Twistor theory and scattering amplitudes on strong curved backgrounds

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Abstract

We apply twistor theory to a variety of problems in quantum field theory in the presence of strong backgrounds, i.e. field configurations solving the classical equations of motions that are treated non-perturbatively. We give all-multiplicity formulae for any tree-level, colour-ordered MHV form factor around self-dual radiative gauge fields. We construct Killing spinors around a self-dual dyon background and use them to construct plane wave-like linear fields of arbitrary helicity propagating on the background. The twistor description of these fields allows to derive a compact formula for the tree-level, colour-ordered gluon MHV amplitude at all multiplicity. In the spirit of the (classical) double-copy, we use these results and a novel twistor sigma model to construct the n-point tree-level graviton MHV amplitude on the self-dual Taub-NUT space-time. As an application, we show that these amplitudes imply that the celestial chiral algebras are undeformed by the backgrounds, both in the gauge theory and gravitational sectors. We conclude by giving a twistorial derivation of the celestial chiral algebra on AdS_4 and propose an extension of this algebra around the unique self-dual metric in the AdS-Taub-NUT family.

Statement of Originality

This thesis presents work developed in the following publications, towards which the author contributed substantially. Chapter 3 is based on

[1] G. Bogna and L. Mason, *Yang-Mills form factors on self-dual backgrounds*, *JHEP* **08** (2023) 165.

Chapters 4 and 5 are based on

[2] T. Adamo, G. Bogna, L. Mason, and A. Sharma, *Scattering on self-dual Taub-NUT*, *Class. Quant. Grav.* **41** (2024) 015030.

[3] T. Adamo, G. Bogna, L. Mason, and A. Sharma, *Gluon scattering on the self-dual dyon*, *Lett. Math. Phys.* **115** (2025) 1, 18.

[4] T. Adamo, G. Bogna, L. Mason, and A. Sharma, *Graviton scattering on self-dual black holes*, arXiv:2507.18605 [hep-th].

Chapter 6 is based on

[5] R. Bittleston, G. Bogna, S. Heuveline, A. Kmec, L. Mason, and D. Skinner, *On AdS_4 deformations of celestial symmetries*, *JHEP* **07** (2024) 010.

[6] G. Bogna and S. Heuveline, *Towards celestial chiral algebras of self-dual black holes*, arXiv:2408.14324 [hep-th].

Chapter 2 is loosely based on the review sections of the publications above, as well as on the author's reports for the Transfer and Confirmation of Status.

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Contents

Chapter 1

Introduction

Over the past two decades, our understanding of quantum field theory and scattering amplitudes has improved dramatically. Sparked by Witten's seminal work [7], the study of scattering amplitudes has developed into a fully fledged and varied research field at the frontier of modern theoretical physics. This has led to the development of cutting edge computational tools [8–15] applications in collider physics [16], the discovery of geometric structures underlying amplitudes [17–19], and, perhaps surprisingly, insights into classical general relativity [20–22]. Although scattering amplitudes are just one class of observables in quantum field theory, they remain invaluable tools for advancing our understanding of quantum field and string theories – see [23–25] and references therein for recent reviews.

An important part of these important development stemmed from twistor theory [26–30]. Penrose initiated the twistor programme in the hope of unifying quantum theory and general relativity in a single, common framework, essentially replacing the role of space-time as the fundamental entity with twistor space – that is, roughly speaking, the space of null self-dual planes. Despite some initial success, it is fair to say that soon enough twistor theory branched out into pure mathematics, where it was the backbone of several important results, mainly in geometry [31–33] and integrable systems [34]. In a surprising turn of the events, with the advent of Witten and Berkovits' twistor strings [7, 35], twistor theory also saw a revival in its applications to physics. Among other things, twistor methods lead to the development of the MHV formalism [36, 37], to the amplitude/Wilson loop duality [38], and from the most compact known expressions for the tree-level graviton amplitudes [39–41], to the development of formulae to all multiplicity and all loops for the (integrand of) the amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills [42–44], as well as to all-multiplicity formulae for form factors [45–47]. Most known techniques developed for the construction of amplitudes have a direct twistorial counterpart – see for example [48] for a formulation of BCFW recursion [13] in twistor space, or [49] for the interplay between the CHY formulae and scattering equations of [15] and the ambitwistor strings.

In view of these successes in the amplitudes program, it is natural to ask whether it's possible to construct scattering amplitudes in the presence of non-trivial background fields, i.e. solutions to the classical equations of motion which are treated non-perturbatively. In gauge theory, such 'strong-field' scattering amplitudes play a key role in the study of non-linear regimes of QED due to intense electromagnetic fields [50–52] and in the high-energy, high-density regime of QCD describing heavy ion collisions [53–55]. We are still far away from having a complete understanding and a reasonable computational control on scattering amplitudes around generic backgrounds, as the traditional approach to this problem is the background field formalism [56–59], which requires establishing the background-coupled Feynman rules first; this preliminary task is rife with technical difficulties, even for highly-symmetric backgrounds where closed-form solutions for the background-coupled free fields are available, such as electromagnetic plane waves in QED [60, 61] or gluonic shockwaves in QCD [62, 63]. The resulting amplitudes are no longer rational functions and typically involve position space integrals which cannot be evaluated analytically and whose number grows with the number of external particles. These features dramatically change around self-dual backgrounds. The self-dual sector of gauge theory has been long known to be classically integrable [30,34], integrability being manifest in the twistor description of these backgrounds via the Ward correspondence [64]. Twistor methods have recently been successfully applied in this setting and lead to the computation of all-multiplicity formulae for gluon scattering around self-dual *radiative* background fields [65, 66],[1] that is self dual fields that are additionally required to be source-free and completely determined by their radiative data at null infinity. After providing a necessary introduction to twistor theory in Chapter 2, we extend this framework to the construction of tree-level MHV form factors in Chapter 3. Remarkably, we show that, much like the MHV gluon amplitude in this class of backgrounds, any tree-level MHV form factor can be obtained by dressing the corresponding expression for the form factor around the vacuum with a single space-time integral. This integral encodes both the kinematical data of the external gluons and the radiative data of the background field. Physically, it replaces the momentum-conserving δ function that is implicitly present around the vacuum, reflecting the explicit breaking of translation invariance by the background. While one might naively expect more complicated modifications involving multiple space-time integrals, it is striking that such a simple structure persists beyond the trivial background. This simplicity hints at the possibility of developing a background-coupled version of the MHV formalism [68].

The next logical step is to study background fields with sources: these were of course excluded from the radiative family by construction, but they are equally - if not more - relevant. The simplest background in this class is the *self-dual dyon*, which we investigate in Chapter 4. This is a solution of the Maxwell equations given by the superposition of a Coulomb field and a magnetic monopole, with equal electric and magnetic charges[2], thus it is the self-dual prototype of more realistic Coulomb profiles.

[1] And, correspondingly, for graviton scattering on self-dual radiative space-times [67].

[2] In Euclidean signature, where the self-dual dyon is a real field. In the Lorentzian slice, the field is complex-valued.

Scattering on Coulombic backgrounds has been studied extensively in the past and suffers the standard difficulties of scattering on curved backgrounds: the background-coupled linearized equations of motion can only be solved using separation of variables and a partial wave expansion, resulting in rather involved expressions for the resulting scattering amplitudes, already for 2-point tree-level amplitudes [69–73]. On the other hand, it is possible to derive *exact* expressions for the gluon wavefunctions around the self-dual dyon and, using a novel twistorial description of the self-dual dyon, these lead to a remarkably compact formula for the n-point, tree-level, colour-ordered MHV amplitude

$$\mathcal{A}_n(1^-,2^-,3^+,\ldots,n^+) \sim \mathrm{g}^{n-2}\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\int \mathrm{d}^4x \prod_{j=1}^{n}\varphi_j(x)\,, \tag{1.1}$$

where φ_j is a scalar wavefunction around the self-dual dyon, with the same quantum numbers as the j^{th} gluon. The amplitude mirrors the Parke-Taylor formula around the vacuum, again modified only by a single spacetime integral capturing the effect of the defect.

We then turn to investigate graviton amplitudes in the presence of black holes, where graviton scattering allows to study important properties both in classical and quantum gravity: in the classical context, the computation of probe or wave scattering off a black hole encodes classical observables including scattering angle, waveform and power emitted [69,74–80]. Indeed, the computation of scattering amplitudes with emission in curved background space-times provides the on-shell building blocks of the self-force expansion [70, 81–84], which systematizes corrections to geodesic motion due to radiation reaction [85, 86]. In the quantum context, scattering near black hole event horizons is a playground where emergent features of quantum gravity in strongly curved space-times can potentially be studied [87–94]. In addition to the usual technical issues already present in gauge theory, graviton scattering on curved space-times poses additional conceptual challenges, since the S-matrix does not exist as a unitary operator in the background QFT, due to the presence of the event horizon. This problem can be circumvented, at least perturbatively, by providing a purely variational definition of tree-level 'scattering amplitudes' as multi-linear pieces of the classical background field action, evaluated on recursively constructed solutions to the background-coupled equations of motion, via the so-called 'perturbiner' approach [95–101]. Even within the perturbiner formalism, an analytic approach which treats the black hole background non-perturbatively is currently out of reach for astrophysical black holes such as the Schwarzschild and Kerr metrics. Consequently, we will focus on self-dual analogues of black hole space-times. The simplest metric in this class is the self-dual Taub-NUT (SDTN) metric, that is the unique metric in the Taub-NUT family that has a vanishing anti-self-dual Weyl tensor [102–104]. In split signature, the metric is known to have a genuine event horizon, so it represents an integrable toy model of a black hole metric [105]; moreover, the SDTN metric can be constructed as the classical double-copy of the self-dual dyon [106], leading to remarkable simplifications in setting up the scattering problem. Being a self-dual Ricci-flat metric in four dimensions, the metric is also automatically hyperkähler, so we can describe it on twistor space using Penrose's non-linear graviton [29, 30]. Using this ingredients, in Chapter 5 we derive the n-point tree-level MHV graviton scattering amplitude on the SDTN space-time, which schematically reads

$$\mathcal{M}_n(1^-,2^-,3^+,\dots,n^+) \sim \kappa^{n-2}\frac{\langle 1\,2\rangle^6}{\langle 1\,i\rangle^2\langle 2\,i\rangle^2}\sum_{t\geq 0}\int \mathrm{d}^4x\,\sqrt{|g|} \\ \times \int_{(S^2)^t}\prod_{\mathrm{m}=1}^{t}\mathrm{d}^2\Omega_{\mathrm{m}}\mathcal{D}[t]|\mathcal{H}[t]^i_i|\prod_{j=1}^{n}\varphi_j(x)\,. \quad (1.2)$$

In this formula, φ_j is again a scalar wavefunction sharing the same quantum numbers of the j^{th} graviton – this time, propagating on the SDTN metric. The sum over the parameter t is a finite sum that, from a physical point of view, encodes tail effects that are expected for scattering processes on curved space-times, as a consequence of the failure of Huygens' principle beyond flat space [99, 107–110]. Each tail contribution contains t integrals over the 2-sphere, with round metric given by $\mathrm{d}^2\Omega$, and is described by a differential operator $\mathcal{D}[t]$ acting on the once-reduced determinant of an $(n+t-2)\times(n+t-2)$ matrix $\mathcal{H}[t]$, whose entries depend on the kinematics and the underlying SDTN metric. In particular, $|\mathcal{H}[t]^i_i|$ is the determinant of the matrix obtained by removing the i^{th} row and column from $\mathcal{H}[t]$, but the overall expression is independent of the choice of i. Equation (1.2) generalizes Hodges' formula [39], to which it reduces in the flat-space limit.

Parallel to these developments, recent years also have seen a reinvigorated interest in understanding quantum gravity in asymptotically flat space-times: the celestial holography programme [111–113] posits that quantum gravity in an asymptotically flat space-time is holographically dual to a 2d CFT living on the celestial sphere at null infinity, aptly named *celestial CFT* (CCFT). One of the greatest successes of the programme has been the description of collinear singularities of scattering amplitudes in the bulk by the celestial OPE of the dual CCFT [114, 115]: this led to the discovery of new infinite-dimensional symmetry algebras of tree-level amplitudes in flat space, respectively the S-algebra for gluons and $L\mathfrak{ham}(\mathbb{C}^2)$ for gravitons [116–118][3]. These symmetry algebras are generated by operators associated to soft positive-helicity gluons and gravitons and have since then been understood as symmetries of the self-dual sector of the respective theories. The symmetry action becomes transparent on twistor space [119], where one can realize the celestial symmetries as gauge and diffeomorphism symmetries, and have been known for quite some time [120, 121]. Most of celestial holography has been developed from bottom-up approaches starting from amplitudes around *flat* space, so it is important to understand whether these

[3]This algebra was also referred to as $Lw_{1+\infty}$ in the literature. The notation $L\mathfrak{ham}(\mathbb{C}^2)$ is perhaps more correct, as the symmetry algebra is the loop algebra of the wedge algebra of $w_{1+\infty}$, and emphasises its geometric origin from twistor space. $Lw_\wedge$ is another notation that occasionally pops up in the literature.

structures acquire non-trivial deformations on curved backgrounds. Several works exist in this direction, for example it is known that the $L\mathfrak{ham}(\mathbb{C}^2)$ is deformed on the Eguchi-Hanson space [122] or on the AdS_4 [123, 124], as well as on non-commutative space-times [125], while both the graviton and gluon symmetry algebras are undeformed by self-dual radiative backgrounds [126]. In the case of Burns space, a fully-fledged holographically dual chiral algebra was identified [127, 128] using methods from twisted holography [129, 130], but the gravitational part of the algebra is much larger than the previously discussed symmetry algebra [117], since the gravitational bulk theory is given by Mabuchi gravity, a self-dual subsector of conformal gravity, rather than Einstein gravity. In any case, Burns holography represents the best and most understood example of flat-space holography. On the gauge theory side, there is now a good understanding about (self-dual) monopoles and line defects [131, 132] and, more generally, on how to compute amplitudes and form factors from suitable chiral algebras [133, 134][4].

In this spirit, in Chapter 6, we investigate whether the gluon and graviton scattering amplitudes found in Chapters 4 and 5 allow for non-trivial deformations of the holographic symmetry algebras by studying their holomorphic collinear limits. Remarkably, the holomorphic splitting functions coincide with their flat-space counterpart, so the S- and $L\mathfrak{ham}(\mathbb{C}^2)$ algebras are not deformed by the self-dual dyon and SDTN metric, essentially in analogy with similar findings for the celestial chiral algebra around self-dual radiative backgrounds [126]. We conclude by describing non-trivial deformations to $L\mathfrak{ham}(\mathbb{C}^2)$ around some specific space-times with a non-vanishing cosmological constant, namely AdS_4 and the Pedersen metric, which can be thought of as the self-dual point inside the Taub-NUT-AdS family. This suggests that a notion of the celestial chiral algebra *persists* beyond asymptotically flat spaces, at least at tree level and in the self-dual sector.

[4]See also [135–144] for related works studying other relevant aspects of celestial holography in curved backgrounds.

Finally, in Chapter 7, we give some concluding remarks and discuss several possible extensions of this work.

The Appendices contain further technical material: Appendix A relates the twistorial descriptions of the self-dual dyon and SDTN metric developed in Chapters 4 and 5 to presentations of these backgrounds that appeared previously in the literature. Appendix B shows how it's possible to extend the construction of the SDTN to other gravitational instantons in the Gibbons-Hawking family. Finally, Appendix C contains some details on the computation of the space-time integrals that appear in the 2- and 3-point amplitudes.

Chapter 2

Background material

In this Chapter, we give an essential introduction to twistor theory. We begin by setting the conventions for spinor variables, then introduce twistor space and discuss several foundational theorems relating physics on space-time with geometry on twistor space, namely the Ward correspondence for gauge theory and the non-linear graviton for gravity. We conclude by reviewing the Penrose transform on flat space and its generalisation to non-trivial backgrounds, with an emphasis on gluon states propagating on self-dual gauge field backgrounds and on graviton states in self-dual space-times. For more comprehensive reviews, see [34, 145–150].

2.1 Spinor-helicity formalism

Let $x^a = (x^0, x^1, x^2, x^3)$ be coordinates on complexified Minkowski space $\mathbb{M}_{\mathbb{C}} = \mathbb{C}^4$, equipped with the standard Lorentzian metric with signature $(+,-,-,-)$. The double cover of the complexified Lorentz group $\mathrm{SO}(4,\mathbb{C})$ is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, in particular the tangent bundle decomposes as $T\mathbb{C}^4 = \mathbb{S} \otimes \tilde{\mathbb{S}}$, where $\mathbb{S}$ and $\tilde{\mathbb{S}}$ are the bundles of undotted and dotted spinors, transforming under the fundamental representation of the left and right $\mathrm{SL}(2,\mathbb{C})$ factors, respectively. These are commonly referred to as negative- and positive-helicity spinors. The decomposition of the tangent bundle further allows us to decompose the space-time coordinates in spinor form as

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}, \tag{2.1}$$

and similarly for any vector field on $\mathbb{M}_{\mathbb{C}}$.

The bundles $\mathbb{S}$ and $\tilde{\mathbb{S}}$ are naturally equipped with symplectic forms $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ that can be used to identify the spinor bundles of a given chirality with their duals, and whose tensor product is identified with the flat metric on $\mathbb{C}^4$. In more practical terms, this means that we can trade any vector index for a pair of one dotted and one undotted spinor indices [23]. Spinor indices are raised and lowered according to [147]

$$\lambda_\alpha = \lambda^\beta \epsilon_{\beta\alpha}, \qquad \lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \qquad \tilde{\lambda}_{\dot{\alpha}} = \tilde{\lambda}^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}}, \quad \tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}, \tag{2.2}$$

where the symplectic forms are given by

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.3}$$

The Levi-Civita symbols satisfy $\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta^\alpha{}_\beta$ and $\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}}$. With these forms, we can define skew-symmetric brackets between spinors of the same chirality, namely

$$\langle\lambda\,\xi\rangle = \lambda^\alpha\xi_\alpha, \qquad [\tilde{\lambda}\,\tilde{\xi}] = \tilde{\lambda}^{\dot{\alpha}}\tilde{\xi}_{\dot{\alpha}}. \tag{2.4}$$

The metric then decomposes as $\mathrm{d}s^2 = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\mathrm{d}x^{\alpha\dot{\alpha}}\mathrm{d}x^{\beta\dot{\beta}}$, so the square of an arbitrary 4-vector v^a satisfies

$$v^a v_a = v^{\alpha\dot{\alpha}}v_{\alpha\dot{\alpha}} = 2\det(v^{\alpha\dot{\alpha}}). \tag{2.5}$$

In particular, a 4-vector is null if and only if the corresponding matrix is the outer product of two 2-spinors. We will always consider scattering states for massless fields, so their momentum vector will be null. We decompose any massless 4-momentum $k^{\alpha\dot{\alpha}}$ as $\kappa^{\alpha}\tilde{\kappa}^{\dot{\alpha}}$, where we adopt the following affine parametrization of κ^{α} and $\tilde{\kappa}^{\dot{\alpha}}$

$$\kappa^{\alpha} = (1, z)\,, \qquad \tilde{\kappa}^{\dot{\alpha}} = \frac{\sqrt{2}\,\omega}{1 + z\tilde{z}}(1, \tilde{z})\,, \tag{2.6}$$

so that the pair $(z, \tilde{z}) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ represents the (complexified) direction at which the massless state reaches null infinity on the (complexified) celestial sphere. Conversely, $\omega = k^0$ is simply the energy of the state. Note the asymmetry between κ^{α} and $\tilde{\kappa}^{\dot{\alpha}}$: while amplitude literature often uses conventions that treat positive- and negative-helicity spinors more symmetrically [23], we adopt this convention because it aligns naturally with the chiral nature of twistor space.

2.1.1 Reality conditions

Throughout this thesis, we primarily work in the complexified setting and remain agnostic about the signature of the space-time metric. This is a natural choice from the perspective of twistor theory, which is inherently holomorphic. Nonetheless, it is useful, and sometimes desirable, to consider real slices – such as Lorentzian, Euclidean, or split signature – and these can all be obtained via appropriate reality conditions on the spinors, which in turn induce corresponding conditions on space-time coordinates.

The Lorentzian slice $\mathbb{M} \subseteq \mathbb{M}_{\mathbb{C}}$ corresponds to $x^a \in \mathbb{R}$, $a = 0, \ldots, 3$, i.e. to Hermitian $x^{\alpha\dot{\alpha}}$. This reality condition amounts to requiring that the space-time point is fixed under the conjugation induced by the standard chirality-changing spinor conjugations

$$\psi^{\alpha} = (\psi^0, \psi^1) \longmapsto \bar{\psi}^{\dot{\alpha}} = (\overline{\psi^0}, \overline{\psi^1})\,, \qquad \chi^{\dot{\alpha}} = (\chi^{\dot{0}}, \chi^{\dot{1}}) \longmapsto \bar{\chi}^{\alpha} = (\overline{\chi^{\dot{0}}}, \overline{\chi^{\dot{1}}}) \tag{2.7}$$

Conversely, the Euclidean slice $\mathbb{E} \subseteq \mathbb{M}_{\mathbb{C}}$ is given by $x^0 \in \mathbb{R}$, $x^i \in \mathrm{i}\mathbb{R}$, $i = 1, 2, 3$, so it is defined by the condition $x^{\alpha\dot{\alpha}} = \hat{x}^{\alpha\dot{\alpha}}$, where we introduced the quaternionic conjugation

$$\psi^\alpha = (\psi^0, \psi^1) \longmapsto \hat{\psi}^\alpha = (-\overline{\psi^1}, \overline{\psi^0}), \qquad \chi^{\dot\alpha} = (\chi^{\dot 0}, \chi^{\dot 1}) \longmapsto \hat{\chi}^{\dot\alpha} = (-\overline{\chi^{\dot 1}}, \overline{\chi^{\dot 0}}) \tag{2.8}$$

The conjugation is quaternionic in the sense that it squares to -1.

Finally, consider the Kleinian slice $\mathbb{K} \subseteq \mathbb{M}_\mathbb{C}$ defined by $x^0, x^1, x^3 \in \mathbb{R}$, $x^2 \in \mathrm{i}\mathbb{R}$. This slice is the fixed locus of the conjugation generated by the component-wise complex conjugation

$$\psi^\alpha = (\psi^0, \psi^1) \longmapsto \psi^{*\,\alpha} = (\overline{\psi^0}, \overline{\psi^1}), \qquad \chi^{\dot\alpha} = (\chi^{\dot 0}, \chi^{\dot 1}) \longmapsto \chi^{*\,\dot\alpha} = (\overline{\chi^{\dot 0}}, \overline{\chi^{\dot 1}}) \tag{2.9}$$

2.2 Twistor theory

The twistor space of $\mathbb{M}_\mathbb{C}$ is a suitable open subset of $\mathbb{CP}^3$. Introduce homogeneous coordinates Z^A, $A = 1, \dots, 4$ on $\mathbb{CP}^3$: the complexified conformal group $\mathrm{SL}(4, \mathbb{C})$ of $\mathbb{M}_\mathbb{C}$ acts linearly on the homogeneous coordinates, in particular we can split them as $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$, where $\alpha = 0, 1$, $\dot\alpha = \dot 0, \dot 1$ are spinor indices of opposite chirality. Twistor space is the subset obtained from $\mathbb{CP}^3$ by removing a sphere

$$\mathbb{PT} = \mathbb{CP}^3 \backslash \mathbb{CP}^1 = \{Z^A \in \mathbb{CP}^3 \,|\, \lambda_\alpha \neq 0\}\,. \tag{2.10}$$

Denoting the holomorphic line bundles over the Riemann sphere whose sections are homogeneous functions of homogeneity degree n in λ_α by $\mathcal{O}(n) \to \mathbb{CP}^1$, we can identify $\mathbb{PT}$ with the total space of the rank-2 bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$, where λ_α are coordinates on the $\mathbb{CP}^1$ base and $\mu^{\dot\alpha}$ are coordinates up the fibres. The pull-back of $\mathcal{O}(n)$ to $\mathbb{PT}$ by the holomorphic projection $\mathbb{PT} \to \mathbb{CP}^1$ will still be denoted by $\mathcal{O}(n)$, as its sections are homogeneous functions of degree n in Z^A, and we will also use the notation $\mathcal{O}(0) \equiv \mathcal{O}$.

The relationship between space-time and twistor space is encoded in the incidence relations

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha \,. \tag{2.11}$$

Points $x \in \mathbb{M}_\mathbb{C}$ correspond to linearly and holomorphically embedded Riemann spheres $X \subset \mathbb{PT}$. We will refer to these spheres as *twistor lines*. Conversely, it's easy to show that any holomorphic and linear embedding of a Riemann sphere in $\mathbb{PT}$ can be put in the form of the incidence relations [147]. In particular, the definition (2.11) can be viewed as the excision of the twistor line $I^0 \subseteq \mathbb{CP}^3$ corresponding to spatial infinity i^0.

On the other hand, points in $\mathbb{PT}$ correspond to α-planes in $\mathbb{M}_\mathbb{C}$ [26], that is, to totally null 2-planes with self-dual tangent bivectors. Equivalently, two twistor lines X', X'' intersect in a twistor $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$ if and only if the corresponding space-time points satisfy $(x' - x'')^{\alpha\dot{\alpha}} = \omega^{\dot{\alpha}}\lambda^\alpha$, for some arbitrary spinor $\omega^{\dot{\alpha}}$; in particular, they are null separated. Thus, the incidence relations give a non-local relation between space-time and twistor space, realising $\mathbb{M}_\mathbb{C}$ as the moduli space of linearly and holomorphically embedded Riemann spheres in $\mathbb{PT}$, and realising $\mathbb{PT}$ as the moduli space of α-planes of $\mathbb{M}_\mathbb{C}$.

We can summarise these properties in terms of a double fibration

$$\begin{tikzcd} & \mathbb{PS} \arrow[dl, "\pi_{\mathbb{PT}}"'] \arrow[dr, "\pi_{\mathbb{M}_\mathbb{C}}"] & \\ \mathbb{PT} & & \mathbb{M}_\mathbb{C} \end{tikzcd}$$

where the *correspondence space* $\mathbb{PS} \cong \mathbb{M}_\mathbb{C} \times \mathbb{CP}^1$ is the projectivised undotted spinor bundle over $\mathbb{M}_\mathbb{C}$. $\pi_{\mathbb{M}_\mathbb{C}}$ is the trivial projection to $\mathbb{M}_\mathbb{C}$, $\pi_{\mathbb{M}_\mathbb{C}} \colon (x^{\alpha\dot{\alpha}}, \lambda_\alpha) \mapsto x^{\alpha\dot{\alpha}}$, while $\pi_{\mathbb{PT}}$ imposes the incidence relations, $\pi_{\mathbb{PT}} \colon (x^{\alpha\dot{\alpha}}, \lambda_\alpha) \mapsto (x^{\beta\dot{\alpha}}\lambda_\beta, \lambda_\alpha)$.

2.2.1 Euclidean twistor space

The holomorphic properties of twistor space are best understood in the Euclidean setting [147, 149, 151], which we now describe. In terms of the spinor conjugation (2.8), a point $x^{\alpha\dot\alpha}$ lies in the Euclidean slice $\mathbb{E} \subseteq \mathbb{M}_{\mathbb{C}}$ if and only if $x^{\alpha\dot\alpha} = \hat{x}^{\alpha\dot\alpha}$, so it follows that for any twistor $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ lying on the line X, the conjugate twistor $\hat{Z}^A = (\hat\mu^{\dot\alpha}, \hat\lambda_\alpha)$ lies on the same line too. Any twistor line can be reconstructed by knowing any two of its points, and indeed, the incidence relations can be readily inverted in Euclidean signature

$$x^{\alpha\dot\alpha} = \frac{\hat\mu^{\dot\alpha}\lambda^\alpha - \mu^{\dot\alpha}\hat\lambda^\alpha}{\langle\lambda\hat\lambda\rangle}\,. \tag{2.12}$$

This map defines a (non-holomorphic) diffeomorphism $\mathbb{PT}_{\mathbb{E}} \cong \mathbb{PS}_{\mathbb{E}} \cong \mathbb{E}\times\mathbb{CP}^1$ between Euclidean twistor space $\mathbb{PT}_{\mathbb{E}}$ and the bundle of projectivised undotted spinors of $\mathbb{E}$. and endows twistor space with a holomorphic fibration $\mathbb{PT}_{\mathbb{E}} \to \mathbb{E}$.[1] The existence of this fibration is rather intuitive since in Euclidean signature there are no null vectors, i.e. two twistor lines never intersect.

In $(x^{\alpha\dot\alpha}, \lambda_\alpha)$ coordinates, a basis of $(0,1)$-forms on $\mathbb{PT}_{\mathbb{E}}$ is given by

$$\bar{e}^0 = \frac{\mathrm{D}\hat\lambda}{\langle\lambda\hat\lambda\rangle^2}\,, \qquad \bar{e}^{\dot\alpha} = \frac{\hat\lambda_\alpha\,\mathrm{d}x^{\alpha\dot\alpha}}{\langle\lambda\hat\lambda\rangle}\,, \tag{2.13}$$

where $\mathrm{D}\hat\lambda = \langle\hat\lambda\,\mathrm{d}\hat\lambda\rangle$. The dual basis of $(0,1)$-vectors is

$$\bar\partial_0 = \langle\lambda\hat\lambda\rangle\lambda_\alpha\frac{\partial}{\partial\hat\lambda_\alpha}\,, \qquad \bar\partial_{\dot\alpha} = \lambda^\alpha\frac{\partial}{\partial x^{\alpha\dot\alpha}}\,. \tag{2.14}$$

Similarly, a basis of $(1,0)$-forms and the corresponding dual basis of $(1,0)$-vectors are

[1] Of course, this fibration exists for the twistor space of complexified Minkowski space too, but Euclidean signature allows for a more explicit discussion.

$$
\begin{aligned}
e^0 &= \mathrm{D}\lambda\,, \qquad e^{\dot\alpha} = \lambda_\alpha\, \mathrm{d}x^{\alpha\dot\alpha}\,, && \text{(2.15a)}\\
\partial_0 &= \frac{\hat\lambda_\alpha}{\langle\lambda\hat\lambda\rangle}\frac{\partial}{\partial\lambda_\alpha}\,, \qquad \partial_{\dot\alpha} = -\frac{\hat\lambda^\alpha}{\langle\lambda\hat\lambda\rangle}\frac{\partial}{\partial x^{\alpha\dot\alpha}}\,, && \text{(2.15b)}
\end{aligned}
$$

with $\mathrm{D}\lambda = \langle\lambda\, \mathrm{d}\lambda\rangle$. This complex structure is compatible with the Euclidean spinor conjugation, as it can be checked straightforwardly that the Dolbeault operator $\bar\partial$ defined by (2.13) and (2.14) coincides with the one computed in $(\mu^{\dot\alpha}, \lambda_\alpha)$ coordinates

$$
\bar\partial = \bar e^0\, \partial_0 + \bar e^{\dot\alpha}\, \partial_{\dot\alpha} = \mathrm{d}\hat Z^A \frac{\partial}{\partial \hat Z^A}\,. \tag{2.16}
$$

2.2.2 Gauge theory on twistor space

In the following Chapters, we will study aspects of gauge theory around non-trivial, self-dual backgrounds. Recall that, in the spinor-helicity formalism, we can represent the curvature of a gauge connection as a field $F_{\alpha\dot\alpha\beta\dot\beta} = -F_{\beta\dot\beta\alpha\dot\alpha}$. Skewness of the curvature implies the decomposition

$$
F_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\alpha\beta}\tilde F_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}\,. \tag{2.17}
$$

$F_{\alpha\beta}$ and $\tilde F_{\dot\alpha\dot\beta}$ represent the anti-self-dual (ASD) and self-dual (SD) components of the curvature, respectively. The self-dual sector corresponds to fields satisfying $F_{\alpha\beta} = 0$. It is possible to impose this condition as the equation of motion of the Chalmers-Siegel action for self-dual Yang-Mills [152, 153]

$$
S_{\text{SDYM}} = \int \mathrm{d}^4x\, \mathrm{tr}\,(B_{\alpha\beta}F^{\alpha\beta})\,. \tag{2.18}
$$

Here, $B_{\alpha\beta}$ is a Lagrange multiplier imposing the self-duality condition. The full, non-self-dual theory admits a perturbative expansion around the self-dual sector, as the ordinary Yang-Mills action can be written as

$$S_{\mathrm{YM}} = \int \mathrm{d}^4x \, \mathrm{tr}\,(B_{\alpha\beta}F^{\alpha\beta}) - \frac{1}{2}\mathrm{g}^2 \int \mathrm{d}^4x \, \mathrm{tr}\,(B_{\alpha\beta}B^{\alpha\beta})\,, \tag{2.19}$$

up to a θ term. The equations of motion

$$F_{\alpha\beta} = \mathrm{g}^2 B_{\alpha\beta}\,, \qquad D^{\alpha\dot{\alpha}}B_{\alpha\beta} = 0\,, \tag{2.20}$$

now identify $B_{\alpha\beta}$ with the ASD component of the field strength.

Self-dual gauge theory admits an elegant description on twistor space, via the classical result of Ward [64][2]

Theorem 1 (Ward [64, 154])**.** *There exists a one-to-one correspondence between*

- *self-dual gauge fields on* $\mathbb{M}_{\mathbb{C}}$*,*
- *holomorphic vector bundles* $E \to \mathbb{PT}$ *such that the restriction* $E|_X$ *to any twistor line is trivial.*

To see how the correspondence arises, given a self-dual field on $\mathbb{M}_{\mathbb{C}}$ and a twistor $Z \in \mathbb{PT}$, the restriction of the curvature of the gauge field to the corresponding α-plane π_Z vanishes, so we can define the fibre $E|_Z$ at Z to be the space of $\mathbb{C}^r$-valued functions that are covariantly constant on π_Z. This space is well-defined and has the structure of an r-dimensional complex vector space, so it naturally defines a holomorphic bundle $E \to \mathbb{PT}$. Given $x \in \mathbb{M}_{\mathbb{C}}$ and an α-plane $Z \in \mathbb{PT}$ through x, the restriction of the bundle to the twistor line X is trivial because for any $s_0 \in \mathbb{C}^r$ we can locally solve for an element $s_Z \in E|_Z$ satisfying $s_Z(x) = s_0$. Varying among

[2] The theorem was originally formulated for gauge fields with gauge group $\mathrm{GL}(r, \mathbb{C})$, but additional constraints allow to consider other gauge groups; for example, we can equip E with a positive real form to restrict the gauge group to $\mathrm{U}(r)$. If $\det E$ is trivial, then the gauge group is further restricted to $\mathrm{SU}(r)$. Similarly, the correspondence can be adapted to gauge fields on suitably chosen regions $R \subseteq \mathbb{M}_{\mathbb{C}}$ and holomorphic vector bundles over the subset of $\mathbb{PT}$ swept out by lines corresponding to points in R [34, 148].

all possible α-planes through z and among a basis of $\mathbb{C}^r$, we can construct r global sections of $E|_X$.

If instead we consider a bundle $E \to \mathbb{PT}$ with gauge group G and associated Lie algebra $\mathfrak{g}$, we can think of E as a complex vector bundle equipped with a $\bar{\partial}$-operator $\bar{D} = \bar{\partial} + \mathsf{a}$, where $\mathsf{a} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O} \otimes \mathfrak{g})$. Holomorphicity of the bundle is equivalent to the integrability condition

$$F^{0,2} = \bar{D}^2 = \bar{\partial}\mathsf{a} + \mathsf{a} \wedge \mathsf{a} = 0\,. \tag{2.21}$$

If $X \subseteq \mathbb{PT}$ is any twistor line, it follows that $E|_X$ is a holomorphic bundle with $\bar{\partial}$ operator $\bar{D}|_X$. In perturbation theory, we can assume that $E|_X$ is *holomorphically* trivial, in particular there exists a frame $\mathsf{H}\colon\ E|_X \to \mathbb{C}^r$ satisfying the Sparling equation [155]

$$\bar{D}\big|_X \mathsf{H}(x, \lambda) = 0\,. \tag{2.22}$$

The holomorphic frame is defined up to right multiplication by a matrix-valued function on spacetime, $\mathsf{H}(x,\lambda) \mapsto \mathsf{H}(x,\lambda)g(x)$. Assuming that the twistor connection a only depends on $(\mu^{\dot\alpha}, \lambda_\alpha, \hat\lambda_\alpha)$, one quickly finds that the restriction $\mathsf{a}|_X$ is annihilated by the (0,1)-vector field $\bar{\partial}_{\dot\alpha}$ in (2.14), as a consequence of the incidence relations (2.11), therefore

$$\bar{\partial}\big|_X (\mathsf{H}^{-1}\lambda^\alpha\partial_{\alpha\dot\alpha}\mathsf{H}) = 0\,. \tag{2.23}$$

This means that $\mathsf{H}^{-1}\lambda^\alpha\partial_{\alpha\dot\alpha}\mathsf{H}$ is a holomorphic function on the sphere with weight $+1$ in the homogeneous coordinate λ_α. An extension of Liouville's theorem then ensures the existence of a gauge field $A_{\alpha\dot\alpha}$ on $\mathbb{M}_\mathbb{C}$ such that

$$\mathsf{H}^{-1}(x,\lambda)\lambda^\alpha\partial_{\alpha\dot\alpha}\mathsf{H}(x,\lambda) = -\mathrm{i}\lambda^\alpha A_{\alpha\dot\alpha}(x)\,. \tag{2.24}$$

The freedom of multiplying the frame by g is now understood as the usual gauge freedom on $\mathbb{M}_{\mathbb{C}}$. Introducing the covariant derivative $D = \mathrm{d} - \mathrm{i}A$ on $\mathbb{M}_{\mathbb{C}}$, we can recast the equation defining $A_{\alpha\dot\alpha}$ as

$$\lambda^\alpha D_{\alpha\dot\alpha}\mathsf{H} = 0\,, \tag{2.25}$$

so that

$$\lambda^\alpha\lambda^\beta F_{\alpha\dot\alpha\beta\dot\beta}\mathsf{H} = [\lambda^\alpha D_{\alpha\dot\alpha}, \lambda^\beta D_{\beta\dot\beta}]\mathsf{H} = 0\,. \tag{2.26}$$

Since this equation holds for any λ^α, we conclude that $A_{\alpha\dot\alpha}$ is self-dual as desired. We will see in Chapter 3 and Chapter 4 how to explicitly reconstruct the space-time gauge field directly from the twistor connection a, for radiative gauge fields and for the self-dual dyon, respectively.

Twistor action for the self-dual sector

The integrability condition (2.21) can be understood as the equation of motion for the action of holomorphic BF theory on twistor space

$$S_{\mathrm{BF}} = \int_{\mathbb{PT}} \mathrm{D}^3 Z \wedge \,\mathrm{tr}\,\left(\mathsf{b} \wedge (\bar\partial \mathsf{a} + \mathsf{a} \wedge \mathsf{a})\right)\,. \tag{2.27}$$

In this action, $\mathrm{D}^3 Z \in \Omega^{1,0}(\mathbb{PT}, \mathcal{O}(4))$ is the weight-4 holomorphic top form

$$\mathrm{D}^3 Z := \frac{1}{4!}\epsilon_{ABCD} Z^A \mathrm{d}Z^B \wedge \mathrm{d}Z^C \wedge \mathrm{d}Z^D\,, \tag{2.28}$$

where ϵ_{ABCD} is the 4-dimensional Levi-Civita symbol, and $\mathsf{b} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathfrak{g})$ in order to have a well-defined integral. The action can also be understood as the $\mathcal{N} = 0$ truncation of the action for holomorphic Chern-Simons theory on supertwistor space $\mathbb{CP}^{3|4}$ [7, 156]. The equation of motion for a simply states that b is closed with respect to $\bar D$. Gauge transformations are generated by $\chi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O} \otimes \mathfrak{g})$ and

$\psi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathfrak{g})$ as

$$\delta \mathsf{a} = \bar{\partial}\chi + [\mathsf{a}, \chi]\,, \qquad \delta \mathsf{b} = [\chi, \mathsf{b}] + \bar{\partial}\psi + [\mathsf{a}, \psi]\,. \tag{2.29}$$

In particular, the field b is an element of the cohomology group $H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathfrak{g})$ when put on-shell, where the cohomology is with respect to the background $\bar{\partial}$-operator $\bar{D}$.

In Euclidean signature, we can dimensionally reduce the twistor action along the fibres of the fibration $\mathbb{PT}_{\mathbb{E}} \to \mathbb{E}$. In Woodhouse gauge [149], we require the fields to be holomorphic up the fibres of the fibration. This allows to reduce the gauge freedom on twistor space to the smaller subset of gauge transformations that only depend on the space-time coordinates. After integrating along the fibres, one finds the Chalmers-Siegel action for the self-dual sector (2.18) [157]. We will see in Chapter 3 that the non-self-dual term in the full action (2.19) is non-local when lifted to twistor space, and generates the gluon MHV amplitude. This is a common feature: self-dual interactions are described by local theories on twistor space and deformations away from self-duality are captured by non-local terms.

2.2.3 Gravity on twistor space

The previous discussion readily extends to (self-dual) gravity. Consider a space-time $\mathscr{M}$ with vanishing cosmological constant and equipped with a Riemannian metric g. Fix a tetrad θ^a, $a = 0, \ldots, 3$ and define[3]

$$\theta^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta^0 + \mathrm{i}\theta^3 & \mathrm{i}\theta^1 + \theta^2 \\ \mathrm{i}\theta^1 - \theta^2 & \theta^0 - \mathrm{i}\theta^3 \end{pmatrix}, \tag{2.30}$$

[3]Note the slightly different convention from the spinor-helicity variables in (2.1). The definition of $\theta^{\alpha\dot{\alpha}}$ is best adapted to Euclidean signature and will also be used in Chapter 5 instead of (2.1).

so that the metric is $g = \theta^{\alpha\dot\alpha} \otimes \theta_{\alpha\dot\alpha}$. We can use this definition to exchange vector indices for pairs of spinor indices of opposite chirality, exactly in the same way as we did in flat space. In particular, the Weyl tensor C_{abcd} associated to the metric admits a decomposition akin to (2.17) into its SD and ASD components $\tilde\psi_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ and $\psi_{\alpha\beta\gamma\delta}$

$$C_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\delta\dot\delta} = \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\tilde\psi_{\dot\alpha\dot\beta\dot\gamma\delta} + \epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta}\psi_{\alpha\beta\gamma\delta}\,. \tag{2.31}$$

The space-time $(\mathscr{M}, g)$ is self-dual if $\psi_{\alpha\beta\gamma\delta} = 0$. In four dimensions, if the metric is also Ricci-flat, then it is also automatically hyperkähler. To see this, first construct bases for the SD and ASD 2-forms on $\mathscr{M}$

$$\tilde\Sigma^{\dot\alpha\dot\beta} = \tilde\Sigma^{(\dot\alpha\dot\beta)} = \theta^{\alpha\dot\alpha} \wedge \theta_\alpha{}^{\dot\beta}\,, \qquad \Sigma^{\alpha\beta} = \Sigma^{(\alpha\beta)} = \theta^{\alpha\dot\alpha} \wedge \theta^\beta{}_{\dot\alpha}\,. \tag{2.32}$$

The curvature of $\mathscr{M}$ is encoded in the SD and ASD spin connection 1-forms $\tilde\Gamma^{\dot\alpha}{}_{\dot\beta}$ and $\Gamma^\alpha{}_\beta$ and, in particular, the curvature $R^\alpha{}_\beta$ of the ASD spin connection decomposes as

$$R_{\alpha\beta} = \mathrm{d}\Gamma_{\alpha\beta} + \Gamma_{\alpha\gamma} \wedge \Gamma_\beta{}^\gamma = \psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta} + \Phi_{\alpha\dot\alpha\beta\dot\beta}\tilde\Sigma^{\dot\alpha\dot\beta} + \frac{1}{12}R\Sigma_{\alpha\beta}\,, \tag{2.33}$$

where $\Phi_{\alpha\dot\alpha\beta\dot\beta}$ is the spinor equivalent of the trace-free component of the Ricci tensor and R is the scalar curvature. From (2.33), it's clear that a self-dual, Ricci-flat 4-dimensional manifold has a flat connection on the undotted spinor bundle; in particular, it has holonomy SU(2) and the metric is hyperkähler. In this case, we can perform a diffeomorphism so that the ASD spin connection vanishes. As a result, the ASD 2-forms are closed. Again, we can formulate general relativity as an expansion around the self-dual sector: introducing the gravitational coupling $\kappa^2 = 16\pi G$, consider the Plebański action [158]

$$S_{\mathrm{GR}}[\theta, \Gamma] = \int_{\mathscr{M}} \Sigma^{\alpha\beta} \wedge (\mathrm{d}\Gamma_{\alpha\beta} + \kappa^2 \Gamma_\alpha{}^\gamma \wedge \Gamma_{\beta\gamma})\,, \tag{2.34}$$

where the fundamental fields are the tetrad and the ASD spin connection. The equations of motion for this action are thus

$$\mathrm{d}\Sigma^{\alpha\beta} = 2\kappa^2 \Gamma^{(\alpha}{}_{\gamma} \wedge \Sigma^{\beta)\gamma}\,, \qquad \theta^{\alpha\dot{\alpha}} \wedge R_{\alpha\beta} = 0\,, \tag{2.35}$$

and are known to be equivalent to the Einstein equations [159]. In the limit $\kappa^2 \to 0$, the action reduces to the action for self-dual gravity

$$S_{\mathrm{SDGR}}[\theta, \Gamma] = \int_{\mathscr{M}} \Sigma^{\alpha\beta} \wedge \mathrm{d}\Gamma_{\alpha\beta}\,, \tag{2.36}$$

where Γ ceases to be interpreted as the ASD spin connection, but it's rather a Lagrange multiplier imposing the closure of the ASD 2-forms. It is also possible to derive actions where the fundamental fields are the ASD 2-forms and the spin connection and with a non-vanishing cosmological constant [160].

On twistor space, there's an analogous construction to Ward's correspondence:

Theorem 2 (Non-linear graviton [29, 30, 161])**.** *There exists a one-to-one correspondence between*

- *(suitably convex regions of) self-dual, Ricci-flat Riemannian 4-manifolds* (M, g)*,*
- *complex 3-folds* $\mathbb{P}\mathscr{T}$ *that are a complex deformations of the neighbourhood of a line in* $\mathbb{PT}$ *and are equipped with*
  - *a holomorphic fibration* $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$*,*
  - *a 4-complex parameter family of holomorphic sections with normal bundle* $\mathcal{O}(1) \oplus \mathcal{O}(1)$*,*
  - *an* $\mathcal{O}(2)$*-valued symplectic form on each fibre,*
  - *an anti-holomorphic involution* $j\colon \mathbb{P}\mathscr{T} \to \mathbb{P}\mathscr{T}$ *that induces the antipodal map on* $\mathbb{CP}^1$ *and picks a 4-real parameter family of sections invariant under*

it.

The proof of this theorem is more involved than the proof for Ward's correspondence, so we will limit ourselves to describe briefly the reconstruction of the space-time and the metric from the holomorphic data on twistor space.

Locally, the deformation of the complex structure is described by a $(0,1)$-form valued in the holomorphic tangent bundle $V \in \Omega^{0,1}(\mathbb{PT}, T^{1,0}\mathbb{PT})$ and satisfying the integrability condition

$$\bar{\partial}V + \frac{1}{2}[V,V] = 0\,, \tag{2.37}$$

where $[-\,,-]$ is the Lie bracket. Modelling the deformation on the flat twistor space with coordinates $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$, we will assume that λ_α are still holomorphic, hence providing the desired fibration $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$. The $\mathcal{O}(2)$-valued symplectic form for flat space Σ can be dually defined in terms of an $\mathcal{O}(-2)$-valued Poisson bracket $\{-\,,-\}$ acting on arbitrary forms ω_1, ω_2

$$\Sigma = [\mathrm{d}\mu \wedge \mathrm{d}\mu]\,, \qquad \{\omega_1, \omega_2\} = \mathcal{L}_{\dot\alpha}\omega_1 \wedge \mathcal{L}^{\dot\alpha}\omega_2\,. \tag{2.38}$$

Here, $\mathcal{L}_{\dot\alpha}$ is the Lie derivative with respect to the vector field $\partial/\partial\mu^{\dot\alpha}$. The symplectic form is holomorphic in the deformed complex structure if we assume $V = \{\mathsf{h}\,, -\}$ to be Hamiltonian with respect to a Hamiltonian $\mathsf{h} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ satisfying

$$\bar{\partial}\mathsf{h} + \frac{1}{2}\{\mathsf{h}, \mathsf{h}\} = 0\,. \tag{2.39}$$

In the flat case, the family of holomorphic sections were simply given by the twistor line defined by (2.11), as they provided a holomorphic embedding of the Riemann sphere after the identification of the coordinate on the sphere with λ_α. The remaining coordinates $\mu^{\dot\alpha}$ are then fixed to be linear functions of λ_α and we parametrised the moduli with points in space-time. In the curved case, the construction is fairly similar:

first, we *define* the (complexified) space-time $\mathscr{M}_{\mathbb{C}}$ as the moduli space of twistor lines, that is as the space of holomorphic sections with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Kodaira's theory ensures that $\mathscr{M}_{\mathbb{C}}$ is four-dimensional [162, 163]. Since λ_α is still holomorphic in the deformed complex structure, we will fix the gauge freedom in the twistor lines to have $\lambda_\alpha = \sigma_\alpha$ – equivalently, we are identifying the base of the fibration $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$ with the Riemann sphere of the projectivised undotted spinor bundle $\mathscr{M}_{\mathbb{C}} \times \mathbb{CP}^1$. Conversely, the coordinates $\mu^{\dot\alpha}$ are no longer holomorphic, and the twistor lines are thus described by a pair of maps $\mathsf{F}^{\dot\alpha}\colon M_{\mathbb{C}} \times \mathbb{CP}^1 \to \mathbb{P}\mathscr{T}$ on the undotted spinor bundle of $\mathscr{M}_{\mathbb{C}}$ satisfying the curved incidence relation

$$\bar\partial\big|_X \mathsf{F}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\, \mathcal{L}_{\dot\beta}\mathsf{h}\big|_X \,. \tag{2.40}$$

This means that a double fibration is also available in the case of a curved space-time

$$\begin{tikzcd} & \mathscr{M}_{\mathbb{C}} \times \mathbb{CP}^1 \arrow[dl, "\pi_{\mathbb{P}\mathscr{T}}"'] \arrow[dr, "\pi_{\mathscr{M}_{\mathbb{C}}}"] & \\ \mathbb{P}\mathscr{T} & & \mathscr{M}_{\mathbb{C}} \end{tikzcd}$$

where $\pi_{\mathscr{M}_{\mathbb{C}}}$ is the trivial projection and $\pi_{\mathbb{P}\mathscr{T}}$ imposes the incidence relations. In order to equip $\mathscr{M}_{\mathbb{C}}$ with a self-dual metric, start by pulling back the $\mathcal{O}(2)$-valued symplectic form by the curved incidence relation (2.40)

$$e^0 \wedge \pi^*_{\mathbb{P}\mathscr{T}}\Sigma = e^0 \wedge \mathrm{d}_x\mathsf{F}^{\dot\alpha} \wedge \mathrm{d}_x\mathsf{F}_{\dot\alpha}\,, \tag{2.41}$$

where d_x denotes the exterior derivative on $\mathscr{M}_{\mathbb{C}}$. This 3-form is holomorphic, since using the incidence relations

$$\begin{aligned} \bar\partial\big|_X (e^0 \wedge \pi^*_{\mathbb{P}\mathscr{T}}\Sigma) &= 2\,\mathrm{D}\lambda \wedge \mathrm{d}_x\mathsf{F}^{\dot\alpha} \wedge \bar\partial\big|_X \mathrm{d}_x\mathsf{F}_{\dot\alpha} \\ &= 2\,\mathrm{D}\lambda \wedge \mathrm{d}_x\mathsf{F}^{\dot\alpha} \wedge \mathrm{d}_x\mathsf{F}^{\dot\beta} \wedge (\mathcal{L}_{\dot\alpha}\mathcal{L}_{\beta}\mathsf{h})\big|_X = 0\,, \end{aligned} \tag{2.42}$$

because $\mathcal{L}_{\dot\alpha}\mathcal{L}_{\dot\beta}\mathsf{h} = \mathcal{L}_{(\dot\alpha}\mathcal{L}_{\dot\beta)}\mathsf{h}$. The holomorphicity and the weight $+1$ in λ_α ensure the existence of a triplet of 2−forms $\Sigma^{\alpha\beta}$ on $\mathscr{M}_{\mathbb{C}}$ satisfying

$$\pi^*_{\mathbb{P}\mathscr{T}}\Sigma = \lambda_\alpha\lambda_\beta\Sigma^{\alpha\beta}(x) \mod e^0 \,. \tag{2.43}$$

It is easily checked that by construction

$$\Sigma^{(\alpha\beta} \wedge \Sigma^{\gamma\delta)} = 0\,, \qquad \mathrm{d}\Sigma^{\alpha\beta} = 0 \tag{2.44}$$

The first equation is known as *simplicity constraint* [160] and is equivalent to the existence of a tetrad $\theta^{\alpha\dot{\alpha}}$ on $\mathscr{M}_{\mathbb{C}}$ such that $\Sigma^{\alpha\beta} = \theta^{\alpha\dot{\alpha}} \wedge \theta^{\beta}{}_{\dot{\alpha}}$. If $\Sigma^{\alpha\beta}$ is indeed derived from a tetrad, then we already know that the second equation in (2.44) ensures that the metric on $\mathscr{M}_{\mathbb{C}}$ is hyperkähler.

