Fiction and Reality

in Aristotle's Philosophy of Geometry

Chiara Martini

St Anne's College

University of Oxford



A Thesis Submitted for the Degree of

Doctor of Philosophy

in Philosophy

September 2022

Table of Contents

Abstract **3**

Acknowledgements **4**

Introduction **6**

Part 1. The Problem of Ascent **13**

Introduction **14**
§1. What geometrical objects cannot be 19
§1.1. Partial Platonism 22
§1.2. Platonism 27
§2. Separation is different from the qua-theory 31
§3. Ta ex aphaireseōs 35

Chapter 1. The *Qua*-Theory **40**
§1. The Qua-Theory as a Methodological Description 41
§2. Answering Question (ii): The Identity of the Sciences 54
§3. Essential/Incidental Troubles 57

Chapter 2. Precision Problem & Other Worries **69**
§1. Precision Problem 71
§2. Contradictory Properties 77
§3. Scarcity Problem 83
§4. Necessarily Masked Potentialities 90

Chapter 3. Separation **95**
§1. Objects and Properties 96
§1.1. Kooky Objects? 101
§2. Separation is not merely considering X qua Y 106
§2.1. The structure of the chapter 107
§2.2. Separation as a consequence of the qua-theory 109
§2.3. Separation involves an element of fiction 113
§3. Separation 116
§4. No falsehood 125

Part 2. The Problem of Descent **134**

Introduction **135**

Chapter 4. Intelligible Matter **139**
§1. Hylomorphic Compounds 140
§2. Intelligible Matter as Extension 149
§3. Extension is not akin to space 160
§4. Finitude of the kosmos 167
§5. The Plurality Problem 173

Chapter 5. Geometrical Changes **176**
§1. Appeals to Change in Geometrical Practice 180
§1.1. Diagrams 180
§1.2. Definitions 184
§1.3. Proofs 186

§2. No Generation in Geometry....188
§3. No Kinesis in Geometry?....195
§4. No error arises....200
§5. Consequences, Limits and Scope....207

Conclusion 213

References.... 216

Abstract

This thesis investigates and attempts to reconstruct Aristotle's philosophy of geometry. The aim of this work is to tell a plausible story as to how, exactly, Aristotle thought mathematics could work without appealing to Platonic entities. This means investigating, on the one hand, how his account could make sense of mathematical truths and could fit with the mathematical practice of his time, and on the other hand, how it made sense of the application of mathematics to the physical world.

The thesis is divided in two parts. The first part offers a close reading of *Metaphysics* M.3, and of its two main claims: that geometry is about sensible objects, but not *qua* sensible, and that geometers are allowed to posit as separate what is in fact not separate. I give a radically anti-Platonist account of geometrical truth, according to which geometry is true in virtue of sensible substances and their properties. However, I claim that Aristotle relies on a moderate form of fictionalism (which only concerns the mode of being of the entities in question) in order to account for geometrical practice.

The second part of the thesis inquires whether Aristotle is justified in assuming that geometry is applicable to sensible reality. I focus in particular on the notion of intelligible matter, and on the role of change in geometry. I find that the applicability of geometry to sensible reality can be granted only by limiting its scope to localised configurations, both in space and in time.

Acknowledgements

I would like to thank my supervisors, Ursula Coope and Michael Peramatzis, for their unwavering support, kindness and generosity, as well as their exceptional guidance and the insightful discussions.

I would also like to thank the Clarendon Fund, Pembroke College, St Anne's College, and the Royal Institute of Philosophy for generously supporting my doctoral studies.

I was very lucky to benefit from the comments and the insights of the participants of conferences, workshops, and reading groups in Berlin, Cambridge, Chicago (virtually!), London, Münich, Oxford, Rome (virtually, again!) and St. Andrews; as well as the 'travelling seminar' across Italy. I have enormously benefitted from conversations with, comments from, or unpublished material generously shared by Francesco Ademollo, Harry Alanen, Andreas Anagnostopoulos, Jonathan Beere, Susanne Bobzien, Alexander Bown, the late Sarah Broadie, Andrea Buongiorno, Luca Castagnoli, Laura Castelli, David Charles, Elisa Citano, Giulia De Cesaris, Luca Dondoni, Marion Durand, David Ebrey, Paolo Fait, Antonio Ferro, Gail Fine, Benedetta Foletti, Lorenzo Giovannetti, Roberto Granieri, Janine Gühler, Lindsay Judson, Emily Katz, Terry Irwin, Joshua Mendelsohn, Beatrice Michetti, Karen Margrethe Nielsen, Demosthenes Patramanis, Sara Pavan, Federico Maria Petrucci, Arianna Piazzalunga, Barbara Sattler, Simon Shogry, Oliver Primavesi, Diana Quarantotto, Christof Rapp, Justin Vlasits, Nicolas Zaks.

Special thanks to Anna Marmodoro, Will Davies, and Roger Crisp for their support and their mentorship. I would also like to thank Annarita Angelini, Carlotta Capuccino, Walter Cavini, Rossella Lupacchini, and Marco Panza for leading me to the crossroads between Ancient Philosophy and Philosophy of Mathematics.

Special thanks to my BPhil family: AJ, Matt, Will, Eilidh, Sybilla, Tomi, Bex, Bea, Milena, Jonathan, Udit, Michael, and Tom, my fellow prisoners in the Cave. Alle mie compagne di paura, delirio, e disperazione, Alesia e Lea – I would not have made it without you.

Thanks to all the friends who have been with me across time and space: Fede, Ric, Papro, Madda, Eugi, Fabri, Sors, Ele, Marco. To Mamma, Babbo, and Nini: evviva! Next year I'll be on holiday with you. To Mic and Dimi, because.

And mostly, to Jack. Puski.

Introduction

Plato's views about mathematics have been immensely influential. The idea that the eternal and necessary truths of mathematics have to be about some immutable separate entities that exist over and above the changing and imperfect sensible reality is still perhaps the default position among mathematicians and philosophers alike to this day. In the contemporary philosophy of mathematics, of course, Mathematical Platonism has its detractors.[1] It has been subjected to criticisms for its inflated and anti-naturalistic ontology,[2] and, most famously, for the mystery connected to our epistemic access to such realm.[3] As a consequence, a whole range of alternative positions has been proposed: from Formalism to Intuitionism, from Constructivism to Fictionalism. It is notable and interesting, however, that the main Ancient alternative to Platonism – namely, Aristotelianism – has mostly been ignored in the recent developments of philosophy of mathematics.[4]

Why has Aristotle's position been largely ignored? This is even more perplexing given the recent overall revival of Aristotelian positions across philosophy. I think there are two main reasons for this absence. The first is surely that Aristotle's philosophy of mathematics is quite obscure and difficult to reconstruct. Aristotle does not offer a systematic analysis, nor does he dedicate a treatise to the topic; on the contrary, his position must be inferred from a series of remarks scattered throughout the *Corpus*, which

[1] Mathematical Platonism is today defined as the conjunction of three theses: Existence (there are mathematical objects); Abstractness (mathematical objects are abstract) and Independence (Mathematical objects are independent of intelligent agents and their language, thought, and practices). It is a metaphysical thesis that does not purport to represent the historical views of Plato himself – but it clearly and explicitly refers back to him as a broad inspiration. See Linnebo (2018).

[2] E.g. Field (1980), Chihara (1990), Azzouni (1994). Ironically, one of the main complaints is that Platonic abstract entities violate the Eleatic Principle, according to which something exists *iff* it exerts some causal influence, and which is inspired by the test proposed by the Eleatic Stranger in Plato's *Sophist* (247d-e).

[3] Benacerraf (1965).

[4] With the notable exception of Franklin (2014). One might argue that Neo-Fregean approaches (e.g. Wright 1983, Hale & Wright 2001; 2009) are close in spirit to the Aristotelian position, but to my knowledge Aristotle's position is not discussed by them and does not seem to have played any influence on the abstractionist position.

are difficult to compose into a unitary and coherent picture. Moreover, mathematics does not take the central stage in Aristotle's philosophy; mathematics is not the first philosophy, and Aristotle takes pain to push back against a dominant, mathematising trend in the Academy.[5]

The other reason is probably that Aristotle's philosophy of mathematics brings together a series of claims that seem to pull in different directions, and thus does not immediately offer a clean and straightforward picture. It tries to pair Platonic truth-realism about mathematics with a rejection of ontological commitments to Platonic special mathematical entities, by tasking sensible objects in the physical world to do the heavy lifting, somehow. However, it is far from clear how sensible objects can perform the task, considering that they are imperfect, imprecise, changeable, and seemingly just not enough in number — for in Aristotle's view the physical world is fundamentally finite (in size, as well as in the number and kind of objects that it contains).

Finally, Aristotle's philosophy of mathematics presents some puzzles also concerning the way in which mathematics is to be applied to the physical world. For Aristotle definitely does not believe, unlike the Pythagoreans or Plato (according to some interpretations of the *Timaeus*), that the world is composed of geometrical entities, and indeed physical reality seems to be rather unlike the sort of environment where Euclidean geometry is conducted: apart from being finite, the world for Aristotle is spatially structured by the powers of places and the natures of the elements into a inhomogeneous and anisotropic whole. And yet, Aristotle seems to find it very obvious and non-problematic that mathematics and geometry have something to tell us about the physical world. Indeed, the relation between mathematics and physical world which emerges from

[5] See *Metaph* A.9, 992a32 and in general the whole argument of books M-N.

Aristotle's remarks on the topic is quite alien to modern sensibility, and difficult to fit with any of the contemporary anti-Platonic philosophies of mathematics.

I believe that this makes it particularly important to investigate Aristotle's philosophy of mathematics in its own terms, especially with respect to the relation between mathematics and physical bodies – both to see if it still has something to offer to the contemporary debate, and because it might provide an interesting contrast to contemporary views, thus putting us in a better position to notice and evaluate our presuppositions and assumptions.

Part of what it means to investigate Aristotle's account on its own terms is also to recognise and adopt some of its least obvious criteria of success. Aside from logical consistency and other standard theoretical virtues, I believe that the ability to fit and make sense of the mathematical practice of the time was an important desideratum for Aristotle, and therefore it should be an important desideratum for the reconstruction of Aristotle's philosophy of mathematics. Unfortunately we do not have many sources for the mathematics of those years. Thus, I will work under the assumption that Euclid's *Elements* can be used as a reliable source of information on the state and practice of geometry in Aristotle's time, despite being composed some time after Aristotle's death, since it is commonly accepted that they constitute a systematisation of pre-existing material.

The aim of this thesis, then, is to tell a plausible story as to how, exactly, Aristotle thought mathematics could work without appealing to Platonic entities. This means investigating on the one hand how his account could make sense of mathematical truths and could fit with the mathematical practice of his time, and on the other hand how it made sense of the application of mathematics to the physical world.

In this thesis I will limit my inquiry to an analysis of Aristotle's philosophy of geometry: I will say very little about arithmetic. This piece-meal approach is justified because Greek arithmetic and geometry can be considered as different and autonomous disciplines, which pose different and distinct sets of problems.[6] Geometry is the obvious starting point for such an analysis, not only because of the special place that geometry occupies in Greek mathematics, nor merely because Aristotle develops his explicit discussion of the ontology of mathematics in *Metaphysics* M.3 having primarily geometry in mind,[7] but also because concentrating on geometry allows us to appreciate fully the specificity of Aristotle's views, and the peculiar way in which he thinks that mathematics is applicable to sensible reality.

In particular, I wish to focus on Aristotle's philosophy of geometry and its relation to the physics of places and of spatially extended magnitudes. Today we think of geometry as the science of space (De Risi 2015), but Aristotle rejects the existence of a space that exists and is structured independently of the bodies that inhabit it (*Physics* IV.1-9). At the same time, he does think that geometry is, somehow, about the physical world.

The relation between geometry and sensible reality poses, then, two complementary problems: going from the world to geometry, and then going from geometry back to the world. I call these, respectively, the Problem of Ascent and the Problem of Descent:

[6] An interpretation of Aristotle's philosophy of mathematics that is able to account for both arithmetic and geometry would surely be preferable, but I believe that it is most important to make sure that each specific set of problems gets answered.

[7] Note that arithmetic is explicitly mentioned at 1078a21-22: presumably, Aristotle takes his account to be valid for both mathematical sciences. However, the examples that he uses are predominantly geometrical in character, which suggests that the account is developed by looking at geometry in the first place.

- The Problem of Ascent: how is it possible to obtain the eternal and necessary truths of geometry, if the sensible world only contains a finite number of imperfect and transient objects?
- The Problem of Descent: even assuming that the truth of geometrical theorems can be grounded, is it really possible to apply them to sensible reality? Can geometry really be used to describe and understand the world which surrounds us?

Accordingly, the thesis will be divided in two parts. Part 1 consists in the analysis of *Metaphysics* M.3, which is the longest and fullest discussion of geometry in the whole *Corpus*. It is almost integrally concerned with the so-called Ontological Question: what is geometry about? After refuting Plato's idea that geometry is about separate, immutable and eternal substances that exist over and above sensible substances, Aristotle puts forward two key claims: (i) geometry is about sensible objects, but not *qua* sensible; (ii) geometers are allowed to 'separate and posit' what is in fact not separate, and study it as such.

I argue that these two statements are not just two different formulations of the same idea, but are genuinely distinct claims, offered as answers to two different questions: the Grounding Question and the Reference Question. The *qua*-theory is offered as an account of geometrical truth, while the doctrine of separation is supposed to account for geometrical practice. I will give a radically anti-Platonist account of geometrical truth, according to which Aristotle does not need to introduce any kind of object that is specific to geometry (not even objects that depend for their existence on sensible substances): geometry is true in virtue of sensible substances and their properties. However, I believe

that Aristotle relies on a moderate form of fictionalism (which only concerns the mode of being of the entities in question) in order to account for geometrical practice.

In particular, Chapter 1 will be dedicated to the analysis of the *qua*-theory, and provide a radically anti-Platonist reading of the grounding of geometrical truths.

Chapter 2 will discuss three problems that threaten a radically anti-Platonist interpretation of Aristotle's philosophy of geometry: the Precision Problem, the Scarcity Problem, and the Problem of Contradictory Properties. All three issues stem from the assumption that geometry is true in virtue of sensible objects and their properties. Aristotle does not provide a solution to these problems, but I think that one can be offered on his behalf; I will rely on an analogy with contemporary discussions in the metaphysics of powers to show that these problems can be solved.

Chapter 3 discusses the last part of *Metaphysics* M.3, in which Aristotle states the doctrine of separation. I argue that separation is not meant to account for geometrical truths, but for the fact that geometers act and speak as if they were studying *objects*, not properties of sensibles. I believe that this step introduces an element of fiction that is absent from the *qua*-theory, and that such fiction regards the mode of existence of the geometrical feature in question. Geometers are allowed to 'separate and posit' in thought the geometrical features of bodies, and study them as if they were independent substances. These 'semi-fictional objects' are the referents of singular terms in geometrical statements, the objects that geometers use and refer to in the course of their proofs. Aristotle claims that geometers are allowed to work on these semi-fictional objects rather than on sensible objects, even if geometrical truth is grounded in sensible objects and their properties.

Is this belief justified? This is the core of the Descent Problem, which can also be called the Applicability Problem, and which constitute the focus of Part 2. It is composed

by two chapters, each of which focuses on a notable feature of the semi-fictional objects which (in my interpretation) are the objects on which geometers conduct their proofs.

Chapter 4 examines the idea that geometrical objects are hylomorphic compounds of geometrical shape and a special kind of matter, called 'intelligible matter'. In particular, I discuss whether identifying intelligible matter with extension is sufficient to grant the applicability of geometry to sensible reality, and how extension should be understood.

Chapter 5 looks into Aristotle's claims that geometrical objects are 'without motion' and that geometry studies movable objects but not *qua* movable, and discusses whether these statements are compatible with the constructive character of Greek geometry.

I find that the applicability of geometry to sensible reality can be granted only by limiting its scope. Geometry can only be used to study specific configurations, which are localised both in a temporal and in a spatial way: geometry is not supposed to study the spatial structure of the universe as a whole, but merely the structural features of determined groups of figures, and only at the instant.

Part 1. The Problem of Ascent

Introduction

One of the most pressing and difficult questions when it comes to philosophy of mathematics is: what is mathematics about? Unsurprisingly, it is to this question that is dedicated the longest and most developed piece of Aristotelian writing on geometry, *Metaphysics* M.3. The first part of this thesis will be dedicated to the analysis of this key text.

My starting point is Netz' (2006: 25-6) insight that the question (and consequently the answer) is fundamentally ambiguous. For a science to be about something, or for something to be the object of a science, might mean either that the science refers to that object, or that the science is true in virtue of that object. We can thus distinguish two different questions:

- The Grounding Question: what are the truth-makers of geometrical theorems?
- The Reference Question: what are the referents of the singular terms that appear in geometrical propositions?

I am going to argue that Aristotle would not give the same reply to these two questions. The deep reason is that the ontology that seems to be implicit in mathematical practice and that is referred to by mathematical propositions is at odds with Aristotle's metaphysical commitments.[8] Rather than questioning the status of geometry as a science, he disentangles the issue of geometrical truth from the legitimacy of geometrical practice. He first shows that geometrical truths can be grounded in sensible reality, and then proceeds to argue that mathematicians can continue to operate as they are used to, even if the ontology that they apparently rely upon is misguided.

[8] Cf. Halper (1989: 275).

To understand the scope, strategy and limits of the theory put forward in M.3, it is essential to pay attention to the context in which it appears. *Metaphysics* M.1 kicks off an ontological inquiry that occupies books M and N, aimed to determine 'whether or not there is any kind of unchanging and eternal substance over and above the perceptible ones, and, if there is, what it is'.[9] As usual, Aristotle starts from a discussion of the views currently on the market; his inquiry into the objects of geometry is prompted not by a specific interest in the nature and features of mathematical objects, but because 'some people say that mathematical objects are substances'[10] and he wishes to examine this claim.

The main reason why mathematical objects might be thought to be 'unchanging and eternal substances' is that the special kind of truth enjoyed by geometrical theorems seems to require the existence of objects that are more stable and more precise than sensible substances. Geometry does not seem to be literally true of physical bodies; moreover, sensible substances are subject to decay, while geometrical truths are supposed to be eternal. In M.1-2 Aristotle rejects the following options:

1. Platonism, according to which the truth-makers of geometrical statements are objects which are separate from sensible substances;
2. Partial Platonism, according to which the truth-makers of geometrical statements are objects which are distinct from, but located in, sensible substances;

As well as:

3. The view that no objects are needed as truth-makers of geometrical statements.

[9] *Metaphysics* M.1, 1076a10-12 (Annas' translation): πότερον ἔστι τις παρὰ τὰς αἰσθητὰς οὐσίας ἀκίνητος καὶ ἀΐδιος ἢ οὐκ ἔστι, καὶ εἰ ἔστι τίς ἐστι.
[10] 1076a17-18 (Annas' translation): τά τε γὰρ μαθηματικά φασιν οὐσίας εἶναί τινες.

But Aristotle is not willing to deny the truth of geometrical theorems, nor to put into question the status of geometry as a science. Thus, he needs to indicate what acts as truth-maker of geometrical propositions. He purports to do that in M.3. Note that he only has two options left: he can argue that the truth-makers of geometrical statements are objects that are distinct from sensible bodies but exist in some special way (neither separate from, nor in, sensible substances), or alternatively that the truth-makers of geometrical statements simply are sensible substances and their properties. The first option yields a moderately anti-Platonist interpretation (where the Platonic assumption that geometry studies a special set of objects is retained, even if those objects are not taken to be independent substances); the second one is a radically anti-Platonist reading.

I am going to argue that Aristotle provides a radically anti-Platonist answer to the Grounding Question: the truth-makers of geometrical statements simply are sensible substances, or properties of sensible substances. In order to make the radically anti-Platonist position viable, it is necessary to offer credible answers to the following three questions:

(i) How can the truth-makers for geometry be sensible objects?

(ii) How can geometry be a distinct science, if its truth-makers are just sensible objects (which are also truth-makers for other sciences, e.g. physics)?

(iii) How can geometrical practice be justified, if the truth-makers of geometrical theorems are sensible objects?

In order to satisfactorily answer question (i), we will have to solve a number of problems that point to the same conclusion: *prima facie*, sensible bodies do not seem able to ground the truth of geometrical statements. There are two especially pressing issues that stand in the way, which I will dub the 'Precision Problem' and the 'Scarcity Problem'. The former stems from the fact that sensible bodies do not appear to instantiate

perfect geometrical properties. The latter, because Aristotle's universe is finite and thus apparently unable to accommodate enough objects (and objects big enough) to ground all geometrical theorems.

Question (ii) originates from the widespread tendency among Greek philosophers to identify and differentiate sciences on the basis of their objects. So, the radical anti-Platonist will have to explain how there can be a science without a corresponding unique set of objects it is about.

Question (iii) arises because the way in which geometers speak of their objects, and the way in which they conduct their proofs, is difficult to square with the idea that geometry is about sensible substances. Geometers do not seem to be talking about objects of everyday experience, and during their proofs they manipulate and construct their objects in ways which are at odds with the normal behaviour of sensible bodies.

In order to defend the radically anti-Platonist reading, I need to answer these three questions, as well as explaining the sense in which (in my view) the objects of geometry exist 'in a certain way' and 'not without qualification'.[11]

In Chapter 1 I discuss problems (i) and (ii), and argue that they are both solved by the *qua*-theory, i.e. the claim that geometry studies sensible substances, but not *qua* sensible.[12] This claim admits of two very different readings, that we can call 'objectual' and 'methodological'. I am going to argue for a methodological interpretation of the *qua*-theory, according to which the *qua*-clause qualifies the way in which geometers approach sensible objects (as opposed to yielding a special kind of objects, the sensible-substances-not-*qua*-sensible). In response to problem (ii), Aristotle claims that sciences can be distinguished on the basis of their methodology rather than their object. Discussing the

[11] *Metaph.* M.2, 1077b16.
[12] *Metaph.* M.3, 1077b21-22; 1077b27-31; *Ph.* II.2, 193b30.

details of the *qua*-theory will also allow me to defend the radically anti-Platonist view, e.g. by clarifying what is meant by existing 'in a certain way' and not *haplōs*.

In Chapter 2 I examine more closely Aristotle's position face to problem (i). I argue that Aristotle himself does not give a clear cut answer to the Precision and Scarcity Problems, but that it is possible to provide one on his behalf – i.e. that he had the resources necessary to address these issues. I also suggest that there are dialectical and contextual reasons for Aristotle's silence on this topic: his goal in M.3 is not to provide a full theory of geometry, but merely to show that his rejection of mathematical Platonism does not jeopardise the possibility of geometry as a science.

Chapter 3 is dedicated to question (iii). I argue that Aristotle's remarks on separation (at lines 1078a19 ff.) provide an answer. In particular, I submit that Aristotle can accommodate this difficulty by distinguishing between the truth-makers of geometrical statements and the referents of the singular terms that appear in them. This marks a shift from the Grounding Question to the Reference Question, and leads to the second part of the thesis.

Before jumping into the reading of M.3, however, it is important to clarify some preliminary points. In the remainder of this introduction I will discuss (§1) why Aristotle rejects options 1., 2., and 3. above, and what options are left open. This will clarify which are the *desiderata* that a viable interpretation of the Aristotelian text must meet. I will then offer (§2) some preliminary evidence for my claim that the *qua*-theory and the material on separation are answering two different questions. In §3 I will briefly discuss the terminology of *aphairesis*, which is strikingly absent from M.3.

§1. What geometrical objects cannot be

Books M-N of the *Metaphysics* are introduced as an inquiry aimed at determining 'whether or not there is any kind of unchanging and eternal substance over and above the perceptible ones, and, if there is, what it is'.[13] Aristotle articulates the plan in three tasks:

1. "We must first consider mathematical objects, without attaching any other nature to them (e.g. whether they are in fact Forms or not, and whether or not they are principles and realities of existing things), but just considering whether, merely as mathematical objects, they exist or not, and, if they exist, in what way they exist";
2. "Then after this we must separately consider Forms themselves";
3. "… [W]e discuss whether the realities and principles of existing things are numbers and Forms".[14]

The first task, which is carried out in chapters 2 and 3, is introduced in the final lines of M.1:

> ἀνάγκη δ', εἴπερ ἔστι τὰ μαθηματικά, ἢ ἐν τοῖς αἰσθητοῖς εἶναι αὐτὰ καθάπερ λέγουσί τινες, ἢ κεχωρισμένα τῶν αἰσθητῶν (λέγουσι δὲ καὶ οὕτω τινές)· ἢ εἰ μηδετέρως, ἢ οὐκ εἰσὶν ἢ ἄλλον τρόπον εἰσίν· ὥσθ' ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου.
>
> If mathematical objects exist, they must either exist in perceptible objects as some say, or separate from perceptible objects (some say this too), or, if neither, then either they do

[13] *Metaphysics* M.1, 1076a10-12 (Annas' translation). ἐπεὶ δ' ἡ σκέψις ἐστὶ πότερον ἔστι τις παρὰ τὰς αἰσθητὰς οὐσίας ἀκίνητος καὶ ἀΐδιος ἢ οὐκ ἔστι, καὶ εἰ ἔστι τίς ἐστι.
[14] 1076a22-31, Annas' translation. σκεπτέον πρῶτον μὲν περὶ τῶν μαθηματικῶν, μηδεμίαν προστιθέντας φύσιν ἄλλην αὐτοῖς, οἷον πότερον ἰδέαι τυγχάνουσιν οὖσαι ἢ οὔ, καὶ πότερον ἀρχαὶ καὶ οὐσίαι τῶν ὄντων ἢ οὔ, ἀλλ' ὡς περὶ μαθηματικῶν μόνον εἴτ' εἰσὶν εἴτε μὴ εἰσί, καὶ εἰ εἰσὶ πῶς εἰσίν· ἔπειτα μετὰ ταῦτα χωρὶς περὶ τῶν ἰδεῶν αὐτῶν […] ἐπισκοπῶμεν εἰ αἱ οὐσίαι καὶ αἱ ἀρχαὶ τῶν ὄντων ἀριθμοὶ καὶ ἰδέαι εἰσίν.

not exist or they exist in some other way; so our debate will be not whether they exist, but in what way they exist.

Metaph. M.1, 1076a32-37 (Annas' translation)

This passage presents four alternatives, which are supposed to be exhaustive of the logical space. It is interesting to note that the taxonomy is bipartite: Aristotle distinguishes two main horns, each divided in two further options:

1. If mathematical objects exist,
    a) They exist in perceptible objects;
    b) They exist separate from perceptible objects.
2. If neither,
    a) They do not exist;
    b) They exist in some other way.

There are two significant aspects to this taxonomy. The first is that the option that geometrical objects do not exist at all is mentioned for the sake of completeness but is not discussed nor ruled out explicitly. It does not seem to be a real, believable option, worthy of being taken seriously. The most obvious reason for this is that Aristotle shares Plato's realist intuition that a true science must be about something that is.[15] This rules out conceptions of truth like theoremhood or internal coherence, as well as purely conditional accounts (even if it might be that the antecedent is not always instantiated, the implication must be grounded in something that is). Thus it is fundamental that a reconstruction of Aristotle's philosophy of geometry yields the conclusion that geometry is about objects

[15] Cf. the *APo* requirement that the subject-matter of a science is an existent item: existence claims are among the principles of each science, and the 'if it is' question is primary to the 'what it is' question. Burnyeat (2012: 153) underlies that this premise is shared by all participants in the debate.

that exist, because the truth of geometrical theorems needs to be grounded in some existing object that acts as truth-maker.

The other notable aspect is that the quadripartition is somehow unsatisfactory: it is not clear why option 2b (that geometrical objects 'exist in some other way') should be alternative to the first horn, i.e. that geometrical objects exist. It seems that Aristotle is operating under some sort of implicit assumption about the way in which geometrical objects are supposed to exist, so that existing 'in some other way' is a real alternative. Note that lines 35-36 (ἢ ἄλλον τρόπον εἰσίν· ὥσθ' ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου) form a syntactic unit, where the *hōste* does not draw a general consequence of the taxonomy but only a consequence of the endmost case. The assumption seems to be that denying the two options that cover the first horn of the dilemma is not equivalent to asserting non-being, but is compatible with a different sort of being.

As a matter of fact, this hidden assumption is brought to light at the very end of *Metaphysics* M.2. The chapter is dedicated to examining (and refuting) the two options in the first horn. The conclusion echoes this quadripartion:

> Ὅτι μὲν οὖν οὔτε οὐσίαι μᾶλλον τῶν σωμάτων εἰσὶν οὔτε πρότερα τῷ εἶναι τῶν αἰσθητῶν ἀλλὰ τῷ λόγῳ μόνον, οὔτε κεχωρισμένα που εἶναι δυνατόν, εἴρηται ἱκανῶς· ἐπεὶ δ' οὐδ'ἐν τοῖς αἰσθητοῖς ἐνεδέχετο αὐτὰ εἶναι, φανερὸν ὅτι ἢ ὅλως οὐκ ἔστιν ἢ τρόπον τινὰ ἔστι καὶ διὰ τοῦτο οὐχ ἁπλῶς ἔστιν· πολλαχῶς γὰρ τὸ εἶναι λέγομεν.
>
> It has been adequately shown that <mathematical objects> are not substances more than bodies are, that they are not prior to perceptible objects in being, but only in definition, and that they cannot have separate existence. Since they could not exist in perceptible

> objects either, clearly either they do not exist at all, or they do so in a certain way and so do not exist without qualification – for we use 'exist' in several senses.
>
> *Metaph.* M.2, 1077b12-17 (Annas' translation)

The two passages are parallel, but for a significant difference: Aristotle takes M.2 to have established not that geometrical objects cannot exist, but that they cannot exist *haplōs*. This explains why Aristotle presents the four options as arranged in two horns: if geometrical objects exist *haplōs*, they must be either in sensible objects or separate from them; if neither, then they either do not exist, or they do not exist *haplōs* but in some other way.

To understand M.3, then, it is necessary to get clear on what Aristotle means when he talks about existing *haplōs*. To do that, we need to look at the arguments against the first two options, presented in M.2. The first one, that Annas (1976) labels 'Partial Platonism', consists in the idea that geometrical objects are distinct from physical objects, but are *in them*. It is not clear who held this position, and Aristotle gives very few details, so that it is not even obvious what sense of 'to be in' is at play. Partial Platonism is usually presented as a naïve theory, and Aristotle's arguments against it are often overlooked. However, I believe that it is important to pay attention to them, and try to determine exactly what is their target: this will be helpful when coming to reconstruct Aristotle's own position, since it is necessary that the proposed interpretation is not liable to fall under these criticisms.

§1.1. Partial Platonism

Aristotle brings two arguments against Partial Platonism:

1. If geometrical objects were in physical bodies, there would be two solids in the same place at the same time, and this is impossible.
2. If geometrical objects were in physical bodies, it would be impossible to divide the latter.

The first argument had already been advanced in the course of the discussion of *aporia* 5 in B.2, 998a11-19. The reasoning is quite easy to understand: it makes use of the core meaning of 'to be in', the locative one. For it to work, it is necessary that the two objects under consideration, the physical and the geometrical one, have the same number of dimensions: it is an argument that concerns the co-location of a three-dimensional physical body and a three-dimensional geometrical solid.[16] The argument would not work if the geometrical object in question were e.g. a line: lower-dimensional objects *can* 'be in' higher-dimensional ones, as their limits. In the same way, the argument would not work if the geometrical entity were an attribute, an aspect, or even a part of the sensible body; *pathē* can be in substances without generating a co-location problem, and the possibility of identifying a part in a whole does not imply that two distinct entities are co-located.

The second argument is less straightforward, and goes like this:

> φανερὸν ὅτι ἀδύνατον διαιρεθῆναι ὁτιοῦν σῶμα· κατ'ἐπίπεδον γὰρ διαιρεθήσεται, καὶ τοῦτο κατὰ γραμμὴν καὶ αὕτη κατὰ στιγμήν, ὥστ' εἰ τὴν στιγμὴν διελεῖν ἀδύνατον, καὶ τὴν γραμμήν, εἰ δὲ ταύτην, καὶ τἆλλα. τί οὖν διαφέρει ἢ ταύτας εἶναι τοιαύτας φύσεις, ἢ αὐτὰς μὲν μή, εἶναι δ' ἐν αὐταῖς τοιαύτας φύσεις; τὸ αὐτὸ γὰρ συμβήσεται· διαιρουμένων γὰρ τῶν αἰσθητῶν διαιρεθήσονται, ἢ οὐδὲ αἱ αἰσθηταί.

[16] It is interesting that Aristotle does not seem to think that the immateriality of the geometrical solid makes any difference when it comes to the impossibility of co-location: two three-dimensionally extended cannot occupy the same place, independently of whether one of the two has or lacks matter. See Pfeiffer (2016), Betegh (2016).

> It is clear that <on that theory> it is impossible for any body to be divided; for it will be divided along a plane, and the plane along a line, and a line at a point, so that if the point cannot be divided, neither can the line, and if the line cannot, the rest cannot either. So what difference does it make whether perceptible bodies are objects of this kind, or whether, while they are not, objects of this kind exist in them? There will be the same result: either they will be divided when perceptible objects are divided, or perceptible objects will not be divided either.
>
> Arist. *Metaph*. M.2, 1076b4-11 (Annas' translation)

If geometrical objects were located in sensible bodies, the division of a sensible body would require the division of the geometrical object located in it. Since a solid is divided at (*kata*) a plane, the plane at a line, and the line at a point, ultimately dividing the sensible body would require dividing a point, which is impossible. The argument is usually given very little attention, probably because it is directed against a 'fanciful' doctrine.[17] Commentators tend to read the argument as directed against the view that points are *actually* present in the line (and thus in the plane and in the body).[18] On this reading, Aristotle would avoid the problem thanks to his theory of continuity, according to which the line is infinitely divisible but it is not composed by an infinity of actual points. The division of the line thus does not 'fall' on an actual point that needs to be divided, but on a potential point that is thus actualised by the division. There are a number of problems with this reading – for one, the argument so reconstructed concerns the relationship between higher- and lower-dimensional *geometrical* objects, rather than between

[17] πλασματίας ὁ λόγος, 1076a39.
[18] See e.g. Crubellier (1994: 100).

geometrical and physical objects. This means that it works as an argument against the Partial Platonist only insofar as she holds the wrong theory of continuity.

It is possible that this is in fact the case, for Aristotle's theory of continuity is revolutionary and ground-breaking. But I still find it quite interesting that the argument could be reconstructed in a different way, as revolving around the different behaviour of geometrical and physical bodies. Read in this way, it cannot be solved merely by adopting the correct account of continuity.

In short, the idea is that if geometrical objects were *substances* that are located in physical bodies and share their diachronic history, absurd consequences would follow when we try to divide the sensible body. For if the geometrical object is a substance co-located with it, it should behave in the same way as the physical body, and undergo the same kind of modifications that the physical body undergoes. But sensible bodies are divided in a very different way from geometrical solids. While geometrical solids are divided 'at a plane', the plane 'at a line', and the line 'at a point', which is itself indivisible but can act as the limit of both parts (being thus one in number but two in being), physical bodies can also be divided into two separate smaller bodies, each comprised of its own limits. This means that we need one more limit than what we need when we divide geometrical objects. Aristotle describes the division as follows:

> τὰς δὲ στιγμὰς καὶ τὰς γραμμὰς καὶ τὰς ἐπιφανείας οὐκ ἐνδέχεται οὔτε γίγνεσθαι οὔτε φθείρεσθαι, ὁτὲ μὲν οὔσας ὁτὲ δὲ οὐκ οὔσας. ὅταν γὰρ ἅπτηται ἢ διαιρῆται τὰ σώματα, ἅμα ὁτὲ μὲν μία ἁπτομένων ὁτὲ δὲ δύο διαιρουμένων γίγνονται· ὥστ' οὔτε συγκειμένων ἔστιν ἀλλ' ἔφθαρται, διῃρημένων τε εἰσὶν αἱ πρότερον οὐκ οὖσαι (οὐ γὰρ δὴ ἥ γ' ἀδιαίρετος στιγμὴ διῃρέθη εἰς δύο).

> It is not possible that points and lines and surfaces should either be coming-to-be or be perishing, even though they at one time exist and at another do not exist. For when bodies come into contact or are divided, <the boundaries, i.e., points, lines, or surfaces> at once become one when <the bodies> are in contact and two when <the bodies> are divided. So that <something, viz., a boundary> does not exist but has been destroyed when <the bodies> have been compounded, and <boundaries> exist which were not formerly existent when <the bodies> have been divided (but it is not the case that a point, which is indivisible, is divided into two).
>
> Arist. *Metaph.* B.5, 1002a32-b4 (Barnes' tr.)

Aristotle's ontology includes only closed bodies: bodies are limited by surfaces on all sides, and there are no bodies without limits.[19] When a body is divided into parts, each resulting part must have its own limits, which surround it completely. Limits depend on their bearer for their existence: they come in and out of existence together with their bearer.[20] Thus, the division of a sensible body into two parts requires bringing into existence of *two* limits, one for each part. On the contrary, the geometrical division of a solid consists in the actualisation of *one* plane, which acts as the limit of both parts. Understood in this way, the argument points to an asymmetry in the *behaviour* of physical and geometrical objects. Supposing that the latter are in the former is unreasonable, because they would need to evolve in the same way through time, and they cannot.[21]

[19] The only case in which it seems that Aristotle might be allowing for open intervals is his discussion of the last instant of change, and only with respect to processes, not to bodies, things or time intervals: 'Let the time be ABC, the object D, which is white at time A, not white at time B. At time C, therefore, it is white and not white. For if in any moment of A it is true to call it white, if it was white for the whole period, and in B not white, but C belongs to both times. We must not, therefore, grant that D is white at all times, but rather at all times except the last now which we designated C. And this already belongs to the last period' (*Ph.* VIII.8, 263b15-21, Graham's translation). This means that Aristotle would not be willing to accept the fix suggested by Pettigrew (2009: 251-2), which involves open intervals.

[20] See Pfeiffer (2018: §6.3.3).

[21] I will discuss this argument again in Chapter 5, because Aristotle's own theory risks being vulnerable to it. I will show that Aristotle can block the argument, but this requires limiting the scope of the applicability of geometry to sensible reality in such a way that it does not track changes.

This means that the argument is directed against the idea that geometrical objects *are in* physical bodies in the sense of being actually present in them, *and* of sharing their diachronic evolution. This suggests that claiming that geometrical objects exist *haplōs* implies that they can persist through time, and undergo some sort of modifications.

§1.2. Platonism

After ruling out Partial Platonism, Aristotle moves on to refute Platonism, i.e. the claim that geometrical objects exist *haplōs* and are separate (*kechōrismenas*) from physical objects. This section is composed by a long series of arguments, whose details are very confused and very much debated. I am not going to offer a detailed reconstruction of each of them; I will focus on the elements that are most important to understand the strategy and the content of M.3.

The main thing to discuss is the key notion of separation, which plays a central role in M.3, too. Since the claim that geometrical objects are separate from sensible substances is offered as an alternative to the view that they 'are in' them, one might be tempted to interpret it as simple separation in place: if this were the case, the view would simply consist in claiming that geometrical objects are not co-located with sensible substances. But this cannot be the whole story.

This appears clearly already from the first of Aristotle's arguments, which shows that positing 'solids over and above perceptible solids, separate from them, distinct from and prior to perceptible solids'[22] leads to an absurd accumulation (ἄτοπος σώρευσις) of entities.[23] The mention of priority signals that the kind of separation that Aristotle has in mind cannot merely be local separation: positing separate geometrical objects is meant to

[22] 1076b13-15.
[23] The details of the argument are difficult to reconstruct – in particular, it is unclear how Aristotle calculates the specific number of entities that he believes the Platonist to be committed to. See Katz (2014) and especially González-Varela (2020) for in-depth discussions.

have an explanatory effect, to account for the geometrical properties of sensible substances.[24] The accumulation is caused by the fact that the entities posited to explain the geometrical features of sensible substances exemplify the very same properties that prompted their introduction; this calls for the introduction of additional separate substances prior to *them* – and so on.[25] This explanatory role can only be played by entities which are *ontologically* prior and separate.

Ontological separation is a non-symmetrical binary relation: schematically, A is separate from B *iff* it can be without B.[26] Priority is the corresponding asymmetrical binary relation: A is prior to B *iff* A can be without B and B cannot be without A.[27]

It is notable that Aristotle uses the term also as akin to monadic predicate, to indicate a mode of being typical of substances.[28] In saying that geometrical objects are separate from sensible substances, Aristotle taps into the operative notion of a self-standing, genuine metaphysical subject, which is not only distinct, bur prior to sensible substances.

That Aristotle has ontological separation in mind becomes evident at the very end of the chapter, where he distinguishes between ontological and definitional priority:

> τῇ μὲν γὰρ οὐσίᾳ πρότερα ὅσα χωριζόμενα τῷ εἶναι ὑπερβάλλει, τῷ λόγῳ δὲ ὅσων οἱ λόγοι ἐκ τῶν λόγων·

[24] For an analysis of the notion of priority, see Peramatzis (2011). Priority and separation are very closely related;
[25] The argument is not a regress *ad infinitum*, though, because of its progressive and decreasing character. For the details, see González-Varela (2020).
[26] But see Katz (2017) who argues that it is an asymmetrical relation.
[27] The chapter contains other arguments by accumulation, which target respectively arithmetic and the subordinate sciences such as astronomy and thus are less relevant for my purposes.
[28] Despite this appearance, though, it is still a binary relation – the second term is merely implicit. How to identify the second term of the relation when substances are said to be *chorista* is at the core of a lively debate. Are substances separate from attributes, from other substances, or both? See for example Fine (1984; 1985); Morrison (1985a).

> Things are prior in reality <to other things> if they continue to exist when separated from the latter, and prior in definition to those things whose definitions are compounded from definitions of them.
>
> *Metaph.* M.2, 1077b2-4 (Annas' translation, modified)

He concedes to Plato that geometrical features might be prior in definition to sensible substances, but – he argues – this does not imply that they are also ontologically prior to them, nor ontologically separate.[29]

The following series of arguments shows that supposing that they are reverses the correct order of priorities. Aristotle argues that if geometrical objects were ontologically separate, they would have to be ontologically prior to sensible substances; but this cannot be right, because geometrical objects can be lower-dimensional (and thus incomplete)[30] and consequently cannot be prior to complete bodies. The same conclusion is reached by pointing out that what is prior in generation is posterior in being, and noting that lower-dimensional objects are used to construct higher-dimensional ones in geometry.

Another argument that Aristotle offers in this chapter, and that is particularly relevant to my discussion of M.3, regards the 'general propositions stated by mathematicians', i.e. theorems that apply to more than one kind of mathematical objects. This is the case, for example, of Eudoxus' theory of proportions, which applies both to numbers and to continuous magnitudes. Aristotle points out that if the Platonist were right in claiming that each science has its proper specific set of objects, there should be another kind of separate objects which are 'neither number nor point nor magnitude nor line' – but this is absurd. I will come back to this argument in the course of Chapter 1.

---

[29] Interestingly, it is in this context that we find the only occurrence of the term *aphairesis* in the whole *Metaphysics* M.1-3. See §3 below.

[30] For the connection between completeness and three-dimensionality, see Betegh, Pedriali & Pfeiffer (2013); Pfeiffer (2018: 76-89). I will discuss three-dimensionality in Chapter 4.

Having ruled out both Platonism and Partial Platonism, Aristotle concludes that geometrical objects cannot exist *haplōs*:

> Ὅτι μὲν οὖν οὔτε οὐσίαι μᾶλλον τῶν σωμάτων εἰσὶν οὔτε πρότερα τῷ εἶναι τῶν αἰσθητῶν ἀλλὰ τῷ λόγῳ μόνον, οὔτε κεχωρισμένα που εἶναι δυνατόν, εἴρηται ἱκανῶς· ἐπεὶ δ' οὐδ'ἐν τοῖς αἰσθητοῖς ἐνεδέχετο αὐτὰ εἶναι, φανερὸν ὅτι ἢ ὅλως οὐκ ἔστιν ἢ τρόπον τινὰ ἔστι καὶ διὰ τοῦτο οὐχ ἁπλῶς ἔστιν· πολλαχῶς γὰρ τὸ εἶναι λέγομεν.
>
> It has been adequately shown that <mathematical objects> are not substances more than bodies are, that they are not prior to perceptible objects in being, but only in definition, and that they cannot have separate existence. Since they could not exist in perceptible objects either, clearly either they do not exist at all, or they do so in a certain way and so do not exist without qualification – for we use 'exist' in several senses.
>
> *Metaph.* M.2, 1077b12-17 (Annas' translation)

This confirms that the argument against Platonism and Partial Platonism is an argument against the hypothesis that geometrical objects exist *haplōs*, and that this here means existing in the mode of a substance: ontologically prior to its attributes; subject of predication; capable of persisting through change.

Where does this leave Aristotle? Having ruled out Platonism, Partial Platonism and the idea that no truth-makers are required, there are only two options left: radical anti-Platonism or moderate anti-Platonism. Even if for the broader purposes of the M-N inquiry the negative results obtained in M.2 might be sufficient (it has been established that geometrical objects are not the immutable and eternal substances that Aristotle is looking for), Aristotle still needs to say something to demonstrate that geometry is about objects that do exist. This is because his denial of Platonism risks jeopardising the truth

of geometrical statements (which cannot lack a truth-maker). This means that the primary goal of M.3 is to answer the Grounding Question.

§2. Separation is different from the qua-theory

It is tempting to think that the *qua*-theory (i.e. the claim that geometry studies sensible objects, but not *qua* sensible) is meant to identify one special kind of objects, that can act both as the truth-makers of geometrical statements and the referents of their singular terms, and which would be the proper objects of geometry. This approach naturally yields a moderately anti-Platonist interpretation, where geometrical objects are identified with special objects, the sensible-objects-*qua*-geometrical, which are distinct from sensible objects but dependent on them for their existence (and thus do not exist *haplōs*). I am going to offer a slightly different interpretation.

Over the next two chapters, I will argue that the *qua*-theory is only concerned with the Grounding Question: its goal is to safeguard the truth of geometrical statements by showing that geometry studies existing objects. Rather than identifying a special set of objects, with a specific dependent mode of existence, Aristotle uses the *qua*-theory to show that geometry is true in virtue of sensible bodies and their properties – it simply treats them in a special way. The implicit assumption that can be read between the lines, but that is not argued for in this context, is that sensible objects do have geometrical features, which belong to them as *pathē*.

The appearance that geometry is about some special objects comes from the way in which geometers talk and act – which gives rise to the Reference Question. Here it is the notion of separation which is at centre-stage, rather than the *qua*-clause.[31] I will argue that Aristotle offers two distinct answers to the Grounding and Reference Questions, and

[31] I will discuss separation in Chapter 3.

that these two distinct answers are underpinned by two different key notions, considering-*qua* and positing as separate, which should therefore not be conflated.

Before looking at the details of my interpretation, I wish to briefly mention why I think it is plausible to understand the *qua*-theory and the doctrine of separation as two distinct claims, and not merely two different formulations of one and the same theory. I will present more detailed reasons in Chapter 3; for the moment, I only wish to point out that this is the picture that Aristotle paints in another passage that is concerned with the relationship between geometrical and sensible objects, in *Physics* II.2.

Physics II.2 aims to determine whether the physicist should focus on matter or on form. The verdict is that physicists should inquire *both* about form and matter, in contrast to geometers, who should not concern themselves with matter. The focus of the inquiry is not on geometry, but on physics; the methodology, requirements and commitments of geometry are taken as understood and established, and used as a comparison to draw conclusions about the correct methodology of natural sciences.[32]

What is interesting is that the passage touches upon all the topics treated in M.3. It starts by presenting the *qua*-theory, then moves on to separation, and finally discusses the status of subordinate sciences such as harmonics, optics and astronomy. Let us take a closer look at the first paragraph, in which the *qua*-theory and separation are introduced:

> (καὶ γὰρ ἐπίπεδα καὶ στερεὰ ἔχει τὰ φυσικὰ σώματα καὶ μήκη καὶ στιγμάς, περὶ ὧν σκοπεῖ ὁ μαθηματικός)· […] περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον· οὐδὲ τὰ συμβεβηκότα θεωρεῖ ᾗ τοιούτοις οὖσι

[32] In this respect, this is the opposite situation than M.3. As we will see, in M.3 Aristotle draws analogies between geometry and various natural sciences to show that geometry is not more problematic than other sciences.

συμβέβηκεν· διὸ καὶ χωρίζει· χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων.

(for natural bodies have planes, solids, lengths, and points, which are the business of the mathematician). […] Both the student of nature and the mathematician deal with these things; but the mathematician does not consider them as boundaries of natural bodies. Nor does he consider things which supervene as supervening on such bodies. That is why he separates them; for they are separable in thought from change, and it makes no difference; no error results.

Physics II.2, 193b24-35 (Charlton's transl.)

