# Fundamental models for the dynamics of electricity prices

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# Abstract

In this work we propose a term structure power price model that, in contrast to widely accepted no-arbitrage based approaches, accounts for the non-storable nature of power. It belongs to a class of equilibrium game-theoretic models with players divided into producers and consumers. Consumers aim to maximize a mean-variance utility function subject to the inelastic demand of their own clients (households, businesses etc.) to whom they sell power. Similarly, producers, who own a portfolio of power plants each defined by a running fuel (gas, coal, oil etc.) and various physical characteristics (efficiency, capacity, ramp up/down times etc.), seek to maximize a mean-variance utility function consisting of power, fuel, and emission prices subject to production constraints. Our goal is to determine the term structure of the power price at which production matches consumption. In this work we show that in such a setting, the equilibrium price exists, and also discuss conditions under which it is unique.

We then extend the model to incorporate information about block contracts, transaction costs and liquidity. Moreover, we propose a tractable quadratic programming formulation for calculating the equilibrium term structure of electricity prices. Our numerical simulations examine the dependence of the term structure on various parameters of the model. Our model is applied to calculate the equilibrium term structure of electricity prices in the UK, by modeling the entire power grid consisting of a few hundred power plants.

We extend the model further by modeling startup costs of power plants. In contrast to other approaches presented in the literature, we incorporate them in a mathematically rigorous manner without relying on ad hoc heuristics. We propose a tractable approach for estimating startup costs of power plants based on their historical production. Through numerical simulations applied to the entire UK power grid, we demonstrate that the inclusion of startup costs is necessary for modeling electricity prices in realistic power systems.

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# Chapter 1

# Introduction

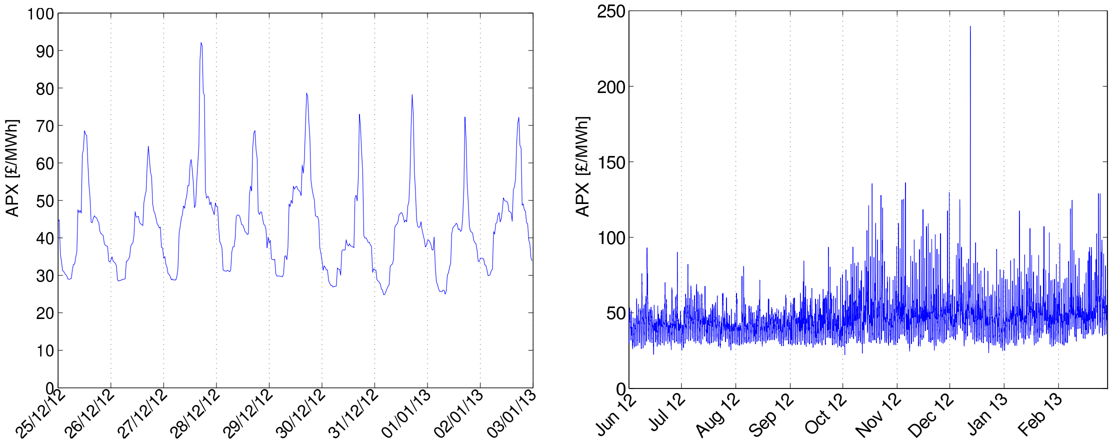
In this chapter, we present state-of-the-art techniques for modeling electricity prices and discuss an important concept in optimization called sensitivity analysis. We finish the chapter by describing the main contributions of this thesis and their relation to the existing literature.

## 1.1 Modeling of electricity prices

Modeling of electricity prices is crucial for effective decision making in the energy sector. It serves as a backbone for most investment and policy decisions. Operation of power plants and the electricity grid, and portfolio and risk management all rely on precise modeling of electricity prices.

The electricity price has some unique features which are distinct from features of stock prices or other commodities. An example of a historical spot electricity price is depicted in Figure 1.1. In Figure 1.1a, one can see strong predictable daily patterns. Since the spot electricity price is driven by demand, it is lower at night and increases during the day peaking in the afternoon. It is affected by holidays (notice the different pattern of the price on New Year's Eve), weekends and even special events such as important football games.

In Figure 1.1b, we depict the historical spot electricity price over a period of nine months. One can see that the spot electricity price is very spiky (i.e. the price can triple or quadruple in just a few hours) and that it possesses a mean-reverting behavior (after large jumps it




(a) Spot price over 9-days period. (b) Spot price over 9-months period.

Figure 1.1: Historical spot electricity price.

always reverts back towards to its mean value). A good overview of the statistical properties of the spot electricity price is available in Lucia and Schwartz (2000).

The literature related to modeling of electricity prices can broadly be divided into non-structural, structural, and game-theoretic approaches. Before we investigate each of them in detail, let us introduce the general modeling framework.

### 1.1.1 The general framework

In this thesis, we model spot and forward electricity prices[1]. In continuous time modeling, the price of a forward electricity contract is denoted by $\Pi(t,T)$, where $t$ and $T$ denote the trading time and delivery time. Since contracts with trading time later than delivery time do not exist, we require $t \leq T$. We are interested in a delivery period $T \in [T_{\min}, T_{\max}]$ and trading period $t \in [t_{\min}, T_{\max}]$. Since some power plants use fuel (coal, natural gas, oil etc.) to produce electricity, we model also fuel prices. A set of all fuels is denoted by $L$ and the price of a forward fuel contract is denoted by $G_l(t,T)$ where $l \in L$ is the particular fuel. Uncertainty is modeled by a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[t_{\min},T_{\max}]}, \mathbb{P}\right)$. The $\sigma$-algebra $\mathcal{F}_t$ represents the information available at time $t$. Electricity prices $\Pi(t,T)$ and fuel prices $G_l(t,T)$, $l \in L$ are assumed to be adapted to the filtration $\{\mathcal{F}_t\}_{t\in[t_{\min},T_{\max}]}$ and

[1] A general term 'price' is used to denote a mid-price. Extension to bid/ask price is addressed in later chapters.

to have finite first and second moments.

Electricity and fuel spot contracts require an immediate delivery of the underlying and using the above notation, their price is denoted by $\Pi(T,T)$ and $G_l(T,T)$, $l \in L$, respectively. However, to simplify the notation, we will often refer to the spot electricity and fuel price as $\Pi(T)$ and $G_l(T)$, $l \in L$.

Besides electricity and fuel prices, we model also the total electricity demand in the power system denoted by $D(T)$, $T \in [T_{\min}, T_{\max}]$. Total electricity demand $D(T)$ is also assumed to be adapted to the filtration $\{\mathcal{F}_T\}_{T\in[T_{\min},T_{\max}]}$ and to have finite first and second moments.

Instead of modeling prices and demand in continuous time, we sometimes in this work focus on discrete delivery and trading periods. We are interested in delivery times $T_j$, $j \in J = \{1, ..., T'\}$, where energy for each delivery time $T_j$ can be traded through numerous forward contracts at times $t_i$, $i \in I_j$. Since contracts with a trading time later than delivery time do not exist, we require $t_{\max\{I_j\}} = T_j$ for all $j \in J$. The number of all forward electricity contracts, $\sum_{j\in J} |I_j|$, is denoted by $N$. Uncertainty in discrete time is modeled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \in I\}, \mathbb{P})$, where $I = \cup_{j\in J} I_j$. The $\sigma$-algebra $\mathcal{F}_t$ represents the information available at time $t$. Similarly, we can also model the total demand $D(T_j)$, $j \in J$ in discrete time.

### 1.1.2 Non-structural approaches

Non-structural approaches model the electricity price directly, without considering the underlying reasons that cause the price to change over time. They heavily rely on standard techniques developed for modeling the prices of other financial securities. The seminal paper by Lucia and Schwartz (2000) studied the suitability of one and multi-factor Ornstein-Uhlenbeck processes to model the spot and log spot price. Their one-factor model for the spot price describes the spot price $\Pi(T)$ as the sum of a deterministic component $f(T)$ and a stochastic component $X(T)$,

$$\Pi(T) = f(T) + X(T), \tag{1.1}$$

where $X(T)$ is a mean-reverting stochastic process satisfying

$$dX(T) = -\kappa X(T)\,dT + \sigma dZ(T), \tag{1.2}$$

with $Z(T)$ a Brownian motion, $\kappa > 0$ a mean-reverting coefficient and $\sigma > 0$ a constant volatility. The deterministic component $f(T)$ is used to capture (yearly, weekly, daily etc.) repetitive patterns.

By examining the qualitative properties of model (1.1), one can see that it admits a nonzero probability of negative prices $\Pi(T)$. Negative prices have recently been observed in some markets (see Schneider (2010)) and with the increasing penetration of wind and solar power, they are expected to become more frequent in most electricity markets.[2] A qualitatively visible drawback of model (1.1) is that price changes $d\Pi(T)$ are independent of the price $\Pi(T)$. On the other hand, a statistical analysis of historical spot prices shows that price changes $d\Pi(T)$ are larger (smaller) when the price $\Pi(T)$ is large (small), see Lucia and Schwartz (2000). This behavior is not captured by (1.1).

The one-factor model (1.1) can, however, be used to price future and forward contracts. Assuming a constant market price of risk, $\widehat{\lambda} \in \mathbb{R}$, and using a risk-neutral valuation, one can calculate the price of a futures contract as

$$\Pi(t,T) = f(T) + (\Pi(t) - f(t))\,e^{-\kappa(T-t)} - \frac{\widehat{\lambda}\sigma}{\kappa}\left(1 - e^{-\kappa(T-t)}\right). \tag{1.3}$$

The first term in (1.3) corresponds to the unconditionally expected future price. The second term accounts for the current displacement of the spot price from the deterministic component. Note that this term decreases exponentially as we increase the delivery time $T$; this is a typical consequence of the mean-reverting behavior. The last term in (1.3) corresponds to the spot price risk. Futures contracts with a delivery date far into the future are less volatile and thus attract a smaller risk premium.

[2] There are two main reasons for negative spot prices. First, solar and wind power plants can produce energy at virtually zero costs and due to high subsidies for renewables, it is often profitable to run these power plants even at negative prices. Second, especially nuclear but also coal and gas power plants have high startup costs and thus, it is sometimes economical for them to produce energy at negative prices to avoid startup costs.

A layer of complexity is added when we take into account that delivery times of electricity futures contracts are not infinitesimally small but instead span longer time intervals. For example, a block month-ahead baseload contract requires delivery of electricity every half hour next month. The price of a future block contract can be calculated as

$$\Pi\left(t,\left[T_1,T_2\right]\right)=\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\Pi\left(t,T\right)dT, \tag{1.4}$$

where $[T_1, T_2]$ denotes the delivery interval.

Lucia and Schwartz (2000) proposed the following model for $\log \Pi(T)$ as an alternative to modeling the spot price $\Pi(T)$ directly:

$$\log\Pi\left(T\right)=f\left(T\right)+X\left(T\right). \tag{1.5}$$

Here $f(T)$ is again a deterministic component and $X(T)$ is defined by (1.2). According to Lucia and Schwartz (2000), the log spot price model (1.5) outperforms the ordinary spot price model (1.1), because it takes into account that price changes $d\Pi(T)$ depend on $\Pi(T)$. However, the spot price $\Pi(T)$ cannot take negative values in (1.5).

By further examining statistical and qualitative properties of electricity prices generated by models (1.1) and (1.5), one can identify (at least) the following drawbacks:

- Price changes $d\Pi(T)$ are not normally distributed in reality (see Lucia and Schwartz (2000)). Thus, the Brownian motion alone is not suitable to accurately model electricity prices.

- Pricing of futures contracts with the approach used in (1.3), implicitly assumes that changes in the spot price and future prices for all maturities are perfectly correlated.

To address the first of the above drawbacks, a combination of the Ornstein-Uhlenbeck and pure jump-processes has been proposed in Hambly et al. (2009). Further developments include using Levy processes (see Meyer-Brandis and Tankov (2008) and García et al. (2011) for example).

Even though models of the spot price might capture the spot price dynamics well, their application to pricing derivatives is sometimes rather questionable. As pointed out in Benth and Meyer-Brandis (2009), a classic buy-and-hold strategy is not applicable to non-storable commodities. Most non-structural models define the price of a futures contract as the expected price of the security at the delivery time under the risk-neutral measure, conditioned on some filtration, which contains all available market information. It is often assumed that all available market information is included in the spot price (i.e. the spot price is a martingale). This is a good approximation for stock prices, but is definitely not valid for non-storable commodities. Information about a huge electricity demand increase in a week, for example, does not have any impact on today's spot price. Note that even if the current spot electricity price is much lower than the expected spot price in a week, one cannot buy the electricity now, store it, and sell it later at a higher price as one would do with stocks. Thus, information on weather forecasts, planned power plant outages etc. must be taken into consideration when pricing futures contracts. If pricing is based solely on the spot price, the outcomes are inconsistent with the prices of futures contracts that are observed in the market.

It can be argued that all relevant market information is factored into the observable prices of forward contracts and other derivatives. The one-factor term structure model in Clewlow and Strickland (1999b) and multi-factor term structure model in Clewlow and Strickland (1999a) were the first to model prices consistent with observable forward prices. The simplest, one-factor model describes the dynamics of a price of a futures contract $\Pi(t,T)$ at time $t$ for delivery at time $T$, $t \leq T$ as

$$\frac{d\Pi(t,T)}{\Pi(t,T)} = \sigma(t,T)\, dZ(t), \tag{1.6}$$

where $Z(t)$ is again a Brownian motion and $\sigma(t,T)$ denotes a variance, which is usually modeled as

$$\sigma(t,T) = \sigma e^{-\alpha(T-t)} \tag{1.7}$$

for some decay constant $\alpha > 0$.

While non-structural models are widely used for short-term risk management and derivatives pricing in practice, they do not cater well for longer-term modeling, where the impact of new power plants must be included. New power plants are frequently connected to the electricity grid, and this has a non-negligible impact on the dynamics of the electricity price. In practice, this implies that non-structural models must frequently be recalibrated to reflect changes in the market.

### 1.1.3 Structural approaches

Structural approaches for modeling electricity prices capture some of the fundamental factors of the electricity market. In particular, they use supply and demand stack to model electricity prices. The supply stack is an increasing monotonic mapping between electricity production and electricity price. The idea is that power plants cannot operate without cost and thus, a power plant will produce electricity only if the electricity price is high enough to cover all costs of production. This relationship is captured in a supply stack. Typical costs of operating a power plant are:

- fuel costs - apart form some renewables such as wind and solar, all power plants consume fuel to produce electricity. For example, a nuclear power plant uses uranium or thorium, a coal fired thermal power plant, coal, a natural gas fired power plant, natural gas etc. Fuel costs include the cost of buying and transporting the required fuel.

- startup costs - to start a power plant is often very energy intensive. Startup costs measure costs of energy that is wasted starting a power plant. For example, starting a coal fired thermal power plant involves heating up the boiler which can often take several hours. The energy used for this does not produce electricity, and is thus wasted.

- operating costs - these cover the maintenance costs of power plants that occur due to the production of electricity, including operates' salaries. In contrast to startup cost, which only exist during a startup phase, operating costs occur during the production of every unit of electricity.

- fixed costs - these cover the costs of building a power plant, fixed maintenance costs and so on.

Similarly, a demand stack is a decreasing monotonic mapping between electricity demand and price. Currently, end users such as households and business do not change their short-term demand for electricity if the price changes. This is because they receive their electricity through utility companies which charge them a fixed, and not a floating wholesale market price for their energy. Thus, electricity demand is inelastic (changes in electricity price do not affect the demand very much). End user demand is mainly driven by the temperature because air conditioning and heating are both relatively energy intensive.

Barlow (2002) produced seminal work that used a supply and demand stack to model electricity prices. This work was further extended by Howison and Coulon (2009) and by Carmona et al. (2013) by taking into account that the supply stack depends on specific fuel (coal, gas, oil etc.) prices. Modeling the supply stack as a piecewise exponential function of spot fuel prices, they calculated the spot electricity price as a point where the supply matches the completely inelastic demand. For each fuel $l \in L$, an exponential supply stack function is defined as

$$b_l\left(D_l\left(T\right), G_l\left(T\right)\right) := G_l\left(T\right) \exp\left(k_l + m_l D_l\left(T\right)\right) \tag{1.8}$$

where $k_l \in \mathbb{R}$ and $m_l > 0$ are experimentally determined constants and $D_l\left(T\right)$ is the demand that is covered by power plants running on fuel $l \in L$. Due to capacity constraints of power plants $D_l\left(T\right) \in [0, D_l^{\max}]$, where $D_l^{\max}$ denotes the maximum production from fuel $l \in L$. By definition, $\sum_{l\in L} D_l\left(T\right) = D\left(T\right)$ for all delivery times $T \in [T_{\min}, T_{\max}]$. The spot electricity price $\Pi\left(T\right)$ and values $D_l\left(T\right)$ $l \in L$ for a fixed $T$, and given $D\left(T\right)$ and $G_l\left(T\right)$ $l \in L$, are determined as the solution of the following optimization problem:

$$\begin{aligned} \min_{D_l(T), l\in L} \quad & \Pi\left(T\right) := \textstyle\sum_{l\in L} G_l\left(T\right) \exp\left(k_l + m_l D_l\left(T\right)\right) \\ \text{s.t.} \quad & 0 \leq D_l\left(T\right) \leq D_l^{\max} \quad \forall l \in L \\ & \textstyle\sum_{l\in L} D_l\left(T\right) = D\left(T\right). \end{aligned} \tag{1.9}$$

From sensitivity analysis of problem (1.9), it follows that $\Pi\left(T\right)$ is a piecewise exponential function of the data (fuel prices, demand, model parameters etc.). For a small number of fuels (one or two), we can write the form of $\Pi(T)$ explicitly, see Howison and Coulon (2009).

We can then calculate the value of a futures contract as

$$\Pi(t,T) = \mathbb{E}^{\mathbb{Q}}\left(\Pi(T)\,|\mathcal{F}_t\right). \tag{1.10}$$

There exists a (lengthy) closed form solution to the above problem, but we omit it in the interest of space. An interested reader can find it together with a derivation in Carmona et al. (2013).

Note that for a fixed delivery date $T$, forward contracts are traded assets (and hence storable) and thus, equation (1.10) does not contradict the non-storability property of electricity. Structural approaches are better suited for derivatives pricing than non-structural approaches, because they do not rely on market completeness and on the buy-and-hold no-arbitrage argument for the spot price. Moreover, the filtration considered is larger and includes information about demand and fuel prices. However, as pointed out in Carmona et al. (2013), the above model does not produce realistic spikes in the spot electricity price, and does not admit negative prices. To get around this problem, one can introduce an artificial fuel with a large value $m_l$ or a regime switching model (as in Mount et al. (2006) for example).

As we will see in this thesis, spikes in the spot electricity price occur mainly because of the physical characteristics of power plants that are not modeled in problem (1.9). Our numerical results show that there are two main reasons for spikes in the spot electricity price:

1. Power plants have tight ramping constraints that limit the speed of increasing/decreasing production. Thus, when large changes in demand occur more power plants must be turned on so that supply can match demand. Since the electricity price must be high enough that the least efficient power plant running is making a profit (or at least no loss), tight ramping constraints together with large changes in the total demand contribute to high spikes in the spot electricity price. The tightness of ramping constraints is highly dependent on the type of power plant. Roughly speaking, natural gas fired power plants can ramp from zero to maximum capacity in hours, coal fired

power plants in hours or days, and nuclear power plants in days or weeks.

2. Power plants have large startup costs. Thus, if a power plant is turned on to cover a short, temporary, increase in demand, startup costs are spread over a short period of time and thus significantly contribute to the temporary increase (i.e. spike) in the spot electricity price. If, on other hand, a power plant produces electricity continually for a long period of time, then startup costs are spread over a longer period and thus have a smaller impact on the spot price.

To understand and model realistic price spikes, one has to examine the behavior of the market participants more closely.

### 1.1.4 Game-theoretic approaches

Game-theoretic approaches model the physical properties and decisions of market participants more accurately. As pointed out in Robinson (2005), models that include ramping constraints of power plants and study the impact of long term contracts on the spot electricity price, are needed in order to prevent and explain disastrous events such as happened in California in 2001, which cost the state as much as \$45 billion. The seminal work on game-theoretic models in electricity markets was produced by Bessembinder and Lemmon (2002): a unique relation between a forward and a spot price is given in a two-stage market involving players who want to maximize their respective mean-variance objective functions. This work has been used to study the benefits of derivatives in the electricity market as in Cavallo and Termini (2005). The model was extended to a multi-stage setting with a dynamic equilibrium by Bühler and Müller-Merbach (2009), and Bühler (2009). De Maere d'Aertrycke and Smeers (2012) have extended the two-stage mean-variance model to any convex risk measure, while also taking into account liquidity constraints.

Bessembinder and Lemmon (2002) model a two-stage market with two types of contracts. The first type is a forward contract $\Pi(t_1)$ and the second, a spot contract $\Pi(t_2)$. It is assumed that there is just one delivery period $T_1$ and thus the delivery period is not explicitly included in the notation. The market consists of a set of producers and consumers denoted by $P$ and $C$. A schematic of the market is depicted in Figure 1.2.

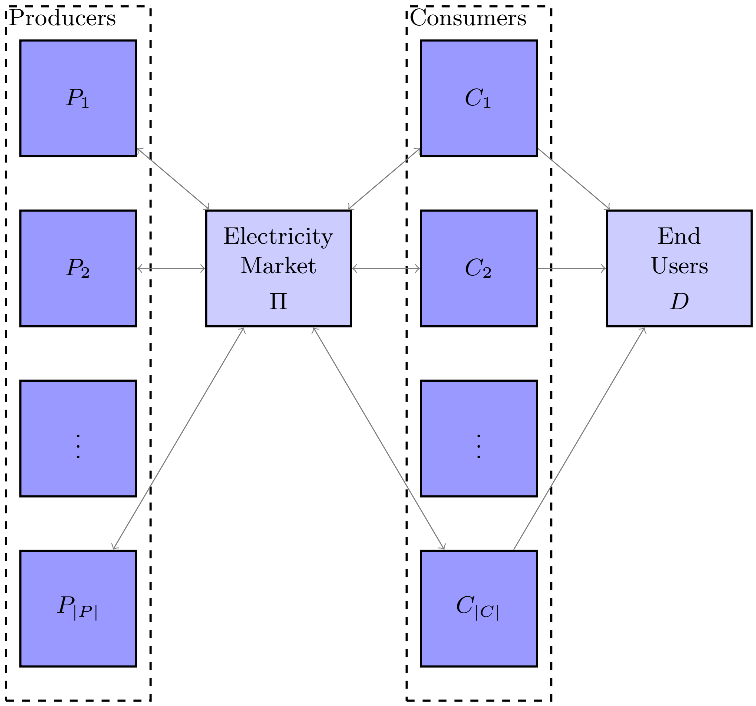



Figure 1.2: Schematic of the electricity market as studied in game-theoretic approaches. Producers use their portfolio of power plants to produce electricity. They sell the electricity at the electricity market. Consumers, on the other hand, buy electricity at the electricity market and sell it at a contractually fixed price to their end users. In reality, consumers, such as utilities, are all market participants that consume electricity and are big enough to have a direct access to the electricity market. On the other hand, end users, such as households and businesses, are smaller and do not have a direct access to the electricity market.

Each producer $p \in P$ owns a portfolio of power plants and the combined costs $C\left(\widehat{V}_p\right)$ of producing $\widehat{V}_p \in \mathbb{R}$ units of electricity is modeled as

$$C\left(\widehat{V}_p\right) := C_{\text{op}}\left(\widehat{V}_p\right) + f_3, \tag{1.11}$$

where $f_3 \in \mathbb{R}$ denotes the sum of all fixed costs of the power plants in the portfolio and $C_{\text{op}}\left(\widehat{V}_p\right)$ denotes all of the costs that depend on the amount of electricity produced. Hence, $C_{\text{op}}\left(\widehat{V}_p\right)$ includes fuel costs, operating costs, startup costs etc. In Bessembinder and Lemmon (2002), it is assumed that $C_{\text{op}}$ can be modeled as

$$C_{\text{op}}\left(\widehat{V}_p\right) := \frac{f_1}{f_2}\left(\widehat{V}_p\right)^{f_2} \tag{1.12}$$

where $f_1 \geq 0$ and $f_2 \geq 2$ are both constants. Note that $C_{\text{op}}$ is a deterministic function of the production volume $\widehat{V}_p$ and does not take into account the stochasticity of fuel prices.

Let $V_k\left(t_i\right)$ denote the number of electricity contracts player $k \in P \cup C$ buys at trading time $t_i$, $i \in \{1, 2\}$. Note that a positive value of $V_k\left(t_i\right)$ implies that the player $k \in P \cup C$ is long in (i.e. is buying) a particular electricity contract and a negative value of $V_k\left(t_i\right)$ implies that the player is short in (i.e. is selling) a contract.

The ex-post profit of a producer $p \in P$ can then be calculated as

$$P_p\left(V_p, \Pi\right) = -\Pi\left(t_1\right) V_p\left(t_1\right) - \Pi\left(t_2\right) V_p\left(t_2\right) - \frac{f_1}{f_2}\left(-V_p\left(t_1\right) - V_p\left(t_2\right)\right)^{f_2} - f_3, \tag{1.13}$$

where $V_k = \left[V_k\left(t_1\right), V_k\left(t_2\right)\right]^\top$, $k \in P \cup C$ and $\Pi = \left[\Pi\left(t_1\right), \Pi\left(t_2\right)\right]^\top$. The first therm in (1.13) denotes the profit from trading the futures contract, the second, the profit from trading the spot contract, and the final two terms denote the costs of producing electricity that was agreed to be delivered through these two contracts.

Similarly, the ex-post profit of a consumer $c \in C$ can be calculated as

$$P_c\left(V_c, \Pi\right) = -\Pi\left(t_1\right) V_c\left(t_1\right) - \Pi\left(t_2\right)\left(p_c D - V_c\left(t_1\right)\right) + s p_c D$$

where $s \in \mathbb{R}$ denotes a constant electricity price which is charged to the end users (i.e.

households and businesses), $D$ is an exogenous variable which denotes the total electricity demand, and $p_c$ denotes the percentage of the total electricity demand that is to be delivered by consumer $c \in C$. Clearly, $V_c(t_1) + V_c(t_2) = p_c D$ and $\sum_{c \in C} p_c = 1$.

Since the electricity price model that we propose in Chapter 2 belongs to the class of game-theoretic models, we investigate this line of research more in detail. We follow Bessembinder and Lemmon (2002) by attempting to solve the problem presented above backwards. At time $t_2$, the forward price $\Pi(t_1)$, the spot price $\Pi(t_2)$ and the decision $V_p(t_1)$ are known, and thus, the optimal decision $V_p(t_2)$ of producer $p \in P$ can be determined from

$$\frac{\partial P_p(V_p, \Pi)}{\partial V_p(t_2)} = -\Pi(t_2) + f_1(-V_p(t_1) - V_p(t_2))^{f_2 - 1} = 0.$$

Solving for $V_p(t_2)$ leads to

$$V_p(t_2) = -V_p(t_1) - \left(\frac{\Pi(t_2)}{f_1}\right)^{\frac{1}{f_2-1}}. \tag{1.14}$$

Since the total production of all producers $p \in P$ must equal the total demand, we get

$$-\sum_{p \in P}(V_p(t_1) + V_p(t_2)) = D. \tag{1.15}$$

By combining (1.14) and (1.15), the expression for $\Pi(t_2)$ is

$$\Pi(t_2) = f_1\left(\frac{D}{|P|}\right)^{f_2-1}, \tag{1.16}$$

where $|P|$ denotes the number of producers, i.e. the cardinality of the set $P$. Using (1.16), the expression (1.14) reads

$$V_p(t_2) = -V_p(t_1) - \frac{D}{|P|}. \tag{1.17}$$

For convenience, let us define profit functions of producers and consumers, when there is no trading on the forward market, i.e. $V_k(t_1) = 0$ for all $k \in P \cup C$. Using (1.17), we get

$$\widehat{P}_p(V_p, \Pi) = \Pi(t_2)\frac{D}{|P|} - \frac{f_1}{f_2}\left(\frac{D}{|P|}\right)^{f_2} - f_3 \tag{1.18}$$

for producers $p \in P$ and

$$\widehat{P}_c(V_c, \Pi) = -\Pi(t_2)\, p_c D + s p_c D \tag{1.19}$$

for consumers $c \in C$.

Players $k \in P \cup C$ would like to maximize their profit subject to a risk budget. Under a mean-variance framework, they are interested in the mean-variance utility

$$\Psi_k(V_k, \Pi) = \mathbb{E}^{\mathbb{P}}[P_k(V_k, \Pi)] - \frac{\lambda}{2}\mathrm{Var}^{\mathbb{P}}[P_k(V_k, \Pi)] \tag{1.20}$$

where $\lambda > 0$ denotes the constant risk preference of producers and consumers. The optimal forward position of producers and consumers $k \in P \cup C$ in this setting, can be calculated as

$$V_k(t_1) = \frac{\mathbb{E}^{\mathbb{P}}[\Pi(t_2)] - \Pi(t_1)}{\lambda \mathrm{Var}^{\mathbb{P}}[\Pi(t_2)]} - \frac{\mathrm{Cov}^{\mathbb{P}}\left[\widehat{P}_k(V_k, \Pi), \Pi(t_2)\right]}{\mathrm{Var}^{\mathbb{P}}[\Pi(t_2)]}, \tag{1.21}$$

where $\widehat{P}_k(V_k, \Pi)$ is defined by (1.18) for producers $k \in P$ and by (1.19) for consumers $k \in C$. A proof of (1.21) can be found in Anderson and Danthine (1980) or Hirshleifer and Subrahmanyam (1993). The same result in a more general setting can also be found in the proof of Proposition 3.3.1 in this work.

Expression (1.21) defines the optimal decisions of all producers and consumers. However, they may not able to execute their calculated optimal trading strategy because they may not find a counterparty to trade with. In reality, each contract consists of a buyer and a seller, which imposes an additional constraint, the market clearing constraint, that matches the number of short and long forward electricity contracts as

$$\sum_{k \in P \cup C} V_k(t_1) = 0. \tag{1.22}$$

Expression (1.22) together with (1.21), leads to

$$\Pi(t_1) - \mathbb{E}^{\mathbb{P}}[\Pi(t_2)] + \frac{\lambda}{|P| + |C|} \sum_{k \in P \cup C} \mathrm{Cov}^{\mathbb{P}}\left[\widehat{P}_k(V_k, \Pi), \Pi(t_2)\right] = 0. \tag{1.23}$$

Using the properties of covariances and expressions (1.18) and (1.19), we can rewrite (1.23) as

$$\Pi\left(t_1\right)=\mathbb{E}^{\mathbb{P}}\left[\Pi\left(t_2\right)\right]-\frac{\lambda}{|P|+|C|}\mathrm{Cov}^{\mathbb{P}}\left[D\left(s-\frac{f_1}{f_2}\left(\frac{D}{|P|}\right)^{f_2-1}\right),\Pi\left(t_2\right)\right]. \tag{1.24}$$

Expression (1.24) allows us to calculate the expected spot electricity price given the current forward price. We notice that the expected spot price depends on total demand, as it did in Howison and Coulon (2009) in the context of structural approaches. An extension of equation (1.24) to multiple delivery periods $T_j$, $j \in J$ is available in Bühler and Müller-Merbach (2009) and Bühler (2009).

In this work we extend the model of Bühler and Müller-Merbach (2009) to include:

- ramping constraints that determine the maximum change (i.e. increase and decrease) of the production of each power plant in the portfolio from one delivery period $T_j$ to the next $T_{j+1}$.
- the dependence of costs $C_{\mathrm{op}}$ (remember definitions (1.11) and (1.12)) on fuel prices. Additionally, instead of modeling the costs of a whole portfolio of power plants, we model costs of each plant separately. This enables us to include startup costs of power plants in our model as well.

Modeling each power plant separately and including capacity and ramping constraints complicates the analysis of the problem. It is trivial to show that the optimal trading volumes as given by (1.21) and expected electricity price as given by (1.24), exist and are unique. However, the addition of inequality constraints (e.g. ramping and capacity constraints), makes the proof of existence and uniqueness of expected electricity prices nontrivial. We provide details in Chapter 2.

## 1.2 Sensitivity analysis of quadratic programming

In the previous section, we analyzed the trading behavior of a market player interested in maximizing a mean-variance objective function (1.20). In Chapter 2, we extend this problem and model the production of power plants more accurately by including their ramping and capacity constraints. As we will see, these constraints can be expressed as linear inequality constraints. Using an optimization framework, we can thus model every market player by a quadratic programming problem with linear constraints.

A standard quadratic programming (QP) problem is defined as

$$\begin{aligned} \max_x \quad & c^\top x - \tfrac{1}{2} x^\top Q x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \tag{1.25}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $Q \in \mathbb{R}^{n\times n}$. In order to ensure concavity of the objective function, we additionally require that matrix $Q$ is positive semidefinite, $Q \succeq 0$. In the mean-variance framework, the linear term and the quadratic term in the objective function of problem (1.25) correspond to the mean maximization and variance minimization, respectively.

In this section we analyze how the optimal solution of (1.25) depends on the constant vector $c$. In our modeling framework the constant vector $c$ denotes a vector of expected electricity prices. By studying the dependence of the optimal trading volumes (1.21) on the expected electricity price, we can try to find an expected electricity price at which the market clearing constraint (1.22) is simultaneously satisfied for all contracts. We will discuss that in more detail in the following few sections. Let us now focus on the sensitivity of the quadratic programming in general.

In the description in this section, we follow the analysis presented in Boot (1963) and Hadigheh et al. (2007). We assume positive definiteness of $Q$, $Q \succ 0$ and non-degeneracy of the quadratic programming problem. Let $x^* \in \mathbb{R}^n$ denote the optimal solution of (1.25) and let $\mathcal{S}$ denote the set of active constraints at $x^*$. Ignoring the non-active constraints, (1.25) can be written as

$$\begin{aligned} \max_x \quad & c^\top x_{\mathcal{S}} - \tfrac{1}{2} x_{\mathcal{S}}^\top Q x_{\mathcal{S}} \\ \text{s.t.} \quad & A_{\mathcal{S}} x_{\mathcal{S}} = b_{\mathcal{S}} \end{aligned} \tag{1.26}$$

where $A_{\mathcal{S}}$ and $b_{\mathcal{S}}$ include only those rows of $A$ and $b$ that belong to the active set constrains.

The necessary and, due to the linearity of constraints, sufficient conditions (as in Lemma

12.7 in Nocedal and Wright (2006)) for $x_{\mathcal{S}} \in \mathbb{R}^n$ to be an optimal solution of problem (1.26) are

$$\frac{\partial \mathcal{L}(x_{\mathcal{S}}, u_{\mathcal{S}})}{\partial x_{\mathcal{S}}} = 0$$

and

$$\frac{\partial \mathcal{L}(x_{\mathcal{S}}, u_{\mathcal{S}})}{\partial u_{\mathcal{S}}} = 0,$$

where $\mathcal{L}(x_{\mathcal{S}}, u_{\mathcal{S}})$ denotes the Lagrangian defined by

$$\mathcal{L}(x_{\mathcal{S}}, u_{\mathcal{S}}) = c^\top x_{\mathcal{S}} - \frac{1}{2} x_{\mathcal{S}}^\top Q x_{\mathcal{S}} - u_{\mathcal{S}}^\top (A_{\mathcal{S}} x_{\mathcal{S}} - b_{\mathcal{S}}).$$

Calculating the necessary and sufficient optimality conditions, we get

$$\frac{\partial \mathcal{L}(x_{\mathcal{S}}, u_{\mathcal{S}})}{\partial x_{\mathcal{S}}} = c - Q x_{\mathcal{S}} - A_{\mathcal{S}}^\top u_{\mathcal{S}}$$

and

$$\frac{\partial \mathcal{L}(x_{\mathcal{S}}, u_{\mathcal{S}})}{\partial u_{\mathcal{S}}} = A_{\mathcal{S}} x_{\mathcal{S}} - b_{\mathcal{S}}.$$

Solving for $x_{\mathcal{S}}$ and $u_{\mathcal{S}}$, we obtain

$$x_{\mathcal{S}} = Q^{-1} c - Q^{-1} A_{\mathcal{S}}^\top u_{\mathcal{S}} \tag{1.27}$$

and

$$u_{\mathcal{S}} = \left(A_{\mathcal{S}} Q^{-1} A_{\mathcal{S}}^\top\right)^{-1} \left(A_{\mathcal{S}} Q^{-1} c - b_{\mathcal{S}}\right). \tag{1.28}$$

Here, we assume that $A_{\mathcal{S}}$ has full rank and thus $\left(A_{\mathcal{S}} Q^{-1} A_{\mathcal{S}}^\top\right)^{-1}$ exists. Inserting (1.28) back into (1.27) leads to

$$x_{\mathcal{S}} = Q^{-1} c - Q^{-1} A_{\mathcal{S}}^\top \left(A_{\mathcal{S}} Q^{-1} A_{\mathcal{S}}^\top\right)^{-1} \left(A_{\mathcal{S}} Q^{-1} c - b_{\mathcal{S}}\right). \tag{1.29}$$

For $x_{\mathcal{S}}$ to be a feasible solution of problem (1.25), the following constraints must be satisfied

$$\begin{aligned} Ax_{\mathcal{S}} &\leq b, \\ u_{\mathcal{S}} &\geq 0. \end{aligned} \tag{1.30}$$

If non-degeneracy is assumed, then constraints (1.30) can be written as

$$\begin{aligned} A_{\mathcal{S}} x_{\mathcal{S}} &= b_{\mathcal{S}}, \\ A_{\overline{\mathcal{S}}} x_{\mathcal{S}} &< b_{\overline{\mathcal{S}}}, \\ u_{\mathcal{S}} &> 0. \end{aligned}$$

where $\overline{\mathcal{S}}$ represent a set of inactive constraints. Due to the assumption of non-degeneracy, $\mathcal{S} \cap \overline{\mathcal{S}} = \emptyset$.

Let us examine how the solution vector $x_{\mathcal{S}}$ changes if the parameters of the objective function $c$ are slightly altered in such a way that the active set $\mathcal{S}$ does not change. Differentiating (1.29), we obtain

$$\frac{\partial x_{\mathcal{S}}}{\partial c} = Q^{-1} - Q^{-1} A_{\mathcal{S}}^{\top} \left( A_{\mathcal{S}} Q^{-1} A_{\mathcal{S}}^{\top} \right)^{-1} A_{\mathcal{S}} Q^{-1}, \tag{1.31}$$

and similarly,

$$\frac{\partial u_{\mathcal{S}}}{\partial c} = \left( A_{\mathcal{S}} Q^{-1} A_{\mathcal{S}}^{\top} \right)^{-1} A_{\mathcal{S}} Q^{-1}. \tag{1.32}$$

We continue with an analysis of matrix $\dfrac{\partial x_{\mathcal{S}}}{\partial c}$.

**Lemma 1.2.1.** *If $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, such that $Q^{-1} = \tilde{Q}^{-\top} \tilde{Q}^{-1}$ exists for some $\tilde{Q}^{-1} \in \mathbb{R}^{n \times n}$, and $A_{\mathcal{S}} \in \mathbb{R}^{m \times n}$, $m < n$ has full rank, then $I - \tilde{Q}^{-1} A_{\mathcal{S}}^{\top} \left( A_{\mathcal{S}} \tilde{Q}^{-\top} \tilde{Q}^{-1} A_{\mathcal{S}}^{\top} \right)^{-1} A_{\mathcal{S}} \tilde{Q}^{-\top}$ is a projection matrix.*

*Proof.* Assume $Q \succ 0$ and symmetric. Then also $Q^{-1} \succ 0$ and symmetric. Write $Q^{-1} = \tilde{Q}^{-\top} \tilde{Q}^{-1}$ for some $\tilde{Q}^{-1} \in \mathbb{R}^{n \times n}$. A matrix $P$ is a projection matrix, if $P = P^2$ holds. It is trivial to check that his indeed holds for $P := I - \tilde{Q}^{-1} A_{\mathcal{S}}^{\top} \left( A_{\mathcal{S}} \tilde{Q}^{-\top} \tilde{Q}^{-1} A_{\mathcal{S}}^{\top} \right)^{-1} A_{\mathcal{S}} \tilde{Q}^{-\top}$. □

**Proposition 1.2.2.** *If $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, and $A_{\mathcal{S}} \in \mathbb{R}^{m \times n}$, $m < n$ has full rank, then $\frac{\partial x_{\mathcal{S}}}{\partial c}$ as given in (1.31) is positive semidefinite.*

*Proof.* Assume $Q \succ 0$ and symmetric. Then also $Q^{-1} \succ 0$ and symmetric. Write $Q^{-1} = \tilde{Q}^{-\top}\tilde{Q}^{-1}$ for some $\tilde{Q}^{-1} \in \mathbb{R}^{n\times n}$ and set $P := \tilde{Q}^{-1}A_{\mathcal{S}}^{\top}\left(A_{\mathcal{S}}\tilde{Q}^{-\top}\tilde{Q}^{-1}A_{\mathcal{S}}^{\top}\right)^{-1}A_{\mathcal{S}}\tilde{Q}^{-\top}$. According to Lemma 1.2.1, $P$ and $I-P$ are both projection matrices. Therefore, we can write

$$\langle x, x\rangle = \langle Px, Px\rangle + \langle (I-P)\,x, (I-P)\,x\rangle$$

for all $x \in \mathbb{R}^n$, and thus

$$\langle x, x\rangle - \langle Px, Px\rangle \geq 0.$$

Since this holds for all $x \in \mathbb{R}^n$, it must hold also for all $y \in \operatorname{Im}\left(\tilde{Q}^{-1}\right)$. Then

$$\begin{aligned}
\langle y, y\rangle - \langle Py, Py\rangle &\geq 0\\
\left\langle \tilde{Q}^{-1}x, \tilde{Q}^{-1}x\right\rangle - \left\langle P\tilde{Q}^{-1}x, P\tilde{Q}^{-1}x\right\rangle &\geq 0\\
x^{\top}Q^{-1}x - x^{\top}\tilde{Q}^{-\top}P^{\top}P\tilde{Q}^{-1}x &\geq 0\\
x^{\top}\left(Q^{-1} - \tilde{Q}^{-\top}PP\tilde{Q}^{-1}\right)x &\geq 0\\
x^{\top}\left(Q^{-1} - \tilde{Q}^{-\top}P\tilde{Q}^{-1}\right)x &\geq 0\\
x^{\top}\left(Q^{-1} - Q^{-1}A_{\mathcal{S}}^{\top}\left(A_{\mathcal{S}}Q^{-1}A_{\mathcal{S}}^{\top}\right)^{-1}A_{\mathcal{S}}Q^{-1}\right)x &\geq 0\\
x^{\top}\tfrac{\partial x_{\mathcal{S}}}{\partial c}x &\geq 0,
\end{aligned}$$

where we used the fact that $P$ is a projection matrix, i.e. symmetric and $P = P^2$. □

If we consider the optimal solution $x_{\mathcal{S}}^*$ as a function of $c$, we can use Proposition 1.2.2 to conclude that the mapping $\frac{\partial x_{\mathcal{S}}}{\partial c}$ is monotone. Proposition 1.2.2 can be used when the active set $\mathcal{S}$ is constant. In the remainder of this section, we investigate properties of the optimal solution $x^*$ at points when the active set changes.

Let $x_{\mathcal{S}}^*$ and $u_{\mathcal{S}}^*$ denote the optimal solution of (1.25) with $\mathcal{S}$ being the set of active constraints. If coefficients of the objective function $c$ are altered to $c + \alpha\triangle c$, $\alpha > 0$ then

$$x_{\mathcal{S}}' = x_{\mathcal{S}}^* + \alpha\frac{\partial x_{\mathcal{S}}}{\partial c}\triangle c$$

and

$$u'_{\mathcal{S}} = u^*_{\mathcal{S}} + \alpha \frac{\partial u_{\mathcal{S}}}{\partial c} \triangle c,$$

where $x'_{\mathcal{S}}$ and $u'_{\mathcal{S}}$ denote the optimal solution of the altered problem. Such a solution is feasible if

$$\begin{aligned} A_{\mathcal{S}} x'_{\mathcal{S}} &= b_{\mathcal{S}}, \\ A_{\overline{\mathcal{S}}} x'_{\mathcal{S}} &< b_{\overline{\mathcal{S}}}, \\ u'_{\mathcal{S}} &> 0. \end{aligned} \tag{1.33}$$

The first constraint is satisfied by construction. The second constraint is satisfied for all $\alpha \leq \alpha^x$, where $\alpha^x$ satisfies

$$\alpha^x < \min_{\substack{i \in \overline{\mathcal{S}}, \\ a_i^\top \frac{\partial x_{\mathcal{S}}}{\partial c} \triangle c > 0}} \frac{b_i - a_i^\top x^*_{\mathcal{S}}}{a_i^\top \frac{\partial x_{\mathcal{S}}}{\partial c} \triangle c}.$$

The third constraint is satisfied for all $\alpha \leq \alpha^u$, where $\alpha^u$ satisfies

$$\alpha^u < \min_{\substack{i \in \mathcal{S}, \\ \left[\frac{\partial u_{\mathcal{S}}}{\partial c} \triangle c\right]_i < 0}} - \frac{[u^*_{\mathcal{S}}]_i}{\left[\frac{\partial u_{\mathcal{S}}}{\partial c} \triangle c\right]_i}.$$

All constraints are thus satisfied for $\alpha \leq \min(\alpha^x, \alpha^u)$. If $\alpha$ increases further, then the active set $\mathcal{S}$ changes. If the second constraint in (1.33) is violated first, then the violating constraint is added to the active set, i.e. $\mathcal{S}' = \mathcal{S} + i$, $i \in \overline{\mathcal{S}}$. Similarly, if the third constraint in (1.33) is violated first, then the violating constraint is removed from the active set, i.e. $\mathcal{S}' = \mathcal{S} - i$, $i \in \mathcal{S}$. We will assume that two constraints cannot be violated for the same $\alpha$ (see Berkelaar et al. (1996) for an analysis without this assumption).

We now will show that optimal solutions $x^*$ and $u^*$ are continuous functions of the data vector $c$. Let us first define

$$P_{\mathcal{S}} := A_{\mathcal{S}} Q^{-1} A_{\mathcal{S}}^\top,$$

$$A_{\mathcal{S}+i} = \begin{bmatrix} A_{\mathcal{S}} \\ \hline a_i \end{bmatrix},$$

and

$$b_{\mathcal{S}+i} = \left[\begin{array}{c} b_{\mathcal{S}} \\ \hline b_i \end{array}\right].$$

$A_{\mathcal{S}+i}$ is a matrix of active constraints with an additional row vector[3] $a_i$ in the last row. Similarly, vector $b_{\mathcal{S}+i}$ corresponds to a vector of active constraints with an additional element $b_i$ in the last row. Then

$$\begin{aligned} P_{\mathcal{S}+i} &= \left[\begin{array}{c} A_{\mathcal{S}} \\ \hline a_i \end{array}\right] Q^{-1} \left[\begin{array}{c} A_{\mathcal{S}} \\ \hline a_i \end{array}\right]^\top \\ &= \begin{bmatrix} P_{\mathcal{S}} & p \\ p^\top & \pi \end{bmatrix}, \end{aligned} \tag{1.34}$$

where

$$p := A_{\mathcal{S}} Q^{-1} a_i^\top, \tag{1.35}$$

and

$$\pi := a_i Q^{-1} a_i^\top. \tag{1.36}$$

**Lemma 1.2.3.** *The inverse of the matrix $P_{\mathcal{S}+i}$ is*

$$P_{\mathcal{S}+i}^{-1} = \begin{bmatrix} P_{\mathcal{S}}^{-1} + \dfrac{P_{\mathcal{S}}^{-1} p p^\top P_{\mathcal{S}}^{-1}}{\pi - p^\top P_{\mathcal{S}}^{-1} p} & -\dfrac{P_{\mathcal{S}}^{-1} p}{\pi - p^\top P_{\mathcal{S}}^{-1} p} \\ -\dfrac{p^\top P_{\mathcal{S}}^{-1}}{\pi - p^\top P_{\mathcal{S}}^{-1} p} & \dfrac{1}{\pi - p^\top P_{\mathcal{S}}^{-1} p} \end{bmatrix}.$$

*Proof.* Using a simple matrix multiplication we get $P_{\mathcal{S}+i} P_{\mathcal{S}+i}^{-1} = I$. Alternatively, consider the blockwise inversion formula. □

**Proposition 1.2.4.** *The optimal values $x^*$ and $u^*$ are continuous, piecewise affine functions of $c$.*

[3] Note the slight abuse of notation in that we omit the transpose operator, $\top$.

*Proof.* Since the first derivatives $\frac{\partial x_{\mathcal{S}}}{\partial c}$ and $\frac{\partial u_{\mathcal{S}}}{\partial c}$ exist and are constant, when the active set is fixed, it is clear that $x^*_{\mathcal{S}}$ and $u^*_{\mathcal{S}}$ are continuous, affine functions of $c$. It remains to show that the continuity holds also at points where the active set $\mathcal{S}$ changes. We consider two possibilities:

1. An inactive constraint $i \in \overline{\mathcal{S}}$ is added to the active set. Then following (1.28) the new $u'_{\mathcal{S}+i} = \begin{bmatrix} u'_{\mathcal{S}} \\ \hline u'_i \end{bmatrix}$, $i \in \overline{\mathcal{S}}$ is calculated as

$$\begin{aligned} u'_{\mathcal{S}+i} &= P^{-1}_{\mathcal{S}+i}\left(A_{\mathcal{S}+i}Q^{-1}c - b_{\mathcal{S}+i}\right) \\ &= P^{-1}_{\mathcal{S}+i}\left(\begin{bmatrix} A_{\mathcal{S}} \\ \hline a_i \end{bmatrix} Q^{-1}c - \begin{bmatrix} b_{\mathcal{S}} \\ \hline b_i \end{bmatrix}\right), \end{aligned}$$

and by using Lemma 1.2.3

$$u'_{\mathcal{S}+i} = \begin{bmatrix} P_{\mathcal{S}}^{-1} + \dfrac{P_{\mathcal{S}}^{-1}pp^\top P_{\mathcal{S}}^{-1}}{\pi - p^\top P_{\mathcal{S}}^{-1}p} & -\dfrac{P_{\mathcal{S}}^{-1}p}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ -\dfrac{p^\top P_{\mathcal{S}}^{-1}}{\pi - p^\top P_{\mathcal{S}}^{-1}p} & \dfrac{1}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \end{bmatrix} \begin{bmatrix} A_{\mathcal{S}}Q^{-1}c - b_{\mathcal{S}} \\ \hline a_iQ^{-1}c - b_i \end{bmatrix}.$$

We now focus on $u'_{\mathcal{S}}$ and $u'_i$ separately. We start with $u'_{\mathcal{S}}$ which can be written as

$$\begin{aligned} u'_{\mathcal{S}} &= P_{\mathcal{S}}^{-1}\left(A_{\mathcal{S}}Q^{-1}c - b_{\mathcal{S}}\right) + \frac{P_{\mathcal{S}}^{-1}pp^\top P_{\mathcal{S}}^{-1}\left(A_{\mathcal{S}}Q^{-1}c - b_{\mathcal{S}}\right) - P_{\mathcal{S}}^{-1}p\left(a_iQ^{-1}c - b_i\right)}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ &= u_{\mathcal{S}} + \frac{P_{\mathcal{S}}^{-1}p\left(p^\top u_{\mathcal{S}} - a_iQ^{-1}c + b_i\right)}{\pi - p^\top P_{\mathcal{S}}^{-1}p}, \end{aligned} \tag{1.37}$$

where (1.28) was used twice. Using definition (1.35) in expression (1.37), we obtain

$$\begin{aligned} u'_{\mathcal{S}} &= u_{\mathcal{S}} + \frac{P_{\mathcal{S}}^{-1}p\left(a_iQ^{-1}A_{\mathcal{S}}^\top u_{\mathcal{S}} - a_iQ^{-1}c + b_i\right)}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ &= u_{\mathcal{S}} + \frac{P_{\mathcal{S}}^{-1}p\left(a_i\left(Q^{-1}A_{\mathcal{S}}^\top u_{\mathcal{S}} - Q^{-1}c\right) + b_i\right)}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ &= u_{\mathcal{S}} + \frac{P_{\mathcal{S}}^{-1}p\left(-a_ix_{\mathcal{S}} + b_i\right)}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \end{aligned}$$

where the last equality holds due to (1.27). For a constraint $i \in \overline{\mathcal{S}}$ that is about to enter the active set, the equality $a_i x_{\mathcal{S}} = b_i$ must hold. Thus, $u'_{\mathcal{S}} = u_{\mathcal{S}}$. Similarly,

$$\begin{aligned} u'_i &= \frac{-p^\top P_{\mathcal{S}}^{-1}\left(A_{\mathcal{S}}Q^{-1}c - b_{\mathcal{S}}\right) + \left(a_i Q^{-1}c - b_i\right)}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ &= \frac{-p^\top u_{\mathcal{S}} + a_i Q^{-1}c - b_i}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ &= \frac{-a_i Q^{-1}A_{\mathcal{S}}^\top u_{\mathcal{S}} + a_i Q^{-1}c - b_i}{\pi - p^\top P_{\mathcal{S}}^{-1}p} \\ &= \frac{a_i x_{\mathcal{S}} - b_i}{\pi - p^\top P_{\mathcal{S}}^{-1}p}, \end{aligned}$$

where the second, third and fourth equality hold due to (1.28), (1.35) and (1.27), respectively. Using the fact that for a constraint $i \in \overline{\mathcal{S}}$ that is to enter the active set, the equality $a_i x_{\mathcal{S}} = b_i$ must hold, we conclude that $u'_i = 0$. This means that the values of $u^*$ do not change when a new constraints is added to the active set. From (1.27) we can see that this implies that $x^*$ does not change either. Thus, we conclude that optimal values $x^*$ and $u^*$ are continuous at points when a new constraint is added to the active set.

2. An active constraint $i \in \mathcal{S}$ is removed from the active set. In this case, we just reverse the argument by noticing the equivalence between $\mathcal{S}' := \mathcal{S} + j$, $j \in \overline{\mathcal{S}}$ and $\mathcal{S}' - i := \mathcal{S}$, $i \in \mathcal{S}$.

□

The sensitivity analysis and slightly generalized versions of Propositions 1.2.2 and 1.2.4 will play a major role in showing the existence and uniqueness of electricity prices in our model in Chapter 2.

## 1.3 Main contributions

In the previous sections of this chapter, we introduced the background material upon which we build the rest of this thesis. In this section, we explain the main contributions of the thesis and how they relate to the existing literature. We consider each chapter separately.

**Chapter 2**

1. We present a new model of electricity prices. Our model belongs to the class of game-theoretic models. It extends the model of Bessembinder and Lemmon (2002) and Bühler (2009) by including ramping and capacity constraints of the power plants. The costs of power plants are modeled more precisely, and in contrast to Bessembinder and Lemmon (2002) and Bühler (2009) (but similar to Howison and Coulon (2009)), they depend on fuel prices. A schematic of our model is depicted in Figure 2.1. These extensions enabled us to develop, according to our knowledge, the most thorough to-date fundamental model of the term structure of electricity prices, while retaining the computational tractability of simpler models.

2. We study the existence and uniqueness of solutions to the model. We show that the existence of solutions depends on properties of the total demand $D(T_j)$, $j \in J$. If total demand is such that it can be satisfied by the existing power plants connected to the electricity grid, then our problem has a solution. This condition requires that the demand is not too large or negative, and that it does not increase or decrease from one delivery period $T_j$ to the next $T_{j+1}$, $j \in J$, too quickly.

   We also study the conditions that ensure the uniqueness of solutions. The analysis relies on the sensitivity analysis of the quadratic programming problem presented in Section 1.2. Roughly speaking, we show that if players are sufficiently risk-averse and if power plants are similar enough (i.e. there are no big differences in efficiency), then the electricity price of all forward and spot contracts is unique. We will see that if the solution is not unique, then it (viz. the price of the contract) may assume any value inside a closed interval.

3. We compare two types of equilibria, namely the competitive equilibrium and the Nash equilibrium, and show the equivalence of their solutions for our model. Both the competitive and Nash equilibrium have attracted a lot of attention in the economics literature. We show that, under a set of assumptions that guarantee the uniqueness of

solutions, these two related equilibria lead to the same solution. However, this is only a theoretical result. Computationally, the calculation of a competitive equilibrium is much more stable than that of a Nash equilibrium.

4. We demonstrate why the conditions for the uniqueness of solutions are not satisfied if players are interested in the mean maximization problem instead of the mean-variance maximization problem.

**Chapter 3**

1. We present a tractable quadratic programming formulation that can be used to solve the model described in Chapter 2.

2. We extend the model to include transaction costs, block contracts and futures contracts. Transaction costs and liquidity constraints have already been studied in De Maere d'Aertrycke and Smeers (2012), but we are not aware of any work which combines transaction costs and block contracts. If one is to model realistic electricity markets, this is very important because some block contracts (such as season-ahead contracts) are less liquid than others (such as day-ahead contracts).

   Bühler (2009) derived a closed form solution for forward electricity prices without taking block contracts into account. However, an inclusion of block contracts is crucial for modeling realistic electricity markets. In reality, it is impossible to trade half hourly electricity contracts seasons or months ahead of delivery. Seasonal and monthly contracts are only available in seasonal and monthly blocks; half hourly contracts are only available close to delivery - usually on the day-ahead market. Block contracts have already been modeled using non-structural approaches (see Lucia and Schwartz (2000)), but we are not aware of any work in the game-theoretic setting that caters for them.

3. We study the impact of various parameters of the model on the term structure of electricity prices. Firstly, we examine the impact of the risk preference constant on the term structure. We show that the risk preference of producers can affect the absolute

level of the price as well as the shape of the term structure, while the risk preference constant of consumers can affect only the term structure. This is a consequence of the fact that consumers must satisfy the total demand of end users, regardless of the absolute level of the electricity price. Consumers can only choose which contracts to trade to satisfy the demand. This gives them the ability to affect the term structure of electricity prices. If the absolute level of the electricity price is not favorable, consumers can maintain their profitability by passing the loss to end users through changing the fixed electricity price of their clients. On the other hand, producers have more flexibility in their decisions and they only generate electricity when it is profitable for them to do so and the risk premium in the price is high enough. Thus, they have the power to affect the level as well as the term structure of electricity prices.

We study also the impact of linear and quadratic transaction costs. We show that, as expected, higher transaction costs lead to reduced trading of a particular contract.

We examine also the impact of the term structure of fuel prices on the term structure of electricity prices. We are not aware of any existing study of this type in the context of game-theoretic models. We show that the term structure of electricity prices is highly dependent on the term structure of fuel prices. Our numerical simulations demonstrate that, qualitatively speaking, the shape of the term structure of electricity prices follows the shape of the term structure of fuel prices, e.g. if the term structure of the fuel prices is increasing (decreasing), the term structure of electricity prices is increasing (decreasing) as well.

4. We calibrate our electricity model using data from the UK power market. For this purpose, we downloaded approximately 70 GB of data about historical production, maintenance schedules, capacity and other physical characteristics of over 400 power plants in the UK from the BM Reports[4] webpage. We used this data, combined with historical coal, natural gas and emission prices, to determine the efficiency and the carbon emission intensity factor of each power plant connected to the UK power grid.

5. We use our calibration results to model the spot and forward electricity price over a

[4] `http://www.bmreports.com/`

four day horizon and in particular consider the impact of ramping constraints. Our experiments show that tightening of the ramping constraints makes the spot electricity price more serrated and slightly higher, on average. Numerical simulations also reveal a few areas to improve the model we presented in Chapter 2. First, the daily variation (i.e. the difference between day and night prices) is not high enough. Second, spikes are not captured by this model. See Figure 1.3a for details. Both problems are typical of fundamental models that explicitly include every power plant in the electricity grid (as in Harvey and Hogan (2002) for instance). We show how to overcome these problems in Chapter 4.

6. We extend the closed form solution of the equilibrium electricity price presented in Bessembinder and Lemmon (2002) and Bühler (2009) to a more general setting, that includes ramping and capacity constraints of power plants.

**Chapter 4**

1. We extend the model from Chapter 3 by incorporating startup costs of power plants. Various methodologies have already been proposed on how to do this (see Martinez (2008), Gribik et al. (2007) and Zhang et al. (2009) for instance). Most of them rely on a price uplift approach, where first the power price without startup costs is calculated. This price is then uplifted to reflect startup costs. In our model, startup costs are included in a mathematically rigorous fashion without relying on the uplift heuristic. We are not aware of any research where the impact of startup costs is studied in the context of term structure electricity price models.