Finally, the involution j can be used to recover a real slice $\mathscr{M} \subseteq \mathscr{M}_{\mathbb{C}}$ with a genuine Riemannian metric in the complexified space-time. In the rest of this work, we will mostly stick to the complexified setting, as all the main results we will derive will be rational functions, so the reality conditions can be safely imposed at the end, if needed.

Twistor action for gravity

We conclude this lighting Section on gravity with a review of the relevant actions. Much like in the gauge theory case, the integrability for h is the EOM for the first-order Poisson-BF action

$$S_{\mathrm{PBF}} = \int_{\mathbb{PT}} \mathrm{D}^3 Z \wedge \mathsf{g} \wedge \left(\bar{\partial}\mathsf{h} + \frac{1}{2}\{\mathsf{h},\mathsf{h}\}\right)\,, \tag{2.45}$$

where $\mathsf{g} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-6))$. Again, we can think of this action as the non-supersymmetric truncation of the holomorphic Chern-Simons theory on $\mathbb{CP}^{3|8}$, which reduces to self-dual $\mathcal{N} = 8$ supergravity on space-time [164]. However, there's an important difference with the gauge theory case: choosing $\mathbb{PT}$ as a background "reference" twistor space, we're implicitly breaking full covariance. The symmetries of the action are generated by fields $\chi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(2))$ and $\psi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(-6))$ as

$$\delta \mathsf{h} = \bar{\partial}\chi + \{\mathsf{h}, \chi\}\,, \qquad \delta \mathsf{g} = \{\chi, \mathsf{g}\} + \bar{\partial}\psi + \{\mathsf{h}, \psi\}\,. \tag{2.46}$$

This means that when we put the fields on-shell, $\bar{\nabla} = \bar{\partial} + \{\mathsf{h}, -\}$ defines an integrable complex structure deformation of $\mathbb{PT}$ and the field g is an element of the cohomology group $H^{0,1}_{\bar{\nabla}}(\mathbb{PT}, \mathcal{O}(-6))$, where the cohomology is computed with respect to $\bar{\nabla}$.

In Euclidean signature, after partially fixing the diffeomorphism gauge symmetry, the action (2.45) reduces to the Plebański action for the self-dual sector (2.36). Away from self-duality, the full action (2.34) admits a non-local uplift to twistor space [165].

2.2.4 Massless states and the Penrose transform

The final ingredient that will appear diffusely in the following Chapters is the Penrose transform. This is a useful tool to describe massless fields on space-time in terms of certain cohomology groups on twistor space. We begin by reviewing the transform on flat space, and then we suitably extend it to non-trivial backgrounds, both in gauge theory and gravity. In the following chapters, we will show its power in deriving exact solutions to the relevant linearised equations of motion.

Recall that helicity-h massless fields on flat space can be described in terms of totally symmetric spinors satisfying the zero-rest-mass (ZRM) equations [26, 28]

$$\partial^{\alpha_1\dot{\alpha}}\phi_{\alpha_1\ldots\alpha_{2|h|}} = 0\,, \qquad \Box\phi = 0\,, \qquad \partial^{\alpha\dot{\alpha}_1}\phi_{\dot{\alpha}_1\ldots\dot{\alpha}_{2h}} = 0\,, \tag{2.47}$$

respectively for negative, zero, and positive helicity. For example, for $h = \pm 1$ the ZRM equations govern the SD/ASD components of the linearised field strength $\tilde{f}_{\dot{\alpha}\dot{\beta}}$ and $f_{\alpha\beta}$. Similarly, the fields corresponding to $h = \pm 2$ are the SD and ASD components of the Weyl tensor, and on the support of the Einstein vacuum equations, the Bianchi identity for the Riemann tensor reduces precisely to the ZRM equations for $h = \pm 2$.

The Penrose transform allows to describe on-shell massless fields in terms of certain (twisted) cohomology groups on twistor space:

Theorem 3 (Penrose transform [28, 166])**.** *There exists an isomorphism between*

- *the solution space of the zero-rest-mass equations* (2.47)
- *the cohomology group*

$$H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2))\,. \tag{2.48}$$

The relevant integral formula is

$$\phi_{\alpha_1\ldots\alpha_{2|h|}}(x) = \frac{1}{2\pi \mathrm{i}} \int_X \mathrm{D}\lambda \wedge \lambda_{\alpha_1} \ldots \lambda_{\alpha_{2|h|}}\, \omega|_X \ , \tag{2.49}$$

for negative-helicity and scalar fields, and

$$\phi_{\dot\alpha_1\ldots\dot\alpha_{2h}} = \frac{1}{2\pi \mathrm{i}} \int_X \mathrm{D}\lambda \wedge \mathcal{L}_{\dot\alpha_1} \ldots \mathcal{L}_{\dot\alpha_{2h}} \omega|_X \ , \tag{2.50}$$

for positive-helicity fields. It is straightforward to show that these fields satisfy the appropriate ZRM equation, noting that, in Euclidean signature, a $(0,1)$-form with components $\omega \ = \ \omega_0 \bar{e}^0 + \omega_{\dot\alpha} \bar{e}^{\dot\alpha}$ is $\bar\partial$-closed if and only if

$$\bar\partial_0 \omega_{\dot\alpha} - \bar\partial_{\dot\alpha} \omega_0 = 0\,, \qquad \bar\partial_{[\dot\alpha} \omega_{\dot\beta]} = 0\,. \tag{2.51}$$

For example, in the negative-helicity case, the derivative of the space-time field is

$$\partial^{\alpha_1 \dot\alpha} \phi_{\alpha_1\ldots\alpha_{2|h|}} = \frac{1}{2\pi \mathrm{i}} \int_X \mathrm{D}\lambda \wedge \bar{e}^0\, \bar\partial_0 (\lambda_{\alpha_2} \ldots \lambda_{\alpha_{2|h|}} \omega^{\dot\alpha})\,, \tag{2.52}$$

and thus vanishes by Stokes' theorem. A similar computation holds for a scalar field.

For positive-helicity fields, we find instead

$$\partial^{\alpha\dot{\alpha}_1}\phi_{\dot{\alpha}_1\ldots\dot{\alpha}_{2h}} = \frac{1}{2\pi\mathrm{i}}\int_X \mathrm{D}\lambda\wedge\bar{e}^0\left(\frac{\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}\bar{\partial}^{\dot{\alpha}_1}\partial_{\dot{\alpha}_1}\ldots\partial_{\dot{\alpha}_{2h}}\omega_0\right.$$
$$\left.+\lambda^\alpha\partial^{\dot{\alpha}_1}\partial_{\dot{\alpha}_2}\ldots\partial_{\dot{\alpha}_{2h}}\omega_{\dot{\alpha}_1}+2h\frac{\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}\bar{\partial}^{\dot{\alpha}_1}\partial_{(\dot{\alpha}_1}\ldots\partial_{\dot{\alpha}_{2h-1}}\omega_{\dot{\alpha}_{2h})}\right). \quad (2.53)$$

Using $\bar{\partial}$-closedness of ω and applying repeatedly $[\bar{\partial}_0,\partial_{\dot{\alpha}}]=\bar{\partial}_{\dot{\alpha}}$, we can rewrite the first term in (2.53) as

$$\frac{\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}\bar{\partial}^{\dot{\alpha}_1}\partial_{\dot{\alpha}_1}\ldots\partial_{\dot{\alpha}_{2h}}\omega_0 = \frac{\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}\bar{\partial}_0(\partial_{\dot{\alpha}_1}\ldots\partial_{\dot{\alpha}_{2h}}\omega^{\dot{\alpha}_1})+2h\frac{\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}\bar{\partial}_{(\dot{\alpha}_1}\partial_{\dot{\alpha}_2}\ldots\partial_{\dot{\alpha}_{2h})}\omega^{\dot{\alpha}_1}\,. \quad (2.54)$$

The third term in (2.53) and the second term in (2.54) cancel each other so that the remaining terms can be rearranged as

$$\partial^{\alpha\dot{\alpha}_1}\phi_{\dot{\alpha}_1\ldots\dot{\alpha}_{2h}} = \frac{1}{2\pi\mathrm{i}}\int_X \mathrm{D}\lambda\wedge\bar{e}^0\bar{\partial}_0\left(\frac{\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}\partial_{\dot{\alpha}_1}\partial_{\dot{\alpha}_1}\ldots\partial_{\dot{\alpha}_{2h}}\omega^{\dot{\alpha}_1}\right)=0\,. \quad (2.55)$$

Twistor representatives can be reconstructed from the fields on space-time as follows: for negative-helicity and scalar fields, we can directly construct the $(0,1)$-form

$$\omega = (2|h|+1)\phi_{\alpha_1\ldots\alpha_{2|h|}}\frac{\hat{\lambda}^{\alpha_1}\ldots\hat{\lambda}^{\alpha_{2|h|}}}{\langle\lambda\hat{\lambda}\rangle^{2|h|}}\bar{e}^0-\partial_{\alpha_1\dot{\alpha}}\phi_{\alpha_2\ldots\alpha_{2|h|+1}}\frac{\hat{\lambda}^{\alpha_1}\ldots\hat{\lambda}^{\alpha_{2|h|+1}}}{\langle\lambda\hat{\lambda}\rangle^{2|h|+1}}\bar{e}^{\dot{\alpha}}\,, \quad (2.56)$$

where the $\bar{e}^0$ component reproduces the space-time field via (2.62) and the $\bar{e}^{\dot{\alpha}}$ component ensures that ω is $\bar{\partial}$-closed on the support of the ZRM equations. For positive-helicity fields, we introduce a potential $\psi_{\dot{\alpha}\alpha_1\ldots\alpha_{2h-1}}$ totally symmetric in its undotted indices and satisfying

$$\phi_{\dot{\alpha}_1\ldots\dot{\alpha}_{\alpha_{2h}}} = \partial^{\alpha_1}_{(\dot{\alpha}_1}\ldots\partial^{\alpha_{2h-1}}_{\dot{\alpha}_{2h-1}}\psi_{\dot{\alpha}_{2h})\alpha_1\ldots\alpha_{2h-1}}\,,\qquad \partial^{\dot{\alpha}(\alpha_1}\psi_{\dot{\alpha}}{}^{\alpha_2\ldots\alpha_{2h})}=0\,, \quad (2.57)$$

and set

$$\omega = \psi_{\dot{\alpha}\alpha_1\dots\alpha_{2h-1}}\lambda^{\alpha_1}\dots\lambda^{\alpha_{2h-1}}\bar{e}^{\dot{\alpha}}\,. \tag{2.58}$$

Note that we can think of the Ward's correspondence and non-linear graviton above as non-linear extensions to the Penrose transform, as the fields b and a in the gauge theory action (2.27) are naturally identified with negative- and positive-helicity spin-1 fields, and similarly the fields g and h in the gravitational action (2.45) are the negative- and positive-helicity components of a spin-2 field.

Massless states on curved backgrounds

When dealing with non-trivial backgrounds, the Penrose transform should be suitably modified. We consider the gauge theory and gravity cases separately. Recall that the two on-shell degrees of freedom of gluons in vacuum are classified by the helicity: the helicity of a gluon corresponds to whether its linearised field strength is self-dual (positive helicity) or anti-self-dual (negative helicity). On a general self-dual background, linear gluon fields still split into positive and negative helicity, but these no longer correspond to SD and ASD perturbations. Suppose we are given a linear perturbation $a_{\alpha\dot{\alpha}}$ to a background gauge field $A_{\alpha\dot{\alpha}}$, with linearised field strength $f_{ab} = f_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \tilde{f}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}$. The linearised equations of motion for the perturbation are

$$D_{\alpha\dot{\alpha}}f^{\alpha\beta} = 0\,, \qquad D_{\alpha\dot{\alpha}}\tilde{f}^{\dot{\alpha}\dot{\beta}} + \mathrm{i}\,[a_{\alpha\dot{\alpha}}, \tilde{F}^{\dot{\alpha}\dot{\beta}}] = 0\,, \tag{2.59}$$

where $\tilde{F}_{\dot{\alpha}\dot{\beta}}$ is the SD field strength of the background field $A_{\alpha\dot{\alpha}}$ and $D = \mathrm{d} - \mathrm{i}A$ is the gauge covariant derivative. Positive helicity gluons are then defined by $f^{\alpha\beta} = 0$; that is, they are SD perturbations to the background gauge field. Conversely, it is inconsistent to set $\tilde{f}^{\dot{\alpha}\dot{\beta}} = 0$ for a negative-helicity gluon, the obstruction being exactly the non-vanishing SD background field strength $\tilde{F}^{\dot{\alpha}\dot{\beta}}$. Thus, negative-helicity gluons are asymmetrically described by the background-coupled zero-rest-mass equation

$$D^{\alpha\dot{\alpha}} f_{\alpha\beta} = 0\,, \tag{2.60}$$

and will generally acquire a SD component of linearised field strength when traversing the SD background. The Penrose transform for background-coupled, negative-helicity gluons is

$$H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End}E) \cong \{a_{\alpha\dot{\alpha}} \text{ on } \mathbb{M} \,|\, D^{\alpha\dot{\alpha}} f_{\alpha\beta} = 0\}\,, \tag{2.61}$$

where $H^{0,1}_{\bar{D}}$ denotes the Dolbeault cohomology defined with respect to $\bar{D}$. Given a twistor representative $b \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End}E)$, the spacetime field $f_{\alpha\beta}$ can be reconstructed via the integral formula

$$f_{\alpha\beta}(x) = \int_X \mathrm{D}\lambda \wedge \lambda_\alpha \lambda_\beta \, \mathsf{H}^{-1}(x,\lambda)\, b|_X \, \mathsf{H}(x,\lambda)\,, \tag{2.62}$$

The insertions of the holomorphic frame ensure that the integrand is trivialised in the E component, while the weights of b and of $\mathrm{D}\lambda\, \lambda_\alpha \lambda_\beta$ give a weight-zero integrand.

The Penrose transform for background-coupled, positive-helicity gluons is given by

$$H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O} \otimes \mathrm{End}E) \cong \{a_{\alpha\dot{\alpha}} \text{ on } \mathbb{M} \,|\, D^{\dot{\alpha}}_{(\alpha} a_{\beta)\dot{\alpha}} = 0\}\,, \tag{2.63}$$

and in this case the linear gauge field $a_{\alpha\dot{\alpha}}$ itself can be constructed via an argument due to Sparling [155]: given a twistor representative $a \in H^{0,1}(\mathbb{PT}, \mathcal{O} \otimes \mathrm{End}E)$, its restriction to a twistor line X is $\bar{\partial}\big|_X$-exact, after a suitable dressing with the holomorphic frame, since $\mathsf{H}^{-1}\, a|_X\, \mathsf{H} \in H^{0,1}(\mathbb{CP}^1, \mathcal{O}) = \emptyset$. Thus, we can write

$$\mathsf{H}^{-1}(x,\lambda) a|_X \mathsf{H}(x,\lambda) = \bar{\partial}\big|_X\, j(x,\lambda)\,, \tag{2.64}$$

for a suitable current j. Noticing that $\lambda^\alpha D_{\alpha\dot{\alpha}} j$ is a holomorphic function of λ_α of weight $+1$, Liouville's theorem gives the space-time perturbation $a_{\alpha\dot\alpha}$ as

$$\lambda^\alpha \partial_{\alpha\dot\alpha} j = -\mathrm{i}\lambda^\alpha\, a_{\alpha\dot\alpha}\,. \tag{2.65}$$

Similarly, on a self-dual space-time, positive-helicity gravitons can still be characterised by metric perturbations $g_{ab} \to g_{ab} + h_{ab}$ such that h_{ab} solves the linearised Einstein equations around the background metric g_{ab} and has purely self-dual linearised Weyl tensor. However, it is not possible to define negative-helicity gravitons in a similar fashion, as a generic infinitesimal diffeomorphism on the self-dual background will give rise to a non-vanishing piece of self-dual linear Weyl curvature. Thus, we define positive- and negative-helicity gravitons to be those metric perturbations h_{ab} satisfying

$$\psi_{\alpha\beta\gamma\delta} = 0\,, \qquad \text{or} \qquad \nabla^{\alpha\dot\alpha}\psi_{\alpha\beta\gamma\delta} = 0\,, \tag{2.66}$$

respectively. Here ∇ is the covariant derivative. For negative-helicity gravitons, the Penrose transform is

$$H^{0,1}_{\bar\nabla}(\mathbb{P}\mathscr{T}, \mathcal{O}(-6)) \cong \{h_{ab} \text{ on } \mathbb{M} \,|\, \nabla^{\alpha\dot\alpha}\psi_{\alpha\beta\gamma\delta} = 0\}\,, \tag{2.67}$$

and it's thus the Dolbeault cohomology of the $\bar\partial$-operator $\bar\nabla$ of the deformed complex structure of the curved twistor space $\mathbb{P}\mathscr{T}$. The integral formula is essentially the same as in flat space, namely

$$\psi_{\alpha\beta\gamma\delta}(x) = \int_X \mathrm{D}\lambda \wedge \lambda_\alpha\, \lambda_\beta\, \lambda_\gamma\, \lambda_\delta\;\, g|_X\,, \tag{2.68}$$

where $g \in H^{0,1}_{\bar\nabla}(\mathbb{P}\mathscr{T}, \mathcal{O}(-6))$ is the twistor representative. The only difference is that now X refers to the curved twistor line associated to the space-time point x via the curved incidence relations (2.40).

In contrast to the negative helicity case, positive-helicity gravitons can be described directly at the level of a metric perturbation. The Penrose transform is now

$$H^{0,1}_{\bar{\nabla}}(\mathbb{P}\mathscr{T}, \mathcal{O}(2)) \cong \{h_{ab} \text{ on } \mathbb{M} \,|\, \psi_{\alpha\beta\gamma\delta} = 0\} \,, \tag{2.69}$$

Given a twistor representative $h \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$, the field on space-time is recovered by restriction to twistor curves:

$$h|_X = \bar{\partial}|_X j(x, \lambda) \,, \tag{2.70}$$

for some function j of homogeneity $+2$ on $\mathbb{P}^1$, as $H^{0,1}(\mathbb{P}^1, \mathcal{O}(2))$ is trivial. Holomorphicity of h on $\mathbb{P}\mathscr{T}$ then implies that $\lambda^\alpha \nabla_{\alpha\dot{\alpha}} j$ is holomorphic on $\mathbb{P}^1$, which in turn means that

$$\lambda^\alpha \, \nabla_{\alpha\dot{\alpha}} j(x, \lambda) = \lambda^\alpha \, \lambda^\beta \, \lambda^\gamma \, \phi_{\dot{\alpha}\alpha\beta\gamma}(x) \,, \tag{2.71}$$

for $\phi_{\dot{\alpha}\alpha\beta\gamma}(x)$ a field on $\mathscr{M}$ which is totally symmetric in its undotted spinor indices. This acts as a potential for a metric perturbation

$$h_{\alpha\dot{\alpha}\beta\dot{\beta}} = \nabla^{\gamma}{}_{(\dot{\alpha}} \phi_{\dot{\beta})\alpha\beta\gamma} \,, \tag{2.72}$$

which is easily seen to be self-dual, and hence positive helicity.

For all the backgrounds we consider in the following Chapters, we will consider scattering states that are plane waves dressed with background-dependent functions, reducing to familiar momentum eigenstates in the limit in which the background is turned off. For this reason, we refer to these solutions as *quasi*-momentum eigenstates, as there is no actual translation invariance in the background.

Chapter 3

Form factors on radiative backgrounds

The first application of twistor theory, as presented in Chapter 2, is quantum field theory around *radiative* backgrounds. The study of these backgrounds has by now a long history [99, 167–170] and led to the construction of all-multiplicity formulae for the tree-level MHV amplitudes of both gluons and gravitons, as well as to conjectural formulae for tree-level amplitudes at higher MHV degree [65–67]. In this Chapter, we show that it is possible to extend this analysis beyond on-shell observables and start to address the construction of off-shell observables on gauge theory radiative backgrounds. We thus consider the computation of *form factors*, that is of matrix elements of local operators between the vacuum and an on-shell state, around this class of backgrounds. Form factors represent partially off-shell observables, so their construction is a first step towards a proper understanding of off-shell QFT around these curved backgrounds.

We begin by reviewing the homogeneous description of null infinity and its relation to twistor space. This will lead to remarkably compact integral representations for the background fields in terms of their radiative data and for the linear fields propagating around these backgrounds. These are then used to lift local operators on space-time to twistor space; the uplifted operators then yield all-multiplicity expressions for the tree-level MHV form factors. Remarkably, any MHV form factor can be obtained as a simple dressing of the corresponding form factor around the vacuum. See [1] for an extension of this construction to super form factors in $\mathcal{N}=4$ super Yang-Mills.

3.1 The homogeneous formalism of null infinity

Recall that, in the Lorentzian slice $\mathbb{M}$, null infinity $\mathscr{I}$ arises as the boundary of the conformal compactification of $\mathbb{M}$ and can be understood as the inversion of the light-cone of the origin of $\mathbb{M}$. Null infinity decomposes into future and past null infinity, $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$, where the two components both have the topology $\mathbb{R}\times S^2$. On the complexified space-time, the conformal compactification gives rise to a partial complexification $\mathscr{I}_{\mathbb{C}} = \mathscr{I}_{\mathbb{C}}^+ \cup \mathscr{I}_{\mathbb{C}}^-$, where future and past null infinity $\mathscr{I}_{\mathbb{C}}^{\pm} = \mathbb{C}\times\mathbb{CP}^1$ are obtained by complexifying the $\mathbb{R}$ factor while keeping the S^2 base.[1] In order to connect with homogeneous coordinates on twistor space, we will use a homogeneous version of the Bondi coordinates. Focussing on future null infinity, we will coordinatize $\mathscr{I}_{\mathbb{C}}^+$ with coordinates $(u, \lambda_\alpha, \bar{\lambda}_{\dot\alpha})$ subject to the equivalence relation

$$(u, \lambda, \bar{\lambda}_{\dot\alpha}) \sim (b\bar{b}u, b\lambda_\alpha, \bar{b}\bar{\lambda}_{\dot\alpha})\,, \tag{3.1}$$

for any $b\in\mathbb{C}^*$. u is a complexification of the standard Bondi retarded time, while $(\lambda_\alpha, \bar{\lambda}_{\dot\alpha})$ are homogeneous coordinates on the celestial sphere thought of as the complex projective line. The (degenerate) Carrollian metric on $\mathscr{I}_{\mathbb{C}}^+$ is

$$\mathrm{d}s^2_{\mathscr{I}_{\mathbb{C}}^+} = 0\times \mathrm{d}u^2 + \mathrm{D}\lambda\,\mathrm{D}\bar{\lambda}\,, \tag{3.2}$$

[1] This will guarantee that each α-plane will intersect $\mathscr{I}_{\mathbb{C}}^{\pm}$ in a unique point.

The homogeneous coordinates allow us to encode spin and conformal weights in terms of homogeneous line bundles $\mathcal{O}(p,q) \to \mathscr{I}^+_{\mathbb{C}}$, where a section of $\mathcal{O}(p,q)$ is represented as a function $f_{p,q}(u,\lambda,\bar{\lambda})$ with weights (p,q) under rescaling of the homogeneous coordinates

$$f_{p,q}(b\bar{b}u, b\lambda, \bar{b}\bar{\lambda}) = b^p \bar{b}^q f_{p,q}(u,\lambda,\bar{\lambda})\,. \tag{3.3}$$

The conformal and spin weights (h,s) are related to (p,q) by

$$h = \frac{p+q}{2}\,, \qquad s = \frac{p-q}{2}\,. \tag{3.4}$$

The vector fields dual to $\mathrm{D}\lambda$ and $\mathrm{D}\bar{\lambda}$ are the edth operators $\eth\colon \mathcal{O}(p,q) \to \mathcal{O}(p-2,q)$ and $\bar{\eth}\colon \mathcal{O}(p,q) \to \mathcal{O}(p,q-2)$ [166,171] – see also [172] for a more modern perspective.

Using the projective formalism, we can construct a fibration $p\colon \mathbb{PT} \to \mathscr{I}^+_{\mathbb{C}}$ where the projection is given by

$$p\colon (\mu^{\dot{\alpha}}, \lambda_\alpha) \longmapsto (u = \mu^{\dot{\beta}}\bar{\lambda}_{\dot{\beta}}, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})\,. \tag{3.5}$$

In the gravitational setting, this construction is known as *asymptotic twistor space* as it allows to encode the complex-structure deformation of twistor space in terms of the asymptotic data of the gravitational radiation at null infinity, e.g. the asymptotic shear [67,145,173–175]. In the same spirit, we will leverage the fibration $\mathbb{PT} \to \mathscr{I}^+_{\mathbb{C}}$ to pull-back asymptotic data for radiative gauge fields to twistor space in the following.

3.2 Twistors for radiative fields

A source-free gauge field is radiative if it extends to null infinity inside the conformal compactification of complexified space-time $\mathbb{M}_{\mathbb{C}}$ and is completely determined by its free characteristic data at either past or future null infinity. Within the projective formalism, the restriction of a gauge field A on $\mathbb{M}_{\mathbb{C}}$ to $\mathscr{I}^+_{\mathbb{C}}$ in temporal gauge $A_u = 0$

is [154, 176–178]

$$A|_{\mathscr{I}^+} = A_-(u,\lambda,\bar\lambda)\,\mathrm{D}\lambda + A_+(u,\lambda,\bar\lambda)\,\mathrm{D}\bar\lambda\,, \tag{3.6}$$

The restriction of the leading components of the SD and ASD parts of the curvature are

$$F^0_+ = \partial_u A_+\,\mathrm{d}u\wedge\mathrm{D}\bar\lambda\,,\qquad F^0_- = \partial_u A_-\,\mathrm{d}u\wedge\mathrm{D}\lambda\,, \tag{3.7}$$

respectively, so A_+, A_- are the free data for the SD and ASD components of the field strength. In terms of the line bundles above, A_+ is a section of $\mathcal{O}(-2,0)\otimes\mathfrak{g}$, while A_- is a section of $\mathcal{O}(0,-2)\otimes\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the gauge group. A self-dual, radiative gauge field is a gauge field completely characterised by the free data $A|_{\mathscr{I}^+_{\mathbb{C}}} = A_+\,\mathrm{D}\bar\lambda$, while $A_- = 0$.

For radiative fields, introducing another description in terms of scalar second potentials will prove to be extremely useful. There are two possible scalar potentials available for a self-dual field, known as the J and K matrices, that can be obtained by partially solving the self-duality equation. Given a dyad $\{\iota_\alpha, o_\alpha\}$ on the undotted spinor bundle normalized such that $\langle\iota\, o\rangle = 1$,[2] the vanishing of the ASD curvature implies flatness in the 2-plane tangent to $\iota^\alpha\beta^{\dot\alpha}$ for an arbitrary dotted spinor $\beta^{\dot\alpha}$, so we can put the gauge connection in the gauge

$$A_{\alpha\dot\alpha} = \iota_\alpha A_{\dot\alpha}\,. \tag{3.8}$$

In particular, the gauge field is in light-cone gauge with respect to any null vector of the form $n^{\alpha\dot\alpha} = \iota^\alpha\beta^{\dot\alpha}$. The pair of equations $\iota^\alpha F_{\alpha\beta} = 0$ then implies the existence of a matrix-valued scalar potential K, the K-matrix, such that

$$A_{\alpha\dot\alpha} = \iota_\alpha\iota^\beta\partial_{\beta\dot\alpha}K\,. \tag{3.9}$$

[2] A common choice is $\iota^\alpha = (1,0)$ and $o^\alpha = (0,-1)$.

The gauge field is automatically in Lorenz gauge as well. If we solve the equation $o^\alpha o^\beta F_{\alpha\beta} = 0$ instead, we can deduce the existence of the scalar potential J, the J-matrix, such that

$$A_{\alpha\dot\alpha} = -\mathrm{i}\,\iota_\alpha J^{-1} o^\beta \partial_{\beta\dot\alpha} J\,. \tag{3.10}$$

The remaining self-duality equations are

$$\Box K + \mathrm{i}[\iota^\alpha \partial_{\alpha\dot\alpha} K, \iota^\beta \partial_\beta{}^{\dot\alpha} K] = 0\,, \qquad \iota^\alpha o^\beta \partial_{\alpha\dot\alpha}(J^{-1}\partial_\beta{}^{\dot\alpha} J) = 0\,, \tag{3.11}$$

respectively, whilst the SD component of the curvature is

$$\begin{aligned} \tilde F_{\dot\alpha\dot\beta} &= \iota^\alpha \partial_{\alpha\dot\alpha}\, \iota^\beta \partial_{\beta\dot\beta} K \\ &= -\mathrm{i}\,\iota^\alpha \partial_{\alpha(\dot\alpha|}(J^{-1} o^\beta \partial_{\beta|\dot\beta)} J)\,, \end{aligned} \tag{3.12}$$

in terms of the two scalar potentials, respectively.

In order to give a twistorial description of radiative fields, we first use the asymptotic gauge field at $\mathscr{I}^+_{\mathbb{C}}$ to construct the $(0,1)$-form

$$\mathsf{a} = p^*(A_+\,\mathrm{D}\bar\lambda) = A_+(\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}, \lambda, \bar\lambda)\mathrm{D}\bar\lambda\,, \tag{3.13}$$

valued in $\Omega^{0,1}(\mathbb{PT}, \mathcal{O}\otimes\mathfrak{g})$ on twistor space. Since a points only along the $\mathrm{D}\bar\lambda$ direction and is holomorphic in $\mu^{\dot\alpha}$, the $\bar\partial$ operator $\bar D = \bar\partial + \mathsf{a}$ trivially satisfies $\bar D^2 = 0$, so it defines a Ward bundle $E \to \mathbb{PT}$ on twistor space and we can use the Ward's correspondence to reconstruct the gauge field in the bulk of space-time. To do so, we first introduce Green's functions for the operator $\bar\partial$ on the sphere. If λ_α is a homogeneous coordinate on $\mathbb{CP}^1$ and $\bar\partial$ is the standard $\bar\partial$-operator on the Riemann sphere, we can invert the equation $\bar\partial g_n = f_n$ for any $f_n \in \Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n)$ to find a solution $g_n \in \Omega^{0,0}(\mathbb{CP}^1, \mathcal{O}(n))$, provided $n \geq -1$. The integral formula for g_n is

$$g_n(\lambda) = \int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda'}{2\pi\mathrm{i}} \frac{1}{\langle \lambda\, \lambda' \rangle} \left(\frac{\langle \iota\, \lambda \rangle}{\langle \iota\, \lambda' \rangle} \right)^{n+1} \wedge f_n(\lambda') \,. \tag{3.14}$$

The integral is over $\mathbb{CP}^1$, on which $\lambda_\alpha, \lambda'_\alpha$ are homogeneous coordinates, and the reference spinor ι_α is used to fix the freedom in adding polynomials of degree n in λ_α to g_n by making it vanish to n^{th} order at ι_α; note that for $n = -1$ the solution is unique, while for $n \geq 0$ the ambiguity in g_n is a consequence of $H^0(\mathbb{CP}^1, \mathcal{O}(n)) \cong \mathbb{C}^{n-1}$. The integral formula (3.14) can be proven straightforwardly by recalling the definition of the holomorphic δ-function

$$\bar{\delta}^2(\lambda_\alpha) = \bigwedge_{\alpha=0,1} \frac{1}{2\pi\mathrm{i}} \bar{\partial} \frac{1}{\lambda_\alpha} \,, \tag{3.15}$$

so that, taking a derivative, we find

$$\bar{\partial} g_n = \int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda'}{2\pi\mathrm{i}} \bar{\delta}(\langle \lambda\, \lambda' \rangle) \left(\frac{\langle \iota\, \lambda \rangle}{\langle \iota\, \lambda' \rangle} \right)^{n+1} \wedge f_n(\lambda') \,. \tag{3.16}$$

The integral has support on the locus where λ'_α and λ_α are proportional to each other, but since the holomorphic δ function has weight -1 in its argument, the integrand has overall weight zero in λ'_α and we can simply set $\lambda'_\alpha = \lambda_\alpha$.

Now let H be the holomorphic frame satisfying the Sparling equation (2.22) and suppose that we fix the gauge on space-time by requiring $\mathsf{H}(x, \iota) = 1$. Differentiating the Sparling equation, we find

$$\bar{\partial}\big|_X \left(\partial_{\alpha\dot{\alpha}} \mathsf{H}^{-1}(x,\lambda)\, \mathsf{H}(x,\lambda)\right) = \lambda_\alpha \mathsf{H}^{-1}(x,\lambda) \left.\frac{\partial \mathsf{a}}{\partial \mu^{\dot{\alpha}}}\right|_X \mathsf{H}(x,\lambda) \,, \tag{3.17}$$

and using the Green's function (3.14)

$$\partial_{\alpha\dot\alpha}\mathsf{H}^{-1}(x,\lambda)\,\mathsf{H}(x,\lambda)=\frac{1}{2\pi\mathrm{i}}\int_X\frac{\mathrm{D}\lambda'}{\langle\lambda\,\lambda'\rangle}\frac{\langle\iota\,\lambda\rangle}{\langle\iota\,\lambda'\rangle}\lambda'_\alpha\mathsf{H}^{-1}(x,\lambda')\left.\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\right|_X\mathsf{H}(x,\lambda')\,. \tag{3.18}$$

The possible ambiguity in adding a constant to the right-hand side at homogeneity degree zero is fixed by the vanishing of both sides of the equation at $\lambda_\alpha=\iota_\alpha$, given the gauge condition $\mathsf{H}(x,\iota)=1$. We can now contract this last equation with λ_α, apply Equation (2.24) and find

$$A_{\alpha\dot\alpha}(x)=\frac{\iota_\alpha}{2\pi}\int_X\frac{\mathrm{D}\lambda}{\langle\iota\,\lambda\rangle}\mathsf{H}^{-1}(x,\lambda)\left.\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\right|_X\mathsf{H}(x,\lambda)\,. \tag{3.19}$$

Equation (2.24) also gives the form of the J and K matrices in terms of the holomorphic frame as

$$J=\mathsf{H}(x,o)\,,\qquad K=\mathrm{i}o^\alpha\left.\frac{\partial\mathsf{H}}{\partial\lambda^\alpha}\right|_{\lambda=\iota}\,. \tag{3.20}$$

In particular, we can identify the integral in Equation (3.19) with the derivative $\iota^\alpha\partial_{\alpha\dot\alpha}K$ of the K-matrix.

The associated field strength can be straightforwardly checked to be self-dual, with SD component

$$\begin{aligned}\tilde F_{\dot\alpha\dot\beta}=&\int_X\frac{\mathrm{D}\lambda_1}{2\pi\mathrm{i}}\mathsf{H}^{-1}(x,\lambda_1)\left.\frac{\partial^2\mathsf{a}}{\partial\mu_1^{\dot\alpha}\,\partial\mu_1^{\dot\beta}}\right|_X\mathsf{H}(x,\lambda_1)\\&-\int_{X^2}\frac{\mathrm{D}\lambda_1\mathrm{D}\lambda_2}{(2\pi\mathrm{i})^2\langle\lambda_1\lambda_2\rangle}\left[\mathsf{H}^{-1}(x,\lambda_1)\left.\frac{\partial\mathsf{a}}{\partial\mu_1^{\dot\alpha}}\right|_X\mathsf{H}(x,\lambda_1),\mathsf{H}^{-1}(x,\lambda_2)\left.\frac{\partial\mathsf{a}}{\partial\mu_2^{\dot\beta}}\right|_X\mathsf{H}(x,\lambda_2)\right]\,,\end{aligned} \tag{3.21}$$

The expression for $\tilde F_{\dot\alpha\dot\beta}$ has the advantage of being now both Lorentz and gauge invariant; the linear Penrose transform would lead to the first term in (3.21) only, but for the fully non-linear field we need also the second, double integral over X in order to ensure gauge invariance.

For the most part, we restrict ourselves further to backgrounds valued in a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Although our methods naturally yield concrete formulae for more general backgrounds, this restriction leads to simpler formulae in which the background is encoded into abelian factors obtainable by quadratures, i.e., direct integral formulae. In this case, we can express the holomorphic frame $\mathsf{H} = e^{-\mathsf{g}}$ explicitly in terms of the Cartan-valued function [64, 66]

$$\mathsf{g}(x, \lambda) = \frac{1}{2\pi \mathrm{i}} \int_X \frac{\mathrm{D}\lambda'}{\langle \lambda \lambda' \rangle} \frac{\langle \iota \lambda \rangle}{\langle \iota \lambda' \rangle} \, \mathsf{a}|_X \ , \tag{3.22}$$

and further use it to provide integral formulae for the background coupled fields. For example, the J- and K-matrices are given by

$$\log J = -\frac{1}{2\pi \mathrm{i}} \int_X \frac{\mathrm{D}\lambda}{\langle \iota \, \lambda \rangle \langle o \, \lambda \rangle} \, \mathsf{a}|_X \ , \tag{3.23}$$

and

$$K = \frac{1}{2\pi} \int_X \frac{\mathrm{D}\lambda}{\langle \iota \, \lambda \rangle^2} \, \mathsf{a}|_X \ . \tag{3.24}$$

Finally, in the following, it will be important to know the Green's function K_X for $\bar{D}\big|_X$ acting on sections of $\mathcal{O}(-1)$. The propagator can be immediately found by noticing that, on each twistor line, we can trivialise the restriction $\mathsf{a}|_X$ of the twistor connection using the holomorphic frame and recalling the definition (3.15) of the holomorphic δ-function, so

$$\mathsf{K}_X(\lambda, \lambda') = \frac{1}{2\pi \mathrm{i}} \frac{\mathsf{H}(x, \lambda) \mathsf{H}^{-1}(x, \lambda')}{\langle \lambda \, \lambda' \rangle} \, . \tag{3.25}$$

3.3 Scattering states

Around the trivial background, the ordinary momentum eigenstates are completely characterised by a null momentum $k^{\alpha\dot\alpha} = \kappa^\alpha \tilde\kappa^{\dot\alpha}$ and a colour $\mathsf{T}^e \in \mathfrak{g}$ and are given by

$$f_{\alpha\beta} = \kappa_\alpha \kappa_\beta \mathsf{T}^e \mathrm{e}^{\mathrm{i}k\cdot x}\,, \qquad a_{\alpha\dot\alpha} = \frac{\xi_\alpha \tilde\kappa_{\dot\alpha}}{\langle \xi\, \kappa\rangle} \mathsf{T}^e \mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{3.26}$$

for negative and positive helicity, respectively. The spinor ξ_α entering the linearised gauge field for a positive-helicity gluon reflects the residual linearised gauge invariance and should drop out of gauge invariant quantities, such as the linearised field strength, and of physical observables, such as amplitudes and form factors. These states can be obtained from twistor space by choosing the twistor representatives

$$b = \mathsf{T}^e \int_{\mathbb{C}^*} \mathrm{d}s\, s^3\, \bar\delta^2(\kappa - s\lambda) \mathrm{e}^{\mathrm{i}s[\mu\tilde\kappa]}\,, \qquad a = \mathsf{T}^e \int_{\mathbb{C}^*} \frac{\mathrm{d}s}{s}\, \bar\delta^2(\kappa - s\lambda) \mathrm{e}^{\mathrm{i}s[\mu\tilde\kappa]}\,. \tag{3.27}$$

and performing the appropriate Penrose transform.

On a radiative background, we should construct momentum eigenstates akin to (3.26) by solving the linearised equations of motion. This task turns out to be extremely hard, even for simple backgrounds. Instead, we use the background-dressed Penrose transform described in Chapter 2. Remarkably, the twistor representatives in Equation (3.27) satisfy

$$\bar D b = 0\,, \qquad \bar D a = 0\,, \tag{3.28}$$

for *any* twistor connection a pulled back from radiative data as in (3.13), because both the background a and the linear fields b, a point along the $\mathrm{D}\bar\lambda$ direction, so the action of the $\bar\partial$-operators $\bar\partial$ and $\bar D$ coincide.

With these, we can apply the background-dressed Penrose transform (2.62) with the first twistor representative in (3.26) and immediately find the negative-helicity linear field

$$f_{\alpha\beta} = \kappa_\alpha \kappa_\beta \mathsf{H}^{-1}(x,\kappa)\mathsf{T}^e \mathsf{H}(x,\kappa)\mathrm{e}^{\mathrm{i}k\cdot x}\,. \tag{3.29}$$

For positive-helicity perturbations, consider a perturbation $\mathsf{a} \mapsto \mathsf{a} + \varepsilon a$ to the background twistor connection, where a is given by the second of (3.26). Using the Green's function (3.14) and integrating against the delta function, we find that the perturbation H to the holomorphic frame to first order in ε is

$$\mathsf{H}(x,\lambda)^{-1} H(x,\lambda) = \frac{\langle \iota\, \lambda\rangle}{\langle \iota\, \kappa\rangle\langle \lambda\, \kappa\rangle} \mathsf{H}^{-1}(x,\kappa)\mathsf{T}^e \mathsf{H}(x,\kappa)\mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{3.30}$$

so the variations δJ and δK of the J- and K-matrices are

$$J^{-1}\,\delta J = -\frac{1}{\langle \iota\, \kappa\rangle\langle o\, \kappa\rangle} \mathsf{H}^{-1}(x,\kappa)\mathsf{T}^e \mathsf{H}(x,\kappa)\mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{3.31}$$

$$\delta K = -\frac{\mathrm{i}}{\langle \iota\, \kappa\rangle^2} \mathsf{H}^{-1}(x,\kappa)\mathsf{T}^e \mathsf{H}(x,\kappa)\mathrm{e}^{\mathrm{i}k\cdot x}\,. \tag{3.32}$$

Either of these can be used to reconstruct the linearised gauge field on space-time

$$a_{\alpha\dot\alpha} = \frac{\iota_\alpha}{\langle \iota\, \kappa\rangle} \mathsf{H}^{-1}(x,\kappa)(\tilde\kappa_{\dot\alpha}\mathsf{T}^e + [\mathsf{g}_{\dot\alpha}(x,\kappa), \mathsf{T}^e])\mathsf{H}(x,\kappa)\mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{3.33}$$

where the function $\mathsf{g}_{\dot\alpha}$ is defined using (2.24)

$$D_{\alpha\dot\alpha}\mathsf{H}^{-1}(x,\lambda) = \lambda_\alpha \mathsf{H}^{-1}(x,\lambda)\mathsf{g}_{\dot\alpha}(x,\lambda)\,. \tag{3.34}$$

Note that the gauge-dependent undotted spinor present in the vector polarisation of a spin-1 momentum eigenstate is taken to be ι_α.

When the background is taken to be Cartan-valued, we can simplify (3.29) and (3.33) further. First, note that we can encode the commutators of any element $\mathsf{T}^e \in \mathfrak{g}$

with the background in terms of Abelian charges e^i, defined by

$$[\mathsf{t}^i, \mathsf{T}^e] = e^i \mathsf{T}^e\,, \tag{3.35}$$

where $\{\mathsf{t}^i\}$ is a basis of the Cartan subalgebra $\mathfrak{h}$. The negative- and positive-helicity states then reduce to

$$b_{\alpha\beta} = \kappa_\alpha \kappa_\beta \mathsf{T}^e \mathrm{e}^{\mathrm{i}k\cdot x + e\mathsf{g}(x,\kappa)}\,, \qquad a_{\alpha\dot\alpha} = \frac{\iota_\alpha}{\langle \iota\, \kappa\rangle}(\tilde\kappa_{\dot\alpha} + e\mathsf{g}_{\dot\alpha}(x,\kappa))\mathsf{T}^e \mathrm{e}^{\mathrm{i}k\cdot x + e\mathsf{g}(x,\kappa)}\,. \tag{3.36}$$

In particular, there is a factorisation between the colour and kinematical degrees of freedom, as the background 'dresses' the dotted component of the momentum as

$$\kappa_\alpha \tilde K_{\dot\alpha}(x) = \kappa_\alpha(\tilde\kappa_{\dot\alpha} + e\mathsf{g}_{\dot\alpha}(x,\kappa))\,, \tag{3.37}$$

while leaving the undotted component invariant. This is expected as a consequence of self-duality. For Cartan-valued backgrounds, the function $\mathsf{g}_{\dot\alpha}$ is given by

$$\mathsf{g}_{\dot\alpha}(x,\lambda) = \frac{1}{2\pi\mathrm{i}} \int_X \frac{\mathrm{D}\lambda'}{\langle \lambda\, \lambda'\rangle} \frac{\partial \mathsf{a}}{\partial \mu'^{\dot\alpha}}\bigg|_X\,. \tag{3.38}$$

3.4 MHV form factors

Given a local operator $\mathscr{O}(x)$, its form factor $\mathcal{F}_{\mathscr{O}} = \mathcal{F}_{\mathscr{O}}(1^{h_1}, \dots, n^{h_n}; q)$ in presence of n external gluons is defined as the Fourier transform of the matrix element of $\mathscr{O}(x)$ between the vacuum and the n-gluon multi-particle state

$$\mathcal{F}_{\mathscr{O}}(1^{h_1}, \dots, n^{h_n}; q) := \int_{\mathbb{M}_\mathbb{C}} \mathrm{d}^4 x\, e^{-\mathrm{i}q\cdot x} \langle 1^{h_1}, \dots, n^{h_n} | \mathscr{O}(x) | 0\rangle\,, \tag{3.39}$$

where we implicitly assumed our gluons to be outgoing plane waves and where h_i denotes the helicity of the i^{th} gluon. Since we focus on MHV form factors, the helicities are almost all positive, the number of negative-helicity gluons being equal to the number of B fields appearing in $\mathscr{O}$. To compute $\mathcal{F}_{\mathscr{O}}$, we will resort to the perturbiner approach [95–98]: in this framework, tree-level amplitudes are computed as multi-linear pieces of the classical action, evaluated on recursively constructed solutions of the non-linear equations of motion with appropriate boundary conditions. Similarly, if $\mathscr{O}$ is a composite operator, it's straightforward to show that its tree-level MHV form factor can be readily computed in self-dual Yang-Mills by simply putting it on-shell [179]: the generating functional for such a form factor is the path integral with action

$$S_{\mathrm{SDYM}} + \int_{\mathbb{M}_{\mathbb{C}}} \mathrm{d}^4x\, \mathscr{J}\, \mathscr{O}\,, \tag{3.40}$$

S_{SDYM} being the action for self-dual Yang-Mills (2.18) and $\mathscr{J}$ being a source for $\mathscr{O}$. At tree-level and in the MHV sector, the generating functional reduces to (the exponential of) the on-shell action in the presence of the source.

Suppose now that we are given a self-dual background $A_{\alpha\dot{\alpha}}$ and a collection of n gluons propagating on this background. Let $\{a^{(i)}_{\alpha\dot{\alpha}}\}_{i\in I}$ be the set of the linearised gauge fields associated to the positive-helicity gluons. Since positive-helicity gluons represent self-dual perturbations to the background, they can be viewed as defining a coherent state and, together with the background $A_{\alpha\dot{\alpha}}$ itself, they define a *new* self-dual field $\mathcal{A}_{\alpha\dot{\alpha}}$, which in general depends non-linearly on each of the $a^{(i)}_{\alpha\dot{\alpha}}$. The negative-helicity gluons then correspond to negative-helicity states $\{b^{(j)}_{\alpha\beta}\}_{j\in J}$ propagating in this background. The data for the self-dual field $\mathcal{A}_{\alpha\dot{\alpha}}$ are nevertheless fairly simple to understand, as at $\mathscr{I}^+_{\mathbb{C}}$ the gauge field reduces to the sum of the asymptotic gauge field (3.6) for $A_- = 0$ and a collection of plane waves, thus the on-shell expression of $\mathscr{O}$ around a general self-dual radiative background is the generating functional for the MHV form factor of $\mathcal{O}$. In practice, $\mathcal{F}_{\mathscr{O}}$ can be computed by considering the bulk gauge field that reduces to

$$A|_{\mathscr{I}^+_{\mathbb{C}}} + \sum_{i\in I} \varepsilon_i a^{(i)}\,, \tag{3.41}$$

at $\mathscr{I}^+_{\mathbb{C}}$, where $\{\varepsilon_i\}_{i\in I}$ are formal parameters, and extracting the term in $\mathscr{O}$ proportional to $\prod_{i\in I}\varepsilon_i$ – see also [180, 181] for a related approach.

3.4.1 Form factor of $\operatorname{tr} B^2$

The computation of such a linear piece of the generating functional using spacetime methods is still rather difficult. However, this computation becomes straightforward when translated to twistor space. Let us show the construction of the operator $\operatorname{tr} B^2$ as an example.