The first sentence quoted paints an explicitly immanentist picture: Aristotle says that natural bodies *have* geometrical objects. It is a very sketchy remark, offered without explanation nor support as evidence that it is necessary to examine in which way the geometer differs from the natural scientist. The upshot seems to be that sensible bodies have or contain features or aspects that are studied by geometry, so that the domains of the two sciences overlaps (at least partially). The verb *echein* does not clarify perfectly what kind of relation there is between sensible bodies and geometrical entities. Presumably it is to be understood in a way that is compatible with the criticism of Partial Platonism: sensible substances might have geometrical objects as their features, or limits.

Indeed, the following sentence describes lower-dimensional geometrical objects as the limits of bodies; the idea seems to be that the limits of sensible objects are *themselves* the objects of geometry, although the geometer differs from the natural scientist in that she does not consider them as boundaries of natural bodies. The difference between the geometer and the natural scientist cannot be reduced to a difference in the domain of objects that they study, but rather to a difference in the way in which they approach them:

the geometer does not consider the limits of sensible bodies *qua* limits of sensible bodies, while we are meant to understand that this is precisely what the natural scientist does.[33]

The following sentence introduces the idea that geometers *separate* their objects. Note that in this passage, separation is introduced as a consequence of the *qua*-theory, and thus as a different operation. The use of the connective *dio* indicates that it is *because* of the *qua*-theory that geometers are allowed to separate their objects. The precise connection between the two is left undetermined, but I take this passage to offer some preliminary reason to (i) take seriously Aristotle's anti-Platonist stance and his claim that geometers study the same objects as natural scientists; (ii) to consider the *qua*-theory and separation as two distinct operations. This second point gains additional support from the following lines, which provide more information about separation. Aristotle claims that the use of separation is only allowed in geometry, not in physics; if sensible features are separated within physics, error results. This is what happens when the Platonists posit Ideas: they are separating something that is not apt to be separated:

> λανθάνουσι δὲ τοῦτο ποιοῦντες καὶ οἱ τὰς ἰδέας λέγοντες· τὰ γὰρ φυσικὰ χωρίζουσιν ἧττον ὄντα χωριστὰ τῶν μαθηματικῶν. γίγνοιτο δ' ἂν τοῦτο δῆλον, εἴ τις ἑκατέρων πειρῷτο λέγειν τοὺς ὅρους, καὶ αὐτῶν καὶ τῶν συμβεβηκότων. τὸ μὲν γὰρ περιττὸν ἔσται καὶ τὸ ἄρτιον καὶ τὸ εὐθὺ καὶ τὸ καμπύλον, ἔτι δὲ ἀριθμὸς καὶ γραμμὴ καὶ σχῆμα, ἄνευ κινήσεως, σὰρξ δὲ καὶ ὀστοῦν καὶ ἄνθρωπος οὐκέτι, ἀλλὰ ταῦτα ὥσπερ ῥὶς σιμὴ ἀλλ' οὐχ ὡς τὸ καμπύλον λέγεται.
>
> Those who talk about ideas do not notice that they too are doing this: they separate physical things though they are less separable than the objects of mathematics. That becomes clear if you try to define the objects and the things which supervene in each

[33] I will discuss the structure and the interpretation of the (very controversial) *qua*-clause in Chapter 1.

> class. Odd and even, straight and curved, number, line, and shape, can be defined without change but flesh, bone, and man cannot. They are like snub nose, not like curved.
>
> 193b35-194a7 (Charlton's translation)

This passage introduces an asymmetry between physics and geometry: while the geometer is allowed to make use of separation, the natural scientist is not. The reason adduced by Aristotle is that geometrical features are more separable than sensible ones. This can be appreciated by looking at their definition: geometrical items can be defined without referring to motion or to matter.[34] What interests me is that in *Metaphysics* M.3, Aristotle introduces the *qua*-theory precisely to show that geometry is no less and no more problematic than other sciences, e.g. medicine. The *qua*-clause can be used to describe the methodology of geometry in the same way as the methodology of medicine – it is sufficient to plug a different feature into the *qua*-clause, so that geometry can be described as the science of sensible substances *qua* continuous in the same way as medicine can be described as the science of sensible substances *qua* healthy. The fact that the *qua*-theory can describe the methodology of all sciences, while separation is only appropriate to geometry, provides additional evidence that the *qua*-theory and the doctrine of separation describe two different operations.

§3. Ta ex aphaireseōs

Throughout the *Corpus*, Aristotle often refers to mathematical objects as *ta ex aphaireseōs*. It is probably because of this formula that Aristotle's philosophy of

[34] It is not completely clear why Aristotle starts by saying that geometrical features are separable in thought form *motion*, and then goes on to give the example of the snub/curved, which is his standard example of something that does not involve matter. It seems clear that Aristotle takes the set of things that have matter to be at the very least co-extensive with the set of things that involve motion. I will discuss the role of motion in geometry, and the claim that geometrical objects are separable from motion and without motion, in Chapter 5.

mathematics is commonly described as 'abstractionist', even if the label is hardly adequate: the use of this expression as label for mathematical objects might suggest that the latter are to be identified with abstractions. Although often translated as 'abstraction', *aphairesis* is most properly understood as 'subtraction' or 'removal'.[35] While speaking of abstraction suggests that one given item is selected and extracted out of a whole, in using the terminology of *aphairesis* Aristotle conveys the opposite image, that of a whole from which the irrelevant features are eliminated.

Note that at times Aristotle uses the expression *ex aphaireseōs* in conjunction with the complementary *ek prostheseōs,* 'by addition'. In these cases it is clear that the term does not describe a psychological theory of abstraction, nor the special mode of existence of mathematical entities, but a logical operation. It is notable that one of these passages appears in *Metaph.* M.2:

> εἰ γὰρ μὴ ἔστι τὰ πάθη παρὰ τὰς οὐσίας, οἷον κινούμενόν τι ἢ λευκόν, τοῦ λευκοῦ ἀνθρώπου τὸ λευκὸν πρότερον κατὰ τὸν λόγον ἀλλ' οὐ κατὰ τὴν οὐσίαν· οὐ γὰρ ἐνδέχεται εἶναι κεχωρισμένον ἀλλ' ἀεὶ ἅμα τῷ συνόλῳ ἐστίν (σύνολον δὲ λέγω τὸν ἄνθρωπον τὸν λευκόν), ὥστε φανερὸν ὅτι οὔτε τὸ ἐξ ἀφαιρέσεως πρότερον οὔτε τὸ ἐκ προσθέσεως ὕστερον· ἐκ προσθέσεως γὰρ τῷ λευκῷ ὁ λευκὸς ἄνθρωπος λέγεται).
>
> For if there are no attributes over and above real objects (e.g. a moving or a white) then white is prior in definition to white man, but not in reality, since it cannot exist separately but only together with the compound (by compound I mean the white man). So clearly the result of abstraction is not prior, nor the result of addition subsequent, because it is by adding to white that the white man is spoken of.
>
> *Metaph.* M.2, 1077b4-11 (Annas' translation)

[35] See Cleary (1985).

This is the only occurrence of the term *aphairesis* in the whole of *Metaphysics* M.1-3, and it is not specifically referred to mathematical objects but also to sensible features. One might thus worry that *Metaphysics* M.3 puts forward a theory of geometrical objects that is disconnected to the rest of the *Corpus*, and possibly incompatible with it. This worry is unjustified.

I believe that the expression *ta ex aphaireseōs* should not be understood as a technical piece of terminology that comes with a full theory of the ontological status of geometrical objects. Cleary (1985) argues convincingly that *prosthesis* and *aphairesis* refer to logical methods of inquiry that were already common in the Academy, and that are not specific to the philosophy of mathematics. This is perfectly coherent with the use of the term in M.2.

Note in particular that the logical method of *aphairesis* can be used to isolate the geometrical features of sensible objects.[36] As such, it can illustrate the same mechanism that is at play in the *qua*-theory, and that I will examine in Chapter 1. The compatibility of these two descriptions can be seen in the following passage:

> ὁ μαθηματικὸς περὶ τὰ ἐξ ἀφαιρέσεως τὴν θεωρίαν ποιεῖται (περιελὼν γὰρ πάντα τὰ αἰσθητὰ θεωρεῖ, οἷον βάρος καὶ κουφότητα καὶ σκληρότητα καὶ τοὐναντίον, ἔτι δὲ καὶ θερμότητα καὶ ψυχρότητα καὶ τὰς ἄλλας αἰσθητὰς ἐναντιώσεις, μόνον δὲ καταλείπει τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ' ἓν τῶν δ' ἐπὶ δύο τῶν δ' ἐπὶ τρία, καὶ τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ, καὶ οὐ καθ' ἕτερόν τι θεωρεῖ, καὶ τῶν μὲν τὰς πρὸς ἄλληλα θέσεις σκοπεῖ καὶ τὰ ταύταις ὑπάρχοντα, τῶν δὲ τὰς συμμετρίας καὶ ἀσυμμετρίας, τῶν δὲ τοὺς λόγους, ἀλλ' ὅμως μίαν πάντων καὶ τὴν αὐτὴν τίθεμεν ἐπιστήμην τὴν γεωμετρικήν)

[36] Cf. *APo* I.5, 74a39-b1.

> [The geometer] investigates abstractions (for in his investigation he eliminates all the sensible qualities, e.g. weight and lightness, hardness and its contrary, and also heat and cold and the other sensible contrarieties, and leaves only the quantitative and continuous sometimes in one, sometimes in two, sometimes in three dimensions, and the attributes of things *qua* quantitative and continuous, and does not consider them in any other respect, and examines the relative positions of some and the consequences of these, and the commensurability and incommensurability of others, and the ratios of others but yet we say there is one and the same science of all these things—geometry).
>
> Arist. *Metaph.* K.3, 1061a28-b3[37]

Note that here the terminology of *aphairesis* is used to clarify what it means for the geometer to consider physical objects *qua quantitative and continuous*. As we will see, M.3 offers a detailed and in-depth analysis of the *qua*-clause: far from being defective, it seems that M.3 offers the more rigorous and precise explanation towards which the expression *ta ex aphaireseōs* gestures.

Note that the terminology of *aphairesis* and the *qua*-operator are complementary descriptions of the same operation of property selection, i.e. of discrimination between aspects of an object that are apt to be studied geometrically and aspects that are not. They complement each other, in that the *qua*-operator positively selects an aspect that is to be kept, while the other aspects are to be removed (*periaireō*). I suspect that one of the reasons why Aristotle avoids using the terminology of *aphairesis* in M.3 might precisely be that he wishes to stress the difference between the two operations that he attributes to the geometer: (i) the *qua-manoeuvre*; (ii) separation. Elsewhere, where he is not as

[37] The authenticity of book K is debated. Assessing it goes beyond the scope of the thesis; see for example Aubenque (1983) against, and Décarie (1983) in favour of the authenticity, and Baldini (2020) for an assessment of the various stratifications of the text. Note that even if the text is not authentic, the combination of *aphairesis* and *qua*-operator shows that in Aristotle's circle the two expressions were used to described the same mechanism.

focused on the specificities and ontology of geometry, he can use the term *ta ex aphaireseōs* to refer to the outcome of both operations, when effectuated in succession.

Chapter 1. The *Qua*-Theory

This chapter examines the first section of M.3. It consists in the presentation of the so-called *qua*-theory, which can be summarised in the deceptively simple slogan 'geometers study sensible objects, although not *qua* sensible'.[38] I am going to argue that the main goal of the *qua*-theory is not to give an ontological account of the special mode of existence of geometrical objects, but rather to show that the rejection of Platonic separate entities as the specific objects of mathematics does not jeopardise the status of geometry as a science.

To do this, Aristotle needs to show that (i) geometry is about objects that exist; (ii) the identity of geometry as a separate science does not depend on the existence of some specific set of objects that are only studied by it. The *qua*-theory provides a solution to both issues: with regard to (i), Aristotle's move is to maintain that geometry studies objects that do exist (because it studies sensible substances and their properties); as to (ii), the claim is that it is a specific and independent science, which can be differentiated by the others on the basis of its specific methodology (it studies them not *qua* sensible, but *qua* quantitative and continuous).

This 'methodological' interpretation of the *qua*-clause, which takes it to qualify the way in which the geometer approaches the study of sensible objects rather than the objects themselves, is instrumental to providing a radically anti-Platonist interpretation of geometrical truth. The truth-makers of geometrical theorems simply are sensible substances which bear geometrical properties, and geometry is characterised by a special approach to those objects.

[38] 1077b21-22; 1077b27-31; *Ph.* II.2, 193b30.

§1. The Qua-Theory as a Methodological Description

To understand the import and goal of the *qua*-theory, we need to start from the last lines of M.2:

> Ὅτι μὲν οὖν οὔτε οὐσίαι μᾶλλον τῶν σωμάτων εἰσὶν οὔτε πρότερα τῷ εἶναι τῶν αἰσθητῶν ἀλλὰ τῷ λόγῳ μόνον, οὔτε κεχωρισμένα που εἶναι δυνατόν, εἴρηται ἱκανῶς· ἐπεὶ δ' οὐδ'ἐν τοῖς αἰσθητοῖς ἐνεδέχετο αὐτὰ εἶναι, φανερὸν ὅτι ἢ ὅλως οὐκ ἔστιν ἢ τρόπον τινὰ ἔστι καὶ διὰ τοῦτο οὐχ ἁπλῶς ἔστιν· πολλαχῶς γὰρ τὸ εἶναι λέγομεν.
>
> It has been adequately shown that <mathematical objects> are not real objects more than bodies are, that they are not prior to perceptible objects in reality, but only in definition, and that they cannot have separate existence. Since they could not exist in perceptible objects either, clearly either they do not exist at all, or they do so in a certain way and so do not exist without qualification – for we use 'exist' in several senses.
>
> *Metaph*. M.2, 1077b13-17 (Annas' translation)

M.2 has shown that mathematical objects are not substances: they do not exist *haplōs*, without qualification; they are not separate from, or prior in being to, sensible objects (even if they might be prior to sensibles in definition, in that they can be defined without mentioning sensible features or matter), nor can they exist *in* them, as distinct but co-located objects. It follows that either they do not exist at all, or they exist *tina tropon*, 'in a certain way' and not without qualification.

As I mentioned in the introduction, the former is not a viable option for Aristotle. This only leaves the latter option. Note that to safeguard the possibility of geometrical truths it is not necessary to provide a fully-fledged account of the mode of existence of

geometrical objects, but it is sufficient to establish *that* they do exist (somehow). Crucially, this is exactly the conclusion that Aristotle draws at 1077b31-34:

> ὥστ' ἐπεὶ ἁπλῶς λέγειν ἀληθὲς μὴ μόνον τὰ χωριστὰ εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἷον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν.
>
> So since it is true to say without qualification not only that separable things exist but also that non-separable things exist (e.g. that moving things exist), it is also true without qualification that mathematical objects exist, and are as they are said to be.

Notice that lines 30-31 echo lines 16-17. This suggests that Aristotle takes himself to have solved the issue.[39]

The formulation τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν might seem puzzling at first, since lines 16-17 clearly state that mathematical objects *do not* exist *haplōs*. However, the word order mismatch between the two occurrences signals that Aristotle is here accomplishing a very refined and sophisticated move: although mathematical objects do not exist without qualification (φανερὸν ὅτι … οὐχ ἁπλῶς ἔστιν, M.2, 1077b16-17), the *qua*-theory allows Aristotle to show that it is nonetheless perfectly legitimate to say that geometry is about something that is, since it is true without qualification to say that geometrical objects exist (τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, M.3, 1077b30-31). In this second occurrence, ἁπλῶς does not qualify ἔστιν, but rather ἀληθὲς εἰπεῖν.[40]

[39] Cf. Netz (2006: 7-9).
[40] See Annas (1976: *ad loc.*). It is not completely clear whether *haplōs* qualifies the whole verbal complex *alēthes eipein* (and *legein alēthes* in the preceding line) or just one of the two words, and in case which one. My best guess is that the qualification applies to *alēthes eipein* as a whole, or just to *alēthes*: it is not so obvious that it would be true to 'say *haplōs*' that mathematical objects exist. Thanks to Michael Peramatzis for pointing this out to me.

So, Aristotle has to show that the objects of geometry (i) exist, but (ii) not *haplōs*. This is where the *qua*-theory comes in. There are two ways two spell out the view – one 'objectual' and one 'methodological'. The objectual reading of the *qua*-clause yields a moderately anti-Platonist interpretation. On this reading, the *qua*-clause introduces a special set of objects, which are distinct from, but dependent on, sensible substances – the sensible-substances-not-*qua*-sensible. The methodological reading, on the other hand, does not introduce any kind of object, and thus yields a radically anti-Platonist interpretation. The truth-makers of geometrical statements simply are sensible substances and their properties, and the *qua*-clause specifies the peculiar way in which geometers approach them, their specific methodology.

To understand how Aristotle takes the *qua*-operator to work and do the job it is supposed to do, we can start by considering two analogies that Aristotle draws. First, he compares geometry to the 'general propositions in mathematics'; then, to natural science. Let us start from the first.

T1 (1077b17-22, Annas' translation)

ὥσπερ γὰρ καὶ τὰ καθόλου ἐν τοῖς μαθήμασιν οὐ περὶ κεχωρισμένων ἐστὶ παρὰ τὰ μεγέθη καὶ τοὺς ἀριθμοὺς ἀλλὰ περὶ τούτων μέν, οὐχ ᾗ δὲ τοιαῦτα οἷα ἔχειν μέγεθος ἢ εἶναι διαιρετά, δῆλον ὅτι ἐνδέχεται καὶ περὶ τῶν αἰσθητῶν μεγεθῶν εἶναι καὶ λόγους καὶ ἀποδείξεις, μὴ ᾗ δὲ αἰσθητὰ ἀλλ' ᾗ τοιαδί.

For just as general propositions in mathematics are not about separate objects over and above magnitudes and numbers, but are about these, only not as having magnitudes or being divisible, clearly it is also possible for there to be statements and proofs about perceptible magnitudes, but not as perceptible but as being of a certain kind.

The analogy is surprisingly dry and scarce of details, and it seems to assume that the reader is familiar with the relation between the 'general propositions in mathematics' and their objects. Aristotle has already discussed the 'general propositions stated by mathematicians' in M.2, during his refutation of mathematical Platonism, but even in that context he does not provide many details. He merely states that their 'application extends beyond these objects',[41] so that if the Platonist wants to hold onto her claim that each science bears on a distinct and specific kind of object, she needs to posit the existence of an additional kind of geometrical objects, which are 'neither number nor point nor magnitude nor time'.[42]

The lack of details and the ease with which Aristotle refers to these propositions show clearly that he is relying on the reader's previous mathematical knowledge, and he assumes that the reference to the 'general propositions' is transparent to his audience. The main example of such general propositions is Eudoxus' theory of proportions, which applies both to geometrical magnitudes and to arithmetical numbers. What is notable about Eudoxus' theory is that it is unitary: one and the same theory of proportions accounts for continuous magnitudes and numbers.

While in M.2 Aristotle is interested in stressing that this does not entail that there are additional general mathematical objects, in M.3 he focuses on the mechanism that allows such level of generality, and refers to it using the *qua*-clause. He claims that the general propositions in mathematics are about magnitudes and numbers, but not *qua* having magnitudes or being divisible. The *qua*-clause is supposed to explain why a proof carried out on one specific kind of objects is valid for all other kinds of mathematicals, so that its 'application extends beyond' the objects on which it is proved.

[41] 1077a10.
[42] 1077a12.

We find more details about the proof of these propositions in *APo* I.5. The context is a discussion of the various ways in which a demonstration might fail to be universal. Aristotle lists three cases: (i) 'when there is nothing higher that can be taken apart from the particular case'; (ii) 'when there is but it is nameless and covers objects of different forms'; (iii) 'when the proof applies to something that is in fact a partial whole'. The theory of proportions is used to exemplify case (ii):

> καὶ τὸ ἀνάλογον ὅτι καὶ ἐναλλάξ, ᾗ ἀριθμοὶ καὶ ᾗ γραμμαὶ καὶ ᾗ στερεὰ καὶ ᾗ χρόνοι, ὥσπερ ἐδείκνυτό ποτε χωρίς, ἐνδεχόμενόν γε κατὰ πάντων μιᾷ ἀποδείξει δειχθῆναι· ἀλλὰ διὰ τὸ μὴ εἶναι ὠνομασμένον τι ταῦτα πάντα ἓν, ἀριθμοί μήκη χρόνοι στερεά, καὶ εἴδει διαφέρειν ἀλλήλων, χωρὶς ἐλαμβάνετο. νῦν δὲ καθόλου δείκνυται· οὐ γὰρ ᾗ γραμμαὶ ἢ ᾗ ἀριθμοὶ ὑπῆρχεν, ἀλλ' ᾗ τοδί, ὃ καθόλου ὑποτίθενται ὑπάρχειν.
>
> Again, it might be thought that proportion alternates for items as numbers and as lines and as solids and as times. In the past this used to be proved separately, although it is possible to prove it of all cases by a single demonstration: because all these items – numbers, lengths, times, solids – do not constitute a single named item and differ in form from one another, they used to be taken separately. Now, however, it is proved universally: what they suppose to hold of them universally does not hold of them as lines or as numbers but as this.
>
> *APo* I.5, 74a17-25 (Barnes' translation)

In contrast to what was the practice in the past, 'now' mathematicians manage to produce a single proof which applies to all cases. Even in this context, Aristotle does not really explain how the proof goes. He assumes that the reader is familiar with it, and once again he uses the *qua*-operator to refer to it. This mechanism, which seems to be a known and

familiar mathematical methodology, is the first term of comparison that Aristotle uses to elucidate the relation that geometry bears to its objects in M.3. If this is right, we need to look at geometrical practice to understand the point that Aristotle wishes to make.

Netz (2006) argues that the mechanism at play in the theory of proportions is actually central in Greek geometrical practice in general. In the theory of proportions, the theorems are proved using one specific kind of magnitudes (segments) and the results apply also to other kinds of magnitudes, because the segments are not considered *qua* segments but more generally *qua* magnitudes. The same mechanism allows Greek geometers to obtain general conclusions while working on one specific object or configuration.

It is useful to take a short detour and look at the analysis of Euclidean theorems provided by Proclus. In his commentary on the first book of Euclid's *Elements*, Proclus identifies six parts of a typical Euclidean demonstration:[43]

1. *Protasis*, enunciation of the proposition to be proved;
2. *Ekthesis*, setting out. The 'given' elements of the proof are exhibited;
3. *Diorismos*, definition of the goal. The general proposition stated in the *protasis* is put forward in terms of the elements introduced in the *ekthesis*. This step might also specify the conditions of possibility for achieving the goal;
4. *Kataskeue*, construction of the diagram;
5. *Apodeixis*, proof proper;
6. *Symperasma*, conclusion (which echoes the *protasis* and is stated in general terms).

Note that the *protasis* and the *symperasma* are stated in general terms: geometrical demonstrations are supposed to prove general truths that hold for all cases. However, the

[43] Note that not all Euclidean theorems actually respect this schema, though; Acerbi (2020: 40).

demonstration itself is carried out on a particular case, which is exhibited in the *ekthesis* and which is represented in the diagram.[44] According to Proclus, this is possible because of the particular way in which geometers use the objects set out in the *ekthesis*; strikingly, he uses the *qua*-operator to describe it:

> μεταβαίνουσι μὲν οὖν ἐπὶ τὸ καθόλου, ἵνα μὴ μερικὸν ὑπολάβωμεν εἶναι τὸ συμπέρασμα. εὐλόγως δὲ μεταβαίνουσιν, ἐπειδὴ τοῖς ἐκτεθεῖσιν, οὐχ ᾗ ταῦτά ἐστιν, ἀλλ' ᾗ τοῖς ἄλλοις ὅμοια, χρῶνται πρὸς τὴν ἀπόδειξιν. οὐ γὰρ ᾗ τοσήδε ἐστὶν ἡ ἐκκειμένη γωνία, ταύτῃ τὴν διχοτομίαν ποιοῦμαι, ἀλλ' ᾗ μόνον εὐθύγραμμος. ἔστι δὲ τὸ μὲν τοσόνδε τῆς ἐκκειμένης ἴδιον, τὸ δὲ εὐθύγραμμον πασῶν τῶν εὐθυγράμμων κοινόν. ἔστω γὰρ ἡ δεδομένη ἡ ὀρθή. εἰ μὲν οὖν τῇ ἀποδείξει τὴν ὀρθότητα παρελάμβανον, οὐκ ἠδυνάμην ἐπὶ πᾶν τὸ εἶδος τῆς εὐθυγράμμου μεταβαίνειν, εἰ δὲ τὸ μὲν ὀρθὸν αὐτῆς οὐ προσποιοῦμαι, τὸ δὲ εὐθύγραμμον σκοπῶ μόνον, ὁμοίως ὁ λόγος ἐφαρμόσει καὶ παρὰ ταῖς εὐθυγράμμοις γωνίαις.
>
> They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not *qua* these particular figures, but *qua* figures resembling others of the same sort. It is not *qua* having such and such a size that the angle before me is bisected, but *qua* being rectilinear and nothing more. Its particular size is a character of the given angle, but its having rectilinear sides is a common feature of all rectilinear angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilinear angles; but if I make no use of its rightness and consider only

[44] But see Acerbi (2020), who argues that geometrical demonstrations are general throughout and do not refer to one particular configuration. I will come back to the role of diagrams in geometrical demonstrations in Chapter 5.

its rectilinear character, the proposition will apply equally to all angles with rectilinear sides.

Proclus, *In Eucl.* 207.12-25 Friedlein (Netz' translation)

In order to achieve generality, the particular features of the individual figure under consideration should not be used in order to reach the conclusion.[45] In Proclus' example, the proof is valid for all angles (and not only for right angles) because the rightness of the right angle under consideration is not relied upon during the demonstration. It is important to stress that Proclus is describing a special *methodology*: it is because the geometer treats her object in a certain way that she can reach a general conclusion, even if she is indeed considering a normal object.[46]

To further illustrate this point, consider that this mechanism of selective attention is also at play when geometers make use of diagrams. I will discuss the role of diagrams in geometrical practice in Chapter 5, but for the moment note that when using a diagram, geometers cannot rely blindly on all of its features: only some features are used, while others are unnecessary and can even get in the way of the desired demonstration. Following Manders (1995/2008), we can distinguish between exact and co-exact features of diagrams. Only the latter can be relied upon in the course of proofs: they are those topological properties that are invariant under a certain set of modifications, and thus are common to all diagrams that represent the same configuration. They include intersections, part-whole relations of regions, segment bounding regions. On the contrary, exact features do not resist any modification of the diagram: they include specific metrical attributions, as well as straightness and circularity, equalities and inequalities. When

[45] This ensures that the proof is repeatable for all other cases; see Netz (1999: ch. 6).
[46] This analysis resonates with Aristotle's own remarks at *APo* I.4, 73b32-74a3 and I.5, 74a32-b4. Note that the second passage used the language of *aphairesis* rather than the *qua*-operator; as I mentioned in the introduction, I believe that the two descriptions are compatible.

doing geometry, these features cannot be read off the diagram, but must be specified by the text of the proof; using a diagram in a proof requires considering the drawing not *qua* drawing, but *qua* diagram. It is this sort of systematic partial attention between different features which, I suggest, lays at the heart of Aristotle's *qua*-theory and his account of geometry.

To come back to Aristotle's own example, it seems clear that also in the case of 'the general propositions of mathematics' mentioned in T1 what matters is the way in which the objects are treated: if the proof is conducted in accordance with the correct methodology, its result will be applicable beyond the singular object on which it has been carried out.

The second analogy drawn in M.3 abandons the field of mathematics, and draws a comparison between geometry and natural sciences – in particular, physics (which is defined as the science of moving things qua moving in *Metaph*. K.4, 1061b27-30):

T2 (1077b22-30, Annas' translation)

ὥσπερ γὰρ καὶ ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί, χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς, καὶ οὐκ ἀνάγκη διὰ ταῦτα ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην, οὕτω καὶ ἐπὶ τῶν κινουμένων ἔσονται λόγοι καὶ ἐπιστῆμαι, οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ σώματα μόνον, καὶ πάλιν ᾗ ἐπίπεδα μόνον καὶ ᾗ μήκη μόνον, καὶ ᾗ διαιρετὰ καὶ ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν καὶ ᾗ ἀδιαίρετα μόνον.

For just as there are many statements about things merely as moving, apart from the nature of each such thing and their incidental properties (and this does not mean that there has to be either some moving object separate from the perceptible objects, or some such entity marked off in them), so in the case of moving things there will be statements and branches

of knowledge about them, not as moving but merely as bodies, and again merely as planes and merely as lengths, as divisible, and indivisible but with position, and merely as indivisible.

Note that T1 and T2 present striking similarities:

1. They are introduced in the same way (ὥσπερ γὰρ καὶ)
2. They are developed along the same lines (περὶ τούτων μέν, οὐχ ᾗ δὲ τοιαῦτα οἷα ἔχειν μέγεθος ἢ εἶναι διαιρετά / ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί)
3. They reach the same conclusion: ἐνδέχεται καὶ περὶ τῶν αἰσθητῶν μεγεθῶν εἶναι καὶ λόγους καὶ ἀποδείξεις, μὴ ᾗ δὲ αἰσθητὰ ἀλλ' ᾗ τοιαδί / ἐπὶ τῶν κινουμένων ἔσονται λόγοι καὶ ἐπιστῆμαι, οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ σώματα μόνον … .

This suggests that the two analogies are meant to make the same point, and presumably to highlight the same mechanism. However, some interpreters worry that the two passages pull in different directions – and in particular, that the analysis of the *qua*-operator as indicating a specific *methodology* does not fit its use in T2.[47] This would, of course, offer evidence for the moderately anti-Platonist interpretation. The reason why T2 can be taken to support the moderately anti-Platonist reading rather than the methodological one is that in T2 the *qua*-operator seems to apply directly to the object under examination, and to select one part or aspect of it. Aristotle says that there are many statements 'about things merely *qua* moving, apart from the nature of each such thing and their incidental properties'. The use of the term *choris* might suggest that it is the object itself that is modified – deprived of some of its features. However, Aristotle hastens to stress that he

[47] E.g. Annas (1976: 148-9).

does not mean that there is a special entity (as the object-*qua*-moving would be) that is either separate from the object itself or 'marked off' in it. This remark refers back to M.2, and to the two versions of Platonism that Aristotle has refuted. I believe that it can also be taken to signal that the *qua*-clause should not be read as modifying the noun, but on the model set by T1: as indicating a special way of treating the objects – not as indicating a special kind of objects. This would also explain why Aristotle feels the need to give *two* analogies that point to one and the same conclusion: the first analogy (which is intra-mathematical and thus not very effective from a dialectical point of view)[48] is supposed to show how the *qua*-operator is to be read; the second analogy (which is extra-mathematical) brings the point home: geometry works in the same way as the other sciences, and so it is as legitimate as they are. The upshot of the *qua*-theory, so interpreted, is that the legitimacy of a science does not depend on the special nature of its objects, but on the way in which it treats them: physics considers its objects *qua* movable, while geometry considers them *qua* planes or solids.

From a grammatical point of view, notice that the *qua*-operator is attached to the *verb*, and not to the noun, in both passages. This construction is the one that is most natural for the Greek, where the particle *hei* maintains its relative value and is thus most naturally used adverbially.[49] The parallel with the theory of proportions gives a clear reading key: 'studying X *qua* Y' simply means that while doing science, the scientist will be allowed only to rely on a subset of the features of the object *X* they are examining. So

[48] Starting the discussion with an analogy to mathematical practice is an unexpected move: after all, if the point of the chapter is to ensure the very possibility of mathematics as a science, one might think that grounding it *on the methodology of mathematics itself* is not going to do the trick. Moreover, the theory of proportions is not an autonomous science; showing that geometry works like Eudoxus' theory of proportions is not sufficient to show that the possibility of geometry as an individual and specific science is not endangered by the rejection of the existence of separate geometrical objects. On the contrary, showing that mathematics works exactly in the same way as the other (less controversial) sciences, and makes the same ontological demands, seems to be a better strategy: it is thus understandable why the mathematical comparison is dropped after T1, and why the following analogies stick to the natural sciences.
[49] See Netz (2006: 9-11).

it is in a certain sense true that the *qua*-clause picks out a certain set of features of the object (as suggested by T2), but this should be read in a methodological perspective rather than in an ontologically loaded way.

Considering *X qua Y* does consist in selecting a subset of the features of the particular object under consideration. However, these properties are not abstracted and considered in a vacuum, nor are the other properties somehow eliminated or removed from the object: they are simply *ignored*. Ignoring a feature is nothing more than avoid using it in the course of the proof; it does not amount to creating an object that lacks such feature. Therefore, I do not think that T2 offers a very compelling justification for preferring the objectual reading of the *qua*-operator; a methodological reading seems perfectly justified in both passages considered.

Now we are in a better position to understand the import of the *qua*-theory, and the conclusion that Aristotle draws from these analogies. Let us have another look at the lines that I quoted above:

T3 (1077b31-34, Annas' translation)

ὥστ' ἐπεὶ ἁπλῶς λέγειν ἀληθὲς μὴ μόνον τὰ χωριστὰ εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἷον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν.

So since it is true without qualification to say not only that separable things exist but also that non-separable things exist (e.g. that moving things exist), it is also true without qualification to say that mathematical objects exist, and are as they are said to be.

As I have already mentioned, the conclusion that Aristotle draws from T1 and T2 is that it is perfectly legitimate to say ('it is true without qualification to say') that geometry

is about something that is. In this final passage, Aristotle is still relying on the analogy with physics. He has just explained that geometry is about sensible things, although not *qua* sensible but *qua* planes or bodies, in the same way as physics is about sensible things, although merely *qua* moving. He now points out that we would never deny that physics is about something that is, even if 'the movings' are not *chorista*.

Again, these last few lines could be interpreted as advocating a moderately anti-Platonist view on which there is a specific kind of objects, *ta mathematika*, that are distinct from sensible bodies but have a special mode of existence – they are not *chorista* but rather they *depend* on sensible bodies for their existence. Such an interpretation would retain the Platonic idea that geometry studies some specific kind of object, and it denies that those objects exist *haplos*, independently of sensible substances.

But this ontological interpretation is not required, and it would be difficult to fit with the first analogy (as underlined by Annas, among others): when it comes to the theory of proportions, it does not seem accurate to say that the mathematician studies a specific object that is distinct from numbers, lengths and times, but dependent on them for its existence. On the contrary, this reading seems in tension with Aristotle's own use of the example in M.2, where appealing to the 'general propositions in mathematics' is successful precisely because these propositions do not bear on a different kind of mathematical objects but on numbers and magnitudes.

Moreover, notice that even if we did want to adopt the moderately anti-Platonist interpretation, it would still be the case that the mode of existence of geometricals is not described as specific to them; on the contrary, Aristotle suggests that we should understand geometricals in the same way as we understand moving things. There is nothing in the world that is *merely* a moving thing: physics nonetheless studies something that is, because physical bodies do move and thus can be considered merely *qua* moving,

without the need to suppose that there is something else, distinct from them, which is 'a moving'. Once again we have a reference back to M.2, where Aristotle has established that geometrical objects cannot exist *haplos* nor *chorista* (1077b4-11). In that context, 'the moving' was used (together with the white) as an example of *pathē* which cannot exist separate from substances. The idea that geometrical features are *pathē* of sensible objects, which is in line with the classification of geometricals as quantities in *Categories* 6, is perfectly compatible with the radically anti-Platonist interpretation that I am suggesting: sensible substances act as truth-makers of geometrical propositions precisely because they do instantiate geometrical properties.[50]

This of course depends on the assumption that sensible bodies *do instantiate* geometrical features. One might very well doubt that this is the case: after all, we never see a perfectly straight line, nor a perfectly bidimensional plane. I will discuss this issue (what I call the Precision Problem) in Chapter 2. For the moment, notice that Aristotle does not seem to be concerned with it in M.3, even if he is obviously aware of the problem (that is mentioned explicitly at e.g. *Metaph*. B.2, 997b32-998a6).

§2. Answering Question (ii): The Identity of the Sciences

In the next section, Aristotle provides more details regarding the identification of each science. Once again, he proceeds by offering an analogy:

> **T4 (1077b34-1078a5, Annas' translation)**
>
> καὶ ὥσπερ καὶ τὰς ἄλλας ἐπιστήμας ἁπλῶς ἀληθὲς εἰπεῖν τούτου εἶναι, οὐχὶ τοῦ συμβεβηκότος (οἷον ὅτι λευκοῦ, εἰ τὸ ὑγιεινὸν λευκόν, ἡ δ' ἔστιν ὑγιεινοῦ) ἀλλ' ἐκείνου

[50] On the contrary, interpretations such as Hussey's (1991) or Mueller's (1970) attribute to geometrical objects a special mode of existence, which is different from that of sensible *pathē*. The former takes geometrical objects to be akin to Kit Fine's arbitrary objects, and thus unable to be fully actualised; the latter takes geometrical objects to 'underlie sensible reality'.

οὗ ἐστὶν ἑκάστη, εἰ <ᾗ> ὑγιεινὸν ὑγιεινοῦ, εἰ δ' ᾗ ἄνθρωπος ἀνθρώπου, οὕτω καὶ τὴν γεωμετρίαν· οὐκ εἰ συμβέβηκεν αἰσθητὰ εἶναι ὧν ἐστί, μὴ ἔστι δὲ ᾗ αἰσθητά, οὐ τῶν αἰσθητῶν ἔσονται αἱ μαθηματικαὶ ἐπιστῆμαι, οὐ μέντοι οὐδὲ παρὰ ταῦτα ἄλλων κεχωρισμένων.

And as it is true to say of other branches of knowledge, without qualification, that they are of this or that – not what is incidental (e.g. not the white, even if the branch of knowledge deals with the healthy, and the healthy is white) but what each branch of knowledge is of, the healthy if <it studies its subject> as healthy, man if <it studies it> as man, likewise with geometry: the mathematical branches of knowledge will not be of perceptible objects just because their objects happen to be perceptible, though not <studied> as perceptible; but nor will they be of other separate objects over and above these.

T4 presents the same structure as T1 and T2. Similarly to T2, it exploits an analogy with some natural science to support an apparently controversial or counter-intuitive claim about geometry. This time, the focus of the analogy is the way in which sciences are individuated. The analogy is not limited to a particular science, but applies in general to the 'other branches of knowledge' – presumably those whose object seems unproblematic, and which might appear to be defined in terms of their object. The specific example that Aristotle uses is the case of medicine: he takes it as uncontroversial that medicine studies the healthy. However, this does not require the existence of some mysterious kind of objects that only have the property of being healthy: it simply means that medicine studies human beings *qua* healthy. Medicine is not the only science that studies human beings; its identity is not given by the specific *objects* that it studies, but by the way in which it studies them – namely, focusing on their health and ignoring the

other features. This being the case, medicine is correctly described as the science *of the healthy* (and not as the science of human beings). In the same way, Aristotle claims, geometry can be rightfully defined as the science *of* geometrical objects, even if in fact it is concerned with physical substances, because it considers those substances not *qua* physical.

In other words, Aristotle wants to show that his *qua*-theory can account for the way in which people normally speak about the sciences and their identity; he explains how his terminology maps on the common way of speaking of his interlocutors. People usually say that a certain science is the science *of* something, using the simple genitive. It is tempting to take it that this 'something' is a special kind of object: this is the starting point of the Platonist 'argument from the sciences'. According to Aristotle, this is a mistake. His *qua*-theory allows him to distinguish between what Barnes (1995) calls the *domain* of a theory, and its *focus*. The domain of a theory is the set of (separate and fully existing) objects that a science is concerned with; the focus is the specific set of features of those objects that a science takes into account. We have seen that Aristotle describes geometry as studying sensible objects, not *qua* sensible but *qua* solids, planes or lines.[51] In this case, the domain is the set of sensible objects, while the focus is expressed by the *qua*-clause: solids, planes and lines. Aristotle uses two different locutions to refer to the domain and to the focus; he claims that geometry is *about* sensible substances (using the formula *peri* + genitive, cf. T1), but that it is the science *of* solids, planes and lines (using the simple genitive).

This allows Aristotle to easily establish that there is the object of geometry, namely, that geometry does not have an empty domain: it is the domain of sensible objects. And,

[51] In M.3 Aristotle is mostly concerned with proving that his rejection of mathematical Platonism does not jeopardise the scientific status nor the truth of geometry, so he is mostly concerned with establishing that geometry is about sensible objects not *qua* sensible; he does not offer a comprehensive positive account. In K.3, 1061a28-b2 we read that geometers consider sensible objects *qua* quantitative and continuous.

at the same time, it allows him to answer question (ii) concerning the identity conditions of geometry as a science. For it is the focus that determines the identity of the science, not its domain. Two sciences which have the same domain (they are concerned with the same set of objects, i.e. sensible substances) might differ because they have a different *focus*, i.e. they have specific methodologies that restrict the attention on different aspects of these objects. So when people say that medicine is the science of the healthy, they are not picking out the *domain* of medicine, but its *focus* – that would be most properly referred to by using the *qua*-clause.[52]

Once again, the upshot is that the identity and possibility of a science do not depend on the existence of a special set of objects that are only studied by that science, but on the existence of a specific methodology that allows each science to focus on different aspects of the same objects (physical substances).[53]

§3. Essential/Incidental Troubles

However, T4 does introduce a new element that was absent from TT1-3: the mention of 'incidental' properties (*tou sumbebekotos*). Aristotle argues that it is true to say that medicine is the science of the healthy, and not of what is incidental; for example, not of the white, even if it might be true that the healthy is in fact white. The idea seems to be that the relation between the object under consideration being white and its being healthy is incidental, so that medicine should not be called the science of the white, even if its

[52] This interpretation also fits Aristotle's characterisation of metaphysics as the science of being *qua* being (Γ.1, 1003a21-22); it seems clear that metaphysics is not the science of some particular object which is 'being *qua* being', but rather the most general science that is concerned with being and studies it *qua* being (rather than having some specific focus). For an overview, see Barnes (1995).

[53] Note that it is not necessary that all sciences have exactly the same domain. In particular, this interpretation does not imply that every science is about *all beings*. I will argue below that it is only possible to study *X qua Y* if *X* is in fact *Y*; this implies that the focus of a science constrains its domain. For example, since medicine is the science of the healthy and only living organisms can be healthy, its domain does not include all sensible bodies but only living organisms.

object might happen to be white. This is confirmed by the last lines of the passage, where Aristotle draws his conclusion about geometry: we should not say that geometry is the science of sensible objects, even if the objects that it studies do in fact happen to be sensible. On the basis of the parallelism with the healthy/white case, we are led to conclude that the relation between the object under consideration being a solid and its being sensible is incidental. However, the two cases do not seem to be perfectly identical. Suppose that the object under consideration is, in all cases, a man.[54] The man has some essential properties (such as being rational) and some incidental properties. Both being healthy and being white are incidental properties of the man, and it is thus not surprising that 'the white' is said to be incidental. When we move on to geometry, the case is slightly different. The properties in question are being a solid (or a plane or a line, or in general quantitative and continuous) on the one hand, and being sensible on the other. But being sensible is *not* an incidental property of man: a man is necessarily sensible.

It seems that in the case of geometry, Aristotle is using the distinction between incidental and essential property in an unconventional way. Lear argues that the *qua*-theory enables Aristotle to relativise the distinction to a certain description of the object.[55] The following passage can be taken either as a justification or as a consequence of this move:

> **T5 (1078a5-10, Annas' translation)**
>
> πολλὰ δὲ συμβέβηκε καθ' αὑτὰ τοῖς πράγμασιν ᾗ ἕκαστον ὑπάρχει τῶν τοιούτων, ἐπεὶ καὶ ᾗ θῆλυ τὸ ζῷον καὶ ᾗ ἄρρεν, ἴδια πάθη ἔστιν (καίτοι οὐκ ἔστι τι θῆλυ οὐδ' ἄρρεν κεχωρισμένον τῶν ζῴων)· ὥστε καὶ ᾗ μήκη μόνον καὶ ᾗ ἐπίπεδα.

[54] Aristotle himself will use the example of the man at 1078a23 ff., discussed below.
[55] Lear (1982: 169) writes: 'In *Metaphysics* M.3, however, the distinction between essential and incidental properties can be made only for an object *under a certain description*. If we are considering *b as being an F*, then every predicate that is not essential to its being *F* is considered incidental, even though it may be essential to *b*'s being the substance that it is'.

> Many properties hold true of things in their own right as being, each of them, of a certain type – for instance there are attributes peculiar to animals as being male or as being female (yet there is no female or male separate from animals). So there are properties holding true of things in their own right merely as lengths or as planes.

Similarly to the previous paragraphs of M.3, also this passage uses an analogy with a natural science (namely biology) to draw a conclusion about geometry; presumably, the goal is to show that geometry studies *kath'auto* features of its objects, and thus correctly qualifies as a science.[56] Note that the wording and the structure of the paragraph are different from the previous ones; this might signal that the function of this section in the overall argument is different, too. The most important difference is that in T5 the *qua*-operator is used to qualify the verb *hyparchein*, while in the previous passages it was used to qualify verbs of attention, such as 'to study' or 'to be about'.[57] So it appears that there are *two* different constructions that involve the *qua*-operator, and both are at play in this chapter:

1. A is about X *qua* Y, or A studies X *qua* Y; e.g. geometry studies sensible bodies not *qua* sensible, but merely *qua* bodies or planes;
2. P belongs to X *qua* Y; e.g. some properties hold true of things in their own right merely *qua* lengths or planes.

The relationship between the two uses of the *qua*-operator is not clear. I am going to argue that the second is more fundamental than the first, and enables it. On the contrary, some scholars (such as Lear and Netz) believe that the first formula is the most important and

[56] Cf. *APo* I.6, 75a29-32 for this requirement.
[57] The terminology is taken from Corkum (2012). In *Ph.* II.2, Aristotle claims that geometry 'studies' (*skopei*) sensible objects but not *qua* sensible.

prior one, the one that grasps the core of the *qua*-theory; the latter is a consequence of the *qua*-theory that can be derived from it.

Lear (1982: 168) writes:

> Let *b* be an Aristotelian substance and let '*b qua F*' signify that *b* is being considered *as an F*. Then a property is said to be true of *b qua F* if and only if *b* is an *F* and its having that property follows of necessity from its being an *F*:
>
> $$G(b \text{ qua } F) \leftrightarrow F(b) \;\&\; \forall x(F(x) \vdash G(x))$$

According to his interpretation, the *qua*-operator works as a property-filter which allows the geometers to ignore the properties that are irrelevant to the inquiry at hand, thus placing themselves behind a sort of veil of ignorance. The application of such a property-filter is what enables the relativisation of the accidental/essential to a certain description of the object: the properties that are essential to *b qua F* are all and only those that follow of necessity from *b*'s being *F*. Relativising the distinction to a certain description of the object is very effective in explaining how it might be true that geometry is about *kath'auta* properties of its objects while avoiding an existential commitment to separate geometrical objects equipped with their own essence.[58] Moreover, this hypothesis seems to do exactly the work that Aristotle needs: some properties, though they are essential to the sensible objects that – we are told – geometers investigate, are nevertheless irrelevant to geometry, and so must not be taken into account when practicing geometry.

[58] Note that Lear believes that Aristotle does introduce fictional geometrical objects as heuristic tools. They are produced by the operation of separation that is presented at lines 1078a17 ff. and have all and only the features that are selected by the property-filter. So the properties that belong *kath'auta* to sensible objects *qua* quantitative and continuous are in fact the essential properties of these fictional geometrical objects, but this is a *consequence* of the relativization of the distinction between essential and incidental properties. I will discuss separation in Chapter 3.

However, many commentators have expressed scepticism about the idea that Aristotle would relativise the essential/incidental distinction to a certain description of the object. Even Annas (1976: 149), who believes that T5 *does* introduce this relativisation, notices that it is more congenial to modern views of necessity than to Aristotle's own conception of essences and necessity, given that modern views 'place necessity in the way we talk about things rather than in the things themselves'. The worry is that on Lear's interpretation, the relativisation depends completely on the way in which the object is described: it is reduced to a linguistic phenomenon. Cleary (1985: 29) describes this idea as 'Quinean' and anachronistic, and he argues that it would be misguided to attribute it to Aristotle, 'especially since he held that the substantial essence of a thing gives one a privileged point of view from which to make the distinction'. He believes that making the essential/incidental distinction depend on the way in which we speak of things gets the order of dependences wrong: it is because things are in such and such a way, that they can be described in one way or the other – not the other way round.

Also Netz (2006) argues that 'studying X *qua* Y' is logically prior to 'P belongs to X *qua* Y', so that once we are equipped with the former notion we can give a full analysis of the latter. In his opinion, studying X *qua* Y simply means to operate in a framework that only has Y-presuppositions – that is to say, where one only assumes that the object under consideration is Y. This leads one to accept only arguments-*qua*-Y, i.e. arguments that only operate under the presupposition that the object is Y. Having established this, the *qua*-predication can be understood as follows: 'P belongs to X *qua* Y' simply means 'P belongs to X, shown by an argument$_Y$(X)' – where an argument$_Y$(X) is an argument which has as its only presupposition that X is Y. This formulation is in fact equivalent to the analysis of the clause given by Lear: $P(X \text{ qua } Y) \leftrightarrow Y(X) \ \& \ \forall x(Y(x) \vdash P(x))$. It has the

advantage of not appealing to the logical notion of 'following of necessity', which risks admitting a number of irrelevant consequences.

