

Quasiconvexity and weak convergence in Nonlinear Analysis



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To my parents,
For setting me on the road;
To Rita,
For cycling through it with me.

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Abstract

The present thesis addresses a broad range of weak convergence problems arising in Nonlinear Analysis. The thesis is divided into three related but essentially independent parts.

In the first part we study the general theory of Compensated Compactness. We begin by giving a new characterization of partial differential operators with constant rank, a mild non-degeneracy assumption which plays an important role in the theory. We then characterize completely the class of nonlinearities which are weakly continuous with respect to constant rank PDEs; in particular, we prove that it agrees with the class of nonlinear operators with Hardy space integrability, answering positively a question by Coifman–Lions–Meyer–Semmes. As an application of this theory we study homogenization problems, both with and without constant rank assumptions. In the constant rank setting we revisit the classical G-closure problem and discuss its connection with quasiconvexity. In the non-constant rank setting we study a homogenization problem for the Einstein vacuum equations in General Relativity: under some symmetry and gauge assumptions, we prove a conjecture by Burnett from 1989 which describes the effective behaviour of a sequence of vacuum space-times. This part of the thesis contains joint work with Jan Kristensen (University of Oxford), Bogdan Raiță (MPI Leipzig), Matthew Schrecker (UCL) and Rita Teixeira da Costa (University of Cambridge).

The second part of this thesis is concerned with quasiconvexity in the classical, curl-free, Calculus of Variations. We contribute to the understanding of the geometry of the class of quasiconvex functions, in particular through its extremal points. We prove a Choquet-type theorem for quasiconvex functions and we provide several examples of extremal quasiconvex functions, proving in particular a conjecture made by Šverák. We then further investigate, through numerical experiments, the relationship between rank-one convexity and quasiconvexity, particularly in low dimensions. We also give a concise proof of Ornstein’s L^1 non-inequality in low dimensions. This part of the thesis contains joint work with Daniel Faraco (Universidad Autónoma de Madrid) and Rita Teixeira da Costa (University of Cambridge).

The third part of this thesis deals with low regularity problems for nonlinear underdetermined PDEs. We mostly focus on the prescribed Jacobian equation, although applications to energy-dissipative solutions of the incompressible Euler equations are also discussed. Concerning the prescribed Jacobian equation, we prove an ill-posedness result for the Dirichlet problem. We also study the uniqueness and symmetry properties of energy minimisers with prescribed Jacobian, concluding that in general they are non-unique and non-symmetric. These results answer several questions posed by Hélein, Hogan–Li–McIntosh–Zhang and Ye in the 1990s and provide some of the first results concerning low regularity solutions of the Jacobian equation. We also prove a nonlinear version of the classical Open Mapping Theorem from Functional Analysis. Our result applies to a wide range of PDEs and, in particular, it applies to the weakly continuous nonlinearities characterized in the first part of this thesis, of which the Jacobian determinant is a particular example. As consequences of this nonlinear Open Mapping Theorem, we prove: i) a partial selection criterion for solutions of the Jacobian equation in the critical Sobolev space, and ii) generic non-existence of weak solutions to the incompressible Euler equations over \mathbb{R}^n with fastly decaying kinetic energy. This part of the thesis contains joint work with Lukas Koch (University of Oxford) and Sauli Lindberg (Aalto University).

Chapter 1

Introduction

This thesis is concerned with weak convergence problems arising in the Calculus of Variations and nonlinear PDEs. In light of its good compactness properties, weak convergence is ubiquitous throughout nonlinear analysis, as it plays a key role in establishing existence of solutions to both variational problems (through the Direct Method) and to nonlinear PDEs (through approximation schemes). However, as this thesis hopefully illustrates, the importance of weak convergence stretches much farther than the existence theory for nonlinear problems: to name just two particularly striking examples, weak convergence plays an important role in establishing nonlinear versions of the Banach–Schauder open mapping principle and, via a host of integral estimates, is intimately tied to several important problems in Harmonic Analysis.

This thesis is divided into three parts, which can be read essentially independently from one another. Nonetheless, each part is closely related to the others: for instance, both Parts II and III can be seen as a detailed study of some of the objects whose general properties are established in Part I. Each part is prefaced by an extended introduction, where the main results are motivated, described in detail, and compared to the existing literature.

In this introduction we will give a broad overview of the original results in this thesis. Only simple, semi-precise versions, will be stated here: the more precise statements, as well as refinements and generalisations, will be discussed in detail in the introduction of the corresponding part of the thesis.

1.1 Compensated Compactness

Part I of this thesis deals with the classical theory of compensated compactness, as developed jointly by MURAT [300, 301] and TARTAR [361, 362]. In their setting, one takes a linear partial differential operator, denoted by \mathcal{A} , which acts on fields $v: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{V}$. Here \mathbb{V} is a finite-dimensional inner product space, which represents the space where the physical fields take their values, and \mathcal{A} is an underdetermined PDE constraint that the fields must obey. This constraint models a basic feature of the physical system: for instance, in incompressible fluid dynamics one takes $\mathcal{A} = \text{div}$, while in nonlinear elasticity one is interested in irrotational fields, so that $\mathcal{A} = \text{curl}$. For the sake of simplicity we assume that \mathcal{A} is homogeneous and also that it has constant coefficients: for some $l \in \mathbb{N}$,

$$\mathcal{A} = \sum_{|\alpha|=l} A_\alpha \partial^\alpha, \quad \text{where } l \in \mathbb{N} \text{ and } A_\alpha \in \text{Lin}(\mathbb{V}, \mathbb{W}) \quad (1.1.1)$$

for another finite-dimensional inner product space \mathbb{W} . As usual, we write $\mathcal{A}(\xi)$ for the symbol of \mathcal{A} evaluated at $\xi \in \mathbb{R}^n \setminus \{0\}$. An important object associated to \mathcal{A} is the *wave cone*

$$\Lambda_{\mathcal{A}} \equiv \{\lambda \in \mathbb{V} : \text{there is } \xi \in \mathbb{R}^n \setminus \{0\} \text{ such that } \mathcal{A}(\xi)\lambda = 0\},$$

which is a set of directions measuring the non-ellipticity of the operator \mathcal{A} .

1.1.1 Compensated Compactness versus Compensated Regularity

A basic aim of the theory of Compensated Compactness is to characterize the nonlinear functions $F: \mathbb{V} \rightarrow \mathbb{R}$ such that

$$\mathcal{A}v_j = 0, v_j \xrightarrow{*} v \text{ in } \mathcal{D}'(\mathbb{R}^n) \quad \implies \quad F(v_j) \xrightarrow{*} F(v) \text{ in } \mathcal{D}'(\mathbb{R}^n). \quad (1.1.2)$$

Here v_j, v are assumed for simplicity to be test functions. These nonlinearities are weakly continuous with respect to the operator \mathcal{A} ; we can think of them as representing physical quantities that are robust to errors in measurements induced from small-scale oscillations.

A seemingly unrelated question is the following: what are the nonlinearities $F: \mathbb{V} \rightarrow \mathbb{R}$ such that

$$\mathcal{A}v = 0 \text{ and } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}) \quad \implies \quad F(v) \in \mathcal{H}^1(\mathbb{R}^n)? \quad (1.1.3)$$

Here $\mathcal{H}^1(\mathbb{R}^n)$ is the real Hardy space, which we may think of as a harmonic-analytical replacement of L^1 composed of functions with additional cancellations; thus (1.1.3) expresses the fact that the nonlinearity (1.1.3) has additional cancellations when tested on \mathcal{A} -free fields. Unsurprisingly, the knowledge that certain nonlinear expressions are in the Hardy space has proved to be extremely useful in PDEs from rather distinct contexts, ranging from Fluid Dynamics [130] to Geometric Analysis [190, 297].

It turns out that the quantities satisfying (1.1.2) are exactly the same as those satisfying (1.1.3). This result was obtained jointly with RAIȚĂ in [182] and answers positively a question of COIFMAN–LIONS–MEYER–SEMMES [87], who first noticed this phenomenon for some particular examples:

Theorem 1.A (Weak continuity \Leftrightarrow Hardy space integrability). *Let $F: \mathbb{V} \rightarrow \mathbb{R}$ be nonlinear, locally bounded and Borel measurable. Let \mathcal{A} be an operator as in (1.1.1) satisfying the non-degeneracy assumptions*

$$0 \neq \xi \mapsto \mathcal{A}(\xi) \text{ has constant rank,} \quad \text{span } \Lambda_{\mathcal{A}} = \mathbb{V}. \quad (1.1.4)$$

Then

$$\mathcal{A}v = 0 \implies F(v) \in \mathcal{H}^1(\mathbb{R}^n) \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$$

holds if and only if, for any sequence $(v_j) \subset C_c^\infty(\mathbb{R}^n, \mathbb{V})$,

$$\mathcal{A}v_j = 0, v_j \xrightarrow{*} v \text{ in } L_{\text{loc}}^\infty(\mathbb{R}^n) \quad \implies \quad F(v_j) \xrightarrow{*} F(v) \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Furthermore, all such functions F are polynomials and can be computed explicitly by solving a linear algebraic system of equations.

Theorem 1.A shows that *compensated compactness*, i.e. the weak convergence of some nonlinear quantities, and *compensated regularity*, i.e. the improved integrability of some nonlinear quantities, are two sides of the same coin.

More general and refined versions of Theorem 1.A can be found in Chapter 5 and in [184], which is joint work with RAIȚĂ and SCHRECKER. Moreover, methods similar to the ones used in the proof of Theorem 1.A allow for improvements of the results of FONSECA–MÜLLER [149], see Chapter 6, based on joint work with KRISTENSEN and RAIȚĂ [181].

Although the constant rank assumption in (1.1.4) is standard in the theory [149, 301], it is worthwhile discussing it here. The constant rank condition is a generalization of ellipticity which allows operators to be underdetermined, while keeping appropriate boundedness properties. To make this statement precise, let \mathcal{A} be as in (1.1.1) and recall that \mathcal{A} is *elliptic of order l* (i.e. its symbol is injective) if and only if, given $p \in (1, \infty)$, there is $C = C(p, \mathcal{A})$ such that

$$\|D^l v\|_{L^p(\mathbb{R}^n)} \leq C \|\mathcal{A}v\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}). \quad (1.1.5)$$

Clearly such an inequality cannot hold if \mathcal{A} has non-trivial kernel. Hence, if \mathcal{A} is underdetermined, it is natural to test (1.1.5) only in the orthogonal complement of $\ker \mathcal{A}$. The following result, obtained in [183] with RAIȚĂ, shows that an operator has constant rank if and only if it satisfies this generalization:

Theorem 1.B (Characterization of constant rank operators). *Let \mathcal{A} be as in (1.1.1) and let $P_{\mathcal{A}}$ be the L^2 -orthogonal projection onto $\ker \mathcal{A}$. Given $p \in (1, \infty)$, \mathcal{A} has constant rank if and only if*

$$\|D^l(v - P_{\mathcal{A}}v)\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{A}v\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}).$$

The reader may find a proof, as well as other properties of constant rank operators, in Chapter 4. We note that the sufficiency of the constant rank condition has been known for a long time [225, 301, 334] and that only the necessity is new.

As a consequence of Theorem 1.B, the methods used to prove Theorem 1.A fail in the absence of the constant rank assumption. In the general case, both weak continuity and \mathcal{H}^1 -integrability are completely open beyond quadratic nonlinearities [261, 361], but see [258, 294, 296] and [310, 379] for weak continuity results for some specific non-constant rank operators.

Surprisingly, the problems of weak continuity in the constant and non-constant rank cases are genuinely different. Indeed, as a consequence of the proof of Theorem 1.A, it follows that

if \mathcal{A} has constant rank and (1.1.2) holds with $v = 0$, then it holds for *any* $v \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$;

however, this is not the case for non-constant rank operators! An example due to TARTAR [366] shows that, for the wave equation, there is a trilinear quantity which is weakly continuous *only at zero*. This example can be described as follows. First, we rewrite the wave equation $\square \equiv -\partial_{tt} + \Delta_x$ as a first-order system: with $v = \nabla_{t,x}u$ and $\operatorname{div}_x v = \sum_{i=1}^n \partial_{x_i} v^i$, we have

$$\square u = 0 \quad \iff \quad \mathcal{A}_{\square} v \equiv (\operatorname{curl}_{t,x} v, (-\partial_t + \operatorname{div}_x)v) = 0.$$

It is easy to see that \mathcal{A}_{\square} does not have constant rank. For any index $\alpha = 0, \dots, n$, the trilinear quantity

$$F(v) \equiv v^\alpha \left(|v^0|^2 - \sum_{i=1}^n |v^i|^2 \right) \tag{1.1.6}$$

is weakly continuous with respect to \mathcal{A}_{\square} only at zero; this is easily proved using the div-curl lemma. We also note that, when $n = 2$, and up to a change of variables, \mathcal{A}_{\square} can be identified with the separate convexity operator [365]. As we will see in the next subsection, the trilinear quantity (1.1.6) plays an important role in some homogenization problems arising in the theory of General Relativity.

1.1.2 Compensated Compactness and Homogenization

The aim of Homogenization Theory is to describe the *effective* macroscopic behaviour of a system which is heterogeneous at a mesoscopic scale. From a mathematical perspective, the problems addressed by the theory of Homogenization amount to computing, or determining in a way as explicit as possible, the weak limits of nonlinear quantities; thus, unsurprisingly, the theory of Compensated Compactness, as described in the previous subsection, is closely connected to Homogenization.

In Chapter 7 we will discuss the homogenization theory of composite, periodic materials. As we will see, the methods introduced in Chapter 5 allow us recover in a unified framework many classical results. We will focus, in particular, on the G-closure problem. The G-closure of a given set \mathbb{M} of tensors (each of which we think of as a *phase*) is the set of all effective tensors that can be obtained by combining tensors of \mathbb{M} in any proportion or shape; in other words, the G-closure of \mathbb{M} describes the set of effective properties of all composite materials made with phases from \mathbb{M} . Calculating the G-closure of a given set is essentially equivalent to calculating the quasiconvex envelope of an associated function and, in general, it is an extraordinarily task. However, whenever the G-closure of \mathbb{M} has empty interior, its calculation becomes much more manageable, and we will review a method, developed by GRABOVSKY, MILTON and collaborators [167, 283], for determining it in this case. While the results in Chapter 7 are not new, there is some novelty in the exposition.

In Chapter 8 we will discuss a homogenization problem arising in the theory of General Relativity. In order to set the stage, fix a four-dimensional manifold \mathcal{M} , and consider a sequence $(g_\varepsilon)_\varepsilon$ of Lorentzian metrics on \mathcal{M} which solve the Einstein vacuum equations, i.e.

$$\operatorname{Ric}(g_\varepsilon) = 0. \tag{1.1.7}$$

As ε decreases, the sequence \mathbf{g}_ε provides a larger and larger scale picture of the space-time \mathcal{M} . The natural homogenization question is what kind of space-time is obtained in the limit and, in particular, what can be said about its Ricci tensor, which is the relevant nonlinear quantity here. It is easy to see that, in general, the effective Ricci tensor will not be zero since, schematically,

$$\mathbf{Ric}(\mathbf{g}) = \mathbf{g} \partial^2 \mathbf{g} + \mathbf{g}^2 \partial \mathbf{g} \partial \mathbf{g},$$

and the second term is not weakly continuous. The Einstein equations read as

$$[\mathbf{Ric}(\mathbf{g})]_{\alpha\beta} = 8\pi \left(\mathbb{T}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \operatorname{tr}_{\mathbf{g}} \mathbb{T} \right) \quad \text{and} \quad \sum_{\alpha=0}^n \nabla^\alpha \mathbb{T}_{\alpha\beta} = 0 \text{ for } \beta = 0, \dots, n, \quad (1.1.8)$$

where \mathbb{T} denotes the *energy-momentum tensor* and ∇ is the covariant derivative with respect to \mathbf{g} . Hence, to identify the Ricci tensor obtained in the limit as *matter*, we must both compute $\mathbf{Ric}(\mathbf{g})$ and show that it verifies the second set of equations in (1.1.8).