2. We develop a tractable approach for estimating the efficiency, carbon emission intensity factor and startup costs of power plants from their historical production. In the existing literature, startup costs are usually estimated using commercially available software (as in Deane et al. (2014)) or their estimation is not described at all (as in Martinez (2008)). We are not aware of any work that provides a tractable methodology for estimating startup costs of power plants from their historical production.

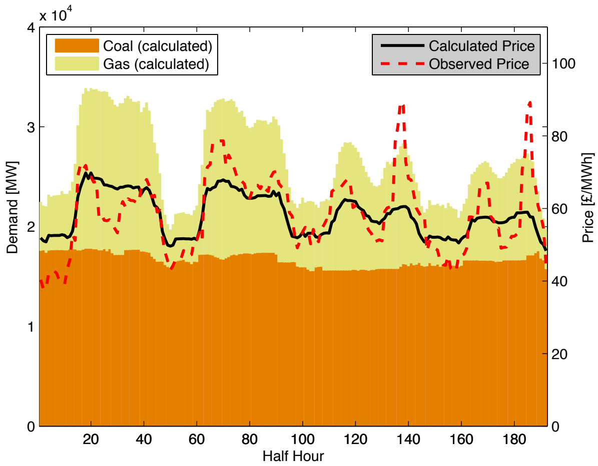
3. We calibrate our electricity model with the UK power market using the above methodology. Approximately 70 GB of data about the historical production of over 400 power plants (publicly available at the BM Reports webpage) is used in combination with historical coal, natural gas and emission prices to determine the efficiency, carbon emission intensity factor and startup costs of each power plant.

4. Through numerical simulations, we demonstrate that startup costs of power plants are the primary reason for high spikes in the spot electricity price. Our model correctly predicts all spikes in the four day horizon of interest. However, our model predicts also many spikes that were not observed in the historical spot price (i.e. false positives). See Figure 1.3b for details.

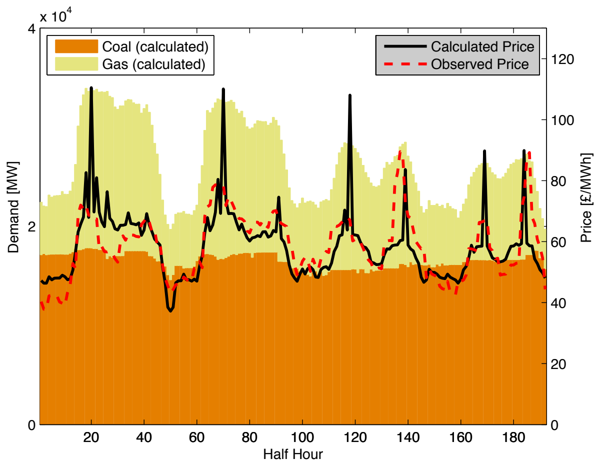
5. We argue that these false positives occur because the actions of the grid operator are not properly captured in our model. In reality, the grid operator is responsible for the robust management of the electricity grid and the reliable delivery of electricity. We conjecture that the actions that help to maintain the reliability of the electricity grid also help to decrease the number and severity of spikes in the spot electricity price. To verify this idea, we incorporate actions of the grid operator in our model though a quadratic penalty function that penalizes all of the players in the electricity market during times of a low stability of the electricity grid. As a result, the number and severity of spikes in the spot price decreases significantly. See Figure 1.3c for details.

## 1.4 Statement of originality

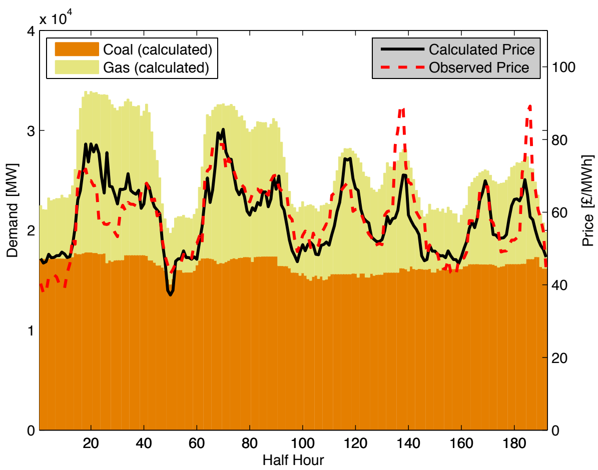
This is to certify that to the best of my knowledge, the content of this thesis is original and is my own work. Chapters 2 - 4 of the thesis are a basis for three research papers which have all been submitted for publication. All the assistance received in preparing the thesis and sources have been acknowledged.




(a) The initial model.




(b) The inclusion of startup costs.




(c) The inclusion of startup costs and actions of the grid operator.

Figure 1.3: Comparison of the calculated and historical electricity price over a four day horizon with half hourly granularity.

# Chapter 2

# Existence and uniqueness of solutions

In this chapter we present our basic model for electricity prices and study the existence and uniqueness of solutions. As mentioned in Chapter 1, our model belongs to the class of game-theoretic approaches. It extends the model of Bühler (2009) to more than one producer and consumer, who maximize their mean-variance utility functions. In contrast to other game-theoretic models, we include capacity and ramping constraints of power plants. Following ideas from structural approaches, the profit of power plants is modeled as the difference between the electricity price and the fuel cost (including emissions). Since we model each power plant directly, we do not have to make any assumptions on the bid stack which, in our case, is determined by the physical properties of power plants. As in Clewlow and Strickland (1999a), our model is consistent with observable fuel and emission prices. We do not focus on a specific emission market scheme, but rather use a generic version of it. For a detailed treatment of the emission market in a game-theoretic setting, see Carmona et al. (2010) and Ludkovski (2011).

The game-theoretic approach attempts to model the decisions made by producers and consumers that participate in the electricity market explicitly. Since a widely used strategy for risk management is delta-hedging, it is important to realize that this strategy can be implemented as a minimum variance strategy in the mean-variance portfolio framework (see Alexander and Nogueira (2006) for details).

This chapter is organized as follows: in Section 2.1, we give a detailed mathematical description of the model and in Section 2.2 we analyze the existence and uniqueness of solutions in a competitive equilibrium setting. In Section 2.3, we extend the analysis to a Nash equilibrium setting.

## 2.1 Problem description

Our market is defined by a set of producers $P$, of cardinality $0 < |P| < \infty$, and a set of consumers $C$, of cardinality $0 < |C| < \infty$. Each of the producers and consumers participates in the electricity market with a goal to maximize their profit, subject to a risk budget under a mean-variance optimization framework. Producers own a number of power plants, which have different physical characteristics and run on different fuels. The set of all fuels is denoted by $L$. We denote by $R^{p,l}$ the set of all of the power plants owned by producer $p \in P$ that run on fuel $l \in L$. The set $R^{p,l}$ may be empty since each producer typically does not own all possible types of power plants. Moreover, this allows us to include non-physical traders such as banks or speculators, who do not own any electricity generation facilities and are without a physical demand for electricity, as producers $p \in P$ with $R^{p,l} = \{\}$ for all $l \in L$.

We are interested in delivery times $T_j$, $j \in J = \{1, ..., T'\}$, where power for each delivery time $T_j$ can be traded through numerous forward contracts at times $t_i$, $i \in I_j$. The electricity price at time $t_i$ for delivery at time $T_j$, is denoted by $\Pi(t_i, T_j)$. Since contracts with a trading time later than the delivery time do not exist, we require $t_{\max\{I_j\}} = T_j$ for all $j \in J$. The number of all forward contracts is denoted by $N := \sum_{j \in J} |I_j|$.

The exogenous variables that appear in our model are:

1. aggregate power demand $D(T_j)$ for each delivery period $j \in J$,
2. prices of forward fuel contracts $G_l(t_i, T_j)$ for each fuel $l \in L$, delivery period $j \in J$, and trading period $i \in I_j$,
3. prices of emissions forward contracts $G_{\text{em}}(t_i, T_j)$, $j \in J$, $i \in I_j$, and
4. the variance of electricity prices $\Pi(t_i, T_j)$, $j \in J$, $i \in I_j$. The expectation of $\Pi(t_i, T_j)$ is considered an endogenous variable and is determined as a solution to our model. This is addressed in more detail in Section 2.1.3.

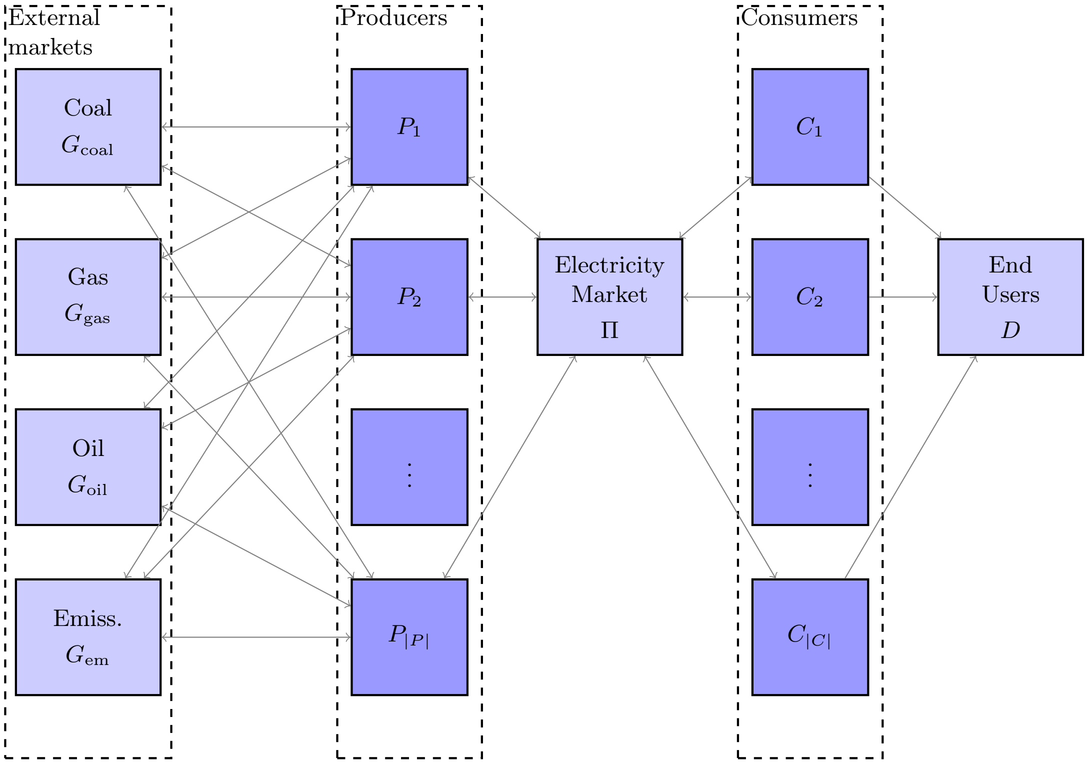



Figure 2.1: A schematic of the market. Producers buy fuels and emission certificates, and use their portfolio of power plants to produce electricity. They sell the electricity at the electricity market. Consumers, on the other hand, buy electricity at the electricity market and sell it at a contractually fixed price to their end users. In reality, consumers, such as utilities, are all market participants that consume electricity and are big enough to have a direct access to the electricity market. On the other hand, end users, such as households and businesses, are smaller and do not have a direct access to the electricity market.

Electricity prices $\Pi(t_i, T_j)$ and all exogenous variables are assumed to be adapted to the filtration $\{\mathcal{F}_t\}_{t \in \cup_{j \in J} I_j}$ and to have finite first and second moments. A schematic of the market in depicted in Figure 2.1.

We remark that through the mechanism of finding the expectation of equilibrium prices $\mathbb{E}^{\mathbb{P}}[\Pi(t_i, T_j)]$, $j \in J$, $i \in I_j$ described in this work, $\mathbb{E}^{\mathbb{P}}[\Pi(t_i, T_j)]$ becomes a function of the exogenous variables $G_l(t_i, T_j)$, $G_{\text{em}}(t_i, T_j)$ and $D(T_j)$, as well as of the model parameters given by the physical characteristics of the power plants and the risk preference of market participants. Since exogenous variables are random, $\mathbb{E}^{\mathbb{P}}[\Pi(t_i, T_j)]$ is also a random variable.

Let $v_k \in \mathbb{R}^{n_k}$, $n_k \in \mathbb{N}$, $k \in K$, and $K = \{1, ..., |K|\}$ be given vectors. For convenience, we define the vector concatenation operator as

$$||_{k\in K}\, v_k = \left[v_1^\top, ..., v_{|K|}^\top\right]^\top .$$

### 2.1.1 Producers

Each producer $p \in P$ has to decide on the number $V_p(t_i, T_j)$ of forward electricity contracts and the number $F_{p,l}(t_i, T_j)$, $l \in L$ of forward fuel contracts to buy, at trading time $t_i$, $i \in I_j$ for delivery at time $T_j$, $j \in J$. We denote the actual production of electricity from fuel $l \in L$ at time $T_j$ from power plant $r \in R^{p,l}$ by $W_{p,l,r}(T_j)$ and the number of emission forward contracts purchased at time $t_i$ for delivery at time $T_j$ by $O_p(t_i, T_j)$. To account for the time value of money (i.e money received earlier is worth more than the same amount received later due to its potential earning capacity), we use a constant interest rate $\hat{r} \in \mathbb{R}$. This interest rate is used for discounting and calculating a net present value (NPV) of future cash flows.

Notation is greatly simplified if decision variables are concatenated into

- electricity purchase vectors $V_p(T_j) = ||_{i\in I_j}\, V_p(t_i, T_j)$ and $V_p = ||_{j\in J}\, V_p(T_j)$,
- fuel purchase vectors $F_p(t_i, T_j) = ||_{l\in L}\, F_{p,l}(t_i, T_j)$, $F_p(T_j) = ||_{i\in I_j}\, F_p(t_i, T_j)$ and $F_p = ||_{j\in J}\, F_p(T_j)$,
- emission purchase vectors $O_p(T_j) = ||_{i\in I_j}\, O_p(t_i, T_j)$ and $O_p = ||_{j\in J}\, O_p(T_j)$,
- electricity production vectors $W_{p,l}(T_j) = ||_{r\in R^{p,l}}\, W_{p,l,r}(T_j)$, $W_p(T_j) = ||_{l\in L}\, W_{p,l}(T_j)$ and $W_p = ||_{j\in J}\, W_p(T_j)$,

and finally $v_p = \left[V_p^\top, F_p^\top, O_p^\top, W_p^\top\right]^\top$.

We proceed similarly with the exogenous variables. We define

- electricity price processes $\Pi(T_j) = ||_{i\in I_j}\, \Pi(t_i, T_j)$, and $\Pi = ||_{j\in J}\, e^{-\hat{r}T_j}\Pi(T_j)$, where $\hat{r} \in \mathbb{R}$ is a constant interest rate,
- fuel price processes $G(t_i, T_j) = ||_{l\in L}\, G_l(t_i, T_j)$, $G(T_j) = ||_{i\in I_j}\, G(t_i, T_j)$ and $G = ||_{j\in J}\, e^{-\hat{r}T_j} G(T_j)$,

- emission price processes $G_{\text{em}}(T_j) = ||_{i\in I_j} G_{\text{em}}(t_i, T_j)$ and $G_{\text{em}} = ||_{j\in J} e^{-\hat{r}T_j} G_{\text{em}}(T_j)$,

and finally $\pi_p = \left[ \Pi^\top, G^\top, G_{\text{em}}^\top, \underbrace{0, ..., 0}_{\dim(W_p)} \right]^\top$, where the number of zeros matches the dimension of vector $W_p$.

The ex-post profit $P_p(v_p, \pi_p)$ of producer $p \in P$ can be calculated as a NPV of discounted cash flows received over all delivery periods $j \in J$ and trading periods $i \in I_j$ as

$$\begin{aligned} P_p(v_p, \pi_p) &= \textstyle\sum_{j\in J} e^{-\hat{r}T_j} \left( \sum_{i\in I_j} P_p^{t_i, T_j}(v_p, \pi_p) \right) \\ &= -\pi_p^\top v_p, \end{aligned} \tag{2.1}$$

where the profit $P_p^{t_i,T_j}(v_p, \pi_p)$ for each $i \in I_j$ and $j \in J$ can be calculated as

$$P_p^{t_i,T_j}(v_p, \pi_p) = -\Pi(t_i, T_j) V_p(t_i, T_j) - G_{\text{em}}(t_i, T_j) O_p(t_i, T_j) - \sum_{l\in L} G_l(t_i, T_j) F_{p,l}(t_i, T_j). \tag{2.2}$$

For each $i \in I_j$ and $j \in J$, the first term in (2.2) denotes the profit or loss (P&L) form trading electricity forward contract, the second term, the P&L from trading emission forward contracts, and the last term, the P&L from trading fuel forward contracts for each fuel $l \in L$.

Ramping up and down constraints for each $j \in \{1, ..., T' - 1\}$, where $T'$ denotes the last delivery period, $l \in L$ and $r \in R^{p,l}$ can be expressed as

$$\triangle\overline{W}_{\min}^{p,l,r} \leq W_{p,l,r}(T_{j+1}) - W_{p,l,r}(T_j) \leq \triangle\overline{W}_{\max}^{p,l,r}. \tag{2.3}$$

For each power plant $r \in R^{p,l}$, $\triangle\overline{W}_{\max}^{p,l,r}$ and $\triangle\overline{W}_{\min}^{p,l,r}$ represent the maximum rates for ramping up and down. Similarly, $\overline{W}_{\max}^{p,l,r}$ denotes the maximum production, and thus we can write the capacity constraints for each power plant $r \in R^{p,l}$ as

$$0 \leq W_{p,l,r}(T_j) \leq \overline{W}_{\max}^{p,l,r}. \tag{2.4}$$

For each $j \in J$ the electricity sold in the forward market must equal the electricity actually produced,

$$-\sum_{i\in I_j} V_p(t_i,T_j) = \sum_{l\in L}\sum_{r\in R^{p,l}} W_{p,l,r}(T_j). \tag{2.5}$$

In addition sufficient fuel $l \in L$ must have been bought for each period $j \in J$,

$$\sum_{r\in R^{p,l}} W_{p,l,r}(T_j)\, c^{p,l,r} = \sum_{i\in I_j} F_{p,l}(t_i,T_j) \tag{2.6}$$

where $c^{p,l,r} > 0$ is the efficiency of power plant $r$. In reality, power plants incur additional losses when starting production; these are neglected in this chapter. They are investigated in more detail in Chapter 4.

The carbon emission obligation constraint can be written as

$$\sum_{j\in J}\sum_{i\in I_j} O_p(t_i,T_j) = \sum_{j\in J}\sum_{l\in L}\sum_{r\in R^{p,l}} W_{p,l,r}(T_j)\, g^l, \tag{2.7}$$

where $g^l > 0$ denotes the carbon emission intensity factor for fuel $l \in L$. This constraint ensures that enough emission certificates have been bought to cover the electricity production over the entire planning horizon.

Producers seek to maximize their profit subject to a risk budget. Under a mean-variance optimization framework they are interested in the mean-variance utility

$$\begin{aligned}\Psi_p(v_p) &= \mathbb{E}^{\mathbb{P}}\left[P_p(v_p,\pi_p)\right] - \tfrac{\lambda_p}{2}\mathrm{Var}^{\mathbb{P}}\left[P_p(v_p,\pi_p)\right] \\ &= -\mathbb{E}^{\mathbb{P}}\left[\pi_p\right]^\top v_p - \tfrac{\lambda_p}{2} v_p^\top Q_p v_p,\end{aligned}$$

where $\lambda_p > 0$ is their risk preference and $Q_p := \mathbb{E}^{\mathbb{P}}\left[\left(\pi_p - \mathbb{E}^{\mathbb{P}}[\pi_p]\right)\left(\pi_p - \mathbb{E}^{\mathbb{P}}[\pi_p]\right)^\top\right]$ an 'extended' covariance matrix. In effect they seek to solve the following optimization problem:

$$\Phi_p = \max_{v_p} \quad \Psi_p(v_p) \tag{2.8}$$

subject to (2.3), (2.4), (2.5), (2.6), and (2.7).

### 2.1.2 Consumers

We make the assumption that energy demand of end users $D(T_j)$, $j \in J$, is completely inelastic, and that each consumer $c \in C$ is responsible for satisfying a proportion $p_c \in [0,1]$ of it. Since $p_c$ is a proportion clearly $\sum_{c \in C} p_c = 1$.

Let us define electricity trading vectors $V_c(T_j) = ||_{i \in I_j} V_c(t_i, T_j)$ and $V_c = ||_{j \in J} V_c(T_j)$. Then, the profit of consumer $c \in C$ can be calculated as

$$\begin{aligned} P_c(V_c, \Pi) &= \sum_{j \in J} e^{-\hat{r} T_j} \left( \sum_{i \in I_j} -\Pi(t_i, T_j) V_c(t_i, T_j) + p_c s_c(T_j) D(T_j) \right) \\ &= -\Pi^\top V_c + \sum_{j \in J} e^{-\hat{r} T_j} p_c s_c(T_j) D(T_j), \end{aligned} \tag{2.9}$$

where $\hat{r} \in \mathbb{R}$ denotes a constant interest rate and $s_c(T_j) \in \mathbb{R}$ denotes a time-dependent, contractually fixed price that consumer $c \in C$ receives for selling electricity to households or businesses (i.e. end users). The demand is expected to be satisfied exactly for each $T_j$,

$$\sum_{i \in I_j} V_c(t_i, T_j) = p_c D(T_j). \tag{2.10}$$

At the time of calculating the optimal decisions, consumers assume that they know the future realization of the demand $D(T_j)$ precisely. If their forecast of the future realization of the demand changes, then they can take recourse by recalculating their optimal decisions with the updated information.

Note further that the contractually fixed price $s_c(T_j)$ only affects the optimal objective value of consumer $c \in C$, but not her optimal solution. Since we are primarily interested in optimal solutions, we simplify the notation and set $s_c(T_j) = 0$ for all $j \in J$. The correct optimal value can always be calculated via post-processing when an optimal solution is already known. This is sometimes needed for risk management purposes.

Consumers seek to maximize their profit subject to a risk budget. Under a mean-variance optimization framework they are interested in the mean-variance utility

$$\begin{aligned}\Psi_c(V_c) &= \mathbb{E}^{\mathbb{P}}\left[P_c(V_c,\Pi)\right] - \tfrac{\lambda_c}{2}\mathrm{Var}^{\mathbb{P}}\left[P_c(V_c,\Pi)\right] \\ &= -\mathbb{E}^{\mathbb{P}}\left[\Pi\right]^\top V_c - \tfrac{\lambda_c}{2} V_c^\top Q_c V_c,\end{aligned}$$

where $\lambda_c > 0$ is their risk preference and $Q_c := \mathbb{E}^{\mathbb{P}}\left[\left(\Pi - \mathbb{E}^{\mathbb{P}}[\Pi]\right)\left(\Pi - \mathbb{E}^{\mathbb{P}}[\Pi]\right)^\top\right]$ a covariance matrix. Their objective is to solve the following optimization problem:

$$\Phi_c = \max_{V_c} \Psi_c(V_c) \tag{2.11}$$

subject to (2.10).

### 2.1.3 Electricity market

In this section we describe the role of the electricity market. First, we focus on a description of the realistic UK electricity market; later we study how to approximate this market, so that it can be modeled in a game theoretic framework as depicted in Figure 2.1.

#### 2.1.3.1 Organization of the UK electricity market

Electricity supply and demand must match continuously in real-time, which makes electricity grids very difficult to manage. At a high level, the UK electricity market can be divided into two different modes of operation: the first mode involves electricity contracts with more distant delivery periods (ranging from seasons to one hour). The second is performed by the grid operator (in the UK, this entity is the National Grid) responsible for the micromanagement of the grid in the last hour before the delivery of electricity.

In the first mode, electricity is traded through forward or futures contracts. Forward contracts can be traded up to four years before delivery. This period is usually referred to as the liquid period. In the beginning of the liquid period, only seasonal contracts are available. Seasonal forward contracts are agreements between a buyer and a seller that the seller will deliver a certain fixed amount of electricity in every half hour during the season of interest, at a price agreed at the time of signing. As the delivery approaches, electricity contracts with finer granularity appear, including quarterly, monthly, weekly, daily and intra-day. Intra-day contracts can cover blocks of twelve hours, four hours, two hours, one hour and the finest granularity is half an hour (see the APX power exchange[1] for details). Various combinations of the above, such as month-ahead peak contracts, which cover all half hours between 7am and 7pm in the following month, are also traded. One could use such a contract to hedge the production of a solar power plant, for example. As we can see a huge variety of different contracts exists. Some of them are traded through an exchange, while others are traded over-the-counter (OTC). According to the Wholesale Market Report[2] issued by Energy UK, a large majority of contracts are traded OTC.

When choosing the right contract to trade, one has to take into consideration liquidity and transaction costs. The Wholesale Market Reports from Energy UK reveal that contracts with delivery after two or more years tend to be very illiquid. Baseload (i.e. midnight to midnight) contracts are much more liquid than peak (i.e. 7am to 7pm) or off-peak (i.e. 7pm to 7am) contracts. The bid-offer spread for different types of contracts between years 2008 and 2011 is available in the Ofgem report.[3]

Besides trading electricity through future and forward contracts, a significant amount of electricity is also traded through auctions. According to the Wholesale Market Report from Energy UK, roughly 15% to 20% of power is traded through day-ahead auctions offered by APX[4] and N2EX[5]. Auctions are not explicitly included in our model and so we will not study them in further detail.[6]

One hour before the physical delivery, at an event called a Gate Closure, trading of future and forward contracts ceases. This is when the grid operator takes over the management of the power grid. All market participants inform the grid operator about their positions in future and forward contracts. They also submit their bids and offers. A bid is a volume/price pair which tells the grid operator at what price a producer can increase production. Similarly, an offer tells the grid operator what compensation the producer is

[1] http://www.apxgroup.com/trading-clearing/apx-power-uk

[2] Report for July 2014 is available at http://www.energy-uk.org.uk/publication/finish/5-research-and-reports/1152-wholesale-market-report-july-2014.html.

[3] https://www.ofgem.gov.uk/ofgem-publications/39661/summer-2011-assessment.pdf

[4] http://www.apxgroup.com/trading-clearing/auction/

[5] https://www.n2ex.com/

[6] An optimal bidding strategy for an auction market is discussed in Anderson and Xu (2002) and Anderson and Xu (2005).

willing to accept from the operator if asked to decrease production. Consumers with flexible consumption capabilities also submit bids and offers. Equipped with this information, the grid operator first compares the demand forecast with the submitted number of traded contracts and adjusts the difference by accepting some of the bids or offers. While doing so, the grid operator must also take the capacity constraints of the transmission lines in the network into account. After the delivery of electricity, the grid operator compares the actual physical production/consumption of electricity by each market participant, with the contracted volumes adjusted by the accepted bids and offers, and calculates the imbalance volume as the difference between the two. If this imbalance volume helped the grid operator match supply and demand, then a fair market index price is used to calculate the imbalance cash flow to/from this market participant. On the other hand, if the imbalance hampered the grid operator in matching supply and demand, then a worse price[7] is applied to calculate the cash flow. Roughly speaking, this price is calculated as the average of the most expensive 500 MWh of accepted bids or offers, adjusted for transmission losses.

The two modes of operation described above are quite different. The first mode of operation can be regarded as a competitive market without a central agent, while the second is controlled by the grid operator who is responsible for matching supply and demand by choosing the cheapest actions. This chapter focuses on the first mode of operation only. A coupling of both modes is investigated in Chapter 4.

#### 2.1.3.2 Modeling of the electricity market

In this section we describe how to model a simplified version of the electricity market in which only forward contract are traded. A trivial extension to futures contracts is discussed in Section 3.2.1.

By definition, each forward contract consists of a buyer and a seller, which imposes a constraint (also called the 'market clearing constraint') that matches the number of short

[7] For a detailed calculation of this price see `http://www.elexon.co.uk/wp-content/uploads/2014/06/imbalance_pricing_guidance_v7.0.pdf`.

and long forward electricity contracts for each $i \in I_j$ and $j \in J$ as

$$\sum_{c \in C} V_c(t_i, T_j) + \sum_{p \in P} V_p(t_i, T_j) = 0. \tag{2.12}$$

In reality, this matching is done through sharing the price and order book information among all market participants. If, at a given price, there is greater demand for long contracts than short contracts, then the price is too low and offers will be submitted at higher prices. The converse occurs if there is more demand for short contracts than long ones. By dynamically adjusting, the price for this forward contract at which the number of long and short contracts matches, is thus found. At such a price, (2.12) is satisfied 'naturally' without explicitly forcing the players to satisfy it. They do so because it is in their best interest, i.e. because their objective functions are jointly maximized.

To include the matching process in our model, we need a mechanism that allows us to adjust forward electricity prices in order to satisfy the market clearing constraints. More precisely, since our initial optimization problem is static, we need a mechanism to adjust the expectation of forward electricity prices. As described in the beginning of Section 2.1, the expectation of $\Pi(t_i, T_j)$ is considered an endogenous variable while higher moments are all given exogenously. In the rest of this section, we take a closer look at the electricity price process $\Pi(t_i, T_j)_{i \in I_j}$, where the expectation of the process is defined by the electricity market in order to match supply and demand, and thus satisfy (2.12).

Since electricity prices $\Pi(t_i, T_j)_{i \in I_j}$ are adapted for any fixed $j \in J$ and $\left\|\mathbb{E}^{\mathbb{P}}[\Pi]\right\| < \infty$, they can be uniquely decomposed, using the Doob Decomposition Theorem, into the sum of a martingale process $M(t_i, T_j)_{i \in I_j}$ and an integrable, predictable process $A(t_i, T_j)_{i \in I_j}$, $A(t_0, T_j) = 0$, such that

$$\Pi(t_i, T_j) = A(t_i, T_j) + M(t_i, T_j)$$

for every $i \in I_j$. Define

$$M(t_i, T_j) := \Pi(t_0, T_j) + \sum_{k=1}^{i} \Big(\Pi(t_k, T_j) - \mathbb{E}^{\mathbb{P}}[\Pi(t_k, T_j) | \mathcal{F}_{k-1}]\Big)$$

and

$$A(t_i,T_j) := \sum_{k=1}^{i} \left( \mathbb{E}^{\mathbb{P}}\left[\Pi(t_k,T_j)\,|\mathcal{F}_{k-1}\right] - \Pi(t_{k-1},T_j) \right),$$

where $\mathcal{F}_k := \mathcal{F}_{t_k}$. It is easy to see that $M(t_i,T_j)$ is a martingale since

$$\mathbb{E}^{\mathbb{P}}\left[M(t_i,T_j) - M(t_{i-1},T_j)\,|\mathcal{F}_{i-1}\right] = 0 \quad \text{a.s.}$$

Moreover, $A(t_i,T_j)$ is predictable, that is, $\mathcal{F}_{i-1}$-measurable.

Note that

$$\mathbb{E}^{\mathbb{P}}\left[\Pi(t_i,T_j)\,|\mathcal{F}_{i-1}\right] = A(t_i,T_j) + M(t_{i-1},T_j),$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\Pi(t_i,T_j)\,|\mathcal{F}_0\right] = \mathbb{E}^{\mathbb{P}}\left[A(t_i,T_j)\,|\mathcal{F}_0\right] + M(t_0,T_j). \tag{2.13}$$

Let us now allow the electricity market to choose

$$\tilde{A}_i(T_j) := \left[\tilde{A}(t_{i+1},T_j), \mathbb{E}^{\mathbb{P}}\left[\tilde{A}(t_{i+2},T_j)\,|\mathcal{F}_i\right], ..., \mathbb{E}^{\mathbb{P}}\left[\tilde{A}\left(t_{\max\{I_j\}},T_j\right)|\mathcal{F}_i\right]\right]^{\top}$$

at time $t_i$, $i \in I_j$ for all $j \in J$. We then model $\Pi(t_i,T_j)$ with a new probability measure $\tilde{\mathbb{P}}$ where $A(t_i,T_j)$ is defined internally, and not by Doob decomposition of $\Pi(t_i,T_j)$, as before. More formally, we define a new probability measure $\tilde{\mathbb{P}} : \mathcal{F} \times \mathbb{R}^N \to [0,1]$ such that for any fixed $j \in J$, $i \in I_j$, and for all $D \in \mathcal{F}_{i-1}$

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\Pi(t_i,T_j) \in D; \tilde{A}_{i-1}(T_j)\right) &= \mathbb{P}\left(M(t_i,T_j) + \tilde{A}(t_i,T_j) \in D\right) \\
&= \mathbb{P}\left(M(t_i,T_j) + A(t_i,T_j) + \tilde{A}(t_i,T_j) - A(t_i,T_j) \in D\right) \\
&= \mathbb{P}\left(\Pi(t_i,T_j) + \tilde{A}(t_i,T_j) - A(t_i,T_j) \in D\right) \\
&= \mathbb{P}\left(\Pi(t_i,T_j) \in \varphi\left(D, \tilde{A}(t_i,T_j) - A(t_i,T_j)\right)\right)
\end{aligned}$$

where $\varphi : \mathcal{F} \times \mathbb{R}^N \to \mathcal{F}$ denotes a translation of a set, i.e. $\varphi(D,\Delta d) := \{d : d + \Delta d \in D\}$. Since

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi(t_k,T_j)\,|\mathcal{F}_i\,; \tilde{A}_i(T_j)\right] = \mathbb{E}^{\mathbb{P}}\left[\tilde{A}(t_k,T_j)\,|\mathcal{F}_i\right] + M(t_i,T_j), \tag{2.14}$$

where $k \in \{i+1, ..., \max\{I_j\}\}$, this selection can be interpreted as determining the term structure of the expected power price relative to the current value of $M(t_i, T_j)$.

At time $t_0$, when players calculate their optimal decisions for the first time, they assume that they will execute their strategies without any future alterations. We allow recourse at a later step, but this is not taken into account at time $t_0$. Therefore,

$$\tilde{A}(t_i, T_j) = \mathbb{E}^{\mathbb{P}}\left[\tilde{A}(t_i, T_j) | \mathcal{F}_k\right] \tag{2.15}$$

for all $k \in \{0, ..., i-1\}$, $j \in J$, and $i \in I_j$. In such a setting, it suffices to determine $\tilde{A}_0(T_j)$, since all other expected prices can be derived from these using (2.14) and (2.15).

The variance under the new probability measure $\tilde{\mathbb{P}}$ can be calculated as

$$\begin{aligned} \mathrm{Var}^{\tilde{\mathbb{P}}}\left(\Pi(t_i, T_j) | \mathcal{F}_0 ; \tilde{A}_i(T_j)\right) &= \mathrm{Var}^{\mathbb{P}}\left(\tilde{A}(t_i, T_j) + M(t_i, T_j) | \mathcal{F}_0\right) \\ &= \mathrm{Var}^{\mathbb{P}}\left(M(t_i, T_j) | \mathcal{F}_0\right). \end{aligned} \tag{2.16}$$

We see that the variance depends only on the process $M(t_i, T_j)$ and cannot be influenced by the electricity market. Using similar reasoning as in (2.16), we also conclude that

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\pi_p - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_p; \tilde{A}_0\right]\right)\left(\pi_p - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_p; \tilde{A}_0\right]\right)^\top ; \tilde{A}_0\right] = Q_p \tag{2.17}$$

and

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\Pi - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi; \tilde{A}_0\right]\right)\left(\Pi - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi; \tilde{A}_0\right]\right)^\top ; \tilde{A}_0\right] = Q_c, \tag{2.18}$$

where $\tilde{A}_0 = ||_{j \in J} \tilde{A}_0(T_j)$.

Without loss of generality, we may set $M(t_0, T_j) = 0$. Then from (2.14) and (2.15),

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi(t_i, T_j) | \mathcal{F}_0 ; \tilde{A}_i(T_j)\right] &= \mathbb{E}^{\mathbb{P}}\left[\tilde{A}(t_i, T_j) | \mathcal{F}_0\right] \\ &= \tilde{A}(t_i, T_j) \end{aligned} \tag{2.19}$$

for all $i \in I_j$ and $j \in J$. Allowing the electricity market to choose $\tilde{A}(t_i, T_j)$ is thus the same as allowing it to choose $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i, T_j\right)|\mathcal{F}_0\,; \tilde{A}\left(t_i, T_j\right)\right]$. In the rest of the thesis, we simplify our notation by writing $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i, T_j\right)\right]$, when we actually mean $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i, T_j\right)|\mathcal{F}_0\,; \tilde{A}\left(t_i, T_j\right)\right]$.

The measure $\tilde{\mathbb{P}}$ corresponds to a physical measure, and $\mathbb{E}^{\tilde{\mathbb{P}}}[\cdot]$ corresponds to a real world expectation. The electricity market is allowed to choose $\tilde{A}\left(t_i, T_j\right)$, and consequently the physical measure $\tilde{\mathbb{P}}$. The aim of the electricity market could be interpreted as finding the physical measure $\tilde{\mathbb{P}}$ that is consistent with the facts (e.g. forward fuel prices, emission prices, demand forecasts) that have been observed in the real world.

### 2.1.4 Recourse

The literature distinguishes between dynamic (i.e. stochastic) and static models. In dynamic models, players adapt to a changing environment (fuel prices, demand etc.) by adapting their decisions. At each stage they determine their current optimal decisions as well as those they will make in the future, under all possible changes in the environment. Note that these changes include forecasts as well. Since future decisions affect present decisions, this is computationally very demanding. In static models on the other hand, players assume that they know the future state of the environment exactly, and can thus stick to an initial plan about future decisions, regardless of changes in the environment. Such approaches are computationally much more tractable, but they do not reflect the reality very well. The model presented in this work is a hybrid of these approaches. The initial optimization problem is static, because players determine all their optimal decisions and assume they will not alter them in the future. However, as the environment changes, players may take recourse by calculating new optimal decisions that take into account both the new state of the environment, as well as decisions they made previously.

It might not be immediately clear from our model why market participants do not execute all of their trades at the beginning of the planning horizon.

1. One reason for spreading out the trading activity is the availability of contracts and their liquidity: contracts with a delivery date far into the future are much less liquid than, for example, day-ahead contracts. Liquidity is included in our model through increased costs of trading (see Section 3.2.3).

2. Another reason is delayed cash flows: when trading is done through futures contracts, some players might wish to enter a position later and thus delay the associated cash flows. In this chapter, we use only forward contracts. A simple extension to futures contracts (assuming a constant interest rate) is presented in Section 3.2.1.

3. The next reason for delayed trading is the exploitation of a trend-following effect in the term structure: due to the risk preference of most players, the term structure of forward or futures contracts with fixed delivery date is usually slightly upward sloping.

4. Another reason is is the lack of information. Since the initial optimization problem is static and assumes that players will not alter their decisions in the future, the trading activity might be delayed until the future state of the environment can be predicted with more certainty.

5. The last reason is transaction costs: when large trades, such as hedging a whole power plant, are executed, they are spread over a longer period of time to decrease transaction costs and market impact.

### 2.1.5 Matrix notation

The analysis of the problem is greatly simplified if a more compact notation is introduced. In this subsection, we will also rewrite our equations using the new probability measure $\tilde{\mathbb{P}}$ instead of $\mathbb{P}$ where appropriate.

The profit of producer $p \in P$ can be written as

$$P_p\left(v_p, \pi_p\right) = -\pi_p^\top v_p.$$

The equality constraints can be expressed as

$$A_p v_p = 0$$

and the inequality constraints,

$$B_p v_p \le b_p$$

for some $A_p \in \mathbb{R}^{|J|(|L|+1)+1\times \dim v_p}$, $B_p \in \mathbb{R}^{n_p\times \dim v_p}$ and $b_p \in \mathbb{R}^{n_p}$, where $n_p$ denotes the number of all inequality constraints of producer $p \in P$. Define a feasible set

$$S_p := \{v_p : A_p v_p = a_p \text{ and } B_p v_p \leq b_p\}.$$

It is useful to investigate the inner structure of the matrices $A_p$. By considering the equality constraints (2.5), (2.6), and (2.7) we see that

$$A_p = \begin{bmatrix} \hat{A}_1 & 0 & \hat{A}_{3,p} \\ 0 & \hat{A}_2 & \hat{A}_{4,p} \end{bmatrix} \tag{2.20}$$

where $\hat{A}_1 \in \mathbb{R}^{|J|\times N}$, $\hat{A}_2 \in \mathbb{R}^{(|J||L|+1)\times N(|L|+1)}$, $\hat{A}_{3,p} \in \mathbb{R}^{|J|\times \dim W_p}$, $\hat{A}_{4,p} \in \mathbb{R}^{(|J||L|+1)\times \dim W_p}$. One can see that matrices $\hat{A}_1$ and $\hat{A}_2$ are independent of producer $p \in P$, whereas the matrices $\hat{A}_{3,p}$ and $\hat{A}_{4,p}$ depend on producer $p \in P$. One can further investigate the structure of $\hat{A}_1$ and see that

$$\hat{A}_1 = \text{Diag}\left(1_{|I_1|}, ..., 1_{|I_{T'}|}\right), \tag{2.21}$$

where $1_n$ is a row vector of ones of length $n$. Similarly,

$$\hat{A}_2 = \text{Diag}\left(\hat{A}_1, ..., \hat{A}_1, 1_N\right), \tag{2.22}$$

where the number of block rows is $|L| + 1$. The first $|L|$ block rows correspond to (2.6), and the last block row corresponds to (2.7).

In a compact notation, the mean-variance utility of producer $p \in P$ can then be written as

$$\begin{aligned} \Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[-\pi_p^\top v_p - \tfrac{1}{2}\lambda_p v_p^\top \left(\pi_p - \mathbb{E}^{\tilde{\mathbb{P}}}[\pi_p]\right)\left(\pi_p - \mathbb{E}^{\tilde{\mathbb{P}}}[\pi_p]\right)^\top v_p\right] \\ &= -\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_p]^\top v_p - \tfrac{1}{2}\lambda_p v_p^\top Q_p v_p, \end{aligned}$$

where (2.17) was used. The structure of matrix $Q_p$ is

$$Q_p = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 & 0 \\ \hat{Q}_2^\top & \hat{Q}_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.23}$$

where $\hat{Q}_1 \in \mathbb{R}^{N\times N}, \hat{Q}_2 \in \mathbb{R}^{N\times(\dim F_p + \dim O_p)} = \mathbb{R}^{N\times N(|L|+1)}, \hat{Q}_3 \in \mathbb{R}^{N(|L|+1)\times N(|L|+1)}$. Clearly, $\hat{Q}_1$, $\hat{Q}_2$, and $\hat{Q}_3$ do not depend on the producer $p \in P$. The size of the larger matrix $Q_p$ depends on the producer $p \in P$, because different producers have different numbers of power plants.

Producer $p \in P$ attempts to solve the optimization problem

$$\Phi_p\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) = \max_{v_p \in S_p} \; -\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_p]^\top v_p - \frac{1}{2}\lambda_p v_p^\top Q_p v_p.$$

The profit of consumer $c \in C$ can be written as

$$P_c(V_c, \Pi) = -\Pi^\top V_c.$$

Note here that we set $s_c(T_j) = 0$ for all $j \in J$, without loss of generality. The equality constraints can be expressed as

$$A_c V_c = a_c$$

where $A_c = \hat{A}_1$ and $a_c \in \mathbb{R}^{|J|}$. Define a feasible set

$$S_c := \left\{V_c \in \mathbb{R}^N : A_c V_c = a_c\right\}.$$

To keep the notation we apply to producers and consumers consistent, we also define an empty matrix of inequality constants for consumers $B_c \in \mathbb{R}^{0\times N}$ and $b_c \in \mathbb{R}^0$.

In a compact notation, the mean-variance utility of a consumer $c \in C$ can be written as

$$\begin{aligned} \Psi_c\left(V_c, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[-\Pi^\top V_c - \tfrac{1}{2}\lambda_c V_c^\top \left(\Pi - \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)\left(\Pi - \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)^\top V_c\right] \\ &= -\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^\top V_c - \tfrac{\lambda_c}{2} V_c^\top Q_c V_c, \end{aligned}$$

where (2.18) was used. Moreover, note that $Q_c = \hat{Q}_1$ for all $c \in C$.

Consumer $c \in C$ attempts to solve the following optimization problem

$$\Phi_c \left( \mathbb{E}^{\tilde{\mathbb{P}}} [\Pi] \right) = \max_{V_c \in S_c} \; - \mathbb{E}^{\tilde{\mathbb{P}}} [\Pi]^\top V_c - \frac{\lambda_c}{2} V_c^\top Q_c V_c.$$

### 2.1.6 Assumptions

In this section we investigate the assumptions that we will use.

In the description of the optimization problems of producers $p \in P$ and consumers $c \in C$, we implicitly assumed that their utility functions can be written in a mean-variance form. We also assumed that all players share the same knowledge of the physical characteristics of power plants (i.e. ramping constraints, efficiency, carbon emission intensity factor), have the same estimates of the covariance matrices, and that the risk preference of each player is known to all market participants. These are all standard assumptions in the economic literature even though they may not be entirely consistent with reality.

We continue with more technical assumptions.

**Assumption 2.1.1.** *For all $p \in P$, there exists a vector $v_p$ such that $A_p v_p = a_p$ a.s. and $B_p v_p < b_p$ a.s., and for all $c \in C$, there exists a vector $V_c$ such that $A_c V_c = a_c$ a.s. and $B_c V_c < b_c$ a.s. Additionally, the vectors $V_p$ and $V_c$ are such that (2.12) is satisfied.*

The next assumption is completely nonrestrictive (see Remark 2.1.3), but it can significantly simplify the analysis.

**Assumption 2.1.2.** *None of the random variables $\Pi$, $G$, and $G_{em}$ can be written as a linear combination of the others a.s.*

This assumption is clearly satisfied in real life applications, where none of the electricity, fuel and emission prices can be precisely determined by knowing the prices of the others. In the rest of this chapter we assume that Assumption 2.1.1 and Assumption 2.1.2 always hold.

**Remark 2.1.3.** *We show that Assumption 2.1.2 is truly completely nonrestrictive. Let $c \in C$. Since $Q_c$ is a covariance matrix, we know that $Q_c \succeq 0$. Moreover, if none of the random variables can be written as a linear combination of the others a.s., then $Q_c \succ 0$.*

*The quadratic term of the objective function of consumer $c \in C$ can be written as*

$$V_c^\top Q_c V_c = \mathbb{E}^{\mathbb{P}}\left[\left(V_c^\top\left(\Pi - \mathbb{E}^{\mathbb{P}}[\Pi]\right)\right)^2\right] \tag{2.24}$$

*where we used the fact that the covariance matrix does not depend on the decisions of the electricity market. Let $\widetilde{\Pi} := \Pi - \mathbb{E}^{\mathbb{P}}[\Pi]$. Then,*

$$V_c^\top Q_c V_c = \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{i=1}^{N} V_c^i \widetilde{\Pi}^i\right)^2\right] \tag{2.25}$$

*where the superscript $i$ denotes an index of a variable in a vector.*

*Assume now that some random variable $\widetilde{\Pi}^k$ can be written as a linear combination of the others a.s. as*

$$\widetilde{\Pi}^k = \sum_{i \neq k} \widetilde{\Pi}^i a^i, \tag{2.26}$$

*for some $a^i \in \mathbb{R}$. Then,*

$$\begin{aligned} V_c^\top Q_c V_c &= \mathbb{E}^{\mathbb{P}}\left[\left(\textstyle\sum_{i\neq k} V_c^i \widetilde{\Pi}^i + \sum_{i\neq k} V_c^k \widetilde{\Pi}^i a^i\right)^2\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\textstyle\sum_{i\neq k} \widetilde{\Pi}^i \left(V_c^i + V_c^k a^i\right)\right)^2\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\textstyle\sum_{i\neq k} \widetilde{\Pi}^i \widetilde{V}_c^i\right)^2\right] \\ &= \widetilde{V}_c^\top \widetilde{Q}_c \widetilde{V}_c \end{aligned} \tag{2.27}$$

*where*

$$\widetilde{V}_c^i := V_c^i + V_c^k a^i, \tag{2.28}$$

*and $\widetilde{Q}_c$ is the matrix $Q_c$ with the $k$-th row and $k$-th column removed. Constraints (2.28) for all $a^i \neq 0$ must be added to the matrix of constraints, $A_c$, by the following rule:*

- *if the new constraint is in the null space of $A_c$, then the entire problem of consumer $c \in C$ can be written in terms of $\widetilde{V}_c$. Thus, effectively the number of constraints in $A_c$ does not change. Remember from (2.20) that consumers $c \in C$ share the same price vector and constraint matrix and thus this re-expression can be done for all consumers* *simultaneously.*

- *if the new constraint is not in the null space of $A_c$, then it is added to matrix $A_c$.*

*One could repeat this procedure until all linear combinations of the random variables are removed. We denote the final matrix of constraints by $\widetilde{A}_c$. Note that by construction, $x^\top \widetilde{Q}_c x > 0$ for all $x \in N\left(\widetilde{A}_c\right)$. We recall from the theory of convex quadratic optimization, that this, together with the Slater condition, is sufficient for showing the uniqueness of a global minimizer.*

*A similar reasoning could also be applied to producers for $v_p := \left[V_p^\top, F_p^\top, O_p^\top\right]^\top$, $\pi := \left[\Pi^\top, G^\top, G_{em}^\top\right]^\top$ and $Q := \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2^\top & \hat{Q}_3 \end{bmatrix}$.*

## 2.2 Competitive equilibrium

We start a discussion of competitive equilibrium (also called Walras equilibrium) by giving its formal definition. Define $V = \left[V_P^\top, V_C^\top\right]^\top$, $V_P = ||_{p \in P} V_p$, $V_C = ||_{c \in C} V_c$, $v_P = ||_{p \in P} v_p$ and $v = \left[v_P^\top, V_C^\top\right]^\top$.

**Definition 2.2.1.** *Competitive Equilibrium (CE)*

*Decisions $v^*$ and $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ constitute a competitive equilibrium if*

1. *For every producer $p \in P$, $v_p^* \in S_p$ is a strategy such that*

$$\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \leq \Psi_p\left(v_p^*, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \tag{2.29}$$

   *for all $v_p \in S_p$;*

2. *For every consumer $c \in C$, $V_c^* \in S_c$ is a strategy such that*

$$\Psi_c\left(V_c, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \leq \Psi_c\left(V_c^*, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \tag{2.30}$$

   *for all $V_c \in S_c$;*

*3. For each $i \in I_j$ and $j \in J$*

$$0 = \sum_{c \in C} V_c\left(t_i, T_j\right) + \sum_{p \in P} V_p\left(t_i, T_j\right) \tag{2.31}$$

*holds.*

Definition 2.2.1 ensures that:

1. Given expected electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, none of the producers $p \in P$ has an incentive to deviate from decision $v_p^* \in S_p$, because none of the remaining feasible solutions $v_p \in S_p$ would improve their utility function.

2. Given expected electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, none of the consumers $c \in C$ has an incentive to deviate from decision $V_c^* \in S_c$, because none of the remaining feasible solutions $V_c \in S_c$ would improve their utility function.

3. Expected electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ is such that decisions $v_p^* \in S_p$, $p \in P$ and $V_c^* \in S_c$, $c \in C$ satisfy the market clearing constraint simultaneously for all forward electricity contracts..

### 2.2.1 Existence of a CE

We first show that optimal solutions $V_p^*$, $F_p^*$, $O_p^*$ for all $p \in P$ and $V_c^*$ for all $c \in C$ are all finite.

**Lemma 2.2.2.** *Let $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$ such that $\left\|\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right\| < \infty$ be given. Then the optimal solutions satisfy $\left\|\left[V_p^{*\top}, F_p^{*\top}, O_p^{*\top}, W_p^{*\top}\right]^\top\right\| < \infty$ for all $p \in P$ and $\|V_c^*\| < \infty$ for all $c \in C$.*

*Proof.* Let $\hat{v}_p = \left[V_p^\top, F_p^\top, O_p^\top\right]^\top$, $\hat{\pi} := \left[\Pi^\top, G^\top, G_{\text{em}}^\top\right]^\top$ and $\hat{Q} := \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2^\top & \hat{Q}_3 \end{bmatrix}$. Then we can write

$$\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) = -\mathbb{E}^{\tilde{\mathbb{P}}}[\hat{\pi}_p]^\top \hat{v}_p - \frac{1}{2}\lambda_p \hat{v}_p^\top \hat{Q}\hat{v}_p, \tag{2.32}$$

where $\hat{Q} \succ 0$ by Assumption 2.1.2 and $0 < \lambda_p < \infty$. Thus, if $\left\|\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right\| < \infty$, then $\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) \to -\infty$ as $\|\hat{v}_p\| \to \infty$ and, using the coercivity argument, we conclude that $\|\hat{v}_p^*\| < \infty$. Clearly, $\|W_p^*\| < \infty$ due to (2.4). A proof for consumers is similar. $\square$

We continue by stating a theorem that shows the existence of a CE. We rely on the shadow price approach that is usually used to calculate equilibrium electricity prices in the linear programming setting (see Liu and Wu (2006) and Milano et al. (2006) for example). A similar approach was also used to show existence of solutions under assumption of positive prices and general utility functions in Jofré et al. (2014).

**Theorem 2.2.3.** *Let Assumption 2.1.1 hold. Then a CE exists.*

*Proof.* Note that Assumption 2.1.1 implies the Slater condition. Thus, necessary and sufficient conditions for all $v_k$, $k \in P \cup C$, and given $\mathbb{E}^{\mathbb{P}}[\Pi] \in \mathbb{R}^N$, to satisfy (2.29) and (2.30) are the following

$$\begin{aligned} -\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_k]^\top - \lambda_k Q_k v_k - B_k^\top \eta_k - A_k^\top \mu_k &= 0, \\ \eta_k^\top (B_k v_k - b_k) &= 0, \\ B_k v_k - b_k &\leq 0, \\ A_k v_k - a_k &= 0, \\ \eta_k &\geq 0. \end{aligned} \tag{2.33}$$

We can combine optimization problems for all producers $p \in P$, and consumers $c \in C$ into one large optimization problem as

$$\begin{aligned} \max_x \quad & -\pi \left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)^\top x - \tfrac{1}{2} x^\top Q x \\ \text{s.t.} \quad & Ax = a \\ & Bx \leq b \end{aligned} \tag{2.34}$$

where

$$A := \text{Diag}\left(A_{p_1}, ..., A_{p_{|P|}}, A_{c_1}, ..., A_{c_{|C|}}\right),\ a := \left[a_{p_1}^\top, ..., a_{p_{|P|}}^\top, a_{c_1}^\top, ..., a_{c_{|C|}}^\top\right]^\top,$$

$$B := \text{Diag}\left(B_{p_1}, ..., B_{p_{|P|}}, B_{c_1}, ..., B_{c_{|C|}}\right),\ b := \left[b_{p_1}^\top, ..., b_{p_{|P|}}^\top, b_{c_1}^\top, ..., b_{c_{|C|}}^\top\right]^\top,$$

$$Q := \text{Diag}\left(\lambda_{p_1} Q_{p_1}, ..., \lambda_{p_{|P|}} Q_{p_{|P|}}, \lambda_{c_1} Q_{c_1}, ..., \lambda_{c_{|C|}} Q_{c_{|C|}}\right),\ x := \left[v_{p_1}^\top, ..., v_{p_{|P|}}^\top, v_{c_1}^\top, ..., v_{c_{|C|}}^\top\right]^\top,$$

and finally

$$\pi\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) := \left[\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_{p_1}]^\top, ..., \mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_{p_{|P|}}\right]^\top, \underbrace{\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^\top, ..., \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^\top}_{|C|}\right]^\top .$$

Let $x^*$ denote an optimal solution of the following problem

$$\begin{aligned}
\max_x \quad & -\pi(0)^\top x - \tfrac{1}{2}x^\top Q x \\
\text{s.t.} \quad & Ax = a \\
& Bx \leq b \\
& \textstyle\sum_{c\in C} V_c + \sum_{p\in P} V_p = 0 \qquad (\text{dual variables } \mu_M),
\end{aligned} \tag{2.35}$$

and let $\mu_M^* \in \mathbb{R}^N$ denote the corresponding dual variables of the last vector constraint. By Assumption 2.1.1, we know that the feasible set of Problem (2.35) is non-empty and since $\left\|\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right\| = 0 < \infty$, we can use Lemma 2.2.2 to conclude that the optimal solution $\|x^*\| < \infty$ exists. Due to the Slater condition, concavity of the objective function of Problem (2.35) and the convexity of the feasible set, we conclude that $\|\mu_M^*\| < \infty$.

The theory of Lagrange multipliers can be utilized to show that the optimal solution of the following problem

$$\begin{aligned}
\max_x \quad & -\pi(0)^\top x - \tfrac{1}{2}x^\top Q x + \left(\textstyle\sum_{c\in C} V_c + \sum_{p\in P} V_p\right)^\top \mu_M^* \\
\text{s.t.} \quad & Ax = a \\
& Bx \leq b
\end{aligned} \tag{2.36}$$

is also $x^*$. One can prove this by showing that every $x$ that satisfies the KKT conditions of (2.35) satisfies also the KKT conditions of (2.36) and vice versa. Note that the objective function of (2.36) can be written as

$$
\begin{aligned}
&-\pi(0)^\top x - \tfrac{1}{2}x^\top Q x + \left(\textstyle\sum_{c\in C} V_c + \sum_{p\in P} V_p\right)^\top \mu_M^* \\
&= -\pi(-\mu_M^*)^\top x - \tfrac{1}{2}x^\top Q x.
\end{aligned}
\tag{2.37}
$$

From (2.35), we can see that $x^*$ satisfies the market clearing constraint (2.31). Moreover, from the KKT conditions of (2.36), we see that $x^*$ satisfies (2.29) and (2.30) as well. Thus, $x^*$ satisfies the conditions of a CE if the expected electricity price is set to $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* = -\mu_M^*$. □

From the proof of Theorem 2.2.3, we see that individual utility maximization problem of multiple players can be expressed as one large maximization problem of a hypothetical central planner with a utility function defined as a sum of the utility functions of individual players. This is a typical result of perfect competition models, which was first observed in Samuelson (1948).

### 2.2.2 Uniqueness of a CE

We start the analysis of the uniqueness of a CE by establishing the uniqueness of the trading decisions of all producers and consumers.

**Lemma 2.2.4.** *Let $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$ be a given price vector. Then the optimal decision vectors $\left[V_p^{*\top}, F_p^{*\top}, O_p^{*\top}\right]^\top$ for all $p \in P$ and the optimal decision vectors $V_c^*$ for all $c \in C$, are unique.*

*Proof.* The objective function of each producer $p \in P$ can be written as

$$
\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) = -\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_p]^\top v_p - \frac{1}{2}\lambda_p v_p^\top Q_p v_p.
$$

Define $v_p' := \left[V_p^\top, F_p^\top, O_p^\top\right]^\top$ and $v_p'' := W_p$. Then

$$
\mathcal{D}_{v_p}\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) = \left[\mathcal{D}_{v_p'}\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)^\top, \mathcal{D}_{v_p''}\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)^\top\right]^\top.
$$

By Assumption 2.1.2 and, consequently, the strict concavity of the expected utility functions in $v_p'$, for any $\hat{v}_p := \left[\hat{v}_p'^\top, \hat{v}_p''^\top\right]^\top$, $\hat{v}_p \in S_p$, and $\tilde{v}_p := \left[\tilde{v}_p'^\top, \tilde{v}_p''^\top\right]^\top$, $\tilde{v}_p \in S_p$, such that $\hat{v}_p' \neq \tilde{v}_p'$, the following strict inequality holds:

$$(\hat{v}_p - \tilde{v}_p)^\top \mathcal{D}_{\tilde{v}_p} \Psi_p \left(\tilde{v}_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) + (\tilde{v}_p - \hat{v}_p)^\top \mathcal{D}_{\hat{v}_p} \Psi_p \left(\hat{v}_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) > 0. \tag{2.38}$$

We continue with a proof by contradiction. Assume that there exist two optimal solutions $\hat{v}'_p \neq \tilde{v}'_p$ for player $p \in P$, given the electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$. Then both must satisfy the KKT conditions, i.e.

$$\mathcal{D}_{\tilde{v}_p} \Psi_p \left(\tilde{v}_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) - B_p^\top \tilde{\eta}_p - A_p^\top \tilde{\mu}_p = 0, \tag{2.39}$$

$$\tilde{\eta}_p^\top (B_p \tilde{v}_p - b_p) = 0, \tag{2.40}$$

$$B_p \tilde{v}_p - b_p \leq 0, \tag{2.41}$$

$$A_p \tilde{v}_p - a_p = 0, \tag{2.42}$$

$$\tilde{\eta}_p \geq 0, \tag{2.43}$$

and

$$\mathcal{D}_{\hat{v}_p} \Psi_p \left(\hat{v}_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) - B_p^\top \hat{\eta}_p - A_p^\top \hat{\mu}_p = 0, \tag{2.44}$$

$$\hat{\eta}_p^\top (B_p \hat{v}_p - b_p) = 0, \tag{2.45}$$

$$B_p \hat{v}_p - b_p \leq 0, \tag{2.46}$$

$$A_p \hat{v}_p - a_p = 0, \tag{2.47}$$

$$\hat{\eta}_p \geq 0. \tag{2.48}$$

Multiplying (2.39) by $(\hat{v}_p - \tilde{v}_p)^\top$ and (2.44) by $(\tilde{v}_p - \hat{v}_p)^\top$, and summing them up, gives the following strict inequality

$$
\begin{aligned}
0 &= (\hat{v}_p - \tilde{v}_p)^\top \mathcal{D}_{\tilde{v}_p}\Psi_p\left(\tilde{v}_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) + (\tilde{v}_p - \hat{v}_p)^\top \mathcal{D}_{\hat{v}_p}\Psi_p\left(\hat{v}_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) \\
&\quad - (\hat{v}_p - \tilde{v}_p)^\top B_p^\top \tilde{\eta}_p - (\tilde{v}_p - \hat{v}_p)^\top B_p^\top \hat{\eta}_p \\
&\quad - (\hat{v}_p - \tilde{v}_p)^\top A_p^\top \tilde{\mu}_p^\top - (\tilde{v}_p - \hat{v}_p)^\top A_p^\top \hat{\mu}_p^\top \\
&> - (\hat{v}_p - \tilde{v}_p)^\top B_p^\top \tilde{\eta}_p - (\tilde{v}_p - \hat{v}_p)^\top B_p^\top \hat{\eta}_p \\
&\quad - (\hat{v}_p - \tilde{v}_p)^\top A_p^\top \tilde{\mu}_p - (\tilde{v}_p - \hat{v}_p)^\top A_p^\top \hat{\mu}_p
\end{aligned} \tag{2.49}
$$

after use of (2.38).

