal operation $- \otimes A$. To get a six-functor formalism, we simply assume that pullback, exceptional pushforward, and tensor products have adjoints: the pushforward, exceptional pullback and internal Hom, respectively.

One can encode this data remarkably efficiently using the ∞-*category of correspondences.* This category encodes the "roof diagrams" one frequently sees in homological algebra. The non-standard terminology of n-rooves is used below.

Definition 4.1. ([34] Definition A.5.1) - A *geometric setup* is a pair $(\mathcal{C}, E)$ of an $(\infty, 1)$-category $\mathcal{C}$ and E a collection of homotopy classes of edges of $\mathcal{C}$, containing all isomorphisms, and stable under compositions and pullbacks.

The simplicial set of *n-rooves* $(\Delta^n)^2_+ \subseteq (\Delta^n)^{\mathrm{op}} \times \Delta^n$ is the subset spanned by pairs of indices (i, j) with $i \geq j$.

The diagrams for $n = 1, 2$ are given below:

$$\begin{tikzcd}
& (1,0) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & (2,0) \arrow[dl] \arrow[dr] & & \\
 & (1,0) \arrow[dl] \arrow[dr] & & (2,1) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2)
\end{tikzcd}$$

The symmetric monoidal ∞-*category of correspondences* $\mathrm{Corr}(C, E)$ is the category with n-morphisms those n-rooves in C such that:

1. The arrows corresponding to increments in the covariant variable (down-right arrows in the diagrams above) are in the class E.
2. The small squares are Cartesian.

By closure over pullbacks, the familiar composition of rooves is well-defined.

The category of correspondences can be equipped with a symmetric monoidal structure via the higher Grothendieck correspondence ([28] Prop 6.1.3).

With the category of correspondences, as promised, three-functor formalisms, and thereby six-functor formalisms, can be defined very elegantly:

Definition 4.2. - A *three-functor* formalism is a lax symmetric monoidal functor $\mathrm{Corr}(C, E) \to \infty - \mathrm{Cat}$. A *six-functor formalism* has adjoints for pullbacks, exceptional pushforwards, and tensor products.

4.2 Constructing Six-Functor Formalisms

The theory of six-functor formalisms is powerful principally because it is so easily applicable: it is remarkably straightforward to specify a six-functor formalism in many algebro-geometric contexts.

Motivated by algebraic geometry, sufficient conditions for such constructions can be imposed on special classes of morphisms I, P, respectively thought of as the *open immersions* and *proper maps.* These classes are assumed to satisfy the expected properties:

Definition 4.3. ([34] Definition A.5.9) - A *suitable decomposition* of a geometric setup $(\mathcal{C}, E)$ is a pair (I, P) of classes of maps $I, P \subseteq E$ satisfying the following conditions:

1. I and P both closed under pullbacks and compositions, contain all isomorphisms. If $f : X \to Y$ and $f' : X' \to Y$ are both in I (resp. P), then a Y-map $g : X \to X'$ between then is also in I (resp. P). All proper open immersions are truncated, and (I, P) defines a weak factorisation system.

2. The pullback f^* along an open immersion f has $f_!$ as left adjoint, satisfying base change and the projection formula.
3. The pullback f^* along a proper map admits f_* as a right adjoint, which also satisfies base change and the projection formula.
4. For the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ Y' \arrow[r, "j"] & Y \end{tikzcd}$$

with j (and hence j') open immersions, and g (and hence g') proper, the natural map $j_! g'_* \to g_* j'_!$ is an isomorphism.

With just these few and commonly-satisfied assumptions, one can derive important familiar isomorphisms.

Proposition 4.1. *([52] Construction 4.3) - The ordinary and exceptional pushforwards of morphisms $f : X \to Y$ in $I \cap P$ are isomorphic.*

Proof - By the assumption that such maps are truncated, this may be proven by induction on homotopical degree. If $n = -2$ then f is an isomorphism. The induction step utilises that the diagonal morphism Δ in the following pullback diagram is $(n-1)$-truncated:

$$\begin{tikzcd} X \arrow[dr, "\Delta"] & & \\ & X \times_Y X \arrow[r, "g"] \arrow[d, "h"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

By pullback stability, g is a proper closed immersion, and therefore also by (1) above, Δ is a proper closed immersion. Therefore the result follows from the following manipulations:

$$f_! = f_! g_! \Delta_* \cong f_! g_* \Delta_! \cong f_* h_! \Delta_! = f_* \tag{126}$$

The second isomorphism follows from assumption (4).

Proposition 4.2. *(Independence of pushforward operations with respect to compactifications, [52] Construction 4.5) - Given $f : X \to Y \in E$, factorised as in assumption (1) in two ways $f = \overline{f} j = \overline{f}' j'$, there is a natural isomorphism of pushforwards.*

$$\overline{f} j \cong \overline{f}' j' \tag{127}$$

Of course most importantly is the following:

Theorem 4.1. *[34] - Under the assumption that such classes I, P exist satisfying the assumptions above, a lax symmetric monoidal functor* $C^{\mathrm{op}} \cong \mathrm{Corr}(C, \mathrm{iso}) \to \infty - \mathrm{Cat}$ *extends to* $\mathrm{Corr}(C, E)$*, defining a six-functor formalism.*

See ([52], Theorem 4.6) for a sketch or ([34], Proposition A.5.10) for a thorough proof.

4.3 Extending Six functor Formalisms

A six functor formalism $D : \mathrm{Corr}(C, E) \to \mathrm{Pr}_\infty$ to presentable ∞-categories can be right Kan extended to presheaves:

$$D(F) = \lim_{X \to F} D(X), \tag{128}$$

where $X \in C$. The class E may then be extended to those maps of presheaves that pull back to morphisms of E on the "affines" $X \in C$.

The six functor formalism can also be extended to sheaves for a suitable choice of topology:

Definition 4.4. ([52] Definition 4.14) - Let $\{f_i : X_i \to Y\}$ be a collection of morphisms in C:

1. The morphisms define a cover in the *canonical topology* if for any morphism $Y' \to Y$ in $\mathcal{C}$, the representable functor $\mathrm{Hom}(-, Z)$ satisfies descent along the pullback family $\{f_i \times_Y Y' : X_i \times_Y Y' \to Y'\}$.
2. The family of maps satisfies *universal $*$-descent* if D, with maps given by $*$-pullback, satisfies descent along these pullback families $\{f_i \times_Y Y' : X_i \times_Y Y' \to Y'\}$.
3. Assume all the maps f_i are in the class E. The maps satisfy *universal !-descent* if, for any (higher) presheaf Y', the functor D, with maps given by !-pullback, satisfies descent along these pullback families $\{f_i \times_Y Y' : X_i \times_Y Y' \to Y'\}$.
4. The family $\{f_i : X_i \to Y\}$ defines a D-cover if they form a cover in the canonical topology and satisfy both forms of descent. D-covers collectively define the *D-topology* on $\mathcal{C}$.

Now as D is a sheaf with respect to the D-topology, it extends to the category $\tilde{\mathcal{C}}$ of $\infty - \mathrm{Grpd}$-valued sheaves on $\mathcal{C}$ with respect to the D-topology:

$$D(\tilde{X}) = \varinjlim_{X \to \tilde{X}} D(X), \tag{129}$$

where the limit runs over $X \in \mathcal{C}$. Let $\tilde{E}_0$ denote the class of maps of such sheaves $\tilde{f} : \tilde{X} \to \tilde{Y}$ universally in E on representable objects, in the following sense: for any $Y \to \tilde{Y}$ with $Y \in \mathcal{C}$, the pullback map $\tilde{X} \times_{\tilde{Y}} Y \to Y$ is in E:

$$\begin{tikzcd}
\tilde{X} \times_{\tilde{Y}} Y \arrow[r, "f \in E"] \arrow[d] & Y \arrow[d] \\
\tilde{X} \arrow[r, "\tilde{f} \in \tilde{E}_0"] & \tilde{Y}
\end{tikzcd}$$

Proposition 4.3. *([34] Proposition A.5.16) - D extends from a six functor formalism on (C, E) to $(\tilde{C}, \tilde{E}_0)$, determined by the formula (129).*

We want to extend the class of exceptionally-pushable morphisms while maintaining a tractable theory. Scholze provides a list of conditions which provide workable theories which can always be obtained:

Definition 4.5. ([52] Definition 4.18) - In the situation above, let $\tilde{E} \supseteq \tilde{E}_0$ be a class of morphisms stable under pullback and composition:

1. $\tilde{E}$ is *stable under disjoint unions* if $\{\tilde{f}_i : \tilde{X}_i \to \tilde{Y}\}$ is a family of maps of $\tilde{E}$, then their disjoint union $\bigsqcup \tilde{X}_i \to \tilde{Y}$ is in $\tilde{E}$.
2. $\tilde{E}$ is *local on the target* if a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ lies in $\tilde{E}$ if the following condition is met: for all $Y \in \mathcal{C}$ with a map $Y \to \tilde{Y}$, the pullback map $\tilde{X} \times_{\tilde{Y}} Y \to Y$ lies in $\tilde{E}$.
3. Assume that D extends uniquely from $(\tilde{C}, \tilde{E}_0)$ to $(\tilde{C}, \tilde{E})$. $\tilde{E}$ is *local on the source* if a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ lies in $\tilde{E}$ if the following condition is met: there exists a map $\tilde{g} : \tilde{X}' \to \tilde{X}$ in $\tilde{E}$, of universal !-descent, such that $\tilde{f} \circ \tilde{g} \in \tilde{E}$.
4. Assume again that D extends uniquely. The class $\tilde{E}$ is *tame* if whenever $\tilde{f} : \tilde{X} \to Y$ is a map of $\tilde{E}$, there is a collection of morphisms $h_i : X_i \to Y$ of E, and a morphism $\tilde{g} : \bigsqcup_i X_i \to \tilde{X}$ over Y, which lies in $\tilde{E}$ and is of universal !-descent:

$$\begin{tikzcd}
\bigsqcup_i X_i \arrow[rr, "\tilde{g} \in \tilde{E}"] \arrow[dr, "\bigsqcup h_i"'] & & \tilde{X} \arrow[dl, "\tilde{f}"] \\
& Y &
\end{tikzcd}$$

Theorem 4.2. *([52] Theorem 4.20) - There is a minimal collection of morphisms $\tilde{E} \supseteq \tilde{E}_0$, such that D extends uniquely to $(\mathcal{C}, \tilde{E})$, and such that $\tilde{E}$ is stable under disjoint unions, local on both source and target, and tame.*

The proof iteratively builds the extension: given a tame extension $\tilde{E}$, one can define $\tilde{E}'$ to be those morphisms which are disjoint unions of maps of $\tilde{E}$ to obtain a tame class stable under disjoint unions (this requires [34] Proposition A.5.12). In particular a minimal such extension can be chosen. In a similar way the other conditions can be imposed.

4.4 A Six-Functor Formalism for Relative Algebraic Geometry

Let $(\mathcal{C}, \otimes)$ be a stable presentable symmetric monoidal $(\infty, 1)$-category. The category of *affine schemes* associated to $(\mathcal{C}, \otimes)$ is the category $\mathrm{Comm}(\mathcal{C})^{\mathrm{op}}$ opposite to internal commutative monoids of $\mathcal{C}$. Then by ([31], Theorem 4.5.3.1, Remark 4.5.3.2) there is a functor

$$\mathrm{QCoh} : \mathrm{Comm}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Comm}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}) \tag{130}$$

$$\mathrm{QCoh}(\mathrm{Spec}(A)) = \mathrm{Mod}_A(\mathcal{C}). \tag{131}$$

Then $(I, P) = (\text{equivalences}, \text{all})$ defines a suitable decomposition, and thus we obtain a six functor formalism for which all maps are exceptionally pushable, and $f_* = f_!$ for all f.

This formalism extends to stacks in the following way: let $\mathrm{Aff} \subseteq \mathrm{Comm}(\mathcal{C})^{\mathrm{op}}$ a full subcategory stable under fibre products and retracts, bestowed with a Grothendieck topology such that QCoh^* defines a sheaf over τ. By evaluation on the monoidal unit, this in particular implies that the topology is subcanonical.

Let $\mathrm{Stk} = \mathrm{Shv}_\tau(\mathrm{Aff}, \infty\mathrm{Grpd})$, and let rep be the class of morphisms in Stk representable in Aff. Then the canonical map of geometric setups $(\mathrm{Aff}, \mathrm{all}) \to (\mathrm{Stk}, \mathrm{rep})$ defines a six-functor formalism by ([34], Proposition A.5.16):

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Stk}, \mathrm{rep}) \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}. \tag{132}$$

Remark - The pushforward of representable morphisms inherit properties of pushforwards of the formalism on $(\mathrm{Comm}(\mathcal{C}), \mathrm{all})$ by descent: pushforwards g_* satisfy base-change and the projection formula, and g_* is conservative ([54] Lemma 3.15).

Moreover, this observation and ([34] Proposition A.5.10), we may construct a six-functor formalism on $(\mathrm{Stk}, \mathrm{rep})$ such that all morphisms $g \in \mathrm{rep}$ satisfy $g_! = g_*$. But both of these formalisms are uniquely extended from QCoh on $(\mathrm{Comm}(\mathcal{C}), \mathrm{all})$, and so the formalisms must be equal.

Applying the extension formalism to the pair $(\mathrm{Aff}, \mathrm{all})$ and $(\mathrm{Stk}, \mathrm{rep})$ we obtain the following:

Theorem 4.3. *([54] Theorem 3.17) - There is a minimal class of edges $E \supseteq \mathrm{rep}$ of* Stk*, such that* QCoh *extends to a six-functor formalism on* (Stk, E) *with the properties of ([52] Definition 4.18).*

In particular, we may take $\mathrm{dAff} = \mathrm{Comm}(D_{\geq 0}(\mathrm{CBorn}_k))^{\mathrm{op}}$, and take τ to be the trivial topology. Then the category of stacks is merely the category of prestacks $\mathrm{PreStk} = \mathrm{Psh}(\mathrm{dAff})$, and we get a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{PStk}, \tilde{E})^{\otimes} \to \mathrm{Pr}^{L,\otimes}_{\mathrm{st}}. \tag{133}$$

where $\tilde{E}$ is a class satisfying all the assumptions of([52] Theorem 4.20).

4.5 Poincaré Duality

Recall the classical statement of Poincaré-Verdier duality: for suitably smooth maps, the exceptional and ordinary pullbacks should agree, up to a twist based on the shift in dimension.

The desired properties are encoded into the notion of D-cohomological smoothness for a given three-functor formalism $D : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$.

Definition 4.6. - A morphism $f : X \to Y$ in E is *D-cohomologically smooth* if:

1. The exceptional pullback $f^!$ exists, and the natural transform

$$f^!(1_Y) \otimes f^*(-) \to f^!(-) \tag{134}$$

given by the adjoint to the isomorphism granted by subsequent use of the projection formula and counit transform $f_! f^! \to \mathrm{id}$:

$$f_!(f^!(1_Y) \otimes f^*(-)) \cong f_! f^!(1_Y) \otimes (-) \to (-) \tag{135}$$

is an isomorphism.

2. The *dualizing complex* $f^!(1_Y)$ is $\otimes$-invertible.

3. The properties above hold for arbitrary base changes of a cohomologically smooth morphism, and for another map $g : Y' \to Y$ the natural map

$$g'^* f^!(1_Y) \to f'^!(1_{Y'}) \tag{136}$$

is an isomorphism, where f' and g' are the base changes of f and g respectively.

Remarks -

1. It is immediate from the definition that the class of cohomologically smooth morphisms is stable under base change and composition, and contains all isomorphisms.

2. Given a suitable decomposition (I, P) for a 3-functor formalism, the class I consists of cohomologically smooth maps, as in this case $f^! = f^*$.

3. Combining these two observations, we see that a morphism which is an open immersion on fibres is cohomologically smooth. In particular, this applies to maps of complex analytic stacks which are open analytic embeddings on each fibre - an example used throughout [51].

In theorem 5.5 of [52], Scholze proves a tractable recognition principle for cohomologically smooth maps, when restricted to a slice category $\mathcal{C}_{/Y}$ with structural morphisms $X \to Y$ in E. The recognition principle then only involves checking certain unit and counit maps involving the diagonal are identities. Since the method of proof is helpful for the following definitions of finiteness, properness and smoothness, we repeat his argument here.

Theorem 4.4. *([52] Theorem 5.5) - Assume all morphisms in $\mathcal{C}$ are in E, and let $f : X \to Y$ be a morphism to the final object Y. Let $\Delta : X \to X \times_Y X$ be the diagonal. Then f is cohomologically smooth if and only if there is a $\otimes$-invertible object $L \in D(X)$ and maps*

$$\alpha : \Delta_! 1_X \to p_2^* L \tag{137}$$

$$\beta : f_! L \to 1_Y \tag{138}$$

such that the composites

$$1_X \xrightarrow{\sim} p_{1!}\Delta_! 1_X \xrightarrow{p_{1!}\alpha} p_{1!}p_2^* L \xrightarrow{\simeq} f^* f_! L \xrightarrow{f^*\beta} 1_X$$

$$L \xrightarrow{\sim} p_{2!}(p_1^* L \otimes \Delta_! 1_X) \xrightarrow{p_2!(p_1^* L \otimes \alpha)} (p_1^* L \otimes p_2^* L) \xrightarrow{\sim} p_{2!}p_1^* L \otimes L \xrightarrow{\sim} f^* f_! L \otimes L \xrightarrow{f^*\beta \otimes L} L$$

are identities.

Proof - Assume first that f is cohomologically smooth, and take $L = f^!(1_Y)$ and β the counit of the adjunction $f_! L = f_! f^!(1_Y) \to 1_Y$. The assumption of cohomological smoothness implies that $p_2^* L = p_2^* f^!(1_Y) = p_1^!(1_X)$, and so α can be taken to be the adjoint to $1_X = \Delta^! p_2^!(1_X)$ with respect to Δ.

The first composition is then clear from the definition, but the other is more subtle. It is in fact easier to express the functors in terms of *Fourier–Mukai transforms*, and calculate with the integral kernels directly.

For $K \in D(X_1 \times X_2)$ define the Fourier-Mukai transform with kernel K:

$$\mathcal{D}(X_1) \to \mathcal{D}(X_1), \tag{139}$$

$$A \mapsto p_{2!}(p_1^* A \otimes K) \tag{140}$$

Just as in classical sheaf theory, the Fourier-Mukai transform encodes various sheaf functors for appropriate choices of kernel. For example, taking $X_1 = X$ and $X_2 = Y$, the identity 1_X corresponds to $f_!$, and taking $X_1 = Y$ and $X_2 = X$, and $K = L$, the associated Fourier-Mukai transform is $G = f^* \otimes L$.

By cohomological smoothness, these functors are adjoints, with associated unit and counit $\alpha_0 : \mathrm{id}_{\mathcal{D}(X)} \to GF$, $\beta : FG \to \mathrm{id}_{\mathcal{D}(Y)}$. Recall that the composition the Fourier-Mukai transforms is itself a Fourier-Mukai transform, given by the convolution of their kernels. We can therefore conclude that the kernel of FG is $f_! L$, the kernel of $\mathrm{id}_{\mathcal{D}(Y)}$ is 1_Y, and $\beta_0 : f_! L \to 1_Y$ is given by β. GF has kernel $p_2^*(L)$, and $\mathrm{id}_{\mathcal{D}(X)}$ has kernel $\Delta_!(1_X)$. For entirely formal reasons, α must induce α_0 (this is just an incarnation of the equality of left and right inverses in the context of adjunctions - see [52] lemma 5.6.).

The composite

$$G \xrightarrow{\alpha_0 G} GFG \xrightarrow{G\beta_0} G$$

is the identity, and unravelling, it is seen to be induced by the second sequence of morphisms of the theorem, finishing the proof of the forward direction.

The proof of the converse requires the introduction of concepts introduced in [29], which will also be useful for the definition of smoothness and properness to follow.

Definition 4.7. - Let $(\mathcal{C}, E)$ be a geometric setup, where $E = \mathrm{Mor}(C)$ is again assumed to consist off all morphisms of $\mathcal{C}$. Define the *Liu-Zheng 2-category* $\mathrm{LZ}_{\mathcal{D}}$ as follows:

1. $\mathrm{Ob}(\mathrm{LZ}_{\mathcal{D}}) = \mathrm{Ob}(\mathcal{C})$.
2. Morphism categories $\mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X') = \mathcal{D}(X \times X')$
3. Identity morphisms are given by $\Delta_!(1_X)$, and composition is given by convolution.

This is the homotopy 2-category associated to the Cat_∞-enriched category $\mathrm{Corr}(\mathcal{C}, E)$ [57].

There is a natural functor $\mathrm{LZ}_{\mathcal{D}} \to \mathrm{Cat}$, sending X to $\mathcal{D}(X)$, and sending a kernel to its associated Fourier-Mukai transform, but it is quite lossful. The category $\mathrm{LZ}_{\mathcal{D}}$ incarnates the principal used above: to calculate with kernels directly, rather than the associated functors.

The benefit of using the category is that adjoints can be demonstrated at the level of kernels. Recall the definition of the adjoint of a morphism $F : X \to Y$ in a 2-category: a triple (G, α, β) of a reverse morphism $G : Y \to X$, and unit and counit morphisms satisfying the usual formulae.

We can now proceed with the converse statement. We of course take the triple (L, α, β), and consider the objects X and Y as objects of $\mathrm{LZ}_{\mathcal{D}}$, and the morphisms

$$F = 1_X \in \mathcal{D}(X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, Y) \tag{141}$$

$$G = L \in \mathcal{D}(X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(Y, X) \tag{142}$$

F is the left adjoint of G in $\mathrm{LZ}_{\mathcal{D}}$. Since F and G correspond to $f_!$ and $f^* \otimes L$, and functors of 2-categories respect adjunctions, this proves that $f_!$ is the left adjoint of $f^* \otimes L$.

Adjointness follows from the assumptions of the theorem: α and β translate to the maps

$$\mathrm{id}_X = \Delta_!(1_X) \to GF = p_2^* L \in \mathcal{D}(X \times_Y X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X) \tag{143}$$

$$FG = f_! L \to \mathrm{id}_Y = 1_Y \in \mathcal{D}(Y)) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(Y, Y) \tag{144}$$

and the commutative diagrams that classify them as adjoints translate to the diagrams in the theorem.

For the other statements, since $f_!$ is given by a kernel, so is $f^!$, and therefore is $\mathcal{D}(Y)$-linear, meaning $f^!(1_Y) \otimes f^* \to f^!$ is an isomorphism. Since $f^!(1_Y) = L$, we see that it is $\otimes$-invertible.

Finally, base change induces functors of Liu-Zheng categories, which encode the same properties for the base change of any morphism of E.

Example ([52] Proposition 5.10) - Consider the geometric setup $(\mathcal{C}, E)$ of locally compact Hausdorff spaces, under all morphisms, and the familiar formalism of derived categories of abelian sheaves. A suitable decomposition is given by the the class of open immersions I and proper maps P (with their usual meanings).

Then, the map $\mathbb{R} \to *$ is cohomologically smooth.

