

Extensions of topological spaces with strongly-discrete remainder

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Received 18 April 1997; received in revised form 11 June 1997

Abstract

The construction of the Alexandroff one-point compactification is extended to provide paracompact extensions of locally compact Hausdorff spaces with strongly-discrete remainder. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Extension; Strongly-discrete; Paracompact; Locally compact; Remainder

AMS classification: Primary 54D35; 54D40, Secondary 54D45; 54D20

One of the most well-known constructions in all topology, of such deceptive simplicity that it is known as widely to undergraduate students of mathematics as to professional workers in the field, is Alexandroff's one-point compactification. The purpose of this paper is to extend his construction in a simple and natural way to provide paracompact extensions of locally compact Hausdorff spaces with 'strongly-discrete' remainder. The work here essentially extends the method employed by Magill in [3] in a different way from his theory in [4].

Suppose that $\{F_p: p \in P\}$ is a family of closed subsets of a noncompact space (X, \mathcal{T}) with $X \cap P = \emptyset$, $P \neq \emptyset$ and such that

- (1) $C_P = \bigcap \{F_p: p \in P\}$ is compact,
- (2) $\bigcap \{F_p: p \in P, p \neq p_0\}$ is not compact for each $p_0 \in P$,
- (3) $F_p \cup F_q = X$ whenever $p \neq q$.

Let \mathcal{T}^* be the topology on $X^* = X \cup P$ with a basis consisting of \mathcal{T} together with all sets U of the form

$$U = \{p\} \cup (X \setminus (F_p \cup C)),$$

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where C is a closed and compact subset of X and p is an element of P . Then X is a dense subspace of X^* and the remainder $X^* - X$ is *strongly-discrete* in the sense that there exists a pairwise-disjoint family of open subsets of X^* covering $X^* - X$ and inducing the discrete topology on it.

Theorem. *If X is locally compact Hausdorff (in which case a singleton set is one possible choice for P), then X^* is locally compact Hausdorff and paracompact, and hence is a disjoint sum of σ -compact spaces.*

Proof. X^* is clearly locally compact and Hausdorff; so, suppose that \mathcal{V} is an open covering of X^* and put

$$S_p = \{p\} \cup (X \setminus F_p) \quad (p \in P)$$

so that, by definition of \mathcal{T}^* , there is for each p in P a compact subset C_p of X and an element V_p of \mathcal{V} such that $S_p \setminus C_p \subseteq V_p$. As X is locally compact, there exist open subsets U_1, U_2 of X such that

$$C_P \subseteq U_1 \subset \overline{U_1}^X \subseteq U_2$$

with both $\overline{U_1}^X, \overline{U_2}^X$ compact. Note that $\overline{S_p}^{X^*} \setminus U_2 = S_p \setminus U_2$ for each p . The compact set $(S_p \setminus U_2) \cap C_p$ is covered by a finite sub-collection \mathcal{V}_p from \mathcal{V} . Put

$$\mathcal{W}_p = \{(V \cap S_p) \setminus \overline{U_1}^X : V \in \mathcal{V}_p\} \quad (p \in P)$$

and select a finite sub-covering \mathcal{V}' of $\overline{U_2}^X$ from \mathcal{V} . We claim that

$$\mathcal{U} = \mathcal{V}' \cup \bigcup \{\mathcal{W}_p : p \in P\} \cup \{S_p \setminus (\overline{U_2}^X \cup C_p) : p \in P\}$$

is a locally finite open refinement of \mathcal{V} . Clearly, proof is only required to show that (a) \mathcal{U} is a covering and (b) \mathcal{U} is locally finite.

To prove (a), note that $p \in (S_p \setminus (\overline{U_2}^X \cup C_p))$ for each p in P . In the case $x \in \overline{U_2}^X$, we have $x \in \bigcup \mathcal{V}'$. Otherwise, $x \notin \overline{U_2}^X \supseteq C_P$, $x \in S_p$ for some p , and either $x \in S_p \setminus (\overline{U_2}^X \cup C_p)$ or $x \in (S_p \setminus U_2) \cap C_p$. Hence,

$$x \in (S_p \setminus \overline{U_1}^X) \cap \bigcup \mathcal{V}_p$$

and therefore $x \in \bigcup \mathcal{W}_p$. Thus \mathcal{U} is a covering.

As to (b), if $x \in S_p$ for some p , then the only elements of \mathcal{U} which have nonempty intersection with the open set S_p are $S_p \setminus (\overline{U_2}^X \cup C_p)$ and members of \mathcal{V}' and \mathcal{W}_p . If $x \notin S_p$ for every p , then $x \in C_P \subseteq U_1$; but the only elements of \mathcal{U} meeting U_1 are elements of \mathcal{V}' .

It is well known that any locally compact, paracompact Hausdorff space is a disjoint sum of σ -compact spaces (for example, see [2]). \square

If in addition P is countable, then X^* is σ -compact, and if P is finite, X^* is an n -point compactification. X^* is always a union of Alexandroff one-point compactifications of suitable subsets.

Problem. It is natural to ask: for which locally compact (paracompact) Hausdorff spaces Y with dense subspace X such that $Y \setminus X$ is strongly discrete do there exist families $\{F_p: p \in Y \setminus X\}$ of closed subsets of X satisfying (1)–(3) above. Of course, in general, the answer must be no.

Example. Let X and Y be the following subspaces of \mathbb{R}^2 (with their usual Euclidean subspace topologies):

$$X = (\mathbb{R} \times \{0\}) \cup (\mathbb{Z} \times [0, 1]), \quad Y = (\mathbb{R} \times \{0\}) \cup (\mathbb{Z} \times [0, 1]).$$

Suppose there exists a family $\{F_p: p \in Y \setminus X\}$ satisfying (1)–(3) and put $S_p = X \setminus F_p$ for each p in P . From (1), $X \setminus \bigcup_p S_p$ is bounded and hence there exists a real number A such that

$$[A, \infty) \times \{0\} \subseteq \bigcup_p S_p.$$

As the family (S_p) is a pairwise-disjoint collection of open sets and $[A, \infty)$ is connected, there is a p_0 such that

$$[A, \infty) \times \{0\} \subseteq S_{p_0}.$$

Moreover, it is clear that for each $(n, 1)$ in P , there exists

$$x_n \in S_{(n,1)} \cap (\{n\} \times (0, 1)).$$

Then, for each $n > A$ and as each $\{n\} \times [0, 1)$ is connected, there exists $y_n \in \{n\} \times (0, 1)$ such that $y_n \notin S_p$ for every p in P . This implies C_P is unbounded, contradicting its assumed compactness.

Of course, when X is connected, so is X^* , and the cardinality of P must not be greater than the cardinality of the set of components of $X^* - D$ for any closed and compact subset D of X containing C_P . In particular, as is well known, the only n -point compactification (and indeed, the only strongly-discrete extension in the above sense) of Euclidean n -space, for $n \geq 2$, is the one-point compactification.

The set of points of local-disconnexion of a locally compact connected Hausdorff space is dense-in-itself (see [1]) implies in the locally compact connected Hausdorff case that X^* is locally connected whenever X is.

I should like to acknowledge a useful discussion with C.A. Hendrie on this topic.

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