Rewriting

$$
\begin{aligned}
(\hat{v}_p - \tilde{v}_p)^\top B_p^\top \tilde{\eta}_p &= (B_p\hat{v}_p - B_p\tilde{v}_p + b_p - b_p)^\top \tilde{\eta}_p \\
&= (B_p\hat{v}_p - b_p)^\top \tilde{\eta}_p - (B_p\tilde{v}_p - b_p)^\top \tilde{\eta}_p \\
&= (B_p\hat{v}_p - b_p)^\top \tilde{\eta}_p
\end{aligned}
$$

and noting that $\tilde{\eta}_p \geq 0$ and $B_p\hat{v}_p - b_p \leq 0$, we conclude that

$$
(\hat{v}_p - \tilde{v}_p)^\top B_p^\top \tilde{\eta}_p \leq 0. \tag{2.50}
$$

Due to symmetry we also have

$$
(\tilde{v}_p - \hat{v}_p)^\top B_p^\top \hat{\eta}_p \leq 0. \tag{2.51}
$$

Rewriting

$$
\begin{aligned}
(\hat{v}_p - \tilde{v}_p)^\top A_p^\top \tilde{\mu}_p &= (A_p\hat{v}_p - A_p\tilde{v}_p + a_p - a_p)^\top \tilde{\mu}_p \\
&= (A_p\hat{v}_p - a_p)^\top \tilde{\mu}_p - (A_p\tilde{v}_p - a_p)^\top \tilde{\mu}_p \\
&= 0
\end{aligned} \tag{2.52}
$$

and by symmetry, also

$$
(\tilde{v}_p - \hat{v}_p)^\top A_p^\top \hat{\mu}_p = 0. \tag{2.53}
$$

Inserting (2.50), (2.51), (2.52), and (2.53) back into (2.49) then gives a contradiction. Hence $\hat{v}'_p = \tilde{v}'_p$. The proof for consumers $c \in C$ is similar. □

We can now state conditions that characterize the uniqueness of a CE.

**Theorem 2.2.5.** *Assume that the strict complementarity slackness holds for Problems (2.8) for all $p \in P$. Assume, moreover, that the optimal production vectors $W_p^*$ are unique for all $p \in P$.*[8] *Then the equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ in a CE is unique if and only if, for every delivery period $j \in J$, there exists at least one power plant with a strictly feasible optimal production.*

*Proof.* Let $x^*$ denote an optimal solution of (2.35). From Lemma 2.2.4, we know that the optimal decision vectors $\left[V_p^{*\top}, F_p^{*\top}, O_p^{*\top}\right]^\top$ for $p \in P$ and the optimal decision vectors $V_c^*$ for $c \in C$, are unique. By assumption the optimal decision vectors $W_p^*$ are unique for all $p \in P$. Then $x^*$ is unique too.

From the proof of Theorem 2.2.3, we can see that the equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ in a CE, is equal to the negative of the vector of Lagrange multipliers of the market clearing constraints in (2.35). Thus, showing the uniqueness of equilibrium electricity prices $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ in a CE is equivalent to showing the uniqueness of the vector of Lagrange multipliers of the market clearing constraints at $x^*$. In Kyparisis (1985), it was shown that the Lagrange multipliers are unique if and only if the Strict Mangasarian-Fromowitz Constraint Qualification (SMFCQ) holds at $x^*$. It is trivial to see that the Linear Independence Constraint Qualification (LICQ) is equivalent to the SMFCQ if strict complementarity slackness holds.

The remaining part of the proof is to show that the LICQ holds if and only if, for every delivery period $j \in J$, there exists at least one power plant with a strictly feasible optimal production. We first notice that for every producer $p \in P$, fuel type $l \in L$, trading time $t_i$, $i \in I_j$ and delivery at time $T_j$, $j \in J$, the decision variables $F_{p,l}(t_i, T_j)$ and $O_p(t_i, T_j)$ appear in the matrix of constraints only once, namely in (2.6) and (2.7), respectively. Thus, they can never be written as a linear combination of other rows in the constraint matrix, and can be eliminated from further consideration.

[8] See Remark 2.2.6, where we show that this assumption is completely nonrestrictive.

Next we see that the row vectors in the matrix of active inequality constraints are linearly dependent only if there exists a power plant with an optimal production such that (2.3) and (2.4) are simultaneously active for two consecutive delivery periods $j, j+1 \in J$, i.e.

- if $W^*_{p,l,r}(T_j) = 0$, $W^*_{p,l,r}(T_{j+1}) = \overline{W}^{p,l,r}_{\max}$ and $W^*_{p,l,r}(T_{j+1}) - W^*_{p,l,r}(T_j) = \triangle\overline{W}^{p,l,r}_{\max}$ or
- if $W^*_{p,l,r}(T_j) = \overline{W}^{p,l,r}_{\max}$, $W^*_{p,l,r}(T_{j+1}) = 0$ and $W^*_{p,l,r}(T_{j+1}) - W^*_{p,l,r}(T_j) = \triangle\overline{W}^{p,l,r}_{\min}$.

The first condition implies that $\overline{W}^{p,l,r}_{\max} = \triangle\overline{W}^{p,l,r}_{\max}$ and the second that $\overline{W}^{p,l,r}_{\max} = \triangle\overline{W}^{p,l,r}_{\min}$. We see that, in this case, (2.3) is redundant and can be removed without affecting the set of feasible solutions. If such constraints are removed, then the row vectors in the matrix of active inequality constraints are always linearly independent. We assume that this is indeed the case.

By considering the inner structure of the matrix of equality constraints given in (2.20), we notice that for (2.35), the LICQ holds if and only if, for every delivery period $j \in J$, there exists at least one power plant with a strictly feasible optimal production. This completes the proof.

□

**Remark 2.2.6.** *In this remark we address a potential non-uniqueness of the optimal decision vectors $W^*_p$, and discuss how this affects the uniqueness of the vector of Lagrange multipliers of the market clearing constraints. When there exists $p \in P$ such that $W^*_p$ is non-unique, the implication is that an element of $W^*_p$ can be written as a non-trivial linear combination of the other elements,*

$$0 = \sum_{j\in J} \sum_{r\in R^{p,l}} a_{p,l,r,j} W^*_{p,l,r}(T_j) \tag{2.54}$$

*for some $a_{p,l,r,j} \in \mathbb{R}$, and this linear combination is in the null space of the active constraints. From the KKT conditions of (2.35), we get*

$$-\pi \left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)^\top - Qx - B^\top \eta - A^\top \mu = 0. \tag{2.55}$$

*Note, by considering the structure of $Q$, that $-\pi \left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)^\top - Qx$ does not depend on $W^*_p$. Thus, non-uniqueness of $W^*_p$ for some $p \in P$ may only affect the set of active constraints*

*$\mathcal{A}(x^*)$ and, consequently, the variables $\eta$. Among all optimal $W_p^*$, we choose one such that all $W_{p,l,r}^*(T_j)$ with $a_{p,l,r,j} \neq 0$ are strictly feasible with respect to (2.3) and (2.4). If $W_p^*$ is non-unique, this is always possible. We require that the LICQ holds at such $x^*$. Clearly, the active sets at all other optimal solutions $W_p^*$ will include the active set at $x^*$ as a subset. Denote by $\eta^*$ and $\mu^*$ the optimal solution of the dual variables at $x^*$. Note that if the LICQ holds at $x^*$, then $\eta^*$ and $\mu^*$ are unique. It is easy to check that the same $\eta^*$ and $\mu^*$ solve the KKT conditions for all optimal $W_p^*$. However, if the active set is different (i.e. larger) than $\mathcal{A}(x^*)$, then the additional active constraints have the corresponding dual variables equal to zero, which contradicts the strict complementarity slackness assumption. Thus, the requirement of the LICQ at $x^*$ is enough to ensure the uniqueness of $\mu_M^*$.*

*It is worth mentioning that, in our numerical experiments, we never encountered a situation in which the optimal $W_p^*$ was non-unique. However, in theory such a case may occur, for example, when there are two power plants, belonging to the same producer, running on the same fuel and having the same constants $g^l$ and $c^{p,l,r}$. In this case it does not matter which of the two produces the electricity because they both have exactly the same costs. Thus, we can, after finding the optimal solution $x^*$, artificially combine the power plants and, for the purpose of calculation, regard them as one power plant with a unique optimal production $W_p^*$. By doing so, we ensure that the optimal $W_p^*$ is unique for all $p \in P$.*

## 2.3 Nash Equilibrium

In this section we investigate a setup that is very similar to the competitive equilibrium discussed in the previous section, but with the electricity market now included as a player with its own objective function. We call this player a 'hypothetical market agent'. This idea was first introduced in De Maere d'Aertrycke and Smeers (2012). Instead of a competitive equilibrium, we are now interested in the Nash equilibrium. In this section, we show that, under certain assumptions, every Nash equilibrium is also a competitive equilibrium and vice versa.

### 2.3.1 Hypothetical Market Agent

Assume now that the electricity market is modeled as a player with decision variables $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$. Our discussion in Section 2.1.3 still applies, except that the decisions of the electricity market are now modeled explicitly through the hypothetical market agent.

Let the hypothetical market agent have the following profit function

$$\begin{aligned} P_M(\Pi, V) &= \sum_{j \in J} e^{-\hat{r} T_j} \left[ \sum_{i \in I_j} \Pi(t_i, T_j) \left( \sum_{c \in C} V_c(t_i, T_j) + \sum_{p \in P} V_p(t_i, T_j) \right) \right] \\ &= \Pi^\top \left( \sum_{c \in C} V_c + \sum_{p \in P} V_p \right) \end{aligned} \tag{2.56}$$

and the expected profit

$$\begin{aligned} \Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi], V\right) &= \mathbb{E}^{\tilde{\mathbb{P}}}[P_M(V, \Pi)] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^\top \left( \sum_{c \in C} V_c + \sum_{p \in P} V_p \right), \end{aligned} \tag{2.57}$$

where $V = \left[V_P^\top, V_C^\top\right]^\top$, $V_P = ||_{p \in P} V_p$, and $V_C = ||_{c \in C} V_c$. Let the hypothetical market agent attempt to solve

$$\Phi_M(V) = \max_{\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N} \Psi_M\left(\mathbb{E}^{\mathbb{P}}[\Pi], V\right). \tag{2.58}$$

### 2.3.2 Definition of an NE

We start our discussion of the Nash equilibrium by stating its formal definition.

**Definition 2.3.1.** *Nash Equilibrium (NE)*

*Decisions $v^*$ and $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ constitute an NE if*

1. *For every producer $p \in P$, $v_p^* \in S_p$ is a strategy such that*

$$\Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \leq \Psi_p\left(v_p^*, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \tag{2.59}$$

*for all $v_p \in S_p$;*

*2. For every consumer $c \in C$, $V_c^* \in S_c$ is a strategy such that*

$$\Psi_c\left(V_c, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \leq \Psi_c\left(V_c^*, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) \tag{2.60}$$

*for all $V_c \in S_c$;*

*3. The price vector $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ maximizes the objective function of the hypothetical market agent,*

$$\Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi], v^*\right) \leq \Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*, v^*\right) \tag{2.61}$$

*for all $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$.*

### 2.3.3 Existence of an NE

We first study the relationship between a CE and an NE.

**Theorem 2.3.2.** *Every CE is also an NE.*

*Proof.* The equilibrium conditions for producers and consumers are clearly the same for both the NE and the CE. It remains to show that (2.31) implies (2.61). Assume a CE holds. Then (2.31) can be inserted into (2.56 - 2.58). It follows that $\Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi], V\right) = 0$ for all $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$, and thus (2.61) is satisfied.

□

The existence of an NE for our Problem 2.3.1 is a trivial consequence of Theorem 2.2.3 and Theorem 2.3.2.

### 2.3.4 Uniqueness of an NE

In this section we investigate conditions for the uniqueness of equilibrium electricity prices, $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, in an NE. In Section 2.2.2 we established the uniqueness of a CE. In this section, we show that every NE is also a CE. Recall that the converse was shown in Theorem 2.3.2.

Let us start with a simple example. Let $|P| = 1$, $|C| = 1$, $T' = 1$, and $I_1 = \{1\}$. Moreover, let producer $p \in P$ own only two gas power plants. Since there is just one trading and one delivery period, we will omit writing $t_i$ and $T_j$. From the optimization problem of

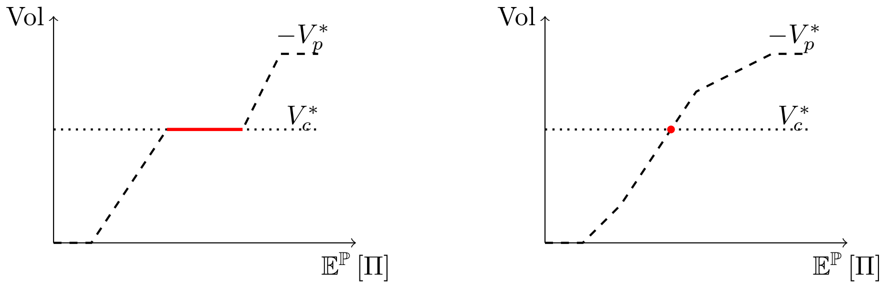



Figure 2.2: An example of non-uniqueness (left) and uniqueness (right) of an equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ in an NE and CE. To visualize the market clearing constraint better, we plot $-V_p^*$ instead of $V_p^*$. Note that $V_p^*$ is a decreasing function of $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$.

the consumer (2.11) and (2.10), one can see that $V_c^* = D$ regardless of the producer's and hypothetical market agent's decisions. On the other hand, the producer's optimal decisions, $V_p^*$, depend on the decisions of the hypothetical market agent, $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$. A sensitivity analysis of non-degenerate quadratic programming problems (see Berkelaar et al. (1996) for the convex case or Proposition 1.2.2 for the strictly convex case) shows that $V_p^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ is a decreasing function as shown in Figure 2.2.

Let us first analyze the CE. Based on the reasoning in Theorem 2.2.5, we can show uniqueness of the equilibrium electricity price, $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, by ensuring the uniqueness of the Lagrange multipliers of the market clearing constraint (2.31).

We claim that the uniqueness of the Lagrange multipliers of the market clearing constraint (2.31) is equivalent to the condition that the mapping $V_p^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ is strictly decreasing for all $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$, in some neighborhood of $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, and decreasing for all $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}$. If this condition holds, we can show that every NE is also a CE. The intuition for the proof is the following.

For any given price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]' \in \mathbb{R}^N$ that is larger than the competitive equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, clearly $V_c^* + V_p^* < 0$ (see Figure 2.2 on the right). From (2.57) it follows that $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]'$ cannot be an NE, because the hypothetical market agent could improve her objective function by decreasing $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]'$. Similarly, if a given price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]''$ is smaller than the competitive equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, then $V_c^* + V_p^* > 0$ (see Figure 2.2 on the right). From (2.57) it again follows that $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]''$ cannot be an NE, because the hypothetical market agent could improve her objective function by increasing $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]''$. From Theorem 2.3.2 we know that an NE in this simplified example exists, and based on the above reasoning, we conclude that, under the assumption of the local strong monotonicity, if the NE holds then $V_c^* + V_p^* = 0$. This proves that every NE is also a CE.

In the rest of this section, we use this intuition to show the uniqueness of the equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ in an NE in a general setting.

Denote by $\mathbb{PC}^k$ the class of piecewise $\mathbb{C}^k$ functions. See Definition 2.3.7 in the next section for a formal description of $\mathbb{PC}^k$.

**Lemma 2.3.3.** *Let $k \in P \cup C$. There exists a $\mathbb{PC}^\infty$ mapping $\tilde{\mathcal{Z}}_k : \mathbb{R}^N \to \mathbb{R}^N$ that maps a given electricity price vector $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$ to an optimal trading vector $V_k^* \in \mathbb{R}^N$.*

*Proof.* Since the proof for producers and consumers is essentially the same, we will write it only for producers. Set $p = k$ for some $p \in P$. The optimization problem for a producer $p$ can be written as

$$\max_{v_p \in S_p} \quad \Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right).$$

For convenience we define $\tilde{\Psi}_p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ as $\tilde{\Psi}_p\left(V_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) := \max_{F_p, O_p, W_p} \Psi_p\left(v_p, \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ subject to (2.3), (2.4), (2.5), (2.6), and (2.7). Using Lemma 2.2.4, we know that optimal trading vectors $V_p^*$ are unique for a given price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$. Thus, there exists a mapping $\tilde{\mathcal{Z}}_p : \mathbb{R}^N \to \mathbb{R}^N$ such that $V_p^* = \tilde{\mathcal{Z}}_p\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$. Results from non-degenerate parametric quadratic programming show that $\tilde{\mathcal{Z}}_p\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ is a continuous and piecewise affine function (and thus $\mathbb{PC}^\infty$), as in the section on non-degenerate parametric CQP problems in Berkelaar et al. (1996). Recall that the same result for strictly convex non-degenerate quadratic programming was proven in Proposition 1.2.4. □

The optimal decisions of all producers and consumers are combined in the objective function of the hypothetical market agent as shown in the following lemma.

**Lemma 2.3.4.** *Define a $\mathbb{PC}^\infty$ mapping*

$$\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) := \sum_{p \in P} \tilde{\mathcal{Z}}_p\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) + \sum_{c \in C} \tilde{\mathcal{Z}}_c\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right). \tag{2.62}$$

*There exists* $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ *such that* $\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) = 0$.

*Proof.* This is an immediate consequence of Theorem 2.2.3 by choosing $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* = -\mu_M^*$. □

Before we state the main theorem of this section, we have to define (strictly) negatively monotone mappings.

**Definition 2.3.5.** *A function* $\tilde{\mathcal{Z}} : S_1 \to \mathbb{R}^N$ *defined on an open set* $S_1 \subseteq \mathbb{R}^N$ *is strictly negatively monotone on* $S_1$ *if*

$$\left(\tilde{\mathcal{Z}}(x) - \tilde{\mathcal{Z}}(y)\right)^\top (x - y) < 0 \tag{2.63}$$

*for all* $x, y \in S_1$ *such that* $x \neq y$. *Similarly, a function* $\tilde{\mathcal{Z}} : S_1 \to \mathbb{R}^N$ *defined on an open set* $S_1 \subseteq \mathbb{R}^N$ *is negatively monotone on* $S_1$ *if*

$$\left(\tilde{\mathcal{Z}}(x) - \tilde{\mathcal{Z}}(y)\right)^\top (x - y) \leq 0 \tag{2.64}$$

*for all* $x, y \in S_1$.

**Theorem 2.3.6.** *Let* $\tilde{\mathcal{Z}}$ *be as in Lemma 2.3.4. If* $\tilde{\mathcal{Z}}$ *is strictly negatively monotone in a neighborhood of* $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ *and negatively monotone on* $\mathbb{R}^N$, *then every NE is a CE. Moreover, the NE is unique.*

*Proof.* The equilibrium conditions for producers and consumers are the same for both the NE and the CE. It remains to show that (2.61) implies (2.31).

Assume the NE holds. By Lemma 2.3.4, we know that there exists $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^* \in \mathbb{R}^N$ such that $\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right) = 0$ holds. $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ is clearly an NE as well as a CE. Moreover, we can show it is the only NE. We use a proof by contradiction. Assume there exists an expected price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]' \in \mathbb{R}^N$, $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]' \neq \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$, which is also an NE. Let $v'$ denote the corresponding vector of optimal solutions of all producers and consumers given $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]'$. Then

$$\Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]', v'\right) \geq \Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*, v'\right)$$

or, equivalently,

$$\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)^{\top}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} \geq \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)^{\top}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}, \tag{2.65}$$

because, otherwise, the hypothetical market agent could improve her objective function by switching form $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}$ to $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}$. By negative monotonicity of $\tilde{\mathcal{Z}}$ on $\mathbb{R}^N$, we see that

$$\left(\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) - \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)\right)^{\top}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right) \leq 0$$

which, using $\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) = 0$, is equivalent to

$$\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)^{\top}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} \leq \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)^{\top}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}. \tag{2.66}$$

Equations (2.65) and (2.66) then give

$$\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)^{\top}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} = \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right)^{\top}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}. \tag{2.67}$$

Define $z_\lambda = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} + \lambda\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right)$ for $\lambda \in (0,1)$. By the negative monotonicity of $\tilde{\mathcal{Z}}$ on $\mathbb{R}^N$ and $\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) = 0$, we get

$$-\tilde{\mathcal{Z}}\left(z_\lambda\right)^{\top}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} - z_\lambda\right) \leq 0$$

which, using the definition of $z_\lambda$, is equivalent to

$$\tilde{\mathcal{Z}}\left(z_\lambda\right)^{\top}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) \leq 0. \tag{2.68}$$

By the negative monotonicity of $\tilde{\mathcal{Z}}$ on $\mathbb{R}^N$, we also get

$$\left(\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right) - \tilde{\mathcal{Z}}\left(z_\lambda\right)\right)^{\top}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} - z_\lambda\right) \leq 0$$

which, by using the definition of $z_\lambda$, is equivalent to

$$\left(\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'}\right) - \tilde{\mathcal{Z}}\left(z_\lambda\right)\right)^{\top}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) \leq 0.$$

By using (2.67) this is equivalent to

$$-\tilde{\mathcal{Z}}\left(z_\lambda\right)^\top \left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) \leq 0. \tag{2.69}$$

Equations (2.68) and (2.69) together imply that

$$\tilde{\mathcal{Z}}\left(z_\lambda\right)^\top \left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) = 0. \tag{2.70}$$

Using the definition of $z_\lambda$ and (2.70), we then get

$$\tilde{\mathcal{Z}}\left(z_\lambda\right)^\top z_\lambda = \tilde{\mathcal{Z}}\left(z_\lambda\right)^\top \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}. \tag{2.71}$$

Denote a neighborhood of $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}$ by

$$\mathcal{N}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}, \delta\right) := \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right] \in \mathbb{R}^N : \left\|\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]\right\| \leq \delta\right\}$$

for some $\delta > 0$. If $\tilde{\mathcal{Z}}$ is strictly negatively monotone in some neighborhood of $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}$, then for all $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right] \in \mathcal{N}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}, \delta\right)$

$$\left(\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) - \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]\right)\right)^\top \left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]\right) < 0$$

which, using $\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right) = 0$, is equivalent to

$$\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]\right)^\top \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right] < \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]\right)^\top \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}. \tag{2.72}$$

Clearly, $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{'} \notin \mathcal{N}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}, \delta\right)$, because otherwise, we would have a contradiction with (2.65) immediately. Letting $\overline{\lambda} := \delta / \left\|\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]'\right\| < 1$, we have

$$\left\|z_\lambda - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*}\right\| = \lambda \left\|\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^{*} - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]'\right\| \leq \delta$$

for all $\lambda \in \left(0, \overline{\lambda}\right]$. Then, by (2.72),

$$\tilde{\mathcal{Z}}\left(z_\lambda\right)^\top z_\lambda < \tilde{\mathcal{Z}}\left(z_\lambda\right)^\top \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*,$$

for all $\lambda \in \left(0, \overline{\lambda}\right]$ which contradicts (2.71). Thus, our initial assumption is false and for all $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]' \in \mathbb{R}^N$, $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]' \neq \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*$, we must have

$$\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]'\right)^\top \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]' < \tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]'\right)^\top \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*,$$

which is equivalent to

$$\Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]', v'\right) < \Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*, v'\right).$$

This shows that $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]'$ is not an NE, because the hypothetical market agent could improve her objective function by changing her decisions to $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*$. Hence, $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*$ is the only NE. This completes the proof.

□

The proof that, if the strict complementarity slackness holds and if for every delivery period $j \in J$ there exists at least one power plant with a strictly feasible optimal production, $\tilde{\mathcal{Z}}$ is indeed strictly negatively monotone in a neighborhood of $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^*$ and negatively monotone for all $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right] \in \mathbb{R}^N$ is addressed in Section 2.3.5.

The same result could also be obtained from the existing literature. In Dontchev and Rockafellar (1996) in Theorem 6, it is shown that, under the strict complementarity slackness assumption, $\tilde{\mathcal{Z}}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]\right)$ is strictly monotone in some neighborhood of $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^* \in \mathbb{R}^N$ if and only if the optimal solution $x^*$ to (2.35) is unique and the LICQ holds at $x^*$. Recall from the proof of Theorem 2.2.5 that the LICQ is equivalent to the condition that, for every delivery period $j \in J$, there exists at least one power plant with a strictly feasible optimal production.

The results of Dontchev and Rockafellar (1996) were extended to general finite-dimensional optimization problems in Mordukhovich et al. (2014).

### 2.3.5 Proof of (strict) negative monotonicity

Before we start with the proof that $\tilde{\mathcal{Z}}$ is strictly negatively monotone in a neighborhood of $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ and negatively monotone on $\mathbb{R}^N$, let us introduce some definitions that are useful for the analysis of piecewise differential functions. A more detailed treatment of this material is available in Clarke (1990) and Ralph and Scholtes (1997).

**Definition 2.3.7.** *A continuous function $\tilde{\mathcal{Z}} : S_1 \to \mathbb{R}^N$ defined on an open set $S_1 \subseteq \mathbb{R}^N$ is $\mathbb{PC}^r$ if for every $x \in S_1$ there exists a finite family of $\mathbb{C}^r$-functions $\tilde{\mathcal{Z}}^i : S_2 \to \mathbb{R}^N$, where $S_2 \subseteq S_1$ is an open neighborhood of $x$ and $i \in \mathcal{I}$, such that $\tilde{\mathcal{Z}}(z) \in \left\{\tilde{\mathcal{Z}}^i(z) : i \in \mathcal{I}\right\}$ for every $z \in S_2$. The $\mathbb{C}^r$-functions $\tilde{\mathcal{Z}}^i$, $i \in \mathcal{I}$, are called selection functions of $\tilde{\mathcal{Z}}$ at $x$.*

**Definition 2.3.8.** *A selection function $\tilde{\mathcal{Z}}^i$ of a $\mathbb{PC}^r$- function $\tilde{\mathcal{Z}}$ at $x$ is essentially active if*

$$x \in \operatorname{cl}\operatorname{int}\left\{z \in S_2 : \tilde{\mathcal{Z}}^i(z) = \tilde{\mathcal{Z}}(z)\right\}. \tag{2.73}$$

*If $\left\{\tilde{\mathcal{Z}}^i : i \in \mathcal{I}\right\}$ is a family of selection functions for $\tilde{\mathcal{Z}}$ at $x$, then the set of indices $i$ of essentially active selection functions $\tilde{\mathcal{Z}}^i$ at $x$ is denoted $\mathcal{I}^e(x)$.*

**Definition 2.3.9.** *Let $\tilde{\mathcal{Z}} : S_1 \to \mathbb{R}^N$ be a continuous $\mathbb{PC}^1$ function defined on an open set $S_1 \subseteq \mathbb{R}^N$. Clarke's generalized Jacobian of $\tilde{\mathcal{Z}}$ at $x$ is defined as*

$$\mathcal{D}^{\mathcal{C}}\tilde{\mathcal{Z}}(x) = \operatorname{conv}\left\{\mathcal{D}\tilde{\mathcal{Z}}^i(x) : i \in \mathcal{I}^e(x)\right\} \tag{2.74}$$

*where* conv *denotes a convex hull.*

Now that we are equipped with definitions, we can investigate some of the properties of Clarke's generalized Jacobian.

**Proposition 2.3.10.** *$\mathcal{D}\tilde{\mathcal{Z}}^i(x) \preceq (\prec) 0$ for all $i \in \mathcal{I}^e(x)$ and all $x \in S_1$ if and only if $\mathcal{J}(x) \preceq (\prec) 0$ for all $\mathcal{J}(x) \in \mathcal{D}^{\mathcal{C}}\tilde{\mathcal{Z}}(x)$.*

*Proof.* This follows directly from (2.74). For any $\mathcal{J}(x) \in \mathcal{D}^{\mathcal{C}}\tilde{\mathcal{Z}}(x)$ and $x \in S_1$, $\mathcal{J}(x)$ can be written as

$$\mathcal{J}(x) = \sum_{i \in \mathcal{I}^e(x)} \alpha_i \mathcal{D}\tilde{\mathcal{Z}}^i(x) \tag{2.75}$$

for some $\alpha_i \geq 0$ such that $\sum_{i \in \mathcal{I}^e(x)} \alpha_i = 1$. The result follows trivially. $\square$

We know that a monotone $\mathbb{C}^1$ function has a positive semidefinite Jacobian. A similar result also holds for $\mathbb{PC}^1$ functions as shown in the following proposition.

**Proposition 2.3.11.** *Let $\tilde{\mathcal{Z}} : S_1 \to \mathbb{R}^N$ be a $\mathbb{PC}^1$ function defined on an open set $S_1 \subseteq \mathbb{R}^N$. If $\mathcal{D}\tilde{\mathcal{Z}}^i(x) \preceq (\prec) \, 0$ for all $i \in \mathcal{I}^e(x)$ and $x \in S_1$, then $\tilde{\mathcal{Z}}$ is (strictly) negatively monotone on $S_1$.*

*Proof.* Assume that $\mathcal{D}\tilde{\mathcal{Z}}^i(x) \prec 0$ for all $i \in \mathcal{I}^e(x)$. By Proposition 2.3.10 also $\mathcal{J}(x) \prec 0$ for all $\mathcal{J}(x) \in \mathcal{D}^{\mathcal{C}}\tilde{\mathcal{Z}}(x)$. By the extended mean-value theorem (see Jeyakumar and Luc (1998))

$$\tilde{\mathcal{Z}}(x) - \tilde{\mathcal{Z}}(y) = \mathcal{J}(y + \delta(x - y))(x - y) \tag{2.76}$$

for some $\mathcal{J}(y + \delta(x - y)) \in \mathcal{D}^{\mathcal{C}}\tilde{\mathcal{Z}}(y + \delta(x - y))$ and $\delta \in (0, 1)$. By multiplying (2.76) on the left hand side by $(x - y)^\top$ and using the assumption of the strict negative definiteness of $\mathcal{J}(x)$, we conclude that

$$\left(\tilde{\mathcal{Z}}(x) - \tilde{\mathcal{Z}}(y)\right)^\top (x - y) < 0. \tag{2.77}$$

It follows similarly that, if $\mathcal{D}\tilde{\mathcal{Z}}^i(x) \preceq 0$ for all $i \in \mathcal{I}^e(x)$ and $x \in S_1$, then $\tilde{\mathcal{Z}}$ is negatively monotone on $S_1$. $\square$

The main problem of this section is to show that there exists a $\delta > 0$ such that $\mathcal{D}\tilde{\mathcal{Z}}^i\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) \prec 0$ for all $i \in \mathcal{I}^e\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ and all $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathcal{N}\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*, \delta\right)$. Moreover, we have to show that $\mathcal{D}\tilde{\mathcal{Z}}^i\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) \preceq 0$ for all $i \in \mathcal{I}^e\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ and all $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \mathbb{R}^N$. In the proof of Lemma 2.3.3, we saw that $\tilde{\mathcal{Z}}^i\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$, $i \in \mathcal{I}$ are all affine functions and thus $\mathcal{D}\tilde{\mathcal{Z}}^i\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ are all constants. Therefore, by the premise of Proposition 2.3.10, it is enough to limit our interest to the prices $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \left\{x \in \mathbb{R}^N : |\mathcal{I}^e(x)| = 1\right\}$. For them, $\mathcal{D}\tilde{\mathcal{Z}}_k\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ exists for all players $k \in P \cup C$. In the rest of this section we simplify the notation and omit the explicit membership of $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \in \left\{x \in \mathbb{R}^N : |\mathcal{I}^e(x)| = 1\right\}$. All the statements hold for any such $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$.

For further argumentation, we introduce $R(\cdot)$ and $N(\cdot)$ that denote a range and a null space.

**Lemma 2.3.12.** *Let $x \in \mathbb{R}^N \setminus \{0\}$ and $c \in C$ be any consumer. Then $x^\top \mathcal{D}\tilde{\mathcal{Z}}_c x \leq 0$. Moreover, $x^\top \mathcal{D}\tilde{\mathcal{Z}}_c x = 0$ if and only if $x \in R\left(\hat{A}_1^\top\right)$.*

*Proof.* The necessary and (due to convexity and the Slater condition in Assumption 2.1.1) sufficient KKT conditions for $V_c$ to be a global maximizer, given the forward electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$, read

$$-\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] - \lambda_c \hat{Q}_1 V_c - \hat{A}_1^\top \mu_c = 0, \tag{2.78}$$

$$\hat{A}_1 V_c = a_c. \tag{2.79}$$

One can solve for $V_c$ from (2.78) by multiplying both sides on the left by $\left(\lambda_c \hat{Q}_1\right)^{-1}$ to obtain

$$V_c = -\frac{1}{\lambda_c}\hat{Q}_1^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] - \frac{1}{\lambda_c}\hat{Q}_1^{-1}\hat{A}_1^\top \mu_c. \tag{2.80}$$

By further multiplying both sides of (2.80) on the left by $\hat{A}_1$, we get

$$\hat{A}_1 V_c = -\frac{1}{\lambda_c}\hat{A}_1\hat{Q}_1^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] - \frac{1}{\lambda_c}\hat{A}_1\hat{Q}_1^{-1}\hat{A}_1^\top \mu_c. \tag{2.81}$$

Since $\hat{A}_1$ has full row rank $\left(\hat{A}_1\hat{Q}_1^{-1}\hat{A}_1^\top\right)^{-1}$ exists. By using (2.79), we arrive at

$$\mu_c = -\left(\hat{A}_1\hat{Q}_1^{-1}\hat{A}_1^\top\right)^{-1}\left(\lambda_c a_c + \hat{A}_1\hat{Q}_1^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right). \tag{2.82}$$

Inserting this back into (2.80), we calculate

$$\frac{\partial V_c}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]} = -\frac{1}{\lambda_c}\hat{Q}_1^{-1} + \frac{1}{\lambda_c}\hat{Q}_1^{-1}\hat{A}_1^\top\left(\hat{A}_1\hat{Q}_1^{-1}\hat{A}_1^\top\right)^{-1}\hat{A}_1\hat{Q}_1^{-1}. \tag{2.83}$$

Since $\hat{Q}_1$ (and consequently also $\hat{Q}_1^{-1}$) is positive definite and symmetric, we can use the Cholesky decomposition to define a matrix $\tilde{Q}_1 \in \mathbb{R}^{N\times N}$ such that $\hat{Q}_1^{-1} = \tilde{Q}_1^{-1}\tilde{Q}_1^{-\top}$. Since

$\tilde{Q}_1$ is invertible, we can rewrite (2.83) as

$$\tilde{P} := -\lambda_c \tilde{Q}_1 \frac{\partial V_c}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]} \tilde{Q}_1{}^\top = I - \tilde{Q}_1{}^{-\top} \hat{A}_1^\top \left( \hat{A}_1 \hat{Q}_1^{-1} \hat{A}_1^\top \right)^{-1} \hat{A}_1 \tilde{Q}_1{}^{-1}. \tag{2.84}$$

It is trivial to check that $\tilde{P} = \tilde{P}^2$ (i.e. $\tilde{P}$ is idempotent) and symmetric. Thus, $\tilde{P}$ is a projection matrix, and as such positive semidefinite. Thus $\frac{\partial V_c}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]} \preceq 0$. Moreover, for any $x \in \mathbb{R}^N \setminus \{0\}$, $x^\top \tilde{P} x = 0$ if and only if $x \in R\left(\tilde{Q}_1{}^{-\top} \hat{A}_1^\top\right)$. Therefore, it follows from (2.84) that $x^\top \frac{\partial V_c}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]} x = 0$ if and only if $x \in R\left(\hat{A}_1^\top\right)$. □

**Lemma 2.3.13.** *Assume that $x^\top \mathcal{D}\tilde{\mathcal{Z}}_p x \leq 0$ for all $p \in P$ and $x \in \mathbb{R}^N$. Then $\mathcal{D}\tilde{\mathcal{Z}} \prec 0$ if and only if $\sum_{p \in P} \hat{A}_1 \mathcal{D}\tilde{\mathcal{Z}}_p \hat{A}_1^\top$ has full rank.*

*Proof.* We write

$$x^\top \mathcal{D}\tilde{\mathcal{Z}} x = \sum_{k \in P \cup C} x^\top \mathcal{D}\tilde{\mathcal{Z}}_k x.$$

By Lemma 2.3.12 and the assumption that $x^\top \mathcal{D}\tilde{\mathcal{Z}}_p x \leq 0$ for all $p \in P$ and $x \in \mathbb{R}^N$, we see that $x^\top \mathcal{D}\tilde{\mathcal{Z}}_k x \leq 0$ for all $k \in P \cup C$. Thus, $x^\top \mathcal{D}\tilde{\mathcal{Z}} x < 0$ if and only if, for each $x \in \mathbb{R}^N \setminus \{0\}$, there exists at least one player $k' \in P \cup C$ such that $x^\top \mathcal{D}\tilde{\mathcal{Z}}_{k'} x < 0$. If $x \notin R\left(\hat{A}_1^\top\right)$, then $x^\top \mathcal{D}\tilde{\mathcal{Z}}_k x < 0$ for all $k \in C$. On the other hand, if $x \in R\left(\hat{A}_1^\top\right)$ then $x^\top \mathcal{D}\tilde{\mathcal{Z}}_k x = 0$ for all $k \in C$. Thus, for all $x \in R\left(\hat{A}_1^\top\right)$, $\mathcal{D}\tilde{\mathcal{Z}} \prec 0$ if and only if $\sum_{p \in P} \hat{A}_1 \mathcal{D}\tilde{\mathcal{Z}}_p \hat{A}_1^\top$ has full rank. □

**Lemma 2.3.14.** *Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times m}$ be real matrices such that $A^{-1}$ and $C - BC^{-1}B^\top$ exist. Then*

$$\begin{bmatrix} A^\top & B^\top \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \end{bmatrix}$$

*and consequently*

$$\begin{bmatrix} A^\top & B^\top \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}^{-1} \begin{bmatrix} A \\ B \end{bmatrix} = A.$$

*Proof.* Since $A^{-1}$ and $C - BC^{-1}B^\top$ exist, we can use the block matrix inverse formula to calculate

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\left(C - B^\top A^{-1}B\right)^{-1} B^\top A^{-1} & -A^{-1}B\left(C - B^\top A^{-1}B\right)^{-1} \\ -\left(C - B^\top A^{-1}B\right)^{-1} B^\top A^{-1} & \left(C - B^\top A^{-1}B\right)^{-1} \end{bmatrix}$$

The result follows immediately. □

**Theorem 2.3.15.** *Let strict complementarily slackness hold for all producers $p \in P$. Then for all $x \in \mathbb{R}^N$ and $p \in P$, $x^\top \mathcal{D}\tilde{\mathcal{Z}}_p x \leq 0$. Moreover, if for each delivery period $j \in J$, there exist at least one power plant that has a strictly feasible optimal production, then $\sum_{p \in P} \hat{A}_1 \mathcal{D}\tilde{\mathcal{Z}}_p \hat{A}_1^\top$ has full rank.*

*Proof.* We can assume without loss of generality that there exists a producer $p \in P$, who owns a set of power plants, among which at least one has a strictly feasible optimal production in each delivery period $j \in J$. In this case, showing that $\sum_{p \in P} \hat{A}_1 \mathcal{D}\tilde{\mathcal{Z}}_p \hat{A}_1^\top$ is full rank simplifies to showing that $\hat{A}_1 \mathcal{D}\tilde{\mathcal{Z}}_p \hat{A}_1^\top$ has full rank.

Assume that the set of active inequality constraints $B_p$ is known. Due to the complementarity slackness assumption and Remark 2.2.6, $B_p$ is unique. The KKT conditions can thus be written as

$$-\mathbb{E}^{\tilde{\mathbb{P}}}[\pi] - \lambda_p \hat{Q} v_p' - \hat{A}_{12}^\top \mu_p' = 0, \tag{2.85}$$

$$-\hat{A}_p^\top \mu_p' - B_p^\top \eta_p' = 0, \tag{2.86}$$

$$\hat{A}_{12} v_p' + \hat{A}_p v_p'' = 0, \tag{2.87}$$

$$B_p v_p'' = b, \tag{2.88}$$

where $v_p' := \left[V_p^\top, F_p^\top, O_p^\top\right]^\top$ and $v_p'' := W_p$. The internal structure of the matrices is illustrated by

$$\hat{Q} =: \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2^\top & \hat{Q}_3 \end{bmatrix}, \quad \tilde{A}_{12} := \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix}, \quad \hat{A}_p := \begin{bmatrix} \hat{A}_{3,p} \\ \hat{A}_{4,p} \end{bmatrix}. \tag{2.89}$$

Since $\hat{Q} \succ 0$, we can express $v_p'$ from (2.85) as

$$v_p' = -\frac{1}{\lambda_p}\hat{Q}^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\pi] - \frac{1}{\lambda_p}\hat{Q}^{-1}\hat{A}_{12}^\top\mu_p' \tag{2.90}$$

and further more

$$\hat{A}_{12}v_p' = -\frac{1}{\lambda_p}\hat{A}_{12}\hat{Q}^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\pi] - \frac{1}{\lambda_p}\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\mu_p'. \tag{2.91}$$

Using (2.87) this reads

$$\hat{A}_p v_p'' = \frac{1}{\lambda_p}\hat{A}_{12}\hat{Q}^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\pi] + \frac{1}{\lambda_p}\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\mu_p'. \tag{2.92}$$

Since $\hat{A}_1$ and $\hat{A}_2$ are both full rank, also $\hat{A}_{12}$ has full rank and thus $\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}$ exists. Then

$$\mu_p' = \left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}\left(\lambda_p\hat{A}_p v_p'' - \hat{A}_{12}\hat{Q}^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\pi]\right). \tag{2.93}$$

Multiplying both sides by $\hat{A}_p^\top$ on the left and using (2.86), we obtain

$$B_p^\top\eta_p' = \hat{A}_p^\top\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}\left(-\lambda_p\hat{A}_p v_p'' + \hat{A}_{12}\hat{Q}^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\pi]\right). \tag{2.94}$$

It is not possible to decompose $v_p''$ in (2.94) because $\operatorname{rank}\left(\hat{A}_p^\top\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}\hat{A}_p\right) \leq |J|\left(|L|+1\right)+1 < \dim W_p$ and thus $\hat{A}_p^\top\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}\hat{A}_p$ is not invertible. We can write (2.94) and (2.88) as the system of linear equations,

$$\begin{bmatrix} \lambda_p\hat{A}_p^\top\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}\hat{A}_p & B_p^\top \\ B_p & 0 \end{bmatrix}\begin{bmatrix} v_p'' \\ \eta_p' \end{bmatrix} = \begin{bmatrix} \hat{A}_p^\top\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}\hat{A}_{12}\hat{Q}^{-1}\mathbb{E}^{\tilde{\mathbb{P}}}[\pi] \\ b_p \end{bmatrix}. \tag{2.95}$$

Before we attempt to solve (2.95), let us try to evaluate $\frac{\partial v_p'}{\partial\mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}$ with our current knowledge.

Evidently,

$$\begin{aligned}\frac{\partial v'_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]} &= -\lambda_p^{-1}\hat{Q}^{-1} - \lambda_p^{-1}\hat{Q}^{-1}\hat{A}_{12}^{\top}\frac{\partial \mu'_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]} \\ &= -\lambda_p^{-1}\hat{Q}^{-1} + \lambda_p^{-1}\hat{Q}^{-1}\hat{A}_{12}^{\top}\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^{\top}\right)^{-1}\left(\hat{A}_{12}\hat{Q}^{-1} - \lambda_p\hat{A}_p\frac{\partial v''_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}\right),\end{aligned}$$

where (2.90) and (2.93) were used to obtain the first and the second equality, respectively. We are interested in the rank of $\hat{A}_1\mathcal{D}\tilde{\mathcal{Z}}_p\hat{A}_1^{\top}$. Define $\tilde{A}_1 = \left[\hat{A}_1 0\right]$, where 0 is a matrix of zeros of size $|J| \times (N\,|L| + 1)$. Then,

$$\begin{aligned}\hat{A}_1\mathcal{D}\tilde{\mathcal{Z}}_p\hat{A}_1^{\top} &= \tilde{A}_1\frac{\partial v'_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}\tilde{A}_1^{\top} \\ &= -\tfrac{1}{\lambda_p}\tilde{A}_1\hat{Q}^{-1}\tilde{A}_1^{\top} + \tfrac{1}{\lambda_p}\tilde{A}_1\hat{Q}^{-1}\hat{A}_{12}^{\top}\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^{\top}\right)^{-1}\hat{A}_{12}\hat{Q}^{-1}\tilde{A}_1^{\top} \\ &\quad -\tilde{A}_1\hat{Q}^{-1}\hat{A}_{12}^{\top}\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^{\top}\right)^{-1}\hat{A}_p\frac{\partial v''_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}\tilde{A}_1^{\top}.\end{aligned} \tag{2.96}$$

Using the internal structure of $\hat{Q}$ and $\hat{A}_{12}$ as defined in (2.89), we evaluate

$$\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^{\top} = \begin{bmatrix} \hat{A}_1\hat{Q}_1\hat{A}_1^{\top} & \hat{A}_1\hat{Q}_2\hat{A}_2^{\top} \\ \hat{A}_2\hat{Q}_2^{\top}\hat{A}_1^{\top} & \hat{A}_2\hat{Q}_3\hat{A}_2^{\top} \end{bmatrix} \tag{2.97}$$

and similarly

$$\tilde{A}_1\hat{Q}^{-1}\hat{A}_{12}^{\top} = \begin{bmatrix} \hat{A}_1\hat{Q}_1\hat{A}_1^{\top} & \hat{A}_1\hat{Q}_2\hat{A}_2^{\top} \end{bmatrix}. \tag{2.98}$$

Using Lemma 2.3.14, (2.97), and (2.98), we rewrite (2.96) as

$$\hat{A}_1\mathcal{D}\tilde{\mathcal{Z}}_p\hat{A}_1^{\top} = -\tilde{A}_1\hat{Q}^{-1}\hat{A}_{12}^{\top}\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^{\top}\right)^{-1}\hat{A}_p\frac{\partial v''_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}\tilde{A}_1^{\top}. \tag{2.99}$$

In order to evaluate (2.99), we return to the system of linear equations (2.95). Using the generalized Bott-Duffin constrained inverse (see Yonglin (1990)), we express $\frac{\partial v''_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}$ from

(2.95) as

$$\frac{\partial v_p''}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]} = \left(\check{A}_p\right)_S^{(+)} \hat{A}_p^\top \left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1} \hat{A}_{12}\hat{Q}^{-1} + P_{N(\check{A}_p)\cap S}\frac{\partial z_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]}, \tag{2.100}$$

where $\left(\check{A}_p\right)_S^{(+)} = P_S\left(\check{A}_p P_S + P_{S^\perp}\right)^{(+)}$, $\check{A}_p := \lambda_p \hat{A}_p^\top \left(\hat{A}_{12} Q_P^{-1} \hat{A}_{12}^\top\right)^{-1} \hat{A}_p$, $S$ is a null space of $B_p$ (i.e. $S = N(B_p)$), $P_S$ a projection on $S$, and $z_p \in \mathbb{R}^{\dim W_p}$, an arbitrary vector. The Moore–Penrose pseudoinverse is denoted by $^{(+)}$. For every $x \in N(\check{A}_p)$,

$$\begin{aligned} 0 &= \check{A}_p x \\ &= \hat{A}_p^\top \left(\hat{A}_{12}\hat{Q}^{-1}A_{12}^\top\right)^{-1} \hat{A}_p x \\ &= x^\top \hat{A}_p^\top \left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1} \hat{A}_p x \\ &= \tilde{Q}\hat{A}_p x \\ &= \hat{A}_p x, \end{aligned} \tag{2.101}$$

where $\tilde{Q} \in \mathbb{R}^{(|J|(|L|+1)+1)\times(|J|(|L|+1)+1)}$ is defined by the Cholesky decomposition such that

$$\tilde{Q}\tilde{Q}^\top = \left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}. \tag{2.102}$$

Since this is true for every $x \in N(\check{A}_p)$, it must also hold for every $x \in N(\check{A}_p) \cap S$ and therefore

$$-\tilde{A}_1\hat{Q}^{-1}\hat{A}_{12}^\top \left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1} \hat{A}_p P_{N(\check{A}_p)\cap S}\frac{\partial z_p}{\partial \mathbb{E}^{\tilde{\mathbb{P}}}[\pi]} = 0. \tag{2.103}$$

Properties of the Bott-Duffin inverse (see Yonglin (1990)) imply that if $\check{A}_p$ is symmetric, then $\left(\check{A}_p\right)_S^{(+)}$ is also symmetric. Moreover, if $\left(v_p''\right)^\top \check{A}_p v_p'' \geq 0$ for all $v_p''$, such that $B_p v_p'' = b_p$, then also $\left(v_p''\right)^\top \left(\check{A}_p\right)_S^{(+)} v_p'' \geq 0$ for all $v_p''$. Thus, we can see from (2.99), (2.100), and (2.103) that $x^\top \mathcal{D}\tilde{\mathcal{Z}}_p x \leq 0$, $p \in P$ for all $x \in \mathbb{R}^N$, which proves the first part of the theorem.

Using (2.103), (2.97) and (2.98), equation (2.99) reads

$$\hat{A}_1\mathcal{D}\tilde{\mathcal{Z}}_p\hat{A}_1^\top = \hat{A}_{3,p}\left(\check{A}_p\right)_S^{(+)} \hat{A}_{3,p}^\top. \tag{2.104}$$

From Yonglin (1990) we know that $R\left(\left(\check{A}_p\right)_S^{(+)}\right) = R\left(P_S\check{A}_p\right)$. Since $\left(\check{A}_p\right)_S^{(+)}$ is symmetric and positive semidefinite, there exists $U \in \mathbb{R}^{\dim W_p \times \dim W_p}$ such that $\left(\check{A}_p\right)_S^{(+)} = UU^\top$. Then,

$$\begin{aligned} R\left(P_S\check{A}_p\right) &= R\left(\left(\check{A}_p\right)_S^{(+)}\right) \\ &= R\left(UU^\top\right) \\ &= R(U) \end{aligned} \tag{2.105}$$

and

$$\begin{aligned} \operatorname{rank}\left(\hat{A}_{3,p}\left(\check{A}_p\right)_S^{(+)}\hat{A}_{3,p}^\top\right) &= \operatorname{rank}\left(\hat{A}_{3,p}U\right) \\ &= \dim R\left(U\right) - \dim N\left(\hat{A}_{3,p}\right) \cap R\left(U\right). \end{aligned} \tag{2.106}$$

The next step is to show that $R\left(P_S\check{A}_p\right) = R\left(P_S\hat{A}_p^\top\right)$. We know that for any matrices $\tilde{A}$ and $\tilde{B}$ of the same size $R\left(\tilde{A}\right) = R\left(\tilde{B}\right)$ if and only if $N\left(\tilde{A}^\top\right) = N\left(\tilde{B}^\top\right)$. Since $\left(\hat{A}_{12}\hat{Q}^{-1}\hat{A}_{12}^\top\right)^{-1}$ is positive definite and symmetric, there exists an invertible matrix $\tilde{Q}$ as defined in (2.102). Then for every $x \in N\left(\left(P_S\check{A}_p\right)^\top\right)$,

$$\begin{aligned} 0 &= \hat{A}_p^\top\tilde{Q}\tilde{Q}^\top\hat{A}_pP_Sx \\ &= x^\top P_S\hat{A}_p^\top\tilde{Q}\tilde{Q}^\top\hat{A}_pP_Sx \\ &= \tilde{Q}^\top\hat{A}_pP_Sx \\ &= \hat{A}_pP_Sx, \end{aligned} \tag{2.107}$$

and thus $R\left(P_S\check{A}_p\right) = R\left(P_S\hat{A}_p^\top\right)$. It follows from (2.105), (2.106) and (2.107) that

$$\begin{aligned} \operatorname{rank}\left(\hat{A}_{3,p}\left(\check{A}_p\right)_S^{(+)}\hat{A}_{3,p}^\top\right) &= \operatorname{rank}\left(\hat{A}_{3,p}P_S\left[\begin{array}{c|c} \hat{A}_{3,p}^\top & \hat{A}_{4,p}^\top \end{array}\right]\right) \\ &= \operatorname{rank}\left(\hat{A}_{3,p}P_S\right), \end{aligned} \tag{2.108}$$

where the last equality holds since $P_S$ is symmetric.

A projection matrix $P_S$ on the subspace $S = N(B_p)$, can be written as

$$P_S = I - B_p^\top \left(B_p B_p^\top\right)^{-1} B_p. \tag{2.109}$$

Taking into account the structure of $B_p$ and $\hat{A}_{3,p}$, it is easy to see that $\hat{A}_{3,p} P_S$ has full rank if and only if, for each delivery period $j \in J$, there exists at least one power plant that has a strictly feasible optimal production (i.e. one of the power plants does not appear in the set of active constraints $B_p$). □

### 2.3.6 Mean maximization

In this section we examine a mean maximization problem, instead of a mean-variance maximization, in the context of the uniqueness of solutions presented in the previous section.

A mean maximization optimization problem for each producer $p \in P$ is defined as

$$\begin{aligned} \max_{v_p} \quad & -\mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_p\right]^\top v_p \\ \text{s.t.} \quad & A_p v_p = a_p \\ & B_p v_p \leq b_p \end{aligned} \tag{2.110}$$

and a mean maximization optimization problem for each consumer $c \in C$ as

$$\begin{aligned} \max_{V_c} \quad & -\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^\top V_c \\ \text{s.t.} \quad & A_c V_c = a_c \\ & B_c V_c \leq b_c. \end{aligned} \tag{2.111}$$

Let us examine the uniqueness of the equilibrium electricity price on a simple example in which $|P| = 1$, $|C| = 1$, $T' = 1$, and $I_1 = \{1\}$. We simplify the notation by setting $V := V_{p_1}$. Sensitivity analysis of the linear programming problem shows that the optimal value $-V^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ for a given price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$, is an increasing piecewise constant function. Since we are looking for a price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ such that $-V^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right) = D(T_1)$, we can distinguish two cases (see Figure 2.3). In the first (graph on the left), the price equilibrium is not unique, while in the second (graph on the right), the price equilibrium is unique. The situation on the

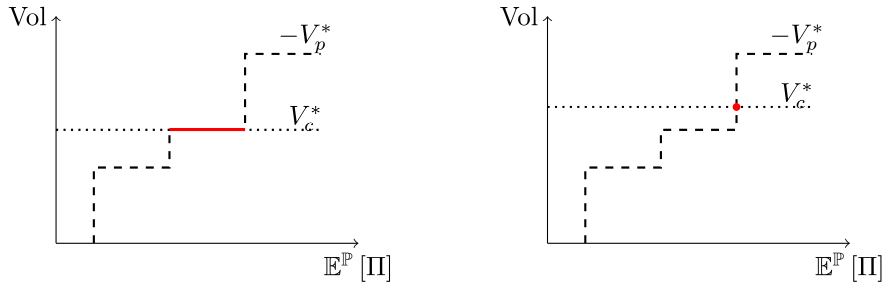



Figure 2.3: Equilibrium types for mean maximization problems.

left-hand side occurs when the optimal solution of the linear programing problem (2.110) lies strictly at the vertex of the polyhedron representing the set of feasible solutions. In this case, a small change in the equilibrium price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$ leads to no change in the optimal solution $V^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$. Hence, if the CE is satisfied for some expected price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ such that the optimal solution $V^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right)$ is strictly at the vertex, then there exists a small neighborhood of expected prices around $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ that satisfy the competitive equilibrium conditions. Thus, the equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ is not unique.

Relating this finding to Theorem 2.3.6, we see that the mapping $V^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\right)$ violates the strict monotonicity requirement at this point. Moreover, relating this analysis to Theorem 2.2.5 and Theorem 2.3.15, we see that the situation in which $V^*\left(\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*\right)$ lies strictly at the vertex of the polyhedron representing the set of feasible solutions, coincides with the condition that none of the power plants has a strictly feasible optimal production.

Even though the uniqueness of the solution does not hold in the mean maximization setting, it is still possible to establish the following unique relation among the prices $\Pi(t_i, T_j)$, $i \in I_j$ for any fixed $j \in J$.:

**Proposition 2.3.16.** *Electricity prices* $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi(t_i, T_j)]$ *that constitute a CE are equal for all* $i \in I_j$ *and any fixed* $j \in J$.

*Proof.* Let $V_k^*$, $k \in P \cup C$ and $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ satisfy the CE conditions in Definition 2.2.1. Moreover, let $j \in J$, $i' \in I_j$, $i'' \in I_j$, $i' \neq i''$ be such that $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi(t_{i'}, T_j)]^* < \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi(t_{i''}, T_j)]^*$. Then for each producer $p \in P$, $V_p^*(t_{i'}, T_j) \geq V_p^*(t_{i''}, T_j)$ because, otherwise, they could improve their objective functions by exchanging $V_p^*(t_{i'}, T_j)$ and $V_p^*(t_{i''}, T_j)$. The same also holds for all consumers $c \in C$. Note that (2.31) implies that

$$\sum_{p \in P} V_p^* \left(t_{i'}, T_j\right) + \sum_{c \in C} V_c^* \left(t_{i'}, T_j\right) = 0$$

and

$$\sum_{p \in P} V_p^* \left(t_{i''}, T_j\right) + \sum_{c \in C} V_c^* \left(t_{i''}, T_j\right) = 0.$$

Therefore, $V_p^* \left(t_{i'}, T_j\right) = V_p^* \left(t_{i''}, T_j\right)$ for all $p \in P$ and $V_c^* \left(t_{i'}, T_j\right) = V_c^* \left(t_{i''}, T_j\right)$ for all $c \in C$. This cannot be a CE, because producer $p \in P$ could improve her objective function by simply increasing $V_p^* \left(t_{i'}, T_j\right)$ for some $\epsilon > 0$ and decreasing $V_p \left(t_{i''}, T_j\right)$ for the same $\epsilon > 0$ without changing any other decision variable or violating any of the constraints. The same reasoning also holds for each consumer $c \in C$. Thus, solution $V_k^* \left(t_{i'}, T_j\right) = V_k^* \left(t_{i''}, T_j\right)$ does not satisfy (2.29) and (2.30). Thus, $\mathbb{E}^{\tilde{\mathbb{P}}} \left[\Pi \left(t_{i'}, T_j\right)\right]^* \geq \mathbb{E}^{\tilde{\mathbb{P}}} \left[\Pi \left(t_{i''}, T_j\right)\right]^*$.