Suppose that $A_{\alpha\dot\alpha}$ is described on twistor space via the Ward correspondence in terms of the partial connection $\bar{D} = \bar\partial + \mathsf{a}$ on the holomorphic vector bundle $E \to \mathbb{PT}$. Suppose further that the j^{th} negative-helicity gluon and i^{th} positive-helicity gluon are described by the twistor representatives $b^{(j)} \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4)\otimes \operatorname{End}E)$ and $a^{(i)} \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}\otimes\operatorname{End}E)$, respectively. Without loss of generality, we can assume that all of the forms a, $\{b^{(r)}_{j\in J}\}$, and $\{a^{(i)}_{i\in I}\}$ point along the $\mathrm{D}\bar\lambda$ direction. A deformed partial connection on E can now be defined by $\tilde{D} = \bar{D} + a$, where

$$a = \sum_{i\in I} \varepsilon_i\, a^{(i)}\,, \tag{3.42}$$

for $\{\varepsilon_i\}$ formal parameters. Under our assumptions, $\tilde{D}^2 = 0$, so the deformed partial connection also defines a holomorphic structure on E. Thus, $\{b^{(j)}\}_{j\in J}$ are also representatives of classes in $H^{0,1}_{\tilde{D}}(\mathbb{PT}, \mathcal{O}(-4)\otimes\operatorname{End}E)$. Let $\tilde{\mathsf{H}}$ and H denote the holomorphic frames associated with the partial connections $\tilde{D}$ and $\bar{D}$, respectively.

The uplift to twistor space of the generating functional of the form factor of $\operatorname{tr} B^2$

is now readily obtained via the Penrose transform (2.62) for negative-helicity fields

$$\mathcal{G}(r,s) = \int_{\mathbb{M}_{\mathbb{C}}} \mathrm{d}^4x \; \mathrm{e}^{-\mathrm{i}q\cdot x} \int_{X^2} \mathrm{D}\lambda_1 \mathrm{D}\lambda_2 \, \langle\lambda_1\,\lambda_2\rangle^2 \; \mathrm{tr}\, \left(\tilde{\mathsf{H}}_1^{-1}\, b_1^{(r)}\, \tilde{\mathsf{H}}_1\, \tilde{\mathsf{H}}_2^{-1}\, b_2^{(s)}\, \tilde{\mathsf{H}}_2\right) . \tag{3.43}$$

where we supposed that the two negative-helicity gluons are the r^{th} and s^{th}. The integrals here are over two copies of the twistor line X, with homogeneous coordinates $\lambda_{1\,\alpha}$ and $\lambda_{2\,\alpha}$ respectively, and there is an additional integration over the moduli of such lines, namely an integral over spacetime. We use the notation $\tilde{\mathsf{H}}_1$ and $b_1^{(r)}$ for $\tilde{\mathsf{H}}(x,\lambda_1)$ and $b^{(r)}\big|_X(x,\lambda_1)$, and similarly for $\tilde{\mathsf{H}}_2$ and $b_2^{(s)}$. The $q\to 0$ limit of the generating functional (3.43) is interpreted as the amplitude for the helicity flip of a single negative-helicity gluon traversing an SD background [182], in particular it is the generating functional of the MHV gluon amplitude around the self-dual background generated by a [66]. In terms of the propagator (3.25), the generating functional can be rewritten as

$$\mathcal{G}(r,s) = 4\pi^2 \int_{\mathbb{M}_{\mathbb{C}}\times X^2} \mathrm{d}^4x\, \mathrm{D}\lambda_1\, \mathrm{D}\lambda_2\, \mathrm{e}^{-\mathrm{i}q\cdot x} \langle\lambda_1\,\lambda_2\rangle^4 \; \mathrm{tr}\, \left(b_1^{(r)}\, \tilde{\mathsf{K}}_X(\lambda_1,\lambda_2)\, b_2^{(s)}\, \tilde{\mathsf{K}}_X(\lambda_2,\lambda_1)\right) . \tag{3.44}$$

Crucially, this representation for the MHV generating functional can now be straightforwardly expanded in a by obtaining a relation between the Green's functions $\tilde{\mathsf{K}}_X$ and K_X. The defining equation for $\tilde{\mathsf{K}}_X$ is

$$\bar{D}\big|_X\, \tilde{\mathsf{K}}_X(\lambda,\lambda') + a|_X\, \tilde{\mathsf{K}}_X(\lambda,\lambda') = \bar{\delta}(\langle\lambda\,\lambda'\rangle)\,, \tag{3.45}$$

which can be integrated to

$$\tilde{\mathsf{K}}_X(\lambda,\lambda') = \mathsf{K}_X(\lambda,\lambda') - \int_X \mathrm{D}\lambda''\, \mathsf{K}_X(\lambda,\lambda'')\; a|_X\,(\lambda'')\, \tilde{\mathsf{K}}(\lambda'',\lambda')\,, \tag{3.46}$$

in terms of the Green's function K_X for $\bar{D}|_X$. This integral formula can be iterated to obtain the Born series

$$\tilde{\mathsf{K}}_X(\lambda,\lambda') = \sum_{n=0}^{\infty}\left(\frac{-1}{2\pi\mathrm{i}}\right)^n \int_{X^n} \mathrm{D}\lambda_1\cdots\mathrm{D}\lambda_n \frac{\mathsf{H}(x,\lambda)\mathsf{H}_1^{-1}a_1\mathsf{H}_1\cdots\mathsf{H}_n^{-1}a_n\mathsf{H}_n\mathsf{H}^{-1}(x,\lambda')}{\langle\lambda\,\lambda_1\rangle\,\langle\lambda_1\,\lambda_2\rangle\cdots\langle\lambda_{n-1}\,\lambda_n\rangle\,\langle\lambda_n\,\lambda'\rangle}\,. \tag{3.47}$$

Here we denote $\mathsf{H}_i = \mathsf{H}(x,\lambda_i)$ and $a_i = a|_X\,(x,\lambda_i)$. Moreover, the $n=0$ term in this series is understood to be $\mathsf{K}_X(\lambda,\lambda')$.

The MHV form factor is now given as the coefficient of $\varepsilon_1\cdots\varepsilon_n$ in the generating functional (3.44), expanded using (3.47):

$$\mathcal{F}_{\mathrm{tr}\,B^2} = \int_{\mathbb{M}_{\mathbb{C}}\times X^n} \mathrm{d}^4x\,\mathrm{D}\lambda_1\cdots\mathrm{D}\lambda_n\,\mathrm{e}^{-\mathrm{i}q\cdot x}\langle\lambda_r\,\lambda_s\rangle^4 \\ \times\frac{\mathrm{tr}\left(\mathsf{H}_1^{-1}a_1^{(1)}\mathsf{H}_1\cdots\mathsf{H}_r^{-1}b_r^{(r)}\mathsf{H}_r\cdots\mathsf{H}_s^{-1}b_s^{(s)}\mathsf{H}_s\cdots\mathsf{H}_n^{-1}a_n^{(n)}\mathsf{H}_n\right)}{\langle\lambda_1\,\lambda_2\rangle\,\langle\lambda_2\,\lambda_3\rangle\cdots\langle\lambda_{n-1}\,\lambda_n\rangle\,\langle\lambda_n\,\lambda_1\rangle}\,, \tag{3.48}$$

where we have written only one of the colour-orderings arising from the perturbative expansion, namely the $(123\cdots n-1\,n)$ ordering. The full form factor is given by a sum over all non-cyclic permutations of this expression, but knowledge of a single colour ordering suffices.

Inserting the twistor representatives (3.27) and the background holomorphic (log) frame (3.22) in (3.48) and localising the integrals over X^n against the holomorphic delta functions present in the twistor representatives, we obtain the form factor for $\mathrm{tr}\,B^2$:

$$\mathcal{F}_{\mathrm{tr}\,B^2} = \frac{\langle r\,s\rangle^4}{\langle 1\,2\rangle\ldots\langle n\,1\rangle}\int_{\mathbb{M}_{\mathbb{C}}} \mathrm{d}^4x\,e^{\mathrm{i}(Q-q)\cdot x+\sum_j e_j\mathsf{g}(x,\kappa_j)}\,, \tag{3.49}$$

where $Q = k_1+\ldots+k_n$ is the sum of the external gluon momenta. As expected, in the limit $q\to 0$ we recover the known expression for the Parke-Taylor formula around a self-dual radiative background [66].

3.4.2 Form factors of $\operatorname{tr} B^k$, $\operatorname{tr} \tilde{F}^2$ and $\operatorname{tr} \tilde{F}^3$

The generalization to the form factor of an arbitrary power $\operatorname{tr} B^k := \operatorname{tr} B_{\alpha_1}{}^{\alpha_2} \dots B_{\alpha_k}{}^{\alpha_1}$ is straightforward

$$\mathcal{F}_{\operatorname{tr} B^k} = \frac{(\langle i_1\, i_2\rangle \dots \langle i_k\, i_1\rangle)^2}{\langle 1\, 2\rangle \dots \langle n\, 1\rangle} \int_{\mathbb{M}_{\mathbb{C}}} \mathrm{d}^4 x\, e^{\mathrm{i}(Q-q)\cdot x + \sum_j e_j \mathsf{g}(x,\kappa_j)}\,, \tag{3.50}$$

where we assumed that the $i_1^{\text{th}}, i_2^{\text{th}}, \dots, i_k^{\text{th}}$ gluons have negative helicity and the other gluons are positive-helicity.

For generic operators containing $\tilde{F}_{\dot\alpha\dot\beta}$ as well, the resulting formulae can still be considerably involved, so in the present section, we first consider the form factor for $\operatorname{tr} \tilde{F}_{\dot\alpha\dot\beta}\tilde{F}^{\dot\alpha\dot\beta}$. Around the flat background, the tree-level colour-ordered MHV form factor is [183]

$$\mathcal{F}_{\operatorname{tr} \tilde{F}^2}|_{A_+=0}\,(1^+, \dots, n^+; q) = \frac{(q^2)^2}{\langle 1\, 2\rangle \dots \langle n\, 1\rangle} \delta^4(Q-q)\,, \tag{3.51}$$

If one interprets the form factor as the amplitude for a massive complex scalar chirally coupled to the SD field strength, this beautiful formula can be proved by Berends-Giele recursion [183]; alternatively, one can compute the parity-conjugate form factor $\mathcal{F}_{\operatorname{tr} B^2}(1^-, \dots, n^-; q)$ as the 'maximally-non-MHV' form factor using the MHV formalism [184]. It's remarkable that such a compact formula exists at all for a maximally googly form factor. We now show that the simplicity of this form factor is a consequence of the existence of the K-matrix and, as a result, that the MHV form factor for $\operatorname{tr} \tilde{F}^2$ around any Cartan-valued self-dual radiative background admits an equally simple expression:

Proposition 3.4.1. *The MHV form factor of the operator* $\operatorname{tr} \tilde{F}^2$ *around any self-dual,* *Cartan-valued radiative background field at arbitrary multiplicity is*

$$\mathcal{F}_{\mathrm{tr}\,\tilde{F}^2}(1^+,\dots,n^+;q) = \frac{(q\cdot Q)^2}{\langle 1\,2\rangle\dots\langle n\,1\rangle}\int_{\mathbb{M}_C} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(Q-q)\cdot x+\sum_j e_j \mathsf{g}(x,\kappa_j)}\,. \tag{3.52}$$

Proof. We begin by expressing the SD field strength in terms of the K-matrix and integrating by parts twice, so that it is straightforward to rewrite the Fourier transform of $\mathrm{tr}\,\tilde{F}^2$ as

$$\int_{\mathbb{M}_\mathbb{C}} \mathrm{d}^4x\, e^{-\mathrm{i}q\cdot x}\iota_\alpha q^{\alpha\dot\alpha}\iota_\beta q^{\beta\dot\beta}\,\mathrm{tr}\,\iota^\gamma\partial_{\gamma\dot\alpha}K\,\iota^\delta\partial_{\delta\dot\beta}K\,. \tag{3.53}$$

Using (3.19) to extract the integral representation of $\iota^\alpha\partial_{\alpha\dot\alpha}K$, the lifting of this expression to twistor space reads

$$\int_{\mathbb{M}} \mathrm{d}^4x\, e^{-\mathrm{i}q\cdot x}\iota_\alpha q^{\alpha\dot\alpha}\iota_\beta q^{\beta\dot\beta} \quad \int_{X^2}\frac{\mathrm{D}\lambda_1\mathrm{D}\lambda_2\,\langle\lambda_1\lambda_2\rangle^2}{\langle\iota\,\lambda_1\rangle\langle\iota\,\lambda_2\rangle}\,\mathrm{tr}\left(\left.\frac{\partial\mathsf{a}_1}{\partial\mu_1^{\dot\alpha}}\right|_X \mathsf{K}_{12}\left.\frac{\partial\mathsf{a}_2}{\partial\mu_2^{\dot\beta}}\right|_X\mathsf{K}_{21}\right)\,, \tag{3.54}$$

so that the Born series (3.47) gives the expression

$$\begin{aligned}\mathcal{F} =& \frac{1}{\langle 1\,2\rangle\dots\langle n\,1\rangle}\int_{\mathbb{M}_\mathbb{C}}\mathrm{d}^4x\, e^{\mathrm{i}(Q-q)\cdot x+\sum_j e_j\mathsf{g}(x,\kappa_j)}\iota_\alpha q^{\alpha\dot\alpha}\iota_\beta q^{\beta\dot\beta}\sum_{i,j}\left\{\frac{\langle i\,j\rangle^2}{\langle\iota\,i\rangle\langle\iota\,j\rangle}\tilde\kappa_{i\dot\alpha}\tilde\kappa_{j\dot\beta}\right.\\ &-2\int_X\frac{\mathrm{D}\lambda}{2\pi\mathrm{i}}\frac{\langle i-1,i\rangle\langle\lambda\,j\rangle^2\tilde\kappa_{j\dot\beta}}{\langle i-1,\lambda\rangle\langle\lambda\,i\rangle\langle\iota\,\lambda\rangle\langle\iota\,j\rangle}\left.\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\right|_X\\ &\left.+\int_{X^2}\frac{\mathrm{D}\lambda\mathrm{D}\lambda'}{(2\pi\mathrm{i})^2}\frac{\langle i-1,i\rangle\langle j-1,j\rangle\langle\lambda\,\lambda'\rangle^2}{\langle i-1,\lambda\rangle\langle\lambda\,i\rangle\langle j-1,\lambda\rangle\langle\lambda\,j\,\rangle\langle\iota\,\lambda\rangle\langle\iota\,\lambda'\rangle}\left.\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\right|_X\left.\frac{\partial\mathsf{a}}{\partial\mu'^{\dot\beta}}\right|_X\right\},\end{aligned} \tag{3.55}$$

for the MHV form factor of $\mathrm{tr}\,\tilde{F}^2$. In principle, the background twistor connection can give derivative contributions to the form factor, namely the second and third line in (3.55). However, one can use the integral

$$\begin{aligned}\int_X\frac{\mathrm{D}\lambda}{2\pi\mathrm{i}}\frac{\lambda_\alpha\lambda_\beta}{\langle\iota\,\lambda\rangle\langle\ell-1,\lambda\rangle\langle\lambda\,\ell\rangle}\left.\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\right|_X =& -\mathrm{i}\left(\frac{\kappa_{\ell-1\,\alpha}\kappa_{\ell-1\,\beta}}{\langle\ell-1,\iota\rangle\langle\ell-1,\ell\rangle}\mathsf{g}_{\dot\alpha}(x,\kappa_{\ell-1})\right.\\ &+\frac{\kappa_{\ell\,\alpha}\kappa_{\ell\,\beta}}{\langle\ell\,\iota\rangle\langle\ell,\ell-1\rangle}\mathsf{g}_{\dot\alpha}(x,\kappa_\ell)\\ &\left.+\frac{\iota_\alpha\iota_\beta}{\langle\iota,\ell-1\rangle\langle\iota\,\ell\rangle}\mathsf{g}_{\dot\alpha}(x,\iota)\right)\,,\end{aligned} \tag{3.56}$$

to realise that the terms containing derivatives of the background actually give a vanishing contribution to the form factor. For generic momenta, this integral can be computed straightforwardly by expanding λ_α in the basis $\{\kappa^\alpha_{\ell-1},\kappa^\alpha_\ell\}$ and recalling the definition (3.38). Using this relation, the sums over i in the second and third line of (3.55) give the contribution

$$\sum_i\left(\frac{\langle i-1,j\rangle^2}{\langle i-1,\iota\rangle}\mathsf{g}_{\dot\alpha}(x,\kappa_{i-1})-\frac{\langle ij\rangle^2}{\langle i\iota\rangle}\mathsf{g}_{\dot\alpha}(x,\kappa_i)+\frac{\langle\iota j\rangle^2\langle i-1,i\rangle}{\langle\iota,i-1\rangle\langle\iota i\rangle}\mathsf{g}_{\dot\alpha}(x,\iota)\right)\,. \tag{3.57}$$

The first two terms obviously cancel in the sum over i. The third one is telescopic in i as well, once we complete ι_α to a basis $\{\iota_\alpha,o_\alpha\}$ of undotted spinors and expand each momentum κ^α_i in this basis. The third term then reduces to

$$\sum_i\frac{\langle i-1,1\rangle}{\langle\iota,i-1\rangle\langle\iota i\rangle}=\sum_i\left(\frac{\langle oi\rangle}{\langle\iota i\rangle}-\frac{\langle o,i-1\rangle}{\langle\iota,i-1\rangle}\right)=0\,. \tag{3.58}$$

Overall, the form factor reads

$$\mathcal{F}=\frac{1}{\langle 12\rangle\ldots\langle n1\rangle}\sum_{i,j}\frac{\langle ij\rangle^2\langle\iota|q|i]\langle\iota|q|j]}{\langle\iota i\rangle\langle\iota j\rangle}\int_{\mathbb{M}}\mathrm{d}^4x\,e^{\mathrm{i}(Q-q)\cdot x+\sum_j e_j\mathsf{g}(x,\kappa_j)}\,. \tag{3.59}$$

Let $\mathcal{S}$ denote the sum over i,j

$$\mathcal{S}=q^{\alpha\dot\alpha}q^{\beta\dot\beta}\sum_{i,j}\frac{\langle ij\rangle^2}{\langle\iota i\rangle\langle\iota j\rangle}\iota_\alpha\iota_\beta\tilde\kappa_{i\dot\alpha}\tilde\kappa_{j\dot\beta}\,. \tag{3.60}$$

Using the Schouten identity and performing one of the sums, $\mathcal{S}$ reduces to

$$\mathcal{S}=q^{\alpha\dot\alpha}q^{\beta\dot\beta}\left(Q_{\gamma\dot\alpha}\sum_i\frac{\iota_\beta\tilde\kappa_{i\dot\beta}\kappa_{i\alpha}\kappa^\gamma_i}{\langle\iota i\rangle}+Q_{\gamma\dot\beta}\sum_i\frac{\iota_\beta\tilde\kappa_{i\dot\alpha}\kappa_{i\alpha}\kappa^\gamma_i}{\langle\iota i\rangle}\right)\,, \tag{3.61}$$

Finally, noticing the identity

$$q^{\alpha\dot\alpha}Q_{\gamma\dot\alpha}=\frac{1}{2}(q^{\alpha\dot\alpha}Q_{\gamma\dot\alpha}-q_{\gamma\dot\alpha}Q^{\alpha\dot\alpha})+\frac{1}{2}\delta^\alpha_\gamma q\cdot Q\,, \tag{3.62}$$

we can further simplify $\mathcal{S}$ down to

$$\mathcal{S} = -(q \cdot Q)^2 \,, \tag{3.63}$$

and the form factor is finally (3.52), up to an overall numerical factor. $\square$

As previously anticipated, the dependence on the gauge spinor ι_α dropped out and the result is fully gauge-invariant. Around a non-trivial background, we expect translations to be broken, and therefore the momentum-conserving δ-function in (3.51) is replaced by the residual integral over space-time in (3.49)-(3.50)-(3.52). This integral cannot be performed analytically for a generic background, but for specific, highly symmetric examples, it is possible to further simplify it. Furthermore, we note that for the form factor around the trivial background, $q^2 = q \cdot Q$ on the support of the momentum conserving δ function.

A similar analysis can be set up for form factors of other polynomials in $\tilde{F}_{\dot\alpha\dot\beta}$ and its derivatives. Let us now consider the cubic operator

$$\operatorname{tr} \tilde{F}^3 \coloneqq \operatorname{tr} \tilde{F}_{\dot\alpha}{}^{\dot\beta} \tilde{F}_{\dot\beta}{}^{\dot\gamma} \tilde{F}_{\dot\gamma}{}^{\dot\alpha} \,. \tag{3.64}$$

Note that this is the unique cubic operator not involving derivatives of the SD field strength, up to a sign. The lifting of this operator to twistor space cannot be simplified using the K-matrix anymore but can be straightforwardly obtained using (3.21) and again the generating functional is the Fourier transform of such lifting. Nevertheless, the perturbative expansion proceeds as before: at arbitrary multiplicity, the tree-level, colour-ordered MHV form factor around a self-dual, Cartan-valued, radiative background is [1]

$$
\begin{aligned}
\mathcal{F}_{\operatorname{tr}\tilde{F}^3} = & \frac{1}{\langle 12\rangle\dots\langle n1\rangle}\int_{\mathbb{M}_{\mathbb{C}}}\mathrm{d}^4x\, e^{\mathrm{i}(Q-q)\cdot x+\sum_j e_j \mathsf{g}(x,\kappa_j)}\times \\
& \times\Bigg(\sum_{i,j,k}\langle ij\rangle\langle jk\rangle\langle ki\rangle[ij][jk][ki] \\
& + 3\sum_{i,j,k,\ell}\langle jk\rangle\langle k\ell\rangle\langle \ell i\rangle[k\ell]([\ell i][jk]+[ik][\ell j]) \\
& + 3\sum_{i,j,k,\ell,m}\langle jk\rangle\langle \ell m\rangle\langle mi\rangle\Big([mi]([jk][\ell m]+[j\ell][km])+(i\leftrightarrow j)\Big) \\
& + \sum_{i,j,k,\ell,m,n}\langle jk\rangle\langle \ell m\rangle\langle ni\rangle\Big([jk]([ni][\ell m]+[mi][\ell n])+(k\leftrightarrow \ell) \\
& + (i\leftrightarrow j)+(i\leftrightarrow j, k\leftrightarrow \ell)\Big)\Bigg)\,.
\end{aligned}
$$

In particular, the colour-ordered formulae obtained with our generating functional match the first few expressions around the trivial background, namely the minimal form factor

$$\mathcal{F}_{\operatorname{tr}\tilde{F}^3}(1^+,2^+,3^+;q) = [12][23][31]\,\delta^4(Q-q)\,, \tag{3.65}$$

and the 4-point form factor [47, 185]

$$\mathcal{F}_{\operatorname{tr}\tilde{F}^3}(1^+,2^+,3^+,4^+;q) = \frac{[12][23][34][41]}{\langle 12\rangle[21]}\left(1+\frac{[31][4|q|3\rangle}{\langle 23\rangle[32][41]}\right)\delta^4(Q-q)+\text{cyclic}\,. \tag{3.66}$$

3.4.3 Generic MHV form factors

The most remarkable feature of the expressions (3.49)-(3.52) and (3.65) is that the form factor around a non-trivial background is obtained by a simple dressing of the form factor around the trivial background, namely one only has to replace the δ function with the space-time integral in the last line of (3.65), while keeping intact the kinematical prefactor. This property is generic of any form factor of a composite operator, as the lifting of any polynomial in $\tilde{F}_{\dot\alpha\dot\beta}$ and its derivatives[3] will be a sum of products of integrals over X of terms of the form $\lambda_i^{\alpha_1}\dots\lambda_i^{\alpha_s}\partial_{\mu_i^{\dot\alpha_1}}\dots\partial_{\mu_i^{\dot\alpha_s}}\mathsf{a}$, intertwined by propagators $\mathsf{K}_X(\lambda_i,\lambda_j)$ between adjacent positions on the sphere X and with some contraction of the spinor indices. Once we expand such an expression in terms of momentum eigenstates, the background connection can contribute to the form factor in two distinct ways: it can either be present only in the holomorphic frames $\mathsf{H}(x,\lambda_i)$, or it can potentially give a contribution when present in a $\mu^{\dot\alpha}$ derivative. The first type of contribution gives a factor of $\exp(e_j\mathsf{g}(x,\kappa_j))$ when the background acts by conjugation on the jth external gluon. Conversely, the second type of contribution is proportional to (possibly a spacetime derivative of)

$$\frac{1}{2\pi\mathrm{i}}\int_X \frac{\mathrm{D}\lambda}{\langle j-1,\lambda\rangle\langle\lambda j\rangle}\lambda_\alpha\,\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\bigg|_X\,, \tag{3.67}$$

when we consider a colour-ordered form factor. Such a term must be inserted in any possible position inside the perturbative expansion, and for a colour-ordered form factor, this means that we must sum over j. Moreover, any of these contributions comes together with a partial Parke-Taylor denominator $1/(\langle 12\rangle\dots\langle\widehat{j-1,j}\rangle\dots\langle n1\rangle)$ from which the factor $1/\langle j-1,j\rangle$ is removed. The integral (3.67) can be evaluated for generic external momenta by considering $\kappa_{j-1,\alpha}$ and $\kappa_{j\,\alpha}$ as a basis of undotted spinors, and it's equal to

$$\frac{1}{2\pi\mathrm{i}}\int_X \frac{\mathrm{D}\lambda}{\langle j-1,\lambda\rangle\langle\lambda j\rangle}\lambda_\alpha\,\frac{\partial\mathsf{a}}{\partial\mu^{\dot\alpha}}\bigg|_X = \mathrm{i}\frac{\kappa_{j-1,\alpha}\mathsf{g}_{\dot\alpha}(x,\kappa_{j-1})-\kappa_{j\,\alpha}\mathsf{g}_{\dot\alpha}(x,\kappa_j)}{\langle j-1,j\rangle}\,. \tag{3.68}$$

[3]The possible presence of $B_{\alpha\beta}$ fields does not bring in any additional issue, as we are working in the MHV sector.

The denominator $\langle j-1, j\rangle$ is precisely the missing factor needed to reconstruct the Parke-Taylor denominator, so that the residual sum over j is telescopic

$$\mathrm{i}\sum_{j=1}^{n}\left(\kappa_{j-1,\alpha}\mathsf{g}_{\dot\alpha}(x,\kappa_{j-1}) - \kappa_{j\,\alpha}\mathsf{g}_{\dot\alpha}(x,\kappa_j)\right) = 0\,. \tag{3.69}$$

Then the only non-vanishing contribution to the form factor around a non-trivial background comes from terms where the background is present only in the holomorphic frames, and this means that the desired form factor coincides with the one around the trivial background, once we replace the δ function with the integral

$$\int_{\mathbb{M}} \mathrm{d}^4x\, e^{\mathrm{i}(Q-q)\cdot x+\sum_j e_j\mathsf{g}(x,\kappa_j)}\,. \tag{3.70}$$

Chapter 4

Gluon amplitudes around the self-dual dyon

Having discussed radiative fields, the next step is to include background sources. In this Chapter, we study gluon amplitudes around point-like sources whose wordlines are supported on line defects in four dimensions. The simplest source is the *self-dual dyon*: this is a solution of the Maxwell equations composed of the superposition of the electric field of a point charge and of the magnetic field sourced by a magnetic monopole, where the electric and magnetic charges are assumed to be equal in order to satisfy the self-duality equation. On the Minkowski slice, the self-dual dyon is a complex-valued gauge field, but on the Euclidean and Kleinian slices it is real-valued.

Scattering on Coulombic sources is a well-known problem and presents remarkable complications already at the level of the scattering states: the Coulomb wave equation can only be solved using separation of variables and a partial wave expansion, with radial modes given by confluent hypergeometric functions. Even the 2-point amplitude (which is only sensitive to the asymptotics of the Coulomb wavefunction) is given by an infinite sum of terms controlled by the radial action [69–73], both for the purely Coulombic background and for generic (non-self-dual) dyonic backgrounds.

However, when the underlying background is self-dual, remarkably simplifications occur: the linearized equations of motion can be solved *exactly* in a closed, compact form in terms of charged Killing spinors. After introducing them, we show how they can be used to give a novel twistorial description of the self-dual dyon using Dolbeault cohomology. As in Chapter 3, the twistor description of the self-dual dyon will allow a compact representation of $\operatorname{tr} B^2$ as the generating functional for the MHV amplitude, for which we derive a compact formula at all multiplicity.

4.1 Spinor-helicity variables for $\mathbb{R}^3$

In this Chapter and in Chapter 5, it will be useful to work with spinor-helicity variables adapted to the Euclidean slice

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + \mathrm{i}x^3 & \mathrm{i}x^1 + x^2 \\ \mathrm{i}x^1 - x^2 & x^0 - \mathrm{i}x^3 \end{pmatrix}, \tag{4.1}$$

and to set up spinor coordinates on the $\mathbb{R}^3$ slices at constant x^0. In terms of the time-like vector $T^{\alpha\dot{\alpha}}$

$$T^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.2}$$

we introduce

$$x^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} T^{(\alpha|\dot{\alpha}|} x^{\beta)\dot{\beta}} = \frac{1}{2} \begin{pmatrix} \mathrm{i}x^1 + x^2 & -\mathrm{i}x^3 \\ -\mathrm{i}x^3 & -\mathrm{i}x^1 + x^2 \end{pmatrix}. \tag{4.3}$$

Let now (r, θ, ϕ) be standard spherical coordinates on $\mathbb{R}^3 - 0 = \mathbb{R}_+ \times S^2$. We will need stereographic coordinates on the sphere at constant r

$$\zeta = \frac{x^1 + \mathrm{i}x^2}{r + x^3} = \mathrm{e}^{\mathrm{i}\phi} \tan\frac{\theta}{2}, \qquad \bar{\zeta} = \frac{x^1 - \mathrm{i}x^2}{r + x^3} = \mathrm{e}^{-\mathrm{i}\phi} \tan\frac{\theta}{2}. \tag{4.4}$$

In terms of the stereographic coordinates, we define the Killing spinors

$$\chi_+^\alpha = \sqrt{\frac{r}{1+\zeta\bar{\zeta}}}\begin{pmatrix} \bar{\zeta} \\ -1 \end{pmatrix}, \qquad \chi_-^\alpha = \sqrt{\frac{r}{1+\zeta\bar{\zeta}}}\begin{pmatrix} 1 \\ \zeta \end{pmatrix}. \tag{4.5}$$

We will study the properties of these spinors extensively below. For the moment, we note that they satisfy

$$x^{\alpha\beta} = \mathrm{i}\chi_+^{(\alpha}\chi_-^{\beta)}\,, \qquad \langle\chi_-\,\chi_+\rangle = r\,. \tag{4.6}$$

4.2 The self-dual dyon

The self-dual dyon (SDD) arises, in the first instance, as a classical solution of Maxwell theory. On the Lorentzian slice, it can be viewed as the gauge field sourced by a point particle carrying electric and magnetic charges of equal magnitude but related by an overall factor of the imaginary unit. This ensures the field strength is self-dual as a 2-form on $\mathbb{M}$. As such, the SDD is a complex gauge field in $\mathbb{M}$, although it can also be viewed as a real gauge field on either $\mathbb{E}$ or $\mathbb{K}$. On the Euclidean slice, the self-dual profile has equal electric and magnetic charges and is captured by the gauge field

$$A = \frac{\mathrm{d}x^0}{r} + \mathrm{a}\,, \qquad \mathrm{a} = (1-\cos\theta)\,\mathrm{d}\phi = \mathrm{i}\frac{\zeta\,\mathrm{d}\bar{\zeta} - \bar{\zeta}\,\mathrm{d}\zeta}{1+\zeta\bar{\zeta}}\,. \tag{4.7}$$

The first term in (4.7) is the Coulomb part of the potential sourced by the electric charge of the dyon, whereas the second term a is the Dirac monopole sourced by the magnetic charge. This has a wire singularity at $\theta = \pi$ which can be moved to $\theta = 0$ by a suitable gauge transformation. In order to construct non-trivial amplitudes on this background, we can embed the SDD into the Cartan subalgebra of the gauge algebra of a generic non-abelian Yang-Mills theory. Let c be an element of the Cartan subalgebra and simply set

$$A = \mathsf{c}\left(\frac{\mathrm{d}t}{r} + \mathrm{a}\right) . \tag{4.8}$$

The condition that it is embedded in a Cartan subalgebra ensures that the self-duality condition of the abelian case is preserved in the full non-abelian theory, as the corresponding field strength reduces to

$$F = \mathrm{d}A + \frac{1}{2}[A, A] = \mathsf{c}\left(\frac{\mathrm{d}t \wedge \mathrm{d}r}{r^2} + \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi\right) . \tag{4.9}$$

This is essentially the same as the abelian field strength, and satisfies the non-abelian self-duality equations. In particular, in 2-spinor variables the field strength is manifestly self-dual

$$F_{\alpha\dot\alpha\beta\dot\beta} = \mathsf{c}\frac{2x_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}}{r^3} , \tag{4.10}$$

where $x^{\dot\alpha\dot\beta} = \epsilon_{\alpha\beta}T^{\alpha(\dot\alpha}x^{|\beta|\dot\beta)}$.

Other gauges for the SDD gauge potential can be found which manifest other useful features of the field. For instance, in the Kerr-Schild gauge the gauge potential is

$$A^{\mathrm{KS}} = \mathsf{c}\,\phi\, q\,, \qquad \phi = \frac{1}{r}\,, \qquad q = \mathrm{d}x^0 + \mathrm{i}\mathrm{d}r - 2\mathrm{i}r\frac{\bar\zeta}{1+\zeta\bar\zeta}\mathrm{d}\zeta\,. \tag{4.11}$$

Here ϕ is a harmonic function, whilst $q = q_a\,\mathrm{d}x^a$ is a 1-form satisfying

$$q^a\, q_a = 0\,, \qquad q^a\,\partial_a q_b = 0\,, \tag{4.12}$$

so that the dual vector field q^a is the tangent vector of a null geodesic in $\mathbb{M}$. Note that this implies that the Kerr-Schild gauge is also a light-cone gauge. This choice of gauge can be used to relate the SDD to the self-dual Taub-NUT metric by the classical double copy [106, 186].

Another natural light-cone gauge can be defined in terms of the Killing spinors

(4.5)

$$A^{\rm LC}_{\alpha\dot\alpha} = \frac{\iota_\alpha T^{\gamma}{}_{\dot\alpha}}{r}\left(\frac{\chi_{+\gamma}}{\langle\iota\,\chi_+\rangle} + \frac{\chi_{-\gamma}}{\langle\iota\,\chi_-\rangle}\right), \tag{4.13}$$

where $\iota^\alpha = (1,0)$ is a constant spinor. The gauge field is in the light-cone gauge $n\cdot A^{\rm LC} = 0$ with respect to the null vector $n^{\alpha\dot\alpha} = \iota^\alpha\bar\iota^{\dot\alpha}$. Note that both the Kerr-Schild and light-cone gauges also happen to be Lorenz gauges.

Finally, the Dirac monopole a can be used to compute the exterior derivatives of the Killing spinors. A direct computation shows

$$\mathrm{d}\chi^\alpha_+ = \left(\frac{x_{\beta\gamma}\mathrm{d}x^{\beta\gamma}}{r^2} - \frac{\mathrm{i}}{2}\mathrm{a}\right)\chi^\alpha_+ - \frac{\mathrm{i}}{r^2}\chi_{+\beta}\chi_{+\gamma}\mathrm{d}x^{\beta\gamma}\chi^\alpha_-\,, \tag{4.14}$$

$$\mathrm{d}\chi^\alpha_- = \left(\frac{x_{\beta\gamma}\mathrm{d}x^{\beta\gamma}}{r^2} + \frac{\mathrm{i}}{2}\mathrm{a}\right)\chi^\alpha_- - \frac{\mathrm{i}}{r^2}\chi_{-\beta}\chi_{-\gamma}\mathrm{d}x^{\beta\gamma}\chi^\alpha_+\,. \tag{4.15}$$

4.3 The twistor quadrille

In Maxwell theory, the self-dual dyon arises from a line bundle $E\to\mathbb{PT}$ over twistor space known as the *twistor quadrille* [187, 188]. The original description of the SDD in twistor theory was based on a Čech formulation [187, 188], briefly discussed in Appendix A; here we develop a new Dolbeault description which is more suited to our later calculations. The two descriptions are, of course, equivalent thanks to the Čech-Dolbeault isomorphism.

Consider the Ward bundle $E\to\mathbb{PT}$ defined by the partial connection

$$\mathsf{a} = -\mathsf{c}\,\bar e^0\,\bar\partial_0 \log\frac{\langle\lambda\,\chi_+\rangle}{\langle\lambda\,\chi_-\rangle}\,, \tag{4.16}$$

where $\chi^\alpha_\pm$ are the charged Killing spinors of the SDD defined by (4.5). The connection is naturally defined in Euclidean signature, where we can employ the space-time coordinates directly on twistor space, whilst it would require a more careful analysis in Lorentzian and split signature. The local nature of a on $\mathbb{PT}$ is clear as the twistor connection it is ill-defined where $\langle\lambda\,\chi_\pm\rangle = 0$, so must be understood as defined on the open patch away from these locii. It is also clear that this partial connection is holomorphic, as it is given by a function on the sphere proportional to $\bar{e}^0$, so $\bar{\partial}\mathsf{a} = 0$ by virtue of $\bar{e}^0 \wedge \bar{e}^0 = 0$. The defining equation (2.22) for the holomorphic frame associated to E with this partial connection implies that a possible choice of H is

$$\mathsf{H}(x,\lambda) = \left(\frac{\langle\lambda\,\chi_+\rangle}{\langle\lambda\,\chi_-\rangle}\right)^{\mathsf{c}}. \tag{4.17}$$

It is easily checked that this holomorphic frame reproduces the SDD field in the gauge (4.8): using (2.24), one obtains

$$-\mathrm{i}\lambda^\alpha A_{\alpha\dot{\alpha}} = \mathsf{c}\lambda^\alpha\left(\frac{\partial_{\alpha\dot{\alpha}}\langle\lambda\,\chi_+\rangle}{\langle\lambda\,\chi_+\rangle} - \frac{\partial_{\alpha\dot{\alpha}}\langle\lambda\,\chi_-\rangle}{\langle\lambda\,\chi_-\rangle}\right). \tag{4.18}$$

Using the exterior derivatives (4.14)-(4.15) and the identity $2\chi_{+[\alpha}\chi_{-\beta]} = r\epsilon_{\alpha\beta}$, we recover

$$A_{\alpha\dot{\alpha}} = \mathsf{c}\left(\mathrm{a}_{\alpha\dot{\alpha}} + \frac{T_{\alpha\dot{\alpha}}}{r}\right). \tag{4.19}$$

Note that, as required, dependence on the open patch $\{\langle\lambda\chi_\pm\rangle \neq 0\} \subset \mathbb{PT}$ drops out in the transform to the spacetime gauge field.

Similar computations show that the holomorphic frames associated to the SDD in Kerr-Schild gauge (4.11) and light-cone gauge (4.13) are given by

$$\mathsf{H}^{\mathrm{KS}} = \left(\frac{\langle\lambda\,\chi_+\rangle}{\langle\lambda\,\chi_-\rangle}\,\frac{r}{1+\zeta\bar{\zeta}}\right)^{\mathsf{c}}, \qquad \mathsf{H}^{\mathrm{LC}} = \left(\frac{\langle\lambda\,\chi_+\rangle\,\langle\iota\,\chi_-\rangle}{\langle\lambda\,\chi_-\rangle\,\langle\iota\,\chi_+\rangle}\right)^{\mathsf{c}}, \tag{4.20}$$

respectively. For instance, the field in light-cone gauge can be obtained by a procedure akin to the construction of the radiative gauge fields of Chapter 3: assuming that the holomorphic frame satisfies the boundary condition $\mathsf{H}^{\mathrm{LC}}(x,\iota) = 1$, for some arbitrary ι_α, one can make the ansatz $\mathsf{H}^{\mathrm{LC}}(x,\lambda) = \exp(-\mathsf{g}^{\mathrm{LC}}(x,\lambda))$ for the holomorphic frame. Using the Green's function (3.14) with the appropriate weight, the function g^{LC} is given by

$$\mathsf{g}^{\mathrm{LC}}(x,\lambda) = \frac{1}{2\pi\mathrm{i}}\int_X \frac{\mathrm{D}\lambda'}{\langle\lambda\,\lambda'\rangle}\wedge\frac{\langle\iota\,\lambda\rangle}{\langle\iota\,\lambda'\rangle}\,\mathsf{a}|_X(x,\lambda')\,. \tag{4.21}$$

This integral formula precisely reproduces the frame H^{LC}:

$$\begin{aligned}
\log\mathsf{H}^{\mathrm{LC}}(x,\lambda) &= -\frac{\mathsf{c}}{2\pi\mathrm{i}}\int_X \frac{\mathrm{D}\lambda'}{\langle\lambda\,\lambda'\rangle}\frac{\langle\iota\,\lambda\rangle}{\langle\iota\,\lambda'\rangle}\wedge\bar\partial'\big|_X\log\frac{\langle\lambda'\,\chi_+\rangle}{\langle\lambda'\,\chi_-\rangle}\\
&= \frac{\mathsf{c}}{2\pi\mathrm{i}}\int_X \mathrm{D}\lambda'\wedge\bar\partial'\big|_X\left(\frac{\langle\iota\,\lambda\rangle}{\langle\lambda\,\lambda'\rangle\langle\iota\,\lambda'\rangle}\right)\log\frac{\langle\lambda'\,\chi_+\rangle}{\langle\lambda'\,\chi_-\rangle}\\
&= \mathsf{c}\int_X \mathrm{D}\lambda'\wedge\left(\frac{1}{\langle\lambda\,\lambda'\rangle}\bar\delta(\langle\iota\,\lambda'\rangle)+\frac{1}{\langle\iota\,\lambda'\rangle}\bar\delta(\langle\lambda\,\lambda'\rangle)\right)\langle\iota\,\lambda\rangle\log\frac{\langle\lambda'\,\chi_+\rangle}{\langle\lambda'\,\chi_-\rangle}\\
&= \mathsf{c}\log\frac{\langle\lambda\,\chi_+\rangle\langle\iota\,\chi_-\rangle}{\langle\lambda\,\chi_-\rangle\langle\iota\,\chi_+\rangle}\,.
\end{aligned} \tag{4.22}$$

The second line follows from Stokes' theorem.

It is also possible to understand these data – in particular, the singularities in the background twistor connection a – as the local data defining a line bundle on a non-Hausdorff twistor space $\mathbb{PT}_Q$, obtained by glueing two copies of $\mathbb{PT}$ over the zero locus of the quadric

$$Q = \mu^{\dot\alpha}T^{\alpha}{}_{\dot\alpha}\lambda_\alpha\,, \tag{4.23}$$

see [189] for the original construction by Bailey and [132] for a more modern discussion. Note in particular that the singularities of the twistor connection a lie exactly on the zero locus of the quadric, as Q intersects the twistor line X on

$$Q|_X = \mathrm{i}\langle\lambda\,\chi_+\rangle\langle\lambda\,\chi_-\rangle\,, \tag{4.24}$$

as it follows from the incidence relations (2.11) and the first equation in (4.6).

4.4 Scattering states

We now construct the scattering states propagating on the SDD background. Since the background is particularly simple, we can address this problem in a variety of ways and it is instructive not to resort to the Penrose transform immediately. We first discuss how the Killing spinors (4.5) can be used to construct charged fields of different helicities and charge. This construction will be extremely valuable in the Chapter 5 as well. We first construct scalar and negative helicity states, and then describe a spin-raising procedure to construct positive helicity states. We then recover the same states from twistor space with the Penrose transform.

4.4.1 Killing spinors for the self-dual dyon

A valence-p ASD Killing spinor on a general curved background is a solution $\chi^{\alpha_1\dots\alpha_p}$ to the equation

$$D^{(\alpha}{}_{\dot\alpha}\chi^{\beta_1\dots\beta_p)} = 0\,. \tag{4.25}$$

Any Killing spinor can be used to raise the helicity of a solution to the helicity $-(n+p)/2$ massless free field equations

$$D^{\alpha\dot\alpha}\psi_{\alpha\alpha_2\dots\alpha_{n+p}} = 0 \tag{4.26}$$

to generate a solution

$$\varphi_{\alpha_1\dots\alpha_n} = \psi_{\alpha_1\dots\alpha_{n+p}}\,\chi^{\alpha_{n+1}\dots\alpha_{n+p}} \tag{4.27}$$

of the helicity $-n/2$ field equations, as it follows from the Leibnitz rule and (4.25). The main result that we will need is:

Proposition 4.4.1. *For the Abelian self-dual dyon in the gauge* (4.7)*, the spinors*

$\chi^\alpha_\pm$ *satisfy the charged Killing spinor equations*

$$\partial^{(\alpha}{}_{\dot\alpha}\chi^{\beta)}_\pm \pm \frac{\mathrm{i}}{2}\, A^{(\alpha}{}_{\dot\alpha}\chi^{\beta)}_\pm = 0\,, \tag{4.28}$$

Thus, $\chi^\alpha_\pm$ *are Killing spinors with charges* $\pm 1/2$.

Proof. This is proved by direct calculation using the exterior derivatives (4.14)-(4.15). Focussing on χ^α_+, we have

$$\partial_{\alpha\dot\alpha}\chi^\beta_+ + \frac{\mathrm{i}}{2}A_{\alpha\dot\alpha}\chi^\beta_+ = \frac{x_{\alpha\gamma}T^\gamma{}_{\dot\alpha}}{r^2}\chi^\beta_+ - \frac{\mathrm{i}}{r^2}\chi^\beta_-\chi_{+\alpha}\chi_{+\gamma}T^\gamma{}_{\dot\alpha} + \frac{\mathrm{i}}{2r}T_{\alpha\dot\alpha}\chi^\beta_+\,, \tag{4.29}$$

so the first of (4.6) implies

$$\partial^{(\alpha}{}_{\dot\alpha}\chi^{\beta)}_+ + \frac{\mathrm{i}}{2}A^{(\alpha}{}_{\dot\alpha}\chi^{\beta)}_+ = \frac{\mathrm{i}}{2r^2}T_{\gamma\dot\alpha}\left(-\chi^\alpha_+\chi^\beta_+\chi^\gamma_- + \chi^{(\alpha}_+\chi^{\beta)}_-\chi^\gamma_+ + r\chi^{(\alpha}_+\epsilon^{\beta)\gamma}\right)\,. \tag{4.30}$$

The RHS is identically vanishing, once one uses the identity $\chi^{[\alpha}_+\chi^{\beta]}_- = r\epsilon^{\alpha\beta}$ and after expanding all symmetrized products. A similar computation holds for χ^α_-. $\square$

The main consequence that we will use is the following:

Corollary 4.4.1. *Given* $n, p, q \in \mathbb{Z}_{\geq 0}$, *let the totally symmetric spinor* $\psi_{\alpha_1\ldots\alpha_{n+p+q}}$ *satisfy the ZRM equation of helicity* $-(n+p+q)/2$ *in flat space. Then the field*

$$\varphi_{\alpha_1\ldots\alpha_n} = \psi_{\alpha_1\ldots\alpha_{n+p+q}}\chi^{\alpha_{n+1}}_+\ldots\chi^{\alpha_{n+p}}_+\chi^{n+p+1}_-\ldots\chi^{n+p+q}_-\,, \tag{4.31}$$

satisfies the ZRM equation coupled to SDD background of helicity $-n/2$ *and charge* $(p-q)/2$.

Proof. For $n > 0$, we have

$$\begin{aligned}
D^{\alpha_1\dot{\alpha}}\varphi_{\alpha_1\ldots\alpha_n} &= (\partial^{\alpha_1\dot{\alpha}}\psi_{\alpha_1\ldots\alpha_{n+p+q}})\chi_+^{\alpha_{n+1}}\ldots\chi_+^{\alpha_{n+p}}\chi_-^{\alpha_{n+p+1}}\ldots\chi_-^{\alpha_{n+p+q}} \\
&\quad + p\psi_{\alpha_1\ldots\alpha_{n+p+q}}(D^{(\alpha_1|\dot{\alpha}|}\chi_+^{\alpha_{n+1})})\chi_+^{\alpha_{n+2}}\ldots\chi_+^{\alpha_{n+p}}\chi_-^{\alpha_{n+p+1}}\ldots\chi_-^{\alpha_{n+p+q}} \\
&\quad + q\psi_{\alpha_1\ldots\alpha_{n+p+q}}\chi_+^{\alpha_{n+1}}\ldots\chi_+^{\alpha_{n+p}}(D^{(\alpha_1|\dot{\alpha}|}\chi_-^{\alpha_{n+p+1})})\chi_-^{\alpha_{n+p+2}}\ldots\chi_-^{\alpha_{n+p+q}}\,,
\end{aligned} \tag{4.32}$$

from the Leibnitz rule and the symmetry of $\psi_{\alpha_1\ldots\alpha_{n+p+q}}$. Each line in (4.32) vanishes independently.

For $n = 0$, suppose without loss of generality $p > 0$ and decompose the field as

$$\varphi = \psi_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q}\chi_+^{\alpha_1}\ldots\chi_+^{\alpha_p}\chi_-^{\beta_1}\ldots\chi_-^{\beta_q} \equiv \chi_+^{\alpha}\varphi_\alpha\,, \tag{4.33}$$

so that

$$D^2\varphi = D^2\chi_+^{\alpha}\,\varphi_\alpha + 2D^{\beta\dot{\beta}}\chi_+^{\alpha}\,D_{\beta\dot{\beta}}\varphi_\alpha + \chi_+^{\alpha}\,D^2\varphi_\alpha\,. \tag{4.34}$$

Since φ_α satisfies the ZRM equation for helicity $h = -1/2$, the second term vanishes in virtue of $D^{(\alpha}{}_{\dot{\alpha}}\chi_+^{\beta)} = 0$ and $D^{[\alpha}{}_{\dot{\alpha}}\varphi^{\beta]} = 0$. The third term vanishes because the background is self-dual, as

$$\begin{aligned}
D^2\varphi_\alpha &= D^{\beta\dot{\beta}}D_{\beta\dot{\beta}}\varphi_\alpha \\
&= D^{\beta\dot{\beta}}D_{\alpha\dot{\beta}}\varphi_\beta \\
&= [D^{\beta\dot{\beta}}, D_{\alpha\dot{\beta}}]\varphi_\beta + D_{\alpha\dot{\beta}}D^{\beta\dot{\beta}}\varphi_\beta = 0
\end{aligned} \tag{4.35}$$

Finally, the first term vanishes because $D^2\chi_+^{\alpha} = 0$ by direct computation, using

$$D^{\alpha\dot{\alpha}}\chi_\pm^{\beta} = \pm\frac{\mathrm{i}}{r}\epsilon^{\alpha\beta}\chi_\pm^{\gamma}T_\gamma{}^{\dot{\alpha}}\,. \tag{4.36}$$

□

Charged quasi-momentum eigenstates of non-positive helicity on the SDD background can now be generated by beginning with uncharged negative-helicity momentum eigenstates on flat space. Using Corollary 4.4.1, we can construct quasi-momentum eigenstates of the form

$$\varphi^{(e,q)}_{\alpha_1\ldots\alpha_n} = \kappa_{\alpha_1}\ldots\kappa_{\alpha_n}\mathsf{T}^e\frac{\langle\kappa\,\chi_+\rangle^{q+e}\langle\kappa\,\chi_-\rangle^{q-e}}{\langle\kappa\,o\rangle^{2q}}\mathrm{e}^{\mathrm{i}k\cdot x}\,. \tag{4.37}$$

These states are defined by a pair of charges (e,q) chosen such that they are simultaneously integer or half-integer, so that the wavefunction is single-valued. T^e is a generator in the Lie algebra of the gauge group such that $[\mathsf{c},\mathsf{T}^e]=e\mathsf{T}^e$. The states have charge e with respect to the background SDD gauge field and are everywhere non-singular on the celestial sphere, provided $q\geq|e|$. The factor of $\langle\kappa\,o\rangle^{-2q}$ has been inserted to give the state the appropriate weight under little-group scaling.

