

AN EXTENDED ERROR ANALYSIS FOR A MESHFREE DISCRETIZATION METHOD OF DARCY'S PROBLEM

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Abstract. Recently ([21]), a new meshfree approximation method for Darcy's problem has been introduced and analyzed. This method is based on a symmetric collocation approach using radial basis functions producing solutions with an analytically divergence-free velocity part. However, the error analysis provided in [21] works only for smooth solutions, where the smoothness is intrinsically linked to the smoothness of the employed basis function. In this paper, we will extend the error analysis to less smooth functions, showing that the approximation order for rougher solutions is determined rather by the smoothness of the solution than the smoothness of the basis function.

Key words. partial differential equations, radial basis functions, high-order method, collocation, error analysis

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1. Introduction. Darcy's problem plays an important role in porous media flow [3, 10]. It can be stated in the following way

$$\mathbf{u} + K\nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.2)$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} \quad \text{on } \partial\Omega. \quad (1.3)$$

Here, $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with boundary $\partial\Omega$ having a unique outer normal vector \mathbf{n} . The right-hand sides \mathbf{f} and $\mathbf{g} \cdot \mathbf{n}$ and the permeability tensor K are given. The boundary function \mathbf{g} must satisfy the compatibility condition

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, dS = 0. \quad (1.4)$$

The tensor K is supposed to be symmetric, $K = K^T$, and strongly elliptic in the sense that there is a constant $\alpha > 0$ such that

$$\xi^T K(\mathbf{x}) \xi \geq \alpha \|\xi\|_2^2, \quad \xi \in \mathbb{R}^d, \mathbf{x} \in \Omega. \quad (1.5)$$

The solution consists of a velocity term $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and a pressure term $p : \Omega \rightarrow \mathbb{R}$.

Recently, a new discretization scheme for Darcy's problem has been developed in [21]. This scheme is based upon symmetric collocation employing matrix-valued “radial” basis functions. It produces analytically divergence-free approximations to the velocity field and, as a meshfree method, is flexible about the shape of the underlying domain.

The error analysis given in [21] applies, unfortunately, only if the solution (\mathbf{u}, p) comes from a specific Hilbert space of smooth functions, which is intrinsically connected to the employed basis function. The Hilbert space is the *reproducing kernel Hilbert space* associated to the basis function. This is in principle not a problem, as long as the smoothness of the solution is known before-hand. Then, the strategy

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would be to choose the basis function accordingly to this smoothness such that the error analysis of [21] applies. However, in the case that the smoothness of the solution is unknown or can only be estimated, it is very likely that the basis of the discretization space is chosen too smoothly. Hence, the natural question that arises is what kind of error estimates remains valid if the solution (\mathbf{u}, p) is less smooth than required by the theory in [21].

In this paper, we will show that in such a situation, an error analysis still holds and gives the expected order, determined by the smoothness of the solution rather than the smoothness of the underlying discretization spaces. It turns out that as natural this result is, its proof is rather complicated and requires deep techniques from multivariate approximation theory.

So far, in the context of meshfree methods based on radial basis functions, such results were only known for classical interpolation and approximation but not for collocation methods for partial differential equations. Error estimates for “rougher” target functions were first presented by NARCOWICH and WARD in [16] in the special situation that the data sites are located on a sphere. Other work on \mathbb{R}^d followed in [5, 12, 13, 18, 17]. An overview can be found in [14]. Recently FUSELIER presented error estimates for interpolation problems with divergence-free or curl-free matrix-valued kernels, where the target function is rougher than required by the classical reproducing kernel Hilbert space theory, see [8].

In this paper, we will show that this concept can be extended to collocation methods for solving partial differential equations. We will establish new Sobolev-type approximation rates for the discretization scheme of Darcy’s problem, where the velocity and the pressure are too rough for the error analysis provided in [21].

For this purpose, the paper is organized as follows. In the rest of this section we collect necessary notation on vector-valued Sobolev spaces. The next section is devoted to our discretization scheme, hence covering matrix-valued kernels, their reproducing kernel Hilbert spaces and optimal recovery as well as stating the approximation scheme. In the third section, we introduce technical tools required for our error analysis, comprising the interpolation and approximation properties of band-limited functions. After this we derive our main result, error estimates for ‘rougher’ solutions to Darcy’s problem. In the final section, we give numerical examples to corroborate our theoretical estimates.

1.1. Sobolev Spaces. We will work with the usual scalar-valued Sobolev spaces. For a domain $\Omega \subseteq \mathbb{R}^d$, a real number $r \geq 1$ or $r = \infty$ and an integer $k \in \mathbb{N}_0$, we denote by $W_r^k(\Omega)$ the space of all functions $f \in L_r(\Omega)$ having weak derivatives $D^\alpha f \in L_r(\Omega)$ for every multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = \alpha_1 + \dots + \alpha_d \leq k$. We will also work with fractional order Sobolev spaces $W_r^\tau(\Omega)$, particularly with $\tau > d/2$ so that we have continuous functions. For the introduction of such fractional order Sobolev spaces we refer, for example, to [1, 4, 23].

Since the pressure p in the solution of (1.1)–(1.3) is determined only up to a constant we will work with the quotient spaces $W_r^\tau(\Omega)/\mathbb{R}$ equipped with the norm

$$\|p\|_{W_r^\tau(\Omega)/\mathbb{R}} := \inf_{c \in \mathbb{R}} \|p + c\|_{W_r^\tau(\Omega)}.$$

We define the vector-valued Sobolev space $\mathbf{W}_r^\tau(\Omega)$ to consist of all vector-valued functions $\mathbf{u} = (u_1, \dots, u_n)^T : \Omega \rightarrow \mathbb{R}^n$, where each component u_j belongs to $W_r^\tau(\Omega)$. A norm on $\mathbf{W}_r^\tau(\Omega)$ can be defined by taking the discrete ℓ_r norm of the $W_r^\tau(\Omega)$ norms

of the components, i.e. by

$$\|\mathbf{u}\|_{\mathbf{W}_r^\tau(\Omega)} = \begin{cases} \left(\sum_{j=1}^n \|u_j\|_{W_r^\tau(\Omega)}^r \right)^{1/r} & \text{for } 1 \leq r < \infty, \\ \max_{1 \leq j \leq n} \|u_j\|_{W_\infty^\tau(\Omega)} & \text{for } r = \infty. \end{cases}$$

Note that we do not use an index to indicate the dimension n since it will become clear from the context. We only distinguish between scalar-valued function spaces and vector-valued ones. Finally, in the case $r = 2$, we will also use the notation $\mathbf{H}^\tau(\Omega) = \mathbf{W}_2^\tau(\Omega)$.

2. The Discretization Scheme. In this section, we will review the necessary material on matrix-valued kernels and the way we will use them for discretizing Darcy's problem (1.1)–(1.3).

First, we will discuss the kernels and their reproducing Hilbert spaces. For this, we will mainly rely on material from [6, 7, 8, 25, 21]. In the last part of this section, we will state the concrete discretization scheme from [21].

2.1. Positive Definite Matrix-valued Kernels. DEFINITION 2.1. *A function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is positive definite if for all $N \in \mathbb{N}$, all pairwise distinct $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^d$ and all $\alpha \in \mathbb{R}^N \setminus \{0\}$, the quadratic form*

$$\sum_{j,k=1}^N \alpha_j \alpha_k \phi(\mathbf{x}_j - \mathbf{x}_k)$$

is positive. More generally, a matrix-valued function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^{n \times n}$ is positive definite if it is even $\Phi(-\mathbf{x}) = \Phi(\mathbf{x})$, symmetric $\Phi(\mathbf{x}) = \Phi(\mathbf{x})^T$ and satisfies

$$\sum_{j,k=1}^N \alpha_j^T \Phi(\mathbf{x}_j - \mathbf{x}_k) \alpha_k > 0$$

for all pairwise distinct $\mathbf{x}_j \in \mathbb{R}^d$ and all $\alpha_j \in \mathbb{R}^n$ such that not all α_j are vanishing.

The theory of the associated function spaces can be formulated for positive definite matrix-valued functions as it can be done for scalar-valued ones. Let $\Omega \subseteq \mathbb{R}^d$ be non-empty. The following definition is taken from [8].

DEFINITION 2.2. *Let \mathbf{H} be a Hilbert space of vector-valued functions $\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$. The space \mathbf{H} is called a reproducing kernel Hilbert space if there exists a continuous $n \times n$ matrix-valued kernel Φ such that*

- (1) $\Phi(\cdot - \mathbf{x})\alpha \in \mathbf{H}$,
- (2) $\alpha^T \mathbf{f}(\mathbf{x}) = (\mathbf{f}, \Phi(\cdot - \mathbf{x})\alpha)_{\mathbf{H}}$

for all $\mathbf{x} \in \Omega$ and $\alpha \in \mathbb{R}^n$. The function Φ is called the reproducing kernel of \mathbf{H} .

It is well known that the reproducing kernel of a reproducing kernel Hilbert space is a positive definite kernel and that every positive definite kernel generates a Hilbert space in a natural way, in which it is the reproducing kernel.

Here, we are interested in the following two reproducing kernel Hilbert spaces (see [15, 6, 7, 25, 21]).

THEOREM 2.3.