Netz believes that it is an advantage of his interpretation that this formula is *derived* from the considering-*qua* clause, rather than introduced independently. This allows him to downplay the significance of T5 in the dialectic of the chapter, and to avoid talking of a relativisation of the incidental/essential distinction. He argues that Aristotle is simply making use of the distinction to indicate that some properties are essential *to the science at hand*, while others are not. This does not change the fact that the object under consideration is one and only one, the sensible object, and has its definite set of essential features; the reason is that, according to Netz, a science does not have *any* ontological commitments: it is identified and characterised by the way in which it operates, not by the kind of objects that it studies.

While Netz is right in stressing that sciences do not need a specific object to be acceptable, and can be properly identified by their methodology, I do believe that Aristotle would think it is necessary to ground the methodology in something real, for the science to be viable. The choice of properties that the science can take into account for its proofs cannot be arbitrary, and I believe that simply saying that it can take into account all and only the properties that are licensed by a proof conducted according to the correct methodology risks being circular, if taken as a justification of the methodology itself. To find a more stable ground, we need to recognise the fundamental role played by T5, and by the *qua*-predication.

I submit that the *qua*-predication is actually more basic than, and grounds the possibility of, considering X *qua* Y. It is *because* some properties belong to X *qua* Y that it is possible to consider X *qua* Y, by restricting attention to all and only those properties. This can be accounted for, I believe, without (i) introducing new objects, nor (ii)

relativising the distinction between essential and incidental properties to a description of the object. If I am right, this is sufficient to avoid Cleary's accusation of arbitrariness.

As in the previous instances, the *qua*-clause modifies the verb of the main sentence. In this case, however, the verb in question is not a verb of attention: what is qualified is not the way in which the observer approaches the object, but directly the relation between a property and its bearer. This is a fact of the matter that is independent of the way in which the object is approached by the various scientists who wish to study it, and can in fact provide a justification for the various scientific approaches (that Aristotle describes using the *qua*-clause to modify a verb of attention). Crucially, I think that this does not require the introduction of any distinct and specifically geometrical objects, not even ontologically dependent on sensible substances.

Before providing an account of the *qua*-predication that can enlighten its role, I wish to stress that the importance of the *qua*-predication for the *qua*-theory can be already seen in the *APo* passage quoted above. When discussing Eudoxos' theory of proportions at *APo* I.5, 74a17-25, Aristotle comments that 'Now, however, it is proved universally: what they suppose to hold of them universally does not hold of them as lines or as numbers but as this'.[59] This comment is presented as a justification of the new, unitary proof: the latter works because the properties that are supposed to hold universally of the various kinds of magnitudes do not in fact hold of each of them *qua* the specific object that they are (e.g. a line or a number), but *qua* 'this' (we would say '*qua* magnitude', but Aristotle lacks an adequate term)[60]. It is because this is the case, that considering the various objects *qua* magnitudes achieves the correct result.

Let us take another look at T5:

[59] νῦν δὲ καθόλου δείκνυται· οὐ γὰρ ᾗ γραμμαὶ ἢ ᾗ ἀριθμοὶ ὑπῆρχεν, ἀλλ' ᾗ τοδί, ὃ καθόλου ὑποτίθενται ὑπάρχειν.

[60] This is the reason why previous geometers did not offer a unitary demonstration: numbers, lengths, times and solids 'do not constitute a single named item'.

T5 (1078a5-10, Annas' translation)

πολλὰ δὲ συμβέβηκε καθ' αὑτὰ τοῖς πράγμασιν ᾗ ἕκαστον ὑπάρχει τῶν τοιούτων, ἐπεὶ καὶ ᾗ θῆλυ τὸ ζῷον καὶ ᾗ ἄρρεν, ἴδια πάθη ἔστιν (καίτοι οὐκ ἔστι τι θῆλυ οὐδ' ἄρρεν κεχωρισμένον τῶν ζῴων)· ὥστε καὶ ᾗ μήκη μόνον καὶ ᾗ ἐπίπεδα.

Many properties hold true of things in their own right as being, each of them, of a certain type – for instance there are attributes peculiar to animals as being male or as being female (yet there is no female or male separate from animals). So there are properties holding true of things in their own right merely as lengths or as planes.

Note that Aristotle uses the expression *hyparchein kath'auto*; I believe that it is fundamental to a proper understanding of the whole passage. For something that belongs *kath'auto* is not only necessary, but grounded in some way or other in the essence of the terms involved.

Aristotle discusses *kath'auto* predication in *APo* I.4, where he lists a number of senses in which one thing can be said to be *kath'auto*. The most relevant for our purposes are the first two:

Καθ' αὑτὰ δ' ὅσα ὑπάρχει τε ἐν τῷ τί ἐστιν, οἷον τριγώνῳ γραμμὴ καὶ γραμμῇ στιγμή (ἡ γὰρ οὐσία αὐτῶν ἐκ τούτων ἐστί, καὶ ἐν τῷ λόγῳ τῷ λέγοντι τί ἐστιν ἐνυπάρχει), καὶ ὅσοις τῶν ὑπαρχόντων αὐτοῖς αὐτὰ ἐν τῷ λόγῳ ἐνυπάρχουσι τῷ τί ἐστι δηλοῦντι, οἷον τὸ εὐθὺ ὑπάρχει γραμμῇ καὶ τὸ περιφερές, καὶ τὸ περιττὸν καὶ ἄρτιον ἀριθμῷ, καὶ τὸ πρῶτον καὶ σύνθετον, καὶ ἰσόπλευρον καὶ ἑτερόμηκες· καὶ πᾶσι τούτοις ἐνυπάρχουσιν ἐν τῷ λόγῳ τῷ τί ἐστι λέγοντι ἔνθα μὲν γραμμὴ ἔνθα δ' ἀριθμός.

> Something holds of an item in itself both if it holds of it in what it is – e.g. line of triangles and point of lines (their essence comes from these items, which inhere in the account which says what they are) – and also if what it holds of itself inheres in the account which shows what it is – e.g. straight holds of lines and so does curved, and odd and even of numbers, and also prime and composite and equilateral and oblong: in all these examples, there inheres in the account which says what they are in the former case line and in the latter number.
>
> Arist. *APo* I.4, 73a34-b3 (Barnes' transl.)

Barnes formalises them as follows:

- A holds of B *kath'auto*$_1$ =$_{def}$ A holds of B and A inheres in the definition of B;
- A holds of B *kath'auto*$_2$ =$_{def}$ A holds of B and B inheres in the definition of A.[61]

What is interesting in this analysis of *kath'auto* is that if A is predicated *kath'auto* of B, this can be either because A inheres in the definition of B, or because B inheres in the definition of A. The locution *kath'auto* indicates that there is a special, strict connection between A and B, that is grasped by and manifested in their definition. It is important to stress that both the subject and the property that belongs to it *kath'auto* are said to have definitions, and it is sufficient that one term appears in the definition of the other for them to be related in such a special way.

Notice that *kath'auto* predication is very strictly associated to the use of the *qua*-operator by Aristotle himself. In *APo* I.4, he remarks that

[61] The other relevant senses of *kath'auto* are the third one, which stresses the causal relation between the terms, and the *symbebekota kath'auta*, which are not discussed in I.4 but seem to include *kath'auto*$_2$ among them, for they are necessary features that follow from the essence of the subject, even if they are not included in it. See Barnes (1993: 114).

τὸ καθ'αὐτὸ δὲ καὶ ᾗ αὐτὸ ταὐτόν, οἷον καθ' αὑτὴν τῇ γραμμῇ ὑπάρχει στιγμὴ καὶ τὸ εὐθύ (καὶ γὰρ ᾗ γραμμή), καὶ τῷ τριγώνῳ ᾗ τρίγωνον δύο ὀρθαί (καὶ γὰρ καθ' αὐτὸ τὸ τρίγωνον δύο ὀρθαῖς ἴσον).

To hold of something in itself and to hold of it as such are the same thing: e.g. point and straight hold of lines in themselves – for they hold of them as lines; and two right angles hold of triangles as triangles – for triangles are in themselves equal to two right angles.

I.4, 73b28-32 (Barnes' translation)

In this passage, Aristotle argues that the *qua*-predication is identical to the *kath'auto* predication, whenever the *qua*-clause is used reflexively. Obviously the use of the *qua*-operator in M.3 is different, for Aristotle uses it precisely to switch the focus from the properties that belong to sensible objects *as sensible* to those that belong to them *as geometrical*. But once we know that the properties that belong to sensible objects *qua* sensible are the ones that belong to them *kath'auto*, we have a better understanding of what the *qua*-operator is supposed to identify: connections among features that are grounded in the essence of the object, or in the essence of the features themselves.

Saying that a property P belongs to S *kath'auto qua* S would be pleonastic, for *kath'auto* and *qua* S are equivalent. But if the *qua*-clause picks out a certain feature of S rather than S itself, then we can combine the two formulae to select a certain subset of S's properties – namely the ones that are connected *kath'auto* to the feature in question. The existence of this *kath'auto* connection is what grants that the relativisation of the essential/incidental distinction is not 'Quinean' or merely linguistic: it is grounded in real facts about the relations between properties.

Let us look at an example. If we say e.g. that being a plane figure belongs *kath'auto* to the face of a pyramidal tea-bag *qua* triangular, I am not attributing to the tea-bag-*qua*-

triangular the essential properties of being a plane figure. Rather, I am pointing out that there is a *kath'auto* connection between being triangular and being a plane figure: whatever is triangular, in virtue of being triangular is also a plane figure. It might well be incidental to the tea-bag as such that its face is triangular – but since it is, it necessarily is a plane figure. This is because properties do not come in isolation, but are connected in complex hierarchical structures that are determined by their essence and manifested in their definition. Obviously properties are not self-standing, but belong to an object. The point that I wish to make is that geometrical properties belong to sensible objects, and might belong incidentally to them,[62] but if they do, then the sensible object will also have a number of other features that are essentially linked to the geometrical property that it happens to have.[63]

What I wish to underline is that this does not require us to think that some *pathē* belong to other *pathē*; ultimately, all properties belong to the sensible substance in question, even if they belong to it *because* of some other features that it has. The *kath'auto* predication does not introduce another subject, different from the sensible substance, but merely establishes a hierarchy in the attribution of properties and features to it. To summarise, the idea is that if P belongs *kath'auto* to X *qua* Y, the ultimate subject of predication is X, the substance. P is a property that belongs to X in virtue of (i) X being Y and (ii) a *kath'auto* connection between Y and P, where the one appears in the definition of the other. This allows Aristotle to hold that geometry studies *kath'auto* properties without having to introduce additional objects in his ontology, nor relativise the distinction between essential and incidental features to a description of the object.

[62] Or might belong to it essentially, as it is essential to the celestial spheres that they are spheres – what matters is that it is irrelevant to geometry whether they belong to a sensible object essentially or incidentally.
[63] Cf. Halper (1989: 262).

The upshot of this analysis is that reading the *qua*-theory as a methodological statement rather than as a description of the specific mode of being of geometrical objects yields a radically anti-Platonist interpretation of Aristotle's philosophy of geometry, according to which geometry is made true by sensible objects and their geometrical features. There is no need to introduce any additional object to account for the possibility of geometry as a science. This interpretation hangs on the assumption that geometrical properties are in fact instantiated by sensible objects; the *qua*-clause only works if the object under consideration (the sensible object, in this case) does in fact have the properties we are interested in. The *qua*-clause does not introduce any falsity nor any idealisation; it merely describes the way in which the geometer approaches the study of sensible objects.

The assumption that if P belongs to X *qua* Y, then P belongs to X *simpliciter* raises a number of problems, that Aristotle does not address here. This is not surprising. M.3 is not supposed to provide a fully-fledged philosophy of geometry, but simply show that the refutation of Platonic geometrical objects does not immediately jeopardise the possibility of geometry as a science. However, if we aim to reconstruct his philosophy of geometry as a whole, we need to determine whether he can fix these problems.

Chapter 2. Precision Problem & Other Worries

According to the interpretation of the *qua*-theory that I have proposed, considering the *Xs qua Ys* does not introduce any new property, nor does it result in any idealisation. It simply consists in filtering out the properties that are not relevant to the issue at hand and restricting attention to the ones that are relevantly related to the target-property *Y*.

This seems to have an unwelcome immediate consequence: that all the properties that belong to the *Xs qua Ys* also belong to the *Xs* simpliciter. For example, if it is true to say that a pyramidal tea-bag, considered *qua* quantitative and continuous, has four perfectly equal sides, it must also be true to say (according to this interpretation) that the tea-bag *simpliciter* has four perfectly equal sides. This is because the property of having four perfectly equal sides is not introduced by the *qua*-operator, nor is it obtained by idealising the property of having four *approximately similar* sides and somehow transforming it into the property of having four *perfectly equal* sides. This seems counterintuitive, and difficult to accept. In general, sensible objects do not seem to have perfect geometrical properties: we do not often stumble upon perfectly straight lines, or perfectly bidimensional planes.

This issue is known in the literature as the Precision Problem, and it is just one of the problematic consequences of the assumption that *Xs* have all the properties that belong to *Xs qua Ys*. In this chapter, I will discuss it together with two other difficulties, that I call the Problem of Contradictory Properties and the Scarcity Problem. These three problems threaten the viability of the main claim that I have attributed to Aristotle in the previous chapter, namely the idea that the truth-makers of geometrical theorems and propositions simply are sensible objects, which bear geometrical properties.

Interestingly, Aristotle does not discuss these problems in detail – he does not even seem to be fully aware of all of them. As we will see in §1, he is aware of the Precision Problem and mentions it on a few occasions, but it is not clear whether he is simply reporting a challenge raised by the Platonists or whether he does endorse its main premise, namely that sensible bodies do not have perfect geometrical features. In any case, it is striking that he does not provide a solution to the problem. As for Contradictory Properties and Scarcity, he does not even acknowledge them, as if he were not conscious of them or did not find them worrying. I believe this apparent lack of interest can be attributed to the context of inquiry. *Metaphysics* M.3 should not be read as a full inquiry into the ontological status of geometrical objects, nor as a full account of Aristotle's philosophy of geometry. On the contrary, Aristotle is mostly concerned with disproving mathematical Platonism, and showing that this does not automatically result in jeopardising the possibility of geometry as a science. To achieve this minimal result, he does not need to take care of all the details nor close all the threads. However, if we want to extrapolate a full philosophy of geometry from Aristotle's remarks, we do need to show that these problems can be solved, and can be solved from an Aristotelian perspective.

Because of Aristotle's silence on these problems, this chapter is bound to be very speculative. I am going to use tools developed in the contemporary debate on the metaphysics of powers and potentialities to show that a solution can be found for all three problems. I think this solution is quite elegant, in that one and the same strategy allows me to block all three issues. I believe this shows that the solution goes straight to the core of the problem and to the common source of these issues, which leads me to believe that it might be able to account also for other problems stemming from the same assumption, if they were to arise. I wish to stress that I am not attributing this solution to Aristotle. I do not pretend to be able to reconstruct what he would have said about problems that he

did not discuss in the works we have. Moreover, even if contemporary metaphysics of powers is deeply influenced by Aristotle and is Aristotelian in nature, it is not completely obvious that its tools and claims can simply be transposed into Aristotle's thought, for they are developed in response to problems that are different from the ones that Aristotle had in mind. The purpose of this chapter is much more limited. I merely wish to show that a solution to these problems can be found, and that my reading of the *qua*-theory cannot be dismissed merely because it gives rise to these three problems.

§1. Precision Problem

The Precision Problem is the most obvious and thus most worrying of the three, for it immediately arises from the common sense and the wide-spread observation that physical objects do not appear to perfectly instantiate geometrical properties. The claim that they do seems to conflict with the phenomena, which need to be accounted for. So, for the proposed interpretation to be viable, I need to show that a solution can be offered, which is compatible with Aristotle's physical and metaphysical commitments.

The Precision Problem is one of the issues that divide the scholarship in two neat and incompatible fields. Some scholars believe that Aristotle agrees with Plato that sensible objects do not instantiate geometrical properties, and thus cannot be what geometry is about.[64] From this point of view, the perceived imprecision of sensible objects is not something to be explained away, but a phenomenon that Aristotle accepts as a datum. This results in understanding Aristotle's philosophy of geometry as a moderate form of anti-Platonism, according to which geometrical objects are in fact different and distinct from sensible bodies (as the Platonists believe), but do not exist *haplōs* and are not ontologically independent of sensible substances. On the contrary, other scholars believe

[64] For example, Mueller (1970), Hussey (1991).

that Aristotle is committed to the claim that geometrical properties are perfectly instantiated by sensible objects, and that the Precision Problem is a challenge that he needs to meet in order to defend his radically anti-Platonist philosophy of geometry.[65] I shall defend a version of the latter view.

Such a wide range of opinions on the issue is due to the fact that Aristotle mentions the Precision Problem in a few passages, but they are very ambiguous and difficult to subsume under a unitary view. In particular, it is not clear whether he reports the Precision Problem as a challenge levelled by a Platonist and that he needs to meet (thus confirming the radical anti-Platonist interpretation), or rather endorses its premise (thus confirming the moderate anti-Platonist interpretation).

A couple of passages are particularly worrying for my proposal, for they seem to voice the opinion that geometry cannot be about sensibles because they are not sufficiently precise. The first one is *Metaph*. B.2, 997b32-998a6, where Aristotle states that 'a hoop touches a straight edge not at a point' and so the claim that a line touches a circle at exactly one point cannot be true of sensible lines. The second is K.1, 1059b7-12, which claims that geometry cannot be about 'the things in this world; for none of these is the sort of thing which the mathematical sciences inquire into'. Both passages are controversial, and Lear (1982: section II) convincingly shows that Aristotle is merely voicing a Platonic worry, and he is not committed to endorsing it.[66] Moreover, these passages are balanced by a series of texts in which Aristotle seems to hold the opposite view. In *DA* III.8, 432a3–6 he claims that *ta ex aphaireseos* (which as I mentioned is the way in which Aristotle refers to mathematical objects outside of *Metaph.* M.3) are in sensible forms, because there is nothing apart and separate in existence from sensible

[65] For example, Lear (1982), Katz (2019; 2022).
[66] See also Katz (2019: §2.1).

spatial magnitudes. As we saw above, the same point in made in *Ph.* II.2, where Aristotle claims that sensible bodies have geometrical features, and that the natural scientist and the geometer study the same objects.

Thus, it can be shown that the radically anti-Platonist interpretation, according to which geometry is true of sensible objects, is at the very least not incompatible with the texts. For it to be a philosophically convincing interpretation, however, it is not sufficient to establish that Aristotle sees the Precision Problem as a problem– it is also necessary to show that he has the means to solve it.

The reason why it is so important for Aristotle to solve the Precision Problem is that he takes shape to be intrinsically perceptible. At *DA* II.6, 418a8–9 he claims that shape is *kath'auto* perceptible, in the same way as colour or heat. If the truth-makers of geometrical statements really are the shapes of sensible objects, and these shapes are perceptible, we should be able to perceive perfect geometrical objects. But this does not seem to be the case.[67]

A promising strategy to solve the Problem is put forward by Katz (2019). Her suggestion is that physical objects do indeed perfectly instantiate geometrical properties, but these properties can be fully appreciated only when we consider the sensible objects *qua* quantitative and continuous; *qua* physical objects, they appear imprecise. The core idea is that the precision of the geometrical properties that a physical object bears *qua* quantitative and continuous is hidden by its matter. According to Katz, this allows Aristotle to hold all of the following:

1. Sensible bodies instantiate perfect geometrical properties;

[67] See Berman (2017) for a reconstruction and formalization of the argument. He shows clearly that even if this passage were only relative to the shapes of physical bodies, the assumption that geometry studies features of sensible bodies is sufficient to get the argument started.

2. Sensible bodies considered *qua* sensible do not manifest precise geometrical shapes;
3. Sensible bodies considered *qua* quantitative and continuous do manifest precise geometrical shapes.

She argues that this is possible because when the geometer considers her objects not *qua* physical, but *qua* quantitative and continuous, she 'removes' matter and thus 'brings to light' the precise geometrical properties.[68] The core idea of this solution is, I think, very interesting and worth exploring. However, the way in which Katz cashes it out is not completely satisfying: how can 'removing matter' bring to light the precise geometrical properties?[69] I have argued that the *qua*-theory does not amount to a modification of the object: it does not deprive it of some of its features, but merely qualifies the way in which the geometer approaches the object. According to this interpretation, 'removing' features does not amount to stripping the object of its characteristics, but simply consists in keeping those features out of the scope of inquiry and ignoring them during the course of the investigation. How can such an operation 'bring to light' the precise geometrical properties? And how can those properties 'be hidden' by matter to begin with?

The problem seems to arise from the basic assumption that shape is *kath'auto* perceptible – that is to say, it is what we would now call an intrinsically powerful property.[70] Powers are irreducibly modal properties for some behaviour (the

[68] See *Metaph*. K.3, 1061a28-b3 for the claim that geometers 'remove' (the verb is *periaireō*) all properties of physical objects apart from those which belong to them *qua* quantitative and continuous.
[69] Katz (2022) expands on this solution, by appealing to the notion of kooky object. I will discuss her revised interpretation in the next chapter. For the moment, it is sufficient to point out that in understanding geometrical properties as kooky objects, she can claim that geometrical objects are not identical to sensible objects, even if they are in a sense (namely: accidentally) the same as them. This provides an easier way out of the problems that I discuss in this chapter. However, it also reverts back to a weaker form of anti-Platonism, in which geometrical objects are distinct from (although dependent on) sensible bodies.
[70] See Berman (2017: 90-91). Note, however, that Berman concludes that geometrical objects are *impure* powers, while I intend to show that we should think of the geometrical features that are not immediately perceptible as *masked* potentialities. Berman quickly dismisses this possibility on the basis of the assumption that geometers strip sensible objects of all their 'affections, products, and capacities'. This assumption, however, is not sufficiently backed up by his reference to *Metaph.* Z.3, 1029a13.

'manifestation' of the power). A standard example is fragility: e.g. a glass is fragile *iff* it is disposed to break, or it can break. This being the case, I believe that a detour into contemporary metaphysics of powers might be useful, to check whether it is in principle possible to solve the Precision Problem.[71] One of the distinctive characteristics of powers and potentialities is that they can be blocked and interfered with: although powers tend to manifest, their manifestation can be prevented. I believe that the case of these so-called 'masked potentialities' presents structural similarities to the case of geometrical properties which are blocked or 'hidden' by matter, and might thus be used as a model to find an adequate solution to the Precision Problem.

A classic example of masked potentiality is the case of a 'nuclear pile':

> [an uranium pile] which is above critical mass has a disposition to chain-react catastrophically. However, the pile has attached a fail-safe mechanism. Heat and radiation sensors detect large increases in radioactivity and allow boron moderating rods to penetrate the pile and by absorbing the radiation to prevent the catastrophic chain-reaction.
>
> Bird (1998: 229)

In the literature on dispositions, entities such as the boron rod and the fail-safe mechanisms are called 'masks': they "override" existing dispositions and prevent them from manifesting by interfering with them.[72] In this case, the mask is *external* to the object that bears the potentiality. The uranium pile is disposed to chain-react, but is prevented from doing so by the external circumstances in which it finds itself. Masks need not be external to the object, though. Think for instance of a soap bubble. Being spherical in

[71] I wish to thank Giacomo Giannini for his help with this section.
[72] See Lewis (1997); Bird (1998).

shape, it has the disposition to roll down an inclined plane. This disposition, however, is masked by another feature of the soap bubble: its potentiality to explode at the minimum touch. If we remove the latter, for example by freezing the soap bubble, then we remove the mask and the bubble becomes able to manifest its potentiality to roll down an inclined plane.

I believe that the case of the soap bubble is relevantly similar to the case of geometrical properties, and might be used as a model to explain how precise geometrical properties can be hidden by the material and sensible features of physical objects. Consider a squared box. According to the interpretation that I have laid out in Chapter 1, the property of being perfectly straight belongs *kath'auto* to the edge of a box *qua* line, because straight is predicated *kath'auto* of line. As a consequence, this perfectly precise property belongs to the box too. But the box also has a lot of other properties – being made of cardboard, which is rough, etc. These properties can mask the straightness of the edge of the box. There is a sense in which the straightness *is* there, even if unmanifested; but for it to be (perfectly) manifested, we would need to remove the sensible features of the box that mask it.

In the case of the soap bubble, the mask could be removed by operating a physical modification of the object: namely, by freezing it. Clearly, the same strategy cannot be adopted to manifest masked geometrical properties, for there is no physical procedure which could physically remove all sensible and material features of a sensible object. Geometrical properties are akin to necessarily unmanifested potentialities.[73] But this is

[73] On necessarily unmanifested potentialities, see Jenkins & Nolan (2012); Vetter & Busse (2022); Werner (2021). For similar cases involving intrinsically finkable dispositions, see Tugby (2016); Choi (2012). One might worry that, although the neo-Aristotelian notion of potentiality is obviously inspired by Aristotle's own notion of *dynamis*, it departs from it in some important ways; one of these is the admissibility of something like a necessary mask. Although the basic mechanism of masks and interferers is compatible with Aristotle's description of *dynameis*, introducing the notion of a necessarily masked potentiality might seem a stretch. After all, *Metaph.* Θ.4 seems to suggest that a potentiality which cannot be actualized is not really a potentiality. I discuss this in §4 below.

not a problem, since it is not necessary that geometrical properties are manifested in the physical world for them to act as truth-makers for geometrical propositions.

Support for this idea can be found in a cryptic remark towards the end of M.3, where Aristotle suggests that geometrical objects might exist *hylikōs* – in a 'matter-like' way:

> ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν· διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ' ὑλικῶς.
>
> This is why geometers speak correctly: they talk about existing things and they really do exist – for what exists does so in one of two senses, in actuality or as matter.
>
> *Metaph.* M.3, 1078a28-31 (Annas' translation)

The expression *hylikōs* is peculiar, and it is not immediately obvious what it means. However, the fact that it is contrasted to *entelecheiai* strongly suggests that it should be read as equivalent to 'potentially', which is the usual contrast with 'actually'.[74] It seems clear that Aristotle believes that statements that describe potentialities and potentially existing things should be regarded as true.

§2. Contradictory Properties

Thinking of geometrical properties under the model of unmanifested potentialities can also help to account for the second problem that I listed above: the Problem of

[74] Cf. e.g. *Metaph.* Z.13, 1039a4-7. This is argued by many commentators, e.g. Lear (1982: 180); Halper (1989: 268); White (1993: 172); Pettigrew (2009: 249). Interpreting *hylikōs* as meaning 'potentially' is especially important for Hussey's (1991) and Pettigrew's (2009) interpretation. Note however that both Hussey and Pettigrew take this remark as meaning that geometricals are a special kind of objects which enjoy a peculiar mode of existence – namely, they can only exist potentially. On the contrary, I argued that geometrical propositions are true in virtue of sensible objects, which do instantiate geometrical properties. Some of these properties might not be fully manifested (and I will argue in the next sections of this chapter that some might even be necessarily masked), but they still are properties that actual sensible objects possess; some other geometrical properties might simply be actual. I will come back to this in Chapter 3, after discussing the doctrine of separation.

Contradictory Properties. To see how this problem arises, let's take a look at the following case, presented in *Metaphysics* M.3:

> ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ ἄνθρωπος· ὁ δ' ἔθετο ἓν ἀδιαίρετον, εἶτ' ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ' ᾗ ἄνθρωπος οὔθ' ᾗ ἀδιαίρετος ἀλλ' ᾗ στερεόν.
>
> A man is one and indivisible as a man, and the arithmetician posits him as one indivisible, and then studies what holds of a man as indivisible; the geometer, on the other hand, studies him neither as man nor as indivisible, but as a solid.
>
> *Metaph*. M.3, 1078a23-30 (Annas' transl.)

This passage is notoriously difficult and ridden with textual and interpretative problems; I will discuss it in more detail in the next chapter. What interests me at this stage is only the setting up of the example. In illustrating the case, Aristotle clearly states that the geometer studies a man *qua* solid, while the arithmetician studies him *qua* 'one indivisible' – that is to say, *qua* unit. The problem arises because the man is said to be divisible *qua* solid. According to my radically anti-Platonist interpretation, the *qua*-operator does nothing more than filter out the irrelevant properties. The properties that are selected as worthy of attention were present in the physical object since the beginning and are not introduced by the *qua*-operator. If this is the case, however, it means that the original physical object (in this case the man) was the bearer of a pair of contradictory properties: he should be both divisible and indivisible. How is this possible?

Interestingly, the attribution of masked potentialities raises a similar problem. Let us go back to the case of the nuclear pile that I introduced above. If the uranium pile is surrounded by the fail-safe mechanism, does it have the potentiality to chain-react or not?

Bird (1998) argues that the answer to this question will depend upon which bearer of the potentiality we are talking about. He invites us to carefully distinguish between

a) The uranium pile alone;

b) The uranium pile plus the fail-safe mechanism.

He argues that (a) the uranium pile alone retains the disposition to chain-react at all times; but (b) the system composed of the uranium and the fail-safe mechanism does not possess such disposition.[75]

However appealing, Bird's solution is not yet sufficient to solve the problem. Its main shortcoming is that it does not account for what Vetter calls 'Type-2 joint potentialities',[76] that is, joint potentialities had by a plurality of entities whose manifestation concern only some of the bearers of the potentiality. Not all the manifestations of potentialities concern the totality of the bearer of the potentiality. For instance, Vetter recognises that (b) the system composed by the uranium pile, the boron rod, and the fail-safe mechanism does have the disposition to be such that the *whole system does not catastrophically chain react.* However, the same system has also a more circumscribed, precise potentiality: it has the potentiality to be such that the *uranium pile does not catastrophically chain react*. That is to say, the system described in (b) has not only a potentiality that concerns all its parts; it also has certain potentialities that concern only some of its parts.

Type-2 joint potentialities show that the problem cannot be solved simply by distinguishing different bearers. Remember: (a) the uranium pile on its own retains the

[75] Note that this is similar in structure to Katz' (2022) solution to the Precision Problem, who argues that the geometrical features should be understood as kooky objects which are merely accidentally the same as their 'parent' sensible objects: she believes that the sensible object *qua* sensible does not have precise geometrical features, while the sensible object *qua* quantitative and continuous does have them.
[76] Vetter (2015: 113).

disposition to chain react. However, (b) the system composed by the uranium pile + failsafe mechanism has the type-2 joint potentiality to be *such that the uranium pile does not chain react,* which means that the system also confers on (a) the uranium pile the disposition not to chain react. Thus, the uranium pile seems to retain two incompatible dispositions: the disposition to chain react, and the disposition not to chain-react.

So, the question is: what disposition does the uranium pile have? More generally: what disposition should we ascribe to the bearer of a *masked potentiality?* Intuitions here diverge and are not clear cut. On the one hand, it seems clear that by surrounding the uranium pile with a fail-safe mechanism, nothing has changed in the intrinsic status of the pile itself – and so, its dispositions should remain intact. On the other hand, it is hard to deny that a uranium pile surrounded by a fail-safe mechanism *is not disposed* to catastrophically chain-react – otherwise, the policy of building nuclear plants would be unequivocally criminal.

What disposition we will end up attributing to the uranium pile (and, relatedly, what modal truth will hold of it)[77] will depend on what kind of background information we will let in, even when we consider only the uranium pile itself. If we consider only the nuclear pile and we attribute a disposition to it while being insensitive to all of its surroundings, then we will be inclined to ascribe to it the disposition to chain-react. If, on the other hand, when considering the nuclear pile we attribute to it a disposition while being sensitive to its surroundings, then if it is placed within a fail-safe mechanism, we will ascribe to it the disposition not to chain-react. Note that 'surroundings' is deliberately vague, and it should be, in keeping with the pragmatics of disposition ascription. If, for

[77] In moving from talk of dispositions to talk of modality I am here assuming that there is a close link between the two along the lines of contemporary Modal Dispositionalism, and that Aristotle recognises such link. Dispositionalism is the view that modal claims of the form "possibly p" are true in virtue of the fact that some actual entity instantiates the relevant disposition or potentiality (see Vetter 2015 for the canonical exposition of the view).

instance, the fail-safe mechanism is itself part of a wider system (the nuclear plant) that is negligently maintained, *that wider system* might be disposed to be such that the uranium pile chain reacts, after all. Ultimately, the only way to warrant perfect stability in our disposition ascriptions while being sensitive to the background context is to consider *all* of the context — that is to say, to attribute dispositions *all things considered.*[78]

Let us now go back to our original Aristotelian problem. The situation is not exactly identical to the nuclear pile case, but it is structurally similar. While in the nuclear pile case the issue is whether the wider system leads us to attribute contradictory dispositions to the pile, which is a part of it, in the Aristotelian example of the man the worry is that *the man* (which is analogous to the system as a whole) ends up having contradictory properties.

We know that being divisible belongs *kath'auto* to the man *qua* solid, because a solid is a bounded three-dimensional continuous extension, and infinite divisibility belongs *kath'auto* to continuity. This is a modal property that the man has – a *dynamis* – and that is apt to be studied geometrically. However, the manifestation of this property is blocked or masked by *other* properties that belong to the man: in particular, his special kind of unity that does not allow him to be divided without ceasing to be a man. Just as the uranium pile's disposition to chain react was masked by the fail-safe mechanism, so the potentiality of the man to be divided is masked by its being a unity. The divisibility *is* present, but it cannot be acted upon without destroying the man: the essential unity of the man makes it such that the divisibility cannot be acted upon.[79]

[78] Cf. the notions of strong and weak possibility in Kimpton-Nye (2018).
[79] Note that the remark about geometrical objects existing *hylikōs* appears at the very end of the discussion of this example, confirming that divisibility can be understood as a masked potentiality of the man. The fact that divisibility is a property that belongs to the man *qua* material body might also explain why Aristotle uses the expression *hylikōs* rather than the more straightforward *dynamei*.

If we are interested in the man, then, and we evaluate the situation *all things considered*, we will say that the man is ultimately *not* divisible, even if the property of divisibility does belong to him as a masked potentiality. However, if we apply the property-filter and restrict attention to the features that are geometrically relevant, we will disregard the catastrophic consequences that the division would have on the man, and conclude that the man, considered *qua* geometric solid, *is* indeed divisible. In the latter case, the *qua*-operator does not *add* a feature, but simply restricts the number of factors that must be taken into account when evaluating the situation.[80]

One might wonder why it is divisibility that is masked, and thus why the man is all things considered *in*divisible, rather than vice versa. Here it is important to remember that properties are arranged in a hierarchy, as I showed in the previous chapter. Indivisibility 'trumps' divisibility and masks it, because it is a property that belongs *kath'auto* to the man *qua* man, while divisibility belongs to it *qua* continuous solid. It follows from the essential definition of man that he is one and indivisible, while divisibility only belongs to him in virtue of his having *other* features – in particular, a material, continuous body. Note that this asymmetry is something that Aristotle is aware of, and that is apparent in the way in which he sets up the case study: he starts by pointing out that the man is one and indivisible, and that arithmetic studies him as such; geometry, on the contrary, does not study the man *qua* man nor *qua* indivisible, but rather *qua* solid, continuous body.[81]

Notice that this model does not require the introduction of any special *qua*-object or incomplete object that co-exists with the physical object under consideration. We do not

[80] Interpreted along these lines, this case has an important upshot and is an example of a more general issue, that I will discuss in the next section: in Aristotle's system, there is not a complete correspondence between physical and geometrical modality, so that some things that might be geometrically possible turn out to be physically impossible. This could be accounted for by saying that geometrical possibility is not concerned with a *strong* possibility (i.e. possibility all things considered), because it deliberately screens out certain *physical features* that play a key role in shaping what is physically possible.
[81] See also Netz (2006: 28).

need to postulate the existence of a man-*qua*-solid and a man-*qua*-man as two distinct but co-located subjects. The model only requires the existence of one object, which has properties that belong to it *qua* physical and properties that belong to it *qua* geometrical. Practitioners of different sciences can consider the object *qua* physical or *qua* geometrical, thus restricting their focus to one set of properties or another. When nothing is specified, it is assumed that one considers the man *qua* man, picking the most general qualification.[82]

§3. Scarcity Problem

Considering geometrical features as akin to masked potentialities of physical objects thus provides the radical anti-Platonist with a way out of the Precision Problem, and allows us to understand how one and the same object can be the bearer of apparently contradictory properties. However, in order to secure the claim that the truth-makers of geometry simply are sensible objects which instantiate geometrical properties, further problems must still be resolved – problems which arise from the volatile and transitory character of all physical objects. One of these is the so-called Scarcity Problem.

This problem stems from the fact that the Aristotelian universe is fundamentally *finite*, together with the assumption that geometrical statements are eternally and necessarily true. Due to the finiteness of the *kosmos*, it might be the case that at a given time, there are not enough objects to make all the theorems of geometry true.

The problem has two distinct versions:

[82] One might wonder why some properties mask others, and not *vice versa*. I believe that the essential properties of each substance have some sort of 'priority' over the others. This sort of hierarchy can be tracked by the *kath'auto* predications.

Quantitative Scarcity Problem: there might not be *enough* items of a certain kind, or items not *big enough* to verify all geometrical propositions. This is because physical objects cannot be bigger than the diameter of the *kosmos* (there is an upper limit to the size of physical objects, although there is no lower limit).

Qualitative Scarcity Problem: there might not be *enough kinds* of physical objects to act as truth-makers for all true geometrical statements. For example, there might not be any physical object that has one thousand sides; however, "the internal angle of a chiliagon measures 179.64°" seems to be a geometrical truth.

A possible way out of these problems is once again suggested by Katz (2019).[83] The central idea is that geometrical theorems are made true by physical objects, but not only the ones that exist actually; also the physical objects that exist *potentially* should be considered. Since matter is continuous and thus infinitely divisible, it can act as an infinite 'reservoir of possibilities': it can be divided in any way whatsoever, so that we can be sure that any possible geometrical objects, of any kind, exist potentially in it. This is sufficient to guarantee the availability of objects to make all geometrical theorems true.[84]

It should be noted that only some of the figures that are not actually present in the physical world are potentially present in it: namely, the possible ones. Thus, while it might be the case that both the chiliagon and the squared circle are not actually present in the physical world at a given time, the two items are not on a par. The chiliagon is potentially present in the *kosmos* while the squared circle is not, because the former is a

[83] A similar strategy might be attributed to Lear (1982), although he does not discuss it so explicitly. See Hussey (1991: 115, fn. 19) for a reading of Lear's strategy along these lines.
[84] Katz (2019: 501-3).

possible figure and the latter is not.[85] The issue is: what kind of modality is at play here? Physical or geometrical?

Recall that the radical anti-Platonist denies that geometry is true in virtue of the existence of additional objects beyond the physical ones. So it seems that when the radical anti-Platonist suggests extending the domain of objects under consideration to include the ones that are merely potentially present in the *kosmos*, she must be thinking of objects that are *physically* possible. On the other hand, since the goal is to secure the existence of truth-makers for all *geometrical* statements, one might think that what needs to be added to the set of actually existing physical objects are all the objects that are *geometrically* possible. Thus, it seems that this strategy to block the Scarcity Problem can only be successful if physical and geometrical modality are extensionally equivalent.[86]

Is it legitimate to assume that this is the case? If we look at the specific characteristics and practice of Greek geometry, we can find a definite, clear-cut and independent criterion for geometrical possibility: namely, constructibility according to the rules of

[85] Aristotle draws the distinction at *Metaph.* Θ.4, 1047b12-14. See Katz (2019: 503).

[86] Here I am adapting Katz' strategy to fit my interpretation. In her own account the difference between invoking geometrical or physical modality is blurred by the fact that she believes that geometrical objects are geometrical properties *of geometrical diagrams*, not of any sensible body. Although I agree that diagrams can instantiate perfect geometrical properties, I do not think that we should restrict the scope of geometry to these specific sensible objects. It seems clear to me that although diagrams are essential to the practice of geometry and to the validity of geometrical proofs, they should not be thought as the *object* of the proof. This is because, as I mentioned in the previous chapter, only a subset of a diagram's features can be relied upon in the course of a proof. Diagrams do not need to be metrically correct, nor do the lines and circles present in them need to be straight or round. Moreover, the fact that also proofs by the absurd are accompanied by diagrams shows quite clearly that diagrams are not supposed to be the object of the proof, but at best a representation of it, which might be more or less accurate. That Aristotle himself thinks of diagrams as representations of the geometrical configuration under examination, and not as the proper object of the proof, is suggested by *APo* I.10, 77a1 ff.

Euclidean geometry.[87] Greek geometry is constructive in character:[88] starting from a small number of basic figures (lines and circles), complex figures are constructed through the repeated application of some moves, licensed by the *Elements'* first three postulates.[89] The geometrical tradition of 'problems' testifies that constructions can be exhibited to prove the admissibility of certain figures.[90] So, it seems reasonable to assume that geometrical possibility can be cashed out in terms of geometrical constructibility.

If this is the case, however, it seems that adding all and only the objects that are *physically* possible might not be sufficient to solve the Scarcity Problem. Since we are

[87] Usually referred to with the slogan "constructibility with straightedge and compass". I will avoid using this label, though; since I am discussing the difference between physical and geometrical modality, it would be misleading to express the geometrical construction rules in physical terms. When I speak of 'constructibility', I mean to refer to the moves that are licenced by the Axioms and Postulates in the *Elements*, that might be understood as moves in a system akin to a natural deduction (see Macbeth 2010) and that are *represented* by the corresponding modification of the diagram. This means that a line longer than the diameter of the *kosmos* counts as constructible, even if it might be physically impossible to draw it using a straightedge. In the same way, I take it that the Euclidean rules of construction license the divisibility of a line beyond what can be discerned by the human eye, and so beyond the level of accuracy that can be obtained in a diagram. More on this below.

[88] See Heath (1908: 124-9); Knorr (1978); Mueller (1969); Macbeth (2010); Beere & Morison (ms.). The constructive character of Greek geometry is particularly evident in the problems, where the goal is explicitly to construct a certain figure. But also the theorems heavily rely on the possibility of constructing auxiliary figures.

[89] The first three postulates are rules licensing basic constructions, while the fourth and the fifth are assertions stating properties of particular geometric objects (see Procl. *In Euc*. 182–4 and 188–93). As Mueller (1969: 290) points out, 'three of the five postulates are not even capable of being true […] they are what might be called licenses to perform certain operations'. The postulates are one of the three sets of unproven assumptions that appear at the beginning of book I of Euclid's *Elements*; the others are definitions and common notions. It is not clear whether pre-Euclidean *Elements* contained the same three sets of assumptions. Aristotle presents a tripartite division of the principles of deductive sciences: 'one is what is proved, the conclusion (this is what applies to some genus per se), one are the axioms (the axioms are from which), the third is the underlying genus, the properties and the per se attributes of which the proof makes clear' (*APo*. I.7, 75a39–b2). Since Proclus, interpreters have tried to map the Aristotelian tripartition onto the Euclidean one, but it is unclear whether this is legitimate; see Acerbi (2013) for a refutation of this identification. For my purposes it is not important whether the three sets of assumptions were codified precisely in this way in pre-Euclidean geometry; it is sufficient that there were precise rules of constructions that geometers had to respect. This much can be safely extrapolated from *Top.* I.1, 101a13–17, where Aristotle describes the 'pseudographer', i.e. someone who makes false proofs in geometry. As Acerbi (2013: 682-4) shows, in presenting the pseudographer as someone who does not draw the diagrams 'as they should be drawn', Aristotle is referring to mathematical principles that guide the construction of figures (that he calls *lemmata*). Acerbi concludes that 'the *Topica* passage attests Aristotle's recognition that there are such 'constructive assumptions', and it is of course not really important whether these λήμματα were explicitly stated as such at the beginning of some pre-Euclidean exposition of 'Elements' or whether they simply featured as fixed formulaic expressions characterizing certain operations on geometrical objects that were only enucleated and given the dignity of ἀρχαί or the status of basic constructive steps in the Elements' (684).

[90] The *Elements* contain two kinds of geometrical propositions: problems and theorems. A problem requires to produce a certain figure, while a theorem requires the geometer to prove that a certain property holds of a given figure. See e.g. Heath (1908: 124-9); Beere&Morison (manuscript).

giving an interpretation of Aristotle, physical possibility should be evaluated on the basis of Aristotle's own physical theories. However, the set of objects that turn out to be physically possible on the basis of Aristotle's physics is not equivalent to the set of objects that are validated as geometrically possible by the Euclidean construction rules. On the contrary, adding all and only the physically possible objects to the domain of objects under consideration seems to under-generate, and thus not be sufficient to satisfactorily solve the Scarcity Problem.

To see why this is the case, it is sufficient to notice that the Aristotelian *kosmos* has a definite length, that no physical object can exceed: there can be nothing bigger than the universe. *A fortiori*, it is physically impossible for a line to be longer than the diameter of the *kosmos*. However, if we take geometrical rules of construction as a criterion for possibility, we should conclude that it is *geometrically possible* to extend a line beyond the length of the diameter of the *kosmos*. There is nothing in Euclid's construction rules that prevents the admissibility of such a figure. On the contrary, this arbitrary restriction seems to be at odds with Euclid's second postulate.[91,92]

[91] It is interesting to note that Aristotle addresses this worry explicitly, in his discussion of the infinite: he provides an argument to the effect that his rejection of the infinite by addition does not falsify geometry. I will discuss this argument in more detail in Chapter 4; for the moment, it is sufficient to remark that Aristotle acknowledges that there is a discrepancy between physical and geometrical possibility (in particular, that there are certain figures that are geometrically possible but not physically possible), and sets out to accommodate this discrepancy in his system, rather than denying it.

[92] One might think that the strategy over-generates, as well. There are configurations that are physically possible, but which cannot be constructed using the rules licensed by Euclid's *Elements.* One example that would have been known to Aristotle is the doubling of the cube, also known as the Delian problem: given a cube, it is surely physically possible that a second cube exists, whose volume is double that of the first. However, this problem is insolvable with the tools of Greek geometry, as was proved by Pierre Wantzel in 1837. Establishing that a solution cannot be found 'by straightedge and compass' is equivalent to the statement that $\sqrt[3]{2}$ is not a constructible number. This being the case, such a cube should be considered geometrically impossible. However, this seems an unwelcome result, that Aristotle might not be willing to accept. The very fact that the Delian problem became so famous, and that attempts to solve it were tried until 1837, shows that geometers tended to look at its solution as *geometrically* possible, and simply difficult to find. Thus one might think that the unresolvability of the Delian problem simply shows the limits of using straightedge and compass as epistemic criterion of geometrical possibility. Similar considerations were probably entertained by those geometers who used non-constructive methods to solve the problem: for example, a solution can be found using a *neusis* construction (sliding marked rulers), and Menaechmus proposed a solution that involves the intersection of two conic curves. These non-constructive methods were debated, but used much more frequently than it is usually imagined; see e.g. Fowler (1999) and Knorr (1978). Aristotle's position face to these non-constructive methods is unclear. He does mention

This discrepancy between physical and geometrical possibility seems to suggest that the Scarcity Problem cannot be solved by considering all and only the objects that are *physically possible.* Instead, it seems that to have enough objects to make all geometrical theorems true, one should add to the actual physical objects also all the objects that are *geometrically possible*.

Is this the end of the radical anti-Platonist interpretation of Aristotle's philosophy of geometry? I do not think so. Using the analogy with powers and potentialities that I have introduced above, I believe that it is possible to modify slightly the strategy proposed by Katz, in such a way that it gives the right outcome – that is, that it provides the radical anti-Platonist with all and only the objects that are *geometrically* possible, without postulating any additional kind of objects, distinct from the physical ones.

The core idea is quite simple. Rather than taking the 'missing objects' to be provided by the (categorical) geometrical features of potential physical objects, I suggest that they are the *potential* geometrical features of *actual* physical objects. Put simply, they are unmanifested geometrical potentialities of actual physical objects.

But what does this mean? To see where this hypothesis comes from, let us go back to the notion of geometrical possibility as constructibility according to the Euclidean rules. According to this picture, geometrical possibility is determined by two things: (i) a small number of basic figures, and (ii) construction rules. The basic figures of Euclidean geometry are lines and circles; interpreting geometrical possibility as constructibility

---

neusis constructions at *APo* 76a31-77a4, which might be taken to imply that pre-Euclidean mathematicians were more liberal in their methods than Euclid himself. As I mentioned in footnote 89, it is not clear that pre-Euclidean geometers accepted exactly the rules of construction that are licensed by the first three postulates, but my argument does not hang on this; it is sufficient that there are some codified rules that are accepted by the scientific community. From a purely philosophical point of view, it is plausible to think that if geometry must be true of sensible bodies, geometrical possibility must be at least as strong as physical possibility; Aristotle might have simply assumed that the Delian problem (and other similar problems that we now know are unsolvable in Euclidean geometry) is solvable with the tools of Greek geometry, even if the solution had not been found yet. This seems to be confirmed by Aristotle's comments about the squaring of the circle at *Cat*. 7, 7b27-33: 'take, for example, the squaring of the circle, supposing it to be knowable; knowledge of it does not yet exist but the knowable itself exists' (Ackrill's translation).

means that all and only the figures that can be constructed out of these basic figures using the admissible rules of construction are geometrically possible.