In [73], BURNETT conjectured that, under suitable assumptions, the effective energy-momentum tensor is of massless Vlasov type; mathematically, this means that lack of compactness in solutions to the Einstein vacuum equations (1.1.7) manifests itself as massless matter which is propagated without collisions along the null directions of spacetime. The main result of Chapter 8, obtained in joint work with TEIXEIRA DA COSTA [177], is a proof of this conjecture, under some symmetry and gauge assumptions:

Theorem 1.C (Burnett's conjecture in $U(1)$ -symmetry). *Let (\mathbf{g}_ε) be a sequence of $U(1)$ -symmetric Lorentzian metrics in elliptic gauge and let \mathbf{g} be another Lorentzian metric. Suppose that, for every compact set $K \subset \mathcal{M}$, there is a sequence $\lambda_\varepsilon \searrow 0$ such that*

$$\|\partial^k (\mathbf{g}_\varepsilon - \mathbf{g})\|_{L^\infty(K)} \lesssim_K \lambda_\varepsilon^{1-k}, \quad k = 0, 1, 2. \quad (1.1.9)$$

Then \mathbf{g} is a $U(1)$ -symmetric solution of the Einstein–massless Vlasov system.

We refer the reader to Chapter 8 for a more precise statement. Theorem 1.C was inspired by recent work of HUNEAU–LUK [200], who proved a weaker version of the same result: they assumed that (1.1.9) holds up to and including $k = 4$. Besides this improvement, the proof presented here in Chapter 8 is also significantly simpler than theirs.

The gauge and symmetry assumptions of Theorem 1.C reduce Burnett's conjecture to a more manageable problem, which we briefly describe. A $U(1)$ -symmetric Lorentzian metric \mathbf{g} on \mathcal{M} can be identified with a triple (g, ψ, ω) , where g is a Lorentzian metric on a three-dimensional manifold \mathcal{M} and ψ and ω are scalar functions in \mathcal{M} . Due to our gauge assumptions, Theorem 1.C reduces to understanding the effective behaviour of wave maps from $(\mathcal{M}, g_\varepsilon)$ into the Poincaré plane:

$$\begin{cases} \square_{g_\varepsilon} \psi_\varepsilon + \frac{1}{2} e^{-4\psi_\varepsilon} g_\varepsilon^{-1} (d\omega_\varepsilon, d\omega_\varepsilon) = 0, \\ \square_{g_\varepsilon} \omega_\varepsilon - 4g_\varepsilon^{-1} (d\psi_\varepsilon, d\omega_\varepsilon) = 0. \end{cases} \quad (1.1.10)$$

Equations (1.1.10) are the Euler–Lagrange equations associated to the action

$$\mathcal{L}_\varepsilon[\psi_\varepsilon, \omega_\varepsilon] \equiv \sum_{\alpha, \beta=0}^2 \int_{\Omega} g_\varepsilon^{\alpha\beta} \mathbb{L}_{\alpha\beta}[\psi_\varepsilon, \omega_\varepsilon] dx, \quad \text{where } \mathbb{L}_{\alpha\beta}[\psi_\varepsilon, \omega_\varepsilon] \equiv 2\partial_\alpha \psi_\varepsilon \partial_\beta \psi_\varepsilon + \frac{1}{2} e^{-4\psi_\varepsilon} \partial_\alpha \omega_\varepsilon \partial_\beta \omega_\varepsilon$$

is the *Lagrangian density*. In order to prove Theorem 1.C, we need to characterize the compactness defects in $\mathbb{L}_{\alpha\beta}[\psi_\varepsilon, \omega_\varepsilon]$. More precisely, suppose that $\mathbf{g}_\varepsilon \equiv (g_\varepsilon, \omega_\varepsilon, \psi_\varepsilon)$ solve (1.1.10) for each $\varepsilon > 0$, and converge weakly to $\mathbf{g} \equiv (g, \omega, \psi)$, according to (1.1.9); in general,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{L}_{\alpha\beta}[\psi_\varepsilon, \omega_\varepsilon] \neq \mathbb{L}_{\alpha\beta}[\psi, \omega]. \quad (1.1.11)$$

The aim is to show that the oscillations and concentrations responsible for the failure of the convergence (1.1.11) *propagate* according to a free transport equation, as per the second equation in (1.1.8). To do so,

we rely on H-measures, introduced independently by GÉRARD [159] and TARTAR [364], as well as on old and new trilinear compensated compactness results. In particular, a variant of the trilinear quantity in (1.1.6) appears naturally in relation to the energy identity for the wave equation. As mentioned above, the quantity in (1.1.6) is weakly continuous only at zero and, as a consequence, due to the coupled nature of the system (1.1.10), compactness defects of a general quadratic quantity on (ψ, ω) propagate in a *complex and non-explicit* way. Nonetheless, due to the variational structure of (1.1.10), compactness defects of the Lagrangian density $\mathbb{L}_{\alpha\beta}[\psi, \omega]$ admit a simple description, as in Theorem 1.C.

1.2 Quasiconvexity and the curl-free Calculus of Variations

Part II of this thesis is mostly concerned with the relation between quasiconvexity and rank-one convexity in the classical curl-free Calculus of Variations: the reader may think of Part II as a detailed study of some of the objects considered in Part I in the case when $\mathcal{A} = \text{curl}$. Although in this section we will only briefly describe the contents of Part II, in Chapter 9 the reader may find a detailed description of the results in Part II, as well as a comprehensive discussion of the existing literature on the topic.

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a function $F: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$, we say that

- (i) F is *quasiconvex* if $0 \leq \int_{\Omega} F(A + D\varphi) - F(A) dx$ for all $\varphi \in C_c^{\infty}(\Omega, \mathbb{R}^m)$ and all $A \in \mathbb{R}^{m \times n}$;
- (ii) F is *rank-one convex* if $t \mapsto F(A + tX)$ is convex for all $A, X \in \mathbb{R}^{m \times n}$ with $\text{rank } X = 1$.

It is well-known that the definition in (i) is independent of Ω and that (i) \Rightarrow (ii), see [98].

Under standard assumptions, F is quasiconvex if and only if the functional

$$\mathcal{F}[u] \equiv \int_{\Omega} F(Du) dx$$

is sequentially weakly lower semicontinuous in an appropriate Sobolev space, and hence quasiconvexity is the natural existence condition in the vectorial Calculus of Variations [28, 285, 286]. However, quasiconvexity also appears naturally in other contexts, such as in Homogenization [168], as discussed in the previous section, and in Geometric Function Theory [207]. It is also closely related to integral estimates, an interesting example in this direction being *Ornstein's non-inequality* [307], of which we give a very short proof in the planar case in Chapter 10:

Theorem 1.D (Ornstein's non-inequality in $\mathbb{R}^{2 \times 2}$). *Let $\mathcal{P}_1, \mathcal{P}_2$ be first-order linear differential operators acting on maps $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The following inequality only holds in trivial cases:*

$$\|\mathcal{P}_1\varphi\|_{L^1(\mathbb{R}^2)} \leq C\|\mathcal{P}_2\varphi\|_{L^1(\mathbb{R}^2)}, \quad \text{for all } \varphi \in C_c^{\infty}(\mathbb{R}^2, \mathbb{R}^2).$$

This proof was obtained jointly with FARACO [133] and is inspired by the one in [233].

Despite being such an old notion, quasiconvexity remains poorly understood: for instance, there are fourth order polynomials whose quasiconvexity cannot be decided [5, 101]. An outstanding problem in the area is whether rank-one convexity implies quasiconvexity; this is known as *Morrey's problem*. ŠVERÁK showed [351] that rank-one convexity does not imply quasiconvexity if $m \geq 3, n \geq 2$ and recently GRABOVSKY [167] gave a completely new example of a rank-one convex non-quasiconvex function when $m = 8, n = 2$. Nonetheless, the case $n \geq m = 2$ remains open, and in fact there is some evidence that the answer might be different in this case, see e.g. [137, 235, 233, 294, 335]. The reader can find an in-depth discussion of Morrey's problem, as well as its connections to Homogenization Theory and Quasiconformal Analysis, in Chapter 9.

Our understanding of the class of quasiconvex integrands remains very incomplete. The main original result of Part II of this thesis is Theorem 1.E below. It was, for the most part, proved in [176], and its proof can be found in Chapter 10. Theorem 1.E contains some geometric results about the class of quasiconvex integrands, in particular concerning *extremal points* of this class. In order to state it, we use the following terminology: a rank-one convex function $F: \mathbb{R}^{n \times n} \rightarrow [0, +\infty)$ is *extremal*¹ if all

¹The definition of extremal rank-one convex functions $F: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ is identical.

decompositions $F = F_1 + F_2$, for rank-one convex functions $F_i: \mathbb{R}^{n \times n} \rightarrow [0, +\infty)$, are of the form $F_i = \lambda_i F$ for some $\lambda_i \geq 0$. We then have:

Theorem 1.E (Extremal rank-one convex functions). *Morrey’s problem is equivalent to deciding whether all extremal rank-one convex functions are quasiconvex.*

The following are examples of extremal rank-one convex functions:

- (i) *the function $M^+ \equiv \max\{M, 0\}$ where $M: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is any minor;*
- (ii) *for $k = 0, \dots, n$, the functions $F_k: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ defined by*

$$F_k(A) = \begin{cases} |\det A| & \text{if } A \text{ has } k \text{ negative eigenvalues,} \\ 0 & \text{otherwise.} \end{cases}$$

This and the previous example prove a conjecture made by ŠVERÁK in [350].

- (iii) *the function $L: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$, introduced by ŠVERÁK in [348], and defined by*

$$L(A) = 1 + \begin{cases} \det A, & |a^+| + |a^-| \leq 1 \\ 2|a^+| - 1, & \text{otherwise} \end{cases},$$

where $A = (a^+, a^-)$ is the conformal/anti-conformal decomposition of a matrix.

We note that the function L is closely related to Burkholder’s function, and thereby to a long-standing conjecture [205] in Quasiconformal Analysis concerning the L^p -norms of the Beurling–Ahlfors transform, see Section 9.1.5 for further details and references.

Despite Theorem 1.E, the class of extremal integrands remains difficult to characterize. Thus, it is natural to look for a more concrete approach to Morrey’s problem. In this direction, we consider a periodic deformation obtained by summing N plane waves with the same shape but different directions, and we look for rank-one convex functions which falsify the quasiconvexity inequality with respect to such deformations. As we have mentioned, when $n \geq 2, m \geq 3$ and $N = 3$, there is a rank-one convex function which does not satisfy the quasiconvexity inequality [351]. However, when $n = m = 2$ and $N = 3$, no such examples exist [335]. In [185], joint with TEIXEIRA DA COSTA, we made the following observation:

Observation 1.F (Numerics for Morrey’s problem). *Numerics suggest that no such examples exist when $n = m = 2$ and $N = 4, 5$. Moreover, the rank-one geometry of the image of the deformation gradient gets much richer as N increases.*

A more detailed discussion of this observation can be found in Chapter 11.

1.3 Nonlinear open map principles and the Jacobian equation

In Part III of this thesis we study low regularity problems for nonlinear underdetermined PDEs. We will focus, in particular, on the Jacobian equation

$$\det Du \equiv Ju = f \quad \text{a.e. in } \Omega \subseteq \mathbb{R}^n, \quad (1.3.1)$$

where f is a given function and $u: \Omega \rightarrow \mathbb{R}^n$ is a map. The underdetermined nature of (1.3.1) is manifest, even among smooth solutions: we can always compose u with an area-preserving diffeomorphism to obtain a new solution. Moreover, as we will see momentarily, some of the new tools developed to study (1.3.1) are applicable to other nonlinear PDEs, such as the equations of incompressible fluid flow.

Equation (1.3.1) appears naturally in several branches of analysis, including Optimal Transport [62, 114], Differential Geometry [19, 287] and Continuum Mechanics [96], and it has a rich geometric flavour: if u is a smooth solution of (1.3.1) then

$$\int_E |f| dx = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) dy \quad \text{for any measurable set } E \subseteq \Omega, \quad (1.3.2)$$

where $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$. We will address two related questions concerning (1.3.1):

- (i) when can we solve (1.3.1), and what is the optimal regularity of solutions?
- (ii) is there a selection criterion for (1.3.1), or is the equation hopelessly underdetermined?

The results presented here are joint work with LUKAS KOCH and SAULI LINDBERG [178, 179, 180].

1.3.1 Non-existence of solutions with high regularity

Let us begin by discussing question (i). For sufficiently smooth data there is a rich well-posedness theory for (1.3.1) which goes back to the classical works of MOSER [287] and DACOROGNA–MOSER [103], see also [94] and the references therein. Nonetheless, prior to the original results presented in this thesis, almost nothing was known concerning (1.3.1) in low regularity, in particular when f is in some Lebesgue-type space.

Low regularity problems for (1.3.1) are natural, but they are also closely related to the results presented in Section 1.1. Recall that, in Theorem 1.A, we characterized the class of operators which map solutions of constant rank PDEs into the real Hardy space; the Jacobian determinant is, of course, in that class. In their seminal paper, COIFMAN–LIONS–MEYER–SEMMES [87] addressed the question of whether the Jacobian determinant is *onto* $\mathcal{H}^1(\mathbb{R}^n)$: in other words, can one always solve (1.3.1) when $f \in \mathcal{H}^1(\mathbb{R}^n)$? Subsequently, IWANIEC [206] generalized this problem to $p > 1$, and in fact conjectured that the answer is positive: (1.3.1) admits a solution $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ whenever $f \in \mathcal{H}^p(\mathbb{R}^n)$. Although all cases of this problem remain open, it is known that $\mathcal{H}^p(\mathbb{R}^n)$ is the smallest Banach space containing the range of the Jacobian, as proved in the original paper [87] for $p = 1$ and in [201] for $p > 1$. We also note that the restriction to $p \in [1, \infty)$ is natural: counter-examples in the $p = \infty$ case were obtained independently in [70, 279], but see also [329] for some positive results at this endpoint.

Besides studying (1.3.1) over \mathbb{R}^n , it is also interesting to study the Dirichlet problem:

$$\begin{cases} Ju = f & \text{in } B, \\ u = \text{id} & \text{on } \partial B, \end{cases} \quad (1.3.3)$$

where $B \subset \mathbb{R}^n$ is the unit ball. In this setting, it is customary to impose the compatibility and non-degeneracy conditions

$$\int_B f \, dx = 1, \quad \text{ess inf}_B f \geq c > 0, \quad (1.3.4)$$

for some $c > 0$; this leads us to the complete metric spaces

$$X_p(B) \equiv \begin{cases} \{f \in L^p(B) : f \text{ satisfies (1.3.4)}\} & \text{if } p > 1, \\ \{f \in L \log L(B) : f \text{ satisfies (1.3.4)}\} & \text{if } p = 1. \end{cases}$$

The special definition of $X_1(B)$ is due to the improved integrability of the Jacobian and indeed one may think of $L \log L$ as the analogue of the Hardy space in this setting: it was proved in [194] that, if $u \in \text{id} + W_0^{1,n}(B, \mathbb{R}^n)$ and $Ju \geq 0$ then $Ju \in L \log L(B)$.

