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Constructing Finitely Presented Simple Groups That Contain Grigorchuk Groups

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We construct infinite finitely presented simple groups that have subgroups isomorphic to Grigorchuk groups. We also prove that up to one possible exception all previously known finitely presented simple groups are torsion locally finite.

Key Words: finitely presented infinite simple groups; Grigorchuk groups; torsion locally finite.

1. INTRODUCTION

In 1965 Thompson gave the first example of a finitely presented infinite simple group (see [16] for a description). This group will be the group $G_{2,1}$ in this work. Higman later generalized this construction to give a countable family of such groups [7] and Brown [3] discovered another family of finitely presented simple groups which are all subgroups of the groups given by Higman. In [8] Higman proved that a finitely generated group is recursively presented if and only if it can be embedded in a finitely presented group. To say that a finitely generated group is recursively presented means precisely that there is a recursively enumerable set of defining relations. The set of all relations (words representing the identity) is recursive if and only if the group has a solvable word problem; thus it seems reasonable to ask if there is an analogue of Higman's theorem which characterizes finitely generated groups with a solvable word problems. Noting that Kuznetsov showed in [9] that any finitely generated subgroup of a finitely presented simple group has a solvable word problem, one might pose the following conjecture (see [2]).

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Higman's Conjecture. A finitely generated group has a solvable word problem if and only if it can be embedded in a finitely presented simple group.

The most comprehensive result in this direction at the moment is a theorem of Thompson [16] saying that a finitely generated group has a solvable word problem if and only if it can be embedded in a finitely generated simple subgroup of a finitely presented group.

With this conjecture in mind, we wish to find new examples of finitely presented simple groups and to enlarge the knowledge of finitely generated subgroups of such groups. Higman [7] showed that the groups he constructed do not have subgroups isomorphic to $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$. In [13] Scott described a method for constructing finitely presented simple groups and used it to embed $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ in such groups, for each $n \in \mathbb{N}$ [14]. Here we generalize her construction to show that any Grigorchuk group (as defined in [6]) which is defined by an almost f-periodic sequence can be embedded in a finitely presented simple group (Theorem 2). Besides yielding new finitely presented simple groups, this also gives a new proof of the solvability of the word problem for these specific Grigorchuk groups.

This work is organized as follows. In Section 2 we give a brief review of the groups $G_{n,r}$ and $\mathscr{G}_{n,r}$, following [13], and define the Grigorchuk groups in this language. Section 3 provides the tools that are used in Section 4 to prove the main result. In Section 5 we establish a result which shows that the groups constructed in this work are very likely to be the first finitely presented simple groups that have infinite finitely generated torsion subgroups (cf. Theorem 3).

2. PRELIMINARIES

In Section 2.1 we will give a brief introduction to the groups $G_{n,r}$ and $\mathscr{G}_{n,r}$. The former were introduced by Higman [7] as generalizations of a finitely presented simple group invented by Thompson in [16]. Our interest in these groups comes from the following fact (Lemma 20 in [13]). Each subgroup K of $\mathscr{G}_{n,r}$ which contains $G_{n,r}$ has a simple commutator subgroup. The simple groups that are constructed in this work arise exactly in this way. We will describe two different ways of representing elements of the groups $G_{n,r}$ and $\mathscr{G}_{n,r}$: one more precise way, the symbols (Section 2.2); and a more visualizable way, tree-diagrams (Section 2.3). In Section 2.4 we will define the Grigorchuk groups as subgroups of $\mathscr{G}_{n,r}$ for suitable r and n.

2.1. *Inescapable Isomorphisms*

For more details of the following we refer the reader to [13]; we mostly use notation from there. In particular, we write all maps on the right.

Let $n, r \in \mathbb{N}$, $n \geq 2$, let W_n be the set of all finite words in the alphabet $\mathscr{A}_n = \{a_1, a_2, \ldots, a_n\}$, and let X_r be a set of distinct symbols $\{x_1, \ldots, x_r\}$ $(a_i \neq x_j)$. If U and V are sets of words we define $UV = \{uv | u \in U, v \in V\}$, where uv denotes the concatenation of u and v.

The binary relation *u is a prefix of v* (written $u \leq v$) equips $X_r W_n$ with a partial order. A subset $U \subset X_r W_n$ is called *independent* if its elements are pairwise $\leq$ -incomparable and is said to be a *basis* if it is maximal among independent subsets. If U is a basis then $S = UW_n$ is an *inescapable subspace* of $X_r W_n$, meaning that $SW_n \subset S$ and for any $v \in X_r W_n$ there exists $w \in W_n$ such that $vw \in S$. Note that any inescapable subspace S has the form $S = B_S W_n$ for a unique basis B_S. If B_S is finite then one says that S is *finitely based*. We also mention that the intersection of two inescapable (finitely based) subspaces is again an inescapable (finitely based) subspace.

If U is a finite basis, $u \in U$, then V defined as $(U \setminus \{u\}) \cup \{ua_1, ua_2, \ldots, ua_n\}$ is also a finite basis. We call V a *simple expansion of U*. Furthermore, the finite basis Z is called an *expansion* of the finite basis U if there exist $m \in \mathbb{N}$ and finite bases $U_0, \ldots, U_m$ such that $U = U_0$, $Z = U_m$, and U_{i+1} is a simple expansion of U_i for $0 \leq i \leq m - 1$. It is not difficult to see that any finite basis is an expansion of the basis X_r.

An *inescapable isomorphism of $X_r W_n$* is a bijective map ϕ between inescapable subspaces S and T such that $(sw)^\phi = s^\phi w$ for all $s \in S$, $w \in W_n$. The inescapable isomorphism $\psi \colon Q \to R$ is an *extension of ϕ* if $S \subset Q$ and the restriction of ψ to S is equal to ϕ. Note that if S' is an inescapable subspace contained in S, then S'^ϕ is an inescapable subspace contained in T. Furthermore, any inescapable isomorphism has a unique maximal extension; see [13].

Let us call S the domain of ϕ (dom ϕ) and T the range of ϕ (rg ϕ) and let ψ be another inescapable isomorphism. As the map $\phi\psi$ is defined on the inescapable subspace $(\mathrm{rg}\,\phi \cap \mathrm{dom}\,\psi)^{\phi^{-1}}$ and behaves like an inescapable isomorphism, we can define a group structure on the set $\mathscr{G}_{n,r}$ of *maximal inescapable isomorphisms* of $X_r W_n$ by defining $\phi\psi$ to be the maximal extension of this map. This is the definition of the group $\mathscr{G}_{n,r}$, and $G_{n,r}$ is defined to be the subgroup of *maximal inescapable isomorphisms between finitely based inescapable subspaces* (these are called maximal cofinite isomorphisms in [13]). Any element $\phi \colon S \to T$ in $\mathscr{G}_{n,r}$ is completely determined by its restriction to the basis B_S (note that $B_S^\phi = B_T$).

2.2. *The Symbol Language*

We are now going to describe a way of representing inescapable isomorphisms. Another way is described in Section 2.3. Throughout this work we shall implicitly identify $X_1 W_n$ with W_n via $x_1 w \mapsto w$.

Let us define a *symbol* to be a scheme of the form

$$\Gamma = \begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ g_1 & g_2 & \cdots & g_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}, \tag{1}$$

where the g_i are elements of $\mathscr{G}_{n,1}$ and $\{u_1, u_2, \ldots, u_s\}$ and $\{v_1, v_2, \ldots, v_s\}$ are finite bases, called the *top row* and *bottom row* of Γ, respectively; we write $\mathrm{top}(\Gamma) = \{u_1, u_2, \ldots, u_s\}$ and $\mathrm{bot}(\Gamma) = \{v_1, v_2, \ldots, v_s\}$. Note that, although rows usually have an order, here they are defined as sets only. This symbol Γ defines the inescapable isomorphism

$$UW_n \to VW_n$$
$$u_i w \mapsto v_i(w)^{g_i}, \qquad \text{whenever } w^{g_i} \text{ is defined}, w \in W_n,$$

the maximalization of which we define to be *the element* $\phi \in \mathscr{G}_{n,r}$ *with symbol* Γ. Observe that there may well be different symbols for the same element; e.g., each permutation of the columns of the symbol Γ from (1) is a symbol which defines the same element. Also, if the element g_i in Γ from (1) has the symbol

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_t \\ h_1 & h_2 & \cdots & h_t \\ z_1 & z_2 & \cdots & z_t \end{pmatrix}$$

then Γ and the symbol

$$\Delta = \begin{pmatrix} u_1 & \cdots & u_{i-1} & u_i y_1 & \cdots & u_i y_t & u_{i+1} & \cdots & u_s \\ g_1 & \cdots & g_{i-1} & h_1 & \cdots & h_t & g_{i+1} & \cdots & g_s \\ v_1 & \cdots & v_{i-1} & v_i z_1 & \cdots & v_i z_t & v_{i+1} & \cdots & v_s \end{pmatrix}$$

define the same inescapable isomorphism. We call Δ an *expansion of* Γ; if $t = n$ then Δ is called a *simple expansion of* Γ.

A more precise notion is that of an E-*symbol*, where E is a subgroup of $\mathscr{G}_{n,1}$; i.e., all the g_i in (1) are elements of E. Note that $G_{n,r}$ consists precisely of those elements that have a 1-symbol, where 1 denotes the trivial group. We also remark that $\mathscr{G}_{n,r}$ embeds in $\mathscr{G}_{n,1}$ for all $n, r \in \mathbb{N}$, showing that $g_i \in \mathscr{G}_{n,1}$ in (1) is no serious restriction.

The next lemma is Lemma 5 in [13].