I believe that the rules of construction can be thought as potentialities of the basic elements. For example, a given line has the potentiality to be extended or divided, as well as combined with other lines so as to construct a more complex geometrical configuration. Since lines, circles, and all the basic geometrical objects are (under the interpretation that I have put forward) nothing else than features of physical objects (in the case of lower-dimensional items, limits of physical objects), it follows that complex geometrical configurations can be seen as the manifestation of geometrical potentialities of sensible bodies. If we allow that these potentialities can act as truth-makers for geometrical sentences even if they are not manifested, then the Scarcity Problem is solved. It is sufficient that a small number of basic geometrical features to be actually instantiated in the sensible world, and this is something that Aristotle can account for.

According to the radically anti-Platonist interpretation of Aristotle that I have presented, the truth-makers of geometrical statements are actual physical objects, which can be studied and considered not *qua* physical, but *qua* quantitative and continuous. The *qua*-operator does not add anything that was not present in the object since the beginning. This means that *actual physical objects* have the potentiality to be manipulated and modified according to the rules of Euclidean geometry. If completely manifested, these potentialities would give rise to all possible geometrical objects. It is physically impossible for all these potentialities to be completely manifested (because of masks such as (a) the finitude of the *kosmos,* as well as (b) the interference of other sensible properties of the object in question, and (c) the tendency of physical objects to decay in accordance with the laws of nature), but this does not mean that the potentialities are not there:

potentialities exist even if they are not manifested, and this is sufficient to ground the truth of all geometrical theorems.

§4. Necessarily Masked Potentialities

The strategy that I have proposed would allow Aristotle to fix the Precision Problem, the Problem of Contradictory Properties, and the Scarcity Problem in one go. This strategy, however, hinges on the assumption that sensible bodies might have some potentialities that are necessarily masked. Is this assumption compatible with Aristotle's own theory of *dynamis*?

The problem caused by necessarily masked potentialities is not merely that they happen never to be enacted; it is an essential feature of potentialities that they do not need to be actualised, and surely Aristotle does not wish to say that all possibilities will sooner or later come to be. This is clear enough from the case of the coat that can but never will be cut, in *De Interpretatione* 9, 18b26-19a22.[93] Rather, the problem of necessarily masked potentialities is that they *cannot* be actualised: is it still legitimate to call them potentialities?

Addressing this question in such general form goes beyond the scope of this work. In order to understand whether Aristotle might have accepted the strategy that I propose, we do not need to establish whether he could buy the idea of necessarily masked potentialities *tout court*. We only need to discuss a very special case, namely the case of a *geometrical* potentiality that is necessarily *physically* masked. It is evident that the problem arises from a mismatch between what is physically possible and what is geometrically possible. As I have argued above, this is a mismatch that Aristotle is

[93] Some interpreters, in particular Hintikka, argue that Aristotle holds a version of the Principle of Plenitude – which, in its crudest form, states that each possibility is realised at some moment in time. Even Hintikka, however, believes that the Principle is only meant to cover only general possibilities, or possibilities for kinds of states of affairs. See Hintikka (1973: 100 ff., 162, 171 ff.). For a criticism, see Waterlow (1982).

committed to: he does recognise that some things are geometrically possible but physically impossible, and he seems ready to accept them as genuine possibilities.[94] For example, he is happy to claim that any continuous magnitude is infinitely divisible, even if it is clear that after a while it becomes physically impossible to keep dividing it, for once the distance between two points is small enough, it becomes impossible to physically distinguish them. The ground for arguing that it is possible to keep dividing is that the magnitude is continuous, and so homogeneous: no point is privileged or 'more divisible' than any other, so that if it is possible to divide at a certain point, it must be possible to divide at any other point.[95]

In cases like this, it is not obvious how Aristotle's test for possibility should be applied. He writes:

> ἔστι δὲ δυνατὸν τοῦτο ᾧ ἐὰν ὑπάρξῃ ἡ ἐνέργεια οὗ λέγεται ἔχειν τὴν δύναμιν, οὐθὲν ἔσται ἀδύνατον.
>
> And this is what is possible – that for which, if the actuality of which it is said to have the capacity obtains, there will be nothing impossible.

[94] I am not suggesting that Aristotle consciously distinguishes between physical and geometrical modality, but that he is committed to such a distinction, which is bound to appear in one way or the other in all reconstructions of his philosophy of geometry. The difference between the two modalities is at the root of the difficulties that I will address in the second part of the thesis, concerned with the applicability of geometry (see in particular Chapter 4). I believe that a good reconstruction of Aristotle's philosophy of geometry should not aim to erase of deflate these difficulties, but bring them to light and aim to solve them. So I do not think that it is *prima facie* problematic if my reconstruction presents some geometrical features as necessarily masked potentialities: the paradoxicality of this formulation might help to cast light on the limits of Aristotle's theory.

[95] The connection between continuity and homogeneity is already present in Parmenides, who presents this as a reason for the indivisibility of being (B 8.5–6, 8.22–24: '[what is] is now, all together, one and continuous since it is all alike; nor is there any more or less of it in one place which might prevent *suneches*) ... Nor is it divisible, it from holding together, but all is full of what is'). The Atomists accepted the connection; from the assumption that a continuous magnitude is not more divisible at a point than at any other, they concluded that allowing for the divisibility of a continuous magnitude would lead to contradiction. Aristotle turned this argument on its head, and concluded that since a continuous magnitude is not more divisible at a point than at any other, it must be everywhere divisible – and thus infinitely divisible. See *GC* I.2, 317a1-17; for a discussion, see Sattler (2020: ch. 7).

Metaph. Θ.3, 1047a24-26 (Makin's translation)

This test provides a way of establishing whether an outcome is *possible*: assume that the outcome obtains, and see whether anything impossible follows from that assumption.[96] The test can be used as a criterion of possibility only once some preconditions are in place. First, it is necessary that there is a set of manifest impossibilities, or evident contradictions, that can immediately be recognised as impossible (so as to be sure that the test stops).[97] Second, it must be possible to operate some additional adjustments; if this were not the case, all falsehoods would come out as impossibilities. Suppose that we want to test whether *p* is possible, and it is currently the case that ¬*p*. If we cannot make any additional adjustment, simply assuming the truth of *p* would result in a contradiction. Precisely which additional adjustments we are allowed to make depends on which type of possibility is being tested for.[98]

Recall that in the interpretation that I am proposing, geometrical potentialities are *geometrical* potentialities *of physical objects.* Thus it is unclear according to which modality they should be evaluated, and accordingly which additional adjustments we are allowed to make. The case of continuity suggests that when evaluating geometrical potentialities we are allowed to make additional adjustments that go beyond what is physically possible. For example, it seems that we are allowed to build into the setting of the thought experiment some counter-factual assumptions, such as supposing that we are able to distinguish points that we are in fact physically unable to differentiate. If this assumption is built into the setting of the test, then nothing impossible results from

[96] Cf. also *APr* I.13, 32a18–20; *Ph.* VII.1, 243a30-1. Note that this claim is not a reductive analysis of the concept of possibility, for it relies on the modal notion of impossibility. Rather, it is presented as a criterion of possibility: a way of determining whether something is possible or not. See Beere (2009: 119-120).
[97] Beere (2009: 120 ff.).
[98] Makin (2006: 77).

supposing the magnitude to have been divided. Accordingly, we will be allowed to say that the physical magnitude does have the potentiality to be divided beyond what could ever be achieved by any physical agent, even if this potentiality cannot as a matter of fact be actualised.

Thus, the structural and geometrical features of sensible bodies should be taken seriously as real features even if their (full) manifestation is physically impossible. This does not mean that anything goes; the counterfactual additional adjustments that are required to validate geometrical potentialities do not completely subvert Aristotelian physics. Note that when it comes to the Scarcity Problem, the configurations that are physically impossible (but that we need to allow for, if we want all of Euclidean geometry to come out as true) are generated by the repeated application of some basic rules that are in fact applicable up to a certain dimension or degree of complexity. This is enough to prove that the simpler items do have the potentiality to be so combined or modified; the fact that such potentiality might not be in fact actualisable in a given condition is not sufficient to invalidate the potentiality (in the same way as the continuous magnitude can be said to be divisible even beyond any physical distinguishability).

The full actualisation of these potentialities requires the removal of the masks that block them – namely, the sensible and material features of physical objects, as well as the physical limitations due to the specific size of the *kosmos*. I have argued that considering the sensible objects not *qua* sensible, but *qua* quantitative or *qua* continuous solids, allows the geometer to bracket the masks and focus on the potentialities. This does not amount to *actualising* such potentialities. According to the interpretation of the *qua*-theory that I have outlined, the *qua*-clause modifies the way in which geometers use and approach sensible objects, not the sensible objects themselves. The fact that the geometer considers

an object in one way or the other does not change the nature or state of the object in any way, and surely cannot bring into actuality its necessarily masked potentialities.

One might worry about the epistemological implications of such a stance. If some geometrical potentialities are necessarily masked and cannot be actualised, how can we get to know anything about them? I believe that this is one of the reasons that led Aristotle to add another part to his theory, that is laid out in the second half of M.3: I call it the doctrine of separation, and I will discuss it in detail in the next chapter.

To anticipate slightly, the core idea is that geometers are allowed to 'posit as separate' the features that they are interested in. This is a mental operation, which results in the creation of a mental object that has all and only the properties that geometers wish to study. For our purposes in this chapter, the important thing is that while the *qua*-operation only instructs geometers to *ignore* the masks, separation allows them to fictionally *remove* them: while the *qua*-theory only allows geometers to consider their objects not *qua* sensible, once they are separated it becomes possible to consider them *qua* not sensible. This creates the adequate conditions for the manifestation (in thought!) of those geometrical potentialities that are necessarily masked in the sensible world. In their imagination, geometers can thus manifest those potentialities, and construct objects that defy the laws of nature. This also allows them to distinguish genuine potentialities from geometrical impossibilities.

Chapter 3. Separation

In the previous chapters I have argued that Aristotle's *qua*-theory in *Metaphysics* M.3 should not be read as providing an ontological account of mathematical objects. Far from identifying some special geometrical objects, distinct from sensible bodies but dependent on such bodies for their existence, the *qua*-theory is meant to show that there is no need to invoke any specifically geometrical object to secure the possibility of geometry as a science. Geometrical statements are true in virtue of sensible bodies and their geometrical properties, while the identity and specificity of geometry as a science is granted by its peculiar methodology.

In this chapter I am going to discuss the final part of M.3's overview of the objects of geometry. After expounding the *qua*-theory, Aristotle dedicates a few lines to the discussion of *akribeia* (precision, 1078a8-13) and of subordinate sciences such as harmonics, optics and mechanics (1078a13-18). As Netz (2006) suggests, these two sections can be considered almost as footnotes to the *qua*-theory, tying up some loose ends and pointing out the consequences of the theory, as well as its relation to the other adjacent sciences. The following section, however, is more substantial. At 1078a18-31, Aristotle argues that geometers are allowed to 'separate and posit' what is not separate, without introducing any error. In this chapter I will examine the role and the motivations of this statement. In short, I will suggest that it should be interpreted as a distinct, additional operation with respect to the *qua-manoeuvre* that I have discussed in the previous two chapters. 'Positing as separate' what is in fact not separate involves an element of fiction that (I have argued) is absent from the *qua*-theory. Although depending on the *qua*-theory for its legitimacy, thus, separation is a distinct and more radical operation.

I believe that Aristotle's motivations for admitting this further step in the geometer's methodological routine should be sought in his attempt to account for the geometers' actual practice. The way in which geometers speak and act seems at odds with the deflationary account provided by the *qua*-theory; Aristotle now needs to show that he can in fact account for the common practice of geometry at his time.

§1. Objects and Properties

The interpretation of the *qua*-theory that I have given in the previous two chapters yields a radically anti-Platonist account of geometrical truths. In particular, I have argued that the truth of geometrical statements and the identity of geometry as a science do not require the existence of any special geometrical objects. Sensible bodies do instantiate geometrical properties, and this is enough to make geometrical propositions true. The *qua*-theory indicates that such sensible bodies can be examined in a specific way (namely, *qua* quantitative and continuous); this peculiar methodology is what distinguishes geometry from other sciences. If my interpretation is correct, it follows that in the sensible world there are precisely instantiated geometrical properties (which might not be completely manifested or actualised, as I have argued in chapter 2), but no geometrical *object* – if by 'geometrical object' we intend a particular individual whose essence is completely exhausted by a geometrical description and which is the proper subject of attributes. That is to say, Aristotle would not admit the existence of *triangles*, but only of *triangular things* – sensible substances which have their own essence and to which triangularity belongs as a property.[99]

[99] The geometrical feature in question might be a necessary and even an essential property of the substance – for example, sphericity is an essential feature of celestial spheres. But what is important is that sphericity does not *exhaust* the essence of a celestial sphere – it is, after all, a *celestial* sphere.

This lines up with some other remarks that are scattered through the *Corpus*, which present geometrical features as properties of sensible objects. One of the most obvious examples is that the shape of sensible objects is listed as a quality in *Categories* 8, 10a10 ff. Aristotle takes pain to distinguish geometrical shape as a specific kind of quality (the fourth kind, in his list), with specific features – for example, he claims that shapes are qualities that do not admit of degrees. It is clear from his examples that Aristotle has *geometrical* shapes in mind (he mentions circles, triangles and squares), and that he considers them as properties of sensible objects. These objects are said to be circular or squared because they possess the quality of circularity or squareness.[100]

But Aristotle is not consistent in his presentation of geometrical features. Already in the *Categories* they receive an unconventional treatment: they are also listed as quantities, and receive a full discussion in *Cat.* 6.[101] Aristotle's treatment of quantities is rather peculiar. He is very clear in stating that quantities are not substances, but he groups together as belonging to the same category items that we would consider properties (such as being 4 feet long) and items that apparently are property-bearers, such as lines or surfaces. That the latter are considered as things rather than properties can be seen by the following passage:

> ἡ δὲ γραμμὴ συνεχές ἐστιν· ἔστι γὰρ λαβεῖν κοινὸν ὅρον πρὸς ὃν τὰ μόρια αὐτῆς συνάπτει, στιγμήν.

[100] *Cat.* 8.10a11-15.

[101] The fact that shapes appear in two categories is peculiar, and in need of an explanation. As it will appear clear from the rest of this chapter, I believe that the explanation lies in the fact that geometrical features receive a double treatment, based on the perspective that Aristotle is adopting at the moment. They are properties of sensible objects, and as such they are qualities that have certain potentialities (e.g. they are visible and causally active). But they can also be considered in their own right, not *qua* properties of sensible bodies. Once we adopt the geometer's perspective, geometrical features can be considered quantities and (as I will argue) object-like. For an interesting take on this issue, which has similarities with my own perspective, see Berman (2017).

> A line, on the other hand, is a continuous quantity. For it is possible to find a common boundary at which its parts join together, a point.
>
> *Cat*. 5a1-2 (Ackrill's translation)

Properties (as we understand them) do not have parts that can be joined together at a point, so it seems difficult to understand the line as a property in this context. In the same way, Aristotle presents geometrical objects as individual, particular objects in *Metaph.* Z.10, 1035a34-b3; 1036a2-12. I will discuss these passages in the next chapter; for the moment, what I am interested in pointing out is that at times, Aristotle seems to consider geometrical entities as what we would call objects, not properties. The oscillation between the two models has been identified also in the way in which Aristotle describes the relationship between sensible and geometrical objects. Commentators noticed that at times Aristotle maintains that geometrical objects lack sensible matter (e.g. *Metaph.* α.3, 995a15-16; Z.10, 1036a9-12), while at other times he argues that geometrical objects are obtained by removing sensible properties (e.g. K.3, 1061a28-b2).[102] The two things seem to be different: as Mueller (1970: 161-2) remarks, 'if abstraction is primarily thought of as eliminating properties, one will think of mathematical objects as physical objects looked at as if they did not have certain properties. On the other hand, if one thinks of abstraction primarily as eliminating matter, one will think of mathematical objects as properties like roundness and triangularity'.[103] But as Katz (2019: 492) points out, in

[102] See Katz (2019: 492); Mueller (1970: 161-3); Corkum (2012: 1060); and Annas (1976: 30).
[103] This is not completely uncontroversial, for Aristotle seems to admit of some sort of matter that is not sensible, that he calls 'intelligible matter'. It follows that removing sensible matter might not amount to removing matter *tout court*, since intelligible matter might be left. I will discuss intelligible matter in the next chapter; for the moment, what interests me is simply the fact that it is not completely clear whether Aristotle considers geometricals as objects or as properties.

Posterior Analytics 1.5, 74a37-b1 both a property (isosceles) *and* sensible matter (bronze) are subtracted.

One might think that Aristotle does not see the difference between object and property in the way that is natural to us, or is indifferent to it. Alternatively, there might be a reason why Aristotle oscillates between these two models. I believe that the oscillation between property- and object-model for geometricals is due to Aristotle's attempt to account for the way in which geometers talk about their objects, and (even more importantly) for the way in which they use them in the course of geometrical proofs and demonstrations. While Aristotle's anti-Platonist account of geometrical truths rejects the existence of geometrical *objects*, and identifies the truth-makers of geometrical theorems with sensible bodies and their geometrical *properties*, the way in which Greek geometry is conducted seems to require the existence of geometrical objects.

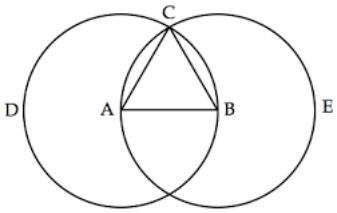
Let me explain. The naïve Platonism of the working geometer is a well-known phenomenon, and something that philosophers have no qualms disregarding. But in the case of Greek geometry, it is important to notice that the fact that geometrical objects are objects and not properties is not something that can be easily dismissed as the mathematician's misguided self-understanding, but is an ineliminable aspect of geometrical practice. By this I mean that:

(i) Geometrical theorems are formulated using objectual language. In particular, they make use of singular terms that are supposed to refer to objects;

(ii) There is a plurality of objects of the same kind;

(iii) The objects that appear in proofs and demonstrations can undergo some (very specific and codified) kind of manipulations.

From a linguistic point of view, words such as 'circle', 'line' or 'square' that appear in the text of geometrical proofs are singular terms that supposedly refer to objects.[104] Such objects are property-bearers: in particular, they have quantitative properties attributed to them. Moreover, it seems clear that those objects are individual (tokens) and not universals or types of objects: during the development of a geometrical proof the geometer might need to make use of more than one object of the same kind.[105] In fact, she might make use of a plurality of qualitatively identical objects of the same kind, such as in the first theorem of the *Elements*, where the construction of an equilateral triangle requires drawing two identical circles (and three identical lines):



Note also that the fourth Euclidean postulate states 'that all right angles are equal to one another':[106] right angles are presented as the kind of thing of which there can be more than one instance, and each instance is identical to the others. If Aristotle's report of Plato's philosophy of mathematics can be trusted, this feature of Greek geometry was well-known and appreciated by philosophers, so much that Plato was led to identify mathematical objects with the Intermediates – entities that are akin to the Forms in that

[104] Note that I am not taking a position on whether these terms refer to individual particulars or to arbitrary objects *à la* Kit Fine, as suggested by Hussey (1991). The point that I am making here is smaller: complete or arbitrary, it seems undoubtable that geometrical objects are *objects*. I take this to be compatible with Acerbi's (2020) claim that there is no direct reference to any singular object in the text of geometrical proofs; even if he is right and reference to geometrical objects is only achieved through indefinite descriptions, it still is the case that those descriptions apply to objects and not properties.

[105] If Aristotle believes in tropes, he might accommodate this point while holding onto the view that geometrical entities are properties of sensibles. However, tropes are individuated by their bearers, and this does not fit with the way in which geometrical objects are used and described by geometers at work.

[106] Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

they are perfect, eternal and unchangeable, but akin to sensible objects in that there is a plurality of them (see *Metaph*. A.6 987b14-18, B.2, M.1-2).[107] As we will see in the next chapter, this opens up the so-called 'Plurality Problem': what makes each token distinct from the others, if there are no qualitative differences? The very fact that this is a problem testifies that geometrical objects are conceived as objects, that require identity conditions.

The most controversial point is (iii), and I ask the reader to please bear with me until Chapter 5, in which I will discuss the role of change and motion in geometry. I will argue extensively that geometrical objects can undergo some (specific kinds of) modifications, such as for example division or extension; if this is the case, they must be underlying persisting objects that admit of contraries.

Taken together, these aspects of geometrical practice suggest that geometrical objects are conceived of as individual particulars, property-bearers and subjects of predications, and able to undergo (some kinds of) modifications. These are characteristics of objects and substance-like individuals, not properties, that are difficult to square with Aristotle's account of geometry as a science concerned with some properties of sensible objects.

§1.1. Kooky Objects?

There might be a way of down-playing the distinction between object and property, by understanding properties as more akin to objects than it is usually done in contemporary metaphysics. Katz (2019; 2022) tries to do exactly this. Starting from the observation that Aristotle describes the geometer at times as removing sensible properties and at times as removing sensible matter from sensible objects, she argues that Aristotle 'does not take this to be a meaningful distinction. To study a sensible object not as sensible but only as

[107] The issue whether Aristotle's report is trustworthy is a complex one, that I will not enter into. Notice that even if Aristotle's report cannot be trusted as a source for Plato's philosophy of mathematics, it still testifies that Aristotle himself thought that this feature of Greek geometry was an important one, that needs to be accounted for.

quantitative and continuous just is to study its geometrical properties alone'.[108] In her (2022: 154) article, Katz explains more in detail what she means by this. She distinguishes two senses of properties:

1. *Property$_1$ (or Accident$_1$)*: A feature or attribute of a subject, for example Socrates' paleness, a mountain's largeness, a sensible triangle's triangularity; and
2. *Property$_2$ (or Accident$_2$)*: The subject itself just insofar as it has a property$_1$ (or accident$_1$, for example the pale thing (Socrates *just* insofar as he is pale), the large thing (the mountain *just* insofar as it is large), the triangular thing (the sensible triangle *just* insofar as it is triangular).

She then proceeds to explain that properties$_2$ are nothing else than Matthew's (2009) 'kooky objects', and thus have a place in Aristotle's ontology. Kooky objects are *sumbebekota*, and so they can be understood as properties, but (contrary to properties$_1$) they are in a certain sense the same as the substance to which they belong.[109]

The notion of kooky object is developed to account for Aristotle's treatment of cases such as the 'Masked Man' (*SE* 179a-b).[110] Suppose that Coriscus puts a mask on, so that it is impossible to recognise him. We then have the following case:

1. Coriscus has the attribute of being known by Socrates to be Coriscus
2. Coriscus and the masked man are the same (man)
3. The masked man does not have the attribute of being known by Socrates to be Coriscus.

[108] Katz (2019: 494).
[109] See White (1986: 479): kooky objects and their 'parent subjects' are 'accidental sames', i.e. they are two distinct entities that can coincide. In Aristotle's terms, they are one in number but two in account.
[110] See Matthews (2009: 227 ff.)

According to Matthews, Aristotle manages to account for the truth of all three propositions by denying that 'Coriscus' and 'the masked man' are co-referential expressions. The former picks out Coriscus, while the latter picks out a kooky object – the masked man – that is merely *accidentally* the same as Coriscus. Matthews argues that accidental sameness is sameness (so that proposition 2 is indeed true), but it is not identity: this is why Coriscus can have the property of being known by Socrates to be Coriscus, while the masked man lacks it.

Discussing the plausibility of kooky objects as an overall interpretation of Aristotle's understanding of contingent properties would require a different thesis, so I am not going to try to assess the theory as a whole.[111] I am only interested in understanding whether it can be used to solve the problem that I have identified in Aristotle's account of geometricals – namely, the fact that geometrical practice seems to require the existence of geometrical *objects*, while the *qua*-theory yields a radically anti-Platonist theory of geometrical truth, which denies the existence of geometrical objects and maintains that the truth-makers of geometrical propositions simply are sensible objects (which do have geometrical *properties*).

By taking geometrical objects to be kooky objects (i.e. properties$_2$), Katz tries to have her cake and eat it, too. On the one hand, she insists that kooky objects are in fact properties, so that the only (full, normal) objects that exist are sensible objects; on the other, kooky objects mostly behave like objects and not like properties: they can be subjects of predication, and be referred to using singular terms and definite descriptions. Moreover, they are merely accidentally the same (and not identical) to their sensible

[111] The main contention regards the notion of 'accidental sameness'. Proponents of kooky objects maintain that it is a relation that falls short of identity, so that kooky objects can have properties that their 'parent objects' lack. On the contrary, critics of kooky objects interpret this relation as contingent identity (see e.g. Barnes 1979).

'parents' objects, which allows Katz to explain why the former might be precise while the latter are not.

Katz' suggestion of understanding geometrical objects as kooky objects seems to be able to account for some of the features that geometers attribute to geometrical objects: kooky objects are individual particulars and can be said to bear properties. By understanding properties in this way, Katz goes some way towards fitting geometrical practice into Aristotle's ontological account. Her kooky objects interpretation can account for 1. and 2., but it does not succeed to make sense of the third motivation for treating geometricals as objects, namely their ability to undergo manipulations. I will discuss the issue in depth in Chapter 5, but here is a short sketchy preview for the sake of the argument. In the course of proofs, geometrical objects might undergo some modifications. For example, they can be divided; Aristotle explicitly states that lines and other continuous magnitudes are said to be *per se* divisible.[112] This means that a line can be divided into parts. If the line is a kooky object (i.e., in Katz' interpretation, the limit of a sensible body, considered only *qua* quantitative and continuous), the division – which counts as a change – is not undergone by the kooky object, but by the parent object. Take the case of a 10cm long stick, and the corresponding kooky object which is supposed to be the 10cm long line. If we divide it in two halves, it is not the kooky object which is divided into two 5cm long parts, but the stick itself, which loses the property of being a 10cm long line. The kooky object that we had identified with the 10cm long line simply disappears, it does not undergo the division. This appears clearly from the way in which Aristotle describes changes in *Physics* I.7: when a man becomes musical,

> καὶ τὸ μὲν ὑπομένει, τὸ δ' οὐχ ὑπομένει· […] ὁ γὰρ ἄνθρωπος ὑπομένει, τὸ μὴ μουσικὸν δὲ καὶ τὸ ἄμουσον οὐχ ὑπομένει, οὐδὲ τὸ ἐξ ἀμφοῖν συγκείμενον, οἷον ὁ ἄμουσος ἄνθρωπος.

[112] καθ' αὑτό, *Ph.* VI.5, 236b8-10.

> This survives but that doesn't […] the man survives, but the not-musical, or unmusical, doesn't survive, nor does the compound of the two, viz., the unmusical man.
>
> *Ph.* I.7, 190a17-21 (Matthew's translation)

According to Matthews, ''the not-musical' is the not-musical person (rather than nonmusicality) and 'the unmusical' is the unmusical person. What Aristotle is telling us is that, when the man becomes musical, the man survives but each of these kooky objects perishes: (i) the not-musical (one); (ii) the unmusical (one); (iii) the unmusical man.'[113] There is no reason to think that, were geometrical objects to be interpreted as kooky objects, they would behave differently.[114] However, this means that the line, if understood as a kooky object, is not the proper object of the division: the parent object undergoes the change, while one kooky object is substituted by another.

The ability to undergo modifications is peculiar to substances, and it cannot be accommodated by any theory of properties, not even one as thick as the kooky objects interpretation. So even if we were to accept the kooky objects interpretation as an interpretation of the *qua*-theory, it is still the case that there is a gap between Aristotle's contention that geometrical objects are properties of sensible bodies, and the way in which geometrical objects are used in geometry.[115] Katz' attempt to understand properties

[113] Matthews (2009: 225).

[114] On the contrary, there are good reasons to expect that geometrical properties behave exactly in this way: when sensible objects change, they acquire or lose geometrical properties – but the properties themselves do not undergo any change, they are simply acquired or lost. I will discuss this issue in detail in Chapter 5. To anticipate the key point, notice that geometrical and sensible objects 'behave' in different ways. For example, the second argument against the Partial Platonist presented in M.2 relies on the assumption that if a geometrical object were actually present in a physical body it would be impossible to divide it, because geometrical objects cannot be divided in the same way as sensible bodies are. One of the consequences that I will draw from this and other observations is that geometricals should not be thought of as objects that persist through time in the same way as sensible bodies, but as properties that can be acquired or lost without undergoing a process of coming in or out of existence.

[115] This means that the decision whether to accept or reject the kooky objects interpretation is very low-stakes from the point of view of reconstructing Aristotle's theory of geometry: it basically amounts to deciding on one's favourite theory of accidental properties. I hope to have shown in the preceding chapters

according to an objectual model falls short of closing this gap. I believe that we should not try to close it, though, because by understanding properties as akin to objects we would risk *eliminating* (rather than solving) a problem that Aristotle recognises as problem, and tries to solve.

I submit that the last part of *Metaphysics* M.3 is in fact dedicated to closing the gap between Aristotle's ontological theory of geometrical objects (according to which the truth of geometrical statements is grounded on the geometrical properties of sensibles) and the common practice of geometers (which takes geometrical objects to be objects). Aristotle recognises that geometers treat their objects in a way that is not perfectly adequate to their ontological status, but wishes to account for their method in such a way that it results valid and compatible with his own ontological theory – namely, by granting them permission to 'posit as separate' what is in fact not separate.

§2. Separation is not merely considering X qua Y

The main contention of this chapter is that the *qua*-theory does not exhaust the content of M.3, nor Aristotle's discussion of geometrical objects. The interpretation of the *qua*-theory that I have given in the previous two chapters explains how geometrical truths can be grounded in sensible reality, without the need to postulate the existence of special geometrical objects. However, it falls short of accounting for geometrical practice and for the way in which geometers describe and treat their objects. I wish to argue that the last part of M.3 is meant to do precisely this: explain why geometers can speak and act as they do, even if the ontology that is implicit in their way of speaking is not the right one.

that there is no need to introduce *qua*-objects to account for Aristotle's *qua*-theory, be they kooky objects or other things; as such, I do not see the need to commit to the existence of kooky objects. But I do not believe that anything I am going to say in this chapter hangs on my rejection of this interpretation.

To make this contention plausible, the first thing that I need to do is show that the second part of M.3 (and in particular lines 1078a18-31) is not part of the exposition of the *qua*-theory, but puts forward a new and different point. There are a few indications that this is the case:

1) The way in which the chapter is organised;
2) Separation is presented as a consequence of the *qua*-theory;
3) Separation involves an element of fiction that is absent from the *qua*-theory.

§2.1. The structure of the chapter

The end of the exposition of the *qua*-theory is marked by a change of topic at line 1078a8, where Aristotle makes a point about the ranking of sciences on the basis of their simplicity and 'accuracy' (*akribeia*):

> καὶ ὅσῳ δὴ ἂν περὶ προτέρων τῷ λόγῳ καὶ ἁπλουστέρων, τοσούτῳ μᾶλλον ἔχει τὸ ἀκριβές (τοῦτο δὲ τὸ ἁπλοῦν ἐστίν), ὥστε ἄνευ τε μεγέθους μᾶλλον ἢ μετὰ μεγέθους, καὶ μάλιστα ἄνευ κινήσεως, ἐὰν δὲ κίνησιν, μάλιστα τὴν πρώτην· ἁπλουστάτη γάρ, καὶ ταύτης ἡ ὁμαλή.
>
> The more it is prior in definition, and the simpler, the greater the accuracy (i.e. simplicity) obtained. So there is more accuracy where there is no magnitude than where there is, and most of all where there is no movement; though if there is movement accuracy is greatest if it is primary movement, this being the simplest, and uniform movement is the simplest form of that.
>
> *Metaph*. M.3, 1078a8-13 (Annas' translation, modified)

It is not surprising that Aristotle wishes to make a point about the position of the various mathematical sciences in the overall system of knowledge. Recall that his main

worry in presenting the *qua*-theory is to show that the rejection of mathematical Platonism does not endanger the possibility of geometry as a science. After explaining why geometrical statements can be true even in the absence of separate geometrical objects, and why geometry can still be considered an autonomous science even if it studies the same objects as other sciences, Aristotle proceeds to show that his account still ranks the sciences in the same way as one would expect.[116]

This passage raises a number of questions that go beyond the scope of this work, and that I will not address. In particular, it is not at all clear what Aristotle means by 'accuracy' (*akribeia*), nor whether he is implying that arithmetic reaches a higher level of certainty than geometry. I will simply note that (in accordance with my reading of the *qua*-theory) I read the claim that 'there is no magnitude' (*aneu*) as meaning that the science in question does not consider magnitude, *not* as meaning that the proper object of the science is an object that lacks magnitude. This reading is validated by the parallel passage at *APo* I.27, 87a34-5, where Aristotle claims that a science is more accurate if it proceeds from fewer 'posits' (*prostheseis*).[117]

This apparently concludes Aristotle's discussion of geometry, for he moves on to make a point about the so-called subordinate sciences:

> ὁ δ' αὐτὸς λόγος καὶ περὶ ἁρμονικῆς καὶ ὀπτικῆς· οὐδετέρα γὰρ ᾗ ὄψις ἢ ᾗ φωνὴ θεωρεῖ, ἀλλ' ᾗ γραμμαὶ καὶ ἀριθμοί (οἰκεῖα μέντοι ταῦτα πάθη ἐκείνων), καὶ ἡ μηχανικὴ δὲ ὡσαύτως
>
> The same account applies to harmonics and optics; neither studies its objects as seeing or utterances, but as lines and numbers (these being proper attributes of the former); and mechanics likewise.

[116] Cf. Plato's *Philebus* 55c–59c.
[117] Cf. also Netz (2006: 22-23); Cleary (1985).

Metaph. M.3, 1078a13-18 (Annas' translation)

His way of speaking suggests that he considers the exposition of the *qua*-theory completed: he refers back to it as a *logos* that has been given and defended, and can now be extended to other disciplines. Subordinate sciences are here treated in the same way as the pure mathematical sciences: the *qua*-operator allows those who study harmonics and optics to study seeing and utterances not *qua* sensible, but *qua* numbers and lines – and this is legitimate because the latter are *pathe*, attributes, of sensibles.[118]

§2.2. Separation as a consequence of the qua-theory

The passage about separation appears immediately after these lines:

> ὥστ' εἴ τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ᾗ τοιαῦτα, οὐθὲν διὰ τοῦτο ψεῦδος ψεύσεται, ὥσπερ οὐδ' ὅταν ἐν τῇ γῇ γράφῃ καὶ ποδιαίαν φῇ τὴν μὴ ποδιαίαν· οὐ γὰρ ἐν ταῖς προτάσεσι τὸ ψεῦδος. ἄριστα δ' ἂν οὕτω θεωρηθείη ἕκαστον, εἴ τις τὸ μὴ κεχωρισμένον θείη χωρίσας, ὅπερ ὁ ἀριθμητικὸς ποιεῖ καὶ ὁ γεωμέτρης.
>
> So if one posits objects separated from what is incidental to them, and studies them as such, one will not for this reason assert a falsehood, just as [a mathematician does not assert a falsehood] when they draw [something] in the sand and say that a [line] that is not one foot long is one foot long; the falsehood is not part of the premises.
>
> The best way of studying each object would be this: to separate and posit what is not separate, as the arithmetician does, and the geometer.
>
> *Metaph.* M.3, 1078a18- 23 (Annas' translation)

[118] I will come back to subordinate sciences in the next subsection, and in Chapter 5.

I will discuss the details of this passage and its philosophical import in the next sections. For the moment, note that the main claim is presented (in a conditional form) as a consequence of the *qua*-theory, thanks to the connective *hoste*.[119] The idea is that since the *qua*-theory holds, it is not a problem if someone 'posits objects separated from what is incidental to them, and studies them as such'. Despite the conditional form of the statement, it appears clearly from the following sentences that Aristotle does believe that this is exactly what geometers and arithmeticians do. The upshot, I take it, is that *since the qua-theory is true*, mathematicians are allowed to proceed as they do, without worrying consequences. This way of introducing the idea of separation suggests that it is different from merely 'considering *qua*'.

As I mentioned in the introduction , support for this reading can be found in the parallel passage in *Physics* II.2, in which the claim that geometers separate their objects is introduced by the connective *dio*:

> περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον· οὐδὲ τὰ συμβεβηκότα θεωρεῖ ᾗ τοιούτοις οὖσι συμβέβηκεν· διὸ καὶ χωρίζει· χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων.
>
> Now the mathematician, though he too treats of these things, nevertheless does not treat of them as the limits of a natural body; nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated.
>
> Arist. *Physics* II.2, 193b30-35 (Charlton's transl.)

[119] I will discuss the analogy and its efficacy in §4.

Before continuing, let me pause for a moment to answer a possible objection. One might think that it is illegitimate to appeal to *Ph.* II.2 in this context, because despite the similarities in structure and topic, the two passages are in tension. This can be appreciated by the *Ph.* II.2 account of subordinate sciences, which apparently differs from the M.3 one:

> ἀνάπαλιν γὰρ τρόπον τιν' ἔχουσιν τῇ γεωμετρίᾳ. ἡ μὲν γὰρ γεωμετρία περὶ γραμμῆς φυσικῆς σκοπεῖ, ἀλλ' οὐχ ᾗ φυσική, ἡ δ' ὀπτικὴ μαθηματικὴν μὲν γραμμήν, ἀλλ' οὐχ ᾗ μαθηματικὴ ἀλλ' ᾗ φυσική.
>
> [Optics, harmonics and astronomy] are in a way the reverse of geometry. Geometry considers natural lines, but not as natural; optics treats of mathematical lines, but considers them not as mathematical but as natural.
>
> *Ph*. II 2, 194a 9–12 (Charlton's transl.)

While M.3 argues that subordinate sciences work in the same way as the pure mathematical ones, *Ph.* II.2 claims that they are 'the reverse' (*anapalin*); the former states that optics and harmonics study sensible phenomena not *qua* sensible but *qua* lines and numbers, while the latter that they study mathematical lines not *qua* mathematical but *qua* natural. It is not impossible that the discrepancy between the two passages reflects an evolution or a shift in Aristotle's thought, as Charlton (1975: 95) suggests. However, this explanation seems unsatisfactory, given the sheer parallelisms between the *Physics* passage and M.3. The *Physics* passage looks like a summary (or an anticipation) of M.3: besides the discussion of subordinate sciences, it features both the *qua*-theory and the idea that geometers 'separate' their objects. As a matter of fact, the tension can be attributed to the different contexts in which the discussion of subordinate sciences

appears, and can be resolved by paying attention to Aristotle's different goal in the two passages.[120] While in M.3 the focus is on accounting in detail for the possibility of geometry, in *Physics* II.2 Aristotle runs through the account of geometry almost as if he could take it for granted, and focuses on what distinguishes mathematical from physical sciences.

With this clear in mind, we can see that *Physics* II.2 does not contradict, but builds on, the account of subordinate sciences that is sketched in M.3. In M.3, Aristotle is concerned with stressing that the *qua*-theory (which explains why mathematics can be true in virtue of sensible bodies) does not introduce any element of idealisation or falsity: it merely states that mathematicians treat their objects in a certain way. Accordingly, he wishes to stress that optics and harmonics can study sensible 'seeing or utterances' *qua* lines and numbers *because* the latter are attributes of the former. Not being particularly interested in the distinction between pure mathematical sciences and subordinate ones, he stops here. In *Physics* II.2, on the contrary, the focus is precisely on this distinction – while the account of geometry is referred to but not discussed in detail. We can thus read the *Physics* II.2 remark as a refinement of the M.3 account, not as an alternative: *Ph.* II.2 takes off from where M.3 stops. Considering mathematical objects *qua* natural amounts to operating an additional *qua*-manoeuvre (which is legitimate because mathematical objects were nothing else than natural objects, considered not *qua* natural).[121]

[120] Cf. also Crubellier (1994: 142).

[121] Cf. Peramatzis (2011): 'subordinate mathematical sciences seek to fit their mathematical objects onto natural material bodies and explain certain features of those bodies on the basis of principles and theorems which are originally about immaterial mathematical forms.' (p. 63). So, in the case of optics there is reverse abstraction in the sense of additional abstraction from objects which are already abstract: e.g. mechanics takes a mathematical line and focuses only on 'those of its geometrical features that can be implemented on perceptible, material spheroids and planes, disregarding any geometrical features that do not apply to its specific subject-matter' (69-70, fn. 18). In Chapter 5 I will argue that this also requires adding some features, in particular motion.

§2.3. Separation involves an element of fiction

Recall that (under my interpretation) the *qua*-theory does not introduce any falsity. Considering *X qua Y* merely amounts to focusing attention to the features that *X* has in virtue of being *Y*, and ignoring the others; this can only be done if *X* is in fact *Y*. On the contrary, separation is presented as involving some sort of falsity or fiction, although as innocuous as drawing a line on the ground and saying that it is a foot long even if it is not – i.e. using a metrically inaccurate diagram to prove a geometrical theorem.[122]

Taking a closer look at Aristotle's formulation confirms that separation cannot be reduced to the *qua*-theory, but is a genuinely new and additional step in the geometers' methodology. He writes: τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ᾗ τοιαῦτα. Crucially, it is not evident what ᾗ τοιαῦτα stands for.[123] Given the proximity of the participle κεχωρισμένα, I submit that ᾗ τοιαῦτα should be read as referring back to it, and thus meaning ᾗ κεχωρισμένα.[124] If this is the case, this use of the *qua*-operator cannot be subsumed under the pattern established by the *qua*-theory. Let me explain.

In M.2, the terminology of separation (*khoriston* and *kekhorismenon*) is used to indicate ontological separation, and it is plausible to think that Aristotle is still using these terms in the same way.[125] Recall that ontological separation is a binary non-symmetrical

[122] I will discuss the analogy and its efficacy in §4.

[123] Crubellier (1994: 132) describes the expression as a "démonstratif quasi-deiptique". Notice that Aristotle has a tendency to complete the *qua*-clause with this kind of expressions: cf. 1077b22, *hei toiadi; APo* I.8, 81b4-5, *hei toiondi hekaston*. In a different context, see also the definition of *kinesis* as ἡ τοῦ δυνάμει ὄντος ἐντελέχεια, ᾗ τοιοῦτον at *Ph.* III.1, 202b10-11.

[124] Alternatively, one might insist that it be interpreted along the model of 1077b22, at the very beginning of the exposition of the *qua*-theory. Aristotle claims that 'it is possible for there to be statements and proofs about perceptible magnitudes, but not as perceptible but as being of a certain kind (*hei toiadi*).' *Toiadi* is used as a placeholder that should be filled by the geometrical property that we want to focus on in each case. In the following paragraph Aristotle lists some options: 'not as moving but merely as bodies, and again merely as divisible, and indivisible but with position, and merely as indivisible' (1077b28-30). It is not impossible to suppose that something similar is going on in the passage at hand, but it is not the most natural reading.

[125] See also *Metaph.* M.4, 1078b30 ff. where Aristotle argues that the Platonists made universals and definitions separate and called them Forms. That this is the same sense is testified by the *Physics* II.2 passage, where Aristotle explicitly stresses that the operation of the geometer is the same as the one that

relation: *X* is separate from *Y* if and only if *X* is (or can be) without *Y*. The second term of the relation is here τῶν συμβεβηκότων: geometricals are posited as separate from what is incidental to them. I take this to be a reference back to line 1077b35, where Aristotle argued that a science can be properly said to be 'about this or that – not what is incidental (e.g. not the white, even if the branch of knowledge deals with the healthy, and the healthy is white)'. 'What is incidental' thus covers all the features that belong to the object under observation but that are irrelevant to the science at hand. In the case of geometry, these features are the material and sensible ones.[126] Thus, I submit that ᾗ τοιαῦτα means *qua* separate from sensible features, which in turn means *without* sensible features – where 'without' is effectively a negation. The upshot is that after separating and positing geometrical features, it becomes possible to consider them *qua not* sensible; before separation, the *qua*-theory only allowed geometers to consider their objects *not qua* sensible.

This shift in the scope of the negation signals that a falsehood or fiction has been introduced, because in M.2 Aristotle made clear that geometrical objects are not in fact separate from sensible substances. This explains why Aristotle rushes to provide an analogy that shows that in some cases it is methodologically sound to use falsities and fiction (provided that 'the falsehood is not part of the premises' – more on this in §4), and also why he uses the verb θέμενος. This verb introduces an element of stipulation or postulation that is not required by, and would be out of place in the context of, the *qua*-

leads the Platonists to posit the Forms. Note that in *Ph.* II.2 Aristotle argues that geometers are allowed to separate because their objects are separate in definition from motion and matter. I read this as meaning that because geometrical features are *in fact* definitionally separate from motion and matter, positing them as ontologically separate does not introduce any error, even if it is in fact false.

[126] If this is right, it further deepens the parallelisms with *Ph.* II.2, where geometrical features are said to be separated from motion and matter.

theory,[127] but fits the idea that separation amounts to positing in thought something that is not in fact separate as if it were separate.

If this is right, then the *qua*-operation and what I have been calling 'separation' are two distinct operations, that are not always necessarily associated. We can think of the geometer's operation as involving two steps: first, she restricts attention to the features that are essential to her study, considering the sensible object not *qua* sensible. Then, once she has selected an adequate set of properties, she can proceed to 'separate' them.[128] It is important to stress that considering *X qua Y* does not always require positing *Y* as separate: this is why for example natural scientists can consider their objects just *qua* moving without falling into the mistake of the Platonists.[129]

In the next section I will examine what separation involves precisely. For the moment, notice that my interpretation allows us to make good sense of the use of the *qua*-operator in the description of separation. Recall that Aristotle argues that it is legitimate to 'posit objects separated from what is incidental to them, *and study them as such*' (1078a18-19, my emphasis). I have argued above that this formulation differs from the preceding uses of the *qua*-clause, in that it involves a change in the scope of the negation. Once it is clear that separation is a distinct operation from the one described in the *qua*-

[127] See Annas (1976: 150); Crubellier (1994: 132).
[128] Cf. Charles (2008: 5, fn. 11), who reads *Physics* II.2 as prescribing three steps:
'Stage 1: abstract away certain features from the object in thought, e.g., perceptual matter (see *Metaph*. K 3, 1061a27ff). This involves not thinking about the perceptual matter in mathematicals but considering only their mathematical aspects.
Stage 2: check that it is OK to proceed with the results of this abstraction: does error occur in one's relevant reasoning as a result of (stage 1) abstraction?
Stage 3: separation occurs when one moves from not thinking of the line as perceptually enmattered to thinking of it as not being perceptually enmattered. (Note the distinction in scope!)'
[129] One might wonder why at 1078a21 Aristotle says that the best way of studying *each object* is to separate, as geometers and arithmeticians do. I believe that the *hekaston* is here limited to the objects of mathematical and subordinate sciences, which are mentioned in the previous paragraph.

theory, we are in a position to appreciate the significance of this clause, and why it is not in contradiction with the *qua*-theory.

Aristotle says that (i) geometers 'separate and posit' their objects, and *then* that (ii) they consider them *qua* separated. The fiction is introduced by (i) the act of positing separate objects. Once this is done, Aristotle can use the *qua*-clause in his usual way: the shift in negation does not imply that the *qua*-clause attributes to X *qua* Y features that X itself does not have, because *in the fiction* introduced by separation, the objects that are considered *qua* separated are in fact separated.

Now that it is clear that 'positing objects separated from what is incidental to them, and studying them as such' is different from studying sensible objects not *qua* sensible, let us take a closer look at what this means and what kind of fiction it involves.

§3. Separation

The notion of ontological separation is a central one in Aristotle's metaphysics. Despite being one of his technical innovations, it is quite difficult to pin down, and this is not the place for a full discussion of the many issues that are linked to this notion.[130] To understand how this notion is used in M.3, it is necessary to distinguish and examine two main questions:

1) Even if at times it seems to be used as a monadic predicate, 'separation' is always a binary relation:[131] some *X* is separate from something *F*. Presumably, what *F* is depends on the context: what is *F* in this case?
2) What does it mean to say that *X* is separable from *F*? E.g. does it mean that *X* can be defined without reference to *F*, or that *X* can exist even if there are no *Fs*?

[130] In particular I will not discuss whether *choriston* should be understood as meaning separate or separable, nor how separation is related to priority. See Peramatzis (2011) for an in-depth discussion of priority.
[131] E.g. *Metaph.* Z.1, 1028a33-34.

Aristotle claims that geometrical objects are not separate but can be posited in thought and then treated as separate. To understand what it means we need to clarify 1) separate from what, and 2) what is meant by separation. A constraint on the answers is that it should turn out that mathematical properties are in fact not separate in this way, but that considering them as such is instrumental to accounting for geometrical practice.

Question 1) has been already partially answered in the previous section. I have suggested that when Aristotle says κεχωρισμένα τῶν συμβεβηκότων, the 'incidental features' from which geometricals are separated are the sensible and material properties of substances. Moreover, in M.2 Aristotle has established that geometrical objects are not in fact separate from sensible substances. The upshot is that in the case of geometrical objects, *F* are sensible and enmattered substances: geometrical features are not in fact separate from sensible substances, but fictionally separating them is useful to do geometry.