With the parametrization (2.6)-(4.5), the general quasi-momentum eigenstate of helicity $-n/2$ is given explicitly by[1]

$$\varphi^{(e,q)}_{\alpha_1\ldots\alpha_n} = \kappa_{\alpha_1}\ldots\kappa_{\alpha_n}\mathsf{T}^e\left(\frac{r}{1+\zeta\bar{\zeta}}\right)^q(\bar{\zeta}z+1)^{q+e}\,(\zeta-z)^{q-e}\,\mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{4.38}$$

A particular feature here is that for smooth solutions on the sphere and non-trivial charge, we see that the $\langle\kappa\,\chi_+\rangle$ factors generate spin-weighted spherical harmonics. These are, in turn, accompanied by powers of r that, for global solutions on the sphere, are higher than those expected for uncharged fields on a flat background.

There are distinguished solutions that are nowhere singular on the celestial sphere and grow with the lowest possible power of r as $r\to\infty$. These are given by

$$\varphi^{+}_{\alpha_1\ldots\alpha_n} = \varphi^{(e,e)}_{\alpha_1\ldots\alpha_n} = \kappa_{\alpha_1}\ldots\kappa_{\alpha_n}\mathsf{T}^e\left(\frac{r}{1+\zeta\bar{\zeta}}\right)^e(\bar{\zeta}z+1)^{2e}\,\mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{4.39}$$

[1] Note in particular that, with the parametrization (2.6), we have $\langle\kappa\,o\rangle=1$.

for positive quantized charge $2e \in \mathbb{Z}_{\geq 0}$, and

$$\varphi^{-}_{\alpha_1 \dots \alpha_n} = \varphi^{(-e,-e)}_{\alpha_1 \dots \alpha_n} = \kappa_{\alpha_1} \dots \kappa_{\alpha_n} \mathsf{T}^e \left(\frac{r}{1+\zeta\bar{\zeta}} \right)^{-e} (\zeta - z)^{-2e} \, \mathrm{e}^{\mathrm{i}k \cdot x} \,, \tag{4.40}$$

for negative quantized charge $2e \in \mathbb{Z}_{\leq 0}$. The $n = 0$ case corresponds to scalar solutions. We refer to these solutions as *minimal fields*, or *minimal states*. The other, non-singular solutions can be obtained as derivatives of the minimal states with respect to the external data, since using (2.6) we either have

$$\varphi^{(e,q)}_{\alpha_1 \dots \alpha_n} = \left(\frac{1 + z\tilde{z}}{2\omega} \, \partial_{\tilde{z}} \right)^{q-e} \varphi^{+}_{\alpha_1 \dots \alpha_n} \qquad \text{or} \qquad \varphi^{(e,q)}_{\alpha_1 \dots \alpha_n} = \left(\frac{1 + z\tilde{z}}{2\omega} \, \partial_{\tilde{z}} \right)^{q+e} \varphi^{-}_{\alpha_1 \dots \alpha_n} \,, \tag{4.41}$$

depending on the sign of the charge e. Observe that the gradient $D_{\alpha\dot{\alpha}}\varphi^{(e,q)}$ of the scalar quasi-momentum eigenstate defines background-dressed null 4-momenta $K^{(e,q)}_{\alpha\dot{\alpha}}(x) = \kappa_\alpha \tilde{K}^{(e,q)}_{\dot{\alpha}}(x)$ via $D_{\alpha\dot{\alpha}}\varphi^{(e,q)} = \mathrm{i}K^{(e,q)}_{\alpha\dot{\alpha}}\varphi^{(e,q)}$, where

$$\tilde{K}^{(e,q)}_{\dot{\alpha}}(x) = \tilde{\kappa}_{\dot{\alpha}} + (q+e)\frac{\chi_{+\gamma} T^{\gamma}{}_{\dot{\alpha}}}{r\langle \kappa \, \chi_+ \rangle} - (q-e)\frac{\chi_{-\gamma} T^{\gamma}{}_{\dot{\alpha}}}{r\langle \kappa \, \chi_- \rangle} \,, \tag{4.42}$$

Thus, $\varphi^{(e,q)}$ satisfies the charged Hamilton-Jacobi equation on the SDD background.

To describe positive-helicity states, we can use the scalar states and an appropriate helicity-raising operator: consider a positive-helicity state with linear gauge field $a_{\alpha\dot{\alpha}}$ and introduce the ansatz

$$a_{\alpha\dot{\alpha}} = \nu_\alpha \nu^\beta D_{\beta\dot{\alpha}} \varphi \,, \tag{4.43}$$

in terms of a scalar potential φ and a reference spinor ν_α. The linearised ASD field strength is

$$b_{\alpha\beta} = \frac{1}{2} \nu_\alpha \nu_\beta D^2 \varphi \,, \tag{4.44}$$

so we can take φ to be a scalar quasi-momentum eigenstate and treat the operator $\nu_\alpha \nu^\beta D_{\beta\dot{\alpha}}$ as a helicity-raising operator. In terms of the dressed momenta, we find the positive-helicity states

$$a^{(e,q)}_{\alpha\dot\alpha} = \frac{1}{\langle \nu\, \kappa\rangle} \nu_\alpha \tilde{K}^{(e,q)}_{\dot\alpha} \varphi^{(e,q)} \,, \tag{4.45}$$

where factors of $\langle \nu\, \kappa\rangle$ have been introduced for normalization. This construction is analogous to the K-matrix formalism of Chapter 3, suitably linearised around the SDD background.

Note that $x^{\alpha\beta} = \mathrm{i}\chi_+^{(\alpha}\chi_-^{\beta)}$ is a valence-2 Killing spinor. However, it has zero charge, so it can only be used to lower the helicity of a state, but it doesn't produce new charged states.

4.4.2 Scattering states from the Penrose transform

Let us now derive the spin-1 quasi-momentum eigenstates (4.37)-(4.45) from the Penrose transform. As it is often the case, constructing scattering states first on twistor space is usually easier and faster than solving directly the background-coupled equations of motion on space-time. Consider the negative-helicity representative

$$b^{(e,q)} = \mathsf{T}^e \int_{\mathbb{C}^*} \mathrm{d}s\, s^3\, \bar\delta^2(\kappa - s\,\lambda) \left(-\frac{\mathrm{i}s^2 \mu^{\dot\alpha}\, T^\alpha{}_{\dot\alpha}\, \lambda_\alpha}{\langle \kappa\, o\rangle^2} \right)^q \mathrm{e}^{\mathrm{i}s\,[\mu\,\tilde\kappa]} \,. \tag{4.46}$$

The representative is in the cohomology group $H^{(0,1)}_{\bar D}(\mathbb{PT}, \mathcal{O}(-4))$ of the background $\bar\partial$-operator $\bar D$ because, as in Chapter 3, both a and $b^{(e,q)}$ point along the $\bar e^0$ direction. Note also that the parameter q governs the number of insertions of the quadric (4.23) in the twistor representative. Applying the standard Penrose transform (2.62) to the representative (4.46) gives back the negative-helicity state (4.37) for $n = 2$, as in the gauge where the holomorphic frame is (4.17) and using (4.24) we have

$$\mathsf{H}^{-1}\, b^{(e,q)}\big|_X\, \mathsf{H} = \mathsf{T}^e \int_{\mathbb{C}^*} \mathrm{d}s\, s^3\, \bar\delta^2(\kappa - s\lambda) \left(\frac{\langle \lambda\, \chi_+\rangle}{\langle \lambda\, \chi_-\rangle} \right)^e \left(\frac{s^2 \langle \lambda\, \chi_+\rangle \langle \lambda\, \chi_-\rangle}{\langle \kappa\, o\rangle^2} \right)^q \mathrm{e}^{\mathrm{i}s\langle\lambda|x|\tilde\kappa]} \,. \tag{4.47}$$

Similarly, the twistor representative for a positive helicity quasi-momentum eigenstate is

$$a^{(e,q)} = \mathsf{T}^e \int_{\mathbb{C}^*} \frac{\mathrm{d}s}{s}\, \bar{\delta}^2(\kappa - s\lambda) \left(-\frac{\mathrm{i}s^2 \mu^{\dot\alpha}\, T^{\alpha}{}_{\dot\alpha}\, \lambda_\alpha}{\langle \kappa\, o\rangle^2}\right)^q \mathrm{e}^{\mathrm{i}s\,[\mu\,\tilde\kappa]}\,. \tag{4.48}$$

The restriction of such a representative to a twistor line X is $\bar\partial\big|_X$-exact, after a suitable dressing with the holomorphic frame, leading to the current

$$j^{(e,q)}(x,\lambda) = \mathsf{T}^e \frac{\langle \lambda\, \nu\rangle}{\langle \kappa\, \nu\rangle \langle \lambda\, \kappa\rangle} \frac{\langle \kappa\, \chi_+\rangle^{q+e} \langle \kappa\, \chi_-\rangle^{q-e}}{\langle \kappa\, o\rangle^{2q}} \mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{4.49}$$

satisfying (2.64). The associated space-time field from (2.65) is exactly (4.45).

4.5 Gluon MHV scattering

As explained in Chapter 3, the generating functional of the gluon MHV amplitude around a self-dual background $A_{\alpha\dot\alpha}$ is the 2-point function of the linearized field strength of the two negative-helicity gluons propagating around the deformed self-dual background defined by $A_{\alpha\dot\alpha}$ and the (non-linear) superposition of the positive-helicity gluons. Following the derivation of Chapter 3, the n-point colour-ordered MHV amplitude around the SDD background is given by

$$\begin{aligned}\mathcal{A}_n = \mathrm{g}^{n-2} \int_{\mathbb{M}\times X^n} & \mathrm{d}^4x\, \mathrm{D}\lambda_1 \cdots \mathrm{D}\lambda_n\, \langle \lambda_r\, \lambda_s\rangle^4 \times \\ & \times \frac{\mathrm{tr}\left(\mathsf{H}_1^{-1} a_1^{(1)} \mathsf{H}_1 \cdots \mathsf{H}_r^{-1} b_r^{(r)} \mathsf{H}_r \cdots \mathsf{H}_s^{-1} b_s^{(s)} \mathsf{H}_s \cdots \mathsf{H}_n^{-1} a_n^{(n)} \mathsf{H}_n\right)}{\langle \lambda_1\, \lambda_2\rangle \langle \lambda_2\, \lambda_3\rangle \cdots \langle \lambda_{n-1}\, \lambda_n\rangle \langle \lambda_n\, \lambda_1\rangle}\,,\end{aligned} \tag{4.50}$$

where g is the gauge coupling.

We now evaluate this formula explicitly on the SDD background – that is, for a given by (4.16) – with each external gluon represented by a quasi-momentum eigenstate. Denote the quantum numbers of the i^{th} gluon as $\{k_i^{\alpha\dot\alpha} = \kappa_i^\alpha \tilde\kappa_i^{\dot\alpha}, e_i, q_i\}$, recalling that e_i is the charge of the gluon with respect to the SDD background while q_i controls the radial growth of the quasi-momentum eigenstate wavefunction. The colour-ordered, tree-level, n-point MHV amplitude reads

$$\mathcal{A}_n = 2\pi\, \mathrm{g}^{n-2}\, \delta(\omega)\, \delta(e)\, \frac{\langle r\, s\rangle^4}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle} \times \\ \times \int \mathrm{d}^3\vec{x}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \prod_{a=1}^{n} \left(\frac{r}{1+\zeta\bar{\zeta}}\right)^{q_a} (\bar{\zeta} z_a + 1)^{e_a+q_a}\, (\zeta - z_a)^{q_a - e_a}\,, \quad (4.51)$$

where the total energy, momentum and charge are defined as

$$\omega = \sum_{i=1}^{n} \omega_i\,, \qquad \vec{k} = \sum_{i=1}^{n} \vec{k}_i\,, \qquad e = \sum_{i=1}^{n} e_i\,, \quad (4.52)$$

respectively. The δ function in the energy arises from the time integral and from the fact that quasi-momentum eigenstates depend on the time coordinate purely through the factor $\mathrm{e}^{-\mathrm{i}\omega_i x^0}$, as a consequence of the time-translation symmetry of the underlying background. The δ function in the charge arises from the trace over the gauge algebra indices, and ensures that the amplitude is invariant under gauge transformations of the SDD background. The presence of the background leads to the integral over the single spatial slice appearing in the second line, which replaces the three spatial momentum conserving delta functions familiar from scattering in trivial backgrounds. Finally, notice that (4.51) apparently violates little group scaling. The little group symmetry just describes the ambiguity in splitting the external null momenta into spinor-helicity variables, so it continues to survive in our background: the violation can be traced back to the fact that the external states now display a somewhat 'anomalous' transformation law. We can make the gluon amplitude have standard little group transformations with weights -2 for positive-helicity gluons and $+2$ for negative-helicity helicity gluons by introducing a reference spinor o as in equation (6.5) in Chapter 6. This is convenient for some purposes, but introduces something akin to a 'gauge choice'. A better alternative is to remove the factors of $\langle a\, o\rangle$ in the product in equation (6.5). In this case, the little group weights becomes $-2+2q_a$ for positive-helicity gluons and $2+2q_a$ for negative-helicity gluons. The $2q_a$ contributions may be thought of as phase anomalies, though we leave their precise physical interpretation to future work. In any case, the amplitude is written in terms of spinor-helicity variables, so it is gauge-invariant by construction.

While the formula (4.51) is already remarkably simple, for special configurations of the quantum numbers of the external gluons, it is possible to further simplify the formula by carrying out the spatial integrals explicitly. For clarity, some technical details of the computation are collected in Appendix C. We will restrict our attention to the scattering of *minimal* quasi-momentum eigenstates, recalling that the amplitude for more general states can be obtained by acting with simple differential operators in the external momenta.

4.5.1 2-point amplitude

First, let us consider the 2-point amplitude, with no positive helicity external gluons. This amplitude can be derived directly from the Chalmers-Siegel action without using twistor theory, as done in [2]. Without loss of generality, suppose gluon 2 has positive charge $e_2 > 0$, so that on the support of charge conservation

$$\mathcal{A}_2 = -2\pi\,\delta(\omega)\,\delta(e)\,\langle 1\,2\rangle^2 \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \left[\frac{r(\zeta - z_1)(\bar{\zeta} z_2 + 1)}{1+\zeta\bar{\zeta}}\right]^{2e_2} . \tag{4.53}$$

The integral can be done with the help of a master Gaussian integral and the integrated form of the amplitude reads

$$\mathcal{A}_2 = -16\pi^2\,e_2\,(2e_2)!\,\langle 1\,2\rangle^2\,\frac{z_{12}\,(-\mathrm{i}\omega_1\,z_{12})^{2e_2-1}}{|\vec{k}_1+\vec{k}_2|^{2+4e_2}}\,, \tag{4.54}$$

where $z_{12} = z_1 - z_2$. This is the unique, non-vanishing 2-point amplitude around the SDD background: the Chalmers-Siegel action is built perturbatively around the integrable self-dual sector described by a BF theory. In particular, the scattering amplitudes in the SD sector vanish identically. As an important consequence, the 2-point amplitudes are zero for the $(+\,+)$ and $(+\,-)$ helicity configurations. This is because such amplitudes do not receive contributions from non-self-dual interactions. On the other hand, since the deformation away from self-duality is quadratic in the negative-helicity field, there is scope for non-vanishing $(-\,-)$ 2-point amplitude, as confirmed by (4.54).

4.5.2 3-point amplitude

The 3-point amplitude in the case where $e_1 \leq 0 \leq e_2, e_3$ is

$$\begin{aligned}\mathcal{A}_3 = 2\pi\,\mathrm{g}\,\delta(\omega)\,\delta(e)\,&\frac{\langle r\,s\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \\ &\times \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2+1)}{1+\zeta\bar{\zeta}}\right]^{2e_2}\left[\frac{r(\zeta - z_1)(\bar{\zeta}z_3+1)}{1+\zeta\bar{\zeta}}\right]^{2e_3}\,,\end{aligned} \tag{4.55}$$

Using the integral in Appendix C, the integrated amplitude reads

$$\begin{aligned}\mathcal{A}_3 = 8\pi^2\mathrm{i}\,\mathrm{g}\,\delta(\omega)\,\delta(e)\,&\frac{\langle r\,s\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle}\,(-1)^{2e_1}\,(2|e_1|)! \\ &\times \frac{(\mathrm{i}\alpha_{21;3})^{2|e_2|}\,(\mathrm{i}\alpha_{31;3})^{2|e_3|}}{|\vec{k}_1+\vec{k}_2+\vec{k}_3|^{2+4|e_1|}}\left(\frac{2|e_2|\,z_{21}}{\alpha_{21}}+\frac{2|e_3|\,z_{31}}{\alpha_{31}}\right)\,,\end{aligned} \tag{4.56}$$

where

$$\begin{aligned}\alpha_{21} &= \omega_{12}z_{12} + \langle 1|T|3]\langle 3\,2\rangle + \langle 2|T|3]\langle 3\,1\rangle\,, \\ \alpha_{31} &= \omega_{13}z_{13} + \langle 1|T|2]\langle 2\,3\rangle + \langle 3|T|2]\langle 2\,1\rangle\,,\end{aligned} \tag{4.57}$$

for $\omega_{ij} = \omega_i - \omega_j$. A similar computation can be done for the 3-point amplitude in other charge configurations and for n-point amplitude in the case where there's a single negative-charge gluon, whilst the rest have positive charge [3].

Chapter 5

Graviton amplitudes on the self-dual Taub-NUT space-time

In the first part of this Chapter, we discuss the twistor description of the self-dual Taub-NUT (SDTN) metric and use the twistor description to construct scattering states. Interestingly, quasi-momentum eigenstates can also be obtained from uplifting the quasi-momentum eigenstates on the self-dual dyon, after identifying the charge e with $2M\omega$, where M is the mass parameter of the SDTN metric and ω is the energy of the state. These states can be used directly to compute 2-point amplitudes around the SDTN metric: we show that the scalar and photon amplitudes are exactly vanishing at tree level, whilst there is a non-zero 2-point amplitude for two negative-helicity gravitons. We then extend this 'MHV' 2-point amplitude to arbitrary multiplicity using the hyperkähler structure of the underlying SDTN metric. The MHV generating functional is the Fourier transform of a deformed Kähler potential describing the superposition of the SDTN metric and the set of positive-helicity gravitons propagating on top of the SDTN background. On twistor space, the deformed Kähler potential is controlled by a twistor sigma model, i.e. a chiral 2d CFT living on a twistor line. In particular, the n-point graviton MHV amplitude can be constructed as the $(n-2)$-point connected, tree-level correlation function of positive-helicity vertex operators in the twistor sigma model. The computation of this correlator can be operationalised using methods from algebraic combinatorics, leading to a remarkably compact formula for the MHV amplitude. We then explore the amplitudes at low-points and discuss the flat-space limit.

5.1 The self-dual Taub-NUT metric

The general Taub-NUT metric depends on two parameters: the ADM mass M and NUT parameter N [103, 104]. The Lorentzian-real Taub-NUT metric is

$$\mathrm{d}s^2 = f(r)(\mathrm{d}t - 2N\mathrm{a})^2 - \frac{\mathrm{d}r^2}{f(r)} - (r^2 + N^2)\mathrm{d}\Omega_2^2\,, \tag{5.1}$$

where a is the Dirac monopole already introduced in (4.7), $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric on the round 2-sphere parametrized by (θ, ϕ), and

$$f(r) = 1 - \frac{M + \mathrm{i}N}{r + \mathrm{i}N} - \frac{M - \mathrm{i}N}{r - \mathrm{i}N}\,. \tag{5.2}$$

Gauge transformations of the monopole $\mathrm{a} \to \mathrm{a} + \mathrm{d}g$, for some function g, are accompanied by coordinate transformations $t \to t + 2Ng$. Globally, this requires the identification $t \sim t + 8\pi\mathrm{i}N$. Setting $N = -\mathrm{i}M$ produces the complex metric

$$\mathrm{d}s^2 = \frac{r - M}{r + M}(\mathrm{d}t + 2\mathrm{i}M\mathrm{a})^2 - \frac{r + M}{r - M}\mathrm{d}r^2 - (r^2 - M^2)\mathrm{d}\Omega_2^2\,, \tag{5.3}$$

which is vacuum and self-dual (in the holomorphic, complexified category). The metric becomes real in either Euclidean or split signature.

Euclidean self-dual Taub-NUT space

When $N \to 0$, the metric (5.1) reduces to the Schwarzschild black hole, but for $N \neq 0$ the metric generically has wire singularities on the celestial sphere at infinity. However, when the mass and NUT parameter are related by $N = -\mathrm{i}\,M$, the metric becomes a complete, Ricci-flat and self-dual metric [102, 190, 191]. The self-dual Taub-NUT (SDTN) space $\mathscr{M}$ is the 4-manifold equipped with the metric

$$\mathrm{d}s^2 = \left(1+\frac{2M}{r}\right)^{-1}(\mathrm{d}t - 2M\mathrm{a})^2 + \left(1+\frac{2M}{r}\right)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2)\,, \tag{5.4}$$

which can be obtained from (5.3) with the replacement of t by $\mathrm{i}t$ and $r - M$ by r. We will refer to this metric as the self-dual Taub-NUT (SDTN) metric. The coordinate ranges are $r \in (0,\infty), (\theta,\phi) \in S^2$, and $t \sim t + 8\pi M$, thus $\mathscr{M}$ is topologically $\mathbb{R}^4$. To verify this, first combine the spherical coordinates $(\theta,\phi) \in S^2$ with the periodic coordinate $t \in S^1$ to find the total space of the Hopf fibration $S^3 \to S^2$ with fibre S^1. Together with $r \in \mathbb{R}_+$, these make up coordinates on $\mathbb{R}^4 - 0 = \mathbb{R}_+ \times S^3$. Since the 3-spheres at constant r approach spheres of radius $\sqrt{2Mr}$ as $r \to 0$, we can complete the spacetime into $\mathbb{R}^4$ by adding back the origin [192]. The locus $r = 0$ is a Euclidean analogue of the black hole horizon; however, in Euclidean signature this is not an actual horizon, but only a single point. The metric has an apparent singularity at $r = 0$, but can be extended across it by a standard change of coordinates. The Kretschmann scalar

$$R_{abcd}R^{abcd} = \frac{96M^2}{(r+2M)^6}\,, \tag{5.5}$$

reveals a true curvature singularity at $r = -2M$, but for $M > 0$, this point is outside of $\mathbb{R}^4$. Equivalently, it can be thought of as being hidden behind the 'horizon'. This intuition becomes precise in split signature, as we briefly mention below. The metric (5.4) is easily verified to be Ricci-flat. For the orientation in which the volume form is $\sqrt{|g|}\mathrm{d}t\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$, (5.4) is also self-dual. This can be verified in many equivalent ways, but perhaps the cleanest is to compute the Newman-Penrose scalars [193] of the metric, finding [194]

$$\widetilde{\Psi}_2 = -\frac{2\,M}{(r+2\,M)^3}\,, \qquad \Psi_2 = 0\,, \tag{5.6}$$

with $\widetilde{\Psi}_2$ the only non-vanishing component of the Riemann curvature tensor. This immediately shows that the metric is vacuum, type D (as only $\widetilde{\Psi}_2 \neq 0$) and self-dual (as $\Psi_2 = 0$).

Note that the metric (5.4) is manifestly in Gibbons-Hawking form. Recall that Gibbons-Hawking metrics [102, 190, 191] are defined by a pair (V, A), where V is a scalar function and A is a 1-form, related by the abelian monopole equation

$$\mathrm{d}V = \star_3\,\mathrm{d}A\,, \tag{5.7}$$

for $\star_3$ the Hodge star on $\mathbb{R}^3$. For the SDTN metric, this pair is given by

$$V = 1 + \frac{2M}{r}\,, \qquad A = 2M\,\mathrm{a}\,, \tag{5.8}$$

The metric (5.4) is then equal to

$$\mathrm{d}s^2 = \frac{(\mathrm{d}t - A)^2}{V} + V\,\mathrm{d}\vec{x}^2\,, \tag{5.9}$$

which is precisely of Gibbons-Hawking form.

The standard tetrad for the metric is

$$\theta^0 = \left(1 + \frac{2M}{r}\right)^{-1/2} (\mathrm{d}t - 2M\mathrm{a})\,, \qquad \theta^i = \left(1 + \frac{2M}{r}\right)^{1/2} \mathrm{d}x^i\,, \tag{5.10}$$

for $i = 1,2,3$. In the 2-spinor variables, the tetrad is

$$\theta^{\alpha\dot{\alpha}} = \left(1 + \frac{2M}{r}\right)^{-1/2} T^{\alpha\dot{\alpha}}\,(\mathrm{d}t - 2M\mathrm{a}) - 2\left(1 + \frac{2M}{r}\right)^{1/2} \mathrm{d}x^{\alpha\beta}\, T_{\beta}{}^{\dot{\alpha}}\,. \tag{5.11}$$

From this tetrad, a basis of ASD 2-forms on $\mathscr{M}$ is given by

$$\Sigma^{\alpha\beta} = 2\,(\mathrm{d}t - 2M\mathrm{a}) \wedge \mathrm{d}x^{\alpha\beta} + 2\left(1 + \frac{2M}{r}\right) \mathrm{d}x^{\alpha\gamma} \wedge \mathrm{d}x^{\beta}{}_{\gamma}\,, \tag{5.12}$$

which will prove useful in later computations.

Kleinian self-dual Taub-NUT space

As the 'true' singularity at $r = -2M$ is not on the Euclidean manifold, one might worry that the SDTN metric is a poor model for a black hole. However, in split signature, the SDTN metric has an event horizon, where the signs of the metric encoding the Kleinian 'causal structure' are interchanged, behind which the metric can be continued up to a true singularity [2, 105, 195].

The Wick rotation that takes (5.4) from Euclidean to split signature is given by $(x^1, x^2) \mapsto (\mathrm{i}x^1, -\mathrm{i}x^2)$, or equivalently $\theta \mapsto \mathrm{i}\theta, \phi \mapsto -\phi$. The resulting metric reads[1]

$$\mathrm{d}s^2 = \left(1 + \frac{2M}{r}\right)^{-1} \left[\mathrm{d}t + 2M\,(1 - \cosh\theta)\,\mathrm{d}\phi\right]^2 + \left(1 + \frac{2M}{r}\right) \mathrm{d}\vec{x}^2\,, \tag{5.13}$$

where now $\mathrm{d}\vec{x}^2 = (\mathrm{d}x^3)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2$. In this signature, the coordinates range over $r, \theta \geq 0$, $\phi \sim \phi + 2\pi$, $t \sim t + 8\pi M$. Once again, the spacetime has a coordinate singularity at $r = 0$. This happens to be a genuine horizon instead of a single point as in Euclidean signature, and the metric can be extended beyond this horizon into the interior of this black hole by continuing r to the negative reals. The maximal extension encounters a curvature singularity at $r = -2M$, where the spacetime ends.

[1] The metric employed in [195] is found by a further change of coordinates $t \mapsto t - 2M$, $r \mapsto r - M$.

This justifies the use of the terminology *self-dual black hole* and viewing the SDTN metric as an integrable toy model of a black hole.

Another important distinction between the geometries is that in Euclidean signature, surfaces of constant t, r are spatial 2-spheres parametrized by θ, ϕ. In split signature, a surface of constant t and constant $r^2 = (x^3)^2 - (x^1)^2 - (x^2)^2$ is a two-dimensional hyperbolic space $\mathbb{H}_2$, also known as the Poincaré disk. So, for $r > 0$, the metric can be thought of as a metric on a circle bundle over $\mathbb{R}_+ \times \mathbb{H}_2$.

5.1.1 Hyperkähler structure of self-dual Taub-NUT metric

The hyperkähler structure of the metric (5.4) can be manifested very cleanly following LeBrun's construction [192]. Let (y, z) be complex coordinates on $\mathbb{C}^2$ and consider the Kähler potential

$$\Omega = 8M(u^2 + v^2) + 2(u^4 + v^4)\,, \tag{5.14}$$

where u, v are two real-valued, positive functions defined implicitly by

$$|y| = \mathrm{e}^{(u^2 - v^2)/4M} u\,, \qquad |z| = uv\,. \tag{5.15}$$

The associated Kähler metric (obtained by implicit differentiation of the Kähler potential) is

$$\mathrm{d}s^2 = 4\left(1 + \frac{2M}{u^2+v^2}\right)|\mathrm{d}z|^2 + 16M^2\left(1 + \frac{2M}{u^2+v^2}\right)^{-1}\left|\frac{\mathrm{d}y}{y} - \frac{v^2}{u^2+v^2}\frac{\mathrm{d}z}{z}\right|^2\,. \tag{5.16}$$

Introducing Cartesian coordinates $\vec{x}$ on $\mathbb{R}^3$ and performing the diffeomorphism

$$y = \sqrt{\frac{r + x^3}{2}}\mathrm{e}^{-\mathrm{i}(t + \mathrm{i}x^3)/4M}\,, \qquad z = \frac{1}{2}(x^1 + \mathrm{i}x^2)\,, \tag{5.17}$$

one finds that

$$u = \sqrt{\frac{r + x^3}{2}}\,, \qquad v = \sqrt{\frac{r - x^3}{2}}\,. \tag{5.18}$$

In these new coordinates, the Kähler metric (5.16) becomes

$$\mathrm{d}s^2 = \left(1 + \frac{2M}{r}\right)^{-1} \left(\mathrm{d}t - \frac{2M}{r}\frac{x^1\mathrm{d}x^2 - x^2\mathrm{d}x^1}{r + x^3}\right)^2 + \left(1 + \frac{2M}{r}\right)\left((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\right), \tag{5.19}$$

which is precisely the SDTN metric (5.4) upon passing to spherical coordinates.

Remarkably, the Kähler potential (5.14) itself is extremely simple when written in the Gibbons-Hawking coordinates:

$$\Omega = r^2 + (x^3)^2 + 8Mr\,. \tag{5.20}$$

However, observe that these coordinates are *not* holomorphic/anti-holomorphic with respect the the associated complex structure. So although the potential (5.20) is simple, taking the appropriate derivatives to arrive at the metric is not.

At this stage, one may note that by working with a single Kähler potential, we have chosen a single complex structure, rather than the 2-sphere's worth associated with a hyperkähler structure. The existence of this S^2 of complex structures is guaranteed by checking that Ω satisfies Plebański's 'first heavenly equation' [158], and in fact the choice of complex structure is generic. Indeed, orientation- and metric-compatible complex structures on SDTN are in one-to-one correspondence with points on the unit sphere, parametrized by $\vec{n} \in \mathbb{R}^3$, $\vec{n}^{\,2} = 1$ [192]. To see this, let z be a complex coordinate on the plane $\vec{n} \cdot \vec{x} = 0$, and $y = u\,\mathrm{e}^{-\mathrm{i}(t + \mathrm{i}\vec{n}\cdot\vec{x})/4M}$. Any such coordinates are related to the coordinates (y, z) defined in (5.17) by SO(3) rotations of $\vec{x}$, which rotate the Kähler potential (5.20) into

$$\Omega = r^2 + (\vec{n} \cdot \vec{x})^2 + 8Mr\,, \qquad \vec{n} \in S^2\,. \tag{5.21}$$

As the SDTN metric is invariant under SO(3) rotations of $\vec{x}$, it follows that the SDTN metric will be recovered from any member of this family of Kähler potentials.

5.2 The Taub-NUT twistor space

The SDTN metric (5.4) is hyperkähler, so it admits a description on twistor space via the non-linear graviton. As reviewed in Chapter 2, if the deformed twistor space $\mathbb{P}\mathscr{T}$ is modelled on the flat twistor space $\mathbb{PT}$, the holomorphic data of the non-linear graviton is represented by an integrable Hamiltonian $\mathsf{h} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$. In the past, the twistor description of the SDTN was mainly discussed in Čech cohomology. For our scope, it will prove more useful to develop a Dolbeault description of the twistor space, which we now describe. In Appendix A, we will also give some details of its equivalence with the original presentation in Čech cohomology.

Introduce the two-set open cover of $\mathbb{PT} = U_0 \cup U_1$ given by

$$U_0 = \{Z^A \in \mathbb{PT} \,|\, \lambda_0 \neq 0\}\,, \qquad U_1 = \{Z^A \in \mathbb{PT} \,|\, \lambda_1 \neq 0\}\,, \tag{5.22}$$

induced by the natural open covering of $\mathbb{CP}^1$ and introduce the notation

$$\mu^- = [\mu\,\tilde{\iota}]\,, \qquad \mu^+ = [\mu\,\tilde{o}]\,, \qquad \eta = \mu^+\mu^-\,, \tag{5.23}$$

where $\mu^{\pm}$ are the decomposition of $\mu^{\dot\alpha}$ with respect to the spinor dyad $\{\tilde{o}_{\dot\alpha},\, \tilde{\iota}_{\dot\alpha}\}$. Now define the $\mathcal{O}(2)$-valued Hamiltonian

$$\mathsf{h} = \frac{\eta^2\,\bar{e}^0}{4M}\,. \tag{5.24}$$

M is a parameter that will be identified with the 'mass' of the SDTN metric.

The complex deformation associated with this Hamiltonian is

$$V = \{\mathsf{h}, \cdot\} = \frac{\eta\, \bar{e}^0}{2M} \wedge \left(\mu^+ \mathcal{L}_+ - \mu^- \mathcal{L}_-\right) , \tag{5.25}$$

where $\mathcal{L}_\pm$ denotes the Lie derivative along $\partial/\partial\mu^\pm$. The obstruction to integrability (2.39) of the complex structure $\bar{\nabla}$ in $\mathbb{CP}^3$ is then

$$\bar{\partial}\mathsf{h} + \frac{1}{2}\{\mathsf{h}, \mathsf{h}\} = \frac{\pi^2\, \eta^2}{M}\, \bar{\delta}^2(\lambda)\,. \tag{5.26}$$

Since we're modelling the complex deformation on the flat-space fibration $\mathbb{PT} \to \mathbb{CP}^1$, $\lambda_\alpha \neq 0$ on $\mathbb{P}\mathscr{T}$. Therefore, $\bar{\nabla}^2 = 0$ as required[2].

In the complex structure (5.25), λ_α are still holomorphic coordinates (as the fibration over $\mathbb{CP}^1$ is undeformed by the complex deformation (5.25)), but the $\mu^{\dot\alpha}$ are no longer holomorphic. Nevertheless, one can construct local holomorphic coordinates on U_0 and U_1 straightforwardly: first, observe that

$$\bar{\nabla}\mu^\pm = \pm\mu^\pm\, \frac{\eta\, \bar{e}^0}{2M}\,, \tag{5.27}$$

so that although $\bar{\nabla}\mu^\pm \neq 0$, the combination $\eta = \mu^+\,\mu^-$ *is* still holomorphic: $\bar{\nabla}\eta = 0$. Then holomorphic coordinates on the fibres of $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$ are given by

$$\rho^\pm = \mu^\pm\, \exp\!\left(\pm\frac{\eta\, f(\lambda)}{2M}\right) , \tag{5.28}$$

[2]It is interesting to contrast this with the analogous twistor space for the Eguchi-Hanson metric [122], where the complex structure is induced by a source on the $\mu^{\dot\alpha} = 0$ locus, which *is* in the twistor space.

where f is defined in the two patches by

$$f(\lambda) = -\frac{1}{\langle\lambda\,\hat{\lambda}\rangle}\begin{cases}\hat{\lambda}_0/\lambda_0\,, & \lambda_\alpha \in U_0 \\ \hat{\lambda}_1/\lambda_1\,, & \lambda_\alpha \in U_1\end{cases}. \tag{5.29}$$

In particular, it can be checked that $\bar{\partial} f = -\bar{e}^0$ globally, i.e. on both U_0 and U_1, from which the holomorphicity of $\rho^\pm$ follows.

The holomorphic coordinates $\rho^\pm$ can be understood as valued in $\mathcal{O}(1)\otimes L^{\pm1}$, where L is a line bundle over the total space of $\mathcal{O}(2) \to \mathbb{CP}^1$ with transition function

$$\phi_{10} = \exp\left(\frac{\eta}{2M\,\lambda_0\,\lambda_1}\right)\,, \tag{5.30}$$

over $U_0 \cap U_1$. Then $\mathbb{P}\mathscr{T}$ is the sub-bundle of $\mathcal{O}(1) \otimes (L \oplus L^{-1})$ defined by

$$\rho^+\,\rho^- = \eta\,. \tag{5.31}$$

5.2.1 Reconstructing the space-time metric

The holomorphic symplectic form on the fibres of $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$ is now

$$\Sigma = 2\,\mathrm{d}\rho^- \wedge \mathrm{d}\rho^+\,. \tag{5.32}$$

It can be checked that Σ is a global section of the exterior square of the conormal bundle: that is, it changes from U_0 to U_1 only by terms proportional to $\mathrm{d}\lambda_\alpha$. Furthermore, Σ coincides with the symplectic form on $\mathbb{PT}$ (2.38), again up to terms proportional to $\mathrm{d}\lambda_\alpha$. Thus, one can continue to work with the homogeneous coordinates $(\mu^{\dot{\alpha}}, \lambda_\alpha)$ at the level of the symplectic form.

To reconstruct the space-time metric, one must first solve for the holomorphic rational curves in twistor space. Recall that these curves are given by $\mathcal{O}(1)$-valued maps satisfying (2.40). The Killing spinors (4.5) can be used to construct an explicit solution for the twistor lines as

$$\mathsf{F}^{\dot\alpha}(x,\lambda) = \tilde\iota^{\dot\alpha}\langle\chi_+\,\lambda\rangle\,\mathrm{e}^{-\mathsf{g}(x,\lambda)} - \tilde o^{\dot\alpha}\langle\chi_-\,\lambda\rangle\,\mathrm{e}^{\mathsf{g}(x,\lambda)}\,, \tag{5.33}$$

where

$$\mathsf{g}(x,\lambda) = \frac{\mathrm{i}}{4M}\left(t + 2\frac{x^{\alpha\beta}\lambda_\alpha\hat\lambda_\beta}{\langle\lambda\,\hat\lambda\rangle}\right)\,. \tag{5.34}$$

To see this, decompose the twistor line as $\mathsf{F} = \mathsf{F}^+\tilde\iota^{\dot\alpha} - \mathsf{F}^-\tilde o^{\dot\alpha}$, so that Equation (2.40) becomes

$$\bar\partial\big|_X\,\mathsf{F}^\pm = \pm\mathsf{F}^\pm\frac{\mathsf{F}^+\mathsf{F}^-}{2M}\bar e^0\,, \tag{5.35}$$

and make the ansatz $\mathsf{F}^\pm = \langle\chi_\pm\,\lambda\rangle\,\mathrm{e}^{\mp\mathsf{g}}$. Next, note that $\bar\partial\big|_X\,\hat\lambda_\alpha = \langle\lambda\,\hat\lambda\rangle\lambda_\alpha\bar e^0$, so g satisfies

$$\bar\partial\big|_X\,\mathsf{g}(x,\lambda) = -\frac{1}{2M}\langle\chi_+\,\lambda\rangle\langle\chi_-\,\lambda\rangle\,, \tag{5.36}$$

where we used Equation (4.6). It follows that

$$\mathsf{F}^\pm = \langle\chi_\pm\,\lambda\rangle\mathrm{e}^{\mp\mathsf{g}(x,\lambda)}\,, \tag{5.37}$$

is a solution to (5.35).

The solution (5.33) for the twistor lines is not unique, as we are free to add to g *any* function that depends on the space-time coordinates and is independent of $(\lambda_\alpha,\hat\lambda_\alpha)$. However, with this choice of g, it is straightforward to check that the curves (5.33) are invariant under the anti-holomorphic involution $j\colon(\mu^{\dot\alpha},\lambda_\alpha)\mapsto(\hat\mu^{\dot\alpha},\hat\lambda_\alpha)$ when $x^{\alpha\beta} = \hat x^{\alpha\beta}$ and $t\in\mathbb{R}$, as required for Euclidean reality conditions. That is, a point $(\mu^{\dot\alpha},\lambda_\alpha)$ lies on a twistor line if and only if the conjugate point $(\hat\mu^{\dot\alpha},\hat\lambda_\alpha)$ lies on the same line as well. This happens because $\hat{\tilde\iota}^{\dot\alpha} = -\tilde o^{\dot\alpha}$ and $\hat\chi_+^\alpha = -\chi_-^\alpha$.

Let $\mathscr{M}$ be the moduli space of the twistor lines. For $x^{\alpha\beta}\neq 0$, $(x^{\alpha\beta},t)$ provide coordinates on a circle bundle over $\mathbb{R}^3 - 0$, the fibre coordinate being $t \in S^1$ of radius $8\pi M$. When $x^{\alpha\beta} = 0$, there is a unique twistor curve $\mathsf{F}^{\dot\alpha}(t, x^{\alpha\beta} = 0, \lambda) = 0$, which just corresponds to adding a single point as the fibre over the origin of $\mathbb{R}^3$. This means that the 4-manifold topology is *still* $\mathbb{R}^4$, as required. To reconstruct the SDTN metric, we pull back the symplectic 2-form Σ by the projection $p\colon \mathscr{M} \times \mathbb{CP}^1 \to \mathbb{P}\mathscr{T}$ from the projectivised undotted spin bundle of $\mathscr{M}$ to twistor space [159].

Lemma 5.2.1. *The pullback of the symplectic 2-form satisfies*

$$p^*\Sigma = \frac{\lambda_\alpha\lambda_\beta}{4M}\Sigma^{\alpha\beta} \mod \mathrm{d}\lambda_\alpha\,, \tag{5.38}$$

where $\Sigma^{\alpha\beta}$ are the ASD 2-forms of the SDTN metric given in (5.12)*.*

Proof. Up to terms proportional to differential forms pointing along the $\mathbb{CP}^1$ directions of $\mathscr{M} \times \mathbb{CP}^1$, we can write

$$p^*\Sigma = 2\,\mathrm{d}_x\mathsf{F}^- \wedge \mathrm{d}_x\mathsf{F}^+ \mod \mathrm{d}\lambda_\alpha\,, \tag{5.39}$$

d_x being the exterior derivative along $\mathscr{M}$. Using the twistor lines (5.33), we find

$$p^*\Sigma = -2\lambda_\alpha\lambda_\beta\left[\mathrm{d}\chi_+^\alpha \wedge \mathrm{d}\chi_-^\beta + \frac{\mathrm{d}x^{\alpha\beta}}{4M} \wedge \left(\mathrm{d}t + 2\frac{\mathrm{d}x^{\gamma\delta}\lambda_\gamma\hat\lambda_\delta}{\langle\lambda\,\hat\lambda\rangle}\right)\right]. \tag{5.40}$$

Now, using the identity

$$\mathrm{d}x^{\alpha\beta} \wedge \mathrm{d}x^{\gamma\delta} = -\frac{1}{2}\epsilon^{\alpha\gamma}\mathrm{d}x^{\beta\eta} \wedge \mathrm{d}x^\delta{}_\eta - \frac{1}{2}\epsilon^{\beta\delta}\mathrm{d}x^{\alpha\eta} \wedge \mathrm{d}x^\gamma{}_\eta\,, \tag{5.41}$$

we can write the pullback in the form (5.38), with

$$\Sigma^{\alpha\beta} = -8M\mathrm{d}\chi_+^{(\alpha} \wedge \mathrm{d}\chi_-^{\beta)} + 2\,\mathrm{d}t \wedge \mathrm{d}x^{\alpha\beta} + 2\,\mathrm{d}x^{\alpha\gamma} \wedge \mathrm{d}x^\beta{}_\gamma\,. \tag{5.42}$$

Finally, the exterior derivatives (4.14)-(4.15) of the Killing spinors imply the identity

$$\mathrm{d}\chi_+^{(\alpha} \wedge \mathrm{d}\chi_-^{\beta)} = \frac{1}{2}\mathrm{d}x^{\alpha\beta} \wedge \mathrm{a} - \frac{1}{2r}\mathrm{d}x^{\alpha\gamma} \wedge \mathrm{d}x^{\beta}{}_{\gamma}\,, \tag{5.43}$$

upon which the 2-forms in (5.42) reduce immediately to the desired ASD 2-forms (5.12). $\square$

This lemma also implies that the exterior derivative $\mathrm{d}_x\mathsf{F}^{\dot\alpha}$ is proportional to $\theta^{\alpha\dot\alpha}\,\lambda_\alpha$ for a tetrad $\theta^{\alpha\dot\alpha}$ on $\mathscr{M}$, up to a frame rotation:

$$\mathrm{d}_x\mathsf{F}^{\dot\alpha} = \mathsf{H}^{\dot\alpha}{}_{\dot\beta}(x,\lambda)\,\theta^{\alpha\dot\beta}\,\lambda_\alpha\,, \tag{5.44}$$

where $\mathsf{H}^{\dot\alpha}{}_{\dot\beta}(x,\lambda)$ is valued in $\mathrm{SL}(2,\mathbb{C})$. The matrix $H^{\dot\alpha}{}_{\dot\beta}(x,\lambda)$ is the gravitational analogue of the holomorphic frame for the Ward's bundle, as it acts as a holomorphic frame for the bundle $\mathcal{N}\otimes\mathcal{O}(-1)$ over the twistor line X, where $\mathcal{N}\cong\mathcal{O}(1)\oplus\mathcal{O}(1)$ is the normal bundle of the curve X in $\mathbb{P}\mathscr{T}$.

For the SDTN tetrad (5.11) in our chosen coordinate system, one finds

$$\begin{aligned}\mathsf{H}^{\dot\alpha}{}_{\dot\beta}(x,\lambda) = &\frac{\mathrm{i}T_{\beta\dot\beta}\tilde\iota^{\dot\alpha}}{2M\sqrt{V}}\left(\frac{\langle\lambda\,\chi_+\rangle}{\langle\lambda\,\hat\lambda\rangle}\hat\lambda^{\beta} - \frac{2M}{r}\chi_+^{\beta}\right)\mathrm{e}^{-\mathsf{g}(x,\lambda)}\\ &+\frac{\mathrm{i}T_{\beta\dot\beta}\tilde o^{\dot\alpha}}{2M\sqrt{V}}\left(\frac{\langle\lambda\,\chi_-\rangle}{\langle\lambda\,\hat\lambda\rangle} - \frac{2M}{r}\chi_-^{\beta}\right)\mathrm{e}^{\mathsf{g}(x,\lambda)}\,.\end{aligned} \tag{5.45}$$

It is a straightforward (albeit somewhat tedious) calculation to verify that this frame obeys (5.44) as well as the unimodularity condition $\mathsf{H}_{\dot\gamma\dot\alpha}(x,\lambda)\,\mathsf{H}^{\dot\gamma}{}_{\dot\beta}(x,\lambda) = \epsilon_{\dot\alpha\dot\beta}$.

It is worth noting that in addition to the special case of the SDTN metric, this basic twistor construction can be easily generalised to describe *any* Gibbons-Hawking gravitational instanton [190, 191]. See Appendix B for the details.

5.3 Scattering states

As in the case for the self-dual dyon, we first discuss how to solve the linearised equations of motion directly on space-time, using the scattering states constructed on the self-dual dyon in Chapter 4. This relation is perhaps not surprising, since the two backgrounds are related by the classical double copy [106]. We then construct suitable twistor representatives and show that their Penrose transform reduces to the desired quasi-momentum eigenstates on space-time.

5.3.1 Uplifting states from the self-dual dyon

As in Chapter 4, a valence-p (ASD) Killing spinor on a general curved spacetime is a solution $\chi^{\alpha_1\dots\alpha_p}$ to the equation

$$\nabla^{(\alpha}{}_{\dot\alpha}\chi^{\beta_1\dots\beta_p)} = 0\,, \tag{5.46}$$

where now $\nabla^{\alpha\dot\alpha}$ denotes the covariant derivative. Let $e_a = (e_0, e_i)$ be the vector fields dual to the tetrad (5.10), which can be written as

$$e_a = V^{-1/2}\delta^\mu_a(\partial_\mu + 2MA^{\mathrm{SDD}}_\mu\partial_t)\,, \tag{5.47}$$

where A^{SDD}_μ are the spacetime components of the Abelian self-dual dyon (4.7) in the Cartesian coordinates $x^\mu = (t, \vec{x})$. In this co-frame, the background ASD spin connection can be checked to vanish exactly. The form of the co-tetrad (5.47) directly implies

Proposition 5.3.1. *Given $n \in \mathbb{Z}_{\geq 0}$, let the totally symmetric $\psi_{\alpha_1\dots\alpha_n}$ be a solution of the ZRM equation propagating on the self-dual Taub-NUT metric* (5.4) *and whose* *dependence on the time coordinates is factorised as*

$$\psi_{\alpha_1\dots\alpha_n}(t,\vec{x}) = \mathrm{e}^{\mathrm{i}\omega t}\phi_{\alpha_1\dots\alpha_n}(\vec{x})\,. \tag{5.48}$$

Then $\psi_{\alpha_1\dots\alpha_n}$ *is also a solution to the ZRM equations around the Abelian self-dual dyon* (4.7) *with charge* $e = 2M\omega$*.*

Proof. For $n > 0$, the proof is trivial as the action of the covariant derivative is[3]

$$\sqrt{V}\,\nabla_a\psi_{\alpha_1\dots\alpha_n} = \delta_a^\mu(\partial_\mu + 2\mathrm{i}\omega M A_\mu^{\mathrm{SDD}})\psi_{\alpha_1\dots\alpha_n}\,, \tag{5.49}$$

directly implying the ZRM equation

$$(\partial^{\alpha_1\dot{\alpha}} + 2\mathrm{i}M\omega A^{\mathrm{SDD}\,\alpha_1\dot{\alpha}})\psi_{\alpha_1\dots\alpha_n} = 0\,. \tag{5.50}$$

For $n = 0$, start from the expression of the Laplacian

$$\Box\psi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\psi)\,, \tag{5.51}$$

where $|g| = V^2$ is the metric determinant. Introducing the flat-space metric η_{ab} and using $g^{\mu\nu} = \eta^{ab}e_a^\mu e_b^\nu$, we can decompose the Laplacian as

$$\Box\psi = \frac{1}{V}\eta^{ab}\sqrt{V}e_a(\sqrt{V}e_b\psi) + \frac{1}{V}\eta^{ab}\partial_\mu(\sqrt{V}e_a^\mu)\,\sqrt{V}e_b\psi\,. \tag{5.52}$$

The components of the tetrad (5.47) are

$$e_a^\mu = \frac{1}{\sqrt{V}}(\delta_a^\mu + 2MA_\nu^{\mathrm{SDD}}\delta_a^\nu\delta_t^\mu)\,, \tag{5.53}$$

[3] Note that we're using self-duality of the background metric, in particular the vanishing of the ASD spin connection, to compute the covariant derivative.

so that

$$\partial_\mu(\sqrt{V}e_a^\mu) = 0\,, \tag{5.54}$$

as the components of the self-dual dyon A^{SDD} are time-independent. Inserting the time dependence of ψ, we conclude

$$\Box\psi = \frac{1}{V}\delta^{\mu\nu}(\partial_\mu + 2\mathrm{i}M\omega A_\mu^{\mathrm{SDD}})(\partial_\nu + 2\mathrm{i}M\omega A_\nu^{\mathrm{SDD}})\psi\,, \tag{5.55}$$

and hence ψ is a solution to the scalar ZRM equation around the self-dual dyon with charge $2M\omega$.