- (1) *Suppose $\phi \in W_1^2(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ is positive definite. Let the matrix-valued kernel $\Phi_{\text{div}} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be defined by $\Phi_{\text{div}} := (-\Delta I + \nabla \nabla^T)\phi$, where Δ is the usual Laplace operator, ∇ denotes the gradient and I the identity matrix.*

Then the associated Hilbert space $\mathcal{H}_{\Phi_{\text{div}}}$ of Φ_{div} consists of all functions $\mathbf{f} \in \mathbf{L}_2(\mathbb{R}^d)$ with

$$\|\mathbf{f}\|_{\mathcal{H}_{\Phi_{\text{div}}}(\mathbb{R}^d)}^2 = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2 \widehat{\phi}(\boldsymbol{\omega})} d\boldsymbol{\omega} < \infty.$$

(2) Let the matrix-valued kernel $\widetilde{\Phi} : \mathbb{R}^d \rightarrow \mathbb{R}^{(d+1) \times (d+1)}$ be defined by $\widetilde{\Phi} = \Phi_{\text{div}} \otimes \psi := \begin{pmatrix} \Phi_{\text{div}} & \mathbf{0} \\ \mathbf{0} & \psi \end{pmatrix}$ with a positive definite function ψ . Then, the corresponding reproducing kernel Hilbert space is given by

$$\mathcal{H}_{\widetilde{\Phi}}(\mathbb{R}^d) = \mathcal{H}_{\Phi_{\text{div}}}(\mathbb{R}^d) \times \mathcal{H}_{\psi}(\mathbb{R}^d)$$

with norm for $\mathbf{f} = (\mathbf{f}_{\mathbf{u}}, f_p)$ given by

$$\begin{aligned} \|\mathbf{f}\|_{\mathcal{H}_{\widetilde{\Phi}}(\mathbb{R}^d)}^2 &= \|\mathbf{f}_{\mathbf{u}}\|_{\mathcal{H}_{\Phi_{\text{div}}}(\mathbb{R}^d)}^2 + \|f_p\|_{\mathcal{H}_{\psi}(\mathbb{R}^d)}^2 \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \left[\frac{\|\widehat{\mathbf{f}}_{\mathbf{u}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2 \widehat{\phi}(\boldsymbol{\omega})} + \frac{|\widehat{f}_p(\boldsymbol{\omega})|^2}{\widehat{\psi}(\boldsymbol{\omega})} \right] d\boldsymbol{\omega}. \end{aligned}$$

We are particularly interested in reproducing kernel Hilbert spaces that are norm equivalent to Sobolev spaces. A scalar-valued RKHS $\mathcal{H}_{\phi}(\mathbb{R}^d)$ is norm equivalent to the Sobolev space $H^{\tau}(\mathbb{R}^d)$ if the kernel function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ has a Fourier transform $\widehat{\phi}$ satisfying

$$c_1(1 + \|\boldsymbol{\omega}\|_2^2)^{-\tau} \leq \widehat{\phi}(\boldsymbol{\omega}) \leq c_2(1 + \|\boldsymbol{\omega}\|_2^2)^{-\tau}, \quad \boldsymbol{\omega} \in \mathbb{R}^d,$$

with two constants $0 < c_1 \leq c_2$, see [24, Corollary 10.13]. This gives for the matrix-valued kernels the following result.

COROLLARY 2.4. *Assume ϕ generates $H^{\tau+1}(\mathbb{R}^d)$ and ψ generates $H^{\rho}(\mathbb{R}^d)$, i.e., $\mathcal{H}_{\phi}(\mathbb{R}^d) = H^{\tau+1}(\mathbb{R}^d)$ and $\mathcal{H}_{\psi}(\mathbb{R}^d) = H^{\rho}(\mathbb{R}^d)$. The associated reproducing kernel Hilbert space of the combined kernel is given by*

$$\mathcal{H}_{\widetilde{\Phi}}(\mathbb{R}^d) = \widetilde{\mathbf{H}}^{\tau}(\mathbb{R}^d; \text{div}) \times H^{\rho}(\mathbb{R}^d).$$

Here,

$$\begin{aligned} \widetilde{\mathbf{H}}^{\tau}(\mathbb{R}^d; \text{div}) &= \left\{ \mathbf{f} \in \mathbf{H}^{\tau}(\mathbb{R}^d; \text{div}) : \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} d\boldsymbol{\omega} < \infty \right\}, \\ \mathbf{H}^{\tau}(\mathbb{R}^d; \text{div}) &= \{ \mathbf{f} \in \mathbf{H}^{\tau}(\mathbb{R}^d) : \nabla \cdot \mathbf{f} = 0 \}. \end{aligned}$$

In addition to the two Sobolev-like spaces above, we are interested in the subspace of curl-free functions of

$$\widetilde{\mathbf{H}}^{\tau}(\mathbb{R}^d) = \left\{ \mathbf{f} \in \mathbf{L}_2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} < \infty \right\}.$$

A function \mathbf{f} is curl-free if and only if we can find a function g such that $\mathbf{f} = \nabla g$. Hence

$$\widetilde{\mathbf{H}}^{\tau}(\mathbb{R}^d, \text{curl}) = \left\{ \mathbf{f} \in \widetilde{\mathbf{H}}^{\tau}(\mathbb{R}^d) : \text{There exists } g \in H^{\tau+1}(\mathbb{R}^d)/\mathbb{R} \text{ such that } \nabla g = \mathbf{f} \right\}.$$

The norm in the spaces $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)$, $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d, \text{div})$ and $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d, \text{curl})$ will be denoted by

$$\|\mathbf{f}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2) d\boldsymbol{\omega}.$$

Finally, since we mainly work on bounded domains, we need for technical reasons to extend our locally defined Sobolev functions to globally defined ones. We will use the following result from [25].

PROPOSITION 2.5. *Let $d = 2, 3$. Let $\tau, \sigma \geq 0$ and let $\Omega \subseteq \mathbb{R}^d$ be a simply-connected domain with $C^{k,1}$ boundary, where $k \geq \tau$ is an integer. Then there exists a continuous operator $\mathbf{E} = (\tilde{\mathbf{E}}_{\text{div}}, E_S) : \mathbf{H}^\tau(\Omega; \text{div}) \times H^\sigma(\Omega) \rightarrow \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\sigma(\mathbb{R}^d)$ such that $\mathbf{E}\mathbf{v}|_\Omega = \mathbf{v}|_\Omega$ for all $\mathbf{v} = (\mathbf{u}, p) \in \mathbf{H}^\tau(\Omega) \times H^\sigma(\Omega)$ and*

$$\|\tilde{\mathbf{E}}_{\text{div}}\mathbf{u}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)} + \|E_S p\|_{H^\sigma(\mathbb{R}^d)} \leq c (\|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)} + \|p\|_{H^\sigma(\Omega)}).$$

The extension operator for the pressure part is the standard Stein extension operator E_S , see [22].

2.2. The Discretization. In [21], the following discretization scheme for solving Darcy's problem numerically has been introduced. It is based upon discretization points $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ in the interior and $Y = \{\mathbf{y}_1, \dots, \mathbf{y}_M\} \subseteq \partial\Omega$ on the boundary. Associated with these discretization points are functionals acting on the combined function $\mathbf{v} = (\mathbf{u}, p)$. They are defined as

$$\lambda_j^{(i)}(\mathbf{v}) = u_i(\mathbf{x}_j) + (K\nabla p)_i(\mathbf{x}_j) \quad (2.1)$$

$$= u_i(\mathbf{x}_j) + \sum_{k=1}^d K_{ik}(\mathbf{x}_j) \partial_k p(\mathbf{x}_j), \quad 1 \leq i \leq d, 1 \leq j \leq N =: N_d \quad (2.2)$$

$$\lambda_j^{(d+1)}(\mathbf{v}) = \sum_{k=1}^d u_k(\mathbf{y}_j) n_k(\mathbf{y}_j), \quad 1 \leq j \leq M =: N_{d+1}, \quad (2.3)$$

and represent on the one hand the partial differential equation and on the other hand the boundary conditions point-wise.

The approximate solution is then given as a linear combination of the Riesz representer of these functionals. Since it is known (see Lemma 3.4 below) that the Riesz representer of a functional is given by the function that results if the functional is applied to one argument of the reproducing kernel, our approximate solution takes the form

$$\mathbf{s}_\mathbf{v}(\mathbf{x}) := \sum_{k=1}^{d+1} \sum_{j=1}^{N_k} \alpha_j^{(k)} (\lambda_j^{(k)})^\mathbf{y} \tilde{\Phi}(\mathbf{x} - \mathbf{y}) \quad (2.4)$$

and the coefficients are determined by enforcing the collocation conditions

$$\lambda_i^{(k)}(\mathbf{s}_\mathbf{v}) = \lambda_i^{(k)}(\mathbf{v}) = \begin{cases} f_k(\mathbf{x}_i) & \text{for } 1 \leq k \leq d, 1 \leq i \leq N =: N_d \\ \mathbf{g} \cdot \mathbf{n}(\mathbf{x}_i) & \text{for } k = d+1, 1 \leq i \leq M =: N_{d+1} \end{cases} \quad (2.5)$$

The following result ensures a unique solution to this linear system. It is taken from [21, Theorem 2.7]

THEOREM 2.6. *Let $\Omega \subseteq \mathbb{R}^d$ with a $C^{1,1}$ boundary. Assume that the building functions $\phi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$ are positive definite and chosen such that $\mathcal{H}_{\tilde{\mathbf{f}}}(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ with $\tau > d/2$. Then, the interpolation function $\mathbf{s}_{\mathbf{v}} = (\mathbf{s}_{\mathbf{u}}, s_p)^T$ from (2.4) is well-defined and uniquely determined by the interpolation conditions (2.5). It satisfies therefore $L\mathbf{s}_{\mathbf{v}}(\mathbf{x}_j) = \mathbf{f}(\mathbf{x}_j)$ with $L\mathbf{v} := \mathbf{u} + K\nabla p$ and $\mathbf{s}_{\mathbf{u}}(\mathbf{y}_j) \cdot \mathbf{n}(\mathbf{y}_j) = \mathbf{g}(\mathbf{y}_j) \cdot \mathbf{n}(\mathbf{y}_j)$. Furthermore, the approximate solution $\mathbf{s}_{\mathbf{u}}$ is analytically divergence free, i.e. $\nabla \cdot \mathbf{s}_{\mathbf{u}} = 0$ in \mathbb{R}^d .*

3. Band-limited Interpolation and Approximation. In this section, we introduce band-limited functions and establish some results regarding the interpolation and the approximation with band-limited functions. These results will be essential for the extended error analysis of the collocation scheme introduced in the last section.