Also question 2) has already been tackled, but it deserves further discussion. In M.2 Aristotle appeals to both ontological separation and definitional separation, and claims that the latter does not imply the former.[132] While summarising his findings at the end of the chapter, Aristotle claims that geometrical objects are definitionally separate from sensible substances and features, but they are not ontologically separate from them. This strongly suggests that the separation in question in M.3 is ontological (and not definitional) separation.

Once again, the parallelism with *Ph.* II.2 is helpful:

[132] To be precise, he states that something can be definitionally prior but ontologically posterior to something else. Since priority implies separation (priority is usually understood as being separation + asymmetry), this means that something can be definitionally separate without being ontologically separate from something else.

διὸ καὶ χωρίζει· χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων. λανθάνουσι δὲ τοῦτο ποιοῦντες καὶ οἱ τὰς ἰδέας λέγοντες· τὰ γὰρ φυσικὰ χωρίζουσιν ἧττον ὄντα χωριστὰ τῶν μαθηματικῶν. γίγνοιτο δ' ἂν τοῦτο δῆλον, εἴ τις ἑκατέρων πειρῷτο λέγειν τοὺς ὅρους, καὶ αὐτῶν καὶ τῶν συμβεβηκότων. τὸ μὲν γὰρ περιττὸν ἔσται καὶ τὸ ἄρτιον καὶ τὸ εὐθὺ καὶ τὸ καμπύλον, ἔτι δὲ ἀριθμὸς καὶ γραμμὴ καὶ σχῆμα, ἄνευ κινήσεως, σὰρξ δὲ καὶ ὀστοῦν καὶ ἄνθρωπος οὐκέτι, ἀλλὰ ταῦτα ὥσπερ ῥὶς σιμὴ ἀλλ' οὐχ ὡς τὸ καμπύλον λέγεται.

That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated. Those who talk about ideas do not notice that they too are doing this: they separate physical things though they are less separable than the objects of mathematics. That becomes clear if you try to define the objects and the things which supervene in each class. Odd and even, straight and curved, number, line, and shape, can be defined without change but flesh, bone and man cannot. They are like snub nose, not like curve.

Ph. II.2, 193b33-194a7 (Charlton's translation)

Aristotle very explicitly states that when a geometer separates (*chorizei*), she is doing the same thing that Platonists do when they separate; we know from *Metaph.* M that Platonists posit Forms as *ontologically* separate.[133] *Ph.* II.2 also mentions definitional separation: Aristotle claims that geometricals can be defined without reference to motion nor matter.[134] Interestingly, this is presented as the sign that positing them as ontologically separate (even if they are not) is a methodologically sound operation, which does not lead to error.

[133] Cf. also *Metaph.* M.9, 1086b5-13.
[134] Aristotle claims that they are definitionally separate from *change*, but the example he provides is his go-to example to illustrate separation from *matter*. I will discuss the role of change in geometrical practice in Chapter 5; for the moment, note that Aristotle takes the two things to be co-extensional.

If this is correct, then saying that geometers are allowed 'to separate and posit what is not separate' amounts to claiming that (i) geometrical features are not ontologically separate from sensible substances, i.e. they cannot exist without being in sensible things; but (ii) they can be posited and treated as separate, i.e. as if they could exist without being in sensible things. Note that since geometers speak of their objects as particular individuals, such objects cannot be said of other things either; it follows that to treat geometrical features as if they could exist without sensible substances amounts to treating them as ultimate subjects of predication – that is to say, as substances.[135]

They can do this by 'positing them as separate', or 'separating in thought'. This consists in creating a mental object whose essence is completely exhausted by the geometrical property that they are interested in: from the triangularity of a sensible object, they obtain a triangle. In so doing, they transform the non-substantial and dependent feature into an individual substance, which is ontologically separate. As Corkum (2012) rightly says, this amounts to a form of fictionalism, but the fiction is limited to the *mode of being* of geometricals. The geometrical features that are posited as separate *do exist*, but they exist *as non-substantial features*, as properties of sensibles.[136] So it is not completely accurate to call these objects 'fictional': I will call them 'semi-fictional' instead, to stress that the fiction is limited to the mode of existence of the object.

This is crucial to understand the difference with contemporary fictionalism in philosophy of mathematics, e.g. Hartry Field's *Science without Numbers*.[137] Field believes that mathematical statements are literally false, because the singular terms that

[135] At least if we accept the *Categories* view that the only options are: present-in-and-said-of; said-of-but-not-present-in; present-in-and-not-said-of; neither present-in-nor-said-of. Thanks to Ursula Coope for helpful suggestions on this point.
[136] This works even if one were to accept the kooky objects interpretation of the *qua*-theory. Positing as separate a kooky object simply amounts to fictionally transforming it into a normal object, a substance. Crucially, this includes attributing it the ability to survive changes. The importance of this aspect will become clear in Chapter 5.
[137] Field (1980).

appear in them lack a referent. On the contrary, Aristotle is adamant that geometrical statements are true – eternally and necessarily true. I wish to argue that in Aristotle's account geometrical singular terms do not lack a referent, but refer to semi-fictional objects which are not the truth-makers of the sentences in which those terms appear.

According to my interpretation, the truth of geometrical statements is grounded in sensible objects and their geometrical properties. The fact that geometers fictionally posit those features as separate, and refer to these fictionally separate objects rather than to the features themselves, is irrelevant to the truth or falsity of the theorems. Separation is not part of the theory of geometrical truth nor of the account of the ontological status of geometricals; it is part of the *epistemology* of geometry. This can be seen by Aristotle's wording at 1078a21, where he claims that positing as separate is 'the best way of studying (θεωρηθείη) each object'. Moreover, it is clear from the way in which he introduces the analogy with the one-foot long line that the worry is not that the fiction makes geometrical statements literally false, but that studying the semi-fictional objects instead of the sensible substances that do in fact instantiate the geometrical properties might lead to mistakes in the geometers' reasoning.

We might wonder whether Aristotle is in fact allowed to pass over the issue of the literal truth of geometrical statements so easily. If my interpretation is correct, it follows that geometrical statements are true in virtue of sensible bodies and their geometrical properties, but contain singular terms that refer not to these sensible bodies but to semi-fictional objects that do not in fact exist *as such*. As I explained, the semi-fictional objects are the result of positing as separate features that are in fact instantiated by sensible bodies, but this fact alone might not be sufficient to dissipate all worries: after all, also the idea of a goat-stag is obtained by positing in thought an object that is obtained by putting together features that are in fact instantiated by sensible substances, but this does

not make claims about the goat-stag any truer. As Lear (1982: 188) recognises, Aristotle's account only works if he can 'explain how mathematical statements can be true in a way that does not depend on the singular terms having any reference'. Lear believes that the key to doing so is recognising that mathematics is *useful*, i.e. that it is applicable: 'For Aristotle, mathematics is true, not in virtue of the existence of separated mathematical objects to which its terms refer, but because it accurately describes the structural properties and relations which actual physical objects do have. […] One can understand how mathematics can be true, Aristotle thinks, by understanding how it is applicable' (191). In reducing mathematical truth to its applicability to sensible reality, this explanation presupposes that Aristotle is right to think that working on semi-fictional geometrical objects does not introduce any falsity, and to assume that the theorems that refer to semi-fictional objects are in fact applicable to sensible reality. As we will see below, this assumption is not as innocent as it might seem; Part 2 of this thesis will be dedicated to discussing whether Aristotle's philosophy of geometry can properly account for its applicability. Lear himself recognises that not all parts of mathematics are applicable, if by this we mean that they describe actual physical states of affairs: he believes that the conclusion that we should draw is that some branches of mathematics are not true, even if they are consistent with mathematical truths; the distinction can only be known *a posteriori*. This is not Aristotle's position, though.[138]

I do not think that invoking applicability to actual physical reality is necessary to account for the truth of geometrical theorems. I have argued that the *qua*-theory establishes that the truth-makers of geometrical statements are sensible objects which possess (either actually or potentially) geometrical features. I believe that this would still

[138] Lear never suggests that it is. He makes the weaker point that 'though not Aristotle's, this philosophy of mathematics is Aristotelian' (1982: 192).

be sufficient to account for the truth of geometrical statements, even if it turned out that some branches of geometry are not 'useful' in that they cannot be used to describe actual sensible substances.

Note that on this interpretation, a science results ontologically committed to its truth-makers, and not necessarily to the referents of the singular terms that appear in its statements. Although it might seem bizarre that the truth-makers of a sentence are not the referents of the singular terms that appear in it, this is a position that is in fact on the market also in the contemporary discussion. For example, this is suggested by Cameron (2008: 4): 'The main question for present purposes is this: what makes existence claims true? Armstrong (2003) says that the truth-maker for <*x* exists> is always *x*. In that case, if a theory T includes the sentence "a exists", the Quinean criterion and the truth-maker criterion will agree that a is an ontological commitment of T. But I reject Armstrong's claim: I think one of the benefits of truth-maker theory is to allow that <*x* exists> might be made true by something other than *x*, and hence that "*a* exists" might be true according to some theory without a being an ontological commitment of that theory.' For example, the claim 'there is a chair' might be true, but this might not carry with it an ontological commitment to chairs. If the truth of the sentence 'there is a chair' is grounded in the existence of atoms assembled chair-wise, a theory can accept the truth of 'there is a chair' but be ontologically committed only to its truth-makers (namely the atoms, not the chair), even while accepting that 'chair' really refers to the chair.

Note that supposing that the truth-makers of geometrical propositions are different from the referents of the singular terms which appear in those propositions (and that these referents are the semi-fictional objects used as a heuristic device to establish those truths) leads to supposing a certain ambiguity in the way in which Aristotle uses expressions such as *ta mathematika*, or when he asks what geometry is about. I believe that this might

account for the oscillations in Aristotle's way of presenting mathematical objects that I have referred to at the beginning of the chapter. Depending on the context, he might focus on the fact that geometricals actually are features and properties of sensible bodies, or on the fact that the objects that mathematicians use to discover and prove geometrical theorems are (semi-fictional) objects that are akin to substances in some key respects.

Before turning to Aristotle's claim that separation does not introduce any falsehood, let me clarify that I do not take separation to consist in the actualisation of geometrical objects that are merely potentially present in sensible substances. This seems to be what Ross has in mind, and it has been more recently suggested by Pettigrew (2009). Pettigrew argues that geometrical entities have a special mode of existence in that they exist *potentially* in sensible reality, and are actualised by the geometer who separates them in thought. On the contrary, I believe that 'to separate and posit what is not in fact separate' means positing in thought, and treating as an independent substance, something that is not in fact a substance and that only exists as a feature of sensible objects. In practical terms, the way in which it is done is by creating a mental object that has all and only the properties that we are interested in – and in particular, whose essence is exhausted by the geometrical features in question. Geometrical features are not potential parts of the sensible object that need to be actualised by the geometrician. They are properties and accidents of sensibles, that might be either actual or potential. Separation does not turn a potential feature into an actual one, but a property into a (fictionally) independent substance.

On the contrary, Pettigrew points at two main reasons why one might think that separation amounts to actualisation:

1. In *Metaphysics* Θ.6, 1047b30-34, Aristotle claims that 'the half-line is potentially in the whole';

2. In *Physics* VIII.8, 262a21-25, Aristotle explains that the division of the line is operated by actualising a point, that was previously merely potential. Such actualisation can be done by a physical act or by a mental act of the observer.

Neither of these passages gives good reasons to think that separation consists in actualising geometrical objects that were previously potentially present in sensible objects. As a matter of fact, both passages describe *intra-mathematical* phenomena, and do not concern the relation between geometrical and sensible objects. The first passage simply states that the half-line is a potential part of the whole line, and can only be actualised by dividing the line. The second passage describes the relationship between lower- and higher-dimensional objects (which is the same both for geometrical and sensible continuous objects, as Aristotle makes clear in *Ph*. VI.2): points are limits of lines, and each continuous line can be potentially divided *ad infinitum*; each division actualises a point that was previously only potentially present in the line. This sheds light on Aristotle's theory of continuity, not on his ontological account of geometrical objects. While it does show that mental acts can have the effect of actualising something that was previously potential, the model of the division of the line should not be used to account for the relationship between sensible and geometrical objects.[139]

However, the reader will recall that in chapter 2 I have suggested that some geometrical properties might be partially masked by the sensible features of physical bodies. In these cases, separation does have the effect of removing the masks, and thus ultimately allows geometrical features to be fully manifested. In these cases, separation *enables* the actualisation of geometrical properties that were previously potential by creating the right circumstances for their manifestation. But it would be a mistake to think

[139] In general, the idea that geometrical objects are *parts* of sensibles (although potential) is problematic: see Katz (2022: 173-192) for a full criticism.

that separation *consists* in such an actualisation. In the next section we will see why this is important for the practice of geometry.

§4. No falsehood

Aristotle claims that positing geometrical features as separate (i.e. as substances) does not introduce any falsehood. This claim is presented as a direct consequence of the *qua*-theory: the idea seems to be that *because* (i) the identity and possibility of geometry as a science do not depend on its objects but on its methodology, and (ii) geometrical statements are true in virtue of sensible bodies and their geometrical properties, *then* geometers can introduce semi-fictional objects and use them to prove their theorems, as heuristic devices.[140] Aristotle recognises that the legitimacy of this procedure might not be obvious, and offers a justification by means of an analogy:

> ὥστ' εἴ τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ᾗ τοιαῦτα, οὐθὲν διὰ τοῦτο ψεῦδος ψεύσεται, ὥσπερ οὐδ' ὅταν ἐν τῇ γῇ γράφῃ καὶ ποδιαίαν φῇ τὴν μὴ ποδιαίαν· οὐ γὰρ ἐν ταῖς προτάσεσι τὸ ψεῦδος.
>
> So if one posits objects separated from what is incidental to them, and studies them as such, one will not for this reason assert a falsehood, any more than if one draws a foot on the ground and calls it a foot long when it is not a foot long; the falsehood is not part of the premises.[141]
>
> *Metaph.* M.3, 1078a18-21 (Annas' translation)

[140] See also Netz (2006: 25).
[141] There is a textual issue on the position of the negation: is it a one-foot long line that is said not to be one-foot long, or vice versa? Netz (2006) accepts the manuscripts' version and rejects Jaeger's correction, which is adopted by Annas and which I have kept. I do not think that this has philosophical implications.

The analogy is spare in details, but it seems clear that Aristotle is referring to the habit of using diagrams that are not metrically accurate to prove geometrical theorems;[142] the mention of premises makes it clear that Aristotle has in mind an argumentative context. Notice that as it happens in the previous sections of M.3, Aristotle uses geometrical practices as a model to solve meta-mathematical issues. Once again, he does not explain the procedure he is referring to: he seems to assume that the reader is familiar with it. It is interesting that the same case is appealed to in a number of places in the *Corpus*:

- *APr.* I 41, 49b35-7: '...like the geometer who calls this a foot-long line, this a straight line, and says that they are breadthless, though they are not, but does not use these things as though he were deducing from them'.
- *APo.* I 10, 76b30-77a3: 'Geometers do not suppose falsehoods, as some have asserted. They say that you should not use falsehoods but that geometers speak falsely when they say that a line which is not a foot long is a foot long or that a drawn line which is not straight is straight. But geometers do not conclude anything from the fact that the lines which they have themselves described are thus and so; rather, they rely on what these lines show'.
- *Metaphysics* N 2, 1089a 21-25: '...like geometers when they assume a line to be a foot long when it is not a foot long. But ... geometers do not make any false assumptions (it is not a premise in their reasoning)'.

The frequency of this example, and the fact that it is referred to without explanation, suggests that Aristotle is hinting at a debate that was ongoing at the time, presumably regarding the use of diagrams in geometrical proofs. It is not a stretch to suppose, as Netz (2006: 25) does, that the debate is inspired by the Divided Line or similar unwritten

[142] This passage might be even read as evidence that Greek geometers drew their diagrams on sand. Other evidence for this can be found in Aristophanes (*Nu.* 77) and Cicero (*Tusc*. V.64; *ND* II.48). See Netz (1999: 14-15).

discussions: the recurring elements are (i) an hypothetical element in mathematics; (ii) the possibility of falsehood deriving from that hypothesis; (iii) an example of declaring a line to be of a certain length. What is interesting is that Aristotle is not participating in this debate, but assuming that it has been successfully solved and that the geometricians' use of metrically inaccurate diagrams is justified. Aristotle's strategy in M.3 is to use the solution to this debate as a model to solve the meta-mathematical problem of the relation between the semi-fictional object which is referred to in geometrical statements and the sensible object that is the truth-maker of such statement. The idea is that the semi-fictional object can be used to find out geometrical truths (even if it is not their truth-maker) in the same way in which geometers allowed to use diagrams in the course of their practice, even if the theorems that are proved by using these diagrams do not precisely hold of them.

Note that the diagrams that we find in the manuscripts tend not to be accurate depictions of the configuration that is described by the proof – and in particular, they tend not to be metrically accurate. Up to a certain level, this is an inevitable feature of geometrical diagrams that we are still familiar with: we would never expect the diagram of a 1km long line to be actually 1km long. But there are also more striking features: for example, they tend to represent special or peculiar cases rather than generic ones (for example, triangles tend to be equilateral or isosceles).[143] Moreover, lines are not always straight and circles are not always round, and lines that are said to be equal are not always drawn equal. This is sufficient to show that the diagram is not the proper object of the proof, and that not everything can be immediately read off from it. But this is not a problem, because the way in which diagrams are used is strictly codified: not all of their

[143] Manders (1995/2008) calls this feature 'atypicality'. But see Acerbi (2017), who suggests that this might be the result of oversymmetrisation during the transmission of the diagrams.

features can be appealed to in the course of the proof (or, to use Aristotle's terms, 'are part of the premises'). As I mentioned above, only co-exact features of diagrams can be used to license inferential steps. Metrical features, on the contrary, are exact and thus need to be provided by the text of the proof rather than being read off the diagram.

The analogy is meant to explain why we can 'separate' the geometrical features and consider them as independently existing: the falsity that the act of separation introduces is not something that geometers rely upon in the course of their demonstrations.

Note that there are two aspects involved in positing as separate something that is not separate. On the one hand, the fiction concerns the mode of existing of the entities, for properties are fictionally transformed in substance-like objects. On the other hand, the separation of geometrical features leads to examining them in isolation from the other properties of the sensible objects to which they belong.

The first aspect is justified by the deflationary ontological import of the *qua*-theory. It is *precisely because* the identity of geometry is not given by the kind of objects that it studies but only by its methodology (as showed by the *qua*-theory), that geometers can conduct their demonstrations on semi-fictional objects without consequences, because the identity and mode of existence of geometrical objects were never meant to be relevant to the science itself.[144] So operating on sensible objects or on semi-fictional geometrical objects leads to the same result, because the specific identity of the object in question is irrelevant. Geometers only care about the things that they find out by using it, not about the nature of the thing itself: the theorems that they prove do not regard the mode of being of their objects.

As for the second aspect, it is important to check that the properties that we wish to separate are of the right kind; for not all kinds of features are apt to be separated. As

[144] Cf. Netz (2006: 29).

Aristotle points out at *Physics* II.2, this is the mistake done by the Platonists: they separate the wrong features, and thus introduce errors. According to Aristotle, the difference can be seen by paying attention to the definitions of each kind of properties: while geometrical features can be defined without reference to matter, sensible features cannot. This explanation is different from the one given in M.3, but the two are complementary: Aristotle apparently believes that because geometrical features can correctly be defined without mentioning matter nor change, the shift in the scope of the negation can be operated without introducing error.

However, Aristotle does not merely say that geometers are allowed to separate their objects. He argues that this is the best way for them to proceed. There are at least two sets of possible reasons. On the one hand, positing as separate in thought results in the creation of a mental image, that is modifiable in the imagination.[145] In Chapter 5 I will argue that geometrical objects undergo modifications in the course of geometrical proofs; while the geometrical properties of sensible objects cannot undergo these modifications, the semi-fictional objects obtained by positing these features as separate can be modified by the imagination.[146]

On the other hand, if I am right and geometrical properties and potentialities are in fact partially masked by the sensible features of physical objects, then it is necessary to separate geometrical properties in order to remove the masks, and allow them to manifest fully. Let us take the example provided by Aristotle in M.3:

[145] Humphreys (2017) argues that it is necessary to subject these mental objects to continuous transformations so as to distinguish between exact and co-exact properties and be able to use diagrams correctly.

[146] Cf. Giardino (2018)'s idea of manipulative imagination: that diagrams prompt us to imagine manipulations and modifications of the geometrical configuration, and this leads to geometrical understanding.

ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ ἄνθρωπος· ὁ δ' ἔθετο ἓν ἀδιαίρετον, εἶτ' ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ' ᾗ ἄνθρωπος οὔθ' ᾗ ἀδιαίρετος ἀλλ' ᾗ στερεόν. ἃ γὰρ κἂν εἰ μή που ἦν ἀδιαίρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν [τὸ δυνατόν], ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν· διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ' ὑλικῶς.

A man is one and indivisible as a man, and the arithmetician posits him as one and indivisible; the geometer, on the other hand, studies him neither as man nor as indivisible, but as a solid object. For clearly properties he would have had even if he had not been indivisible can belong to him without them.[147] That is why geometers speak correctly: they talk about existing things, and they really do exist – for what exists does so in one of two senses, in actuality or as matter.

Metaph. M.3, 1078a23-31 (Annas' translation)

The arithmetician and the geometer are said to study one and the same object, a man, but with a different focus: the arithmetician studies him as one and indivisible, while the geometer studies him as a solid object, i.e. as continuous and divisible. In the previous chapter I have discussed how Aristotle's account of the *qua*-operator can be defended against the Contradictory Properties Objection. I have suggested that although the property of divisibility does belong to the man in virtue of the man's having a three-dimensional and continuous body, the property's full manifestation is blocked by the fact

[147] The example is difficult to follow because this key sentence presents a number of textual and interpretative problems. Τούτων at line 28 refers to humanity and indivisibility, i.e. the focus of the arithmetician, who at line 26 is said to study the man *qua* man and indivisible. Alexander may have read τούτου instead, as referring only to indivisibility; from a philosophical point of view it makes no difference. I agree with Ross that τὸ δυνατόν, which Alexander takes to be the subject of ὑπάρχειν, is probably the gloss (probably on ὑλικῶς at line 31) of a copyist who took τούτων to be the antecedent of ἃ and thought that ὑπάρχειν needed a subject. The words are omitted by Γ and are difficult to square with the meaning of the passage.

that the man is indivisible *qua* man. Here, I wish to stress that the geometer, in order to be able to properly study the man *qua* continuous and divisible, is not only allowed, but *needs* to separate these features and study them as separate. For otherwise the divisibility, that does belong to the man in virtue of his being a bodily entity, could not be manifested: it is blocked by the fact that the man is indivisible. Separation allows the geometer to actualise in thought the potential divisibility of the man, and thus visualise and study its manifestation. Positing geometrical features as separate enables the geometer not only to ignore, but to properly remove the masks: the mental image that results from separation can thus be modified in ways that would not be allowed in the sensible world, and this leads to gaining knowledge.

Despite the textual problems, it seems clear that when Aristotle speaks of the 'properties he would have had even if he had not been indivisible' he is envisaging a counterfactual situation in which we are not only ignoring the actual indivisibility of the man, but the man does not in fact have this feature.[148] This counterfactual situation is precisely the one that is obtained (in thought) thanks to separation, in which irrelevant features are not merely bracketed but are removed by the imagination. As such, the divisibility of the man can be acted upon, and the man can be manipulated in such a way as to make its geometrical features fully manifest and knowable. The assumption on which Aristotle's theory rests is that the features discovered in this way are in fact features *of the original sensible object*, in this case the man.

Is this assumption justified? Is it certain that the analysis conducted on the separated, semi-fictional object is applicable to the sensible body we started from? One might doubt it. For example, Netz (2006: 34) argues that even if the only difference between the separate semi-fictional objects and the sensible ones is that the former lack some defining

[148] See Netz (2006: 29) for the idea that Aristotle is referring to a counterfactual situation.

property of the latter, this is not sufficient to grant that whatever holds of the separate objects also holds of the sensible ones. For adding a property might result in the destruction of a phenomenon. He considers the following example:

> Consider for instance 'enmassed' and 'positively charged'. One of the consequents of 'enmassed' is 'mutually attracted': two pieces of enmassed matter attract each other. So:
>
> *Mutual attraction* **is** among *consequents (enmassed)*.
>
> At the same time, 'mutual attraction' is not among the consequents of 'positively charged'. In fact, two pieces of matter with positive electric charge repel each other. So:
>
> *Mutual attraction* **is not** among *consequents (positively charged)*.
>
> Clearly, then,
>
> *Mutual attraction* is **not among** *consequents (enmassed and positively charged)*
>
> Netz (2006: 34-5)

Netz concludes that Aristotle's philosophy of geometry is ultimately indefensible, because his assumption of monotonicity is ungrounded and jeopardises the applicability of geometry to the sensible world. It is not always the case that results obtained by studying separated properties are in fact true of the objects out of which those properties have been separated; without an additional argument, Aristotle is not allowed to simply assume that this is the case.

However, he also stresses that Aristotle's theory is (in historic terms) a valid response to Platonism:

> What is remarkable is not that Aristotle failed, but that he was so nearly right: there is no reason why he should have come up with the analysis of the non-compositionality of perspectives and, without this analysis, he was, I suggest, justified in his position.

Netz (2006: 38)

I believe that he is at the same time too lenient and too hard with Aristotle. While it is true that we should not expect Aristotle to come up with an analysis of the non-compositionality of perspective in the same terms as Cartwright (1983), it should be stressed that Aristotle himself claims that separation is allowed only insofar as 'the falsehood does not enter the premises'. If shifting the scope of the negation, and eliminating the masks, makes such a difference to the outcome of the analysis, it means that the falsehood has indeed entered the premises: the conclusion strictly depends on a false assumption. Thus, I believe that if really geometry turned out not to be applicable to the sensible world, Aristotle himself would have to accept that his view is flawed. At the same time, I think that Aristotle's faith in the applicability of geometry might be (at least partially) justified, even if this might require a renegotiation of the notion and the scope of applicability.

Part 2. The Problem of Descent

Introduction

Part 1 offered an analysis of *Metaphysics* M.3, the key passage on the ontology of geometry. I argued that Aristotle gives a radically anti-Platonist account of geometrical truth, according to which no additional, specifically geometrical objects are needed to ground the truth of geometrical theorems. Geometry is true of the sensible world, for sensible substances do instantiate geometrical properties (even if they might be hidden and masked by the sensible and material aspects of physical bodies). Thus, geometers simply study sensible substances, although considering them not *qua* sensible but *qua* quantitative and continuous. This is sufficient to account for the truth of geometrical theorems, and for the identity and specificity of geometry. However, it seems in tension with the geometrical practices of the time. The way in which geometers act and speak seems to presuppose a Platonist ontology, according to which geometry is about properly geometrical objects (and not geometrical properties of sensibles). I suggested that in order to account for this, Aristotle appeals to the doctrine of separation.

In Chapter 3 I argued that Aristotle allows geometers to 'separate and posit' the geometrical features of sensible objects, and to study them 'as separate' even if in fact they are not separate from sensible substances. In particular, I claimed that this operation involves introducing an element of fiction that is absent from the *qua*-theory, and that is reflected in a shift in the scope of the negation. While the *qua*-theory states that geometers study sensible objects not *qua* sensible, separating a geometrical feature and considering it *qua* separate amounts considering their object *qua* not sensible. This would not be allowed by the *qua*-theory, which assumes that it is only possible to consider *X qua Y* if *X* is in fact *Y*; the shift in the scope of the negation signals that a fiction has been introduced. This does not mean that geometry studies objects that do not exist, though,

for the fiction concerns the mode of existence of the features under consideration; what is in fact a property of a sensible substance is taken to exist in its own right, as a substance.

The objects that are produced via separation are mental objects. I have called them 'semi-fictional' objects to stress that the features they instantiate are actually present in the sensible world and are not introduced as idealisations or full-on inventions; the fictional element is limited to their mode of existence. While they are real (existent) properties, they are not real objects but are treated as non-sensible real objects.

I claimed that these semi-fictional objects are supposed to account for the actual practice of geometry, and for the way in which geometers act and speak. What emerged from my analysis is that such objects are the referents of the singular terms that appear in geometrical statements, and are the objects that geometers work on when they carry out proofs and demonstrations, but they are not the truth-makers of geometrical statements. Geometry is true because it tells something true about the sensible world, its substances and the geometrical features that they instantiate; geometers are allowed to work with mental models that they obtain by separating the features they are interested in from the sensible substances in which they inhere, but this operation does not affect in any way the truth-value of geometrical statements.

I have also pointed out that Aristotle seems to assume that working on such semi-fictional objects will yield results that are still applicable to sensible reality. This assumption is not as innocent and uncontroversial as Aristotle seems to suppose. The very same thing that makes it necessary to resort to semi-fictional objects – namely, that some geometrical potentialities might not be (fully) actualisable in normal conditions – makes it difficult to believe that the findings obtained by working on these objects are

automatically and unproblematically applicable to the sensible substances that we started from.[149]

In Part 2, I will leave on the side the issue of the ontological status of geometrical objects and the way in which the truth of geometrical statements is grounded in sensible objects and their properties. I will rather focus on the features that Aristotle attributes to these semi-fictional objects, and the use that he makes of them. In so doing, I hope to be able to show that his faith in the applicability of geometry is not completely misplaced, but can be defended and (partially) salvaged. However, I will conclude that this can only be done by imposing some constraints on the notion of applicability; in particular, we will see that its scope must be limited. For Aristotle, geometry is applicable to the sensible world only at a strictly *local* level, both from a spatial and from a temporal point of view: geometry can be used to study the structural properties and the spatial relations of small configurations of objects, and only at a certain instant. It cannot be used to describe the global spatial structure of the Aristotelian universe, nor to account for the changes and modifications of sensible objects through time.[150]

Chapter 4 focuses on the notion of intelligible matter, and on the idea that the semi-fictional geometrical objects appealed to in the course of geometrical proofs are 'thises' which have a hylomorphic structure. Chapter 5 examines whether and in which way such objects are able to undergo manipulations and modifications, given that Aristotle claims that they are 'separate from motion'.[151] With respect to the Applicability Problem, the analysis conducted in Chapter 4 will explain why I take geometry to be applicable at a

[149] This is the problem that Netz (2006) raises in terms of compositionality of perspectives, and that can also be formulated in terms of monotonicity. Introducing semi-fictional objects allows the geometer to work in a fictional world in which there are no masks. Why should the results that she gets in this way still be valid once we reintroduce masks and interferers?
[150] Note that this only applies to geometry, not to subordinate sciences such as mechanics or astronomy. I will discuss these sciences in Chapter 5.
[151] *Ph.* II.2, 193b34.

mere localised level (as opposed to global) in the *spatial* sense; in Chapter 5, I will show that the applicability of geometry is constrained from a *temporal* point of view, too.

Recall that I have identified an ambiguity in Aristotle's use of the term *ta mathematika* (and in general, in the way in which he talks about what geometry is about): he might mean either the truth-makers of geometrical propositions (sensible substances and their geometrical properties) or the referents of the singular terms that appear in geometrical statements, i.e. the semi-fictional objects that geometers use in the course of their work (that result from the operation of separation). If not otherwise stated, in these last two chapters I will use the terms *ta mathematika*, 'geometrical objects', and cognates to indicate the latter. I will assume the perspective of the working geometer who is not interested in the ontological status of geometrical objects, but simply studies them and uses them to acquire and prove geometrical truths. I am not questioning the ontological results that emerged from the previous chapters, but approaching the issue from the other side: are geometers really justified in taking their objects (which we know to be semi-fictional objects) at face value, and use them to learn something about the geometrical features of the sensible world?

Chapter 4. Intelligible Matter

Metaphysics M.3 does not say anything about the specific features of the objects that result from separation. On my reading of the chapter, the reason for this silence is that M.3 is mainly concerned with establishing that the truth of geometrical statements and the possibility of geometry as a science are not jeopardised by Aristotle's rejection of eternal and immutable geometrical objects that exist independently of, and over and above, sensible substances. Discussing the specific features of the semi-fictional objects that geometers appeal to in the course of their everyday practice goes beyond the scope of Aristotle's inquiry in M.3. It is therefore not surprising that he does not say much about this.[152] We find more information on this issue in other parts of the *corpus*.

I will start by looking at the first notable fact about these semi-fictional geometrical objects: they are conceived of as individual particulars akin to substances. In *Metaphysics* Z.10 and 11 we are told that geometrical objects are hylomorphic compounds that have a special sort of matter – that Aristotle calls 'intelligible matter', *hylē noētē*. The formula 'intelligible matter' is in itself peculiar, and occurs only three times in the whole *Corpus*. After introducing the relevant passages in §1, I examine the idea that the role of intelligible matter for geometrical objects can be played by extension (§2), and the hypothesis that understanding intelligible matter in this way can grant the applicability of geometry to sensible reality. I argue that the notion of extension should not be modelled after our concept of space, but can only be the extension of individual bodies (§3).

[152] A similar reasoning is valid for *Ph.* II.2, which is concerned with distinguishing the enterprise of the geometer from that of the physicist, even if they study the same objects. Clearly, the objects in question are the truth-makers of geometrical statements, not the semi-fictional objects that result from separation. The passage gives quite a few details as to why geometers are allowed to separate while physicists are not, and this involves stressing that geometrical features are separable in definition from motion and matter. But Aristotle does not need to go further than that to make his point, and in particular he does not need to give any information about the characteristics of the objects that result from separation.

Euclidean geometry is not apt to describe the spatial structure of Aristotle's *kosmos*, which is anisotropic, inhomogeneous and (most importantly) finite; this implies that geometry can only be applicable in a localised way, to describe determinate configurations of bodies and not the structure of the universe as a whole (§4). To conclude, I examine whether identifying the intelligible matter of geometrical objects with extension is sufficient to solve the Plurality Problem (§5).

§1. Hylomorphic Compounds

In books Z and H of the *Metaphysics*, there are three occurrences of a strange expression that does not appear anywhere else in the Aristotelian *Corpus*: ὕλη νοητή, intelligible matter.[153] The expression itself is puzzling: matter is *per se* unknowable, so that the combination sounds like a contradiction in terms. The way in which it is used is even more puzzling: despite appearing only three times in total, it is difficult to give a coherent account of its usage that fits all three the occurrences. While in Z.10 and Z.11 intelligible matter is presented as a special kind of matter (namely the matter of intelligible objects such as geometrical objects), the H.6 occurrence seems difficult to square with this picture: according to the mainstream interpretation, in that context 'intelligible matter' refers to the genus in a definition. It is thus unsurprising that most of the (scarce!) literature on the topic focuses on how to square the three occurrences.[154] This is a pressing and challenging issue, but I will not be tackling it. Solving this question would require considering the general dialectic and the internal consistency of the Central Books – a task beyond the scope of this book. My interest in intelligible matter is instrumental to highlighting Aristotle's conception of geometrical objects. Thus, I will focus on the first

[153] *Metaph.* Z.10, 1036a8-11; *Metaph.* Z.11, 1037a5; *Metaph.* H.6, 1045a34-36 and 45b1.
[154] See e.g. Gaukroger (1980), Jones (1983), Helmig (2007), Michetti (2022).

two instances (in Z.10 and Z.11), where it is clear that intelligible matter is supposed to be a special kind of matter that belongs to intelligible objects – among which figure geometrical objects.[155] I wish to stress from the beginning of this analysis that my interest in this context is only in the intelligible matter *of geometrical objects*. Aristotle does not explain what intelligible matter is, nor what are its specific features. My goal in this chapter is to understand what can play the role of intelligible matter for geometrical objects, but I have no ambition to give an account that can be extended to other kinds of intelligible objects (not even to other mathematical objects: determining whether numbers have intelligible matter, and in case what it is, would require additional investigation).[156]

Let us look at the relevant passages and their context. Z.10's central question is whether the definition of the parts is contained in the definition of the whole. Aristotle starts by drawing a distinction between formal and material parts. The material parts of an item are the ones 'into which it can be divided as into its matter';[157] for example, bones, sinews and flesh are material parts of a man but are not parts of its substance. 'These are rather their matter, and are parts of the combined whole, but not parts of the form or of what has the formula. Hence they do not occur in the formula'.[158] Only the formal parts of an object appear in its definition, while the material parts do not. For our purposes, it is interesting that Aristotle applies the same distinction between material and formal parts both to sensible and to geometrical objects. For example, he stresses that 'the formula of a circle does not contain that of its segments',[159] because the segments are material parts

[155] See Michetti (2022) for a defence of the compatibility of the H.6 occurrence with this reading.
[156] As will become clear shortly, I believe that in Z.10-11 Aristotle does not give a detailed theory of a special kind of matter that he calls 'intelligible matter'; rather, he argues that also in the case of intelligible objects there must be something that plays the role of matter. It is not necessary that one and the same thing plays the role of matter for all different kinds of intelligible objects. Thanks to David Ebrey for a helpful conversation on this point.
[157] 1035b12, εἰς ἃ διαιρεῖται ὡς ὕλην.
[158] 1035a20-22, ἀλλ' ὡς ἐξ ὕλης, καὶ τοῦ μὲν συνόλου μέρη, τοῦ εἴδους δὲ καὶ οὗ ὁ λόγος οὐκέτι· διόπερ οὐδ' ἐν τοῖς λόγοις.
[159] 1035a9-10, ὁ μὲν τοῦ κύκλου λόγος οὐκ ἔχει τὸν τῶν τμημάτων.

of the circle, something out of which it is composed as of its matter and into which it can be divided.

After establishing that only the formal parts appear in the definition, Aristotle proceeds to state that the definition is of the universal and not of the particular. Here we have another remark regarding geometrical objects: Aristotle stresses that the distinction between universal and particular applies to them, too. He points this out because a lack of specific terminology might create confusion:

> …ἔτι δὲ ὁ κύκλος εἰς τὰ τμήματα· ἔστι γάρ τις ὃς συνείληπται τῇ ὕλῃ· ὁμωνύμως γὰρ λέγεται κύκλος ὅ τε ἁπλῶς λεγόμενος καὶ ὁ καθ’ ἕκαστα διὰ τὸ μὴ εἶναι ἴδιον ὄνομα τοῖς καθ’ ἕκαστον.
>
> …and again the circle [is resolved] into its segments; for there is a sense of ‘circle’ in which it involves matter. For ‘circle’ is used homonymously, meaning both the circle in general and the individual circle, because there is no name proper to the individuals.
>
> *Metaph.* Z.10, 1035a34-b3 (Barnes’ translation)

This passage is important, for it not only confirms that Aristotle takes geometrical objects to be individual particulars, but also explicitly connects their being particulars to their having matter. He then proceeds to state that (properly speaking) the definition is only of the universal and not of the particulars, i.e. not of the individual compounds:

> τοῦ δὲ συνόλου ἤδη, οἷον κύκλου τουδὶ καὶ τῶν καθ’ ἕκαστά τινος ἢ αἰσθητοῦ ἢ νοητοῦ—λέγω δὲ νοητοὺς μὲν οἷον τοὺς μαθηματικούς, αἰσθητοὺς δὲ οἷον τοὺς χαλκοῦς καὶ τοὺς ξυλίνους—τούτων δὲ οὐκ ἔστιν ὁρισμός.

> But when we come to the compound, e.g. *this* circle, i.e. one of the individual circles, whether sensible or intelligible (I mean by intelligible circles the mathematical, and by sensible circles those of bronze and of wood), of these there is no definition.
>
> *Metaphysics* Z.10, 1036a2-5 (Barnes's translation, modified)

Interestingly, this passage presents both the sensible and the intelligible particular circles as hylomorphic compounds. In what follows, Aristotle explains that there are two different kinds of matter, sensible and intelligible:

> ἀλλ' ἀεὶ λέγονται καὶ γνωρίζονται τῷ καθόλου λόγῳ. ἡ δ' ὕλη ἄγνωστος καθ' αὑτήν. ὕλη δὲ ἡ μὲν αἰσθητή ἐστιν ἡ δὲ νοητή, αἰσθητὴ μὲν οἷον χαλκὸς καὶ ξύλον καὶ ὅση κινητὴ ὕλη, νοητὴ δὲ ἡ ἐν τοῖς αἰσθητοῖς ὑπάρχουσα μὴ ᾗ αἰσθητά, οἷον τὰ μαθηματικά.
>
> They are always known and spoken of by means of the universal formula, for matter is in itself unknowable. (Some matter is perceptible, e.g. bronze, wood, and all changeable matter, while some is intelligible, namely that which is present in perceptible things but not *qua* perceptible, such as the objects of mathematics.)
>
> 1036a7-12 (Bostock's translation, modified)

Let us stop for a second on the labels 'intelligible matter' and 'perceptible matter'. Having just read that 'matter is in itself unknowable', the idea that some matter might be intelligible looks like a contradiction in terms. Notice that also the other pole of the dichotomy is equally paradoxical: the matter of perceptible objects can only be perceived insofar as it is informed. It is not matter itself, but the material compound, which is perceptible. It seems clear that 'intelligible' and 'perceptible' are not meant to qualify the two kinds of matter, but the two kinds of objects to which such matter may belong. There

is a certain kind of matter that belongs to sensible objects, and a different one that belongs to intelligible objects. The latter set comprises (but might not be limited to, as the use of *hoion* indicates) the objects of geometry.[160] We are not given any elucidation as to the specific characteristics of intelligible matter, nor as to the difference between the two kinds of matter. Aristotle merely gives two examples of sensible matter, and then generalises quickly to 'all matter that is changeable'; intelligible matter is only characterised by reference to the objects it belongs to. It is not even clear that they are really two *kinds* of matter, if by this we mean that they are species of one and the same genus. Note that the material parts of an object can be distinguished from its formal parts not because we know what (intelligible) matter is and we recognise it in the object, but merely on the basis of their function; material parts are the ones in which the object can be divided, and out of which it can be composed. The upshot is that Aristotle is not interested here in giving a theory of matter and of its different kinds. His goal is merely to stress that intelligible and sensible objects are not so different when it comes to their definition: their material parts are not included in their *logos*.

This conclusion is made explicit in Z.11:

> περὶ δὲ τὰ μαθηματικὰ διὰ τί οὐκ εἰσὶ μέρη οἱ λόγοι τῶν λόγων, οἷον τοῦ κύκλου τὰ ἡμικύκλια; οὐ γάρ ἐστιν αἰσθητὰ ταῦτα. ἢ οὐθὲν διαφέρει; ἔσται γὰρ ὕλη ἐνίων καὶ μὴ αἰσθητῶν· καὶ παντὸς γὰρ ὕλη τις ἔστιν ὃ μὴ ἔστι τί ἦν εἶναι καὶ εἶδος αὐτὸ καθ' αὑτὸ ἀλλὰ τόδε τι. κύκλου μὲν οὖν οὐκ ἔσται τοῦ καθόλου, τῶν δὲ καθ' ἕκαστα ἔσται μέρη ταῦτα, ὥσπερ εἴρηται πρότερον· ἔστι γὰρ ὕλη ἡ μὲν αἰσθητὴ ἡ δὲ νοητή.

[160] Note the use of the formula ἡ ἐν τοῖς αἰσθητοῖς ὑπάρχουσα μὴ ᾗ αἰσθητά: saying that intelligible matter inheres in sensible objects not *qua* sensible, Aristotle refers directly to the *qua*-theory and the account of geometrical objects provided in M.3. As we have seen, however, in that context there is no mention of intelligible matter.

> Yet the objects of mathematics are not perceptible objects. So why is it that here the formulae of the parts (e.g. the semicircles) are not parts of the formula of the whole (e.g. the circle)? But in fact there is no difference, for some non-perceptible objects also have matter. Indeed everything has matter of some sort unless it is not a this but a what-being-is and a form itself in its own right. Accordingly the semicircles will not be part of the universal circle, but they will be part of particular circles, as was said before. For matter may be either perceptible or intelligible.
>
> 1036 b32-1037 a5 (Bostock's translation)

Aristotle is here anticipating and blocking a possible objection: if the formal parts of an object are included in its definition, why aren't the parts of geometrical objects all included in their definition? After all (the objection goes), geometrical objects do not have sensible parts. Aristotle blocks this objection by distinguishing between sensible and material parts: not all material parts are sensible, because there are two kinds of matter – sensible and intelligible. Intelligible matter is not sensible but it does count as matter of individual geometrical objects; as a consequence, geometrical objects can have material parts that do not appear in the definition, even if they are not sensible.[161]

Once again, we have a close connection between having matter and being an individual particular. In this passage Aristotle is particularly explicit in claiming that everything that is a 'this' has matter: any individual *substance* comes with matter.[162] But why is this the case? *Metaphysics* Z.10-11 does not offer any precise reason. In Z.8,

[161] This does not imply that mathematical objects does not essentially include *any* intelligible matter. According to Z.11, 1036b7-13, continuity seems to be included in the definition of both the universal compound (e.g. triangle) and the form (triangularity). Intelligible objects do not differ in this respect from sensible objects. The material parts of individual sensible substances are not included in the definition of the correspondent universal compound, but matter might still be referred to in such definition.

[162] Not everything that is particular is necessarily a substance and a 'this'. For example, tropes are particulars but are not substances, and they do not have matter.

however, Aristotle seems to suggest that matter provides the principle of individuation for sensible things:

> τὸ δ' ἅπαν ἤδη, τὸ τοιόνδε εἶδος ἐν ταῖσδε ταῖς σαρξὶ καὶ ὀστοῖς, Καλλίας καὶ Σωκράτης· καὶ ἕτερον μὲν διὰ τὴν ὕλην (ἑτέρα γάρ), ταὐτὸ δὲ τῷ εἴδει (ἄτομον γὰρ τὸ εἶδος).
>
> The complete result, such a kind of form in this flesh and bones, is Callias or Socrates. They are different because of their matter, which is different; but they are the same in form, since their form is indivisible.
>
> *Metaphysics* Z.8, 1034a5-8 (Bostock's translation, modified)

Socrates and Callias share the same form (both are men), and thus they also have the same definition, for the definition is only of the form. However, they are two distinct individuals: what differentiates them is that the same form is applied to two different chunks of matter. The interpretation of this passage is controversial, and it is difficult to pin down precisely the role of matter in providing a principle of individuation. It is certain that matter does play a role in individuating particulars, but it is highly debated whether it really can achieve this result alone without requiring any appeal to form. Aristotle appeals to matter in order to distinguish between two individuals which share the same form. Lloyd correctly points out that a principle of *unity* (i.e. what makes this one thing rather than many) is required to put oneself in the position to appeal to matter as a principle of *individuation* (i.e. what makes this thing the same as or different from that thing): 'the flesh and bones have to have made one man, the metal one penny, before either can be or not be the same man or penny as some other man or penny' (1970: 519).

As Whiting (1986) argues, this principle of unity can only be formal.[163] Moreover, it should also be noted that the ostensive appeal to matter in the sense of 'composed of different stuff' is only effective in differentiating two things that are present *at the same time*.[164] If we want to move from synchronic to diachronic individuation, we need more complex tools – if only because it is unlikely that one individual is throughout its existence composed by one and the same chunk of matter (surely the identity of living beings survives changes in their material composition!).

If matter is what provides a principle of individuation, thus allowing one to distinguish two different individuals of the same kind, then it becomes clear why having matter is a necessary condition for being a particular, individual 'this'. But how is intelligible matter supposed to fulfil this function? The example of Callias and Socrates does not appear immediately generalisable to intelligible objects, which presumably are not made up of 'chunks' of intelligible matter – whatever intelligible matter is supposed to be. Determining whether intelligible matter can provide a principle of individuation for geometrical objects is one of the goals of this chapter. The idea that geometrical objects are individual particulars seems to presuppose the existence of a principle of individuation by which one can distinguish the various instances of the same kind of geometrical object. The issue is known in the literature as the Plurality Problem, and it is tempting to suppose that the doctrine of intelligible matter is meant to provide a solution.

But in the case of sensible objects, the link between having matter and being a 'this' is not limited to the fact that matter might provide a principle of individuation. Crucially, matter acts as a substratum to which properties and features belong. It is also required to

[163] Aristotle does not distinguish systematically between principle of unity and principle of individuation, and this causes a lot of confusion in the scholarship. For an illustrious example, see Anscombe et. al. (1953): as Popper notes in his intervention, Łukasiewicz' and Anscombe's papers are concerned with two different questions – the former focuses on the principle of unity, while the latter on the principle of individuation.
[164] See Anscombe (in Anscombe et. al. 1953).

account for the persistence of an object throughout changes: it provides the persistent substratum that undergoes and survives the change. Finally, matter is presented as 'that out of which things come to be',[165] and as the 'stuff' that makes up the objects. As we have seen, it is on the basis of this idea that the material parts of an object are distinguished from its formal parts, in the case of both sensible and intelligible objects.

So there are at least five ways in which sensible matter plays a role in accounting for the particularity of sensible substances:

1. Providing a principle of individuation;
2. Providing a substratum that persists through changes;
3. Providing a substratum to which properties belong;
4. Providing that out of which particulars come to be;
5. Providing the 'stuff' of which they are composed.