□

We can therefore lift the solutions (4.37) from Chapter 4 via the simple replacement $e \to 2M\omega$, leading to the negative-helicity and scalar quasi-momentum eigenstates

$$\varphi^{(q)}_{\alpha_1\ldots\alpha_n} = \kappa_{\alpha_1}\ldots\kappa_{\alpha_n}\frac{\langle\kappa\,\chi_+\rangle^{q+2M\omega}\langle\kappa\,\chi_-\rangle^{q-2M\omega}}{\langle\kappa\,o\rangle^{2q}}\mathrm{e}^{\mathrm{i}k\cdot x}\,, \tag{5.56}$$

with minimal states now corresponding to the choice $q = 2M|\omega|$ for q. The quantization condition $4M\omega \in \mathbb{Z}$ follows from either from charge quantization $2e \in \mathbb{Z}$ around the self-dual dyon, or from the periodicity of the time coordinate in the SDTN metric; more precisely, $2M\omega$ and q must be simultaneously integers or half-integers, as it was the case for the self-dual dyon. The background-dressed dotted momentum can be defined in the gravitational setting too; in this case, the definition is given by

$$\mathrm{d}\varphi^{(q)} = \mathrm{i}\kappa_\alpha\tilde{K}^{(q)}_{\dot\alpha}\varphi^{(q)}\theta^{\alpha\dot\alpha}\,, \tag{5.57}$$

and the expression for the dotted momentum is very similar to the one found around the self-dual dyon

$$\tilde{K}^{(q)}_{\dot\alpha} = \frac{1}{\sqrt{V}}\left(\tilde{\kappa}_{\dot\alpha} + (q+2M\omega)\frac{\chi_{+\gamma}T^\gamma{}_{\dot\alpha}}{r\langle\kappa\,\chi_+\rangle} - (q-2M\omega)\frac{\chi_{-\gamma}T^\gamma{}_{\dot\alpha}}{r\langle\kappa\,\chi_-\rangle}\right)\,. \tag{5.58}$$

In the following, it will be convenient to express the dressed momentum in terms of a dressing matrix acting on the bare momentum

$$\tilde{K}^{(q)}_{\dot\alpha} = \tilde\kappa_{\dot\beta} G^{\dot\beta}{}_{\dot\alpha}(x,\kappa,q)\,, \tag{5.59}$$

where

$$G^{\dot\alpha}{}_{\dot\beta}(x,\kappa,q) = \frac{1}{\sqrt{V}}\left(\delta^{\dot\alpha}{}_{\dot\beta} + (q+2M\omega)\frac{\kappa_\alpha \chi_{+\,\beta} T^{\alpha\dot\alpha} T^{\beta}{}_{\dot\beta}}{\omega r\langle\kappa\,\chi_+\rangle} - (q-2M\omega)\frac{\kappa_\alpha \chi_{-\,\beta} T^{\alpha\dot\alpha} T^{\beta}{}_{\dot\beta}}{\omega r\langle\kappa\,\chi_-\rangle}\right). \tag{5.60}$$

It is straightforward to show that the dressing matrix is unimodular and satisfies

$$G^{\dot\alpha}{}_{\dot\beta}(x,\kappa,q)\epsilon_{\dot\alpha\dot\gamma}G^{\dot\gamma}{}_{\dot\delta}(x,\kappa,q) = \epsilon_{\dot\beta\dot\delta}\,, \qquad \mathrm{d}(\kappa_\alpha G^{\dot\alpha}{}_{\dot\beta}(x,\kappa,q)\theta^{\alpha\dot\beta}) = 0\,. \tag{5.61}$$

Note that in proving Proposition 5.3.1, we did not use the explicit form of the Bogomolny pair (V,A) defining the SDTN metric, so this construction can be applied to any Gibbons-Hawking metric to obtain background-coupled massless fields from charged fields with respect to the static SD Maxwell field $\tilde{A}_\mu = (V-1, A)$.

Positive-helicity states

Given our choice of co-frame (5.47), the ASD spin connection vanishes identically, so any constant spinor ν_α on $\mathbb{R}^4$ is also covariantly constant. Hence, the operator $\nu_\alpha\nu^\beta\nabla_{\beta\dot\alpha}$ can still be used as a helicity-raising operator [196], essentially in the same way as in Chapter 4. Positive-helicity photons propagating on the SDTN metric are thus described by

$$a^{(q)}_{\alpha\dot\alpha} = \nu_\alpha\nu^\beta\nabla_{\beta\dot\alpha}\varphi^{(q)}\,, \tag{5.62}$$

and similarly positive-helicity gravitons are given by the metric perturbations

$$h^{(q)}_{\alpha\dot\alpha\beta\dot\beta} = \nu_\alpha\nu_\beta\nu^\gamma\nu^\delta\nabla_{\gamma\dot\alpha}\nabla_{\delta\dot\beta}\varphi^{(q)}\,. \tag{5.63}$$

On the support of the Klein-Gordon equation for $\varphi^{(q)}$, the linearised ASD field strength of (5.62) and linearised ASD Weyl tensor of (5.63) vanish. For the positive-helicity graviton, this is a linearization of the formulation of hyperkähler metrics in terms of Plebański's second heavenly potential [158].

5.3.2 Scattering states from the Penrose transform

The Penrose transform on the SDTN background presents some additional subtleties. We begin by reviewing the transform for scalar states to elucidate these points. The twistor representative for the quasi-momentum eigenstate (5.56) for $n = 0$ is

$$\Phi^{(q)}(Z) = \int_{\mathbb{C}^*} \mathrm{d}s\, s\, \bar{\delta}^2(s\lambda - \kappa)\, \left(s\, \mu^+\right)^{q+2M\omega} \left(s\, \mu^-\right)^{q-2M\omega}\, \mathrm{e}^{-\xi s^2 \eta}\,, \tag{5.64}$$

where

$$\xi = \frac{\sqrt{2}\,[\bar{\kappa}\,\tilde{\kappa}]}{\langle\kappa\,\hat{\kappa}\rangle}\,, \tag{5.65}$$

and $\hat{\kappa}^\alpha$, $\bar{\kappa}^{\dot{\alpha}}$ are the Euclidean and Lorentzian conjugates of κ^α. It follows immediately that $\Phi^{(q)}$ is a cohomology class of the appropriate weight, since both $\bar{\nabla}\mu^{\pm}$ and the holomorphic δ function $\bar{\delta}^2(s\,\lambda - \kappa)$ are proportional to $\bar{e}^0$, so that $\Phi^{(q)} = 0$ is $\bar{\nabla}$-closed. To see that (5.64) does indeed give rise to the quasi-momentum eigenstate (5.56), one first pulls the representative back to the holomorphic rational curve in twistor space corresponding to the point $x \in \mathscr{M}$. Using the formulae (5.33) for the twistor lines, one finds

$$\begin{aligned}\Phi^{(q)}\big|_X = \langle\chi_+\,\kappa\rangle^{q+2M\omega}\, \langle\chi_-\,\kappa\rangle^{q-2M\omega}\, \frac{\langle\nu\,\kappa\rangle}{\langle\nu\,\lambda\rangle}\, \bar{\delta}(\langle\lambda\,\kappa\rangle) \\ \times\, \exp\left[\mathrm{i}\,\omega\, t + \mathrm{i}\, x^{\alpha\beta}\left(2\omega\, \frac{\kappa_\alpha\, \hat{\kappa}_\beta}{\langle\kappa\,\hat{\kappa}\rangle} + \frac{\sqrt{2}\,[\bar{\kappa}\,\tilde{\kappa}]\,\kappa_\alpha\,\kappa_\beta}{\langle\kappa\,\hat{\kappa}\rangle}\right)\right]\,.\end{aligned} \tag{5.66}$$

Here, we have used the definition of ξ from (5.65), the support of the remaining holomorphic delta function $\bar{\delta}(\langle\lambda\,\kappa\rangle)$, and ν_α appears as part of a Jacobian upon performing the s-integral. The exponential can be further simplified by using (2.6) to deduce that

$$\omega = \langle\kappa|T|\tilde{\kappa}]\,, \qquad 2\omega\,\kappa^{(\alpha}\hat{\kappa}^{\beta)} + \sqrt{2}\,[\bar{\kappa}\,\tilde{\kappa}]\,\kappa^{\alpha}\,\kappa^{\beta} = -2\,\langle\kappa\,\hat{\kappa}\rangle\,\kappa^{(\alpha}\,T^{\beta)\dot{\alpha}}\,\tilde{\kappa}_{\dot{\alpha}}\,, \tag{5.67}$$

so that

$$\omega t + x^{\alpha\beta}\left(2\omega\,\frac{\kappa_\alpha\,\hat{\kappa}_\beta}{\langle\kappa\,\hat{\kappa}\rangle} + \frac{\sqrt{2}\,[\bar{\kappa}\,\tilde{\kappa}]\,\kappa_\alpha\,\kappa_\beta}{\langle\kappa\,\hat{\kappa}\rangle}\right) = k\cdot x\,. \tag{5.68}$$

With this identity, the quasi-momentum eigenstate scalar field on SDTN is recovered by the usual integral formula. The Penrose transform implies an important relationship between the holomorphic frame (5.45) arising on twistor space and the dressing frame (5.60) for the dotted momentum spinor. Observe that, by (5.44)

$$\mathrm{d}\varphi^{(q)}(x) = \theta^{\alpha\dot{\beta}}\int_X \mathrm{D}\lambda\wedge\mathsf{H}^{\dot{\alpha}}{}_{\dot{\beta}}(x,\lambda)\,\lambda_\alpha\,\left.\frac{\partial\Phi^{(q)}}{\partial\mu^{\dot{\alpha}}}\right|_X\,, \tag{5.69}$$

but upon comparison with (5.57) this implies that

$$\mathsf{H}^{\dot{\alpha}}{}_{\dot{\beta}}(x,\lambda)\,\lambda_\alpha\,\left.\frac{\partial\Phi^{(q)}}{\partial\mu^{\dot{\alpha}}}\right|_X = \mathrm{i}\,\kappa_\alpha\,\tilde{K}^{(q)}_{\dot{\beta}}(x)\;\Phi^{(q)}\big|_X\,. \tag{5.70}$$

In particular, this means that derivatives of the twistor quasi-momentum eigenstate representatives, contracted with the frame $\mathsf{H}^{\dot{\alpha}}{}_{\dot{\beta}}$ can be replaced by contractions of the 4-momentum with the dressing matrix $G^{\dot{\alpha}}{}_{\dot{\beta}}$. In other words, these frame-contracted derivatives are effectively exponential in nature. This will have important consequences in our later calculations of scattering amplitudes.

Negative helicity gravitons

Let $\psi^{(q)}_{\alpha\beta\gamma\delta}$ be the quasi-momentum eigenstate (5.56) for $n = 4$, that is the quasi-momentum eigenstate associated to a negative-helicity graviton. The suitable twistor representative is obtained by simply modifying the projective weight of the scalar representative:

$$g^{(q)}(Z) = \int_{\mathbb{C}^*} \mathrm{d}s\, s^5\, \bar{\delta}^2(s\,\lambda - \kappa)\, \left(s\,\mu^+\right)^{q+2M\omega} \left(s\,\mu^-\right)^{q-2M\omega} \exp\left(-\xi\, s^2\,\eta\right)\,, \tag{5.71}$$

with ξ given by (5.65) as before. The wavefunction in (5.56) can be recovered with the standard integral formula (2.68). However, in the following we will also need the perturbation to the ASD spin connection corresponding to such a negative helicity graviton. Let the triplet of 1-forms $\gamma^{(q)}_{\alpha\beta}$ denote linear perturbation to the (vanishing) ASD spin connection of SDTN. This perturbation is related to the ZRM field $\psi^{(q)}_{\alpha\beta\gamma\delta}$ by the linearization of the equation of motions (2.34) around the SDTN metric

$$\mathrm{d}\gamma^{(q)}_{\alpha\beta} = \psi^{(q)}_{\alpha\beta\gamma\delta}\, \Sigma^{\gamma\delta}\,. \tag{5.72}$$

The spin connection perturbation is given by

$$\gamma^{(q)}_{\alpha\beta} = \frac{2\mathrm{i}}{[\tilde{\nu}\,\tilde{\kappa}]}\, \kappa_\alpha\, \kappa_\beta\, \kappa_\gamma\, \tilde{\nu}_{\dot{\delta}}\, G^{\dot{\delta}}{}_{\dot{\gamma}}(x,\kappa,q)\, \varphi^{(q)}\, \theta^{\gamma\dot{\gamma}}\,, \tag{5.73}$$

for $\tilde{\nu}_{\dot{\alpha}}$ a fixed reference spinor amounting to a gauge choice for the spin connection. It follows that this satisfies (5.72) upon using the identities (5.61) for the dressing matrix. There exists a suitable version of the Penrose transform adapted to the linearised ASD spin connection [197]: the linearised ASD spin connection$\gamma_{\alpha\beta}$ is represented by a field $f \in H^{0,1}_{\bar{\nabla}}(\mathbb{P}\mathscr{T}, \Omega^{1,0} \otimes \mathcal{O}(-4))/\sim$, where $\sim$ is the equivalence relation

$$f \sim f + \partial c_{-4} + c_{-6} \wedge e^0\,, \tag{5.74}$$

for $c_{-k} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-k))$. This additional freedom can be used to fix the gauge

$$f = f_{\dot{\alpha}}\, \mathrm{d}\mu^{\dot{\alpha}}\,, \qquad \frac{\partial}{\partial \mu^{(\dot{\alpha}}} f_{\dot{\beta})} = 0\,, \tag{5.75}$$

where $f_{\dot{\alpha}}$ is a pair of $(0,1)$-forms valued in $\mathcal{O}(-5)$. In this gauge, the Penrose integral for the components of $\gamma_{\alpha\beta} = \gamma_{\alpha\beta\gamma\dot{\gamma}}\theta^{\gamma\dot{\gamma}}$ is

$$\gamma_{\alpha\beta\gamma\dot{\gamma}}(x) = \int_X \mathrm{D}\lambda \wedge \lambda_\alpha\, \lambda_\beta\, \lambda_\gamma\, \mathsf{H}^{\dot{\alpha}}{}_{\dot{\gamma}}(x,\lambda)\;\; f_{\dot{\alpha}}|_X\;. \tag{5.76}$$

For the quasi-momentum eigenstates, consider the twistor representative

$$\begin{aligned} f^{(q)}_{\dot{\alpha}} = -\frac{1}{4M\omega\xi}\int_{\mathbb{C}^*} \mathrm{d}s\, s^4 \bar{\delta}^2(s\lambda-\kappa)(s\mu^+)^{q+2M\omega}(s\mu^-)^{q-2M\omega}\mathrm{e}^{-\xi s^2\eta}\times \\ \times\left(\frac{q+2M\omega}{s\mu^+}\tilde{o}_{\dot{\alpha}} + \frac{q-2M\omega}{s\mu^-}\tilde{\iota}_{\dot{\alpha}}\right)\,. \end{aligned} \tag{5.77}$$

Note that this representative is related to the twistor representative (5.71) for the ASD Weyl spinor by

$$g^{(q)} = \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial f^{(q)}_{\dot{\alpha}}}{\partial \mu^{\dot{\beta}}}\,. \tag{5.78}$$

Upon inserting the representative $f^{(q)}_{\dot{\alpha}}$ in (5.76), the integral formula reduces to

$$\gamma_{\alpha\beta\gamma\dot{\gamma}} = -\frac{1}{4M\omega\xi}\varphi^{(q)}_{\alpha\beta\gamma}\mathsf{H}^{\dot{\alpha}}{}_{\dot{\gamma}}(x,\kappa)\left(\frac{q+2M\omega}{\langle\chi_+\,\kappa\rangle}\tilde{o}_{\dot{\alpha}}\mathrm{e}^{g(x,\kappa)} + \frac{q-2M\omega}{\langle\chi_-\,\kappa\rangle}\tilde{\iota}_{\dot{\alpha}}\mathrm{e}^{-g(x,\kappa)}\right)\,, \tag{5.79}$$

where $\varphi^{(q)}_{\alpha\beta\gamma}$ is a spin-3/2 negative-helicity ZRM field. Finally, inserting the holomorphic frame (5.45) further simplifies this expression to

$$\gamma_{\alpha\beta\gamma\dot{\gamma}} = \frac{2\mathrm{i}T_{\beta\dot{\alpha}}\hat{\kappa}^\beta}{\langle\kappa\,\hat{\kappa}\rangle}G^{\dot{\alpha}}{}_{\dot{\gamma}}(x,\kappa,q)\varphi^{(q)}_{\alpha\beta\gamma}\,. \tag{5.80}$$

This formula agrees with the space-time field (5.73), upon choosing the reference spinor $\tilde{\nu}_{\dot{\alpha}} = \hat{\kappa}^\alpha T_{\alpha\dot{\alpha}}$.

Positive-helicity gravitons

As in the case of positive-helicity gluons around the self-dual dyon background, we can use Sparling's argument to recover the metric perturbation (5.63) from a first potential. Consider the twistor representative

$$h^{(q)} = \int_{\mathbb{C}^*} \frac{\mathrm{d}s}{s^3} \bar{\delta}^2(s\lambda - \kappa)(s\,\mu^+)^{q+2M\omega}(s\,\mu^-)^{q-2M\omega}\mathrm{e}^{-\xi s^2\eta} \,. \tag{5.81}$$

Upon restriction to the curved twistor lines (5.33) and following the procedure given by (2.70)-(2.71), we find the potential

$$\phi^{(q)}_{\alpha\dot{\alpha}\beta\gamma} = \mathrm{i}\frac{\nu_\alpha\nu_\beta\nu_\gamma}{\langle\nu\,\kappa\rangle^3}\tilde{K}^{(q)}_{\dot{\alpha}}\varphi^{(q)} \,, \tag{5.82}$$

whose relation with the metric perturbation (5.63) is exactly captured by (2.72) as required.

5.4 2-point amplitudes

The previous Section already gives us the tools to compute 2-point amplitudes around the SDTN metric. We first compute the scalar, photon and graviton 2-point amplitudes using space-time methods exclusively, before introducing the last necessary tools from twistor theory to extend the graviton amplitude to arbitrary multiplicity in the MHV sector. As in Chapters 3 and 4, we work in the perturbiner framework, which posits that the tree-level 2-point amplitude is captured by the on-shell kinetic term evaluated on a linear combination of external states. In all cases, energy conservation require the two particles to have equal and opposite energies, so we will assume that particle 2 has energy $\omega_2 = \omega > 0$ and particle 1 has energy $\omega_1 = -\omega$. Moreover, for spinning particles, the 2-point amplitude can be non-vanishing only in the $(-,-)$ helicity configuration, i.e. at MHV degree, as a consequence of the integrability of the underlying SDTN metric. We focus on minimal states, as the amplitudes for non-minimal states can be computed using (4.41). Throughout this and the next Sections, we denote the quantum numbers of the i^{th} particle as $\{\kappa_{i\,\alpha}, \tilde{\kappa}_{i\,\dot{\alpha}}, q_i\}$ and introduce the notation

$$[\![i\,j]\!] = \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{K}^{(q_i)}_{i\,\dot{\beta}}(x) \tilde{K}^{(q_j)}_{j\,\dot{\alpha}}(x)\,, \tag{5.83}$$

for contractions of background-dressed momenta.

5.4.1 Scalar amplitude

The free scalar action is

$$S_{\text{scalar}} = \int_{\mathscr{M}} \mathrm{d}^4x\, \sqrt{|g|}\, g^{\mu\nu} \nabla_\mu \varphi\, \nabla_\nu \varphi\,. \tag{5.84}$$

Using (5.58), the scalar 2-point amplitude is

$$\mathcal{S}_2 = -2\pi \langle 1\,2 \rangle \delta(\omega_1 + \omega_2) \int \mathrm{d}^3\vec{x}\, V \langle 1\,\chi_- \rangle^{4M\omega} \langle 2\,\chi_+ \rangle^{4M\omega} [\![1\,2]\!] \mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}}\,, \tag{5.85}$$

and, as we show in appendix C, the integral can be evaluated explicitly and vanishes on the support of energy conservation. This is consistent with a variety of observations: vanishing of $\mathcal{S}_2$ turns out to be closely related to gauge invariance of the 2-graviton $(--)$ amplitude. It is also expected from the fact that the free scalar action uplifts to a local holomorphic action on twistor space, and the spacetime states lift to twistor space states that do not scatter. The latter is essentially a consequence of the geometric fact that twistor space contains multiple copies of spacetime, and the two scalar states live on disjoint copies which do not interact [130]. A similar computation shows that the scalar 2-point amplitude around the self-dual dyon background vanishes as well [2].

5.4.2 Photon amplitude

The standard QED action on a general curved spacetime can be used to compute the 2-point $(--)$ photon amplitude on SDTN. In terms of the states (5.56), the on-shell QED action reads

$$S_{\mathrm{QED}} = \int_{\mathscr{M}} \mathrm{d}^4x \, V \, \varphi_{1\,\alpha\beta}\varphi_2^{\alpha\beta} \,, \tag{5.86}$$

Upon inserting the expressions for the quasi-momentum eigenstates and performing the integral along the time direction, the 2-point amplitude reads

$$\mathcal{A}_2^{\mathrm{ph}} = 2\pi\langle 1\,2\rangle^2\delta(\omega_1+\omega_2)\int \mathrm{d}^3\vec{x}\, V\langle 1\,\chi_-\rangle^{4M\omega}\langle 2\,\chi_+\rangle^{4M\omega}\mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \,. \tag{5.87}$$

This integral can be evaluated by means of the techniques developed in appendix C and it vanishes.

5.4.3 Graviton amplitude

The Plebański action (2.34) can be used to derive the 2-point amplitude: the on-shell action is compactly given by

$$\mathcal{M}_2 = \int_{\mathscr{M}} \Sigma^{\alpha\beta}\wedge\gamma_{1\,\alpha}{}^{\gamma}\wedge\gamma_{2\,\beta\gamma} \,, \tag{5.88}$$

in terms of the linearised ASD spin connections $\gamma_{1\,\alpha\beta}$ and $\gamma_{2\,\alpha\beta}$ associated to the two negative-helicity gravitons. Inserting the fields (5.73), the 2-point amplitude reads

$$\mathcal{M}_2 = -\frac{2\pi\langle 1\,2\rangle^3\tilde{\nu}_{1\,\dot{\alpha}}\tilde{\nu}_{2\,\dot{\beta}}}{[\tilde{\nu}_1\,1][\tilde{\nu}_2\,2]}\delta(\omega_1+\omega_2)\int \mathrm{d}^3\vec{x}\, V G_{1\,\dot{\gamma}}^{\dot{\alpha}} G_2^{\dot{\beta}\dot{\gamma}}\langle 1\,\chi_-\rangle^{4M\omega}\langle 2\,\chi_+\rangle^{4M\omega}\mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \,. \tag{5.89}$$

We suppressed the dependence of the various parameters of the dressing matrices, $G_{i\;\dot{\beta}}^{\dot{\alpha}} = G^{\dot{\alpha}}{}_{\dot{\beta}}(x,\kappa_i,q_i)$. The 2-point amplitude should be independent of the choice of the spinors $\tilde{\nu}_{i\,\dot{\alpha}}$ and, as previously anticipated, this gauge invariance is closely related to the vanishing of the scalar amplitude around SDTN. Recalling the first equation (5.59) relating the background-dressed momenta and the dressing matrices, we can express the scalar and graviton amplitudes as

$$\mathcal{S}_2 \propto \tilde{\kappa}_{1\,\dot{\alpha}}\tilde{\kappa}_{2\,\dot{\beta}} \int \mathrm{d}^3\vec{x}\, V G_{1}^{\dot{\alpha}}{}_{\dot{\gamma}} G_2^{\dot{\beta}\dot{\gamma}} \langle 1\,\chi_-\rangle^{4M\omega}\langle 2\,\chi_+\rangle^{4M\omega} \mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}}\,, \tag{5.90}$$

$$\mathcal{M}_2 \propto \frac{\tilde{\nu}_{1\,\dot{\alpha}}\tilde{\nu}_{2\,\dot{\beta}}}{[\tilde{\nu}_1\,1][\tilde{\nu}_2\,2]} \int \mathrm{d}^3\vec{x}\, V G_{1}^{\dot{\alpha}}{}_{\dot{\gamma}} G_2^{\dot{\beta}\dot{\gamma}} \langle 1\,\chi_-\rangle^{4M\omega}\langle 2\,\chi_+\rangle^{4M\omega} \mathrm{e}^{\mathrm{i}(\vec{k}_1+\vec{k}_2)\cdot\vec{x}}\,, \tag{5.91}$$

up to kinematical prefactors. Gauge invariance of the graviton 2-point amplitude requires the integral to be proportional to $\tilde{\kappa}_1^{\dot{\alpha}}\tilde{\kappa}_2^{\dot{\beta}}$, and this in turn implies the vanishing of the scalar 2-point amplitude.

Finally, the evaluation of the integral in (5.89) leads to the following expression for the 2-point amplitude

$$\mathcal{M}_2 = 4\pi^2 M(4M\omega)!\,\frac{(-2\omega)^{4M\omega-2}\langle 1\,2\rangle^{4+4M\omega}}{|\vec{k}_1+\vec{k}_2|^{2+8M\omega}(\kappa_1^0\kappa_2^0)^{4M\omega}}\,\delta_{-\omega_1,\omega_2}\,. \tag{5.92}$$

Again the factor of $(\kappa_1^0\,\kappa_2^0)^{-4M\omega_2}$ comes from the original wave functions and can be dropped to manifest rotational invariance at the price of introducing an anomalous little-group weight.

5.5 Graviton MHV scattering

We now derive the tree-level, MHV amplitude for an arbitrary number of gravitons propagating on the SDTN metric. As in the case of the form factors and gluon MHV amplitude generating functionals, we can view the positive-helicity gravitons as self-dual perturbations to the background metric, so we can understand the 2-point tree-level amplitude of two negative-helicity gravitons around a generic self-dual background as the generating functional of the graviton MHV amplitude [67,175,197]. Thus, given the self-dual background space-time $\mathscr{M}$ with metric $g_{\mathscr{M}}$, consider the space-time M with metric

$$g_{M\,\alpha\dot{\alpha}\beta\dot{\beta}} = g_{\mathscr{M}\,\alpha\dot{\alpha}\beta\dot{\beta}} + \sum_{i=3}^{n} \varepsilon_i h_{i\,\alpha\dot{\alpha}\beta\dot{\beta}}\,, \tag{5.93}$$

where $h_{i\,\alpha\dot{\alpha}\beta\dot{\beta}}$ is the metric perturbation associated to the i^{th} graviton, $\varepsilon_3, \ldots, \varepsilon_n$ are formal parameters, and we supposed that graviton 1 and 2 have negative helicity. The MHV generating functional is the 2-point function calculated around the background g_M, so (5.88) immediately gives

$$\mathcal{G}(1,2) = \int_M \Sigma^{\alpha\beta} \wedge \gamma_{1\alpha}^{\gamma} \wedge \gamma_{2\,\beta\gamma}\,. \tag{5.94}$$

The n-point MHV amplitude around the background space-time $\mathscr{M}$ is then given by

$$\mathcal{M}_n = \left.\frac{\partial^{n-2}\mathcal{G}(1,2)}{\partial\varepsilon_3 \ldots \partial\varepsilon_n}\right|_{\varepsilon_3=\ldots=\varepsilon_n=0}\,, \tag{5.95}$$

that is, the piece of the generating functional which is multi-linear in the positive-helicity gravitons on the scattering background. Note that because both M and $\mathscr{M}$ are vacuum SD, $\gamma_{1,2}^{\alpha\beta}$ represent negative-helicity gravitons on both. For the rest of this Section, we will specialize the discussion to the case where $g_{\mathscr{M}}$ is the SDTN metric.

While the generating functional (5.94) gives a beautiful geometric interpretation of MHV scattering on any SD background, it is practically difficult to work with. For starters, the generating functional is not manifestly gauge invariant: the negative helicity particles enter at the level of their corresponding spin connection perturbations (i.e., as a potential rather than as zero-rest-mass fields), whereas any resulting amplitude must be gauge invariant. How this gauge invariance for the amplitude emerges from the expansion of the generating functional is not obvious. Secondly – and perhaps more importantly – it is not at all clear how to operationalise the perturbative expansion of (5.94) to obtain the MHV amplitude in practice. Remarkably, these two problems can be solved simultaneously. As we are interested in extracting scattering amplitudes, let us consider the negative-helicity gravitons in (5.94) to be quasi-momentum eigenstates of the form (5.73) with on-shell (undressed) 4-momenta and charges $k_1^{\alpha\dot\alpha} = \kappa_1^\alpha \tilde\kappa_1^{\dot\alpha}$, q_1 and $k_2^{\alpha\dot\alpha} = \kappa_2^\alpha \tilde\kappa_2^{\dot\alpha}$, q_2, respectively. Now, introduce a new set of coordinates $y^{\alpha\dot\alpha} = (y_1^{\dot\alpha}, y_2^{\dot\alpha})$ on M adapted to the spinor dyad $(\kappa_1^\alpha, \kappa_2^\alpha)$, defined by

$$y_i^{\dot\alpha} = \kappa_{i\,\alpha} x^{\alpha\dot\alpha} - \mathrm{i}\frac{q_i + 2M\omega_i}{\omega_i}\kappa_{i\,\alpha} T^{\alpha\dot\alpha}\log\langle i\,\chi_+\rangle - \mathrm{i}\frac{q_i - 2M\omega_i}{\omega_i}\kappa_{i\,\alpha} T^{\alpha\dot\alpha}\log\langle i\,\chi_-\rangle\,, \tag{5.96}$$

for $i = 1, 2$. By construction, these coordinates satisfy

$$\varphi_i^{(q_i)} = \mathrm{e}^{\mathrm{i}[y_i\,i]}\,, \tag{5.97}$$

and one can show that

$$\mathrm{d}y_i^{\dot\alpha} = G^{\dot\alpha}{}_{\dot\beta}(x, \kappa_i, q_i)\kappa_{i\,\beta}\theta^{\beta\dot\beta}\,, \tag{5.98}$$

so these coordinates have the two-fold advantage of locally solving for the closure of the dressing matrix and rendering the scalar quasi-momentum eigenstates of the two negative-helicity fields as a pure phase. In particular, this means that the negative-helicity perturbations to the ASD spin connection can be written in these new coordinates as

$$\gamma_{i\,\alpha\beta}^{(q_i)} = -2\mathrm{i}\frac{[\tilde\nu_i\,\mathrm{d}y_i]}{[\tilde\nu_i\,i]}\kappa_{i\,\alpha}\kappa_{i\,\beta}\,\mathrm{e}^{\mathrm{i}[y_i\,i]}\,. \tag{5.99}$$

The projections of the ASD 2-forms $\Sigma^{\alpha\beta}$ onto the spinor dyad $\{\kappa_1^\alpha, \kappa_2^\alpha\}$ also assume a particularly simple and helpful form. First, (5.98) and the unimodularity of the dressing matrix imply that

$$\mathrm{d}y_i^{\dot\alpha} \wedge \mathrm{d}y_{i\,\dot\alpha} = \kappa_{i\,\alpha}\kappa_{i\,\beta}\Sigma^{\alpha\beta}\,, \tag{5.100}$$

for each of $i = 1, 2$. The third projection is instead governed by the first Plebański scalar for the hyperkähler structure

$$\kappa_{1\,\alpha}\kappa_{2\,\beta}\Sigma^{\alpha\beta} = \langle 1\,2\rangle \mathrm{d}y_1^{\dot\alpha} \wedge \mathrm{d}y_2^{\dot\beta} \frac{\partial^2\Omega}{\partial y_1^{\dot\alpha}\partial y_2^{\dot\beta}}\,. \tag{5.101}$$

Implementing these details at the level of the generating functional (5.94) and integrating by parts twice (once with respect to $y_1^{\dot\alpha}$ and once with respect to $y_2^{\dot\alpha}$) gives

$$\begin{aligned}\mathcal{G}(1,2) &= -\frac{\langle 1\,2\rangle^4}{[\tilde\nu_1\,1]\,[\tilde\nu_2\,2]} \int_M \mathrm{d}^2y_1 \wedge \mathrm{d}^2y_2\, \tilde\nu_1^{\dot\alpha}\, \tilde\nu_2^{\dot\beta}\, \frac{\partial^2\Omega}{\partial y_1^{\dot\alpha}\,\partial y_2^{\dot\beta}}\, \mathrm{e}^{\mathrm{i}\,([y_1\,1]+[y_2\,2])} \\ &= \langle 1\,2\rangle^4 \int_M \mathrm{d}^2y_1 \wedge \mathrm{d}^2y_2\, \Omega\, \mathrm{e}^{\mathrm{i}([y_1\,1]+[y_2\,2])}\,.\end{aligned} \tag{5.102}$$

This expression (5.102) for the generating functional is now manifestly gauge invariant: all dependence on the spinors $\tilde\nu_{1,2}^{\dot\alpha}$ has dropped out. This resolves the first difficulty associated with extracting a (gauge-invariant) scattering amplitude from the MHV generating functional, but does not make the actual perturbative expansion in terms of positive helicity gravitons on $\mathscr{M}$ appear any easier. The last ingredient to resolve this issue is, once again, coming from twistor theory, in the form of a *twistor sigma model*.

5.5.1 The Taub-NUT twistor sigma model

As M is simply a deformation of the SDTN space $\mathscr{M}$ by self-dual radiative data (i.e., by a collection of positive helicity gravitons), it also has a twistor description via the non-linear graviton theorem. Consequently, the twistor space of M is described by a weighted Hamiltonian of the form

$$\mathsf{h} + \sum_{i=3}^{n} \varepsilon_i\, h_i \equiv \mathsf{h} + h\,, \tag{5.103}$$

where h is the SDTN Hamiltonian (5.24) and each h_i is a class in $H^{0,1}_{\bar\nabla}(\mathbb{P}\mathscr{T},\mathcal{O}(2))$ of the form (5.81), where we suppressed for brevity the dependence on the quantum numbers $\{q_i\}$. Holomorphic curves corresponding to points in M are then described by maps

$$F^{\dot\alpha}(x,\lambda)=\mathsf{F}^{\dot\alpha}(x,\lambda)+m^{\dot\alpha}(x,\lambda)\,. \tag{5.104}$$

Continuing to denote holomorphic curves in the SDTN twistor space $\mathbb{P}\mathscr{T}$ by X, let $\mathcal{X}$ denote holomorphic curves in the twistor space of M. The deformed holomorphic curves, defined by (5.104), must then satisfy

$$\bar\partial m^{\dot\alpha}=\left.\frac{\partial h}{\partial\mu_{\dot\alpha}}\right|_{\mathcal{X}}+\left.\frac{\partial \mathsf{h}}{\partial\mu_{\dot\alpha}}\right|_{\mathcal{X}}-\left.\frac{\partial \mathsf{h}}{\partial\mu_{\dot\alpha}}\right|_{X}\,, \tag{5.105}$$

where the $\bar\partial$-operator is understood to be the one along the curve[4] and we have used the equation (2.40) for the holomorphic curves in the twistor space of SDTN. To ensure that the deformation $m^{\dot\alpha}(x,\lambda)$ does not introduce any new moduli (i.e., that the curves $\mathcal{X}$ still form a 4-dimensional family), one must impose boundary conditions on $F^{\dot\alpha}$. We do this in a way which is compatible with the coordinates (5.96), setting

$$F^{\dot\alpha}(x,\kappa_1)=\mathsf{F}^{\dot\alpha}(x,\kappa_1)=y_1^{\dot\alpha}\,,\qquad F^{\dot\alpha}(x,\kappa_2)=\mathsf{F}^{\dot\alpha}(x,\kappa_2)=y_2^{\dot\alpha}\,. \tag{5.106}$$

This is equivalent to saying that the deformation $m^{\dot\alpha}(x,\lambda)$ has zeros at $\lambda_\alpha=\kappa_{1,2\,\alpha}$, which removes any additional moduli associated with the deformation.

Now, the differential equation (5.105) defining the holomorphic curves in the deformed twistor space can be obtained as the Euler-Lagrange equations of the ac-

[4]More precisely, $\bar\partial$ is the standard $\bar\partial$-operator on the copy $\mathbb{CP}^1$ at fixed $x\in M$ inside the projectivised undotted spinor bundle.

tion [67, 175]

$$S[m] = \frac{1}{\hbar} \int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda}{\langle \lambda\, 1\rangle^2 \langle \lambda\, 2\rangle^2} \left\{ [m\, \bar{\partial} m] + 2\, h|_{\mathcal{X}} + 2 \left(\mathsf{h}|_{\mathcal{X}} - \mathsf{h}|_{X} - \frac{\partial \mathsf{h}}{\partial \mu^{\dot\alpha}}\bigg|_X m^{\dot\alpha} \right) \right\} , \tag{5.107}$$

where $\hbar$ is a formal parameter. Note that this action functional constitutes a well-posed variational problem thanks to the boundary conditions (5.106). Similarly, the term $\mathsf{h}|_X$ in the action ensures that the on-shell action is well-defined and non-singular, as we can expand the action as

$$S[m] = \frac{1}{\hbar} \int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda}{\langle \lambda\, 1\rangle^2 \langle \lambda\, 2\rangle^2} \left([m\, \bar{\partial} m] + 2\, h|_{\mathcal{X}} + \sum_{p=2}^{4} \frac{2}{p!} \frac{\partial^p \mathsf{h}}{\partial \mu^{\dot\alpha_1} \cdots \partial \mu^{\dot\alpha_p}}\bigg|_X m^{\dot\alpha_1} \cdots m^{\dot\alpha_p} \right) . \tag{5.108}$$

This action defines a classical, chiral 2d conformal field theory (CFT) on the Riemann sphere governing holomorphic rational maps to twistor space, and as such is referred to as a *twistor sigma model*. The key fact about this twistor sigma model is that its on-shell value encodes the Plebański scalar Ω of the hyperkähler manifold M [67,175,198]. More precisely,

$$\Omega = \Omega_{\mathrm{SDTN}} - \frac{\hbar}{4\pi \mathrm{i}} \, S[m]\bigg|_{\mathrm{on-shell}} , \tag{5.109}$$

where Ω_{SDTN} is the Plebański scalar of the SDTN metric and $S[m]|_{\mathrm{on-shell}}$ denotes the twistor sigma model action (5.108) evaluated on solutions of its equations of motion (5.105). This means that the generating functional (5.102) is governed by the on-shell action of the twistor sigma model, therefore the perturbative expansion of M in positive-helicity gravitons on $\mathscr{M}$ is equivalent to classical expansion of the on-shell twistor sigma model action in the twistor representatives of the positive-helicity gravitons. Finally, the piece of the on-shell sigma model action that is multi-linear in each of the formal parameters $\{\varepsilon_i\}$ is given by

$$\frac{\partial^{n-2} S[m]|_{\mathrm{on-shell}}}{\partial \varepsilon_3 \cdots \partial \varepsilon_n}\bigg|_{\varepsilon_3 = \cdots = \varepsilon_n = 0} = \left\langle \prod_{i=3}^{n} V_i \right\rangle_{\mathrm{SDTN}}^{\mathrm{conn.,\, tree}} , \tag{5.110}$$

where the quantity on the right-hand-side of this equation is the connected, tree-level (i.e., $O(\hbar^0)$ in this case) correlation function of vertex operators

$$V_i=\int_{\mathbb{CP}^1}\frac{\mathrm{D}\lambda_i}{\langle 1\,\lambda_i\rangle^2\,\langle 2\,\lambda_i\rangle^2}\,h_i(\mathsf{F}+m,\lambda_i)\,, \tag{5.111}$$

in the 2d CFT

$$\begin{aligned}S_{\mathrm{SDTN}}[m]=\int_{\mathbb{CP}^1}\frac{\mathrm{D}\lambda}{\langle 1\,\lambda\rangle^2\,\langle 2\,\lambda\rangle^2}\Bigg[&m^{\dot\alpha}\left(\epsilon_{\dot\beta\dot\alpha}\,\bar\partial+\frac{\partial^2\mathsf{h}}{\partial\mu^{\dot\alpha}\partial\mu^{\dot\beta}}(\mathsf{F},\lambda)\right)m^{\dot\beta}\\&+\frac{\bar e^0}{24M}\left([\tilde o\,m]^2\,[\tilde\iota\,m]\,[\tilde\iota\,\mathsf{F}]+[\tilde\iota\,m]^2\,[\tilde o\,m]\,[\tilde o\,\mathsf{F}]+3\,[\tilde\iota\,m]^2\,[\tilde o\,m]^2\right)\Bigg]\,,\end{aligned} \tag{5.112}$$

on the Riemann sphere. The terms in the second line of (5.112), which can be thought of as explicit ‘background’ terms associated with SDTN, mean that this 2d CFT is not free. However, as we are only interested in computing tree-level (or semi-classical) correlation functions (5.110), these terms can be treated with perturbation theory:

$$\left\langle\prod_{i=3}^{n}V_i\right\rangle_{\mathrm{SDTN}}^{\text{conn., tree}}=\sum_{t=0}^{\infty}\sum_{p+q+r=t}\left\langle\prod_{i=3}^{n}V_i\prod_{a=1}^{p}U_a^+\prod_{b=1}^{q}U_b^-\prod_{c=1}^{r}U_c^0\right\rangle_{\text{free}}^{\text{conn., tree}}\,, \tag{5.113}$$

where the correlator is evaluated in the free 2d CFT

$$S_{\mathrm{free}}=\int_{\mathbb{CP}^1}\frac{\mathrm{D}\lambda}{\langle 1\,\lambda\rangle^2\,\langle 2\,\lambda\rangle^2}\left[m^{\dot\alpha}\left(\epsilon_{\dot\beta\dot\alpha}\,\bar\partial+\frac{\partial^2\mathsf{h}}{\partial\mu^{\dot\alpha}\partial\mu^{\dot\beta}}(\mathsf{F},\lambda)\right)m^{\dot\beta}\right]\,, \tag{5.114}$$

and the effect of the ‘non-free’ background terms from S_{SDTN} is encapsulated by the insertion of the background vertex operators

$$U_a^+=\frac{1}{24\,M}\int_{\mathbb{CP}^1}\frac{\varpi_a}{\langle 1\,\lambda_a\rangle^2\,\langle 2\,\lambda_a\rangle^2}\,\mathsf{F}^+(\lambda_a)\,[m\,\tilde o]\,[m\,\tilde\iota]^2\,, \tag{5.115}$$

$$U_a^-=\frac{1}{24\,M}\int_{\mathbb{CP}^1}\frac{\varpi_a}{\langle 1\,\lambda_a\rangle^2\,\langle 2\,\lambda_a\rangle^2}\,\mathsf{F}^-(\lambda_a)\,[m\,\tilde\iota]\,[m\,\tilde o]^2\,, \tag{5.116}$$

and

$$U_a^0 = \frac{1}{8\,M} \int_{\mathbb{CP}^1} \frac{\varpi_a}{\langle 1\,\lambda_a\rangle^2\,\langle 2\,\lambda_a\rangle^2}\,[m\,\tilde{o}]^2\,[m\,\tilde{\iota}]^2\,, \tag{5.117}$$

where

$$\varpi = \frac{\mathrm{D}\lambda \wedge \mathrm{D}\hat{\lambda}}{\langle \lambda\,\hat{\lambda}\rangle^2}\,, \tag{5.118}$$

is the Kähler form on $\mathbb{CP}^1$.

5.5.2 Computing the twistor sigma model correlator

The computation of (5.113) can be approached using fairly standard methods in 2d CFT and graph theory. To begin, we decompose the correlator depending on the number of insertions of background operators by defining

$$\mathcal{C}_n[p,q,r] = \left\langle \prod_{i=3}^{n} V_i \prod_{a=1}^{p} U_a^+ \prod_{b=1}^{q} U_b^- \prod_{c=1}^{r} U_c^0 \right\rangle_{\text{free}}^{\text{conn., tree}}\,. \tag{5.119}$$

Let us first consider the correlator $\mathcal{C}_n[0,0,0]$, which receives no contributions from the background operators. The only quantum mechanical in the 2d CFT (5.114) field is $m^{\dot\alpha}(\lambda)$, whose propagator is immediately constructed as

$$\langle m^{\dot\alpha}(\lambda) m^{\dot\beta}(\lambda')\rangle = \frac{\mathsf{H}^{\dot\alpha}{}_{\dot\gamma}(\lambda)\mathsf{H}^{\dot\beta\dot\gamma}(\lambda')}{\langle \lambda\,\lambda'\rangle}\langle 1\,\lambda\rangle\langle 2\,\lambda\rangle\langle 1\,\lambda'\rangle\langle 2\,\lambda'\rangle\,, \tag{5.120}$$

where $\mathsf{H}^{\dot\alpha}{}_{\dot\beta}(x,\lambda)$ is the holomorphic frame (5.45) and where all dependence on x has been suppressed. Indeed, Equations (2.40) and (5.44) imply that the holomorphic frame satisfies

$$\bar\partial\big|_X\,\mathsf{H}^{\dot\alpha}{}_{\dot\beta}(x,\lambda) = \left.\frac{\partial^2 \mathsf{h}}{\partial\mu^{\dot\alpha}\partial\mu^{\dot\beta}}\right|_X \mathsf{H}^{\dot\gamma}{}_{\dot\beta}(x,\lambda)\,, \tag{5.121}$$

on any twistor curve, thus diagonalising the kinetic operator of the free action (5.114). In particular, the propagator (5.120) directly implies that the Wick contraction between two of the positive-helicity vertex operators (5.111) is

$$\langle V_i\,V_j\rangle = \int_{(\mathbb{CP}^1)^2} \frac{\mathrm{D}\lambda_i\,\mathrm{D}\lambda_j}{\langle 1\,\lambda_i\rangle\langle 2\,\lambda_i\rangle\langle 1\,\lambda_j\rangle\langle 2\,\lambda_j\rangle} \frac{\mathsf{H}^{\dot\alpha}{}_{\dot\gamma}(\lambda_i)\mathsf{H}^{\dot\beta\dot\gamma}(\lambda_j)}{\langle\lambda_i\,\lambda_j\rangle} \left.\frac{\partial h_i}{\partial\mu^{\dot\alpha}}\right|_X(\lambda_i)\left.\frac{\partial h_j}{\partial\mu^{\dot\beta}}\right|_X(\lambda_j)\,. \tag{5.122}$$

We can now insert the twistor representatives (5.81), apply the identity (5.70) connecting the holomorphic frame and the dressing matrix to finally find

$$\langle V_i\,V_j\rangle = -\int_{(\mathbb{CP}^1)^2} \frac{\mathrm{D}\lambda_i\,\mathrm{D}\lambda_j}{\langle 1\,\lambda_i\rangle\langle 2\,\lambda_i\rangle\langle 1\,\lambda_j\rangle\langle 2\,\lambda_j\rangle} \frac{s_i s_j [\![i\,j]\!]}{\langle\lambda_i\,\lambda_j\rangle}\, h_i|_X(\lambda_i)\, h_j|_X(\lambda_j)\,. \tag{5.123}$$

We have abused notation by writing explicit powers of the scaling parameters s_i, s_j, which are integrated over inside of the twistor representatives. What is meant by this is that one multiplies the measure inside h_i by s_i prior to integration over $\mathbb{C}^*$; this definition is unambiguous and saves having to rewrite all powers of s_i and its measure in h_i every time a Wick contraction is taken.

Now, all of the Feynman diagrams contributing to $C_n[0,0,0]$ are spanning tree graphs on $n-2$ vertices, where the edge between the i^{th} and j^{th} vertex is weighted precisely by the Wick contraction (5.123). The weighted matrix-tree theorem of algebraic combinatorics (cf., [199–201] for textbook treatments) computes this weighted sum over spanning tree graphs as the determinant of a matrix. Let W be the weighted Laplacian matrix associated to the totally connected graph on all of the positive-helicity graviton vertex operators: this is the $(n-2)\times(n-2)$ matrix with entries

$$W_{jk} = \begin{cases} -\dfrac{s_j s_k [\![j\,k]\!]}{\langle\lambda_j\,\lambda_k\rangle}\langle 1\,\lambda_j\rangle\langle 2\,\lambda_j\rangle\langle 1\,\lambda_k\rangle\langle 2\,\lambda_k\rangle\,, & j\neq k \\ \displaystyle\sum_{\ell\neq j}\frac{s_j s_\ell [\![j\,\ell]\!]}{\langle\lambda_j\,\lambda_\ell\rangle}\langle 1\,\lambda_j\rangle\langle 2\,\lambda_j\rangle\langle 1\,\lambda_\ell\rangle\langle 2\,\lambda_\ell\rangle\,, & j=k\,. \end{cases} \tag{5.124}$$

By definition, this matrix has co-rank 1, and the matrix-tree theorem states that the weighted sum over all spanning tree graphs on the set of graviton vertex operators is given by taking the determinant of the one-reduced minor $|W_i^i|$, where W_i^i denotes the weighted Laplacian matrix with row and column i removed. The matrix-tree theorem also ensures that the value of the determinant $|W_i^i|$ is independent of the choice of i. We thus conclude

$$\mathcal{C}_n[0,0,0] = \int_{(\mathbb{CP}^1)^{n-2}} \prod_{j=3}^{n} \frac{\mathrm{D}\lambda_j \wedge h_j|_X(\lambda_j)}{\langle 1\,\lambda_j\rangle^2 \langle 1\,\lambda_j\rangle^2} \, |W_i^i| \, . \tag{5.125}$$

Inserting the twistor representatives (5.81) and performing all the scale and $\mathbb{CP}^1$ integrals against the holomorphic delta functions, we are finally left with

$$\mathcal{C}_n[0,0,0] = \frac{|\mathbb{H}_i^i|}{\langle 1\,i\rangle^2 \langle 2\,i\rangle^2} \prod_{j=3}^{n} \varphi_j^{(q_j)}(x) \, , \tag{5.126}$$

where the matrix $\mathbb{H}$ is the matrix with entries

$$\mathbb{H}_{jk} = \begin{cases} -\dfrac{[\![j\,k]\!]}{\langle j\,k\rangle} \, , & j \neq k \\ \displaystyle\sum_{\ell \neq j} \frac{[\![j\,\ell]\!] \langle 1\,\ell\rangle \langle 2\,\ell\rangle}{\langle j\,\ell\rangle \langle 1\,j\rangle \langle 2\,j\rangle} \, , & j = k \, . \end{cases} \tag{5.127}$$

This matrix can be constructed from the weighted Laplacian matrix by removing a factor of $\langle 1\,\lambda_j\rangle\langle 2\,\lambda_j\rangle$ from the j^{th} row and column, after performing all the scale and sphere integrals.