One can apply similar reasoning by assuming that $\mathbb{E}^{\tilde{\mathbb{P}}} \left[\Pi \left(t_{i'}, T_j\right)\right]^* > \mathbb{E}^{\tilde{\mathbb{P}}} \left[\Pi \left(t_{i''}, T_j\right)\right]^*$. Thus, $\mathbb{E}^{\tilde{\mathbb{P}}} \left[\Pi \left(t_{i'}, T_j\right)\right]^* = \mathbb{E}^{\tilde{\mathbb{P}}} \left[\Pi \left(t_{i''}, T_j\right)\right]^*$. □

# Chapter 3

# Calculation of a power price equilibrium

In this chapter, we extend the model from Section 2.1 to include transaction costs, liquidity constraints and block electricity and fuel contracts. We also propose a tractable quadratic programming formulation to solve the model numerically. Numerical simulations are used to examine the dependence of the term structure of electricity prices on various parameters in the model. The algorithm is applied in a realistic setting by incorporating the entire UK power grid, consisting of several hundred power plants.

This chapter is organized as follows: a quadratic programming formulation that can be used to calculate the competitive price equilibrium is described in Section 3.1. In Section 3.2 we continue with various realistic extensions of the model. In Section 3.3, we illustrate the forecasting power of our model through numerical experiments by modeling the entire UK power grid. In the first part of this final section, we study the term structure of electricity prices in a simple example. In the second part, we calculate the equilibrium electricity price for block as well as spot contracts, and in the third, we study a closed form relation among equilibrium electricity prices.

## 3.1 Quadratic programming formulation

The traditional approach to solving competitive equilibrium optimization problems in a linear programming framework is via shadow prices (see De Maere d'Aertrycke and Smeers (2012) and Milano et al. (2006)). In Theorem 2.2.3, it was shown how to extend this approach to the quadratic programming framework to calculate a competitive equilibrium. In this section we describe a variation of the shadow price approach, which is mathematically equivalent to the approach presented in Theorem 2.2.3, but relies on a Nash equilibrium, and thus offers some additional insights into the problem.

A naïve approach to solving equilibrium optimization problems would be to choose an expected price vector $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$, and then calculate optimal solutions for each producer $p \in P$ and each consumer $c \in C$. If at such a price $\left\| \sum_{c \in C} V_c + \sum_{p \in P} V_p \right\|$ is close to zero, then the solution is found and $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$ is an equilibrium expected price vector. Otherwise, we have to adjust the expected price vector and repeat the procedure. We can see that such an algorithm is costly, because it involves solving a large optimization problem (to calculate the optimal solutions of each producer and each consumer) several times. In the section below, we show that we can do much better than this naïve approach. Using the reformulation we propose, a large optimization problem need only be solved once.

### 3.1.1 Reformulation of the hypothetical market agent's problem

The equivalence of the competitive equilibrium (Definition 2.2.1) and the Nash equilibrium (Definition 2.3.1) as shown in Theorems 2.3.2 and 2.3.6 is a theoretical result that has to be applied with caution in an algorithmic framework. The formulation of the hypothetical market agent's problem in (2.58) is clearly unstable since only a small mismatch in the market clearing constraint sends the prices to plus or minus infinity. Thus, a stable formulation of the hypothetical market agent's problem must be found. Let us now analyze the hypothetical market agent with the following, slightly altered, optimization problem

$$\begin{aligned} \max_{\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]\in\mathbb{R}^N} \quad & \Psi_M\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right], V\right) \\ \text{s.t.} \quad & \textstyle\sum_{c\in C} V_c + \sum_{p\in P} V_p = 0 \\ & \mu_M = 0, \end{aligned} \tag{3.1}$$

where $\mu_M$ denotes the dual variables of the equality constraint in (3.1). It is trivial to check that the optimality conditions for (3.1) correspond to (2.58). (3.1) is clearly stable, because the market clearing constraint is satisfied precisely. The equality constraint on the dual variables ensures that the optimal solution remains the same if the market clearing constraint is removed after calculation of the optimal solution. (3.1) is, in fact, used as the definition of the hypothetical market agent in the rest of this chapter.

### 3.1.2 Calculation of the Nash equilibrium

Since Assumption 2.1.1 implies the Slater condition, it follows that necessary and sufficient conditions for all $v_k$, $k \in P \cup C$ and $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]$ to constitute an NE are,

$$\begin{aligned} -\mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_k\right]^\top - \lambda_k Q_k v_k - B_k^\top \eta_k - A_k^\top \mu_k &= 0, \\ \eta_k^\top\left(B_k v_k - b_k\right) &= 0, \\ B_k v_k - b_k &\leq 0, \\ A_k v_k - a_k &= 0, \\ \eta_k &\geq 0, \\ \textstyle\sum_{c\in C} V_c + \sum_{p\in P} V_p &= 0. \end{aligned} \tag{3.2}$$

The last equation corresponds to the market clearing constraint (2.31) as well as the KKT conditions of the hypothetical market agent.

We can now interpret (3.2) as the set of KKT conditions of one large optimization problem. To see this, we combine all of the decision variables into one vector $x := \left[v^\top, \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]^\top\right]^\top$ and rewrite

- the equality constraints as $Ax = a$ with $a := \left[a_{p_1}^\top, ..., a_{p_{|P|}}^\top, a_{c_1}^\top, ..., a_{c_{|C|}}^\top, \underbrace{0, ..., 0}_{N}\right]^\top$ where the number of trailing zeros is $N$, and

$$A := \begin{bmatrix} A_{p_1} & 0 & & & & & 0 \\ 0 & \ddots & 0 & & & & \vdots \\ & 0 & A_{p_{|P|}} & 0 & & & 0 \\ & & 0 & A_{c_1} & 0 & & 0 \\ & & & 0 & \ddots & 0 & \vdots \\ & & & & 0 & A_{c_{|C|}} & 0 \\ M_{p_1} & \cdots & M_{p_{|P|}} & I & \cdots & I & 0 \end{bmatrix},$$

where $M_p \in \mathbb{R}^N \times \mathbb{R}^{\dim v_p}$ is a matrix defined as

$$M_p = \left[ \operatorname{diag}\left( \underbrace{1, ..., 1}_{N} \right) \middle| \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} \right],$$

- the inequality constraints as $Bx \leq b$ with $b := \left[ b_{p_1}^\top, ..., b_{p_{|P|}}^\top, b_{c_1}^\top, ..., b_{c_{|C|}}^\top \right]^\top$, and

$$B := \begin{bmatrix} B_{p_1} & 0 & & & & & 0 \\ 0 & \ddots & 0 & & & & \vdots \\ & 0 & B_{p_{|P|}} & 0 & & & 0 \\ & & 0 & B_{c_1} & 0 & & 0 \\ & & & 0 & \ddots & 0 & \vdots \\ & & & & 0 & B_{c_{|C|}} & 0 \end{bmatrix},$$

- the objective function as $-\pi^\top x - \frac{1}{2} x^\top Q x$, with

$$\pi := \left[ \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \pi_{0,p_1} \right]^\top, ..., \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \pi_{0,p_{|P|}} \right]^\top, \underbrace{0, ..., 0}_{(|C|+1)N} \right]^\top,$$

where $\pi_{0,p}$ is $\pi_p$ with the elements of $\Pi$ set to zero, and

$$Q := \begin{bmatrix} \lambda_{p_1} Q_{p_1} & 0 & & & & & & M_{p_1}^\top \\ 0 & \ddots & 0 & & & & & \vdots \\ & 0 & \lambda_{p_{|P|}} Q_{p_{|P|}} & 0 & & & & M_{p_{|P|}}^\top \\ & & 0 & \lambda_{c_1} Q_{c_1} & 0 & & & I \\ & & & 0 & \ddots & 0 & & \vdots \\ & & & & 0 & \lambda_{c_{|C|}} Q_{c_{|C|}} & & I \\ M_{p_1} & \cdots & M_{p_{|P|}} & I & \cdots & I & & 0 \end{bmatrix}, \tag{3.3}$$

- the dual variables as

$$\eta := \left[\eta_{p_1}^\top, ..., \eta_{p_{|P|}}^\top, \eta_{c_1}^\top, ..., \eta_{c_{|C|}}^\top\right],$$

and

$$\mu := \left[\mu_{p_1}^\top, ..., \mu_{p_{|P|}}^\top, \mu_{c_1}^\top, ..., \mu_{c_{|C|}}^\top, \mu_M^\top\right].$$

In this setting we can reformulate the KKT conditions (3.2) as follows,

$$\begin{aligned} -\pi - Qx - B^\top \eta - A^\top \mu &= 0, \\ \eta^\top (Bx - b) &= 0, \\ Bx - b &\leq 0, \\ Ax - a &= 0, \\ \eta &\geq 0, \\ \mu_M &= 0. \end{aligned} \tag{3.4}$$

**Lemma 3.1.1.** *$Q \succeq 0$ for all $x$ that satisfy the market clearing constraint (2.31).*

*Proof.* We have

$$
\begin{aligned}
x^\top Q x &= v^\top \begin{bmatrix} \lambda_{p_1} Q_{p_1} & 0 & & & & \\ 0 & \ddots & 0 & & & \\ & 0 & \lambda_{p_{|P|}} Q_{p_{|P|}} & 0 & & \\ & & 0 & \lambda_{c_1} Q_{c_1} & 0 & \\ & & & 0 & \ddots & 0 \\ & & & & 0 & \lambda_{c_{|C|}} Q_{c_{|C|}} \end{bmatrix} v \\
&\quad + 2\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^\top \left( \textstyle\sum_{c \in C} V_c + \sum_{p \in P} V_p \right) \\
&= \textstyle\sum_{p \in P} \lambda_p v_p^\top Q_p v_p + \sum_{c \in C} \lambda_c v_c^\top Q_c v_c \\
&\geq 0,
\end{aligned}
$$

since $\lambda_p > 0$, $\lambda_c > 0$, $Q_c \succ 0$, $Q_p \succeq 0$ for all $p \in P$ and $c \in C$. ☐

**Proposition 3.1.2.** *The KKT conditions (3.4) are equivalent to the KKT conditions (3.2).*

*Proof.* We will consider each equation separately. Let us start with $Ax - a = 0$. Writing the equation in matrix form

$$
\begin{bmatrix} A_{p_1} & 0 & & & & & 0 \\ 0 & \ddots & 0 & & & & \vdots \\ & 0 & A_{p_{|P|}} & 0 & & & 0 \\ & & 0 & A_{c_1} & 0 & & 0 \\ & & & 0 & \ddots & 0 & \vdots \\ & & & & 0 & A_{c_{|C|}} & 0 \\ M_{p_1} & \cdots & M_{p_{|P|}} & I & \cdots & I & 0 \end{bmatrix}
\begin{bmatrix} v_{p_1} \\ \vdots \\ v_{p_{|P|}} \\ v_{c_1} \\ \vdots \\ v_{c_{|C|}} \\ \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \end{bmatrix}
-
\begin{bmatrix} a_{p_1} \\ \vdots \\ a_{p_{|P|}} \\ a_{c_1} \\ \vdots \\ a_{c_{|C|}} \\ 0 \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}
$$

leads to

$$\begin{bmatrix} A_{p_1} v_{p_1} \\ \vdots \\ A_{p_{|P|}} v_{p_{|P|}} \\ A_{c_1} v_{c_1} \\ \vdots \\ A_{c_{|C|}} v_{c_{|C|}} \\ \sum_{p\in P} V_p + \sum_{c \in C} V_c \end{bmatrix} - \begin{bmatrix} a_{p_1} \\ \vdots \\ a_{p_{|P|}} \\ a_{c_1} \\ \vdots \\ a_{c_{|C|}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

as required.

We continue with the expression $-\pi - Qx - B^\top \eta - A^\top \mu = 0$. We first focus on $-\pi - Qx$ only. Writing the equation in matrix form

$$-\pi - \begin{bmatrix} \lambda_{p_1} Q_{p_1} & 0 & & & & & M_{p_1}^\top \\ 0 & \ddots & 0 & & & & \vdots \\ & 0 & \lambda_{p_{|P|}} Q_{p_{|P|}} & 0 & & & M_{p_{|P|}}^\top \\ & & 0 & \lambda_{c_1} Q_{c_1} & 0 & & I \\ & & & 0 & \ddots & 0 & \vdots \\ & & & & 0 & \lambda_{c_{|C|}} Q_{c_{|C|}} & I \\ M_{p_1} & \cdots & M_{p_{|P|}} & I & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} v_{p_1} \\ \vdots \\ v_{p_{|P|}} \\ v_{c_1} \\ \vdots \\ v_{c_{|C|}} \\ \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \end{bmatrix}$$

$$= - \begin{bmatrix} \mathbb{E}^{\tilde{\mathbb{P}}}[\pi_{0,p_1}] \\ \vdots \\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_{0,p_{|P|}}\right] \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \lambda_{p_1} Q_{p_1} v_{p_1} + M_{p_1}^\top \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \\ \vdots \\ \lambda_{p_{|P|}} Q_{p_{|P|}} v_{p_{|P|}} + M_{p_{|P|}}^\top \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \\ \lambda_{c_1} Q_{c_1} v_{c_1} + \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \\ \vdots \\ \lambda_{c_{|C|}} Q_{c_{|C|}} v_{c_{|C|}} + \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \\ \sum_{p\in P} V_p + \sum_{c\in C} V_c \end{bmatrix},$$

and noting that $-\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_{0,p}] - M_p^\top \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] = -\mathbb{E}^{\tilde{\mathbb{P}}}[\pi_p]$ for all $p \in P$, leads to

$$-\pi - Qx = -\begin{bmatrix} \lambda_{p_1} Q_{p_1} v_{p_1} + \mathbb{E}^{\tilde{\mathbb{P}}}[\pi_{p_1}] \\ \vdots \\ \lambda_{p_{|P|}} Q_{p_{|P|}} v_{p_{|P|}} + \mathbb{E}^{\tilde{\mathbb{P}}}[\pi_{p_P}] \\ \lambda_{c_1} Q_{c_1} v_{c_1} + \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \\ \vdots \\ \lambda_{c_{|C|}} Q_{c_{|C|}} v_{c_{|C|}} + \mathbb{E}^{\tilde{\mathbb{P}}}[\Pi] \\ \sum_{p\in P} V_p + \sum_{c\in C} V_c \end{bmatrix}.$$

Similarly, writing the remainder $-B^\top \eta - A^\top \mu$ in matrix form leads to

$$-\begin{bmatrix} B_{p_1}^\top & 0 & & & & & \\ 0 & \ddots & 0 & & & & \\ & 0 & B_{p_{|P|}}^\top & 0 & & & \\ & & 0 & B_{c_1}^\top & 0 & & \\ & & & 0 & \ddots & 0 & \\ & & & & 0 & B_{c_{|C|}}^\top & \\ 0 & & 0 & 0 & & 0 & \end{bmatrix} \begin{bmatrix} \eta_{p_1} \\ \vdots \\ \eta_{p_{|P|}} \\ \eta_{c_1} \\ \vdots \\ \eta_{c_{|C|}} \end{bmatrix} = -\begin{bmatrix} B_{p_1}^\top \eta_{p_1} \\ \vdots \\ B_{p_{|P|}}^\top \eta_{p_{|P|}} \\ B_{c_1}^\top \eta_{c_1} \\ \vdots \\ B_{c_{|C|}}^\top \eta_{c_{|C|}} \\ 0 \end{bmatrix}$$

and

$$-\begin{bmatrix} A_{p_1}^\top & 0 & & & & & M_{p_1}^\top \\ 0 & \ddots & 0 & & & & \vdots \\ & 0 & A_{p_{|P|}}^\top & 0 & & & M_{p_{|P|}}^\top \\ & & 0 & A_{c_1}^\top & 0 & & I \\ & & & 0 & \ddots & 0 & \vdots \\ & & & & 0 & A_{c_{|C|}}^\top & I \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{p_1} \\ \vdots \\ \mu_{p_{|P|}} \\ \mu_{c_1} \\ \vdots \\ \mu_{c_{|C|}} \\ \mu_M \end{bmatrix} = -\begin{bmatrix} A_{p_1}^\top \mu_{p_1} \\ \vdots \\ A_{p_{|P|}}^\top \mu_{p_{|P|}} \\ A_{c_1}^\top \mu_{c_1} \\ \vdots \\ A_{c_{|C|}}^\top \mu_{c_{|C|}} \\ 0 \end{bmatrix}$$

where $\mu_M = 0$ was used. Combining these results, we have

$$-\begin{bmatrix} \lambda_{p_1} Q_{p_1} v_{p_1} + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_{p_1}\right] \\ \vdots \\ \lambda_{p_{|P|}} Q_{p_{|P|}} v_{p_{|P|}} + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_{p_{|P|}}\right] \\ \lambda_{c_1} Q_C v_{c_1} + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right] \\ \vdots \\ \lambda_{c_{|C|}} Q_C v_{c_{|C|}} + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right] \\ \sum_{p\in P} V_p + \sum_{c\in C} V_c \end{bmatrix} - \begin{bmatrix} B_{p_1}^\top \eta_{p_1} \\ \vdots \\ B_{p_{|P|}}^\top \eta_{p_{|P|}} \\ B_{c_1}^\top \eta_{c_1} \\ \vdots \\ B_{c_{|C|}}^\top \eta_{c_{|C|}} \\ 0 \end{bmatrix} - \begin{bmatrix} A_{p_1}^\top \mu_{p_1} \\ \vdots \\ A_{p_{|P|}}^\top \mu_{p_{|P|}} \\ A_{c_1}^\top \mu_{c_1} \\ \vdots \\ A_{c_{|C|}}^\top \mu_{c_{|C|}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

as required.

The proofs for $\eta^\top (Bx - b) = 0$, $Bx - b \leq 0$ and $\eta \geq 0$ are similar. $\square$

Since the additional constraints $\mu_M = 0$ on the dual variables of (3.4) cannot be handled by most of the available quadratic programming software solvers, we reformulate the problem in a dual form. We start by formulating the optimization problem using the KKT conditions (3.4) as

$$\begin{aligned} \max_x \quad & -\pi^\top x - \tfrac{1}{2} x^\top Q x \\ \text{s.t.} \quad & Ax = a \\ & Bx \leq b \\ & \mu_M = 0 \end{aligned} \tag{3.5}$$

and define the Lagrangian as

$$\mathcal{L}(x, \mu, \eta) = \begin{cases} -\tfrac{1}{2} x^\top Q x - \pi^\top x - (Ax - a)^\top \mu - (Bx - b)^\top \eta & \text{if } \eta \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

By the virtue of Lemma 3.1.1, $Q \succeq 0$, and $\mathcal{L}(x, \mu, \eta)$ is therefore a smooth and concave function. The unconstrained maximizer can be determined by setting $\mathcal{D}_x \mathcal{L}(x, \mu, \eta) = 0$.

Calculating

$$\mathcal{D}_x \mathcal{L}(x, \mu, \eta) = -Qx - \pi - A^\top \mu - B^\top \eta$$

and inserting $\pi$ back into the Lagrangian, an equivalent formulation is obtained as follows,

$$\mathcal{L}(x, \mu, \eta) = \begin{cases} \frac{1}{2} x^\top Q x + a^\top \mu + b^\top \eta & \text{if } \eta \geq 0 \text{ and } -Qx - \pi - A^\top \mu - B^\top \eta = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Relating the latter to a maximization optimization problem, the following formulation is obtained

$$\begin{aligned} \max_{x,\mu,\eta} \quad & -\tfrac{1}{2} x^\top Q x - \mu^\top a - \eta^\top b \\ \text{s.t.} \quad & Qx + A^\top \mu + B^\top \lambda + \pi = 0 \\ & \eta \geq 0 \\ & \mu_M = 0. \end{aligned} \tag{3.6}$$

(3.6) is equivalent to (3.5), but it can be solved using any quadratic programming algorithm. The numerical results in this thesis are calculated using Gurobi (see Gurobi Optimization (2014)).

## 3.2 Extensions of the model

In this section we describe a few realistic extensions of the model from Section 2.1, that are needed to make the model applicable in practice. Note that all extensions can be incorporated into the quadratic programming framework, and thus the reformulation of Section 3.1 still applies.

### 3.2.1 Futures contracts

Thus far, we have assumed that market participants trade only forward contracts. A more realistic model would include also futures contracts. In this section we explain how the model can be extended in this regard.

Let $\acute{\Pi}$ and $\grave{\Pi}$ denote the prices of forward and futures contracts. A no-arbitrage argument allows one to calculate $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\grave{\Pi}\right]$ as a solution of the following triangular system of linear equations for each $j \in J$,

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\acute{\Pi}\left(t_i, T_j\right)\right] =$$
$$\begin{cases} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\grave{\Pi}\left(t_i, T_j\right)\right] & i = \max\left\{I_j\right\}, \\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[\grave{\Pi}\left(t_i, T_j\right)\right] + \sum_{k=1}^{\max\{I_j\}-1} \left(\left[\mathbb{E}^{\tilde{\mathbb{P}}}\left[\grave{\Pi}\left(t_{k+1}, T_j\right)\right] - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\grave{\Pi}\left(t_k, T_j\right)\right]\right] \frac{e^{-rt_{k+1}}}{e^{-rT_j}}\right) & i < \max\left\{I_j\right\}. \end{cases}$$

Recall from the introduction that $t_{\max\{I_j\}} = T_j$.

### 3.2.2 Block contracts

In the previous sections, we implicitly assumed that each contract covers one delivery period. As described in Section 2.1.3.1, this is clearly not true in reality, where contracts often cover multiple delivery periods.

A contract that covers many delivery periods $J' \subseteq J$ and is traded at $t_i$, $i \in \cap_{j' \in J'} I_{j'}$ can be incorporated in our model by specifying that

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i, T_{j'}\right)\right] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i, T_{j''}\right)\right] \tag{3.7}$$

and

$$V_k\left(t_i, T_{j'}\right) = V_k\left(t_i, T_{j''}\right) \tag{3.8}$$

for all $k \in P \cup C$ and all $j', j'' \in J'$.

The same reasoning can also be applied to forward and future fuel (gas, coal, oil etc.) and emission contracts.

### 3.2.3 Costs of trading

In the previous sections we assumed that players can change their positions without cost. The reality is of course different, because trading costs are an important factor in trading power. The precise formulation of trading costs depends on the market micro-structure and is a research area itself. In this thesis, we adopt the view of Almgren and Chriss (2001) and assume that the cost of trading (purchasing or selling) one unit of power is approximated by the linear function

$$h\left(V_p\left(t_i, T_j\right)\right) = \epsilon_{ij}\,\mathrm{sign}\left(V_p\left(t_i, T_j\right)\right) + \upsilon_{ij} V_p\left(t_i, T_j\right). \tag{3.9}$$

The first term represents the fixed cost of trading that occurs due to the bid-offer spread and trading fees. A good estimate for $\epsilon_{ij}$ is half of the bid-offer spread, plus the trading fee. The second term approximates the micro-structure of the order book. Selling a large volume $V_p\left(t_i, T_j\right)$ exhausts the supply of liquidity, which causes a short-term decrease in the price. The factor $\upsilon_{ij}$ thus represents half of the change in price caused by selling a volume of $V_p\left(t_i, T_j\right)$. We assume that this decrease in price is only temporary, and that the price returns to the equilibrium level at the next trading time, $t_{i+1}$.

(3.9) represents the total cost of trading a unit of power. The cost of trading a volume $V_p\left(t_i, T_j\right)$ is then

$$V_p\left(t_i, T_j\right) h\left(V_p\left(t_i, T_j\right)\right) = \epsilon_{ij}\left|V_p\left(t_i, T_j\right)\right| + \upsilon_{ij} V_p\left(t_i, T_j\right)^2. \tag{3.10}$$

### 3.2.4 Recourse

Let $\widehat{V}_k\left(T_j\right)$ denote the number of forward electricity contracts that player $k \in P \cup C$ has already traded before taking recourse. Then, for every $p \in P$ and $j \in J$, (2.5) must be replaced by

$$-\sum_{i \in I_j} V_p\left(t_i, T_j\right) - \widehat{V}_p\left(T_j\right) = \sum_{l \in L} \sum_{r \in R^{p,l}} W_{p,l,r}\left(T_j\right).$$

Similarly, for every $c \in C$, (2.10) must be replaced by

$$\sum_{i \in I_j} V_c\left(t_i, T_j\right) + \widehat{V}_c\left(T_j\right) = p_c D\left(T_j\right).$$

Fuel constraints (2.6) and emission constraints (2.7) must be altered in a similar fashion for every producer $p \in P$.

The objective functions of producers and consumers must be altered to reflect the profit or loss that has occurred or will occur due to the trading that happened before recourse was taken. This can be achieved by adding a constant to the objective function, which does not have any impact on the optimal solution. Thus, any alteration of the objective functions can be done via post processing, if the correct optimal value is needed.

## 3.3 Numerical results

In this section we describe numerical results obtained by our model. First we investigate the calibration of power plant parameters from historical production data. Then we study the term structure of electricity prices in a simple setting. We then extend the setting to include block contracts and multiple spot contracts. Finally, we study a closed form relation among equilibrium electricity prices.

### 3.3.1 Calibration

If our model is applied in practice, one has to estimate the physical characteristics of the power plants such as capacity, ramp-up and ramp-down constraints, efficiency and the carbon emission intensity factor.

In the UK, all power plants are required to submit their available capacity as well as ramp-up and ramp-down constraints to the grid operator on a half hourly basis. This data is publicly available at the Elexon website. For a description and a few samples of the data used in this thesis see Appendix A.

The carbon emission intensity factor $g^l$ for fuel $l \in L$ can either be estimated from historical production data (in this case it would be plant-dependent and thus modeled as $g^{p,l,r}$) or obtained from various energy market reports. We tested both approaches, but we did not observe any significant difference in the calibration results. Hence, for simplicity, we employ the latter approach[1] and set $g^{\text{coal}} = 0.973$ $tCO_2$/MWh and $g^{\text{gas}} = 0.412$ $tCO_2$/MWh as given in Platts Report.[2]

A more challenging problem is to estimate the efficiency of each power plant. As we explained in Section 2.1.3.1, all market participants submit their expected production/con-

[1]In what follows we use $tCO_2$/MWh, where $tCO_2$ denotes the amount of $CO_2$ emissions in tones.

[2]For details see `https://www.platts.com/IM.Platts.Content/methodologyreferences/methodologyspecs/european_power_methodology.pdf`.

sumption to the grid operator one hour before the delivery. Our goal is to find the efficiency that best describes the historical production of each plant.

We first normalize the historical production of each power plant by calculating the ratio of production to available capacity for each half hour. We denote the normalized historical production by $\tilde{W}_{p,l,r}(T_j)$ for each delivery period $j \in J$ and power plant $r \in R^{p,l}$. Normalization makes sure that $\tilde{W}_{p,l,r}(T_j) \in [0,1]$. For the purpose of calibration, we assume that producers are risk-neutral and set the risk preference $\lambda_p = 0$ for all $p \in P$. Furthermore, we neglect the ramp-up and ramp-down constraints (2.3). With these simplifications, we see that a power plant will produce at time $T_j$, if and only if, the income from selling electricity at the spot price is greater or equal to the costs of purchasing the required fuel and emission certificates at the current spot price. In other words, for a power plant that runs on fuel $l \in L$ and produces electricity at time $T_j$,

$$\Pi(T_j, T_j) - c^{p,l,r} G_l(T_j, T_j) - g^l G_{\text{em}}(T_j, T_j) \geq 0 \tag{3.11}$$

must hold for production to take place.

It is immediately clear why (3.11) must hold when only spot contracts are available. Let us investigate (3.11) if forward and future electricity contracts are also available. At any trading time $t_i$, $i \in I_j$, a rational producer could simultaneously enter into a short forward electricity contract and a long fuel and emission forward contract if

$$\Pi(t_i, T_j) - c^{p,l,r} G_l(t_i, T_j) - g^l G_{\text{em}}(t_i, T_j) \geq 0. \tag{3.12}$$

At delivery time $T_j$, this producer the has two options:

1. Acquire the fuel and emission certificates bought at trading time $t_i$ and produce electricity. In this case, she earns the following profit

$$\widehat{P_1}(T_j) = \Pi(t_i, T_j) - c^{p,l,r} G_l(t_i, T_j) - g^{p,l,r} G_{\text{em}}(t_i, T_j). \tag{3.13}$$

2. Produce no electricity, but instead, close the forward electricity, fuel and emission

contracts. In this case, she earns the following profit

$$\begin{aligned}\widehat{P_2}(T_j) &= [\Pi(t_i,T_j)-\Pi(T_j,T_j)] - c^{p,l,r}[G_l(t_i,T_j)-G_l(T_j,T_j)] \\ &\quad -g^l[G_{\mathrm{em}}(t_i,T_j)-G_{\mathrm{em}}(T_j,T_j)].\end{aligned} \tag{3.14}$$

Obviously, power plant $r \in R^{p,l}$ will run at $T_j$ if and only if

$$\widehat{P_1}(T_j) \geq \widehat{P_2}(T_j). \tag{3.15}$$

With some reordering of the terms, it is clear that inequality (3.15) is equivalent to inequality (3.11).

To adjust inequality (3.11) for the neglected risk premium and trading costs, we add another term $\tilde{c}^{p,l,r} \geq 0$ so that (3.11) reads

$$\Theta\left(c^{p,l,r}, g^l, \tilde{c}^{p,l,r}; T_j\right) \geq 0, \tag{3.16}$$

where

$$\Theta\left(c^{p,l,r}, g^l, \tilde{c}^{p,l,r}; T_j\right) := \Pi(T_j,T_j) - c^{p,l,r}G_l(T_j,T_j) - g^l G_{\mathrm{em}}(T_j,T_j) - \tilde{c}^{p,l,r}. \tag{3.17}$$

In last step of the calibration, we use $\Theta\left(c^{p,l,r}, g^l, \tilde{c}^{p,l,r}; T_j\right)$ in a logistic regression analysis. The efficiency $c^{p,l,r}$ that best explain the historical production of power plant $r \in R^{p,l}$ is found as an optimal solution to

$$\min_{c^{p,l,r},\tilde{c}^{p,l,r}} \sum_{j\in J}\left(\tilde{W}_{p,l,r}(T_j) - \frac{1}{1+\exp\left(-\Theta\left(c^{p,l,r}, g^l, \tilde{c}^{p,l,r}; T_j\right)\right)}\right)^2. \tag{3.18}$$

For each power plant we used over 5 000 training samples obtained from the period between 01-Jan-2012 and 01-Jan-2013.

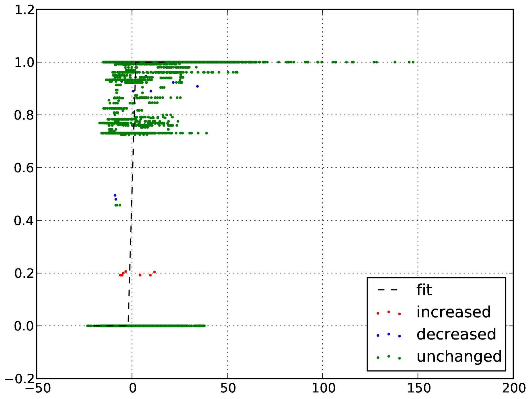
Figures 3.1 and 3.2 show a subset of the calibration results for coal and gas power plants. We see that power plants base their production decisions on the value of $\Theta\left(c^{p,l,r}, g^l, \tilde{c}^{p,l,r}; T_j\right)$: they produce more when their profit is higher and less when it is lower. However, it is also clear that $\Theta\left(c^{p,l,r}, g^{l}, \tilde{c}^{p,l,r}; T_j\right)$ cannot contain all of the information used to determine the production schedule in reality. In our model in Section 2.1, we have already seen that power plants have ramp-up and ramp-down constraints which were not taken into account in the current calibration exercise. These constraints restrict power plants' ability to respond immediately to price signals. Another important factor not taken into account in this model, is startup costs. If a power plants is not running, then a significant costs occur when it is turned on. Thus, power plants sometimes produce even when it is not profitable for them to do so, because production helps to avoid startup costs. Similarly, a power plant will only start production if the operators believe that their profit will exceed the cost of starting the plant. Similar to ramp-up and ramp-down constraints, startup costs also prevent power plants from responding immediately to price signals. This explains the multiple green horizontal lines in Figure 3.1. This effect is more visible for coal power plants, since they have tighter ramping constraints and higher startup costs. Startup costs are addressed in Chapter 4.

In this section, we apply our model to the entire system of UK power plants. We focus on coal, gas and oil power plants, because these plants adapt their production to cover changes in demand, and are thus responsible for setting the electricity price. Nuclear power plants do not have to be modeled explicitly, because their ramp-up and ramp-down constraints are so tight that their production is almost constant. They usually deviate from their maximum production only for maintenance reasons. Renewable sources and interconnectors are not modeled explicitly, because they require a different treatment not discussed in this thesis. In this section, we define demand $D\left(T_j\right)$ for all $j \in J$ as

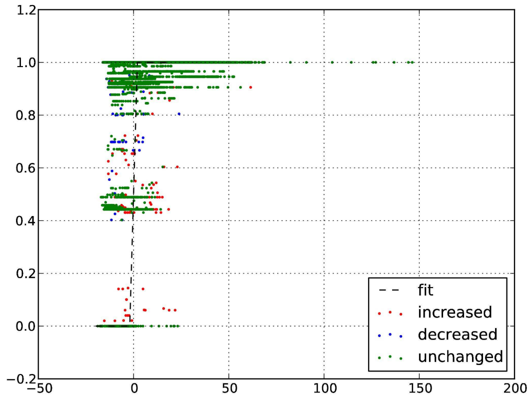
$$D\left(T_j\right) := D_{\text{act}}\left(T_j\right) - P_{\text{renw}}\left(T_j\right) - P_{\text{inter}}\left(T_j\right) \tag{3.19}$$

where $D_{\text{act}}\left(T_j\right)$ denotes the actual demand on the system, $P_{\text{renw}}\left(T_j\right)$ denotes the production from all renewable sources including wind, solar, biomass, hydro and pumped storage, and $P_{\text{inter}}\left(T_j\right)$ denotes the inflow of power into the UK from other countries through interconnectors[3]. To make this model useful in practice one should really model each of these terms

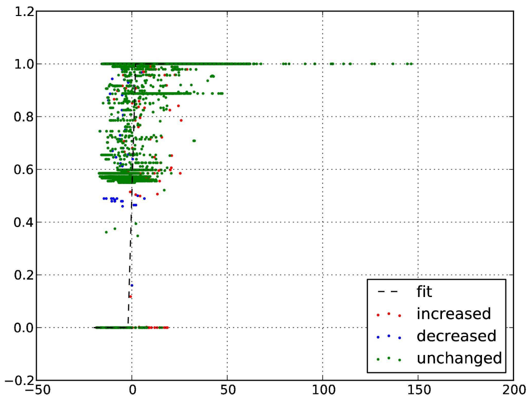
[3]See Appendix A for a description of the existing interconnectors in the UK.



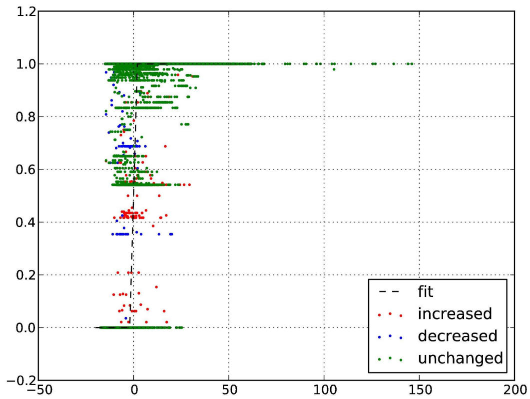

(a) BMU: T_ABTH7.




(b) BMU: T_COTPS-4.

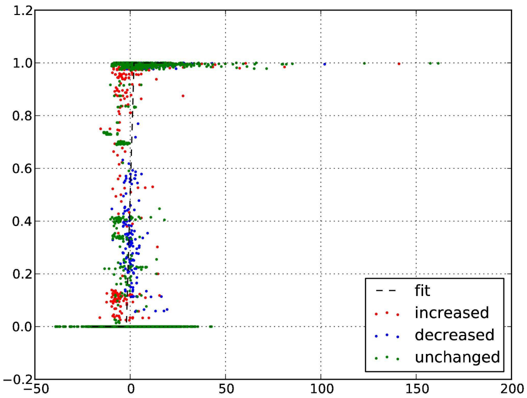



(c) BMU: T_RATS-3.

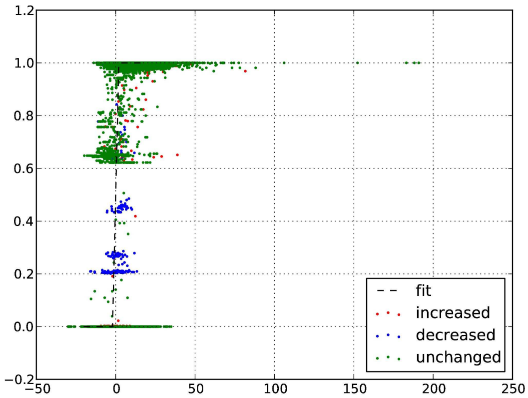



(d) BMU: T_WBUPS-4.

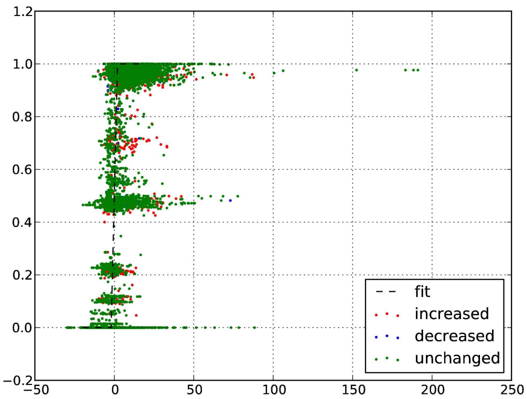
Figure 3.1: The goodness-of-fit for four coal power plants. The $x$-axis on the graphs represents the value of $\Theta\left(c^{p,l,r}, g^{l}, \tilde{c}^{p,l,r}; T_j\right)$ as defined in (3.17) and the $y$-axis represents $\tilde{W}_{p,l,r}\left(T_j\right)$. Each point in the graph corresponds to one delivery period $T_j$. Colors are used to denote the relationship between $\tilde{W}_{p,l,r}\left(T_{j+1}\right)$ and $\tilde{W}_{p,l,r}\left(T_j\right)$. If $\tilde{W}_{p,l,r}\left(T_j\right) = \tilde{W}_{p,l,r}\left(T_{j+1}\right)$, then green color is used. Similarly, if $\tilde{W}_{p,l,r}\left(T_j\right) < \tilde{W}_{p,l,r}\left(T_{j+1}\right)$, then red color is used. Otherwise, blue color is used. Power plants can be uniquely identified by their Balancing Mechanism Unit (BMU). Ramping constraints of the above and also the rest of coal power plants are listed in Appendix C.



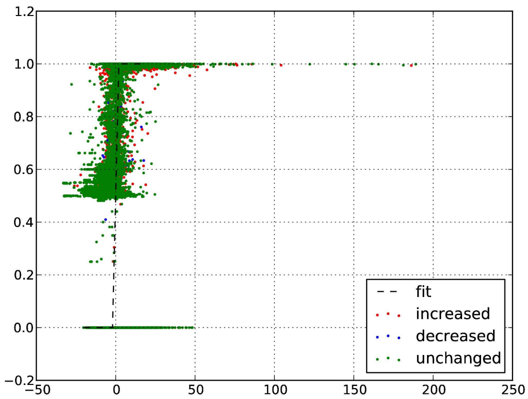

(a) BMU: T_BARKB2.




(b) BMU: T_GRAI-6.




(c) BMU: T_LAGA-1.




(d) BMU: T_ROCK-1.

Figure 3.2: The goodness-of-fit for four gas power plants. The $x$-axis on the graphs represents the value of $\Theta\left(c^{p,l,r}, g^l, \tilde{c}^{p,l,r}; T_j\right)$ as defined in (3.17) and the $y$-axis represents $\tilde{W}_{p,l,r}\left(T_j\right)$. Each point in the graph corresponds to one delivery period $T_j$. Colors are used to denote the relationship between $\tilde{W}_{p,l,r}\left(T_{j+1}\right)$ and $\tilde{W}_{p,l,r}\left(T_j\right)$. If $\tilde{W}_{p,l,r}\left(T_j\right) = \tilde{W}_{p,l,r}\left(T_{j+1}\right)$, a green dot is used. Similarly, if $\tilde{W}_{p,l,r}\left(T_j\right) < \tilde{W}_{p,l,r}\left(T_{j+1}\right)$, we use red. Otherwise, blue is used. Power plants can be uniquely identified by their Balancing Mechanism Unit (BMU). Ramping constraints of the above and also the rest of gas power plants are listed in Appendix C.

separately, but this is beyond the scope of this thesis.

We calibrate the model and for each power plant $r \in R^{p,l}$, $l \in L$ and $p \in P$, estimate the efficiency $c^{p,l,r}$, the maximum capacity $\overline{W}^{p,l,r}_{\max}$, the ramp-up rate $\triangle\overline{W}^{p,l,r}_{\max}$, and the ramp-down rate $\triangle\overline{W}^{p,l,r}_{\min}$ using historical production data as described above. The covariance matrices were also estimated from historical data using the shrinkage approach described in Ledoit and Wolf (2003).

### 3.3.2 Term structure of the price

In this section, we study the term structure of the electricity price. We focus on a simple problem with one delivery period and five trading periods. Our goal is to calculate the electricity term structure with the information available on 11-Feb-2013. We are interested in the half hourly delivery period from 04-Apr-2013 00:00 to 04-Apr-2013 00:30. Trading periods are labelled with a number from 1 to 5 corresponding, in sequence, to two month-ahead, one month-ahead, one week-ahead, one day-ahead and the spot price. We use the future prices of coal, gas and oil as available on 11-Feb-2013. Since the historical demand forecast is not available, we use realized demand as a proxy for this, which is standard practice in the literature. By doing so, we neglect the impact that potential errors in the demand forecast have on the electricity price. For example, if a day-ahead demand forecast is poor, then inefficient and expensive (but flexible) Open Cycle Gas Turbines, must be turned on a few hours before delivery to satisfy the extra demand. This usually causes a high increase in the spot electricity price. By using realized demand as a proxy for the demand forecast, this effect is not properly captured. To use our model in practice, one could use a demand forecast available at the Elexon web page or develop a new approach. Since we do not have information about plant ownership, we assume that there is only one producer who owns all of the plants connected to the UK grid, and only one consumer that is responsible for satisfying the demand of the end users. In reality, market participants have more information about ownership that can be incorporated into the model. To keep the effects of the various parameters of the model on the term structure of the electricity price clearly visible, we set the risk-free interest rate $\hat{r}$ to zero.

As described in the previous section, the covariance matrix is estimated from historical data. However, the contracts used in this example cover only one delivery period and thus, apart from spot contracts, they are not traded in the market. The covariance matrix in this example is estimated from the corresponding, actually traded, baseload contracts that span multiple delivery periods instead.

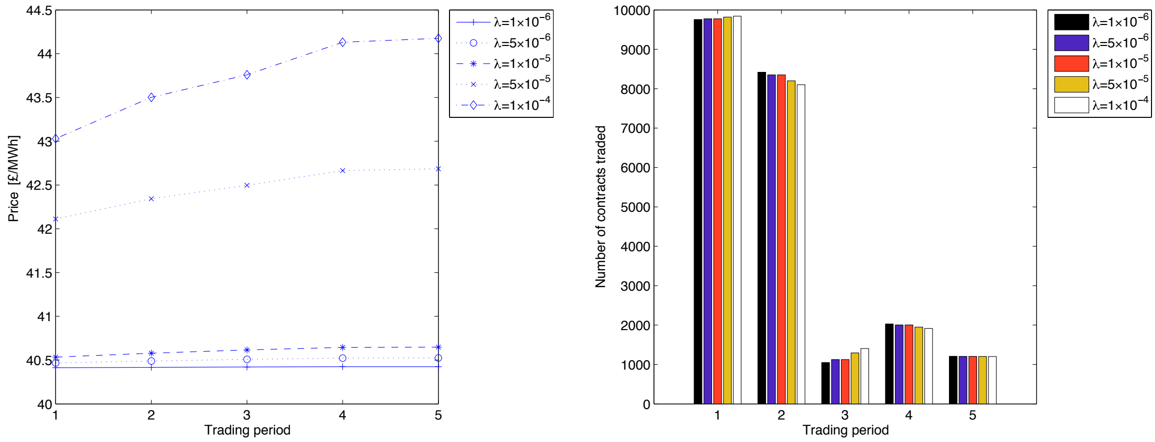
From historical data, we know that the spot price on 04-Apr-2013 00:00 was £40.33/MWh, which broadly determines the ranges of the parameters studied.

Figure 3.3 shows how the term structure of the electricity price and the number of contracts traded depend on the risk preference of the players involved. We see that the market is usually in normal backwardation (i.e. the term structure is upward sloping) and that the slope and the price increase when the risk preference of players increases. The increase in the price is due to the increased risk premium of the producers; the risk premium for risky contracts is obviously higher than that of less risky contracts.

Figure 3.3b shows that most trading happens in the first two periods through the two-month-ahead and month-ahead contracts. This is due to the fact that monthly contracts have a much smaller volatility, and are thus preferred by risk averse investors. However, we can see that the volume traded in the third trading period is slightly smaller than that traded in the fourth and fifth periods. This seems counterintuitive, because the volatility of a week-ahead contract is smaller than that of day-ahead and spot contracts. A more detailed investigation shows, that this effect is a consequence of the small (statistically insignificant) negative correlation between the week-ahead electricity contract and the two-months-ahead gas contract. Even though the correlation is small, it still has a visible impact on the number of contracts traded. Note that the number of contracts traded is shown from the perspective of a consumer; from a producer's perspective the number is the same, but with the opposite sign.

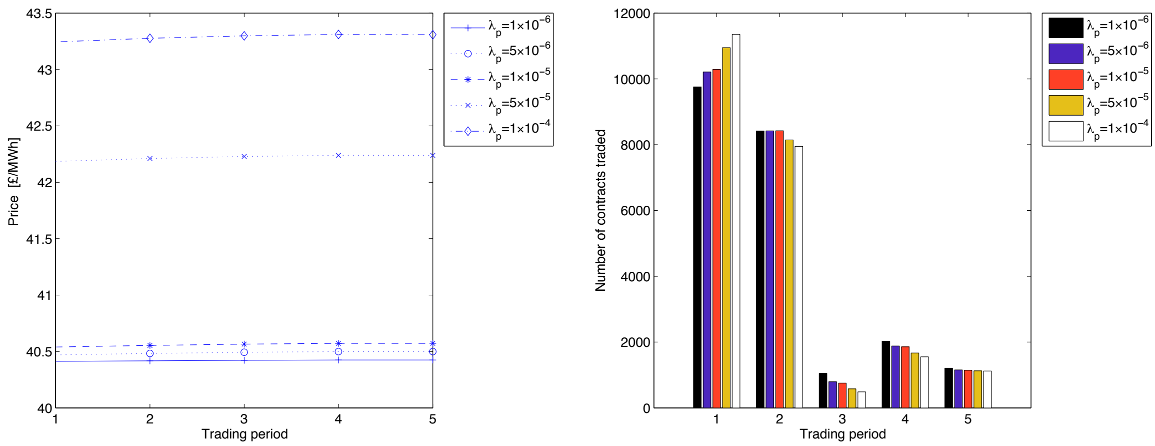
In Figure 3.4, we study what happens when only the risk preference of producers is increased, while that of consumers is kept constant. As we can see, the risk preference of producers does not have any significant impact on the shape of the term structure of the price. However, the price of all contracts increases significantly due to the increased risk premium of producers.

In Figure 3.5, we study the converse: the risk preference of consumers is increased while



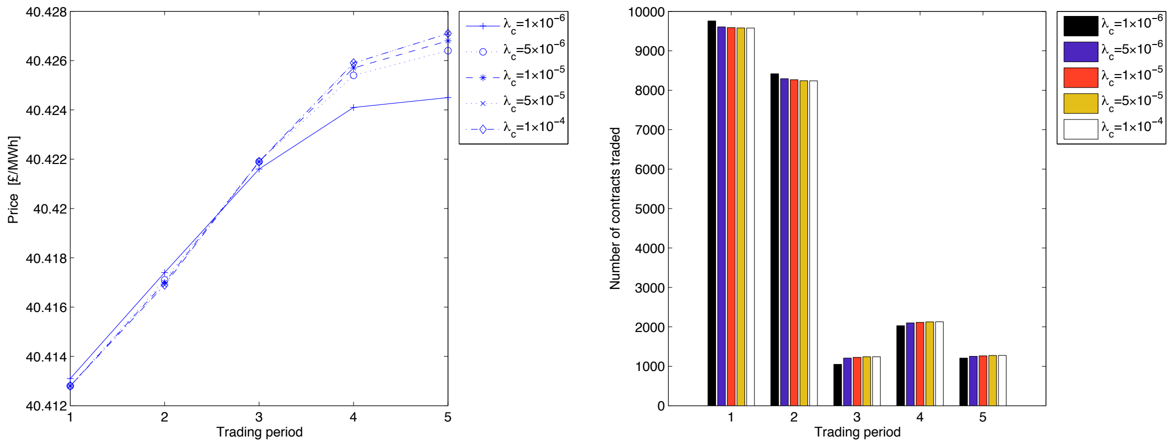

(a) (b)

Figure 3.3: $\upsilon_{i1} = 0$, $\epsilon_{i1} = 0$ for all $i \in \{1, ..., 5\}$, and increasing $\lambda_k$ for all $k \in P \cup C$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.




(a) (b)

Figure 3.4: $\upsilon_{i1} = 0$, $\epsilon_{i1} = 0$ for all $i \in \{1, ..., 5\}$, $\lambda_c = 10^{-6}$ for all $c \in C$, and increasing $\lambda_p$ for all $p \in P$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.



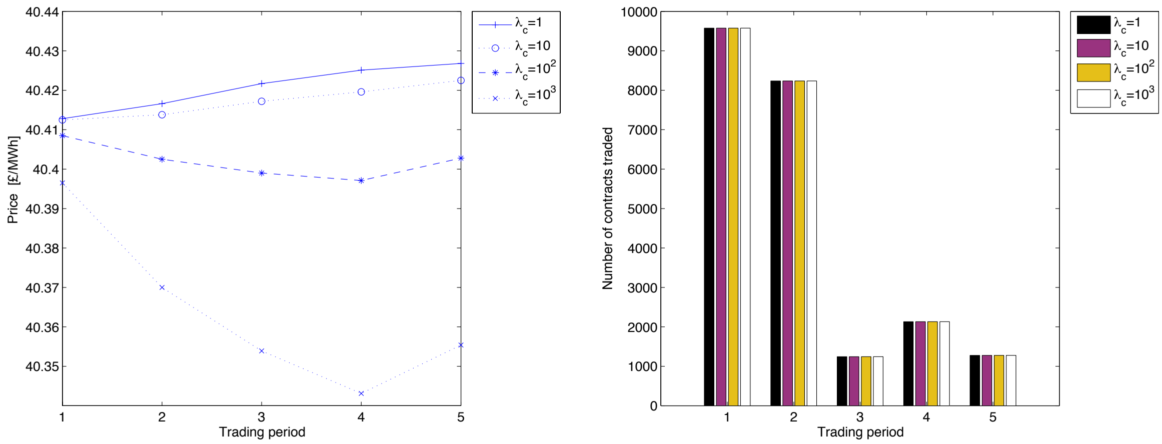

(a) (b)

Figure 3.5: $\upsilon_{i1} = 0$, $\epsilon_{i1} = 0$ for all $i \in \{1, ..., 5\}$, $\lambda_p = 10^{-6}$ for all $p \in P$, and increasing $\lambda_c$ for all $c \in C$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

that of producers is kept constant. We see that this has a small impact on the term structure of the price. In contrast with producers, the increased risk premium of consumers is not reflected in an increased price. Thus, consumers can only maintain their profitability by increasing the fixed price they charge to end users.

One must increase the consumers' risk preference parameter $\lambda_c$ significantly to observe a qualitatively different impact. Such a hypothetical case is depicted in Figure 3.6. We see that the term structure of electricity prices changes from the normal backwardation (i.e. upward sloping) to contango (i.e. downward sloping) when the risk preference of consumers significantly increases. An increased risk preference increases consumers' interest in less risky contracts and decreases their interest in more risky contracts. Since the contracts closer to delivery are usually riskier, the term structure flips from normal backwardation to contango. A small kink at the fourth trading period is the consequence of a calibration of the covariance matrix from historical values.

It may not be immediately clear why the change in the risk preference of producers and consumers has such a different impact on the term structure of the price and why the level of the price is defined by producers only. Let us consider a simplified setting in which there




(a) (b)

Figure 3.6: $\upsilon_{i1} = 0$, $\epsilon_{i1} = 0$ for all $i \in \{1, ..., 5\}$, $\lambda_p = 10^{-6}$ for all $p \in P$, and increasing $\lambda_c$ for all $c \in C$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

is one trading period and one delivery period. Suppose there is just one producer and one consumer. In such a situation, the profit of the consumer can be calculated as

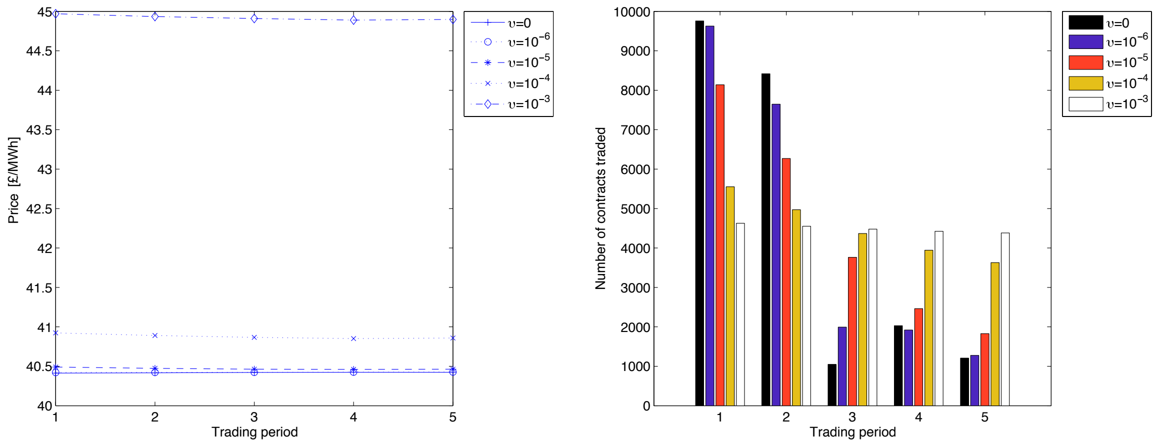
$$P_c(V_c, \Pi) = e^{-\hat{r}T_1}\left(-\Pi(t_1, T_1) + s_c(T_1)\right) D(T_1) \tag{3.20}$$

where the constraint $V_c(t_1, T_1) = D(T_1)$ was used. We see that the profit function does not depend on the consumer's decision variables $V_c$. Thus the consumer does not have any power to influence the price. She has to buy a sufficient amount of electricity to satisfy the demand of her end users, regardless of the price. On the other hand, the producer has much more flexibility. Her profit can be calculated as

$$\begin{aligned} P_p(V_p, \Pi) = e^{-\hat{r}T_1}\Bigg[ & -\Pi(t_1, T_1) V_p(t_1, T_1) - G_{\text{em}}(t_1, T_1) O_p(t_1, T_1) \\ & - \sum_{l \in L} G_l(t_1, T_1) F_{p,l}(t_1, T_1)\Bigg]. \end{aligned} \tag{3.21}$$

Obviously a feasible solution to her optimization problem (2.8) is

$$V_p(t_1, T_1) = O_p(t_1, T_1) = F_{p,l}(t_1, T_1) = 0 \tag{3.22}$$




(a) (b)

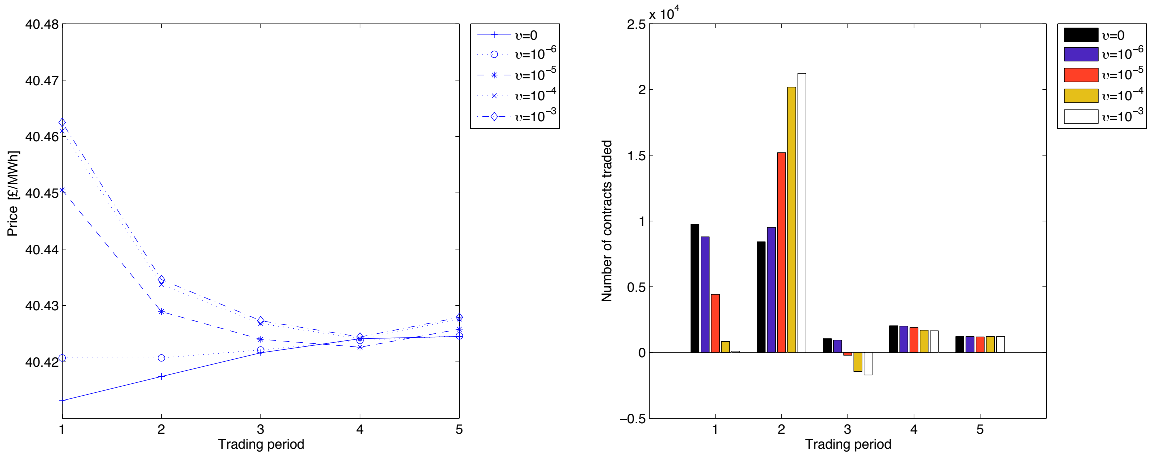
Figure 3.7: $\lambda_k = 10^{-6}$ for all $k \in P \cup C$, $\epsilon_{i1} = 0$ and increasing $\upsilon_{i1}$ for all $i \in \{1, ..., 5\}$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

for all $l \in L$. This leads to the objective value $\Phi_p = 0$. Thus, clearly, for any given price $\Pi(t_1, T_1)$, the objective value satisfies $\Phi_p \geq 0$. A producer will only produce power if the price is high enough to cover all production and trading costs, and the risk premium. Similar reasoning can be extended to a multi-period setting.

In Figures 3.7, 3.8, 3.9 and 3.10, we study the impact of liquidity on the term structure of the electricity price. In Figures 3.7 and 3.8 the impact of the market micro-structure (the quadratic term $\upsilon$) is examined, while in Figures 3.9 and 3.10 the impact of the bid-offer spread and trading fees (the linear term $\epsilon$) is depicted.

Increasing $\upsilon$ for all of the contracts simultaneously as in Figure 3.7, increases the price without changing the shape of the term structure. The increase of $\upsilon$ has a large impact on the optimal trading strategy of the players. When $\upsilon$ is large, players spread the number of contracts traded in each period equally among all of the available trading periods.

When $\upsilon$ is changed for only the first trading period (the two-month-ahead contract), there is a significant change in the term structure of the price. As we see in Figure 3.8, such an increase in $\upsilon$ changes the term structure from normal backwardation to contango. As expected, the players also trade a much smaller number of contracts in the first trading



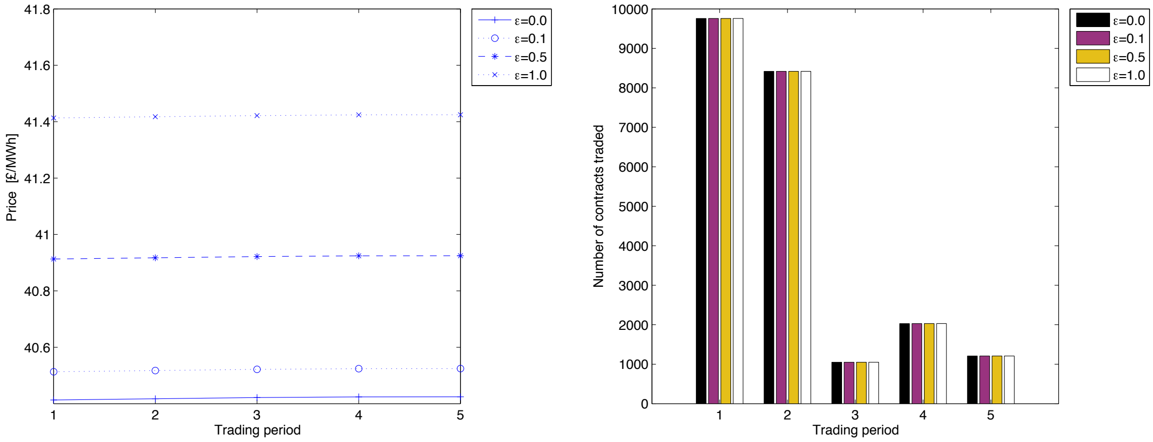

(a) (b)

Figure 3.8: $\epsilon_{i1} = 0$ for all $i \in \{1, ..., 5\}$, $\lambda_k = 10^{-6}$ for all $k \in P \cup C$, and increasing $\upsilon_{11}$. For the other trading periods $i \in \{2, ..., 5\}$, $\upsilon_{i1} = 0$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

period. A small negative trading position is a consequence of a small (statistically insignificant) negative correlation between the week-ahead electricity contract and the week-ahead gas contract, as mentioned in the beginning of this section.