Proof - We know from the previous proof that we must have $L = f^!(1_*) = f^!(\mathbb{Z}) = \mathbb{Z}[1]$. α and β are intuitively defined: $\alpha : \Delta_! \mathbb{Z} \to p_2^* \mathbb{Z}[1]$ corresponds to compactly supported sections on the diagonal valued in $\mathbb{Z}[1]$, and $\mathbb{R}\Gamma_c(\mathbb{R}, \mathbb{Z}[1]) \cong \mathbb{Z}$ as we know from the familiar integral short exact sequence. Let $j : \mathbb{R}^2 \setminus \Delta \hookrightarrow \mathbb{R}^2$ be the complementary open inclusion to the diagonal. The triangle $0 \to \Delta_! \mathbb{Z} \to \mathbb{Z} \to j_! \mathbb{Z}$ implies that

$$\mathbb{R}\mathrm{Hom}(\Delta_! \mathbb{Z}[-1], \mathbb{Z}) \cong \mathbb{Z} \tag{145}$$

as it is the cone of $\mathbb{R}\Gamma(\mathbb{R}^2, \mathbb{Z}) \to \mathbb{R}\Gamma(\mathbb{R}^2 \setminus \Delta, \mathbb{Z})$, which is the diagonal embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$, corresponding to those global sections that agree on the two connected components. Q.E.D

4.6 Finiteness, Properness and Smoothness

For the ensuing discussion of analytic de Rham stacks and $\mathcal{D}$-modules, it will be profitable to compare these to their algebraic counterparts. Various maps between relevant quotient stacks will turn out to have pleasant familiar cohomological properties, enabling classical sheaf-theoretic manipulations. In particular, following [52], notions of properness and smoothness of a structure sheaf are crystallized from familiar properties of smooth and proper maps in topology and algebraic geometry: smooth maps ought to have agreement of ordinary and exceptional pullback, and proper maps ought to have agreement of pushforwards, up to dimension shifting.

Let $(\mathcal{C}, E)$ be a geometric setup, once again with E consisting of all morphisms, and let $\mathcal{D} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$ be a six-functor formalism. Let $\mathrm{LZ}_\mathcal{D}$ be the associated Liu-Zheng category.

Definition 4.8. ([52] Definition 6.1) - Let $f : X \to Y$ be a morphism to the final object, and $A \in \mathcal{D}(X)$.

$A \in \mathcal{D}(X)$ is *f-smooth* if, regarded as a morphism $A \in \mathrm{Hom}_{\mathrm{LZ}_\mathcal{D}}(X, Y)$, it's a left adjoint. It is *f-proper* if, regarded as a morphism $A \in \mathrm{Hom}_{\mathrm{LZ}_\mathcal{D}}(Y, X)$, it's a right adjoint.

These are formally dual notions, under the equivalence of the correspondence category with their opposites, and these notions are stable by base change. This duality manifests as an abstract form of Verdier duality.

Concretely, let f be as above, and let $A \in \mathcal{D}(X) = \mathrm{Hom}_{LZ_\mathcal{D}}(X, Y)$. Then A is f-smooth if and only if has a left adjoint $B \in \mathrm{Hom}_{LZ_\mathcal{D}}(Y, X) = \mathcal{D}(X)$, with maps

$$\alpha : \Delta_!(1_X) \to p_1^* A \otimes p_2^* B \tag{146}$$

$$\beta : f_!(A \otimes B) \to 1_Y \tag{147}$$

such that the following composite, and the similar one with A and B reversed by autoduality, are identities:

$$\begin{aligned} B &\cong p_{2!}(p_1^* B \otimes \Delta_!(1_X)) \\ &\xrightarrow{\alpha} p_{2!}(p_1^* B \otimes p_1^* A \otimes p_2^* B) \\ &\cong p_{2!} p_1^* (B \otimes A) \otimes B \\ &\cong f^* f_!(B \otimes A) \otimes B \\ &\xrightarrow{\beta} f^*(1_Y) \otimes B \cong B \end{aligned}$$

The second isomorphism (third line) arises from the projection formula. Let us see how this notion implies some familiar results:

1. The right adjoint B to f-smooth $A \in D(X)$ is also f-smooth with right adjoint A.
2. There is a natural isomorphism

   $$B \otimes f^*(-) \cong \mathcal{H}\mathrm{om}(A, f^!(-)), \tag{148}$$

   as both of these are right adjoint to $f_!(A \otimes (-))$. This can be seen similarly to before, checking the adjunction at the level of kernels.
3. Applying this to 1_Y implies immediately, as expected, that B is the Verdier dual $\mathcal{H}\mathrm{om}(A, f^!(1_Y))$ of A.
4. Applying this observation twice yields the statement of Verdier duality:

   $$A \cong \mathbb{D}(\mathbb{D}(A)) \tag{149}$$
5. Verdier duality commutes with base change, since right adjoints do.

Formally, proper maps have similar properties and fit into similar diagrams.

Recall that in many geometric contexts, properness is sufficient to identify ordinary and exceptional pushforward functors. In the abstract theory the conditions necessary are captured by the notion of *cohomologically proper morphism.* This definition is due to Scholze to generalize this theory to perfectoid spaces, where one does not have the familiar topological or algebro-geometric definitions. Dually, smooth maps can be re-imagined cohomologically, resulting in *cohomologically étale morphisms*:

Definition 4.9. - A map $f : X \to Y$ in $\mathcal{C}$ is *cohomologically proper* if the following conditions hold:

1. It is n-truncated for some n.
2. The monoidal unit $1_X \in \mathcal{D}(X)$ is f-proper.
3. $\Delta_f : X \to X \times_Y X$ is cohomologically proper

Note that the diagonal is $(n-1)$-truncated, so this definition is valid by induction. The map is *cohomologically étale* if the corresponding conditions are met, with smoothness substituted for properness.

Proposition 4.4. *[52] - Let $f : X \to Y$ $\mathcal{C}$ have cohomologically proper diagonal. Then there is a natural transform $f_! \to f_*$, and this transform is an isomorphism if and only if f is cohomologically proper.*

Moreover, to verify that f is cohomologically proper, it suffices to check this on the monoidal unit.

Properness and smoothness are closely related to familiar finiteness properties, such as compactness and dualizability. We briefly summarise these results:

1. id_X-smoothness, id_X-properness and dualizability are all equivalent for $A \in \mathcal{D}(X)$.
2. For $f : X \to Y$ with $A \in \mathcal{D}(X)$ f-smooth, if $\Delta_!(1_X)$ is compact then A is compact.

   This holds because the Hom set $\mathrm{Hom}(A, B)$ can be manipulated into the Hom set

   $$\mathrm{Hom}(\Delta_!(1_X), p_!^* B \otimes p_2^* \mathbb{D}_f(A)). \tag{150}$$

   As pullbacks and Verdier duals commute with filtered colimits, so does this composite functor, and hence A is compact.
3. For $f : X \to Y$ with A f-proper, if 1_Y is compact, A is compact. The reasoning is very similar to the above.
4. Properness and smoothness are stable under retracts ([52] Proposition 6.17).

4.7 Examples of Six Functor Formalisms

We briefly discuss the examples of six functor formalisms from [52]: abelian sheaves on topological spaces, étale sheaves, coherent sheaves and $\mathcal{D}$-modules.

4.7.1 Abelian Sheaves on Topological Spaces

This is of course the prototypical example of a six functor formalism, and the principal subject of many texts on sheaf theory, such as Kashiwara-Schapira [26].

Theorem 4.5. *([52] Theorem 7.4) - Let $\mathcal{C}$ denote the category of locally compact Hausdorff spaces. The functor*

$$X \mapsto \mathcal{D}(X) = \mathrm{Sh}(X, \mathcal{D}(\mathbb{Z})) \tag{151}$$

defines a six-functor formalism, with respect to the suitable decomposition (I, P) of open immersions and proper maps.

Proof - All that needs to be shown is that (I, P) is a suitable decomposition. The necessary properties are easily checked, except for proper base change. So it must be checked that proper maps of locally compact Hausdorff spaces have cocontinuous pushforwards, and satisfy the projection formula, and satisfy the Base change with respect to any other map. Throughout what follows, let $f : X \to Y$ be a proper map of Hausdorff spaces.

Regarding sheaves as systems on closed sets rather than open ones, the pushforward of $f : X \to Y$ is defined simply as $(f_*\mathcal{F})(K) = \mathcal{F}(f^{-1}(K))$, from which cocontinuity is clear.

Regarding the projection formula:

$$f_*A \otimes B \to f_*(A \otimes f^*B), \tag{152}$$

let $A \in \mathcal{D}(X)$ and $B \in \mathcal{D}(Y)$. The functors involved in the projection formula are cocontinuous, and therefore the formula only needs to be verified for generators of the category of sheaves: that is, on sheaves freely generated on open subsets $j_!\mathbb{Z}$ for $j : U \hookrightarrow X$. This verification needs to be checked on U and it's complement. On U the formula is clear - both sides compute the restriction of f_*A to U. On the complement Z, the each side of the equation vanishes, by expressing it as a colimit on a system of open neighbourhoods of Z shrinking onto Z.

Concerning base change, let $g : Y' \to Y$ be another map, and consider the base change transform

$$g^*f_* \to f'_*g'^* : \mathcal{D}(X) \to \mathcal{D}(Y) \tag{153}$$

It suffices to verify this over all compact Y', and therefore we can pass to compact subsets, and verify the formula with g_* applied. By repeated application of the projection formula, we are reduced to checking only the constant sheaf, which can be checked stalkwise. Q.E.D

We finish by summarising some of the properties of smooth and proper maps in this formalism.

Proposition 4.5. *([15] Proposition 7.7) - Any topological manifold bundle is cohomologically smooth.*

Proof - Cohomological smoothness is local on the source, stable under pullbacks and passing to open subsets. We can therefore reduce to a Euclidean chart and thus a vector bundle, and thereby reduce to a line bundle. Moreover we can reduce to a contractible base neighbourhood and thereby a point. Therefore we are reduced to $\mathbb{R} \to *$, which is proven in Proposition 5.10 of [52]. Q.E.D

Proposition 4.6. *([15] Proposition 7.8-7.9) - For a projection $f : X \to *$ of a Euclidean neighbourhood retract, $\mathbb{Z}$ is f-smooth.*

Moreover, assume the space X has naturally defined orientations, in the sense that $H_i(X, X \setminus \{x\}; \mathbb{Z})$ vanishes in all degrees except d, for which the relative homology is free on a single generator. Then $f^!\mathbb{Z}$ is isomorphic to the shifted constant sheaf $\mathbb{Z}[d]$. Consequently it is invertible, and therefore such projections are cohomologically smooth.

As a corollary, suitably constructible sheaves on a Euclidean neighbourhood retract are smooth for the projection to a point: by locality and stability under triangles, this reduces to the individual strata, so the condition we require is that the closure of the strata are Euclidean neighbourhood retracts.

Finally, proper maps can be characterised completely:

Proposition 4.7. *([52] Proposition 7.11) - For X a locally compact Hausdorff space, the maps proper for the projection $f : X \to *$ are exactly the compact objects, or equivalently those sheaves locally isomorphic to constant perfect complexes of sheaves supported on a compact subset.*

Proof - If A is f-proper then A is compact ([52] Proposition 6.16).

So, let A be compact. We will show that this implies the last equivalent condition. It is sufficient to prove that A is dualizable, as then it is locally isomorphic to the constant sheaf on a perfect complex, and has compact support by assumption.

So, we may restrict to the support of A and assume that X is compact. But then $D(X)$ is a rigid dualizable presentable symmetric monoidal stable ∞-category, and in such categories dualizable and compact objects agree.

One characterization of such categories is that the unit is compact, ω_1-compactly generated, and any ω_1-compact B can be expressed as a sequential colimit $\varinjlim B_n$, such that for all other objects C the natural map

$$\mathcal{H}\mathrm{om}(C, \mathbb{Z}) \otimes B \to \mathrm{colim}\mathcal{H}\mathrm{om}(C, B_n) \tag{154}$$

is an isomorphism. In the setting above, the ω_1-generators can be taken to be $B = j_!\mathbb{Z}$, such that j is the sequential limit of open embeddings $j_n : U_n \hookrightarrow X$ with $\overline{U_n} \subseteq U_{n+1}$, and $B_n = j_{n!}\mathbb{Z}$. The B_n form a pro-constant system, and the equivalence for $C = B$ proves that B is dualizable. Q.E.D

4.7.2 Coherent Sheaves

On the category of quasicompact and quasiseparated derived schemes, define the quasicoherent ∞-category $\mathcal{D}_{\mathrm{qc}}(X)$ by Kan extension from the affine case. There are various choices of suitable decompositions defining six-functor formalisms of quasicoherent sheaves.

Since right adjoints f_* always satisfy base change in this context, one may take all maps as proper, and then the open immersions to be isomorphisms. Alternatively, if any one wishes to include non-isomorphisms in the class I, they must be proper maps of finite Tor-dimension, and proper maps can be taken to be open immersions.

The abstract notions are mismatched against geometric intuition, so one may choose to dualize to remedy this. That is, quasicoherent sheaves may be viewed as inductive systems of perfect complexes, and so its opposite category is the Pro-category of perfect complexes. Then, one may restrict to Pro(Coh) on Noetherian schemes, and then dualize again to return to the Ind(Coh)-sheaves studied by Gaitsgory-Rosenblyum.

4.7.3 Algebraic $\mathcal{D}$-modules

In [18] a six-functor formalism of $\mathcal{D}$-modules is studied in terms of Ind-coherent sheaves on de Rham stacks. In what follows, k will denote a field of characteristic zero.

Definition 4.10. - On the category $\mathcal{C}$ of separated (derived) schemes, the *category of crystals* is defined by extension from the affine case:

$$\mathrm{Crys}(X) = \varinjlim_{R, \mathrm{Spec}(R_{\mathrm{red}}) \to X} D(R) \tag{155}$$

With respect to the suitable decomposition (I, P) of proper maps and open immersions, respectively, this defines a six-functor formalism of crystals. In this formalism, for any $f : X \to Y$, the exceptional pushforward of the associated map of de Rham spaces $f_{\mathrm{dR}!}$ exists. If f is étale then $f_{\mathrm{dR}!}$ is cohomologically proper, and if f is proper, $f_{\mathrm{dR}!}$ is cohomologically étale.

This chapter details the construction of this formalism following the appendix to lecture VIII of [52].

IndCoh and D_{qc} are extended to stacks on $\mathcal{C}$ via the extension formalism, and there is a natural transform $D_{\mathrm{qc}}(X) \to \mathrm{IndCoh}(X)$ which is usually fully faithful.

These formalisms are not nil-invariant, unlike the other formalisms. They can be made nil-invariant by the precomposition of the quotient to the de Rham prestack X_{dR}:

$$X_{\mathrm{dR}} : A \to \varinjlim_{I \subseteq \pi_0 A} (\pi_0 A/I) \tag{156}$$

where I runs over the nilpotent ideals of the classical ring $\pi_0(A)$. This quotient respects schemes almost of finite type, and so if we restrict to that subcategory we acquire the simpler description:

$$X_{\mathrm{dR}}(A) = X(A_{\mathrm{red}}), \tag{157}$$

the evaluation on the reduced quotient (the quotient ring by the nilradical).

The de Rham stack may be viewed as the quotient by Grothendieck's relation of *"infinitesimal closeness"*, and so sheaves on the de Rham stack can be viewed as sheaves admitting a form of infinitesimal *"parallel transport"*. For this reason, left and right $\mathcal{D}$-modules are defined as sheaves on this quotient prestack:

Definition 4.11. ([52] Definition 8.27, [19]) - The categories of left and right $\mathcal{D}$-modules on X are defined as follows:

$$\begin{aligned} \mathrm{Dmod}^{\mathrm{L}}(X) &= D_{\mathrm{qc}}(X_{\mathrm{dR}}) \\ \mathrm{Dmod}^{\mathrm{R}}(X) &= \mathrm{IndCoh}(X_{\mathrm{dR}}) \end{aligned}$$

The comparison $D_{\mathrm{qc}} \to \mathrm{IndCoh}$ therefore induces a cocontinuous symmetric monoidal functor from left to right $\mathcal{D}$-modules, which is in fact an equivalence.

The following theorem, along with the equivalence between left and right $\mathcal{D}$-modules, proves the existence of the six functor formalism above.

Theorem 4.6. *([52] Theorem 8.27) - For any map $f : X \to Y$ of finite-type derived schemes over k, the induced map $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is $!$-pushable in the IndCoh formalism. If f is étale then f_{dR} is cohomologically proper, and if f is proper, f_{dR} is cohomologically étale.*

The general proof requires several special cases:

Proposition 4.8. *([52] Proposition 8.30) - Let X be a derived scheme almost of finite type over k, and let Z be a closed subscheme. Let $X^{\wedge}_{Z} \subseteq X$ be the formal completion of Z in X: this is the subfunctor representing maps* $\mathrm{Spec}(A) \to X$ *that factor set-theoretically through Z.*

Then the embedding $j : X^{\wedge}_{Z} \hookrightarrow X$ admit exceptional pushforwards in the IndCoh formalism, and is cohomologically étale: in fact, it can be identified with the image of the cohomologically-étale map $Z \to X$.

Proof - The question is local on X, and so we can take X to be an affine scheme, such that Z is the vanishing locus of some functions $f_1, \cdots, f_n$ on X. By induction on n and stability under pullback, we can reduce to the case $n = 1$. This case itself follows via pullback from the case of $X = \mathrm{Spec}(k[T])$ and $Z = \mathrm{Spec}(\mathrm{k})$ embedded at the origin. This map is proper and thus cohomologically étale, and so its image is a cohomologically étale monomorphism.

It remains only to show that this image is indeed $X^{\wedge}_{Z}$. We can make the same assumptions above about X and Z.

The image is trivially contained in $X^{\wedge}_{Z}$. Conversely, let $Y = \mathrm{Spec}(B) \to X^{\wedge}_{Z}$ be a mapping from an affine derived scheme. Then it factors over the fat point $\mathrm{Spec}(k[T]/T^n)$ for some n, and we can therefore take $\mathrm{Spec}(B)$ to be this fat point. Therefore it must be shown that $\mathrm{Spec}(k[T]/T^n) \to \mathrm{Spec}(k[T])$ is contained in the image of $\mathrm{Spec}(k) \to \mathrm{Spec}(k[T])$. In other words, $\mathrm{Spec}(k) \to \mathrm{Spec}(k[T]/T^n)$ is a D-cover. This follows from proposition 6.18 of [52].

Proposition 4.9. *([52] Proposition 8.31) - The exceptional pushforward*

$$j_! : \mathrm{IndCoh}(X^{\wedge}_{Z}) \to \mathrm{IndCoh}(X) \tag{158}$$

is fully faithful, and its essential image are those objects that vanish outside Z.

Proof - For any cohomologically étale monomorphism j, by base change, $j^* j_! = \mathrm{id}$. Therefore $j_!$ is fully faithful.

The kernel of j^* are those modules that vanish on $X^{\wedge}_{Z}$, or equivalently, vanish after pullback along $Z \to X$. Therefore the essential image of $j_!$ is contained in $\mathrm{IndCoh}(X|_Z)$, and as $\mathrm{IndCoh}(X|_Z) \to \mathrm{IndCoh}(Z)$ is conservative, the essential image is all of $\mathrm{IndCoh}(X|_Z)$. Q.E.D

Proposition 4.10. *([52] Proposition 8.32) - For finite type X the quotient map $X \to X_{\mathrm{dR}}$ admits an exceptional pushforward in the IndCoh formalism which is cohomologically étale and surjective.*

Proof - The statement is local in X, so X can be assumed to be affine. By compactification X can also be assumed to be proper.

The definition of cohomologically étale maps of stacks is extended from derived schemes by universality: for all derived schemes Y almost of finite type over k and maps $Y \to X_{\mathrm{dR}}$, we must show that the resulting maps $X \times_{X_{\mathrm{dR}}} Y \to Y$ admit exceptional pushforwards and are cohomologically étale and surjective.

The monomorphism $X \times_{X_{\mathrm{dR}}} Y \to X \times_k Y$ admits an exceptional pushforward and is cohomologically étale: the graph of $Y_{\mathrm{red}} \to X$ defines a closed subset $Z \subseteq X \times_k Y$, and

$$X \times_{X_{\mathrm{dR}}} Y = (X \times_k Y)^{\wedge}_{Z}. \tag{159}$$

Therefore by ([52] proposition 8.30), recalled above, the statement is proven in this case. Then since $X \times_k Y \to Y$ is also cohomologically étale (since X is proper), this proves the universal cohomological étaleness as required, and surjectivity of $X \to X_{\mathrm{dR}}$ is clear.

We now sketch the proof of theorem 8.29 of [52]:
Proof Sketch -

1. First we show $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$. It suffices verify this for $X \to X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$, which is equivalently the composite $X \to Y \to Y_{\mathrm{dR}}$, which both admit exceptional pushforwards by assumption and the previous results.

2. If f is proper then f_{dR} is cohomologically étale: similarly considering the pullback along $X \to X_{\mathrm{dR}}$, $X \to Y$ is cohomologically étale by properness, and $Y \to Y_{\mathrm{dR}}$ is cohomologically étale by the previous proposition.

3. If f is étale then f_{dR} is cohomologically proper: it suffices to verify this after pullback along $Y \to Y_{\mathrm{dR}}$, which yields the cohomologically proper map $X = X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y \to Y$.

The proof of the formalism is concluded, and we now discuss the relation to the classical theory of algebraic $\mathcal{D}$-modules. Firstly, the six-functor formalism easily yields an analogue of Kashiwara's equivalence:

Proposition 4.11. *(Kashiwara's equivalence, [52] Proposition 8.33) - Let X be a finite type k-scheme with closed subscheme Z and open complement U. Then any right $\mathcal{D}$-module $A \in \mathrm{Dmod}^{\mathrm{R}}(X)$ fits into an exact triangle*

$$j_! j^* A \to A \to i_* i^* A \tag{160}$$

Proof - The triangle of right $\mathcal{D}$-modules, viewed as modules over the de Rham stack, can be pulled back along $X \to X_{\mathrm{dR}}$, and it is enough to verify the resulting decomposition on X. But then the strata defined by this triangle are $X \times_{X_{\mathrm{dR}}} Z_{\mathrm{dR}} = X^{\wedge}_{Z}$ and $X \times_{X_{\mathrm{dR}}} U_{\mathrm{dR}} = U$, and so the discussion follows from

([52] Proposition 8.31) above.

Remarks

1. Kashiwara's equivalence can be used to produce an equivalence between left and right $\mathcal{D}$-modules ([52] Theorem 8.34). Briefly, the proof in loc. cit. appeals to universal descent to de Rham stacks to reduce to the case of a smooth embedding $X \subseteq \tilde{X}$, and the equivalence $\mathrm{IndCoh}(\tilde{X}|_X)$

2. The proof of the equivalence of left and right $\mathcal{D}$-modules on X describes left $\mathcal{D}$-modules as quasicoherent sheaves on X with a descent datum that encodes infinitesimal equivariance, equivalent to a left $\mathcal{O}_X$-linear action of the algebra $\mathcal{D}_X$ of differential operators on X.

   Similarly, right $\mathcal{D}$-modules yield modules $M \in \mathrm{IndCoh}(X)$ with a similar action. Now, one has a choice of equivalences between $\mathrm{Ind}(\mathrm{Coh})(X)$ and $D_{\mathrm{qc}}(X)$, depending on whether one takes the twist in the equivalence or not. If one does not, taking the equivalence $\mathrm{Ind}(\mathrm{Coh})(X) = D_{\mathrm{qc}}(X)$ induced by $\mathrm{Coh}(X) = \mathrm{Perf}(X)$, then the action unravels to a right $\mathcal{O}_X$-linear $\mathcal{D}_X$-action.