A *band-limited function* is a function $f \in L_2(\mathbb{R}^d)$ with a compactly supported Fourier transform \hat{f} . For $\sigma > 0$ we denote the d -variate ball with centre $\mathbf{0}$ and radius $\sigma > 0$ by $B(\mathbf{0}, \sigma)$. Then, the set of all band-limited functions with band-width σ will be denoted by

$$\mathcal{B}^\sigma := \left\{ f \in L_2(\mathbb{R}^d) : \text{supp } \hat{f} \subseteq B(\mathbf{0}, \sigma) \right\}.$$

The concept of band-limited functions can be extended to vector-valued functions. We are interested in two different kinds of vector-valued functions: Divergence-free and curl-free band-limited functions. We define the following spaces:

$$\begin{aligned} \mathcal{B}^\sigma &:= \left\{ \mathbf{f} \in \mathbf{L}_2(\mathbb{R}^d) : \text{supp } \hat{\mathbf{f}} \subseteq B(\mathbf{0}, \sigma) \right\}, \\ \tilde{\mathcal{B}}^\sigma &:= \left\{ \mathbf{f} \in \mathcal{B}^\sigma : \int_{\mathbb{R}^d} \frac{\|\hat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} d\boldsymbol{\omega} < \infty \right\}, \\ \tilde{\mathcal{B}}_{\text{div}}^\sigma &:= \left\{ \mathbf{f} \in \tilde{\mathcal{B}}^\sigma : \boldsymbol{\omega}^T \hat{\mathbf{f}}(\boldsymbol{\omega}) = 0 \right\}. \end{aligned}$$

In the first case, the requirement $\text{supp } \hat{\mathbf{f}} \subseteq B(\mathbf{0}, \sigma)$ is meant component-wise. Since band-limited functions have a compactly supported Fourier transform, they belong to the reproducing kernel Hilbert spaces of all relevant (radial) basis functions.

We will use band-limited function to approximate a given function, which does not belong to the reproducing kernel Hilbert space and then approximate the band-limited function using our collocation scheme to derive error estimates for functions outside the reproducing kernel Hilbert space.

We will use a general concept to derive approximation results for band-limited functions, which can be stated in a quite general form. Its proof can be found in [17]. It shows the existence of an abstract interpolant, which is comparable to a best approximation. To state it, we use the distance between an element y of a linear space \mathcal{Y} and a subspace \mathcal{V} of \mathcal{Y} , which is defined by

$$\text{dist}_{\mathcal{Y}}(y, \mathcal{V}) := \min_{v \in \mathcal{V}} \|y - v\|_{\mathcal{Y}}.$$

PROPOSITION 3.1. *Let \mathcal{Y} be a (possibly complex) Banach space, \mathcal{V} be a subspace of \mathcal{Y} , and Z^* be a finite dimensional subspace of \mathcal{Y}^* , the dual of \mathcal{Y} . If for every $\lambda^* \in Z^*$ and some $\beta > 1$, β independent of λ^* ,*

$$\|\lambda^*\|_{\mathcal{Y}^*} \leq \beta \|\lambda^*|_{\mathcal{V}}\|_{\mathcal{V}^*},$$

then for $y \in \mathcal{Y}$ there exists $v \in \mathcal{V}$ such that v interpolates y on Z^* ; that is, $\lambda^*(y) = \lambda^*(v)$ for all $\lambda^* \in Z^*$. In addition, v approximates y in the sense that

$$\|y - v\|_{\mathcal{Y}} \leq (1 + 2\beta) \text{dist}_{\mathcal{Y}}(y, \mathcal{V}).$$

The following lemma was proven by FUSELIER in [8, Lemma 1]. It shows that every $\mathbf{f} \in \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div})$ can be approximated by a band-limited function $\mathbf{f}_\sigma \in \tilde{\mathcal{B}}_{\text{div}}^\sigma$. We give a slightly extended version, since we need it for functions in $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\rho(\mathbb{R}^d)$, its proof can be done analogously.

LEMMA 3.2. *Let $\tau \geq \beta \geq 0$ and $\sigma > 0$. For every $\mathbf{f} \in \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div})$ there exists a function $\mathbf{g}_\sigma \in \tilde{\mathcal{B}}_{\text{div}}^\sigma$ with*

$$\|\mathbf{f} - \mathbf{g}_\sigma\|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d)} \leq \sigma^{\beta-\tau} \|\mathbf{f}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}.$$

Moreover, for every $f \in H^\tau(\mathbb{R}^d)$ there exists a function $g_\sigma \in \mathcal{B}^\sigma$ with

$$\|f - g_\sigma\|_{H^\beta(\mathbb{R}^d)} \leq \sigma^{\beta-\tau} \|f\|_{H^\tau(\mathbb{R}^d)}.$$

Obviously, the first statement of the previous lemma would also hold for curl-free functions.

The Sobolev-like space $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)$ and its subspaces $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div})$ and $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{curl})$ are reproducing kernel Hilbert spaces with canonical reproducing kernels given by $\tilde{\mathcal{K}}^\tau := -\Delta \mathcal{K}^{\tau+1}$, $\tilde{\mathcal{K}}_{\text{div}}^\tau := (-\Delta I + \nabla \nabla^T) \mathcal{K}^{\tau+1}$ and $\tilde{\mathcal{K}}_{\text{curl}}^\tau := -\nabla \nabla^T \mathcal{K}^{\tau+1}$ respectively, where \mathcal{K}^s is the canonical reproducing kernel of $H^s(\mathbb{R}^d)$ defined by its Fourier transform

$$\widehat{\mathcal{K}}^s(\boldsymbol{\omega}) := (1 + \|\boldsymbol{\omega}\|_2^2)^{-s}.$$

The following lemma is essentially lemma 2 in [8]. The case of the divergence-free functions was proven there, the case of the curl-free functions can be done in an analogous way. A detailed proof of this is given in [20].

LEMMA 3.3. *Let $q_X = \frac{1}{2} \min_{j \neq k} \|\mathbf{x}_j - \mathbf{x}_k\|_2$ be the separation radius of the discrete set $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$. Let $\mathbf{g} = \sum_{j=1}^N \tilde{\mathcal{K}}_{\text{div}}^\tau(\cdot - \mathbf{x}_j) \boldsymbol{\alpha}_j$ or $\mathbf{g} = \sum_{j=1}^N \tilde{\mathcal{K}}_{\text{curl}}^\tau(\cdot - \mathbf{x}_j) \boldsymbol{\alpha}_j$, $\tau > d/2$, respectively and define \mathbf{g}_σ by $\mathbf{g}_\sigma = \mathbf{g} \chi_\sigma$, where χ_σ is the characteristic function of the ball $B(\mathbf{0}, \sigma)$. Then, there exists a constant $\kappa \geq 24 \left(\frac{\pi(d+2)(d+3)d}{4(d-1)} \Gamma^2 \left(\frac{d+2}{2} \right) \right)^{1/(d+1)} =: \tilde{C}$, which is independent of X and the $\boldsymbol{\alpha}_j$'s, such that for $\sigma = \kappa/q_X$ the following inequality holds:*

$$I_\sigma := \|\mathbf{g} - \mathbf{g}_\sigma\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)} \leq \frac{1}{2} \|\mathbf{g}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}.$$

The following theorem is an extension of [24, Theorem 16.7] for vector-valued reproducing kernel Hilbert spaces. Its proof is exactly the same as the proof in the scalar-valued case. It is the already mentioned fact on Riesz representers.

LEMMA 3.4. *Suppose that \mathbf{H} is a real, vector-valued Hilbert space of functions with reproducing matrix-valued kernel $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^{n \times n}$ and dual \mathbf{H}^* . For every $\lambda \in \mathbf{H}^*$, we have that $\lambda^\mathbf{y}(\Phi(\cdot - \mathbf{y})\boldsymbol{\alpha}) \in \mathbf{H}$ and*

$$\lambda(\mathbf{f}) = (\mathbf{f}, \lambda^\mathbf{y}(\Phi(\cdot - \mathbf{y})\boldsymbol{\alpha}))_{\mathbf{H}} \quad (3.1)$$

for all $\mathbf{f} \in \mathbf{H}$ and all $\boldsymbol{\alpha} \in \mathbb{R}^n$. Moreover,

$$\|\lambda\|_{\mathbf{H}^*} = \|\lambda^{\mathbf{y}}(\Phi(\cdot - \mathbf{y})\boldsymbol{\alpha})\|_{\mathbf{H}}. \quad (3.2)$$

We now state and prove the main result of this section, which is central for the proof of the extended error estimates later on. It guarantees the existence of a band-limited function, which approximates the true solution of Darcy's problem and also gives a bound for the error.

THEOREM 3.5. *Let $\tau, \rho, t, r \in \mathbb{R}$ with $\tau > d/2$, $\rho > d/2 + 1$ and $t, r \geq 0$. Let $\Omega \subseteq \mathbb{R}^d$ be bounded with a $C^{\lceil \tau \rceil + 1, 1}$ boundary and outer unit normal vector \mathbf{n} . Let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ and $Y = \{\mathbf{y}_1, \dots, \mathbf{y}_M\} \subseteq \partial\Omega$ be discrete sets with separation radius $q := q_{X \cup Y}$. For a smooth, combined function $\mathbf{v} = (\mathbf{u}, p)$ we define the operator $L\mathbf{v} := \mathbf{u} + K\nabla p$, where $K_{ij} \in H^\rho(\mathbb{R}^d)$ for all $1 \leq i, j \leq d$.*

For a given $\mathbf{v} = (\mathbf{u}, p) \in \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\rho(\mathbb{R}^d)$ there exists a function $\mathbf{v}_\sigma = (\mathbf{u}_{\sigma_u}, p_{\sigma_p}) \in \tilde{\mathcal{B}}_{\text{div}}^{\sigma_u} \times \mathcal{B}^{\sigma_p}$ such that

$$L\mathbf{v}|_X = L\mathbf{v}_\sigma|_X, \quad \mathbf{u} \cdot \mathbf{n}|_Y = \mathbf{u}_\sigma \cdot \mathbf{n}|_Y$$

and

$$\|\mathbf{v} - \mathbf{v}_\sigma\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} \leq 5 \left(\sigma_u^{-2t} \|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\tau+t}(\mathbb{R}^d)}^2 + \sigma_p^{-2r} \|p\|_{H^{\rho+r}(\mathbb{R}^d)}^2 \right)^{1/2},$$

whenever $\sigma_u, \sigma_p > \frac{\tilde{C}}{q}$, where \tilde{C} is the constant from lemma 3.3.