In Chapter 14 we show that the maximal regularity of solutions to (1.3.3) is generically quite low:

Theorem 1.G (Non-existence of solutions with high regularity). *Fix $c \in (0, 1)$, $p \in (1, \infty)$.*

There is $f \in X_p(B)$ such that (1.3.3) has no solution $u \in W^{1,np}(B, \mathbb{R}^n)$. In fact, for a Baire-generic $f \in X_p(B)$, (1.3.3) has no solution in the space $\bigcup_{n \leq q, p < q} W^{1,q}(B, \mathbb{R}^n)$.

Theorem 1.G answers in the negative questions posed in [194] and [372] and it shows that the analogue of IWANIEC's conjecture for the Dirichlet problem is drastically false. Theorem 1.G should also be compared with the non-existence results of [372]. From Sobolev embeddings, it is not difficult to see that

$$u \in W^{k+1,p} \implies Ju \in W^{k,p} \quad \text{for } k \geq 1, p \in (n/k, +\infty]$$

and this shows that, for data with Sobolev regularity, we cannot hope for derivatives with higher integrability. Although this simple argument fails for $k = 0$, Theorem 1.G shows that a similar conclusion also holds in that case. In order to prove the theorem, we consider data which is ill-behaved near ∂B and we use a geometric argument to estimate the Sobolev regularity of an arbitrary solution. Unfortunately, as the proof relies heavily on the boundary condition, we are not able to extend Theorem 1.G to \mathbb{R}^n .

1.3.2 Energy minimisation as a selection criterion

Let us now discuss question (ii) above. In Chapter 15 we consider the possibility of finding a selection criterion for (1.3.1) through energy minimisation, an idea which was proposed in [206]. Since we are interested in solutions with Sobolev regularity, the natural energy to consider is the q -Dirichlet energy: we say that a solution u of (1.3.1) is a q -energy minimiser for f if

$$\int_{\Omega} |Du|^q dx = \inf \left\{ \int_{\Omega} |Dv|^q : Jv = f \right\},$$

with similar definitions for the Dirichlet problem (1.3.3). Due to the weak continuity of the Jacobian, if there is a solution of (1.3.1) in $\dot{W}^{1,q}$ then f has a q -energy minimiser, whenever $q \geq n$.

Whenever f is spherically symmetric, in the sense that $f(x)$ only depends on x through $r \equiv |x|$, (1.3.1) admits a unique formal symmetric solution, known as a radial stretching in the literature, see for instance [372, Theorem 5]; it takes the form $\phi(x) \equiv \rho(r) \frac{x}{r}$ for some $\rho: \mathbb{R}^+ \rightarrow \mathbb{R}^+$. It is thus natural to wonder whether energy minimisation selects the symmetric solutions for symmetric data. The next result gives a rather complete answer to this question, posed originally by HÉLEIN [372]:

Theorem 1.H (Symmetry vs symmetry breaking). *Energy minimisers are symmetric for some data but not for other. More precisely, let $f \in X_p(B)$ be spherically symmetric and define*

$$\lambda[f] \equiv \sup_{r \in [0,1]} \frac{f(r)}{\int_{B_r(0)} f dx}.$$

We have two regimes:

- (i) if $\lambda[f] \leq 1$ then energy minimisers for the Dirichlet problem are symmetric, and hence unique;
- (ii) for some data with $\lambda[f] \gg 1$, there are uncountably many energy minimisers for the Dirichlet problem, none of which is symmetric.

It follows that energy minimisation is not an appropriate selection criterion for (1.3.1). Concerning the somewhat strange-looking quantity $\lambda[f]$, note that, as data in $X_p(B)$ is bounded uniformly away from zero, one has $\lambda[f] \lesssim_c \|f\|_{\infty}$ for $f \in X_p(B)$. However, $\lambda[f]$ is genuinely different from the supremum norm: for instance, taking $f_{\alpha}(r) \equiv \frac{2-\alpha}{2} r^{-\alpha}$, one computes

$$\lambda[f_{\alpha}] = \frac{2-\alpha}{2} \quad \text{for } \alpha < 2$$

and in particular $\lambda[f_{\alpha}] \leq 1$ for $\alpha \in [0, 2)$.

1.3.3 A priori estimates as a selection criterion: an open mapping principle

In this subsection we consider a different (partial) selection criterion, based on a priori estimates.

For linear PDE, the method of a priori estimates can be justified through the classical open mapping theorem from Functional Analysis, which essentially says that a linear PDE can be solved if and only if it can be solved *with an estimate*; for applications to nonlinear analysis it would be interesting to have a nonlinear version of this result. In the late 60s, RUDIN [331] asked whether the open mapping theorem extends to bilinear operators. Shortly thereafter, counterexamples due to COHEN [86] and HOROWITZ [197] showed that the answer is negative, *even if all the spaces involved are finite dimensional!*

Despite the previous counterexamples, in Chapter 16 we prove that the Jacobian determinant does have an open mapping principle:

Theorem 1.I (Open map principle for the Jacobian). *Fix $1 \leq p < \infty$. The following are equivalent:*

- (i) $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ is surjective;
- (ii) there is a bounded operator $E: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ such that $J \circ E = \text{Id}$;

(iii) J is open at zero: for all $f \in \mathcal{H}^p(\mathbb{R}^n)$ there is $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ such that $Ju = f$ and

$$\|Du\|_{L^{np}(\mathbb{R}^n)}^n \lesssim \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}. \quad (1.3.5)$$

In fact, Theorem 1.I extends to a wide range of weakly- $*$ continuous PDEs. We will return to this point in Section 1.3.4, but for now we just mention that Theorem 1.I is valid almost verbatim for any of the operators characterised in Theorem 1.A.

Despite its theoretical interest, one may also think of Theorem 1.I as providing the basis for a (partial) selection criterion to (1.3.1): if (1.3.1) is always solvable in the appropriate regularity regime then one may use the a priori estimate (1.3.5) to obtain solutions for less regular data through a regularization procedure. This is particularly striking when $p = 1$, since there are continuous maps u in $\dot{W}^{1,n}$ such that

$$\mathcal{L}^n(E) = 0 \quad \not\Rightarrow \quad \mathcal{L}^n(u(E)) = 0;$$

such maps clearly cannot satisfy the change of variables formula (1.3.1). Hence, the geometric interpretation of (1.3.1) in the *critical* space $\dot{W}^{1,n}$ is, in general, not valid. Nonetheless, as a consequence of Theorem 1.I, it follows that if $J: \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^1(\mathbb{R}^n)$ is surjective then one can select, through the a priori estimate, solutions which verify the change of variables formula under natural assumptions.

1.3.4 Applications to the incompressible Euler equations

To conclude this introduction we briefly discuss more general versions of Theorem 1.I. Roughly speaking, the open mapping principle is valid for nonlinear systems of constant-coefficient PDEs posed either over \mathbb{R}^n or \mathbb{R}^{1+n} , in regularity regimes where the equations are preserved under weak- $*$ convergence. We refer the reader to Part III, as well as [179], for further details. Here we will instead discuss an application of this theory to the incompressible Euler equations.

As we saw in the last subsection, the open mapping principle is a powerful tool in the study of *underdetermined* PDEs, as it can act as a sort of selection criterion. We now consider the Cauchy problem for the incompressible Euler equations, which model the behaviour of ideal fluids:

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) - \nabla p = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ \operatorname{div} u = 0 & \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \quad (1.3.6)$$

It is well-known that there is at most one smooth solution of (1.3.6). Nonetheless, it is a deep and surprising fact that, for some initial data, there are multiple weak solutions of (1.3.6): hence, in a low regularity setting, one may view the Euler equations as an underdetermined system of PDEs. Low regularity solutions of (1.3.6) have been studied extensively in connection to the so-called Onsager conjecture, see [68, 112, 204] and the references therein.

Before proceeding further, we note that the incompressibility constraint $\operatorname{div} u = 0$ will often be codified through the subscript div : thus $L_{\operatorname{div}}^2 \equiv \{u \in L^2 : \operatorname{div} u = 0\}$. By a theorem of SZÉKELYHIDI–WIEDEMANN [357], for a dense set of initial data in L_{div}^2 there exist infinitely many *admissible* solutions $u \in L_t^\infty L_{\operatorname{div},x}^2$ of (1.3.6), that is, weak solutions such that

$$\int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq \int_{\mathbb{R}^n} |u_0(x)|^2 dx \quad \text{for a.e. } t \geq 0.$$

For some data $0 \neq u_0 \in L_{\operatorname{div}}^2$ there are even *admissible compactly supported* solutions. In [333] SCHEFFER constructed solutions of (1.3.6) which are compactly supported and square integrable in space-time and a systematic study via convex integration was initiated by DE LELLIS and SZÉKELYHIDI in [112, 113].

Nevertheless, the general nonlinear version of the open mapping theorem alluded to above easily implies that, for a Baire-generic datum in $L_{\operatorname{div}}^2(\mathbb{R}^n)$, the kinetic energy of weak solutions cannot decay too quickly. To state the result, let us say that a weak solution of (1.3.6) is *strongly dissipative* if

$$\int_{\mathbb{R}^n} |u(x, t)|^2 dx = O(t^{-1/2+\varepsilon}) \quad \text{as } t \rightarrow \infty$$

for some $\varepsilon > 0$. We then have:

Theorem 1.J (Generic non-existence of strongly dissipative solutions). *For a Baire-generic $u_0 \in L^2_{\text{div}}$, the Cauchy problem (1.3.6) does not admit strongly dissipative weak solutions in $L_t^\infty L_x^2$.*

We note that analogues of Theorem 1.J also hold for other models of incompressible fluid mechanics, such as the MHD equations.

Chapter 2

Notation and the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$

We begin by fixing some notation that will be used throughout this thesis. Unless stated otherwise, p will denote a real number in the interval $(1, \infty)$ and \mathcal{F} denotes the Fourier transform. As usual, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain and we write $\mathcal{M}(\Omega)$ for the space of Radon measures in Ω , $\mathcal{M}^+(\Omega)$ denotes positive measures in $\mathcal{M}(\Omega)$ and $\mathcal{M}_1^+(\Omega)$ denotes probability measures. The n -dimensional Lebesgue measure will be denoted by \mathcal{L}^n . Given a measurable set $E \subset \mathbb{R}^n$ and $f \in L^1(E)$, we write

$$\int_E f \, dx \equiv \frac{1}{|E|} \int_E f \, dx.$$

Boldface letters denote manifolds: $\mathbb{V}, \mathbb{U}, \mathbb{W}$ are reserved for finite-dimensional inner product vector spaces, \mathbb{M} denotes a (non-linear) manifold, and we write \mathbb{T}^n for the n -dimensional torus. Similarly, given numbers $0 < r < R$, we write

$$\mathbb{A}(r, R) \equiv \{x \in \mathbb{R}^n : r < |x| < R\}.$$

If $\mathbb{U} \subset \mathbb{V}$, $\text{Proj}_{\mathbb{U}} : \mathbb{V} \rightarrow \mathbb{U}$ denotes the orthogonal projection onto \mathbb{U} . We write $\odot^k(\mathbb{R}^n, \mathbb{U})$ for the space of all \mathbb{U} -valued symmetric k -linear maps on \mathbb{R}^n ; thus, for a C^k map $u : \Omega \rightarrow \mathbb{U}$, we have that $D^k u \in \odot^k(\mathbb{R}^n, \mathbb{U})$.

The rest of this chapter contains basic material on real Hardy spaces that we gathered here for the convenience of the reader. The reader may find proofs, as well as much more information, in [89, 336, 345].

2.1 The real Hardy Space

Classically, Hardy spaces were studied in connection with the boundary behavior of harmonic and holomorphic functions in the unit disk, and they appear naturally in the theory of functions of one complex variable. Nonetheless, in extending the theory to higher dimensions, a purely real-variable theory has emerged, in particular concerning the so-called real Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. In this thesis only the real-variable theory will be discussed and we will ignore the complex-analytical origins of the subject, referring instead the interested reader to [241]. We also note that the real-variable theory of $\mathcal{H}^1(\mathbb{R}^n)$ extends to the much more general setting of spaces of homogeneous type [89].

Fix a function $\phi \in C_c^\infty(\mathbb{R}^n)$ with non-zero mean and let $\phi_t(x) \equiv t^{-n}\phi(x/t)$ for $t > 0$. We begin with the following definition:

Definition 2.1.1. Let $p \in (0, \infty]$. The space $\mathcal{H}^p(\mathbb{R}^n)$ is the space of those $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathcal{M}_\phi f(x) \equiv \sup_{t>0} |f * \phi_t| \in L^p(\mathbb{R}^n).$$

We write $\|f\|_{\mathcal{H}^p(\mathbb{R}^n)} \equiv \|\mathcal{M}_\phi f\|_{L^p(\mathbb{R}^n)}$.

We note that the definition of $\mathcal{H}^p(\mathbb{R}^n)$ is independent of the choice of ϕ [141]. When $p > 1$, the Hardy–Littlewood maximal inequality implies that the Hardy spaces reduce to the usual Lebesgue spaces:

Proposition 2.1.2. For $p > 1$, the space $\mathcal{H}^p(\mathbb{R}^n)$ is isometrically isomorphic to $L^p(\mathbb{R}^n)$.

We now discuss the much more interesting case $p \leq 1$. We begin by noting that $\mathcal{H}^1(\mathbb{R}^n) \subsetneq L^1(\mathbb{R}^n)$:

Lemma 2.1.3. *Let $p \in (0, 1]$. If $f \in L^1(\mathbb{R}^n) \cap \mathcal{H}^p(\mathbb{R}^n)$ then $\int_{\mathbb{R}^n} x^\beta f(x) dx = 0$ whenever $|\beta| \leq n(\frac{1}{p} - 1)$. In particular, functions in $\mathcal{H}^1(\mathbb{R}^n)$ have zero mean over \mathbb{R}^n .*

The space $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space and it is in fact a dual space. To state the result precisely, we recall the in the next definition two standard function spaces:

Definition 2.1.4. The space $\text{BMO}(\mathbb{R}^n)$ of functions of *bounded mean oscillation* is defined as the space of those functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\text{BMO}} \equiv \lim_{\delta \rightarrow \infty} M_\delta(f) < \infty, \quad \text{where } M_\delta(f) \equiv \sup_{|B| < \delta} \int_B \left| f - \int_B f \right| dx$$

and the supremum runs over balls in \mathbb{R}^n . The space $\text{VMO}(\mathbb{R}^n)$ of functions of *vanishing mean oscillation* is defined as the space those functions in $f \in \text{BMO}(\mathbb{R}^n)$ such that

$$\lim_{\delta \rightarrow 0} M_\delta(f) = 0.$$

We then have the following celebrated theorem:

Theorem 2.1.5 ([140, 332]). $\text{VMO}(\mathbb{R}^n)^* \cong \mathcal{H}^1(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)^* \cong \text{BMO}(\mathbb{R}^n)$.

In particular, we can consider $\mathcal{H}^1(\mathbb{R}^n)$ as equipped with the weak* topology induced from $\text{VMO}(\mathbb{R}^n)$. In this direction, we have the following classical result [220]:

Theorem 2.1.6 (Jones–Journé). *If a sequence f_j is bounded in $\mathcal{H}^1(\mathbb{R}^n)$ and it converges a.e. to f then $f \in \mathcal{H}^1$ and in fact $f_j \xrightarrow{*} f$ in \mathcal{H}^1 .*

Notice that if we replace $\mathcal{H}^1(\mathbb{R}^n)$ bounds by $L^1(\mathbb{R}^n)$ bounds, while still assuming pointwise convergence, then the conclusion of the theorem does not hold; in this case, we have that

$$f_j \rightharpoonup f \text{ in } L^1(\mathbb{R}^n) \iff (f_j) \text{ is equi-integrable.}$$

The difference between \mathcal{H}^1 and L^1 convergence will be used crucially in Lemma 5.3.4 below.