An effect of a simultaneous change in the linear trading costs $\epsilon$ for all contracts, is depicted in Figure 3.9. We see that this change has no effect on the trading strategy or the shape of the term structure. However, the price of all contracts is increased to cover producers' losses resulting from the increased linear cost of trading. Consumers cover their losses from an increased linear cost of trading by increasing the prices they charge end users. A simultaneous change in the linear trading cost $\epsilon$ for all the contracts evidently does not impact the optimal trading strategy of the players.

Changing $\epsilon$ for only the first trading period (the two-month-ahead contract) has a significantly effects on price in the first trading period. The price of the two-month-ahead contract is increased to cover producers' losses resulting from increased linear cost of trading. If $\epsilon$ is increased significantly, this makes the two-month-ahead contract so unattractive that no contracts are traded (as in Figure 3.10b). As already mentioned, a small negative trading




(a) (b)

Figure 3.9: $\lambda_k = 10^{-6}$ for all $k \in P \cup C$, $\upsilon_{i1} = 0$ and increasing $\epsilon_{i1}$ for all $i \in \{1, ..., 5\}$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

position is a consequence of a small (statistically insignificant) negative correlation between the week-ahead electricity contract and the week-ahead gas contract.

In the remainder of this section, we investigate the effect of the term structure of fuel and emission prices on the term structure of electricity prices. Initially, we set[4]

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}}\left[G_1\left(t_1, T_1\right)\right] &= \ldots = \mathbb{E}^{\tilde{\mathbb{P}}}\left[G_1\left(t_5, T_1\right)\right] = 69.30 \text{ [p/therm]}, \\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[G_2\left(t_1, T_1\right)\right] &= \ldots = \mathbb{E}^{\tilde{\mathbb{P}}}\left[G_2\left(t_5, T_1\right)\right] = 57.87 \text{ [£/tonne]}, \\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[G_{\text{em}}\left(t_1, T_1\right)\right] &= \ldots = \mathbb{E}^{\tilde{\mathbb{P}}}\left[G_{\text{em}}\left(t_5, T_1\right)\right] = 3.883 \text{ [£/tonne]}, \end{aligned} \tag{3.23}$$

where $l = 1$ and $l = 2$ denote gas and coal prices. The electricity price is quoted in £/MWh.

We conduct two types of experiments: in the first, we perform parallel shifts of the term structure of fuel/emission prices. In the second, we change the shape of the term structure of fuel/emission prices. Since the results for all fuels and emissions are very similar, we present them for gas only.

Figure 3.11 depicts the effect of various parallel shifts in the term structure of gas prices. As expected, an increase/decrease in the gas price causes an increase/decrease in the elec-

[4]In what follows we use p/therm, where p is the penny sterling and therm, a non-SI unit of heat energy.

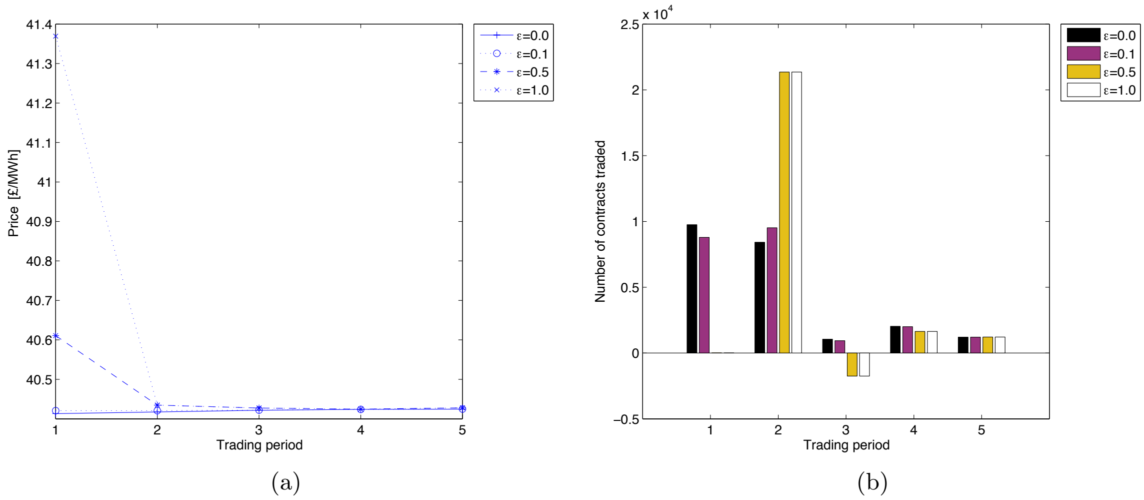
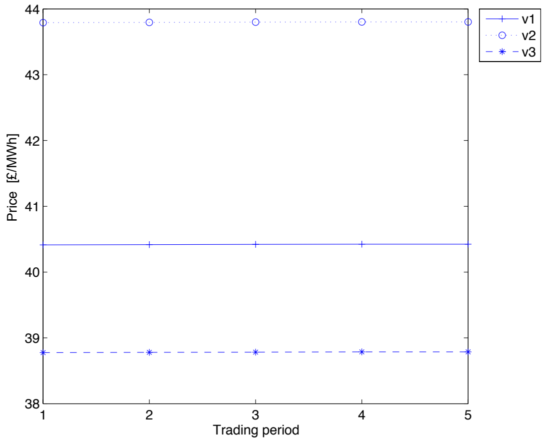



Figure 3.10: $\upsilon_{i1} = 0$ for all $i \in \{1, ..., 5\}$, $\lambda_k = 10^{-6}$ for all $k \in P \cup C$, and increasing $\epsilon_{11}$. For the other trading periods $i \in \{2, ..., 5\}$, $\epsilon_{i1} = 0$. The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

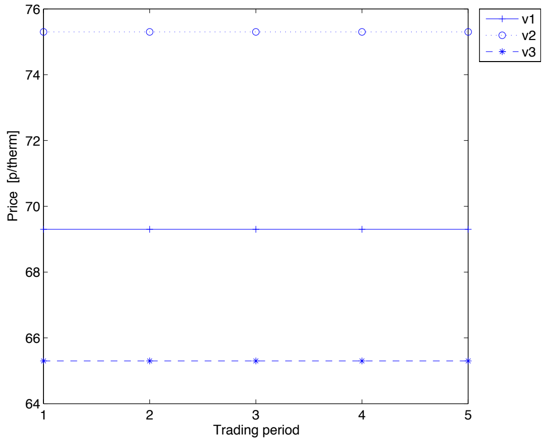
tricity price.

The second experiment is more interesting. Here we calculate the term structure of electricity prices for three different shapes of the term structure of gas prices. In Figure 3.12, we present results for a constant term structure of gas prices, normal backwardation and contango. We see that the term structure of gas prices has a large impact on the term structure of electricity prices. When the term structure of gas prices is in contango, we see that the term structure of electricity prices also changes to contango (a small kink at the fourth trading period is a consequence of a calibration of the covariance matrix from historical values). Note that most of the trading of gas contracts in this example occurs in the later trading period. Since gas contracts in these trading periods are highly correlated with electricity contracts from the same periods, the electricity price lowers. Recall that producers are short electricity contracts and long gas contracts, and so a high correlation between gas and electricity prices has the effect of reducing portfolio risk. This is visible also from Proposition 3.3.1 in Section 3.3.4.

We see that the change of the term structure of gas prices from constant to normal



(a) Electricity prices.



(b) Gas prices.

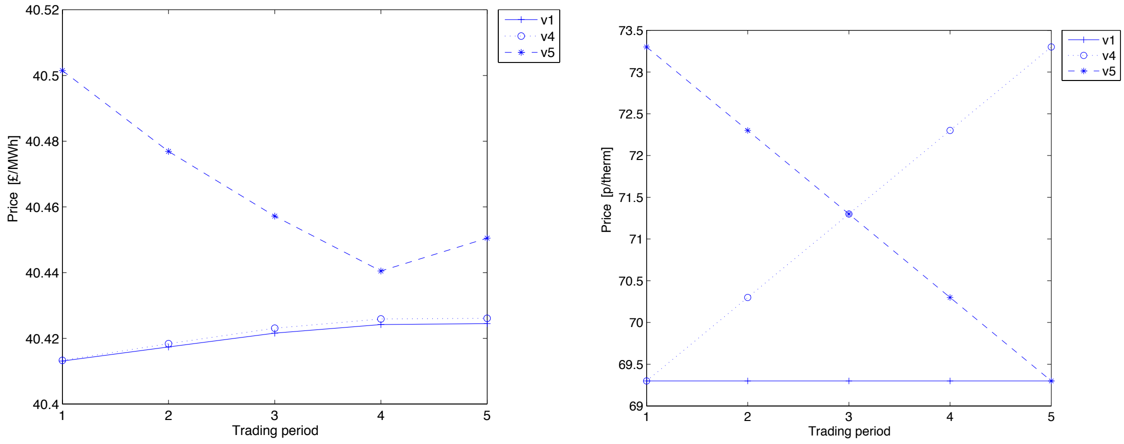
Figure 3.11: Effect of parallel shifts in the term structure of gas prices (right) on the term structure of electricity prices (left). The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

backwardation does not have any visible impact on the term structure of electricity prices. This is as expected, since this change does not have any impact on the fuel trading strategy. In both cases, most of the gas is bought in the first period, since these contracts are less risky (and in case of the normal backwardation, cheaper) and are thus preferable for risk-averse producers.

### 3.3.3 Block and spot contracts

Our goal is to calculate the spot electricity price with the information as at 11-Feb-2013 for delivery period from 04-Apr-2013 00:00 to 08-Apr-2013 00:00. We assume that there are two types of power contract available. The first is a month-ahead contract traded on 15-Mar-2013 17:00 that covers delivery over all four days. The second is a spot contract that requires immediate delivery and is traded separately for each half hour. We use the future prices of coal, gas and oil as at 11-Feb-2013. Since the historical demand forecast is not available, we once again use the realized demand as a proxy. We make the same assumptions as we did in Section 3.3.2 in respect of plant ownership.

The numerical results in Figures 3.13 - 3.16 are all calculated using $\lambda_k = 10^{-7}$ for every




(a) Electricity prices.

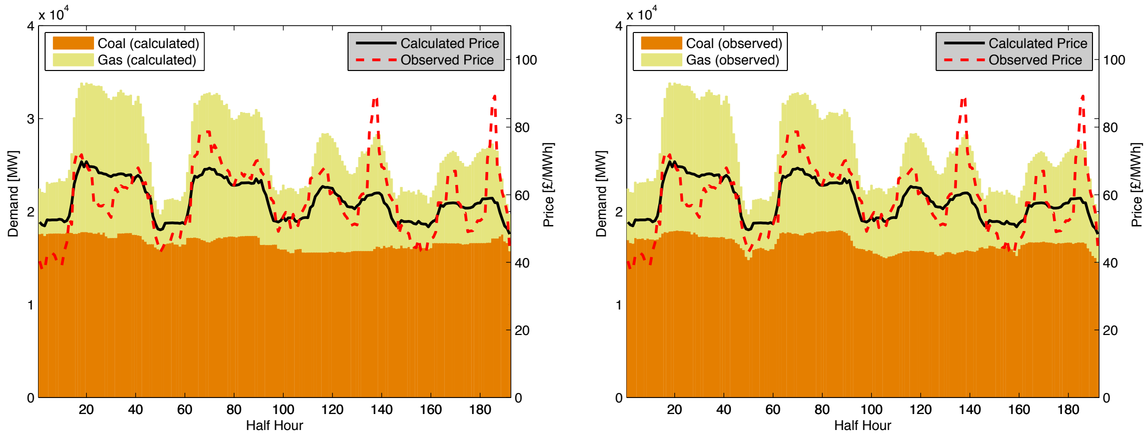
(b) Gas prices.

Figure 3.12: Dependence of the term structure of electricity prices (left) on the shape of the term structure of gas prices (right). The trading periods labelled from 1 to 5 correspond to a two-month-ahead, month-ahead, week-ahead, day-ahead and the spot price.

$k \in P \cup C$, and $\epsilon_{ij} = 0.1$ for all of the contracts. Since forward contracts are less liquid than spot contracts, we used $\upsilon_{ij} = 10^{-3}$ for forward and $\upsilon_{ij} = 10^{-4}$ for spot contracts. Each figure on the left-hand side depicts the calculated energy mix of coal and gas power plants, while the one on the right depicts the observed energy mix. Each figure also contains the spot price calculated by our model as well as the observed spot price. Plotting the energy mix and the electricity price in the same figure helps us to visually examine the relationship between the two.

Figure 3.13 shows that our model predicts the energy mix very closely. The daily pattern of the electricity price predicted by our model is also similar to that observed. Figure 3.14 shows the optimal number of forward and spot contracts traded. We see that approximately the same amount of electricity is delivered through forward and spot contracts. This happens because forward contracts have smaller volatility but higher transaction costs than spot contracts, and so the contribution of these two factors approximately cancel each other out. The calculated equilibrium price of the month-ahead forward contract is £58.31/MWh.

Figure 3.15 shows how the results change when we tighten the ramp-up and ramp-down constraints by 20% from their initial value. We notice that the price becomes more serrated and is slightly higher because more expensive power plants must be turned on to cover




(a) (b)

Figure 3.13: $\lambda_k = 10^{-7}$ for all $k \in P \cup C$, $\epsilon = 0.1$, $\upsilon = 10^{-4}$ for spot contracts and $\upsilon = 10^{-3}$ for the forward contract. The figure on the left-hand side depicts the calculated energy mix of coal and gas power plants, while the one on the right depicts the observed energy mix. Both figures also contains the spot price calculated by our model as well as the observed spot price.

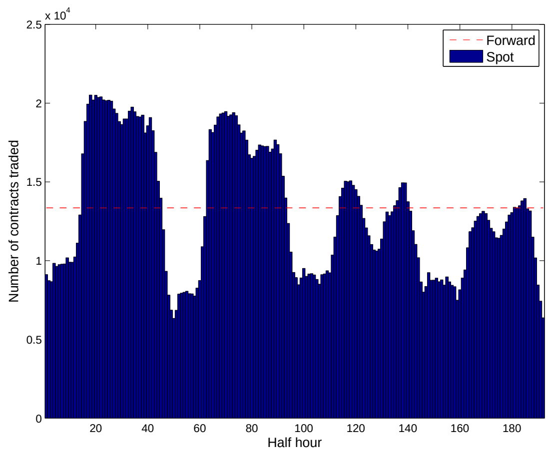
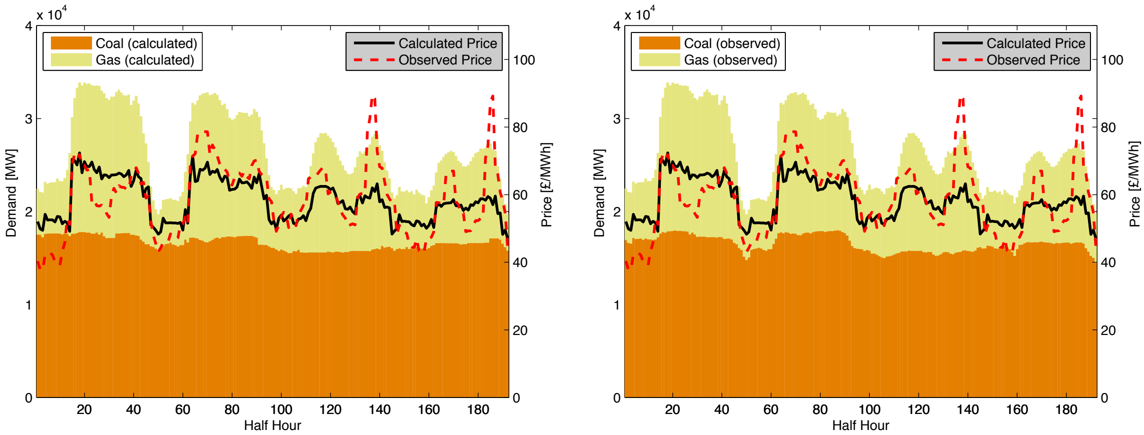



Figure 3.14: $\lambda_k = 10^{-7}$ for all $k \in P \cup C$, $\epsilon = 0.1$, $\upsilon = 10^{-4}$ for spot contracts and $\upsilon = 10^{-3}$ for the forward contract. This image depicts the calculated optimal number of forward and spot contracts traded.



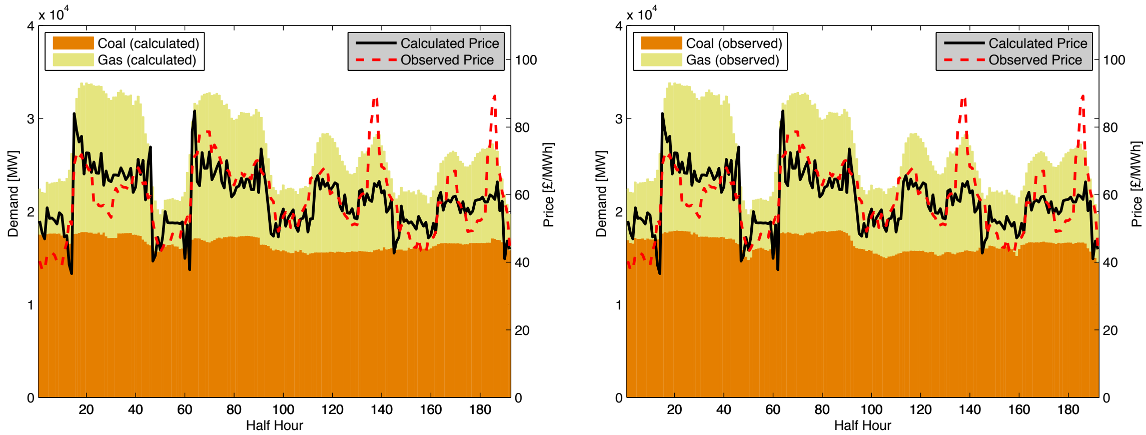

(a) (b)

Figure 3.15: Ramp-up and ramp-down constraints are tightened by 20% from their initial value. The figure on the left-hand side depicts the calculated energy mix of coal and gas power plants, while the one on the right depicts the observed energy mix. Both figures also contains the spot price calculated by our model as well as the observed spot price.

the changes in demand. This change has a very small impact on the number of contracts traded, so the results from Figure 3.14 still apply. The calculated equilibrium price of the month-ahead forward contract is £58.32/MWh.

Similarly, Figure 3.16 shows how the results change when we tighten the ramp-up and ramp-down constraints by 50% from the initial value. This change has a very small impact on the number of contracts traded, so the results from Figure 3.14 still apply. The calculated equilibrium price of the month-ahead forward contract is £58.36/MWh. From simulations above, we see that tighter ramping constraints lead to a slightly higher equilibrium price of the forward contract.

Figure 3.13 reveals a few shortcomings of our model when we use it to forecast spot electricity prices. First, we see that the daily variation in the calculated price is smaller than observed. Second, two spikes in the observed price are not captured by our model. Our hypothesis is that the first problem occurs because our model does not incorporate startup costs of power plants correctly. This is addressed in Chapter 4. As for the second problem, we believe that the demand forecast was erroneous. Spikes in the spot price occur when there is an unexpected change in demand. Since only a few, usually rather inefficient




(a) (b)

Figure 3.16: Ramp-up and ramp-down constraints are tightened by 50% from the initial value. The figure on the left-hand side depicts the calculated energy mix of coal and gas power plants, while the one on the right depicts the observed energy mix. Both figures also contains the spot price calculated by our model as well as the observed spot price.

Open Cycle Gas Turbine, power plants are flexible enough to cover large shifts in demand, they require a very high electricity price in order to be turned on. Such spikes cannot be forecasted two months before delivery.

### 3.3.4 Closed form solutions

In this section we derive a closed form solution for calculating the term structure of electricity prices from the spot price. Closed form solutions are convenient, because they are computationally inexpensive and are easy to analyze. It is possible to establish the following unique closed form relation among the prices $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i, T_j\right)\right]$, $i \in \{1, 2, ..., n\}$ for any fixed $j \in J$.

**Proposition 3.3.1.** *Let* $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\right]$ *denote a competitive equilibrium price vector and let* $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_n\right)\right]$ *be a given spot price. Then other competitive equilibrium prices* $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i\right)\right]$ $i \in \{1, ..., n-1\}$, *can be derived using*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i\right)\right]=\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_n\right)\right]-\lambda^{-1}\operatorname{Cov}\left[\Pi\left(t_i\right)-\Pi\left(t_n\right),\sum_{p\in P}C_p\left[k_p,F_p,G\right]-\left(\sum_{c\in C}s_cp_c\right)D\right] \tag{3.24}$$

*where* $\lambda^{-1}=\sum_{p=1}^{P}\frac{1}{\lambda_p}+\sum_{c=1}^{C}\frac{1}{\lambda_c}$ *and* $k_p=\sum_{i=1}^{n}V_p\left(t_i\right)$*.* $C_p\left(k_p,F,G\right)$ *denotes the costs of producing* $k_p$ *units of electricity with the fuel trading strategy* $F$ *at fuel prices* $G$*.*

*Proof.* To simplify the notation we introduce $\widetilde{V}_k=\left[V_k\left(t_1\right),...,V_k\left(t_{n-1}\right)\right]^\top$ for all $k\in P\cup C$ and $\widetilde{\Pi}=\left[\Pi\left(t_1\right),...,\Pi\left(t_{n-1}\right)\right]^\top$.

The profit of producer $p\in P$ can be written in this setting as

$$P_p\left(V_p,F_p,\Pi\right)=-\sum_{i=1}^{n}\Pi\left(t_i\right)V_p\left(t_i\right)-C_p\left[k_p,F_p,G\right] \tag{3.25}$$

where

$$k_p=\sum_{i=1}^{n}V_p\left(t_i\right). \tag{3.26}$$

The profit of consumer $c\in C$ can be written as

$$P_c\left(V_c,\Pi\right)=s_cp_cD-\sum_{i=1}^{n}\Pi\left(t_i\right)V_c\left(t_i\right).$$

We see that for any producer $p\in P$, there is no constraint on small changes to $V_p\left(t_i\right)$, but there is a capacity constraint on changes to $k_p$. Thus, by keeping $k_p$ fixed, (2.3) and (2.4) cannot be violated by small changes in the volumes $V_p\left(t_i\right)$. By expressing $V_p\left(t_n\right)$ from (3.26), inserting (3.25) into a mean-variance objective function, and taking a derivative with respect to $\widetilde{V}_p$, we obtain the following expression

$$\begin{aligned}\frac{\partial\Psi_p\left(P_p\left(V_p,F_p,\Pi\right)\right)}{\partial\widetilde{V}_p} &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_n\right)\right]e-\mathbb{E}^{\tilde{\mathbb{P}}}\left[\widetilde{\Pi}\right]-\lambda_p\widetilde{Q}\widetilde{V}_p \\ &\quad +\lambda_p\operatorname{Cov}\left(\Pi\left(t_n\right)e-\widetilde{\Pi},k_p\Pi\left(t_n\right)e+C_p\left[k_p,F_p,G\right]e\right),\end{aligned}$$

where $e$ is a column vector of ones of length $n-1$, and $\widetilde{Q}$ is defined as

$$\widetilde{Q} := \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\widetilde{\Pi} - \mathbb{E}^{\mathbb{P}}\left[\widetilde{\Pi}\right]\right)\left(\widetilde{\Pi} - \mathbb{E}^{\mathbb{P}}\left[\widetilde{\Pi}\right]\right)^{\top}\right].$$

Cov, the covariance operation, is applied element-wise. By setting $\frac{\partial \Psi_p(P_p(V_p,B_p,\Pi))}{\partial \widetilde{V}_p} = 0$ and expressing $\widetilde{V}_p$, we obtain

$$\widetilde{V}_p = \frac{1}{\lambda_p}\widetilde{Q}^{-1}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_n\right)\right]e - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\widetilde{\Pi}\right]\right) + \widetilde{Q}^{-1}\mathrm{Cov}\left(\Pi\left(t_n\right)e - \widetilde{\Pi}, k_p\Pi\left(t_n\right)e + C_p\left[k_p, F_p, G\right]e\right). \tag{3.27}$$

Since $\widetilde{Q}$ is a covariance matrix and none of the electricity contracts can be written as a linear combination of the others a.s., $\widetilde{Q}$ is invertible. Similarly, for each consumer $c \in C$

$$\widetilde{V}_c = \frac{1}{\lambda_c}\widetilde{Q}^{-1}\left(\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_n\right)\right]e - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\widetilde{\Pi}\right]\right) + \widetilde{Q}^{-1}\mathrm{Cov}\left(\Pi\left(t_n\right)e - \widetilde{\Pi}, k_c\Pi\left(t_n\right)e - s_c p_c D e\right), \tag{3.28}$$

where $k_c = \sum_{i=1}^{n} V_c\left(t_i\right)$. The market clearing condition (2.31) requires

$$\sum_{p\in P}\widetilde{V}_p + \sum_{c\in C}\widetilde{V}_c = 0. \tag{3.29}$$

Inserting expressions (3.27) and (3.28) into (3.29), we obtain the result

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_i\right)\right] \;=\; \mathbb{E}^{\tilde{\mathbb{P}}}\left[\Pi\left(t_n\right)\right] - \lambda^{-1}\mathrm{Cov}\left[\Pi\left(t_i\right) - \Pi\left(t_n\right), \sum_{p\in P} C_p\left[k_p, F_p, G\right] - \left(\sum_{c\in C} s_c p_c\right) D\right]$$

where $\lambda = \sum_{p=1}^{P}\frac{1}{\lambda_p} + \sum_{c=1}^{C}\frac{1}{\lambda_c}$. □

The analytical expression obtained in Proposition 3.3.1 matches the expression in Bessembinder and Lemmon (2002), but extends it to cover multiple delivery periods $T_j$. In addition, it also matches the result in Bühler and Müller-Merbach (2009), and Bühler (2009), but does not assume that all producers and consumers are modeled by one representative agent each, and instead, allows multiple producers and consumers to be modeled directly. In contrast to the existing literature, it also takes into consideration capacity and ramping constraints of power plants.

By examining (3.24), one can study the impact of the ramp-up and ramp-down constraints on the term structure of the electricity price. These constraints affect the energy mix and consequently $\sum_{p \in P} C_p [k_p, F_p, G]$. Since coal power plants have tighter ramping constraints than gas power plants, one can conclude that during a period of a large increase in demand, there will be more gas production than one would predict solely on the basis cost. Similarly, as demand decreases substantially, we might expect more coal production that would be predicted solely on the basis of cost. From (3.24) we see that the correlation between the electricity price and the energy mix plays an important role in determining the term structure of the electricity price. During periods of large demand increase (decrease), the correlation between gas (coal) and electricity prices plays a more important role than one would predict without taking ramping constraints into account.

(3.24) can also be used to explain the numerical example illustrated in Figure 3.12.

# Chapter 4

# Startup costs and the grid operator

Our numerical simulations in Chapter 3 reveled that our model has a tendency to underestimate spot prices during peak hours, and overestimate them during off-peak hours. We hypothesize that this may be because startup costs are not included in the model.

In this chapter, we extend the model presented in Chapter 2 to include startup costs. Various methodologies have been proposed as to how to include startup costs (see Martinez (2008), Gribik et al. (2007) and Zhang et al. (2009)). Most of them rely on a price uplift approach where the power price without startup costs is first calculated; this price is then uplifted to reflect startup costs. In our model, startup costs will be included in a mathematically rigorous fashion, without relying on the uplift heuristic.

We will show that startup costs alone cause too many spikes in spot electricity prices. To reduce this number, we model the grid operator, who is responsible for micromanaging the electricity grid, in the second part of this chapter.

This chapter is organized as follows: in Section 4.1, we give a detailed mathematical description of the extended model, and in Section 4.2 we present numerical results after including startup costs. These motivate us to introduce the grid operator in Section 4.3.

## 4.1 Extending the model

In this section, we amend a description of producers' optimization problems, presented in Chapter 2, to include startup costs of power plants.

### 4.1.1 Producers

A producer may participate in the market by buying and selling forward and spot contracts. The number of forward electricity contracts that producer $p \in P$ buys at trading time $t_i$, $i \in I_j$, for delivery at time $T_j$, $j \in J$, is denoted by $V_p(t_i, T_j)$. Similarly, the number of fuel and emission forward contracts that producer $p \in P$ buys at trading time $t_i$, $i \in I_j$, for delivery at time $T_j$, $j \in J$, is denoted by $F_{p,l}(t_i, T_j)$, $l \in L$ and $O_p(t_i, T_j)$, respectively. Producers usually own a non-empty portfolio of power plants. The actual production of electricity from power plant $r \in R^{p,l}$ at delivery time $T_j$, $j \in J$, is denoted by $\widehat{W}_{p,l,r}(T_j)$.

#### 4.1.1.1 Production variables

In this section we investigate the production of power plants more closely. Each power plant $r \in R^{p,l}$, $p \in P$, $l \in L$ has a maximum export limit and minimum stable limit denoted by $\overline{W}^{p,l,r}_{\max}(T_j)$ and $\overline{W}^{p,l,r}_{\min}(T_j)$, respectively. The maximum export limit defines the maximum production capacity of a power plant, and the minimum stable limit defines the minimum production that a power plant can maintain for a longer period of time. We allow each of these parameters to be time-dependent to account for the maintenance of power plants. In particular, we identify a maintenance shutdown with $\overline{W}^{p,l,r}_{\max}(T_j) = 0$.

The stable production of each power plant must satisfy

$$\widehat{W}_{p,l,r}(T_j) \in \{0\} \cup \left[\overline{W}^{p,l,r}_{\min}(T_j), \overline{W}^{p,l,r}_{\max}(T_j)\right] \tag{4.1}$$

for each $j \in J$. It is possible for a power plant to have production below the minimum stable limit, $\widehat{W}_{p,l,r}(T_j) \in \left(0, \overline{W}^{p,l,r}_{\min}(T_j)\right)$, but only for a very short period of time (i.e. during a ramp-up and ramp-down phase). To formulate these constraints in an optimization framework, we introduce the new decision variables $W^{(k)}_{p,l,r}(T_j)$, $k \in \{1, ..., 6\}$ with the following meaning:

- $W^{(1)}_{p,l,r}(T_j)$, $j \in J$, is a continuous variable that is 1 if the power plant is fully ramped up at time $T_j$ and 0 if the power plant is not producing. If $W^{(1)}_{p,l,r}(T_j) \in (0,1)$, then the power plant is in the ramp-up or ramp-down phase. In an optimization framework,

$W^{(1)}_{p,l,r}(T_j)$ is constrained by

$$W^{(1)}_{p,l,r}(T_j) \in [0,1]\,. \tag{4.2}$$

- $W^{(2)}_{p,l,r}(T_j)$, $j \in J$, is a binary variable that is 1 if the power plant is fully ramped up at time $T_j$ and 0 otherwise. In an optimization framework, $W^{(2)}_{p,l,r}(T_j)$ is constrained by

$$\begin{aligned} W^{(2)}_{p,l,r}(T_j) &\leq W^{(1)}_{p,l,r}(T_j) \\ W^{(2)}_{p,l,r}(T_j) &\in [0,1] \end{aligned} \tag{4.3}$$

  and

$$W^{(2)}_{p,l,r}(T_j) \in \mathbb{Z}. \tag{4.4}$$

- $W^{(3)}_{p,l,r}(T_j)$, $j \in J\backslash\{1\}$, is a continuous variable that denotes the increase in $W^{(1)}_{p,l,r}(T_j)$ from time $T_{j-1}$ to time $T_j$. In an optimization framework, $W^{(3)}_{p,l,r}(T_j)$ is constrained by

$$W^{(3)}_{p,l,r}(T_j) \geq W^{(1)}_{p,l,r}(T_j) - W^{(1)}_{p,l,r}(T_{j-1}) \tag{4.5}$$

  and

$$W^{(3)}_{p,l,r}(T_j) \in [0,1]\,. \tag{4.6}$$

- $W^{(4)}_{p,l,r}(T_j)$, $j \in J\backslash\{1\}$, is a binary variable that is 1 if the power plant is in the ramp-up phase and 0 otherwise. In an optimization framework, $W^{(4)}_{p,l,r}(T_j)$ is constrained by

$$\begin{aligned} W^{(4)}_{p,l,r}(T_j) &\geq W^{(1)}_{p,l,r}(T_j) - W^{(1)}_{p,l,r}(T_{j-1}) \\ W^{(4)}_{p,l,r}(T_j) &\in [0,1] \end{aligned} \tag{4.7}$$

  and

$$W^{(4)}_{p,l,r}(T_j) \in \mathbb{Z}. \tag{4.8}$$

- $W^{(5)}_{p,l,r}(T_j)$, $j \in J\backslash\{1\}$, is a binary variable that is 1 if the power plant is in the ramp-down phase and 0 otherwise. In an optimization framework, $W^{(5)}_{p,l,r}(T_j)$ is constrained by

$$W^{(5)}_{p,l,r}(T_j) \geq W^{(1)}_{p,l,r}(T_{j-1}) - W^{(1)}_{p,l,r}(T_j)$$
$$W^{(5)}_{p,l,r}(T_j) \in [0,1] \tag{4.9}$$

and

$$W^{(5)}_{p,l,r}(T_j) \in \mathbb{Z}. \tag{4.10}$$

- $W^{(6)}_{p,l,r}(T_j)$, $j \in J$, is a continuous variable such that

$$\widehat{W}_{p,l,r}(T_j) = W^{(1)}_{p,l,r}(T_j)\,\overline{W}^{p,l,r}_{\min}(T_j) + W^{(6)}_{p,l,r}(T_j)\left(\overline{W}^{p,l,r}_{\max}(T_j) - \overline{W}^{p,l,r}_{\min}(T_j)\right) \tag{4.11}$$

where

$$W^{(6)}_{p,l,r}(T_j) \leq W^{(2)}_{p,l,r}(T_j) \tag{4.12}$$

and

$$W^{(6)}_{p,l,r}(T_j) \in [0,1]. \tag{4.13}$$

Variable $W^{(1)}_{p,l,r}(T_j)$ tells us whether the power plant is running at time $T_j$. If it isn't, then by (4.3) and (4.12), $W^{(6)}_{p,l,r}(T_j) = 0$ and by (4.11), $\widehat{W}_{p,l,r}(T_j) = 0$. On the other hand, if the power plant is fully ramped up time $T_j$, then $W^{(1)}_{p,l,r}(T_j) = 1$ and $W^{(6)}_{p,l,r}(T_j) \in [0,1]$, and thus $\widehat{W}_{p,l,r}(T_j) \in \left[\overline{W}^{p,l,r}_{\min}(T_j), \overline{W}^{p,l,r}_{\max}(T_j)\right]$.

#### 4.1.1.2 Maximum ramp-up and ramp-down constraints

Producer $p \in P$ is not able to arbitrarily choose her decision variables because there are constraints that limit her feasible set. The change in the production of each power plant from one delivery period to the next is limited by ramp-up and ramp-down constraints. For each $j \in \{1, ..., T' - 1\}$, where $T'$ denotes the last delivery period, these constraints can be expressed as

$$\triangle\overline{W}^{p,l,r}_{\min}(T_j) \leq \widehat{W}_{p,l,r}(T_{j+1}) - \widehat{W}_{p,l,r}(T_j) \leq \triangle\overline{W}^{p,l,r}_{\max}(T_j), \tag{4.14}$$

where $l \in L$, $r \in R^{p,l}$, and $\triangle\overline{W}^{p,l,r}_{\max}$ and $\triangle\overline{W}^{p,l,r}_{\min}$ represent the maximum rates for ramping up and down. Ramping rates are highly dependent on the type of the power plant. Some gas power plants can increase production from zero to the maximum in a few minutes, while the same action may take days or weeks for a nuclear power plant.

Using (4.11), we can rewrite (4.14) for all $j \in \{1, ..., T' - 1\}$ as

$$\begin{aligned} \overline{W}_{\min}^{p,l,r}(T_j) &\leq W_{p,l,r}^{(1)}(T_{j+1})\,\overline{W}_{\min}^{p,l,r}(T_{j+1}) + W_{p,l,r}^{(6)}(T_{j+1})\left(\overline{W}_{\max}^{p,l,r}(T_{j+1}) - \overline{W}_{\min}^{p,l,r}(T_{j+1})\right) \\ &\quad - W_{p,l,r}^{(1)}(T_j)\,\overline{W}_{\min}^{p,l,r}(T_j) - W_{p,l,r}^{(6)}(T_j)\left(\overline{W}_{\max}^{p,l,r}(T_j) - \overline{W}_{\min}^{p,l,r}(T_j)\right) \\ &\leq \triangle\overline{W}_{\max}^{p,l,r}(T_j). \end{aligned} \tag{4.15}$$

Additionally, if the power plant is in a ramp-up phase, then it has to increase production and finish the ramp-up phase as quickly as possible. Such a requirement can be enforced via

$$W_{p,l,r}^{(1)}(T_{j+1}) \geq \min\left\{W_{p,l,r}^{(1)}(T_j) + \frac{\triangle\overline{W}_{\max}^{p,l,r}(T_j)}{\overline{W}_{\min}^{p,l,r}(T_j)}, 1\right\} \tag{4.16}$$

where $j \in \{1, ..., T' - 1\}$. Since this constraint is relevant only during the ramp-up phase, we reformulate it for $j \in \{1, ..., T' - 1\}$ as

$$W_{p,l,r}^{(1)}(T_{j+1}) \geq \min\left\{W_{p,l,r}^{(1)}(T_j) + \frac{\triangle\overline{W}_{\max}^{p,l,r}(T_j)}{\overline{W}_{\min}^{p,l,r}(T_j)}, 1\right\} - M_1\left(1 - W_{p,l,r}^{(4)}(T_j)\right), \tag{4.17}$$

where $M_1 \geq 1 + \frac{\triangle\overline{W}_{\max}^{p,l,r}(T_j)}{\overline{W}_{\min}^{p,l,r}(T_j)}$. Most of the available optimization solvers are not able to handle constraints that include min or max functions. Thus, we apply a well-established approach to handle logical constraints, and introduce a new binary decision variable, $W_{p,l,r}^{(7)}(T_j)$, defined as

$$W_{p,l,r}^{(7)}(T_j) \in [0, 1] \tag{4.18}$$

and

$$W_{p,l,r}^{(7)}(T_j) \in \mathbb{Z}, \tag{4.19}$$

where $j \in J$, that makes sure that at least one of

$$W_{p,l,r}^{(1)}(T_{j+1}) \geq W_{p,l,r}^{(1)}(T_j) + \frac{\triangle\overline{W}_{\max}^{p,l,r}(T_j)}{\overline{W}_{\min}^{p,l,r}(T_j)} - M_1\left(1 - W_{p,l,r}^{(4)}(T_j)\right) - M_2 W_{p,l,r}^{(7)}(T_j) \tag{4.20}$$

and

$$W^{(1)}_{p,l,r}(T_{j+1}) \geq 1 - M_1\left(1 - W^{(4)}_{p,l,r}(T_j)\right) - M_2\left(1 - W^{(7)}_{p,l,r}(T_j)\right), \tag{4.21}$$

where $M_2 \geq 1$, is enforced.

Similarly, if a power plant is in the ramp-down phase, then it has to decrease production and finish the ramp-down phase as quickly as possible. Such a requirement can be enforced via

$$W^{(1)}_{p,l,r}(T_{j+1}) \leq \max\left\{W^{(1)}_{p,l,r}(T_j) - \frac{\left|\triangle\overline{W}^{p,l,r}_{\min}(T_j)\right|}{\overline{W}^{p,l,r}_{\min}(T_j)}, 0\right\} + M_1\left(1 - W^{(5)}_{p,l,r}(T_j)\right) \tag{4.22}$$

for $j \in \{1, ..., T' - 1\}$. We apply the same approach as for the ramp-up phase and introduce a new binary decision variable $W^{(8)}_{p,l,r}(T_j)$ defined as

$$W^{(8)}_{p,l,r}(T_j) \in [0, 1] \tag{4.23}$$

and

$$W^{(8)}_{p,l,r}(T_j) \in \mathbb{Z}, \tag{4.24}$$

where $j \in J$, that makes sure that at least one of

$$W^{(1)}_{p,l,r}(T_{j+1}) \leq W^{(1)}_{p,l,r}(T_j) - \frac{\left|\triangle\overline{W}^{p,l,r}_{\min}(T_j)\right|}{\overline{W}^{p,l,r}_{\min}(T_j)} + M_1\left(1 - W^{(5)}_{p,l,r}(T_j)\right) + M_2 W^{(8)}_{p,l,r}(T_j) \tag{4.25}$$

and

$$W^{(1)}_{p,l,r}(T_{j+1}) \leq M_1\left(1 - W^{(5)}_{p,l,r}(T_j)\right) + M_2\left(1 - W^{(8)}_{p,l,r}(T_j)\right) \tag{4.26}$$

is enforced.

#### 4.1.1.3 Equality constraints

There are a number of equality constraints that connect power plant production with electricity, fuel, and emission trading. For each $j \in J$, the sum of electricity sold in the forward and spot market must equal the actually produced electricity,

$$-\sum_{i\in I_j} V_p\left(t_i, T_j\right) = \sum_{l\in L}\sum_{r\in R^{p,l}} \widehat{W}_{p,l,r}\left(T_j\right). \tag{4.27}$$

Each producer $p \in P$ has to make sure that a sufficient amount of fuel $l \in L$ has been bought to cover production for each delivery period $j \in J$ . This constraint can be expressed as

$$\sum_{r\in R^{p,l}} \widehat{W}_{p,l,r}\left(T_j\right) c^{p,l,r} = \sum_{i\in I_j} F_{p,l}\left(t_i, T_j\right) \tag{4.28}$$

where $c^{p,l,r} > 0$ is the efficiency of power plant $r \in R^{p,l}$.

The carbon emission obligation constraint can be written as

$$\sum_{j\in J}\sum_{i\in I_j} O\left(t_i, T_j\right) = \sum_{j\in J}\sum_{l\in L}\sum_{r\in R^{p,l}} \widehat{W}_{p,l,r}\left(T_j\right) g^l, \tag{4.29}$$

where $g^l > 0$ denotes the carbon emission intensity factor of fuel $l \in L$. This constraint ensures that sufficient emission certificates have been bought to cover production over the whole planning horizon.

#### 4.1.1.4 Producers' optimization problems

Decision variables are greatly simplified if they are concatenated into:

- electricity trading vectors, $V_p\left(T_j\right) = ||_{i\in I_j} V_p\left(t_i, T_j\right)$ and $V_p = ||_{j\in J} V_p\left(T_j\right)$,
- fuel trading vectors, $F_p\left(t_i, T_j\right) = ||_{l\in L} F_{p,l}\left(t_i, T_j\right)$, $F_p\left(T_j\right) = ||_{i\in I_j} F_p\left(t_i, T_j\right)$, and $F_p = ||_{j\in J} F_p\left(T_j\right)$,
- emission trading vectors, $O_p\left(T_j\right) = ||_{i\in I_j} O_p\left(t_i, T_j\right)$ and $O_p = ||_{j\in J} O_p\left(T_j\right)$,
- electricity production vectors, $W_{p,l,r}\left(T_j\right) = ||_{k\in\{1,..,8\}} W^{(k)}_{p,l,r}\left(T_j\right)$, $W_{p,l}\left(T_j\right) = ||_{r\in R^{p,l}} W_{p,l,r}\left(T_j\right)$, $W_p\left(T_j\right) = ||_{l\in L} W_{p,l}\left(T_j\right)$, and $W_p = ||_{j\in J} W_p\left(T_j\right)$,

and finally $v_p = \left[V_p^\top, F_p^\top, O_p^\top, W_p^\top\right]^\top$ is the concatenation of all decision variables.

Similarly, the notation we use for prices is greatly simplified if they are concatenated into:

- electricity price vectors, $\Pi(T_j) = ||_{i\in I_j} \Pi(t_i, T_j)$, and $\Pi = ||_{j\in J} e^{-\hat{r}T_j}\Pi(T_j)$, where $\hat{r} \in \mathbb{R}$ is a constant interest rate,

- fuel price vectors, $G(t_i, T_j) = ||_{l\in L} G_l(t_i, T_j)$, $G(T_j) = ||_{i\in I_j} G(t_i, T_j)$, and $G = ||_{j\in J} e^{-\hat{r}T_j} G(T_j)$,

- emission price vectors, $G_{\text{em}}(T_j) = ||_{i\in I_j} G_{\text{em}}(t_i, T_j)$, and $G_{\text{em}} = ||_{j\in J} e^{-\hat{r}T_j} G_{\text{em}}(T_j)$,

- startup costs vectors, $\widehat{s}^{p,l,r} = \left[0, 0, s^{p,l,r}, 0, 0, 0, 0, 0\right]^\top$, $s^{p,l} = ||_{r\in R^{p,l}} \widehat{s}^{p,l,r}$, $s^p = ||_{l\in L} s^{p,l}$, and $\widehat{s}^p = ||_{j\in J} e^{-\hat{r}T_j} s^p$, where $s^{p,l,r} \geq 0$ denotes the startup costs of power plant $r \in R^{p,l}$,

and finally

$$\pi_p = \left[\Pi^\top, G^\top, G_{\text{em}}^\top, (\widehat{s}^p)^\top\right]^\top$$

is the concatenation of all of the prices.

Producer's goal is to maximize her expected profit, subject to a risk budget. In this work, we assume that this budget is expressed in a mean-variance framework.

The profit $P_p(v_p, \pi_p)$ of producer $p \in P$ can be calculated as

$$P_p(v_p, \pi_p) = \sum_{j\in J} e^{-\hat{r}T_j} \left( \sum_{i\in I_j} P_p^{t_i,T_j}(v_p, \pi_p) - \sum_{l\in L} \sum_{r\in R^{p,l}} s^{p,l,r} W_{p,l,r}^{(3)}(T_j) \right), \tag{4.30}$$

where the profit $P_p^{t_i,T_j}(v_p, \pi_p)$ for each $i \in I_j$ and $j \in J$ can be calculated as

$$P_p^{t_i,T_j}(v_p, \pi_p) = -\Pi(t_i, T_j) V_p(t_i, T_j) - G_{\text{em}}(t_i, T_j) O_p(t_i, T_j) - \sum_{l\in L} G_l(t_i, T_j) F_{p,l}(t_i, T_j).$$

Under a mean-variance optimization framework, producers are interested in the mean-variance utility,

$$\begin{aligned} \Psi_p(v_p) &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[P_p(v_p, \pi_p)\right] - \tfrac{\lambda_p}{2} \text{Var}^{\tilde{\mathbb{P}}}\left[P_p(v_p, \pi_p)\right] \\ &= -\mathbb{E}^{\tilde{\mathbb{P}}}\left[\pi_p\right]^\top v_p - \tfrac{1}{2}\lambda_p v_p^\top Q_p v_p, \end{aligned}$$

where $\lambda_p > 0$ is their risk preference parameter and $Q_p := \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\pi_p - \mathbb{E}^{\mathbb{P}}[\pi_p]\right)\left(\pi_p - \mathbb{E}^{\mathbb{P}}[\pi_p]\right)^\top\right]$ a new 'extended' covariance matrix. Their objective is to solve the following optimization problem

$$\Phi_p = \max_{v_p} \quad \Psi_p(v_p) \tag{4.31}$$

subject to (4.2) - (4.10), (4.12), (4.13), (4.15), (4.18) - (4.21), and (4.23) - (4.29). We denote the feasible set of (4.31) as $S_p$.

A standard approach to solving an optimization problem with integrality constraints is to consider its continuous relaxation. We define the continuous relaxation of (4.31) as

$$\Phi_p = \max_{v_p} \quad \Psi_p(v_p) \tag{4.32}$$

subject to (4.2), (4.3), (4.5), (4.6), (4.7), (4.9), (4.12), (4.13), (4.15), (4.18), (4.20), (4.21), (4.23), (4.25), (4.26), (4.27), (4.28), and (4.29). (4.32) is the same as problem (4.31) except that it does not include the integrality constraints (4.4), (4.8), (4.10), (4.19) and (4.24). We denote the feasible set of Problem (4.32) as $\widetilde{S}_p$.

Note that consumers' optimization problems (2.11), and the hypothetical market agent's optimization problem (3.1) are not affected by production startup costs.

### 4.1.2 Continuous relaxation

The integrality constraints (4.4), (4.8), (4.10), (4.19) and (4.24) of each producer significantly complicate the analysis of (4.31), and so we focus on the continuous relaxation (4.32) instead. We show through various numerical results in Sections 4.2.2 and 4.3.2, that the integrality constraints listed above do not have a significant impact on the equilibrium electricity price.

We are interested in finding a competitive equilibrium and Nash equilibrium as in Definitions 2.2.1 and 2.3.1, using the continuous relaxation (4.32) instead of (2.8).

To estimate the error caused by the continuous relaxation, we use the following procedure:

1. We calculate the equilibrium electricity price $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]^*$ by solving the continuous (relaxed) problem (4.32) for producers, and the original problem (2.11) for consumers.

2. Using this price, we calculate optimal trading vectors $V_p^*$, from the discrete producers' problem (4.31), and optimal trading vectors $V_c^*$ from the original consumers' problem (2.11).

3. We have taken a continuous relaxation of a discrete problem. There are two potential sources of error in this procedure:

   (a) the error caused by neglecting the integrality constraints and considering the continuous relaxation of the problem,

   (b) potential errors that may occur due to numerical instabilities when solving the problem.

   Since both of these contribute to the mismatch in the market clearing constraint, we define the error of this mixed integer quadratic programming (MIQP) approach as

$$\text{MIQP} := \sum_{c \in C} V_c^* + \sum_{p \in P} V_p^*. \tag{4.33}$$

   To reiterate, the MIQP error would be zero if the problem was solved exactly.

In order to verify the procedure above and estimate the numerical contribution to the MIQP error, we apply the following procedure:

1. As for step 1 above.

2. As for step 2 above, except that we replace the discrete producers' problem (4.31) by its continuous counterpart (4.32), to calculate optimal trading vectors $\widetilde{V}_p^*$.

3. We calculate the error of this quadratic programming (QP) approach, again, as the mismatch in the market clearing constraint,

$$\text{QP} := \sum_{c \in C} V_c^* + \sum_{p \in P} \widetilde{V}_p^*. \tag{4.34}$$

   The QP error indicates the mismatch in the market clearing constraint caused by numerical instabilities only.

In Sections 4.2.2 and 4.3.2, we present the MIQP and QP errors when modeling the entire UK power grid.

## 4.2 Numerical results

In this section, we discuss our numerical results and apply the model from Section 4.1 to study the UK power grid.

### 4.2.1 Estimation of parameters

To start, we estimate the parameters in our revised model.

As mentioned previously, in Section 3.3.1, in the UK, all power plants are required to submit their available capacity as well as ramp-up and ramp-down constraints to the grid operator on a half hourly basis.[1] For a description and a few samples of the data used in this thesis, see Appendix A.

As in Section 3.3.1, the carbon emission intensity factor $g^l$ for fuel $l \in L$ can either be estimated from historical production data (in this case it would be plant-dependent and thus modeled as $g^{p,l,r}$) or obtained from various energy market reports. We tested both approaches, but we did not observe any significant difference in the calibration results. Hence, for simplicity, we employ the latter approach and set $g^{\text{coal}} = 0.973$ $\text{tCO}_2/\text{MWh}$ and $g^{\text{gas}} = 0.412$ $\text{tCO}_2/\text{MWh}$ as given in Platts Report.[2]

A more challenging problem is to estimate the efficiency factors $c^{p,l,r}$, and startup costs $s^{p,l,r}$ for each power plant $r \in R^{p,l}$. In order to calibrate the model, we assume that all producers are risk-neutral and set $\lambda_p = 0$. We also neglect the ramp-up and ramp-down constraints (4.15), (4.20), (4.21), (4.25), and (4.26). Since each power plant is treated separately, we avoid writing sub/superscripts $p, l, r$.

In Section 3.3.1, we showed why there is no need to consider forward contracts when calibrating the physical parameters of power plants. Similarly, here, we can conclude that it suffices to consider spot electricity, fuel and emission contracts only in order to determine

[1]This data is publicly available at the BM Reports website `http://www.bmreports.com/`

[2]For details see `https://www.platts.com/IM.Platts.Content/methodologyreferences/methodologyspecs/european_power_methodology.pdf`.

the stack. By taking into account startup costs and the equations described in Section 4.1.1, the profit maximization problem of each power plant can be written as

$$\max_{W^{(2)},W^{(4)},W^{(6)}} \sum_{j\in J} \widehat{W}\left(T_j\right)\overline{P}\left(T_j\right) - W^{(4)}\left(T_j\right)s \tag{4.35}$$

subject to

$$W^{(4)}\left(T_j\right) \geq W^{(2)}\left(T_j\right) - W^{(2)}\left(T_{j-1}\right), \forall j \in J \backslash \{1\} \tag{4.36}$$

$$W^{(6)}\left(T_j\right) \leq W^{(2)}\left(T_j\right), \forall j \in J \tag{4.37}$$

$$W^{(k)}\left(T_j\right) \in [0,1], \forall j \in J,\ k \in \{2,4,6\} \tag{4.38}$$

$$W^{(2)}\left(T_j\right) \in \mathbb{Z}, \forall j \in J, \tag{4.39}$$

where

$$\widehat{W}\left(T_j\right) = W^{(2)}\left(T_j\right)\overline{W}_{\min}\left(T_j\right) + W^{(6)}\left(T_j\right)\left(\overline{W}_{\max}\left(T_j\right) - \overline{W}_{\min}\left(T_j\right)\right),\ \forall j \in J \tag{4.40}$$

and

$$\bar{P}\left(T_j\right) = \Pi\left(T_j, T_j\right) - cG\left(T_j, T_j\right) - gG_{\text{em}}\left(T_j, T_j\right),\ \forall j \in J. \tag{4.41}$$

Note that we do not have to impose the integrality constraints of variable $W^{(4)}\left(T_j\right)$ $j \in J$, because they are implied by (4.36) and (4.39).

As in Section 3.3.1, to account for the neglected risk premium, trading and maintenance costs, we introduce an additional constant $m > 0$ and include it in (4.41) as

$$\bar{P}\left(T_j\right) = \Pi\left(T_j, T_j\right) - cG\left(T_j, T_j\right) - gG_{\text{em}}\left(T_j, T_j\right) - m,\ \forall j \in J. \tag{4.42}$$

We wish to know how the optimal solution of (4.35) depends on the parameters $c$, $g$, $m$, and $s$. Let $\widehat{W}^*\left(T_j; c, g, m, s\right)$ denote the optimal production of (4.35). Our task is to find $c$, $m$, and $s$ (and possibly also $g$) that satisfy

$$\min_{c,g,m,s} \Xi\left(c, g, m, s\right) := \sum_{j\in J} \left(\widehat{W}^*\left(T_j; c, g, m, s\right) - \tilde{W}\left(T_j\right)\right)^2 \tag{4.43}$$

where $\tilde{W}(T_j)$ denotes the observed historical production of a power plant. This is a bi-level optimization problem where (4.43) corresponds to the outer and (4.35) corresponds to the inner problem. Traditionally, such problems have been very difficult to solve because they are highly non-convex and the process of finding the optimal solution of the outer problem involves many expensive evaluations of the inner integer programming problem. However, we can show that in our case, a difficult integer programming problem can be replaced by a tractable linear programming problem, without affecting the optimal solution.

By the following proposition, we see that the optimal solution of (4.35) can be found by its linear programming relaxation.

**Proposition 4.2.1.** *The matrix of constraints (4.36), (4.37), and (4.38) for (4.35) is totally unimodular.*

*Proof.* Let us write the matrix of inequality constraints (4.36) and (4.37) as

$$[A_1\ A_2\ A_3] \begin{bmatrix} W^{(2)} \\ W^{(6)} \\ W^{(4)} \end{bmatrix} \leq 0 \tag{4.44}$$

for some block matrices $A_1$, $A_2$, and $A_3$. We will first show that matrix $[A_1\ A_2]$ is totally unimodular. Note that all of its entries are from the set $\{-1, 0, 1\}$. Moreover, each row contains exactly two non-zero entries. One of the entries is 1 and the other is $-1$. These are sufficient conditions for matrix $[A_1\ A_2]$ to be totally unimodular. It is trivial to see that $A_3 = P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q$ for some permutation matrices $P$ and $Q$, of the appropriate size. Hence, matrix $[A_1\ A_2\ A_3]$ is totally unimodular. The bound constraints (4.38) can be included by using a similar argument. □

By virtue of Proposition 4.2.1, we can relax the integrality constraints and reformulate

(4.35) as a linear programming problem:

$$\begin{aligned} \max_{W^{(2)},W^{(4)},W^{(6)}} \quad & \textstyle\sum_{j\in J} \widehat{W}\left(T_j\right)\overline{P}\left(T_j\right) - W^{(4)}\left(T_j\right)s \\ \text{s.t.} \quad & W^{(4)}\left(T_j\right) \geq W^{(2)}\left(T_j\right) - W^{(2)}\left(T_{j-1}\right),\ \forall j \in J\backslash\left\{1\right\} \\ & W^{(6)}\left(T_j\right) \leq W^{(2)}\left(T_j\right),\ \forall j \in J \\ & W^{(k)}\left(T_j\right) \in \left[0,1\right],\ k \in \left\{2,4,6\right\},\ \forall j \in J. \end{aligned} \tag{4.45}$$

We use a combination of a particle swarm algorithm (as in Vaz and Vicente (2007)) for the outer problem, and Gurobi (as in Gurobi Optimization (2014)) for the inner problem, to solve the bi-level optimization problem (4.45) in practice.

Our initial calibration results showed that if (4.43) is used to calibrate the physical properties of power plants, then the solution $\widehat{W}^*\left(T_j; c, g, m, s\right) = 0$ or $\widehat{W}^*\left(T_j; c, g, m, s\right) = \overline{W}_{\max}\left(T_j\right)$ for all $j \in J$ is sometimes found as optimal. Even though such a solution may correspond to a local minimum of (4.43), it is usually not the global minimum. To help our calibration procedure find a more useful optimal solution, we alter the outer optimization problem (4.43) as

$$\min_{c,g,m,s} \Xi\left(c, g, m, s\right)\left(1 + z\theta_{\text{startups}}\right) \tag{4.46}$$

where

$$\theta_{\text{startups}} = \frac{\left|\#\text{ of observed startups} - \#\text{ of forecasted startups}\right|}{\#\text{ of observed startups}}$$

and $z > 0$ is a constant. The new objective function encourages the calibration procedure to find a local minimum in which the number of startups in the calculated and observed production does not differ too much. Our calibration results show that by setting $z := 0.5$, we rarely converge to the undesired minimums mentioned before.

For each power plant, we use over 5 000 training samples obtained for the period 01-Jan-2012 to 01-Jan-2013. Figures 4.1 and 4.2 depict a subset of the calibration results for coal and gas power plants.[3] For each power plant, we show in MW:

1. The final physical notification (as observed). This value is denoted by $\tilde{W}_{p,l,r}\left(T_j\right)$ in

[3]For the rest, see Appendix B.

(4.43).

2. Calculated optimal production using efficiency $c^{p,l,r}$, maintenance costs $m^{p,l,r}$ and startup costs $s^{p,l,r}$ from the calibration exercise (as calculated). This value is denoted by $\widehat{W}^*\left(T_j; c^{p,l,r}, g^l, m^{p,l,r}, s^{p,l,r}\right)$ in (4.43).

3. The difference between observed and calculated (as observed minus calculated).

4. The maximum export limit (the 'MEL'). This value is denoted by $\overline{W}^{p,l,r}_{\max}(T_j)$ in (4.40).

5. The stable export limit (the 'SEL'). This value is denoted by $\overline{W}^{p,l,r}_{\min}(T_j)$ in (4.40).

The final physical notification data, the MEL and the SEL are described in detail in Appendix A.1.2.