Proof. To prove this result, we will use proposition 3.1 with

$$\mathcal{Y} := \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\rho(\mathbb{R}^d), \quad \mathcal{V} := \tilde{\mathcal{B}}_{\text{div}}^{\sigma_u} \times \mathcal{B}^{\sigma_p}$$

and $Z^* := \text{span}\{Z_X^* \cup Z_Y^*\}$ with

$$\begin{aligned} Z_X^* &:= \{\lambda(\mathbf{v}) = u_i(\mathbf{x}) + (K\nabla p)_i(\mathbf{x}) : \mathbf{x} \in X, \mathbf{v} = (\mathbf{u}, p) \in \mathcal{Y}, 1 \leq i \leq d\}, \\ Z_Y^* &:= \{\lambda(\mathbf{v}) = \mathbf{n}(\mathbf{x})^T \mathbf{u}(\mathbf{x}) : \mathbf{x} \in Y, \mathbf{v} = (\mathbf{u}, p) \in \mathcal{Y}\}. \end{aligned}$$

Obviously, \mathcal{Y} is a Banach space, \mathcal{V} is a subspace of \mathcal{Y} , Z^* is finite dimensional and every $\lambda \in Z^*$ is linear. Since \mathcal{Y} is a reproducing kernel Hilbert space, the point evaluation functionals are in \mathcal{Y}^* . Furthermore, the Sobolev embedding theorem, our assumption on K and the boundary of Ω guarantee that the functions \mathbf{u}, p, K and \mathbf{n} are sufficiently smooth. Thus, all functionals $\lambda \in Z^*$ are indeed continuous on \mathcal{Y} .

We will now show that for every $\lambda \in Z^*$ we have

$$\|\lambda\|_{\mathcal{Y}^*} \leq 2\|\lambda|_{\mathcal{V}}\|_{\mathcal{V}^*}. \quad (3.3)$$

First of all, we will calculate the Riesz representer and express the norms of the dual space in terms of the original space. Then we can bound $\|\lambda\|_{\mathcal{Y}^*}$ and show that (3.3) holds.

Let $\mathbf{x}_{N+j} := \mathbf{y}_j$ for all $1 \leq j \leq M$. Let $\mathbf{f} = (\mathbf{f}_u, f_p) \in \mathcal{Y}$. We pick an arbitrary element $\lambda \in Z^*$, which can be written as

$$\lambda(\mathbf{f}) = \sum_{j=1}^N \alpha_j^T [\mathbf{f}_u(\mathbf{x}_j) + K(\mathbf{x}_j) \nabla f_p(\mathbf{x}_j)] + \sum_{j=N+1}^{N+M} \alpha_j \mathbf{n}(\mathbf{x}_j)^T \mathbf{f}_u(\mathbf{x}_j),$$

for all $\mathbf{f} = (\mathbf{f}_u, f_p) \in \mathcal{Y}$, where $\alpha_j \in \mathbb{R}^d$ for $1 \leq j \leq N$ and $\alpha_j \in \mathbb{R}$ for $N < j \leq N+M$. If we define

$$\gamma_j := \begin{cases} \alpha_j, & \text{if } 1 \leq j \leq N \\ \alpha_j \mathbf{n}(\mathbf{x}_j), & \text{if } N < j \leq N+M \end{cases}$$

and $\zeta_j^T := \alpha_j^T K(\mathbf{x}_j)$ then we can write the functional as $\lambda(\mathbf{f}) := \lambda_u(\mathbf{f}_u) + \lambda_p(f_p)$, where $\lambda_u(\mathbf{f}_u) := \sum_{j=1}^{N+M} \gamma_j^T \mathbf{f}_u(\mathbf{x}_j)$ and $\lambda_p(f_p) := \sum_{j=1}^N \zeta_j^T \nabla f_p(\mathbf{x}_j)$.

The reproducing kernel of the space $\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\rho(\mathbb{R}^d)$, equipped with the inner product

$$(\mathbf{f}, \mathbf{g})_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} = (\mathbf{f}_u, \mathbf{g}_u)_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)} + (f_p, g_p)_{H^\rho(\mathbb{R}^d)},$$

is given by

$$\mathcal{K}_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\rho(\mathbb{R}^d)}(\mathbf{x} - \mathbf{y}) := \begin{pmatrix} \tilde{\mathcal{K}}_{\text{div}}^\tau(\mathbf{x} - \mathbf{y}) & \mathbf{0} \\ \mathbf{0} & \mathcal{K}^\rho(\mathbf{x} - \mathbf{y}) \end{pmatrix}.$$

This means that we can work out the Riesz representer \mathbf{g}_u and g_p for the functionals $\lambda_u \in \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div})$ and $\lambda_p \in H^\rho(\mathbb{R}^d)$ separately to derive the Riesz representer \mathbf{g}_λ for the functional λ .

In the first case, Lemma 3.4 gives

$$\begin{aligned} \mathbf{g}_u &= \sum_{j=1}^{N+M} \tilde{\mathcal{K}}_{\text{div}}^\tau(\cdot - \mathbf{x}_j) \gamma_j, \\ \|\lambda_u\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div})}^2 &= \sum_{j,k=1}^{N+M} \gamma_j^T \tilde{\mathcal{K}}_{\text{div}}^\tau(\mathbf{x}_j - \mathbf{x}_k) \gamma_k. \end{aligned}$$

In the second case, the same lemma shows that the representer for λ_p is given by $g_p = \sum_{j=1}^N \zeta_j^T \nabla \mathcal{K}^\rho(\cdot - \mathbf{x}_j)$. Since $\zeta_k^T \nabla p$ is a scalar, we have

$$\begin{aligned} \|\lambda_p\|_{H^\rho(\mathbb{R}^d)}^2 &= \|g_p\|_{H^\rho(\mathbb{R}^d)}^2 = \sum_{j,k=1}^N \zeta_j^T \nabla_{\mathbf{x}} \left(\zeta_k^T \nabla_{\mathbf{y}} \mathcal{K}^\rho(\mathbf{x}_j - \mathbf{x}_k) \right)^T \\ &= \sum_{j,k=1}^N \zeta_j^T \nabla_{\mathbf{x}} \nabla_{\mathbf{y}}^T \mathcal{K}^\rho(\mathbf{x}_j - \mathbf{x}_k) \zeta_k. \end{aligned}$$

Altogether we have derived that

$$\|\lambda\|_{\mathcal{Y}^*}^2 = \|\mathbf{g}_\lambda\|_{\mathcal{Y}}^2 = \sum_{j,k=1}^{N+M} \gamma_j^T \tilde{\mathcal{K}}_{\text{div}}^\tau(\mathbf{x}_j - \mathbf{x}_k) \gamma_k + \sum_{j,k=1}^N \zeta_j^T \nabla_{\mathbf{x}} \nabla_{\mathbf{y}}^T \mathcal{K}^\rho(\mathbf{x}_j - \mathbf{x}_k) \zeta_k.$$

The next step is to show that $\|\lambda|_{\mathcal{V}}\|_{\mathcal{V}^*} = \|\mathbf{g}_\sigma\|_{\mathcal{V}}$, where $\mathbf{g}_\sigma = (\mathbf{g}_{u,\sigma_u}, g_{p,\sigma_p})$ is the Riesz representer from the band-limited space. Since \mathcal{V} is a subspace of \mathcal{Y} the norms are the same for every element in \mathcal{V} . Again we first deal with λ_u . Let $\mathbf{f} \in \tilde{\mathcal{B}}_{\text{div}}^{\sigma_u}$ and

$\mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}}$ be defined by $\widehat{\mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}}} = \widehat{\mathbf{g}_{\mathbf{u}}} \chi_{\sigma_{\mathbf{u}}}$. This gives

$$\begin{aligned}
\lambda_{\mathbf{u}}(\mathbf{f}) &= (\mathbf{f}, \mathbf{g}_{\mathbf{u}})_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{\mathbf{g}_{\mathbf{u}}}(\boldsymbol{\omega})^* \widehat{\mathbf{f}}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} d\boldsymbol{\omega} \\
&= (2\pi)^{-d/2} \int_{\|\boldsymbol{\omega}\|_2^2 \leq \sigma_{\mathbf{u}}} \frac{\widehat{\mathbf{g}_{\mathbf{u}}}(\boldsymbol{\omega})^* \widehat{\mathbf{f}}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} d\boldsymbol{\omega} \\
&= (2\pi)^{-d/2} \int_{\|\boldsymbol{\omega}\|_2^2 \leq \sigma_{\mathbf{u}}} \frac{\widehat{\mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}}}(\boldsymbol{\omega})^* \widehat{\mathbf{f}}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} d\boldsymbol{\omega} \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{\mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}}}(\boldsymbol{\omega})^* \widehat{\mathbf{f}}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\tau+1} d\boldsymbol{\omega} \\
&= (\mathbf{f}, \mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}})_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d)},
\end{aligned}$$

where we used the fact that $\widehat{\mathbf{f}}$ vanishes outside the ball $B(\mathbf{0}, \sigma_{\mathbf{u}})$. This equality and the same idea as in the proof of lemma 3.4 lead us to

$$\|\lambda_{\mathbf{u}}|_{\tilde{\mathcal{B}}_{\text{div}}^{\sigma_{\mathbf{u}}}}\|_{(\tilde{\mathcal{B}}_{\text{div}}^{\sigma_{\mathbf{u}}})^*} = \|\lambda_{\mathbf{u}}|_{\tilde{\mathcal{B}}_{\text{div}}^{\sigma_{\mathbf{u}}}}\|_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d; \text{div})^*} = \|\mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}}\|_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d)}.$$

Using similar arguments and g_{p, σ_p} defined by $\widehat{g_{p, \sigma_p}} = \widehat{g_p} \chi_{\sigma_p}$ allows us to derive

$$(f, g_p)_{H^{\rho}(\mathbb{R}^d)} = (f, g_{p, \sigma_p})_{H^{\rho}(\mathbb{R}^d)}$$

and

$$\|\lambda_p|_{\mathcal{B}^{\sigma_p}}\|_{\mathcal{B}^{\sigma_p}*} = \|\lambda_p|_{\mathcal{B}^{\sigma_p}}\|_{H^{\rho}(\mathbb{R}^d)*} = \|g_{p, \sigma_p}\|_{H^{\rho}(\mathbb{R}^d)}.$$