The next result shows that $\mathcal{H}^1(\mathbb{R}^n)$ is a natural harmonic analytical replacement of $L^1(\mathbb{R}^n)$:

Proposition 2.1.7. *Let f be a tempered distribution which is restricted at infinity in the sense that, for all $r < \infty$ sufficiently large,*

$$f * \varphi \in L^r(\mathbb{R}^n) \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Then $f \in \mathcal{H}^1(\mathbb{R}^n)$ if and only if both f and $R_j(f)$, for $j = 1, \dots, n$, are in $L^1(\mathbb{R}^n)$, where the R_j are the Riesz transforms.

To conclude this chapter we discuss the atomic characterization of the spaces $\mathcal{H}^p(\mathbb{R}^n)$, $p \in (0, 1]$.

Definition 2.1.8. Let $p \in (0, 1]$. A measurable function $a: \mathbb{R}^n \rightarrow \mathbb{C}$ is an $\mathcal{H}^p(\mathbb{R}^n)$ -atom if:

- (i) for some ball $B \subset \mathbb{R}^n$, $\text{supp } a \subset B$ and $|a| \leq 1/|B|^{-1/p}$ a.e.;
- (ii) $\int_{\mathbb{R}^n} x^\beta a(x) dx = 0$ for all multi-indices such that $|\beta| \leq n(\frac{1}{p} - 1)$.

Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution. A sequence of numbers $\lambda_k \in \mathbb{C}$ and $\mathcal{H}^p(\mathbb{R}^n)$ -atoms a_k such that $f = \sum_{k=1}^{\infty} \lambda_k a_k$, where the series converges in the sense of tempered distributions, is called an *atomic decomposition* of f .

Note that condition (ii) of Definition 2.1.8 is the same as in Lemma 2.1.3. In fact, we have the following alternative characterization of $\mathcal{H}^p(\mathbb{R}^n)$:

Theorem 2.1.9. *Let $p \in (0, 1]$. Then*

$$\|f\|_{\mathcal{H}^p(\mathbb{R}^n)} \approx \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k|^p : \sum_{k=1}^{\infty} \lambda_k a_k \text{ is an atomic decomposition of } f \right\}.$$

Finally, note that, for $p < 1$, the expression $\|\cdot\|_{\mathcal{H}^p(\mathbb{R}^n)}$ is not a norm, since it is p -homogeneous. In fact, the spaces $\mathcal{H}^p(\mathbb{R}^n)$ are not locally convex, although they are complete metric spaces under the distance induced by $\|\cdot\|_{\mathcal{H}^p(\mathbb{R}^n)}$.

Part I

Compensated Compactness

Chapter 3

Introduction

In Part I of this thesis we will study problems coming from the theories of Homogenization and Compensated Compactness. In particular, we will be concerned with the following type of question:

Question 3.0.1. Consider a sequence of vector fields which converges weakly in some Lebesgue space. What are the nonlinear quantities which are preserved in the limit? How do we characterize the behaviour of those quantities which are not preserved in the limit?

The relevance of this question is easily justified by the ubiquitous role played by weak convergence in nonlinear analysis, but there is also a simple physical motivation behind it. Recall that, if $p \in (1, \infty)$, a sequence of fields converges weakly in $L^p(\Omega)$, denoted $v_j \rightharpoonup v$, if and only if $\sup_j \|v_j\|_{L^p(\Omega)} < \infty$ and

$$\int_B v_j \, dx \rightarrow \int_B v \, dx \quad \text{for all balls } B \subset \Omega.$$

In other words, a bounded sequence converges weakly if and only if its averages converge. Since one cannot measure a physical field at a *point*, but only in regions of small but positive measure (and, anyway, fields in L^p do not have pointwise values), one can think of weak convergence as *convergence in measurement*. Intuitively, one may think that the nonlinear function in Question 3.0.1 is a macroscopic quantity of interest, which we loosely think of as an energy, and that, as j increases, v_j represents an increasingly larger-scale version of the physical system. Thus, Question 3.0.1 asks for a description of the *effective* large-scale behaviour of the macroscopic energy of the system.

In practice, in addressing Question 3.0.1 we will not consider *arbitrary* sequences of vector fields: it is often the case that, in Continuum Mechanics and elsewhere, the sequences of vector fields one is interested in satisfy an *underdetermined PDE constraint*. Important examples of constrained vector fields are found in incompressible fluid flow, where the fluid velocity is divergence-free, and in nonlinear elasticity, where the energy density depends on the deformation gradient—such a vector field is necessarily curl-free.

In Chapters 4 and 5 we will study exhaustively the first part of Question 3.0.1, characterizing the nonlinear quantities which are preserved under weak convergence. We address this problem in a general framework, pioneered by MURAT and TARTAR [300, 301, 361, 362], in which one can study oscillations and concentrations in sequences of solutions to variational problems and nonlinear PDEs, see Chapter 6 for further details. In their framework the PDE constraint is *linear*, as in the above examples. The original results of Chapters 4 to 6 consist of joint work with JAN KRISTENSEN [181], BOGDAN RAIȚĂ [182, 183], and MATTHEW SCHRECKER [184] and will be discussed in further detail below.

In Chapters 7 and 8 we will deal with the second and more difficult part of Question 3.0.1, concerning quantities which are not preserved in the limit: if this is the case, it may be very challenging to obtain a satisfactory description of the limit behaviour of the system, even in simple-looking situations. In Chapter 7 we will discuss classical results about the G-closure problem; although the results detailed in that chapter are not new, there is some novelty in the exposition. In Chapter 8 we will present joint work with RITA TEIXEIRA DA COSTA [177] about a homogenization problem for the Einstein vacuum equations; in that case, the nonlinear quantity one is interested in is the Ricci curvature.

3.1 The constant rank \mathcal{A} -free setting

In this section we discuss the general \mathcal{A} -free setting of MURAT–TARTAR’s theory of Compensated Compactness. In the last four decades, the theory was developed much further, having found applications in Continuum Mechanics [120, 121, 130], Homogenization [60, 65, 282, 283] and Nonlinear Analysis [23, 115, 146, 219, 299]. We also refer the reader to the recent papers [13, 93, 109, 243, 317].

As mentioned above, we are interested in sequences of vector fields which satisfy a PDE constraint, denoted here by \mathcal{A} . Take finite-dimensional inner product spaces \mathbb{V}, \mathbb{W} , a collection of operators $A_\alpha \in \text{Lin}(\mathbb{V}, \mathbb{W})$, and set

$$\mathcal{A}v \equiv \sum_{|\alpha|=l} A_\alpha \partial^\alpha v, \quad \text{where } v: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{V}. \quad (3.1.1)$$

We note that the assumption that \mathcal{A} has constant coefficients is far from essential and is made just for the sake of simplicity. The symbol of \mathcal{A} , denoted by $\mathcal{A}(\xi)$, is a $\text{Lin}(\mathbb{V}, \mathbb{W})$ -valued homogeneous polynomial.

A fundamental object in Compensated Compactness theory is the vector bundle

$$V_{\mathcal{A}} \equiv \left\{ (\lambda, \xi) \in \mathbb{V} \times \mathbb{S}^{n-1} : \mathcal{A}(\xi)\lambda = 0 \right\}$$

which contains the frequencies and amplitudes along which \mathcal{A} is not elliptic. Its projection onto \mathbb{V} is known as the *wave cone* associated to \mathcal{A} , which we denote by

$$\Lambda_{\mathcal{A}} \equiv \bigcup_{\xi \in \mathbb{S}^{n-1}} \ker \mathcal{A}(\xi).$$

A typical non-degeneracy assumption in the theory, which will be used throughout this section, is that \mathcal{A} has *constant rank*. This condition will be examined in detail in the next subsection.

3.1.1 Constant rank operators

Given an operator \mathcal{A} as in (3.1.1), we say that:

- (i) \mathcal{A} is (*overdetermined*) *elliptic* if $\mathbb{S}^{n-1} \ni \xi \mapsto \mathcal{A}(\xi)$ is injective;
- (ii) \mathcal{A} has *constant rank* if $\mathbb{S}^{n-1} \ni \xi \mapsto \text{rank } \mathcal{A}(\xi)$ is constant.

Note that \mathcal{A} has constant rank if and only if $V_{\mathcal{A}}$ is a *smooth vector bundle* over \mathbb{S}^{n-1} . Moreover, the constant rank condition can also be thought of as a general tameness condition that is natural extension of ellipticity to allow for underdeterminedness.

To make the above remark precise, recall that \mathcal{A} is elliptic if and only if, for a given $p \in (1, \infty)$, there is a constant C_p such that

$$\|D^l v\|_{L^p(\mathbb{R}^n)} \leq C_p \|\mathcal{A}v\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}). \quad (3.1.2)$$

We remark that the estimate (3.1.2) only holds in trivial cases when $p = 1$ [307, 233] and $p = \infty$ [51, 111, 284], see also Theorem 10.5.4 for a short proof of the $p = 1$ case. We define, for $v \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$, the operator

$$\widehat{P}_{\mathcal{A}} v(\xi) \equiv \text{Proj}_{\ker \mathcal{A}(\xi)} \widehat{v}(\xi).$$

Note that $P_{\mathcal{A}} v \in L^2(\mathbb{R}^n, \mathbb{V})$ whenever $v \in L^2(\mathbb{R}^n, \mathbb{V})$ and that $P_{\mathcal{A}}$ is the orthogonal projection onto $\ker \mathcal{A} \subseteq L^2(\mathbb{R}^n, \mathbb{V})$. It is thus natural to wonder whether (3.1.2) holds if we test it only in the orthogonal complement of $\ker \mathcal{A} \subseteq L^2(\mathbb{R}^n, \mathbb{V})$, and the following is the main new result of Chapter 4:

Theorem 3.A (Characterization of constant rank operators). *Given $p \in (1, \infty)$, an operator \mathcal{A} as in (3.1.1) has constant rank if and only if there is a constant $C = C(p, \mathcal{A}) > 0$ such that*

$$\|D^l(v - P_{\mathcal{A}}v)\|_{L^p(\mathbb{R}^n)} \leq C \|\mathcal{A}v\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}). \quad (3.1.3)$$

The sufficiency of the constant rank condition for the estimate (3.1.3) is classical and seems to go back to the work of SCHULENBERGER–WILCOX [334], at least for the case $p = 2$, see also [225, 301]. The necessity of this condition, however, had remained unnoticed.

Note that, when $p = 1$ or $p = \infty$, (3.1.3) never holds except in trivial cases: this is recovered from the classical results mentioned above by considering $v = \mathcal{A}^*w$ for a test function w , since in that case $P_{\mathcal{A}}v = 0$, see the proof of the theorem below; here \mathcal{A}^* denotes the formal adjoint of \mathcal{A} . On the other hand, strong type estimates on lower order derivatives in the spirit of (3.1.3) can be proved, see [321], building on [319, 369]. Finally, we remark that the constant rank condition is not necessary for estimates on lower order derivatives, as can be seen from the simple example $\|u\|_{L^\infty} \leq \|\partial_1 \partial_2 u\|_{L^1}$ for $u \in C_c^\infty(\mathbb{R}^2)$.

Theorem 3.A shows that constant rank operators are precisely those operators for which the projection onto their kernel is well-behaved. Another very useful characterisation of constant rank operators had been previously proved by RAIȚĂ [319], who showed that an operator has constant rank if and only if it admits an exact potential operator, see Theorem 4.3.1 below. Using RAIȚĂ's result, we will also see in Chapter 4 that elements in $\ker \mathcal{A}$ have differential structure, a fact that will be extremely useful in Chapters 4 to 7 of the thesis.

3.1.2 Compensated compactness and compensated regularity

As discussed in the preamble to this introduction, we are interested in studying the behaviour of nonlinear functions along weakly convergent sequences which satisfy a linear PDE constraint. We formulate a precise version of the type of question we are interested in:

Question 3.1.1. Are there special quantities $F: \mathbb{V} \rightarrow \mathbb{R}$ which are well-behaved with respect to solutions of the system $\mathcal{A}v = 0$? In particular:

- For solutions of $\mathcal{A}v = 0$, does $F(v)$ benefit from *compensated regularity*?, e.g.

$$v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}) \text{ and } \mathcal{A}v = 0 \quad \implies \quad F(v) \in \mathcal{H}^1(\mathbb{R}^n). \quad (3.1.4)$$

- For solutions of $\mathcal{A}v = 0$, does $F(v)$ benefit from *compensated compactness*?, e.g.

$$v_j \overset{*}{\rightharpoonup} v \text{ in } \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \text{ and } \mathcal{A}v_j = 0 \quad \implies \quad F(v_j) \overset{*}{\rightharpoonup} F(v) \text{ in } \mathcal{D}'(\mathbb{R}^n). \quad (3.1.5)$$

If there are such quantities, how do we characterize and compute them?

It is clear that, for the first part of Question 3.1.1, one should look for *nonlinear* quantities, since otherwise $F(v)$ has precisely the same regularity as v . In (3.1.4), $\mathcal{H}^1(\mathbb{R}^n)$ denotes the real Hardy space, which can be thought of as a proper subspace of $L^1(\mathbb{R}^n)$ whose elements have cancellations at all scales and, therefore, have additional integrability, but see Chapter 2 and the references therein for further details. Being able to identify L^1 -quantities that in fact have Hardy space integrability is often important in PDE: this has been useful in Fluid Dynamics [130, 134] as well as Differential Geometry [190, 297] and we refer the reader to [87, 292] for further examples and references.

For the second part of Question 3.1.1 one is again interested only in nonlinear functions, as otherwise (3.1.5) holds trivially. Functions satisfying (3.1.5) will be referred to as *weakly continuous*, and they can be thought of as representing physical quantities that are robust to errors in measurements induced from small-scale oscillations and concentrations. We call these quantities *null Lagrangians* [28] or \mathcal{A} -quasiaffine functions [98] and they play an important role in the theory.