LEMMA 1. *If*

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ g_1 & \cdots & g_s \\ v_1 & \cdots & v_s \end{pmatrix} \quad \textit{and} \quad \Delta = \begin{pmatrix} v_1 & \cdots & v_s \\ h_1 & \cdots & h_s \\ w_1 & \cdots & w_s \end{pmatrix}$$

are symbols for the inescapable isomorphisms ϕ and ψ, respectively, then

$$\begin{pmatrix} u_1 & \cdots & u_s \\ g_1 h_1 & \cdots & g_s h_s \\ w_1 & \cdots & w_s \end{pmatrix}$$

is a symbol for $\phi\psi$.

For symbols Γ and Δ we say that *the combination* $\Gamma\Delta$ *exists* if $\mathrm{bot}(\Gamma) = \mathrm{top}(\Delta)$. In this case $\Gamma\Delta$ is the symbol for $\phi\psi$ given in Lemma 1. Similarly, we say that the combination $\Gamma_1 \cdots \Gamma_m$ of the symbols $\Gamma_1, \dots, \Gamma_m$ exists if $\mathrm{bot}(\Gamma_k) = \mathrm{top}(\Gamma_{k+1})$ for $1 \leq k \leq m-1$ and we identify this combination with the symbol given by repeated application of Lemma 1. Observe that this definition ignores the order of the columns of the involved symbols as they appear on this paper. This suggests thinking of symbols as collections of columns.

2.3. *Tree-Diagrams*

Here we will describe the group $G_{2,1}$ in a different way, namely, in terms of tree-diagrams. It will be clear that all the groups $G_{n,r}$ can be described in this fashion but the diagrams get a little more complex. This should encourage readers to draw tree-diagrams whenever they fear getting lost in the symbol notation. The precise development of this approach can be found in [5].

It is not difficult to see that the set of finite bases of W_2 is in bijection with the set of finite rooted binary trees. Figure 1, for example, shows the tree corresponding to the basis $\{a_1a_1a_1, a_1a_1a_2, a_1a_2, a_2a_1, a_2a_2\}$.

Since a finitely based inescapable isomorphism $\phi\colon S \to T$ is uniquely determined by its restriction to B_S and, what is more, $B_S^{\phi} = B_T$, we can

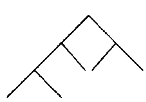

FIGURE 1

describe ϕ using the two trees corresponding to B_S and B_T, this time with the leaves labelled in such a way that $b \in B_S$ and $c \in B_T$ have the same label if $b^\phi = c$. The element defined by the symbol

$$\Delta = \begin{pmatrix} a_1 & a_2 a_1 & a_2 a_2 \\ 1 & 1 & 1 \\ a_1 a_2 & a_1 a_1 & a_2 \end{pmatrix}$$

can be regarded as the tree-diagram in Fig. 2.

If $B_S = B_T$ then ϕ induces a permutation of B_S. In this case we sometimes use a tree-diagram with only one tree and arrows indicating the action of ϕ. Figure 3 gives such an example for the element defined by the symbol

$$\Delta = \begin{pmatrix} a_1 a_1 & a_1 a_2 & a_2 \\ 1 & 1 & 1 \\ a_1 a_2 & a_1 a_1 & a_2 \end{pmatrix}.$$

2.4. *Grigorchuk Groups*

Since from now on we are only concerned with the case $r = 1$, we use the set W_n instead of $X_1 W_n$, identifying w with $x_1 w$. We shall also use the inductively defined notation $w^{n+1} = w^n w$, where $w^1 = w$ and $n \in \mathbb{N}$.

For our purpose it is sufficient to know that any Grigorchuk group is defined in the following way by two infinite sequences $\overline{\omega}$ and $\tilde{\omega}$ with values in $\{0, 1, \ldots, p-1\}$, where p is a fixed prime. Given an infinite sequence $\omega = \omega(0)\omega(1)\cdots$ with values in $\{0, 1, \ldots, p-1\}$, we define the

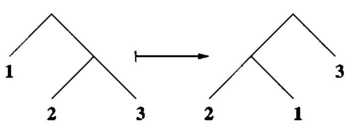

FIGURE 2

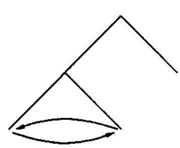

FIGURE 3

inescapable isomorphism b_ω of W_p by its action on the basis

$$\left\{a_p^r a_1 a_1, \dots, a_p^r a_1 a_p, a_p^r a_2, \dots, a_p^r a_{p-1} | r \geq 0\right\};$$

the action is

$$\begin{aligned} a_p^r a_1 a_i &\mapsto a_p^r a_1\left(a_i \pi^{\omega(r)}\right), && 1 \leq i \leq p, \\ a_p^r a_j &\mapsto a_p^r a_j, && 2 \leq j \leq p-1, \end{aligned}$$

where π denotes the cyclic permutation $(a_1 a_2 \cdots a_p)$. See Fig. 4 for an example.

Now given a sequence ω with values in $\{0, 1, \dots, p\}$ define the two maps $\bar{\ }$ and $\tilde{\ }: \{0, \dots, p\} \to \{0, \dots, p-1\}$ as in [6], i.e.,

$$\bar{x} = \begin{cases} 0, & \text{if } x = p \\ 1, & \text{if } x \neq p \end{cases} \quad \text{and} \quad \tilde{x} = \begin{cases} 1, & \text{if } x = p \\ x, & \text{if } x \neq p. \end{cases}$$

The *Grigorchuk group* G_ω is isomorphic to the subgroup of $\mathscr{G}_{p,1}$ generated by $b_{\bar{\omega}}, b_{\tilde{\omega}}$, and the inescapable isomorphism γ with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ 1 & 1 & \cdots & 1 & 1 \\ a_2 & a_3 & \cdots & a_p & a_1 \end{pmatrix}. \tag{2}$$

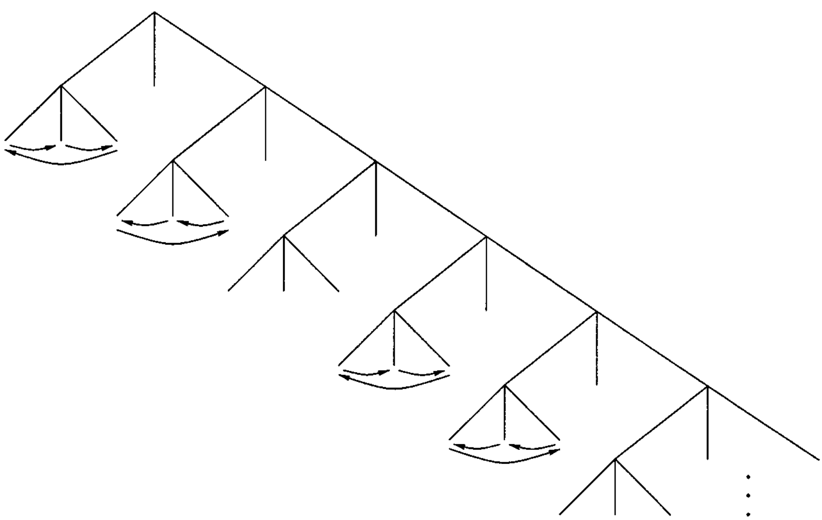

FIG. 4. The element b_ω with $\omega = 120120120\cdots$ and $p = 3$.

The curious reader will find that in [6] the groups G_ω are described by their action on the set $\mathscr{C}$ of all open intervals in the real line of the form $(t/p^k, (t+1)/p^k)$ with $0 \leq t \leq p^k - 1$ and $1 \leq k \in \mathbb{N}$. Later it became popular to describe the groups G_ω as subgroups of the automorphism group of a rooted p-regular tree (see, for example, [1, 12]). The isomorphism between G_ω as described above and G_ω as described in [6] is induced by the bijection between W_p and the set $\mathscr{C}$ that lets $a_{i_1} a_{i_2} \cdots a_{i_k} \in QW_p$ correspond to $(t/p^k, (t+1)/p^k)$, where $t = \sum_{j=1}^{k} (i_j - 1)/p^j$. It is also worth noting that the nature of the maps $\bar{\ }$ and $\tilde{\ }$ is absolutely irrelevant to the arguments in this work. They were used enormously in [6] to deduce that G_ω is a finitely generated infinite torsion group of intermediate growth in case each of the letters $0, 1, \ldots, p$ occurs infinitely many times in ω.

3. THE FUNDAMENTALS

In [13] Scott introduced the notion of E-expansibility to show that certain subgroups of $\mathscr{G}_{n,r}$ are finitely presented. Let us state an equivalent definition of E-expansibility.

Let E and H be subgroups of $\mathscr{G}_{n,r}$ with $E \subset H$. Then H is *E-expansible* if there exists a generating system Y for H such that, if h is in H and $h = y_1 \cdots y_m$ with $y_i \in Y^{\pm 1}$ for $1 \leq i \leq m$, then there are E-symbols $\Gamma_1, \ldots, \Gamma_m$ for $y_1, \ldots, y_m$, respectively, such that the combination $\Gamma_1 \cdots \Gamma_m$ exists.

In Section 3.1 we introduce fake-E-symbols and the notion of fake-E-expansibility which we then relate to E-expansibility (Propositions 1 and 2). We then give an example of a fake-E-expansible group E (Section 3.2) which will be used in Section 4 to construct the finitely presented simple groups we are looking for.