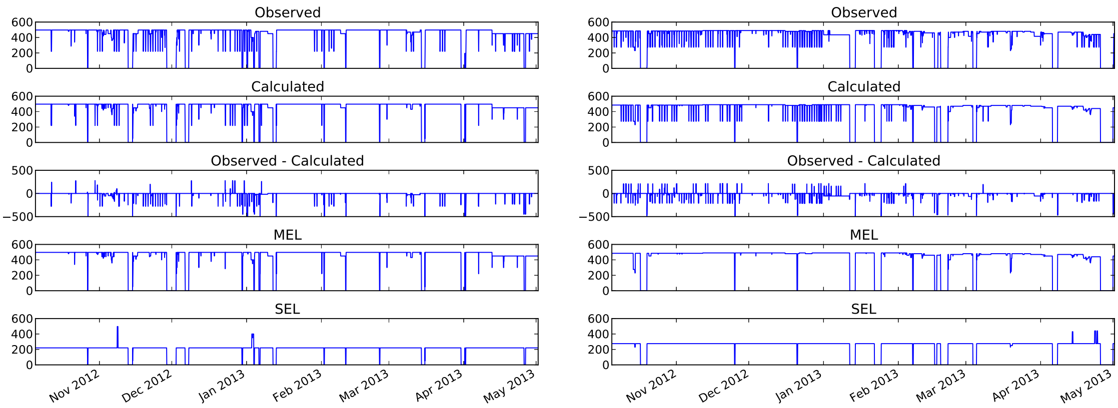
The results in Figures 4.1 and 4.2 demonstrate that the calculated production matches the observed one very closely. Model (4.45) is clearly able to capture the main features of plant production behavior (daily cycling between zero and the MEL, or between the SEL and the MEL, constant production at the MEL, not producing etc.) most of the time. As in reality, coal power plants tend to cover baseload demand and produce most of the time. Since they have relatively high startup costs, they are rarely turned off; when the electricity price is low, they produce at the MEL to avoid startup costs. On the other hand, gas power plants predominantly cover peak demand; since they have relatively low startup costs they produce during the day when the electricity price is high, and turn off at night, when the price is low.

Our model (4.45) is clearly not able to exactly recreate historical production. In our opinion the following factors mainly contribute to the mismatch:

1. In reality, startup costs depend on the shutdown period. The longer a power plant has been offline, the higher the startup costs. Hence, startup costs are usually classified as hot, warm, and cold.[4] This aspect of startup costs was not included in (4.45).

2. We assume that power plant operators have perfect price foresight. In reality, the operation of a power plant is never optimal due to errors in price forecasts.

[4]For more information see `http://wind.nrel.gov/public/wwis/aptechfinalv2.pdf`.




(a) BMU: T_COTPS-4, $e = 0.43$, $s = 103910$ and $m = 0.0$.

(b) BMU: T_RATS-3, $e = 0.39$, $s = 9681$ and $m = 4.4$.

Figure 4.1: The calibration results for two coal power plants (all values are in MW). Model (4.45) is clearly able to capture the main features of plant production behavior (daily cycling between zero and the MEL, or between the SEL and the MEL, constant production at the MEL, not producing etc.) most of the time. As in reality, coal power plants tend to cover baseload demand and produce most of the time. Since they have relatively high startup costs, they are rarely turned off; when the electricity price is low, they produce at the MEL to avoid startup costs. For the rest of the results, see Appendix B.

3. In reality, the efficiency of a power plant depends on the production. A power plant achieves its highest efficiency when producing at the MEL and any deviation from this leads to a lower efficiency. This is usually referred to as the no-load cost.[5]

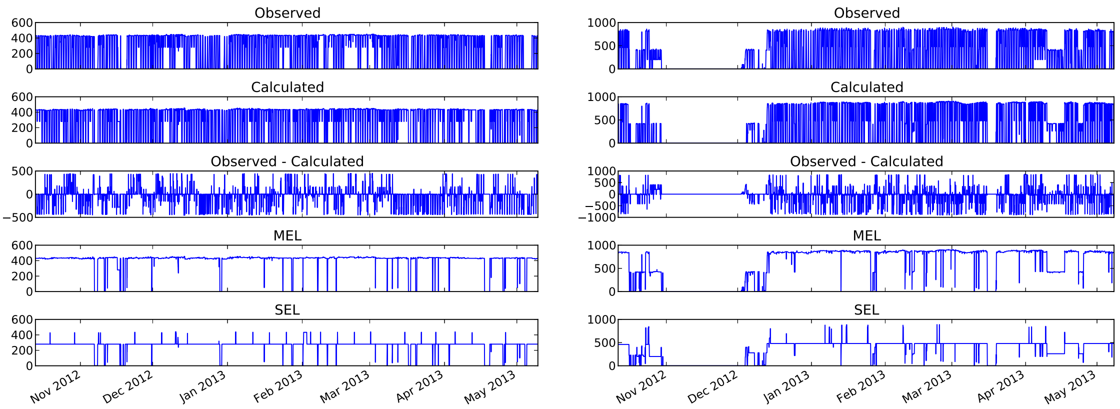
4. Ramping constraints are neglected in (4.45).

For the calibration results of the remaining gas and coal power plants, see Appendix B.

### 4.2.2 UK power grid

The numerical results in this section are calculated by solving the problem described in Section 3.3.3 with the same constants as specified there. The only difference is that in this section, startup costs are included and a new technique, as described in Section 4.2.1, is used to estimate parameters $c$, $m$, and $s$ for each power plant.

[5] For more information see `http://www.pjm.com/~/media/documents/manuals/m15.ashx`.




(a) BMU: T_GRAI-6, $e = 0.57$, $s = 12338$ and $m = 4.2$.

(b) BMU: T_LAGA-1, $e = 0.58$, $s = 13291$ and $m = 7.1$.

Figure 4.2: The calibration results for two gas power plants (all values are in MW). Model (4.45) is clearly able to capture the main features of plant production behavior (daily cycling between zero and the MEL, or between the SEL and the MEL, constant production at the MEL, not producing etc.) most of the time. As in reality, gas power plants predominantly cover peak demand; since they have relatively low startup costs they produce during the day when the electricity price is high, and turn off at night, when the price is low. Note that T_LAGA-1 was offline in November 2012 for maintenance. For the rest of the results, see Appendix B.

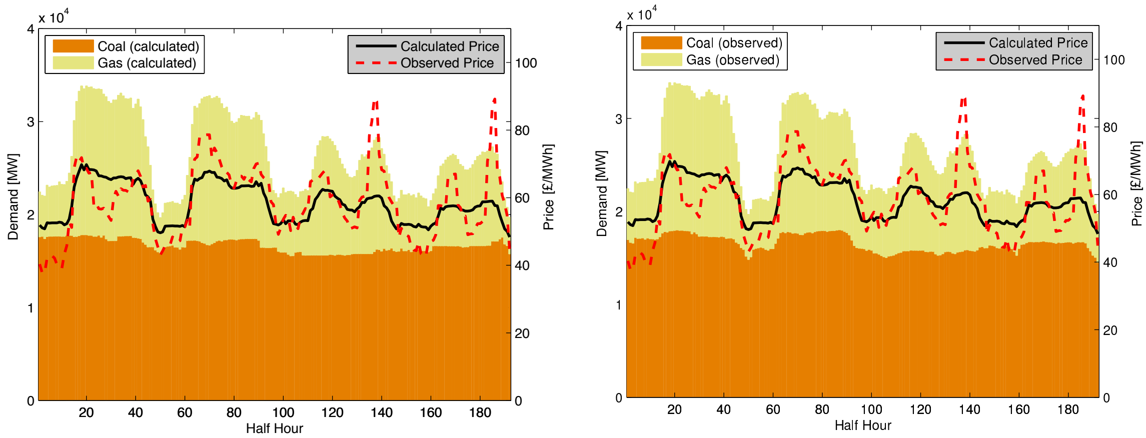
To motivate the inclusion of startup costs, we first investigate a simplified version of our model which neglects them (by setting them to zero). Figure 4.3 shows the output of such a model. Figure 4.3a depicts the calculated energy mix between coal and gas power plants, while Figure 4.3b depicts the observed energy mix. Both figures also contain the spot price calculated by our model as well as the observed price. The daily pattern of the electricity price predicted by our model is similar to observation. The model correctly predicts that the electricity price is higher during peak hours than off-peak hours. Furthermore, the calculated price has two daily peaks that are noticeable also in the observed price. The calculated equilibrium price of the month-ahead forward contract is £58.18/MWh.

Figure 4.4 shows the optimal number of forward contracts traded. We see that approximately the same amount of electricity is delivered through forward as through spot contracts. This occurs because spot contracts have high volatility but small transactions costs, while forward contracts have low volatility but higher transaction costs. Both factors approximately balance each other. The optimal number of contracts traded in this simulation is similar to what we saw in Section 3.3.3 (Figure 3.14).

The difference between the calculated and observed power production from each fuel is depicted in Figure 4.5. We see that our model predicts the energy mix quite closely. Contrasting the error with the total production of gas and coal power plants as depicted in Figure 4.3, we see that the error is a few percent of the total production.

The graphs also reveal a few weaknesses of our model. First, we can see that our model underestimates spot prices during peak hours, but overestimates them during off-peak hours. A similar result was also noted in Harvey and Hogan (2002). Second, the extreme spikes in the observed price were not predicted by our model. This motivated us to extend our model to incorporate startup costs.

Calculated equilibrium prices and the energy mix with startup costs included are depicted in Figure 4.6. The calculated equilibrium price of the month-ahead forward contract is £58.41/MWh. By comparing Figure 4.3 and Figure 4.6, we see that the calculated equilibrium price captures the daily variations of the observed price much more closely. It correctly predicts some of the spikes, but also forecasts several false positives. Figure 4.7 shows that the inclusion of startup costs does not have a significant impact on the prediction of the




(a) (b)

Figure 4.3: Comparison of the calculated and historical electricity price, and the energy mix when startup costs are excluded (i.e. set to zero). We see that the daily pattern of the electricity price predicted by our model is similar to observation; the model correctly predicts that the electricity price is higher during peak hours than off-peak hours. However, our model underestimates spot prices during peak hours, but overestimates them during off-peak hours. It also does not predict the extreme spikes well.

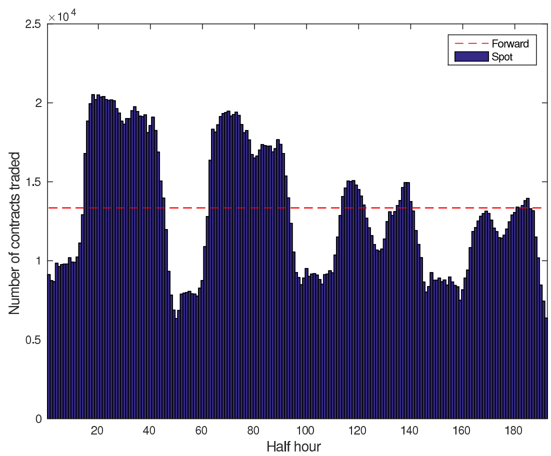



Figure 4.4: The optimal number of forward and spot contracts traded when startup costs are excluded.

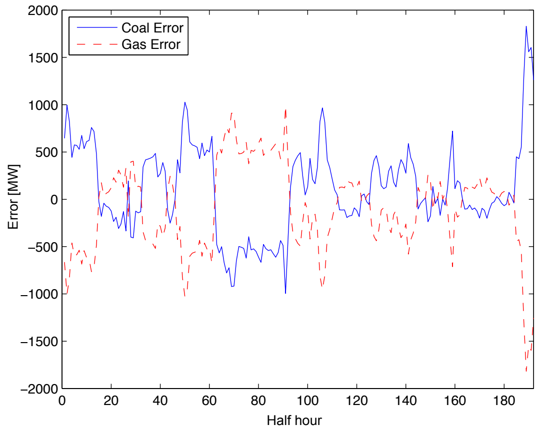



Figure 4.5: The difference between the calculated and observed gas and coal production. We see that our model predicts the energy mix quite closely. Contrasting the error with the total production of gas and coal power plants as depicted in Figure 4.3, we see that the error is approximately 3% of the total production.

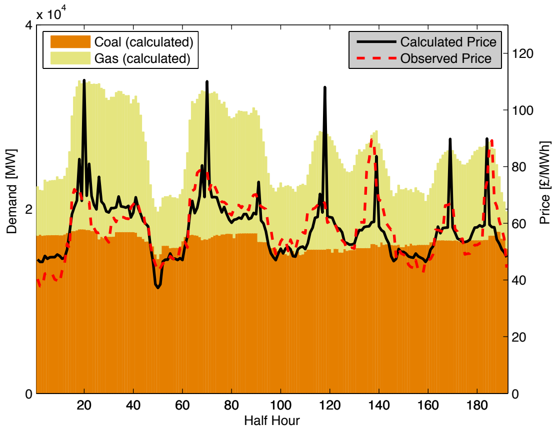
energy mix. The inclusion of startup costs does not have any impact on the optimal number of contracts traded, and so the results from Figure 4.4 still apply.

It is interesting to explore the state of the grid when spikes in the electricity price occur. A very descriptive parameter in this regard is standing reserve $SR(T_j)$, $j \in J$, which is defined as

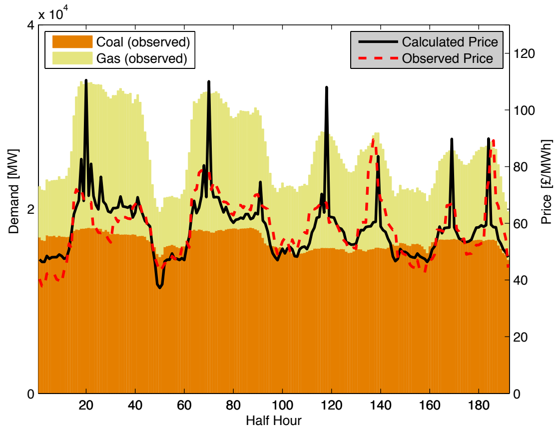
$$SR(T_j) = \sum_{p \in P} \sum_{l \in L} \sum_{r \in R^{p,l}} \left[ W^{(2)}_{p,l,r}(T_j) - W^{(6)}_{p,l,r}(T_j) \right] \left[ \overline{W}^{p,l,r}_{\max}(T_j) - \overline{W}^{p,l,r}_{\min}(T_j) \right]. \qquad (4.47)$$

Standing reserve quantifies by how much the power plants that are running at time $T_j$ can increase their production before a new power plant must be turned on. Since most power plants have severe constraints on startup times, low standing reserve usually implies low stability of the electricity grid, because a small change in demand can result in an electricity outage.

Figure 4.8 depicts the calculated standing reserve over the forecast horizon. We see that all of the price spikes in our model occur when the standing reserve is close to zero. In such situations, a new power plant must be turned on (and off quickly afterwards) to cover the temporary extra demand. The associated startup costs are then spread over a very short




(a)




(b)

Figure 4.6: Comparison of the calculated and historical electricity price and energy mix when startup costs are included. The calculated price now captures the daily variations of the observed price much more closely. It correctly predicts some of the spikes, but also forecasts several false positives.

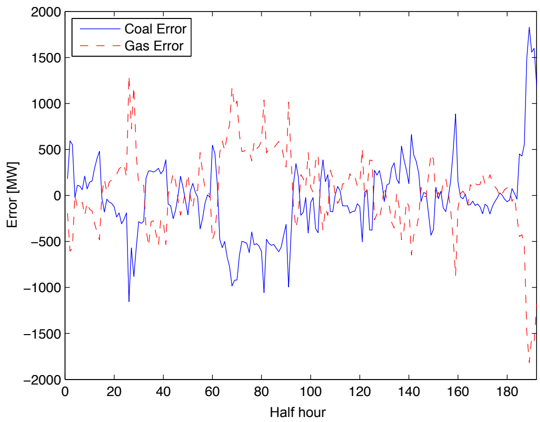



Figure 4.7: The difference between the calculated and observed gas and coal production after including startup costs. Comparing the error here and in Figure 4.5, we see that the inclusion of startup costs does not have a significant impact on the prediction of the energy mix.

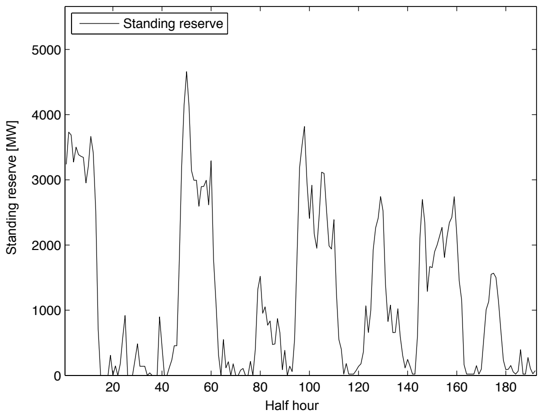



Figure 4.8: The calculated standing reserve over the forecast horizon. By examining Figure 4.6, we see that all of the price spikes in our model occur when the standing reserve is close to zero.

period of time, and a high electricity price is required for such an action to be profitable. However, in reality, times of a low standing reserve are rare. The grid operator is responsible for ensuring a reliable electricity supply and preventing instances of a low standing reserve. This is achieved by incentivizing some power plants to produce even when it may not be profitable for them to do so. The costs of such actions are distributed among all market participants. In the next section, we discuss how to include the actions of the grid operator in our model.

In the remaining part of this section, we evaluate the error caused by neglecting the integrality constraints and considering the continuous relaxation of the problem (4.31). We follow the procedure described in Section 4.1.2. The MIQP error is depicted by a dashed line in Figure 4.9. To estimate the effect of numerical errors, we also calculated the QP error which is shown in Figure 4.9 as a solid line.

We see from Figure 4.9 that $\left\|\sum_{c\in C} V_c^* + \sum_{p\in P} V_p^*\right\|_\infty \approx 400\,\text{MW}$ and that the largest deviations occur during peak hours when total electricity demand is high. Since the demand during peak hours is approximately 40 GW we conclude that MIQP error is within $\pm 1\%$ of the total demand.

In reality, it is also clear that production and consumption do not match exactly. This mismatch is reflected in changes in the power line frequency. In the UK, the nominal power

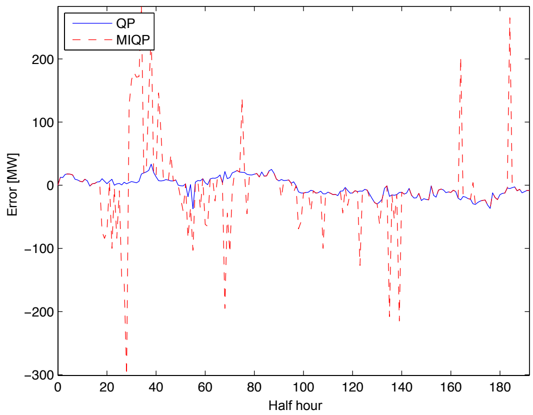



Figure 4.9: The error caused by neglecting the integrality constraints and considering the continuous relaxation of the problem (4.31).

line frequency is 50 Hz. The National Grid, is responsible for keeping the actual frequency within ±1% of this nominal frequency.[6]

Our model neglects losses of electricity in transmission and distribution lines. According to the World Bank[7] transmission and distribution losses in the UK account for approximately 7.5% (maximum 8.5% in 2004 and minimum 7.0% in 2010) of total electricity production. The losses clearly vary in time and hence a small mismatch in supply and demand can also be attributed to this.

For the reasons discussed above, we believe that it is possible to model realistic power prices by considering the continuous relaxation of (4.31) and neglecting the integrality constraints.

## 4.3 Grid operator

In Section 4.1, we included startup costs in our model. The equilibrium price we calculated contained many spikes, which in reality are prevented by actions of the grid operator. At times when the standing reserve is low, the grid operator financially incentivizes some power plants to start production, which increases the standing reserve and makes the delivery of electricity more reliable. In this section, we investigate how to incorporate the actions of

[6]See `http://www2.nationalgrid.com/uk/services/balancing-services/frequency-response/`.
[7]See `http://data.worldbank.org/indicator/EG.ELC.LOSS.ZS/countries/GB?display=graph`.

the grid operator into our model.

### 4.3.1 Quadratic programming formulation

The costs of the grid operator's actions are distributed among all market participants. They are collectively penalized in situations when the standing reserve is low. To include this effect in our model, we propose a quadratic penalty function $\Upsilon\left(SR\left(T_j\right)\right)$ defined as

$$\Upsilon\left(SR\left(T_j\right)\right) := \alpha\left(\max\left\{0, \beta - SR\left(T_j\right)\right\}\right)^2, \tag{4.48}$$

where $\alpha > 0$ and $\beta > 0$ are both used to describe the risk preference of the grid operator. Parameter $\beta$ specifies the level of the standing reserve at which the grid operator starts to take action. Parameter $\alpha$ tells us how much the grid operator is willing to incentivize power plants to start production.

It might not immediately be clear, how to write the penalty term (4.48) in a quadratic programming framework. We follow an approach that is widely used in the linear programming literature and introduce a decision variable $z\left(T_j\right)$ with the following constraints

$$\begin{aligned} z\left(T_j\right) &\geq 0, \\ z\left(T_j\right) &\geq \beta - SR\left(T_j\right), \end{aligned}$$

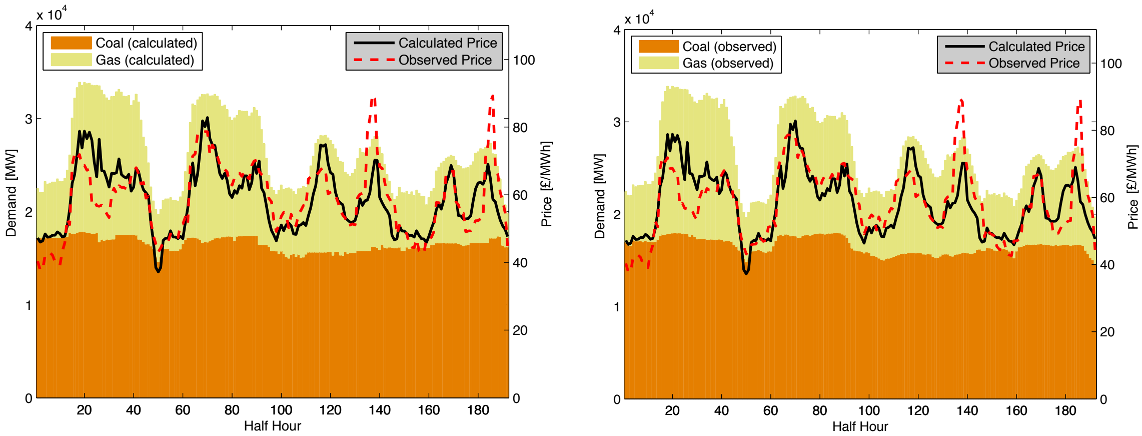
which hold for each $j \in J$. The penalty function $\Upsilon\left(SR\left(T_j\right)\right)$ can be written as a function of $z\left(T_j\right)$ as

$$\Upsilon\left(z\left(T_j\right)\right) := \alpha z\left(T_j\right)^2, \tag{4.49}$$

which fits into the quadratic programming framework. It can be easily incorporated into the algorithm we presented in Chapter 3 to calculate equilibrium electricity prices.

### 4.3.2 Numerical results

In this section, we investigate numerical results after inclusion of the grid operator. We set $\alpha = 0.01$ and $\beta = 1500$. Calculated equilibrium prices and the energy mix are depicted in Figure 4.10. Figure 4.10a depicts the calculated energy mix between coal and gas power




(a) (b)

Figure 4.10: Comparison of the calculated and historical electricity price, and the energy mix with startup costs and the grid operator included.

plants, while Figure 4.10b depicts the actually observed energy mix. Both figures contain the spot price calculated by our model and the observed spot price. The calculated equilibrium price of the month-ahead forward contract is £58.46/MWh. The determination of the optimal standing reserve is a challenging problem which has received a lot of attention in the literature (see Ela et al. (2010) and De Vos and Driesen (2014) for example).

By comparing Figures 4.6 and 4.10, we see that the calculated equilibrium electricity price in Figure 4.10 follows the observed daily variations much more closely. The calculated equilibrium electricity price does not contain any extreme spikes, because the grid operator prevented them by managing the standing reserve. In our model, we assume that the players (and the grid operator) have a perfect demand forecast. However, in reality this is never the case. The grid operator is not able to predict demand perfectly, and corrective actions are frequently required. When large corrective action is required close to delivery, then only a few (usually rather inefficient Open Cycle Gas Turbine) power plants are flexible enough to supply the additional demand, which causes extreme spikes in the electricity price.

The inclusion of the grid operator does not have any impact on the optimal number of contracts traded, and so the results from Figure 4.4 still apply.

Figure 4.11 shows that the inclusion of the grid operator did not have any significant impact on the error in the energy mix.

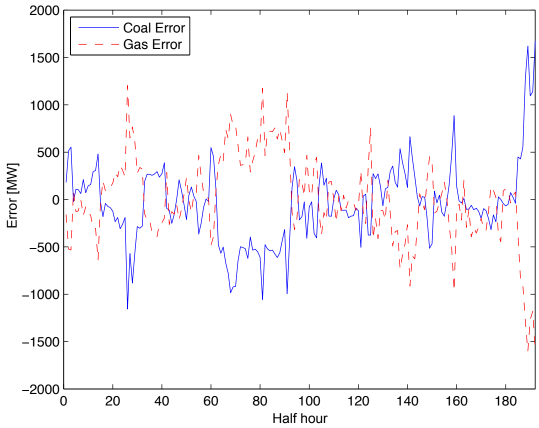



Figure 4.11: The difference between the calculated and observed gas and coal production after including the grid operator.

Figure 4.12 shows the standing reserve after the inclusion of the grid operator. The standing reserve never reaches zero since the grid operator coerces power plants to start production to ensure stability of the electricity grid. This makes the spot price smoother and significantly decreases the number of extreme spikes. Figure 4.12 also depicts the negative of the observed costs (quoted in £/MWh) that market participants had to pay the grid operator for managing the standing reserve.[8] We see that periods of low standing reserve coincide closely with high costs paid by market participants to the grid operator.

Figure 4.13 depicts the MIQP and QP errors after inclusion of the grid operator. By comparing Figures 4.9 and 4.13, we can see that the inclusion of the grid operator has a small impact on the errors, which remained approximately within ±400 MW and consequently ±1% of the total demand.

[8]The observed costs are available at the BM Reports webpage, see `http://www.bmreports.com/bsp/NETBSADFrame.htm`.

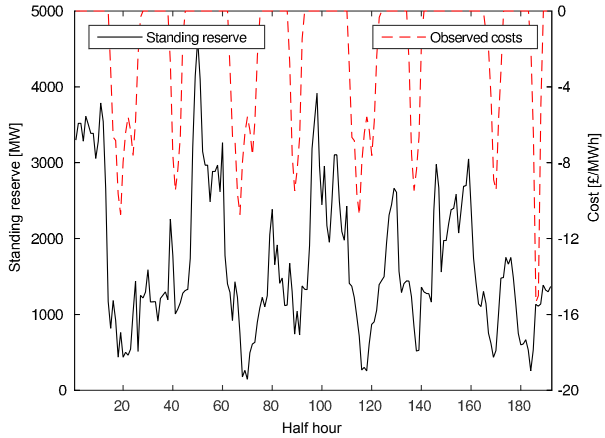



Figure 4.12: The standing reserve (black line) and the negative of observed historic costs (red dashed line) over the forecast horizon, after including the grid operator.

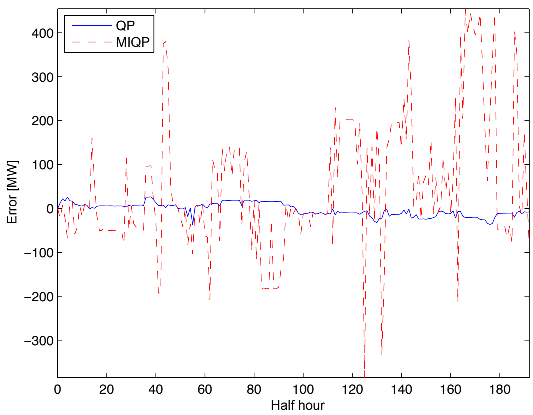



Figure 4.13: The error caused by neglecting the integrality constraints and considering the continuous relaxation of the problem (4.31), after including the grid operator.

# Chapter 5

# Conclusions

In this chapter, we first summarize the main topics covered by this thesis and then describe a few promising directions for future work.

## 5.1 Summary

We started this work by describing three approaches to modeling electricity prices, namely nonstructural, structural and game-theoretic. In this section, we first discuss the advantages and disadvantages of each approach and then provide a summary of the core chapters of the thesis.

### 5.1.1 Three approaches to modeling electricity prices

1. Nonstructural approach

   - The main advantage of the nonstructural approach is that it is computationally inexpensive and thus many scenarios can be simulated in a short time.
   - The main disadvantage of this approach is that it does not cope well with the quickly changing characteristics of electricity markets such as the introduction of new, intermittent, renewable power plants. Thus, such models, when used in practice, must frequently be recalibrated. Another important disadvantage of these models, when used in combination with the risk neutral valuation for

pricing forward contracts and other derivatives, is that they do not account for the non-storable nature of electricity.

2. Structural approach

    - The main advantage of the structural approach is that it is computationally relatively inexpensive (more expensive than nonstructural; less so than game-theoretic approach) and thus, a reasonably large number of scenarios can be simulated. Since it takes into account some of the fundamental properties of electricity, such as the supply and demand stack, it can be used to price various derivatives that depend on electricity as well as fuel prices (sparks spread, clean sparks spread etc.). It also takes into account the non-storable nature of electricity.
    - The main disadvantage, as with to the nonstructural approach, is that it is not able to model the changes caused by new power plants in the electricity market. Thus, structural models must often be recalibrated to include these alterations. Moreover, structural approaches are unable to simulate realistic spikes in the spot electricity price and thus, artificial tweaks are often required when such models are used in practice.

3. Game-theoretic approach

    - This is the approach that we studied and extend in this thesis. Its main advantage is that it captures the complexity of electricity markets in the greatest detail. Game-theoretic models can produce realistic spikes by modeling the startup costs of power plants as well as actions of the grid operator. They cope very well with the quickly changing electricity market and can be used to study the impact of a new power plant, or a failure of an existing one, on the electricity price. Moreover, they can be used to study how the statistical properties of the electricity price change when, for instance, more renewable capacity or energy storage facilities are added to the energy mix. They can also be used to predict the future relationship between fuel and electricity prices.

- The main disadvantage of the game-theoretic approach is that it is computationally more expensive than the other two.

### 5.1.2 Summary of each chapter of the thesis

In this section we summarize the core chapters of the thesis.

#### Chapter 2

1. We presented a new game-theoretic model for modeling the term structure of electricity prices. In contrast to other game-theoretic models in the literature, our model takes into account ramping and capacity constraints of power plants, and prices of fuel and emission certificates. Our model thus includes many important characteristics of electricity markets, which, to our knowledge, have never previously been modeled together. It is important to stress that even though our model is more detailed than other game-theoretic models, it remains computationally tractable.

2. Due to the capacity and ramping constraints that appear in our model, we had to find conditions that ensure existence and uniqueness of the equilibrium term structure of electricity prices. We proved existence if and only if the total demand of end users can be satisfied by the existing set of power plants connected to the grid. This condition ensures that there is sufficient capacity to cover the total demand, and that power plants can change their production quickly enough to cover the variation in the total demand.

   The conditions that ensure the uniqueness of the equilibrium term structure of electricity prices depend on physical characteristics of power plants and on the risk preference of players participating in the electricity market. Roughly speaking, if the players are sufficiently risk averse and power plants are similar enough, then the equilibrium term structure of electricity prices is unique. We showed that when the equilibrium electricity price of a particular contract is not unique, it may assume any value inside a closed interval.

3. We analyzed our model in the competitive equilibrium and Nash equilibrium settings. We established the equivalence of these two equilibria under an assumption of the uniqueness of the equilibrium term structure of electricity prices.

4. We analyzed the equilibrium term structure of electricity prices if all market players are interested in maximizing their expected profit without considering risk. We proved that, in this case, the term structure of electricity prices is a constant. Moreover, we showed that the term structure of electricity prices is non-unique (up to small parallel shifts) for some values of the total demand.

**Chapter 3**

1. We extended the model from the previous chapter by including block contracts, transactions costs, and futures contracts. Block contracts have already been studied in the context of nonstructural approaches, but, to our knowledge, this is the first game-theoretic model that takes them into account. We are not aware of any other game-theoretic model that examines the impact of transaction costs.

2. We presented a tractable quadratic programming formulation to calculate the equilibrium term structure of electricity prices.

3. We calibrated our model to the UK electricity grid. We downloaded and analyzed over 70 GB of data about the historical production of over 400 power plants in the UK.

4. We used results from the calibration exercise to calculate and predict the equilibrium term structure of electricity prices in the UK electricity market. We studied the impact of various parameters of the model. The most important qualitative conclusions are the following:

    (a) The risk preference of producers affects the absolute level as well as shape of the term structure. The risk preference of consumers, on the other hand, affects only the shape of the term structure. This happens because producers only produce electricity when the absolute level of the electricity price is high enough. On the other hand, consumers have to satisfy total demand of end users, regardless of the absolute level of the electricity price.

(b) The shape of the term structure of electricity prices in our simulations qualitatively matches that of fuel prices. We are not aware of any such studies in the literature.

(c) Tightening of ramping constraints makes the spot electricity price more serrated and slightly higher, on average. Similar to other fundamental models in the literature, our model produced spot electricity prices that were too flat and did not contain any spikes. This motivated us to include startup costs of power plants in the model.

**Chapter 4**

1. We further extended our model by including startup costs of power plants.

2. We developed a tractable bi-level optimization problem to estimate the startup costs of power plants from historical production data. We are not aware of any literature that addresses the estimation of startup costs of power plants, even though this is a very important task when modeling realistic electricity markets.

3. We demonstrated that startup costs cause high spikes in the spot electricity price. These occur when a power plant is turned on to cover only a short, temporary increase in the total demand. In this case, the startup costs of a power plant are spread over a short period of time, and thus a very high spot electricity price is required to encourage the power plant to start production.

4. Through numerical simulations, we demonstrated that the mixed integer quadratic programming problem that must be solved to calculate the term structure of electricity prices in the presence of startup costs, can be replaced by its continuous counterpart, without significantly affecting the solution. Note that even in reality, there is always a small mismatch between electricity supply and demand. We showed that the error caused by replacing the mixed integer programming problem with its continuous coun-

terpart is of the same order of magnitude as the allowed supply and demand mismatch in the UK grid.

5. By modeling the entire UK electricity grid, we demonstrated that the frequency and severity of spikes in the spot electricity price can be decreased by the intervention of the grid operator, responsible for managing the grid and for a reliable delivery of electricity. In our model, the main responsibility of the grid operator is to prevent the times of a low standing reserve at minimal cost. We are not aware of any other study where actions of the grid operator are taken into consideration when modeling the electricity price. In this work, we show why inclusion of the grid operator is crucial for modeling realistic electricity markets.

## 5.2 Main conclusions

In this section, we briefly describe the main conclusions of this thesis.

1. Our game-theoretic model for modeling electricity prices has a solution if and only if the total demand of end users can be satisfied by the existing set of power plants connected to the grid. Uniqueness of solutions is not guaranteed, but we showed that the conditions for uniqueness depend on the risk preferences of market players and physical properties of power plants.

2. Our model can be formulated as a tractable convex quadratic programming problem.

3. The risk preference of producers affects the equilibrium electricity price level, as well as the shape of the term structure. On the contrary, the risk preference of consumers affects only the shape of the term structure. Consumers can maintain profitability only by altering the pre-agreed price they charge end users.

4. Qualitatively speaking, the shape of the term structure of electricity price follows the one of fuel prices.

5. Startup costs and actions of the grid operator must be considered if one is to model realistic power markets. The fundamental models in the existing literature tend to

produce spot prices with a trend that is too flat (i.e. too high during the night and too low during the day).

6. Startup costs can be estimated precisely from historical power production data using our methodology.

7. Low standing reserve and the severity of spikes in the spot electricity price are closely related. Effective decision making of the grid operator can significantly decrease the frequency and severity of spikes.

8. Integrality constraints do not play a major role in modeling realistic electricity markets and can be removed without significantly affecting the solution.

## 5.3 Future work

In this section we explore a few possible extensions of this thesis that we find particularly interesting.

### 5.3.1 Finding the optimal grid operator's strategy

Our model can be used to study the frequency of electricity outages as a function of the grid operator's strategy (defined by $\alpha$ and $\beta$) for maintaining the standing reserve. A more risk-averse strategy implies less outages, but a higher electricity price, and vice versa. After estimating the economic value of an outage, it should be possible to find $\alpha$ and $\beta$ so as to optimize this tradeoff.

### 5.3.2 Modeling of renewables and interconnectors

In numerical simulations, we modeled the output of natural gas and coal power plants only, because they set the electricity price. However, to calculate or predict the demand satisfied by those two sources, one has to first predict the total demand on the system, and then subtract all interconnector flows, the production from renewable sources (wind, solar and hydro), and the production from nuclear power plants as described in (3.19). If our model is to be used in practice, these components must be forecasted and included in the model.

Moreover, actions of the grid operator depend on the amount of intermittent renewable energy in the system and thus, it would be interesting to investigate how the parameters $\alpha$ and $\beta$ defined in (4.48) depend on the amount of renewable generation in the energy mix.

### 5.3.3 Network constraints and the balancing market

In this work we assumed that, all electricity is delivered to one physical location to which all end users of electricity are connected. However, in reality the grid operator has to make sure that the electricity supplied is delivered to end users through the electricity network, which consists of numerous transmission and distribution lines with a limited capacity. As noted in Section 2.1.3.1, the trading of electricity occurs in two markets, namely the financial market and the balancing market. In the financial market, players trade forward and future electricity contracts without considering the limited capacity of transmission lines. One hour before the delivery of electricity all players inform the grid operator about their positions in forward and futures contracts. At this time, the grid operator takes over the management of the electricity grid and makes sure that forward and futures contracts are actually delivered. If the grid operator is not able to deliver one or more of the contracts due to the capacity constraints of transmission lines, then bids and offers, which players submit to the grid operator together with their positions in future and forward contracts, must be used to balance the electricity grid through the so-called balancing market. Accepting bids and offers in the balancing market could result that in one of the regions a previously non-profitable power plant, must increase/start production and in the other region, a previously profitable power plant, must decrease/stop production. By accepting bids and offers the grid operator introduces different electricity prices in different regions. This creates arbitrage opportunities between the financial and the balancing market. In some regions, it might be better to sell electricity in the balancing market instead of the financial market; for others, the opposite may hold. By extending our model to include network constraints, one could model the balancing market and hence explore arbitrage opportunities that occur between the financial and balancing markets and study the impact of these on the robustness of the electricity grid.

### 5.3.4 Extension to general concave utility functions

In this work we assumed that all producers and consumers have quadratic, i.e. mean-variance, objective functions. We believe, it would be interesting to study how results are impacted when we allow players to optimize their portfolio using different utility functions, such as the Hyperbolic Absolute Risk Aversion (HARA).

A few competing approaches can be used to approximate integrals and calculate expected utility functions. The most widely used approximation schemes are the quadrature, the Monte Carlo approximations and the Taylor series expansion. As argued in Garlappi and Skoulakis (2011), the Taylor series expansion often proves superior to the other techniques especially in the context of optimal portfolio choice.

Note that the Taylor series expansion of the expected utility function does not necessarily converge to its exact counterpart. Consider, for instance, a heavy-tailed distribution such as the Student's t-distribution, whose higher moments are all infinite. It is clear that one cannot use a Taylor series expansion with such a distribution. Determination of conditions that guarantee the convergence of Taylor series approximations has received a lot of attention in the literature (see Jurczenko and Maillet (2006) and Garlappi and Skoulakis (2011), for example). As argued in Garlappi and Skoulakis (2011), one can establish uniform convergence of the Taylor series approximations if, inter alia, the utility function is from the HARA class, and the distribution of the return has a bounded support.

Under these assumptions, it would be interesting to investigate how the proofs of the existence and uniqueness of solutions change if players are assumed to maximize their HARA expected utility functions. Most of our lemmas and theorems can be extended to this context. We provide some illustrative insights:

#### Analysis of the competitive equilibrium

- Lemma 2.2.2: the proof is no longer valid, but it might be possible to show the same result by using the analysis in Černý et al. (2012).

- Theorem 2.2.3: the proof is no longer valid, because it relies on the fact that only the linear term in the optimization problem (2.35) depends on the expected equilibrium price, $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$. If a Taylor series is used to approximate the expectation of a HARA utility, then all of the terms in the infinite sum depend on $\mathbb{E}^{\tilde{\mathbb{P}}}[\Pi]$.

- Lemma 2.2.4: the proof is valid for the HARA utility.
- Theorem 2.2.5: the proof is valid for the HARA utility.

**Analysis of the Nash equilibrium**

- Theorem 2.3.2: the proof is valid for the HARA utility.
- Lemma 2.3.3: the proof is no longer valid, but it might be possible to show the same result using the analysis in Robinson and Day (1974), and Bonnans and Shapiro (1998).
- Lemma 2.3.4: the proof is valid for the HARA utility.
- Theorem 2.3.6: the proof is valid for the HARA utility.

In our opinion, the major challenge of this extension would be the adjustment of Theorem 2.2.3.

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# Appendix A

# Data

In this appendix, we describe the UK power plant data that we downloaded and analyzed to calibrate our model. Broadly speaking, we split our required data into two categories, namely price and electricity grid data. We obtained the first one through the Bloomberg Terminal at the Oxford-Man Institute of Quantitative Finance. Since this is proprietary data, we do not discuss it here. Instead we focus on the electricity grid data.

UK electricity grid data is available at the BM Reports website[1] which is maintained and operated by Elexon. Their data is grouped into four categories:

1. National data: demand forecasts (day, day-ahead, 2-14 days-ahead and 2-52 weeks-ahead), production capacity forecasts (2-49 days-ahead, 1-5 years-ahead), system prices, costs of maintaining the standing reserve, the energy mix etc. are available in this section.

2. Zonal data: the UK electricity grid consists of 17 zones.[2] Demand forecasts (day and day-ahead) and production capacity forecasts (2-14 days-ahead, 2-52 weeks-ahead, 1-5 years-ahead) are available in this section.

3. Balancing Mechanism Unit (BMU) data: BMUs are used as units of trade within the Balancing Mechanism (BM). Each BMU accounts for a collection of plant and/or apparatus, and is considered the smallest grouping that can be independently controlled.

[1] `http://www.bmreports.com/`

[2] See a map of the zones at `http://www.bmreports.com/bsp/staticdata/System_Zone_map.pdf`.

As a result, most BMUs contain either a generating unit or a collection of consumption meters. To simplify the already heavy terminology in this thesis, we often refer to a BMU simply as 'a power plant'. Currently, there are 2 259 registered BMUs in the UK; 383 of these are the production and generation units[3] that we analyzed in this work. For each BMU the following data is available:

(a) Physical data: (final) physical notification, quiescent physical notification, maximum import limit, maximum export limit and bid-offer acceptance level at half hourly granularity. Each of these terms is defined and analyzed in Appendix A.1.2.

(b) Dynamic data: run up rate export, run down rate export, run up rate import, run down rate import, minimum zero time, minimum non-zero time, stable export limit, stable import limit, notice to deliver offers and notice to deliver bids at half hourly granularity. Each of these is defined and analyzed in Appendix A.1.3.

(c) Bid-offers data: bids and offers for every BMU submitted to the grid operator one hour before delivery of electricity at half hourly granularity. Bids and offers inform the grid operator of the prices at which players are willing to deviate from their current production submitted as the final physical notification (see physical data).

4. Transparency data: all data that is required under the new EU Transparency Regulation, which came into active effect in December 2014. The following data can be found:

   (a) unavailability of consumption units;

   (b) unavailability of production and generation units; and

   (c) unavailability of transmission infrastructure and transmission expansion projects.

We used physical and dynamic data at the BMU level, and energy mix data at the national level. We now investigate these in more detail.

[3]For mapping information between production units and zones see `http://www.bmreports.com/bsp/staticdata/bm_unit_to_oc2_zone_mapping.xls?rand=721`.

| Type | Comment | Number |
|---|---|---|
| CCGT | Combined Cycle Gas Turbines | 71 |
| COAL | Coal power plant | 54 |
| OCGT | Open Cycle Gas Turbine | 44 |
| NUCLEAR | Nuclear power plant | 22 |
| WIND | Wind farm | 114 |
| OIL | Oil power plant | 7 |
| PS | Pumped-storage | 16 |
| NPSHYD | Non-pumped-storage hydropower plant | 43 |
| INTEW | The East-West Interconnector (500 MW) | 1 |
| INTFR | The Anglo/French Interconnector (2 GW) | 1 |
| INTIRL | The Moyle Interconnector (500 MW) | 1 |
| INTNED | BritNed Interconnector (1 GW) | 1 |
| OTHER | Other types not listed in this table | 8 |

Table A.1: Summary classification of 383 generation BMUs by type.

## A.1 BMU data

All BMU data (other than the BMU type) is available at `http://www.bmreports.com/servlet/com.logica.neta.bwp_PanBMUData`. In this section, we examine type, physical and dynamical data in more detail. We also give a few illustrative examples to help us understand the data better.

### A.1.1 BMU type

The type of each production BMU can be found at `http://www.bmreports.com/bsp/staticdata/BMUFuelType.xls`. Currently, there are 383 production BMUs. A summary of all of the BMUs classified by type is presented in Table A.1.

In the rest of this section, we focus on eight BMUs (each of a different type) and investigate their physical and dynamic data more in detail. The data entities and definitions that follow are drawn from the BM Reports website.

### A.1.2 Physical data

Physical data is submitted to the grid operator one hour before the delivery of electricity on a half hourly basis and is made publicly available shortly thereafter. A file of physical data, for a particular half hour, typically contains between three and four thousand entries. In this section, we analyze the entries submitted on 11-Jul-2014 at 11:30 for delivery at 12:30 for eight BMUs. The operator of a BMU is required to submit the following items:

- FPN - (Final) Physical Notification. An FPN is the best estimate of the level of generation or demand that a participant in the BM expects a BMU to export or import in a particular half hour. FPNs have to be submitted to the grid operator as MW values by gate closure (one hour before delivery) for each half hour. A sample set of FPN values appears in Table A.2. The FPN values for all CCGT and coal BMUs, over a period of seven months, are presented in Appendices B.1 and B.2.

| BMU | Type | FPN (in MW) at 12:00 | FPN (in MW) at 12:30 |
|---|---|---|---|
| T_GRAI-6 | CCGT | 420 | 422 |
| T_DRAXX-2 | COAL | 645 | 645 |
| T_FAWL1G | OCGT | 0 | 0 |
| T_HUNB-7 | NUCLEAR | 503 | 503 |
| T_GNFSW-1 | WIND | 36 | 33 |
| T_FAWL1 | OIL | 0 | 0 |
| T_DINO-3 | PS | 300 | 300 |
| T_FASN-1 | NPSHYD | 0 | 0 |

Table A.2: Sample FPN data submitted on 11-Jul-2014 at 11:30 for delivery at 12:30. Zero values imply that the BMU was not producing at this time. This may occur when the electricity price is not high enough or due to maintenance.

- QPN - Quiescent Physical Notification. The QPN is a series of MW values and associated times, expressing the volume of generation or demand expected to be generated or consumed by an underlying process that forms part of the operation of a particular BMU. QPN values are not used in settlement, but if submitted to the grid operator, are deducted from FPN to determine the net operating level to which the dynamic data of a BMU apply at a particular time. The submission of QPNs is optional. A sample set of QPN values appears in Table A.3.

| BMU | Type | QPN (in MW) at 12:00 | QPN (in MW) at 12:30 |
|---|---|---|---|
| T_GRAI-6 | CCGT | 0 | 0 |
| T_DRAXX-2 | COAL | 0 | 0 |
| T_FAWL1G | OCGT | 0 | 0 |
| T_HUNB-7 | NUCLEAR | 0 | 0 |
| T_GNFSW-1 | WIND | 0 | 0 |
| T_FAWL1 | OIL | 0 | 0 |
| T_DINO-3 | PS | 0 | 0 |
| T_FASN-1 | NPSHYD | 0 | 0 |

Table A.3: Sample QPN data submitted on 11-Jul-2014 at 11:30 for delivery at 12:30. It is typical that most BMUs submit zero QPN values. However, QPN values must always be downloaded and analyzed for every half hour; if a QPN value deviates from zero, then it must be deducted from the FPN value to calculate the net production of a BMU. We collected historic QPNs spanning multiple years, and none of the values differed from zero.

- MIL - Maximum Import Limit. The MIL denotes the maximum power import level of a particular BMU at a particular time. It is submitted as a series of non-positive point MW values and associated times. A sample set of MIL values appears in Table A.4.

- MEL - Maximum Export Limit. The MEL denotes the maximum power export level of a particular BMU at a particular time. It is submitted as a series of point MW values and associated times. In this thesis, MEL is usually referred to as the maximum capacity, $\overline{W}^{p,l,r}_{\max}(T_j)$. A sample set of MEL values appears in Table A.5. The MEL values for all CCGT and coal BMUs, over a period of seven months, are presented in Appendices B.1 and B.2.

- BOAL - Bid-Offer Acceptance Level. Bid-offer acceptance is a formalized representation of the purchase and/or sale of offers and/or bids by the grid operator in its operation of the BM. Each bid-offer acceptance is shown as a MW level of operation

| BMU | Type | MIL (in MW) at 12:00 | MIL (in MW) at 12:30 |
|---|---|---|---|
| T_GRAI-6 | CCGT | 0 | 0 |
| T_DRAXX-2 | COAL | 0 | 0 |
| T_FAWL1G | OCGT | 0 | 0 |
| T_HUNB-7 | NUCLEAR | 0 | 0 |
| T_GNFSW-1 | WIND | 0 | 0 |
| T_FAWL1 | OIL | 0 | 0 |
| T_DINO-3 | PS | -287 | -287 |
| T_FASN-1 | NPSHYD | 0 | 0 |

Table A.4: Sample MIL data submitted on 11-Jul-2014 at 11:30 for delivery at 12:30. Note that generation BMUs cannot import electricity from the grid and thus, their MIL is zero. Among the BMUs in this sample, only T_DINO-3 can import electricity. Pumped-storage BMUs are able to import electricity form the grid to pump water from a lower elevation reservoir to a higher elevation.

at the beginning and at the end (and for any intermediate points deemed necessary) of the acceptance period. The start and end time of each acceptance is also shown, from which the bid-offer acceptance volumes (in MWh) can be calculated. A sample set of BOAL values appears in Table A.6.

| BMU | Type | MEL (in MW) at 12:00 | MEL (in MW) at 12:30 |
|---|---|---|---|
| T_GRAI-6 | CCGT | 422 | 422 |
| T_DRAXX-2 | COAL | 635 | 635 |
| T_FAWL1G | OCGT | 18 | 18 |
| T_HUNB-7 | NUCLEAR | 503 | 503 |
| T_GNFSW-1 | WIND | 36 | 33 |
| T_FAWL1 | OIL | 0 | 0 |
| T_DINO-3 | PS | 300 | 300 |
| T_FASN-1 | NPSHYD | 22 | 22 |

Table A.5: Sample MEL data submitted on 11-Jul-2014 at 11:30 for delivery at 12:30. All of the BMUs in our sample can generally produce electricity and thus submit non-zero MEL values, unless they cannot run due to maintenance or a very low electricity price. Oil fired power plants, such as T_FAWL1, are usually very inefficient, but flexible, and are only turned on when the electricity price is very high. We suspect that a low electricity price was the reason T_FAWL1 submitted zero MEL values in this sample. In this thesis, MEL is usually referred to as the maximum capacity, $\overline{W}^{p,l,r}_{\max}(T_j)$.

| BMU | Type | Request ID | Issued | Start time | End time | From (in MW) | To (in MW) |
|---|---|---|---|---|---|---|---|
| T_DRAXX-2 | COAL | 117566 | 23:43 | 23:45 | 23:50 | 645 | 600 |
| T_DRAXX-2 | COAL | 117566 | 23:43 | 23:50 | 23:55 | 600 | 600 |
| T_DRAXX-2 | COAL | 117566 | 23:43 | 23:55 | 00:00 | 600 | 645 |
| T_DRAXX-2 | COAL | 117567 | 23:51 | 23:53 | 23:57 | 600 | 600 |
| T_DRAXX-2 | COAL | 117567 | 23:51 | 23:57 | 00:02 | 600 | 645 |

Table A.6: Sample BOAL data on 03-May-2014 from 23:30 to 04-May-2014 00:00. In this sample, the grid operator accepted two offers identified by two request ID numbers. In both cases, the request was issued two minutes before the required change in production (compare this time to the notice to deliver offers and notice to deliver bids values in Tables A.15 and A.16 in the next section). The first request (117566) requires the BMU to decrease production from 645 MW to 600 MW (compare this decrease to the run down rate export in Table A.7 in the next section), then keep production at 600 MW for five minutes and finally increase the production back up to 645 MW (compare this increase to the run up rate export in Table A.7 in the next section). During the execution of the first request, the grid operator issued a new request (117567) that required the BMU to keep production at 600 MW for two minutes longer than initially planned. Note that BOAL only contains volumes of accepted bids and offers. Prices can be found in bid-offer data category as described in the beginning of this appendix.

### A.1.3 Dynamic data

- RURE - Run Up Rate Export. The RURE expresses the rate of increase in active power production for a particular BMU which is exporting power to the grid within a particular operating range. It is defined by a piecewise linear function with at most two kinks called 'elbows'. The initial rate, $\text{Rate}_1$, applies to production between zero and $\text{Elbow}_2$, $\text{Rate}_2$ to production between $\text{Elbow}_2$ and $\text{Elbow}_3$, and $\text{Rate}_3$ to production between $\text{Elbow}_3$ and the MEL. Rates are submitted as MW/min, and elbows as MW. In this work, RURE is usually referred to as the maximum rate for ramping-up, $\triangle\overline{W}^{p,l,r}_{\max}$. A sample set of RURE values appears in Table A.7.

| BMU | Type | Last updated | $\text{Rate}_1$ | $\text{Elbow}_2$ | $\text{Rate}_2$ | $\text{Elbow}_3$ | $\text{Rate}_3$ |
|---|---|---|---|---|---|---|---|
| T_GRAI-6 | CCGT | 10-Jul-2014 | 3.6 | 180 | 0.2 | 182 | 12 |
| T_DRAXX-2 | COAL | 28-Jun-2014 | 60 | 60 | 10 | 300 | 10 |
| T_FAWL1G | OCGT | 29-Oct-2012 | 0.5 | 2 | 17 | / | / |
| T_HUNB-7 | NUCLEAR | 17-Nov-2010 | 10 | 100 | 10 | 172 | 5 |
| T_GNFSW-1 | WIND | 03-Jan-2014 | 108 | 1 | 108 | 1 | 108 |
| T_FAWL1 | OIL | 15-Jan-2013 | 3.5 | 160 | 0.2 | 166 | 5 |
| T_DINO-3 | PS | 07-Apr-2003 | 999 | / | / | / | / |
| T_FASN-1 | NPSHYD | 31-Mar-2005 | 6 | 7 | 6 | 22 | 6 |

Table A.7: Sample RURE data submitted on 11-Jul-2014 at 11:30. RURE is defined by a piecewise linear function with at most two kinks called 'elbows'. The initial rate, $\text{Rate}_1$, applies to production between zero and $\text{Elbow}_2$, $\text{Rate}_2$ to production between $\text{Elbow}_2$ and $\text{Elbow}_3$, and $\text{Rate}_3$ to production between $\text{Elbow}_3$ and the MEL. Rates are submitted as MW/min, and elbows as MW. Some BMUs define their RURE by piecewise linear functions with only one elbow (or none), which can be identified by empty values marked with '/' above. In this work, RURE is usually referred to as the maximum rate for ramping-up, $\triangle\overline{W}^{p,l,r}_{\max}$. Since our model assumes a constant ramp-up rate, we did not use RURE values to calibrate the model, but instead made use of the largest positive changes in historical FPN production.

- RDRE - Run Down Rate Export. The RDRE expresses the rate of decrease in active power production for a particular BMU, which is exporting power to the grid within a particular operating range. It is defined by a piecewise linear function with at most two kinks called 'elbows'. The initial rate, $\text{Rate}_1$, in this case applies to production between the MEL and $\text{Elbow}_2$, $\text{Rate}_2$ to production between $\text{Elbow}_2$ and $\text{Elbow}_3$, and $\text{Rate}_3$ to production between $\text{Elbow}_3$ and zero. Rates are submitted as MW/min, and elbows as MW. In this work RDRE is usually referred to as the maximum rate for ramping-down, $\triangle\overline{W}^{p,l,r}_{\min}$. A sample set of RDRE values appears in Table A.8.

| BMU | Type | Last updated | $\text{Rate}_1$ | $\text{Elbow}_2$ | $\text{Rate}_2$ | $\text{Elbow}_3$ | $\text{Rate}_3$ |
|---|---|---|---|---|---|---|---|
| T_GRAI-6 | CCGT | 15-Apr-2014 | 12 | 200 | 13.4 | 120 | 17 |
| T_DRAXX-2 | COAL | 24-Jun-2014 | 10 | 300 | 10 | 200 | 99 |
| T_FAWL1G | OCGT | 01-Apr-2014 | 18 | / | / | / | / |
| T_HUNB-7 | NUCLEAR | 17-Nov-2010 | 4 | 550 | 4 | 175 | 3 |
| T_GNFSW-1 | WIND | 03-Jan2014 | 108 | 2 | 108 | 1 | 108 |
| T_FAWL1 | OIL | 20-Oct-2009 | 25 | / | / | / | / |
| T_DINO-3 | PS | 07-Apr-2003 | 999 | / | / | / | / |
| T_FASN-1 | NPSHYD | 31-Mar-2005 | 6 | 22 | 6 | 7 | 6 |

Table A.8: Sample RDRE data submitted on 11-Jul-2014 at 11:30. RDRE is defined by a piecewise linear function with at most two kinks called 'elbows'. $\text{Rate}_1$ applies to production between the MEL and $\text{Elbow}_2$, $\text{Rate}_2$ to production between $\text{Elbow}_2$ and $\text{Elbow}_3$, and $\text{Rate}_3$ to production between $\text{Elbow}_3$ and zero. Rates are submitted as MW/min, and elbows as MW. Some BMUs define their RURE by piecewise linear functions with only one elbow (or none), which can be identified by empty values marked with '/' above. In this work, RURE is usually referred to as the maximum rate for ramping-down, $\triangle\overline{W}^{p,l,r}_{\min}$. Since our model assumes a constant ramp-down rate, we did not use RDRE values to calibrate the model, but instead made use of the largest negative changes in historical FPN production.

- RURI - Run Up Rate Import. The same as RURE, except that these increase constraints apply for the import of electricity from the grid. A sample set of RURI values appears in Table A.9.

- RDRI - The same as RDRE, except that these decrease constraints apply for the

| BMU | Type | Last updated | $Rate_1$ | $Elbow_2$ | $Rate_2$ | $Elbow_3$ | $Rate_3$ |
|---|---|---|---|---|---|---|---|
| T_GRAI-6 | CCGT | / | / | / | / | / | / |
| T_DRAXX-2 | COAL | / | / | / | / | / | / |
| T_FAWL1G | OCGT | / | / | / | / | / | / |
| T_HUNB-7 | NUCLEAR | / | / | / | / | / | / |
| T_GNFSW-1 | WIND | / | / | / | / | / | / |
| T_FAWL1 | OIL | / | / | / | / | / | / |
| T_DINO-3 | PS | 07-Apr-2003 | 999 | / | / | / | / |
| T_FASN-1 | NPSHYD | / | / | / | / | / | / |

Table A.9: Sample RURI data submitted on 11-Jul-2014 at 11:30. Since production BMUs do not have import capabilities, they do not submit RURI data. The only BMU in our sample with import capabilities is T_DINO-3. Pumped-storage can increase consumption very quickly and thus, a value of 999 MW/min is submitted.

import of electricity from the grid. A sample set of RDRI values appears in Table A.10.

- MZT - Minimum Zero Time. The MZT is the minimum time, in minutes, that a BMU, which has been exporting, must operate at zero or import, before returning to export; whereas if the BMU has been importing, the MZT indicates the minimum time that it must operate at zero or export, before returning to import, if action by the grid operator (i.e. a bid-offer acceptance) places it at such a level. A sample set of MZT values appears in Table A.11.

- MNZT - Minimum Non-Zero Time. The MNZT represents the minimum time, in minutes, that a BMU can operate at a non-zero level as a result of a bid-offer acceptance. A sample set of MNZT values appears in Table A.12.