To address Question 3.1.1 we will assume that

$$\mathcal{A} \text{ has constant rank} \quad \text{and} \quad \text{span } \Lambda_{\mathcal{A}} = \mathbb{V}. \quad (3.1.6)$$

The prototypical example of an operator satisfying these assumptions is $\mathcal{A}(B, E) = (\text{div } B, \text{curl } E)$, where $E, B: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are fields in $L^2(\Omega)$, which we think of as the electric and the magnetic fields respectively. COIFMAN–LIONS–MEYER–SEMMES [87] proved that (3.1.4) holds for the inner product, i.e.

$$\text{div } B = 0, \text{ curl } E = 0 \quad \implies \quad B \cdot E \in \mathcal{H}^1(\mathbb{R}^n). \quad (3.1.7)$$

The implication (3.1.7) was inspired by a surprising and remarkable result of MÜLLER [290] and it can be proved through the COIFMAN–ROCHBERG–WEISS commutator theorem [88], see also [260] for a different approach and [105] for local, non-homogeneous versions. The quantity $E \cdot B$ is also weakly continuous for the system $(\operatorname{div}, \operatorname{curl})$, a fact which goes back to the pioneering work of MURAT and TARTAR [361]: if $d = n^2 + 1$, then (3.1.5) holds, i.e.

$$\left. \begin{array}{l} (B_j, E_j) \rightharpoonup (B, E) \quad \text{in } L^2(\Omega, \mathbb{R}^{2n}) \\ (\operatorname{div} B_j, \operatorname{curl} E_j) \rightarrow (\operatorname{div} B, \operatorname{curl} E) \quad \text{in } H_{\operatorname{loc}}^{-1}(\Omega, \mathbb{R}^d) \end{array} \right\} \implies B_j \cdot E_j \xrightarrow{*} B \cdot E \text{ in } \mathcal{D}'(\Omega). \quad (3.1.8)$$

In light of this example, one is led to the following natural question, which was raised in [87]: is the simultaneous weak continuity and Hardy space integrability of the inner product a general phenomenon, i.e. is it the case that (3.1.4) and (3.1.5) are *equivalent*? The main theorem of Chapter 5 shows that, under the standard assumption (3.1.6), this is indeed the case:

Theorem 3.B (Hardy integrability \Leftrightarrow weak continuity). *Assume (3.1.6) and let $F: \mathbb{V} \rightarrow \mathbb{R}$ be a locally bounded, Borel function that is not affine. Then*

$$v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}) \text{ and } \mathcal{A}v = 0 \quad \implies \quad F(v) \in \mathcal{H}^1(\mathbb{R}^n)$$

holds if and only if

$$v_j \xrightarrow{*} v \text{ in } L_{\operatorname{loc}}^\infty(\mathbb{R}^n, \mathbb{V}) \text{ and } \mathcal{A}v_j = 0 \quad \implies \quad F(v_j) \xrightarrow{*} F(v) \text{ in } \mathcal{D}'(\mathbb{R}^n)$$

and, in that case, we have:

- F is a polynomial of degree $s \leq \min\{n, \dim \mathbb{V}\}$ and it is \mathcal{A} -quasiaffine, i.e. F and $-F$ are both \mathcal{A} -quasiconvex;
- if moreover F is s -homogeneous, there is an estimate

$$\|F(v)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|v\|_{L^s(\mathbb{R}^n)} \quad \text{for all } v \in L^s(\mathbb{R}^n) \text{ such that } \mathcal{A}v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

The class of such polynomials can be computed explicitly by solving an algebraic system of linear equations.

Theorem 3.B shows that compensated compactness and compensated regularity are two facets of the algebraic cancellations in the nonlinearity, which compensate the lack of ellipticity of \mathcal{A} . We also refer the reader to Section 3.1.3 below for the definition of \mathcal{A} -quasiconvexity.

When F is linear, it is possible to make a statement similar to the one in Theorem 3.B, c.f. Theorem 5.3.1, although we show that there is no estimate in that case. We would also like to highlight that we provide an effective way of computing the \mathcal{A} -quasiaffine functions. MURAT [301] derived the algebraic identity (5.2.2) that characterizes these functions but, as he was already aware, it is in general not feasible to decide which nonlinear polynomials, if any, satisfy this identity. In order to deal with this issue, we crucially rely on the work of BALL–CURRIE–OLVER [26]. We deduce that all \mathcal{A} -quasiaffine functions can be written as coefficients of differential forms, which answers in the positive a question of ROBBIN–ROGERS–TEMPLE [330, §5] under the assumption (3.1.6).

In the setup of Theorem 3.B, it is natural to wonder whether the convergence in (3.1.5) can be improved. TARTAR [367, Lemma 7.3] showed that one cannot upgrade weak- $*$ convergence in measures to weak convergence in L^1 , i.e. one cannot test the convergence against L^∞ functions. However, as a by product of Theorem 3.B, one can test the convergence against functions in $\operatorname{VMO}(\mathbb{R}^n)$; this a space which is neither contained nor contains $L^\infty(\mathbb{R}^n)$:

Theorem 3.C (Improved and quantified convergence). *As before assume (3.1.6) holds and let $F: \mathbb{V} \rightarrow \mathbb{R}$ be \mathcal{A} -quasiaffine and s -homogeneous for some $s \geq 2$. Then*

$$v_j \rightharpoonup v \text{ in } L^s(\mathbb{R}^n, \mathbb{V}) \text{ and } \mathcal{A}v_j = 0 \quad \implies \quad F(v_j) \xrightarrow{*} F(v) \text{ in } \mathcal{H}^1(\mathbb{R}^n). \quad (3.1.9)$$

Moreover, let $p \in (s-1, \infty)$ and $q \in (1, \infty)$ be such that $\frac{s-1}{p} + \frac{1}{q} = 1$. For \mathcal{A} -free fields $v_1, v_2 \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$ and any $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have the uniform estimate

$$\left| \int_{\mathbb{R}^n} \varphi (F(v_1) - F(v_2)) \, dx \right| \leq C \|v_1 - v_2\|_{\dot{W}^{-1,q}} (\|v_1\|_{L^p} + \|v_2\|_{L^p})^{s-1} \|\operatorname{D}\varphi\|_{L^\infty}.$$

The last part of Theorem 3.C generalizes the quantitative statements in the $\mathcal{A} = \text{curl}$ case of [64] and [210, §8], see also [145, 211, 213]. It shows that, under weaker integrability hypothesis, *distributional* \mathcal{A} -quasiaffine quantities are still weakly continuous, c.f. [184].

We conclude this section by discussing the standard assumption (3.1.6). The spanning assumption is natural since weakly continuous quantities are completely unconstrained along directions not in $\text{span } \Lambda_{\mathcal{A}}$. The constant rank assumption is, by Theorem 3.A, equivalent to an L^p -estimate which plays a crucial role in studying weak continuity. In fact, very little is known about weak continuity in the non-constant rank case, an important exception being the cases of wave operators, to which we will return in Section 3.3, and of the separate convexity operator [294, 296, 310, 365], which was proposed by TARTAR as a toy model to study rank-one convexity. Apart from these isolated examples, there is no general theory in the non-constant rank case, except when F is quadratic: for quadratic nonlinearities there is a satisfactory theory for weak continuity [361] as well as for Hardy space integrability [87, 261]. Surprisingly, as we will see at the end of Section 3.3, the weak continuity problem is fundamentally different in the constant and non-constant rank cases.

3.1.3 \mathcal{A} -quasiconvexity and sequential weak lower semicontinuity

In the previous section we presented a complete characterization of the nonlinear functions which are sequentially weakly continuous with respect to a constant rank constraint. However, in view of applications to the Calculus of Variations, it is also interesting to consider the class of functions which are sequentially weakly *lower semicontinuous*; this is, essentially, the class of \mathcal{A} -quasiconvex functions. Recall that a locally bounded, Borel measurable function $F: \mathbb{V} \rightarrow \mathbb{R}$ is \mathcal{A} -quasiconvex if, for all $z \in \mathbb{V}$,

$$F(z) \leq \int_{\mathbb{T}^n} F(z + v(x)) \, dx, \quad \text{for all } v \in C^\infty(\mathbb{T}^n, \mathbb{V}) \text{ such that } \int_{\mathbb{T}^n} v \, dx = 0 \text{ and } \mathcal{A}v = 0.$$

In Theorems 3.B and 3.C we dealt with \mathcal{A} -quasiaffine functions, which are necessarily polynomials, and hence do not have a fixed sign. It is thus natural to attempt to establish a weak lower semicontinuity result for *general* \mathcal{A} -quasiconvex integrands without a sign. We note that this case is not covered by the influential work of FONSECA–MÜLLER [149] (see also [144]), where only positive integrands are studied. In fact, when the integrand changes sign one needs to deal with the possibility of concentrations of the sequence on the boundary of the domain. When this happens, weak lower semicontinuity breaks down: such is the case already when $\mathcal{A} = \text{curl}$, as an example due to TARTAR shows [27]. It follows that the convergence should be tested against functions which vanish on the boundary of the domain. In Chapter 5, we prove the following result:

Theorem 3.D (Weak lower semicontinuity). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1, \infty)$, and let $F: \mathbb{V} \rightarrow \mathbb{R}$ be an \mathcal{A} -quasiconvex function such that, for all $v \in \mathbb{V}$, $|F(v)| \leq C(|v|^p + 1)$. As before assume (3.1.6). Then, for all $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$,*

$$\left. \begin{array}{l} v_j \rightharpoonup v \quad \text{in } L^p(\Omega, \mathbb{V}) \\ \mathcal{A}v_j \rightarrow \mathcal{A}v \quad \text{in } W_{\text{loc}}^{-l,p}(\Omega, \mathbb{V}) \end{array} \right\} \implies \liminf_{j \rightarrow \infty} \int_{\Omega} \varphi F(v_j) \, dx \geq \int_{\Omega} \varphi F(v) \, dx.$$

This result is sharp in the sense that φ cannot be taken to be in $C^\infty(\overline{\Omega})$, nor even $\varphi \equiv 1$.

The methods used to prove Theorem 3.D are distinct than the ones from [149], where the general case of Carathéodory integrands is addressed. Nonetheless, in Chapter 6, we will revisit the methods of [149], which rely on Young measures.

Young measures are a very classical yet powerful tool to describe the effective limits of energy functionals. In their original formulation in [373], but see also [24, 31, 47, 83, 295], Young measures were used to describe oscillation phenomena in the Calculus of Variations. They have been subsequently used in the study of PDEs modeling numerous problems in Continuum Mechanics [32, 119, 361] and some of their extensions, such as the DiPerna–Majda measures [121], were developed to also account for concentration effects. The notion of Young measures that we use is closely related to that of ALIBERT–BOUCHITTÉ [4], see also [252, 357].

In Chapter 6 we will prove further refinements of aspects of the results of FONSECA–MÜLLER [149], including a variant of Theorem 3.D. We will also present a very precise characterization of L^p -Young measures, for $1 < p < \infty$, which are generated by sequences satisfying a constant rank constraint: the characterization is by duality with Jensen inequalities for \mathcal{A} -quasiconvex functions. Our analysis relies on a refined decomposition lemma which separates the gradient structure of oscillation and concentration effects. This result is analogous to that of [150], where varifolds are used, in the case of weak convergence of gradients. A result related to ours, pertaining to DiPerna–Majda measures, was proved in [144]. Such characterizations for (oscillation) Young measures were first proved in [149, 227, 228].

3.1.4 \mathcal{A} -quasiconvexity and other semiconvexity notions

\mathcal{A} -quasiconvexity is a *necessary* condition for sequential weak lower semicontinuity and, as shown for instance by Theorem 3.D, under natural assumptions it is also a *sufficient* condition. Thus, in view of the Direct Method, \mathcal{A} -quasiconvexity is the fundamental mathematical assumption on the integrands in the Calculus of Variations. The important case where $\mathcal{A} = \text{curl}$ [28, 97, 295] will be discussed in-depth in Part II of this thesis, but here we note that, even in this classical setting, the concept of quasiconvexity is still poorly understood. In particular, it is not known whether rank-one convex functions defined on $\mathbb{R}^{2 \times N}$, for $N \geq 2$, are quasiconvex. This last question is known as Morrey’s problem and it remains an outstandingly difficult problem.

In the curl-free case, quasilinear functions play an important role in the study of quasiconvexity, for instance through the notion of polyconvexity; we note, however, that in the general \mathcal{A} -free setting, there are several distinct competing notions of polyconvexity, c.f. Section 5.2. The case of quasilinear functions in the general \mathcal{A} -free setup is interesting in relation to Morrey’s problem: Morrey’s problem was solved, in sufficiently high dimensions, much earlier for higher order gradients [26] than for first order gradients [351]. Furthermore, the example in [351] has many similarities with an older example of TARTAR [361] of a $\Lambda_{\mathcal{A}}$ -affine function which is not \mathcal{A} -quasilinear, where $\mathcal{A}u = (\partial_1 u_1, \partial_2 u_2, (\partial_1 + \partial_2)u_3)$ for $u: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Therefore, even from the perspective of the classical Calculus of Variations, it is interesting to study weak continuity and lower semicontinuity in a larger class of operators [97].

3.2 Quadratic homogenization under constant rank constraints

The basic aim of homogenization theory is to describe the macroscopic behaviour of a system which is heterogeneous at a small scale. From a mathematical perspective, the problems addressed by the theory of homogenization amount to computing, or determining in a way as explicit as possible, the weak limits of nonlinear quantities; thus, unsurprisingly, the theory of compensated compactness, as described in the previous section, plays an important role in homogenization.

In Chapter 7 of this thesis we will focus on the homogenization theory of composite periodic materials in the simple case of convex, quadratic homogenization. The methods we use extend by standard means to more general situations, see e.g. [60], but we focus on this basic setting since we are mostly interested in discussing the G-closure problem.

The G-closure problem is, loosely speaking, the following question: consider a fixed set \mathbb{M} of phases, which can be finely mixed to obtain a composite material; what is the range of effective properties of the composites that can be obtained from \mathbb{M} ? Determining this range of properties amounts to calculating the G-closure of \mathbb{M} , which we denote by $\mathbf{G}(\mathbb{M})$. There is a deep connection between the G-closure and quasiconvexification:

Theorem 3.2.1. *As before, assume (3.1.6). Consider a compact subset \mathbb{M} of the space of symmetric positive-definite operators on \mathbb{V} and let $F: \mathbb{V} \rightarrow \mathbb{R}$ be defined by*

$$F(v) \equiv \min_{S \in \mathbb{M}} \langle Sv, v \rangle.$$

The \mathcal{A} -quasiconvex envelope of F , i.e. the largest quasiconvex function below F , is the function

$$v \mapsto \min_{S \in \mathbf{G}(\mathbb{M})} \langle Sv, v \rangle.$$

The above theorem is well-known, see e.g. [283], and the reader can find a proof, as well as a more complete statement, in Theorem 7.3.5 below.

In general, calculating the G-closure of a set is extremely difficult. A special non-generic case where calculations become easier is when $G(\mathbb{M})$ has empty interior; in that case, we say that \mathbb{M} is an *exact relation*. The theory of exact relations is by now well-developed due to the work of GRABOVSKY, MILTON and collaborators, see e.g. the book [167], and it combines in an elegant way analysis, geometry and algebra. In Chapter 7 we present an overview of this theory, explaining how to calculate exact relations in practice. The results in this chapter cannot be considered new, although there is some novelty in the presentation. Our aim is to give a concise, self-contained introduction to the theory of exact relations, and to explain in some detail the recent results of GRABOVSKY [168, 169]. These will also be discussed in more detail in Part II, in particular in Chapter 9.

3.3 Homogenization for the Einstein equations under symmetry

In this section we will discuss a homogenization problem arising in the theory of General Relativity. As we will see below, many of the tools discussed in the previous sections will be extremely useful.

Fix a four-dimensional manifold \mathcal{M} , and consider a sequence $(g_\varepsilon)_\varepsilon$ of Lorentzian metrics on \mathcal{M} which solve the Einstein vacuum equations, i.e.

$$\mathbf{Ric}(g_\varepsilon) = 0. \quad (3.3.1)$$

Intuitively, as ε decreases, the sequence g_ε provides an increasingly larger scale picture of spacetime \mathcal{M} . The natural homogenization question is what kind of space-time is obtained in the limit. It is easy to see that, in general, the effective Ricci tensor will not be zero since, schematically,

$$\mathbf{Ric}(g) = g \partial^2 g + \partial g \partial g.$$

Writing the most well-known Einstein equation as

$$[\mathbf{Ric}(g)]_{\alpha\beta} = 8\pi \left(\mathbb{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \operatorname{tr}_g \mathbb{T} \right), \quad (3.3.2)$$

where \mathbb{T} denotes the *energy-momentum tensor*, we are tempted to identify the Ricci tensor obtained in the limit as *matter*. However, this is only the case if our candidate matter verifies the second Einstein equation,

$$\nabla^\alpha \mathbb{T}_{\alpha\beta} = 0, \quad (3.3.3)$$

where ∇ is the covariant derivative with respect to g ; in this section, as well as in Chapter 8, we also use throughout the Einstein summation convention. In other words, for (\mathcal{M}, g) to generate a solution to Einstein's equations in General Relativity and thus to be considered an effective space-time, we must both compute $\mathbf{Ric}(g)$ and show that it verifies the propagation equation (3.3.3). We note that (3.3.3) can also be seen as a compatibility condition, and it follows from the Bianchi identities.