3.1. *Fake Symbols and Fake Expansibility*

Let S_n be the subgroup of $G_{n,1}$ consisting of the elements which have a symbol of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \\ a_1\pi & a_2\pi & \cdots & a_n\pi \end{pmatrix}, \tag{3}$$

where π is some permutation of the set $\{a_1, \ldots, a_n\}$. Let E and H be subgroups of $\mathscr{G}_{n,1}$ with $E \subset H$. Then a *fake-E-symbol for* $h \in H$ is a symbol for h of the form

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ x_1 & x_2 & \cdots & x_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

where $x_i \in E \cup S_n$ for $1 \le i \le s$. Furthermore we call H *fake-E-expansible* if there exists a set of generators Y for H such that, if h is in H and $h = y_1 \cdots y_m$ with $y_i \in Y^{\pm 1}$ for $1 \le i \le m$, then h has a symbol which is the combination $\Gamma_1 \cdots \Gamma_m$ of fake-E-symbols $\Gamma_1, \dots, \Gamma_m$ for $y_1, \dots, y_m$, respectively. Note that this combination is rather a $\langle E, S_n \rangle$-symbol than a fake-E-symbol.

PROPOSITION 1. *Let E and H be subgroups of $\mathscr{G}_{n,1}$ and $E \subset H$. If H is fake-E-expansible and every $e \in E$ has a fake-E-symbol of the form*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ x_1 & x_2 & \cdots & x_n \\ a_1\pi & a_2\pi & \cdots & a_n\pi \end{pmatrix} \tag{4}$$

for some permutation π (possibly depending on e) of the set $\{a_1, \dots, a_n\}$, then H is E-expansible.

Proof. We must show that there is a generating system Y for H such that, if $y_1 \cdots y_m$ is any word in $Y^{\pm 1}$, then there are E-symbols $\Gamma_1, \dots, \Gamma_m$ for $y_1, \dots, y_m$, respectively, such that their combination $\Gamma_1 \cdots \Gamma_m$ exists. By (Section 2.2, Lemma 1), this combination is then an E-symbol for $y_1 \cdots y_m \in H$.

To this end let Y be a generating system for H such that H is fake-E-expansible with respect to Y. Let $y_1 \cdots y_m$ be any word in $Y^{\pm 1}$ and choose fake-E-symbols $\Gamma_1, \dots, \Gamma_m$ for $y_1, \dots, y_m$, respectively, such that the combination $\Gamma_1 \cdots \Gamma_m$ exists. Observe that all of the Γ_k have the same number of columns, C say, and that we can assume (by reordering if necessary) that the bottom row of Γ_k is the same as the top row of Γ_{k+1} as ordered sets, for $1 \le k \le m-1$. Now enumerate the columns of each Γ_j, $1 \le j \le m$, from left to right by $1, \dots, C$ and apply the following procedure.

Start with Step 1 and stop when you reach Step $m+1$, where for $1 \le i \in \mathbb{N}$ Step i is defined as follows.

Step i. Let X be the set of all numbers of columns whose middle row entry in Γ_i is a non-trivial element of S_n.

(a) If X is empty then go to Step $i+1$.

(b) If X is not empty then do the following. For $1 \le j \le m$ simply expand all the columns numbered by some $x \in X$ in Γ_j and denote the resulting symbol by Γ_j again. Now reorder the columns in $\Gamma_1, \dots, \Gamma_m$ so that the bottom row of Γ_k is equal to the top row of Γ_{k+1} as ordered sets, for $1 \le k \le m-1$. Replace C by $C + |X|(n-1)$ and go to Step 1.

Note that (3) and (4) ensure we can execute part (b) and that C will again be the number of columns of each of the Γ_i. To prove that the algorithm gives the required result we will show, by induction on $i \le m+1$, that

(α) Step i will be reached and

(β) at the time of reaching Step i all the symbols $\Gamma_1, \dots, \Gamma_{i-1}$ are E-symbols.

Part (β) holds because the only way to reach Step i is via Step $i-1$ part (a), $i \ge 2$.

As Step 1 is reached by definition, to prove (α) it suffices to show that Step i will finally lead to Step $i+1$, $1 \le i \le m$. First let $i = 1$. As part (a) of Step 1 leads immediately to Step 2, we only need to check part (b). By the definition of X and (3) it is clear that at the end of Step 1 part (b) Γ_1 is an E-symbol, and the procedure continues with Step 1 part (a), which leads to Step 2.

Now consider i with $m \ge i > 1$. Again Step i part (a) leads to Step $i+1$. If, on the other hand, we have to execute Step i part (b) then the definition of X and (3) show that, at the end of this, Γ_i is an E-symbol. Furthermore, using (β), if $1 \le c \le C$ and for some j with $1 \le j \le i-1$ the cth column of Γ_j has a now non-trivial middle row entry in S_n, then the cth column of Γ_i has trivial middle row entry. Thus, while going from Step 1 up to Step i again, only columns with trivial middle row entry in Γ_i are affected. Hence, using the induction hypothesis, we will come to Step i part (a), which leads to Step $i+1$. This completes the proof of the proposition.

The next result is to show that with a little more care we can even deduce the E-expansibility of $\mathscr{H} = \langle H, G_{n,1} \rangle$.

PROPOSITION 2. *Let E and H be subgroups of $\mathscr{G}_{n,1}$ with $E \subset H$ and let H be fake-E-expansible. Assume that there is a generating system Y for H such that $E \subset Y$ and every $y \in Y$ has a fake-E-symbol of the form*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ x_1 & x_2 & \cdots & x_n \\ a_1\pi & a_2\pi & \cdots & a_n\pi \end{pmatrix}$$

for some permutation π *of the set* $\{a_1, \dots, a_n\}$. *Then* $\mathcal{H} = \langle H, G_{n,1} \rangle$, *the subgroup of* $G_{n,1}$ *generated by* H *and* $G_{n,1}$, *is* E*-expansible.*

Proof. In the light of Proposition 1, it suffices to prove that $\mathcal{H}$ is fake-E-expansible with respect to the generating system $Z = Y \cup G_{n,1}$. Let $h = z_1 \cdots z_m$, $z_i \in Z^{\pm 1}$, $1 \le i \le m$. The proof is by induction on m. If $m = 1$ it follows from the assumptions that $h = z_1$ has a fake-E-symbol. So we assume that there are fake-E-symbols $\Gamma_1, \dots, \Gamma_{m-1}$ for $z_1, \dots, z_{m-1}$, respectively, such that the combination $\Gamma_1 \cdots \Gamma_{m-1}$ exists.

First let $z_m \in Y$. Then z_m has a fake-E-symbol Γ of the form (4), and, as $E \subset Y$, any middle row entry of Γ has such a fake-E-symbol too. An induction on the number of simple expansions needed to go from $\{a_1, \dots, a_n\}$ to $\mathrm{bot}(\Gamma_{m-1})$ now shows that we can expand Γ to a fake-E-symbol Γ_m for z_m such that the combination $\Gamma_1 \cdots \Gamma_m$ exists.

In the case $z_m \in G_{n,1}$, let Γ be a 1-symbol for z_m. Then there is a common expansion $\mathcal{U}$ of the two finite bases $\mathrm{top}(\Gamma)$ and $\mathrm{bot}(\Gamma_{m-1})$. The argument in Proposition 1 also shows that we can expand all the Γ_k, $1 \le k \le m-1$, to obtain fake-E-symbols Δ_k for z_k such that the combination $\Delta_1 \cdots \Delta_{m-1}$ exists and $\mathrm{bot}(\Delta_{m-1}) = \mathcal{U}$. There is also an expansion Δ_m of Γ with $\mathrm{top}(\Delta_m) = \mathcal{U}$, because 1 has the symbol

$$\begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \\ a_1 & \cdots & a_n \end{pmatrix}.$$

Hence, the combination $\Delta_1 \cdots \Delta_m$ exists and therefore $\mathcal{H}$ is in fact E-expansible, as required.

3.2. *The Example*

Here we describe the groups which are used to prove the main result in the subsequent section. In the following let p be a fixed prime, f a positive integer, and Ω the set of all countably infinite sequences with values in $\{0, \dots, p-1\}$. We will write $\omega(i)$ for the $(i+1)$st component of $\omega \in \Omega$, $i \ge 0$. For $n \in \mathbb{N}$ define the sequence $[n]$ by

$$[n](i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

The "forget" map κ is defined by

$$\begin{aligned} \kappa: \quad & \Omega \to \Omega \\ & \omega = \omega(0)\,\omega(1) \cdots \quad \mapsto \quad \omega^{\kappa} = \omega(1)\,\omega(2) \cdots . \end{aligned}$$

Define $\Omega_0 = \{[pf]^{\kappa^i} | 0 \le i \le pf - 1\}$ and let $E_{f,p}$ be the subgroup of $\mathscr{G}_{p,1}$ generated by the set $\{b_\omega | \omega \in \Omega_0\}$, where b_ω is as in Section 2.4. Observe that for $0 \le i \le pf - 1$

$$[pf]^{\kappa^i}(j) = \begin{cases} 1 & \text{if } j \equiv -i \pmod{pf} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if we define the sum of the two sequences ω, ω' by $(\omega + \omega')(i) = (\omega(i) + \omega'(i)) \pmod p$ then we have $b_\omega b_{\omega'} = b_{\omega+\omega'}$. And a glance at Fig. 4 tells us that $b_\omega^p = 1$ for all $\omega \in \Omega_0$. Furthermore, b_ω has the symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ \gamma^{\omega(0)} & 1 & \cdots & 1 & b_{\omega^\kappa} \\ a_1 & a_2 & \cdots & a_{p-1} & a_p \end{pmatrix}, \tag{5}$$

where γ is the "top spin" as defined in (2) Section 2.4. Since $\gamma \in S_p$ and $[pf]^{\kappa^i} = [pf]^{\kappa^{pf+i}}$ for $i \in \mathbb{N}$, we have that $E_{f,p}$ is fake-$E_{f,p}$-expansible (with respect to the generating system $E_{f,p}$) and any element $e \in E_{f,p}$ has a fake-$E_{f,p}$-symbol of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ \gamma^k & 1 & \cdots & 1 & e' \\ a_1 & a_2 & \cdots & a_{p-1} & a_p \end{pmatrix}, \tag{6}$$

where $0 \le k \le p - 1$, $e' \in E_{f,p}$. Applying Proposition 2 with $H = Y = E = E_{f,p}$ we get the following lemma.