In Z.10-11, intelligible matter is introduced as analogous to sensible matter and as a necessary condition for being an individual 'this' in the case of intelligible objects. Aristotle does not provide any information as to its nature or characteristics. It is not even obvious that he thinks that one and the same thing is the intelligible matter of *all* intelligible objects. But it is clear that something must play this role, fulfilling at least some of the functions that I have outlined. In the following section I will try to determine whether in Aristotle's ontology there is something that can do the job in the case of geometrical objects. As I mentioned above, I do not wish to suggest that this analysis extends to other intelligible objects. Aristotle has only established that there must be something that plays the role of matter for each kind of intelligible object. It is not necessary that one and the same thing plays this role in all cases.

[165] *Metaph.* Z.7, 1032a17, τὸ δ' ἐξ οὗ γίγνεται, ἣν λέγομεν ὕλην.

§2. Intelligible Matter as Extension

From Alexander of Aphrodisias onwards, it has been suggested that the intelligible matter of geometrical objects might be spatial extension. This interpretation has become traditional, and I think that there are good reasons to accept it, provided that we understand 'extension' in the right way. In this section I will go through these reasons, examine what we mean by 'extension' and explain why the identification of intelligible matter and spatial extension is compatible with Aristotle's physics.

To understand what intelligible matter is, we can start by getting clearer on the other pole of the hylomorphic compound, that is the form of geometrical objects. Intuitively, it seems plausible to think that the form of a geometrical object simply is its *shape*, its *topological* form. If this is the case, then the matter of geometrical objects has to be what is delimited and determined by the shape – that is to say, extension. We can find a first element in support of this interpretation in *Metaph.* Δ.17, where Aristotle discusses the notion of limit (*peras*). He distinguishes various senses of 'limit' or ways in which something can be said to be a limit; the first definition that he provides is 'the terminus of a thing, i.e. the first thing outside which there is nothing to be found and the first thing inside which everything is to be found'.[166] The second characterisation is the one that is most interesting for us, for Aristotle identifies the limit of a magnitude with its form:

> καὶ ὃ ἂν ᾖ εἶδος μεγέθους ἢ ἔχοντος μέγεθος.
>
> [the limit is] that which is the form (εἶδος) of a magnitude or of something having a magnitude.
>
> *Metaphysics* Δ.17, 1022a5-6 (Pfeiffer's translation)

[166] *Metaphysics* Δ.17, 1022a4-5, Kirwan's translation. Πέρας λέγεται τό τε ἔσχατον ἑκάστου καὶ οὗ ἔξω μηδὲν ἔστι λαβεῖν πρώτου καὶ οὗ ἔσω πάντα πρώτου.

Since this is a definition of limit, it is clear that by 'form' (*eidos*), here Aristotle means *shape*: the topological form of the object, which is given by its outer boundary.[167] This is immediately picked up by Alexander, who comments that 'according to the present definition he means that figure (*schēma*) is the limit'.[168] In this passage Aristotle is not very precise in identifying the material element of the couple: he simply states that the limit is the form 'of a magnitude or of something having a magnitude', without even discussing the difference between the two options.

The same association between extension and matter can be found also in a radically different context: the discussion of place in *Physics* IV:

> εἰ δή ἐστιν ὁ τόπος τὸ πρῶτον περιέχον ἕκαστον τῶν σωμάτων, πέρας τι ἂν εἴη, ὥστε δόξειεν ἂν τὸ εἶδος καὶ ἡ μορφὴ ἑκάστου ὁ τόπος εἶναι, ᾧ ὁρίζεται τὸ μέγεθος καὶ ἡ ὕλη ἡ τοῦ μεγέθους· τοῦτο γὰρ ἑκάστου πέρας. οὕτω μὲν οὖν σκοποῦσιν ὁ τόπος τὸ ἑκάστου εἶδός ἐστιν· ᾗ δὲ δοκεῖ ὁ τόπος εἶναι τὸ διάστημα τοῦ μεγέθους, ἡ ὕλη· τοῦτο γὰρ ἕτερον τοῦ μεγέθους, τοῦτο δ' ἐστὶ τὸ περιεχόμενον ὑπὸ τοῦ εἴδους καὶ ὡρισμένον, οἷον ὑπὸ ἐπιπέδου καὶ πέρατος, ἔστι δὲ τοιοῦτον ἡ ὕλη καὶ τὸ ἀόριστον· ὅταν γὰρ ἀφαιρεθῇ τὸ πέρας καὶ τὰ πάθη τῆς σφαίρας, λείπεται οὐδὲν παρὰ τὴν ὕλην.
>
> Then, if place is the first thing surrounding each body, it will be a kind of limit. So place would appear to be the form and the shape of each thing, by which the magnitude, and the matter of the magnitude, are bounded; for that is what the limit of each thing is. If one considers it in this way, then, place is the form of each thing. But [if one considers place] in the way in which place is thought to be the extension of the magnitude, [place is] the matter. For this [extension of the magnitude] is different from the magnitude; this is that which is surrounded and bounded by the form, as by a surface and a limit; and it is this

[167] See Pfeiffer (2018: 96), Mendell (1987).
[168] Al. *in Metaph.* 413.29-30. See also Pfeiffer (2018: 96).

> kind of thing that matter and the indefinite are. For when the limit and the properties of the sphere are removed, nothing is left but the matter.
>
> Arist. *Ph.* IV.2, 209b1-11 (Hussey's translation)

This passage can help us to understand the way in which form and matter are related, and how Aristotle is conceiving of each of them, in the case of magnitudes – and thus, of geometrical objects. Aristotle is listing and examining the candidates for the role of place. The first two options that he considers (and rules out) are (i) the limit of a magnitude and (ii) 'that which is surrounded and bounded by the form', which he calls matter and 'extension', *diastēma*. Let us look at them in turn.

The first candidate is the limit of the magnitude: the first thing surrounding it, that 'outside which there is nothing to be found'. In accordance with the definition of limit given above, Aristotle immediately identifies this object with the shape (μορφὴ) of the magnitude, and with its form (εἶδος). In associating the form of the object with its shape, it is clear that Aristotle is talking about the topological body of the object, that is its magnitude.[169] In general, the form of a sensible substance is not identical to its shape – for example, the form of a book is not its being a parallelepiped. In the case of geometrical objects, however, the identification between form and shape is legitimate.

In accordance with Aristotle's theory of hylomorphism, form is imposed on matter and makes it determinate. If form is the shape of the magnitude, and thus its outer limit,

[169] Pfeiffer (2018: 97). Mendell (1987: 213) argues that 'this passage is quite clearly drawing a contrast between the matter and the form of solid magnitude, i.e. of mathematical bodies. Matter here is the extension of a mathematical body. Failure to understand this point has led to much confusion and many unfair criticisms of Aristotle's treatment of Plato'. Pfeiffer wishes to distinguish between the topological body of a sensible substance and the mathematical object, which is the topological body considered not *qua* the body of a sensible object. For my purposes it is not necessary to dwell on this disagreement, since I am using the *Physics* IV.2 passage as a way to get insights on the hylomorphic conception of geometrical objects: I take it that it does not make much of a difference if the passage is immediately about mathematical objects (as Mendell believes) or if the analogy is once removed (as Pfeiffer argues), since the way in which matter and form are related is supposed to be the same.

it follows that 'that which is surrounded and bounded' by it is the matter.[170] This is precisely how Aristotle describes the second option: he argues that Plato identifies place with matter, i.e. *diastema* bounded by the form (*eidos*). *Diastema* is a difficult word to translate,[171] but here Aristotle gives enough details to make it clear that he is thinking of the spatial extension between the limits of a body. He stresses that even if the magnitude and *diastema* are bound by the same limit, they are not the same thing: the difference lies in their relation to the limit. While the limit is not a part of the magnitude, it belongs to it in a strong way, in the sense that it is ontologically dependent on the magnitude and cannot be divided from it.[172] By contrast, the limit does not belong in this way to the *diastema*. The latter is what is left when the limit/form of the magnitude, as well as the other properties, are removed. It is *ahoriston*, indeterminate.

The fact that Aristotle associates limit and shape with form, and extension with matter, provides some support for the claim that the intelligible matter of geometrical objects is extension. But there is a worry that needs to be addressed: in the rest of *Physics* IV, Aristotle argues *against* the identification of place and extension, on the grounds that a spatial extension that is independent of bodies cannot exist. Thus there is an asymmetry between the strategy used by Aristotle to rule out the idea that place is the limit of the magnitude (the first hypothesis presented above) and the idea that place is the extension

[170] In most cases, the topological shape of an object is its shape. Monodimensional objects such as lines do not fit perfectly this model: they are limited by two points, which are not identical to their shape. However, the points do limit the extension of the line and determine its length, thus contributing to its shape. Anyway, for my purposes it is not necessary that the identity between limit and shape holds in all cases; the main point that I want to make is that what is limited is extension, and the association between limit and form supports this conclusion even if it is not a perfect identity that holds for all kinds of objects. Note that in *Ph.* IV Aristotle has very good reasons not to consider lines while looking for candidates for the role of place.

[171] The term *diastema* is used to refer to the radium of a circle, to the extension between two limits, as well as to the three directional axes of the universe (up-down; left-right; in front-behind. See *Ph.* IV.1, 208b8-23 and below, §3). See Sattler (2020b: 17-18), Sidoli (2004), Algra (1995: 54).

[172] Limits are not parts of bodies, because Aristotle has very strong conditions on parthood: (i) parts can measure the whole; (ii) parts can make up the whole; (iii) the whole can be divided into parts. Limits do not fulfil these conditions, but they do belong to the body that they delimit in a very strong sense: they are ontologically dependent on their bearer and cannot be divided from it. See Pfeiffer (2018: §6.3) for a discussion.

of the magnitude (the second hypothesis presented above). In the former case, he accepts that the limit exists but denies that it can be identified with the place (the limit would move with the body, thus being inadequate to act as place of the body); in the latter case, he straightforwardly denies that such an extension can exist.[173] So can we still identify intelligible matter with extension, given Aristotle's arguments against its existence?

I believe we can, because these arguments are aimed at refuting the existence of an extension that is *independent* of bodies;[174] they do not prove that bodies themselves are deprived of extension.[175] On the contrary, that bodies have extension is stated very clearly in *Physics* III.5, 204b20: 'body is what has extension in all dimensions'.[176]

This is sufficient for our purposes, for the intelligible matter of geometrical objects does not need to be independent of bodies. Recall that geometrical objects are semi-fictional objects obtained by separating features that in fact belong to sensible substances – it is unsurprising that their matter turns out to be a feature of bodies themselves, dependent on them for its existence and characteristics.[177] This is of fundamental

[173] Quarantotto (2017: 177).
[174] The main argument is presented at 211b9-19 and is very complex and difficult to interpret. Discussing it goes unfortunately beyond the scope of this work. For our purposes, the important thing is that the argument (however it is supposed to work) is aimed at refuting the existence of an *independently existing* extension, not at extension in general. See Quarantotto (2017: ch. 4) for a convincing interpretation.
[175] On the contrary, at least one of the arguments against the existence of a separate extension relies on the implicit assumption that bodies themselves do have extension. It is the so-called argument of the wooden cube, at *Ph*. IV 8, 216a26–b12. As Pfeiffer (2016; 2018: §6.4.4.2) points out, this argument works by showing that if there were an empty and independent extension that might be filled by bodies, the way in which a body penetrates it would be structurally analogous to the coincidence of two bodies. This leads to absurdity: if we allow for two bodies to coincide, what prevents any number of things being in the same place? The intuition is that if we allow for the coincidence of two bodies 'we lose a proper principle of individuation and cannot explain why there aren't even more things coinciding' (2016: 380). The two bodies in question are the extension external to the body, and the 'magnitude' (μέγεθος) or 'volume' (ὄγκος, also translated as 'bulk') of the body. What is interesting for our purposes is that the argument only works if the two bodies cannot be distinguished in any other way: the volume of the body does not have any peculiar feature that would distinguish it from the void that it occupies. This means that the *ogkos* itself is conceived as an extension, even if it is not external to the body but depends on it for its existence (as Aristotle says, it is 'not separable' from the body and its attributes). Thus, the argument indirectly proves that extension does exist – but it is not independent of bodies: bodies themselves have extension.
[176] σῶμα μὲν γάρ ἐστιν τὸ πάντῃ ἔχον διάστασιν.
[177] Note that if intelligible matter is identified with the extension of sensible bodies, it follows that sensible bodies have intelligible matter as well as sensible matter. This is ok; as I mentioned above, intelligible and sensible are not two distinct species of the kind matter. Intelligible matter needs not be something that

importance when it comes to evaluating the role of intelligible matter in granting the applicability of geometry.

To sum up, we now know that (i) Aristotle accepts the notion of extension, even if he stresses that it is not independent of bodies; (ii) he believes that such an extension is *ahoriston* and is determined by the imposition of limit/form; and (iii) he is willing to express this relation in hylomorphic terms. This is sufficient to establish that extension might fulfil at the very least one of the functions of matter that I have identified above – namely, it can act as substratum for properties. But it is still not enough to establish that intelligible matter really should be understood in terms of extension, nor does it allow us to evaluate the role of this identification in securing the applicability of geometry to the sensible world. To do this, we need to get clearer on the notion of extension itself.

Even if in *Ph.* IV.2 Aristotle maintains that *diastema* is what is left once the limit and all the properties of the sphere are removed, the extension of sensible bodies is not completely deprived of all features.[178] I will now show that it has two, which are structural and apt to be studied geometrically: dimensionality and continuity.[179] This can be appreciated already from the definition of body that I have quoted above: if body is what has extension in all dimensions, it means that extension is the kind of thing that has dimensionality. In particular, we will see that it can have up to three dimensions: body is said to be extended in *all* dimensions, and in *De Caelo* I.1 Aristotle states clearly that there cannot be more than three. On the other hand, lower-dimensional items such as the

sensible bodies do not have. It merely needs to be different from sensible matter. And in fact extension is not sensible matter, but a feature of sensible matter. See Pfeiffer (2018: 133).

[178] This means that it should not be thought as the pure potentiality, bare 'stuff', completely lacking all positive qualifications, that some interpreters take to be the matter of the elements, and which is supposed to make elemental change possible. What I say about extension is independent of the debate on prime matter.

[179] Note that if this were not the case, the argument of the wooden cube at *Ph*. IV.8, 216a26-b12 would not work. See fn. 173 above.

limits of bodies can be extended in fewer than three dimensions: surfaces are two-dimensional, while edges only have one dimension.

Continuity and dimensionality are closely connected: as in this passage, Aristotle often expresses the idea that magnitudes have one, two, or three dimensions by saying that they are *continuous* in one, two, or three ways or directions. Continuity applies to a broader set of magnitudes than the ones that have extension (for example, time and changes are continuous without being spatially extended), but there is a special way of being continuous that belongs to spatially extended magnitudes that Aristotle singles out in *Categories* 6, 5a15 ff., where he specifies that only a subset of the continuous magnitudes have parts which have a position.[180] This special kind of dimensional continuity is the extension that we are looking for. We will shortly return to the analysis of dimensionality and continuity, and argue that extension has precisely these characteristics because it is a feature of the matter of sensible bodies. But first, note that dimensionality and continuity are precisely the characteristics that the intelligible matter of geometrical objects should have; this makes extension a perfect fit for the role.

Note that Aristotle explicitly associates the notion of intelligible matter and continuity in a passage of Z.11 that is sandwiched between the two occurrences of the expression *hylē noētē*:

> ἀποροῦσί τινες ἤδη καὶ ἐπὶ τοῦ κύκλου καὶ τοῦ τριγώνου ὡς οὐ προσῆκον γραμμαῖς ὁρίζεσθαι καὶ τῷ συνεχεῖ, ἀλλὰ πάντα καὶ ταῦτα ὁμοίως λέγεσθαι ὡσανεὶ σάρκες καὶ ὀστᾶ τοῦ ἀνθρώπου καὶ χαλκὸς καὶ λίθος τοῦ ἀνδριάντος· καὶ ἀνάγουσι πάντα εἰς τοὺς ἀριθμούς, καὶ γραμμῆς τὸν λόγον τὸν τῶν δύο εἶναί φασιν.

[180] See Pfeiffer (2018: §5.3).

> Some have raised the problem even in the case of the circle and the triangle. They suggest that the circle and the triangle should not be defined by reference to lines and continuity, but that these too should rather be understood as the flesh and bones of a man, or the bronze or wood of a statue. Thus they reduce everything to numbers, and say that the formula of a line is simply the formula of two.
>
> *Metaph.* Z.11, 1036b8-13 (Bostock's translation)

The problem that Aristotle is referring to is how to distinguish the essential features of an object, that need to appear in its definition. Aristotle points out that whenever we see the same feature instantiated in things 'that differ in form' – he gives the example of a circle which might be made of bronze, stone or wood – it is easy to see that the latter are not meant to appear in its definition. But it is difficult to make the distinction, if something is never found separate from a certain kind of matter. For example, the form of man is always found in flesh and blood, but these should not appear in the definition of man.

The context does not make it easy to understand whether Aristotle is willing to accept the association between continuity and matter that he attributes to his opponents. They argue that since continuity and lines play the role of matter for triangle and circle, they should not appear in the definition (in the same way as flesh and blood do not appear in the definition of man): line should simply be defined as duality. Aristotle rejects this conclusion, claiming that such definitions would be inadequate ('there will be one form of many things which evidently differ in form'[181]). But the next step in his argument is not to deny the connection between continuity and matter; rather, he goes on to allow for definitions of the universal compounds as well as of forms. Such definitions include

[181] 1036b17-18, ἕν τε πολλῶν εἶδος εἶναι ὧν τὸ εἶδος φαίνεται ἕτερον.

mention of matter; in the case of geometrical objects, they include mention of continuity and extension. See for example *DA* III.4, 429b18-19: 'in the case of things which are by abstraction, the straight is as the snub is, since it is with extension'.[182]

Compare also the way in which Aristotle describes the result of the *aphairesis* operated by geometers in *Metaph.* K.3:

> καθάπερ δ' ὁ μαθηματικὸς περὶ τὰ ἐξ ἀφαιρέσεως τὴν θεωρίαν ποιεῖται (περιελὼν γὰρ πάντα τὰ αἰσθητὰ θεωρεῖ, οἷον βάρος καὶ κουφότητα καὶ σκληρότητα καὶ τοὐναντίον, ἔτι δὲ καὶ θερμότητα καὶ ψυχρότητα καὶ τὰς ἄλλας αἰσθητὰς ἐναντιώσεις, μόνον δὲ καταλείπει τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ' ἓν τῶν δ' ἐπὶ δύο τῶν δ' ἐπὶ τρία, καὶ τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ.
>
> The mathematician theorizes about abstractions, for he theorizes having removed all sensibles such as weight and lightness, hardness and its opposite, heat and cold, and the other sensible opposites. He leaves only the quantitative and continuous in one, two, or three [dimensions] and the properties of these as quantitative and continuous.
>
> *Metaphysics* K.3.1061a28-35 (*ROT*)

Continuity in one, two or three dimensions is presented as the substratum which underlies geometrical features. This passage also confirms that it is the same extension which underlies sensible features, too: extension is primarily the extension of bodily and sensible substances, and only secondarily is it the intelligible matter of the semi-fictional geometrical objects which are obtained by separating some features of sensible substances. Note that also the dimensionality and continuity of extension are features that belong to extension primarily because it is the extension of sensible substances;

[182] ἐπὶ τῶν ἐν ἀφαιρέσει ὄντων τὸ εὐθὺ ὡς τὸ σιμόν· μετὰ συνεχοῦς γάρ.

geometrical objects obtained by separation 'inherit' such features from the sensible objects to which the geometrical features in fact belong.

Take continuity. It is a topological property that can be geometrically studied, and determines important mathematical features such as infinite divisibility. But continuity is first and foremost a particular kind of unity that is attributed to substances on the basis of physical and metaphysical facts, not because of any underlying geometrical constraint.[183]

That is to say, it is because of physical (and metaphysical – but not geometrical) reasons that bodies are continuous and three-dimensional, and consequently that their extension has these features too. As Pfeiffer (2018) demonstrates, continuity is a topological property that sensible substances have in virtue of having a specific kind of unity: the attribution of this property is grounded on physical and metaphysical facts, not on any underlying geometrical constraint.

With regard to dimensionality, in *De Caelo* I.1 Aristotle offers some arguments to establish that there cannot be more than three spatial dimensions; the context makes it clear that the discussion is conducted from the perspective of natural science, and not of geometry. Aristotle explains that magnitudes can be extended in one, two or three dimensions, and then stresses that 'there is no other magnitude beyond these, since the three [dimensions] are all and the thrice is in every way'.[184] As a consequence, physical bodies are said to be complete (and/or perfect: *teleion*) because they are three-dimensional, while lines and planes are incomplete because they are lower-dimensional (cf. also *Metaph.* M.2, 1077a26-36). Aristotle brings three observations in support of his claim that there cannot be more than three dimensions:

[183] See Pfeiffer (2018: §7.4), who argues that the difference between continuity and contact is not topological, but grounded in the ontology of the objects involved. Continuity is a relation that holds between the parts of a single object, while contact is a relation that holds between ontologically distinct objects.
[184] *Cael.* I.1, 268a9-10, Pfeiffer's translation. καὶ παρὰ ταῦτα οὐκ ἔστιν ἄλλο μέγεθος διὰ τὸ τὰ τρία πάντα εἶναι καὶ τὸ τρὶς πάντῃ.

1. the Pythagorean doctrine of the completeness of number three (268a10 ff.);
2. the fact that the number three has a special role in religious ceremonies (268a13 ff.);
3. Greek grammar's distinction between the dual and the plural, which makes it the case that the locution 'all' can only be applied to groups of three or more (268a15 ff.).

Commentators have been very puzzled by these arguments, and it is not completely clear whether Aristotle thinks he has *proved* that there cannot be objects with more than three dimensions. What I am interested in, however, is merely to notice that he offers no geometrical argument in support of the three-dimensionality of bodies (which would be stronger than the ones he does offer!),[185] nor does he appeal to the spatial structure of the universe. This shows that he takes the three-dimensionality of bodies to be primary and most fundamental.[186]

This analysis of the properties of extension shows that it is indeed a very good candidate to act as intelligible matter for geometrical objects. To sum up, we have seen that it is a feature of the matter of sensible bodies, and that it can be individuated thanks to the same mental operation that allows geometers to isolate the geometrical features of sensible bodies. In particular, it is continuous and thus determines the infinite divisibility of geometrical objects; if the division of geometrical objects can be considered a change

[185] Notice that *n*-dimensional geometrical objects can be constructed out of (*n*-1)-dimensional ones, or be introduced as limits or divisions of (*n*+1)-dimensional ones. There are no good geometrical arguments to explain why *n* shouldn't be bigger than three, as shown by the fact that nowadays there are geometries with more than three dimensions. The fact that Euclidean geometry only allows three dimensions seems to be due not to internal geometrical constraints, but because of the assumption that geometry is true of sensible things, which can have three dimensions at most.

[186] Cf. *Cael* I.1, 268a30-b5: 'there is no transition to another kind of magnitude, as we passed from length to surface, and from surface to body. For if we could, it would cease to be true that body is complete magnitude. We could pass beyond it only in virtue of a defect in it; and that which is complete cannot be defective, since it is in all ways'. Ἀλλ' ἐκεῖνο μὲν δῆλον, ὡς οὐκ ἔστιν εἰς ἄλλο γένος μετάβασις, ὥσπερ ἐκ μήκους εἰς ἐπιφάνειαν, εἰς δὲ σῶμα ἐξ ἐπιφανείας· οὐ γὰρ ἂν ἔτι τὸ τοιοῦτον τέλειον εἴη μέγεθος· ἀνάγκη γὰρ γίγνεσθαι τὴν ἔκβασιν κατὰ τὴν ἔλλειψιν, οὐχ οἷόν τε δὲ τὸ τέλειον ἐλλείπειν· πάντῃ γάρ ἐστιν.

(I will discuss this in the next chapter), this means that intelligible matter can also act as a substratum for change. Moreover, the fact that extension is a property of bodies, and that its features are structural features that belong to it for physical reasons, supports the idea that the identification between intelligible matter and spatial extension plays some role in justifying the applicability of geometry. But is it sufficient to do so?

§3. Extension is not akin to space

On the interpretation that I have suggested, the Applicability Problem arises from Aristotle's apparent assumption that examining the semi-fictional objects resulting from separation allows geometers to learn geometrical truths that are true in virtue of features of the sensible world. The same problem arises for a whole family of interpretations that deny the radically anti-Platonist reading of the *qua*-theory that I have put forward in Chapter 1, and simply argue that geometry is in fact about some special kind of objects distinct from sensible substances, which exist in some special, subordinate way and not as independent substances. Among those readings, one of the most sophisticated is presented by Mueller (1970). Mueller argues that geometrical properties are in fact not perfectly instantiated by sensible objects, and that geometers do not study sensible objects at all. Far from believing that the *qua*-theory aims at accounting for the truth of geometrical statements, he argues that the *qua*-manoeuvre is just an epistemological tool that geometers use to isolate the geometrical shapes that bodies possess imperfectly; the real objects of geometry are obtained by imposing such shapes onto intelligible matter. Despite the radical differences between this and my own reading of the ontology of geometry, the Applicability Problem seems to arise for both interpretations in a similar

way.[187] If geometers reason about mental objects that they obtain by imposing geometrical properties on intelligible matter, how can Aristotle be sure that geometrical truths can be applied to the sensible world?

Interestingly, Mueller does not seem worried by this issue, and he puts forward a very simple explanation as to why.[188] He suggests that identifying the intelligible matter of geometrical objects with extension is sufficient for explaining the applicability of geometry. This is because extension is 'three-dimensional underlying sensible properties in the physical world' (166, *sic*), so that 'Aristotle explains [the] applicability [of geometry to the physical world] by considering mathematical objects to underlie physical reality' (Mueller 1970: 161). But what does it mean to say that extension 'underlies physical reality'?

To elucidate the notion of extension, he appeals to the contemporary notion of space. He writes: 'a modern analogue of Aristotelian extension would be space, which is formally nothing but a class of elements (points) related in certain ways but which is also in some sense an object of perception or intuition.'[189] I believe that this association should be resisted, because it risks introducing anachronistic links. As we saw, on Aristotle's view extension is the extension of bodies, not the spatial manifold in which these bodies are embedded. This means that if extension 'underlies physical reality', it is not in the

[187] It might even be more worrying for Mueller than for my interpretation. If my analysis up to here is correct, there is a clear connection between sensible reality and geometry that has already been established and secured, for the truth-makers of geometrical statements are sensible bodies and their geometrical properties. In Mueller's system, on the contrary, the connection between sensible reality and geometry is much more tenuous, and risks being severed if his solution to the Applicability Problem does not work. See Halper (1989: 267, fn. 34).

[188] Notice that Mueller does not present the Applicability Problem as a problem that requires an articulated solution; thus, he puts forward his claim without a detailed argument. I believe that the issue deserves further attention, and I have tried to reconstruct Mueller's reasons for his claim, and to provide an argument in support of his solution. It goes without saying that this work of reconstruction might have led me to misrepresent Mueller's idea. This section is not supposed to be a criticism of Mueller's actual theory, but of a Mueller-inspired strategy to solve the Applicability Problem in the context of my own interpretation.

[189] Mueller (1970: 171). He is not the only one who uses the notion of space in this context; another example is Popper (in Anscombe et al. 1953: 107): '[Aristotle's] intelligible matter can be identified with space'.

same way as space does: extension does not pre-exist bodies, and does not host them.[190] It is true that since Aristotle's universe is full and continuous, we can consider it as a whole and unitary body.[191] The extension of the universe so considered is indeed a unitary three-dimensional, continuous and finite manifold, that comprises all possible positions in Aristotle's universe, and might be said to 'underlie physical reality'. But thinking of this 'underlying manifold' as space would be misleading.

I suspect that part of the reason why presenting extension in terms of space might be tempting is that this association suggests an easy way out of both the problems we are facing – the Applicability Problem and the Plurality Problem. If extension can be thought of as space, then there is a clear sense in which identifying it with intelligible matter can be sufficient to grant the applicability of geometry: studying geometrical objects would amount to studying the structure of physical space, and as a consequence the findings of geometry would certainly be applicable to the physical bodies that inhabit such space. It would also be easy to solve the Plurality Problem, for if extension were space it would surely be able to provide a principle of individuation for geometrical objects: two objects would be distinct if they occupy different regions of space.

Unfortunately, it appears that the Aristotelian notion of extension cannot fulfil both these roles at the same time. The reason is that the three-dimensional continuous manifold by itself is not sufficient to provide a principle of individuation; it would need to be supplemented with a frame of reference. In Aristotle's cosmology, such frame of reference is provided by the theory of the appropriate places of the elements.[192] The

[190] I am not suggesting that Mueller himself falls prey to these confusions. But he does not explain how he takes extension to underlie physical reality, and presenting extension in terms of space might suggest a wrong and anachronistic picture. Thus, I think it might be helpful to explicitly rule out the space-model.
[191] Aristotle allows for this possibility in his discussion of the transmission of the prime motion in *Physics* VIII. Although the sensible substances that populate the universe are individual substances which are continuous among their limits and separate from each other, Aristotle stresses that the universe can still be considered continuous by contact.
[192] They are usually called 'natural places' in the literature, but the expression does not appear in Aristotle, who rather calls them *oikeios* at 211a5 and 212b33. See Quarantotto (2017: 73) for a full discussion.

resulting spatial structure, however, would be difficult to describe using Euclidean geometry: it is anisotropic and inhomogeneous, as well as finite.[193] As a consequence, it is not adequate as a geometrical space.

Note that Aristotle does not draw a distinction between physical and geometrical space, but I will argue that he is committed to it (or better: that he would be, if he were concerned with the notion of geometrical space). His discussion of place is evidently conceived as a physical analysis, centred around the necessity of accounting for local motion; the spatial structure that results from this analysis is difficult to square with Euclidean geometry. In this section I will focus on anisotropy and inhomogeneity; I will discuss the finitude of the universe in the next section.

Aristotle believes that places play an active causal role in the natural motion of the elements: places are not all identical, but have different powers. In particular, the appropriate places of the elements are the places where each natural element tends to go if unimpeded. They determine a fixed and stable frame of reference organised along six *diastemata*, directions or dimensions:

> ἔτι δὲ αἱ φοραὶ τῶν φυσικῶν σωμάτων καὶ ἁπλῶν, οἷον πυρὸς καὶ γῆς καὶ τῶν τοιούτων, οὐ μόνον δηλοῦσιν ὅτι ἐστί τι ὁ τόπος, ἀλλ' ὅτι καὶ ἔχει τινὰ δύναμιν. φέρεται γὰρ ἕκαστον εἰς τὸν αὑτοῦ τόπον μὴ κωλυόμενον, τὸ μὲν ἄνω τὸ δὲ κάτω· ταῦτα δ' ἐστὶ τόπου μέρη καὶ εἴδη, τό τε ἄνω καὶ τὸ κάτω καὶ αἱ λοιπαὶ τῶν ἓξ διαστάσεων. ἔστι δὲ τὰ τοιαῦτα οὐ μόνον πρὸς ἡμᾶς, τὸ ἄνω καὶ κάτω καὶ δεξιὸν καὶ ἀριστερόν· ἡμῖν μὲν γὰρ οὐκ ἀεὶ τὸ αὐτό, ἀλλὰ κατὰ τὴν θέσιν, ὅπως ἂν στραφῶμεν, γίγνεται (διὸ καὶ ταὐτὸ πολλάκις δεξιὸν καὶ ἀριστερὸν καὶ ἄνω καὶ κάτω καὶ πρόσθεν καὶ ὄπισθεν), ἐν δὲ τῇ φύσει διώρισται χωρὶς ἕκαστον. οὐ γὰρ ὅ τι ἔτυχέν ἐστι τὸ ἄνω, ἀλλ' ὅπου φέρεται τὸ πῦρ καὶ τὸ κοῦφον· ὁμοίως δὲ καὶ τὸ κάτω οὐχ

[193] Euclidean spaces, on the contrary, are homogeneous, isotropic and infinite. 'Homogeneity' means that something is uniform throughout, while 'isotropy' pertains to what exhibits equal properties in all directions.

ὅ τι ἔτυχεν, ἀλλ' ὅπου τὰ ἔχοντα βάρος καὶ τὰ γεηρά, ὡς οὐ τῇ θέσει διαφέροντα μόνον ἀλλὰ καὶ τῇ δυνάμει.

Again, the locomotions of the natural simple bodies (such as fire and earth and the like) not only show that place is something but also that it has some power, since each body, if not impeded, moves to its own place, some above and some below. These are the parts and kinds of place: above, below, and the rest of the six directions. These are not just relative to us. Relatively to us, they – above, below, right, left – are not always the same, but come to be in relation to our position, according as we turn ourselves about, which is why, often, right and left are the same, and above and below, and ahead and behind. But in nature each is distinct and separate. 'Above' is not anything you like, but where fire, and what is light, move [to]. Likewise, 'below' is not anything you like, but where heavy and earth-like things move [to]. So they differ not by position alone but in power too.

Physics IV.1, 208b8-23 (Hussey's translation)

The existence of natural places of the elements is enough to make the *kosmos* inhomogeneous: the place of fire is different from the place of air, so that various positions in the universe are qualitatively different the one from the other. One might think that the behaviour of bodies is not due to geometrical features of space, and hence that these differences do not pose problems for geometry; after all, Newton believed that space was homogeneous *and* that bodies naturally fall to the ground because of gravity. However, it is important to stress that Aristotle's view of natural places is different from Newton's understanding of gravity. First of all, places themselves are said to have a power, *dynamis*. Second, Aristotle seems to think that the 'dynamic field structure'[194] that

[194] Jammer (1954/1993: 19).

governs the behaviour of bodies is only determined by the spatial conditions, and not by the distribution of matter:

> διαπορήσειε δ' ἄν τις, ἐπεὶ ταὐτὸν ἀμφοτέρων ἐστὶ τὸ μέσον, πρὸς πότερον φέρεται τὰ βάρος ἔχοντα καὶ τὰ μόρια τῆς γῆς κατὰ φύσιν· πότερον ὅτι τοῦ παντός ἐστι μέσον, ἢ διότι τῆς γῆς. Ἀνάγκη δὴ πρὸς τὸ τοῦ παντός· καὶ γὰρ τὰ κοῦφα καὶ τὸ πῦρ εἰς τοὐναντίον φερόμενα τοῖς βάρεσι πρὸς τὸ ἔσχατον φέρεται τοῦ περιέχοντος τόπου τὸ μέσον. Συμβέβηκε δὲ ταὐτὸ μέσον εἶναι τῆς γῆς καὶ τοῦ παντός· φέρεται γὰρ καὶ ἐπὶ τὸ τῆς γῆς μέσον, ἀλλὰ κατὰ συμβεβηκός, ᾗ τὸ μέσον ἔχει ἐν τῷ τοῦ παντὸς μέσῳ.
>
> It might be asked, since the centre of both (i.e., the earth and the universe) is the same point, in which capacity the natural motion of heavy bodies, or parts of the earth, is directed towards it; whether as centre of the universe or of the earth. But it must be towards the centre of the universe that they move, seeing that light bodies like fire, whose motion is contrary to that of the heavy, move to the extremity of the region which surrounds the centre. It so happens that the earth and the universe have the same centre, for the heavy bodies do move also towards the centre of the earth, yet only incidentally, because it has its centre at the centre of the universe.
>
> *Cael*. II.14, 296b6-18 (Leggatt's translation)

As Jammer (1954/1993: 20) notes, this passage shows that Aristotle takes the centre of the universe to be qualitatively differentiated from the other places; the difference cannot be reduced to the fact that it is occupied by earth.

As a consequence, the appropriate places of the elements determine a fixed and stable frame of reference that is independent of the distribution of matter in the universe at any given moment. The existence of privileged directions (the six 'parts of place', arranged along three directional axes: up/down, left/right, and in front/behind) makes the spatial

structure of the *kosmos* anisotropic. Notice that Aristotle specifies that the directions are not only relative to us, but that 'in nature' there is a univocal, fixed and objective up and down, for example. These directions are essential to the spatial structure of the universe, and are not simply abstracted out of regularities in the motion of the elements: this is why they are properly called 'parts of space'.

The upshot is that the spatial structure of the Aristotelian *kosmos* is not apt to be described by Euclidean geometry, and thus cannot be the space of geometrical proofs. Aristotle does not seem at all worried by this, though.[195] This is very surprising for contemporary readers, influenced by the post-Kantian idea that geometry is the science of space. But this idea is alien to the Greek context. Greek geometry is a geometry of figures, which studies the properties and relations of configurations of objects – not the characteristics of the manifold in which they are embedded.[196] What I am suggesting is that the reason why Aristotle does not distinguish between physical and geometrical space is not that he assumes them to be identical, but that the idea of geometrical space is not thematised nor discussed at the time.

Moreover, there are reasons specific to Aristotle's philosophy of geometry that allow him to see the anisotropy and inhomogeneity of the *kosmos* as unproblematic for the applicability of geometry: he believes that geometry should ignore the motions and changes of sensible bodies.[197] The anisotropy and inhomogeneity of the *kosmos* are so strictly linked to the natural motion of the elements that it is plausible to suppose that the geometer must abstract from these features as well. This is perfectly compatible with

[195] In particular, he does not seem worried by the anisotropy and inhomogeneity of the *kosmos*. On the contrary, he is conscious that its finitude might create problems to geometry; I discuss this in the next section.
[196] Cf. De Risi (2015), Mendell (2015), Ranocchia et. al. (2014).
[197] See e.g. *Metaph.* M.3, 1077b28; *Ph.* II.2, 193b34. I will discuss the details and implications of Aristotle's claim that geometers consider sensible objects not *qua* movable in Chapter 5.

intelligible matter being extension, as long as it is clear that extension is the extension of bodies, and not supposed to model physical space.

As I mentioned above, Aristotle seems to allow for the possibility of considering the *kosmos* as a unitary continuous whole. It is surely possible to study it geometrically, for example considering its spherical shape. In focusing on its continuous extension, geometers ignore not only the motions of the individual bodies that inhabit the *kosmos*, but also the spatial structure that determines such motions, and even the distinctions between those bodies. In this way the *kosmos* can be studied geometrically as a continuous object on a par with the others, not as space.

To sum up, I have argued that Aristotle's extension cannot be understood as what we would call the space of the Aristotelian universe. The spatial characteristics that Aristotle attributes to the *kosmos* have certain features that are incompatible with geometrical extension. So geometrical extension cannot be identified (or even be like) Aristotle's physical space. On the contrary, extension always is the extension *of bodies*. This explains why Aristotle believes that the inhomogeneity and anisotropy *of space* are not problematic for his account of geometry: geometry was never meant to describe the spatial structure of the universe, but only the structural features of bodies.

§4. Finitude of the kosmos

The finitude of the *kosmos* raises a more pressing problem. Even if we do not identify extension with space nor with the spatial structure of the universe, it is still the case that bodily extension is finite according to Aristotle. The consequence is that Euclidean geometry cannot straightforwardly be applied to the sensible world, because some of its theorems are falsified by the spatial limitedness of the universe.[198] In particular, Euclid's

[198] See Chapter 2 above.

second postulate is invalidated: if the *kosmos* is finite, it is *not* always possible to extend a line.[199]

Aristotle is well aware of the problem, and discusses it briefly in *Physics* III, right after having denied the possibility of an actual infinite by addition:

> οὐκ ἀφαιρεῖται δ' ὁ λόγος οὐδὲ τοὺς μαθηματικοὺς τὴν θεωρίαν, ἀναιρῶν οὕτως εἶναι ἄπειρον ὥστε ἐνεργείᾳ εἶναι ἐπὶ τὴν αὔξησιν ἀδιεξίτητον· οὐδὲ γὰρ νῦν δέονται τοῦ ἀπείρου (οὐ γὰρ χρῶνται), ἀλλὰ μόνον εἶναι ὅσην ἂν βούλωνται πεπερασμένην· τῷ δὲ μεγίστῳ μεγέθει τὸν αὐτὸν ἔστι τετμῆσθαι λόγον ὁπηλικονοῦν μέγεθος ἕτερον. ὥστε πρὸς μὲν τὸ δεῖξαι ἐκείνοις οὐδὲν διοίσει τὸ [δ'] εἶναι ἐν τοῖς οὖσιν μεγέθεσιν.
>
> This reasoning does not deprive the mathematicians of their study, either, in refuting the existence in actual operation of an untraversable infinite in extent. Even as it is they do not need the infinite, for they make no use of it; they need only that there should be a finite line of any size they wish. But another magnitude of any size whatever can be divided in the same proportion as the greatest magnitude; so that, at least for the purpose of proof, it will make no difference to them that [a magnitude] is among the magnitudes that exist.
>
> *Ph.* III.6, 207b27-33 (Hussey's translation)

Note that the finitude of the universe raises two distinct problems:

(i) geometers appeal to the notion of infinite, and they seem to mean it as an actual infinity;

[199] Ugaglia (2018) notices that actually Aristotle has the resources to verify the second postulate literally in his physical system, by constraining the way in which lines can be extended: if at every step they are extended by a fraction of the extension added in the previous step (by the procedure of converse increasing), then the line will keep getting closer and closer to the edge of the *kosmos*, without ever reaching it. However (i) the resulting geometry would not be Euclidean, but hyperbolic; and (ii) Aristotle does not in fact take this route.

(ii) geometers make use of figures and constructions that are bigger than the dimensions of the *kosmos*.

Aristotle is adamant in rejecting the view that (i) raises a genuine problem. He argues that geometers never make use of an actual infinity, but merely of the possibility of indeterminate extension. What is important to geometry is the possibility of taking a line or a magnitude of any dimension whatsoever, and this can be accommodated by Aristotle's theory of potential infinity. This is consistent with what we know of the geometry of the time: each proof deals with a limited number of objects, that all have limited dimension. It is essential to geometry that these figures can be as big or as small as one wishes, and that they can be extended as much as it might be needed – but there is no figure which is actually infinite.[200] Once again, we see that Greek geometry is concerned with limited groups of figures and their spatial relationships, not with the structure of the manifold that embeds them. The fact that these figures can be indeterminately extended, however, is sufficient to create problems for Aristotle, for they might end up being bigger than the size of the *kosmos*.

Crucially, Aristotle's reply does not consist in trying to explain how physical extension might be stretched or expanded to accommodate these cases,[201] nor does he try to find a way around this limitation.[202] On the contrary, he simply seems to accept that

[200] Cf. Netz (1999: 32).

[201] Aristotle never discusses the precise size of the *kosmos*; he is mostly concerned with proving that the universe must be finite and spherical. But it is unlikely that he would consider the possibility that the universe expands while keeping its shape unchanged. For one, Aristotle believes that outside the universe there is nothing at all – not even place nor void: the universe would have literally nowhere to expand to. It seems that this hypothesis is also contradictory with the idea that the appropriate places of the natural elements are fixed and absolute. For if the universe expands, presumably its centre would stay fixed – but the appropriate places of the elements would expand accordingly, so that the sphere of water might end up partially overlapping with the place that was previously the sphere of air, and so on. Since Aristotle seems to consider the appropriate places of the elements as fixed, and never mentions anything similar to an expansion of the universe, I believe that we can safely conclude that this is not an option that he might consider.

[202] Notice that Aristotle would have the theoretical tools to do that. For example, he might argue that intelligible matter is supposed to be identified with extension *taken per se*, without the form/shape that limits it. As we saw above, extension is *ahoriston*, which can be understood as meaning that it is *unlimited*, and so (potentially) infinite. An argument in support of the idea that extension is *per se* potentially infinite

there is an asymmetry between physical and geometrical space: while the dimension of physical bodies is constrained by the dimension of the universe, geometrical configurations are not.

Having accepted the premises of the problem, Aristotle proceeds to offer a solution. The last line of the passage quoted states that, for the purposes of the proof, it does not make a difference whether the configuration that the geometers are examining is an existing one. I take this to mean that the geometrical proof can be valid even if the geometrical object on which it is carried out does not in fact exist. This is what we would expect Aristotle to say, given the interpretation that I have provided. Geometrical proofs are conducted on semi-fictional objects that can be modified in thought and extended beyond the limits of the *kosmos*, for in the imagination all masks are removed: not only the sensible features of material objects, but also the physical limitations imposed by the dimensions of the universe and by the theory of places. According to the solution to the Scarcity Problem that I have suggested in Chapter 2, the truth of the resulting theorems was simply grounded in the potentialities of the basic items that compose the complex (and physically impossible) configuration. Here, Aristotle goes one step further. He argues that the resulting theorem is itself applicable to sensible reality, for 'another magnitude of any size whatever can be divided in the same proportion as the greatest

---

could maybe be built on Aristotle's remarks regarding continuity. In *Physics* VIII.8, in the course of the discussion of the first motion, Aristotle draws a very strict connection between the circular motion's unity, its continuity, and its eternity. In this context, it appears clear that even if also motions that are limited and continuous between their limits can be said to be continuous, the only *properly* continuous motion is the one that goes on *ad infinitum*. In the same way, time, being one and continuous and not limited by anything, goes on *ad infinitum*. Since Aristotle claims that time, motion, and *megethos* (both the *megethos* of the moving body, and the trajectory of its motion) are all continuous in the same way (as it is established in *Ph.* VI.2), it is not too much of a stretch to suppose that, *if one were to remove the form/shape that delimits the extension of the body,* the extension would go on *ad infinitum* exactly as time and the primary motion. This is not to deny that there is an important asymmetry in how Aristotle treats time and spatial extension: while time and change are potentially infinite, the spatial extension of the *kosmos* is finite. Moreover, it is not finite because limited from the outside; it is intrinsically finite. But one might still insist that it is the *universe* which is intrinsically finite – not its extension, when considered *per se* and in abstraction from the form. The *kosmos* is a hylomorphic compound composed by matter and form: its finiteness might be due to its form and not to its matter.

magnitude': the complex figure can be 'scaled down' to an acceptable dimension, so as to fit into the sensible universe.

This means not only that the mode of existence of the semi-fictional object, but also its absolute size, is irrelevant to the proof.[203] To Aristotle's contemporaries this would have sounded less controversial than it does for us now, because they are used to ignoring the size of the diagrams (together with all other exact properties).[204] The fact that the absolute size of the semi-fictional object on which the proof is conducted is irrelevant to the truth and the applicability of the theorem supports the idea that these mental objects are models that can be used in the same way as diagrams, rather than being the proper objects or the truth-makers of the theorems. What grants the applicability of the theorems obtained by examining these mental models is the possibility of creating a smaller but isomorphic object, of which the theorem is literally true.

We have here a further confirmation that the space-model of extension should be rejected. Aristotle seems to accept that the 'space of the proof' is not the same as the physical space of the universe, because the geometrical proof can go beyond the limits of the universe and then be scaled back to an acceptable dimension. This discrepancy between the physical and geometrical space is not worrying for Aristotle, because the idea of a unitary and infinite geometrical space that hosts all possible geometrical constructions is alien to Greek sensibility, and is introduced in modern times. Each proof and each demonstration should be considered as its own specific 'universe' (to use Netz' (1999) suggestive terminology), determined by the objects that are involved in the demonstration (both the ones that are given in the enunciation of the proof, and the ones

[203] The relative size of the components of the configuration, on the contrary, is important: Aristotle stresses that the same configuration can be reproduced in the same proportion at a smaller scale.
[204] Cf. Manders (1995/2008) and above, p. 43. I will come back to diagrams also in Chapter 5.

that are constructed in the course of the demonstration). Each 'universe of proof' is as big as needed (there is no upper limit for its size) but finite.[205]

The consequences of this for the Applicability Problem are striking: what matters for the applicability of geometry is not that there is a global correspondence between geometrical space and the physical universe, but rather that a local isomorphism can be established between a geometrical configuration and a physical one. The finitude of the *kosmos* is not a problem in this context, because (i) applicability is always localised, and (ii) geometrical configurations that are too big to be put in direct correspondence with a physical configuration can be 'scaled down' to an acceptable dimension.

Note that the possibility of scaling down the configuration is secured thanks to the continuity of matter and (consequently) of extension. Continuous extension is homogeneous and infinitely divisible, and this means that it can be divided anywhere.[206] It is because extension is continuous that 'zooming in or out' does not reveal any structural difference, and the bigger and the smaller configurations can be perfectly isomorphic. This means that identifying intelligible matter with extension does play an essential role in granting the applicability of geometry: it ensures that an isomorphism between the semi-fictional object and a (potential) physical object can be found also when the semi-fictional object is too big to be actually instantiated in the physical *kosmos*. But it is not in itself sufficient. What is really important is that there is an isomorphism between the physical and the geometrical configuration. If this is right, then it is necessary that sensible bodies do instantiate geometrical properties for geometry to be applicable.[207]

---

[205] Netz (1999: 31-32).
[206] Cf. *GC* I.2, 316b20-317a17.
[207] This is granted by my interpretation, but denied by Mueller. He jumps from the claim that extension is a three-dimensional manifold underlying sensible reality to the conclusion that *geometrical objects* underlie sensible reality. But this does not follow, if we reject the space-model. There needs to be a direct connection between the particular geometrical configurations and sensible bodies, and Mueller's system is not able to account for it.

§5. The Plurality Problem

Before concluding our analysis of intelligible matter, there is one issue left to discuss: we have not yet established a principle of individuation for geometrical objects. Extension taken in its own right is not enough, because it is unclear what would distinguish one 'bit of extension' from the other. In order to do so it is necessary to appeal to a frame of reference.