To compute $\mathcal{C}_n[p,q,r]$ for generic values of p,q,r, we adopt a similar strategy and reduce the correlator to the problem of summing over weighted spanning tree graphs, now on an enhanced set of vertices which includes not only the $n-2$ graviton vertex operators, but also p of the background vertex operators U^+, q of the U^-, and r of the U^0. However, as it stands, the matrix-tree theorem will produce an *overcounting* of the Feynman diagrams. Indeed, each of the background vertex operators is a polynomial in $m^{\dot\alpha}$ and must have a precise number of edges attached to it in any given Feynman diagram – otherwise, the diagram's contribution to the correlator is zero because $m^{\dot\alpha}$ is a field with no zero modes. Simply applying the matrix-tree theorem sums *all* of the spanning tree graphs on the $n+t-2$ vertices, though, including those whose contribution to the correlator must be zero. Fortunately, it's easy to pick out those graphs with the correct number of edges by weighted each Wick contraction involving a background vertex operator with a further formal parameter α. The determinant arising from the matrix-tree theorem will be a polynomial in α, with the order k terms corresponding to those trees in which the vertex corresponding to the background vertex operator has valence k. By isolating those terms of the appropriate order in α, one then ensures that only the correct graphs are included in the counting. Actually, since the two components $m^+ = [m\,\tilde{o}]$ and $m^- = [m\,\tilde{\imath}]$ enter the background vertex operators asymmetrically, we need a pair of formal parameters $\alpha^\pm$ for each of the background vertex operators. We thus write the correlator as

$$\begin{aligned}\mathcal{C}_n[p,q,r] = \int_{(\mathbb{CP}^1)^t} \prod_{\mathrm{m}=1}^{t} \frac{\varpi_m}{\langle 1\,\lambda_{\mathrm{m}}\rangle^2 \langle 2\,\lambda_{\mathrm{m}}\rangle^2} \prod_{a=1}^{p} \frac{\mathsf{F}^+(\lambda_a)}{24M} \frac{\partial}{\partial\alpha_a^+} \frac{\partial^2}{\partial{\alpha_a^-}^2} \prod_{b=1}^{q} \frac{\mathsf{F}^-(\lambda_b)}{24M} \frac{\partial^2}{\partial{\alpha_b^+}^2} \frac{\partial}{\partial\alpha_b^-} \times \\ \times \prod_{c=1}^{r} \frac{1}{8M} \frac{\partial^2}{\partial{\alpha_c^+}^2} \frac{\partial^2}{\partial{\alpha_c^-}^2} \left\langle \prod_{j=3}^{n} V_j \prod_{\mathrm{n}=1}^{t} \mathrm{e}^{\alpha_{\mathrm{n}}^+[m\,\tilde{o}] + \alpha_{\mathrm{n}}^-[m\,\tilde{\imath}]} \right\rangle \Bigg|_{\alpha_1^\pm = \ldots \alpha_t^\pm = 0}\end{aligned} \tag{5.128}$$

where $t = p+q+r$. The correlator in the second line can now be computed using the standard matrix-tree theorem and, after performing as many scale and $\mathbb{CP}^1$ integrals against the holomorphic δ functions as possible, one arrives at

$$\begin{aligned}\mathcal{C}_n[p,q,r] = \int_{(\mathbb{CP}^1)^t} \prod_{\mathrm{m}=1}^{t} \frac{\varpi_{\mathrm{m}}}{\langle 1\,\lambda_{\mathrm{m}}\rangle^2 \langle 2\,\lambda_{\mathrm{m}}\rangle^2} \prod_{a=1}^{p} \frac{\mathsf{F}^+(\lambda_a)}{24M} \frac{\partial}{\partial\alpha_a^+} \frac{\partial^2}{\partial{\alpha_a^-}^2} \prod_{b=1}^{q} \frac{\mathsf{F}^-(\lambda_b)}{24M} \frac{\partial^2}{\partial{\alpha_b^+}^2} \frac{\partial}{\partial\alpha_b^-} \times \\ \times \prod_{c=1}^{r} \frac{1}{8M} \frac{\partial^2}{\partial{\alpha_c^+}^2} \frac{\partial^2}{\partial{\alpha_c^-}^2} \; \frac{|\mathcal{H}[\mathbf{t}]_i^i|}{\langle 1\,i\rangle^2 \langle 2\,i\rangle^2} \Bigg|_{\alpha_1^\pm = \ldots \alpha_t^\pm = 0} \prod_{j=3}^{n} \varphi_j^{(q_j)}(x)\,.\end{aligned} \tag{5.129}$$

Here, $\mathbf{t} = (p,q,r)$ such that $p+q+r = t$ and $\mathcal{H}[\mathbf{t}]$ is the $(n+t-2) \times (n+t-2)$

matrix which has a block decomposition

$$\mathcal{H}[\mathbf{t}] = \begin{pmatrix} \mathbb{H}[\mathbf{t}] & \mathfrak{h}[\mathbf{t}] \\ \mathfrak{h}[\mathbf{t}]^T & \mathbb{T}[\mathbf{t}] \end{pmatrix}, \tag{5.130}$$

with $\mathbb{H}[\mathbf{t}]$, $\mathfrak{h}[\mathbf{t}]$, and $\mathbb{T}[\mathbf{t}]$ being $(n-2)\times(n-2)$, $(n-2)\times t$, and $t\times t$ matrices, respectively. Introducing the notation

$$\boldsymbol{\alpha}_{\mathrm{m}\,\dot\alpha} = \mathsf{H}^{\dot\beta}{}_{\dot\alpha}(\lambda_{\mathrm{m}})(\alpha^+_{\mathrm{m}}\tilde{o}_{\dot\beta} + \alpha^-_{\mathrm{m}}\tilde{\iota}_{\dot\beta})\,, \qquad [\![j\,\mathrm{m}]\!] = \tilde{K}^{\dot\alpha}_j\boldsymbol{\alpha}_{\mathrm{m}\,\alpha}\,, \qquad [\![\mathrm{m}\,\mathrm{n}]\!] = \boldsymbol{\alpha}^{\dot\alpha}_{\mathrm{m}}\boldsymbol{\alpha}_{\mathrm{n}\,\dot\alpha}\,, \tag{5.131}$$

their entries are given by

$$\mathbb{H}_{jk}[\mathbf{t}] = \begin{cases} -\dfrac{[\![j\,k]\!]}{\langle j\,k\rangle}\,, & j\neq k \\ \displaystyle\sum_{\ell\neq j}\frac{[\![j\,\ell]\!]\langle 1\,\ell\rangle\langle 2\,\ell\rangle}{\langle j\,k\rangle\langle 1\,j\rangle\langle 2\,j\rangle} - \mathrm{i}\sum_{\mathrm{m}=1}^{t}\frac{[\![j\,\mathrm{m}]\!]\langle 1\,\lambda_{\mathrm{m}}\rangle\langle 2\,\lambda_{\mathrm{m}}\rangle}{\langle j\,\lambda_{\mathrm{m}}\rangle\langle 1\,j\rangle\langle 2\,j\rangle}\,, & j=k \end{cases} \tag{5.132}$$

$$\mathfrak{h}_{j\mathrm{m}}[\mathbf{t}] = \mathrm{i}\frac{[\![j\,\mathrm{m}]\!]\langle 1\,\lambda_{\mathrm{m}}\rangle\langle 2\,\lambda_{\mathrm{m}}\rangle}{\langle j\,\lambda_{\mathrm{m}}\rangle}\,, \tag{5.133}$$

$$\mathbb{T}_{\mathrm{mn}}[\mathbf{t}] = \begin{cases} \dfrac{[\![\mathrm{m}\,\mathrm{n}]\!]\langle 1\,\lambda_{\mathrm{m}}\rangle\langle 2\,\lambda_{\mathrm{m}}\rangle\langle 1\,\lambda_{\mathrm{n}}\rangle\langle 2\,\lambda_{\mathrm{n}}\rangle}{\langle\lambda_{\mathrm{m}}\,\lambda_{\mathrm{n}}\rangle}\,, & \mathrm{m}\neq\mathrm{n} \\ -\displaystyle\sum_{\mathrm{p}\neq\mathrm{m}}\frac{[\![\mathrm{m}\,\mathrm{p}]\!]\langle 1\,\lambda_{\mathrm{m}}\rangle\langle 2\,\lambda_{m}\rangle\langle 1\,\lambda_{\mathrm{p}}\rangle\langle 2\,\lambda_{\mathrm{p}}\rangle}{\langle\lambda_{\mathrm{m}}\,\lambda_{\mathrm{p}}\rangle} - \mathrm{i}\sum_{j=3}^{n}\frac{[\![j\,\mathrm{m}]\!]\langle 1\,\lambda_{\mathrm{m}}\rangle\langle 2\,\lambda_{\mathrm{m}}\rangle}{\langle j\,\lambda_{\mathrm{m}}\rangle\langle 1\,j\rangle\langle 2\,j\rangle}\,, & \mathrm{m}=\mathrm{n} \end{cases} \tag{5.134}$$

for $j,k=3,\ldots n$ and $\mathrm{m},\mathrm{n}=1,\ldots,t$.

A few further comments about the structure of the result (5.129) are in order. Firstly, there remain t integrals over the Riemann sphere – one corresponding to each background vertex operator – which have not been performed analytically due to the complicated dependence on the vertex operator insertion points in the reduced determinant $|\mathcal{H}[\mathbf{t}]^i_i|$. Unlike the graviton vertex operator insertions, these are not localised against delta functions. One can confirm that these integrals are projectively well-defined on each copy of $\mathbb{CP}^1$: effectively, each formal parameter $\alpha^{\pm}_{\mathrm{m}}$ carries scaling weight -1 with respect to the homogeneous coordinate λ_{m}, so that the entries of the matrix $\mathcal{H}[\mathsf{t}]$ are weightless in λ_{m}. The integral measure

$$\frac{\varpi_{\mathrm{m}}}{\langle 1\, \lambda_{\mathrm{m}}\rangle^2 \, \langle 2\, \lambda_{\mathrm{m}}\rangle^2}\,, \tag{5.135}$$

is weight -4, which is then balanced by the weight $+4$ differential operator in the formal parameters which extracts the appropriate terms from the reduced determinant. Equivalently, it is easy to see that once the formal parameters have been removed, the remaining terms from the determinant are weight $+4$.

The correlator (5.113) that underpins the gravitational MHV amplitude on SDTN is given by a sum – in principle, an infinite sum – over the building blocks (5.129), graded by the number of background vertex operator insertions. However, this sum is actually finite for any given n. Each background vertex operator must absorb a minimum of three Wick contractions, with the resulting Feynman graph restricted to be a spanning tree on the set of all vertices. An inductive argument easily shows that the maximum number of background insertions t for which this is possible for fixed n is given by $n-4$.

So we have finally established an explicit formula for the full correlation function of interest:

$$\begin{aligned}\left\langle \prod_{i=3}^{n} V_i \right\rangle_{\mathrm{SDTN}}^{\mathrm{conn.,\,tree}} &= \sum_{t=0}^{n-4} \sum_{p+q+r=t} \int_{(\mathbb{CP}^1)^t} \prod_{\mathrm{m}=1}^{t} \frac{\varpi_{\mathrm{m}}}{\langle 1\, \lambda_{\mathrm{m}}\rangle^2 \, \langle 2\, \lambda_{\mathrm{m}}\rangle^2} \prod_{a=1}^{p} \frac{\mathsf{F}^+(\lambda_a)}{24\, M}\, \frac{\partial}{\partial \alpha_a^+}\, \frac{\partial^2}{\partial \alpha_a^{-\,2}} \\ &\times \prod_{b=1}^{q} \frac{\mathsf{F}^-(\lambda_b)}{24\, M}\, \frac{\partial^2}{\partial \alpha_b^{+\,2}}\, \frac{\partial}{\partial \alpha_b^-} \prod_{c=1}^{r} \frac{1}{8\, M}\, \frac{\partial^2}{\partial \alpha_c^{+\,2}}\, \frac{\partial^2}{\partial \alpha_c^{-\,2}}\, \frac{|\mathcal{H}[\mathsf{t}]^i_i|}{\langle 1\, i\rangle^2 \, \langle 2\, i\rangle^2} \Bigg|_{\alpha_1^\pm = \ldots \alpha_{\mathrm{m}}^\pm = 0} \prod_{j=3}^{n} \varphi_j^{(q_j)}(x)\,,\end{aligned} \tag{5.136}$$

which encapsulate the perturbative expansion of the MHV generating functional on the SDTN metric.

5.5.3 The MHV amplitude

At this point, the result (5.136) for the twistor sigma model correlator can be fed back into (5.102) to obtain a final expression for the graviton MHV amplitude on SDTN. After re-writing the generating functional in the Gibbons-Hawking coordinates, we are left with:

$$\mathcal{M}_n = \frac{\langle 1\,2\rangle^6}{\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\int_{\mathscr{M}} \mathrm{d}^4x\,\sqrt{|g|}\sum_{t=0}^{n-4}\sum_{p+q+r=t}\int_{(\mathbb{CP}^1)^t}\prod_{\mathrm{m}=1}^{t}\frac{\varpi_{\mathrm{m}}}{\langle 1\,\lambda_{\mathrm{m}}\rangle^2\,\langle 2\,\lambda_{\mathrm{m}}\rangle^2}$$
$$\times\prod_{a=1}^{p}\frac{\mathsf{F}^{+}(\lambda_a)}{24\,M}\frac{\partial}{\partial\alpha_a^{+}}\frac{\partial^2}{\partial{\alpha_a^{-}}^{2}}\prod_{b=1}^{q}\frac{\mathsf{F}^{-}(\lambda_b)}{24\,M}\frac{\partial^2}{\partial{\alpha_b^{+}}^{2}}\frac{\partial}{\partial\alpha_b^{-}}\prod_{c=1}^{r}\frac{1}{8\,M}\frac{\partial^2}{\partial{\alpha_c^{+}}^{2}}\frac{\partial^2}{\partial{\alpha_c^{-}}^{2}}$$
$$\times\left|\mathcal{H}[\mathbf{t}]^i_i\right|\Big|_{\boldsymbol{\alpha}=0}\prod_{j=1}^{n}\varphi_j^{(q_j)}(x)\,. \tag{5.137}$$

where $|g|$ is the determinant of the SDTN metric and the gravitons 1 and 2 have negative helicity, while $3,\ldots,n$ have positive helicity. By inserting the scalar quasi-momentum eigenstates (5.56) and the metric determinant, this formula can be made more explicit as

$$\mathcal{M}_n = 2\pi\,\kappa^{n-2}\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\sum_{t=0}^{n-4}\sum_{p+q+r=t}\int_{\mathbb{R}^3\times(\mathbb{CP}^1)^t}\mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot\vec{x}}\left(1+\frac{2\,M}{r}\right)$$
$$\times\prod_{\mathrm{m}=1}^{t}\frac{\varpi_{\mathrm{m}}}{\langle 1\,\lambda_{\mathrm{m}}\rangle^2\,\langle 2\,\lambda_{\mathrm{m}}\rangle^2}\prod_{j=1}^{n}\left(\frac{r}{1+\zeta\,\bar{\zeta}}\right)^{q_j}(\zeta-z_j)^{q_j-2M\omega_j}\,(\bar{\zeta}\,z_j+1)^{q_j+2M\omega_j}$$
$$\times\prod_{a=1}^{p}\frac{\mathsf{F}^{+}(\lambda_a)}{24\,M}\frac{\partial}{\partial\alpha_a^{+}}\frac{\partial^2}{\partial{\alpha_a^{-}}^{2}}\prod_{b=1}^{q}\frac{\mathsf{F}^{-}(\lambda_b)}{24\,M}\frac{\partial^2}{\partial{\alpha_b^{+}}^{2}}\frac{\partial}{\partial\alpha_b^{-}}\prod_{c=1}^{r}\frac{1}{8\,M}\frac{\partial^2}{\partial{\alpha_c^{+}}^{2}}\frac{\partial^2}{\partial{\alpha_c^{-}}^{2}}\left|\mathcal{H}[\mathbf{t}]^i_i\right|\Big|_{\boldsymbol{\alpha}=0}, \tag{5.138}$$

where we have reinstated the appropriate powers of the gravitational coupling constant κ and abbreviated

$$\omega=\sum_{j=1}^{n}\omega_j\,,\qquad \vec{k}=\sum_{j=1}^{n}\vec{k}_j\,. \tag{5.139}$$

As the sum over t in this formula corresponds to the number of background vertex operator insertions in the twistor sigma model, it is clear that these contributions to the amplitude encode the external gravitons scattering off the non-trivial geometry of the SDTN metric itself. This is a ubiquitous feature of gravitational scattering in curved spacetimes, known as *tails*, resulting from the violation of Huygens' principle. Indeed, for scalar scattering it was established long ago that the only metrics which do not lead to tails are Minkowski space and vacuum plane waves [107], while for gravitons only flat space scattering does not produce tails [99, 108–110]. However, it should be emphasized that while the $t \geq 1$ terms in the amplitude are unambiguously tail effects, tails are present even in the $t = 0$ contributions, through the dressed momentum spinors and quasi-momentum eigenstate wavefunctions.

5.5.4 Properties of the amplitude

At this stage, we can comment on some general features of the MHV amplitude formula (5.138), including its explicit expansion for small n and flat space limit.

Low-point examples: for the number of external gravitons $n \leq 4$, the sum over explicit tail terms is absent, and the formula for the MHV amplitude simplifies to

$$\begin{aligned}\mathcal{M}_{n\leq 4} = 2\pi\,\kappa^{n-2}\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot\vec{x}}\left(1+\frac{2\,M}{r}\right)& \\ \times\,\left|\mathbb{H}^i_i\right|\prod_{j=1}^{n}\left(\frac{r}{1+\zeta\,\bar{\zeta}}\right)^{q_j}(\zeta-z_j)^{q_j-2M\omega_j}\,(\bar{\zeta}\,z_j+1)^{q_j+2M\omega_j}\,,& \quad (5.140)\end{aligned}$$

where $\mathbb{H}$ is the $(n-2)\times(n-2)$ matrix with entries (5.127).

To further simplify matters, we can restrict our attention to the scattering of minimal quasi-momentum eigenstates, for which $q_i = 2M|\omega_i|$; as usual, the amplitudes for more general configurations will follow by acting with differential operators in momentum space. Let $\mathfrak{n}_{\pm}$ be the set of positive- and negative-energy gravitons: the amplitude becomes

$$\mathcal{M}_{n\leq 4} = 2\pi\,\kappa^{n-2}\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot\vec{x}}\left(1+\frac{2\,M}{r}\right)$$
$$\times\,\left|\mathbb{H}^i_i\right|\prod_{j\in\mathfrak{n}_+}\left(\frac{r}{1+\zeta\,\bar{\zeta}}\right)^{2M\omega_j}(\bar{\zeta}\,z_j+1)^{4M\omega_j}\prod_{k\in\mathfrak{n}_-}\left(\frac{r}{1+\zeta\,\bar{\zeta}}\right)^{-2M\omega_k}(\zeta-z_k)^{-4M\omega_k}\,. \tag{5.141}$$

Clearly, overall energy conservation implies that there must be at least one external graviton of both positive and negative frequency in order to obtain a non-vanishing amplitude.

For $n=3$, one can assume without loss of generality that graviton 1 has negative frequency, so that

$$\mathcal{M}_3 = 2\pi\,\kappa\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot\vec{x}}\left(1+\frac{2\,M}{r}\right)$$
$$\times\left[\frac{r\,(\zeta-z_1)\,(\bar{\zeta}\,z_2+1)}{1+\zeta\,\bar{\zeta}}\right]^{4M\omega_2}\left[\frac{r\,(\zeta-z_1)\,(\bar{\zeta}\,z_3+1)}{1+\zeta\,\bar{\zeta}}\right]^{4M\omega_3}\,, \tag{5.142}$$

It is possible to carry out all the integrals, as explained in Appendix C. The integrated amplitude reads

$$\mathcal{M}_3 = 4\pi^2\,\mathrm{i}\,\kappa\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2}\,\frac{(-1)^{4M\omega_1}\,(\mathrm{i}\,\alpha_{21})^{4M\omega_2}\,(\mathrm{i}\,\alpha_{31})^{4M\omega_3}}{|\vec{k}|^{2+8M|\omega_1|}}$$
$$\times\left[2M+(4M|\omega_1|)!\left(\frac{4M\omega_2\,z_{21}}{\alpha_{21}}+\frac{4M\omega_3\,z_{31}}{\alpha_{31}}\right)\right]\,, \tag{5.143}$$

Already at $n=4$, the structure of the amplitude becomes more complicated; in some sense, this is inherited from flat space: unlike its gauge theory cousin, the MHV graviton amplitude depends on both angle and square bracket kinematic invariants beyond 3-point. However, in SDTN the square brackets are themselves dressed by the background, leading to more terms which contribute to the integral over $\mathbb{R}^3$.

For the 4-point scattering of minimal quasi-momentum eigenstates and graviton 1 the only negative-frequency state, the MHV amplitude is given by

$$\mathcal{M}_4 = 2\pi\,\kappa^2\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,3\rangle\,\langle 1\,4\rangle\,\langle 2\,3\rangle\,\langle 2\,4\rangle\,\langle 4\,3\rangle} \times \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\left(1+\frac{2M}{r}\right)[\![4\,3]\!]\prod_{i=2}^{4}\left(\frac{r\,(\zeta-z_1)\,(\bar{\zeta}\,z_i+1)}{1+|\zeta|^2}\right)^{4M\omega_i} . \tag{5.144}$$

The last integral can be evaluated using the same methods as at 3-points, although the resulting expressions are not particularly enlightening.

It is also illustrative to consider the case $n = 5$, where the first explicit tail contributions appear in the amplitude thanks to the insertion of background vertex operators in the twistor sigma model. In this case, only the $(p, q, r) = (1, 0, 0)$ and $(p, q, r) = (0, 1, 0)$ terms contribute to the amplitude beyond $t = 0$. For instance, the former is given by

$$\frac{\pi\,\kappa^3}{12\,M}\,\delta(\omega)\,\frac{\langle 1\,2\rangle^6}{\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2}\int_{\mathbb{R}^3\times S^2}\frac{\mathrm{d}^3\vec{x}\,\mathrm{D}\lambda\wedge\mathrm{D}\hat{\lambda}}{\langle\lambda\,\hat{\lambda}\rangle^2\,\langle 1\,\lambda\rangle^2\,\langle 2\,\lambda\rangle^2} \times\left(\mathsf{F}^+(\lambda)\,\frac{\partial}{\partial\alpha^+}\,\frac{\partial^2}{\partial{\alpha^-}^2}+\mathsf{F}^-(\lambda)\,\frac{\partial^2}{\partial{\alpha^+}^2}\,\frac{\partial}{\partial\alpha^-}\right)\left|\mathcal{H}^3_3[1]\right|\Big|_{\alpha^\pm=0} , \tag{5.145}$$

with the minor given by

$$\left|\mathcal{H}^3_3[1]\right| = \mathbb{H}_{44}\left(\mathbb{T}\,\mathbb{H}_{55}-\mathfrak{h}_5^2\right)-\mathbb{H}_{45}\left(\mathbb{H}_{45}\,\mathbb{T}-\mathfrak{h}_4\,\mathfrak{h}_5\right)+\mathfrak{h}_4\left(\mathbb{H}_{45}\,\mathfrak{h}_5-\mathbb{H}_{55}\,\mathfrak{h}_4\right) , \tag{5.146}$$

the tail index on matrix entries being suppressed as it is irrelevant in this case.

The matrix entries $\mathbb{H}_{45}$ contain no powers of the formal parameters, while all other entries appearing in (5.146) are at most linear in $\alpha^\pm$, making it clear that only

$$\mathbb{H}_{44}\,\mathbb{H}_{55}\,\mathbb{T}-\mathbb{H}_{55}\,\mathfrak{h}_5^2 , \tag{5.147}$$

can give non-vanishing contributions to (5.145). Introducing the notation

$$[\![i\,\mathsf{H}^{\pm}]\!] = \tilde{K}_i^{\dot\alpha}\,\mathsf{H}^{\pm}{}_{\dot\alpha}(\lambda)\,, \qquad \mathsf{H}^{+}{}_{\dot\alpha}(\lambda) = \mathsf{H}^{\dot\beta}{}_{\dot\alpha}(\lambda)\,\tilde{o}_{\dot\beta}\,, \quad \mathsf{H}^{-}{}_{\dot\alpha}(\lambda) = \mathsf{H}^{\dot\beta}{}_{\dot\alpha}(\lambda)\,\tilde{\iota}_{\dot\beta}\,, \tag{5.148}$$

the coefficient of $\alpha^{+}\,(\alpha^{-})^{2}$ can be extracted from (5.147) to give

$$\begin{aligned} \left|\mathcal{H}_3^3[1]\right|\Big|_{\alpha^{+}(\alpha^{-})^{2}} &= \frac{\langle 1\,\lambda\rangle^{3}\,\langle 2\,\lambda\rangle^{3}}{\langle 3\,\lambda\rangle\,\langle 4\,\lambda\rangle\,\langle 5\,\lambda\rangle\,\langle 1\,3\rangle\,\langle 2\,3\rangle\,\langle 1\,4\rangle\,\langle 2\,4\rangle\,\langle 1\,5\rangle\,\langle 2\,5\rangle} \\ &\quad \times \left([\![3\,\mathsf{H}^{+}]\!]\,[\![4\,\mathsf{H}^{-}]\!]\,[\![5\,\mathsf{H}^{-}]\!] + [\![3\,\mathsf{H}^{-}]\!]\,[\![4\,\mathsf{H}^{+}]\!]\,[\![5\,\mathsf{H}^{-}]\!] + [\![3\,\mathsf{H}^{-}]\!]\,[\![4\,\mathsf{H}^{-}]\!]\,[\![5\,\mathsf{H}^{+}]\!]\right)\,. \end{aligned} \tag{5.149}$$

As expected, this corresponds precisely to the sum of tree diagrams in the twistor sigma model including three external graviton vertex operators and a single insertion of the background vertex operator U^{+}. Extracting the coefficient of $(\alpha^{+})^{2}\alpha^{-}$ from (5.147) gives a similar result with a single insertion of U^{-}.

Flat-space limit: of course, a basic consistency check on the amplitude (5.138) is that it should be equal to the MHV graviton scattering amplitude on flat space when the curvature of the background metric is turned off. Naively, one might assume that the flat limit of SDTN corresponds to $M \to 0$; indeed, the metric (5.4) certainly becomes flat in this limit. However, the resulting flat manifold has topology $S^1 \times \mathbb{R}^3$, rather than $\mathbb{R}^4$: this is because the Euclidean time coordinate is compactified (recall that $t \sim t + 8\pi M$) to a circle with radius $4M$, so in the $M \to 0$ limit this circle becomes infinitesimally small.

To compare with flat space scattering amplitudes, one requires a flat limit with trivial topology – that is, resulting in $\mathbb{R}^4$. This requires de-compactifying the Euclidean time coordinate, which (somewhat non-intuitively) actually corresponds to taking the $M \to \infty$ limit of SDTN. From the perspective of its twistor description, this is clearly the correct limit, as $\mathsf{h} \to 0$ when $M \to \infty$ and thus the complex structure reduces to that of the flat twistor space $\mathbb{PT}$. To see that this is indeed the correct limit from the metric perspective, consider the rescaling

$$t \to M\,t\,, \qquad r \to \frac{r}{M}\,, \tag{5.150}$$

under which the SDTN metric goes to

$$\mathrm{d}s^2 = \left(\frac{1}{M^2}+\frac{2}{r}\right)^{-1}(\mathrm{d}t-2(1-\cos\theta)\mathrm{d}\phi)^2+\left(\frac{1}{M^2}+\frac{2}{r}\right)\left(\mathrm{d}r^2+r^2\,\mathrm{d}\Omega_2^2\right)\,. \tag{5.151}$$

In these rescaled coordinates, it follows that

$$\lim_{M\to\infty}\mathrm{d}s^2 = \frac{r}{2}\,(\mathrm{d}t-2(1-\cos\theta)\mathrm{d}\phi)^2+\frac{2}{r}\left(\mathrm{d}r^2+r^2\,\mathrm{d}\Omega_2^2\right)\,, \tag{5.152}$$

which is precisely the flat hyperkähler metric on $\mathbb{R}^4$ in Gibbons-Hawking coordinates [102, 190]. This highlights the fact that although M resembles the ADM mass parameter of a black hole metric, at the self-dual point it is also conflated with the topological NUT charge, making the flat space limit somewhat non-intuitive.

This flat limit can now be implemented at the level of the MHV amplitude (5.138). Under the scaling (5.150), it follows that

$$\omega \to \frac{\omega}{M}\,, \qquad \vec{k} \to \frac{\vec{k}}{M}\,, \tag{5.153}$$

to ensure that the quantity $k\cdot x$ remains finite. Under this scaling, the important scalings of the MHV amplitude are captured by the collection:

$$\begin{aligned}&\delta(\omega)\,\mathrm{d}^3\vec{x}\,\left(1+\frac{2M}{r}\right)\prod_{j=1}^{n}\left(\frac{r}{1+\zeta\,\bar{\zeta}}\right)^{q_j}(\zeta-z_j)^{q_j-2M\omega_j}\,(\bar{\zeta}\,z_j+1)^{q_j+2M\omega_j}\\&\quad\to\delta(\omega)\,\mathrm{d}^3\vec{x}\,\left(\frac{1}{M^2}+\frac{2}{r}\right)\prod_{j=1}^{n}M^{-q_j}\,\left(\frac{r}{1+\zeta\,\bar{\zeta}}\right)^{q_j}(\zeta-z_j)^{q_j-2\omega_j}\,(\bar{\zeta}\,z_j+1)^{q_j+2\omega_j}\,.\end{aligned} \tag{5.154}$$

Now, in the $M \to \infty$ limit the topology of the metric becomes trivial, so all topological charges vanish, meaning that (after the rescaling) $q_j \pm 2\omega_j \to 0$ and $\sum_j q_j \to \pm 2 \sum_j \omega_j = 0$ in the flat limit.

Thus, one finds that

$$\lim_{M\to\infty} \mathcal{M}_n = 4\pi\, \kappa^{n-2}\, \delta(\omega)\, \frac{\langle 1\, 2\rangle^6}{\langle 1\, i\rangle^2\, \langle 2\, i\rangle^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{x}}{r}\, \mathrm{e}^{\mathrm{i}\,\vec{k}\cdot\vec{x}}\, |\mathbb{H}^i_i|\,, \tag{5.155}$$

with all $t > 0$ terms suppressed as powers of $1/M$. After performing the diffeomorphism from the flat Gibbons-Hawking coordinates to standard spherical polar coordinates on $\mathbb{R}^3$, the remaining integrals can be performed trivially to give

$$\lim_{M\to\infty} \mathcal{M}_n = (2\pi)^4\, \kappa^{n-2}\, \delta(\omega)\, \delta^3(\vec{k})\, \frac{\langle 1\, 2\rangle^6}{\langle 1\, i\rangle^2\, \langle 2\, i\rangle^2}\, |\mathbb{H}^i_i|\,, \tag{5.156}$$

which is precisely Hodges' formula for n-graviton MHV scattering in Minkowski spacetime [39]. One might worry that we have seemingly obtained a formula for scattering in Lorentzian Minkowski space from scattering on Euclidean $\mathbb{R}^4$. In flat space there is, in fact, no distinction, as the amplitude is a rational function whose analytic continuation to the complex momenta of the Euclidean setting is trivial.

Chapter 6

Deformations of celestial chiral algebras

In this final Chapter, we apply the results of the previous Chapter to study deformations of celestial symmetry algebras induced by curved backgrounds. We begin by studying the holomorphic collinear limits of the gluon MHV amplitude (4.51) and of the graviton MHV amplitude (5.138). We find that the holomorphic splitting functions coincide with the holomorphic splitting functions around the vacuum; these are related, via a Mellin transform, to the celestial OPE in the dual celestial CFT, so the celestial OPE is similarly undeformed by the presence of the self-dual dyon or on the self-dual Taub-NUT space-time. We then briefly review how to introduce a cosmological constant on twistor space and give a twistorial derivation of the celestial chiral algebra on AdS_4, which was recently constructed by Taylor and Zhu [123]. We conclude by discussing a further deformation of the chiral algebra on the Pedersen metric, that is a self-dual metric in the AdS-Taub-NUT family.

6.1 Symmetries in celestial holography

The basic correspondence in the celestial holographic dictionary[1] is between scattering amplitudes in the bulk and correlators of the CCFT: to any massless state with helicity h and with momentum $k^{\alpha\dot{\alpha}}$ decomposed as in (2.6), we can associate an operator $\mathcal{O}_{\Delta,h}(z,\bar{z})$ on the celestial sphere [202, 203]. Scattering amplitudes in the bulk can then be computed as the Mellin transform of correlators of the corresponding dual operators in the CCFT. This is equivalent to computing the $\mathcal{S}$-matrix elements in a basis of eigenstates of the Lorentz group, rather than the standard momentum eigenstates that diagonalize the translation operators [204–206]. It is then fairly natural to connect the holomorphic collinear limits of the scattering amplitudes and the *celestial OPE* of the dual CFT [114, 115]. This correspondence has been made precise for tree-level scattering amplitudes and unveiled a rich symmetry structure: there are infinite towers of symmetries associated to soft positive-helicity gluons and graviton operators [115–117, 207], corresponding to the residuals of the operators $\mathcal{O}_{\Delta,+1}$ for $\Delta = 1, 0, -1, \ldots$ and of the operators $\mathcal{O}_{\Delta,+2}$ for $\Delta = 2, 1, 0, -1, \ldots$, respectively. These soft operators act as currents on the celestial sphere and their modes can be organized into holographic symmetry algebras. Introducing the modes $\phi^a_{p,m,r}$, where a is a colour index, $p \pm m - 1 \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z} + p$, the symmetry algebra arising from the gluon celestial OPE is the S-algebra

$$[\phi^a_{p,m,r}, \phi^b_{q,n,s}] = \mathrm{i} f^{ab}{}_c \phi^c_{p+q-1,m+n,r+s} \,. \tag{6.1}$$

Geometrically, the S-algebra can be understood as the loop algebra of the Lie algebra of holomorphic maps $\mathbb{C}^2 \to \mathfrak{g}$, with r being the loop parameter. Similarly, introducing the modes $w^p_{m,a}$ for the soft graviton operators, subject to $p \pm m - 1 \in \mathbb{Z}_{\geq 0}$ and

[1] See [111–113] for more comprehensive reviews on celestial holography.

$a \in \mathbb{Z} + p$, the gravitational symmetry algebra is the $L\mathfrak{ham}(\mathbb{C}^2)$ algebra

$$[w^p_{m,a}, w^q_{n,b}] = (m(q-1) - n(p-1))w^{p+q-2}_{m+n,a+b}\,. \tag{6.2}$$

This is the loop algebra of Poisson diffeomorphisms of the plane.

On twistor space, the celestial chiral algebras arise as gauge symmetries and diffeomorphism symmetries of the Ward's bundle and non-linear graviton, respectively [119]. In particular, the modes can be identified with

$$\phi^a_{p,m,n} = \frac{(\mu^{\dot{0}})^{p+m-1}(\mu^{\dot{1}})^{p-m-1}}{2\lambda_0^{p-n-1}\lambda_1^{p+n-1}}\mathsf{T}^a\,, \tag{6.3}$$

and

$$w^p_{m,a} = \frac{(\mu^{\dot{0}})^{p+m-1}(\mu^{\dot{1}})^{p-m-1}}{2\lambda_0^{p-a-2}\lambda_1^{p+a-2}}\,. \tag{6.4}$$

These generators form bases of gauge connections $\mathsf{a} \in H^{0,1}(\mathbb{PT}, \mathcal{O} \otimes \mathfrak{g})$ and Hamiltonians $\mathsf{h} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ that are regular near $\mu^{\dot{\alpha}} = 0$, but can be singular in the λ_α directions, which are interpreted as (homogeneous) loop parameters. More precisely, (6.3)-(6.4) represent Čech cohomology classes, and are related to the usual twistor representatives by the Čech-Dolbeault isomorphism. The S-algebra (6.1) is recovered simply by taking the Lie algebra bracket between two generators from (6.3). The $L\mathfrak{ham}(\mathbb{C}^2)$ algebra is instead constructed as the Poisson bracket (2.38) of two generators of (6.4).

6.2 Holomorphic collinear limits on the self-dual dyon and self-dual Taub-NUT

In this Section, we show that the celestial chiral algebras are undeformed by the presence of the self-dual dyon and around the self-dual Taub-NUT metric. In the gauge theory case, this seems to be due primarily to the fact that the background is Cartan-valued [132]. We show this by explicitly computing the splitting functions arising from the amplitudes (4.51) and (5.138) and by showing that they coincide with their counterparts around the trivial backgrounds – see [126] for a similar analysis of amplitudes around radiative backgrounds.

6.2.1 Gluon holomorphic collinear limit

To calculate the holomorphic collinear limits around the self-dual dyon, it is useful to express the holomorphic frames in terms of spinor variables and to not integrate over the time direction, so that the appropriate presentation of the MHV amplitude is

$$\mathcal{A}_n = \mathrm{g}^{n-2}\,\frac{\langle r\,s\rangle^4}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}\,\mathrm{tr}\,(\mathsf{T}^{e_1}\ldots\mathsf{T}^{e_n}) \\ \times\int \mathrm{d}^4x\,\prod_{a=1}^{n}\mathrm{e}^{\mathrm{i}k_a\cdot x}\,\frac{\langle a\,\chi_+\rangle^{q_a+e_a}\,\langle a\,\chi_-\rangle^{q_a-e_a}}{\langle a\,o\rangle^{2q_a}} + \text{perms.}\,, \tag{6.5}$$

where +perms denotes the sum over all non-cyclic permutations of the external gluons. In particular, this is the *full*, non-colour-ordered amplitude. Now, consider the case where the i^{th} and j^{th} gluons become holomorphically collinear; this limit is implemented by writing

$$k_i + k_j = P + \varepsilon^2\,y\,, \tag{6.6}$$

where $P^{\alpha\dot{\alpha}} = \kappa_P^{\alpha}\tilde{\kappa}_P^{\dot{\alpha}}$ is the on-shell collinear momentum, $y^{\alpha\dot{\alpha}} = \nu^{\alpha}\,\tilde{\nu}^{\dot{\alpha}}$ is a reference null vector, and the holomorphic limit corresponds $\langle i\,j\rangle \sim \varepsilon \to 0$ whilst $[i\,j]$ is held fixed. This means that we take

$$\kappa_i^{\alpha} = \frac{\langle \nu\,i\rangle}{\langle \nu\,P\rangle}\,\kappa_P^{\alpha} + O(\varepsilon)\,, \qquad \kappa_j^{\alpha} = \frac{\langle \nu\,j\rangle}{\langle \nu\,P\rangle}\,\kappa_P^{\alpha} + O(\varepsilon)\,, \tag{6.7}$$

and the holomorphic splitting functions in the SDD background are extracted as the coefficients of the $\langle i\,j\rangle^{-1}$ singularity in the MHV amplitude. The calculation entails considering two cases: where the collinear gluons have the same or opposite helicities. We now consider each case in turn.

Same helicity: the only colour orderings that can give a singular contribution in the holomorphic collinear limit are those where the i^{th} and j^{th} gluons are adjacent, so that

$$\begin{aligned}\mathcal{A}_n = \mathrm{g}^{n-2}\Bigg[&\frac{\langle r\, s\rangle^4}{\langle 1\, 2\rangle\cdots\langle i-1\, i\rangle\langle i\, j\rangle\langle j\, j+1\rangle\cdots\langle n\, 1\rangle}\,\mathrm{tr}\,(\mathsf{T}^{e_1}\cdots\mathsf{T}^{e_i}\mathsf{T}^{e_j}\cdots\mathsf{T}^{e_n})+\\ &+\frac{\langle r\, s\rangle^4}{\langle 1\, 2\rangle\cdots\langle i-1\, j\rangle\langle j\, i\rangle\langle i\, j+1\rangle\cdots\langle n\, 1\rangle}\,\mathrm{tr}\,(\mathsf{T}^{e_1}\cdots\mathsf{T}^{e_i}\mathsf{T}^{e_j}\cdots\mathsf{T}^{e_n})\Bigg]\\ &\times\int \mathrm{d}^4x\prod_{a=1}^{n}\mathrm{e}^{\mathrm{i}k_a\cdot x}\,\frac{\langle a\,\chi_+\rangle^{q_a+e_a}\,\langle a\,\chi_-\rangle^{q_a-e_a}}{\langle a\, o\rangle^{2q_a}}+O(\varepsilon)\,. \end{aligned} \tag{6.8}$$

The leading order singularity is then simply

$$\begin{aligned}\mathcal{A}_n = &\frac{\mathrm{g}\,\langle \nu\, P\rangle^2}{\langle i\, j\rangle\,\langle \nu\, i\rangle\,\langle \nu\, j\rangle}\,\frac{\mathrm{g}^{n-3}\,\langle r\, s\rangle^4}{\langle 1\, 2\rangle\cdots\langle i-1\, P\rangle\langle P\, j+1\rangle\cdots\langle n\, 1\rangle}\,\mathrm{tr}\,(\mathsf{T}^{e_1}\cdots[\mathsf{T}^{e_i},\mathsf{T}^{e_j}]\cdots\mathsf{T}^{e_n})\\ &\times\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}P\cdot x}\,\frac{\langle P\,\chi_+\rangle^{q_P+e_P}\,\langle P\,\chi_-\rangle^{q_P-e_P}}{\langle P\, o\rangle^{2q_P}}\prod_{\substack{a=1\\ a\neq i,j}}^{n}\mathrm{e}^{\mathrm{i}k_a\cdot x}\,\frac{\langle a\,\chi_+\rangle^{q_a+e_a}\,\langle a\,\chi_-\rangle^{q_a-e_a}}{\langle a\, o\rangle^{2q_a}}\,, \end{aligned} \tag{6.9}$$

where $e_P = e_i + e_j$ and $q_P = q_i + q_j$. Note that $[\mathsf{T}^{e_i},\mathsf{T}^{e_j}]$ has precisely charge e_P, as implied by the Jacobi identity. The holomorphic collinear limit is thus

$$\begin{aligned}\mathcal{A}_n(1^+,\dots,i^+,j^+,\dots,n^+)\to\ &\mathrm{Split}(i^{+,e_i,q_i},j^{+,e_j,q_j}\to P^{+,e_P,q_P})\times\\ &\times\mathcal{A}_{n-1}(1^+,\dots,P^+,\dots,n^+)\,,\end{aligned} \tag{6.10}$$

with splitting function

$$\mathrm{Split}(i^{+,e_i,q_i},j^{+,e_j,q_j}\to P^{+,e_p,q_p}) = \frac{\mathrm{g}\,\langle \nu\, P\rangle^2}{\langle i\, j\rangle\langle \nu\, i\rangle\,\langle \nu\, j\rangle}\,\delta_{q_P,q_i+q_j}\,\delta_{e_P,e_i+e_j}\,. \tag{6.11}$$

This is the holomorphic splitting function that arises for the same-helicity collinear limit in a trivial background [208–210], so we see that the SDD leaves the splitting function invariant. After a Mellin transform, this implies that the associated celestial OPE coefficient and chiral algebra are similarly un-altered.

Opposite helicity: in the case where the two collinear gluons have opposite helicity, the computation proceeds along the same lines as before. Suppose we take the collinear limit between the r^{th} and i^{th} gluon, so that the leading singularity in the n-point amplitude is

$$\begin{aligned}\mathcal{A}_n = &\frac{\mathrm{g}\,\langle\nu\,r\rangle^3}{\langle r\,i\rangle\,\langle\nu\,i\rangle\,\langle\nu\,P\rangle^2}\,\frac{\mathrm{g}^{n-3}\,\langle P\,s\rangle^4}{\langle 1\,2\rangle\cdots\langle r-1\,P\rangle\,\langle P\,i+1\rangle\ldots\langle n\,1\rangle}\,\mathrm{tr}\left(\mathsf{T}^{e_1}\cdots[\mathsf{T}^{e_r},\mathsf{T}^{e_i}]\cdots\mathsf{T}^{e_n}\right)\\ &\times\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}P\cdot x}\,\frac{\langle P\,\chi_+\rangle^{q_P+e_P}\,\langle P\,\chi_-\rangle^{q_P-e_P}}{\langle P\,o\rangle^{2q_P}}\prod_{\substack{a=1\\ a\neq i,r}}^{n}\mathrm{e}^{\mathrm{i}k_a\cdot x}\,\frac{\langle a\,\chi_+\rangle^{q_a+e_a}\,\langle a\,\chi_-\rangle^{q_a-e_a}}{\langle a\,o\rangle^{2q_a}}\,,\end{aligned} \tag{6.12}$$

so that the splitting function is now

$$\mathrm{Split}(i^{+,e_i,q_i},r^{-,e_r,q_r}\to P^{-,e_P,q_P}) = \frac{\mathrm{g}\,\langle\nu\,r\rangle^3}{\langle r\,i\rangle\langle\nu\,i\rangle\,\langle\nu\,P\rangle^2}\,\delta_{q_P,q_r+q_i}\,\delta_{e_P,e_r+e_i}\,. \tag{6.13}$$

Again, the splitting function is un-deformed by the presence of the SDD background [208–210].

6.2.2 Graviton holomorphic collinear limit

A similar computation can be done starting from the graviton MHV amplitude (5.138). There are however some new features due to the non-trivial topology of the SDTN metric: introduce the longitudinal momentum fraction $t\in[0,1]$ as

$$\omega_i = t\,\omega_P\,,\qquad \omega_j = (1-t)\,\omega_P\,, \tag{6.14}$$

for ω_P the frequence of the collinear momentum $P^{\alpha\dot{\alpha}}$. In order to preserve the topological quantization condition $q_P \pm 2M\omega_P$, it follows that the topological charges must follow the same parametrization

$$q_i = t\, q_P\,, \qquad q_j = (1-t)\, q_P\,, \tag{6.15}$$

in the collinear limit where $q_i + q_j = q_P$.

Let us focus on the holomorphic collinear limit between two positive-helicity gravitons. The central ingredient in each term of (5.138) is the once-reduced determinant of the matrix $\mathcal{H}[\mathbf{t}]$, with the row and column removed to create the minor being arbitrarily chosen. As such, we can freely choose to remove the row and column corresponding to one of the two holomorphically collinear gravitons, say j. By expanding the resulting minor along the i^{th} row and exploiting the properties of the holomorphic collinear limit, it follows that

$$\left|\mathcal{H}^j_j[\mathbf{t}]\right| = \frac{[\![i\,j]\!]}{\langle i\,j\rangle}\,\left|\mathcal{H}^{ij}_{ij}[\mathbf{t}]\right| + O(\varepsilon^0)\,, \tag{6.16}$$

exposes the leading holomorphic collinear singularity of the amplitude; all other ingredients of $\mathcal{M}_n$ are regular as $\langle i\,j\rangle \to 0$.

Now, recall that

$$[\![i\,j]\!] = \tilde{\kappa}_{i\,\dot{\beta}}\,\tilde{\kappa}_{j\,\dot{\gamma}}\,G^{\dot{\beta}\dot{\alpha}}(x;k_i,q_i)\,G^{\dot{\gamma}}{}_{\dot{\alpha}}(x;k_j,q_j)\,, \tag{6.17}$$

for the dressing matrix $G^{\dot{\alpha}}{}_{\dot{\beta}}$ given by (5.60). The dressing matrix is a homogeneous function of the undotted momentum spinor, so in the holomorphic collinear limit, it follows that

$$G^{\dot{\beta}\dot{\alpha}}(x;k_i,q_i) = G^{\dot{\beta}\dot{\alpha}}(x;P,q_P) + O(\varepsilon)\,, \tag{6.18}$$

and hence that

$$\left|\mathcal{H}^j_j[\mathbf{t}]\right| = \frac{[i\,j]}{\langle i\,j\rangle}\,\left|\mathcal{H}^{ij}_{ij}[\mathbf{t}]\right| + O(\varepsilon^0)\,, \tag{6.19}$$

in the holomorphic collinear limit.

In the minor $|\mathcal{H}^{ij}_{ij}[\mathbf{t}]|$, the only remaining dependence on the collinear momenta is through the diagonal entries of the matrix $\mathcal{H}$, since the rows and columns corresponding to the collinear momenta have been removed. This dependence is controlled in the diagonal entries $\mathbb{H}_{kk}[\mathbf{t}]$ through linear combinations of the form

$$\begin{aligned}\frac{[\![k\,i]\!]\,\langle 1\,i\rangle\,\langle 2\,i\rangle}{\langle k\,i\rangle\,\langle 1\,k\rangle\,\langle 2\,k\rangle} &+ \frac{[\![k\,j]\!]\,\langle 1\,j\rangle\,\langle 2\,j\rangle}{\langle k\,j\rangle\,\langle 1\,k\rangle\,\langle 2\,k\rangle} \\ &= \frac{\langle 1\,P\rangle\,\langle 2\,P\rangle}{\langle k\,P\rangle\,\langle 1\,k\rangle\,\langle 2\,k\rangle\,\langle \nu\,P\rangle}\left([\![k\,i]\!]\,\langle \nu\,i\rangle + [\![k\,j]\!]\,\langle \nu\,j\rangle\right) + O(\varepsilon)\,, \end{aligned} \tag{6.20}$$

with a similar formula controlling the dependence of the diagonal entries of the block $\mathbb{T}[\mathbf{t}]$.

In order to further simplify such expressions, one must account for the fact that the dressed momentum spinors $\tilde{K}^{\dot\alpha}_{i,j}$ depend on both the frequencies and topological charges of the dressed momenta. In particular,

$$\begin{aligned}[\![k\,i]\!]\,\langle \nu\,i\rangle+[\![k\,j]\!]\,\langle \nu\,j\rangle = \frac{\tilde{K}^{\dot\alpha}_k}{\sqrt{V}}\Bigg[&\langle \nu\,i\rangle\,\tilde\kappa_{i\,\dot\alpha} + \langle \nu\,j\rangle\,\tilde\kappa_{j\,\dot\alpha} + (q_i + 2M\omega_i)\,\frac{\langle P|T|_{\dot\alpha}\,\langle \chi_+|T|i]\,\langle \nu\,i\rangle}{\omega_i\,r\,\langle P\,\chi_+\rangle} \\ + (q_j + 2M\omega_j)\,\frac{\langle P|T|_{\dot\alpha}\,\langle \chi_+|T|j]\,\langle \nu\,j\rangle}{\omega_j\,r\,\langle P\,\chi_+\rangle} &- (q_i - 2M\omega_i)\,\frac{\langle P|T|_{\dot\alpha}\,\langle \chi_-|T|i]\,\langle \nu\,i\rangle}{\omega_i\,r\,\langle P\,\chi_-\rangle} \\ &-(q_j - 2M\omega_j)\,\frac{\langle P|T|_{\dot\alpha}\,\langle \chi_-|T|j]\,\langle \nu\,j\rangle}{\omega_j\,r\,\langle P\,\chi_-\rangle}\Bigg] + O(\varepsilon)\,, \end{aligned} \tag{6.21}$$

having abbreviated $\langle P|T|_{\dot\alpha} = \kappa^\alpha_P\,T_{\alpha\dot\alpha}$ and $\langle \chi_\pm|T|i] = \chi^\alpha_\pm\,T_\alpha{}^{\dot\alpha}\,\tilde\kappa_{i\,\dot\alpha}$, etc. Using the collinear parametrizations (6.14), (6.15), this simplifies to

$$\frac{\tilde{K}_k^{\dot\alpha}}{\sqrt{V}}\Bigg[\langle\nu\, i\rangle\,\tilde\kappa_{i\,\dot\alpha}+\langle\nu\, j\rangle\,\tilde\kappa_{j\,\dot\alpha}+(q_P+2M\omega_P)\,\frac{\langle P|T|_{\dot\alpha}\left(\langle\chi_+|T|i]\,\langle\nu\, i\rangle+\langle\chi_+|T|j]\,\langle\nu\, j\rangle\right)}{\omega_P\, r\,\langle P\,\chi_+\rangle}$$
$$-(q_P-2M\omega_P)\,\frac{\langle P|T|_{\dot\alpha}\left(\langle\chi_-|T|i]\,\langle\nu\, i\rangle+\langle\chi_-|T|j]\,\langle\nu\, j\rangle\right)}{\omega_P\, r\,\langle P\,\chi_-\rangle}\Bigg]+O(\varepsilon)\,. \quad (6.22)$$

Observing that in the collinear limit (6.6) – (6.7),

$$\langle\nu\, i\rangle\,\tilde\kappa_i^{\dot\alpha}+\langle\nu\, j\rangle\,\tilde\kappa_j^{\dot\alpha}=\langle\nu\, P\rangle\,\tilde\kappa_P^{\dot\alpha}\,, \quad (6.23)$$

it then follows that

$$[\![k\, i]\!]\,\langle\nu\, i\rangle+[\![k\, j]\!]\,\langle\nu\, j\rangle=[\![k\, P]\!]\,\langle\nu\, P\rangle+O(\varepsilon)\,, \quad (6.24)$$

so that (6.20) simplifies to

$$\frac{[\![k\, i]\!]\,\langle 1\, i\rangle\,\langle 2\, i\rangle}{\langle k\, i\rangle\,\langle 1\, k\rangle\,\langle 2\, k\rangle}+\frac{[\![k\, j]\!]\,\langle 1\, j\rangle\,\langle 2\, j\rangle}{\langle k\, j\rangle\,\langle 1\, k\rangle\,\langle 2\, k\rangle}=\frac{[\![k\, P]\!]\,\langle 1\, P\rangle\,\langle 2\, P\rangle}{\langle k\, P\rangle\,\langle 1\, k\rangle\,\langle 2\, k\rangle}+O(\varepsilon)\,, \quad (6.25)$$

to leading order in the holomorphic collinear limit. A similar simplification occurs in the diagonal entries of $\mathbb{T}[\mathbf{t}]$.