LEMMA 2. *The group $E_{f,p}$ is an elementary abelian p-group of rank pf and the group $\mathscr{H}_{f,p} = \langle E_{f,p}, G_{p,1} \rangle$ is $E_{f,p}$-expansible.*

4. THE MAIN RESULTS

The goal of this section is to show that the group $\mathscr{H}_{f,p}$ from the example above has a finitely presented simple commutator subgroup which contains an isomorphic copy of each Grigorchuk group G_ω if ω is periodic of period f. We first obtain a finite presentation for $\mathscr{H}_{f,p}$ (Sections 4.1 and 4.2). In Section 4.3 we then prove the commutator subgroup to have finite index and to contain the required subgroups.

4.1. *Relations for* $\mathscr{H}_{f,p}$

For $g \in \mathscr{G}_{n,1}$, let σ_g denote the element with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

and describe the set $\{\eta_2, \dots, \eta_s\}$ as *of type s* if there is a finite basis $\{a_1, w_2, \dots, w_s\}$ such that η_i has the symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{i-1} & w_i & w_{i+1} & \cdots & w_s \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_i & w_2 & \cdots & w_{i-1} & a_1 & w_{i+1} & \cdots & w_s \end{pmatrix}.$$

Let $E_{f,p}$ and $\mathscr{H}_{f,p}$ be the groups defined in the example (Section 3.2), define $E^*_{f,p} = \{\sigma_e | e \in E_{f,p}\}$, and let X be a finite generating set for $G_{p,1}$. Observe that $E^*_{f,p} \cong E_{f,p}$. All that follows will be with respect to the generating set $E^*_{f,p} \cup X$ for $\mathscr{H}_{f,p}$. That this is a generating set is shown in Lemma 8 in [13]. We now define four sets A, B, C, and D of relations.

Relation-Set A. This consists of a fixed set of defining relations of $G_{p,1}$ and a fixed set of defining relations of $E^*_{f,p}$.

Relation-Set B. Let F denote the subgroup of $G_{p,1}$ that fixes all words of the form $a_1 w$, $w \in W_p$. Then $B = \{\alpha\sigma_e = \sigma_e \alpha | e \in E_{f,p},\ \alpha \in F\}$.

Relation-Set C. Let δ be the element with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p a_1 & a_p a_2 & \cdots & a_p a_p \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_p a_1 & a_2 & \cdots & a_{p-1} & a_1 & a_p a_2 & \cdots & a_p a_p \end{pmatrix}.$$

Then $C = \{\delta\sigma_e \delta\sigma_{e'} = \sigma_{e'} \delta\sigma_e \delta | e, e' \in E_{f,p}\}$.

Relation-Set D. Let $e \in E_{f,p}$ and let

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ e_1 & \cdots & e_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

be an $E_{f,p}$-symbol for σ_e. Let $S = \{\eta_2, \ldots, \eta_s\}$ be a set of type s with corresponding basis $\{a_1, w_2, \ldots, w_s\}$ and define τ and ϵ by the symbols

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

respectively. Furthermore, let $R_{e,\Gamma,S}$ denote the relation

$$\sigma_e = \tau \sigma_{e_1} \eta_2 \sigma_{e_2} \cdots \eta_s \sigma_{e_s} \eta_s \cdots \eta_2 \epsilon .$$

Then D is the set of relations $R_{e,\Gamma,S}$ for all possible choices of subscripts.

Putting Theorem 1 and Lemmas 8 and 13 from [13] together with Lemma 2 we obtain the following result.

LEMMA 3. *Let $\chi = A \cup B \cup C \cup D$, where A, B, C, and D are as above. Then χ is a set of defining relations of the group $\mathscr{H}_{f,p}$. Furthermore, $A \cup B \cup C$ is finitely based; i.e., there is a finite subset of $A \cup B \cup C$ which implies all the relations in $A \cup B \cup C$.*

We need to define two more sets of relations.

Relation-Set D''. Let $e \in E_{f,p}$ and let

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ e_1 & \cdots & e_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

be a fake-$E_{f,p}$-symbol for σ_e. Let $S = \{\eta_2, \ldots, \eta_s\}$ be a set of type s with corresponding basis $\{a_1, w_2, \ldots, w_s\}$ and define τ and ϵ by the symbols

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

respectively. Furthermore, let $R''_{e,\Gamma,S}$ denote the relation

$$\sigma_e = \tau \sigma_{e_1} \eta_2 \sigma_{e_2} \cdots \eta_s \sigma_{e_s} \eta_s \cdots \eta_2 \epsilon .$$

Then D'' is the set of relations $R''_{e,\Gamma,S}$ for all possible choices of subscripts.

Note that this differs from the definition of the set D only in the use of "fake-E-symbol" in place of "E-symbol." But clearly each E-symbol for σ_e is also a fake-$E_{f,p}$-symbol for σ_e so that we have $D \subset D''$.

Relation-Set D'. Let $e \in E_{f,p}$ and let

$$\Gamma = \begin{pmatrix} a_1 & \cdots & a_p \\ x_1 & \cdots & x_p \\ a_1\pi & \cdots & a_p\pi \end{pmatrix}$$

be the fake-$E_{f,p}$-symbol (6) from Section 3.2 for e. Let δ be the same as for the relation set C and let δ_i, $1 \le i \le p$ act on the basis

$$\{a_1, a_2, \dots, a_{p-1}, a_p a_1 a_1, \dots, a_p a_1 a_p, a_p a_2, \dots, a_p a_p\}$$

as the involution that interchanges a_1 with $a_p a_1 a_i$ and fixes all the other elements. Let R'_e denote the relation

$$\sigma_e = \delta\delta_1 \sigma_{x_1} \delta_2 \sigma_{x_2} \cdots \delta_p \sigma_{x_p} \delta_p \cdots \delta_1 \delta \sigma_\pi . \tag{7}$$

Then $D' = \{R'_e | e \in E_{f,p}\}$.

The rest of this subsection is devoted to the proof of the following theorem up to a lemma which is proved in Section 4.2.

THEOREM 1. *Let $\chi' = A \cup B \cup C \cup D'$. Then χ' is a set of defining relations for $\mathscr{H}_{f,p}$. Hence $\mathscr{H}_{f,p}$ is finitely presented.*

Recall that $A \cup B \cup C$ is finitely based (Lemma 3). Since D' is clearly finite, Theorem 1 will follow immediately from the next lemma.

LEMMA 4. *Let $\chi'' = A \cup B \cup C \cup D''$. Then each relation in χ' follows from relations in χ and each relation in χ'' is a consequence of relations in χ'.*

Before we prove Lemma 4 in Section 4.2 we need two more facts. The next result is Lemma 12 in [13].

LEMMA 5. *If $\{\eta_2, \dots, \eta_s\}$ is of type s and μ is a permutation of the set $\{m, \dots, r\}$, $1 \le m \le r \le s$, then the relation*

$$\sigma_{g_m} \eta_{m+1} \sigma_{g_{m+1}} \cdots \eta_r \sigma_{g_r} = \alpha_\mu \sigma_{g_{m\mu}} \eta_{m+1} \sigma_{g_{(m+1)\mu}} \cdots \eta_r \sigma_{g_{r\mu}} \beta_\mu$$

is a consequence of $A \cup B \cup C$, where α_μ and β_μ have symbols

$$\begin{pmatrix} w_2 & \cdots & w_m & x_m & x_{m+1} & \cdots & x_r & w_{r+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_m & a_1 & w_{m+1} & \cdots & w_r & w_{r+1} & \cdots & w_s \end{pmatrix}$$

and

$$\begin{pmatrix} w_2 & \cdots & w_m & w_{m+1} & \cdots & w_r & a_1 & w_{r+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_m & y_{m+1} & \cdots & y_r & y_{r+1} & w_{r+1} & \cdots & w_s \end{pmatrix}$$

where $x_{m\mu^{-1}} = a_1$, $x_j = w_{j\mu}$ $(j\mu \neq m)$, *and* $y_{i+1} = w_{i\mu+1}$ $(i\mu \neq r)$.

LEMMA 6. *Any fake-*$E_{f,p}$*-symbol for* σ_e *is an expansion of*

$$\Gamma = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ e & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}.$$

Proof. Let Σ be the fake-$E_{f,p}$-symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ x_1 & \cdots & x_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

for σ_e. Then $\{u_1, \ldots, u_2\}$ is a finite basis and hence an expansion of $\{a_1, \ldots, a_p\}$. As any expansion of $\{a_1, \ldots, a_p\}$ leads to an expansion Δ of Γ such that Δ is a fake-$E_{f,p}$-symbol for σ_e, there is a fake-$E_{f,p}$-symbol Δ for σ_e of the form

$$\begin{pmatrix} u_1 & \cdots & u_s \\ y_1 & \cdots & y_s \\ z_1 & \cdots & z_s \end{pmatrix}$$

which is an expansion of Γ. It follows, by Lemma 4 in [13], that $x_i = y_i$ and $z_i = v_i$ for $1 \leq i \leq s$, which implies $\Delta = \Sigma$. Thus the lemma is proved.