Aristotle's theory of place does provide such a frame of reference, which is generated by the three directional axes of natural motion and by the theory of natural places. This absolute frame of reference provides the principle of individuation for places, so as to allow one to recognise a place after the body that occupied it has moved (thus causing the disappearance of the inner limit of the containing body that had been identified with its place) and it has been occupied by a different body, with a different shape (so that we can legitimately say that the second body has taken the place of the first, even if the inner limit of the containing body is not exactly the same in virtue of their different shapes).[208] Presumably, this fixed frame of reference can also be appealed to in order to distinguish two chunks of sensible matter from each other, thus allowing Aristotle to use sensible matter as a principle of individuation for sensible objects.

Unfortunately, this frame of reference cannot be appealed to when we are dealing with geometrical objects; as we have seen above, the spatial structure that is determined by this frame of reference is not Euclidean, and the frame of reference itself is deeply embedded with the natural motion of the elements, which has no place in geometry. As a

[208] See Quarantotto (2017: §4.4; 2022). For a slightly different account that still stresses the importance of a fixed frame of reference, see Sattler (2020b: 46-48).

consequence, I do not think that intelligible matter can fulfil this function, and provide a solution to the Plurality Problem.

Luckily, as Katz (2019: 508) argues, we do not need to invoke intelligible matter to do this. There is a spatial notion that Aristotle can appeal to in order to provide a principle of individuation for geometrical objects: position (*thesis*).[209] Geometrical objects do not have a place (*topos*),[210] but they are said to have a position; not an absolute one, but a position with respect to us.[211] Note that Aristotle explicitly states that points have position (*Metaph*. Δ.6, 1016b29-30), too. If the principle of individuation were provided by intelligible matter, we would not be able to account for the distinguishability of points; invoking position allows us to include the points in the scope of the criterion.

The relationship between *thesis* and *topos* in Aristotle's physics is unclear, and it is not certain that it is consistent throughout the *Corpus*.[212] The notion of position is appealed to in the *Categories* to distinguish two different kinds of continuous magnitudes: the ones whose parts have position and the ones whose parts do not.[213] At *Cat.* 6, 5a24-25 he implies that having position can be understood as being situated somewhere. This is different from, and independent of, being in a place.[214] The notion of position is directly associated to a frame of reference:[215] at *Physics* IV.1, 208b8-23 quoted above, absolute position is directly associated to the six directions. Importantly, Aristotle distinguishes

[209] Cf. *DA* I.4, 409a20-21: 'what difference could there be in the case of a unit point except position?'
[210] Geometrical properties have a place indirectly, in virtue of the fact that the sensible substances that instantiate them are in a place. But they do not have a place in their own right. The semi-fictional geometrical objects that I am considering in this chapter do not have a place either, for they are mental objects.
[211] *Ph.* VI.1, 208b22-27.
[212] In particular there seems to be a discrepancy between *Physics* IV.1, 208b8-27 and *GC* I.6, 322b32-323a3
[213] *Cat*. 6, 5a15-23. Cf. Pfeiffer (2018: §5.3).
[214] The conditions for being in a place are stricter than for having position. Only movable things are in a place, and the parts of a whole do not have a place either (they are said to be in it as parts, and not as in a place. See *Ph.* IV.4, 211a30-31). On the contrary, we have seen that the parts of continuous magnitudes have position, and that geometrical objects are said to have position but not space.
[215] While place is indirectly associated with a frame of reference: Aristotle defines proper places before, and independently of, introducing the theory of natural places that gives rise to the absolute frame of reference.

this absolute sense of position, which is defined in relation to the fixed frame of reference provided by the natural places, from a weaker sense of position: position with respect to us. In this latter case, the frame of reference with respect to which each position is identified is not the absolute one, but is centred on the subject (and so it changes if the subject moves: something can first be at my right and then at my left). This latter sense is the one that can be applied to geometrical objects.

This means that *thesis*-relative-to-us is independent of the non-Euclidean spatial structure that I have described above, but sufficient for distinguishing one object from the other. The fact that geometrical objects only have position *with respect to us* is just what one would expect, given my claim that the objects in question are semi-fictional geometrical objects, and thus mental objects. It also fits very well the feature of Greek geometry that we have already mentioned above: the universe of the proof is self-contained, and determined each time by the geometers. Only the objects that are explicitly introduced at the beginning of the proof, or that are constructed in the course of the proof, are part of the universe of the proof. Since each proof has its own universe, it does not make sense to ask whether the objects of one proof are the same as or different from the objects of another proof. So the principle of individuation that we are looking for needs to be something localised and not global – the notion of *thesis* with respect to us provides precisely this.

Chapter 5. Geometrical Changes

In the previous chapter I examined some problems that threaten Aristotle's account of geometry and its applicability to the sensible world – problems arising from his cosmology and his theory of places, as well as from his belief in the finitude of the spatial extension of the universe. I argued that these are indeed sources of tension in Aristotle's account, but they can be accommodated by restricting the scope of the applicability of geometry. Geometry is not meant to describe the spatial structure of the universe (as we might think, being influenced by the post-Kantian understanding of geometry as the science of space), but only the spatial and structural features of locally constrained configurations.

In this chapter I will turn to problems that arise with respect to time, and to the diachronic development of sensible substances. Physical bodies move, change and decay in a characteristic way, due to their natures and to their interaction with other objects in the world. Geometrical objects do not. They do not fall to the ground nor clash with each other, they do not shrink nor decay nor lose sharpness due to the passing of time. As we saw in Part 1, Aristotle argues that geometers study sensible objects 'not *qua* movable' (*Metaph.* M.3, 1077b28) and that the objects of geometry are 'separable in thought from motion' (*Ph*. II.2, 193b34). In the secondary literature, these remarks are usually interpreted as meaning that geometrical objects are completely unchangeable.[216]

In the framework of the interpretation that I have put forward, this would mean not only that geometers ignore the natural motion of sensible substances, but also that the semi-fictional objects that arise from the separation in thought of the geometrical

[216] See e.g. Katz (2019: 507): 'Mathematical objects are unmoving and do not undergo processes of change'; Mendell (2019): '[mathematical objects] lack generation, destruction, and change (with appropriate provisos for kinematics and mathematical astronomy)'; Iwata (2021: 373): 'his view that geometrical objects do not change'.

properties of sensibles and their fictional transformation into object-like individuals are not the kind of objects that can undergo motion or change. However, the way in which Greek geometers act and speak suggests that they consider the objects of their study as apt to be manipulated and changed, even if not subject to natural decay. As we will see in §1, Greek geometry is constructive in character, and it makes constant and ineliminable appeals to motion and change. Can Aristotle's philosophy of geometry account for these practices?

This way of speaking, and in general the constructive character of Greek geometry, seems to be generally problematic for philosophers – not only for Aristotle. Plato, for example, acknowledges that geometers 'talk as if they were doing something and making all their terms to fit their activity: they talk about making the square, applying and adding, and similarly with everything else',[217] but describes this way of talking as 'quite ridiculous', although 'unavoidable'.[218] The reason, he argues, is that 'geometry is the knowledge of the eternally real'.[219] According to Plato, geometry is about separate, eternal and unchangeable objects: the constructive talk of geometers should not be taken seriously, for it is impossible to bring into existence, modify, or generally act on those objects.

The most obvious reason why the constructive and dynamic character of Greek geometrical practice is so difficult to account for is that it seems to contrast with some assumptions about truth and knowledge that are widely shared by Greek philosophers. In particular, it seems difficult to square with the idea that geometrical truths are eternal and necessary, and independent of what we do or know. Allowing change and motion into

[217] ὡς γὰρ πράττοντές τε καὶ πράξεως ἕνεκα πάντας τοὺς λόγους ποιούμενοι λέγουσιν τετραγωνίζειν τε καὶ παρατείνειν καὶ προστιθέναι καὶ πάντα οὕτω φθεγγόμενοι.
[218] μάλα γελοίως τε καὶ ἀναγκαίως. For the interpretation of ἀναγκαίως as 'unavoidable', see Burnyeat (2000: 33 ff.).
[219] τοῦ γὰρ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν. See *Republic* VII, 527a-b (Grube's translation).

geometry, and in particular allowing that geometrical objects might be manipulated at will by geometers, risks jeopardising the status of geometry as a science; Plato's reaction consists in explaining away the practice and the language of geometers as misleading. Although geometers *think* that they are manipulating geometrical objects, Plato argues that in fact that do not do such thing: geometrical objects are immutable and eternal, and geometers can only manipulate their sensible representations.[220]

Aristotle shares many of these assumptions, but (I argue) his position faced with the constructive character of Greek geometry must be different from Plato's. First of all, we have seen that Aristotle does not think that geometry is about some special, perfect, immutable and unchangeable objects that exist independently of physical substances. On the contrary, he devotes the whole of *Metaphysics* M.2 to refuting this Platonic position, and states quite clearly that geometry is supposed to be about sensible objects (although considered not *qua* sensible). This means that he cannot explain away the geometers' talk of constructions and changes in geometry on the same ground as Plato. Note that there are clear signs that Aristotle does not wish to reject the geometers' characteristic methodology nor way of speaking: he considers geometrical methodology as a model for scientific reasoning, and frequently refers to it to clarify his theory of demonstration.[221] So we would expect his philosophy of geometry to try and account for common geometrical practices, rather than dismissing them.

But what, then, should we make of his claims that geometrical objects are 'separable in thought from motion' (*Ph.* II.2), 'immovable' (*Metaph.* E.1) and 'without motion' (*Metaph.* A.8)? Can we understand them in such a way that they are compatible with the

[220] Cf. *Resp.* VI, 510d-e.
[221] See Ugaglia (2015) for a discussion of Aristotle's use of geometrical practice as a model for his theory of knowledge. It is also important to stress that the uncharitable picture of Aristotle as ignorant of mathematics, that was once widespread among commentators, is nowadays commonly rejected. See Mendell (2004) for an overview of the literature.

use of constructive methods in the practice of geometry? Answering this question is the first goal of this chapter. It will be fundamental to keep in mind the ambiguity of the expression 'to be about' and the distinction between the two uses of the expression *ta mathematika* that I have individuated in Part 1. The expression can pick out either the truth-makers of geometrical statements, or the referents of the singular terms that appear in such statements. These two kinds of entities are very differently related to change: geometrical properties (the truth-makers of geometrical statements) stand in one relation to change, whereas semi-fictional mathematical objects (the referents of the singular terms in geometrical statements) stand in another, quite different, relation to change. As the previous chapter, this chapter tries to determine whether Aristotle's philosophy of geometry can account for the common geometrical practice of his time. This means that the analysis will proceed by adopting the point of view of the working mathematician: the question whether geometrical objects can undergo change should thus be understood as relative to the semi-fictional objects that are produced by the operation of separation.

There is an obvious reason why one might want to deny that geometrical objects can change: to preserve the stability and eternity of geometrical truth. I will argue that this is not Aristotle's main worry when he claims that geometry ignores motion. Thanks to the distinction between truth-makers of geometrical statements and referents of singular terms, Aristotle's theory can account for the production and introduction of new objects in proofs without having to admit that the truth-makers of geometrical statements are *generated* by the geometer's actions (§2). That Aristotle's worry is not related to the possibility that the truth-makers of geometrical theorems come into existence thanks to the actions of geometers can be seen by the fact that he stresses that there is no *kinesis* in geometry, but does not seem very concerned with the possibility of *genesis*. In §3 I will discuss what makes *kinesis* so problematic for geometry, and examine whether the very

specific and limited range of manipulations and modifications that are allowed in geometry (and that pertain to the semi-fictional objects that result from separation, not to the geometrical properties of sensible objects which act as truth-makers of geometrical theorems) should be understood as a distinctive type of changes. Stressing the differences between physical and geometrical changes, however, points to a potential problem for the applicability of geometry (§4): if geometrical objects are allowed to be changed and manipulated, but in a *different* way than sensible objects, can geometry really be applicable to the sensible world? The last section (§5) addresses objections and draws conclusions.

§1. Appeals to Change in Geometrical Practice

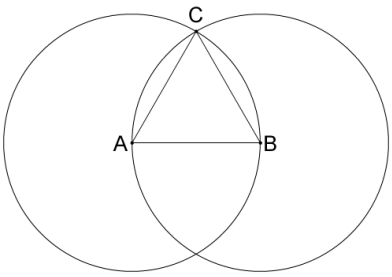
That Greek geometrical practice involves some dynamical and constructive elements is clear already from Plato's brief discussion of it in *Republic* VII. As Plato understands, this is an ineliminable aspect of Greek geometrical practice. To have a clearer idea of the role that these dynamical elements play, however, we need to turn to proper mathematical treatises. It appears that Greek geometrical practice appeals to change in three main ways: at the level of diagrams, of definitions, and of proofs. Let us look at them in turn.

§1.1. Diagrams

The first way in which motions and changes are appealed to in the practice of geometry is as obvious as it is apparently irrelevant: Greek geometry makes an extensive use of diagrams, and motion is required to draw them. This is an undeniable fact, but it might seem very easy to dismiss: surely the diagram is not the proper object of the proof, but merely a representation of it?

The role of diagrams in Greek geometrical practice has been recently re-evaluated.[222] In particular, it has been stressed that diagrams are not merely essential to the *discovery* of geometrical truths, but they also constitute an essential part of the geometrical *proofs*: some inferential steps are licensed not by the text alone, but are read off the diagram (through diagrammatic inferences). For example, an appeal to the diagram can be seen in *Elements* I.1, where the existence of the point of intersection between the two circles seems to be taken for granted.

Instructing the reader on how to construct an equilateral triangle on a given straight line AB, Euclid takes it for granted that the two circles drawn with the two extremities A, B as centres intersect each other in a point C distinct from A and B:



This assumption is not warranted by Euclid's explicit stipulations: it seems to rely on the presupposition that the circles are continuous, so that their point of intersection exists. The property of continuity, however, is never discussed by Euclid, nor is the Principle of Continuity stated anywhere.

From Leibniz onwards, this has been taken as a proof that the *Elements* are 'incomplete': some inferential steps (especially the ones that require the assumption of the existence of some specific object, as in this case) seem unjustified. A certain tradition of mathematicians, culminating with Hilbert (1899), has tried to fill the gaps in the

[222] See in particular the seminal studies of Manders (1995/2008); Netz (1999).

deductions by adding axioms (such as the axioms of continuity, which grant the existence of the intersection point). In more recent years, the focus of attention has shifted to try to understand why Euclid and his contemporaries did not think that their proofs were incomplete, and how it is that they got their proofs (mostly) right even if by our standards they were jumping to conclusions. It has been argued that an explicit discussion of continuity was not felt necessary by Euclid, because information about continuity was provided by the diagram;[223] the point of intersection is a co-exact feature that can be read off the diagram. Thus, recognising that diagrams are *part* of the proof and not merely an illustration of it goes a long way towards re-evaluating Greek geometrical practice.[224]

But one might object that this merely proves that the *diagram* is essential to the proofs; not that the *act of drawing the diagram* is.[225] The motion that is required to draw the diagram seems exactly the kind of motion that we can ignore while doing geometry, although it is necessary to put us in the position to start doing geometry. One might even suggest that the drawing of the diagram should not really be considered a proper part of Greek geometrical practice, but only a preparatory step. As a matter of fact, geometers did not always draw the diagrams while doing geometry: given the absence of adequate

[223] Panza (2012).

[224] Acerbi (2020: 52, fn. 57) dismisses the role of diagrams in geometrical proofs, pointing out that an incorrect diagram does not invalidate the proof. He claims that 'the alleged gaps in ancient geometric proofs are a matter of axiomatic incompleteness, not of some information being necessarily drawn from the diagram'. Although it is obviously possible that the gaps in ancient geometrical proofs are just a sign of incompleteness, I believe that an explanation that manages to save them is *ceteris paribus* preferable. As for incorrect diagrams, notice that Netz (1999: 17-19) allows for the possibility of 'schematic' drawings: diagrams that are not correct but should be read as indications and guidelines to produce one's own precise diagram, either on paper or in the imagination.

[225] Actually I do not think that this objection takes off. Relying merely on the appearance of the diagram cannot ensure continuity. Not only because sensible objects such as diagrams are not precise enough to provide this kind of information, but especially because even in perfect conditions there would be no perceptible difference between a continuous and a discrete system for an insider observer (Dedekind 1872: 38). The real reason why we know that the diagram is continuous, I believe, is because it has been drawn by a continuous stroke. The continuity of the diagram is ensured by the way in which it is constructed, and in particular by the continuity of the motion that draws it. Proving this suggestion would require a deeper analysis than what I can give now, though, so I will leave it on the side (especially because my point goes through even if we accept the objection for the sake of argument).

mediums, diagrams were often prepared in advance.[226] Geometers would then go through the proof with a diagram already made in front of them. Accordingly, the suggestion goes, one could ignore the fact that the diagrams were actually drawn by someone and required motion, and simply treat them as immobile and immutable figures. After all, once drawn, they cannot be moved.

If we look more closely to the way in which the diagram is supposed to provide the essential information that grants the validity of the proof, however, we see that it is not so easy to dismiss the dynamic elements required for its construction. It is (for us) striking that the diagrams found in Greek manuscripts tend not to be metrically correct: they might represent as different in length lines that are said to be equal in the body of the proof. As Netz (1999: 54) argues, the diagram is reliable not because of its appearance, but because it encodes information about its construction.[227] The diagram is 'schematic', is a 'token gesture, a make-believe [...] It acknowledges the shadow of a possible construction without actually performing it'[228]. In short, the diagram represents the geometrical construction: not merely its outcome (the completely constructed geometrical object), but also the steps necessary to get to it. Which means that, contrary to what the objection suggested, the actual picture is not as important as the way in which it is constructed.

This should not come as a surprise to the reader, as one well-known feature of Greek geometry is that the way in which geometrical diagrams are drawn is strictly codified: policing the way in which the diagram is drawn is a form of control on the acceptability

[226] Netz (1999: 16).
[227] See also Sidoli (2004: 10): 'Using a *diastēma* [i.e. a compass] to draw a circle tells us how the circle has been drawn in a way that introduces necessity into the construction itself. The use of a *diastēma* allows us to then say that certain lines are equal because that is how the circle was drawn.'
[228] See also Netz (1999: 181): 'Why is the diagram reliable? First, because references to it are references to a construction'.

of the corresponding geometrical object.[229] The underlying assumption seems to be that the relevant geometrical object is constructed in the same way as the diagram is.

§1.2. Definitions

That change and motion are required to construct the geometrical objects themselves, and not only the diagrams that represent them, can be appreciated also by looking at the way in which some geometrical objects are defined.[230] Some geometrical objects are defined by appealing to motion, others as changeable.

As for the first case, consider the solids of revolution. These solids, which are studied in Book XI of Euclid's *Elements*, are presented as the result of the revolution of a given plane figure. See for example the definition of the sphere:

> Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.
>
> When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *sphere*.
>
> *El.* XI, def. 14 (Heath's translation)

[229] This is even more evident when considering the ancient debate about non-standard methods of construction which are essentially based on motion, e.g. the *neusis*. See Knorr (1986), Fowler (1987), Heath (1921).

[230] This way of appealing to motion and change in geometry is particularly interesting for our purposes, for Aristotle explicitly states that geometrical objects can be defined without appealing to motion (*Ph.* II.2, 193b35 ff.). Of course, this might mean that Aristotle has in mind a different set of definitions than the ones that can be found in Euclid's *Elements* – after all, the *Elements* are still to be written when Aristotle composes his *Physics*. Unfortunately, the scarcity of sources at our disposal makes it difficult to have access to the precise set of definitions that Aristotle might have in mind. However, it is commonly accepted that the *Elements* mostly consist in a systematization of pre-existing material; so, I believe that it would be better if we could interpret Aristotle's remark in such a way as to be compatible with the *Elements'* definitions.

Definitions 15, 18, 19, 20, 21, 22, 23 follow the same schema: they define a solid by explaining how it is created. In all these cases, a plane figure is presented as able to undergo motion, thus determining a three-dimensional area that is meant to be the area of the solid.[231]

As for geometrical objects defined as changeable, the most striking and obvious example is the straight line, which is by definition extendable *ad libitum*. The extendibility of the line is stated by Euclid's Postulate 2, which gives licence 'to produce a finite straight line continuously in a straight line'.[232]

[231] Another very interesting case is provided by curves. As the geometry of curves is codified quite late in Greek geometry, this case study is somehow less compelling than that of the solids of revolution. It is notable, however, that an appeal to motion seems to be unavoidable when defining curves. Indeed, Greek geometers have two main ways to define a curve: stereometrically and kinematically. When curves are defined kinematically, they are presented as the result of a point subjected to a double motion: the trajectory of the moving point constitutes the curve. When curves are defined stereometrically, they are presented as the result of the intersection of a plane and a solid. The solid is cut by the plane, thus generating the curve. Once again, we see motion and change at play: in order to cut the solid, the plane must be movable; at the same time, cutting the solid amounts to subjecting it to a radical modification – from a unitary thing, two smaller items are created. That this operation is considered a genuine change, and not just as an inadequate way of describing a configuration that is in reality static and unchangeable, can be seen already in the way in which Democritus problematizes it: see Plut. *CN* 1079E; Sorabji (1983: 341).

[232] Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν. The formulation of the Postulate is quite obscure, and its interpretation is object of controversy. The most intuitive way to understand the process of extending a line is as a stretching of the original line, operated through the fluxion of the final point. According to this model, extending a line requires both the motion of the final point, and the change of the line. Recently, however, De Risi (manuscript: 30) suggested that 'the procedure of extending a straight segment consists in drawing a second segment and placing it in such a way that the two are continuous'. Aristotle did certainly take the line to be extendable (*DC* I.2, 269a20 ff.) but it is unclear how he thought that the extension was to be operated. He mentions the hypothesis that the line is generated by the fluxion of a point at *DA* I.4, 409a4, but interpretations of the passage disagree as to whether Aristotle embraces this possibility. One of the problems is that Aristotle believes that zero-dimensional and indivisible objects such as points cannot move *per se*, but only *per accidens* (i.e. by being the limit of something that is in motion). Even if De Risi is right, however, and the extension of the line consists in nothing else than the addition of a second segment continuous with the given one, this process still involves a change: when the two segments become *continuous*, it means that they merge in a unitary object. Discussing Aristotle's theory of continuity goes beyond the scope of this work, but it is clear that Aristotle strictly associates the notions of continuity and of unity. In *Physics* V.3, he concludes his discussion by claiming that 'something is continuous rather than consecutive when the limits by which the two objects are in contact have become identical and, as the word implies, enable one object to continue into the other. This is impossible where there are two separate limits. It is clear from this definition that continuity is a property of things which naturally form a unity by their contact with one another' (227a10 ff. Bostock's translation). This passage is notoriously controversial, but at the very least it shows that continuity is taken to be strictly associated with unity. This association also results clearly from *Physics* VIII.8, where the unity and the continuity of the first motion go hand in hand. See Pfeiffer (2018: §7.4).

§1.3. Proofs

The most widespread and interesting appeals to motion and change appear in the course of geometrical proofs and demonstrations, which are fundamentally constructive in character. This is particularly evident in the tradition of problems, which require the geometer to construct a geometrical object. Problems start with the instruction to perform an action, and are solved when the action is concluded and it has been proved that it achieved the desired effect.[233] But constructive steps are essential in theorems, too.

First, notice that in the course of the demonstration, geometers can add objects to the system under consideration. In the *Elements*, this is reflected in the difference between objects which are 'given' since the beginning and objects which are 'produced' at later stages of the proof.[234] The introduction of new elements is licensed by the Postulates, which give permission to draw additional lines and circles to the existing diagrams: these can be combined in such a way as to construct higher-dimensional objects, or made to intersect so as to individuate points.

Moreover, the introduction of new elements in the configuration of the proof results in the modification of the objects that were already present in it. An obvious case of modification is the introduction of a line that is incident to a given one: this operation (i) divides the given line in two parts, and (ii) determines the point of intersection.[235] The

[233] For an analysis of problems as involving a mathematical form of knowing how, see Beere & Morison (manuscript).

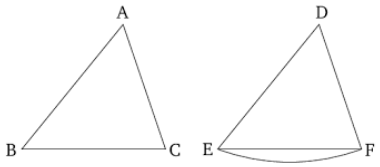
[234] It is controversial how to understand what it means that some objects are 'given'; for a discussion see Taisbak (2003), Netz (1999), Panza (2012). For my purposes, it is not indispensable to clarify it: what is notable is that *other* geometrical objects are *produced* during the proof, and are added to the configuration under exam.

[235] As I mentioned, this is not the place for a full discussion of Aristotle's theory of continuity. It is important to stress, however, that we should not think of the line as composed by an infinity of potential points, each of them placed in a definite position and just waiting to be actualized (see Coope (2005: 10-11); White (1992: 22) for a discussion). The line is a unitary and homogeneous object which can be divided *ad infinitum* and in any way that one might want; *not* a collection of points, be they actual or potential. Saying that the line is infinitely divisible means that wherever the division is operated, a point will be actualised; the determinate location of the point, however, depends on the act of division. Thus, the act of division counts as a real and proper change, in that it determines and actualizes one specific configuration over all possible others and it breaks the continuity of the line. For an alternative account, according to which there

modifications caused by the introduction of new objects in the proof are a special case of a more general phenomenon: in the course of geometrical demonstrations, some objects can be manipulated and undergo modifications. In particular, geometers routinely (a) divide their objects, and (b) extend their lines.

In addition to these standard modifications, the *Elements* also include a couple of theorems that seem to rely on the possibility of moving geometrical objects around. In particular, Euclid appeals to an intuitive idea of congruence according to which two objects are congruent if and only if they would coincide when superposed. This intuition comes to light in a couple of proofs were locomotion is directly appealed to. This is the case of Euclid, *Elements*, I.4, which aims to prove the side-angle-side condition for congruence of triangles.[236]

The proof is based on the possibility of rigidly displacing a triangle, as if it were a configuration of rigid bars: the core of the proof is the claim that 'if the triangle ABC be applied to the triangle DEF, and if the point A be placed on the point D, and the straight line AB on DE, then the point B will also coincide with E, because AB is equal to DE'.[237]



It is evident that this kind of change concerns the geometrical object itself, and not its representation in the diagram (which is indeed static).

is no fact of the matter whether the final configuration should count as a unitary continuous line, or as two contiguous segments, see Pfeiffer (2018: 147, fn. 2).

[236] Another example is theorem I.8. The translation is by Heath (1926).

[237] Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Δ σημεῖον τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΔΕ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Ε διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΔΕ.

The analysis of Greek geometry and of its practice that I have sketched shows that dynamical elements are involved at three different levels: they are necessary for the diagrams, they enter the definition of geometrical objects, and they are pervasive in proofs and demonstrations. Such a widespread appeal to change cannot be explained away merely by reforming the language of geometry, but is deeply engrained in the practice. It seems that Netz is right when he claims that 'Greek geometry is the study of spatial action, not visual representation'.[238] Can Aristotle account for it? Is it possible to interpret his remarks on the absence of change in geometry so as to be compatible with the role of change in geometrical practice?

§2. No Generation in Geometry

The first thing to do is understand what is the problem with change, and why Aristotle might want to ban it from geometry. The most obvious reason is that the constructive character of Greek geometry, and the three kinds of appeal to change that I have just outlined, are difficult to square with the realist conception of stable and necessary mathematical truth that is widely shared by Greek philosophers. As Burnyeat underlines, all parties to the debate would agree that the truth of a science requires the existence of the entities that said science is about.[239] An apparently immediate consequence is that

[238] Netz (1999: 60).

[239] This formulation is clunky, and less elegant than Burnyeat's own claim that 'the entities *referred to* in a given science are entities whose existence is necessary for the theorems of the science to be true' (2012: 153, my italics). The reason why I opt for the formula 'to be about' rather than adopting the jargon of reference is not merely to remain stylistically closer to the Greek expression περὶ + genitive, which Aristotle uses in *Metaph.* M.3, but mostly to preserve its ambiguity. As I argued in Part 1, I believe that the Greek formula is ambiguous and might pick out (a) the referents of the singular terms that appear in the theorems, or (b) the truth-makers of the theorems. As I showed, these two items do not necessarily coincide. Speaking of 'being about' rather than 'referring to' allows me to maintain this ambiguity and to maintain that geometry is indeed about existing entities, even if geometers refer to semi-fictional objects.

since geometrical truth is necessary and eternal, geometrical objects must exist in a stable way. This seems to be the reasoning behind Socrates' comments in *Republic* VII:

> Λέγουσι μέν που μάλα γελοίως τε καὶ ἀναγκαίως· ὡς γὰρ πράττοντές τε καὶ πράξεως ἕνεκα πάντας τοὺς λόγους ποιούμενοι λέγουσιν τετραγωνίζειν τε καὶ παρατείνειν καὶ προστιθέναι καὶ πάντα οὕτω φθεγγόμενοι, τὸ δ' ἔστι που πᾶν τὸ μάθημα γνώσεως ἕνεκα ἐπιτηδευόμενον.
>
> Παντάπασι μὲν οὖν, ἔφη.
>
> Οὐκοῦν τοῦτο ἔτι διομολογητέον;
>
> Τὸ ποῖον;
>
> Ὡς τοῦ ἀεὶ ὄντος γνώσεως, ἀλλὰ οὐ τοῦ ποτέ τι γιγνομένου καὶ ἀπολλυμένου.
>
> Εὐομολόγητον, ἔφη· τοῦ γὰρ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν.
>
> "I think the way they argue is quite absurd and is forced on them: I mean, they talk as if they were doing something and making all their terms to fit their activity: they talk about making the square, applying and adding, and similarly with everything else; but in my view the subject as a whole is studied for the sake of knowledge."
>
> "I agree entirely," he said.
>
> "So do we still need to agree fully on this?"
>
> "In what respect?"
>
> "That it is the knowledge of the eternally real and not what comes into being and then passes away."
>
> "That's easy to do," he said. "Geometry after all is the knowledge of the eternally real."
>
> *Republic* VII, 527a6-b (Grube's translation)

Aristotle would agree with Plato that geometrical truths are eternal and independent of what geometers know or do – in contemporary terms, we would say that Aristotle is a

truth-realist. For this reason, he could not accept that the truth-makers of geometrical theorems are brought into existence by the actions of geometers, and that geometrical constructions result in the coming-into-being of such items. Aristotle cannot hold that adding objects to the configuration of a proof amounts to bringing into existence the truth-makers of the theorems.[240]

If the interpretation that I have outlined in Part 1 is correct, though, this worry has nothing to do with the question whether the semi-fictional objects to which geometers refer can undergo changes. I argued that geometrical propositions are true in virtue of the fact that sensible bodies have geometrical properties (either actually or potentially). The objects that geometers refer to and manipulate during their activities, on the other hand, are semi-fictional objects that result from separating those properties and positing them as substances; they are not the truth-makers of geometrical statements (even if they are relevantly related to them).[241] The truth of the theorems does not in any way depend on the existence or state of these objects, even if they are essential to the work of the practising geometer. The truth of geometrical statements depends only on sensible objects and their (geometrical) properties – exactly as is the case for all other sciences.

Sensible objects are changeable and corruptible, and come in and out of existence, independently of whether we accept constructive methods in geometry. The changes that they undergo lead them to acquire or lose geometrical properties. But I have argued that also *potential* properties can be the truth-makers of geometrical statements, so the stability of geometrical truth is not endangered by this.[242] The interpretation that I have

[240] Accordingly, it would be a mistake to attribute to Aristotle anything similar to modern day mathematical constructivism or intuitionism. Cf. *Cat*. 7.7b27-33, where Aristotle clearly states that the knowable is prior to knowledge: 'For if the knowable does not exist, there can be no knowledge – since there will no longer be knowledge of anything – but the non-existence of knowledge is no obstacle to the existence of the knowable' (Heath's translation).
[241] See Chapter 3.
[242] See Chapter 2.

provided in Part 1 stresses the analogy between geometry and other sciences: in the same way as natural sciences are proper sciences even if their objects are changeable and perishable, so is geometry. The reading of the *qua*-theory that I have put forward takes Aristotle not to be committed to the existence of a special kind of unmovable geometrical objects to ground the truth of geometrical statements. This is independent of whether or not we accept constructive methods and language in the practice of geometry.

Under the interpretation that I have suggested, the issue of what makes geometrical statements true (the Grounding Question) and the attempt to account for the practice of geometry and how we get to know geometrical truths (the Epistemological Question) are kept distinct. The truth-makers of geometrical statements are sensible objects which instantiate geometrical properties. These sensible objects *are* changeable, and the fact that we might want to consider them not *qua* changeable does not make any difference to the truth of geometrical propositions – nor to the fact that the sensible objects are in fact changeable. When we do geometry (i.e. when we embark in the epistemological project of finding out truths about geometry), we consider these sensible objects not *qua* sensible and we separate them, thus fictionally transforming the properties into objects and treating them as independent and persistent substances.

This second step introduces an element of fiction, for it involves treating properties as if they were ultimate subjects of predication. The question that I am examining now is whether *these* semi-fictional objects are movable, changeable and generable/perishable or not. However we answer this question, it will not affect the truth-value of geometrical sentences, for these objects do not contribute towards the truth of the theorems that are proved by appealing to them. We are on the epistemological side, not on the ontological.

And indeed, when Aristotle describes the geometers' modification of diagrams and geometrical configurations he does not seem worried that this might require bringing

geometrical objects into existence, thus endangering the stability of geometrical truths. Consider the following passage:

> εὑρίσκεται δὲ καὶ τὰ διαγράμματα ἐνεργείᾳ· διαιροῦντες γὰρ εὑρίσκουσιν. εἰ δ' ἦν διηρημένα, φανερὰ ἂν ἦν· νῦν δ' ἐνυπάρχει δυνάμει. διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίῳ ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα ὀρθή, ἰδόντι δῆλον τῷ ἐκεῖνο εἰδότι. ὥστε φανερὸν ὅτι τὰ δυνάμει ὄντα εἰς ἐνέργειαν ἀγόμενα εὑρίσκεται· αἴτιον δὲ ὅτι ἡ νόησις ἐνέργεια· ὥστ' ἐξ ἐνεργείας ἡ δύναμις, καὶ διὰ τοῦτο ποιοῦντες γιγνώσκουσιν (ὕστερον γὰρ γενέσει ἡ ἐνέργεια ἡ κατ' ἀριθμόν).
>
> And the constructions[243] are discovered in actuality; for they discover them by dividing. If they had been divided they would have been evident; but as it is they are in there potentially. Why is the triangle two right angles? Because the angles around one point are equal to two right angles. So if the line parallel to the side had been drawn up, it would have been clear immediately on seeing it. Why is there universally a right angle in the semi-circle? Because if three lines are equal, the two which are the base and the one dropped straight from the centre, it is clear on seeing it to the person who knows that. So that it is evident that the things which are potentially are discovered when they are drawn out into actuality; the explanation is that thinking is the actuality; so that the potentiality is from actuality, and because of this they know by making (for the individual actuality is posterior in coming to be).
>
> *Metaphysics* Θ.9, 1051a21-31 (Makin's translation)

[243] The term διαγράμματα is usually understood as referring to geometrical constructions or diagrams, but see Iwata (2021) for a different interpretation, according to which διαγράμματα refers to the propositional content of geometrical proofs. Even if he is right, however, it is still the case that such propositional content is discovered by manipulating diagrams and geometrical constructions, and that this operation amounts to actualizing elements which were merely potentially present.

In this very dense and much-discussed passage, Aristotle describes the way in which geometers operate and how they come to know geometrical truths. He focuses on two specific examples: the so-called '2R' theorem (which states that the internal angles of a triangle add up to two right angles), and the theorem that all triangles inscribed in a semi-circle are right angled. His description of the diagrams involved in the proofs is very sketchy, so much that it is quite difficult to reconstruct precisely the diagrammatic configurations that he might have been referring to.[244] I am not going to discuss the details of the proofs; for my purposes, what matters is that it is clear that Aristotle is fascinated by the 'construction viewed in its process of development, that is, in the figure a geometer draws, and gradually modifies, when he tries to solve a problem'.[245] He mentions the geometers drawing and dividing lines, and he does not at all seem bothered by these practices. On the contrary, he refers to these practices approvingly, and uses them as a model to illustrate some aspects of his theory of knowledge:[246] the act of modifying the diagram by introducing new elements in it is what leads the geometer to 'see' the solution of the problem. Aristotle does not seem worried that the introduction of new elements in the diagram amounts to bringing geometrical objects into existence; on the contrary, he argues that geometers can discover geometrical truths by bringing into actuality (ἐνεργείᾳ) something that was merely potentially present (ἐνυπάρχει δυνάμει). This means that adding an element to a geometrical configuration does not amount to bringing it into *existence*, but brings into *actuality* what was previously merely potentially present.

Note that this passage can be read as regarding the manipulation of the diagram as well as of the corresponding geometrical object (which, in my reading, is a semi-fictional object resulting from the separation of the geometrical features of the sensible object).

[244] See Ugaglia (2015) for review of the literature and a possible reconstruction of the diagrams.
[245] Ugaglia (2015: 51).
[246] In particular, Ugaglia (2015) suggests that the construction and manipulation of diagrams are correlated to the process known as ἐπαγωγή, which allows to recognize an individual as falling under a universal.

The ambiguity is due to the fact that they are supposed to track one another: the geometrical object is modified in the same way as the diagram that represents it. Both levels can be accounted for.

If we focus on the manipulation of the diagram, we should bear in mind that it is itself a sensible object that instantiates geometrical properties, and thus can act as truth-maker for geometrical theorems.[247] The modification of the diagram does not endanger the stability of geometrical truth, because this is grounded not only in actual but also in potential geometrical features of the sensible world.

On the epistemological side, the question is whether Aristotle would agree that the semi-fictional objects that result from separation can undergo the same modifications as the diagram. I think he would. Note that at times, Aristotle seems to allow for some sort of generation of geometrical objects. For example, in *Metaph.* M.2, 1077a24-30 we read that higher-dimensional objects are posterior in generation to lower-dimensional ones, for 'first length is generated, then breadth, finally depth'.[248] It is not completely clear whether Aristotle endorses this claim, or if he is simply adopting the Platonists' point of view for dialectical reasons.[249] But the idea that there is some sort of generation of higher-dimensional objects from lower-dimensional ones appears also in *De Caelo* I.1, 268a30-b5. Aristotle's understanding of continuity leads him to maintain that it is impossible to obtain a continuous higher-dimensional object by putting together a number of lower-dimensional items, so the process in question cannot be one of composition. Pfeiffer (2018: 85) calls it a 'quasi-generation': Aristotle has probably in mind the geometrical

[247] Cf. Katz (2019), who argues that geometry studies the geometrical properties not of any sensible object, but specifically of diagrams. I think that this restriction is too narrow, but I agree that diagrams are among the objects that are studied by geometry.
[248] *Metaph.* M.2, 1077a24-26: πρῶτον μὲν γὰρ ἐπὶ μῆκος γίγνεται, εἶτα ἐπὶ πλάτος, τελευταῖον δ' εἰς βάθος.
[249] The context is one of the arguments against Mathematical Platonism, and in particular against the idea that mathematical objects might be prior to sensible things. For if this were the case, lower-dimensional objects should be prior in reality to three-dimensional bodies, but this is absurd: lower-dimensional objects are prior in generation, so they must be posterior in reality.

construction of plane figures by drawing lines, or of solids by assembling planes. Note that we can also have the reverse process: in the same way as we obtain a higher-dimensional object by drawing its lower-dimensional limits, we can obtain lower-dimensional items by dividing a higher-dimensional object. It is striking that while in physics Aristotle assigns a certain priority to three-dimensional bodies over lower-dimensional entities, in geometry such hierarchy is completely absent. Items of all dimensions have the same ontological status.[250] This supports my claim that separation introduces an element of fiction, which regards specifically the mode of existence of the object: in geometry it is legitimate to treat as substances also items that are not properly speaking substances. It seems to me that this being the case, Aristotle would have no qualms allowing some sort of generation of geometrical objects *via* geometrical construction, because the semi-fictional objects that are generated in this way are not the truth-makers of geometrical theorems.

§3. No Kinesis in Geometry?

Aristotle's observations about the unchangeability of geometrical objects do not seem primarily concerned with denying that the truth-makers of geometrical statements are brought into being by the actions of the geometer. Let us take a closer look at the passages in question. In *Metaphysics* A.8, 989b32-33 he states that 'mathematical objects, except those of astronomy, are without motion' (τὰ γὰρ μαθηματικὰ τῶν ὄντων ἄνευ κινήσεώς ἐστιν ἔξω τῶν περὶ τὴν ἀστρολογίαν), and similar terms are employed in multiple other passages throughout the *Corpus*:

[250] If anything, plane figures are privileged in Greek geometry.

περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ’ οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον· οὐδὲ τὰ συμβεβηκότα θεωρεῖ ᾗ τοιούτοις οὖσι συμβέβηκεν· διὸ καὶ χωρίζει· χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων.

Now the mathematician, though he too treats of these things, nevertheless does not treat of them as the limits of a natural body; nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated.

Physics II.2, 193b30-35 (Charlton’s translation)

ἀλλ’ ἔστι καὶ ἡ μαθηματικὴ θεωρητική· ἀλλ’ εἰ ἀκινήτων καὶ χωριστῶν ἐστί, νῦν ἄδηλον, ὅτι μέντοι ἔνια μαθήματα ᾗ ἀκίνητα καὶ ᾗ χωριστὰ θεωρεῖ, δῆλον. […] τῆς δὲ μαθηματικῆς ἔνια περὶ ἀκίνητα μὲν οὐ χωριστὰ δὲ ἴσως ἀλλ’ ὡς ἐν ὕλῃ·

Mathematics also is theoretical; but whether its objects are immovable and separable from matter, is not at present clear; it is clear, however, that it considers some mathematical objects qua immovable and qua separable from matter. […] some parts of mathematics deal with things which are immovable, but probably not separable, but embodied in matter

Metaphysics E.1, 1026a7-15 (*ROT*)[251]

It is important to stress that in all these passages, Aristotle uses the language of *kinesis* and not of *genesis*. He is not primarily concerned with denying that geometrical objects come into being, but with the appropriateness of describing them as movable or

[251] The part of this passage that I have not quoted, which regards natural science, is the object of a scholarly controversy. While according to the manuscript tradition Aristotle writes that natural science studies objects that are *separate* but not unmoving, Schwegler (1848, vol. 4: 16) emendates *chōrista* in *achōrista*. See Griffin (2022) for a defence of the manuscript reading.

changeable, and of associating them with the notion of *kinesis*.[252] Why? If the problem is not (just) the stability of geometrical truth, why should Aristotle be concerned with the presence of *kinesis* in geometry?

The notion of *kinesis* has a very important and central place in Aristotle's philosophy, and in particular in Aristotle's physics. Physics is the study of nature, and nature is an inner principle of motion and rest.[253] *Kinesis* is the characteristic feature of sensible substances, so much so that sensible features are characterised as those that belong to 'things in motion *qua* in motion'.[254] This already goes a long way towards explaining why Aristotle stresses that geometers need to remove motion as well as the sensible aspects and the nature of the sensible body under consideration: motion is so closely connected to them that considering a natural object not *qua* sensible requires also abstracting from its natural motion.[255]

Kinesis also seems to bring about problems in its own right. As we have seen in the previous chapter, the natural motion of the elements depends on, and embeds, a spatial structure that is at odds with the requirements of Euclidean geometry. In more general terms, *kinesis* is the result of a complex system of causes that cannot be fully expressed nor comprehended in geometrical terms, and that might even act as masks and hide some geometrical features.

These observations all contribute to understanding why it is necessary for the geometer to ignore the motion of the sensible objects if she wants to study their geometrical features and properties. The reason seems to be more related to the division of labour among the different sciences than to the preservation of the stability of

[252] '*Kinesis'* is at times used as an umbrella term to cover all changes that are compatible with the persistence of the subject of change (as opposed to radical changes), and at times with the more restricted sense of motion. This is why I speak both of motion and of change.
[253] *Ph*. II.1, 192b20–23.
[254] *Metaph.* Δ.14, 1020b9-11; 17-18.
[255] See Pfeiffer (2018: 38).

geometrical truth. In the next section, I will suggest that there is also another, more fundamental, reason. Before getting there, the question that we now need to ask is: does this ban on *kinesis* imply that the semi-fictional objects that result from the separation of the geometrical features of a sensible body cannot be subjected to the modifications and manipulations that are licensed by the postulates and axioms of geometry?

The manipulations that are allowed in geometry seem to be of a very special kind, so much that it is dubious whether they can be properly considered *kineseis*. They are not natural motions nor the kind of changes that are proper to natural substances; they are not spontaneous changes nor degradations, but mechanical manipulations that need to be willingly enforced and are completely under the control of the geometer (the geometrical system does not itself have any inner principle of changing or remaining at rest).[256]

The peculiar nature of geometrical manipulations can be appreciated from the case of division. While both physical and geometrical objects are said to be divisible (and this divisibility is granted by one and the same property that they share, namely continuity), the way in which they are divisible is radically different. Physical bodies are divisible into spatially distinct and separate parts, while the division of geometrical objects does not lead to spatial separation, but simply to the actualisation of parts in a whole.

It should be noted, however, that also some physical magnitudes undergo division in the same way as geometrical lines. Changes and times cannot be divided into spatially separated parts, but the act of identifying a point or limit does count as dividing, and thus as a change. Aristotle's solution to Zeno's paradox in book VIII of the *Physics* depends on the possibility of dividing the motion into parts, even if the division cannot for obvious reasons result in spatially distinct parts. The division only requires the actualisation of a

[256] This might be (part of) why Aristotle claims that the sciences that are 'without motion' are more precise and accurate (*Metaph.* M.3, 1078a11-13).

limit that can act as the end of one part and the beginning of the other; this can be done by stopping the mover or even just counting the limit twice.[257] A mental operation of division into parts is compared to a physical change in the process. More importantly, Aristotle himself uses cases of geometrical division to illustrate changes in the whole: for example, in *Metaph.* Z.13 he argues that the half-line can only be actualised when the whole is divided. He seems to consider the division of the line as a proper change, analogous to physical divisions.[258] Similarly, at *DA* III.6 430b6-20 he states that parts of geometrical objects are actualised by noetic activity of the mathematician.

So, there are good reasons to understand geometrical changes as changes, even if they constitute a subset of the changes that might take place in sensible reality, and they are not spontaneous changes but rather manipulations operated by the geometer on the semi-fictional objects in her imagination. As I have argued in §1, these changes are essential to geometrical practice. The Θ.9 passage quoted above proves that Aristotle believes that there is some value in the geometers' way of 'knowing by doing', and recognises that sometimes it is just necessary to add a line or modify slightly the configuration to 'see' the solution. This way of reasoning is difficult to fit into a syllogistic model, but it is essential to geometrical knowledge.

If this is correct, the fact that geometrical reasoning needs to be able to modify the configuration under examination explains why Aristotle needs to allow geometers to 'separate' their objects. As a matter of fact, the geometrical properties of sensible substances cannot themselves be subjected to those modifications; the sensible objects,

[257] *Ph.* VIII.8, 263a23-27: 'for if one divides a continuous line into two halves, one treats a single point as two – for one makes it both a beginning and an end. And both counting and bisecting produce this result. And if one divides in this way, neither the line nor the motion will be continuous' (Graham's translation). ἐὰν γάρ τις τὴν συνεχῆ διαιρῇ εἰς δύο ἡμίση, οὗτος τῷ ἑνὶ σημείῳ ὡς δυσὶ χρῆται· ποιεῖ γὰρ αὐτὸ ἀρχὴν καὶ τελευτήν. οὕτω δὲ ποιεῖ ὅ τε ἀριθμῶν καὶ ὁ εἰς τὰ ἡμίση διαιρῶν. οὕτω δὲ διαιροῦντος οὐκ ἔσται συνεχὴς οὔθ'ἡ γραμμὴ οὔθ' ἡ κίνησις.
[258] Z.13, 1039a5-7.

at the same time, undergo different modifications due to their nature and the interaction with the surrounding bodies. The operation of separation allows the geometer to have at her disposal an object that only has the relevant properties, whose manifestation is not hindered by any masks, and that she can modify at will according to the operations licensed by the Postulates.[259] Aristotle believes that this will not result in any error. But is he justified in assuming that this is the case?

§4. No error arises

I argued that geometry does in fact ignore the natural motions of sensible bodies: the semi-fictional geometrical objects do not have a natural motion, they do not grow or decay. The modifications and changes that are allowed in geometry are very specific and codified, and differ from the motions that sensible objects would undergo. This means that a certain sensible object differs from the semi-fictional object that results from the separation not only because the former has more features than the latter, but also in their development: the sensible object will move of natural motion and develop according to the laws of physics, while the semi-fictional geometrical objects can only be manipulated according to the laws of geometry. How can Aristotle be sure that this operation does not lead to any falsity?

To appreciate the extent of the problem, it is useful to briefly come back to an argument that we have already met in the introduction to Part 1: Aristotle's second argument against Partial Platonism in M.2. Recall that Partial Platonists believe that

[259] Humphreys (2017) notices another very important role played by the possibility of modifying geometrical objects in the mind: it allows the geometer to test which diagrammatic properties she can rely on. This is because the co-exact properties which can be used to license inferential steps are the ones that are invariant under a certain set of transformations; thanks to separation we can operate such transformations in the imagination.

geometrical objects exist *haplōs*, as substances, but are located in sensible bodies. Aristotle maintains that if this were the case, impossible consequences would result.