Before discussing some examples, we present a final theorem on duality:

Theorem 4.7. *([52] Theorem 8.35) - Let X be a finite type k-scheme:*

1. *The category* $\mathrm{Dmod}(X)$ *is compactly generated. The compact objects are exactly the f_{Dmod}-proper objects for the projection $f : X \to *$ to a point. Therefore compact objects are self-dual.*

2. *If X is smooth of dimension d, then the f_{Dmod}-proper dual of $1_{\mathrm{Dmod}(X)}$ is the shift $1_{\mathrm{Dmod}(X)}[-2d]$.*

3. *For proper $f : X \to Y$, $f_{!,\mathrm{Dmod}}$ preserves compact objects and commutes with proper duality.*

Example - The affine line - Let $X = \mathbb{A}^1 = \mathrm{Spec}(k[T])$, and let $j : Z = \mathrm{Spec}(k) \hookrightarrow X$ be the origin with complement $i : U = \mathrm{Spec}(k[T]/T^{-1}) \hookrightarrow X$. Consider the associated exact triangle of $\mathcal{D}$-modules.

$$j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(Z)}) \to 1_{\mathrm{Dmod}(X)} \to j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(U)}) \tag{161}$$

Since the embedding i is open, it's cohomologically proper, and so $i_{*,\mathrm{Dmod}} = i_{!,\mathrm{Dmod}}$. As j is proper, it is cohomologically étale in the $\mathcal{D}$-module formalism, and so $j_{!,\mathrm{Dmod}}$ is the left adjoint of j^*_{Dmod}.

Under the realization $\mathrm{Dmod} \to \mathcal{D}_{\mathrm{qc}}(X)$, this is the usual realization as left $\mathcal{D}$-modules. The functor $i_{*,\mathrm{Dmod}}$ pulls back to i_* in the quasicoherent formalism, because $U = X \times_{X_{\mathrm{dR}}} U_{\mathrm{dR}}$. Therefore the last two terms of the sequence are the global functions $k[T]$ and $k[T^{\pm 1}]$ on the affine line and punctured disc, respectively. Therefore we can complete the whole triangle to

$$(k[T^{\pm 1}]/k[T])[-1] \to k[T] \to k[T^{\pm 1}] \tag{162}$$

The IndCoh realization is given by the following composition:

$$\mathrm{Dmod}(X) \cong \mathrm{Dmod}^{\mathrm{R}}(X) = \mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X) = \mathcal{D}_{\mathrm{qc}}(X) \tag{163}$$

This sends $1_{\mathrm{Dmod}(X)}$ to the dualizing complex $\omega_X = (k[T]dT)[-1]$. For any open immersion i, the pullback i^*_{Dmod} is the naive pullback i^* in any realization, and so its adjoint i_* also has naive realization. Therefore the triangle becomes

$$(k[T^{\pm 1}]/k[T])dT \to (k[T]dT)[1] \to k[T^{\pm 1}]dT[1] \tag{164}$$

which is just the twisting of the first realization by the dualizing complex.

5 The Six-Functor Formalism of Quasicoherent Sheaves

5.1 Introduction

We define a formalism of quasicoherent sheaves on derived analytic stacks suitable for applications to geometric representation theory.

The first section introduces quasicoherent sheaves on derived dagger Stein domains in a tautological manner, and then introduce the category of stacks to which the formalism will be extended. For this, the theory of locales associated to symmetric monoidal $(\infty, 1)$-categories is recalled from ([14], Chapter 5), and the initial class of maps for which exceptional pushforwards, the *quasicompact quasiseparated morphisms* are defined, and the necessary properties verified.

The next section is devoted to analyzing this formalism. Important examples of infinite covers satisfying !-descent are discussed, local cohomology is defined and discussed, and Zariski closed and open immersions are defined.

The third section is dedicated to infinitesimal objects. We begin by recalling the theory of Betti stacks, and exploring their relationship with quasicoherent sheaves. Then germs and their quotient stacks, ∞-*de Rham stacks*, are defined and studied. The section ends with a discussion of the relationships between ∞-de Rham stacks, $\mathcal{D}^{\infty}$-modules, and Betti stacks.

5.2 Quasicoherent Sheaves on Derived Analytic Spaces

5.2.1 Quasicoherent Sheaves on Derived Dagger Steins

Lemma 5.1. *([54] Lemma 4.2) -*

1. $\mathrm{dStnAlg}^{\dagger}$ *is stable under pushouts in* $\mathrm{Alg}(D_{\geq 0}(\mathrm{CBorn}_k))$.
2. $\mathrm{dStnAlg}^{\dagger}$ *is stable under finite products and retracts in* $\mathrm{Comm}(D_{\geq 0}(\mathrm{CBorn}_k))$.

Proof -

1. Let $A \leftarrow B \rightarrow C$ be a diagram of derived dagger Stein algebras, and form the pushout diagram in $\mathrm{Alg}(D_{\geq 0}(\mathrm{CBorn}_k))$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & B\hat{\otimes}^{\mathbb{L}}_{A}C \end{tikzcd}$$

By Proposition 5.4.112 of [7], we can choose geometric presentations $(A, \overline{A}_n, A_n), (B, \overline{B}_n, B_n), (C, \overline{C}_n, C_n)$ of A, B and C, respectively, such that the maps $A \to B, A \to C$ are given by systems of maps $\overline{A}_n \to \overline{B}_n, \overline{A}_n \to \overline{C}_n$.

Motivated by the proof of Theorem 5.4.117 of [7], we use the bar complexes $|B_n \widehat{\otimes}^{\mathbb{L}}_{A_n} C_n|$, as n varies, to produce a geometric presentation of $B \otimes^{\mathbb{L}}_A C$. Note that each term $B_n \widehat{\otimes}^{\mathbb{L}}_{A_n} C_n$ is a derived dagger affinoid by lemma 4.2 of [54]. The homology of each term of the bar complex $|B_n \widehat{\otimes}^{\mathbb{L}}_{A_n} C_n|$ is $\varprojlim$-acyclic - as all terms are homotopy flat:

$$\pi_*(B_n \widehat{\otimes}^{\mathbb{L}}_k A_n^{\widehat{\otimes}^{\mathbb{L}}_k i} \widehat{\otimes}^{\mathbb{L}}_k C_n) \cong \pi_*(B_n) \widehat{\otimes}_k \pi_*(A_n)^{\widehat{\otimes}_k n} \widehat{\otimes}_k \pi_*(C_n). \tag{165}$$

This is a sequence of nuclear Fréchet spaces with dense transition maps, and it is therefore $\varprojlim$-acyclic by the bornological Mittag-Leffler theorem ([5], Corollary 3.80). We see that $B_i \widehat{\otimes}^{\mathbb{L}}_{A_i} C_i$ furnishes a geometric presentation[1] of $B \widehat{\otimes}^{\mathbb{L}}_A C$ as a derived dagger Stein algebra by the following manipulations:

$$\begin{aligned}
\varprojlim_n B'_n \widehat{\otimes}^{\mathbb{L}}_{A'_n} C'_n &\cong \varprojlim_n |B'_n \widehat{\otimes} {A'_n}^{\widehat{\otimes}_k i} \widehat{\otimes}_k C'_n| \\
&\cong |B' \widehat{\otimes}^{\mathbb{L}} {A'}^{\widehat{\otimes}^{\mathbb{L}}_k i} \widehat{\otimes}^{\mathbb{L}}_k C'| \\
&\cong B' \widehat{\otimes}^{\mathbb{L}}_{A'} C'.
\end{aligned}$$

2. π_* commutes with finite products and respects retracts, and so the second statement follows immediately. Q.E.D

Definition 5.1. 1. Define the category of *derived dagger Steins* $\mathrm{dStn}^\dagger$ as opposite to the category of derived dagger Stein algebras $\mathrm{dStnAlg}^\dagger$.

2. A morphism $\mathrm{Spec}(B) \to Spec(A)$ defines a *Stein subdomain* if the corresponding morphism $A \to B$ is a dagger Stein localization.

3. The *small weak analytic site* on a derived dagger Stein space $X = \mathrm{Spec}(A)$ is the $(\infty, 1)$-site with underlying $(\infty, 1)$-category the category of Stein subdomains, and covering sieves generated by finite families of Stein subdomains $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(B)\}$, such that the family on underlying dagger Steins $\{\pi_0(\mathrm{Spec}(B_i)) \to \pi_0(\mathrm{Spec}(B))\}$ is a classical cover.

4. The *big weak analytic site* on $\mathrm{dStn}^\dagger$ has $\mathrm{dStn}^\dagger$ as the underlying $(\infty, 1)$-category, and covering sieves generated by finite families of Stein subdomains as in the weak analytic site.

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[1] Note here that we have tacitly used some stability properties of Weierstrass localizations and interior embeddings of Berkovich spectra.

Definition 5.2. - For Spec(A) are derived dagger Stein space, the ∞-category of quasicoherent sheaves is simply the module category over A:

$$\mathrm{QCoh}(\mathrm{Spec}(A)) \cong \mathrm{Mod}_A(D(\mathrm{CBorn}_k)). \tag{166}$$

A morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ corresponds to a map of derived dagger Stein algebras $B \to A$, which induces a restriction of scalars map

$$f_* : \mathrm{Mod}_A = \mathrm{QCoh}(\mathrm{Spec}(A)) \to \mathrm{QCoh}(\mathrm{Spec}(B)) = \mathrm{Mod}_B \tag{167}$$

with right adjoint f^*.

Remark - Base change and the projection formula arise from associativity of the tensor product of modules:

1. For a morphism of derived Steins $f : \mathrm{dSp}(A) \to \mathrm{dSp}(B)$, the canonical natural transform

   $$f_* \widehat{\otimes}^{\mathbb{L}}_B \mathrm{id} \to f_*(\mathrm{id} \widehat{\otimes}^{\mathbb{L}}_A f^*) \tag{168}$$

   corresponds to the canonical morphism for A-modules M and B-modules N:

   $$M \widehat{\otimes}^{\mathbb{L}}_B N \to M \widehat{\otimes}^{\mathbb{L}}_A A \widehat{\otimes}^{\mathbb{L}}_B N \tag{169}$$

   which is an isomorphism by associativity of the tensor product.

2. Given a Cartesian square

   $$\begin{tikzcd} \mathrm{dSp}(C \otimes^{\mathbb{L}}_B A) \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{dSp}(C) \arrow[d, "g"] \\ \mathrm{dSp}(A) \arrow[r, "f"] & \mathrm{dSp}(B) \end{tikzcd}$$

   the Beck-Chevalley transform $g^* f_* \to f'_* g'^*$ corresponds to the transform of C-modules:

   $$C \widehat{\otimes}^{\mathbb{L}}_B M \to C \widehat{\otimes}^{\mathbb{L}}_B A \widehat{\otimes}^{\mathbb{L}}_A M \tag{170}$$

   which, again, is an isomorphism by associativity of the derived monoidal product.

Lemma 5.2. *([54] Definition 4.15, Lemma 4.16) - The $(\infty,1)$-category of sheaves on* $\mathrm{dStn}^{\dagger}$ *valued in the category of ∞-groupoids with respect to the weak topology is denoted*

$$\mathrm{Shv}_{\mathrm{weak}}(\mathrm{dStn}^{\dagger}) = \mathrm{Shv}_{\mathrm{weak}}(\mathrm{dStn}^{\dagger}, \infty\mathrm{Grpd}). \tag{171}$$

The weak topology is subcanonical: for each derived Stein $\mathrm{Spec}(A)$ *the presheaf it represents is a weak sheaf.*

Moreover, QCoh *is also a weak sheaf.*

Proof - We detail the argument used in Lemma 4.16 of [54] since the facts concerning derived strength of localisations, transversality of higher homotopy modules to localisations (by virtue of coherence), and descent of coherent sheaves on affines apply in both contexts. Coherent descent in our context corresponds to Cartan's theorem B.

Consider a cover in the weak topology $\{\mathrm{dSp}(B_i) \to \mathrm{dSp}(B)\}$. The underlying cover of $\mathrm{Sp}(\pi_0(B))$ is a finite open cover, and so the associated Čech complex is acyclic. The Čech complex is the normalised chain complex associated to the system of modules

$$\pi_0(B_{i_1})\widehat{\otimes}_{\pi_0(B)}\pi_0(B_{i_2})\widehat{\otimes}_{\pi_0(B)}\cdots\widehat{\otimes}_{\pi_0(B)}\pi_0(B_{i_k}) \tag{172}$$

indexed by finite subsets of I, viewed as a category under inclusions. Moreover, π_0 commutes with tensor products: in other words, the connected components of a derived tensor product is the tensor product of connected components. Therefore, by the Dold-Kan correspondence, $\pi_0(B)$ is the inductive limit

$$\pi_0(B) \simeq \varprojlim_I \pi_0(B_{i_1})\widehat{\otimes}_{\pi_0(B)}\pi_0(B_{i_2})\widehat{\otimes}_{\pi_0(B)}\cdots\widehat{\otimes}_{\pi_0(B)}\pi_0(B_{i_k}) \tag{173}$$

$$\simeq \varprojlim_I \pi_0(B_{i_1}\widehat{\otimes}^{\mathbb{L}}_B B_{i_2}\widehat{\otimes}^{\mathbb{L}}_B\cdots\widehat{\otimes}^{\mathbb{L}}_B B_{i_k}) \tag{174}$$

By derived strength of $B \to B_{i_1}\widehat{\otimes}^{\mathbb{L}}_B B_{i_2}\widehat{\otimes}^{\mathbb{L}}_B\cdots\widehat{\otimes}^{\mathbb{L}}_B B_{i_k}$, the functor $\pi_q(B)\otimes^{\mathbb{L}}_{\pi_0(B)}$ $(-)$ respects this equivalence: since the limit is finite and the categories are stable, this commutes with the projective limit, and so we obtain a similar equivalence for q^{th} homotopy:

$$\pi_q(B) \simeq \varprojlim_I \pi_q(B_{i_1}\widehat{\otimes}^{\mathbb{L}}_B B_{i_2}\widehat{\otimes}^{\mathbb{L}}_B\cdots\widehat{\otimes}^{\mathbb{L}}_B B_{i_k}). \tag{175}$$

This implies that the spectral sequence (afforded by cohomological filtration - see chapter 1.2 of [32] and chapter 2.3 of [7])

$$H^p(\varprojlim_I \pi_q(B_{i_1}\widehat{\otimes}^{\mathbb{L}}_B B_{i_2}\widehat{\otimes}^{\mathbb{L}}_B\cdots\widehat{\otimes}^{\mathbb{L}}_B B_{i_k})) \Rightarrow \pi_{q-p}(\varprojlim_I B_{i_1}\widehat{\otimes}^{\mathbb{L}}_B B_{i_2}\widehat{\otimes}^{\mathbb{L}}_B\cdots\widehat{\otimes}^{\mathbb{L}}_B B_{i_k}) \tag{176}$$

degenerates. In particular, applying $p = 0$ yields the isomorphism

$$\pi_q(\varprojlim_I B_{i_1} \widehat{\otimes}^{\mathbb{L}}_B B_{i_2} \widehat{\otimes}^{\mathbb{L}}_B \cdots \widehat{\otimes}^{\mathbb{L}}_B B_{i_k}) \cong \mathrm{eq}(\prod \pi_q(B_i) \rightrightarrows \prod \pi_q(B_i \widehat{\otimes}^{\mathbb{L}}_B B_j)). \tag{177}$$

The maps $B \to B_{i_1} \widehat{\otimes}^{\mathbb{L}}_B B_{i_2} \widehat{\otimes}^{\mathbb{L}}_B \cdots \widehat{\otimes}^{\mathbb{L}}_B B_{i_k}$ are derived strong, and $\pi_q(B)$ is transversal to Stein localisations. Therefore the calculation can be transferred to underlying Stein spaces, where Cartan's theorem B applies to give the isomorphism:

$$\mathrm{eq}(\prod \pi_q(B_i) \rightrightarrows \prod \pi_q(B_i \widehat{\otimes}^{\mathbb{L}}_B B_j)) \tag{178}$$

$$\cong \mathrm{eq}(\prod \pi_q(B) \widehat{\otimes}_{\pi_0(B)} \pi_0(B_i) \rightrightarrows \prod \pi_q(B) \widehat{\otimes}_{\pi_0(B)} \pi_0(B_i) \widehat{\otimes}_{\pi_0(B)} \pi_0(B_j)) \tag{179}$$

$$\cong \pi_q(B). \tag{180}$$

Therefore, it has been shown that the map $B \to B_{i_1} \widehat{\otimes}^{\mathbb{L}}_B B_{i_2} \widehat{\otimes}^{\mathbb{L}}_B \cdots \widehat{\otimes}^{\mathbb{L}}_B B_{i_k}$ is a weak equivalence. It follows that $f : B \to \prod B_i$ is *descendable* in the sense of Mathew [35]. Recall that descendability of this map means that B is the limit of partial totalisations of the bar complex associated to the map $B \to \prod B_i$. It then follows by Proposition 3.22 of [35] that the adjunction between B-modules and $\prod B_i$-modules is comonadic:

$$\prod B_i \otimes^{\mathbb{L}}_B (-) : \mathrm{Mod}_B \leftrightarrows \mathrm{Mod}_{\prod B_i} : |-| \tag{181}$$

In the language of quasicoherent sheaves on affines, the map $B \to \prod B_i$ corresponds to the morphism $\bigsqcup \mathrm{dSp}(B_i) \to \mathrm{dSp}(B)$, and descendability translates to the natural equivalence

$$\mathrm{QCoh}(X) \simeq \varprojlim_{[m]\in\Delta} \mathrm{QCoh}(Y^{\times_X (m+1)}). \tag{182}$$

This proves that QCoh is a sheaf. Moreover, we can conclude that the site is subcanonical: considering the trivial B-module B in the above equation, we deduce the equivalence

$$B \simeq \varprojlim_{[m]\in\Delta} (\prod B_i)^{\otimes^{\mathbb{L}}_B (m+1)} \tag{183}$$

Applying $\mathrm{Map}(A, -)$ for a dagger Stein algebra A implies descent for representable, as required. Q.E.D

5.2.2 Weak Sheaves and Derived Analytic Spaces

We now introduce the category of stacks to be used in what follows.

Definition 5.3. -

1. An *analytic subspace* of a derived dagger Stein space $X = \mathrm{Spec}(A)$ is a weak subsheaf $U \subseteq \mathrm{Spec}(A)$, admitting an effective epimorphism
$$\coprod \mathrm{Spec}(A_i) \to U \tag{184}$$
such that the compositions $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ are Stein subdomains.

2. For $X \in \mathrm{Shv}_{\mathrm{weak}}(\mathrm{dStn}^\dagger)$ an arbitrary weak sheaf, an *analytic subspace* of X is a weak subsheaf $Y \subseteq X$, such that for any derived dagger Stein $\mathrm{Spec}(A)$, the pullback $Y \times_X \mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(A)$ is an analytic subspace in the above sense:
$$\begin{tikzcd} Y \times_X \mathrm{Spec}(A) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

3. A *derived complex analytic space* is a weak sheaf admitting an effective epimorphism $\coprod_{i\in I} \mathrm{Spec}(A_i) \to X$, where each map $\mathrm{Spec}(A_i) \to X$ is an affine analytic subspace. We denote the category of derived analytic spaces by dAn_k

4. For $X \in \mathrm{dAn}_k$, the *strong analytic site of* X has underlying $(\infty, 1)$-category the analytic subspaces of X, and covering sieves given by small families of analytic subspaces
$$\{Y_i \hookrightarrow X\} \tag{185}$$
such that $\coprod Y_i \to X$ is an effective epimorphism.

5.2.3 Locales of Analytic Subspaces

We recall the discussion of locales of analytic subspaces from Scholze [14], and explain along the same lines as [54] how to construct an underlying topological space of any derived analytic space.

The context of the construction of [14] is a closed symmetric monoidal stable $(\infty, 1)$-category $\mathcal{C}$. Consider the collection of all idempotent algebra objects in $\mathcal{C}$ - objects $A \in \mathcal{C}$ equipped with a homotopy epimorphism $1 \to A$, so that $A \otimes A \cong A$. The collection of such idempotent algebras form a partially ordered set, as there is at most one map $A \to A'$ over the structure morphisms $1 \to A$, $1 \to A'$.

This poset in fact forms a locale $\mathcal{S}(\mathcal{C})$: regarding the idempotent algebras A_i as closed subsets denoted Z_i, the intersection $\bigcap_i Z_i$ corresponds to $\varinjlim_i A_i$, and the union of two closed subsets Z, Z' corresponds to the equalizer of the pair of maps $A \oplus A' \to A \otimes A'$:

$$\begin{tikzcd}
Z \cap Z' \arrow[r] \arrow[d] & Z' \arrow[d] \\
Z \arrow[r] & Z \sqcup_{Z \cap Z'} Z
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A' & A' \arrow[l] \\
A \arrow[u] & A \times_{A \otimes A'} A' \arrow[l] \arrow[u]
\end{tikzcd}$$

We then define a sheaf on this locale in the following way: to a closed subset Z with corresponding idempotent algebra A, associate the module category $\mathcal{C}(Z) = \mathrm{Mod}_A = \{X \in \mathcal{C}, X \otimes A \cong X\}$. If we regard A as the function algebra of Z, then $\mathcal{C}(Z)$ is the module category "over Z".

The module category $\mathcal{C}(U)$ on the formal "complementary open" U of Z is then defined to be the Verdier quotient $\mathcal{C}(U)$. This defines a sheaf of $(\infty, 1)$-categories on the locale $\mathcal{S}(\mathcal{C})$.

We now consider the case that $\mathcal{C}$ is a category of modules over a complex analytic space X. The algebras that correspond to the closed subspaces Z of X are the *overconvergent holomorphic functions* around Z.

We reproduce the sample calculation of proposition 5.6 of [14] in the case of closed discs and annuli in the complex affine line:

Proposition 5.1. - *The algebras of overconvergent functions on the closed unit disc* $D^1_{\leq 1,\mathbb{C}}$ *and closed annulus* $D^1_{\geq 1,\mathbb{C}}$ *are idempotent, with corresponding localizations the holomorphic functions on their complements in the complex affine line: the open annulus of radii* $(1, \infty)$ *and the open unit disc, respectively:*

$$A(\{|T| \leq 1\}) = \bigcup_{r>1} \left\{ \sum_{n=0}^{\infty} a_n T^n : a_n r^n \to 0 \right\} \tag{186}$$

$$A(\{|T| \geq 1\}) = \bigcup_{k} \bigcup_{r<1} \left\{ \sum_{n=-\infty}^{k} a_n T^n : a_n r^n \to 0 \right\} \tag{187}$$

$$\mathcal{O}(\{|T| > 1\}) = \bigcup_{m \geq 0} \bigcap_{r>1} \left\{ \sum_{n=m}^{-\infty} a_n T^n : a_n r^n \to 0 \right\} \tag{188}$$

$$\mathcal{O}(\{|T| < 1\}) = \bigcap_{r>1} \left\{ \sum_{n=0}^{\infty} a_n T^n : a_n r^n \to 0 \right\} \tag{189}$$

$$\tag{190}$$

Proof - The first two algebras are related by the base change $T \to T^{-1}$, so it suffices to prove the first algebra A is idempotent. The algebra $A \otimes A$

$$\bigcup_{r>1} \left\{ \sum_{n,m=0}^{\infty} a_{n,m} T^n U^m : a_{n,m} r^{n+m} \to 0 \right\} / (U - T). \tag{191}$$

The map $T, U \mapsto T$ defines an isomorphism $A \otimes A \cong A$. It is clear that the map is surjective by considering those series in which U does not appear, and clearly any element of the kernel admits a division by $(T - U)$.

The formulae for the localizations can be proven using Verdier duality arguments similar to those in chapter 2. The proof found in loc. cit. uses similar reasoning. Q.E.D

By base change we similarly obtain idempotent algebras $\{|f| \leq 1\}$ and $\{|f| \geq 1\}$, and this sample calculation can be globalized to complex analytic spaces.