4.2. *Proof of Lemma* 4

We now turn to the proof of Lemma 4. The second part of the proof is almost a verbatim copy of the proof of Lemma 19 in [13].

Let us first show that any relation in χ' follows from the relations in χ. To this end let $e \in E_{f,p}$ have the fake-$E_{f,p}$-symbol (6) (see Section 3.2) and let π denote the permutation of $\{a_1, \dots, a_n\}$ induced by γ^k. Then

$$\Gamma = \begin{pmatrix} a_pa_1 & a_pa_2 & \cdots & a_pa_p & a_1a_1a_1 & \cdots & a_1a_1a_p \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_pa_1 & a_pa_2 & \cdots & a_pa_p & a_1a_1(a_1\pi) & \cdots & a_1a_1(a_p\pi) \end{pmatrix.$$

$$\begin{matrix} a_1a_2 & \cdots & a_1a_{p-1} & a_1a_p & a_2 & \cdots & a_{p-1} \\ 1 & \cdots & 1 & e' & 1 & \cdots & 1 \\ a_1a_2 & \cdots & a_1a_{p-1} & a_1a_p & a_2 & \cdots & a_{p-1} \end{pmatrix}$$

is an $E_{f,p}$-symbol for σ_e. Now consider the relation $R_{e,\Gamma,S}$ in D corresponding to the ordered basis $\{a_1, a_pa_2(= w_2), \dots, a_pa_p, a_pa_1a_1(a_1\pi), \dots, a_pa_1a_1(a_p\pi), a_pa_1a_2, \dots, a_pa_1a_p(= w_{3p-1}), a_2, \dots, a_{p-1}\}$ for the set $S = \{\eta_2, \dots, \eta_{4p-3}\}$ of type $4p - 3$, namely,

$$\sigma_e = \tau\eta_2 \cdots \eta_{3p-1}\sigma_{e'}\eta_{3p} \cdots \eta_{4p-3}\eta_{4p-3} \cdots \eta_2\epsilon. \tag{8}$$

Because $\eta_r\eta_m\eta_r$ fixes a_1 for $m \neq r$, $\eta_m\eta_r\sigma_{e'}\eta_r = \eta_r\sigma_{e'}\eta_r\eta_m$ is a consequence of $A \cup B$. Since also $\eta_m^2 = 1$, the relation (8) implies $\sigma_e = \tau\eta_{3p-1}\sigma_{e'}\eta_{3p-1}\epsilon$. Observe that $\epsilon = \delta$ and $\eta_{3p-1} = \delta_p$. Furthermore, $\delta\delta_1\sigma_{\gamma^k}\delta_1 = \tau$ is a consequence of A (see the Appendix). Thus $\sigma_e = \delta\delta_1\sigma_{\gamma^k}\delta_1\delta_p\sigma_{e'}\delta_p\delta$ is a consequence of χ. Using that $\{\delta_1, \dots, \delta_p\} \subset \{\eta_2, \dots, \eta_{4p-3}\}$, we may repeat the above argument to obtain

$$\sigma_e = \delta\delta_1\sigma_{\gamma^k}\delta_2 \cdots \delta_{p-1}\delta_p\sigma_{e'}\delta_p \cdots \delta_1\delta$$

as a consequence of χ. But this is exactly the relation in D' corresponding to the fake-$E_{f,p}$-symbol of the form (6) for e. As e was arbitrary, this proves the first part of the lemma.

To show the second part of the lemma, let

$$\Delta = \begin{pmatrix} u_1 & \cdots & u_s \\ x_1 & \cdots & x_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

be a fake-$E_{f,p}$-symbol for σ_e, $e \in E_{f,p}$. Then, by Lemma 6, Δ is an expansion of the symbol in Lemma 6, and by Lemma 5 we can, and will, assume that the columns of Δ are in any order which suits us. The proof is by induction on the number, m, of columns of the form

$$\begin{pmatrix} a_1u \\ l \\ a_1v \end{pmatrix}$$

in Δ. The pattern of the induction is as follows. We first prove the cases $m = 1$ and $m = p$ and then turn to the induction step.

If $m = 1$, then

$$\Delta = \begin{pmatrix} a_1 & x_2 & \cdots & x_s \\ e & 1 & \cdots & 1 \\ a_1 & x_2 & \cdots & x_s \end{pmatrix}$$

and the corresponding relation in χ'' is $\sigma_e = \tau\sigma_e\epsilon$. But $\tau\epsilon = 1$ is a consequence of A, in this case, and τ fixes a_1, so the result is a consequence of $A \cup B$.

If $m = p$, then

$$\Delta = \begin{pmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_{p-1} & a_1a_p & x_{p+1} & \cdots & x_s \\ \gamma^k & 1 & \cdots & 1 & f & 1 & \cdots & 1 \\ a_1(a_1\pi) & a_1(a_2\pi) & \cdots & a_1(a_{p-1}\pi) & a_1(a_p\pi) & x_{p+1} & \cdots & x_s \end{pmatrix},$$

by (6) ($f = e'$ and $\pi = 1$), and the corresponding relation in $D'' \subset \chi''$ is

$$\sigma_e = \tau\sigma_{\gamma^k}\eta_2 \cdots \eta_p\sigma_f\eta_p \cdots \eta_2\epsilon \tag{9}$$

after the obvious cancellations implied by $\eta_i^2 = 1$, $p < i \le s$, which are consequences of A. Let α be the element with symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_p & w_{p+1} & \cdots & w_{s-1} & w_s \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_pa_1a_2 & \cdots & a_pa_1a_p & z_{p+1} & \cdots & z_{s-1} & a_pa_1a_1 \end{pmatrix},$$

for some $z_{p+1}, \ldots, z_{s-1}$. The relations $\alpha^{-1}\eta_i\alpha = \delta_i$, $2 \le i \le p$, are consequences of A and α fixes a_1, so $\sigma_e = \delta\delta_1\alpha^{-1}\sigma_{\gamma^k}\eta_2 \cdots \eta_p\sigma_f\eta_p \cdots \eta_2\alpha\delta_1\delta\sigma_\pi$ is obtained from R'_e (7) as a consequence of χ'. The element $\tau\alpha\delta_1\delta$ fixes a_1, because

$$a_1a_i \overset{\tau}{\mapsto} w_i \overset{\alpha}{\mapsto} a_pa_1a_i \overset{\delta_1}{\mapsto} a_pa_1a_i \overset{\delta}{\mapsto} a_1a_i, \qquad \text{if } 2 \le i \le p,$$
$$a_1a_1 \mapsto a_1 \mapsto a_1 \mapsto a_pa_1a_1 \mapsto a_1a_1.$$

Hence, using B and the consequences $\delta^2 = \delta_1^2 = 1$ of A,

$$\sigma_e = \tau\alpha\delta_1\delta\delta\delta_1\alpha^{-1}\sigma_{\gamma^k}\eta_2 \cdots \eta_p\sigma_f\eta_p \cdots \eta_2\alpha\delta_1\delta\sigma_\pi\delta\delta_1\alpha^{-1}\tau^{-1}$$

is a consequence of χ'. Furthermore, the relation $\alpha\delta_1\delta\sigma_\pi\delta\delta_1\alpha^{-1}\tau^{-1} = \epsilon$ follows from A (see the Appendix) and hence (9) is a consequence of χ'.

We now turn to the inductive step. Consider the simple expansion

$$\begin{pmatrix} u_1 & \cdots & u_{s-1} & u_s a_1 & \cdots & u_s a_p \\ x_1 & \cdots & x_{s-1} & y_1 & \cdots & y_p \\ v_1 & \cdots & v_{s-1} & v_s(a_1\pi) & \cdots & v_s(a_p\pi) \end{pmatrix}$$

of Δ. For a given set of type $r = s + p - 1$ we have to show that the relation

$$\sigma_e = \tau\sigma_{x_1}\eta_2\sigma_{x_2} \cdots \eta_{s-1}\sigma_{x_{s-1}}\eta_s\sigma_{y_1} \cdots \eta_r\sigma_{y_p}\eta_r \cdots \eta_2\epsilon$$

follows from χ'. By the inductive hypothesis we can assume that, for any set $\{\nu_2, \dots, \nu_s\}$ of type s,

$$\sigma_e = \tau'\sigma_{x_1}\nu_2\sigma_{x_2} \cdots \nu_s\sigma_{x_s}\nu_s \cdots \nu_2\epsilon' \tag{10}$$

is a consequence of χ'. Let β be the element with symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{s-1} & w_s & w_{s+1} & \cdots & w_r \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1 & z_2 & \cdots & z_{s-1} & z_s a_1 & z_s a_2 & \cdots & z_s a_p \end{pmatrix}$$

for some $z_2, \dots, z_s$ and define $\nu_i = \beta^{-1}\eta_i\beta$, $2 \le i \le s-1$. Then there exists ν_s such that $\{\nu_2, \dots, \nu_s\}$ is of type s with corresponding basis $\{a_1, z_2, \dots, z_s\}$. With the symbol

$$\begin{pmatrix} z_2 & \cdots & z_s & a_1 a_1 & \cdots & a_1 a_p \\ 1 & \cdots & 1 & y_1 & \cdots & y_p \\ z_2 & \cdots & z_s & a_1(a_1\pi) & \cdots & a_1(a_p\pi) \end{pmatrix}$$

for σ_{x_s}, the case $m = p$ implies that

$$\sigma_{x_s} = \tau''\eta_2 \cdots \eta_s\sigma_{y_1}\eta_{s+1} \cdots \sigma_{y_p}\eta_r \cdots \eta_2\epsilon'' \tag{11}$$