In [73], BURNETT conjectured that, under suitable assumptions, the effective energy-momentum tensor is of massless Vlasov type; mathematically, this means that lack of compactness in solutions to the Einstein vacuum equations (3.3.1) manifests itself as massless matter which is propagated without collisions along the null directions of spacetime:

Conjecture 3.3.1 (Burnett). Let (g_ε) be a sequence of smooth solutions to (3.3.1) and suppose that there is another Lorentzian metric g such that, for every compact set $K \subset \mathcal{M}$, there is a sequence $\lambda_\varepsilon \searrow 0$ with

$$\|\partial^k (g_\varepsilon - g)\|_{L^\infty(K)} \lesssim_K \lambda_\varepsilon^{1-k}, \quad k = 0, 1, 2. \quad (3.3.4)$$

Then g is a solution to the Einstein–massless Vlasov system.

Conjecture 3.3.1 suggests that the Einstein–massless Vlasov system provides an effective model for a scenario in which high-frequency but small amplitude gravitational waves propagate on a fixed background; such a scenario was studied earlier in the physics literature [66, 85, 203, 202, 268]. BURNETT

also conjectured that all Einstein–massless Vlasov systems can be realized as weak limits of vacuum spacetimes, see [198, 199] for some progress in that direction. GREEN–WALD [171] generalized BURNETT’s framework to sequences of non-vacuum spacetimes; several examples of sequences of vacuum and non-vacuum spacetimes satisfying (3.3.4) are given in [172, 195, 358, 359].

In this thesis we will only address Conjecture 3.3.1 under the assumption that all the metrics involved have $U(1)$ -symmetry, a condition which is preserved under weak limits. To be specific, we assume that $\mathcal{M} = \mathcal{M} \times \mathbb{R}$ and $\mathcal{M} = (0, T) \times \mathbb{R}^2$, for some fixed $T > 0$, and we further assume that \mathcal{M} is covered by global coordinates $(t \equiv x^0, x^1, x^2)$, which we denote with greek indices, and \mathcal{M} by coordinates (x^0, x^1, x^2, x^3) . We take \mathbf{g} to be a Lorentzian metric with $U(1)$ -symmetry, i.e.

$$\mathbf{g} \equiv e^{-2\psi} g + e^{2\psi} \left(dx^3 + \mathfrak{A}_\alpha dx^\alpha \right)^2, \quad (3.3.5)$$

where g is a Lorentzian metric on \mathcal{M} and ψ and $\mathfrak{A} \equiv \mathfrak{A}_\alpha dx^\alpha$ are, respectively, a real-valued function and a 1-form on \mathcal{M} .

In the $U(1)$ -setting, a recent breakthrough by HUNEAU–LUK [200] provided a proof Conjecture 3.3.1, albeit under additional assumptions: they require that (3.3.4) holds up to and including $k = 4$ and that the metrics \mathbf{g}_ε and \mathbf{g} satisfy some gauge assumptions, see Remark 3.3.3. In Chapter 8 we analyze more closely the mechanisms behind their proof, isolating the special structures that allow for the cancellation phenomena to take place. By doing so, we are able to obtain stronger, more general statements, with simpler proofs. In particular, we prove:

Theorem 3.E (Burnett’s conjecture in $U(1)$ -symmetry). *Let (\mathbf{g}_ε) be a sequence of $U(1)$ -symmetric Lorentzian metrics in elliptic gauge which satisfy (3.3.4). Then \mathbf{g} is a $U(1)$ -symmetric solution of the Einstein–massless Vlasov system.*

We refer the reader to [177] for a precise statement of the result. Concerning the proof of Theorem 3.E, and since we assume that \mathcal{M} has trivial topology, one can check that

$$[\mathbf{Ric}(\mathbf{g})]_{\alpha 3} = 0 \quad \implies \quad d\mathfrak{A} = e^{-4\psi} \star_{\mathbf{g}} d\omega,$$

for a real-valued scalar field ω on \mathcal{M} . It turns out that, under $U(1)$ -symmetry, Einstein’s vacuum equations are *equivalent* to the following Einstein-wave map system:

$$\begin{cases} \square_{\mathbf{g}} \psi + \frac{1}{2} e^{-4\psi} g^{-1}(d\omega, d\omega) = 0, \\ \square_{\mathbf{g}} \omega - 4g^{-1}(d\psi, d\omega) = 0, \\ [\mathbf{Ric}(\mathbf{g})]_{\alpha\beta} = 2\partial_\alpha \psi \partial_\beta \psi + \frac{1}{2} e^{-4\psi} \partial_\alpha \omega \partial_\beta \omega. \end{cases} \quad (3.3.6)$$

We again refer the reader to [177] for further details.

A simple but important observation concerning the proof of Theorem 3.E is that the assumption of elliptic gauge is sufficient to lead to a decoupling of the wave map and Einstein parts of (3.3.6) (respectively the first two and the last equations), in practice rendering this *quasilinear* problem into a *semilinear* one. In fact, our approach handles *any* wave map from a Lorentzian domain $(\mathcal{M}, g_\varepsilon)$ onto a target Riemannian domain $(\mathcal{N}, \mathfrak{g})$:

$$\square_{g_\varepsilon} u_\varepsilon^I + \Gamma_{JK}^I(u_\varepsilon) g_\varepsilon^{-1}(du_\varepsilon^J, du_\varepsilon^K) = 0, \quad u_\varepsilon^I, f_\varepsilon^I : \mathcal{M} \rightarrow \mathbb{R}, \quad I, J, K \in \{1, \dots, N\}, \quad (3.3.7)$$

where Γ_{JK}^I are the Christoffel symbols of \mathfrak{g} . We refer the reader to [337] for an introduction to wave maps and their well-posedness theory. For our purposes here it suffices to recall that (3.3.7) is the Euler–Lagrange equation associated with the action

$$\mathcal{L}[u] = \int_{\mathcal{M}} g^{\alpha\beta} \mathbb{L}_{\alpha\beta}[u] d\text{Vol}_g, \quad \text{where } \mathbb{L}_{\alpha\beta}[u] \equiv \mathfrak{g}_{IJ}(u) \partial_\alpha u^I \partial_\beta u^J. \quad (3.3.8)$$

Note that the first two equations in (3.3.6) are a particular case of (3.3.7) when $N = 2$, and \mathcal{N} is the Poincaré plane equipped with $\mathbf{g} = 2 \, d\psi \otimes d\psi + \frac{1}{2} e^{-4\psi} \, d\omega \otimes d\omega$, while the last equation in (3.3.2) is

$$[\text{Ric}(g)]_{\alpha\beta} = \mathbb{L}_{\alpha\beta}[(\psi, \omega)].$$

Under appropriate assumptions, the system (3.3.7) is amenable to the methods of Compensated Compactness. Indeed, in the simple case where $g_\varepsilon = g$ is a *fixed* metric and u_ε is bounded in $W_{\text{loc}}^{1,4}$, it follows that $\square_g u_\varepsilon$ is bounded in L^2 . Moreover, a version of the div-curl lemma yields

$$\left. \begin{array}{l} u_\varepsilon \rightharpoonup u \text{ in } W^{1,2} \\ \square_g u_\varepsilon \text{ is bounded in } L^2 \end{array} \right\} \implies g^{-1}(du_\varepsilon, du_\varepsilon) \rightharpoonup g^{-1}(du, du) \text{ in } \mathcal{D}'$$

so that the equations (3.3.7) will be preserved in the limit. Similarly, a trilinear Compensated Compactness result, which will be discussed below, plays an important role in the proof of Theorem 3.E.

Let us now state the main result concerning the general wave map system (3.3.7). For simplicity, we take $\mathcal{M} \subset \mathbb{R}^{1+n}$ and $\mathcal{N} \subset \mathbb{R}^N$ to be domains; however, in light of the assumptions ensuing, this restriction is without loss of generality. Indeed, we will assume:

Hypotheses 3.3.2. Let $\mathcal{U}_\varepsilon \equiv (g_\varepsilon, (u_\varepsilon^I)_{I=1}^N)$ and $\mathcal{U} \equiv (g, (u^I)_{I=1}^N)$ satisfy:

- (a) the eigenvalues of g_ε are uniformly bounded from above and away from zero and g is a smooth metric such that $g_\varepsilon \rightarrow g$ entrywise in C_{loc}^0 ; furthermore, one of the following two conditions holds:

(a₁) $g_\varepsilon \rightarrow g$ in $W_{\text{loc}}^{1,\infty}$, or

(a₂) g_ε is bounded in $W_{\text{loc}}^{1,\infty}$, $\partial_0(g_\varepsilon)_{ij} \rightarrow \partial_0(g_\varepsilon)_{ij}$ strongly in L_{loc}^4 , $\delta^{ij} \partial_{ij} g_\varepsilon^{\alpha\beta}$ is bounded in L_{loc}^2 ;

- (b) $u_\varepsilon^I \rightharpoonup u^I$ in $W_{\text{loc}}^{1,4}$ and $\|g_\varepsilon^{\alpha\beta} - g^{\alpha\beta}\|_{L^\infty(K)} \|\partial^2(u_\varepsilon^I - u^I)\|_{L^p(K)} \lesssim_K 1$ for every compact $K \subset \mathcal{M}$.

Remark 3.3.3. We note that the boundedness of second derivatives of the metric components and the strong convergence of $\partial_0 \tilde{g}_\varepsilon^{ij}$ is ensured if g is in elliptic gauge, see [177] for further details.

The convergence of \mathcal{U}_ε assumed in Hypotheses 3.3.2 is strong enough to ensure (g, u^1, \dots, u^N) is itself a wave map. This is a substantially more difficult result under weaker hypotheses, see for instance [22, 156, 157, 160] for several examples of oscillation and concentration effects in semilinear wave equations in lower regularity, albeit in a setting where $g_\varepsilon = g$ is the Minkowski metric. On the other hand, Hypotheses 3.3.2 are weak enough that general quadratic quantities in the solutions, such as the Lagrangian density $\mathbb{L}_{\alpha\beta}[u_\varepsilon]$, are not continuous. The last condition in Hypothesis 3.3.2(b) is imposed by analogy with Conjecture 3.3.1. We then have:

Theorem 3.F (Oscillations in wave maps). *Let Γ_{JK}^I be continuous Christoffel symbols arising from a Riemannian metric*

$$\mathbf{g} = \mathbf{g}_{IJ}(y) \, dy^I \otimes dy^J.$$

Let \mathcal{U}_ε be a sequence of solutions to (3.3.7) satisfying Hypotheses 3.3.2 with $u_\varepsilon^I \rightarrow u^I$ in C_{loc}^0 . There exists a Radon measure ν on the cosphere bundle $S^\mathcal{M}$ such that:*

- (a) **Limit equation.** \mathcal{U} is a distributional solution of (3.3.7) and its Lagrangian energy density satisfies, for all $Y \in C_0^\infty(\mathcal{M})$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} \mathbb{L}_{\alpha\beta}[u_\varepsilon] Y^\alpha Y^\beta \, d\text{Vol}_{g_\varepsilon} = \int_{\mathcal{M}} \mathbb{L}_{\alpha\beta}[u] Y^\alpha Y^\beta \, d\text{Vol}_g + \int_{S^*\mathcal{M}} \xi_\alpha \xi_\beta Y^\alpha Y^\beta \sqrt{|g|} \, d\nu.$$

- (b) **Energy density.** *The measure ν is a (radially averaged) measure-valued solution of a massless Vlasov equation with respect to g :*

- (b₁) Support property: ν is supported on the zero mass shell of g , i.e. for all $\varphi \in C_0^\infty(\mathcal{M})$

$$\int_{S^*\mathcal{M}} \varphi(x) g^{\alpha\beta} \xi_\alpha \xi_\beta \, d\nu = 0.$$

(b₂) Propagation property: for all $\tilde{a} \in C_0^\infty(S^*\mathcal{M})$, extended as a positively 1-homogeneous function to $T^*\mathcal{M} \setminus \{0\}$, the measure ν satisfies

$$\int_{S^*\mathcal{M}} \left[g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\mu} g^{\alpha\beta} \xi_\alpha \xi_\beta \partial_{\xi_\mu} \tilde{a} \right] d\nu = 0. \quad (3.3.9)$$

Remark 3.3.4. The transport equation in part (b₂) of Theorem 3.F naturally inherits a suitable set of initial conditions in terms of initial conditions for (3.3.7), see [364, Section 3.4] as well as [154] for a detailed study of the initial value formulation for the linear wave equation. In other words, the compactness singularities seen in the evolution may be characterized in terms of the compactness singularities of the initial data.

As we mentioned above, Theorem 3.E follows readily from Theorem 3.F. Strictly speaking, in Theorem 3.F one may need to pass to a subsequence in \mathcal{U}_ε , which we do not relabel. Likewise, throughout Chapter 8 we work modulo subsequences.

Let us now sketch the proof of Theorem 3.F. We rely crucially on the theory of H-measures, developed independently by GÉRARD [159] and TARTAR [364] and summarized here in Section 8.2. Since $\mathbb{L}_{\alpha\beta}$ is quadratic, H-measures provide a way of calculating the discrepancy between $\mathbb{L}_{\alpha\beta}[u]$ and $\lim_{\varepsilon \rightarrow 0} \mathbb{L}_{\alpha\beta}[u_\varepsilon]$; moreover, unlike other popular tools to study weak limits such as Young measures (see [327] for a comparison), H-measures capture the *propagation of compactness singularities*, as in Theorem 3.F.

For flat targets and a fixed Lorentzian metric, a full characterization of the H-measure associated to the linear wave equation is already essentially contained in TARTAR's original paper [364], as well as in [154]. The case of flat targets but with *oscillating* metrics g_ε is, as predicted by FRANCFORT–MURAT [154], much more involved. An obvious difficulty is that it is not clear what is the appropriate notion of convergence for the metrics: if the convergence of g_ε is too weak, oscillations in its derivatives may drastically affect propagation of compactness singularities. Though this is an interesting question, we do not investigate it here: Hypotheses 3.3.2 provide sets of convergence conditions under which the oscillations of g_ε *do not* contribute to the propagation of non-compactness. With stronger conditions on the rates of convergence, as mentioned above, this remarkable fact is one of the key observations of HUNEAU–LUK [200], and it served as inspiration for Theorem 3.E.