Consequently, in the holomorphic collinear limit it follows that the two collinear gravitons are effectively replaced by the single collinear graviton:

$$\left|\mathcal{H}_j^j[\mathbf{t}]\right|=\frac{[i\, j]}{\langle i\, j\rangle}\,\left|\widehat{\mathcal{H}}_P^P[\mathbf{t}]\right|+O(\varepsilon^0)\,, \quad (6.26)$$

where $\widehat{\mathcal{H}}[\mathbf{t}]$ is the matrix defined for the set of external positive helicity gravitons in which i, j have been removed and replaced by P with its associated collinear quantum numbers. The other ingredients in the n-point MHV amplitude (5.138) can likewise be evaluated in the strict collinear limit, in essentially the same way as we treated the gluon amplitude around the self-dual dyon. The only possible subtlety lies in the range of sum over tails in the amplitude (5.138). However, it can be seen that the $t = n - 4$ terms in the MHV amplitude cannot contain the holomorphic collinear singularity (6.26), using an identical argument to the self-dual radiative case [126]. Thus, only terms with t running from zero to $n-5$ actually contribute to the leading holomorphic collinear limit.

Pulling all of these pieces together, one obtains that

$$\mathcal{M}_n \to \text{Split}(i^{+,q_i}, j^{+,q_j} \to P^{+,q_P})\mathcal{M}_{n-1}\,, \tag{6.27}$$

where the holomorphic collinear splitting function is

$$\text{Split}(i^{+,q_i},\, j^{+,q_j} \to P^{+,q_P}) = \kappa\,\frac{[i\,j]}{\langle i\,j\rangle}\,\frac{\langle \nu\,P\rangle^4}{\langle \nu\,i\rangle^2\,\langle \nu\,j\rangle^2}\delta_{q_i+q_j,q_P}\,. \tag{6.28}$$

Again, this holomorphic collinear splitting function matches the tree-level holomorphic collinear splitting function of positive helicity gravitons in flat space [211–213]. It is far from obvious that this should be the case from background field theory in the SDTN metric, but the twistor theory of SDTN in fact hints that this should be the case. In [126], it was shown that holomorphic collinear splitting functions on any self-dual radiative metric are un-deformed from flat space. While the interpretation of SDTN as a self-dual black hole makes it apparently non-radiative, the complex structure defined by the Hamiltonian (5.24) takes the same functional form as that of a self-dual radiative metric.

6.3 Twistors with a cosmological constant

From the results of the previous Section, it appears that both the gauge and gravitational sources are not able to deform the celestial chiral algebras. Interestingly, there's a simple ingredient to add to our construction that results in non-trivial deformations: the cosmological constant. Taylor and Zhu [123] derived a Λ-deformed version of the $L\mathfrak{ham}(\mathbb{C}^2)$ algebra starting from the AdS graviton amplitudes constructed in [214–217]. Their derivation is however a bit artificial, as they have to introduce additional terms in the graviton OPE in order to satisfy the Jacobi identity. Moreover, their approach is perturbative in the cosmological constant and they only derive the first-order correction in Λ to the chiral algebra. Instead, we start from the twistorial description of self-dual Einstein manifolds – i.e., of self-dual manifolds satisfying Einstein's equations with a cosmological constant – and show how to derive the deformed $L\mathfrak{ham}(\mathbb{C}^2)$ from a suitably modified Poisson bracket on twistor space. By construction, our result satisfies the Jacobi identity and is exact at all orders in Λ. We begin with an elementary review of the necessary twistorial ingredients.

As mentioned in Chapter 2, $\mathbb{CP}^3$ naturally carries an action of the conformal group of $\mathbb{M}$, so the construction of flat twistor space $\mathbb{PT}$ is conformally invariant. A conformal scale is encoded by a choice of skew bitwistor I_Λ^{AB} known as the *infinity twistor*. As our notation emphasises, this bitwistor depends on the cosmological constant Λ, and is normalized to obey

$$I_\Lambda^{AB} I^\Lambda_{CB} = \Lambda\, \delta^A_C\,, \tag{6.29}$$

where $I^\Lambda_{AB} = \frac{1}{2}\epsilon_{ABCD} I_\Lambda^{CD}$ is the dual of I_Λ^{AB}. A common choice for the infinity twistor is

$$I_\Lambda^{AB} = \begin{pmatrix} \epsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & \Lambda\, \epsilon_{\alpha\beta} \end{pmatrix}, \qquad I^\Lambda_{AB} = \begin{pmatrix} \Lambda\, \epsilon_{\dot\alpha\dot\beta} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix} \tag{6.30}$$

The infinity twistor defines an $\mathcal{O}(-2)$-valued deformed Poisson bracket as

$$\{\omega_1, \omega_2\}_\Lambda = I_\Lambda^{AB} \mathcal{L}_A \omega_1 \wedge \mathcal{L}_B \omega_2\,, \tag{6.31}$$

where $\mathcal{L}_A$ is the Lie derivative along $\partial/\partial Z^A$ and ω_1, ω_2 are arbitrary forms. Given an $\mathcal{O}(2)$-valued Hamiltonian h, we can construct the deformed complex structure $\bar{\nabla}_\Lambda = \bar{\partial} + \{\mathsf{h}\,,\cdot\}_\Lambda$ as in the $\Lambda = 0$ case; the integrability condition is formally identical to (2.39), after the replacement of $\{\cdot\,,\cdot\}$ with $\{\cdot\,,\cdot\}_\Lambda$. Let $\mathbb{P}\mathscr{T}$ be the deformed twistor space, as usual. A basis of $(1,0)$-forms in the deformed complex structure is given by

$$e^A = \mathrm{d}Z^A + I_\Lambda^{AB}\mathcal{L}_B\mathsf{h}\,. \tag{6.32}$$

We can define the holomorphic contact structure $\tau \in \Omega^{1,0}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$ and its exterior derivative $\Sigma \in \Omega^{1,0}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$ as

$$\tau = -I_{AB}Z^A e^B\,, \qquad \Sigma = \mathrm{d}\tau = -I_{AB}e^A \wedge e^B\,, \tag{6.33}$$

where the second equality in the expression for the 2-form follows from the integrability condition for $\bar{\nabla}_\Lambda$. Using these ingredients, Theorem 2 was extended by Ward as follows[2]

Theorem 4 (Ward's non-linear graviton [220])**.** *There is a 1-to-1 correspondence between*

- *(suitably convex regions of) complex, self-dual Einstein 4-manifolds* (M, g)
- *complex 3-folds* $\mathbb{P}\mathscr{T}$ *that are a complex deformation of the neighbourhood of a line in* $\mathbb{PT}$ *and preserve the contact 1-form* τ_Λ *and the volume 3-form* $\tau \wedge \Sigma$.

Here by Einstein manifold we mean a manifold (M, g) whose metric g satisfies Einstein's equations with a cosmological constant. The normalization that is commonly used in the twistor literature is $R = 24\Lambda$ for the scalar curvature.

In the simple case of AdS_4, the complex structure is the trivial one, $\mathsf{h} = 0$, and the rank-4 infinity twistor is the only new element. As in the case of the non-

[2]See also [33, 218, 219] for extensions and variations of this construction.

linear graviton for $\Lambda = 0$, space-time arises as the moduli space of holomorphic rational curves sitting inside $\mathbb{PT}$. To recover the metric, consider the projection $p\colon \mathrm{AdS}_4 \times \mathbb{CP}^1 \to \mathbb{PT}$ given by the modified incidence relations

$$F^A(x,\sigma) = \Omega^{-1/2}(x^{\beta\dot{\alpha}}\sigma_\beta, \sigma_\alpha)\,, \qquad \Omega(x) = 1 + \frac{1}{2}\Lambda x^2\,. \tag{6.34}$$

The space-time function Ω allows to normalize the incidence relations (6.34), so that they satisfy $I^\Lambda_{AB}F^A\partial_0 F^B = -1$. With this normalization, the contact and 2-forms on twistor space pull back to the forms

$$p^*\tau = \langle\sigma\,\mathrm{d}\sigma\rangle + \Gamma^{\alpha\beta}\sigma_\alpha\sigma_\beta\,, \qquad p^*\tau\wedge p^*\Sigma = \Lambda p^*\tau\wedge\Sigma^{\alpha\beta}\sigma_\alpha\sigma_\beta\,, \tag{6.35}$$

on the projectivised undotted spinor bundle, where

$$\Gamma^{\alpha\beta} = \Omega^{-1}x^{(\alpha|\dot{\alpha}|}\mathrm{d}x^{\beta)}{}_{\dot{\alpha}}\,, \qquad \Sigma^{\alpha\beta} = \Omega^{-2}\mathrm{d}x^{\alpha\dot{\alpha}}\wedge\mathrm{d}x^{\beta}{}_{\dot{\alpha}}\,. \tag{6.36}$$

$\Gamma^{\alpha\beta}$ and $\Sigma^{\alpha\beta}$ are precisely the ASD spin connection and ASD 2-forms for the ball model of (Euclidean) AdS_4, where the metric is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^{\alpha\dot{\alpha}}\mathrm{d}x_{\alpha\dot{\alpha}}}{(1+\frac{1}{2}\Lambda x^2)^2}\,, \tag{6.37}$$

and AdS_4 is the interior of the ball $x^2 < -2/\Lambda$. We can characterize the space-time boundary as the locus $\mathscr{I}_\Lambda$ where the holomorphic contact form is singular.

6.4 The celestial chiral algebra on AdS_4

The infinitesimal symmetries of AdS_4 are naturally defined by holomorphic functions of homogeneity degree 2 on twistor space, given as $h = \frac{1}{2}h_{AB}Z^AZ^B$ for a constant, symmetric h_{AB}. The associated Hamiltonian vector fields $X_h = \{h\,,\cdot\}_\Lambda$ generate the corresponding motions of twistor space. By construction, flows along such Hamiltonian vector fields preserves the Poisson bracket and its dual contact 1-form. In space-time, this motion induces the standard isometries of AdS_4, arising as the spin group $\mathrm{Sp}(2)$ of the more usual $\mathrm{SO}(3,2)$. These isometries provide the starting-point for the full celestial chiral algebra of AdS_4. Using the Λ-deformed Poisson bracket $\{\cdot,\,\cdot\}_\Lambda$ and the generators (6.4), the flat space celestial chiral algebra (6.2) is deformed to

$$\{w^p_{m,a}, w^q_{n,b}\}_\Lambda = (m(q-1)-n(p-1))w^{p+q-2}_{m+n,a+b} - \Lambda(a(q-2)-b(p-2))w^{p+q-1}_{m+n,a+b}\,, \quad (6.38)$$

by the presence of a cosmological constant. As above, for $2-p \leq a \leq p-2$ the $w^p_{m,a}$ are the quadratic Hamiltonians generating the standard AdS_4 isometries. We denote this algebra as $\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$.

The algebra (6.38) agrees with that found by Taylor and Zhu [123], who considered the holomorphic collinear limit of the Mellin transform of the leading order in Λ correction to the 4-point tree-level graviton amplitude in AdS_4. This Mellin amplitude was computed in [214–217]. Interestingly, [123] found that the true 4-point amplitude does *not* quite exhibit this algebra; the form of the $O(\Lambda)$ correction needs to be modified in order to ensure the Jacobi identity holds. For us, the fact that the algebra arises from a Poisson bracket immediately ensures that the Jacobi identity is satisfied. The twistor construction suggests that the algebra (6.38) is a true celestial symmetry of self-dual gravity on AdS_4, at least at the classical level. Note that $\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$ does *not* arise as the loop algebra of a deformation of $\mathfrak{ham}(\mathbb{C}^2)$ itself. Indeed, the unique Lie algebra deformation of the latter is the Weyl algebra corresponding to a non-commutative self-dual gravity [125].

6.4.1 Celestial chiral algebras as vertex algebras

Lie algebras of local symmetries on twistor space are closely associated with vertex algebras supported on twistor lines [119, 130]. We can understand the vertex algebra corresponding to (6.38) in the following way. Suppose we couple the twistor uplift of self-dual gravity on AdS_4 to a 2d holomorphic theory living on a twistor line. In general such a coupling will take the form

$$\sum_{p\pm m\in\mathbb{N}} \frac{2}{(p+m-1)!\,(p-m-1)!}\int_X \langle\lambda\,\mathrm{d}\lambda\rangle\wedge\, w^p_m(\lambda)\,\partial^{p+m-1}_{\mu^{\dot 0}}\partial^{p-m-1}_{\mu^{\dot 1}}H\ , \tag{6.39}$$

for 2d operators $w^p_m(\lambda)$ depending meromorphically on λ and labelled by p, m with the same ranges as above. Here $H\in\Omega^{0,1}(\mathbb{PT},\mathcal{O}(2))$ is a Dolbeault representative corresponding to the Čech cocycle $[h]$. In order for the integrand to have vanishing homogeneity the operators $w^p_m(\lambda)$ must take values in $\mathcal{O}(2p-6)$.

Our notation for the operators $w^p_m(\lambda)$ can be justified by choosing H corresponding to the basis (6.4). Explicitly, we fix $H = w^p_{m,a}\,\bar\partial B$ for B a bump function on $\mathbb{CP}^1_x$ taking the value 1 in a neighbourhood of $\lambda_0 = 0$, 0 in a neighbourhood of $\lambda_1 = 1$ and non-constant on an annulus disconnecting λ_0, λ_1. Introducing an affine coordinate z on X, in the limit of an arbitrarily narrow annulus we may take $B = \Theta(|z|^2 < 1)$ for Θ the Heaviside step function. Substituting into (6.39) gives

$$\oint \frac{\langle\lambda\,\mathrm{d}\lambda\rangle}{\lambda_0^{p-a-2}\lambda_1^{p+a-2}}\, w^p_m(\lambda) = \oint \mathrm{d}z\, z^{a+2-p} w^p_m(z)\,. \tag{6.40}$$

In this way the Hamiltonians $w^p_{m,a}$ are naturally identified with the modes of the operators $w^p_m(z)$.

Invariance of the coupling (6.39) under the local holomorphic diffeomorphism symmetry on twistor space necessitates the following OPE

$$w^p_m(z_1)w^q_n(z_2) \sim \frac{m(q-1)-n(p-1)}{z_{12}} w^{p+q-2}_{m+n}(z_2) \\ - \frac{\Lambda}{z_{12}^2}\big((p+q-4)w^{p+q-1}_{m+n}(z_2) + z_{12}(p-2)\partial_z w^{p+q-1}_{m+n}(z_2)\big) . \quad (6.41)$$

This is the (tree-level) celestial chiral algebra of SD gravity on AdS_4 represented as a vertex algebra. We remark that the field $T(z) = w^1_0(z)/\Lambda$ plays the role of a stress tensor, with OPEs

$$T(z_1)w^p_m(z_2) \sim \frac{1}{z_{12}^2}\big((3-p)w^p_m(z_2) + z_{12}\partial_z w^p_m(z_2)\big) . \quad (6.42)$$

The corresponding central charge vanishes. Furthermore, $T(z)$ is the field of conformal spin 2 in a vertex subalgebra generated by $w^p_0(z)/\Lambda$ for $p \in \mathbb{Z}_{\geq 1}$. This resembles the w_∞ vertex algebra which has the same defining operator products, but for labels taking values in the range $p \in \mathbb{Z}_{\leq 1}$. w_∞ is generated by fields of all integer conformal spins $s \geq 2$ rather than $s \leq 2$.

The fields $w^p_m(z)$ can be conveniently organised into hard generating functions depending on an auxiliary right-handed spinor $\tilde\lambda_{\dot\alpha}$, defined by

$$w(\tilde\lambda, z) = \sum_{p\pm m\in\mathbb{Z}_{\geq 1}} \frac{2\tilde\lambda_{\dot 0}^{p+m-1}\tilde\lambda_{\dot 1}^{p-m-1}}{(p+m-1)!(p-m-1)!} w^p_m(z) . \quad (6.43)$$

In terms of these hard generators the vertex algebra (6.41) reads

$$w(\tilde\lambda_1, z_1)w(\tilde\lambda_2, z_2) \sim \frac{[12]}{z_{12}} w(\tilde\lambda_1+\tilde\lambda_2, z_2) - \\ - \frac{\Lambda}{z_{12}^2}\Big(\big(\tilde\lambda_{\dot\alpha}\partial_{\tilde\lambda_{\dot\alpha}} - 4\big)w(\tilde\lambda, z_2) + z_{12}\big(\tilde\lambda_{1\dot\alpha}\partial_{\tilde\lambda_{\dot\alpha}} - 2\big)\partial_z w(\tilde\lambda, z_2)\Big)\Big|_{\tilde\lambda=\tilde\lambda_1+\tilde\lambda_2} . \quad (6.44)$$

We recognise the coefficient $[12]/z_{12}$ as the tree graviton splitting function on flat space [221]. On the other hand, the Λ dependent coefficients are not simply functions of the spinor-helicity variables, instead taking the form of differential operators. This results from the loss of supertranslation invariance on AdS_4.

6.4.2 Variations and extensions

The choice (6.30) for the infinity twisor is not unique. Introduce the constant vector

$$Z^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}(o^{\alpha}\tilde{o}^{\dot{\alpha}} - \iota^{\alpha}\tilde{\iota}^{\dot{\alpha}}) \,, \tag{6.45}$$

and define the coordinate $\mu^{\alpha\dot{\alpha}} = Z^{\alpha}{}_{\dot{\alpha}}\mu^{\dot{\alpha}}$ on $\mathbb{PT}$. It can be checked straightforwardly that the infinity twistor

$$I^{\Lambda}_{AB} = \sqrt{\Lambda}\begin{pmatrix} 0 & \epsilon^{\alpha\beta} \\ -\epsilon_{\alpha\beta} & 0 \end{pmatrix} \,, \tag{6.46}$$

gives rise via Theorem 4 to the AdS_4 metric in the Poincaré patch

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^{\alpha\dot{\alpha}}\mathrm{d}x_{\alpha\dot{\alpha}}}{\Lambda z^2} \,, \tag{6.47}$$

where $z = x^{\alpha\dot{\alpha}}Z_{\alpha\dot{\alpha}}$. The Poisson bracket is modified accordingly and gives the algebra

$$\begin{aligned}\{\hat{w}^p_{m,a}, \hat{w}^q_{n,b}\}^P_{\Lambda} = {}& ((p+m-1)(q-b-2)-(q+n-1)(p-a-2))\hat{w}^{p+q-3/2}_{m+n-1/2,a+b-1/2} \\ & + ((p-m-1)(q+b-2)-(q-n-1)(p+a-2))\hat{w}^{p+q-3/2}_{m+n+1/2,a+b+1/2} \,,\end{aligned} \tag{6.48}$$

where the generators of the algebra have been redefined to

$$\hat{w}^p_{m,a} \coloneqq \frac{2}{\sqrt{\Lambda}}w^p_{m,a}, \tag{6.49}$$

Although these again provide an extension of the AdS_4 symmetries, they are not suitable for expansion around $\Lambda = 0$. They are however well adapted to soft limits for momentum eigenstates based on translations of the Poincaré patch. We emphasise that the Lie algebra (6.48) is *not* isomorphic to (6.38). The difference is essentially the choice of the location of the line $\lambda_\alpha = 0$: up to AdS_4 isometries there are two such choices, the first where the line is in the complement of the unit ball, and the second here where the line corresponds to a point of $\mathscr{I}_\Lambda$. These two choices provide the two algebras (6.38) and (6.48), respectively. On the other hand, the choices of the sets U_0 and U_1 used to define our basis $\{w^p_{m,n}\}$ are more associated to the choice of cohomology representation. Indeed, these can be made canonically in split signature.

Both the Lie algebra adapted to the ball (6.38) and the Poincaré patch (6.48) can easily be extended to incorporate free fields propagating on the gravitational background using the Penrose transform. Representing such fields in the Čech language by holomorphic functions on $U_0 \cap U_1$, with homogeneity $2s-2$, regularity near $\mu^{\dot\alpha} = 0$ allows us to decompose a helicity-s field into modes

$$x^p_{m,a} = \frac{(\mu^{\dot 0})^{p+m-1}(\mu^{\dot 1})^{p-m-1}}{2\lambda_0^{p-a-s}\lambda_1^{p+a-s}} \tag{6.50}$$

where as above $p \pm m - 1 \in \mathbb{Z}_{\geq 0}$. For $s = 1$ and $s = 2$ we recover (6.3) and (6.4), respectively. The Hamiltonians $w^p_{m,a}$ naturally act on these modes via the Poisson bracket, furnishing us with modules for the Lie algebras (6.38) and (6.48). For example, in the case of the ball model this action reads

$$\{w^p_{m,a}, x^q_{n,b}\}_\Lambda = (m(q-1) - n(p-1))x^{p+q-2}_{m+n,a+b} - \Lambda(a(q-s) - b(p-2))x^{p+q-1}_{m+n,a+b}\,. \tag{6.51}$$

Extending by these modules gives symmetry algebras for the coupled systems. As we can see in equation (6.51), for $\Lambda \neq 0$ the structure constants of such extended algebras depend explicitly on s. This reflects the fact that for general fluctuations

h the resulting curved twistor space does not holomorphically fibre over $\mathbb{CP}^1$. The complex structure of the line bundles $\mathcal{O}(2s-2)$ is deformed to $\mathcal{K}^{(1-s)/2}$ for $\mathcal{K}$ the canonical bundle of the curved twistor space. The s-dependent term in (6.51) is generated by this shift. To incorporate self-dual Yang-Mills is no more difficult. Conformal invariance ensures that the S-algebra is undeformed on AdS_4, and the natural action of (6.38) and (6.48) outlined above distributes over its commutators.

Finally, we mention that it is possible to understand these algebras as Noether charges for the gauge symmetries of the Poisson-BF action (2.45), after the replacement of the Poisson bracket by $\{\cdot\,,\cdot\}_\Lambda$ – see also [172, 222] for related discussion – and to find supersymmetrical extensions of (6.38)-(6.48).

6.5 The Pedersen metric

We conclude this thesis by describing a further deformation to the celestial chiral algebra (6.2) that contains both a cosmological constant and a mass parameter. The deformation is conjectural, in that we don't have yet a complete understanding of the underlying twistor space, but passes several consistency checks and satisfies the Jacobi identity.

Among the Plebański-Demiański solutions of Einstein's equations [223], consider the 3-parameter family of Taub-NUT-AdS black holes with vanishing electromagnetic charges, acceleration and rotation parameters given by the Lorentzian metrics [224, 225]

$$\mathrm{d}s^2 = \frac{\Delta}{r^2+N^2}(\mathrm{d}t - 2N\mathrm{a})^2 - \frac{r^2+N^2}{\Delta}\mathrm{d}r^2 - (r^2+N^2)\mathrm{d}\Omega_2^2\,, \tag{6.52}$$

where

$$\Delta = r^2 - 2mr - N^2 + \frac{r^4+6N^2r^2-3N^4}{\ell^2}\,, \tag{6.53}$$

and a is the usual Dirac monopole (4.7). The metric (6.52) solves Einstein's equations with a cosmological constant $\Lambda = -1/(2\ell^2)$. The remaining parameters m and N are related to the mass and NUT charge of the black hole. In the Newman-Penrose formalism, the only non-vanishing components of the Weyl tensor are [224]

$$\tilde{\psi}_2 = \frac{4N^3 + l^2(\mathrm{i}m + N)}{l^2(N + \mathrm{i}r)^3}\,, \qquad \psi_2 = \frac{4N^3 + l^2(-\mathrm{i}m + N)}{l^2(N - \mathrm{i}r)^3}\,, \tag{6.54}$$

We can set either of them to zero by imposing the condition[3]

$$N = \pm \mathrm{i}M\,, \qquad m = M\left(1 - \frac{4M^2}{l^2}\right)\,, \tag{6.56}$$

which gives an (anti-)self-dual metric. The Euclidean version of the metric

$$\mathrm{d}s^2 = \frac{f_\ell}{V}(\mathrm{d}t - 2M\mathrm{a})^2 + V\left(\frac{\mathrm{d}r^2}{f_\ell} + r^2\mathrm{d}\Omega_2^2\right)\,, \qquad f_\ell = 1 + \frac{r(r+4M)}{\ell^2}\,, \tag{6.57}$$

can be obtained via $t \mapsto \mathrm{i}t$ and $r \mapsto r + M$ and was first constructed by Pedersen [226, 227], so we will refer to it as the *Pedersen metric*. The range of the coordinates is $r \in (0,\infty), (\theta,\phi) \in S^2$, and $t \sim t + 8\pi M$, while V is the scalar potential (5.8) already appearing in the SDTN metric.

Other forms of the Pedersen metric

The Pedersen metric was studied in various contexts [228–231] and in different coordinate systems, so it's useful to derive different presentations of it. Consider the diffeomorphism

$$\rho = \sqrt{\frac{2M\ell^2 r}{\ell^2 + 2Mr}}\,, \qquad \psi = \frac{t}{2M} - \phi \tag{6.58}$$

[3]Note that, in contrast with the Kerr-Taub-NUT solution, the self-duality condition is modified for spinning black holes with a non-zero spinning parameter a, as the self-dual point is given by

$$N = \pm \mathrm{i}M\,, \qquad m = M\left(1 - \frac{a^2 + 4M^2}{l^2}\right)\,. \tag{6.55}$$

and introduce the parameter

$$\nu^2 = \frac{1}{4M^2} - \frac{1}{\ell^2}\,. \tag{6.59}$$

The coordinate transformation (6.58) brings the metric (6.57) into the triaxial form

$$\mathrm{d}s^2 = \frac{4}{(1-\rho^2/\ell^2)^2}\left(\frac{1+\nu^2\rho^2}{1+\nu^2\rho^4/\ell^2}\mathrm{d}\rho^2 + \rho^2(1+\nu^2\rho^2)(\sigma_1^2+\sigma_2^2) + \rho^2\frac{1+\nu^2\rho^4/\ell^2}{1+\nu^2\rho^2}\sigma_3^2\right) \tag{6.60}$$

where σ_i are the standard SU(2) left-invariant 1-forms

$$\sigma_1 \pm \mathrm{i}\sigma_2 = \frac{1}{2}\mathrm{e}^{\mp\mathrm{i}\psi}(\mathrm{d}\theta \pm \mathrm{i}\sin\theta\,\mathrm{d}\phi)\,, \quad \sigma_3 = \frac{1}{2}(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi)\,. \tag{6.61}$$

This is actually the form of the metric that was investigated by Pedersen. From (6.60), we can see that the conformal boundary is a biaxially squashed 3-sphere, known as Berger's sphere, with boundary metric

$$\mathrm{d}s_3^2 = \sigma_1^2 + \sigma_2^2 + \frac{1}{1+\nu^2\ell^2}\sigma_3^2\,. \tag{6.62}$$

Since $(1+\nu^2\ell^2) \geq 1$, the boundary metric is an oblate squashing of the 3-sphere and we will also refer to the metric (6.60) as the *oblate Pedersen metric.* It is also possible to analytically continue the metric by $\nu \mapsto \mathrm{i}\nu$. Provided that $\nu\ell < 1$, the metric is still complete on the open ball $\rho < \ell$. For obvious reasons, we will refer to this metric as the *prolate Pedersen metric.*

The Pedersen metric can be also put in generalized Gibbons-Hawking form. Consider the diffeomorphism

$$\rho = \sqrt{\frac{\ell}{\nu}\tan\chi}\,. \tag{6.63}$$

This brings the metric (6.60) into the form

$$\mathrm{d}s^2 = \frac{\nu\ell^3}{(\sin\chi - \nu\ell\cos\chi)^2}(V^{-1}(\mathrm{d}\psi - A)^2 + V\mathrm{d}\Omega_3^2)\,, \tag{6.64}$$

where $\mathrm{d}\Omega_3^2 = \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega_2^2$ is the round metric on the 3-sphere and

$$V = -\nu\ell + \cot\chi\,, \qquad A = -\cos\theta\,\mathrm{d}\phi\,. \tag{6.65}$$

It's straightforward to check that (V, A) is a Bogomolny pair with respect to the Hodge operator $\star_{S^3}$ induced by the round metric on the 3-sphere. For the prolate metric, the diffeomorphism (6.63) is obtained by the analytical continuation $\nu \mapsto \mathrm{i}\nu$ and requires $\chi \mapsto \mathrm{i}\chi$, so that the metric $\mathrm{d}\Omega_3^2$ on the base space is replaced by the standard metric on hyperbolic 3-space $\mathrm{d}s_{\mathbb{H}^3} = \mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Omega_2^2$.

Limits of the Pedersen metric

The Pedersen metric is particularly attractive because it encompasses most of the curved backgrounds studied in recent years in celestial holography: it's clear that the metric is an interpolating metric between the self-dual Taub-NUT metric, which is recovered from (6.57) in the limit $\ell \to \infty$, and the AdS_4 metric, which can be reached from (6.60) in the limit $\nu \to 0$.[4]

If we instead consider the limit $\nu \to 0$, $\ell \to 0$ of the prolate Pedersen metric, with $c^2 \equiv \ell/\nu$ held constant, we obtain the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}\varrho^2}{1 - c^4/\varrho^4} + \varrho^2(\sigma_1^2 + \sigma_2^2) + \varrho^2\left(1 - \frac{c^4}{\varrho^4}\right)\sigma_3^2\,, \tag{6.66}$$

after the inversion

$$\varrho = \frac{c^2}{\rho}\,. \tag{6.67}$$

The space-time metric can be extended to the region $\varrho > c$ and coincides with the Eguchi-Hanson metric. More precisely, the singularity at $\varrho = c$ is not removable without taking a $\mathbb{Z}_2$ quotient, so the space-time is actually a singular double cover of

[4]In particular, this 'flat-space' limit corresponds to $M \to \infty$, as expected from Chapter 5.

Eguchi-Hanson space [232].

Instead, setting $\nu\ell = 1$ in the prolate Pedersen metric, we recover the Burns space metric [128] up to a conformal prefactor. In fact, performing the change of coordinates

$$\rho^2 = \frac{R^2}{2 - R^2/l^2}\,, \tag{6.68}$$

in the triaxial form of the metric, we find

$$\mathrm{d}s^2 = \frac{2}{(1 - R^2/\ell^2)^2}(\mathrm{d}R^2 + R^2\sigma_3^2) + \frac{2}{1 - R^2/\ell^2}(\sigma_1^2 + \sigma_2^2)\,, \tag{6.69}$$

which is the Fubini-Study metric on a non-compact version of $\mathbb{CP}^2$ commonly referred to as Bergmann space and denoted by $\widetilde{\mathbb{CP}^2}$. As a homogeneous space, it is described by $\widetilde{\mathbb{CP}^2} = SU(2,1)/U(2)$ and it is the non-compact dual of $\mathbb{CP}^2 = SU(3)/U(2)$ Both $\widetilde{\mathbb{CP}^2}$ and $\mathbb{CP}^2$ with a point removed are conformally equivalent to Burns space [128]. Note that the oblate Pedersen metric is generically a complete metric on a 4-dimensional ball. This changes as we approach $\nu\ell = 1$. $\widetilde{\mathbb{CP}^2}$ does not have the same topology as the 4-dimensional ball, by having a non-trivial 2-cycle. In fact, we only find the Fubini-Study metric an open subset of $\widetilde{\mathbb{CP}^2}$ which then can then be extended to all of $\widetilde{\mathbb{CP}^2}$ including this 2-cycle.

6.6 The celestial chiral algebra of the Pedersen metric

The twistor space of the metric (6.57) was first constructed by Pedersen in [226, 227]. Since the metric admits a (generalized) Gibbons-Hawking form, the twistor space necessarily fibres over the minitwistor space $\mathbb{CP}^1 \times \mathbb{CP}^1$ of S^3. Pedersen leveraged this property to construct the twistor space as the total space of the line bundle

$\mathcal{O}(1,-1)^{\mathrm{i}\nu\ell} \otimes L \to \mathbb{CP}^1 \times \mathbb{CP}^1$, where L has transition function

$$\phi_{01} = \frac{(\zeta_1 - \zeta_2)^2}{\zeta_1\zeta_2}\,, \tag{6.70}$$

in terms of affine coordinates (ζ_1, ζ_2) on the two factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Consider again the Hamiltonian (5.24) that gives rise to the SDTN metric, with a minor modification in the overall coefficient

$$\mathsf{h} = \frac{1}{4}\nu^2\eta^2\bar{e}^0\,. \tag{6.71}$$

This Hamiltonian is still integrable on $\mathbb{PT}$, since it is holomorphic away from the locus $\lambda_\alpha = 0$ and $\{\mathsf{h},\mathsf{h}\}_\Lambda = 0$. The associated complex structure deformation is

$$V_\Lambda = \{\mathsf{h},\cdot\}_\Lambda = \frac{1}{2}\nu^2\eta\bar{e}^0 \wedge (\mu^+\mathcal{L}_+ - \mu^-\mathcal{L}_-) + \frac{1}{2}\Lambda\nu^2\eta^2\frac{\hat{\lambda}_\alpha}{\langle\lambda\,\hat{\lambda}\rangle^2}\bar{e}^0 \wedge \mathcal{L}^\alpha\,, \tag{6.72}$$

where $\mathcal{L}^\alpha$ is the Lie derivative along $\partial/\partial\lambda_\alpha$. Now neither $\mu^{\dot{\alpha}}$ nor λ_α are holomorphic, while η still is.

Lemma 6.6.1. *Holomorphic coordinates with respect to the deformed complex structure $\bar{\nabla}_\Lambda = \bar{\partial} + V_\Lambda$ are*

$$Y^{\alpha\beta} = \lambda^\alpha\lambda^\beta - \frac{1}{2}\Lambda\nu^2\eta^2\frac{\hat{\lambda}^\alpha\hat{\lambda}^\beta}{\langle\lambda\,\hat{\lambda}\rangle^2}\,, \tag{6.73}$$

and

$$\rho^\pm = \mu^\pm\left(\frac{\sqrt{\frac{2}{\Lambda\nu^2}} + \eta f(\lambda)}{\sqrt{\frac{2}{\Lambda\nu^2}} - \eta f(\lambda)}\right)^{\pm\sqrt{\frac{\nu^2}{8\Lambda}}}\,, \tag{6.74}$$

where f is the function (5.29).

Proof. Holomorphicity of $Y^{\alpha\beta}$ follows immediately from

$$\bar{\nabla}_\Lambda\lambda_\alpha = \frac{1}{2}\Lambda\eta^2\nu^2\frac{\hat{\lambda}_\alpha}{\langle\lambda\,\hat{\lambda}\rangle}\bar{e}^0\,, \qquad \bar{\nabla}_\Lambda\frac{\hat{\lambda}_\alpha}{\langle\lambda\,\hat{\lambda}\rangle} = \lambda_\alpha\bar{e}^0\,, \tag{6.75}$$

and from holomorphicity of η. Holomorphicity of $\rho^{\pm}$ follows from the identity

$$\frac{\hat{\lambda}_\alpha}{\langle\lambda\,\hat{\lambda}\rangle}\mathcal{L}^\alpha f = f^2\,, \tag{6.76}$$

which holds globally on both U_0 and U_1. □

Some of these coordinates are redundant, as they satisfy the identities

$$\rho^+\rho^- = \eta\,, \qquad Y^{\alpha\beta}Y_{\alpha\beta} = -\Lambda\nu^2\eta^2\,. \tag{6.77}$$

We can view $(Y^{\alpha\beta},\eta)$ as homogeneous coordinates on $\mathbb{CP}^3$, where the quadric

$$Q_{\Lambda\nu^2} = \left\{(Y^{\alpha\beta},\eta)\in\mathbb{CP}^3\,:\,Y^{\alpha\beta}Y_{\alpha\beta}+\Lambda\nu^2\eta^2=0\right\}\hookrightarrow\mathbb{CP}^3\,, \tag{6.78}$$

can be identified with $\mathbb{CP}^1\times\mathbb{CP}^1$. $\rho^{\pm}$ define bundles over the quadric with transition functions

$$\left(\frac{Y^{01}-\sqrt{\frac{\Lambda\nu^2}{2}}\eta}{Y^{01}+\sqrt{\frac{\Lambda\nu^2}{2}}\eta}\right)^{\pm\sqrt{\frac{\nu^2}{8\Lambda}}}\,. \tag{6.79}$$

This is essentially the same transition function found by Pedersen [227]. We thus arrive at the following conjecture

Conjecture 1. *The defining data of the Dolbeault description of the twistor space of the Pedersen metric* (6.57) *are the Hamiltonian* (6.71) *and the infinity twistor* (6.30)*.*

Further evidence for this conjecture is given by the fact that the Beltrami differential (6.72) has the correct limits, as it is clear that the limit $\ell\to\infty$ gives the SDTN Beltrami differential, whilst the $\nu\to 0$ simply turns off the complex structure deformation and recovers the AdS_4 case. In the Eguchi-Hanson limit, we recover the correct differential after the exchange $\lambda_\alpha \longleftrightarrow \mu^{\dot\alpha}$, which is expected given the inversion (6.67). In this limit, (6.73) also reduce to the correct triplet of holomorphic coordinates on the Eguchi-Hanson twistor space, see [122].

Celestial symmetries from twistor space

As before, the celestial chiral algebra arises as the Poisson bracket on twistor space of a basis of $\mathcal{O}(2)$-valued Hamiltonians. Here we follow a strategy similar to [122] to construct such a basis in terms of the holomorphic coordinates $\rho^{\pm}$ and

$$X = Y^{00}\,, \qquad Y = Y^{11}\,, \qquad Z = Y^{01}\,. \tag{6.80}$$

Let us first derive the Poisson bracket of the holomorphic coordinates. It's straightforward to derive

$$\{X, Y\}_\Lambda = 4\Lambda Z\,, \qquad \{X, Z\}_\Lambda = 2\Lambda X\,, \qquad \{Y, Z\}_\Lambda = -2\Lambda Y\,. \tag{6.81}$$

It is also straightforward to derive the relations

$$\{\rho^{\pm}, \eta\}_\Lambda = \pm\rho^{\pm}\,, \qquad \{Y_{\alpha\beta}, \eta\}_\Lambda = 0\,, \qquad \{\rho^{+}, \rho^{-}\}_\Lambda = 1\,. \tag{6.82}$$

Demanding now that the subspace generated by the second constraint in (6.77) is a Poisson ideal, that the bracket obeys the Jacobi identity and that the remaining brackets are symmetric in X and Y yields the remaining commutators

$$\{\rho^{\pm}, X\}_\Lambda = \mp\frac{\Lambda\nu^2\rho^{\pm}\eta}{2Y}\,, \qquad \{\rho^{\pm}, Y\}_\Lambda = \mp\frac{\Lambda\nu^2\rho^{\pm}\eta}{2Y}\,, \qquad \{\rho^{\pm}, Z\}_\Lambda = 0\,. \tag{6.83}$$

We hope to give a more direct derivation of (6.83) in the near future. This discussion leads to

Conjecture 2. *The Poisson bracket in the holomorphic coordinates* $(Y_{\alpha\beta}, \eta, \rho^{\pm})$ *is* *given by*

$$\begin{aligned}\{\omega_1\,,\omega_2\}_\Lambda = {} & 2\Lambda X\mathcal{L}_X\omega_1\wedge\mathcal{L}_Z\omega_2 - 2\Lambda Y\mathcal{L}_Y\omega_1\wedge\mathcal{L}_Z\omega_2 + 4\Lambda Z\mathcal{L}_X\omega_1\wedge\mathcal{L}_Y\omega_2\\ & + \mathcal{L}_{\rho^+}\omega_1\wedge\mathcal{L}_{\rho^-}\omega_2 + \rho^+\mathcal{L}_{\rho^+}\omega_1\wedge\mathcal{L}_\eta\omega_2 - \rho^-\mathcal{L}_{\rho^-}\omega_1\wedge\mathcal{L}_\eta\omega_2\\ & + \frac{\Lambda\nu^2\eta\rho^-}{2X}\mathcal{L}_{\rho^-}\omega_1\wedge\mathcal{L}_Y\omega_2 + \frac{\Lambda\nu^2\eta\rho^-}{2Y}\mathcal{L}_{\rho^-}\omega_1\wedge\mathcal{L}_X\omega_2\\ & - \frac{\Lambda\nu^2\eta\rho^+}{2X}\mathcal{L}_{\rho^+}\omega_1\wedge\mathcal{L}_Y\omega_2 - \frac{\Lambda\nu^2\eta\rho^+}{2Y}\mathcal{L}_{\rho^+}\omega_1\wedge\mathcal{L}_X\omega_2 + (\omega_1\longleftrightarrow\omega_2)\,.\end{aligned} \tag{6.84}$$

We can now use the Poisson bracket to derive the deformed celestial chiral algebra of the Pedersen metric. Given the first constraint in (6.77), a basis of $\mathcal{O}(2)$-valued Hamiltonians is given by

$$w^{a,\,b}_{c,\,d,\,e} = (\rho^+)^a(\rho^-)^b X^c Y^d Z^e\,, \tag{6.85}$$

where $a, b\in\mathbb{Z}_{\geq 0}$ and $c, d, e\in\mathbb{Z}$ and where the generators are subject to the second constraint in (6.77) and to the homogeneity condition

$$a + b + 2c + 2d + 2e = 2\,. \tag{6.86}$$

The second constraint in (6.77) allows to restrict $e\in\{0,1\}$, whilst the homogeneity condition could be used to reduce the number of free indices down to three, as in (6.4). However, keeping the generators in the form (6.85) allows a cleaner derivation of the algebra. Inserting two generators in the Poisson bracket (6.84), we find the algebra

$$\begin{aligned}\{w^{p,\,q}_{i,\,j,\,k}, w^{r,\,s}_{l,\,m,\,n}\}_\Lambda = {} & (ps-qr)w^{p+r-1,\,q+s-1}_{i+l,\,j+m,\,k+n} + 4\Lambda(im-jl)w^{p+r,\,q+s}_{i+l-1,\,j+m-1,\,k+n+1}\\ & + 2\Lambda((i-j)n-(l-m)k)w^{p+r,\,q+s}_{i+l,\,j+m,\,k+n-1}\\ & - \frac{1}{2}\Lambda\nu^2((p-q)(l+m)-(r-s)(i+j))w^{p+r+1,\,q+s+1}_{i+l-1,\,j+m-1,\,k+n}\end{aligned} \tag{6.87}$$

This is the unique algebra that respects the expected symmetries, obeys the Jacobi identity and reduces to the expected algebras in the known limits of the Pedersen metric. Indeed, it can be seen to be a further deformation of the Λ-deformed algebra (6.38). In fact, we obtain a $\mathbb{Z}_2$-quotient of the AdS_4 algebra in the limit $\nu \to 0$, which is an expected feature due to the quadratic nature of the coordinates (6.73). After rescaling the generators by Λ and sending $\Lambda \to \infty$ and $M \to \infty$, keeping $c^2 = \frac{\Lambda\nu^2}{2}$ constant, the algebra becomes the expected algebra of Eguchi-Hanson space considered in equation (4.8) of [233]. The role of $\mu^{\dot{\alpha}}$ and λ_α is exchanged as expected from the conformal inversion in equation (6.67). In the self-dual Taub-NUT limit, $\ell \to \infty$, we find an undeformed $L\mathfrak{ham}(\mathbb{C}^2)$, in agreement with the holomorphic collinear limit (6.27)-(6.28).

Chapter 7

Outlook

Throughout this thesis, we have been able to apply classical methods from twistor theory to various problems in QFT and celestial holography around strong backgrounds. We have extended the formalism developed for amplitudes around radiative backgrounds of [66] to the construction of form factors by exploiting the representation of the Ward correspondence at null infinity [154, 155] to obtain formulae on such non-trivial self-dual radiative backgrounds. We have then developed novel descriptions in Dolbeault cohomology of the self-dual dyon and self-dual Taub-NUT metric and used them to derive compact all-multiplicity formulae for the tree-level MHV scattering amplitudes of gluons and gravitons around these backgrounds. We further showed that the celestial chiral algebras are undeformed around both the self-dual dyon and self-dual Taub-NUT metric, and concluded by discussing deformations to the graviton celestial chiral algebra induced by a non-vanishing cosmological constant, proving that the algebra is deformed in AdS_4 and providing strong evidence for a conjectural deformed algebra around the Pedersen metric. There are several possible extensions of this work.

N^{k}MHV observables: the main restriction of this thesis was the focus on MHV amplitudes and form factors. In order to construct higher MHV degree observables on non-trivial backgrounds, the first step is the construction of the twistor space propagator on such curved backgrounds. Around the vacuum, it is possible to find compact expressions [38, 234, 235] for the propagator in CSW gauge, forming the backbone of the MHV formalism in momentum space [36,68]. It might be possible to construct a similar expression in Euclidean signature, starting from the holomorphic frame on twistor space, perhaps inspired by the construction of space-time Green's functions done by Page [236] for the SDTN and by Zoubos [229] for the Pedersen metric. The appearance of a single residual space-time integral at all multiplicity in any of the MHV form factors and MHV amplitudes could signal that a background-couple version of the MHV formalism is indeed possible, at least in the case of gluon amplitudes.

Even without a proper understanding of the background-coupled MHV rules on twistor space, we can still make an educated guess based on similar formulae around the trivial background [7, 46, 47, 65, 237, 238]. Focussing on the gluon amplitudes around the self-dual dyon, let $\sigma^{\mathbf{a}} = (\sigma^{\mathbf{0}}\sigma^{\mathbf{1}})$ be homogeneous holomorphic coordinates on $\mathbb{CP}^1$, so that a holomorphic map $Z\colon \mathbb{CP}^1 \to \mathbb{PT}$ of degree $k+1$ can be parametrized as a degree-$(k+1)$ polynomial in the homogeneous coordinates depending on $4(k+2)$ moduli $U^A_{\mathbf{a}_1\ldots\mathbf{a}_{k+1}}$

$$Z^A(\sigma) = U^A_{\mathbf{a}_1\ldots\mathbf{a}_{k+1}}\sigma^{\mathbf{a}_1}\ldots\sigma^{\mathbf{a}_{k+1}}\,. \tag{7.1}$$

Letting $\mathbf{g}$ and $\tilde{\mathbf{g}}$ be the sets of positive- and negative-helicity gluons, respectively, we can make the following conjecture for the colour-ordered N^{k}MHV partial amplitude

$$\mathcal{A}_{n,k} = \int \frac{\mathrm{d}^{4(k+2}U}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\prod_{\substack{i<j\\ i,j\in\tilde{\mathbf{g}}}}(i\,j)^4\prod_{i=1}^{n}\frac{\mathrm{D}\sigma_i}{(i\,i+1)}\times$$
$$\times\prod_{j\in\mathbf{g}}a_j(Z(\sigma_j))\mathrm{e}^{e_j\mathrm{g}(\sigma_j)}\prod_{k\in\tilde{\mathbf{g}}}b_k(Z(\sigma_k))\mathrm{e}^{e_k\mathrm{g}(\sigma_k)}\,. \tag{7.2}$$

In this formula, the volume factor fixes the residual $\mathrm{GL}(2,\mathbb{C})$ redundancy in the holomorphic map Z in the standard Faddeev-Popov fashion, $(i\,j) = \sigma_i^{\mathbf{a}}\sigma_j^{\mathbf{b}}\epsilon_{\mathbf{ba}}$, $\mathrm{D}\sigma = (\sigma\,\mathrm{d}\sigma)$, a_j and b_k are the twistor representatives (4.48)-(4.46) of the j^{th} positive-helicity gluon and k^{th} negative-helicity gluon, respectively, and g is related to the twistor connection (4.16) by

$$\mathsf{g}(\sigma) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma'}{(\sigma\,\sigma')}\frac{(\iota\,\sigma)}{(\iota\,\sigma')}\mathsf{a}(Z(\sigma'))\,, \tag{7.3}$$

so that it's the generalization of (3.22) at higher degree. While the formula (7.2) is mathematically well-defined and has the correct 'flat-space' limit, we have not yet been able to compute any of the moduli integrals for $k > 0$. Furthermore, there is no first-principles derivation of (7.2). It would thus be very interesting to try to simplify this formula and further test (or prove) its veracity.