- SEL - Stable Export Limit. The SEL is a positive MW value, expressing the minimum stable operating level at which a particular BMU can export power to the transmission system. In this thesis, SEL is usually referred to as $\overline{W}^{p,l,r}_{\min}(T_j)$. A sample set of SEL values appears in Table A.13. The SEL values for all CCGT and coal BMUs, over a period of seven months are presented in Appendices B.1 and B.2.

| BMU | Type | Last updated | $Rate_1$ | $Elbow_2$ | $Rate_2$ | $Elbow_3$ | $Rate_3$ |
|---|---|---|---|---|---|---|---|
| T_GRAI-6 | CCGT | / | / | / | / | / | / |
| T_DRAXX-2 | COAL | / | / | / | / | / | / |
| T_FAWL1G | OCGT | / | / | / | / | / | / |
| T_HUNB-7 | NUCLEAR | / | / | / | / | / | / |
| T_GNFSW-1 | WIND | / | / | / | / | / | / |
| T_FAWL1 | OIL | / | / | / | / | / | / |
| T_DINO-3 | PS | 07-Apr-2003 | 999 | / | / | / | / |
| T_FASN-1 | NPSHYD | / | / | / | / | / | / |

Table A.10: Sample RDRI data submitted on 11-Jul-2014 at 11:30. Since production BMUs do not have import capabilities, they do not submit RURI. The only BMU in our sample with import capabilities is T_DINO-3. Pumped-storage can decrease consumption very quickly and thus, a value of 999 MW/min is submitted.

- SIL - Stable Import Limit. The SIL is a negative MW value, expressing the minimum stable operating level at which a particular BMU can import power from the transmission system. A sample set of SIL values appears in Table A.14.
- NDO - Notice to Deliver Offers. The NDO indicates the length of time, in minutes, between the issuing of a bid-offer acceptance and when a BMU begins to deliver offer volumes. A sample set of NDO values appears in Table A.15.
- NDB - Notice to Deliver Bids. The NBD indicates the length of time, in minutes, between the issuing of a bid-offer acceptance and when a BMU begins to deliver bid volumes. A sample set of NDB values appears in Table A.16.

| BMU | Type | Last updated | MZT (in min) |
|---|---|---|---|
| T_GRAI-6 | CCGT | 11-Jun-2014 | 360 |
| T_DRAXX-2 | COAL | 24-Jun-2014 | 360 |
| T_FAWL1G | OCGT | 01-Apr-2014 | 5 |
| T_HUNB-7 | NUCLEAR | 17-Nov-2010 | 999 |
| T_GNFSW-1 | WIND | / | / |
| T_FAWL1 | OIL | 20-Oct-2009 | 60 |
| T_DINO-3 | PS | 10-Jul-2014 | 5 |
| T_FASN-1 | NPSHYD | 09-Jul-2011 | 999 |

Table A.11: Sample MZT data submitted on 11-Jul-2014 at 11:30. Pumped-storage and OCGT power plants can be turned on and off frequently, because they have a small MZT. On the other hand, nuclear power plants cannot restart production soon after being turned off. It is surprising that such a high value was submitted for T_FASN-1. One could speculate that this non-pumped-storage hydropower plant was going through a maintenance period, because hydropower plants can usually turn off and on very quickly. MZT value does not apply to wind farms, because they are not operated in the same way as other power plants. Coal and CCGT power plants, when turned off, must usually stay offline for a few hours (six for the plants in this sample).

| BMU | Type | Last updated | MNZT (in min) |
|---|---|---|---|
| T_GRAI-6 | CCGT | 21-Jun-2014 | 360 |
| T_DRAXX-2 | COAL | 28-Jun-2014 | 240 |
| T_FAWL1G | OCGT | 29-Oct-2012 | 5 |
| T_HUNB-7 | NUCLEAR | 17-Nov-2010 | 999 |
| T_GNFSW-1 | WIND | / | / |
| T_FAWL1 | OIL | 15-Jan-2013 | 240 |
| T_DINO-3 | PS | 10-Jul-2014 | 180 |
| T_FASN-1 | NPSHYD | 03-Jul-2008 | 60 |

Table A.12: Sample MNZT data submitted on 11-Jul-2014 at 11:30. Similar conclusions apply as in Table A.11.

| BMU | Type | Last updated | SEL (in MW) |
|---|---|---|---|
| T_GRAI-6 | CCGT | 18-Jun-2014 | 230 |
| T_DRAXX-2 | COAL | 24-Jun-2014 | 300 |
| T_FAWL1G | OCGT | 01-Apr-2014 | 18 |
| T_HUNB-7 | NUCLEAR | 08-Aug-2014 | 470 |
| T_GNFSW-1 | WIND | / | / |
| T_FAWL1 | OIL | 01-Oct-2009 | 220 |
| T_DINO-3 | PS | 21-Jun-2014 | 150 |
| T_FASN-1 | NPSHYD | 29-Apr-2014 | 7 |

Table A.13: Sample SEL data submitted on 11-Jul-2014 at 11:30. Power plants can only produce between the SEL and MEL values. A production below the SEL may only occur during the ramp-up or ramp-down phase. Comparing the SEL and MEL values for PS, COAL and CCGT power plants, we see that the SEL accounts for approximately 50% of the MEL. On the other hand, the SEL accounts for 100% of the MEL for OCGT and 93% for NUCLEAR, in this sample. Wind power plants do not submit SEL data, because they can produce at any value between zero and the MEL, depending on wind speed.

| BMU | Type | Last updated | SIL (in MW) |
|---|---|---|---|
| T_GRAI-6 | CCGT | / | / |
| T_DRAXX-2 | COAL | 12-Sep-2012 | 0 |
| T_FAWL1G | OCGT | 20-Jul-2008 | 0 |
| T_HUNB-7 | NUCLEAR | / | / |
| T_GNFSW-1 | WIND | / | / |
| T_FAWL1 | OIL | 16-Apr-2009 | 0 |
| T_DINO-3 | PS | 12-Feb-2013 | -265 |
| T_FASN-1 | NPSHYD | / | / |

Table A.14: Sample SIL data submitted on 11-Jul-2014 at 11:30. Since production BMUs do not have import capabilities, they do not submit SIL data. The only BMU in our sample with import capabilities is T_DINO-3. Comparing the SIL and MIL data, we see that T_DINO-3, when importing electricity from the grid, imports at least of 92% of its MIL.

| BMU | Type | Last updated | NDO (in min) |
|---|---|---|---|
| T_GRAI-6 | CCGT | 27-Aug-2010 | 2 |
| T_DRAXX-2 | COAL | 24-Jun-2014 | 2 |
| T_FAWL1G | OCGT | 01-Apr-2014 | 2 |
| T_HUNB-7 | NUCLEAR | 17-Nov-2014 | 2 |
| T_GNFSW-1 | WIND | 11-Sep-2012 | 2 |
| T_FAWL1 | OIL | 20-Oct-2009 | 2 |
| T_DINO-3 | PS | 01-Jun-2006 | 0 |
| T_FASN-1 | NPSHYD | 31-Mar-2005 | 2 |

Table A.15: Sample NDO data submitted on 11-Jul-2014 at 11:30. Power plants, when producing, can change production quickly. If the grid operator accepts their offers, they can respond to this request immediately (T_DINO-3) or in two minutes (the rest). Note that even wind power plants can submit offers to decrease their production, when needed.

| BMU | Type | Last updated | NDB (in min) |
|---|---|---|---|
| T_GRAI-6 | CCGT | 27-Aug-2010 | 2 |
| T_DRAXX-2 | COAL | 24-Jun-2014 | 2 |
| T_FAWL1G | OCGT | 01-Apr-2014 | 2 |
| T_HUNB-7 | NUCLEAR | 17-Nov2014 | 2 |
| T_GNFSW-1 | WIND | / | / |
| T_FAWL1 | OIL | 20-Oct-2009 | 2 |
| T_DINO-3 | PS | 01-Jun-2006 | 0 |
| T_FASN-1 | NPSHYD | 31-Mar-2005 | 2 |

Table A.16: Sample NDB data submitted on 11-Jul-2014 at 11:30. Power plants, when producing, can change production quickly. If the grid operator accepts their bids, they can react to this request immediately (T_DINO-3) or in two minutes (the rest). Note that wind power plants cannot generally submit bids, because they are not able to increase production.

## A.2 National data

National data contains all data that is aggregated at the national level. In our model, we are particularly interested in the energy mix, which we use in two ways. First, the sum of production from CCGT and coal power plants is used as a proxy for total demand, as in (3.19). This allows us to focus only on CCGT and coal power plants, because they effectively set the electricity price. Second, historical energy mix data was compared to our calculated mix to verify the results of our model.

### A.2.1 Energy mix

The National Grid measures production from generation BMUs connected to the high voltage transmission system in real-time. To provide market participants with information regarding the levels of generation, this metering data is aggregated into fuel type categories. The data is provided as half hourly time averages per BMU type, in MW, and is available at `http://www.bmreports.com/bsp/additional/soapserver.php`. A sample set of energy mix appears in Table A.17.

| HH | CCGT | OIL | COAL | NUC. | WIND | PS | NPSHYD | OCGT | OTH. | INTFR | INTIRL | INTNED | INTEW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 374 | 0 | 8 068 | 7 139 | 1 079 | 0 | 256 | 0 | 757 | 1 998 | 132 | 992 | 72 |
| 2 | 6 319 | 0 | 8 106 | 7 129 | 911 | 0 | 253 | 0 | 757 | 1 998 | 132 | 1 000 | 80 |
| 3 | 6 379 | 0 | 8 235 | 7 118 | 798 | 0 | 234 | 0 | 755 | 1 998 | 132 | 1 000 | 80 |
| 4 | 6 663 | 0 | 8 367 | 7 101 | 845 | 0 | 233 | 0 | 754 | 1 998 | 132 | 1 000 | 80 |
| 5 | 6 581 | 0 | 8 304 | 7 095 | 982 | 0 | 234 | 0 | 754 | 1 998 | 128 | 1 000 | 80 |
| 6 | 6 467 | 0 | 8 109 | 7 074 | 922 | 0 | 227 | 0 | 782 | 1 998 | 72 | 1 000 | 80 |
| 7 | 6 631 | 0 | 8 059 | 7 058 | 880 | 0 | 228 | 0 | 783 | 1 998 | 72 | 616 | 34 |
| 8 | 6 673 | 0 | 8 045 | 7 047 | 798 | 0 | 227 | 0 | 782 | 1 998 | 72 | 614 | 26 |
| 9 | 6 479 | 0 | 7 955 | 7 043 | 850 | 0 | 225 | 0 | 783 | 1 998 | 72 | 614 | 26 |
| 10 | 6 517 | 0 | 7 972 | 7 041 | 906 | 0 | 226 | 0 | 783 | 1 998 | 72 | 612 | 26 |

Table A.17: Sample energy mix on 01-May-2015, starting at midnight. The first column denotes the half hour of the day. All other numerical values are in MW.

# Appendix B

# Calibration results

In this chapter, we present calibration results for all CCGT and coal power plants. A description of the figures is available in Section 4.2.1.

## B.1 Gas power plants

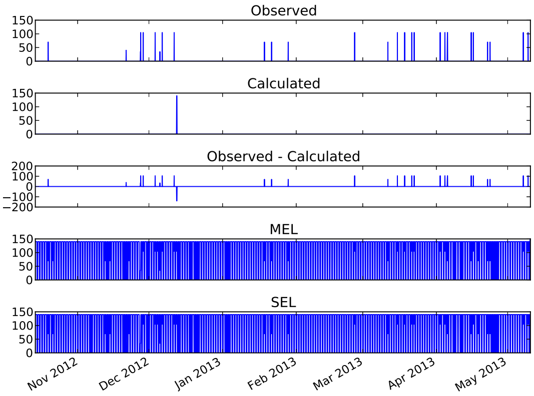


Figure B.1.1: BMU: E_BRGG-1, $e = 1.0$, $s = 32886$ and $m = 5.6$.

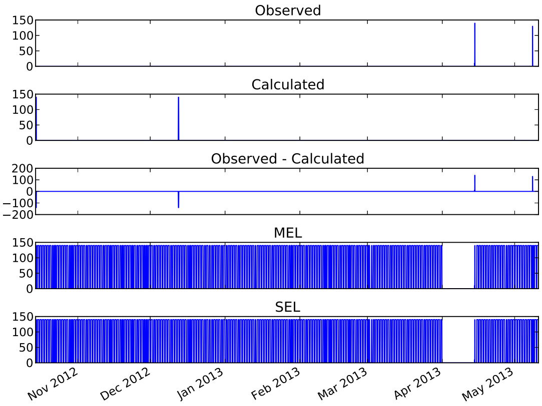


Figure B.1.2: BMU: E_BRYP-1, $e = 1.0$, $s = 3500$ and $m = 8.0$.

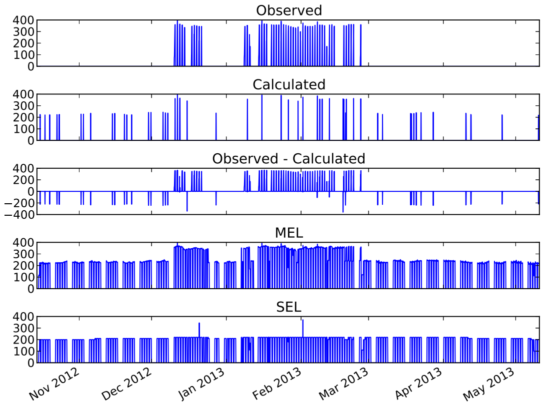


Figure B.1.3: BMU: E_CORB-1, $e = 0.95$, $s = 8289$ and $m = 9.0$.

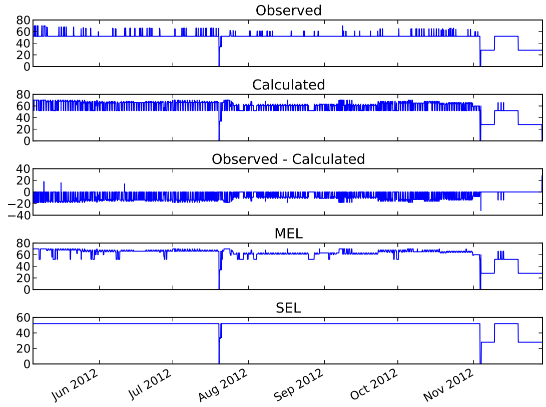


Figure B.1.4: BMU: E_DERW-1, $e = 0.5$, $s = 1750$ and $m = 6.9$.

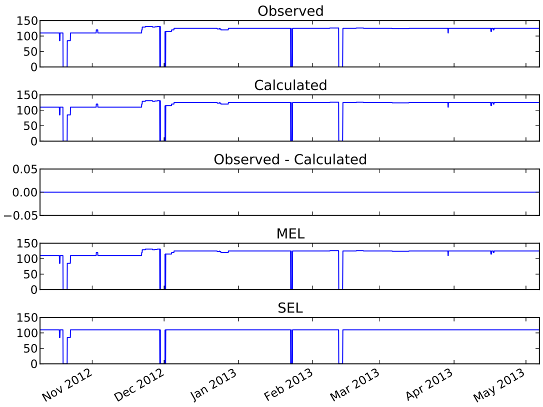


Figure B.1.5: BMU: E_FAWN-1, $e = 0.29$, $s = 17297$ and $m = 0.0$.

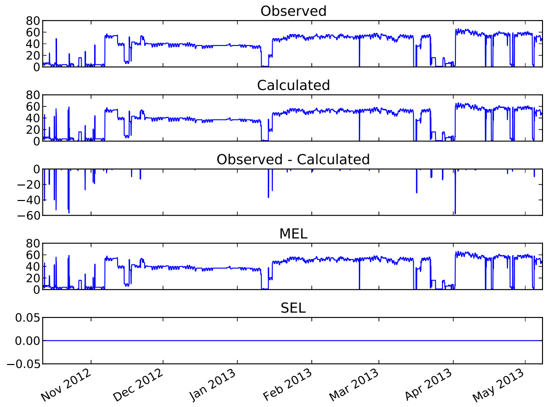


Figure B.1.6: BMU: E_FELL-1, $e = 0.29$, $s = 39$ and $m = 0.0$.

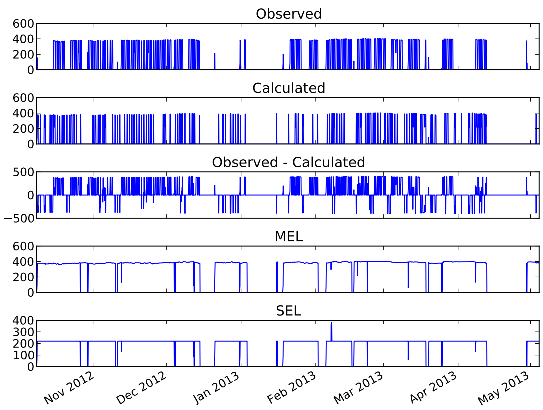


Figure B.1.7: BMU: E_GYAR-1, $e = 0.72$, $s = 2083$ and $m = 6.7$.

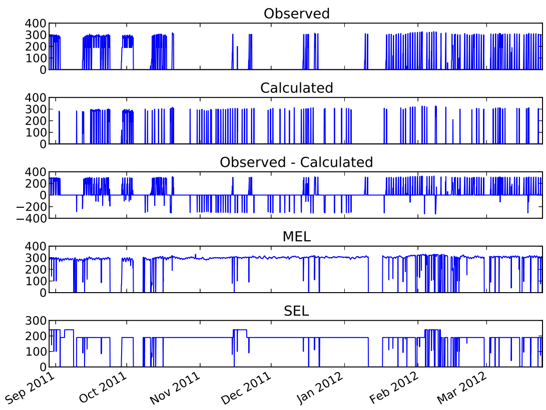


Figure B.1.8: BMU: E_KLYN-A-1, $e = 0.96$, $s = 6385$ and $m = 0.0$.

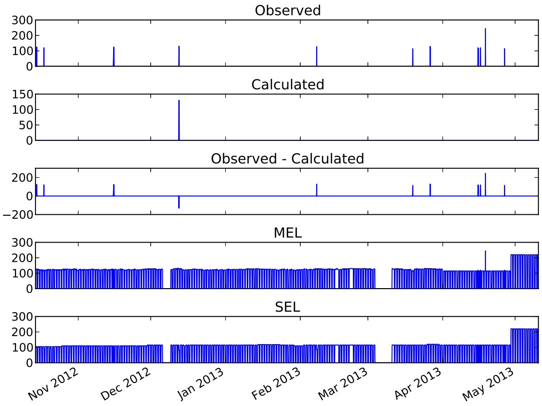


Figure B.1.9: BMU: E_PETEM1, $e = 0.99$, $s = 5248$ and $m = 6.1$.

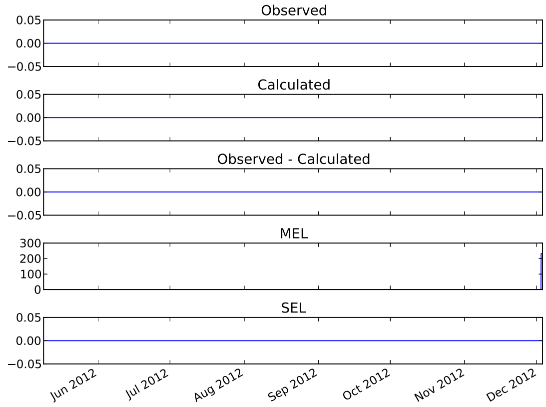


Figure B.1.10: BMU: E_ROOS-1, $e = 0.85$, $s = 5759$ and $m = 6.4$.

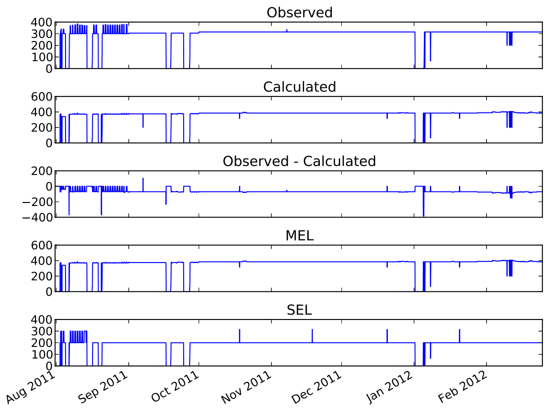


Figure B.1.11: BMU: E_SHOS-1, $e = 0.32$, $s = 8608$ and $m = 1.3$.

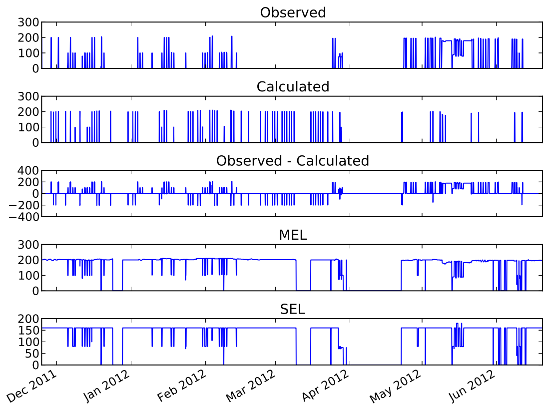


Figure B.1.12: BMU: E_SHOT-1, $e = 0.93$, $s = 126$ and $m = 6.6$.

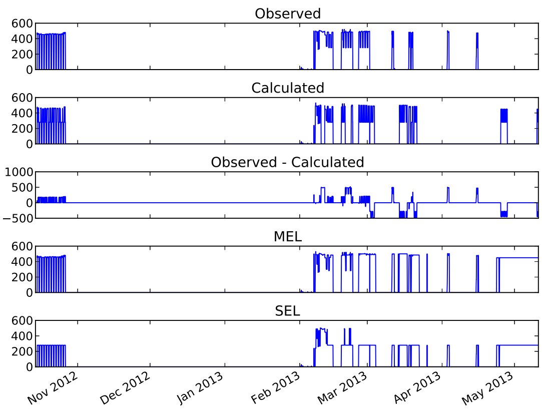


Figure B.1.13: BMU: T_BAGE-1, $e = 0.74$, $s = 13250$ and $m = 0.0$.

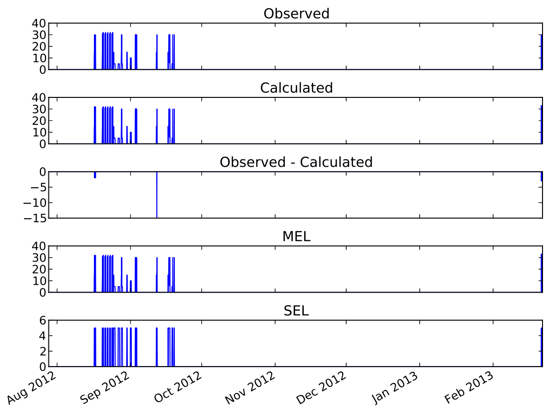


Figure B.1.14: BMU: T_BAGE-2, $e = 0.32$, $s = 6745$ and $m = 3.7$.

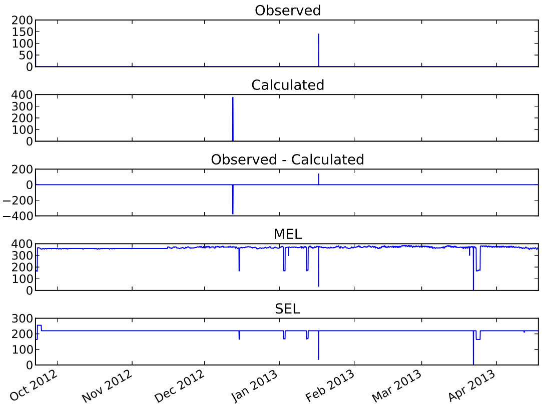


Figure B.1.15: BMU: T_BARK-1, $e = 0.98$, $s = 83800$ and $m = 4.6$.

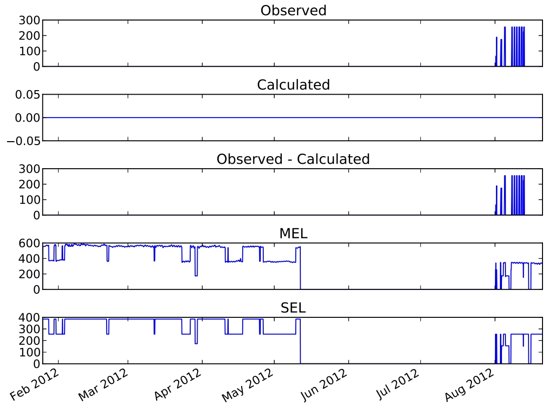


Figure B.1.16: BMU: T_BARKB2, $e = 0.9$, $s = 137534$ and $m = 9.0$.

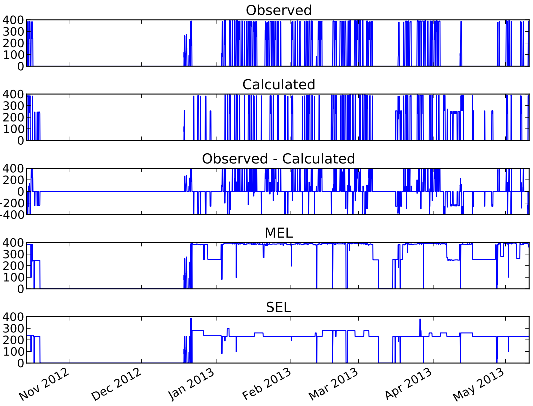


Figure B.1.17: BMU: T_CDCL-1, $e = 0.74$, $s = 2289$ and $m = 3.8$.

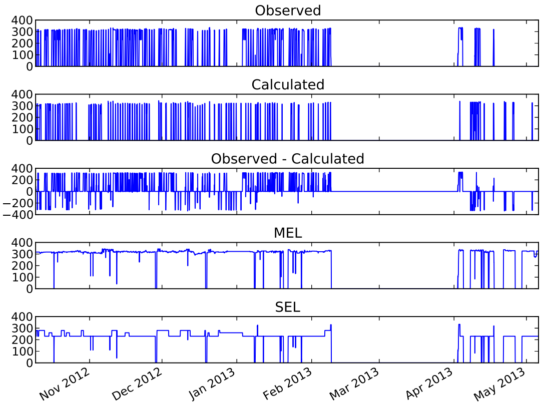


Figure B.1.18: BMU: T_CNQPS-1, $e = 0.79$, $s = 14889$ and $m = 2.2$.

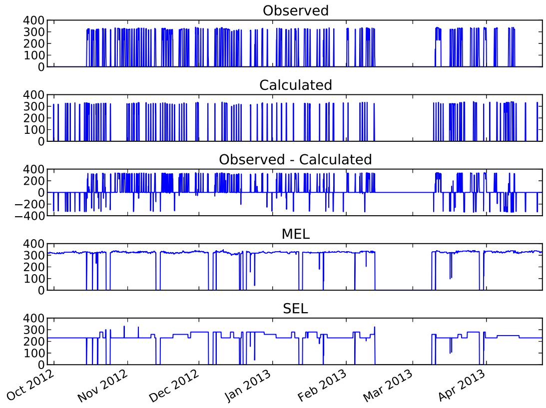


Figure B.1.19: BMU: T_CNQPS-2, $e = 0.76$, $s = 19964$ and $m = 5.1$.

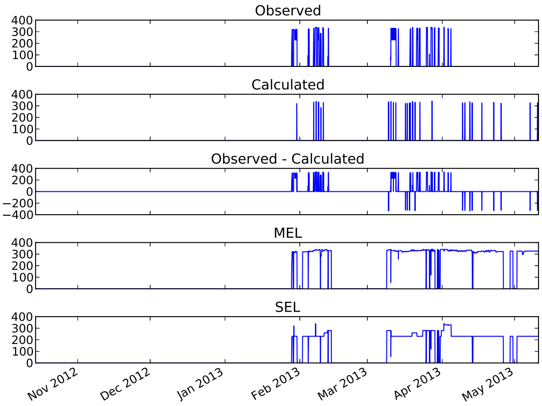


Figure B.1.20: BMU: T_CNQPS-3, $e = 1.0$, $s = 205$ and $m = 9.0$.

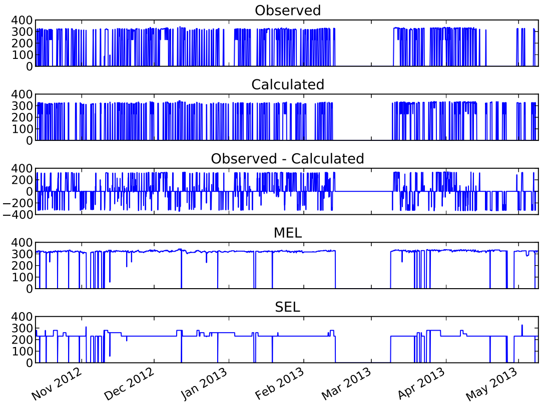


Figure B.1.21: BMU: T_CNQPS-4, $e = 0.67$, $s = 22136$ and $m = 4.8$.

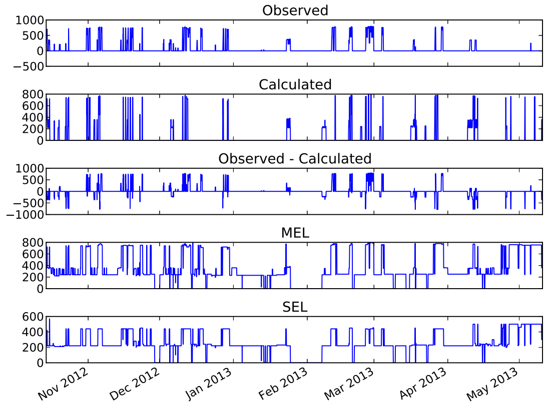


Figure B.1.22: BMU: T_COSO-1, $e = 0.78$, $s = 64354$ and $m = 2.8$.

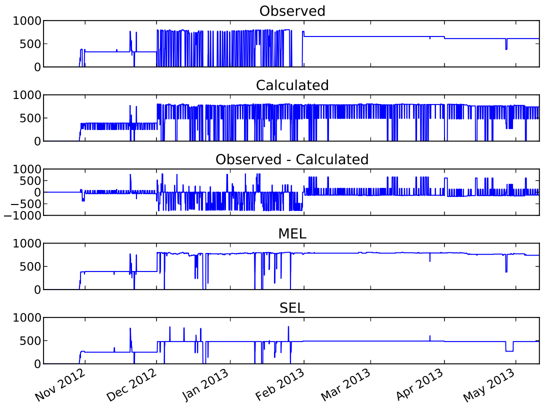


Figure B.1.23: BMU: T_DAMC-1, $e = 0.52$, $s = 69283$ and $m = 7.8$.

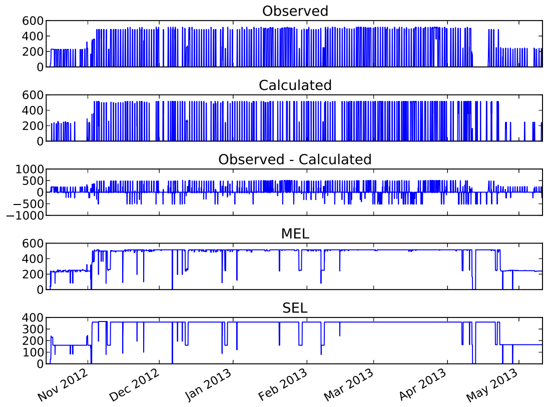


Figure B.1.24: BMU: T_DEEP-1, $e = 0.7$, $s = 9408$ and $m = 8.1$.

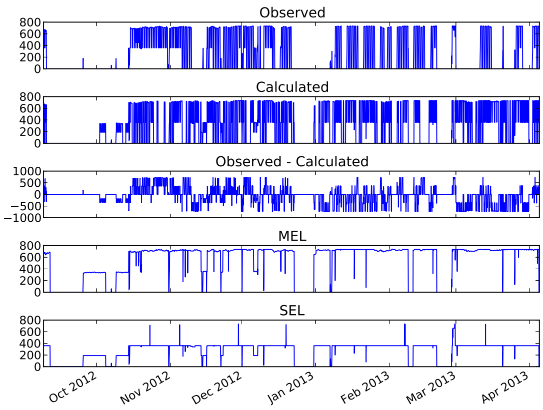


Figure B.1.25: BMU: T_DIDCB5, $e = 0.62$, $s = 64164$ and $m = 5.6$.

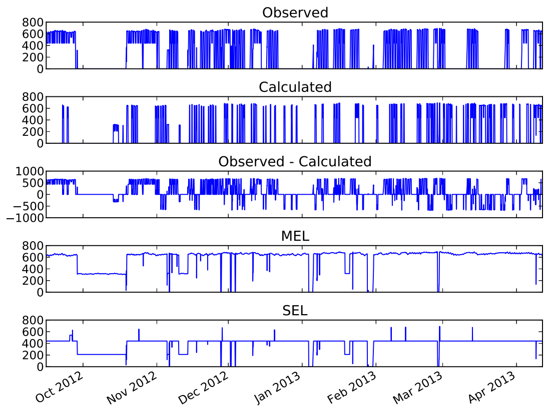


Figure B.1.26: BMU: T_DIDCB6, $e = 0.69$, $s = 67313$ and $m = 5.6$.

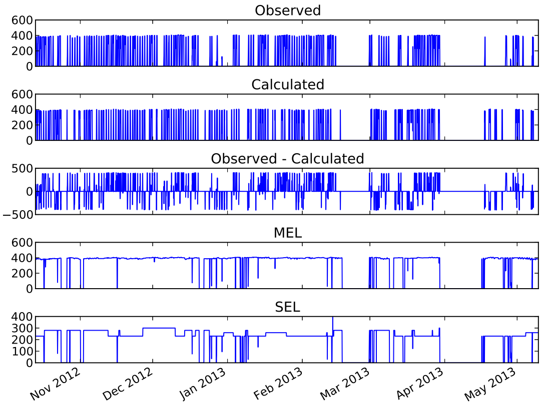


Figure B.1.27: BMU: T_EECL-1, $e = 0.74$, $s = 20213$ and $m = 2.7$.

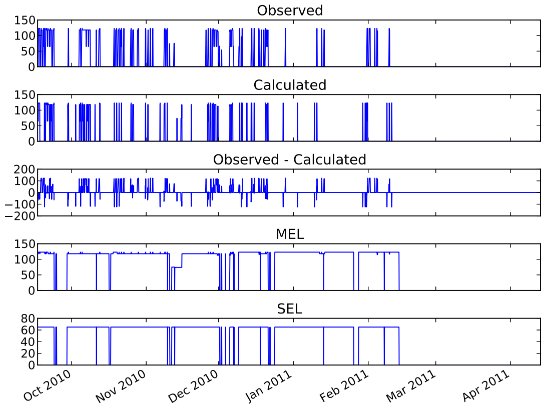


Figure B.1.28: BMU: T_FIFE-1, $e = 0.93$, $s = 6410$ and $m = 0.0$.

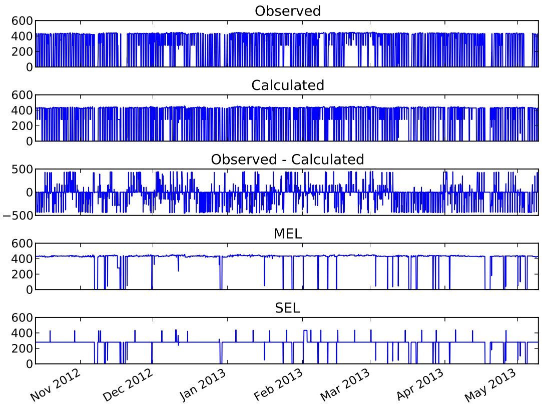


Figure B.1.29: BMU: T_GRAI-6, $e = 0.57$, $s = 12338$ and $m = 4.2$.

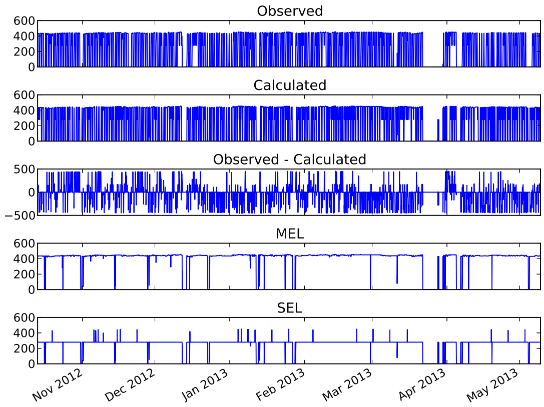


Figure B.1.30: BMU: T_GRAI-7, $e = 0.64$, $s = 9407$ and $m = 2.2$.

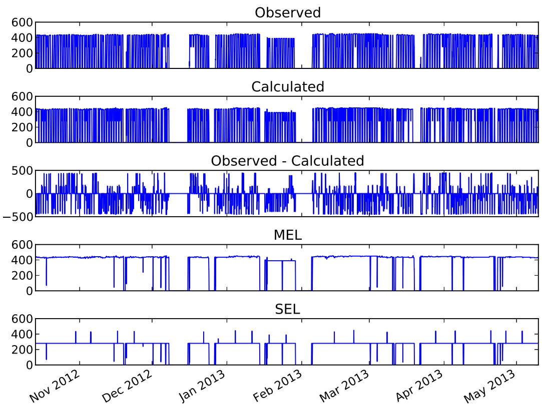


Figure B.1.31: BMU: T_GRAI-8, $e = 0.65$, $s = 5397$ and $m = 0.8$.

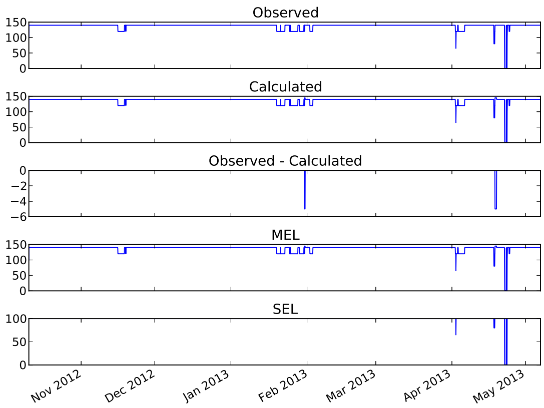


Figure B.1.32: BMU: T_GRMO-1, $e = 0.29$, $s = 22124$ and $m = 0.0$.

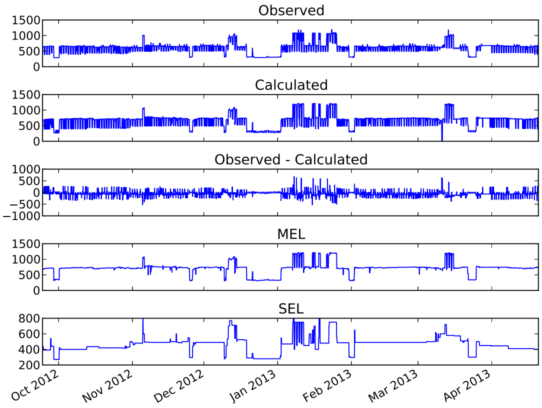


Figure B.1.33: BMU: T_HUMR-1, $e = 0.52$, $s = 155824$ and $m = 6.2$.

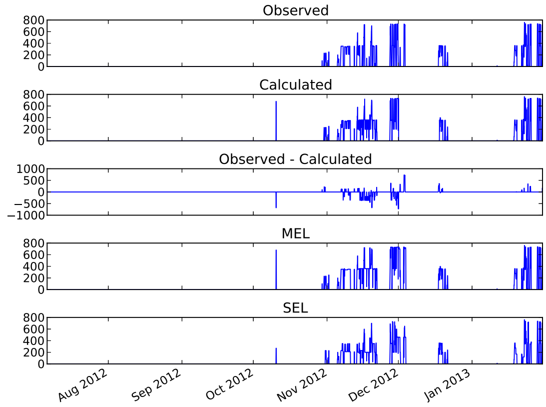


Figure B.1.34: BMU: T_KEAD-1, $e = 0.62$, $s = 125862$ and $m = 4.4$.

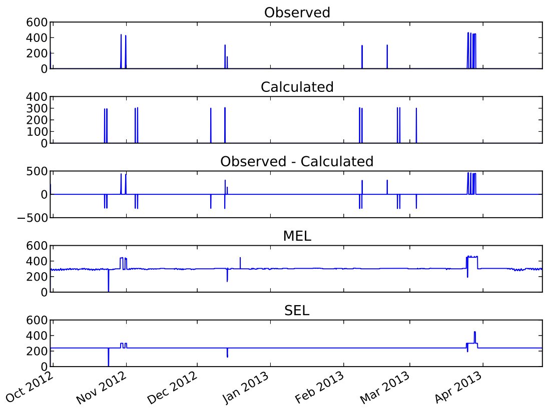


Figure B.1.35: BMU: T_KILLPG-1, $e = 1.0$, $s = 37923$ and $m = 8.2$.

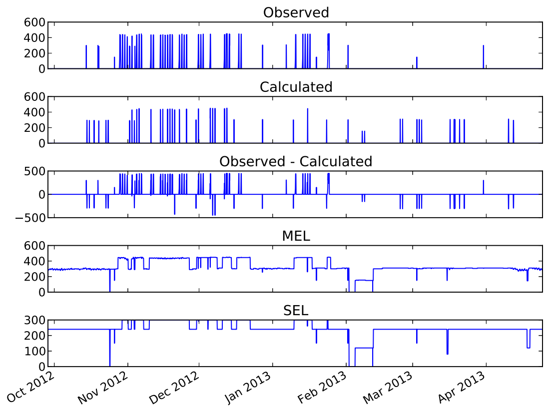


Figure B.1.36: BMU: T_KILLPG-2, $e = 1.0$, $s = 16043$ and $m = 4.5$.

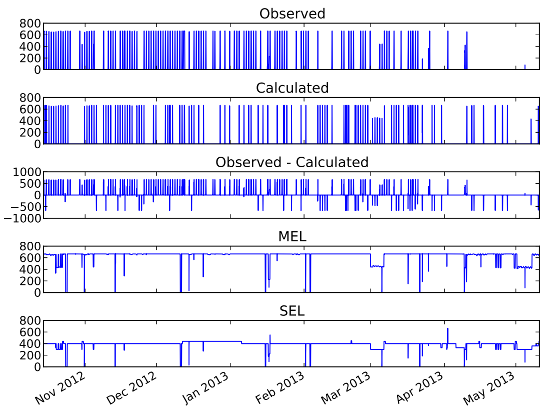


Figure B.1.37: BMU: T_KILNS-1, $e = 1.0$, $s = 5681$ and $m = 4.6$.

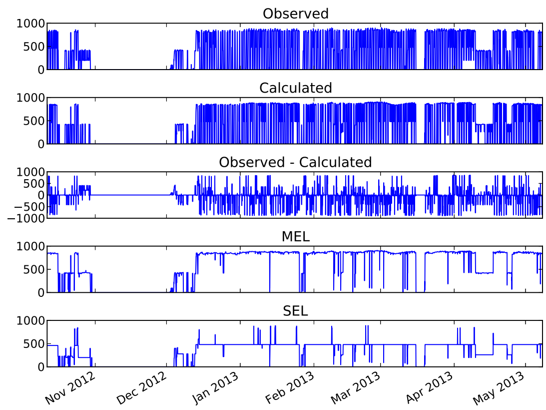


Figure B.1.38: BMU: T_LAGA-1, $e = 0.58$, $s = 13291$ and $m = 7.1$.

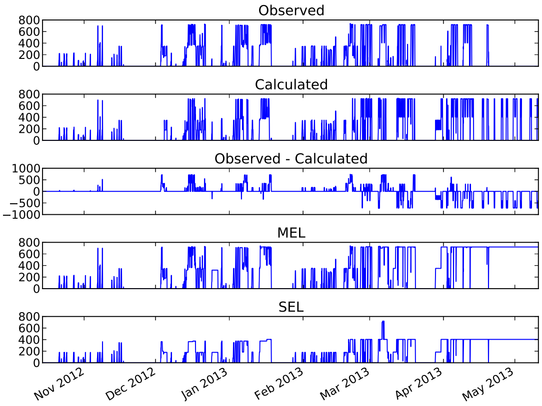


Figure B.1.39: BMU: T_LBAR-1, $e = 0.6$, $s = 86007$ and $m = 9.0$.

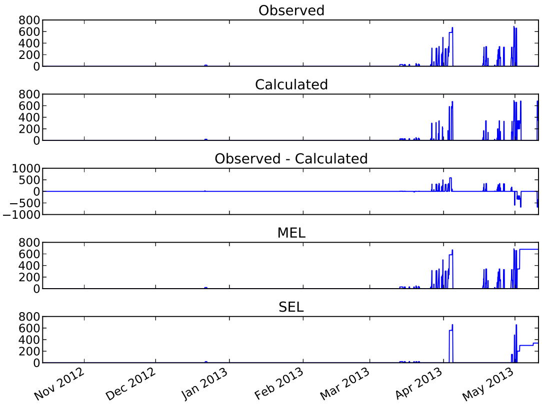


Figure B.1.40: BMU: T_MEDP-1, $e = 0.68$, $s = 17250$ and $m = 4.7$.

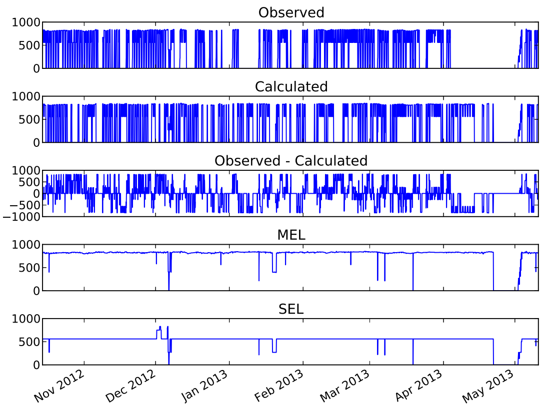


Figure B.1.41: BMU: T_MRWD-1, $e = 0.65$, $s = 77337$ and $m = 4.8$.

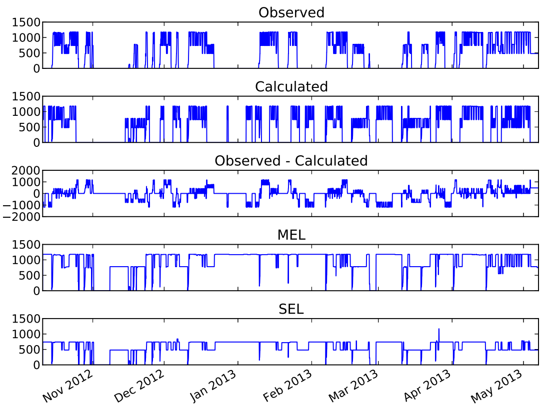


Figure B.1.42: BMU: T_PEHE-1, $e = 0.62$, $s = 29500$ and $m = 6.9$.

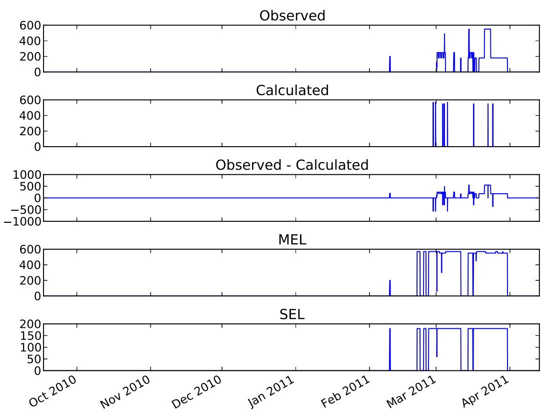


Figure B.1.43: BMU: T_PEHE-2, $e = 1.0$, $s = 342$ and $m = 9.0$.

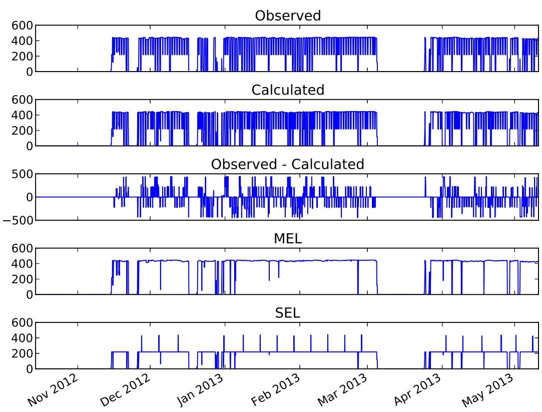


Figure B.1.44: BMU: T_PEMB-11, $e = 0.58$, $s = 28069$ and $m = 3.7$.

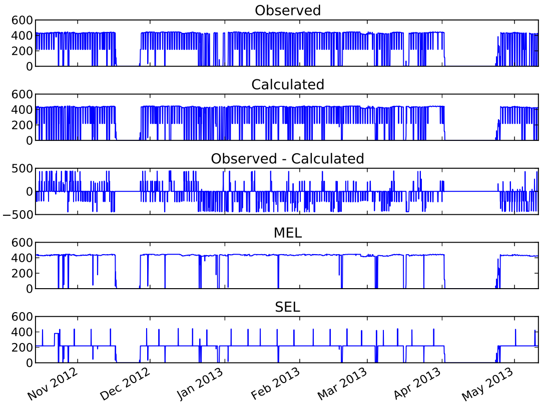


Figure B.1.45: BMU: T_PEMB-21, $e = 0.47$, $s = 21834$ and $m = 9.0$.

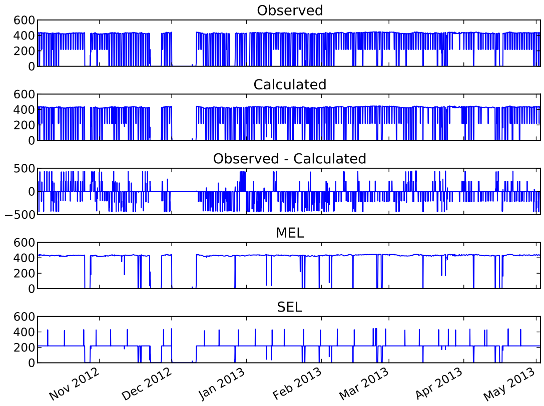


Figure B.1.46: BMU: T_PEMB-31, $e = 0.55$, $s = 16416$ and $m = 2.7$.

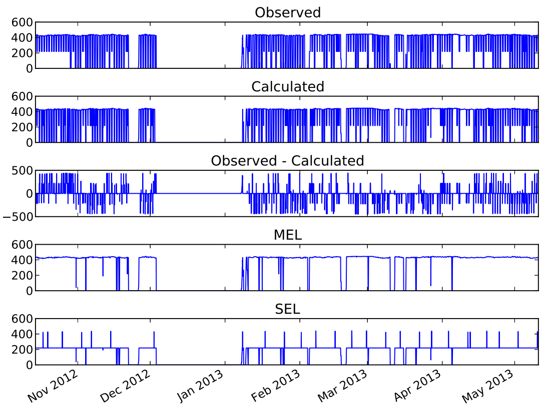


Figure B.1.47: BMU: T_PEMB-41, $e = 0.48$, $s = 1477$ and $m = 9.0$.

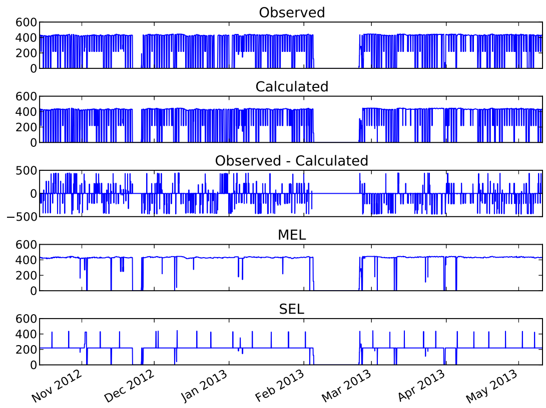


Figure B.1.48: BMU: T_PEMB-51, $e = 0.52$, $s = 16793$ and $m = 5.4$.

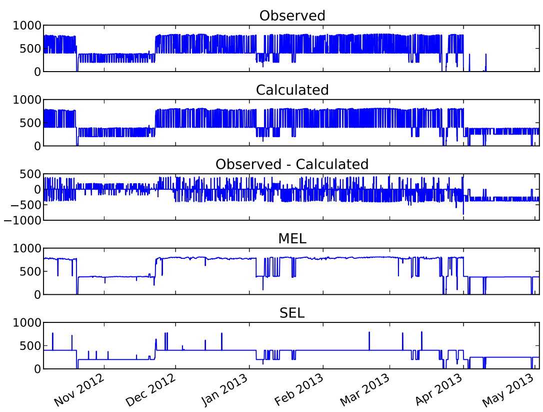


Figure B.1.49: BMU: T_ROCK-1, $e = 0.62$, $s = 20250$ and $m = 3.4$.

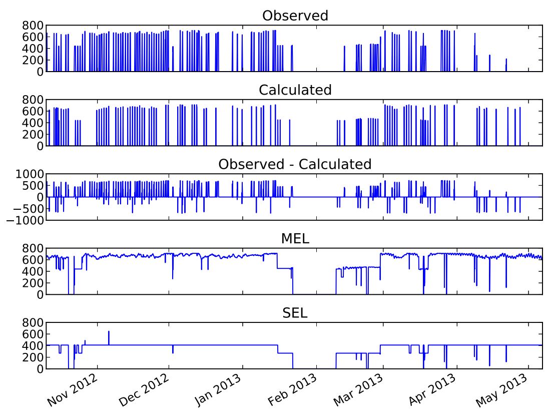


Figure B.1.50: BMU: T_RYHPS-1, $e = 1.0$, $s = 426$ and $m = 6.0$.

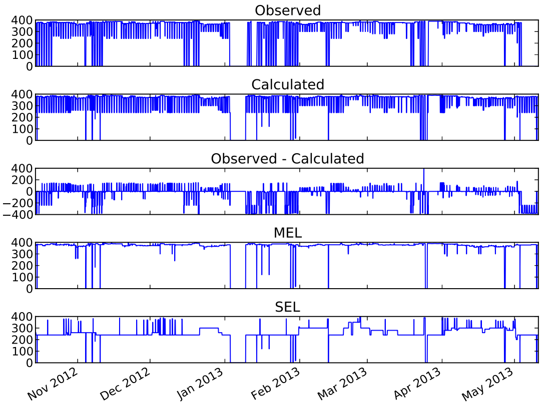


Figure B.1.51: BMU: T_SCCL-1, $e = 0.55$, $s = 87855$ and $m = 3.5$.

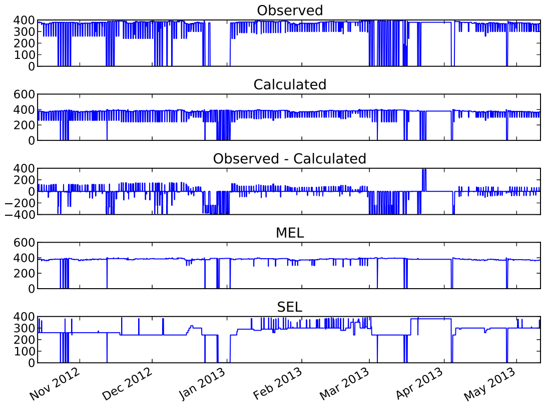


Figure B.1.52: BMU: T_SCCL-2, $e = 0.45$, $s = 8664$ and $m = 9.0$.

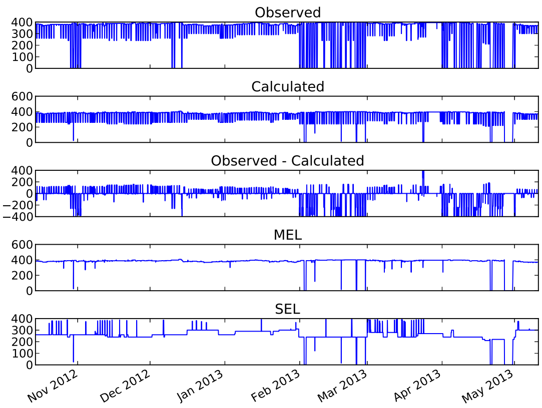


Figure B.1.53: BMU: T_SCCL-3, $e = 0.52$, $s = 10175$ and $m = 5.8$.

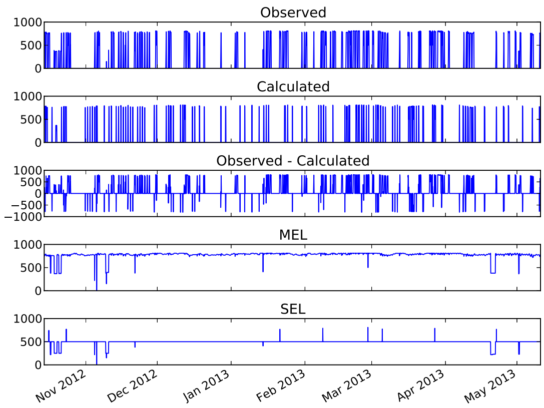


Figure B.1.54: BMU: T_SEAB-1, $e = 0.8$, $s = 43302$ and $m = 5.9$.

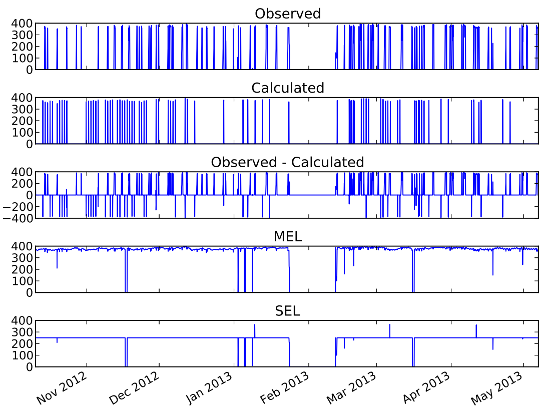


Figure B.1.55: BMU: T_SEAB-2, $e = 1.0$, $s = 4730$ and $m = 9.0$.

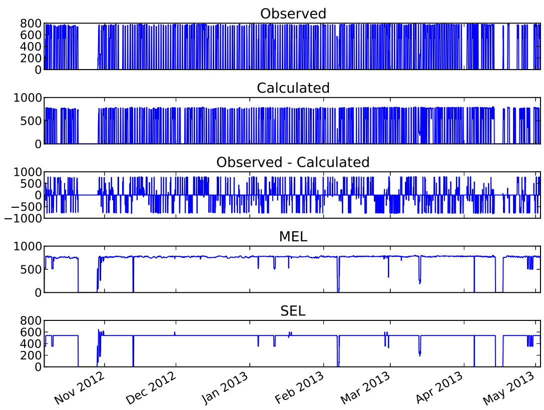


Figure B.1.56: BMU: T_SHBA-1, $e = 0.64$, $s = 21794$ and $m = 7.2$.

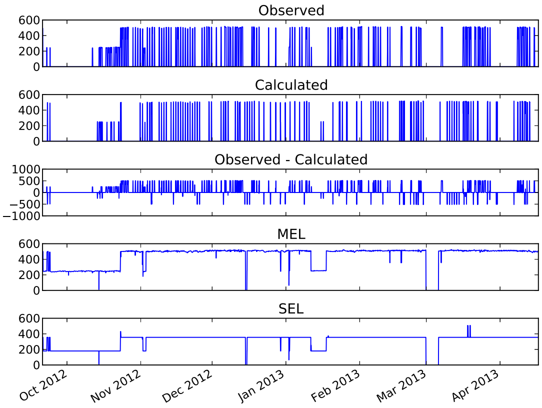


Figure B.1.57: BMU: T_SHBA-2, $e = 0.85$, $s = 16046$ and $m = 5.0$.

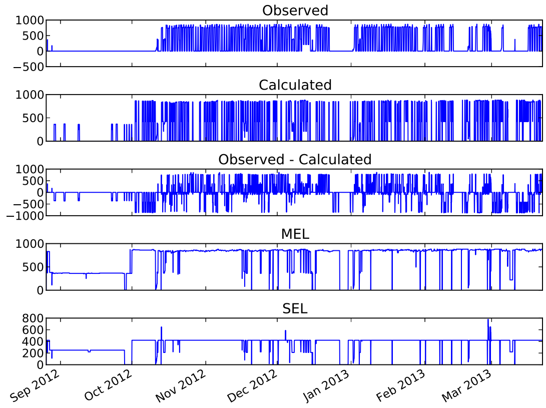


Figure B.1.58: BMU: T_SPLN-1, $e = 0.72$, $s = 55416$ and $m = 1.5$.