Applying this to the symmetric monoidal $(\infty, 1)$-category $\mathrm{Shv}_{\mathrm{weak}}(\mathrm{dStn}^{\dagger})_X$, for $X \in \mathrm{dAn}_k$, we obtain a locale of idempotent weak subsheaves.

Lemma 5.3. *([54] Lemma 4.23) - The join and meet operations of the locale associated to a derived analytic space X respect the analytic subspaces of X:*

Proof - The meet of two analytic subspaces of X is their usual intersection:

$$U \cap V = U \times_X V. \tag{192}$$

Finite meets are determined by pairwise meets which are given by the formula

$$U \cup V = U \coprod_{U \cap V} V. \tag{193}$$

Arbitrary joins are given by the colimit of finite subjoins - for $\{U_i\}_{i \in \mathcal{I}}$ a small family of analytic subspaces of X, their join is given by

$$\bigcup_{i \in \mathcal{I}} U_i = \varinjlim_{I \subseteq \mathcal{I}} \bigcup_{i \in I} U_i \tag{194}$$

where the colimit runs over finite subsets of $\mathcal{I}$. It must be verified that this is also an analytic subspace of X. This will follow from the fact that colimits are universal in any $(\infty, 1)$-topos (this is one of the Giraud-Rezk-Lurie axioms, [32] Theorem 6.1.0.6).

Let $Y = \mathrm{Spec}(A) \to X$ be a map from a derived dagger Stein, and $\{U_i\}_{i \in \mathcal{I}}$ a small family of analytic subspaces of X. Then

$$\left(\bigcup_{i \in \mathcal{I}} U_i\right) \times_X Y = \bigcup_{i \in \mathcal{I}} (U_i \times_X Y), \tag{195}$$

and each $U_i \times_X Y$ is an analytic subspace of Y. Then by (192) (193) the formulae the join in (195) is an analytic subspace of Y, furnishing $\bigcup_{i \in \mathcal{I}} U_i$ as an analytic subspace of X. Q.E.D

Denote the locale of analytic subspaces of X by $\mathrm{An}(X)$. Then sheaves valued in an $(\infty,1)$-category $\mathcal{D}$ with small limits on this locale are exactly $\mathcal{D}$-valued sheaves on the strong site:

$$\mathrm{Shv}_{\mathtt{Strong}}(X,\mathcal{D}) \cong \mathrm{Shv}(\mathrm{An}(X),\mathcal{D}). \tag{196}$$

The same argument as in the discussion of lemma 4.23 of [54] demonstrates that $\mathrm{An}(X)$ is locally compact: in the context of that paper analytic spaces are built out of affinoids, which are compact. Rather than take the dagger Stein algebras directly, we can take the algebras on closed discs that define them to draw the same conclusions. Therefore, by Hoffman-Lawson duality [25], the locale is spatial. We therefore associate to X the topological space

$$|X| = \mathrm{pt}(\mathrm{An}(X)), \tag{197}$$

where the points of the locale are the *completely prime filters* on it. The constructions are all functorial, so we get a functor $|\cdot| : \mathrm{dAn}_k \to \mathrm{Top}$ from derived analytic spaces to (sober) topological spaces.

The underlying topological space is indifferent to the derived structure of X, or in other words, X_0 carries all the geometric information about X:

Lemma 5.4. *([54] Lemma 4.24(i)) - For* $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$ *a finite collection of derived Stein subdomains, the induced map* $\bigsqcup |\mathrm{Spec}(A_i)| \to |\mathrm{Spec}(A)|$ *is a surjection if and only if* $\bigsqcup |\mathrm{Spec}(\pi_0(A_i))| \to |\mathrm{Spec}(\pi_0(A))|$ *is a surjection.*

Proof - The covering sieves are determined by maps on π_0. Q.E.D

From this we immediately obtain the following:

Lemma 5.5. *([54] Lemma 4.24(ii)) - For analytic subspaces* $U, V \subseteq X$ *of a weak derived analytic stack* X, $|U| \subseteq |V|$ *if and only if* $|U_0| \subseteq |V_0|$. *In particular,* $|U| = |V|$ *if and only if* $|U_0| = |V_0|$.

Lemma 5.6. *([54] Lemma 4.25, Corollary 4.26) - For* $X \in \mathrm{dAn}_k$, *any Stein subdomain* $V \subseteq X_0$, *lifts to a derived Stein subdomain* $U \subseteq X$, *such that* $U_0 = V$.

Consequently, for $X \in \mathrm{dAn}_k$ *and any open* $V \subseteq |X_0|$, *there exists an analytic subspace* $U \subseteq X$ *with* $|U_0| = V$.

Proof - The first statement follows from ([7] Corollary 2.1.36), since localizations are formally étale. The second statement is local, and so can be reduced to the case that X is a derived dagger Stein. But then dagger Stein subdomains form a basis the topology, so this follows from the first statement. Q.E.D

Theorem 5.1. *For* $X \in \mathrm{dAn}_{\mathbb{C}}$, $|X_0| \cong |X|$.

Proof - The previous lemmas show that the lattices of opens of $|X|$ and $|X_0|$ are equivalent, and therefore so are their underlying spaces. Q.E.D

5.2.4 Quasicompact and Quasiseparated Morphisms

To define six-functor formalisms we require a *geometric setup* in the sense of Mann [34]. This consists of a pair $(\mathcal{C}, E)$ of an ∞-category $\mathcal{C}$, and a class of morphisms E, for which exceptional pushforwards will be defined. This class must contain all equivalences, be stable under composition and arbitrary pullbacks. The class we opt for is the class of *quasicompact quasiseparated morphisms*, as in Definition 4.29 of [54]:

Definition 5.4.

1. A derived analytic stack X is *quasicompact* if all covers admit a finite subcover. A morphism is quasicompact if for all quasicompact Z and morphisms $Z \to Y$, the fibre product $X \times_Y Z$ is quasicompact.
2. A morphism is *quasiseparated* if its diagonal is quasicompact. A stack X is quasicompact if $X \to \mathrm{Spec}(\mathbb{C})$ is quasicompact.
3. Quasicompact quasiseparated morphisms will be called qcqs.

Lemma 5.7. *([54] Lemma 4.30) - Quasicompact and quasiseparated morphisms are stable under base change and composition. Moreover for morphisms* $f : Y \to X$ *and* $g : Z \to Y$*, if* fg *is quasicompact and* f *is quasiseparated, then* g *is quasicompact .*

Proof - Quasicompactness of morphisms is a statement concerning pullbacks, and compositions of pullbacks are pullbacks. Therefore, any base change of a quasicompact morphism is quasicompact.

The base change of a diagonal morphism is the diagonal of a base change, and so quasiseparated morphisms are also stable under base change.

Base change is functorial, and from this is follows that quasicompact and quasiseparated morphisms are stable under composition.

Let f, g be as above. Factorize g as $Z \to Z \times_X Y \to Y$. The first map is the base change of Δ_f along $g \times_Y \mathrm{id}$, and the second is the base change of fg along f. By stability under base change and composition, g is quasicompact. Q.E.D

Remark - The weak topology is certainly finitary (covers have finite refinements, [33]), and so all derived Steins are both quasicompact and quasiseparated when viewed as weak sheaves ([33] Proposition 3.19).

Quasicompact and quasiseparated morphisms can be characterized as follows:

Lemma 5.8. *([54] Lemma 4.32, Lemma 4.33) -*

1. *Quasicompactness of morphisms is local in the following sense: a morphism $f : X \to Y \in \mathrm{dAn}_{\mathbb{C}}$ is quasicompact, if and only if for every dagger Stein subdomain $U \subseteq X$ the pullback $U \times_X Y$ is quasicompact.*

   Moreover, it is sufficient to find a single Stein cover $\mathcal{U} = \{U_i \subseteq Y\}$ for which all the pullbacks $U_i \times_Y X$ are quasicompact.

2. *A morphism $f : X \to Y$ is quasiseparated, if and only if for every pair of subdomains $U, V \subseteq X$, each mapping to a common subdomain of Y, the intersection $U \times_X V$ admits a finite cover by dagger Stein subdomains of X.*

   Again, this property is equivalent to the existence of a Stein subdomain cover such that this property holds for all pairs.

Proof - Since base change respects quasicompactness, all that is required is to show that the existence of such a Stein cover implies quasicompactness. So, let $f : X \to Y$ be the morphism, $\mathcal{U} = \{U_i \to Y\}$ the Stein cover, and $g : Z \to Y$ a morphism from quasicompact Z. Then $\mathcal{U}$ pulls back to a cover along g, which admits a finite subcover $\{V_j\}$ by quasicompactness. Therefore, the cover $\{X \times_Y V_j\}$ is a finite cover of $X \times_Y Z$ by quasicompact subspaces, and is therefore itself quasicompact, as required.

Similarly, to prove the statement concerning quasiseparatedness, it suffices to prove that the existence of such a cover implies quasiseparatedness. But such a cover induces a decomposition of the diagonal Δ_f into restrictions to Stein subdomains, which is quasicompact since Stein domains are quasiseparated. Therefore by the characterization of quasicompactness, Δ_f is quasicompact, and so f is quasiseparated. Q.E.D

To prove that QCoh defines a six-functor formalism, the exceptionally pushable morphisms are first defined as the qcqs morphisms. The pair (Equivalences, qcqs) defines a *suitable decomposition* in the sense of [34] - for this we just need to show the following:

Lemma 5.9. *([54] Lemma 4.34, Corollary 4.35) - For a pair of composable morphisms f, g of derived analytic stacks, if fg is quasi-separated, then so is g.*

Consequently, if fg and f are both qcqs, so is g.

Proof - Write $g : Z \to Y$, $f : Y \to X$. Given Stein subdomains $U, V \subseteq Z$ mapping to a common subdomain of Y, we can decompose the image in X into subdomains. Pulling this cover back to Z provides covers $\{U_i \subseteq U\}, \{V_i \subseteq V\}$, such that for each i, U_i and V_i map to a common Stein subdomain. By quasiseparatedness and the previous lemma, the fibre products $U_i \times_X V_i$ admit finite covers by Stein subdomains. Taking the union over i implies the condition of the lemma for g: that is, g is quasiseparated.

5.2.5 Definition of the Formalism

We may now introduce the six functor formalism of quasicoherent sheaves. As in [54], the definition is extended from derived Steins to complex stacks via right Kan extension. The previous lemmas on the Stein site, on QCoh, and on qcqs morphisms prove that QCoh defines a six functor formalism in which qcqs morphisms are exceptionally pushable.

On derived dagger Steins, QCoh simply gives the associated category of modules:

$$\mathrm{QCoh}(\mathrm{Spec}(A)) = \mathrm{Mod}_A \tag{198}$$

QCoh can be extended from derived dagger Steins to derived stacks by along the following diagram:

$$\mathrm{dAn}_k \hookrightarrow \mathrm{Sh}_{\mathrm{weak}}(\mathrm{dStn}_k^\dagger) \leftrightarrows \mathrm{Psh}(\mathrm{dStn}_k^\dagger) \hookleftarrow \mathrm{dStn}_k^\dagger \tag{199}$$

Here the right arrows are inclusions, the arrow from presheaves to sheaves is the sheafification and the embedding of derived dagger Steins is the Yoneda embedding. For any ∞-category admitting small limits, we obtain a series of functors:

$$\mathrm{Sh}(\mathrm{Sh}_{\mathrm{weak}}(\mathrm{dStn}_k^\dagger), \mathcal{D}) \hookrightarrow \mathrm{Fun}_0(\mathrm{Psh}(\mathrm{dStn}_k^\dagger)^{\mathrm{op}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{dStn}_k^\dagger, \mathcal{D}) \tag{200}$$

from precomposition of sheafification, and the fact that presheaves are the free cocompletion. The sheaves on weak sheaves are taken with respect to the effective epimorphism topology [33].

As QCoh is an weak sheaf, functorial under pullback, it belongs to $\mathrm{Sh}_{\mathrm{an}}(\mathrm{dStn}_k^\dagger, \mathrm{Comm}(\mathrm{Pr}_{\mathrm{St}}^L))$. Since the site is subcanonical, QCoh restricts along the embedding of the sequence (200) to a functor

$$\mathrm{QCoh} : \mathrm{Sh}_{\mathrm{weak}}(\mathrm{dStn}_k^\dagger) \to \mathrm{Comm}(\mathrm{Pr}_{\mathrm{St}}^L)) \tag{201}$$

This extension ultimately comes from the right Kan extension of QCoh along the embedding of derived Steins in analytic sheaves: for a complex analytic stack X:

$$\mathrm{QCoh}(X) \xrightarrow{\sim} \varprojlim_{Y \in \mathrm{dStn}_{\mathbb{C}}^{\mathrm{op}}/X} \mathrm{QCoh}(Y). \tag{202}$$

Thus we see the importance of the site being subcanonical.

QCoh further restricts from weak sheaves to derived analytic spaces. By construction the pullback functor f^* has been extended from derived Steins. Rather than extend f_* also we use categorical reasoning to show the right adjoint to f^* still exists: QCoh is a symmetric monoidal cocontinuous functor to presentable categories, and so the adjoint functor theorem implies formally that it has a right adjoint f_*. It must be demonstrated that this new pushforward satisfies the same pleasant properties as the base f_* (base change, projection formula etc.) when f is qcqs:

Lemma 5.10. *([54] Lemma 3.2) – Let $f : X \to Y$ be a qcqs morphism:*

1. *Base change equivalence: given a Cartesian square in* $\mathrm{dAn}_{\mathbb{C}}$*:*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

   the associated Beck-Chevalley transformation $g^* f_* \to f'_* g'^*$ is an equivalence.

2. *The projection formula holds: the canonical morphism $f_* \widehat{\otimes}_Y \mathrm{id} \to f_*(\mathrm{id} \widehat{\otimes}_X f^*)$ is an equivalence.*

3. *f_* is cocontinuous, and so by the adjoint functor theorem, each pushforward functor f_* for f qcqs admits a right adjoint $f^!$.*

Proof - A convenient intermediate step in proving the theorem is proving that the Beck-Chevalley morphism is an equivalence in the specific case that g is an inclusion of a dagger Stein. So, let $f : X \to Y$ be a qcqs map of derived analytic spaces, $g : Y' \to Y$ a dagger Stein subspace embedding, fitting into a Cartesian square as above. We follow the steps of ([34], Lemma 2.4.16):

Proof of intermediate lemma:

1. First assume that both X and Y are derived dagger Stein spaces. Choose a covering $\{V'_j \to Y\}_{\mathcal{J}}$. For each subset $J \subseteq \mathcal{J}$, let $V'_J = \bigcap_{j \in J} V'_j$, and let $U'_J = X' \times_{Y'} V'_J$ denote the pullback to X'. Then $g^* f_*$ may be expressed as the inverse limit of push-pull operators $g^*_J f_*$ as J ranges over the finite subsets of $\mathcal{J}$. The Beck-Chevalley transformation is an equivalence for these restricted operators, and so the inverse limit is an equivalence too. This proves the Beck-Chevalley equivalence in this case.

2. Assume instead that X is an analytic subspace of a dagger Stein space Z. Then X admits a finite cover by Stein subdomains of Z, and f_* is the inverse limit of pushforwards restricted to these subdomains. g^* commutes with finite limits, the Beck-Chevalley transformation is also an equivalence in this case.

3. For arbitrary X, there is a finite cover $\{U_i \to X\}$ by dagger Stein subdomains. Then $U_I = \bigcap_{i \in I} U_i$ is an analytic subspace of the Stein space U_i, and the pushforward is again an inverse limit of these pushforwards, and so the equivalence is proven in this case.

4. The lemma extends in much the same way for Y.

We now consider the general Beck-Chevalley transformation, for $g : Y' \to Y$ an arbitrary morphism, following the proof of proposition 2.4.21 of [34]:

1. Let $\mathcal{U}$ and $\mathcal{U}'$ be dagger Stein covers of Y and Y' respectively, such that $\mathcal{U}'$ refines the pullback of $\mathcal{U}$ along g. The collection of pullback functors along the inclusions of $\mathcal{U}'$ form a conservative family, and so the Beck-Chevalley equivalence need only be checked after pullback to the Stein subdomains. By the convenient intermediate lemma, we can therefore reduce to the case that both Y and Y' are affine.
2. By the characterization of quasicompact and quasiseparated morphisms, as f and f' are qcqs maps to affines, X and X' are also qcqs.
3. By the same reasoning as in the intermediate lemma, by descent to rational covers, the Beck-Chevalley equivalence is extended sequentially from affine X to analytic subspaces to affines and then to arbitrary X. This concludes the proof of the Beck-Chevalley equivalence.

The projection formula and cocontinuity of pushforwards also reduces to the affine case by equivalent reasoning.

Proposition 5.2. - *QCoh defines a six-functor formalism on the category of derived analytic stacks, with qcqs morphisms as exceptionally pushable morphisms.*

Proof - The properties of qcqs morphisms imply that (Equivalences, qcqs) defines a *suitable decomposition* in the sense of ([34] Definition A.5.9). The previous proposition proves that this decomposition satisfies the conditions for ([34] Proposition A.5.10), and so it follows that QCoh indeed defines a six functor formalism. Moreover, because the qcqs morphisms form the projective morphisms of the suitable decomposition, they all satisfy $f_! = f_*$. Q.E.D

Remark - In defining the six-functor formalism $\mathrm{QCoh} : \mathrm{Corr}(\mathrm{dStn}_k^{\dagger}, \mathrm{qcqs}) \to$, we restricted to derived analytic spaces from weak sheaves, and proposed a class qcqs with respect to which exceptional pushforwards can be defined. We could of course have instead constructed the full six-functor formalism on weak sheaves:

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Sh}_{\mathrm{weak}}(\mathrm{dStn}_k^{\dagger}), \tilde{E}) \to \mathrm{Pr}^L_{\mathrm{st}}. \tag{203}$$

Here, $\tilde{E}$ is the class of exceptionally-pushable morphisms, containing representable maps and satisfying the assumptions afforded by the extension formalism. The following theorem shows that the extension of $(\mathrm{dAn}_k^{\dagger}, qcqs)$ is unique:

Theorem 5.2. *([54] Theorem 4.40) - Every qcqs morphism between derived analytic spaces is in the class E, and thereby there is a map of geometric setups*

$$(\mathrm{dAn}_k, \mathrm{qcqs}) \to (\mathrm{Sh}_{\mathrm{weak}}(\mathrm{dStn}^{\dagger}), E) \tag{204}$$

and the restriction of QCoh *along this map is exactly the formalism on derived analytic spaces defined above. In particular, it follows that* $f_! \simeq f_*$ *for all qcqs* f.

5.3 Properties of the Formalism

Following sections 4.6-4.9, we discuss properties of the formalism QCoh on derived analytic spaces. The first section discusses some particularly important examples of infinite covers satisfying !-descent, ensuring that the formalism is workable for a broad class of spaces. Local cohomology is then examined, and as a side product we discover a particularly important class of exceptionally-pushable open embeddings. The last section discusses Zariski-closed and open immersions, which are particularly well-behaved and convenient for the infinitesimal constructions to follow.

5.3.1 Examples of !-Descent

We demonstrate !-descent for finite covers by quasicompact analytic subspaces, and for infinite covers of a particular kind. This in particular applies to coverings of affine space by discs of increasing radius.

Lemma 5.11. *([54] Lemma 4.41, Corollary 4.42) - Finite covers* $\{t_i : U_i \to X\}_{i \in I}$ *by analytic subspaces, such that all inclusions* $t_I : \bigcap_{i\in I} U_i \to X$ *are quasicompact, define morphisms* $\bigsqcup_{i\in I} U_i \to X$ *of universal* !*-descent.*

Proof - The unit 1_X decomposes as the inverse limit $\varprojlim_I t_{I*}$, and as the t_I are qcqs, $t_I = t_*$. Therefore by adjunction

$$\begin{aligned} M &\simeq \underline{\mathrm{Hom}}_X(1_X, M) \\ &\simeq \underline{\mathrm{Hom}}_X(\varprojlim_I t_{I*}, M) \\ &\simeq \varinjlim_I \underline{\mathrm{Hom}}_X(t_{I*}, M) \\ &\simeq \varinjlim_I \underline{\mathrm{Hom}}_X(1_{U_I}, t_I^! M) \\ &\simeq \varinjlim_I t_{I*} t_I^! M. \end{aligned}$$

Note that the limits are finite, and so commute with Hom, yielding the third equivalence. This shows that the counit of the adjunction between $\mathrm{QCoh}^!(X)$ and $\varprojlim_I \mathrm{QCoh}^!(U_i)$ is an equivalence. The unit is also an equivalence by splitness of the cover after base change.

The canonical morphism

$$\mathrm{QCoh}^!(X) \to \varprojlim_{m\in\Delta} \mathrm{QCoh}^!(Y^{\times_X (m+1)}) \tag{205}$$

is an equivalence by the Lurie-Beck-Chevalley condition (Higher Algebra, 4.7.5.3). We briefly recall the statement: let $\mathcal{C}^\bullet : N(\Delta_+) \to \mathrm{Cat}_\infty$ be an augmented cosimplicial $(\infty, 1)$-category, with $G : \mathcal{C}^{-1} \to \mathcal{C}^0$ the canonical functor. Suppose $\mathcal{C}^{-1} = \mathcal{C}^\bullet(-1)$ admits geometric realizations of G-split simplicial objects, those geometric realizations are preserved by G, and all the transition morphisms in the system $\mathcal{C}^\bullet$ are left-adjointable.

Then the canonical map $\theta : \mathcal{C}^{-1} \to \varprojlim_{n\in\Delta} \mathcal{C}^n$ admits a fully faithful left adjoint, which is moreover an equivalence when G is conservative. This is a consequence of the Lurie-Barr-Beck theorem, recalled in the appendix.

Let (M_m) be a $t^!$-split simplicial object in $\mathrm{QCoh}(X)$. Then in particular it is $t_*t^!$-split, and moreover $t_{I*}t_I^!$-split for all I. Therefore the system of skeleta $(\mathrm{sk}_k(t_{I*}t_I^* M_\bullet))_{k\geq 0}$ is Ind-constant, and this system is equivalent, by the equivalence above, to the system $(\mathrm{sk}_k M_\bullet)_{k\geq 0}$. In this case it is clear that $t^!$ commutes with the geometric realization of $M_\bullet$.

The second condition of the Lurie-Barr-Beck theorem follows from base change and conservativity of $t^!$. Q.E.D

Let X be a derived analytic space, $S \subseteq |X|$ a closed subset with open complement $V = |X| \setminus S$, again regarded as an analytic subspace $j : V \hookrightarrow X$. $\Gamma_S\mathrm{QCoh}$ is defined as before, and we define $\mathrm{L}_S\mathrm{QCoh}$ to be the full subcategory of $\mathrm{QCoh}(X)$ spanned by objects M with $j^! M \simeq 0$ ([54], Section 4.6).

Lemma 5.12. *([54] 4.46-4.49) - Let $X \in \mathrm{dAn}_k$, and let $V_1 \supseteq V_2 \supseteq \cdots$ and $U_1 \subseteq U_2 \subseteq \cdots$ be sequences of analytic subspaces. Assume the morphisms $V_n \to X$ and $U_n \to X$ are quasicompact, $U_n \cup V_n = X$, and $U_n \cap V_{n+1} = \emptyset$. Write $S_n = |X| \setminus V_n$ for the closed complement of V_n, and write $\Gamma_n\mathrm{QCoh}(X)$ for $\Gamma_{S_n}\mathrm{QCoh}(X)$:*

1. *There is an equivalence of pro-systems*

$$``\varprojlim_n" \mathrm{QCoh}^!(U_n) \simeq ``\varprojlim_n" \Gamma_n \mathrm{QCoh}(X), \tag{206}$$

2. *If in addition $U_n \cap V_n$ is quasicompact, we get an equivalence of pro-systems*

$$``\varprojlim_n" \mathrm{QCoh}^*(U_n) \simeq ``\varprojlim_n" L_n \mathrm{QCoh}(X) \tag{207}$$

3. *If $X = \bigcup_{i\in\mathbb{Z}_{\geq 1}} U_i$, then $\mathrm{QCoh}^!(X) \to \varprojlim_n \mathrm{QCoh}^!(U_n)$ is an equivalence. In particular, $\bigsqcup_{i\geq 1} U_i \to X$ is of universal $!$-descent.*

Example - The main example to have in mind is the cover of the affine line by a family $\{U_i\}$ of overconvergent discs of radius (centred at the origin, say), and the V_n overconvergent annuli of suitably-chosen complementary radii. For example, the U_n can be chosen to be overconvergent discs of radius n, and the V_n can be overconvergent annuli of radii $(n - \frac{1}{2}, \infty)$.