is a consequence of χ'. The relations $\tau''\eta_2 \cdots \eta_{s-1} = \nu_s \beta^{-1}$, $\epsilon''\nu_s \cdots \nu_2 \epsilon' = \epsilon$, and $\tau'\beta^{-1} = \tau$ are consequences of A (see the Appendix) and hence

$$\begin{aligned}
\sigma_e &\overset{(10)}{=} \tau'\sigma_{x_1}\nu_2\sigma_{x_2} \cdots \nu_s\sigma_{x_s}\nu_s \cdots \nu_2\epsilon' \\
&= \tau'\beta^{-1}\sigma_{x_1}\eta_2\sigma_{x_2} \cdots \eta_{s-1}\sigma_{x_{s-1}}\beta\nu_s\sigma_{x_s}\nu_s \cdots \nu_2\epsilon' \\
&\overset{(11)}{=} \tau'\beta^{-1}\sigma_{x_1}\eta_2 \cdots \sigma_{x_{s-1}}\beta\nu_s\tau''\eta_2 \cdots \eta_s\sigma_{y_1}\eta_{s+1} \cdots \sigma_{y_p}\eta_r \cdots \eta_2\epsilon''\nu_s \cdots \nu_2\epsilon' \\
&= \tau\sigma_{x_1}\eta_2 \cdots \sigma_{x_{s-1}}\eta_s\sigma_{y_1}\eta_{s+1} \cdots \sigma_{y_p}\eta_r \cdots \eta_2\epsilon
\end{aligned}$$

is a consequence of χ'. Here the equalities follow from (10), the definition of the ν_i and B as β fixes a_1, (11), and the argument of the preceding paragraph, respectively. This completes the proof of Lemma 4 and Theorem 1.

4.3. *The Commutator Subgroup of* $\mathscr{H}_{f,p}$

Let us call the infinite sequence ω *almost periodic* if there exist $k, f \in \mathbb{N}$, $f \geq 1$, such that $\omega(i) = \omega(i+f)$ for $k \leq i \in \mathbb{N}$. In this case the least such f is called the *period* of ω. We also say that the sequence ω is *almost f-periodic*. We can now establish the main theorem.

THEOREM 2. *Let* $\mathscr{H}_{f,p}$ *be as in Theorem* 1. *The derived subgroup* $\mathscr{H}'_{f,p}$ *of* $\mathscr{H}_{f,p}$ *is a finitely presented simple group and contains every Grigorchuk group* G_ω *which is defined by an almost f-periodic sequence* ω.

Proof. The simplicity of $\mathscr{H}'_{f,p}$ follows from Lemma 20 in [13]. To show that $\mathscr{H}'_{f,p}$ is finitely presented, using Theorem 1, we only need to check that it has finite index in $\mathscr{H}_{f,p}$ (Corollary 2.8 in [10]). By a theorem of Higman [7] we have $|G_{p,1} : G'_{p,1}| \leq 2$ so $x^{2p} \in \mathscr{H}'_{f,p}$ for all $x \in G_{p,1} \cup E_{f,p}$. As $G_{p,1} \cup E_{f,p}$ contains a finite generating system for $\mathscr{H}_{f,p}$, we see that $\mathscr{H}_{f,p}/\mathscr{H}'_{f,p}$ is a finitely generated abelian group of exponent at most $2p$ and hence finite.

Now it only remains to show that $\mathscr{H}'_{f,p}$ contains the required Grigorchuk groups. First note that if ω is any fp-periodic sequence then $b_\omega \in E_{f,p}$. Thus $\mathscr{H}_{f,p}$ contains $\sigma_{b_{\hat{\omega}}}$, where $\hat{\omega}$ is fp-periodic and begins with

$$\underbrace{\underbrace{10\cdots0}_{f}\,\underbrace{20\cdots0}_{f}\cdots\underbrace{(p-1)0\cdots0}_{f}\,\underbrace{0\cdots0}_{f}}_{pf}.$$

Let γ be the element with symbol

$$\begin{pmatrix} a_2a_1 & \cdots & a_2a_{p-1} & a_1 & a_2a_p & a_3 & \cdots & a_p \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ a_1a_1 & \cdots & a_1a_{p-1} & a_1a_p & a_2 & a_3 & \cdots & a_p \end{pmatrix}$$

and observe that $\gamma^{-1}\sigma_{b_\omega}\gamma = \sigma_{b_{0\omega}}$ and hence $\gamma^{-f}\sigma_{b_{\hat{\omega}}}^{-1}\gamma^f\sigma_{b_{\hat{\omega}}} = \sigma_{b_{[f]}} \in \mathscr{H}'_{f,p}$. Furthermore, for any f-periodic sequence ω, $\sigma_{b_\omega} \in \mathscr{H}'_{f,p}$.

As $E^* \cong E$ for any subgroup $E \subset \mathscr{G}_{p,1}$ (see Section 4.1 for the definition of E^*), to complete the proof it suffices to exhibit σ_γ and $\sigma_{b_{\nu\bar{0}}}$ as elements of $\mathscr{H}'_{f,p}$ for every finite sequence ν in $\{0,1,\ldots,p-1\}$, where $\bar{0} = 000\cdots$ and γ is as in (2). Because $G'_{2,1} = G_{2,1}$ (see [16, 7]) and $\sigma_\gamma \in \mathrm{Alt}(\{a_1a_1,\ldots,a_1a_p,a_2,\ldots,a_p\}) \subset G'_{p,1}$ if p is odd, where $\mathrm{Alt}(Y)$ denotes the alternating group on the set Y, we have $\sigma_\gamma \in \mathscr{H}'_{f,p}$ and similarly $\sigma_{\sigma_\gamma} \in \mathscr{H}_{f,p}$. From $\sigma_{b_{1\bar{0}}} = \sigma_{\sigma_\gamma}$ and the last paragraph it follows that $\sigma_{b_{\nu\bar{0}}} \in \mathscr{H}'_{f,p}$ for every finite sequence ν in $\{0,1,\ldots,p-1\}$. This completes the proof of Theorem 2.

5. TORSION SUBGROUPS OF FINITELY PRESENTED SIMPLE GROUPS

In this section we show that all the groups $G_{n,r}$ and the groups constructed by Scott in [14] are torsion locally finite (Theorem 3). A group is called *torsion locally finite* (t.l.f.) if any torsion subgroup is locally finite. We believe this result to be of independent interest. Our main interest, though, comes from the fact that it shows that the groups constructed in this paper are genuinely new finitely presented simple groups. This is because the Grigorchuk groups are finitely generated infinite p-groups provided that each of $0,1,\ldots,p$ occurs infinitely often in the defining sequence ω (see [6]).

5.1. *Generalities about Bases and Columns*

Fix $n, r \in \mathbb{N}$ and let B and C be finite bases of X_rW_n. We write $B \preccurlyeq C$ ($C \succcurlyeq B$) if C is an expansion of B. And if U is a finitely based subspace of X_rW_n, then B_U denotes its basis, i.e., $U = B_UW_n$. Equivalently, B_U is the set of those elements $b \in U$ such that for $u \in U$ it never is the case that $u < b$. We recall that $v \leq w$ ($v < w$) denotes that "v is a (proper) prefix of w," $v, w \in W_n$.

LEMMA 7. *Let B_i, $1 \leq i \leq m$, be a finite family of finite bases of X_rW_n. If $B = B_{\bigcap_{i=1}^m B_iW_n}$, then $B \subset \bigcup_{i=1}^m B_i$.*

Proof. First observe that $\bigcap_{i=1}^{m} B_i W_n$ is a finitely based subspace. Without loss of generality we assume $m = 2$. To obtain a contradiction, let $d \in B$ and assume $d \notin B_1 \cup B_2$. As $B \subset B_1 W_n \cap B_2 W_n$, there are $b \in B_1$, $c \in B_2$, and $w, v \in W_n \setminus \{\varnothing\}$ such that $d = bw = cv$. It follows that $w = xa_i$ and $v = ya_i$ for some $1 \leq i \leq n$ and hence $z = bx = cy \in B_1 W_n \cap B_2 W_n$. But $z < d$ contradicts the assumption that B is the basis of $B_1 W_n \cap B_2 W_n$.

LEMMA 8. *Let B_i, $1 \leq i \leq m$, be a finite family of finite bases of $X_r W_n$. If $w \in B_j \cap (\bigcap_{i=1}^{m} B_i W_n)$ for some $1 \leq j \leq m$, then $w \in B_{\bigcap_{i=1}^{m} B_i W_n}$.*

Proof. Again we can assume that $m = 2$, and we lose nothing by setting $j = 1$. Assume that $w \in (B_1 W_n \cap B_2 W_n) \setminus B_{B_1 W_n \cap B_2 W_n}$. Then $w = bx$ for some $\varnothing \neq x \in W_n$ and $b \in B_{B_1 W_n \cap B_2 W_n}$. In particular $b \in B_1 W_n \cap B_2 W_n$. But $b < w \in B_1$ implies $b \notin B_1 W_n$, which contradicts our assumption.

Let $\mathscr{H}$ be an H-expansible subgroup of $\mathscr{G}_{n,r}$. We say that $k \in \mathscr{H}$ *has the column*

$$\begin{pmatrix} u \\ h \\ v \end{pmatrix},$$

where $h \in H$, $u, v \in X_r W_n$, if, whenever w^h is defined for $w \in W_n$, then $(uw)^k$ is defined and equal to $v(w^h)$. It is shown in [13] that, if

$$\begin{pmatrix} u \\ h_1 \\ v_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u \\ h_2 \\ v_2 \end{pmatrix}$$

are both columns of k, then $h_1 = h_2$ and $v_1 = v_2$. Noting that an H-symbol is nothing but a collection of columns, the following lemmas also give information about columns, which will be used later.