This all together means that

$$\begin{aligned}
\|\lambda|_{\mathcal{V}}\|_{\mathcal{V}^*}^2 &= \|\lambda|_{\mathcal{V}}\|_{\mathcal{Y}^*}^2 = \|\lambda_{\mathbf{u}}|_{\tilde{\mathcal{B}}_{\text{div}}^{\sigma_{\mathbf{u}}}}\|_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d)*}^2 + \|\lambda_p|_{\mathcal{B}^{\sigma_p}}\|_{H^{\rho}(\mathbb{R}^d)*}^2 \\
&= \|\mathbf{g}_{\mathbf{u}, \sigma_{\mathbf{u}}}\|_{\tilde{\mathbf{H}}^{\tau}(\mathbb{R}^d)}^2 + \|g_{p, \sigma_p}\|_{H^{\rho}(\mathbb{R}^d)}^2 = \|\mathbf{g}_{\lambda, \sigma}\|_{\mathcal{Y}}^2.
\end{aligned}$$

Later we want to apply lemma 3.3 to bound $\|g_p - g_{p, \sigma_p}\|_{H^{\rho}(\mathbb{R}^d)}$ by $\|g_p\|_{H^{\rho}(\mathbb{R}^d)}$. Before doing so, we need to show that $\|g_p\|_{H^{\rho}(\mathbb{R}^d)} = \|\mathbf{g}_{\text{curl}}\|_{\tilde{\mathbf{H}}^{\rho-1}(\mathbb{R}^d)}$ and $\|g_p - g_{p, \sigma_p}\|_{H^{\rho}(\mathbb{R}^d)} = \|\mathbf{g}_{\text{curl}} - \mathbf{g}_{\text{curl}, \sigma_p}\|_{\tilde{\mathbf{H}}^{\rho-1}(\mathbb{R}^d; \text{curl})}$, where $\mathbf{g}_{\text{curl}} = \sum_{j=1}^N \tilde{\mathcal{K}}_{\text{curl}}^{\rho-1}(\cdot - \mathbf{x}_j) \boldsymbol{\zeta}_j$ and g_{p, σ_p} , $\mathbf{g}_{\text{curl}, \sigma_p}$ are defined by $\widehat{g_{p, \sigma_p}} = \widehat{g_p} \chi_{\sigma_p}$ and $\widehat{\mathbf{g}_{\text{curl}, \sigma_p}} = \widehat{\mathbf{g}_{\text{curl}}} \chi_{\sigma_p}$ respectively. We have that $\tilde{\mathcal{K}}_{\text{curl}}^{\rho-1} = -\nabla \nabla^T \mathcal{K}^{\rho}$, where \mathcal{K}^{ρ} is the reproducing kernel of $H^{\rho}(\mathbb{R}^d)$. Therefore

$$\mathbf{g}_{\text{curl}} = \sum_{j=1}^N \tilde{\mathcal{K}}_{\text{curl}}^{\rho-1}(\cdot - \mathbf{x}_j) \boldsymbol{\zeta}_j = -\sum_{j=1}^N \nabla \nabla^T \mathcal{K}^{\rho}(\cdot - \mathbf{x}_j) \boldsymbol{\zeta}_j = -\nabla g_p.$$

With the identity above, we can derive

$$\begin{aligned}
\|\mathbf{g}_{\text{curl}} - \mathbf{g}_{\text{curl}, \sigma_p}\|_{\tilde{\mathbf{H}}^{\rho-1}(\mathbb{R}^d)}^2 &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|(\widehat{\mathbf{g}_{\text{curl}}} - \widehat{\mathbf{g}_{\text{curl}, \sigma_p}})(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\rho} d\boldsymbol{\omega} \\
&= (2\pi)^{-d/2} \int_{\|\boldsymbol{\omega}\|_2^2 \geq \sigma_p} \frac{\|\widehat{-\nabla g_p}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\rho} d\boldsymbol{\omega} \\
&= \|g_p - g_{p, \sigma_p}\|_{H^{\rho}(\mathbb{R}^d)}^2.
\end{aligned} \tag{3.4}$$

and

$$\|\mathbf{g}_{\text{curl}}\|_{\tilde{\mathbf{H}}^{\rho-1}(\mathbb{R}^d)} = \|g_p\|_{H^\rho(\mathbb{R}^d)}. \quad (3.5)$$

We add the term $\mathbf{g} - \mathbf{g}_\sigma$ and apply the inverse triangle inequality to establish

$$\begin{aligned} \|\mathbf{g}_\sigma\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} &= \|\mathbf{g} - (\mathbf{g} - \mathbf{g}_\sigma)\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} \\ &\geq \left| \|\mathbf{g}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} - \|\mathbf{g} - \mathbf{g}_\sigma\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} \right| \end{aligned}$$

To bound the norm above, we apply (3.4) and (3.5) together with lemma 3.3,

$$\begin{aligned} \|\mathbf{g} - \mathbf{g}_\sigma\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} &= \left(\|\mathbf{g}_\mathbf{u} - \mathbf{g}_{\mathbf{u}, \sigma_\mathbf{u}}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}^2 + \|\mathbf{g}_{\text{curl}} - \mathbf{g}_{\text{curl}, \sigma_p}\|_{\tilde{\mathbf{H}}^{\rho-1}(\mathbb{R}^d)}^2 \right)^{1/2} \\ &\leq \left(\frac{1}{4} \|\mathbf{g}_\mathbf{u}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}^2 + \frac{1}{4} \|\mathbf{g}_{\text{curl}}\|_{\tilde{\mathbf{H}}^{\rho-1}(\mathbb{R}^d)}^2 \right)^{1/2} \\ &= \frac{1}{2} \|\mathbf{g}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)}. \end{aligned}$$

Thus we have

$$\begin{aligned} \|\lambda|_{\mathcal{V}}\|_{\mathcal{V}^*} &= \|\mathbf{g}_\sigma\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} \geq \left| \|\mathbf{g}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} - \frac{1}{2} \|\mathbf{g}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} \right| \\ &= \frac{1}{2} \|\mathbf{g}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d) \times H^\rho(\mathbb{R}^d)} = \frac{1}{2} \|\lambda\|_{\mathcal{V}^*}. \end{aligned}$$

Hence we have proven that $\|\lambda\|_{\mathcal{V}^*} \leq 2\|\lambda|_{\mathcal{V}}\|_{\mathcal{V}^*}$ for every $\lambda \in Z^*$, i.e., all assumptions of theorem 3.1 are satisfied with $\beta = 2$. Thus for every $\mathbf{v} \in \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^\rho(\mathbb{R}^d)$ there exists a $\mathbf{v}_\sigma = (\mathbf{u}_{\sigma_\mathbf{u}}, p_{\sigma_p}) \in \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}} \times \mathcal{B}^{\sigma_p}$ such that \mathbf{v}_σ interpolates \mathbf{v} on Z^* , i.e., $\lambda(\mathbf{v}) = \lambda(\mathbf{v}_\sigma)$ for all $\lambda \in Z^*$. In addition, \mathbf{v}_σ approximates \mathbf{v} in the sense that

$$\|\mathbf{v} - \mathbf{v}_\sigma\|_{\mathcal{Y}}^2 = \|\mathbf{u} - \mathbf{u}_{\sigma_\mathbf{u}}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}^2 + \|p - p_{\sigma_p}\|_{H^\rho(\mathbb{R}^d)}^2 \leq 5^2 \text{dist}_{\mathcal{Y}}(\mathbf{v}, \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}} \times \mathcal{B}^{\sigma_p})^2.$$

Next, we will bound the distance between \mathbf{v} and $\tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}} \times \mathcal{B}^{\sigma_p}$. We will do this again component-wise, since the definition of the distance gives that

$$\begin{aligned} \text{dist}_{\mathcal{Y}}(\mathbf{v}, \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}} \times \mathcal{B}^{\sigma_p})^2 &= \min_{\mathbf{v}_\sigma \in \mathcal{V}} \left\{ \|\mathbf{u} - \mathbf{u}_{\sigma_\mathbf{u}}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}^2 + \|p - p_{\sigma_p}\|_{H^\rho(\mathbb{R}^d)}^2 \right\} \\ &= \min_{\mathbf{u}_{\sigma_\mathbf{u}} \in \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}}} \|\mathbf{u} - \mathbf{u}_{\sigma_\mathbf{u}}\|_{\tilde{\mathbf{H}}^\tau(\mathbb{R}^d)}^2 + \min_{p_{\sigma_p} \in \mathcal{B}^{\sigma_p}} \|p - p_{\sigma_p}\|_{H^\rho(\mathbb{R}^d)}^2 \\ &= \text{dist}_{\tilde{\mathbf{H}}(\mathbb{R}^d)}(\mathbf{u}, \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}})^2 + \text{dist}_{H^\rho(\mathbb{R}^d)}(p, \mathcal{B}^{\sigma_p})^2. \end{aligned}$$

Note that, to simplify notation, we have again used the notation $\mathbf{u}_{\sigma_\mathbf{u}}$ and p_{σ_p} , though, this time, they denoted arbitrary elements of the respective spaces of band-limited functions. We proceed now by bounding both terms separately. We have by lemma 3.2 that

$$\text{dist}_{\tilde{\mathbf{H}}(\mathbb{R}^d)}(\mathbf{u}, \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}})^2 \leq \sigma_\mathbf{u}^{2(\tau - (\tau+t))} \|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\tau+t}(\mathbb{R}^d)}^2 \leq \sigma_\mathbf{u}^{-2t} \|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\tau+t}(\mathbb{R}^d)}^2.$$

and

$$\text{dist}_{H^\rho(\mathbb{R}^d)}(p, \mathcal{B}^{\sigma_p})^2 \leq \sigma_p^{2(\rho - (\rho+r))} \|p\|_{H^{\rho+r}(\mathbb{R}^d)}^2 \leq \sigma_p^{-2r} \|p\|_{H^{\rho+r}(\mathbb{R}^d)}^2$$

and thus

$$\text{dist}_{\mathcal{Y}}(\mathbf{v}, \tilde{\mathcal{B}}_{\text{div}}^{\sigma_\mathbf{u}} \times \mathcal{B}^{\sigma_p})^2 \leq \sigma_\mathbf{u}^{-2t} \|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\tau+t}(\mathbb{R}^d)}^2 + \sigma_p^{-2r} \|p\|_{H^{\rho+r}(\mathbb{R}^d)}^2,$$

which finishes the proof. \square

4. Error Analysis. Our error analysis is mainly based on a 'shift' type theorem for the analytical solution of Darcy's problem. It is taken from [21, Theorem 3.2].