Proof - The proof of the equivalences (206) and (207) are similar, so we only present the first. Write $j_n : V_n \hookrightarrow X$, $t_n : U_n \hookrightarrow X$, and $\mathrm{incl}_n : \Gamma_n\mathrm{QCoh}(X) \hookrightarrow \mathrm{QCoh}(X)$. The projective systems factor through each other:

$$\begin{tikzcd}
& \Gamma_{n+1}\mathrm{QCoh}(X) \arrow[dr, "t^!\mathrm{incl}_{n+1}"'] & \mathrm{QCoh}(U_{n+1}) \arrow[l, "\Gamma_{n+1}t_{n+1,*}"'] & \cdots \arrow[l] \\
\cdots & \Gamma_n\mathrm{QCoh}(X) \arrow[l] & \mathrm{QCoh}(U_n) \arrow[l, "\Gamma_n t_{n,*}"] &
\end{tikzcd}$$

It is therefore sufficient to verify that this projective system is equivalent to both: i.e the compositions of pairs of consecutive maps of this system are equivalent to the maps of the individual systems:

$$t_n^!\mathrm{incl}_{n+1}t_{n+1,*} \simeq t_{n,n+1}^! \tag{208}$$

$$\Gamma_n t_{n,*}t_n^!\mathrm{incl}_{n+1} \simeq \Gamma_{n,n+1} \tag{209}$$

Now by the assumption $U_n \cap V_{n+1} \simeq 0$ we get $j_{n+1}^*t_{n,*} \simeq 0$. Therefore $t_{n,*}$ factors through $\Gamma_{n+1}\mathrm{QCoh}$, and so $t_{n,*}$, viewed as a functor to $\Gamma_{n+1}\mathrm{QCoh}(X)$, is left adjoint to $t_n^!\mathrm{incl}_{n+1}$. So the required equivalence (208) and (209) can be checked after passing to adjoint functors:

$$t_{n+1}^*\mathrm{incl}_{n+1}t_{n,*} \simeq t_{n,n+1,*} \tag{210}$$

$$t_{n,*}t_n^*\mathrm{incl}_n \simeq \mathrm{incl}_{n,n+1}. \tag{211}$$

The first follows from base change, and the second from descent along the cover $X = U_n \cup V_n$.

By the first lemma, the last lemma follows from the equivalence of $\mathrm{QCoh}(X)$ with the limit of local cohomologies on the S_n:

$$\mathrm{QCoh}(X) \simeq \varprojlim_n \Gamma_n\mathrm{QCoh}(X) \tag{212}$$

Here $M \mapsto (\Gamma_n M)_n$, and this functor is right adjoint to $(M_n)_n \mapsto \varinjlim M_n$. The counit of this adjunction is given by

$$\varinjlim_n \mathrm{incl}_n\Gamma_n M \to M. \tag{213}$$

This is an equivalence: to verify this it is sufficient to pull back along t_m^* for all m by descent. But t_m^* commutes with colimits, and $t_m^* \mathrm{incl}_n \Gamma_n M \cong t_m^* M$ for n sufficiently large, as required. in an entirely similar way, (209) is an equivalence: it must be checked that

$$M_m \mapsto \Gamma_m(\varinjlim_n \mathrm{incl}_n M_n) \tag{214}$$

is an equivalence. But as Γ_m also commutes with colimits (as a right adjoint), we need only see that $M_m \simeq \varinjlim_n \Gamma_m \mathrm{incl}_n M_n$.

5.3.2 Local Cohomology

The six-functor formalism QCoh : $(\mathrm{dAn}_k, \mathrm{qcqs})$ provides the flexibility to speak of sections that vanish on an open subset, by imposing that the pullback along the embedding of that open subset vanishes. This motivates the following definitions:

Definition 5.5. ([54] Section 4.7) - Let $X \in \mathrm{dAn}_k$, and let $S \subseteq |X|$ be a closed subset of the underlying space of X. Let $U = |X| \setminus S$. Recall that such complementary opens determine and are determined by an open analytic subspace which we will also denote by U.

The category of quasicoherent sheaves *local to S* is the full subcategory

$$\mathrm{incl}_S : \Gamma_S \mathrm{QCoh}(X) \hookrightarrow \mathrm{QCoh}(X) \tag{215}$$

spanned by those $M \in \mathrm{QCoh}(X)$ such that $j^* M \simeq 0$. j^* is cocontinuous, and therefore so is incl_S: an inductive limit of objects vanishing under j^* will also vanish under j^*. Therefore by the adjoint functor theorem incl_S has a right adjoint:

$$\mathrm{incl}_S : \Gamma_S \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}(X) : \Gamma_S. \tag{216}$$

Assume that j belongs to the class of exceptionally-pushable morphisms. Then there is a canonical map $j^! \to j^*$:

$$j^! \to j^! j_* j^* \simeq j^*. \tag{217}$$

It is the composition of the unit morphism $\mathrm{id} \to j_* j^*$ and base change. Assume for the rest of this chapter that the morphism (217) is an equivalence, so that Γ_SQCoh can equivalently be described as those M for which $j^! M \simeq 0$. Since $j^!$ is *continuous*, then incl_S is similarly seen to be continuous and therefore admits a *left* adjoint:

$$\mathrm{L}_S : \Gamma_S \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}(X) : \mathrm{incl}_S. \tag{218}$$

Proposition 5.3. *([54] Proposition 4.51) - Under the assumptions above:*

1. *$\Gamma_S \simeq \mathrm{Fib}(\mathrm{id} \to j_*j^*)$*
2. *$\mathrm{L}_S \simeq \mathrm{Cofib}(j_!j^! \to \mathrm{id}) \simeq \mathrm{Cofib}(j_!j^!1_X \to 1_X)\widehat{\otimes}_X\mathrm{id}$. In particular, as the functor is given by tensoring with an idempotent algebra, this equips $\Gamma_S\mathrm{QCoh}(X)$ with a symmetric monoidal structure, with tensor unit $\mathrm{Cofib}(j_!j^!1_X \to 1_X)$.*
3. *$\Gamma_S\mathrm{incl}_S \simeq \mathrm{id}$ and $j^*j_* \simeq \mathrm{id}$.*
4. *$L_Sj_! \simeq 0$, $j^!j_! \simeq \mathrm{id}$ and $L_S\mathrm{incl}_S \simeq \mathrm{id}$.*

Proof -

1. Let F denote $\mathrm{Fib}(\mathrm{id} \to j_*j^*)$. Then

$$j^*F = \mathrm{Fib}(j^* \to j^*j_*j^*) \simeq \mathrm{Fib}(j^* \to j^!j_*j^*) \simeq \mathrm{Fib}(j^* \to j^*) \simeq 0 \quad (219)$$

   The second equivalence is given by the equivalence $j^! \simeq j^*$, as j is a subspace immersion, and the third is by base change. $j^*\mathrm{incl}_S \simeq 0$ by definition, and so $F\mathrm{incl}_S \simeq \mathrm{id}$.Then F is realized as the right adjoint of incl_S: the counit is induced by the canonical morphism $\mathrm{Fib}(\mathrm{id} \to j_*j^*)$, and the unit by the equivalence $\mathrm{id} \simeq F\mathrm{incl}_S$.

2. The proof that $\mathrm{L}_S \simeq \mathrm{Cofib}(j_!j^! \to \mathrm{id})$ is formally similar to the equivalence above. The second equivalence follows from the following:

$$j_!j^!1_X\widehat{\otimes}_X\mathrm{id} \simeq j_!(j^!1_X\widehat{\otimes}_Uj^*) \simeq j_!j^! \quad (220)$$

   The first equivalence is the projection formula, and the second follows since $j^!(= j^*)$ is symmetric monoidal. Then the remaining statement follows from the expression of the idempotent monad $\mathrm{incl}_S\Gamma_S$ as tensoring with the idempotent algebra object $\mathrm{Cofib}(j_!j^!1_X \to 1_X)$.

3. The remaining equivalence $j^*j_* \simeq \mathrm{id}$ follows from $j^! \simeq j^*$ and base change.

4. The equivalence $L_Sj_! \simeq 0$ follows from the adjoint equivalence $j^!\mathrm{incl}_S \simeq 0$, and $j^!j_! \simeq$ similarly follows from $j^! \simeq j^*$ and base change.

Example ([54] Proposition 4.52) - Let $X \in \mathrm{dAn}_k$, $S \subseteq |X|$ a closed subset of the underlying space, with open complement U. Suppose U admits an open exhaustive filtration $U_1 \subseteq U_2 \subseteq \cdots \subseteq U$ and S has a shrinking series of open neighbourhoods $V_1 \supseteq V_2 \supseteq \cdots \supseteq S$ such that

1. The maps $k_n : U_n \to X$, $l_n : V_n \to X$ are quasicompact.
2. $U_n \cup V_n = X, U_n \cap V_{n+1} = \emptyset$.

Then:

1. The inclusion $j : U \to X$ admits an exceptional pushforward.
2. There are equivalences

$$\varinjlim k_{n,*}k_n^* \simeq \mathrm{Cofib}(j_!j^! \to \mathrm{id}) \tag{221}$$

$$\varprojlim k_{n,*}k_n^! \simeq \mathrm{Fib}(\mathrm{id} \to j_*j^*) \tag{222}$$

3. The canonical morphism $j^! \simeq j^*$ is an isomorphism.

The natural setting is a shrinking family of open analytic neighbourhoods V_n of a zero locus in an affine space $\mathrm{Spec}(A)$, and appropriately defined annuli U_n.

5.3.3 Zariski Closed and Open Immersions

Zariski-closed immersions are a particularly well-behaved class of closed embeddings, with respect to which constructions such as germs are particularly well-behaved. Zariski-open immersions are their open complements, and we show that they are exceptionally pushable, and their ordinary and exceptional pullbacks agree ([54] Section 4.9).

Definition 5.6. ([54] Definition 4.54) - A morphism $f : X \to Y$ is a *Zariski closed immersion* if it is locally represented by closed immersions: there is a cover $\{U_i \to Y\}$ of Y by dagger Stein subspaces, such that each $X \times_Y U_i$ is a derived dagger Stein, and the pullback maps $X \times_Y U_i \to U_i$ correspond to maps of derived dagger Stein algebras which are surjective on π_0.

Lemma 5.13. *([54] Lemma 4.55) - Zariski closed immersions are stable under composition and base change, and are always quasicompact.*

Proof - The stability is by the same reasoning as for quasicompact and quasiseparated morphisms. They are quasicompact by the characterization of quasicompact morphisms as locally quasicompact on affine subspaces of the target ([54] Lemma 2.22).

Lemma 5.14. *([54] Lemma 4.56) - The underlying map $|f| : |X| \to |Y|$ of a Zariski closed immersion has closed image.*

Proof - This reduces immediately to the affine case, which follows from Stein duality. Q.E.D

Definition 5.7. ([54]) - A morphism $f : X \to Y$ of derived analytic spaces is a *Zariski-open immersion* if it is the open complement of a Zariski-closed immersion.

Proposition 5.4. *([54] Proposition 4.58) - Zariski-open immersions $j : X \to Y$ admit exceptional pushforwards and satisfy $j^* \simeq j^!$.*

5.4 Infinitesimal Theory

In this section we develop the theory of quasicoherent sheaves on infinitesimal objects to define crystals and $\mathcal{D}$-modules of derived stacks. Because some of these objects cannot be expressed as derived analytic spaces, we need to work in the broader context of algebraic geometry relative to $D_{\geq 0}(\mathrm{CBorn}_k)$.

5.4.1 Čech Stacks and Betti Stacks

Betti stacks are the stackifications of constant prestacks valued in a homotopy type. They encode local systems on spaces with these homotopy types, as quasicoherent sheaves on Betti stacks K_{Betti} correspond to functors $K \to \mathrm{Mod}_{\mathbb{C}}$. We first review the theory of Čech stacks from section 6.2.1 of [7], and then recall the theory of Betti stacks from [40], and reproduce results of [51] relating them to de Rham stacks and crystals.

Definition 5.8. (Čech Stacks, [7], Definition 6.2.7) - Let $(\mathcal{C}, \tau)$ be an infinity site. A *Čech cover* in $\mathcal{C}$ is a hypercover $K_\bullet \to U$ arising as the nerve of some cover of τ.

The category of *Čech stacks* $\breve{\mathrm{Stk}}(\mathcal{C}, \tau) \subseteq \mathrm{PreStk}(\mathcal{C}, \tau)$ is the subcategory of the category of groupoid-valued prestacks that satisfies descent along Čech covers.

Like ordinary stacks, Čech stacks sit as a full reflective subcategory of the category of prestacks on M:

$$\breve{i} : \breve{\mathrm{Stk}}(\mathcal{C}, \tau) \leftrightarrows \mathrm{PreStk}(\mathcal{C}) : \breve{\pi} \tag{223}$$

The left adjoint $\mathrm{PreStk}(\mathcal{C}) \to \breve{\mathrm{Stk}}(\mathcal{C}, \tau)$ is the *Čech stackification functor*. For many topologies of interest, it is not immediately clear that they are subcanonical, but much easier to prove that they are for Čech covers (that is to say, the functors representing objects of the site descend along Čech covers.)

Definition 5.9. (Betti Stacks, [40], Section 3.1) - Let $\mathcal{S}$ denote the $(\infty, 1)$-category of spaces, and dAn_k the $(\infty, 1)$-site of derived k-Steins, equipped with the weak topology. There is an adjunction

$$\pi^* : \mathcal{S} \leftrightarrows \mathrm{PreStk}(\mathrm{dStn}_k) : \pi_* \tag{224}$$

π^* sends a space S to the constant prestack valued S, and π_* sends a prestack to its value on k, the base field.

The *Betti stack* S_{Betti} associated to a space S is the Čech stackification of the constant prestack valued K:

$$S_{\mathrm{Betti}} = \breve{\pi}\pi^*(K). \tag{225}$$

For a prestack X, we use the shorthand $X_{\text{Betti}} = X(k)_{\text{Betti}} = \breve{\pi}\pi^*\pi_*(X)$.

1. $*_{\text{Betti}} = \mathrm{Spec}(\mathbb{C})$.
2. Quasicoherent sheaves on Betti stacks K_{Betti} correspond to representations of S - for a derived complex space X (or perhaps even a weak sheaf on $\mathrm{dStn}_{\mathbb{C}}$):

$$\mathrm{QCoh}(S_{\text{Betti}} \times X) \cong \mathrm{Fun}(K, \mathrm{QCoh}(X)). \tag{226}$$

Proof - The first point is clear since Spec(k) represents the constant prestack valued $*$, which already satisfies Čech descent. Alternatively, as in [40], one can show that the adjunction between spaces and Čech stacks defines a geometric morphism of $(\infty, 1)$-topoi, and therefore the Betti stackification preserves initial objects.

K may be expressed as a colimit of its points over its fundamental groupoid $\pi(S)$. We may then make the following manipulations:

$$\begin{aligned}
\mathrm{QCoh}(S_{\text{Betti}}) &\cong \mathrm{QCoh}((\varinjlim_{\pi(K)} *)_{\text{Betti}}) \\
&\cong \mathrm{QCoh}(\varinjlim_{\pi(S)} *_{\text{Betti}}) \\
&\cong \varprojlim_{\pi(S)} \mathrm{QCoh}(*_{\text{Betti}}) \\
&\cong \varprojlim_{\pi(S)} \mathrm{QCoh}(\mathrm{Spec}(\mathbb{C})) \\
&\cong \varprojlim_{\pi(S)} \mathrm{Mod}_{\mathbb{C}} \\
&\cong \mathrm{Fun}(S, \mathrm{Mod}_{\mathbb{C}})
\end{aligned}$$

The second isomorphism follows from the commutativity of the Betti stack functor with colimits, and the third by commutativity of QCoh with small colimits. We may repeat these manipulations by replacing S_{Betti} with $S_{\text{Betti}} \times X$ to similarly obtain the isomorphism $\mathrm{QCoh}(K_{\text{Betti}} \times X) \cong \mathrm{Fun}(K, \mathrm{QCoh}(X))$. Q.E.D

Remarks - In the case that X is an affine scheme we see the connection between Betti stacks and local systems:

$$\mathrm{QCoh}(K_{\text{Betti}} \times \mathrm{Spec}(\mathrm{A})) \cong \mathrm{Fun}(K, \mathrm{Mod}_A). \tag{227}$$

In particular for underived rings A, such functors correspond to representations of the fundamental groupoid of K, and thereby local systems on K.

We now investigate maps to Betti stacks. This is a modification of ([51] proposition II.1.3) to our setting:

Proposition 5.5. - *Let $S \in \mathcal{S}$ be a finite homotopy type. Maps $X \to S_{\text{Betti}}$ determine and are determined by the pullback functor*

$$f^* : \text{QCoh}(S_{\text{Betti}}) \to \text{QCoh}(X). \tag{228}$$

These functors are characterized by being k-linear [2]*, cocontinuous and symmetric monoidal, and satisfying weak Čech descent.*

General QCoh($*$)-linear cocontinuous symmetric monoidal functors QCoh(S_{Betti}) $\to$ QCoh(X) correspond to maps of corresponding locales of idempotent algebras. The idempotent algebras of QCoh(S_{Betti}) correspond to closed subsets of S, and so such a functor is the specification of idempotent algebras $A_Z \in \text{QCoh}(X)$ for closed $Z \subseteq X$. This corresponds to a map $X \to S_{\text{Betti}}$ if and only if it satisfies weak Čech descent.

Proposition 5.6. *(Analogue of [51] II.1.4) - For a complex analytic space X, the unit $X \to X(\mathbb{C})_{\text{Betti}}$ is surjective.*

Proof - Regard X and $X(\mathbb{C})$ as their functors of points. The adjunction (**??**) yields the following for any derived stacks X, Y:

$$\begin{aligned} \text{Hom}(X(\mathbb{C})_{\text{Betti}}, Y) &\cong \text{Hom}(\pi^*\pi_*X, Y) \\ &\cong \text{Hom}(\pi_*X, \pi_*Y) \cong \text{Hom}(X(\mathbb{C}), Y(\mathbb{C})). \end{aligned}$$

The statement will then follow if the precomposition $\text{Hom}(X(\mathbb{C})_{\text{Betti}}, Y) \to \text{Hom}(X, Y)$ is injective, and this follows if every homotopy class of maps between homotopy types of global sections of X and Y arises from an analytic map $X \to Y$. By Weierstrass theory this can even be achieved with polynomials, and so can certainly be done with analytic maps. Q.E.D

Remark - In [51] the result follows formally from the conventions of condensed mathematics. In our context some nontrivial function theory is necessary.

5.4.2 Germs, Internal Groupoids and the ∞-de Rham Stack

Definition 5.10. ([54] Definition 4.60) - Let $X = \text{Spec}(A) \in \text{dStn}^{\dagger}$ be a derived dagger Stein domain, and let $Z = \text{Spec}(B) \hookrightarrow X$ be a closed immersion. The *algebra of germs along Z in X* is the colimit of algebras of open neighbourhoods of Z:

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[2] By this we mean linear over the higher category of k-modules, or equivalently, quasicoherent sheaves on the point, QCoh($*$).

$$A^\dagger_Z = \varinjlim_{U \supseteq Z} A_U. \tag{229}$$

Remarks - In [54] the basic algebraic objects are derived affinoid algebras, which are not closed under colimits of this form. However in our setting of overconvergent analysis, this algebra may be obtained from a cofinal system of Weierstrass localizations, and therefore remains an algebra in the Lawvere theory generated by Stein algebras.

This observation causes some results of [54] to follow more easily. For example, ([54], lemma 4.66) proves $*$-descent for quasicoherent sheaves on the germ of Z along affinoid covers of $|Z|$. But in our setting a derived dagger Stein cover $\{U_i\}$ of $|Z|$ defines a derived dagger Stein cover $\{U_i^\dagger\}$ of the germ of Z, which follows from ordinary $*$-descent.

Lemma 5.15. - *The embedding* $(Z \subseteq X)^\dagger \hookrightarrow X$ *is a homotopy monomorphism.*

Proof - The corresponding map of algebras is a homotopy epimorphism by the results of [5]. Alternatively, as in [54], we can express $A \to A^\dagger_Z$ as a colimit of homotopy epimorphisms. Q.E.D

Remark - With the notations above, the quasicoherent sheaves on the germ are given by the local cohomology of X at Z:

$$\mathrm{QCoh}((Z \subseteq X)^\dagger) \simeq \Gamma_Z \mathrm{QCoh}(X). \tag{230}$$

as both are the modules over $A^\dagger_Z$.

Definition 5.11. ([54] Definition 4.64) - The category Pairs is defined as the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{dStn}^\dagger)$ of maps $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ such that $\pi_0(A) \to \pi_0(B)$ is surjective.

Lemma 5.16. *([54] Lemma 4.65) - The functor* Pairs $\to \mathrm{dStn}^\dagger$*, sending a pair* $Z \to X$ *to the germ* $(Z \subseteq X)^\dagger$ *preserves fibre products.*

Proof - First note that the category of pairs does indeed have fibre products:

$$(Z \to X) \times_{(Z'' \to X'')} (Z' \to X') = (Z \times_{Z''} Z' \to X \times_{X''} X'). \tag{231}$$

The method of proof is then equivalent to that of [54] - write $(Z \to X) \to (\mathrm{Spec}(B) \to \mathrm{Spec}(A))$, $(Z' \to X') \to (\mathrm{Spec}(B') \to \mathrm{Spec}(A'))$, $(Z'' \to X'') \to (\mathrm{Spec}(B'') \to \mathrm{Spec}(A''))$. The ideals defining the $\pi_0(B)$ and $\pi_0(B')$ are finite ideals $(f_1, \cdots, f_n)$ and $(f'_1, \cdots, f'_m)$, respectively. Then the quotient map $\pi_0(A \otimes_{A''} A) \to \pi_0(B \otimes_{B''} B)$ is given by the ideal

$$(f_1 \otimes 1, \cdots, f_n \otimes 1, 1 \otimes f'_1, \cdots, 1 \otimes f'_m)). \tag{232}$$

Cofinal systems for open neighbourhoods of Z and Z' are given by neighbourhoods for which the f_i and f'_i, respectively, take shrinking values. Let V_n, V''_n be such system of open neighbourhoods. Then $V_n \times_X V'_n$ defines a system of shrinking neighbourhoods of $Z \times_{Z''} Z'$. The equality

$$\varinjlim_n A_{V_n} \widehat{\otimes}^{\mathbb{L}}_{A''} A'_{V'_n} = \varinjlim_n (A \widehat{\otimes}^{\mathbb{L}}_{A''} A')_{V_n \times_X V''_n}. \tag{233}$$

demonstrates that germ respects the pushout. Q.E.D

Definition 5.12. - For X a derived analytic space, the *germ of the diagonal* is the colimit over all open analytic neighbourhoods of the diagonal, just as for dagger Steins.