We recall the construction due to Scott [13] of the groups H_θ defined by a homomorphism $\theta\colon F \to F \wr \mathrm{Sym}_{n-1}$ from a free group F to the permutational wreath product of F by the symmetric group on $n - 1$ objects, i.e., the base group of this wreath product is the direct product of $n - 1$ copies of F. In fact, all we need here is that H_θ embeds in $\mathscr{G}_{n,1}$, that $\mathscr{H}_\theta = \langle G_{n,1}, H_\theta \rangle$ is H_θ-expansible and each $h \in H_\theta$ has an H_θ-symbol of the form

$$\begin{pmatrix} a_1 & \cdots & a_n \\ h_1 & \cdots & h_n \\ a_1\pi & \cdots & a_n\pi \end{pmatrix},$$

where π is a permutation of the set $\{a_1, \ldots, a_n\}$ (Lemmas 14, 15, and 17 in [13]). This implies that H_θ acts in a length preserving manner on W_n. The following is now an easy exercise.

LEMMA 9. *Let Δ and Γ be H_θ-symbols for some $k \in \mathscr{H}_\theta$. Suppose that Δ is an expansion of Γ (write $\Delta \succcurlyeq \Gamma$) and let B and C be finite bases in W_n with $B \succcurlyeq \operatorname{top}(\Delta)$ and $C \succcurlyeq \operatorname{bot}(\Delta)$. Then the following hold.*

(i) $\operatorname{top}(\Delta) \succcurlyeq \operatorname{top}(\Gamma)$ *and* $\operatorname{bot}(\Delta) \succcurlyeq \operatorname{bot}(\Gamma)$.

(ii) *There are unique H_θ-symbols Δ_1 and Δ_2 for k such that* $\operatorname{top}(\Delta_1) = B$ *and* $\operatorname{bot}(\Delta_2) = C$. *Furthermore, $\Delta \preccurlyeq \Delta_1$ and $\Delta \preccurlyeq \Delta_2$.*

5.2. *Finite Subgroups of $\mathscr{H}_\theta$*

Here we examine the structure of finite subgroups of $\mathscr{H}_\theta$.

PROPOSITION 3. *Let $K \subset \mathscr{H}_\theta$ be a finite subgroup. Then there exists a finite basis B of W_n such that each $k \in K$ has an H_θ-symbol Δ with* $\operatorname{top}(\Delta) = \operatorname{bot}(\Delta) = B$.

Proof. It suffices to find a finite basis B such that, for every $b \in B$, each $k \in K$ has a column

$$\begin{pmatrix} b \\ h \\ c \end{pmatrix} \qquad \text{with } c \in B \text{ and } h \in H_\theta.$$

To this end let $K = \{k_1, \ldots, k_t\}$ $(t = |K|)$ and for $1 \le i \le t$ let Γ_i be an H_θ-symbol for k_i. Let C be a finite basis of W_n contained in $\bigcap_{i=1}^t \operatorname{top}(\Gamma_i) W_n$. In particular $C \succcurlyeq \operatorname{top}(\Gamma_i)$ for $1 \le i \le t$, and hence there are (unique) H_θ-symbols Δ_i for k_i with $\operatorname{top}(\Delta_i) = C$, by Lemma 9(ii). Let $U = \bigcap_{i=1}^t \operatorname{bot}(\Delta_i) W_n$ and $B = B_U$. We claim that B has the required properties.

First let $u \in U$. Then $u \in \operatorname{bot}(\Delta_i) W_n$ for $1 \le i \le t$ and, by Lemma 9(i), there are $v_i \in CW_n$ and $h_i \in H_\theta$ such that k_i has the column

$$\begin{pmatrix} v_i \\ h_i \\ u \end{pmatrix}.$$

Choose $k \in K$ and note that $B \succcurlyeq C$. By Lemma 9, k has the column

$$\begin{pmatrix} u \\ h \\ w \end{pmatrix} \qquad \text{with (unique) } w \in W_n \text{ and } h \in H_\theta.$$

Hence

$$\begin{pmatrix} v_i \\ h_i h \\ w \end{pmatrix}$$

is a column of $k_{j_i} = k_i k$, and $v_i \in CW_n$ implies $w \in \operatorname{bot}(\Delta_{j_i})W_n$ for $1 \leq j_i \leq t$. Thus $w \in U$.

Now let $b \in B$. Then, by Lemma 7, $b \in \operatorname{bot}(\Delta_j)$ for some j, $1 \leq j \leq t$. Hence

$$\begin{pmatrix} c \\ h \\ b \end{pmatrix}$$

is a column of k_j, where $c \in C$, $h \in H_\theta$. For $k \in K$ with column

$$\begin{pmatrix} b \\ h' \\ w \end{pmatrix},$$

we see that

$$\begin{pmatrix} c \\ hh' \\ w \end{pmatrix}$$

is a column of Δ_l where $k_l = k_j k$. Together with the previous paragraph it follows that $w \in U \cap \operatorname{bot}(\Delta_l)$. Lemma 8 now implies $w \in B$, as required.

Addendum. The above proof actually establishes a more general result. Namely, if D is a finite basis contained in $\bigcap_{i=1}^{t} \operatorname{top}(\Gamma_i)W_n$, then

$$\overline{D}^K = B_{\bigcap_{i=1}^{t} \operatorname{bot}(\Sigma_i)W_n}$$

is such that each $k \in K$ has an H_θ-symbol Δ with $\operatorname{top}(\Delta) = \operatorname{bot}(\Delta) = \overline{D}^K$, where Σ_i is an H_θ-symbol for k_i with $\operatorname{top}(\Sigma_i) = D$. We call $\overline{D}^K$ *the K-closure of D* and any finite basis with this property is called a *K-basis*.

In the following we investigate subgroups K of $\mathscr{H}_\theta$ that admit a K-basis.

LEMMA 10. *Let $K \subset \mathscr{H}_\theta$ have a K-basis B. Then K embeds in the permutational wreath product $H_\theta \wr \operatorname{Sym}(B)$ of H_θ by the symmetric group on the set B.*

Proof. Let $B = \{b_1, \dots, b_s\}$. Then each $k \in K$ has an H_θ-symbol of the form

$$\begin{pmatrix} b_1 & b_2 & \cdots & b_s \\ h_1(k) & h_2(k) & \cdots & h_s(k) \\ b_1\pi_k & b_2\pi_k & \cdots & b_s\pi_k \end{pmatrix}$$

with $h_i(k) \in H_\theta$ and $\pi_k \in \mathrm{Sym}(B)$. We leave it to the reader to check that

$$\begin{aligned} \phi\colon K &\to H_\theta \wr \mathrm{Sym}(B) \\ k &\mapsto (h_1(k), h_2(k), \dots, h_s(k))\pi_k \end{aligned}$$

is an embedding.

5.3. *Embeddings Which Preserve Torsion Local Finiteness*

Recall from the introduction to this section that a group is called *torsion locally finite* (t.l.f.) if each torsion subgroup is locally finite.

LEMMA 11. *Let G be a group, let $N \subset G$ be a normal subgroup, and let $\pi\colon G \to G/N$ denote the natural projection. If N and G/N are torsion locally finite, then so is G.*

Proof. Let S be a finitely generated torsion subgroup of G. We have to show that S is finite. Now S^π is a finitely generated torsion subgroup of G/N and therefore finite. Let $K = N \cap S$ be the kernel of the restriction of π to S. Then $|S : K|$ is finite. Thus K is a finitely generated torsion subgroup of N and hence finite. So S is finite, as an extension of the finite group K by the finite group S^π.

PROPOSITION 4. *Let H_θ be a t.l.f. group defined by a homomorphism θ: $F \to F \wr \mathrm{Sym}_{n-1}$ as in* [13]. *Then $\mathscr{H}_\theta = \langle G_{n,1}, H_\theta \rangle$ is t.l.f.*

Proof. By Lemma 11, $H_\theta \wr \mathrm{Sym}(B)$ is t.l.f. for any finite set B. Thus, it suffices to show that any finitely generated torsion subgroup K of $\mathscr{H}_\theta$ admits a K-basis. The proof is by induction on the minimal number, d, of generators for K; the case $d = 1$ follows from Proposition 3.

Let $K = \langle k_1, \dots, k_d \rangle$, $d > 1$, and let $G = \langle k_1, \dots, k_{d-1} \rangle$ and $X = \langle k_d \rangle$. By the induction hypothesis, G and X are finite and have G- and X-bases A_0 and A_1, respectively. Let $A = B_{A_0W_n \cap A_1W_n}$ and define inductively, for $i \geq 0$,

$$B_0 = \overline{A}^X, \qquad C_i = \overline{B}_i^G, \quad \text{and} \quad B_{i+1} = \overline{C}_i^X.$$

Note that the B_i and C_i are (finite) X-(respectively G-)bases. Furthermore, the proof of Proposition 3 shows that for each $b_i \in B_i$, $i \geq 1$, there are $c_i \in C_{i-1}$, $h_i \in H_\theta$, and $x_i \in X$ such that

$$\begin{pmatrix} c_i \\ h_i \\ b_i \end{pmatrix}$$

is a column of x_i. Similarly, for each $c_i \in C_{i-1}$, $i \geq 1$, there are $d_i \in B_{i-1}$, $l_i \in H_\theta$, and $g_i \in G$ such that

$$\begin{pmatrix} d_i \\ l_i \\ c_i \end{pmatrix}$$

is a column of g_i. Repetition of these arguments shows that for all $i \geq 0$ and $b \in B_i$ there are $\bar{b} \in B_0$, $h_b \in H_\theta$, and an element $k_b \in K = \langle G, X \rangle$ such that

$$\begin{pmatrix} \bar{b} \\ h_b \\ b \end{pmatrix}$$

is a column of k_b.