PROPOSITION 4.1. *Let Ω be a bounded open subset of \mathbb{R}^d with a $C^{[\tau]+1,1}$ boundary $\partial\Omega$. Assume that the data $\mathbf{f} \in \mathbf{W}_r^{\tau+1}(\Omega)$ and $\mathbf{g} \in \mathbf{W}_r^{\tau+1-1/r}(\partial\Omega)$ for $1 < r < \infty$ and satisfies $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, dS = 0$. Assume that the permeability tensor $K = (K_{ij})$ satisfies (1.5), $K = K^T$ and $K_{ij} \in W_r^{\tau+1}(\overline{\Omega})$. Then there exist a velocity $\mathbf{u} \in \mathbf{W}_r^{\tau+1}(\Omega)$ and a pressure $p \in W_r^{\tau+2}(\Omega)/\mathbb{R}$, solutions to (1.1)–(1.3), which satisfy*

$$\|\mathbf{u}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|p\|_{W_r^{\tau+2}(\Omega)/\mathbb{R}} \leq c \left(\|\mathbf{f}\|_{\mathbf{W}_r^{\tau+1}(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{W_r^{\tau+1-1/r}(\partial\Omega)} \right).$$

Besides the shift type theorem, we will apply so called sampling inequalities. To state them, we have to introduce a measure for the data density on Ω and $\partial\Omega$. In the first case we introduce the "fill distance"

$$h_{X,\Omega} := \sup_{\mathbf{x} \in \Omega} \min_{\mathbf{x}_j \in X} \|\mathbf{x} - \mathbf{x}_j\|_2.$$

The following result has precursors in [2, 18, 19] and comes in its vector-valued form from [25].

LEMMA 4.2. *Let $1 < r < \infty$, and $\tau, \eta \in \mathbb{R}$ with $\tau > d/2$ and $0 \leq \eta \leq \tau - d(1/2 - 1/r)_+$. Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain having a Lipschitz boundary. Let $X \subseteq \Omega$ be a discrete set with fill distance $h_{X,\Omega}$ sufficiently small. Assume that $\mathbf{u} \in \mathbf{H}^\tau(\Omega)$ satisfies $\mathbf{u}|_X = \mathbf{0}$. Then we also have*

$$\|\mathbf{u}\|_{\mathbf{W}_r^\eta(\Omega)} \leq ch_{X,\Omega}^{\tau-\eta-d(1/2-1/r)_+} \|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)}.$$

To introduce a measure on the boundary, we follow ideas from [9, 25]. Let $\partial\Omega = \cup_{j=1}^J V_j$, where $V_j \subseteq \partial\Omega$ are relatively open sets. Furthermore,

$$\varphi_j : B \rightarrow V_j,$$

where φ_j is a $C^{k,s}$ -diffeomorphism and $B = B(\mathbf{0}, 1)$ denotes the unit ball in \mathbb{R}^{d-1} . We will measure the density of the points Y on $\partial\Omega$ by introducing

$$h_{Y,\partial\Omega} := \max_{1 \leq j \leq J} h_{T_j, B}$$

with $T_j = \varphi_j^{-1}(Y \cap V_j) \subseteq B$ analogously to the definition of the fill distance. We assume that the atlas V_j is fixed, i.e., we do not have to worry about the dependence of $h_{Y,\partial\Omega}$ on the atlas.

To derive the estimate on the boundary, we need a similar result as lemma 4.2 on manifolds. This has been done in [11] for the special case of $\partial\Omega$ being the sphere in \mathbb{R}^d and in a more general context in [9]. We give an extended version which also deals with non-integer orders η . Its proof can be found in [25].

LEMMA 4.3. *Let $1 < r < \infty$ and $\tau = k + s > d/2$, where $k \in \mathbb{N}_0$ and $0 < s \leq 1$. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain having a $C^{k,s}$ smooth boundary. Assume that $Y \subseteq \partial\Omega$ with $h_{Y,\partial\Omega}$ sufficiently small. Then there is a constant $c > 0$ such that for all $\mathbf{u} \in \mathbf{H}^\tau(\Omega)$ with $\mathbf{u}|_Y = 0$ we have for $0 \leq \eta \leq \tau - 1/2 - (d-1)(1/2 - 1/r)_+$ that*

$$\|\mathbf{u}\|_{\mathbf{W}_r^\eta(\partial\Omega)} \leq ch_{Y,\partial\Omega}^{\tau-1/2-\eta-(d-1)(1/2-1/r)_+} \|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)}.$$

The standard trace theorem establishes that if $u \in H^\tau(\Omega)$ then $u \in H^{\tau-1/2}(\partial\Omega)$, cf. [26, Theorem 8.7]. If $\tau > d/2$, then this guarantees in combination with the Sobolev embedding theorem that u is continuous on the boundary $\partial\Omega$.

The following result gives the error estimates for smooth target functions, i.e., target functions form the associated reproducing kernel Hilbert space. It was the main result in [21].

THEOREM 4.4. *Let Ω be a bounded, simply connected, open subset of \mathbb{R}^d , $d = 2, 3$, with a $C^{\lceil\tau\rceil+1,1}$ boundary $\partial\Omega$. Suppose that Φ is chosen such that its associated Hilbert space is $\mathcal{H}_\Phi(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ and the permeability tensor $K = K_{ij}$ satisfies (1.5), $K = K^T$ and $K_{ij} \in H^{\tau+1}(\overline{\Omega})$. Furthermore, assume that the data satisfy $\mathbf{f} \in \mathbf{H}^{\tau+1}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\tau+1/2}(\partial\Omega)$ and $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} dS = 0$, where $\tau > d/2$. Then, the error between the true solution and the collocation approximation can be bounded by*

$$\begin{aligned} \|\mathbf{u} - \mathbf{s}_\mathbf{u}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p - s_p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} &\leq c(h_{X,\Omega}^{\tau-\eta-1-d(1/2-1/r)_+} + \\ &\quad + h_{Y,\partial\Omega}^{\tau-\eta-1-1/2+1/r-(d-1)(1/2-1/r)_+}) (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\tau-1/2}(\partial\Omega)}) \end{aligned}$$

for $1 < r < \infty$ and $0 \leq \eta \leq \tau - d(1/2 - 1/r)_+ - 1$. If $r \geq 2$ and $h = h_{X,\Omega} \approx h_{Y,\partial\Omega}$ this reduces to

$$\begin{aligned} \|\mathbf{u} - \mathbf{s}_\mathbf{u}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p - s_p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \\ \leq ch^{\tau-\eta-1-d(1/2-1/r)_+} (\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\tau-1/2}(\partial\Omega)}). \end{aligned}$$

We now state and prove the error estimates for rougher target functions. Besides a similar approach as in the standard error analysis, the main idea is to find a band-limited function \mathbf{v}_σ , which approximates the true solution. Then we can add the term $\mathbf{v}_\sigma - \mathbf{v}$ to the difference between the true solution \mathbf{v} and our approximating function $\mathbf{s}_\mathbf{v}$. With the triangle inequality the norm can be split into two. The difference between the true solution and the band-limited function can be bounded with theorem 3.5. The difference between the band-limited function and the approximating function can be bounded with standard error analysis, since $\mathbf{s}_\mathbf{v}$ also approximates \mathbf{v}_σ and both functions are sufficiently smooth.

THEOREM 4.5. *Let Ω be a bounded, simply connected, open subset of \mathbb{R}^d , $d = 2, 3$, with a $C^{\lceil\beta\rceil+1,1}$ boundary $\partial\Omega$. Suppose that Φ is chosen such that its associated Hilbert space is $\mathcal{H}_\Phi(\mathbb{R}^d) = \tilde{\mathbf{H}}^\tau(\mathbb{R}^d; \text{div}) \times H^{\tau+1}(\mathbb{R}^d)$ and the permeability tensor $K = K_{ij}$ satisfies (1.5), $K = K^T$ and $K_{ij} \in H^{\beta+1}(\overline{\Omega})$. Furthermore, assume that the data satisfy $\mathbf{f} \in \mathbf{H}^{\beta+1}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\beta+1/2}(\partial\Omega)$ and $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} dS = 0$, where $\tau \geq \beta > d/2$. Then, the error between the true solution and the collocation approximation can be bounded by*

$$\begin{aligned} \|\mathbf{u} - \mathbf{s}_\mathbf{u}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \|p - s_p\|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \\ \leq c \frac{h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta}}{q^{\tau-\beta}} \left(h_{X,\Omega}^{\beta-\eta-1-d(1/2-1/r)_+} + h_{Y,\partial\Omega}^{\beta-\eta-1-1/2+1/r-(d-1)(1/2-1/r)_+} \right) \times \\ \times (\|\mathbf{f}\|_{\mathbf{H}^\beta(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\beta-1/2}(\partial\Omega)}) \end{aligned}$$

for every $1 < r < \infty$ and $0 \leq \eta \leq \beta - d(1/2 - 1/r)_+ - 1$ and separation radius

$q := q_{X \cup Y}$. If $r \geq 2$ and $h = h_{X, \Omega} \approx h_{Y, \partial \Omega}$ this reduces to

$$\begin{aligned} & \| \mathbf{u} - \mathbf{s}_{\mathbf{u}} \|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \| p - s_p \|_{W_r^{\eta+2}(\Omega)/\mathbb{R}} \\ & \leq ch^{\beta-\eta-1-d(1/2-1/r)} \left(\frac{h}{q} \right)^{\tau-\beta} \left(\| \mathbf{f} \|_{\mathbf{H}^\beta(\Omega)} + \| \mathbf{g} \cdot \mathbf{n} \|_{H^{\beta-1/2}(\Omega)} \right). \end{aligned}$$

Proof. First of all, we pick a representer p of the pressure such that $\|p\|_{W_r^{\beta+2}(\Omega)} = \|p\|_{W_r^{\beta+2}(\Omega)/\mathbb{R}}$.