Remark - The local identification of the germ $\Delta(X)^\dagger$ with the spectrum of the algebra of overconvergent functions, furnishes an atlas of $\Delta(X)^\dagger$, realizing it as a derived analytic space.

Despite their concrete description in our setting, it remains convenient to study germs and de Rham stacks using the formal constructions of chapters 4.12-13 of [54]:

Definition 5.13. -

1. Let $\mathcal{C}$ be an $(\infty, 1)$-category admitting all fibre products. A *groupoid object* in $\mathcal{C}$ is a simplicial object of $\mathcal{C}$, such that for all subsets $S, S' \subseteq [n]$ with $S \cup S' = [n]$ and $S \cap S' = *$, the following diagram is Cartesian:

$$\begin{tikzcd} X([n]) \arrow[r] \arrow[d] & X(S) \arrow[d] \\ X(S') \arrow[r] & X(*) \end{tikzcd}$$

   A groupoid object $X_\bullet$ is *effective* if it is equivalent to the nerve of $X_0 \to X_{-1}$, where $X_{-1} = \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} X_n$. In any $(\infty, 1)$-topos, all groupoid objects are effective: this is one of the Giraud-Rezk-Lurie axioms characterizing $(\infty, 1)$-topoi ([32] Theorem 6.1.0.1.6).

2. Let $\mathcal{C}$ be an $(\infty, 1)$-category admitting fibre products. A morphism $X_\bullet \to Y_\bullet$ is a *homotopy Kan fibration* if it satisfies the following horn-filler condition: for all $n \geq 1$ and $0 \leq k \leq n$ the canonical map

$$X(\Delta^n) \to X(\Lambda^n_k) \times_{Y(\Lambda^n_k)} Y(\Delta) \tag{234}$$

is an effective epimorphism. For $X_\bullet$ and $Y_\bullet$ groupoid objects, it is sufficient to verify this in the case $n = 1$ since for $n \geq 2$, $X(\Delta^n) \simeq X(\Delta^n_k)$ from the underlying homotopy.

3. In any $(\infty, 1)$-topos $\mathcal{C}$, pullbacks along homotopy Kan fibrations respect geometric realizations: for $X_\bullet \to Z_\bullet$ and $Y_\bullet \to Z_\bullet$ maps of s$\mathcal{C}$, with $X_\bullet \to Z_\bullet$ a homotopy Kan fibration, the canonical map

   $$|X_\bullet \times_{Z_\bullet} Y_\bullet| \cong |X_\bullet| \times_{|Z_\bullet|} |Y_\bullet|. \tag{235}$$

   For ∞-groupoids this is proven in ([36], Corollary 6.7). From this the result extends to presheaves since colimits are computed objectwise, and from there it extends to localizations of presheaf categories left exactness. It therefore applies to all $(\infty, 1)$-topoi ([54] Proposition 4.73, [32] Theorem 6.1.0.1.6).

   We can define a model structure on s$\mathcal{C}$, characterized by taking fibrations to be homotopy Kan fibrations, and weak equivalences those morphisms whose geometric realizations are weak equivalences. Then using the resulting weak factorization systems, given a pair of maps $X_\bullet \to Z_\bullet$ and $Y_\bullet \to Z_\bullet$ of s$\mathcal{C}$, we can factorize $X_\bullet \to Z_\bullet$ as $X_\bullet \to X'_\bullet \to Z_\bullet$, where the first map is a weak equivalence, and the latter is a homotopy Kan fibration. The fibre product of geometric realizations can then be computed as follows, by the formula (235):

   $$|X_\bullet| \times_{|Z_\bullet|} |Y_\bullet| \simeq |X'_\bullet| \times_{|Z_\bullet|} |Y_\bullet| \simeq |X'_\bullet \times_{Z_\bullet} Y_\bullet|. \tag{236}$$

Definition 5.14. ([54] Definition 4.75) - For $f : X \to Y$ a map of derived analytic spaces, the *infinitesimal groupoid of* f, $\mathrm{Inf}(X/Y)$ is the simplicial object of analytic spaces with components:

$$\mathrm{Inf}(X/Y)_n = (X \subseteq X^{n+1/Y})^\dagger \tag{237}$$

and the evident structural morphisms. The ∞-*de Rham stack* of f is the colimit $(X/Y)_{\mathrm{dR}_\infty} = \varinjlim_{n \in \Delta^{\mathrm{op}}} \mathrm{Inf}(X/Y)_n$. The infinitesimal groupoid and ∞-de Rham stack of X are defined to be the relative notions for $X \to *$.

Examples and Remarks -

1. The constructions define functors

   $$\mathrm{Inf}(-/Y) : \mathrm{dAn}^\dagger_{k/Y} \to \mathrm{sdAn}^\dagger_{k/Y},\ (-/Y)_{\mathrm{dR}_\infty} : \mathrm{dAn}_{k/Y} \to \mathrm{dAn}_{k/Y}. \tag{238}$$

2. Since dAn_k is an $(\infty,1)$-topos, we see that a map $X \to Y$ over Z defines a homotopy Kan fibration $\mathrm{Inf}(X/Z) \to \mathrm{Inf}(Y/Z)$ if and only if the morphisms

$$\mathrm{Inf}(X/Z)(\Delta^1) \to \mathrm{Inf}(X/Z)(\Lambda^1_i) \times_{\mathrm{Inf}(Y/Z)(\Lambda^1_i)} \mathrm{Inf}(Y/Z)(\Delta^1) \tag{239}$$

are effective epimorphisms, for $i = 0, 1$. These maps correspond to the maps $(X \subseteq X \times_Z X)^\dagger \to (X \subseteq X \times_Z Y)^\dagger$ induced by (id, f), and $(X \subseteq X \times_Z X)^\dagger \to (X \subseteq Y \times_Z X)^\dagger$ induced by (f, id). Both of these maps can be regarded as the *graph* of f.

3. ∞-de Rham stacks are compatible with homotopy Kan fibrations: let $f : X \to Z$ and $g : Y \to Z$ be maps in dAn_k, with $\mathrm{Inf}(X) \to \mathrm{Inf}(Z)$ a homotopy Kan fibration. Let $X' = X \times_Z Y$. Then
   (a) $(X/Z)_{\mathrm{dR}_\infty} \to X_{\mathrm{dR}_\infty} \times_{Z_{\mathrm{dR}_\infty}} Z$ is an equivalence, where $Z \to Z_{\mathrm{dR}_\infty n}$ is induced by the map $Z \to \mathrm{Inf}(Z)$ given by diagonal maps:

   $$Z \times_{\mathrm{Inf}(Z)_n} \mathrm{Inf}(X)_n \simeq (X \subseteq X^{n/Z})^\dagger = \mathrm{Inf}(X/Z)_n. \tag{240}$$

   (b) By a similar computation, $X'_{\mathrm{dR}_\infty} \simeq X_{\mathrm{dR}_\infty} \times_{Z_{\mathrm{dR}_\infty}} Y_{\mathrm{dR}_\infty}$.
   (c) $\mathrm{Inf}(X') \to \mathrm{Inf}(Y)$ is also a homotopy Kan fibration since it is the base change of $\mathrm{Inf}(X) \to \mathrm{Inf}(Z)$. Again by a similar computation to those above, $(X'/Y)_{\mathrm{dR}_\infty} \to (X/Z)_{\mathrm{dR}_\infty} \times_Z Y$ is an equivalence.

4. For any map $X \to Y$ of derived dagger Stein spaces, the map $X \to (X/Y)_{\mathrm{dR}_\infty}$ is representable: Let Z be a derived dagger Stein, with map $Z \to (X/Y)_{\mathrm{dR}_\infty}$. We must prove that the fibre product $Z \times_{(X/Y)_{\mathrm{dR}_\infty}} X$ is a derived dagger Stein:

$$\begin{tikzcd}
Z \times_{(X/Y)_{\mathrm{dR}_\infty}} X \arrow[r] \arrow[d] & X \arrow[d] \\
Z \arrow[r] \arrow[ur, dashed] & (X/Y)_{\mathrm{dR}_\infty}
\end{tikzcd}$$

A lift as in the diagram above exists because Z is representable and therefore projective. But then we may express the fibre product as a fibre product of derived dagger Steins, proving the result:

$$Z \times_{(X/Y)_{\mathrm{dR}_\infty}} X \simeq Z \times_X X \times_{(X/Y)_{\mathrm{dR}_\infty}} X \simeq Z \times_X (X \subseteq X \times_Y X)^\dagger \tag{241}$$

.

This expresses the fibre product as a fibre product of two derived dagger Steins.

5.4.3 Crystals

We conclude this chapter by justifying the definition ∞-de Rham stacks by relating them to crystals and $\mathcal{D}$-modules.

Definition 5.15. - For X a derived analytic space, $\mathrm{Crys}(X)$ is the category of quasicoherent sheaves on the ∞-de Rham stack of X:

$$\mathrm{Crys}(X) = \mathrm{QCoh}(X_{\mathrm{dR}_\infty}) \tag{242}$$

This can be made into a six-functor formalism, but not with respect to qcqs morphisms on $\mathrm{dAn}_k^\dagger$. We require class of morphisms E can be defined such that the ∞-de Rham functor defines a map of geometric setups

$$(-)_{\mathrm{dR}_\infty} : (\mathrm{dAn}_k, E) \to (\mathrm{PreStk}, \tilde{E}) \tag{243}$$

where the latter setup is the geometric setup of relative algebraic geometry (133). The class can be constructed as follows:

Definition 5.16. ([54] Definition 4.80, 4.83) - A map $f : X \to Y$ between derived dagger Steins is *good* if $\mathrm{Inf}(X) \to \mathrm{Inf}(Y)$ is a homotopy Kan fibration, and $X \to (X/Y)_{\mathrm{dR}_\infty}$ is of universal !-descent. Note this makes sense because $X \to (X/Y)_{\mathrm{dR}_\infty}$ is representable.

A morphism $f : X \to Y$ of derived analytic spaces is good if it is universally good on all affines: for any $Y' \to Y$ with Y' a derived dagger Stein, the pullback map $X \times_{Y'} Y \to Y$ is good.

Lemma 5.17. *([54], 4.81-4.83) - The class of good morphisms between derived dagger Steins is stable under base change and composition. The functor* $(-)_{\mathrm{dR}_\infty}$ *sends good morphisms to exceptionally-pushable maps of prestacks, and therefore defines a map of geometric setups*

$$(\mathrm{dStn}^\dagger, \mathrm{good}) \to (\mathrm{PreStk}, \tilde{\mathrm{E}}). \tag{244}$$

and thereby a six-functor formalism

$$\mathrm{Crys} : \mathrm{Corr}(\mathrm{dStn}_k^\dagger, \mathrm{good}) \to \mathrm{Pr}_{\mathrm{st}}^L \tag{245}$$

and by construction the formalism extends to derived analytic spaces by ([34], Proposition A.5.16)

$$\mathrm{Crys} : \mathrm{Corr}(\mathrm{dAn}_k, \mathrm{good}) \to \mathrm{Pr}_{\mathrm{st}}^L. \tag{246}$$

5.4.4 Differential Operators and Jets as Monads and Comonads

We conclude this chapter by connecting de Rham stacks and crystals to differential operators.

Recall from the algebraic theory of crystals that algebraic $\mathcal{D}$-modules on a smooth scheme X can be understood as algebraic crystals: quasicoherent sheaves on the algebraic de Rham stack, which is the quotient stack of X by the relation of *infinitesimal closeness* - two R-points $x, y : \operatorname{Spec}(R) \to X$ are infinitesimally close if their associated reduced points are equal. Intuitively, the connection comes from *parallel transport* isomorphisms.

We recall the connection to algebraic $\mathcal{D}$-modules, following the discussion in [17]. So, let F be a crystal of quasicoherent sheaves on a smooth scheme X. Given a pair $x, y : \operatorname{Spec}(R) \to X$ of R-points, they are infinitesimally close if and only if the pair $(x, y) : \operatorname{Spec}(R/I) \to X \times X$ factors through the diagonal, or equivalently, $(x, y) : \operatorname{Spec}(R) \to X \times X$ factors through the diagonal set-theoretically, or equivalently again, through the *formal completion* of the diagonal. This is to say that, as expected, the quotient by the relation of (algebraic) infinitesimal closeness is given by the formal completion of the diagonal.

If $\pi_1, \pi_2 : (X \times X)^\vee \to X$ are the projections of the formal diagonal, to give an isomorphism $\pi_1^* F \cong \pi_2^* F$ is equivalent to giving a compatible family of isomorphisms $(\pi_1^n)^* F \cong (\pi_2^n)^* F$, where these are the projections of the n^{th} infinitesimal neighbourhood of the diagonal $\Delta(X) \subseteq X \times X$. By adjunction this corresponds to an isomorphism $F \cong (\pi_1^n)_* (\pi_2^n)^* F$, or equivalently base change to the structure sheaf of the n^{th} infinitesimal neighbourhood:

$$(\pi_1^n)_* (\pi_2^n)^* F \cong \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} F \tag{247}$$

But then there is a perfect pairing between $\mathcal{O}_{X^{(n)}}$ and algebraic differential operators on X of order at most n. Therefore the data of a crystal is a sheaf F with suitably compatible actions of $\mathcal{D}_X^{\leq n}$: exactly an algebraic $\mathcal{D}_X$-module.

It is therefore expected that crystals with respect to the *germ of the diagonal* ought to correspond to another category of $\mathcal{D}$-modules. In this section we present an analogous result, connecting ∞-de Rham stacks with *infinite-order differential operators.*

Definition 5.17. ([54] Definition 4.91) - Let $f : X = \operatorname{Spec}(A) \to \operatorname{Spec}(B) = Y$ be a morphism of derived dagger Steins, $p_{X/Y} : X \to (X/Y)_{\mathrm{dR}_\infty}$ the canonical map. The *comonad of jets* and *monad of infinite-order differential operators* are defined respectively as

$$\mathcal{J}^{\infty}_{X/Y} = p^*_{X/Y} p_{X/Y,*} \tag{248}$$

$$\mathcal{D}^{\infty}_{X/Y} = p^!_{X/Y} p_{X/Y,!} \tag{249}$$

Remarks -

1. As $p_{X/Y}$ is a representable map of prestacks, $p_{X/Y,!} = p_{X/Y,*}$. Consequently, we see that $\mathcal{D}^{\infty}_{X/Y}$ is right adjoint to $\mathcal{J}^{\infty}_{X/Y}$.
2. Applying base-change to $p_{X/Y}$ along itself, letting $\tilde{\pi}_1, \tilde{\pi}_2$ be the projections $(X \subseteq X \times_Y X)^{\dagger} \to X$:

$$\mathcal{J}^{\infty}_{X/Y} = \tilde{\pi}_{1,*}\tilde{\pi}_2^* \simeq (A\widehat{\otimes}^{\mathbb{L}}_B A)\widehat{\otimes}^{\mathbb{L}}_A - \tag{250}$$

$$\mathcal{D}^{\infty}_{X/Y} = \tilde{\pi}_{2,!}\tilde{\pi}_1^! \simeq \mathbb{R}\underline{\mathrm{Hom}}_A((A\widehat{\otimes}^{\mathbb{L}}_B A)^{\dagger}_{\Delta}, -) \tag{251}$$

Lemma 5.18. *([54] Lemma 4.94) - The adjunction $p^*_{X/Y} \dashv p_{X/Y_*}$ is comonadic, and if $X \to (X/Y)_{\mathrm{dR}_\infty}$ is of !-descent, then $p_{X/Y_!} \dashv p^!_{X/Y}$ is monadic. We therefore obtain a series of equivalences*

$$\mathrm{Mod}_{\mathcal{D}^{\infty}_{X/Y}}\mathrm{QCoh}(X) \simeq \mathrm{QCoh}((X/Y)_{\mathrm{dR}_\infty}) \simeq \mathrm{Comod}_{\mathcal{J}^{\infty}_{X/Y}}\mathrm{QCoh}(X). \tag{252}$$

Proof - All statements follow from ([12], Proposition 3.1.27): a morphism $f : X \to Y$ of $*$-descent for a six-functor formalism Q induces an equivalence $Q(Y) \simeq \mathrm{comod}_{f^*f_*}Q(X)$, and a morphism $f : X \to Y$ of !-descent induces an equivalence $Q(Y) \simeq \mathrm{mod}_{f^!f_!}Q(X)$. Q.E.D

We end this section with a conjectural connection between algebras for the monad D^{∞} and traditional D^{∞}-modules, under suitable circumstances. We contend that one can follow the steps of chapter 4.16 of [54], but the verifications are yet to be completed:

1. We begin by describing the algebra of the germ of the diagonal for a classical dagger Stein $X = \mathrm{Spec}(A)$ admitting an étale morphism[3] $X \to \mathbb{A}^r$. There is an isomorphism of complete bornological k-algebras:

$$\varinjlim_{m\in\mathbb{N}} A\widehat{\otimes}_k k\left\langle m dx\right\rangle \to \varinjlim_{U\supseteq\Delta X} (A\widehat{\otimes}_k A)_U \tag{253}$$

induced by $a\otimes 1 \mapsto a\otimes 1$ and $dx_i \mapsto 1\otimes x_i - x_i\otimes 1$. This is the analogue of proposition 4.96 of [54], replacing the unbounded p-adic sequence $\{1/p^m : m \in \mathbb{N}\}$ with the unbounded archimedean sequence $\{m \in \mathbb{N}\}$.

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[3] In [54], X is assumed to be a classical affinoid space equipped with an étale morphism $X \to \mathbb{D}^r_k$, from which standard vector fields are pulled back to define the tangent bundle. In the Archimedean setting we can use affine space for the same purpose.

As the formula and intuition suggests, one might think of this as arising from the change of coordinates $(x_i, y_i) \mapsto (x_i - y_i, x_i + y_i)$. It also suggests that the germ of X in $X \times X$ can be replaced by the germ of X in another space which captures the same infinitesimal behaviour.

2. Motivated by this last observation, Soor considers the *germ of the zero section* of $X = \operatorname{Spec}(A)$:

   $$(X \subseteq TX)^{\dagger} = \operatorname{Spec}(A \widehat{\otimes}_k k \langle \infty dx \rangle) \tag{254}$$

   Here TX is the tangent bundle, obtained from pullback along the étale map $X \to \mathbb{A}^r$, and $k \langle \infty dx \rangle$ is the analogue of $k \langle dx/p^{\infty} \rangle$, of series of radius of convergence 0. The isomorphism of the previous item can be applied to produce a groupoid object $\exp(T_X)$ equivalent to $\operatorname{Inf}(X)$.

   $$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & (X \subseteq TX)^{\dagger} \times_X (X \subseteq TX)^{\dagger} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (X \subseteq TX)^{\dagger} \arrow[r, shift left=1] \arrow[r, shift right=1] & X
\end{tikzcd}$$

   Therefore, letting $q : X \to X/\exp(TX)$, the monad D_X^{∞} identifies with $q^{!}1q!$. The morphisms of this groupoid arise from the maps $\sigma, \tau : A \to A \widehat{\otimes}_k k \langle \infty dx \rangle$ of underlying algebras:

   $$\sigma = \operatorname{id} \otimes 1 \tag{255}$$

   $$\tau(a) = \sum_{\underline{\alpha} \in \mathbb{N}^r} \frac{1}{\underline{\alpha}!} \partial^{\underline{\alpha}}(a) \otimes (dx)^{\underline{\alpha}} \tag{256}$$

   σ is the evident map with trivial germ component, and τ evaluates the Taylor series of a.

3. Recall from [10] that for a complex affine space U, the infinite order differential operators on U, $D^{\infty}(U)$, is determined by a sequence $\{g_{\alpha} \in \mathcal{O}(U)\}$ subject to a *rapid decay* condition, defining the operator $\sum g_{\alpha} \partial / \partial x^{\alpha}$.

   The rapid decay condition is usually imposed on compact subsets, but since we are working with Steins we can impose it on global sections. In this way we obtain a ring of infinite-order differential operators $D^{\infty}(X)$ on these Steins, that will agree with the classical notion for complex manifolds.

4. Similarly to the algebraic case, $D^{\infty}(X)$ is the A-linear bornological dual of $A \widehat{\otimes}_k k \langle \infty dx \rangle$ via the pairing[4] $(\partial^{\beta}, (dx)^{\alpha}) = \underline{\alpha}! \delta_{\alpha\beta}$. Using this we can make the following manipulations:

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[4] Here ! denotes multi-index factorial and δ denotes the Kronecker delta.

$$\begin{aligned} q^! q_! 1_X &\simeq \mathbb{R}\underline{\mathrm{Hom}}_A (A \widehat{\otimes}_k k \left\langle \infty dx \right\rangle, A) && (257) \\ &\simeq \mathbb{R} \varprojlim_{m \in \mathbb{N}} \mathbb{R}\underline{\mathrm{Hom}}_A (A \widehat{\otimes}_k k \left\langle m dx \right\rangle, A) && (258) \\ &\simeq \mathbb{R} \varprojlim_{m \in \mathbb{N}} A \widehat{\otimes}_k k(\partial / m) && (259) \\ &\simeq \varprojlim_{m \in \mathbb{N}} A \widehat{\otimes}_k k(\partial / m) && (260) \end{aligned}$$

The third equivalence follows from the pairing and cofinality, and the last equivalence by Mittag-Leffler result ([11] Theorem 5.26). This final algebra identifies with the algebra of infinite order differential operators[5]. This completes the identification for the unit, which then extends routinely to perfect complexes.

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[5] This is the analogue of Lemma 3.4 of [1]

6 Locally Analytic Representations and the Beilinson-Bernstein Localization Theorem

In this chapter discuss applications of six-functor formalisms on complex analytic stacks to a derived analytic Beilinson-Bernstein localization theorem, building on the work of Ben-Zvi & Nadler [9] and Scholze [51]. We emphasize that this chapter is expository in style: the results from [51] are stated in the context of condensed mathematics, and while it is strongly expected that the results will be compatible, the verification of this is not the focus of this work.

6.1 Locally Analytic Representations

Following chapter III of [51], we describe how maps of quotient stacks are used to compare representations of various infinitesimal objects associated to a real Lie group G, with Lie algebra $\mathfrak{g}$. Let G^{la} denote the corresponding analytic stack. To define this, one can take the underlying real analytic manifold of G, and glue the algebras of complex-valued analytic functions on G on open subsets of G into an analytic stack, and bestow this gluing with a group object structure through duality arguments between group schemes and Hopf algebras of structural algebras. However, since G is a Lie group, the local analytic functions are bestowed with a translation action from G, and this contains all the gluing data. We therefore expect that one can simply take G^{la} to be the spectrum of global analytic functions[6][7].

Definition 6.1. - The category of locally analytic G-representations is $\mathrm{QCoh}(*/G^{\mathrm{la}})$.

Proposition 6.1. *([51], Proposition III.1.2) - $*/G^{\mathrm{la}} \to *$ is cohomologically smooth. The dualizing complex is the modulus character concentrated in degree 0.*

Proof - The map $(*/(1 \subseteq G^{\mathrm{la}})^{\dagger}) \to */G^{\mathrm{la}}$ is cohomologically smooth, and $*/(1 \subseteq G^{\mathrm{la}})^{\dagger} \to *$ is cohomologically smooth, both of dimension $\dim(G)$. The pulls back to $G_{\mathrm{Betti}} \to *$ which is cohomologically smooth, and the latter fits into a factorization

$$\tilde{G}^{\mathrm{an}}_{\mathrm{dR}} \to */(1 \subseteq G^{\mathrm{la}})^{\dagger} \to * \tag{261}$$

where the total map is cohomologically smooth. The dualizing complex can be calculated via "linearization" - it must be trivial if it's isomorphic to a real vector space, by induction on the Künneth formula we can reduce to dimension 1, and the dualizing complex defines a character on $\mathbb{R}$ that is invariant under automorphisms, which must be trivial. A choice of such a trivialization produces

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[6] Note that this is not adequate for incarnating general real analytic manifolds as analytic stacks.