Similarly, it's possible to conjecture a formula for the graviton S-matrix around SDTN in all MHV sectors starting from the Cachazo-Skinner formula [239, 240]. We can introduce a higher-degree analogue of the curved twistor lines (5.33) as the solution to

$$\bar{\partial}\mathsf{F}^{\dot\alpha}(U,\sigma) = \frac{\partial\mathsf{h}}{\partial\mu_{\dot\alpha}}(\mathsf{F}(U,\sigma),\sigma)\,. \tag{7.4}$$

Perturbations to this equation induced by a perturbation h of the Hamiltonian can be interpreted as solutions to the equations of motion of the higher-degree twistor sigma model

$$\begin{aligned} S^{(k)}[m] = \frac{1}{\hbar}\int_{\mathbb{CP}^1}\frac{\mathrm{D}\sigma}{\prod_{j\in\tilde{\mathbf{h}}}(\sigma\,j)^2}\Bigg(&[m\,\bar{\partial}m] + 2h(\mathsf{F}(U,\sigma)+m,\sigma) \\ &+\sum_{p=2}^{4}\frac{2}{p!}\frac{\partial^p\mathsf{h}}{\partial\mu^{\dot\alpha_1}\ldots\partial\mu^{\dot\alpha_p}}(\mathsf{F}(U,\sigma),\sigma)m^{\dot\alpha_1}\ldots m^{\dot\alpha_p}\Bigg)\,, \end{aligned} \tag{7.5}$$

where $\tilde{\mathbf{h}} \subseteq \{1,\ldots,n\}$ is the subset of negative-helicity gravitons. The connected, tree-level correlation function of the positive-helicity vertex operators

$$V_i = \int \frac{\mathrm{D}\sigma}{\prod_{j\in\tilde{\mathbf{h}}}(\sigma\, j)^2} h_i(\mathsf{F}(U,\sigma)+m,\sigma)\,, \tag{7.6}$$

then computes the desired amplitude. As in the case of the MHV amplitude, we can instead construct the correlator in a free theory in the presence of suitable background vertex operators and use the matrix-tree theorem, leading to the formula

$$\begin{aligned}\mathcal{M}_{n,k} = \sum_{t=0}^{n-k-4} \sum_{p+q+t=r} \int \frac{\mathrm{d}^{4(k+2)}U}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \int_{(\mathbb{P}^1)^t} \prod_{\mathrm{m}=1}^{t} \mathrm{D}\sigma_{\mathrm{m}} \wedge \mathrm{D}\bar{\lambda}(\sigma_{\mathrm{m}})\ \mathrm{det}'(\mathbb{H}^\vee)& \\ \times \prod_{a=1}^{p} \frac{\mathsf{F}^+(U,\sigma_a)}{24M} \frac{\partial}{\partial\alpha_a^+} \frac{\partial^2}{\partial\alpha_a^{-\,2}} \prod_{b=1}^{q} \frac{\mathsf{F}^-(U,\sigma_b)}{24M} \frac{\partial^2}{\partial\alpha_b^{+\,2}} \frac{\partial}{\partial\alpha_b^-}& \\ \times \prod_{c=1}^{r} \frac{1}{8M} \frac{\partial^2}{\partial\alpha_c^{+\,2}} \frac{\partial^2}{\partial\alpha_c^{-\,2}} \left. \frac{|\mathcal{H}[\mathbf{t}]_i^i|}{\prod_{j<k\in\tilde{\mathbf{h}}\cup\{i\}}(j\,k)} \right|_{\alpha_1^\pm=\ldots=\alpha_t^\pm=0}& \\ \times \prod_{i=1}^{n} \frac{\mathrm{D}\sigma_i\,\mathrm{d}s_i}{s_i^3} \bar{\delta}^2(\kappa_i - s_i\lambda(\sigma_i))(s_i\mathsf{F}^+(U,\sigma_i))^{q_i+2M\omega_i}(s_i\mathsf{F}^-(U,\sigma_i))^{q_i-2M\omega_i}\mathrm{e}^{-\xi_i s_i^2 \mathsf{F}^+(U,\sigma_i)\mathsf{F}^-(U,\sigma_i)}\,,&\end{aligned} \tag{7.7}$$

for the N^{k}MHV tree-level graviton scattering amplitude around the SDTN metric. In this formula, $\mathrm{det}'(\mathbb{H}^\vee)$ is the resultant of the map components $\lambda_\alpha(\sigma)$ [240]. The $\mathcal{H}[\mathbf{t}]$ is a matrix with the same block decomposition as in (5.130) and entries

$$\mathbb{H}_{jk}[\mathbf{t}] = \begin{cases} -s_i s_j \dfrac{[\![j\,k]\!]}{(j\,k)}\,, & j \neq k \\ \displaystyle\sum_{\ell\neq j} s_j s_\ell \frac{[\![j\,\ell]\!]}{(j\,\ell)} \prod_{i\in\tilde{\mathbf{h}}} \frac{(i\,\ell)}{(i\,j)} - \mathrm{i} \sum_{\mathrm{m}=1}^{t} s_j \frac{[\![j\,\mathrm{m}]\!]}{(j\,\mathrm{m})} \prod_{i\in\tilde{\mathbf{h}}} \frac{(i\,\mathrm{m})}{(i\,j)}\,, & j = k \end{cases} \tag{7.8}$$

$$\mathfrak{h}_{j\mathrm{m}}[\mathbf{t}] = \mathrm{i}\, s_j \frac{[\![j\,\bar{\lambda}(\sigma_{\mathrm{m}})]\!]}{(j\,\mathrm{m})} \prod_{i\in\tilde{\mathbf{h}}} (i\,\mathrm{m})\,, \tag{7.9}$$

$$
\mathbf{T}_{\mathrm{mn}}[\mathbf{t}] = \begin{cases} \dfrac{[\![\bar{\lambda}(\sigma_{\mathrm{m}})\,\bar{\lambda}(\sigma_{\mathrm{n}})]\!]}{(\mathrm{m}\,\mathrm{n})}\,, & \mathrm{m} \neq \mathrm{n} \\ -\displaystyle\sum_{\mathrm{p}\neq\mathrm{m}} \frac{[\![\bar{\lambda}(\sigma_{\mathrm{m}})\,\bar{\lambda}(\sigma_{\mathrm{p}})]\!]}{(\mathrm{m}\,\mathrm{p})} \prod_{i\in\tilde{\mathbf{h}}} \frac{(i\,\mathrm{p})}{(i\,\mathrm{m})} - \mathrm{i}\sum_{j=1}^{n} s_j \frac{[\![j\,\bar{\lambda}(\sigma_{\mathrm{m}})]\!]}{(j\,\mathrm{m})} \prod_{i\in\tilde{\mathbf{h}}} \frac{(i\,j)}{(i\,\mathrm{m})}\,, & \mathrm{m} = \mathrm{n} \end{cases} \tag{7.10}
$$

The same caveats apply as in the case of higher MHV degree gluon amplitudes: Equation (7.7) has the correct flat-space limit, but proving it from first principles will probably be fairly difficult.

Loops: in the same spirit, another obvious avenue of further exploration would be the construction of loop amplitudes and form factors around non-trivial backgrounds. If a background-coupled version of the MHV formalism can be constructed, then it should be possible to apply it to construct loop observables too. Alternatively, one could also try to find *ad hoc* generating functionals for loop amplitudes in a given helicity sector. In both gauge theory and gravity, the simplest loop amplitude would be the 1-loop all-plus amplitude, which is known to be a rational function around the vacuum, as a consequence of the vanishing of the all-plus and single-minus tree-level amplitudes. A generating functional for the gluon all-plus one-loop amplitude was proposed in [241]: it would be interesting to test it on non-trivial backgrounds.

Other gravitational backgrounds: in principle, the techniques discussed in Chapter 5 can be applied to other gravitational instantons in the Gibbons-Hawking family, in view of the generalisation of the Taub-NUT twistor space discussed in Appendix B. It should be possible to derive a formula akin to (5.138) describing the MHV scattering of gravitons around any gravitational instanton; namely, one should only replace the twistor holomorphic frame and dressing matrices with the appropriate ones, as well as modify the structure of the background vertex operators appropriately. For generic gravitational instantons, these will probably resemble the background vertex operators that appear in the gravitational S-matrix on a radiative space-time [67].

In [4], we also derived formulae for scattering amplitudes around *spinning* black holes: recall that is possible to obtain the Kerr metric from the Schwarzschild metric using the Newman-Janis shift [242], that is via a complex coordinate transformation which is essentially a complex shift by the spin vector $\vec{a}$. The metric is real in Lorentzian signature only if it is superimposed with the metric obtained with the complex conjugate shift in a specific, non-linear way. However, in the self-dual sector the non-linearity disappears and the self-dual Kerr-Taub-NUT metric [243–245] can actually be constructed by a complex linear translation of the SDTN metric. This means that the simple shift $\vec{x} \to \vec{x} - \mathrm{i}\,\vec{a}$ in the SDTN metric leads to the real-valued Euclidean self-dual Kerr-Taub-NUT metric. By the same argument, we can compute quantities such as wavefunctions and amplitudes first in the SDTN metric, and the shift $\vec{x} \to \vec{x} - \mathrm{i}\,\vec{a}$ to obtain the corresponding quantities around the SDKTN metric.

It would be an extremely interesting problem to study whether our results on SDTN and SDKTN can be extended away from self-duality. While astrophysical black holes are definitely not self-dual, they do possess a Hermitian structure which is a Lorentzian version of the conformal Kähler condition [246, 247]. The hermitian structure endows the black hole metrics with a 2-dimensional twistor space [248, 249] and allows the construction of a Penrose transform for the Teukolsky system [250, 251]. Apart from an initial attempt at solving the wave equation on Kerr perturbatively around the self-dual point [252], it appears that none of these facts have been used so far to investigate amplitudes on these more realistic metrics.

Other directions: while we studied the leading holomorphic collinear limits of the gluon and graviton MHV amplitudes (4.51)-(5.138), it should also be possible to obtain an *all-order* holomorphic collinear expansion of the amplitudes using methods similar to flat space [253, 254]. In contrast to the undeformed leading splitting functions, one expects that such an all-orders collinear expansion would be sensitive to the SDD and SDTN backgrounds through the sub-leading terms, as shown for MHV scattering on SD radiative backgrounds [126].

The twistor sigma models of [175] can be extended to Einstein manifolds, essentially by promoting the square bracket on twistor space to a generalised skew-symmetric product constructed with the infinity twistor. Correlation functions of the vertex operators of the sigma model then give rise to correlators with a non-vanishing cosmological constant: their interpretation as genuine graviton amplitudes is not clear, so it would be interesting to study the relation between these correlators and AdS 'amplitudes', especially in view of graviton amplitudes constructed in [214–217] and used to derive the Λ-deformed chiral algebra in [123]. It would be interesting to generalise these correlators to correlators around the Pedersen metric, and ideally match these results in the Burns space limit with the computations performed in [127,128]. A related, but distinct, question is how to incorporate spin when $\Lambda \neq 0$: we already know that the self-duality condition of the spinning Kerr-Taub-NUT metric is actually slightly modified in the presence of both a cosmological constant and a spinning parameter, so it would be nice to explore whether and how the correlators get modified in this setting.

Finally, the Poisson-BF theory on twistor space is anomalous for $\Lambda = 0$ [255], and one should expect a similar anomaly to be present for $\Lambda \neq 0$. Identifying this anomaly and ways to successfully cancel it could lead to quantum counterparts of the tree level celestial chiral algebras of Chapter 6 along the lines of [255, 256].

Appendix A

Alternative descriptions of the twistor quadrille and of the Taub-NUT twistor space

The self-dual dyon and self-dual Taub-NUT metric are some of the oldest examples of the Ward's correspondence and non-linear graviton construction, and as such they have been treated many times in the twistor literature. While all descriptions must be equivalent up to gauge transformations and diffeomorphisms, respectively, they are not all equally easy to work with from a computational point of view. In this Appendix, we review the Čech description of the twistor quadrille and two possible presentations of the Taub-NUT twistor space which have appeared in the literature.

A.1 The twistor quadrille

Given a twistor line $X \subseteq \mathbb{PT}$, we can construct two possible open covers of X: the first one is given by the restriction

$$U_i^x = \{Z^A \in \mathbb{PT} \,|\, \mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha \,, \lambda_i \neq 0\}\,, \qquad i = 0, 1\,, \tag{A.1}$$

of $\{U_0, U_1\}$ to the line, while the second one is given by the sets

$$U^x_{\pm} = \{Z^A \in \mathbb{PT} \,|\, \mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha \,, \langle\lambda\, \chi_{\pm}\rangle \neq 0\} \,. \tag{A.2}$$

Let us define a refined open cover by $U^x_{i\pm} = U^x_i \cap U^x_{\pm}$. As we continuously vary x in space-time, this open cover uplifts to an open cover of $\mathbb{PT}$, which we denote as $\{U_{i\pm}, i = 0, 1\}$. Consider now the functions c^{kl}_{ij} defined on $U_{ij} \cap U_{kl}$, for $i, k \in \{0, 1\}$ and $j, l \in \{\pm\}$, by

$$\begin{aligned} c^{0+}_{0-} &= \frac{Q}{\lambda_1^2} \,, & c^{1-}_{0-} &= \frac{\lambda_0}{\lambda_1} \,, & c^{1+}_{0-} &= \frac{Q}{\lambda_0\lambda_1} \,, \\ c^{1-}_{0+} &= \frac{\lambda_0\lambda_1}{Q} \,, & c^{1+}_{0+} &= \frac{\lambda_1}{\lambda_0} \,, & c^{1+}_{1-} &= \frac{Q}{\lambda_0^2} \,, \end{aligned} \tag{A.3}$$

and by $c^{kl}_{ij} = (c^{ij}_{kl})^{-1}$ for the remaining combinations of indices. Q is the quadric (4.23): since its restriction (4.24) to any line X vanishes if and only if $\langle\lambda\, \chi_{\pm}\rangle = 0$, each one of the c^{ij}_{kl} is holomorphic on $U_{ij} \cap U_{kl}$. It's also straightforward to check that they satisfy

$$c^{kl}_{ij} c^{mn}_{kl} = c^{mn}_{ij} \,, \tag{A.4}$$

so these functions are the transition function of a holomorphic bundle $L \to \mathbb{PT}$. Using again the restriction (4.24), we also find that on X the transition functions decompose as

$$c^{kl}_{ij}(Z)\big|_X = \mathsf{H}_{ij}(x, \lambda)\mathsf{H}^{-1}_{ij}(x, \lambda) \,, \tag{A.5}$$

where

$$\mathsf{H}_{0-} = \frac{\langle\lambda\, \chi_-\rangle}{\lambda_0} \,, \qquad \mathsf{H}_{0+} = \frac{\lambda_0}{\langle\lambda\, \chi_+\rangle} \,, \qquad \mathsf{H}_{1-} = \frac{\langle\lambda\, \chi_-\rangle}{\lambda_0} \,, \qquad \mathsf{H}_{1+} = \frac{\lambda_1}{\langle\lambda\, \chi_+\rangle} \,. \tag{A.6}$$

Every H_{ij} is holomorphic over U_{ij}, so the restriction $L|_X$ to any twistor line is trivial and L is a Ward bundle. H_{ij} are the Cech analogous of the holomorphic frame. Any of them can be used to restruct the space-time gauge field via

$$\mathsf{H}_{ij}^{-1}\lambda^{\alpha}\partial_{\alpha\dot{\alpha}}\mathsf{H}_{ij} = -\mathrm{i}\lambda^{\alpha}A_{\alpha\dot{\alpha}}\,, \tag{A.7}$$

as holomorphicity of the transition functions ensures that this relation is independent of the open set U_{ij} and thus produces a well-defined space-time field. The resulting gauge field is the Abelian self-dual dyon in Kerr-Schild gauge.

A.2 Hitchin's constrained twistor space

Let us now consider the SDTN metric. Hitchin [257] described its twistors space using the fact that the SDTN metric is a complete hyperkähler metric on $\mathbb{R}^4 \cong \mathbb{C}^2$ of Gibbons-Hawking form. Since such a metric is specified by a solution of the Bogomolny monopole equations on $\mathbb{R}^3$, it follows that the associated twistor space $\mathbb{P}\mathscr{T}$ must be a bundle over the minitwistor space $\mathbb{MT}$ of $\mathbb{R}^3$. This minitwistor space is the space of oriented geodesics in $\mathbb{R}^3$, which can be parametrized as the total space of $\mathcal{O}(2) \to \mathbb{P}^1$ [258]. The twistor description used throughout Chapter 5 clearly fits into this framework, with $\eta = \mu^+\mu^-$ acting as a coordinate on the fibres of $\mathbb{MT}$.

Hitchin's construction instead takes the coordinate Q given in Equation (4.23) as coordinate on the $\mathcal{O}(2)$ fibres. One can then consider a line bundle $L \to \mathbb{MT}$ with partial connection

$$\bar{D}_L = \bar{\partial}_{\mathbb{MT}} + \frac{Q}{2M}\,\bar{e}^0\,, \tag{A.8}$$

and the corresponding patching function on $U_1 \cap U_0$ is easily seen to be

$$\phi_{10} = \exp\left(\frac{Q}{2M\,\lambda_0\,\lambda_1}\right)\,, \tag{A.9}$$

in agreement with (5.30). The twistor space is the sub-bundle of $(L \oplus L^{-1}) \otimes \mathcal{O}(1)$

defined by

$$\mu^+ \, \mu^- = Q \,, \tag{A.10}$$

where $\mu^\pm$ are elements of the fibres of $L^{\pm 1} \otimes \mathcal{O}(1)$, respectively. The Euclidean structure in this presentation of twistor space corresponds to the anti-holomorphic involution

$$(\mu^+,\, \mu^-,\, Q,\, \lambda_\alpha) \mapsto \left(\overline{\mu^-},\, -\overline{\mu^+},\, -\overline{Q},\, \hat{\lambda}_\alpha\right) \,. \tag{A.11}$$

This is a global, albeit constrained thanks to (A.10), description of the twistor space.

The fibration $\mathbb{P}\mathscr{T} \to \mathbb{P}^1$ is found by composing $\mathbb{P}\mathscr{T} \to \mathbb{MT}$ with $\mathbb{MT} \to \mathbb{P}^1$. To write a holomorphic symplectic form on the fibers of $\mathbb{P}\mathscr{T} \to \mathbb{P}^1$, one introduces local trivializations ρ^+, ρ^- of L, L^{-1} on the patch U_0, and $\tilde{\rho}^+, \tilde{\rho}^-$ on the patch U_1:

$$\begin{aligned} \mu^+ &= \begin{cases} \rho^+ \, \exp\left(\frac{-Q\,\hat{\lambda}_0}{2M\,\lambda_0\,\langle\lambda\,\hat{\lambda}\rangle}\right) & \lambda_0 \neq 0 \\ \tilde{\rho}^+ \, \exp\left(\frac{-Q\,\hat{\lambda}_1}{2M\,\lambda_1\,\langle\lambda\,\hat{\lambda}\rangle}\right) & \lambda_1 \neq 0 \end{cases} \,, \\ \mu^- &= \begin{cases} \rho^- \, \exp\left(\frac{Q\,\hat{\lambda}_0}{2M\,\lambda_0\,\langle\lambda\,\hat{\lambda}\rangle}\right) & \lambda_0 \neq 0 \\ \tilde{\rho}^- \, \exp\left(\frac{Q\,\hat{\lambda}_1}{2M\,\lambda_1\,\langle\lambda\,\hat{\lambda}\rangle}\right) & \lambda_1 \neq 0 \end{cases} \,, \end{aligned} \tag{A.12}$$

where $\rho^\pm, \tilde{\rho}^\pm$ are $\mathcal{O}(1)$-valued. These coordinates are related by the transition functions

$$\tilde{\rho}^+ = \rho^+ \, \exp\left(\frac{Q}{2M\,\lambda_0\,\lambda_1}\right) \,, \qquad \tilde{\rho}^- = \rho^- \, \exp\left(\frac{-Q}{2M\,\lambda_0\,\lambda_1}\right) \,, \tag{A.13}$$

on the overlap $U_0 \cap U_1$.

The holomorphic symplectic form on U_0 is given by

$$\Sigma = \frac{1}{2}\,\mathrm{d}\log\left(\frac{\rho^+}{\rho^-}\right) \wedge \mathrm{d}Q = \frac{1}{2}\,\frac{\rho^-\,\mathrm{d}\rho^+ - \rho^+\,\mathrm{d}\rho^-}{\rho^+\,\rho^-} \wedge \mathrm{d}Q \,, \tag{A.14}$$

which appears to be non-singular only when $\rho^+\,\rho^- \neq 0$. However, upon using the constraint $Q = \mu^+ \mu^- = \rho^+ \rho^-$, it follows that

$$\Sigma = \mathrm{d}\rho^+ \wedge \mathrm{d}\rho^- \,, \tag{A.15}$$

which extends across $\rho^+ \rho^- = 0$. Using the transition functions (A.13) along with the relation $\rho^+ \rho^- = Q$, it is easily checked that this takes the same form on the other patch:

$$\Sigma = \mathrm{d}\rho^+ \wedge \mathrm{d}\rho^- = \mathrm{d}\tilde{\rho}^+ \wedge \mathrm{d}\tilde{\rho}^- \quad \mod \mathrm{d}\lambda_\alpha \,. \tag{A.16}$$

Thus, Σ is global on the fibres of $\mathbb{P}\mathscr{T} \to \mathbb{P}^1$ and $\rho^\pm, \tilde{\rho}^\pm$ both provide Darboux coordinates on their respective domains of definition. To reconstruct the metric, one follows the usual method [159] of solving for the holomorphic rational curves in twistor space, then pulling Σ back to $\mathscr{M} \times \mathbb{P}^1$ and reading off a basis of ASD 2-forms.

A.3 Sparling's non-linear graviton

In fact, the earliest description of the SDTN metric was by Sparling [259] (see also Section 3.4 of [188]), in terms of a Čech description of twistor space built on a four-set open cover. This is closely related to Hitchin's construction. For $i = 0, 1$, let $\rho_i^\pm$ be local coordinates for $\rho^\pm$, defined by setting

$$\rho^+ = \begin{cases} \rho_0^+ \exp\left(-\dfrac{Q\hat{\lambda}_0}{2M\langle\lambda\,\hat{\lambda}\rangle\lambda_0}\right) , & \lambda_\alpha \in U_0 \\ \rho_1^+ \exp\left(-\dfrac{Q\hat{\lambda}_1}{2M\langle\lambda\,\hat{\lambda}\rangle\lambda_1}\right) , & \lambda_\alpha \in U_1 \end{cases} , \tag{A.17}$$

$$\rho^- = \begin{cases} \rho_0^- \exp\left(\dfrac{Q\hat{\lambda}_0}{2M\langle\lambda\,\hat{\lambda}\rangle\lambda_0}\right) , & \lambda_\alpha \in U_0 \\ \rho_1^- \exp\left(\dfrac{Q\hat{\lambda}_1}{2M\langle\lambda\,\hat{\lambda}\rangle\lambda_1}\right) , & \lambda_\alpha \in U_1 \end{cases} . \tag{A.18}$$

Now define the four sets

$$U_{0\pm} = \{\lambda_0 \neq 0, \rho_0^\pm \neq 0\}\,, \qquad U_{1\pm} = \{\lambda_1 \neq 0, \rho_1^\pm \neq 0\}\,, \tag{A.19}$$

so that $U_{i+} \cup U_{i-} = U_i$ for $i = 0, 1$. On each open set, one of the $\rho_i^\pm$ coordinates can be eliminated using the constraint $\rho_i^+ \rho_i^- = Q$. For instance, local holomorphic coordinates on U_{0+} are given by $(\rho_0^+, Q, \lambda_\alpha)$ with $\rho_0^- = Q/\rho_0^+$ fixed by the constraint.

Now, defined the local coordinates

$$\mu^{\dot\alpha}\big|_{U_{0+}} = 2T^{1\dot\alpha}\,\frac{Q}{\lambda_0} - 4M\,T^{\alpha\dot\alpha}\,\lambda_\alpha\,\log\frac{\rho_0^+}{\lambda_0}\,, \tag{A.20}$$

$$\mu^{\dot\alpha}\big|_{U_{0-}} = 2T^{1\dot\alpha}\,\frac{Q}{\lambda_0} + 4M\,T^{\alpha\dot\alpha}\,\lambda_\alpha\,\log\frac{\rho_0^-}{\lambda_0}\,, \tag{A.21}$$

$$\mu^{\dot\alpha}\big|_{U_{1+}} = -2T^{0\dot\alpha}\,\frac{Q}{\lambda_1} - 4M\,T^{\alpha\dot\alpha}\,\lambda_\alpha\,\log\frac{\rho_1^+}{\lambda_1}\,, \tag{A.22}$$

$$\mu^{\dot\alpha}\big|_{U_{1-}} = -2T^{0\dot\alpha}\,\frac{Q}{\lambda_1} + 4M\,T^{\alpha\dot\alpha}\,\lambda_\alpha\,\log\frac{\rho_1^-}{\lambda_1}\,. \tag{A.23}$$

It can be checked that $\mu^{\dot\alpha}T^{\alpha}{}_{\dot\alpha}\lambda_\alpha = Q$ holds globally. Moreover, the six transition functions are

$$U_{0+}\cap U_{0-} \quad : \quad \mu^{\dot\alpha}\big|_{U_{0+}} - \mu^{\dot\alpha}\big|_{U_{0-}} = -4MT^{\alpha\dot\alpha}\lambda_\alpha\log\frac{Q}{\lambda_0^2}\,, \tag{A.24}$$

$$U_{0+}\cap U_{1+} \quad : \quad \mu^{\dot\alpha}\big|_{U_{0+}} - \mu^{\dot\alpha}\big|_{U_{1+}} = -4MT^{\alpha\dot\alpha}\lambda_\alpha\log\frac{\lambda_1}{\lambda_0}\,, \tag{A.25}$$

$$U_{0+}\cap U_{1-} \quad : \quad \mu^{\dot\alpha}\big|_{U_{0+}} - \mu^{\dot\alpha}\big|_{U_{1-}} = -4MT^{\alpha\dot\alpha}\lambda_\alpha\log\frac{Q}{\lambda_0\lambda_1}\,, \tag{A.26}$$

$$U_{0-}\cap U_{1+} \quad : \quad \mu^{\dot\alpha}\big|_{U_{0-}} - \mu^{\dot\alpha}\big|_{U_{1+}} = -4MT^{\alpha\dot\alpha}\lambda_\alpha\log\frac{\lambda_0\lambda_1}{Q}\,, \tag{A.27}$$

$$U_{0-}\cap U_{1-} \quad : \quad \mu^{\dot\alpha}\big|_{U_{0-}} - \mu^{\dot\alpha}\big|_{U_{1-}} = -4MT^{\alpha\dot\alpha}\lambda_\alpha\log\frac{\lambda_0}{\lambda_1}\,, \tag{A.28}$$

$$U_{1+}\cap U_{1-} \quad : \quad \mu^{\dot\alpha}\big|_{U_{1+}} - \mu^{\dot\alpha}\big|_{U_{1-}} = -4MT^{\alpha\dot\alpha}\lambda_\alpha\log\frac{Q}{\lambda_1^2}\,. \tag{A.29}$$

These are precisely the patching functions prescribed by Sparling [259]. This construction should be viewed as a gravitational analogue of the twistor quadrille.

Appendix B

Twistor theory for Gibbons-Hawking instantons

In this appendix, we demonstrate how our presentation of the SDTN twistor space can easily be generalised to give any Gibbons-Hawking gravitational instanton [102, 190, 191] in Euclidean signature. See [67] for an analogous construction for radiative Gibbons-Hawking metrics in the complexified setting or for split signature.

Consider a complex deformation of $\mathbb{PT}$ of the form $\bar\nabla = \bar\partial + \{\mathsf{h}, \cdot\}$ for $\mathcal{O}(2)$-valued Hamiltonian

$$\mathsf{h} = f(\eta, \lambda)\, \bar{e}^0 \,, \tag{B.1}$$

where f is any function that has total homogeneity 4 in $(\mu^{\dot\alpha}, \lambda_\alpha)$ but depends on $\mu^{\dot\alpha}$ only through η. Denoting $\partial_\eta f$ as $\dot f$, holomorphic rational curves $(\mu^{\dot\alpha} = \mathsf{F}^{\dot\alpha}(x,\lambda), \lambda_\alpha)$ in the complex structure of (B.1) are defined by

$$\bar\partial\big|_X \mathsf{F}^\pm = \pm \mathsf{F}^\pm \; \dot f\Big|_X \bar{e}^0 \,. \tag{B.2}$$

These equations are solved by

$$\mathsf{F}^{\pm}(x,\lambda) = \langle \chi_{\pm}\,\lambda\rangle\, \exp\!\left(\mp\frac{2\pi\mathrm{i}}{p}\,\tau \pm \mathsf{g}(x,\lambda)\right)\,, \tag{B.3}$$

where $\tau \sim \tau + p$ is a coordinate on a circle of radius $p/2\pi$ and g is a weight-0 function which can be constructed via the Green's function (3.14) as

$$\mathsf{g}(x,\lambda) = \int_X \mathrm{d}^2\lambda'\,\frac{\langle \iota\,\lambda\rangle}{\langle \iota\,\lambda'\rangle\langle \lambda\,\lambda'\rangle}\,\dot{f}\Big|_X\,, \qquad \mathrm{d}^2\lambda \equiv \frac{\langle \lambda\,\mathrm{d}\lambda\rangle\wedge \bar{e}^0}{2\pi\mathrm{i}}\,. \tag{B.4}$$

In this integral formula, $\dot{f}|_X \equiv \dot{f}(\langle\chi_+\,\lambda'\rangle\langle\chi_-\,\lambda'\rangle,\lambda')$ is the restriction of $\dot{f}$ to the holomorphic rational curve X, where $\eta = -\mathrm{i}x^{\alpha\beta}\lambda_\alpha\lambda_\beta$.

Let $\mathscr{M}$ denote the real, 4-dimensional moduli space of these holomorphic curves. The pullback of the weighted symplectic form Σ to $\mathscr{M}\times\mathbb{P}^1$ given by the holomorphic curves is

$$p^*\Sigma = -2\lambda_\alpha\,\lambda_\beta\left(\mathrm{d}\chi_+^\alpha\wedge\mathrm{d}\chi_-^\beta - \frac{2\pi}{p}\,\mathrm{d}\tau\wedge\mathrm{d}x^{\alpha\beta} - \mathrm{i}\,\mathrm{d}_x\mathsf{g}\wedge\mathrm{d}x^{\alpha\beta}\right) \mod \mathrm{d}\lambda_\alpha\,, \tag{B.5}$$

where $\mathrm{d}_x\mathsf{g}$ is the differential of g with λ held constant. At this stage, it is not immediately possible to identify a triplet of ASD 2-forms on $\mathscr{M}$, as $\mathrm{d}_x\mathsf{g}$ depends on λ_α. To proceed, note the identity

$$\lambda_\alpha\lambda_\beta\lambda'_\gamma\lambda'_\delta\,\mathrm{d}x^{\alpha\beta}\wedge\mathrm{d}x^{\gamma\delta} = \langle\lambda\,\lambda'\rangle\lambda_\alpha\lambda'_\beta\,\mathrm{d}x^{\alpha\gamma}\wedge\mathrm{d}x^{\beta}{}_{\gamma}\,, \tag{B.6}$$

so that we can write

$$\lambda_\alpha\,\lambda_\beta\,\mathrm{d}x^{\alpha\beta}\wedge\mathrm{d}_x\mathsf{g} = \lambda_\alpha\,\lambda_\beta\,\varpi^{\alpha\beta}(x)\,, \tag{B.7}$$

where $\varpi^{\alpha\beta}$ is the 2-form on $\mathscr{M}$ defined by

$$\varpi^{\alpha\beta}(x) = -\mathrm{i}\,\iota^{(\alpha}\mathrm{d}x^{\beta)\gamma}\wedge\mathrm{d}x^{\delta}{}_{\gamma}\int_X \mathrm{d}^2\lambda\,\frac{\lambda_\delta}{\langle\iota\,\lambda\rangle}\;\ddot{f}\Big|_X\,. \tag{B.8}$$

The 2-form $\varpi^{\alpha\beta}$ can be decomposed further as

$$\varpi^{\alpha\beta} = \mathrm{i}\,\tilde{V}\,\mathrm{d}x^{\alpha\gamma}\wedge\mathrm{d}\vec{x}^{\beta}{}_{\gamma} + \mathrm{i}\,\mathrm{d}x^{\alpha\beta}\wedge\tilde{\mathrm{a}}\,, \tag{B.9}$$

for the scalar $\tilde{V}$ and 1-form $\tilde{\mathrm{a}}$

$$\tilde{V}(x) = -\frac{1}{2}\int_X \mathrm{d}^2\lambda\ \ddot{f}\Big|_X\,, \qquad \tilde{\mathrm{a}} = \mathrm{d}x^{\alpha\beta}\int_X \mathrm{d}^2\lambda\frac{\iota_\alpha\lambda_\beta}{\langle\iota\,\lambda\rangle}\ \ddot{f}\Big|_X\,. \tag{B.10}$$

In terms of these data, we can now write

$$p^*\Sigma = \frac{2\pi}{p}\,\lambda_\alpha\,\lambda_\beta\,\Sigma^{\alpha\beta}(x) \qquad \mathrm{mod}\ \mathrm{d}\lambda_\alpha\,, \tag{B.11}$$

for

$$\Sigma^{\alpha\beta} = 2\left[\mathrm{d}\tau - \frac{p}{2\pi}\left(\tilde{\mathrm{a}} - \frac{\mathrm{a}}{2}\right)\right]\wedge\mathrm{d}x^{\alpha\beta} + \frac{p}{\pi}\left(\tilde{V} + \frac{1}{2r}\right)\mathrm{d}x^{\alpha\gamma}\wedge\mathrm{d}x^{\beta}{}_{\gamma}\,. \tag{B.12}$$

a triplet of ASD 2-forms on $\mathscr{M}$.

Identifying a tetrad from $\Sigma^{\alpha\beta} = \theta^{\alpha\dot{\alpha}}\wedge\theta^{\beta}{}_{\dot{\alpha}}$, one sees that these are the ASD 2-forms associated to the Gibbons-Hawking metric

$$\mathrm{d}s^2 = \frac{(\mathrm{d}\tau + A)^2}{V} + V\,\mathrm{d}\vec{x}^2\,, \tag{B.13}$$

with

$$V = \frac{p}{2\pi}\left(\tilde{V} + \frac{1}{2r}\right)\,, \qquad A = \frac{p}{2\pi}\left(\frac{\mathrm{a}}{2} - \tilde{\mathrm{a}}\right)\,. \tag{B.14}$$

It can be checked immediately that the pair (V, A) satisfies the Bogomolny equation $\mathrm{d}V = \star_3\,\mathrm{d}A$ thanks to the identity

$$\star_3\,\mathrm{d}x^{\alpha\beta} = \mathrm{d}x^{\alpha\gamma}\wedge\mathrm{d}x^{\beta}{}_{\gamma}\,, \tag{B.15}$$

so the metric is vacuum and self-dual (i.e., hyperkähler) as required.

Note that the entire construction can be run in reverse: given a Gibbons-Hawking metric defined by (V, A), we can reconstruct the associated $f(\eta, \lambda)$ and hence the complex structure on twistor space. The SDTN case studied throughout the paper is obtained as the special case where $p = 8\pi M$ and $f = \eta^2/(4M)$, for which $\tilde{V} = 1/(4M)$ is a constant and ã is pure gauge, hence it can be reabsorbed and set to zero by a shift in τ.

Appendix C

Integration of the 2- and 3-point amplitudes

In this Appendix, we give some details on the integration of the low-point amplitudes presented in Chapters 4 and 5. Given a positive integer $n \in \mathbb{Z}_{\geq 0}$, a 3-momentum $\vec{k}$ and a pair of points z_1, z_2 on $\mathbb{CP}^1$, consider the six basis integrals

$$\mathscr{H}_n(\vec{k}, z_1, z_2) = \int \mathrm{d}^3\vec{x}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^n , \tag{C.1}$$

$$\mathscr{I}_n^0(\vec{k}, z_1, z_2) = \int \mathrm{d}^3\vec{x}\, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{1 + \zeta\bar{\zeta}} \left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n-1} , \tag{C.2}$$

$$\mathscr{I}_n^\zeta(\vec{k}, z_1, z_2) = \int \mathrm{d}^3\vec{x}\, \frac{\zeta\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{1 + \zeta\bar{\zeta}} \left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n-1} , \tag{C.3}$$

$$\mathscr{I}_n^{\bar{\zeta}}(\vec{k}, z_1, z_2) = \int \mathrm{d}^3\vec{x}\, \frac{\bar{\zeta}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{1 + \zeta\bar{\zeta}} \left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n-1} , \tag{C.4}$$

$$\mathscr{I}_n^{\zeta\bar{\zeta}}(\vec{k}, z_1, z_2) = \int \mathrm{d}^3\vec{x}\, \frac{\zeta\bar{\zeta}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{1 + \zeta\bar{\zeta}} \left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n-1} , \tag{C.5}$$

$$\mathscr{K}_n(\vec{k}, z_1, z_2) = \int \mathrm{d}^3\vec{x}\, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{r} \left[\frac{r(\zeta - z_1)(\bar{\zeta}z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n-1} . \tag{C.6}$$

These integrals will decompose the 2-point amplitudes. Similarly, for the 3-point amplitudes we will need the integrals

$$\mathscr{V}_{n_2,n_3}(\vec{k}, z_1, z_2, z_3) = \int \mathrm{d}^3\vec{x}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \left[\frac{r(\zeta - z_1)(\bar{\zeta} z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n_2} \left[\frac{r(\zeta - z_1)(\bar{\zeta} z_3 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n_3}, \tag{C.7}$$

$$\mathscr{W}_{n_2,n_3}(\vec{k}, z_1, z_2, z_3) = \int \mathrm{d}^3\vec{x}\, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{r} \left[\frac{r(\zeta - z_1)(\bar{\zeta} z_2 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n_2} \left[\frac{r(\zeta - z_1)(\bar{\zeta} z_3 + 1)}{1 + \zeta\bar{\zeta}}\right]^{n_3}, \tag{C.8}$$

where z_3 is a third point on $\mathbb{CP}^1$ and we assume $n_2, n_3 \in \mathbb{Z}_{\geq 0}$. We will compute such integrals assuming $\vec{k}$ to be real and analytically continue the final results to complex $\vec{k}$. This is equivalent to Wick rotating $t \mapsto \mathrm{i}t$ to turn the SDTN metric into a complex metric on a real slice on which the scattering particles have Lorentzian momenta. This follows the computation of scattering amplitudes on instantons via Wick rotation, first developed in [260, 261]. The integrals (C.1)-(C.8) are oscillatory integrals so we need to use an iε prescription to make the integrals convergent. Thus we introduce a Gaussian factor of $\mathrm{e}^{-\varepsilon(|\zeta|^2 + r)}$ into the integrand, and take the limit $\varepsilon \to 0$. We will ignore the distributional terms supported on $|\vec{k}| = 0$ that will also be present which correspond to forward scattering contributions. We discard these taking merely the *naïve* $\varepsilon \to 0$ limit.

Decomposition of the amplitudes

The gluon amplitude (4.53) is simply given by

$$\mathcal{A}_2 = -\pi\delta(\omega_1 + \omega_2)\delta_{-e_1,e_2}\langle 1\,2\rangle^2 \mathscr{H}_{2e_2}(\vec{k}, z_1, z_2)\,. \tag{C.9}$$

In terms of the dyad $\iota^\alpha = (1,0)$, $o^\alpha = (0,-1)$, the scalar amplitude (5.85) can be expressed as

$$
\begin{aligned}
\mathcal{S}_2 = 2\pi\delta_{-\omega_1,\omega_2}\langle 1\,2\rangle \Big[& -[12]\mathscr{H}_{4M\omega_2}(\vec{k},z_1,z_2) - \frac{1}{2}(4M\omega_2)^2\mathscr{K}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& - 4M\omega_2(\iota_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{2\,\dot{\alpha}} + z_1 o_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{1\,\dot{\alpha}})\mathscr{J}^{0}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + (o_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{1\,\dot{\alpha}} + o_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{2\,\dot{\alpha}})\mathscr{J}^{\zeta}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& - (z_1\iota_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{1\,\dot{\alpha}} + z_2\iota_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{2\,\dot{\alpha}})\mathscr{J}^{\bar{\zeta}}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + (z_2 o_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{2\,\dot{\alpha}} + \iota_\alpha T^{\alpha\dot{\alpha}}\tilde{\kappa}_{1\,\dot{\alpha}})\mathscr{J}^{\zeta\bar{\zeta}}_{4M\omega_2}(\vec{k},z_1,z_2)\Big] \,.
\end{aligned} \tag{C.10}
$$

Similarly, the photon amplitude (5.87) is

$$
\mathcal{A}_2^{\mathrm{ph}} = 2\pi\langle 1\,2\rangle^2\delta_{-\omega_1,\omega_2}(\mathscr{H}_{4M\omega_2}(\vec{k},z_1,z_2) + 2M\mathscr{K}_{1+4M\omega_2}(\vec{k},z_1,z_2))\,. \tag{C.11}
$$

The graviton amplitude (5.89) decomposes as

$$
\begin{aligned}
\mathcal{M}_2 = -\frac{\langle 1\,2\rangle^3\eta_{1\,\dot{\alpha}}\eta_{2\,\dot{\beta}}}{[\eta_1\,1][\eta_2\,2]}2\pi\delta_{-\omega_1,\omega_2}\Big(& \epsilon^{\dot{\alpha}\dot{\beta}}\mathscr{H}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + 4MT^{\alpha\dot{\alpha}}T^{\beta\dot{\beta}}(\kappa_{1\,\alpha}\iota_\beta - z_1 o_\alpha\kappa_{2\,\beta})\mathscr{J}^{0}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + 4MT^{\alpha\dot{\alpha}}T^{\beta\dot{\beta}}(-\kappa_{1\,\alpha}o_\beta + o_\alpha\kappa_{2\,\beta})\mathscr{J}^{\zeta}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + 4MT^{\alpha\dot{\alpha}}T^{\beta\dot{\beta}}(z_2\kappa_{1\,\alpha}\iota_\beta - z_1\iota_\alpha\kappa_{2\,\beta})\mathscr{J}^{\bar{\zeta}}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + 4MT^{\alpha\dot{\alpha}}T^{\beta\dot{\beta}}(-z_2\kappa_{1\,\alpha}o_\beta + \iota_\alpha\kappa_{2\,\beta})\mathscr{J}^{\zeta\bar{\zeta}}_{4M\omega_2}(\vec{k},z_1,z_2) \\
& + 8M^2T^{\alpha\dot{\alpha}}\kappa_{1\,\alpha}T^{\beta\dot{\beta}}\kappa_{2,\beta}\mathscr{K}_{4M\omega_2}(\vec{k},z_1,z_2)\Big) \,,
\end{aligned} \tag{C.12}
$$

The 3-point gluon amplitudes (4.55) is

$$
\mathcal{A}_3 = 4\pi\mathrm{i}g\,\delta(\omega)\,\delta(e)\frac{\langle r\,s\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\mathscr{V}_{2e_2,2e_3}(\vec{k},z_1,z_2,z_3)\,. \tag{C.13}
$$

Finally, the 3-point graviton amplitude (5.142) admits the decomposition

$$\mathcal{M}_3 = 4\pi \mathrm{i}\kappa\, \delta(\omega)\, \delta(e) \frac{\langle 1\, 2\rangle^6}{\langle 1\, 3\rangle^2 \langle 2\, 3\rangle^2} (\mathscr{V}_{4M\omega_2, 4M\omega_3}(\vec{k}, z_1, z_2, z_3) + 2M \mathscr{W}_{4M\omega_2, 4M\omega_3}(\vec{k}, z_1, z_2, z_3) \,. \tag{C.14}$$

In all cases, $\vec{k}$ denotes the total 3-momentum, that is $\vec{k}_1 + \vec{k}_2$ for the 2-point amplitudes and $\vec{k}_1 + \vec{k}_2 + \vec{k}_3$ for the 3-point amplitudes.

Computation of $\mathscr{H}_n$

We will now give a detailed derivation of

$$\mathscr{H}_n(\vec{k}, z_1, z_2) = 4\pi n\, n! \frac{z_{12}(\mathrm{i}Q)^{n-1}}{(\vec{k}^2)^{n+1}} \,, \tag{C.15}$$

where $z_{12} = z_1 - z_2$ and we introduced the quantity

$$Q = k^3(w - z_1 - z_2 - \bar{w} z_1 z_2) \,, \qquad w = \frac{k^1 + \mathrm{i}k^2}{k^3} \,, \qquad \bar{w} = \frac{k^1 - \mathrm{i}k^2}{k^3} \,, \tag{C.16}$$

in terms of the components of $\vec{k}$. We can parameterize the 3-vectors $\vec{x}$ and $\vec{k}$ as

$$\vec{x} = \frac{r}{1 + \zeta\bar{\zeta}} (\zeta + \bar{\zeta}, \mathrm{i}(\bar{\zeta} - \zeta), 1 - \zeta\bar{\zeta}) \,, \qquad \vec{k} = \frac{1}{2} k^3 (w + \bar{w}, \mathrm{i}(\bar{w} - w), 2) \,, \tag{C.17}$$

so that

$$\vec{k} \cdot \vec{x} = \frac{k^3 r}{1 + \zeta\bar{\zeta}} (\zeta\bar{w} + \bar{\zeta} w + 1 - \zeta\bar{\zeta}) \,, \qquad \vec{k}^2 = (k^3)^2 (1 + w\bar{w}) \,. \tag{C.18}$$

Upon performing the rescaling $r \mapsto r(1 + \zeta\bar{\zeta})$, the integral (C.1) reduces to

$$\mathscr{H}_n(\vec{k}, z_1, z_2) = 2\mathrm{i} \int_0^\infty \mathrm{d}r\, r^2 \int_{\mathbb{C}} \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, (1 + \zeta\bar{\zeta}) [r(\zeta - z_1)(\bar{\zeta} z_2 + 1)]^n \mathrm{e}^{\mathrm{i}k^3 r(\zeta\bar{w} + \bar{\zeta} w + 1 - \zeta\bar{\zeta})} \,, \tag{C.19}$$

so that introducing a formal parameter t, we can write

$$\mathscr{H}_n(\vec{k}, z_1, z_2) = (-\mathrm{i}\partial_t)^n \, \hat{\mathscr{H}}(\vec{k}, z_1, z_2; t)\Big|_{t=0} \,, \tag{C.20}$$

in terms of the 'master integral'

$$\hat{\mathscr{H}}(\vec{k}, z_1, z_2; t) = 2\mathrm{i}\int_0^\infty \mathrm{d}r\, r^2 \int_{\mathbb{C}} \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, (1+\zeta\bar{\zeta})\mathrm{e}^{\mathrm{i}k^3 r(\zeta\bar{w}+\bar{\zeta}w+1-\zeta\bar{\zeta})+\mathrm{i}tr(\zeta-z_1)(\bar{\zeta}z_2+1)} \,. \tag{C.21}$$

The advantage in computing $\hat{\mathscr{H}}$ instead of evaluating $\mathscr{H}_n$ directly is manifest, as the former is a simple Gaussian integral over the stereographic coordinates: introducing the parameters

$$A = tz_2 - k^3\,, \quad B = k^3\bar{w} + t\,, \quad C = k^3 w - tz_1 z_2\,, \quad D = k^3 - tz_1\,, \tag{C.22}$$

we can express the master integral as

$$\hat{\mathscr{H}}(\vec{k}, z_1, z_2; t) = 2\mathrm{i}\int_0^\infty \mathrm{d}r\, r^2 \int_{\mathbb{C}} \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, (1+\zeta\bar{\zeta})\mathrm{e}^{\mathrm{i}r(A\zeta\bar{\zeta}+B\zeta+C\bar{\zeta}+D)} \,. \tag{C.23}$$

As we will set $t = 0$ at the end of the calculation, we are free to think of t to be small enough so that the imaginary part of tz_2 does not dominate over the (implicit) $\mathrm{i}\varepsilon$ regulator.

Performing the Gaussian integral leaves us with a Mellin integral over the radial variable

$$\hat{\mathscr{H}}(\vec{k}, z_1, z_2; t) = \frac{4\pi\mathrm{i}}{A^3}\int_0^\infty \mathrm{d}r\, (r(A^2+BC)+\mathrm{i}A)\mathrm{e}^{\mathrm{i}r(AD-BC)/A} \,. \tag{C.24}$$

Since we are interested in taking derivatives of the master integral at $t = 0$ and since the function $(AD - BC)/A$ has an imaginary part at least linear in t, the implied $\mathrm{i}\varepsilon$ prescription allows us to make this residual Mellin integral convergent as well. In this way, we finally find

$$\hat{\mathscr{H}}(\vec{k}, z_1, z_2; t) = -\frac{4\pi\mathrm{i}(A+D)}{(AD-BC)^2} = \frac{4\pi\mathrm{i}tz_{12}}{(\vec{k}^{\,2}+tQ)^2}\,, \tag{C.25}$$

where in the second line we plugged in the values (C.22). Upon taking n derivatives, we recover (C.15).

Other basis integrals

The remaining integrals (C.2)-(C.8) can be obtained exactly with the same methods, – see the Appendices of [2,3] for the full details. Here, we will simply state the results

$$\mathscr{J}_n^0(\vec{k}, z_1, z_2) = -4\pi(n-1)!\frac{(\mathrm{i}Q)^{n-2}}{(\vec{k}^2)^n}\left(\frac{nk^3Q}{\vec{k}^2} + (n-1)z_2\right)\,, \tag{C.26}$$

$$\mathscr{J}_n^{\zeta}(\vec{k}, z_1, z_2) = -4\pi(n-1)!\frac{(\mathrm{i}Q)^{n-2}}{(\vec{k}^2)^n}\left(\frac{nk^3wQ}{\vec{k}^2} + (n-1)z_1z_2\right)\,, \tag{C.27}$$

$$\mathscr{J}_n^{\bar{\zeta}}(\vec{k}, z_1, z_2) = -4\pi(n-1)!\frac{(\mathrm{i}Q)^{n-2}}{(\vec{k}^2)^n}\left(\frac{nk^3\bar{w}Q}{\vec{k}^2} - (n-1)\right)\,, \tag{C.28}$$

$$\mathscr{J}_n^{\zeta\bar{\zeta}}(\vec{k}, z_1, z_2) = 4\pi(n-1)!\frac{(\mathrm{i}Q)^{n-2}}{(\vec{k}^2)^n}\left(\frac{nk^3Q}{\vec{k}^2} + (n-1)z_1\right)\,, \tag{C.29}$$

$$\mathscr{K}_n(\vec{k}, z_1, z_2) = 4\pi(n-1)!\frac{(\mathrm{i}Q)^{n-1}}{(\vec{k}^2)^n}\,, \tag{C.30}$$

$$\mathscr{V}_{n_2,n_3}(\vec{k}, z_1, z_2, z_3) = -4\pi(n_2+n_3)!\frac{(\mathrm{i}\alpha_{21})^{n_2}(\mathrm{i}\alpha_{31})^{n_3}}{(\vec{k}^2)^{n_2+n_3+1}}\left(\frac{n_2z_{21}}{\alpha_{21}} + \frac{n_3z_{31}}{\alpha_{31}}\right)\,, \tag{C.31}$$

$$\mathscr{W}_{n_2,n_3}(\vec{k}, z_1, z_2, z_3) = -4\pi(n_2+n_3)!\frac{(\mathrm{i}\alpha_{21})^{n_2}(\mathrm{i}\alpha_{31})^{n_3}}{(\vec{k}^2)^{n_2+n_3+1}}\,. \tag{C.32}$$

In (C.31)-(C.32), the coefficients α_{21} and α_{31} are precisely the ones given in (4.57). Feeding these integrals back into (C.9)-(C.14) and, in the case of the 2-point amplitudes, using the on-shell expressions

$$\vec{k}^{\,2} = -\frac{4z_{12}\tilde{z}_{12}\omega_2^2}{(1+z_1\tilde{z}_1)(1+z_2\tilde{z}_2)}\,, \tag{C.33}$$

$$k^3 = -2\mathrm{i}\omega_2\frac{z_1\tilde{z}_1 - z_2\tilde{z}_2}{(1+z_1\tilde{z}_1)(1+z_2\tilde{z}_2)}\,, \tag{C.34}$$

$$k^3 w = -2\mathrm{i}\omega_2\frac{-z_{12} + \tilde{z}_{12}z_1z_2}{(1+z_1\tilde{z}_1)(1+z_2\tilde{z}_2)}\,, \tag{C.35}$$

$$k^3 \bar{w} = -2\mathrm{i}\omega_2\frac{-\tilde{z}_{12} + z_{12}\tilde{z}_1\tilde{z}_2}{(1+z_1\tilde{z}_1)(1+z_2\tilde{z}_2)}\,, \tag{C.36}$$

$$Q = -2\mathrm{i}\omega_1 z_{12}\,. \tag{C.37}$$

valid on the support of $\omega_1 + \omega_2 = 0$, we finally recover the vanishing of the scalar and photon amplitudes around SDTN, as well as the expressions (4.53)-(4.55)-(5.89)-(5.142).

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