Let $\mathbf{v} = (\mathbf{u}, p)$. Since all norms on \mathbb{R}^d are equivalent, we have that $\| \mathbf{u} - \mathbf{s}_{\mathbf{u}} \|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \| p - s_p \|_{W_r^{\eta+2}(\Omega)}$ is equivalent to $\| \mathbf{v} - \mathbf{s}_{\mathbf{v}} \|_{\mathbf{W}_r^{\eta+1}(\Omega) \times W_r^{\eta+2}(\Omega)/\mathbb{R}}$.

We now apply proposition 4.1 to the difference $\mathbf{v} - \mathbf{s}_{\mathbf{v}}$ instead of \mathbf{v} , i.e., we derive

$$\begin{aligned} & \| \mathbf{u} - \mathbf{s}_{\mathbf{u}} \|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \| p - s_p \|_{W_r^{\eta+2}(\Omega)} \\ & \leq c \left(\| L\mathbf{v} - L\mathbf{s}_{\mathbf{v}} \|_{\mathbf{W}_r^{\eta+1}(\Omega)} + \| (\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n} \|_{W_r^{\eta+1-1/r}(\partial \Omega)} \right), \end{aligned}$$

for all $0 \leq \eta \leq \beta$. To estimate the two terms on the right hand side of the last equation, we first observe that we have

$$\begin{aligned} (L\mathbf{v} - L\mathbf{s}_{\mathbf{v}})(\mathbf{x}_j) &= \mathbf{0}, \quad 1 \leq j \leq N, \\ (\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n}(\mathbf{y}_j) &= 0, \quad 1 \leq j \leq M. \end{aligned}$$

Hence, we are dealing with smooth functions that have a large number of zeros. In the first case we have functions defined on a bounded region of \mathbb{R}^d , while in the second case we are dealing with functions on a manifold. For such functions, we can apply the sampling inequalities. We will now bound $\|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{W}_r^{\eta+1}(\Omega)}$ and $\|(\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n}\|_{W_r^{\eta+1-1/r}(\partial \Omega)}$ separately.

We will start with the estimate in the interior. The function $L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}$ has many zeros, i.e., we can apply the sampling inequality lemma 4.2, such that

$$\|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{W}_r^{\eta+1}(\Omega)} \leq ch_{X, \Omega}^{\beta-\eta-1-d(1/2-1/r)+} \|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{H}^\beta(\Omega)}.$$

Obviously,

$$\begin{aligned} \|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{H}^\beta(\Omega)} &\leq c \left(\| \mathbf{u} - \mathbf{s}_{\mathbf{u}} \|_{\mathbf{H}^\beta(\Omega)} + \| p - s_p \|_{H^{\beta+1}(\Omega)} \right) \\ &\leq c \| \mathbf{v} - \mathbf{s}_{\mathbf{v}} \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)}. \end{aligned}$$

To bound (4.1), we apply the extension operator \mathbf{E} to \mathbf{v} and extend K component-wise with Stein's extension operator E_S , see proposition 2.5. Then there exists a band-limited function \mathbf{v}_σ which approximates the extension of \mathbf{v} , see theorem 3.5. Adding $\mathbf{v}_\sigma - \mathbf{v}_\sigma$ and using the triangle inequality leads to

$$\begin{aligned} \| \mathbf{v} - \mathbf{s}_{\mathbf{v}} \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)} &= \| \mathbf{E}\mathbf{v} - \mathbf{s}_{\mathbf{E}\mathbf{v}} \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)} \\ &\leq \| \mathbf{E}\mathbf{v} - \mathbf{v}_\sigma \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)} + \| \mathbf{v}_\sigma - \mathbf{s}_{\mathbf{E}\mathbf{v}} \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)}. \end{aligned} \tag{4.1}$$

The first part of (4.1) can be bounded by theorem 3.5 with $t, r = 0$ and the properties of the extension operator:

$$\begin{aligned} \| \mathbf{E}\mathbf{v} - \mathbf{v}_\sigma \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)} &\leq \| \mathbf{E}\mathbf{v} - \mathbf{v}_\sigma \|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d) \times H^{\beta+1}(\mathbb{R}^d)} \\ &\leq c \left(\| \tilde{\mathbf{E}}_{\text{div}} \mathbf{u} \|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d)} + \| E_S p \|_{H^{\beta+1}(\mathbb{R}^d)} \right) \\ &\leq c \left(\| \mathbf{u} \|_{\mathbf{H}^\beta(\Omega)} + \| p \|_{H^{\beta+1}(\Omega)} \right) \\ &\leq c \| \mathbf{v} \|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)}. \end{aligned} \tag{4.2}$$

To bound the second part of (4.1) we can apply theorem 4.4 with $\mathbf{f} := L\mathbf{v}_\sigma$, $\mathbf{g} := \mathbf{u}_\sigma$, $r := 2$ and $\eta := \beta - 1$, since all functions are sufficiently smooth. The definition of \mathbf{v}_σ provides that $L\mathbf{v}_\sigma = L\mathbf{v}$ on X and $\mathbf{v}_\sigma \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ on Y . This means in particular that $\mathbf{s}_{\mathbf{v}_\sigma} = \mathbf{s}_{\mathbf{E}\mathbf{v}}$, i.e., the reconstruction of \mathbf{v}_σ is the same as the reconstruction of $\mathbf{s}_{\mathbf{E}\mathbf{v}}$. Furthermore, \mathbf{v}_σ is a smooth target function such that we can indeed apply theorem 4.4. If we use also the trace theorem and $\|L\mathbf{v}_\sigma\|_{\mathbf{H}^\tau(\Omega)} \leq c\|\mathbf{v}_\sigma\|_{\mathbf{H}^\tau(\Omega) \times H^{\tau+1}}$ we can derive

$$\begin{aligned} \|\mathbf{v}_\sigma - \mathbf{s}_{\mathbf{E}\mathbf{v}}\|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)} &\leq c \left(h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta} \right) (\|L\mathbf{v}_\sigma\|_{\mathbf{H}^\tau(\Omega)} + \|\mathbf{u}_\sigma \cdot \mathbf{n}\|_{H^{\tau-1/2}(\partial\Omega)}) \\ &\leq c \left(h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta} \right) \|\mathbf{v}_\sigma\|_{\mathbf{H}^\tau(\Omega) \times H^{\tau+1}(\Omega)}. \end{aligned}$$

Since $\mathbf{v}_\sigma = (\mathbf{u}_{\sigma_u}, p_{\sigma_p})$ is band-limited, we have Bernstein estimates of the form

$$\begin{aligned} \|\mathbf{v}_\sigma\|_{\mathbf{H}^\tau(\Omega) \times H^{\tau+1}(\Omega)} &\leq \|\mathbf{v}_\sigma\|_{\mathbf{H}^\tau(\mathbb{R}^d) \times H^{\tau+1}(\mathbb{R}^d)} \\ &\leq c (\|\mathbf{u}_{\sigma_u}\|_{\mathbf{H}^\tau(\mathbb{R}^d)} + \|p_{\sigma_p}\|_{H^{\tau+1}(\mathbb{R}^d)}) \\ &\leq c (\sigma_u^{\tau-\beta} \|\mathbf{u}_{\sigma_u}\|_{H^\beta(\mathbb{R}^d)} + \sigma_p^{\tau-\beta} \|p_{\sigma_p}\|_{H^{\beta+1}(\mathbb{R}^d)}) \\ &\leq cq^{\beta-\tau} \|\mathbf{v}_\sigma\|_{\mathbf{H}^\beta(\mathbb{R}^d) \times H^{\beta+1}(\mathbb{R}^d)} \\ &\leq cq^{\beta-\tau} \|\mathbf{v}_\sigma\|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d) \times H^{\beta+1}(\mathbb{R}^d)}, \end{aligned}$$

where we have also used the fact that σ_u and σ_p are proportional to $1/q$.

Adding and subtracting $\mathbf{E}\mathbf{v}$ and applying the triangle inequality gives in combination with (4.2) and the properties of the extension operator that

$$\begin{aligned} \|\mathbf{v}_\sigma\|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d) \times H^{\beta+1}(\mathbb{R}^d)} &\leq \|\mathbf{E}\mathbf{v} - \mathbf{v}_\sigma\|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d) \times H^{\beta+1}(\mathbb{R}^d)} + \|\mathbf{E}\mathbf{v}\|_{\tilde{\mathbf{H}}^\beta(\mathbb{R}^d) \times H^{\beta+1}(\mathbb{R}^d)} \\ &\leq c\|\mathbf{v}\|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)}. \end{aligned} \quad (4.3)$$

With (4.3), we can bound the second part of (4.1) by

$$\|\mathbf{v}_\sigma - \mathbf{s}_{\mathbf{E}\mathbf{v}}\|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)} \leq cq^{\beta-\tau} \left(h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta} \right) \|\mathbf{v}\|_{\mathbf{H}^\beta(\Omega) \times H^{\beta+1}(\Omega)}.$$

Since $q \leq q_X \leq h_{X,\Omega}$ and $\beta \leq \tau$, we have that $\frac{h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta}}{q^{\tau-\beta}}$ is greater than or equal to one. Combining the above inequalities and applying proposition 4.1 gives

$$\begin{aligned} \|L\mathbf{v} - L\mathbf{s}_{\mathbf{v}}\|_{\mathbf{H}^\beta(\Omega)} &\leq c \frac{h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta}}{q^{\tau-\beta}} (\|\mathbf{u}\|_{\mathbf{H}^\beta(\Omega)} + \|p\|_{H^{\beta+1}(\Omega)/\mathbb{R}}) \\ &\leq c \frac{h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta}}{q^{\tau-\beta}} (\|\mathbf{f}\|_{\mathbf{H}^\beta(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\beta-1/2}(\Omega)}). \end{aligned} \quad (4.4)$$

We now bound the boundary part. The proof of [21, proposition 3.6] establishes that there exists an extension $\tilde{\mathbf{n}} \in \mathbf{H}^{[\beta]}(\Omega)$ of the normals \mathbf{n} to the interior of Ω with $\tilde{\mathbf{n}}|_{\partial\Omega} = \mathbf{n}|_{\partial\Omega}$.