[7] My supervisor has also suggested that one can argue this by embedding the underlying real analytic manifold of G inside a general linear group.

a map $\mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}^*$, trivial on $\mathrm{SL}_n(\mathbb{R})$, and therefore given by a character of $\mathbb{R}^*$, and this yields the norm character.

For general G, one can degenerate G to $\mathfrak{g}$ by an $\mathbb{R}$-parameterized family via a "deformation to the normal cone". This family is invariant under conjugation of G, and the character can be computed as $G \to \mathrm{GL}(G) \to \mathbb{R}_{\geq 0}$. The first map is the adjoint representation, the second is the norm of the determinant in the same way as above. But this computes the modulus character. Q.E.D

Representation of infinitesimal algebraic objects are then compared using the seqeunce of maps

$$(1 \subseteq G^{\mathrm{la}})^{\wedge} \to (1 \subseteq G^{\mathrm{la}}) \to (K \subseteq G^{\mathrm{la}})^{\dagger} \to G^{\mathrm{la}} \tag{262}$$

The associated representations will correspond respectively to Lie algebra representations, locally analytic representations, locally analytic (g, K)-modules, and locally analytic G-representations.

A long series of properties of the maps between these classifying stacks is proven in chapter III.1 of [51], summarised below. Let G be a real Lie group:

Comparing quotients by analytic germs and the whole group - The projections

$$*/(1 \subseteq G^{\mathrm{la}})^{\dagger} \to */G^{\mathrm{la}} \to * \tag{263}$$

are cohomologically smooth - that is, the first map is a cohomologically smooth cover, and the composite is cohomologically smooth, both of dimension $\dim(G)$. From this it follows that the second projection is also cohomologically smooth, with dualising complex concentrated in degree 0. The character associated to this representation is the modulus character.

Proof - The first map pulls back to the projection $G_{\mathrm{Betti}} \to *$, which is cohomologically smooth.

The germ $(1 \subseteq G^{\mathrm{la}})^{\dagger}$ acts on a smooth infinitesimal neighbourhood $\tilde{G}$ of G, embedded in a complex manifold. The de Rham stack $\tilde{G}_{\mathrm{dR}_\infty} = \tilde{G}/(1 \subseteq G)^{\dagger}$ therefore admits a smooth projection

$$\tilde{G}_{\mathrm{dR}_\infty} \to */(1 \subseteq G)^{\dagger} \tag{264}$$

As the de Rham stack is a Betti stack, it is cohomologically smooth over the point.

Quotients by formal neighbourhoods correspond to representations of the Lie algebra/universal enveloping algebra - Let $\mathfrak{g} = \mathrm{Lie}(G)$, with universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The quotient map

$$a : * \to */(1 \subseteq G^{\mathrm{la}})^{\wedge} \tag{265}$$

is cohomologically smooth and surjective, inducing the structure of an associative algebra object on $A = a^! a_!(1)$. This A is naturally isomorphic to $\mathcal{U}(\mathfrak{g})$, and the exceptional pullback $a^!$ then induces an equivalence:

$$\mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^{\wedge}) \xrightarrow{\sim} \mathcal{D}(\mathcal{U}(\mathfrak{g})) \tag{266}$$

Proof - To show a is cohomologically smooth, by base change stability, it suffices to prove that $(1 \subseteq G^{\mathrm{la}})^{\wedge} \to *$ is cohomologically smooth:

$$\begin{tikzcd}
(1 \subseteq G^{\mathrm{la}})^{\wedge} \arrow[r] \arrow[d] & * \arrow[d, "a"] \\
* \arrow[r, "a"] & */(1 \subseteq G^{\mathrm{la}})^{\wedge}
\end{tikzcd}$$

By definition, this is the formal infinitesimal thickening of the identity of G, which is the Ind-scheme associated to truncations of the free polynomial algebras valued in generators of $\mathfrak{g}$. Therefore this formal neighbourhood is isomorphic to the formal spectrum of a free complex power series in $d = \dim(G)$ variables:

$$(1 \subseteq G^{\mathrm{la}})^{\wedge} \cong \mathrm{Spec}(\mathbb{C}[[X_1, \cdots, X_n]]) \tag{267}$$

This admits an open immersion into complex projective d-space, which is itself smooth and proper.

The properties of cohomological smoothness then ensure linearity over $D(\mathbb{C})$, and thus by the Lurie-Barr-Beck theorem, the resulting monad restricted to the identity defines an algebra $A = a^! 1_!(1)$ such that

$$\mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^{\dagger}) \cong D(A) \tag{268}$$

It remains to prove that $A \cong \mathcal{U}(\mathfrak{g})$. A can be computed as the compactly supported cohomology of the dualizing complex of $\mathrm{Spec}(\mathbb{C}[X_1, \cdots, X_d])$, and this is given by the continuous dual of $\mathbb{C}[X_1, \cdots, X_d]$. The canonical map $\mathfrak{g} \to A$, sending a tangent vector to its evaluation at the identity, defines such a dual element. This sends a bracket to the associator of the operators, and thus extends to $\mathcal{U}(\mathfrak{g})$. Monomials in the generators of $\mathfrak{g}$ map to independent operators, and span the continuous dual, and so the result follows from the Poincaré-Birkhoff-Witt theorem.

Comparing the quotients by the formal neighbourhood and analytic germ - The quotient

$$b : */(1 \subseteq G^{\mathrm{la}})^{\wedge} \to */(1 \subseteq G^{\mathrm{la}})^{\dagger} \tag{269}$$

is such that the structure sheaf $\mathcal{O}_{*/(1\subseteq G^{\mathrm{la}})^\wedge}$ is b-proper, with invertible b-proper dual. The pullback b^* is fully faithful, and therefore defines an embedding into the derived category of $\mathcal{U}(\mathfrak{g})$-modules by the previous item:

$$\mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^\dagger) \xrightarrow{b^*} \mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^\dagger) \xrightarrow{\sim} \mathcal{D}(\mathcal{U}(\mathfrak{g})) \tag{270}$$

As an open embedding, the image can be identified with those modules annihilated by tensoring with a certain idempotent algebra.

Additionally, containment in the essential image of b^* can be checked on the 1-parameter subgroups of G: let $X_1, , ..., X_n \in \mathfrak{g}$ form a basis, and let $(\mathbb{G}_a^\dagger)_i \subseteq G^{\mathrm{la}}$ be the analytic germs of the 1-parameter groups that they infinitesimally generate. In other words, if G_i is the group generated by X_i, then

$$(\mathbb{G}_a^\dagger)_i \subseteq G^{\mathrm{la}} = (0 \subseteq \mathbb{G}_i)^\dagger. \tag{271}$$

A quasicoherent sheaf lies in the image of b^* if and only if, for all i, its restriction to $(\mathbb{G}_a^\wedge)_i$ lies in the essential image of

$$\mathrm{QCoh}(*/(\mathbb{G}_a^\dagger)_i) \hookrightarrow \mathrm{QCoh}(*/(\mathbb{G}_a^\wedge)_i) \tag{272}$$

Proof - The comparison of the algebraic and analytic quotient stacks is a generalization of the arguments given for the affine line. We consider the pair of quotients:

$$* \xrightarrow{\ a\ } */(1 \subseteq G)^\wedge \xrightarrow{\ b\ } */(1 \subseteq G)^\dagger$$

The constant sheaf $\mathbb{C}$ is $b \circ a$-proper, and so $a_!\mathbb{C}$ is b-proper. $a_!\mathbb{C}$ is a quasicoherent sheaf on the formal neighbourhood, corresponding to the regular representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The constant sheaf on $*/(1 \subseteq G)^\wedge$ is also b-proper, from which it follows that the b-proper dual of $a_!\mathbb{C}$ is invertible, and so the pushforward b_* satisfies the projection formula. From this, it suffices to prove $b_*\mathcal{O} = \mathcal{O}$, which is entirely analogous to the case of the affine line.

Since b^* is fully faithful, so is any base change, and being in the essential image of b^* can be checked after pullback. In particular we can pull back to the classifying spaces of 1-parameter subgroups G_i. Consider the pullback diagrams

$$\begin{tikzcd}
*/G_i^\wedge \arrow[r, "\tilde{b}_i"] \arrow[d] & */G_i^\dagger \arrow[d] \\
*/(1 \subseteq G^{\mathrm{la}})^\wedge \arrow[r, "b_i"] & */(G_i \subseteq G^{\mathrm{la}})^\dagger
\end{tikzcd}$$

The structure sheaf is proper for the top horizontal map, from the affine case, and therefore it is also for the bottom horizontal map. We can express the push-pull functor b^*b_* as the composite of push-pulls associated to these 1-parameter subgroups:

$$b^* b_* = b_d^* b_{d*} \cdots b_1^* b_{1*} \tag{273}$$

This functor is the identity on the collective essential image of the b_i^*, and so an object is in the essential image of b^* if and only if its restriction to G_i is in the essential image of b_i.

Analyzing the integral kernels associated to these functors, we see that this boils down to an isomorphism of the germ of the identity with the product of the germs of the identity in the 1-parameter subgroups:

$$(1 \subseteq G^{\mathrm{la}})^\wedge \cong \prod_{i=1}^{d} (1 \subseteq G_i)^\dagger \tag{274}$$

Both of these are isomorphic to the germ of the identity in affine d-space, and the tangent map is an isomorphism, so the result follows from the implicit function theorem.

Quasicoherent sheaves on the quotient by the analytic germ are modules over distribution algebras - The quotient map

$$c : * \to */(1 \subseteq G^{\mathrm{la}})^\dagger \tag{275}$$

is proper and surjective. By Barr-Beck, $c^* c_* 1 = \mathcal{O}((1 \subseteq G^{\mathrm{la}})^\dagger$ acquires the structure of a coalgebra, and $\mathrm{QCoh}((1 \subseteq G^{\mathrm{la}})^\dagger)$ is naturally equivalent to the category of comodules over $\mathcal{O}((1 \subseteq G^{\mathrm{la}})^\dagger$.

The action of $\mathcal{U}(\mathfrak{g})$ can be extended to the *algebra of formal distributions at* $1 \subseteq G$ - this is the algebra of functionals on the analytic germ:

$$\mathcal{D}(1 \subseteq G) = (\mathcal{O}(1 \subseteq G^{\mathrm{la}})^\dagger) \tag{276}$$

Therefore, this comodule structure induces a module structure over the distribution algebra. The resulting functor

$$\mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^\dagger) \to D(\mathcal{D}(1 \subseteq G))^* \tag{277}$$

is fully faithful, with image modules annihilated by a certain idempotent $D(\mathcal{D}(1 \subseteq G))$-algebra.

Proof - Only the second paragraph requires further comment. Consider the diagram

$$\mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^\wedge) \to \mathrm{QCoh}(*/(1 \subseteq G^{\mathrm{la}})^\dagger) \to D(\mathcal{D}(1 \subseteq G)) \tag{278}$$

The first map is the right adjoint to the inclusion of formal group representations into analytic representations. By the previous result, the statement reduces to showing that the algebra of distributions at the identity of G becomes equivalent to the pullback of $\mathcal{U}(\mathfrak{g})$.

The equivalence $a^!$ sends $\mathcal{D}(1 \subseteq G)$ to $b^! c_*(1)$, as $a^! b^! c_*(1) = c^! c_*(1)$ is dual to $c^* c_*(1)$ by properness, and this corresponds to analytic germ representations $\mathcal{O}(1 \subseteq G^{\mathrm{la}})^\dagger$, and $\mathcal{U}(\mathfrak{g})$ corresponds to $a_!(1)$. So it must be shown that the natural map $a_!(1) \to b^! c_*(1)$ becomes an isomorphism after applying $b_* = b_!$. This is equivalent to showing that $c_*(1) \to b_! b^! c_*(1)$ is an isomorphism. As b is proper, it has a fully faithful left adjoint, and so $b^!$ is also fully faithful, and $b_! b^!$ is an equivalence, as desired.

Comparison of the analytic germ at the origin to the germ at a maximal compact - Recall that G is a real Lie group, and therefore admits maximal compact subgroup. Let K be such a subgroup. The quotient map

$$*/(1 \subseteq G)^\dagger \to */(K \subseteq G)^\dagger \tag{279}$$

is a pullback of the quotient map $* \to */K_{\mathrm{Betti}}$. By base change stability, this quotient is therefore cohomologically smooth and proper.

Similarly to the previous item, quasicoherent sheaves over the classifying stack of the analytic germ of K is equivalent to the ∞-category of $\mathcal{O}((K \subseteq G)^\dagger)$-comodules, and can be embedded within the derived category of distribution algebras as a subcategory of "opens", in the sense that they are annihilated by some idempotent algebra:

$$\mathrm{QCoh}(*/(K \subseteq G^{\mathrm{la}})^\dagger) \subseteq D(\mathcal{D}(K \subseteq G)) \tag{280}$$

Membership in the essential image can be checked at the identity. Moreover, there is a unique t-structure in $\mathrm{QCoh}(*/(K \subseteq G^{\mathrm{la}})^\dagger)$that makes d^* t-exact and fully faithful if K is connected, identifying the heart with the germ representations that extend to a locally analytic representation of K.

Proof - The statemments appearing before the final paragraph can be proven along the same lines as the previous propositions. The derived category $D(\mathcal{D}(K \subseteq G))$ admits truncations in the usual sense, and the forgetful functor to $D(\mathcal{D}(1 \subseteq G))$ is clearly t-exact for the canonical t-structure. The truncation functors on this latter derived category preserve $\mathrm{QCoh}(*/(1 \subseteq G)^\dagger)$, and so the truncation functors also preserve the image of the subcategory $\mathrm{QCoh}(*/(K \subseteq G)^\dagger)$. Therefore t^* is exact, as required.

For the last point, connectivity of K, full faithfulness of $(d^*)^\heartsuit$ follows from 1-connectivity of the cone of $d_\sharp \mathcal{O} \to \mathcal{O}$, where $d_\sharp$ is the left adjoint to d^*. This can be verified by pullback, where it comes down to the homology of the Betti stack of K, and so this follows from the connectivity assumption. The essential image can be checked after pullback to the embedding of the classifying stack of the germ of the identity in K into the classifying stack of K, and this corresponds to integrating an analytic germ representation to a full analytic representation.

Comparison of analytic germ at K with the whole group - The quotient

$$e : */(K \subseteq G^{\mathrm{la}}) \to */G^{\mathrm{la}} \tag{281}$$

is cohomologically smooth, with fully faithful pullback satisfying the projection formula.

Proof - e^* is also pullback of a Betti stack projection $(G/K)_{\mathrm{Betti}} \to *$ and therefore cohomologically smooth. G/K is homeomorphic to real affine space and is therefore contractible. By Poincaré duality (see notes on six functors [52]), a shift of $f_!$ is left-adjoint to f^*, and the essential image of f^* is stable under truncation.

6.2 Distribution Algebras

Pullbacks of quasicoherent sheaves on the locally analytic quotient stack $*/G^{\mathrm{la}}$ (which correspond to locally analytic representations) naturally refine to functors to $\mathcal{D}_c(G)$ of compactly-supported distributions on G.

Definition 6.2. - Let $Z \subseteq G$ be a compact Stein. The *distributions on G supported on Z* is

$$\mathcal{D}(Z \subseteq G) = (\mathcal{O}(Z \subseteq G^{\mathrm{la}})^{\dagger})^*. \tag{282}$$

The Hopf algebra of *compactly supported distributions* is the union of all such compactly supported distributions

$$\mathcal{D}_c(G) - \cup_{Z \subseteq G} \mathcal{D}(Z \subseteq G) \tag{283}$$

Proposition 6.2. *([51] Proposition III.2.2) - The $*$-pullback functor*

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D_{\mathrm{qc}}(*) = D(\mathbb{C}) \tag{284}$$

refines to a functor valued in the derived category of compactly supported distributions:

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D_{\mathrm{qc}}(\mathcal{D}_c(G)) \tag{285}$$

This refinement is fully faithful, with essential image the subcategory of objects annihilated by an idempotent $\mathcal{D}_c(G)$-algebra. Moreover, containment in the essential image can be checked after restriction to distributions supported at the origin.

Proof - Consider the Cartesian square

$$\begin{tikzcd}
G_{\mathrm{Betti}} = G^{\mathrm{la}}/(1 \subseteq G^{\mathrm{la}})^{\dagger} \arrow[r, "h"] \arrow[d, "\pi"] & 1/(1 \subseteq G^{\mathrm{la}})^{\dagger} \arrow[d, "f"] \\
* \arrow[r, "g"] & */G^{\mathrm{la}}
\end{tikzcd}$$

$D_{\mathrm{qc}}(*/G^{\mathrm{la}})$ can be identified with modules in $D(*/(1 \subseteq G^{\mathrm{la}})^{\dagger})$ over $A = f^! g_! \mathcal{O}$ by Barr-Beck. But then by smooth base change, this is equal to $h_! \pi^! \mathcal{O} = h_! \mathbb{D}_{G_{\mathrm{Betti}}}$, and this can be written as the filtered colimit over all compact Stein subsets of G^{la},

$$h_{Z,!} \mathbb{D}_{Z \subseteq G} = h_{Z,*} \mathbb{D}_{Z \subseteq G}. \tag{286}$$

These correspond to $\mathcal{D}(Z \subseteq G)$, and so their filtered colimit is $\mathcal{D}_c(G)$. Q.E.D

6.3 Globalizations and $(\mathfrak{g}, K)$-modules

Globalizations naturally arise from the consideration of f^* and $f^!$ for the quotient map $* \to */G^{\mathrm{la}}$, and for correspondence diagrams of such quotients.

Parabolic induction is an example of such a correspondence: let G^{alg} be an algebraic real reductive group, P^{alg} a parabolic subgroup with Levi subgroup M^{alg}. Then the diagram

$$\begin{tikzcd}
 & & G^{\mathrm{la}}/P^{\mathrm{la}} \arrow[dl, "f'"] \arrow[dr, "p'"] & & \\
 & */P^{\mathrm{la}} \arrow[dl, "q"] \arrow[dr, "p"] & & * \arrow[dl, "f"] & \\
*/M^{\mathrm{la}} & & */G^{\mathrm{la}} & &
\end{tikzcd}$$

has q cohomologically smooth and p proper, and so there is an associated push-pull functor

$$p_!q^* : D_{\mathrm{qc}}(*/M^{\mathrm{la}}) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \tag{287}$$

from M-representations to G-representations, corresponding to the classical construction of *parabolic induction*. Here $p_! = p_*$, while $q^!$ and q_* differ by a twist, so there are no meaningfully distinct other parabolic induction functors.

Let $f : * \to */G$ be the quotient, and consider the realizations of parabolic induction functors under $f^!$ and f^*. By proper base change,

$$f^*p_!q^*(1) \simeq p'_!f^*q^*(1) \simeq \mathcal{O}((G/P)^{\mathrm{la}})$$

is the space of locally analytic functions on G/P, the minimal globalization of the principal series representation of $\mathrm{Ind}_P^G(1)$.

Under the $f^!$ realization

$$f^!p_!q^*(1) \simeq f^!p_*(1) \simeq p'_*f'^!(1) \tag{288}$$

then corresponds to the twist of $\mathcal{O}((G/P)^{\mathrm{la}})$ by the dualising complex on $*/P^{\mathrm{la}}$, which itself corresponds to the modulus character.

A Haar measure on G furnishes an isomorphism $f^!(1) \simeq 1$, and the natural map $f^* \to f^!$ corresponds to the embedding of locally analytic functions on G/P into hyperfunctions, the maximal globalization.

Alternatively, we may consider the Verdier quotient

$$D(\mathcal{D}_c(G)) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \tag{289}$$

with left adjoint f^* and right adjoint $f^! \otimes (f^!(1))^{-1}$ ([51] Proposition III.3.2). Now the objects of $D(\mathcal{D}_c(G))$ are representations of G, equipped with a canonical functor to $D_{\mathrm{qc}}(*/G)$. Distinct representations may map to the same quasicoherent sheaf, and the two adjoints correspond to minimal and maximal globalizations.

$(\mathfrak{g}, K)$-modules can be studied using similar correspondences. We incarnate $(\mathfrak{g}, K)$-modules as quasicoherent sheaves on the formal neighbourhood of K^{alg} in G^{alg}, $D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})^\wedge)$. Then the relation to locally analytic G-representations arises from the correspondence:

$$\begin{tikzcd}
& */(K^{\mathrm{la}} \subseteq G^{\mathrm{la}})^\wedge \arrow[dl, "a"'] \arrow[dr, "b"] & \\
*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})^\wedge & & */G^{\mathrm{la}}
\end{tikzcd}$$

a is proper and a^* is fully faithful, and b^* is fully faithful: b can be factored through the overconvergent neighbourhood:

$$*/(K^{\mathrm{la}} \subseteq G^{\mathrm{la}})^\wedge \xrightarrow{c} */(K^{\mathrm{la}} \subseteq G^{\mathrm{la}})^\dagger \xrightarrow{d} */G^{\mathrm{la}}. \tag{290}$$

The unit is c-proper and $c_*\mathcal{O} = \mathcal{O}$, so c^* is fully faithful, and $c_!$ and c_* agree up to a twist. d^* is also fully faithful, with left adjoint a twist of $d_!$.

However, the essential images of a^* and b^* are quite different, and so the push-pull operation must be localized to behave well. The correspondence diagram is naturally linear over the Harish-Chandra centre $Z(\mathcal{U}(\mathfrak{g}))$. It suffices to verify this for the middle term $D_{\mathrm{qc}}(*/(K^{\mathrm{la}} \subseteq G^{\mathrm{la}})^\wedge)$ since the maps are fully faithful, and this is a full subcategory of the category of modules over the algebra of locally analytic distributions on G supported on K. $\mathcal{U}(\mathfrak{g})$ naturally maps to this algebra and has dense image, and so a central element of $\mathcal{U}(\mathfrak{g})$ maps to an element that commutes with a dense subset, and hence the entire algebra.

The following theorem says that we can transfer between the two sides of the correspondence after localizing to the *bounded part* of $Z(\mathcal{U}(\mathfrak{g}))$: that is, the analytification of $\mathrm{Spec}(Z(\mathcal{U}(\mathfrak{g})))$.

Theorem 6.1. *([51] Theorem III.4.1) - The push-pull functor*

$$D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})^\wedge) \to D_{qc}(*/G^{\mathrm{la}}) \tag{291}$$

becomes a t-exact equivalence when localized to the bounded part of the Harish-Chandra centre $Z(\mathcal{U}(\mathfrak{g}))$, with inverse a_*b^*.

Proof Sketch -

1. It suffices to show that, after localization, the composition

$$a_*b_*b'_!a^* : D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})^\wedge) \to D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})) \tag{292}$$

   is equivalent to the identity. The functor is $Z(\mathcal{U}(\mathfrak{g}))$-linear and commutes with tensoring with finite-dimensional G-representations, and the source category is compactly generated by finite-dimensional K-representations along

$$*/K^{\mathrm{alg}} \to */(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}}) \tag{293}$$

   So, it suffices to show that it's the identity functor on these generating representations.