Let us briefly explain our strategy to deal with the oscillations of g_ε . If Hypotheses 3.3.2 hold, we can use the parity of the H-measure to reduce the contributions of the metric oscillations to the commutators

$$\partial_\alpha u_\varepsilon [g_\varepsilon^{\alpha\beta} - g^{\alpha\beta}, A] \partial_\beta e_0 u_\varepsilon, \quad e_0 \equiv \partial_0 + \frac{g^{0i}}{g^{00}} \partial_i \quad (3.3.10)$$

where A is the 0th order pseudodifferential operator associated to the test function \tilde{a} in (3.3.9) and the upper indices denote components of the inverse metrics; we drop the superscripts on u_ε for simplicity. We recall that here, and throughout Chapter 8, we use the Einstein summation convention with roman indices in $\{1, \dots, n\}$ and greek indices starting at 0. By the Calderón commutator estimate, if $g_\varepsilon \rightarrow g$ strongly in $W_{\text{loc}}^{1,\infty}$, i.e. when Hypothesis 3.3.2(a₁) holds, (3.3.10) vanishes in the limit. However, even if all derivatives but one converge strongly, this simple proof fails, as the Calderón commutator estimate requires Lipschitz bounds. This is the case in Hypothesis 3.3.2(a₂): the assumptions imply that spatial derivatives of g_ε converge strongly, with the convergence of $\partial_0 g_\varepsilon$ holding only weakly. We examine the failure of compactness in $e_0 g_\varepsilon$ in Fourier space, and we denote by Λ the region where the symbol of e_0 vanishes. This naturally induces a partition of Fourier space:

- *Low frequencies.* In bounded regions of frequency space, $W_{\text{loc}}^{1,2}$ and L_{loc}^2 norms are comparable, hence $e_0 g_\varepsilon$ is, in fact, *compact* in this range.
- *High frequencies away from Λ .* Noting that e_0 is invertible, we write $g_\varepsilon - g = e_0 e_0^{-1} (g_\varepsilon - g)$. The trilinear form of (3.3.10) is then key: integrating by parts brings the extra e_0 derivative onto u_ε and, relying on the parity of the H-measure and the structure of the commutator, we can use it to fashion $\square_g u_\varepsilon$ out of the second derivatives on u_ε which appear. Then, *lack of compactness* of $e_0 g_\varepsilon$ is *compensated* by noting the following differential condition on u_ε : unlike general second order derivatives of u_ε , $\square_g u_\varepsilon$ is uniformly bounded in L_{loc}^p , see Hypotheses 3.3.2(b).

- *High frequencies close to Λ .* In this region, e_0 is no longer invertible, so the fact that u_ε appear in the commutator does not help. We instead *compensate* for the *lack of compactness* in $e_0 g_\varepsilon$ by imposing a differential condition on g_ε : we require the spatial laplacians of their entries to be bounded in L^2_{loc} , see Hypotheses 3.3.2(a₂).

It is natural to ask whether $W^{1,\infty}_{\text{loc}}$ -bounds on g_ε in can be weakened to $W^{1,q}_{\text{loc}}$ -bounds, for some $q < \infty$. This would affect the expected regularity of g , which would drop below C^1 . Such a level of regularity seems problematic: indeed, the integrand in the left-hand side of (3.3.9) is the Poisson bracket between the symbol of \square_g and \tilde{a} , which in turn is the symbol of a commutator between the corresponding pseudo-differential operators; such a commutator ought to be at least bounded, c.f. Remark 8.3.4.

Finally, for the case of non-flat targets, the proof follows from standard Compensated Compactness results due to MURAT and TARTAR [300, 366] (see also [330] and Chapter 5). The main additional difficulty in this case is that the propagation of compactness singularities for each component u^I is coupled to that of u^J for $I \neq J$. Such a coupling is essentially due to the fact that the implication

$$\left. \begin{array}{l} u_\varepsilon \rightharpoonup u \text{ in } W^{1,3}_{\text{loc}} \\ \square_g u_\varepsilon \text{ is compact in } W^{-1,3}_{\text{loc}} \end{array} \right\} \implies \partial u_\varepsilon g^{-1}(du_\varepsilon, du_\varepsilon) \overset{*}{\rightharpoonup} \partial u g^{-1}(du, du) \text{ in } \mathcal{D}'$$

holds *if and only if* $u = 0$! Here ∂ denotes an arbitrary derivative. In other words, the trilinear quantity $F(du) = \partial u g^{-1}(du, du)$ is weakly continuous exactly at zero. That such quantities even exist is only possible because \square_g , thought of as a first-order operator acting on ∂u , does not have constant rank, c.f. Chapter 5 and Remark 8.3.7. The upshot is that compactness singularities in general quadratic quantities associated to wave maps do not admit a simple characterization. Nonetheless, due to the variational structure of (3.3.7), the propagation of compactness singularities in the Lagrangian density does.

Chapter 4

Linear operators with constant rank

The purpose of this chapter is to develop the general theory of constant rank operators. The results here were obtained in collaboration with BOGDAN RAIȚĂ and can be found in [182, 183]. After giving some examples in Section 4.1, we will prove in Section 4.2 a new characterisation of constant rank operators. Section 4.3 concerns the existence and properties of potential operators, while Section 4.4 uses these potentials to establish very useful properties of Lebesgue-type spaces associated with a constant rank operator, including a Helmholtz decomposition. Finally, in Section 4.5 we gather a few elementary definitions from the non-convex Calculus of Variations.

4.1 Examples

In this section we gather a few examples of operators which satisfy (3.1.6).

- Example 4.1.1.** (i) Unconstrained fields: if $\mathcal{A} = 0$ then $\Lambda_{\mathcal{A}} = \mathbb{V}$ and \mathcal{A} -quasiconvexity is just ordinary convexity.
- (ii) Curl-free fields: if $\mathcal{A} = \text{curl}$ then \mathcal{A} -free vector fields have a potential over simply connected domains, i.e. they can be written as the gradient of some other vector field. One can also consider other variants $\tilde{\mathcal{A}}$ of the curl so that $\tilde{\mathcal{A}}$ -free fields correspond to k -th order gradients; see [26] or [149] for details.
- (iii) Divergence-free fields: the constraint $\mathcal{A} = \text{div}$ appears, for instance, in Fluid Dynamics, where the velocity field of an incompressible fluid is divergence-free.
- (iv) Linear elasticity: in this case one studies integrands which depend only on the symmetric gradient $\mathcal{E}(u) \equiv \frac{1}{2}(Du + (Du)^T)$ of the displacement $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$. A sufficiently regular vector field $v: \Omega \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$ is a symmetric gradient if and only if it is (curl curl)-free, where

$$(\text{curl curl } v)_{i,j,k,l} \equiv \partial_{kl}^2 v_{ij} + \partial_{ij}^2 v_{kl} - \partial_{jk}^2 v_{il} - \partial_{il}^2 v_{jk}$$

is the Saint-Venant compatibility operator.

- (v) Coupling of constraints: by combining several admissible constraints one obtains a new operator. For instance, by coupling (iii) and (ii) we have the equations of Electrostatics:

$$\text{div } B = 0, \quad \text{curl } E = 0.$$

If furthermore we couple these equations with (iv) we have the system of piezoelectricity. See [283] for more examples.

Important examples where the constant rank assumption does not hold are the operator associated to the incompressible Euler equations [112, 113, 357], the operator associated to separate convexity [294, 365] and the wave operator, c.f. Chapter 8. An example related to separate convexity can be found in [296].

4.2 An L^p -estimate and the necessity of constant rank

The purpose of this section is to prove Theorem 3.A, which we restate here:

Theorem 4.2.1. *Let $p \in (1, \infty)$. An operator \mathcal{A} as in (3.1.1) has constant rank if and only if*

$$\|D^l(v - P_{\mathcal{A}}v)\|_{L^p(\mathbb{R}^n)} \leq C_p \|\mathcal{A}v\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{V}). \quad (4.2.1)$$

Recall that, for $A \in \text{Lin}(\mathbb{V}, \mathbb{W})$, the *Moore–Penrose generalized inverse* of A , sometimes called the *pseudoinverse*, is the unique $A^\dagger \in \text{Lin}(\mathbb{W}, \mathbb{V})$ such that $AA^\dagger = \text{Proj}_{\text{im } A}$ and $A^\dagger A = \text{Proj}_{\text{im } A^*}$. Equivalently, we may define

$$A^\dagger \equiv \left(A|_{(\ker A)^\perp} \right)^{-1} \text{Proj}_{\text{im } A}.$$

We refer the reader to [77] for these and numerous other properties of generalized inverses.

The proof of the theorem is based on two observations, that we record as separate lemmas.

Lemma 4.2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $A: \Omega \rightarrow \text{Lin}(\mathbb{V}, \mathbb{W})$ be smooth. Then $A^\dagger: \Omega \rightarrow \text{Lin}(\mathbb{W}, \mathbb{V})$ is locally bounded if and only if $\text{rank } A$ is constant in Ω . In that case, A^\dagger is also smooth.*

Proof: Let $|\cdot|$ be the operator norm on $\text{Lin}(\mathbb{V}, \mathbb{W})$. We have that, for $\xi_1, \xi_2 \in \Omega$,

$$\text{rank}(A(\xi_1)) > \text{rank}(A(\xi_2)) \quad \implies \quad |A^\dagger(\xi_1)| \geq \frac{1}{|A(\xi_1) - A(\xi_2)|}. \quad (4.2.2)$$

Indeed, if the hypothesis holds then there exists $v \in \ker A(\xi_2) \cap (\ker A(\xi_1))^\perp$ with $|v| = 1$. Thus $A^\dagger(\xi_1)(A(\xi_1) - A(\xi_2))v = A^\dagger(\xi_1)A(\xi_1)v = v$ and so

$$1 \leq |A^\dagger(\xi_1)(A(\xi_1) - A(\xi_2))| \leq |A^\dagger(\xi_1)||A(\xi_1) - A(\xi_2)|.$$

Now suppose that $\text{rank } A$ is not constant, so we can pick a point $\xi_0 \in \Omega$ and a sequence $\xi_n \rightarrow \xi_0$ such that $\text{rank}(A(\xi_n)) \neq \text{rank}(A(\xi_0))$. It follows from (4.2.2) that A^\dagger is not bounded near ξ_0 .

Conversely, assuming that $\text{rank } A$ is constant, A^\dagger is smooth. Indeed, and as in [319], this is easily deduced from DECELL's formula [116]

$$A^\dagger = -\frac{1}{a_r} A^* \left(\sum_{i=1}^r a_{i-1} (AA^*)^{r-i} \right),$$

where $r = \text{rank } A$, $d = \dim \mathbb{W}$ and $p(\lambda) = (-1)^d \sum_{j=0}^d a_j \lambda^{d-j}$ is the characteristic polynomial of A ; note that $a_j = 0$ for $j > r$ and $a_r \neq 0$ away from zero. Since the coefficients a_i depend polynomially on A , it follows that A^\dagger is smooth. \square

In order to deduce the theorem from Lemma 4.2.2, we need the following auxiliary result:

Lemma 4.2.3. *If (4.2.1) holds for some $1 \leq p \leq \infty$, there is a constant C such that*

$$|\xi|^l |\mathcal{A}^*(\xi)w| \leq C |\mathcal{A}(\xi)\mathcal{A}^*(\xi)w| \quad \text{for all } w \in \mathbb{W}, \xi \in \mathbb{R}^n \setminus \{0\}. \quad (4.2.3)$$

An argument in a similar spirit, but concerning (3.1.2), is outlined in [51].

Proof: Fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and $w \in \mathbb{W}$ and let $g \in C_c^\infty(B_1(0))$ be such that $0 \leq g \leq 1$ and $g = 1$ in $B_{1/2}(0)$. Set $v_\varepsilon(x) \equiv \mathcal{A}^*(g(\varepsilon x)e^{ix \cdot \xi} w)$ for $\varepsilon \in (0, 1)$, so that

$$\begin{aligned} v_\varepsilon(x) &= g(\varepsilon x)e^{ix \cdot \xi} \mathcal{A}^*(\xi)w + \sum_{|\alpha|=l} \sum_{\beta < \alpha} \binom{\alpha}{\beta} \varepsilon^{l-|\beta|} (i\xi)^\beta (\partial^{\alpha-\beta} g)(\varepsilon x) e^{ix \cdot \xi} A_\alpha^* w \\ &\equiv g(\varepsilon x)e^{ix \cdot \xi} \mathcal{A}^*(\xi)w + \varepsilon F_\varepsilon(x), \end{aligned}$$

where $F_\varepsilon \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$ is supported inside $B_{1/\varepsilon}(0)$ and is bounded independently of ε by $C_0(\mathcal{A}, g, \xi, w)$, say.. On the other hand, $P_{\mathcal{A}}v_\varepsilon = 0$: indeed, $\ker \mathcal{A}(\xi) = (\text{im } \mathcal{A}^*(\xi))^\perp$ and so, writing $\eta(x) \equiv g(\varepsilon x)e^{ix \cdot \xi}w$,

$$\mathcal{F}(P_{\mathcal{A}}v_\varepsilon) = \text{Proj}_{\ker \mathcal{A}(\xi)} \mathcal{A}^*(\xi) \hat{\eta}(\xi) = 0.$$

We can analogously obtain

$$\mathcal{A}v_\varepsilon(x) = g(\varepsilon x)e^{ix \cdot \xi} \mathcal{A}(\xi) \mathcal{A}^*(\xi) w + \varepsilon G_\varepsilon(x),$$

where $G_\varepsilon \in C_c^\infty(\mathbb{R}^n, \mathbb{W})$ is supported inside $B_{1/\varepsilon}(0)$ and can be assumed to be bounded independently of ε by C_0 , so

$$|\mathcal{A}v_\varepsilon(x)| \leq |g(\varepsilon x)| |\mathcal{A}(\xi) \mathcal{A}^*(\xi) w| + \varepsilon |G_\varepsilon(x)|. \quad (4.2.4)$$

A similar calculation yields

$$|D^l v(x)| \geq |g(\varepsilon x)| |\xi|^l |\mathcal{A}^*(\xi) w| - \varepsilon |H_\varepsilon(x)| \quad (4.2.5)$$

for another smooth function H_ε having the same properties as G_ε . Clearly we can assume that $\mathcal{A}^*(\xi)w \neq 0$ for otherwise there is nothing to prove. We take ε small enough such that $|\xi|^l |\mathcal{A}^*(\xi)w| \geq C_0 \varepsilon$, so the right hand side of (4.2.5) is non-negative inside $B_{1/(2\varepsilon)}(0)$. Thus, for $1 \leq p < \infty$, combining (4.2.4) and (4.2.5) with (4.2.1) we find

$$\mathcal{L}^n(B_{1/(2\varepsilon)}) \left(|\xi|^l |\mathcal{A}^*(\xi)w| - \varepsilon C_0 \right)^p \leq C \mathcal{L}^n(B_{1/\varepsilon}) \left(|\mathcal{A}(\xi) \mathcal{A}^*(\xi)w| + \varepsilon C_0 \right)^p.$$

Dividing by $\mathcal{L}^n(B_{1/\varepsilon})$ and sending $\varepsilon \rightarrow 0$ we arrive at the conclusion. The case $p = \infty$ is similar, but easier. \square

Proof of Theorem 4.2.1: Note that, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\hat{v}(\xi) - \text{Proj}_{\ker \mathcal{A}(\xi)} \hat{v}(\xi) = \mathcal{A}^\dagger(\xi) \widehat{\mathcal{A}v}(\xi)$. Thus, by the definition of $P_{\mathcal{A}}$, we have that

$$D^l(v - P_{\mathcal{A}}v) = \mathcal{F}^{-1}(\mathcal{A}^\dagger(\xi) \widehat{\mathcal{A}v}(\xi) \otimes \xi^{\otimes l})$$

and the ‘‘if’’ direction follows from Lemma 4.2.2 and the Hörmander–Mihlin multiplier theorem.