The function $\mathbf{u} - \mathbf{s}_{\mathbf{u}}$ has many zeros, i.e., we can apply the sampling inequality lemma 4.3. Therefore

$$\|(\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \mathbf{n}\|_{W_r^{\eta+1-1/r}(\partial\Omega)} \leq ch_{Y,\partial\Omega}^{\beta-1-1/2-\eta+1/r-(d-1)(1/2-1/r)+} \|(\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \tilde{\mathbf{n}}\|_{H^\beta(\Omega)}.$$

The proof of [21, proposition 3.6] also establishes

$$\|(\mathbf{u} - \mathbf{s}_{\mathbf{u}}) \cdot \tilde{\mathbf{n}}\|_{H^\beta(\Omega)} \leq \|\tilde{\mathbf{n}}\|_{\mathbf{H}^\beta(\Omega)} \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{H}^\beta(\Omega)} \leq c \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{H}^\beta(\Omega)}.$$

With (4.1) and (4.4) we have

$$\|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{H}^\beta(\Omega)} \leq c \frac{h_{X,\Omega}^{\tau-\beta} + h_{Y,\partial\Omega}^{\tau-\beta}}{q^{\tau-\beta}} \left(\|\mathbf{f}\|_{\mathbf{H}^\beta(\Omega)} + \|\mathbf{g} \cdot \mathbf{n}\|_{H^{\beta-1/2}(\Omega)} \right),$$

which finishes the proof. \square

In the case that $\beta = \tau$, the result above is identical to the one in theorem 4.4.

Note that the limitation of this result to the dimensions $d = 2, 3$ is only due to the fact that the extension operator has not yet been proven for a general $d > 3$.

5. Numerical Examples. We will now illustrate our theory by considering a numerical example for rougher target functions.

In all computations, the compactly supported Wendland functions $\phi_{d,\ell}$ are chosen for the underlying functions ϕ and ψ of $\tilde{\Phi}$. Thus they are reproducing kernels of

$$\mathcal{H}_{\tilde{\Phi}}(\mathbb{R}^d) = \mathbf{H}^{\ell+1/2}(\mathbb{R}^d; \text{div}) \times H^{\ell+3/2}(\mathbb{R}^d).$$

We will focus on the L_2 and L_∞ errors. Let $\mathbf{f} \in \mathbf{H}^{\beta+1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\beta+1/2}(\partial\Omega)$, where $d/2 < \beta \leq \tau := d/2 + \ell + 1/2$. Theorem 4.5 shows that we have to expect the following behaviour of the error:

$$\begin{aligned} \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{H}^\eta(\Omega)} + \|p - s_p\|_{H^{\eta+1}(\Omega)} &\leq c_{\mathbf{f},\mathbf{g}} \left(\frac{h}{q} \right)^{\tau-\beta} h^{\beta-\eta}, \\ \|\mathbf{u} - \mathbf{s}_{\mathbf{u}}\|_{\mathbf{W}_\infty^\eta(\Omega)} + \|p - s_p\|_{W_\infty^{\eta+1}(\Omega)} &\leq c_{\mathbf{f},\mathbf{g}} \left(\frac{h}{q} \right)^{\tau-\beta} h^{\beta-\eta-d/2}. \end{aligned}$$

Note that the convergence order does not depend on the smoothness τ of the employed kernel if the separation radius is comparable to the fill distance, i.e., on quasi-uniform data sets.

We choose \mathbf{f} and \mathbf{g} such that the true solution of the velocity and the pressure are

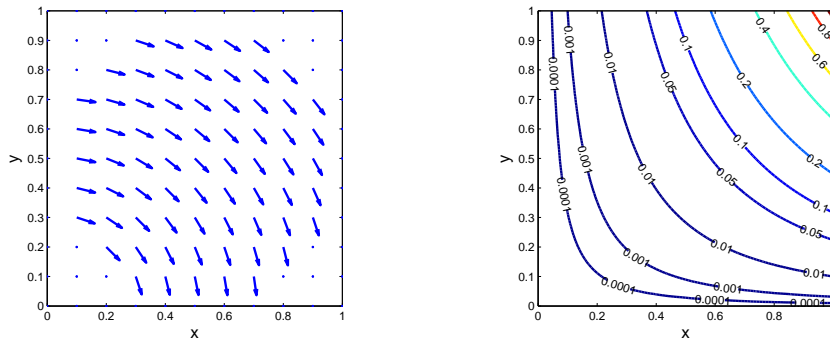
$$\mathbf{u}(x, y) = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix} \phi_{2,1}(r), \quad p(x, y) = x^3 y^2,$$

where $r := \sqrt{(x - x_0)^2 + (y - y_0)^2}/\gamma$ with $x_0 = y_0 = \gamma = 0.5$. Furthermore, we pick $K = I$ the identity matrix. Figure 5.1 shows the velocity field and the contour lines of the pressure.

The Wendland function $\phi_{d,\ell}$ is an element of all Sobolev spaces $H^\alpha(\mathbb{R}^d)$ with $\alpha < 2\tau - d/2 = d + 2\ell + 1 - d/2$. Therefore the function $\phi_{2,1}$ is in $H^\alpha(\Omega)$ with $\alpha < 4$. Due to our choice of the velocity, we have $\mathbf{u} \in H^{\beta+1}(\Omega)$ for all $\beta < 2$. Thus \mathbf{u} is not an element of the associated Hilbert space of $\phi_{2,3}$, where $\tau = 3.5$. We chose $\beta = 2$ which is the supremum of the smoothness, to work out the theoretical approximation orders.

In all cases the notation $\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{s}_{\mathbf{u}}$ and $e_p = p - s_p$ is used. The numerical tests were run on a sequence of equidistant grids. The computational approximation orders are given by

$$\frac{\log(E_n/E_{2n})}{\log(1/2)},$$



(a) The velocity field for the homogeneous example visualized with unit vectors. (b) The contour lines of the pressure field.

FIGURE 5.1. The true solution of the homogeneous example.

n	$\ \mathbf{e}_u\ _{L_2}$	$\ \mathbf{e}_u\ _{L_\infty}$	$\ \mathbf{e}_u\ _{H^1}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
4	1.2836e+00	2.2024e+00	9.8131e+00	2.0342e-01	1.5629e+00
8	7.9192e-02	3.8168e-01	1.6632e+00	3.4692e-02	2.1824e-01
16	8.7058e-03	5.1376e-02	3.9872e-01	2.9099e-03	3.0136e-02
32	7.4455e-04	1.1950e-02	7.9047e-02	2.7432e-04	3.0789e-03
64	6.5388e-05	2.8183e-03	1.6406e-02	2.8833e-05	7.2737e-04

TABLE 5.1

Approximation errors for the example with rougher target functions with $\phi = \psi = \phi_{2,3}$.

where E_n is one of the errors for given $n \times n$ input grid.

All results are displayed in tables 5.1 and 5.2. Here, the values in the brackets give the approximation orders if the target functions were in the reproducing kernel Hilbert space. Figure 5.2 illustrates the numerical approximation errors. From table 5.2 it can be seen that the numerical approximation orders more than match the theoretical ones. Moreover, some of them even match the approximation orders for smoother target functions.

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	$\ e_u\ _{L_2}$	$\ e_u\ _{L_\infty}$	$\ e_u\ _{H^1}$	$\ \nabla e_p\ _{L_2}$	$\ \nabla e_p\ _{L_\infty}$
computed	4.0187	2.5286	2.5607	2.5518	2.8403
	3.1853	2.8932	2.0605	3.5756	2.8564
	3.5475	2.1041	2.3346	3.4070	3.2910
	3.5093	2.0841	2.2685	3.2501	2.0817
estimated	2 (3.5)	1 (2.5)	1 (2.5)	2 (3.5)	1 (2.5)

TABLE 5.2

Approximation orders for the example with rougher target functions with $\phi = \psi = \phi_{2,3}$, where the values in the brackets give the approximation orders if the target function would be in the reproducing kernel Hilbert space.

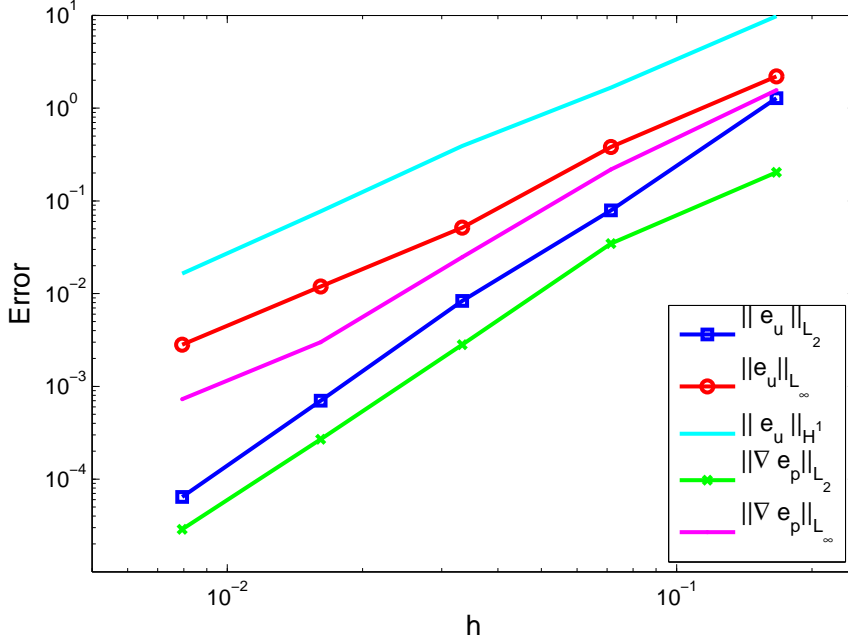


FIGURE 5.2. Approximation errors of example with rougher target functions, where $\phi = \psi = \phi_{2,3}$.

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