Assume now, to obtain a contradiction, that the chain

$$A \preccurlyeq B_0 \preccurlyeq C_0 \preccurlyeq B_1 \preccurlyeq C_1 \preccurlyeq \cdots \tag{12}$$

does not terminate. Then we can find a not eventually constant sequence $b_0, b_1, b_2, \ldots$ of elements of W_n with $b_i \in B_i$ and $b_i \leq b_{i+1}$ for all $i \geq 0$. Since B_0 is finite there exist $i > j$ such that $b_i \neq b_j$ but $\overline{b_i} = \overline{b_j}$, showing that $k = k_{b_j}^{-1} k_{b_i}$ has the column

$$\begin{pmatrix} b_j \\ h_{b_j}^{-1} h_{b_i} \\ b_i \end{pmatrix}.$$

Using $b_i > b_j$ and the fact that H_θ acts in a length preserving manner, it is easy to prove that k has infinite order, which is a contradiction. Thus the chain (12) terminates and it is obvious that its last element is a K-basis. This completes the proof of the proposition.

Note that subgroups of t.l.f. groups are also t.l.f. and that all the groups H_θ used in [14] are either torsion-free or isomorphic to $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ and therefore t.l.f. (see, for example, [11]). Using also the fact that $G_{n,r}$ embeds in $G_{n,1}$ (see [7]), we have the following.

THEOREM 3. *All the groups $G_{n,r}$, $G'_{n,r}$ and all the finitely presented simple groups constructed by Scott in* [14] *are torsion locally finite*.

At this point we should also note that the finitely presented simple groups recently constructed by Burger and Mozes [4] are t.l.f., indeed torsion-free. This follows immediately from their presentation as a free product with amalgamation $F *_A F$, where F and A are finitely generated free groups. Therefore, Theorem 3 tells us that none of the previously known finitely presented simple groups can be isomorphic to any of the groups $\mathscr{H}'_{f,p}$ with one exception: the finitely presented simple group with unsolvable conjugacy problem from [15]. Unfortunately, neither does the method of this section apply to this group nor were we able to prove that each $\mathscr{H}'_{f,p}$ has a solvable conjugacy problem.

APPENDIX

Here we present detailed calculations of relations used in the proof of Lemma 4. Since we are dealing with elements of $G_{p,1}$ which all have 1-symbols we do not print the ones in the middle rows. Permutations π of $\{a_1, \ldots, a_p\}$ are also viewed as permutations of the set $\{1, \ldots, p\}$ in the obvious way.

We first show how to obtain the relation $\delta\delta_1\sigma_{\gamma^k}\delta_1 = \tau$ in the first part of the proof of Leema 4. We omit the columns

$$\begin{pmatrix} a_p a_j \\ 1 \\ a_p a_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_i \\ 1 \\ a_i \end{pmatrix}$$

for $2 \leq j \leq p$ and $1 \leq i \leq p-1$ as they are not altered by the elements under consideration.

$$\left.\begin{array}{ccccccc}
a_p a_1 & a_1 a_1 a_1 & \cdots & a_1 a_1 a_p & a_1 a_2 & \cdots & a_1 a_p \\
 & & & & & & \downarrow \delta \\
a_1 & a_p a_1 a_1 a_1 & \cdots & a_p a_1 a_1 a_p & a_p a_1 a_2 & \cdots & a_p a_1 a_p \\
 & & & & & & \downarrow \delta_1 \\
a_p a_1 a_1 & a_1 a_1 & \cdots & a_1 a_p & a_p a_1 a_2 & \cdots & a_p a_1 a_p \\
 & & & & & & \downarrow \sigma_{\gamma^k} \\
a_p a_1 a_1 & a_1(a_1\pi) & \cdots & a_1(a_p\pi) & a_p a_1 a_2 & \cdots & a_p a_1 a_p \\
 & & & & & & \downarrow \delta_1 \\
a_1 & a_p a_1 a_1(a_1\pi) & \cdots & a_p a_1 a_1(a_p\pi) & a_p a_1 a_2 & \cdots & a_p a_1 a_p
\end{array}\right\} \tau .$$

Computing the relation $\alpha\delta_1\delta\sigma_\pi\delta\delta_1\alpha^{-1}\tau^{-1} = \epsilon$ in the case $m = p$, assume that $a_i\pi = a_1$, that is, $i\pi = 1$.

$$\left.\begin{array}{cccccccccc}
a_1 & w_2 & \cdots & w_i & \cdots & w_p & w_{p+1} & \cdots & w_{s-1} & w_s \\
 & & & & & & & & & \downarrow \alpha \\
a_1 & a_pa_1a_2 & \cdots & a_pa_1a_i & \cdots & a_pa_1a_p & z_{p+1} & \cdots & z_{s-1} & a_pa_1a_1 \\
 & & & & & & & & & \downarrow \delta_1 \\
a_pa_1a_1 & a_pa_1a_2 & \cdots & a_pa_1a_i & \cdots & a_pa_1a_p & z_{p+1} & \cdots & z_{s-1} & a_1 \\
 & & & & & & & & & \downarrow \delta \\
a_1a_1 & a_1a_2 & \cdots & a_1a_i & \cdots & a_1a_p & z_{p+1} & \cdots & z_{s-1} & a_pa_1 \\
 & & & & & & & & & \downarrow \sigma_\pi \\
a_1(a_1\pi) & a_1(a_2\pi) & \cdots & a_1a_1 & \cdots & a_1(a_p\pi) & z_{p+1} & \cdots & z_{s-1} & a_pa_1 \\
 & & & & & & & & & \downarrow \delta \\
a_pa_1(a_1\pi) & a_pa_1(a_2\pi) & \cdots & a_pa_1a_1 & \cdots & a_pa_1(a_p\pi) & z_{p+1} & \cdots & z_{s-1} & a_1 \\
 & & & & & & & & & \downarrow \delta_1 \\
a_pa_1(a_1\pi) & a_pa_1(a_2\pi) & \cdots & a_1 & \cdots & a_pa_1(a_p\pi) & z_{p+1} & \cdots & z_{s-1} & a_pa_1a_1 \\
 & & & & & & & & & \downarrow \alpha^{-1} \\
w_{1\pi} & w_{2\pi} & \cdots & a_1 & \cdots & w_{p\pi} & w_{p+1} & \cdots & w_{s-1} & w_s \\
 & & & & & & & & & \downarrow \tau^{-1} \\
a_1(a_1\pi) & a_1(a_2\pi) & \cdots & a_1a_1 & \cdots & a_1(a_p\pi) & x_{p+1} & \cdots & x_{s-1} & x_s
\end{array}\right\} \epsilon.$$

Computing the relation $\tau''\eta_2 \cdots \eta_{s-1} = \nu_s\beta^{-1}$,

$$\begin{array}{ccccccccc}
z_2 & z_3 & \cdots & z_{s-1} & z_s & a_1a_1 & \cdots & a_1a_p & \\
 & & & & & & & & \downarrow \tau'' \\
a_1 & w_2 & \cdots & w_{s-2} & w_{s-1} & w_s & \cdots & w_r & \\
 & & & & & & & & \downarrow \eta_2 \cdots \eta_{s-1} \\
w_2 & w_3 & \cdots & w_{s-1} & a_1 & w_s & \cdots & w_r & \\
 & & & & & & & & \downarrow \beta \\
z_2 & z_3 & \cdots & z_{s-1} & a_1 & z_sa_1 & \cdots & z_sa_p & \\
 & & & & & & & & \downarrow \nu_s^{-1} \\
z_2 & z_3 & \cdots & z_{s-1} & z_s & a_1a_1 & \cdots & a_1a_p &
\end{array}$$

Computing the relation $\epsilon'' \nu_s \cdots \nu_2 \epsilon_2 = \epsilon$,

$$\left.\begin{array}{cccccccccl}
a_1 & w_2 & \cdots & w_{s-2} & w_{s-1} & w_s & \cdots & w_r & & \\
 & & & & & & & & & \downarrow \epsilon'' \\
z_2 & z_3 & \cdots & z_{s-1} & z_s & a_1(a_1\pi) & \cdots & a_1(a_p\pi) & & \\
 & & & & & & & & & \downarrow \nu_s \\
z_2 & z_3 & \cdots & z_{s-1} & a_1 & z_s(a_1\pi) & \cdots & z_s(a_p\pi) & & \\
 & & & & & & & & & \downarrow \nu_{s-1} \cdots \nu_2 \\
a_1 & z_2 & \cdots & z_{s-2} & z_{s-1} & z_s(a_1\pi) & \cdots & z_s(a_p\pi) & & \\
 & & & & & & & & & \downarrow \epsilon' \\
v_1 & v_2 & \cdots & v_{s-2} & v_{s-1} & v_s(a_1\pi) & \cdots & v_s(a_p\pi) & &
\end{array}\right\} \epsilon.$$

Computing the relation $\tau' \beta^{-1} = \tau$,

$$\left.\begin{array}{ccccccccl}
u_1 & u_2 & \cdots & u_{s-1} & u_s a_1 & \cdots & u_s a_p & & \\
 & & & & & & & & \downarrow \tau' \\
a_1 & z_2 & \cdots & z_{s-1} & z_s a_1 & \cdots & z_s a_p & & \\
 & & & & & & & & \downarrow \beta^{-1} \\
a_1 & w_2 & \cdots & w_{s-1} & w_s & \cdots & w_r & &
\end{array}\right\} \tau.$$

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