2. This is verified using the zig-zag diagram

$$a_*b^*b'_!a^* \to a_*b'^!b'_!a^* \leftarrow a_*a^* = \mathrm{id} \tag{294}$$

   where $b'^!$ is the right adjoint of $b'_!$, and therefore my be written as $(c_*)^R d^*$. The transform $c^* \to (c_*)^R$ induces a natural transform $b^* \to b'^!$. Then the equivalence is shown on the compact generators of the category $D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}}))$, i.e the category of $(\mathfrak{g}, K)$-modules. Again, all functors commute with tensoring with finite-dimensional G-representations, and all finite-dimensional K-representations are restrictions of G-representations, so it suffices to verify the equivalence only for the free $(\mathfrak{g}, K)$-module on the trivial representation of K, and this corresponds to the algebra of formal distributions on the algebraic flag variety $X^{\mathrm{alg}} = G^{\mathrm{alg}}/K^{\mathrm{alg}}$:

$$\mathcal{D}((1 \subseteq X^{\mathrm{alg}})^\wedge). \tag{295}$$

3. The composite associates to this module the K-finite vectors in the compactly supported cohomology of the analytic flag variety (under fixed Haar measures on G and K):

$$\mathbb{R}\Gamma_c(X^{\mathrm{la}}, \mathcal{O}). \tag{296}$$

   The middle term is (the maximal globalization of) the space $\mathcal{D}_c(X)$ of compactly supported locally analytic distributions on X, and so the zig-zag diagram on the generating representation becomes

$$\mathbb{R}\Gamma_c(X^{\mathrm{la}}, \mathcal{O}) \to \mathcal{D}_c(X) \leftarrow \mathcal{D}((1 \subseteq X^{\mathrm{alg}})^\wedge) \tag{297}$$

4. The first map can be understood as the application of $\mathbb{R}\Gamma_c$ to the map from locally analytic functions to hyperfunctions. This becomes an isomorphism when localized to $Z(\mathcal{U}(\mathfrak{g}))$ and passed to K-finite vectors since $Z(\mathcal{U}(\mathfrak{g}))$ act as elliptic operators on X. The second map can be shown to be an isomorphism using the Cartan decomposition $G = KA_+K$ to make explicit the K-finite vectors on each side, and use this to describe the action of elliptic operators.

5. For the statement concerning the inverse equivalence, the counit of a yields the transformation

   $$\beta : b'_! a^* a_* b^* \to b'_! b^* \cong \mathrm{id}. \tag{298}$$

   The other composite associated to the zig-zag diagram is the identity:

   $$\alpha : \mathrm{id} \xrightarrow{\sim} b'_! a^* a_* b^*. \tag{299}$$

   These observations are sufficient to demonstrate that $a_* b^*$ is the right adjoint of $b'_! a^*$. It must be checked that the standard unit-counit composites are identities:

   $$b'_! a^* \xrightarrow{b'_1 a^* \alpha} b_! a^* a_* b^* b'_! a^* \xrightarrow{\beta b'_! a^*} b'_! a^*, \tag{300}$$

   $$a_* b^* \xrightarrow{\alpha a_* b^*} a_* b^* b'_! a^* a_* b^* \xrightarrow{\beta b'_! a^*} b'_! a^*. \tag{301}$$

   In the first composite, the second map factors as

   $$b_! a^* a_* b^* b'_! a^* \to b_! a^* a_* b'^! b'_! a^* \to b'_! b'^! b'_! a^* \cong b'_! a^* \tag{302}$$

   and so only the partial composite

   $$b'_! a^* \to b_! a^* a_* b^* b'_! a^* \to b_! a^* a_* b'^! b'_! a^* \tag{303}$$

   needs to be identified, and by construction of α it's given by the composite

   $$b'_! a^* \xrightarrow{\sim} b'_! a^* a_* a^* \to b'_! a^* a_* b'^! b'_! a^*. \tag{304}$$

   All the involved morphisms are compositions of unit and counits and so the composite is the identity, as desired. The second composite is treated similarly, although a little more is needed, reflecting the existence of various globalizations.

6. To finish the proof, it must be shown that the image of $b'_! a^*$ generates $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$. This will follow from later results pertaining to the analytic Beilinson-Bernstein correspondence. Q.E.D

6.4 An Analytic Beilinson-Bernstein Localization Theorem

Theorem 6.2. *There is a diagram*

$$*/G^{\mathrm{la}} \overset{a}{\leftarrow} \mathrm{Fl}/G^{\mathrm{la}} \overset{b}{\rightarrow} \mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}} \tag{305}$$

The resulting push-pull functor a_*b^* induces an equivalence between quasicoherent sheaves on the quotient Betti stack, and locally analytic representations of G with trivial infinitesimal character:

$$D_{\mathrm{qc}}(\mathrm{Fl}_{\mathrm{Betti}}/G_{\mathrm{Betti}}) \simeq D_{\mathrm{qc}}(*/G^{\mathrm{la}})^{\chi=1} = D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \otimes_{Z(\mathcal{U}(\mathfrak{g}))} \mathbb{C} \tag{306}$$

The tensor product on the right hand side of (306) is along the map $Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ given by the action of $\mathfrak{g}$ on the trivial representation. The push-pull functor a_*b^* sends a Betti sheaf $\mathcal{F}$ to $\mathbb{R}\Gamma(\mathrm{Fl}, \mathcal{F} \otimes \mathcal{O})$.

Recall the algebraic correspondence: $\mathfrak{g}$ is a complex reductive Lie algebra, Fl is the flag variety of all Borel subalgebras. The universal Cartan quotient $\mathfrak{b} \mapsto \mathfrak{t}$ defines a constant sheaf of abelian Lie algebras over Fl, and thereby the universal Cartan algebra $\mathfrak{h}$.

$\mathfrak{g}$ integrates to a smooth formal group $\widehat{G}$, whose function algebra is dual to the universal enveloping algebra. The Borel $\mathfrak{b}$ and its unipotent radical $\mathfrak{u}$ integrate to $\widehat{B}$ and $\widehat{U}$, respectively.

Lemma 6.1. *([51] Lemma IV.1.1) - The action*

$$\widehat{G} \times \mathrm{Fl} \to \mathrm{Fl} \tag{307}$$

factors through the quotient by the formal completion of any characteristic subgroup: in particular, through each of the universal unipotent radical and Borel $\widehat{U} \subseteq \widehat{B} \subseteq \widehat{G} \times \mathrm{Fl}$:

$$(\widehat{G} \times \mathrm{Fl})/\widehat{U} \to \mathrm{Fl} \tag{308}$$

$$(\widehat{G} \times \mathrm{Fl})/\widehat{B} \to \mathrm{Fl} \tag{309}$$

Proof - The factorization through the universal unipotent and Borel means that they act trivially on Fl, and the second statement means that the composition maps for the groupoid descent along the quotient. These are both clear from the fact that, as subgroups, they normalize the universal Borel. Q.E.D

Denote by $\mathrm{Fl}/(\widehat{G}/\widehat{U})$ and $\mathrm{Fl}/(\widehat{G}/\widehat{B})$ the associated quotient stacks. Then $\mathrm{Fl}/(\widehat{G}/\widehat{U}) \to \mathrm{Fl}/(\widehat{G}/\widehat{B})$ is an $\widehat{H}$-gerbe, and the canonical map

$$(\mathrm{Fl} \times \widehat{G})/\widehat{B} \to \mathrm{Fl} \times \mathrm{Fl} \tag{310}$$

factors through the formal completion of the diagonal, and this is in fact an isomorphism, inducing an equivalence $\mathrm{Fl}/(\widehat{G}/\widehat{B}) \simeq \mathrm{Fl}_{\mathrm{dR}}$. To verify the isomorphism, as it arises from a map of smooth formal schemes, it is enough to verify it on reduced subschemes and first infinitesimal neighbourhoods.

Any character $\widehat{H} \to \mathbb{C}$ induces a twisted sheaf on $\mathrm{Fl}_{\mathrm{dR}}$, and therefore a kind of twisted D-module on Fl. In the correspondence

$$*/\widehat{G} \xleftarrow{a} Fl/\widehat{G} \xrightarrow{b} \mathrm{Fl}/(\widehat{G}/\widehat{U}) \tag{311}$$

a is proper and cohomologically smooth, as Fl is, and b is cohomologically proper, and the structure sheaf of $Fl/\widehat{G}$ is b-proper with b-proper dual, as the fibres are $*/\widehat{U}$.

The two sides of this correspondence are linear over different, but related algebras: $D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$ is linear over $\mathcal{U}(\mathfrak{h})$, and $D_{\mathrm{qc}}(*/\widehat{G})$ is linear over $Z(\mathcal{U}(\mathfrak{g})) \cong \mathcal{U}(\mathfrak{h})^W$ (this Weyl group ambiguity is resolved via derived methods in [9]).

The actions are compatible in the following sense: the correspondence is encoded in the $Z(\mathcal{U}(\mathfrak{g})) \otimes \mathcal{U}(\mathfrak{h})$-linear object

$$(a,b)_! \mathcal{O}_{\mathrm{Fl}/\widehat{G}} \in D_{\mathrm{qc}}(*/\widehat{G} \times \mathrm{Fl}/(\widehat{G}/\widehat{U})). \tag{312}$$

The pullback to the flag variety has fibres equal to the compactly supported cohomology of $\widehat{G}/\widehat{U}$ in degree $\dim(G/U)$, and the $Z(\mathcal{U}(\mathfrak{g}))$ -action agrees with the restriction of the $\mathcal{U}(\mathfrak{h})$-action.

Therefore, the push-pull functor lifts to the base change:

$$D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U})) \to D_{\mathrm{qc}}(*/\widehat{G}) \otimes_{Z(\mathcal{U}(\mathfrak{g}))} \mathcal{U}(\mathfrak{h}) \tag{313}$$

with left adjoint $b_\sharp a^*$ (recall that $(-)_\sharp$ denotes the appropriate shift of the exceptional pushforward).

Theorem 6.3. *(The Beilinson-Bernstein Localization Theorem, ([6], [51] Theorem IV.1.3, [9] Proposition 3.8) - This left adjoint is fully faithful, and so the push-pull functor $a_* b^*$ is a Verdier quotient. Its kernel is the full subcategory of quasicoherent sheaves on* $\mathrm{Fl}/(\widehat{G}/\widehat{U})$ *with vanishing global sections on pullback to the flag variety:*

$$\mathbb{R}\Gamma(\mathrm{Fl}, M_{\mathrm{Fl}}) = 0. \tag{314}$$

Restricting to a regular weight, this kernel vanishes, and the functor is an equivalence. Restricting to a weakly dominant weight, $a_* b^*$ is t-exact.

Proof Sketch -

1. To verify that the left adjoint is fully faithful, we check that the unit of the adjunction is an equivalence. Since all the functors involved are cocontinuous, it suffices to verify this only for a generator: the regular representation.

   Applying $b_\sharp a^*$ to the regular representation and pulling back to the flag variety yields a sheaf with fibres $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{u})} \mathbb{C}$. Denote it by $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{u})} \mathcal{O}_{\mathrm{Fl}}$. Then the natural map

   $$\mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathcal{U}(\mathfrak{g}))} \mathcal{U}(\mathfrak{h}) \to \mathbb{R}\Gamma(\mathrm{Fl}, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{u})} \mathcal{O}_{\mathrm{Fl}}) \tag{315}$$

   is an isomorphism. This is an argument detailed in ([38], Chapter 2, Theorem 6.5) and recalled after this proof. We can filter both sides so that the associated graded algebras become commutative $\mathcal{U}(\mathfrak{h})$-algebras. The left-hand algebra becomes the nilpotent cone in $\mathfrak{g}^*$ base changed to $\mathcal{U}(\mathfrak{h})$, and the right-hand algebra becomes the cotangent bundle $T^*\mathrm{Fl}$, also base changed to $\mathcal{U}(\mathfrak{h})$. But then this associated graded map is the Springer resolution, which is a rational resolution.

2. Let V be a finite-dimensional representation of $\widehat{G}$ (and thus an element of $*/\widehat{G}$) with highest weight λ. On pullback to $\mathrm{Fl}/\widehat{G}$, the sheaf acquires a filtration such that $\widehat{U}$ acts trivially on the graded pieces. Any $M \in D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$ pulls back to a module on $\mathrm{Fl}/\widehat{G}$ with trivial $\widehat{U}$-action, and so $M_{\mathrm{Fl}/\widehat{G}} \otimes V_{\mathrm{Fl}/\widehat{G}}$ is filtered by objects of the form

   $$(M \otimes V_\chi)_{\mathrm{Fl}/\widehat{G}} \tag{316}$$

   where V_χ are finite-dimensional vector spaces on which $\widehat{H}$ acts with weight χ. Upon pushforward to $*/\widehat{G}$, $a_*b^*M \otimes V$ is filtered by $a_*b^*(M \otimes V_\chi)$.

3. So, if M has regular weight μ, with Weyl group element w such that $\omega \cdot \mu$ is dominant, the term $M \otimes V_{w^{-1}\lambda}$ in $a_*b^*M \otimes V$ can be isolated as a weight space for the infinitesimal character. Therefore if $a_*b^*M = 0$, then $a_*b^*(M \otimes V_{w^{-1}\cdot\lambda})$ for all highest weight modules. But then the cohomology of M_{Fl} vanishes under twisting by a collection of generating line bundles for $D(\mathrm{Fl})$, and so $M_{\mathrm{Fl}} = 0$.

4. Let M have dominant weight μ, and denote by w_0 the longest Weyl group element. Then the object

   $$a_*b^*(M \otimes V_{-w_0\cdot\lambda}) \otimes V \tag{317}$$

   admits a filtration by objects of the form

$$a_* b^*(M \otimes V_{-w_0 \cdot \lambda + \chi}) \otimes V \tag{318}$$

The term for $\chi = w_0\lambda$ is isolated similarly to the previous case, and $a_* b^* M$ is a summand of $a_* b^*(M \otimes V_{-w_0 \cdot}) \otimes V$ for all λ. Thus by twisting with various very ample line bundles, all higher cohomologies can be annihilated. Q.E.D

In the course of the proof we used the last item in the following theorem, which assumes some familiarity with the cohomology of twisted $\mathcal{D}$-modules. We refer the reader to the source for most of the notation and terminology:

Theorem 6.4. *([38], Chapter 2, Lemmas 1.2-1.5, Theorem 6.5) - Let* $\mathcal{D}_{\mathfrak{h}} = \mathcal{U}^\circ / \mathfrak{n}^\circ \mathcal{U}^\circ$*, where* $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ *is the universal enveloping algebra,* $\mathfrak{n}$ *is the nilpotent radical, and* $\mathcal{U}^\circ$*,* $\mathfrak{n}^\circ$ *denote the sheaves of local sections associated to the trivial bundles* $X \times \mathcal{U}$ *and* $X \times \mathfrak{n}$*, respectively.*

1. *The natural morphism* $\mathfrak{g}^\circ \to \mathcal{D}_{\mathfrak{h}}$ *induces* $\mathfrak{h}^\circ \to \mathcal{D}_{\mathfrak{h}}$*, and thereby* $\phi : \mathcal{U}(\mathfrak{h}) \to \Gamma(\mathcal{D}(\mathfrak{h}))$*. The* G*-action on* $\mathfrak{h}$ *is trivial, and so* ϕ *maps to the subalgebra of* G*-invariants.*

   In fact, ϕ *induces an isomorphism* $\mathcal{U}(\mathfrak{h})$ *with* $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$*, and* ϕ *maps to the centre of* $\mathcal{D}_{\mathfrak{h}}$*.*

2. *The Bruhat decomposition induces an isomorphism*

   $$\psi : (\mathcal{O}_U \times_{\mathbb{C}} \mathcal{U}(\overline{\mathfrak{n}})) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}) \to \mathcal{D}_{\mathfrak{h}}|_U \tag{319}$$

   where $\mathfrak{b}_x$ *is a chosen Borel in* X*,* $\overline{\mathfrak{n}}$ *is the nilpotent radical opposite to* $\mathfrak{b}_x$*, and* U *is the open* $\overline{N}$*-orbit of* x*.*

   As a consequence, $\mathcal{D}_{\mathfrak{h}}$ *is locally free.*

3. $\Gamma(X, \mathcal{D}_{\mathfrak{h}}) = \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{h})$*, and* $H^i(X, \mathcal{D}_{\mathfrak{h}}) = 0$ *for all* $i > 0$*.*

Proof Sketch -

1. To see that ϕ is injective, notice that the composition of ϕ with the evaluation of a global section at a point $x \in X$

   $$\mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{b}_x)/\mathfrak{n}_x \mathcal{U}(\mathfrak{b}_x) \to \mathcal{U}(\mathfrak{g})/\mathfrak{n}_x \mathcal{U}(\mathfrak{g}) \tag{320}$$

   and this morphism is injective by the PBW theorem, so ϕ is injective. We omit the proof that it's an isomorphism - see Lemma 1.5 of loc. cit.

   The elements of $\phi(\mathcal{U}(\mathfrak{h}))$ commute with the image of $\mathfrak{g}$ in $\mathcal{D}_{\mathfrak{h}}$, which generates $\mathcal{D}_{\mathfrak{h}}$ along with $\mathcal{O}_X$. Therefore the image commutes with all of $\mathcal{D}_{\mathfrak{h}}$.

2. See Lemma 1.3 and corollary 1.4 of loc. cit.

3. The complex $\mathcal{C}^\bullet = \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^j \mathfrak{n}^\circ$ defines an acyclic resolution of $\mathcal{D}_\mathfrak{h}$ as a $(\mathfrak{g} \times \mathfrak{g}, G)$-module, so the higher cohomology of $\mathcal{D}_\mathfrak{h}$ vanishes. The higher cohomologies $H^i(X, \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^j \mathfrak{n}^\circ)$ vanish for $i \neq j$, and

   $$H^i(X, \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^i \mathfrak{n}^\circ) \cong \bigoplus_{|W(i)|} \mathcal{U}(\mathfrak{g}) \tag{321}$$

   where $W(i)$ is the subset of Weyl group elements of length j. This follows from induction and the Borel-Weil-Bott theorem, given in ([38] Chapter 2, Lemma 5.2).

   Consequently, the spectral sequence arising from the resolution $\mathcal{C}^\bullet$ degenerates on its second page, and from it we deduce that the higher cohomology of $\mathcal{D}_\mathfrak{h}$ vanishes, and $\Gamma(X, \mathcal{D}_\mathfrak{h})$ has a finite filtration

   $$0 = F_0\Gamma(X, \mathcal{D}_\mathfrak{h}) \subseteq F_1\Gamma(X, \mathcal{D}_\mathfrak{h}) \subseteq \cdots \subseteq F_n\Gamma(X, \mathcal{D}_\mathfrak{h}) = \Gamma(X, \mathcal{D}_\mathfrak{h}) \tag{322}$$

   with each of the graded pieces $F_k\Gamma(X, \mathcal{D}_\mathfrak{h})/F_{k-1}\Gamma(X, \mathcal{D}_\mathfrak{h})$ isomorphic to $\bigoplus_{|W(k)|} \mathcal{U}(\mathfrak{g})$.

   This filtration is a G-module filtration and therefore descends to the G-invariants, which via ϕ are isomorphic to $\mathcal{U}(\mathfrak{h})$. As G is reductive, and the action on $\Gamma(X, \mathcal{D}_\langle)$ is algebraic, the module is fully reducible. Therefore $\mathrm{Gr}\phi(\mathcal{U}(\mathfrak{h})) = \mathrm{Gr}\Gamma(X, \mathcal{D}_\mathfrak{h})$, and

   $$F_k\phi(\mathcal{U}(\mathfrak{h}))/F_{k-1}\phi(\mathcal{U}(\mathfrak{h})) \cong \left(\bigoplus_{|W(k)|} \mathcal{U}(\mathfrak{g})\right)^G \cong \bigoplus_{|W(k)|} Z(\mathfrak{g}) \tag{323}$$

   Consequently, $\mathcal{U}(\mathfrak{h})$ is a free $Z(\mathfrak{g})$-module of rank $|W|$.

   We can now prove the statement. The morphism $\Psi : \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{h}) \to \Gamma(X, \mathcal{D}_\mathfrak{h})$ is constructed as the tensor product of the infinitesimal action of $\mathcal{U}(\mathfrak{g})$ and ϕ. The filtration on $\Gamma(X, \mathcal{D}_\mathfrak{h})$ is restricted and transferred to $\mathcal{U}(\mathfrak{h})$ via ϕ, and trivially imposed on $\mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{h})$:

   $$F_p(\mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{h})) = \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} F_p(\mathcal{U}(\mathfrak{h})) \tag{324}$$

   Ψ is compatible with these filtrations, inducing

   $$\mathrm{Gr}\Psi : \mathrm{Gr}(\mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{h})) = \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \bigoplus_W Z(\mathfrak{g}) \to \mathrm{Gr}(\Gamma(X, \mathcal{D}_\mathfrak{h})) = \bigoplus_W \mathcal{U}(\mathfrak{g}), \tag{325}$$

and this is evidently an isomorphism. Therefore, so is Ψ itself, as required. Q.E.D

This concludes the stacky presentation of algebraic Beilinson-Bernstein localization, and we now move on to an analytic analogue. With the previous assumptions, the formal group $\widehat{G}$ integrates uniquely to the overconvergent group $(1 \subseteq G)^\dagger$ (we contract to $G^\dagger$ as in [51]). Then we form the analogous correspondence:

$$*/G^\dagger \overset{a}{\leftarrow} Fl/G^\dagger \overset{b}{\rightarrow} \mathrm{Fl}/(G^\dagger/U^\dagger) \tag{326}$$

Then similarly, $\mathrm{Fl}/(G^\dagger/U^\dagger) \to \mathrm{Fl}/(G^\dagger/B^\dagger)$ is an $H^\dagger$-gerbe, and by the Riemann-Hilbert correspondence of [51], $\mathrm{Fl}/(G^\dagger/B^\dagger)$ is the analytic de Rham stack $\mathrm{Fl}^{\mathrm{an}}_{\mathrm{dR}}$.

The analytic Beilinson-Bernstein localization can be deduced from the algebraic theorem by comparing the correspondences:

$$\begin{tikzcd}
*/\widehat{G} \arrow[d, "p_1"] & \mathrm{Fl}/\widehat{G} \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "p_2"] & \mathrm{Fl}/(\widehat{G}/\widehat{U}) \arrow[d, "p_3"] \\
*/G^\dagger & \mathrm{Fl}/G^\dagger \arrow[l, "a"'] \arrow[r, "b"] & \mathrm{Fl}/(G^\dagger/U^\dagger)
\end{tikzcd}$$

The left square is Cartesian, and all the projections p_i induce fully faithful pullbacks on quasicoherent sheaves. By proper base change, that pulling back the bottom pull-push functor along p_1 is equivalent to the top pull-push functor applied to the pullback along p_3.

Analytic Beilinson Bernstein Localization ([51]) - The analytic pull-push functor $b^\dagger_\sharp a^{\dagger,*}$ is fully faithful, its left adjoint a Verdier quotient with Kernel those quasicoherent sheaves with vanishing global sections.

It is an equivalence when restricted to regular weights. In particular, for trivial infinitesimal characters, there is an equivalence

$$D_{\mathrm{qc}}(\mathrm{Fl}_{\mathrm{Betti}}) \cong D_{\mathrm{qc}}(*/G^\dagger) \otimes_{Z(\mathcal{U}(\mathfrak{g}))} \mathbb{C} \tag{327}$$

Proof - One need only show that all the functors involved commute with the fully faithful embeddings to quasicoherent sheaves on the algebraic stacks. This is clear for $a_* b^*$ by the proper base change observation above.

It must be shown that its adjoint $b_\sharp a^*$ sends $D_{\mathrm{qc}}(*/G^\dagger)$ to $D_{\mathrm{qc}}(\mathrm{Fl}/(G^\dagger/U^\dagger))$. Its fibres are given by the homology of $\widehat{U}$, but the representation comes from the restriction of $U^\dagger$, and so the homology agrees with that of $U^\dagger$.

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