> φανερὸν ὅτι ἀδύνατον διαιρεθῆναι ὁτιοῦν σῶμα· κατ'ἐπίπεδον γὰρ διαιρεθήσεται, καὶ τοῦτο κατὰ γραμμὴν καὶ αὕτη κατὰ στιγμήν, ὥστ' εἰ τὴν στιγμὴν διελεῖν ἀδύνατον, καὶ τὴν γραμμήν, εἰ δὲ ταύτην, καὶ τἆλλα. τί οὖν διαφέρει ἢ ταύτας εἶναι τοιαύτας φύσεις, ἢ αὐτὰς μὲν μή, εἶναι δ' ἐν αὐταῖς τοιαύτας φύσεις; τὸ αὐτὸ γὰρ συμβήσεται· διαιρουμένων γὰρ τῶν αἰσθητῶν διαιρεθήσονται, ἢ οὐδὲ αἱ αἰσθηταί.
>
> It is clear that <on that theory> it is impossible for any body to be divided; for it will be divided along a plane, and the plane along a line, and a line at a point, so that if the point cannot be divided, neither can the line, and if the line cannot, the rest cannot either. So what difference does it make whether perceptible bodies are objects of this kind, or whether, while they are not, objects of this kind exist in them? There will be the same result: either they will be divided when perceptible objects are divided, or perceptible objects will not be divided either.
>
> Arist. *Metaph*. M.2, 1076b4-11 (Annas' transl.)

The argument is usually interpreted as simply directed against the view that points are *actually* present in the line, i.e. against a naïve and non-Aristotelian theory of continuity. However, I suggested that it can be construed in a different way, as a (more subtle) argument revolving around the different behaviour of geometrical and physical bodies. Read in this way, it cannot be solved merely by adopting the correct account of continuity. I believe that this argument provides a good illustration of the risks that Aristotle might have to face.

Recall that the idea is that if geometrical objects were *substances* that are located in physical bodies and share their diachronic history, absurd consequences would follow when we try to divide the sensible body. The reason is that physical bodies undergo division in a radically different way from geometrical solids. While the division of a geometrical solid only brings into actuality *one* plane (which acts as the limit of both parts of the solid), the division of a physical body brings into existence *two* planes, one for each of the resulting parts.

This argument (independently of whether it is indeed an argument that Aristotle himself puts forward in M.2 or not) is built using purely Aristotelian premises, and leads to contradiction using only the assumption that geometrical objects undergo some kinds of changes but not others. I believe that this is a problem that Aristotle needs to take seriously. If geometrical objects behave differently from sensible bodies, is it true that 'no error arises' if we do geometry on the semi-fictional objects resulting from separation, rather than on the sensible objects that are the truth-makers of geometry?

The worry that I am outlining regards the applicability of geometry. If geometry is supposed to study sensible objects (although not *qua* sensible), but proceeds by examining semi-fictional objects and subjecting them to diachronic modifications that do not track the changes of the sensible bodies, how can geometry really be applicable to the sensible world?

I believe that a cryptic passage from *De Caelo* I.10 gestures towards a solution. It is worth quoting it in full:

> Ἤν δέ τινες βοήθειαν ἐπιχειροῦσι φέρειν ἑαυτοῖς τῶν λεγόντων ἄφθαρτον μὲν εἶναι γενόμενον δέ, οὐκ ἔστιν ἀληθής· ὁμοίως γάρ φασι τοῖς τὰ διαγράμματα γράφουσι καὶ σφᾶς εἰρηκέναι περὶ τῆς γενέσεως, οὐχ ὡς γενομένου ποτέ, ἀλλὰ διδασκαλίας χάριν ὡς μᾶλλον γνωριζόντων, ὥσπερ τὸ διάγραμμα γιγνόμενον θεασαμένους. Τοῦτο δ' ἐστίν, ὥσπερ

λέγομεν, οὐ τὸ αὐτό· ἐν μὲν γὰρ τῇ ποιήσει τῶν διαγραμμάτων πάντων τεθέντων εἶναι ἅμα τὸ αὐτὸ συμβαίνει, ἐν δὲ ταῖς τούτων ἀποδείξεσιν οὐ ταὐτόν, ἀλλ' ἀδύνατον· τὰ γὰρ λαμβανόμενα πρότερον καὶ ὕστερον ὑπεναντία ἐστίν· ἐξ ἀτάκτων γὰρ τεταγμένα γενέσθαι φασίν, ἅμα δὲ ἄτακτον εἶναι καὶ τεταγμένον ἀδύνατον, ἀλλ' ἀνάγκη γένεσιν εἶναι τὴν χωρίζουσαν καὶ χρόνον· ἐν δὲ τοῖς διαγράμμασιν οὐδὲν τῷ χρόνῳ κεχώρισται. Ὅτι μὲν οὖν ἀδύνατον ἅμ' ἀΐδιον αὐτὸν εἶναι καὶ γενέσθαι, φανερόν.

Some of those who say that the world came to be but is imperishable try to win for themselves support which is untenable. For they maintain that they, with their talk of generation, are like those who construct geometrical figures:[260] it is not that they hold that the world came to be at some time, rather their talk of generation is for the sake of instruction, since more is gained thereby, just like those who have seen a geometrical figure being produced. Yet this, as we say, is not the same; for in the production of geometrical figures, if everything involved is supposed to exist simultaneously, the same result is achieved, whereas in the proofs these people give the result is not the same. Nor can it be, for the assumptions made at an earlier stage are at odds: they maintain that the world came to be once its elements had been put into order following their initial disorder. Yet order and disorder cannot exist simultaneously; rather, there has to be a process of generation separating the two states of affairs, as well as a period of time. In geometrical figures, however, no element is separate in time. It is clear, then, that the world cannot both have come to be and be everlasting.

Cael. I.10, 279b32-280a11 (Leggatt's translation)

This passage constitutes the fourth argument against the thesis that the *kosmos* had a beginning. Aristotle's target is a non-literal interpretation of the *Timaeus,* according to

[260] This could also be translated as: demonstrating geometrical theorems; see Iwata (2021). If this is the case, this passage is way less problematic for my interpretation. So I will discuss the text assuming that it is talking about constructing diagrams and figures: if I manage to prove my point with this interpretation, I shouldn't have any problems with the other.

which the talk of generation found in the dialogue is only 'for the sake of instruction'.[261] The idea is that one can speak of things *as if* they were generated in order to convey information about them, even if *in fact* they are not generated: the case of geometry testifies that this is possible, since geometers do that all the time. Aristotle objects to this kind of argument by showing that the analogy does not work: the case of the cosmos differs from the case of geometry in such a way that makes the analogy improper. The difference is that 'in the production of geometrical figures, if everything involved is supposed to exist simultaneously, the same result is achieved', because 'in geometrical figures no element is separate in time'. No contradiction nor any error ensues; on the contrary, natural generation violates the principle of non contradiction, for there would be order and disorder at the same time.

The fact that 'no element is separate in time' is presented as the reason why it is legitimate to use constructions to discover and prove geometrical theorems, even if those constructions involve artificial motions and changes that are different from the natural motion and development of sensible substances. I believe that this is the key to understanding how geometry is meant to be applicable to the sensible world, and also why Aristotle puts so much emphasis on geometry's indifference to change. In short, I take his point to be that geometry is not supposed to study physical bodies as they move and evolve in time, but *at an instant.*

Aristotle maintains that change requires time; at the instant there can be no change nor rest.[262] I wish to suggest that 'separating from change' or 'removing change' does not mean that geometry studies special objects that are unchangeable, but rather that it amounts to considering only the instantaneous configuration of sensible bodies. Such

[261] Leggatt (1995: 209). Plutarch and Simplicius ascribe this interpretation of the *Timaeus* to Xenocrates.
[262] *Ph.* VI.3, 234a24-b9; VI.6, 237a14-15.

configuration is static not because geometrical objects cannot undergo motions, but because no change can happen at an instant. I take this to be a restriction on the application of geometry to sensible reality – not a statement about the features and characteristics of the geometrical semi-fictional objects. These semi-fictional objects are the result of separation: geometrical features that belong to a certain sensible body at one instant are considered in their own right and posited as separate. I wish to suggest that such semi-fictional objects can be modified and manipulated in thought; the only condition is that these manipulations do nothing more than actualising features that were already potentially present in the instantaneous configuration that we are examining. If this condition is respected, it means that the artificial modification of the semi-fictional objects resulting from separation widens our understanding of the instantaneous state of the world that we are considering, without introducing changes in the world.

Take the case of a spherical lump of clay that is moulded into a cube. The correct way of analysing the situation from a physical point of view is by saying that the lump loses the property of sphericity and acquires the property of being a cube. The lump itself undergoes a change, while its geometrical features cannot be said to change: being properties, they can be lost or acquired by various objects, but they do not undergo change themselves. My suggestion is that geometry considers it not *qua* changeable, and separates from change, because it can study only the initial and the final instantaneous configuration, but not the transition from the one to the other. When studying e.g. the initial configuration, in which the lump is spherical, geometers separate the sphericity and posit it in thought as a substance; having done this, they can reason in a narrative and diachronic way, manipulating and modifying this semi-fictional sphere. These modifications and manipulations, which must be operated in accordance with geometrical postulates and rules of construction, do not track the physical changes undergone by the

spherical lump of clay – but they were never meant to. The goal of geometrical manipulations is to put into light (by bringing into actuality) the geometrical properties of the configuration that is instantiated in the sensible world at the instant in which the examination starts.

The geometrical reasoning develops by using changes and motions, and so in a diachronic way. However, the special kinds of changes that are allowed in geometry are such that there is no contradiction if all the stages of the change are supposed to exist at the same time. This allows geometers to apply the results of their diachronic reasoning to the initial state of the physical system.

This is possible, I believe, because geometrical truths are in a certain sense indifferent to the distinction between actual and potential existence. We have already seen that geometrical theorems can be true in virtue of features that are merely potentially present in physical substances. After separation, such potential features can be actualised in thought, thanks to geometrical modifications and constructions carried out by the imagination.[263] Actualising these features results into the production of knowledge that is still relative to the initial state of the system, but does not impact the truth-value of geometrical statements. The result is that geometry is applicable at the instant and not diachronically, even if its practice appeals to constructions and changes.

The upshot of this analysis is that the applicability of geometry can be saved from Netz' accusations, provided that it is limited at one instantaneous configuration. In Aristotle's system the applicability of geometry is strictly localised, from a spatial as well as from a temporal point of view.

[263] Cf. Chapter 2, and the solution to the Scarcity Problem.

§5. Consequences, Limits and Scope

If my reasoning is correct, we can also use it to extract a criterion that allows us to determine which kinds of changes are allowed in geometry – namely, those whose stages can all be coexistent, if we ignore the actuality/potentiality distinction or if we treat everything in a diagram as actually existing at the same time.[264]

It also marks a difference in the status and applicability of geometry with respect to mechanics. According to my reading, geometry can use instrumental and artificial changes to examine a static configuration, and is not meant to trace nor model the motion of sensible objects. Mechanics, on the contrary, is supposed to be the mathematical treatment of the motion of sensibles. If I am correct, Aristotle should treat geometry and mechanics in different ways.[265]

Indeed, Aristotle does list mechanics together with astronomy in *Metaph.* M.3, and specifies that astronomy has a different status than geometry. Recall that in *Physics* II.2, Aristotle claims that astronomy, optics and harmonics are in a certain sense the 'reverse' of geometry:

> ἀνάπαλιν γὰρ τρόπον τιν' ἔχουσιν τῇ γεωμετρίᾳ. ἡ μὲν γὰρ γεωμετρία περὶ γραμμῆς φυσικῆς σκοπεῖ, ἀλλ' οὐχ ᾗ φυσική, ἡ δ' ὀπτικὴ μαθηματικὴν μὲν γραμμήν, ἀλλ' οὐχ ᾗ μαθηματικὴ ἀλλ' ᾗ φυσική.

[264] Another way of cashing out this intuition is by saying that the changes that are allowed are the ones that do not destroy the object we started from. This would give Aristotle a reasoned motive to reject some of the non-constructive methods suggested for example by Archytas, or the generation of lines as flux of points, all while accepting that some changes can be appealed to in geometry.

[265] This issue is complicated to assess because it is not completely clear whether mechanics was already a systematised field of knowledge at Aristotle's time. It is most unlikely that the *Mechanics* is actually by Aristotle, but he is surely aware of some principles of mechanics such as the lever principle and the principle of the moving radius. See De Groot (2008; 2014), Berryman (2009).

> [Optics, harmonics and astronomy] are in a way the reverse of geometry. Geometry considers natural lines, but not as natural; optics treats of mathematical lines, but considers them not as mathematical but as natural.
>
> *Ph*. II 2, 194a 9–12 (Charlton's transl.)

I have argued in Chapter 3 that treating mathematical objects *qua* natural requires operating an additional *qua*-manoeuvre. Note that in order to treat a mathematical line as physical, it would seem necessary to *add* some features (such as materiality, sensibility, and motion). This is why optics is 'in a way the reverse of geometry': the optician takes the mathematical object which results from separation and 'adds to these geometrical items certain physical properties relevant to the understanding of visual appearances, first and foremost motion'.[266] This interpretation of the reverse of *aphairesis* as addition of features is supported by the fact that Aristotle contrasts *aphairesis* to *prosthesis*, addition: this happens most of all in the *Topics*,[267] but the two terms are also associated in *Metaph.* M.2, 1077b9-11.[268]

We are now in a position to see that this addition of motion in astronomy, optics and mechanics can only be fictional, for (i) the geometrical objects in question are the semi-fictional objects that result from the 'separation' of geometrical properties; (ii) these

[266] Burnyeat (2005: 36). One might wonder whether the 'reverse operation' should be rather understood as a more literal 'reversal' of the *aphairesis* operation: the optician could *undo* the *aphairesis*, playing it backwards. However, this interpretation would make the *Ph*. II.2 passage incompatible with the characterisation of optics that Aristotle gives at *Metaph*. M.3, 1078a14-6: ὁ δ' αὐτὸς λόγος καὶ περὶ ἁρμονικῆς καὶ ὀπτικῆς· οὐδετέρα γὰρ ᾗ ὄψις ἢ ᾗ φωνὴ θεωρεῖ, ἀλλ' ᾗ γραμμαὶ καὶ ἀριθμοί (οἰκεῖα μέντοι ταῦτα πάθη ἐκείνων): 'The same account applies to harmonics and optics; neither studies its objects as seeing or as utterance, but as lines and numbers (these being proper attributes of the former)' (Annas' transl.). To avoid attributing a blatant inconsistency to Aristotle, we must suppose that the 'reverse operation' that generates optical objects out of geometrical objects does not undo the previous operation that leads from the physical to the geometrical objects. Rather, it is a further operation that is applied to the results of the first one.

[267] Cf. Cleary (1985).

[268] ὥστε φανερὸν ὅτι οὔτε τὸ ἐξ ἀφαιρέσεως πρότερον οὔτε τὸ ἐκ προσθέσεως ὕστερον· ἐκ προσθέσεως γὰρ τῷ λευκῷ ὁ λευκὸς ἄνθρωπος λέγεται: 'so clearly the result of abstraction is not prior, nor the result of addition subsequent, because it is by adding to white that the white man is spoken of' (Annas' transl.).

motions do not stem from the nature of the geometrical object. The natural motions that mechanics is supposed to model are not the kind of changes that are allowed in geometry: they are not manipulations meant to put into evidence features of an instantaneous configuration, but artificial models of natural motions. As such, their applicability is limited. Aristotle is acutely aware of this, as it appears from his use of mechanical models to describe the motion of animals' joints in *De Motu Animalium*:[269]

> δεῖ γάρ, ἂν κινῆταί τι τῶν μορίων, ἠρεμεῖν τι· καὶ διὰ τοῦτο αἱ καμπαὶ τοῖς ζῴοις εἰσίν. ὥσπερ γὰρ κέντρῳ χρῶνται ταῖς καμπαῖς, καὶ γίνεται τὸ ὅλον μέρος, ἐν ᾧ ἡ καμπή, καὶ ἓν καὶ δύο, καὶ εὐθὺ καὶ κεκαμμένον, μεταβάλλον δυνάμει καὶ ἐνεργείᾳ διὰ τὴν καμπήν. καμπτομένου δὲ καὶ κινουμένου τὸ μὲν κινεῖται σημεῖον τὸ δὲ μένει τῶν ἐν ταῖς καμπαῖς, ὥσπερ ἂν εἰ τῆς διαμέτρου ἡ μὲν Α καὶ ἡ Δ μένοι, ἡ δὲ Β κινοῖτο, καὶ γίνοιτο ἡ ΑΓ. ἀλλ' ἐνταῦθα μὲν δοκεῖ πάντα τρόπον ἀδιαίρετον εἶναι τὸ κέντρον (καὶ γὰρ τὸ κινεῖσθαι, ὡς φασί, πλάττουσιν ἐπ' αὐτῶν· οὐ γὰρ κινεῖσθαι τῶν μαθηματικῶν οὐδέν), τὰ δ' ἐν ταῖς καμπαῖς δυνάμει καὶ ἐνεργείᾳ γίνεται ὁτὲ μὲν ἓν ὁτὲ δὲ διαιρετά.
>
> For if one of the parts moves, there must be some part at rest; and it is for this reason that animals have joints. For they use their joints like a centre, and the whole section containing the joint becomes both one and two, both straight and bent, changing potentially and actually by reason of the joint. And when the part is bending and being moved, one of the points in the joints is moved and the other remains at rest, just as if on a diameter AD should remain fixed and B be moved, so as to give AC. But in the geometrical example the centre appears to be indivisible in every respect (for the motion they speak of is also a fiction: no mathematical entity moves); whereas in the case of joints the centres become, both potentially and actually, now one, now divided.

[269] I wish to thank Oliver Primavesi for bringing this passage to my attention, and Christof Rapp, Laura Castelli, and all the participants to the MusaΦ Colloquium for an invaluable discussion. Thanks are due also to Ursula Coope for her help.

MA 1, 698a17-b1 (Nussbaum's translation)

The parenthetical remark might seem, on the face of it, to invalidate my suggestion that geometrical objects are apt to undergo some kinds of changes. But once we pay attention to the broader context of the passage, and to the way in which Aristotle uses the analogy, we realise that this is not the case.

First of all, notice that there is a textual issue to be discussed. At line 698a26 there is a variant: where **β**E**θ**Ca read κινεῖσθαι, (**ι**) has κινεῖται. If we read κινεῖσθαι with the majority of the manuscripts, we take the remark to be indirect speech depending on ὡς φασί. This implies that the same people who ascribe motion to the geometrical objects recognise that strictly speaking this is just a fiction, because such objects cannot undergo motion. According to this reading, the remark is made from the point of view of the mathematicians, and thus regards the semi-fictional objects to which they refer. This seems problematic for my interpretation, because I have suggested that geometers do think that the objects to which they refer can undergo some manipulations.

Nussbaum (1978: 282-3) prefers to read κινεῖται, and claims that Aristotle is criticising the talk of movable geometrical objects in his own voice. She argues that this is an attack on Plato's astronomy and his theory of mathematical intermediates. Rapp (2020: 230) believes that 'criticizing other people's theories at this point of the argument would only distract from the main argument'.[270] I agree that it is a stretch to read a reference to Plato into this remark. In the theoretical framework of my interpretation, however, it would make sense to adopt Nussbaum's preferred reading. If Aristotle is speaking in his own voice, then this passage can be easily reconciled with my proposal: Aristotle does think that motion in geometry is fictional, since it is the motion of semi-

[270] He suggests that the remark makes reference to those geometers (such as Archytas) that introduce motion in geometrical diagrams in a very conscious way, being aware that this can only be fictional.

fictional objects. The 'real' objects of geometry, its truth-makers, cannot change (because they are properties and not substances). This reading is not very convincing, though. Why should Aristotle be thinking about the geometrical properties of sensible objects, which act as truth-makers of geometrical statements? The remark seems completely off topic.

Thus, I believe we should follow Rapp and read κινεῖσθαι. But I do not think that this is a problem for my interpretation. Aristotle is not here talking about the use of constructive methods in geometry,[271] because he is not talking about geometry at all. The model that he is proposing is a *mechanical* model, in which geometrical objects are supposed to mimic the behaviour of sensible objects.[272]

Aristotle uses the same model twice in *MA*, once in chapter 1 and once in chapter 9; in addition, in chapter 7 he makes reference to the movable theatres and the toy wagon, which employ the same principle of the lever.[273] In all three cases, the mechanical model is employed as an analogy to make a point that is not about the dynamics of the motion. In chapters 1 and 9, Aristotle wishes to stress that all motion requires that there is something at rest; in chapter 7, he is discussing the way in which small changes in the soul produce bigger and more varied effects in bodily actions. The mechanical model is used as an analogy to illustrate the specific point that Aristotle wishes to make; it is useful because it is simpler than the real physical phenomenon, but its limits are immediately underlined. The simplicity comes from the fact that geometers ignore some of the factors that complicate the picture, such as the material elements and the causal network and relations in which each body is embedded in the sensible world. But these elements are

[271] Not even the use of non-constructive methods such as *noesis*, in fact. Note that Archytas is one of the mathematicians that introduce such non-constructive and dynamical methods, but he is also known as the founder of mechanics (DL VIII 83). It is clear from the context that Aristotle has mechanics, and not geometry, in mind.
[272] See Bodnár (2004); De Groot (2008; 2014: §§2, 5); Berryman (2009) for arguments in support of the claim that Aristotle is consciously using mechanical principles.
[273] See De Groot (2008; 2014: §§2, 5).

in fact necessary to properly account for physical motion: removing them yields a deceptively simple picture. This picture might be useful, but it has its limits. For example, in the passage quoted above the over-simplification leads to suppose that an indivisible point should be divided. In chapter 9, Aristotle will refine the model and stress that the centre must have some positive dimension, if it is to be part in motion and part unmoved. As Bénatouïl (2004: 96) remarks, the model can (properly speaking) only provide a representation of the final and initial configuration of the motion. This is perfectly in line with my findings. The model cannot properly account for the motion itself, because it can only grasp the formal cause of the motion and not its material and efficient causes. As a model *of natural motion,* it is necessarily defective (even if possibly useful).

The shortcomings of mathematical mechanics, however, do not say anything about the status of geometry nor the possibility of using constructive and diachronic reasoning to put into light the geometrical features of a certain physical configuration, studied at the instant. Also in this case the changes are fictional, in the sense that they only apply to the semi-fictional objects that result from the separation of the geometrical features. But they do not introduce any approximation nor error.

Conclusion

The picture of Aristotle's philosophy of geometry that comes out of my analysis is of a delicate (and possibly precarious) equilibrium between forces that pull in different directions.

I argued that Aristotle offers two very different answers to the two questions that might be merged under the formula 'what is geometry about?' – namely the Grounding Question and the Reference Question. He gives an ontologically parsimonious account of geometrical truth, arguing that it does not require the introduction of any specific geometrical object: sensible objects really do instantiate geometrical properties, and this is sufficient to ground the truth of geometrical statements.

At the same time, he allows for a moderate form of fictionalism in order to account for the practice of geometry, and for the way in which geometers speak of their objects. The fiction only regards the mode of existence of the entities in question, that are in fact properties of sensibles but are posited in thought as separate from them, and as independent substances. The fiction is allowed only for heuristic reasons, and does not impact the truth of geometrical theorems: the features that are posited as separate were already in existence, although not as substances.

Is this two-tier account satisfactory? To conclude, I wish to point explicitly at a source of tension that runs through the analysis that I have proposed.

In Chapter 2 I argued that this radically anti-Platonist account of geometrical truth can be defended from a number of objections that arise from one and the same source: Aristotle's assumption that whatever one proves of sensible objects not *qua* sensible, was already true of sensible objects in the first place. This assumption is fundamental to ground the truth of geometrical statements in sensible reality, without introducing special geometrical objects. However, it seems to clash with the phenomena and common-sense:

sensible objects do not appear to instantiate perfect geometrical properties, and there might not be enough of them to make all the statements of geometry true. I argued that these problems might be solved by allowing potentially existing objects and features to act as truth-makers on the same level as actually existing ones. This is not very controversial nor original, and it can find confirmation in Aristotle's own cryptic remarks on the 'matter-like' existence of geometrical objects. But this is not all. Careful consideration of the Problems of Precision, Scarcity and Contradictory Properties shows that Aristotle needs to allow as truth-makers also come objects and features that can *only* exist potentially: something similar to necessarily masked potentialities in contemporary metaphysics of powers.

The reason for this is that there is a discrepancy between physical and geometrical modality, that sits uneasily with Aristotle's attempt to ground geometrical truth in physical reality alone. It is this discrepancy, I believe, that makes it necessary for Aristotle to allow the possibility of separating geometrical features: it is only by removing physical masks that geometrical potentialities can be fully expressed (even if just in thought) and thus studied.

Aristotle is aware of the discrepancy between physical and geometrical modality, as can be seen from his (admittedly very sketchy) discussion of the role of the infinite in mathematics. I pointed out that Aristotle recognises that figures bigger than the dimensions of the *kosmos* are geometrically possible even if physically impossible, and wishes to accommodate this discrepancy in his account (namely, by allowing geometrical statements to be true in virtue of an object that is smaller than, but isomorphic to, the semi-fictional one that they are considering) rather than deny it or explain it away.

I believe that this discrepancy risks creating problems for Aristotle's account. Not only when it comes to grounding the truth of geometry, but also on the epistemological

side: how can we get to know what is geometrically possible, if geometrical potentialities are masked by the material features of reality? I have suggested that what is geometrically possible is what is licensed by the rules of construction of Euclidean geometry, but this merely pushes the question slightly further: how do we get to know which rules are correct? Aristotle is aware that changing the principles of geometry would lead to completely different theorems,[274] but he does not take this possibility into account in a serious way.

I believe that the reason why Aristotle can disregard the tension between physical and geometrical modality without incurring in immediate devastating consequences is that he constrains the applicability of geometry to strictly localised configurations, both at the spatial and the temporal level. If geometry can only be used to describe the structural features of an instantaneous, finite configuration, the discrepancy between physical and geometrical modality can be kept at bay.

[274] Cf. *Ph.* II.9, 200a15-19.

References

Acerbi, F. (2020). "Mathematical Generality, Letter-Labels, and All That". *Phronesis* 65: 27-75.

Ackrill, J.L. (1963). *Aristotle:* Categories *and* De Interpretatione. Oxford: Clarendon Press.

Algra, K. (1995). *Concepts of Space in Greek Thought*. Leiden: Brill.

Alvarez, C. (2003). "Two Ways of Reasoning and Two Ways of Arguing in Geometry. Some Remarks concerning the Application of Figures in Euclidean Geometry". *Synthese* 134: 289-323.

Annas, J. (1976). *Aristotle's Metaphysics: Books M and N*. Oxford: Oxford University Press.

Anscombe, G.E.M., Lukasiewicz, J., and Popper, K. (1953). "Symposium: The Principle of Individuation". *Aristotelian Society Supplementary* 27 (1): 69 - 120.

Aubenque, P. (1983). 'Inauthenticité de *Metaphysique* K'. In P. Moraux and J. Wiesner (edd.). *Symposium Aristotelicum, 'Zweifelhaftes im Corpus Aristotelicum'*. Berlin: De Gruyter.

Azzouni, J. (1994). *Metaphysical Myths, Mathematical Practice: The Ontology and Epistemology of the Exact Sciences*. Cambridge: Cambridge University Press.

Bäck, A. (2015). *Aristotle's Theory of Abstraction*. Dordrecht: Springer.

Baldini, C.L. (2020). *Aristotle's* Metaphysics *Book K*. DPhil Thesis, University of Oxford.

Barnes, J. (1979). "Review of Hartman, *Substance, Body, and Soul*". *Philosophical Books* 20: 59.

Barnes, J. (1993[2]). *Aristotle:* Posterior Analytics. Oxford: Oxford University Press.

Barnes, J. (1995). "Metaphysics". In Barnes, J. (ed). *The Cambridge Companion to Aristotle*. Cambridge: Cambridge University Press: 66-108.

Bartha, P. (2001). "Monstrous Neighbors or Curious Coincidence: Aristotle on Boundaries and Contact". *History of Philosophy Quarterly*, 18/1: 1-16.

Beere, J. (2009). *Doing and Being: An Interpretation of Aristotle's* Metaphysics *Theta*. Oxford: Oxford University Press.

Beere, J. & Morison, B. (ms). "A Mathematical Form of Knowing How: the Nature of Problems in Euclid's Geometry".

Benacerraf, P. (1965). "What Numbers Could Not Be". *Philosophical Review* 74: 47-73.

Berman, B. (2017). "Why Can't Geometers Cut Themselves on the Acutely Angled Objects of Their Proofs? Aristotle on Shape as an Impure Power". *Méthexis* 29: 89-106.

Berryman, S. (2009). *The Mechanical Hypothesis in Ancient Greek Natural Philosophy*. Cambridge: Cambridge University Press.

Betegh, G., Pedriali F. & Pfeiffer, C. (2013). "The Perfection of Bodies Aristotle's *De Caelo* I 1". *Rhizomata*, 1 (1): 30-62.

Betegh, G. (2016). "Colocation". In T. Buchheim, D. Meissner & N. Wachsmann (ed.). *ΣΩΜΑ. Körperkonzepte und körperliche Existenz in der antiken Philosophie und Literatur*. Hamburg: Felix Meiner Verlag: 393-421.

Bird, A. (1998). Dispositions and Antidotes. *The Philosophical Quarterly* 48: 227-34.

Bodnár, I. (2004). "The Mechanical Principles of Animal Motion", in Laks, A. & Rashed, M. *Aristote et le mouvement des animaux: Dix études sur le De motu animalium.* Villeneuve d'Ascq.: 137–48.

Bostock, D. (2006). "Aristotle on Continuity in *Physics* VI". In *Space, Time, Matter and Form. Essays on Aristotle's* Physics. Oxford: Oxford University Press.

Burnyeat, M. F., ed. (1979). *Notes on Book Zeta of Aristotle's 'Metaphysics'*. Study Aids (University of Oxford: Sub-Faculty of Philosophy) Oxford.

Burnyeat, M. (2000). "Why is Mathematics Good for the Soul?". In T. Smiley (ed.), *Mathematics and Necessity: Essays in the History of Philosophy*: 1-81.

Berryman, S. (2009). *The Mechanical Hypothesis in Ancient Greek Natural Philosophy*. Cambridge: Cambridge University Press.

Bird, A. (1998). Dispositions and Antidotes. *The Philosophical Quarterly* 48: 227-34.

Burnyeat, M. (2005). "Archytas and Optics". *Science in Context* 18: 35-53.

Burnyeat, M. (2012). “Platonism and mathematics: A prelude to discussion”. In *Explorations in Ancient and Modern Philosophy* (pp. 145-172). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511974069.009

Cameron, R.P. (2008). “Truthmakers and ontological commitment: or how to deal with complex objects and mathematical ontology without getting into trouble”. *Philosophical Studies* 140: 1-18.

Charles, D. (2008). “Aristotle’s Psychological Theory”, *Proceedings of the Boston Area Colloquium in Ancient Philosophy* 24: 1–30.

Charles, D. & Peramatzis, M. (in progress). *Aristotle’s* Metaphysics*: the Philosophical Project of the Central Books.*

Charlton, W. (1970). *Aristotle’s* Physics *I, II. Translated with Introduction and Notes*. Oxford: Clarendon Press.

Charlton, W. (1991). “Aristotle’s Potential Infinites”. In L. Judson (ed). *Aristotle’s Physics: A Collection of Essays*. Oxford: Clarendon Press.

Chihara, C. (1990). *Constructibility and Mathematical Existence*. Oxford: Oxford University Press.

Choi, S. (2012). “Intrinsic Finks and Dispositional/Categorical Distinction”. *Noûs* 46 (2):289-325.

Cleary (1985). “On the Terminology of ‘Abstraction’ in Aristotle”. *Phronesis* 30: 13-45.

Cohen, S. M. (2008). “Kooky Objects Revisited: Aristotle’s Ontology”. *Metaphilosophy* 39 (1): 3-19.

Coliva, A. (2012). “Human Diagrammatic Reasoning and Seeing-As”. *Synthese* 186: 121-148.

Corkum, P. (2012). “Aristotle on Mathematical Truth”. *British Journal for the History of Philosophy* 20 (6): 1057-76.

Coope, U. (2005). *Time for Aristotle*. Oxford: Oxford University Press.

Crubellier, M. (1994). *Les livres “Mu” et “Nu” de la* Métaphysique *d’Aristote. Traduction et Commentaire*. Thèse de doctorat en Philosophie sous la direction de André Laks, Université Charles de Gaulle, Lille 3.

Crubellier, M. & A. Laks, ed. (2009). *Aristotle: Metaphysics Beta*. Oxford: Oxford University Press.

De Groot, J. (2008). "Dunamis and the Science of Mechanics: Aristotle on Animal Motion". *Journal of the History of Philosophy* 46 (1):43-67.

De Groot, J. (2014) *Aristotle's Empiricism: Experience and Mechanics in the 4th Century BC.* Las Vegas (NV): Parmenides Publishing.

De Risi, V. (2015). *Mathematizing Space*. Basel: Birkhäuser.

De Risi, V. (manuscript). *Continuity of Space and Theories of Intersections from Euclid to Leibniz.*

Décarie, V. (1983). 'L' authenticité du livre K de la *Metaphysique*'. In P. Moraux and J. Wiesner (edd.). *Symposium Aristotelicum, 'Zweifelhaftes im Corpus Aristotelicum'*. Berlin: De Gruyter.

Dedekind, R. (1963) [1888]."The Nature and Meaning of Numbers" in Berman, W.W. (ed.) *Essays on the Theory of Numbers.* New York: Dover Press: 31-115.

Field, H. (1980). *Science Without Numbers*. Princeton: Princeton University Press.

Fine, G. (1984). "Separation". *Oxford Studies in Ancient Philosophy* 2: 31-87.

Fine, G. (1985). "Separation: A Reply to Morrison". *Oxford Studies in Ancient Philosophy* 3: 159-65.

Fine, G. (1993). *On Ideas: Aristotle's Criticism of Plato's Theory of Forms*. Oxford: Clarendon Press.

Fine, K. (1985). *Reasoning With Arbitrary Objects*. Oxford: Blackwell.

Fowler, D.H. (1987). *The Mathematics of Plato's Academy: A New Reconstruction*. Oxford: Oxford University Press.

Franklin, J. (2014). *An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure*. London/New York: Palgrave MacMillan.

Friedlein, G. (1873). *Procli Diadochi in primum Euclidis Elementorum librum commentarii*. Leipzig: Teubner.

Furley, D. (1967). *Two Studies in Greek Atomism.* Princeton: Princeton University Press.

Gaukroger, S. (1980). “Aristotle on Intelligible Matter”. *Phronesis* 25: 187-97.

González-Varela, J.E. (2020). “The Platonist Absurd Accumulation of Geometrical Objects: *Metaphysics* M.2”. *Phronesis* 65: 75-115.

Grube, G. M. A. & Reeve, C. D. C. (1992). *Plato*: Republic. Indianapolis: Hackett Publishing Press.

Hale, B. & Wright, C. (2001). *The Reason’s Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics*. Oxford: Clarendon Press.

Hale, B. & Wright, C. (2009). ‘The Metaontology of Abstraction’, in D. Chalmers, D. Manley, & R. Wasserman, eds. *Metametaphysics: New Essays on the Foundations of Ontology*. Oxford: Oxford University Press: 178–212.

Halper, E. (1989). “Some Problems in Aristotle’s Mathematical Ontology”. *Proceedings of the Boston Area Colloquium in Ancient Philosophy* 5: 247-276.

Heath, T. (1921). *A History of Greek Mathematics.* 2 vols. Oxford: Oxford University Press.

Heath, T. (1926). *Euclid. The Thirteen Books of the Elements*. 3 vols. Cambridge: Cambridge University Press.

Heath, T. (1949). *Mathematics in Aristotle*. New York: Routledge.

Helmig, C. (2007). “Aristotle’s Notion of Intelligible Matter”. *Quaestio* 7 (1):53-78.

Hilbert, D. (1959) [1899]. *Foundations of Geometry.* La Salle (IL): Open Court.

Huffman, C. (2020). “Archytas”. *The Stanford Encyclopedia of Philosophy* (Winter 2020 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2020/entries/archytas/>.

Humphreys, J. (2017). “Abstraction and Diagrammatic Reasoning in Aristotle’s Philosophy of Geometry”. *Apeiron* 50 (2): 197–224.

Hussey, E. (1991). “Aristotle on Mathematical Objects”. *Apeiron* 24: 105-33.

Iwata, N. (2021). “Aristotle on Geometrical Potentialities”. *Journal of the History of Philosophy* 59 (3): 371-397.

Jaeger, W., ed. (1957). *Aristotelis Metaphysica*. Oxford: Clarendon Press.

Jammer, M. (1954). *Concepts of Space*: *The History of Theories of Space in Physics.* Cambridge (MA): Harvard University Press.

Jenkins, C. S. & Nolan, D. (2012). "Disposition Impossible". *Noûs* 46: 732-52.

Katz, E. (2014). "An Absurd Accumulation: Metaphysics M.2, 1076b11-36". *Phronesis Phronesis* 59 (4): 343-368.

Katz, E. (2017). "Ontological Separation in Aristotle's *Metaphysics*". *Phronesis* 62: 26-68.

Katz, E. (2019). "Geometrical Objects as Properties of Sensibles: Aristotle's Philosophy of Geometry". *Phronesis* 64 (4): 465-513.

Katz, E. (2022). "Aristotle's Philosophy of Geometry: A Philosophical Defence". *OSAP* LXI: 143-200.

Kirwan, C. (1993). *Aristotle:* Metaphysics *Γ, Δ, E*. Oxford: Oxford University Press.

Knorr, W.R. (1986). *The Ancient Tradition of Geometric Problems.* Boston/Basel/Stuttgart: Birkhäuser.

Lear, J. (1980). *Aristotle and Logical Theory*. Cambridge: Cambridge University Press.

Lear, J. (1982). "Aristotle's Philosophy of Mathematics", *The Philosophical Review* 91: 161–92.

Leggatt, S. (1995). *Aristotle:* On the Heavens *Books I & II.* Oxford: Oxbow Books.

Lewis, D. (1997). "Finkish Dispositions". *Philosophical Quarterly* 47: 143-58.

Linnebo, Ø. (2018) "Platonism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/spr2018/entries/platonism-mathematics/>.

Lloyd, A. C. (1970). "Aristotle's Principle of Individuation". *Mind* 79 (316): 519-529.

Lloyd, G. E. R. (2011). "Mathematics and Narrative: An Aristotelian Perspective".

Macbeth, D. (2010). "Diagrammatic Reasoning in Euclid's *Elements*". In In B. Van Kerkhove, J. P. van Bendegem, and J. de Vuyst (eds.), *Philosophical Perspectives on Mathematical Practice* 12: 235-267.

Madigan, A. (1999). *Aristotle:* Metaphysics *Book B and Book K 1-2*. Oxford: Clarendon Press.

Makin, S. (2006). *Aristotle:* Metaphysics *Book Θ*. Oxford: Clarendon Press.

Manders, K. (1995/2008). "The Euclidean Diagram". In P. Mancosu (ed.) *The Philosophy of Mathematical Practice*. Oxford: Oxford University Press: 80-133.

Matthews, G. B. (1982). "Accidental Unities". In M. Schofield and M. C. Nussbaum (eds.). *Language and Logos.* Cambridge: Cambridge University Press: 223–40.

Mendell, H. (1987). "Topoi on Topos. The Development of Aristotle's Concept of Place". *Phronesis* 32 (1):206-231.

Mendell, H. (2004). "Aristotle and Mathematics", *The Stanford Encyclopedia of Philosophy* (Fall 2019 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/fall2019/entries/aristotle-mathematics/>.

Mendell, H. (2015). "Putting Aristotle's *Physics* in its Place. A Discussion of Benjamin Morison, *On Location*". *Oxford Studies in Ancient Philosophy* 28:327-366.

Mendell, H. (2019). "Aristotle and Mathematics", *The Stanford Encyclopedia of Philosophy* (Fall 2019 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/fall2019/entries/aristotle-mathematics/>.

Menn, S. (*unpublished*). The Argument Structure of MN and their Place in the *Metaphysics*. In: *The Aim and the Argument of Aristotle's Metaphysics*, ch. Ig3 www.philosophie.huberlin.de/de/lehrbereiche/antike/mitarbeiter/menn/texte/ig3

Michetti, B. (2022). "«The Matter Present in Sensibles but not qua Sensibles». Aristotle's Account of Intelligible Matter as the Matter of Mathematical Objects". *Methexis* 34: 42-70.

Mignucci, M. (1987). "Aristotle's Arithmetic". In: A. Graeser (ed.), *Mathematik und Metaphysik bei Aristoteles: Akten des X. Symposium Aristotelicum, Sigriswil, 6.-12. September 1984,* Bern: Berner Reihe philosophischer Studien: 213-40.

Miller, F. D. Jr. (1982). "Aristotle Against the Atomists". In N. Kretzmann (ed). *Infinity and Continuity in Ancient and Medieval Thought*. Ithaca: Cornell University Press.

Morrison, D. (1985a). "Separation in Aristotle's Metaphysics". *Oxford Studies in Ancient Philosophy* 3: 125-57.

Morrison, D. (1985b) "χωριστός in Aristotle". *Harvard Studies in Classical Philology* 89: 89-105.

Mueller, I. (1969) “Euclid’s *Elements* and the Axiomatic Method”. *The British Journal for the Philosophy of Science* 20 (4): 289-309.

Mueller, I. (1970). “Aristotle on Geometrical Objects”, *Archiv für Geschichte der Philosophie*, 52: 156–71.

Mueller, I. (2009). “*Aporia* 12 (and 12bis)”. In M. Crubellier and A. Laks (ed.). *Aristotle: Metaphysics Beta*. Oxford: Oxford University Press.

Mumford, S. & Anjum, R. L. (2011). *Getting Causes from Powers*. Oxford: Oxford University Press.

Netz, R. (1999). *The Shaping of Deduction in Greek Mathematics*. Cambridge: Cambridge University Press.

Netz, R. (2006). “Aristotle’s *Metaphysics* M.3: Realism and the Philosophy of QUA”, *Princeton/Stanford Working Papers in Classics Paper No. 120602*. Available at: http://ssrn.com/abstract=1426876

Newstead, A. G. J. (2001). “Aristotle and Modern Mathematical Theories of Continuum”. In Sfendoni-Mentzou, D. *Aristotle and Contemporary Science*. New York: Lang.

Nussbaum, M. C. (1986). *Aristotle’s De Motu Animalium*. Princeton: Princeton University Press.

Panza, M. (2012). ‘The twofold role of diagrams in Euclid’s plane geometry’. *Synthese*, Springer Verlag (Germany) 186 (1), pp.55-102. <10.1007/s11229-012-0074-2>. <hal-00192165v2>

Peramatzis, M. (2008) “Aristotle’s Notion of Priority in Nature and Substance”. *Oxford Studies in Ancient Philosophy* 35: 187-247.

Peramatzis, M. (2011). *Priority in Aristotle’s Metaphysics*. Oxford: Oxford University Press.

Pettigrew, R. (2009). “Aristotle on the Subject Matter of Geometry”. *Phronesis*, 54, 239-260.

Pfeiffer, C. (2016). “Aristotle on Being in the Same Place”. In T. Buchheim, D. Meissner & N. Wachsmann (ed.). *ΣΩΜΑ. Körperkonzepte und körperliche Existenz in der antiken Philosophie und Literatur*. Hamburg: Felix Meiner Verlag: 363-93.

Pfeiffer, C. (2018). *Aristotle’s Theory of Bodies*. Oxford: Oxford University Press.

Pfeiffer, C. (2018b). “Aristotle and the Thesis of Mereological Potentialism”. *Philosophical Inquiry* 42/3-4, 28-66.

Quarantotto, D. (2017). *L’Universo Senza Spazio. Aristotele e la Teoria del Luogo*. Napoli: Bibliopolis.

Quarantotto, D. (2022). “Aristotle's Account of the Immobility of Place: A Cold Case”. *Phronesis* (online)*:* 1-41.

Ranocchia, G., Helmig, C. & Horn, C. (eds.) (2014). *Space in Hellenistic Philosophy.* Berlin: De Gruyter.

Rapp, C., and Primavesi, O. (eds.) (2020). *Aristotle’s De Motu Animalium: Symposium Aristotelicum*. Oxford: Oxford University Press.

Ross, W. D. (1953). *Aristotle’s* Metaphysics. *A Revised Text with Introduction and Commentary.* (2 vols.). Oxford: Clarendon Press.

Ross, W. D. (1956). *Aristotle’s* Physics. *A Revised Text with Introduction and Commentary.* Oxford: Clarendon Press.

Saito, K. (2012). “Traditions of the Diagram, Tradition of the Text: a Case Study”. *Synthese* 186: 7-20.

Sattler, B. (2020). *The Concept of Motion in Ancient Greek Thought.* Cambridge: Cambridge University Press.

Sattler, B. (2020b). “Space in Ancient Times. From the Beginning to Aristotle”, in. A. Janiak (ed.), *Space. A History*. Oxford: Oxford University Press.

Schrenk, M. (2010). “Antidotes for Dispositional Essentialism”, in A. Marmodoro (ed.), *The Metaphysics of Powers*. London: Routledge.

Schwegler, A. (1847). *Die Metaphysik des Aristoteles*, 4 vols. Tübingen.

Sidoli, N. (2004). “On the Use of the Term *Diastēma* in Ancient Greek Constructions”. *Historia Mathematica* 31: 2-10.

Sidoli, N. & Saito, K. (2009). “The Role of Geometrical Construction in Theodosius’s Spherics”. *Archive for History of Exact Sciences* 63: 581-609.

Sorabji, R. (1983). *Time, Creation, and the Continuum: Theories in Antiquity and the Early Middle Ages.* London: Duckworth.

Taisbak, C.M. (2003). *ΔΕΔΟΜΕΝΑ. Euclid's Data or The Importance of Being Given*. Copenhagen: Museum Tusculanum Press.

Tiles, J. E. (1983). "Why the Triangle has Two Right Angles Kath'Auto". *Phronesis* 28 (1): 1-16.

Tugby, M. (2016). "On the Reality of Intrinsically Finkable Dispositions". *Philosophia* 44: 623-31.

Ugaglia, M. (2015). "*Knowing by Doing*: The Role of Geometrical Practice in Aristotle's Theory of Knowledge". *Elenchos* 36 (1): 45-88.

Ugaglia, M. (2018). "Existence vs. Conceivability in Aristotle: Are Straight Lines Infinitely Extendible?". In M. Piazza, G. Pulcini (eds.), *Truth, Existence and Explanation*. Boston Studies in the Philosophy and History of Science 334: 249-272.

Ugaglia, M. (2012). *Aristotele, Fisica-Libro III, Introduzione, traduzione e commento.* Roma: Carocci.

Ugaglia, M. (2012b). *Modelli idrostatici del moto da Aristotele a Galileo*. Roma, Lateran University Press

Vetter, B. (2015). *Potentiality*. *From Dispositions to Modality.* Oxford: Oxford University Press.

Vetter, B. & Busse, R. (2022). "Modal Dispositionalism and Necessary Perfect Masks". *Analysis* 82 (1):84-94.

Waterlow, S. (1982). *Passage and Possibility. A Study of Aristotle's Modal Concepts.* Oxford: Oxford University Press.

Waterlow, S. (1988). *Nature, Change, and Agency in Aristotle's Physics: A Philosophical Study*. Oxford: Clarendon Press.

Werner, J. (2021). "Plenitude and Necessarily Unmanifested Dispositions". *Thought: A Journal of Philosophy* 10 (3):169-177.

White, M. J. (1992). *The Continuous and the Discrete. Ancient Physical Theories from a Contemporary Perspective.* Oxford: Clarendon Press.

White, M. J. (1993). “The Metaphysical Location of Aristotle’s Μαθηματικά”. *Phronesis* 38: 166-82.

White, N. (1986). “Identity, Modal Individuation, and Matter in Aristotle”. *Midwest Studies in Philosophy.* 11: 475–94.

Whiting, J. E. (1986). “Form and Individuation in Aristotle”. *History of Philosophy Quarterly* 3(4): 359-377.

Wieland, W. (1970). *Die aristotelische Physik: Untersuchungen über die Grundlegung der Naturwissenschaft und die sprachlichen Bedingungen der Prinzipienforschung bei Aristoteles*. Göttingen: Vandenhoeck & Ruprech.

Williams, N. E. (2019). *The Powers Metaphysics*. Oxford: Oxford University Press.

Witt, C. (2003). *Ways of Being: Potentiality and Actuality in Aristotle’s Metaphysics*. Ithaca.

Wright, C. (1983). *Frege’s Conception of Numbers as Objects.* Aberdeen: The Scots Philosophical Monograph Series, Aberdeen University Press.