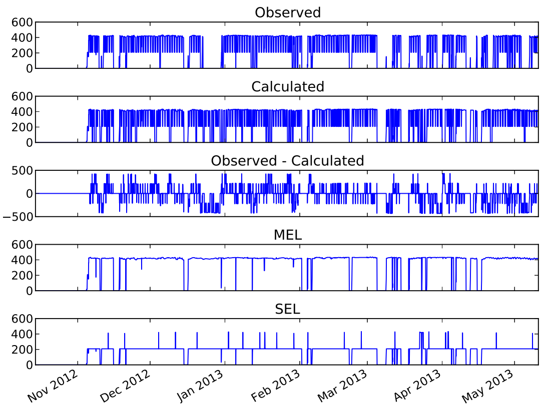


Figure B.1.59: BMU: T_STAY-1, $e = 0.63$, $s = 38378$ and $m = 1.6$.

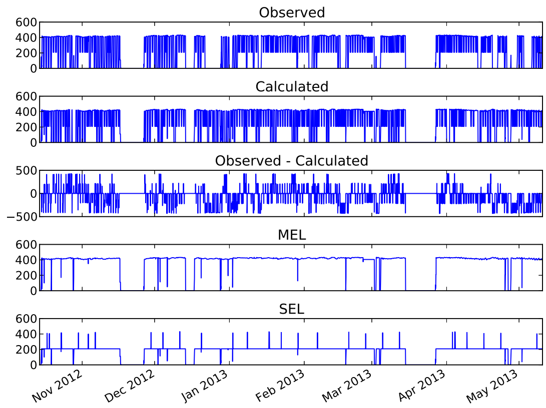


Figure B.1.60: BMU: T_STAY-2, $e = 0.62$, $s = 31080$ and $m = 1.6$.

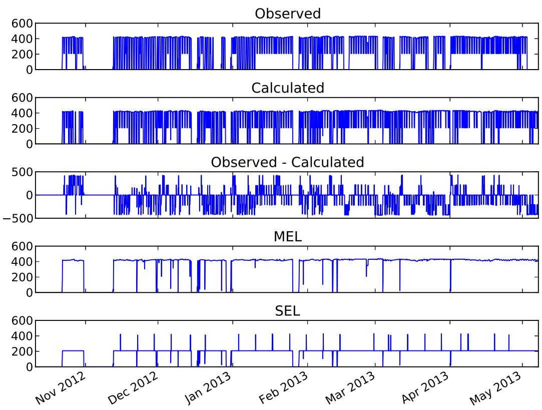


Figure B.1.61: BMU: T_STAY-3, $e = 0.59$, $s = 26000$ and $m = 3.3$.

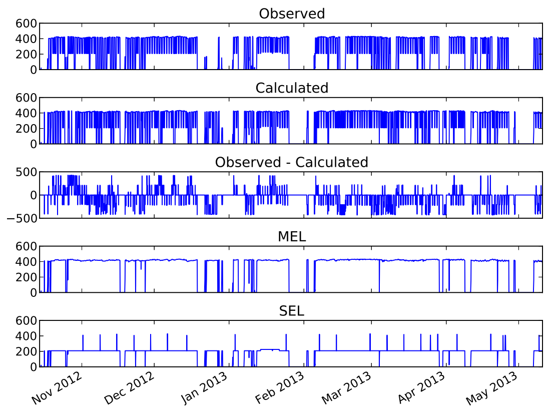


Figure B.1.62: BMU: T_STAY-4, $e = 0.63$, $s = 31523$ and $m = 0.8$.

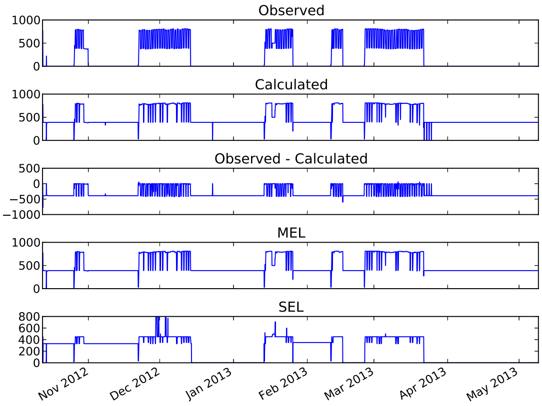


Figure B.1.63: BMU: T_SUTB-1, $e = 0.33$, $s = 4900$ and $m = 2.4$.

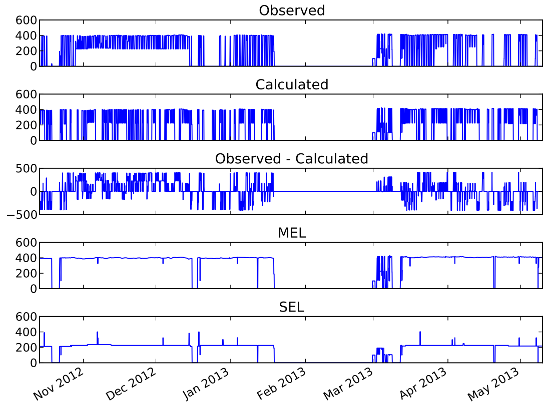


Figure B.1.64: BMU: T_SVRP-10, $e = 0.64$, $s = 42497$ and $m = 5.0$.

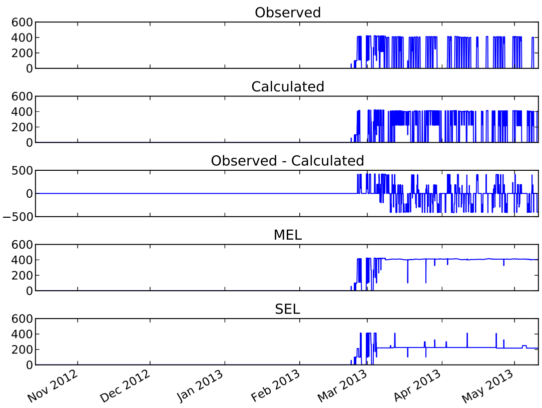


Figure B.1.65: BMU: T_SVRP-20, $e = 0.65$, $s = 28844$ and $m = 4.8$.

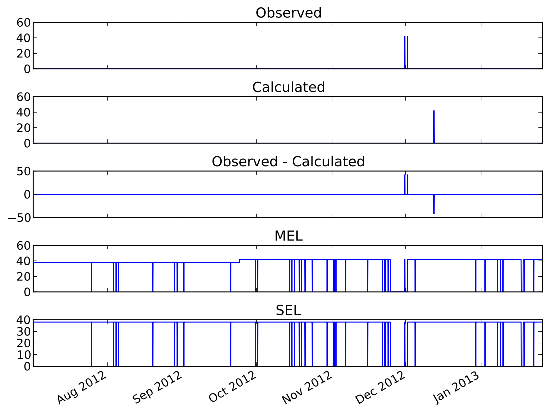


Figure B.1.66: BMU: T_TESI-2, $e = 1.0$, $s = 10291$ and $m = 4.7$.

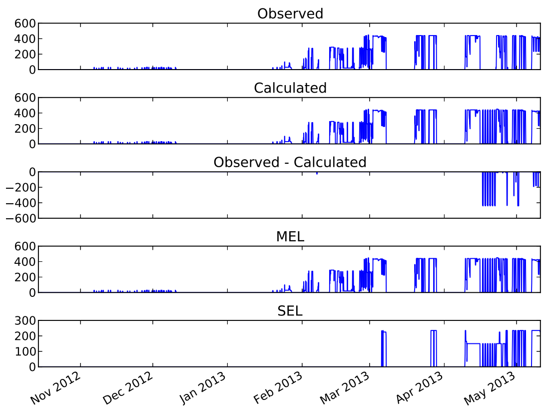


Figure B.1.67: BMU: T_WBURB-1, $e = 0.29$, $s = 42653$ and $m = 0.0$.

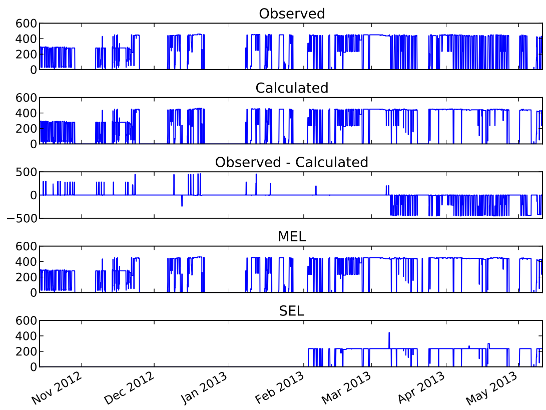


Figure B.1.68: BMU: T_WBURB-2, $e = 0.36$, $s = 51626$ and $m = 5.5$.

## B.2 Coal power plants

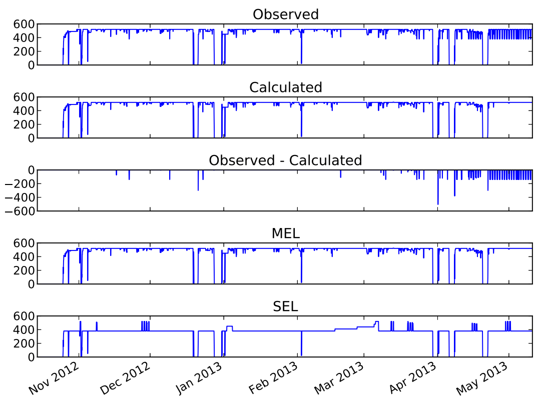


Figure B.2.1: BMU: T_ABTH7, $e = 0.32$, $s = 56759$ and $m = 0.0$.

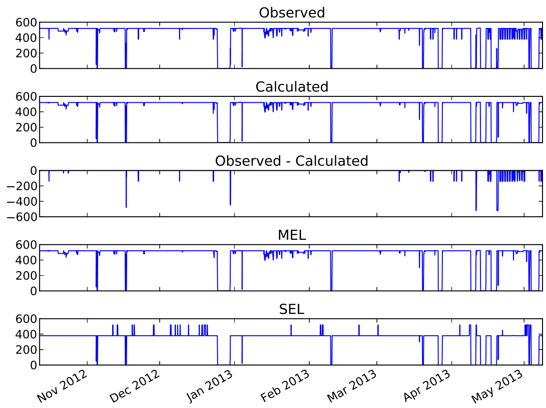


Figure B.2.2: BMU: T_ABTH8, $e = 0.32$, $s = 103104$ and $m = 0.0$.

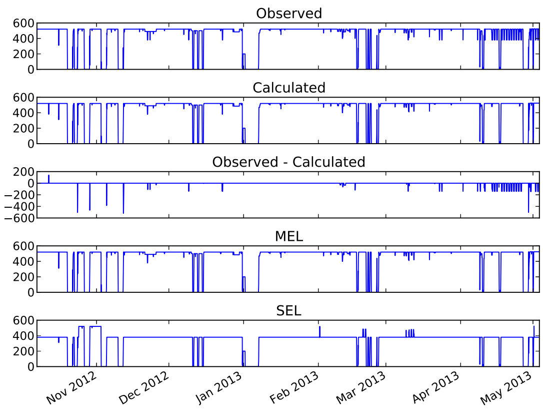


Figure B.2.3: BMU: T_ABTH9, $e = 0.32$, $s = 93807$ and $m = 0.7$.

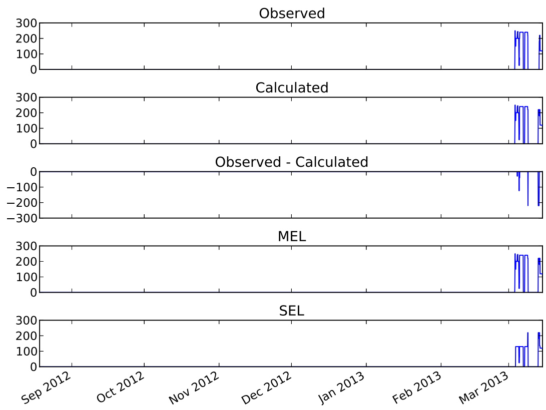


Figure B.2.4: BMU: T_COCK-1, $e = 0.36$, $s = 2186$ and $m = 2.0$.

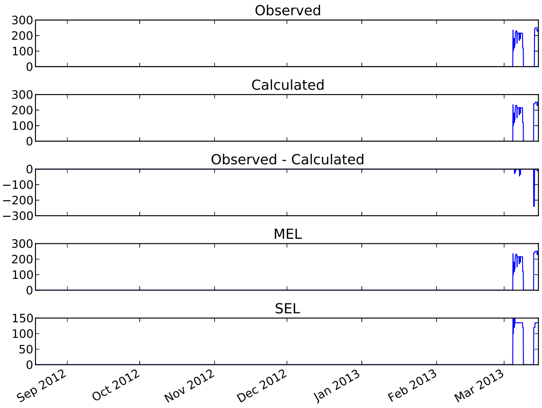


Figure B.2.5: BMU: T_COCK-2, $e = 0.3$, $s = 62132$ and $m = 4.9$.

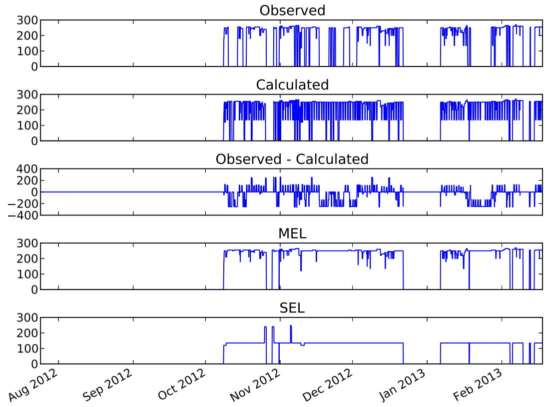


Figure B.2.6: BMU: T_COCK-3, $e = 0.64$, $s = 17719$ and $m = 0.8$.

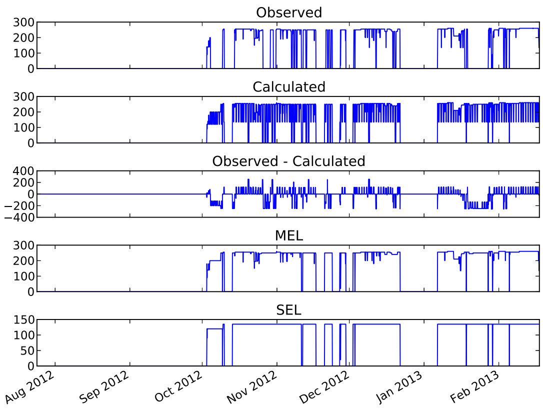


Figure B.2.7: BMU: T_COCK-4, $e = 0.56$, $s = 20087$ and $m = 5.7$.

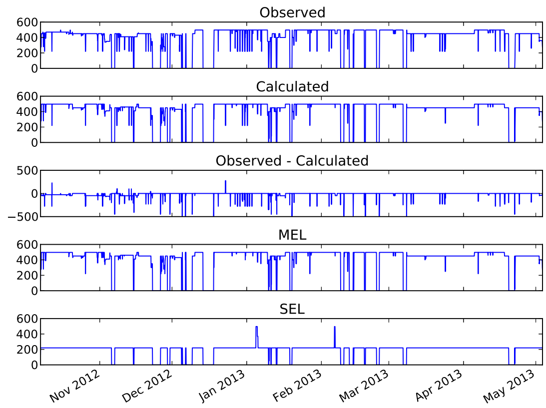


Figure B.2.8: BMU: T_COTPS-1, $e = 0.35$, $s = 46442$ and $m = 3.6$.

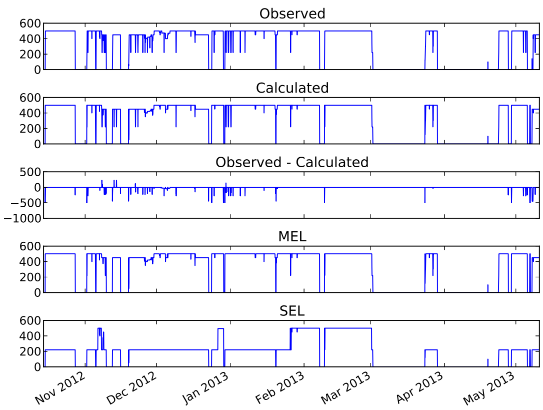


Figure B.2.9: BMU: T_COTPS-2, $e = 0.39$, $s = 110680$ and $m = 1.2$.

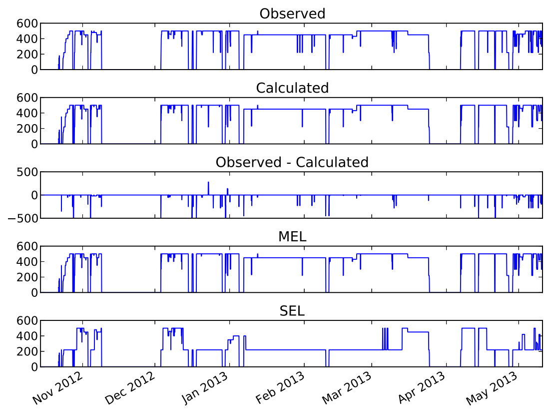


Figure B.2.10: BMU: T_COTPS-3, $e = 0.38$, $s = 4790$ and $m = 1.4$.

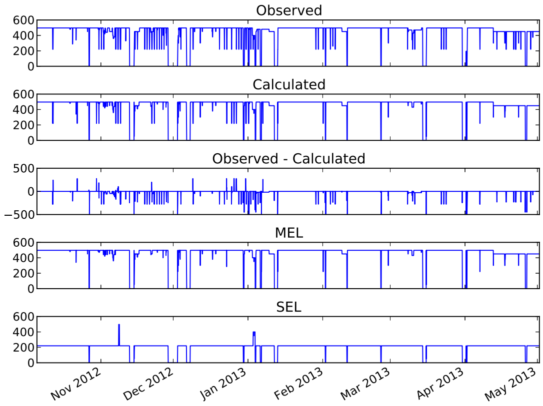


Figure B.2.11: BMU: T_COTPS-4, $e = 0.43$, $s = 103910$ and $m = 0.0$.

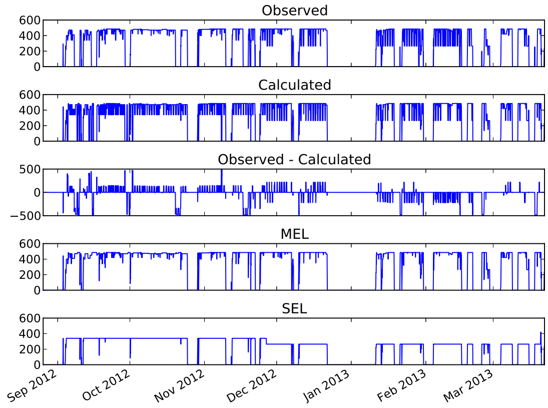


Figure B.2.12: BMU: T_DIDC1, $e = 0.53$, $s = 49456$ and $m = 5.0$.

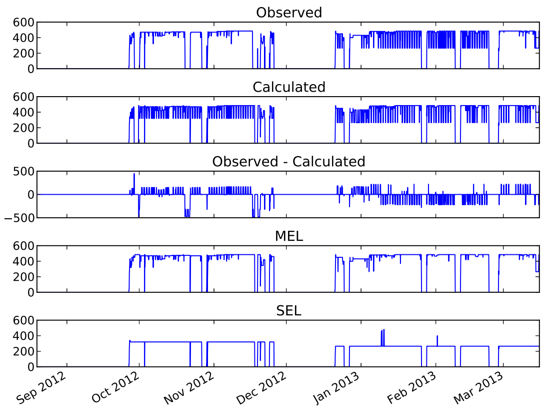


Figure B.2.13: BMU: T_DIDC2, $e = 0.53$, $s = 86177$ and $m = 5.2$.

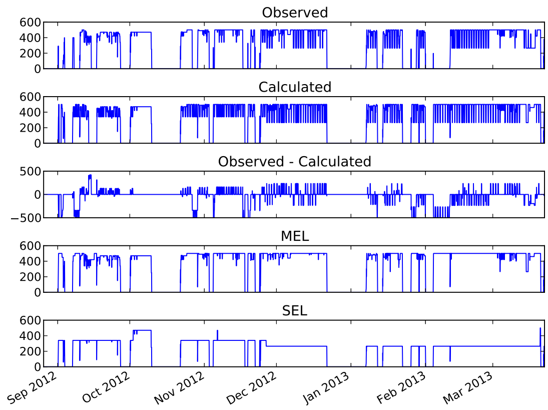


Figure B.2.14: BMU: T_DIDC3, $e = 0.51$, $s = 10745$ and $m = 8.8$.

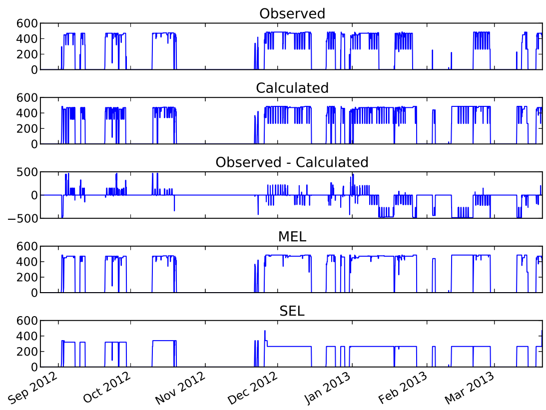


Figure B.2.15: BMU: T_DIDC4, $e = 0.55$, $s = 34202$ and $m = 2.2$.

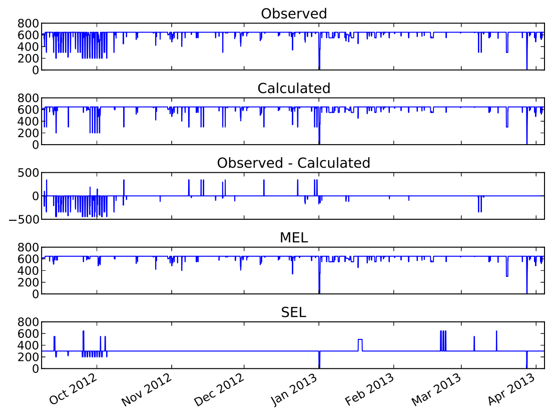


Figure B.2.16: BMU: T_DRAXX-1, $e = 0.3$, $s = 142410$ and $m = 6.1$.

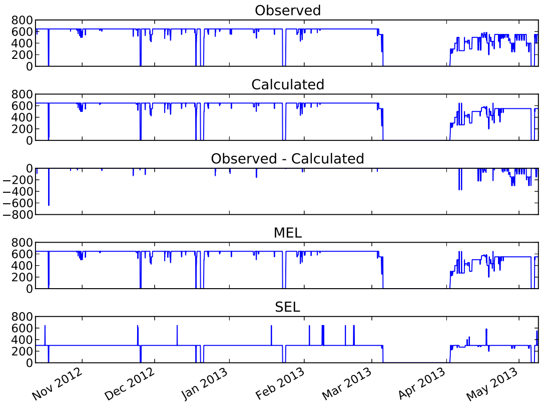


Figure B.2.17: BMU: T_DRAXX-2, $e = 0.32$, $s = 46796$ and $m = 0.0$.

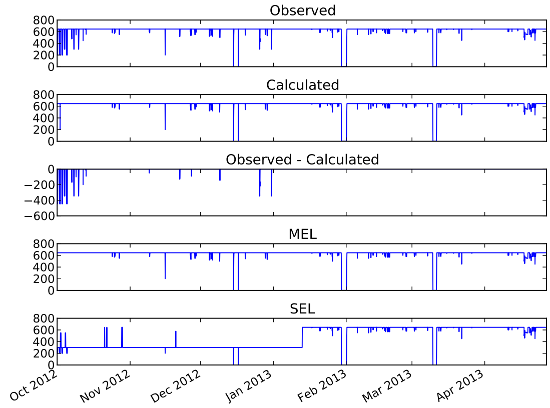


Figure B.2.18: BMU: T_DRAXX-3, $e = 0.3$, $s = 116917$ and $m = 0.1$.

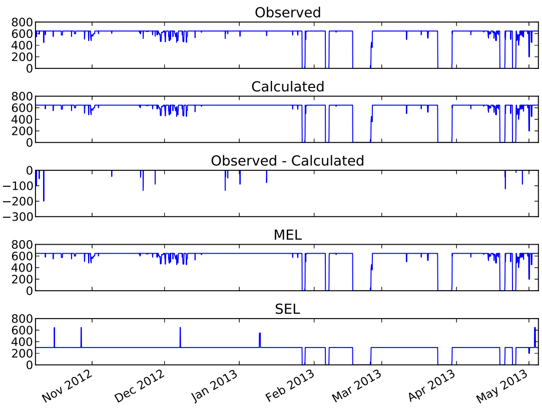


Figure B.2.19: BMU: T_DRAXX-4, $e = 0.32$, $s = 88688$ and $m = 0.0$.

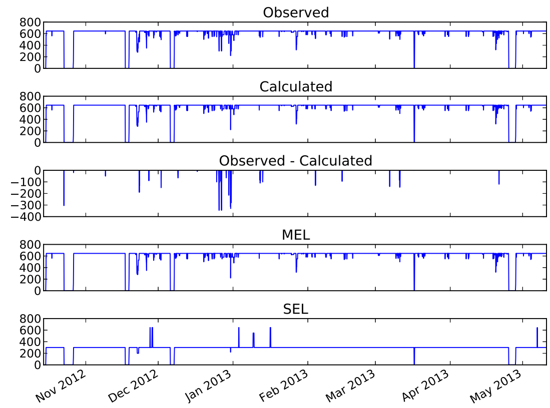


Figure B.2.20: BMU: T_DRAXX-5, $e = 0.32$, $s = 116300$ and $m = 0.0$.

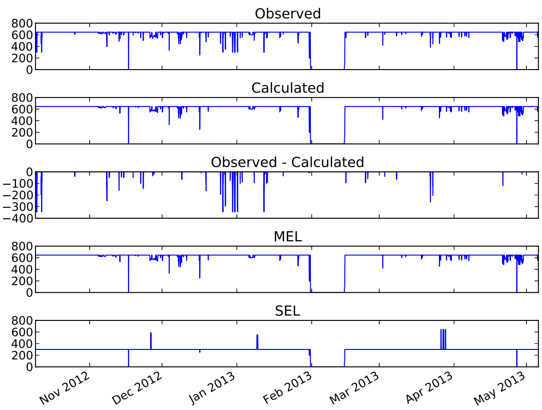


Figure B.2.21: BMU: T_DRAXX-6, $e = 0.32$, $s = 10841$ and $m = 0.0$.

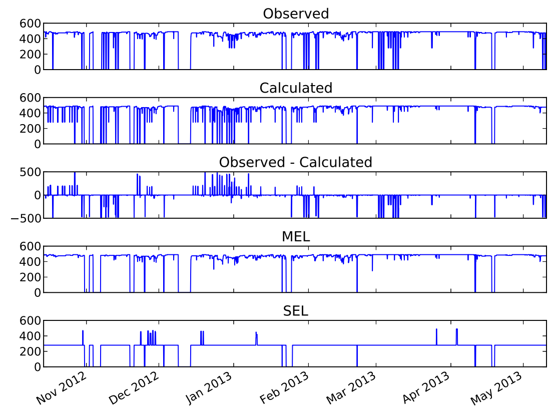


Figure B.2.22: BMU: T_EGGPS-1, $e = 0.32$, $s = 294$ and $m = 8.3$.

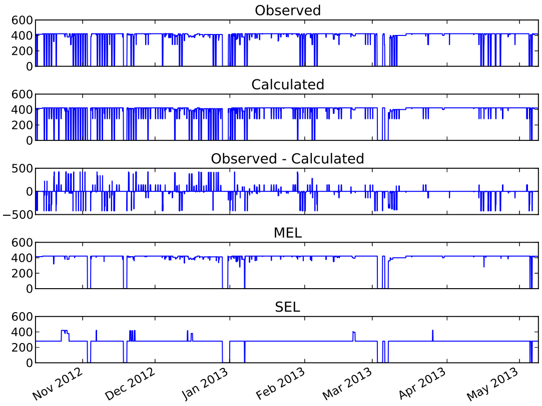


Figure B.2.23: BMU: T_EGGPS-2, $e = 0.44$, $s = 12112$ and $m = 6.8$.

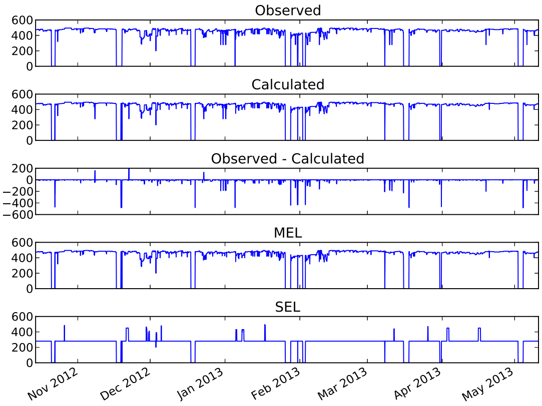


Figure B.2.24: BMU: T_EGGPS-3, $e = 0.38$, $s = 34609$ and $m = 0.0$.

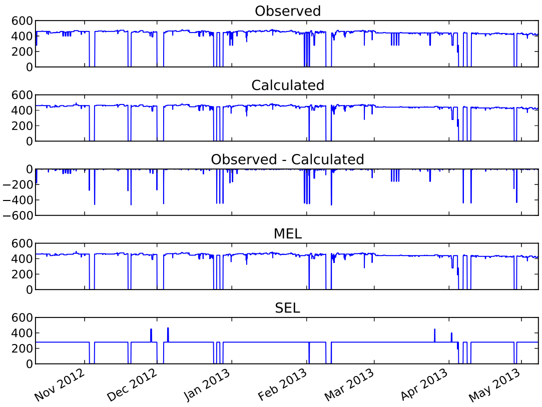


Figure B.2.25: BMU: T_EGGPS-4, $e = 0.32$, $s = 4363$ and $m = 0.0$.

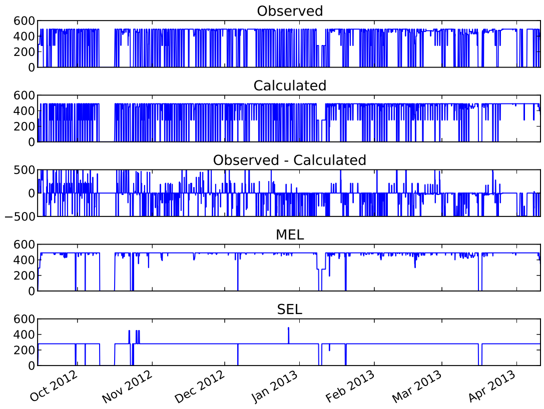


Figure B.2.26: BMU: T_FERR-1, $e = 0.62$, $s = 12586$ and $m = 2.5$.

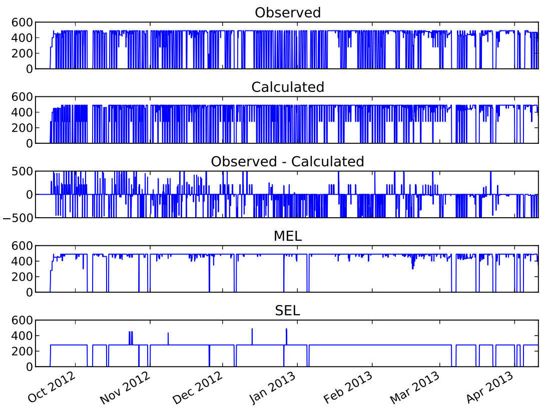


Figure B.2.27: BMU: T_FERR-2, $e = 0.59$, $s = 12863$ and $m = 4.4$.

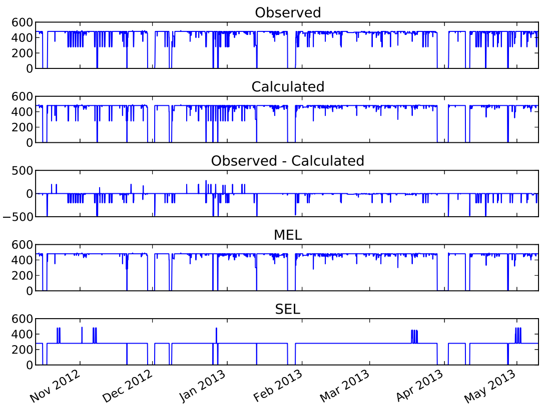


Figure B.2.28: BMU: T_FERR-3, $e = 0.32$, $s = 9891$ and $m = 6.7$.

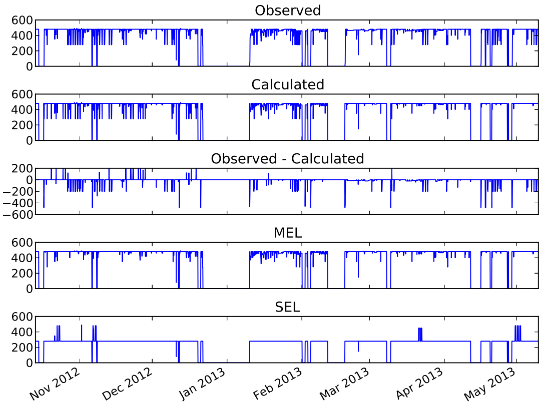


Figure B.2.29: BMU: T_FERR-4, $e = 0.32$, $s = 55158$ and $m = 8.1$.

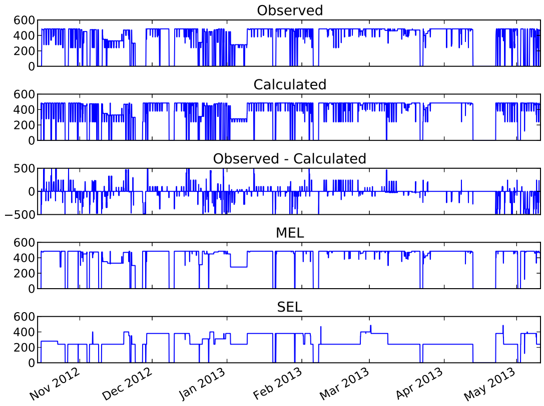


Figure B.2.30: BMU: T_FIDL-1, $e = 0.5$, $s = 29524$ and $m = 7.1$.


Figure B.2.31: BMU: T_FIDL-2, $e = 0.43$, $s = 8131$ and $m = 6.1$.

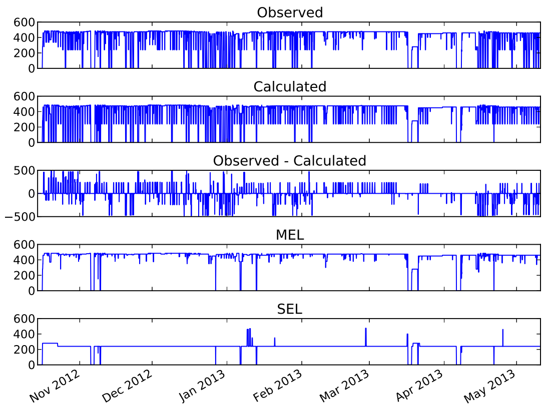


Figure B.2.32: BMU: T_FIDL-3, $e = 0.47$, $s = 23224$ and $m = 8.7$.

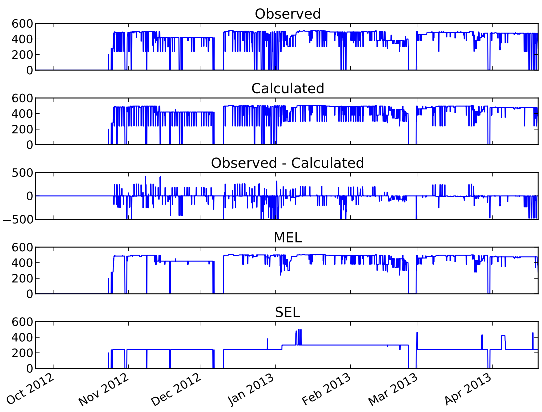


Figure B.2.33: BMU: T_FIDL-4, $e = 0.45$, $s = 17701$ and $m = 5.4$.

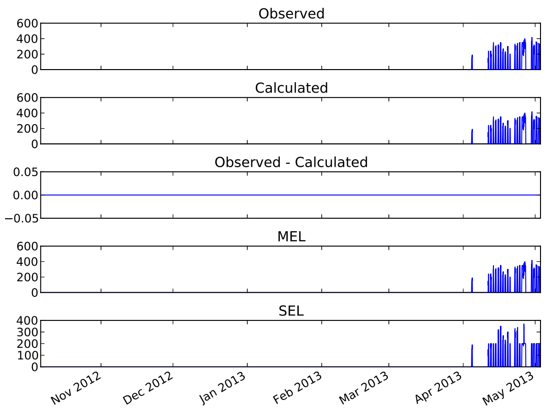


Figure B.2.34: BMU: T_IRNPS-1, $e = 0.5$, $s = 249$ and $m = 3.0$.

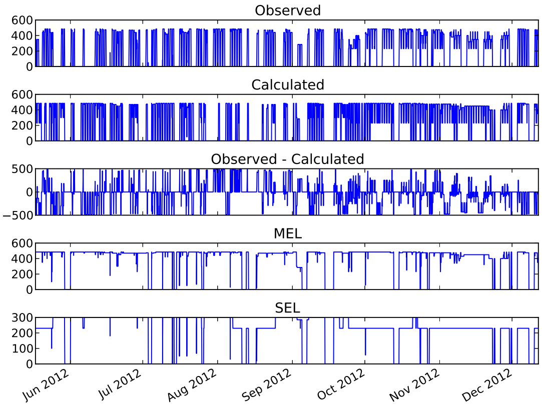


Figure B.2.35: BMU: T_KINO-1, $e = 0.56$, $s = 2873$ and $m = 4.6$.

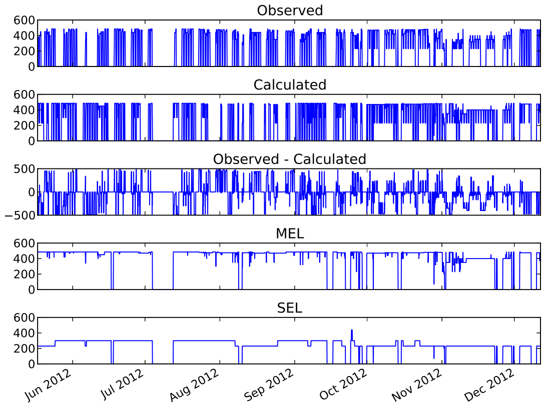


Figure B.2.36: BMU: T_KINO-2, $e = 0.53$, $s = 28815$ and $m = 6.3$.

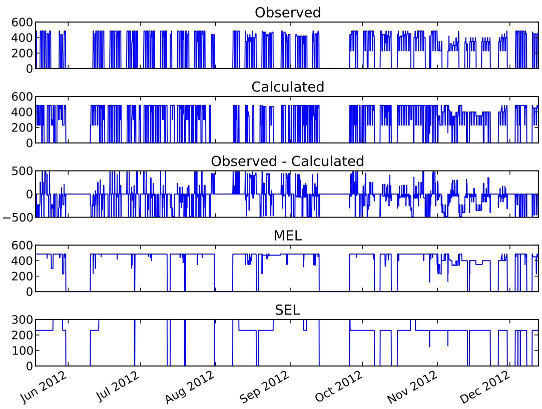


Figure B.2.37: BMU: T_KINO-3, $e = 0.57$, $s = 24822$ and $m = 2.6$.

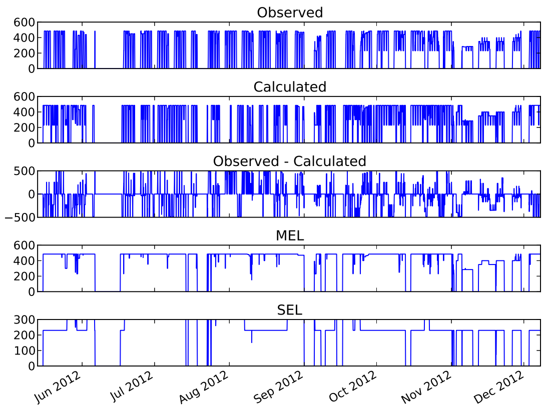


Figure B.2.38: BMU: T_KINO-4, $e = 0.54$, $s = 25528$ and $m = 4.6$.

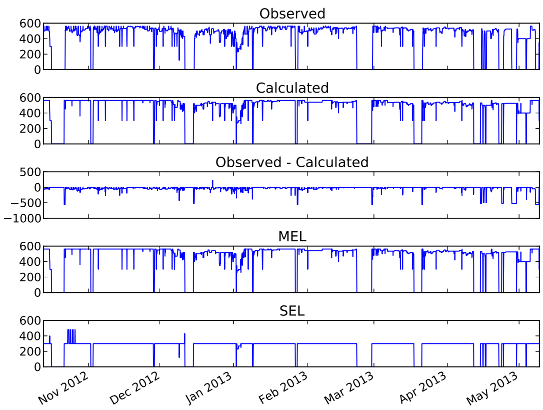


Figure B.2.39: BMU: T_LOAN-1, $e = 0.32$, $s = 38088$ and $m = 0.9$.

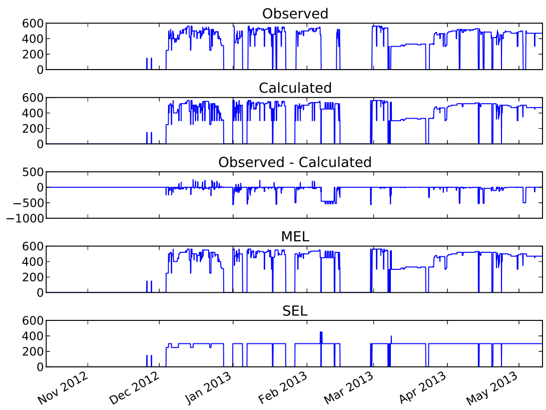


Figure B.2.40: BMU: T_LOAN-2, $e = 0.33$, $s = 76737$ and $m = 8.0$.

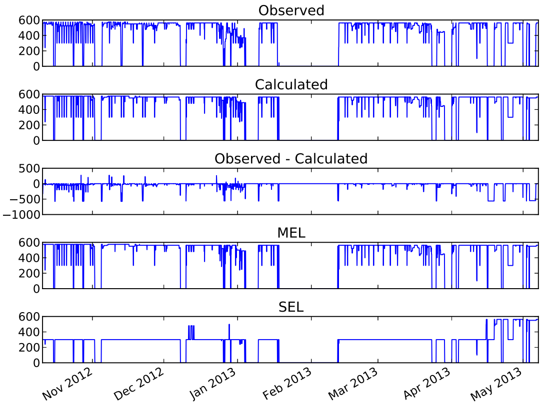


Figure B.2.41: BMU: T_LOAN-3, $e = 0.32$, $s = 91352$ and $m = 5.6$.

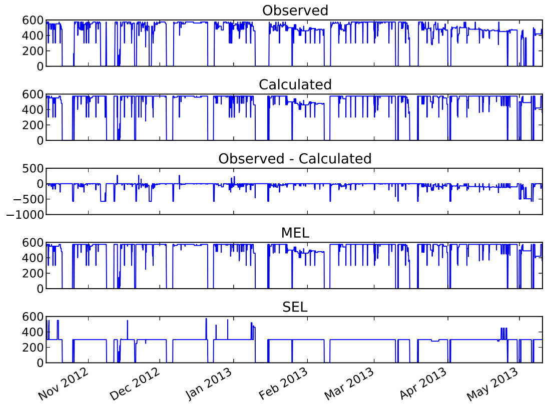


Figure B.2.42: BMU: T_LOAN-4, $e = 0.35$, $s = 124113$ and $m = 3.0$.

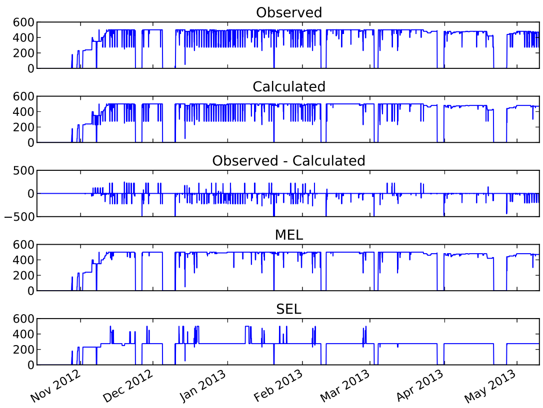


Figure B.2.43: BMU: T_RATS-1, $e = 0.41$, $s = 48266$ and $m = 4.9$.

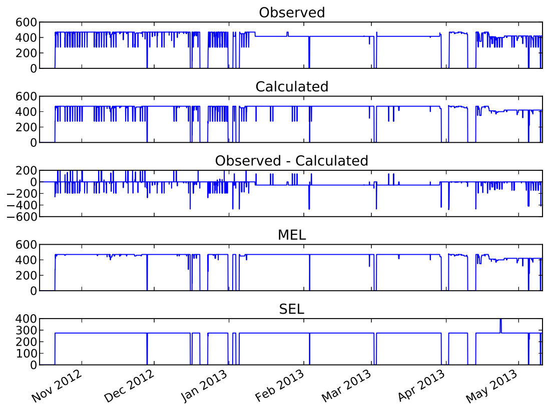


Figure B.2.44: BMU: T_RATS-2, $e = 0.43$, $s = 78263$ and $m = 2.4$.

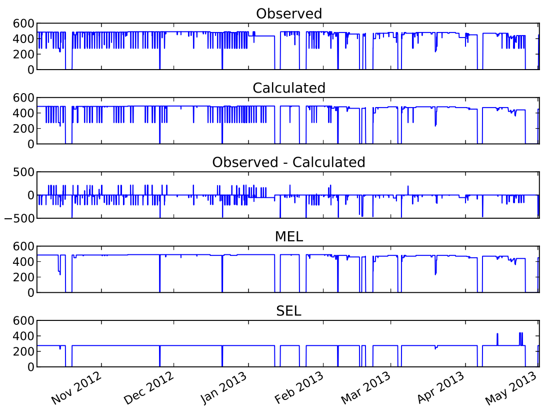


Figure B.2.45: BMU: T_RATS-3, $e = 0.39$, $s = 9681$ and $m = 4.4$.

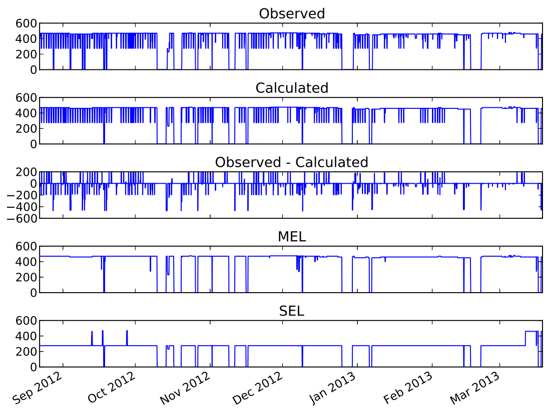


Figure B.2.46: BMU: T_RATS-4, $e = 0.43$, $s = 80116$ and $m = 3.7$.

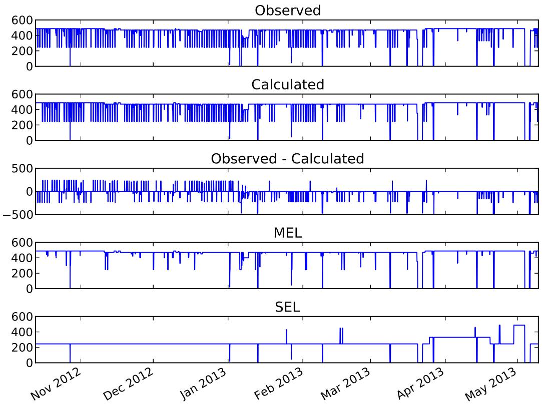


Figure B.2.47: BMU: T_RUGPS-6, $e = 0.4$, $s = 88499$ and $m = 6.6$.

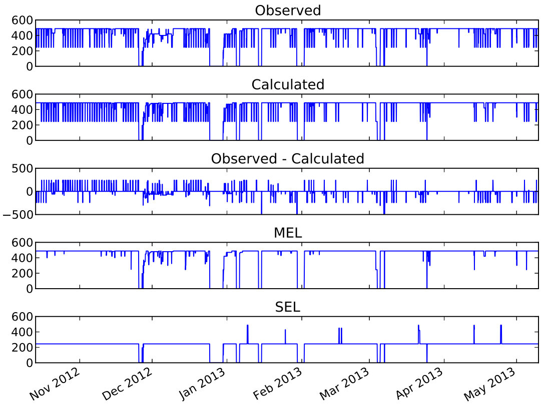


Figure B.2.48: BMU: T_RUGPS-7, $e = 0.42$, $s = 12200$ and $m = 7.5$.

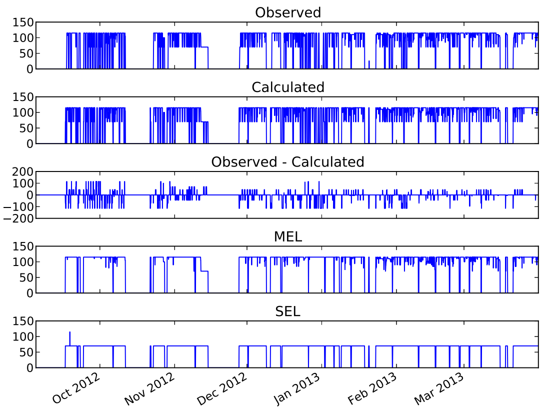


Figure B.2.49: BMU: T_USKM-13, $e = 0.46$, $s = 7043$ and $m = 9.0$.

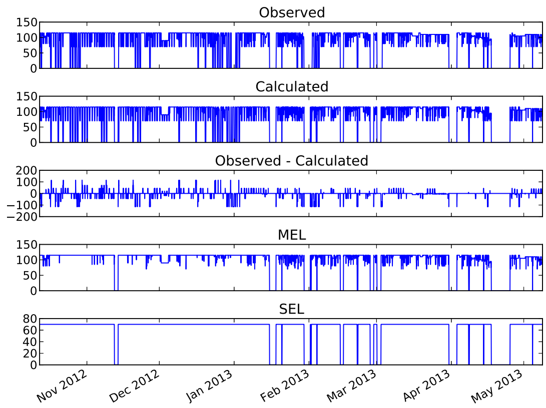


Figure B.2.50: BMU: T_USKM-14, $e = 0.55$, $s = 6873$ and $m = 3.8$.

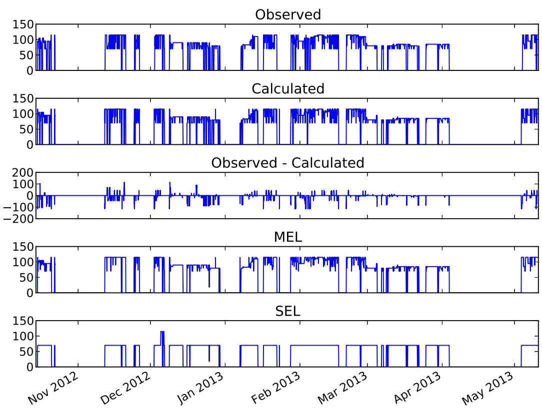


Figure B.2.51: BMU: T_USKM-15, $e = 0.56$, $s = 5240$ and $m = 2.4$.

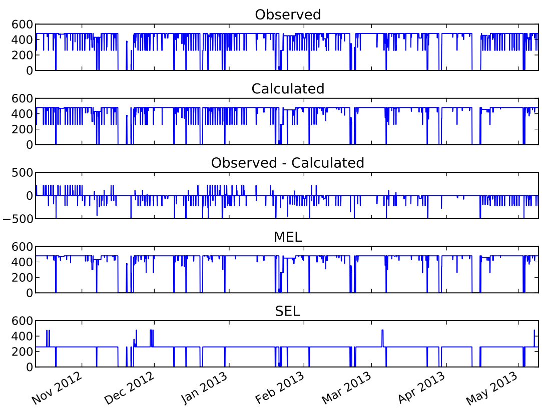


Figure B.2.52: BMU: T_WBUPS-1, $e = 0.48$, $s = 1973$ and $m = 0.1$.

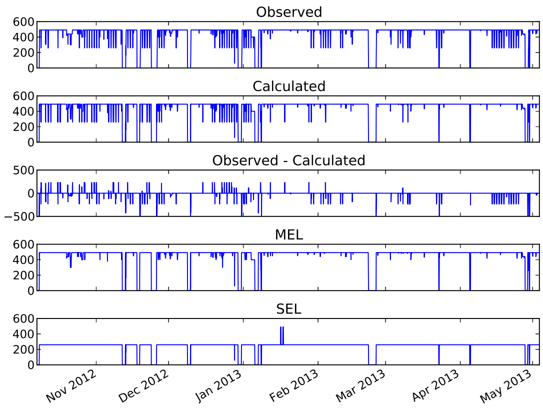


Figure B.2.53: BMU: T_WBUPS-2,

$e = 0.43$, $s = 48644$ and $m = 1.9$.

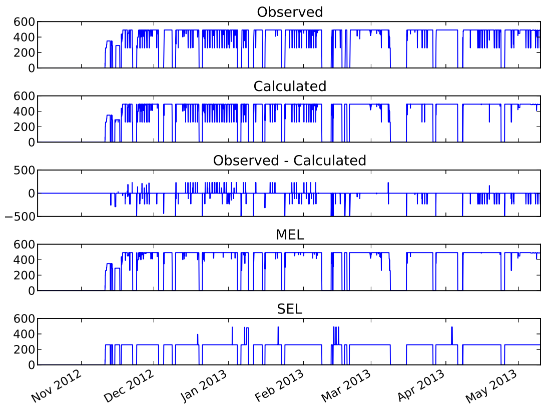


Figure B.2.54: BMU: T_WBUPS-3,

$e = 0.48$, $s = 90412$ and $m = 0.7$.

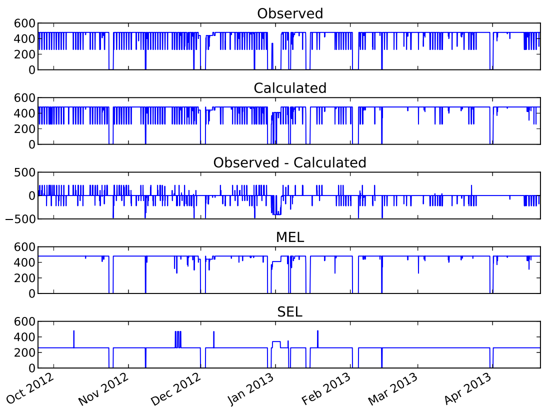


Figure B.2.55: BMU: T_WBUPS-4,

$e = 0.51$, $s = 26081$ and $m = 0.0$.

# Appendix C

# Input data to our model

In this appendix, we present the ramping constraints of power plants that we used to produce our results in Chapters 3 and 4. Other physical characteristics are presented in Appendix B.

## C.1 Gas power plants

Table C.1: Ramping constraints of CCGT BMUs.

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ | BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|---|---|---|
| E_BRGG-1 | 121 | 162 | T_KILLPG-1 | 203 | 203 |
| E_BRYP-1 | 168 | 98 | T_KILLPG-2 | 209 | 314 |
| E_CORB-1 | 196 | 255 | T_KILNS-1 | 385 | 466 |
| E_DERW-1 | 46 | 105 | T_LAGA-1 | 600 | 525 |
| E_FAWN-1 | 90 | 91 | T_LBAR-1 | 378 | 420 |
| E_FELL-1 | 40 | 43 | T_MEDP-1 | 490 | 490 |
| E_GYAR-1 | 154 | 265 | T_MRWD-1 | 578 | 584 |
| E_KLYN-A-1 | 199 | 214 | T_PEHE-1 | 826 | 826 |

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|
| E_PETEM1 | 175 | 210 |
| E_ROOS-1 | 160 | 148 |
| E_SHOS-1 | 203 | 168 |
| E_SHOT-1 | 125 | 138 |
| T_BAGE-1 | 294 | 336 |
| T_BAGE-2 | 22 | 21 |
| T_BARK-1 | 140 | 147 |
| T_BARKB2 | 196 | 210 |
| T_CDCL-1 | 251 | 277 |
| T_CNQPS-1 | 238 | 228 |
| T_CNQPS-2 | 161 | 237 |
| T_CNQPS-3 | 225 | 238 |
| T_CNQPS-4 | 230 | 244 |
| T_COSO-1 | 315 | 126 |
| T_DAMC-1 | 321 | 294 |
| T_DEEP-1 | 225 | 223 |
| T_DIDCB5 | 296 | 336 |
| T_DIDCB6 | 242 | 384 |
| T_EECL-1 | 182 | 280 |
| T_FIFE-1 | 41 | 86 |
| T_GRAI-6 | 314 | 315 |

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|
| T_PEHE-2 | 168 | 168 |
| T_PEMB-11 | 229 | 189 |
| T_PEMB-21 | 234 | 189 |
| T_PEMB-31 | 228 | 294 |
| T_PEMB-41 | 232 | 189 |
| T_PEMB-51 | 235 | 190 |
| T_ROCK-1 | 284 | 126 |
| T_RYHPS-1 | 210 | 210 |
| T_SCCL-1 | 144 | 275 |
| T_SCCL-2 | 137 | 276 |
| T_SCCL-3 | 267 | 280 |
| T_SEAB-1 | 426 | 538 |
| T_SEAB-2 | 258 | 105 |
| T_SHBA-1 | 526 | 537 |
| T_SHBA-2 | 361 | 364 |
| T_SPLN-1 | 378 | 386 |
| T_STAY-1 | 188 | 190 |
| T_STAY-2 | 180 | 287 |
| T_STAY-3 | 181 | 181 |
| T_STAY-4 | 179 | 190 |
| T_SUTB-1 | 273 | 245 |

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ | BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|---|---|---|
| T_GRAI-7 | 315 | 315 | T_SVRP-10 | 298 | 298 |
| T_GRAI-8 | 315 | 315 | T_SVRP-20 | 297 | 298 |
| T_GRMO-1 | 0 | 0 | T_TESI-2 | 189 | 189 |
| T_HUMR-1 | 364 | 438 | T_WBURB-1 | 308 | 308 |
| T_KEAD-1 | 532 | 515 | T_WBURB-2 | 315 | 322 |

## C.2 Coal power plants

Table C.2: Ramping constraints of coal BMUs.

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ | BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|---|---|---|
| T_ABTH8 | 105 | 164 | T_FERR-4 | 336 | 343 |
| T_ABTH9 | 105 | 164 | T_FIDL-1 | 340 | 336 |
| T_COCK-1 | 84 | 88 | T_FIDL-2 | 340 | 340 |
| T_COCK-2 | 84 | 98 | T_FIDL-3 | 340 | 340 |
| T_COCK-3 | 84 | 168 | T_FIDL-4 | 354 | 340 |
| T_COCK-4 | 84 | 175 | T_IRNPS-1 | 258 | 340 |
| T_COTPS-1 | 194 | 194 | T_KINO-1 | 179 | 340 |
| T_COTPS-2 | 161 | 350 | T_KINO-2 | 210 | 340 |
| T_COTPS-3 | 196 | 196 | T_KINO-3 | 209 | 340 |
| T_COTPS-4 | 161 | 175 | T_KINO-4 | 209 | 340 |
| T_DIDC1 | 105 | 147 | T_LOAN-1 | 105 | 105 |

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|
| T_DIDC2 | 105 | 105 |
| T_DIDC3 | 105 | 105 |
| T_DIDC4 | 105 | 105 |
| T_DRAXX-1 | 210 | 245 |
| T_DRAXX-2 | 452 | 452 |
| T_DRAXX-3 | 242 | 452 |
| T_DRAXX-4 | 210 | 452 |
| T_DRAXX-5 | 242 | 452 |
| T_DRAXX-6 | 210 | 210 |
| T_EGGPS-1 | 147 | 172 |
| T_EGGPS-2 | 137 | 172 |
| T_EGGPS-3 | 151 | 214 |
| T_EGGPS-4 | 151 | 214 |
| T_FERR-1 | 343 | 343 |
| T_FERR-2 | 343 | 343 |
| T_FERR-3 | 315 | 315 |

| BMU | $\triangle\overline{W}^{p,l,r}_{\max}$ | $\triangle\overline{W}^{p,l,r}_{\min}$ |
|---|---|---|
| T_LOAN-2 | 105 | 371 |
| T_LOAN-3 | 105 | 385 |
| T_LOAN-4 | 105 | 364 |
| T_RATS-1 | 193 | 350 |
| T_RATS-2 | 189 | 347 |
| T_RATS-3 | 186 | 347 |
| T_RATS-4 | 172 | 343 |
| T_RUGPS-6 | 170 | 342 |
| T_RUGPS-7 | 170 | 202 |
| T_USKM-13 | 49 | 81 |
| T_USKM-14 | 81 | 81 |
| T_USKM-15 | 81 | 81 |
| T_WBUPS-1 | 315 | 336 |
| T_WBUPS-2 | 182 | 344 |
| T_WBUPS-3 | 202 | 224 |
| T_WBUPS-4 | 182 | 196 |