For the ‘‘only if’’ direction, suppose that (4.2.1) holds. Thus Lemma 4.2.3 shows that (4.2.3) must hold as well and this easily implies that \mathcal{A} has constant rank. Indeed, (4.2.3) shows that the spectrum of $\mathcal{A}(\xi)|_{\text{im } \mathcal{A}^*(\xi)}$ is bounded away from zero uniformly in ξ ; equivalently,

$$\mathbb{S}^{n-1} \ni \xi \mapsto \left(\mathcal{A}(\xi)|_{\text{im } \mathcal{A}^*(\xi)} \right)^{-1} \text{ is bounded.}$$

The definition of \mathcal{A}^\dagger , together with Lemma 4.2.2, show that \mathcal{A} has constant rank. \square

In fact, our observation can be improved when $p = 2$:

Corollary 4.2.4. *The operator \mathcal{A} has constant rank if and only if there is a constant C such that*

$$\inf \left\{ \|D^l(v - \tilde{v})\|_{L^2(\mathbb{R}^n)} : \mathcal{A}\tilde{v} = 0, \tilde{v} \in C_c^\infty(\mathbb{R}^n, \mathbb{V}) \right\} \leq C \| \mathcal{A}v \|_{L^2(\mathbb{R}^n)} \quad (4.2.6)$$

for all $v \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$. In particular, \mathcal{A} has constant rank if and only if the operator

$$\mathcal{A}: \mathcal{W}^{\mathcal{A}, 2}(\mathbb{R}^n) \equiv \text{clos}_{v \mapsto \|\mathcal{A}v\|_{L^2}} C_c^\infty(\mathbb{R}^n, \mathbb{V}) \rightarrow L^2(\mathbb{R}^n, \mathbb{W})$$

has closed range.

Proof: Note that the infimum in (4.2.6) is attained with $\psi = P_{\mathcal{A}}\varphi$, by Plancherel’s theorem and the minimization properties of orthogonal projections. Hence the first part follows from the theorem, while the second statement is an immediate consequence of general results on unbounded linear operators, see for instance [63, §2.7, Remark 18]. \square

4.3 Potentials for constant rank operators

Using a result of DECELL [116], RAIȚĂ [319] gave the following characterisation of constant rank operators:

Theorem 4.3.1. *A linear homogeneous differential operator \mathcal{A} with constant coefficients has constant rank if and only if there is a linear homogeneous differential operator \mathcal{B} with constant coefficients such that*

$$\operatorname{im} \mathcal{B}(\xi) = \ker \mathcal{A}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (4.3.1)$$

We also refer the reader to [251] for a very direct proof of Theorem 4.3.1. We will write, for some $B_\alpha \in \operatorname{Lin}(\mathbb{U}, \mathbb{V})$,

$$\mathcal{B}u = \sum_{|\alpha|=k} B_\alpha \partial^\alpha u, \quad u: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{U}; \quad (4.3.2)$$

equivalently, there is $T \in \operatorname{Lin}(\odot^k(\mathbb{R}^n, \mathbb{U}), \mathbb{V})$ such that

$$\mathcal{B} = T \circ \mathcal{D}^k. \quad (4.3.3)$$

We would like to emphasize that the construction of \mathcal{B} given in [319] is computable and that in fact one can always take $\mathbb{U} = \mathbb{V}$. We will refer to the potential operator \mathcal{B} simply as the *potential* and to \mathcal{A} as the *annihilator*, although this terminology is not standard.

4.3.1 Cocanceling operators

In order to discuss further properties of constant rank operators it will be convenient to employ simple algebraic properties of cocanceling operators, which for the reader's convenience we prove in this section.

Definition 4.3.2. The operator \mathcal{B} is said to be *cocanceling* if $\mathbb{I}_\mathcal{B} \equiv \bigcap_{\xi \in \mathbb{S}^{n-1}} \ker \mathcal{B}(\xi) = \{0\}$.

This notion was introduced by VAN SCHAFTINGEN in [369] and is equivalent to a critical linear L^1 -estimate for \mathcal{B} -free fields. Typical examples of cocanceling operators are the divergence, the exterior derivative and the Saint–Venant compatibility operator, c.f. Example 4.1.1(iv).

We recall a fundamental characterization of cocanceling operators [369, Proposition 2.1]:

Lemma 4.3.3. *The following are equivalent:*

- (i) \mathcal{A} is cocanceling;
- (ii) $\int v = 0$ for all $v \in C_c^\infty(\Omega, \mathbb{V})$ such that $\mathcal{A}v = 0$;
- (iii) If $v_0 \in \mathbb{V}$ such that $\mathcal{A}(\delta_0 v_0) = 0$, then $v_0 = 0$.

For our purposes, the relevance of cocancellation stems from the following simple result:

Lemma 4.3.4. *Let \mathcal{B} be as in (4.3.2) and let \mathbb{J} be a subspace which is such that $\mathbb{U} = \mathbb{I}_\mathcal{B} \oplus \mathbb{J}$. Then there is a choice of coordinates of \mathbb{U} such that \mathcal{B} can be represented as a block matrix*

$$\mathcal{B} = \begin{bmatrix} 0_{\operatorname{Lin}(\mathbb{I}_\mathcal{B}, \mathbb{V})} & \tilde{\mathcal{B}} \end{bmatrix}$$

where $\tilde{\mathcal{B}}(\xi): \mathbb{J} \rightarrow \mathbb{V}$ is cocanceling.

An immediate consequence of Lemma 4.3.4 is that the space of \mathcal{B} -free fields contains $C_c^\infty(\mathbb{R}^n, \mathbb{I}_\mathcal{B})$. This space is trivial if and only if \mathcal{B} is cocanceling.

Proof: The proof relies on [369, Proposition 2.5]. Using the notation in (4.3.2), we first claim that

$$\mathbb{I}_{\mathcal{B}} = \bigcap_{|\alpha|=k} \ker B_{\alpha}.$$

On one hand, if $B_{\alpha}v_0 = 0$ for all α , then $\mathcal{B}(\xi)v_0 = 0$ for all $\xi \in \mathbb{R}^n$, so that $v_0 \in \mathbb{I}_{\mathcal{B}}$. On the other hand, if $\sum_{|\alpha|=k} \xi^{\alpha} B_{\alpha}v_0 = 0$ for all $\xi \in \mathbb{R}^n$, by identifying coefficients, we obtain that $B_{\alpha}v_0 = 0$ for all α .

We choose a basis of \mathbb{U} such that the matrices B_{α} can be written as $B_{\alpha} = [0_{\text{Lin}(\mathbb{I}_{\mathcal{B}}, \mathbb{V})} \tilde{B}_{\alpha}]$ and define $\tilde{\mathcal{B}}(\xi) = \sum_{|\alpha|=k} \xi^{\alpha} \tilde{B}_{\alpha}$. It is then clear that $\bigcap_{|\alpha|=k} \ker \tilde{B}_{\alpha} = \{0\}$, which implies that $\tilde{\mathcal{B}}$ is cocanceling. \square

These results suggest that one can reduce statements about non-cocanceling operators to statements about cocanceling operators, as often Lemma 4.3.4 can be used to perform reductions. As a side note, we also record the following consequence:

Corollary 4.3.5. *With the notation of Lemma 4.3.4, we have that $\Lambda_{\mathcal{B}} = \mathbb{I}_{\mathcal{B}} \times \Lambda_{\tilde{\mathcal{B}}}$.*

4.3.2 Further properties of potentials

We shall now consider the following question: is there any meaningful sense in which the potential \mathcal{B} associated with the operator \mathcal{A} is unique? To find a canonical potential \mathcal{B} , one must take into account the following:

- (i) \mathcal{B} should have minimal order (for instance, if \mathcal{B} is a potential, so is $|\xi|^2 \mathcal{B}(\xi)$);
- (ii) \mathcal{B} is at best unique only modulo isomorphisms: if $Q \in \text{GL}(\mathbb{U})$, then $\mathcal{B}Q$ is another potential;
- (iii) \mathcal{B} should be cocanceling, since adding columns of zeroes does not change $\text{im } \mathcal{B}$ and hence preserves the exactness (4.3.1), see Lemma 4.3.4.

While for many of the operators that occur in applications these conditions seem to suffice to single out a canonical potential (modulo isomorphisms of \mathbb{U}), in general they are not enough:

Proposition 4.3.6. *There is a first order constant rank operator \mathcal{A} which admits two cocanceling potentials $\mathcal{B}_1, \mathcal{B}_2$ of minimal order which moreover satisfy $\mathcal{B}_1 \neq \mathcal{B}_2 Q$ for all $Q \in \text{Lin}(\mathbb{U}, \mathbb{U})$.*

The proof of the proposition proceeds by construction of an explicit example; the reader can find the proof in the appendix of [182]. The example in the appendix is also one where it is not possible to choose \mathcal{B} to have the order of \mathcal{A} . It seems to have been known for quite some time that this is generically the case, see for instance [238, page 445]. A simpler example with this property can be found by considering the symmetric gradient of maps $u: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which only has annihilators of order two or higher, see also [300, Remarque 4]. On the other hand, there is an example [149, Example 3.10(d)] of a first order annihilator for which the only known potential is D^k . To sum up, we remark that one cannot make any assumption on the relation between the orders of \mathcal{A} and \mathcal{B} .

From our perspective, Proposition 4.3.6 implies that, in the general, the operator \mathcal{B} associated to the constant rank operator \mathcal{A} has no physical content and is instead a useful mathematical tool. The potential is simply a polynomial parametrization of the wave cone; the physically relevant object is $\ker \mathcal{A}(\xi)$. This is already apparent in the Hilbert space axiomatization of MILTON [282] for composite materials, where the author postulates an orthogonal decomposition of the form

$$\mathbb{V} = \mathcal{E}_{\xi} \oplus \mathcal{J}_{\xi}, \quad \xi \neq 0;$$

the subspaces \mathcal{E}_{ξ} and \mathcal{J}_{ξ} correspond to the constraints satisfied by the applied and induced fields, respectively—these would be, for instance, the electric field and current in the case of conductivity, hence the choice of notation. In practice, these constraints come from a partial differential equation and we have $\mathcal{E}_{\xi} = \ker \mathcal{A}(\xi)$ and $\mathcal{J}_{\xi} = \ker \mathcal{B}^*(\xi)$ for some suitable operators. We will return to this point in Chapter 7.

4.4 Function spaces

In this subsection we gather some notation for function spaces associated with linear operators and prove some basic properties of these spaces. For our purposes it will be important to consider the space of \mathcal{A} -free test fields, i.e.

$$C_{c,\mathcal{A}}^\infty(\Omega) \equiv \{v \in C_c^\infty(\Omega, \mathbb{V}) : \mathcal{A}v = 0\}.$$

In the general case where \mathcal{A} is cocanceling (but does not necessarily have constant rank) it is unclear whether this space contains non-zero functions, while it always does in the non-cocanceling case as per Lemma 4.3.4. Related to this we have the following simple lemma (see also [369, Proposition 2.1]):

Lemma 4.4.1. *The space $C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ is contained in $\mathcal{H}^1(\mathbb{R}^n)$ if and only if \mathcal{A} is cocanceling.*

Proof: Suppose that $C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ is contained in the Hardy space; since functions in $\mathcal{H}^1(\mathbb{R}^n)$ have zero mean then so do functions in $C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ and this happens if and only if \mathcal{A} is cocanceling. Moreover, test functions with zero mean are contained in $\mathcal{H}^1(\mathbb{R}^n)$ —in fact, they are dense there—and this proves the other direction. \square

For $1 \leq p \leq \infty$, we have the L^p -type spaces

$$L_{\mathcal{A}}^p(\Omega) \equiv \{v \in L^p(\Omega, \mathbb{V}) : \mathcal{A}v = 0\}.$$

Associated with \mathcal{B} , we define the \mathcal{B} -Sobolev-type spaces

$$\mathcal{W}^{\mathcal{B},p}(\Omega) \equiv \text{clos}_{u \rightarrow \|\mathcal{B}u\|_p} C_c^\infty(\Omega, \mathbb{U}). \quad (4.4.1)$$

General properties of the $\mathcal{W}^{\mathcal{B},p}$ -spaces can be found in the recent works [61, 163].

When \mathcal{A} is a constant rank operator and $1 < p < \infty$ we have that $C_{c,\mathcal{A}}^\infty$ is dense in $L_{\mathcal{A}}^p$; it is unclear whether this holds for non constant rank operators. In fact, we have:

Proposition 4.4.2. *If \mathcal{B} is a potential for \mathcal{A} , we have*

$$\mathcal{B}(\mathcal{W}^{\mathcal{B},p}(\mathbb{R}^n)) = \mathcal{B}(\dot{W}^{k,p}(\mathbb{R}^n, \mathbb{U})) = L_{\mathcal{A}}^p(\mathbb{R}^n), \quad (4.4.2)$$

where $\dot{W}^{k,p}(\mathbb{R}^n, \mathbb{U})$ denotes the usual homogeneous Sobolev space.

Proposition 4.4.2 follows from the following Helmholtz–Hodge decomposition:

Proposition 4.4.3. *Let $1 < p < \infty$. A vector field $v \in L^p(\mathbb{R}^n, \mathbb{V})$ can be uniquely¹ decomposed as*

$$v = \mathcal{B}u + \mathcal{A}^*w$$

for some $u \in \mathcal{W}^{\mathcal{B},p}(\mathbb{R}^n)$, $w \in \mathcal{W}^{\mathcal{A}^*,p}(\mathbb{R}^n)$. Moreover, this decomposition is continuous:

$$\|\mathcal{B}u\|_{L^p(\mathbb{R}^n)} \leq C\|v\|_{L^p(\mathbb{R}^n)}, \quad \|\mathcal{A}^*w\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{A}v\|_{\dot{W}^{-l,p}(\mathbb{R}^n)}.$$

Proposition 4.4.3 follows by standard methods from Theorem 4.3.1, see for instance [149, 162]. We will in fact construct $u \in \dot{W}^{k,p}(\mathbb{R}^n, \mathbb{U})$, $w \in \dot{W}^{l,p}(\mathbb{R}^n, \mathbb{W})$.

Proof: We begin by remarking that, once we have the decomposition, uniqueness follows straightforwardly from orthogonality. Indeed, consider a decomposition of zero, $0 = \mathcal{B}u + \mathcal{A}^*w$. If p' denotes the Hölder conjugate of p , let $\varphi \in L^{p'}(\mathbb{R}^n, \mathbb{V})$ be arbitrary and write $\varphi = \mathcal{B}\chi + \mathcal{A}^*\psi$ for $\chi \in \mathcal{W}^{\mathcal{B},p'}$, $\psi \in \mathcal{W}^{\mathcal{A}^*,p'}$. Then

$$\int_{\mathbb{R}^n} \langle \mathcal{B}u, \varphi \rangle = \int_{\mathbb{R}^n} \langle \mathcal{B}u, \mathcal{B}\chi \rangle + \int_{\mathbb{R}^n} \langle \mathcal{B}u, \mathcal{A}^*\psi \rangle = \int_{\mathbb{R}^n} \langle \mathcal{B}u, \mathcal{B}\chi \rangle = - \int_{\mathbb{R}^n} \langle \mathcal{A}^*w, \mathcal{B}\chi \rangle = 0$$

where we used twice the fact that $\int \langle \mathcal{B}b, \mathcal{A}^*a \rangle = 0$ for all $b \in \mathcal{W}^{\mathcal{B},p}$, $a \in \mathcal{W}^{\mathcal{A}^*,p'}$ in view of (4.3.1).

¹Here we do not mean that u, w are unique, but rather their images $\mathcal{B}u, \mathcal{A}^*w$.

We assume that $\text{ord } \mathcal{B} = k \geq l = \text{ord } \mathcal{A}$, for otherwise we can replace \mathcal{B} by $|\xi|^{2m}\mathcal{B}(\xi)$ for m sufficiently large. Let $j = k - l$ and consider the homogeneous k -th order operator

$$\square \equiv \mathcal{B}\mathcal{B}^* + \mathcal{A}^*\mathcal{A}\Delta^j;$$

by the exactness relation (4.3.1), this operator is elliptic, meaning that $\square(\xi) \in \text{GL}(\mathbb{V})$ for all $0 \neq \xi \in \mathbb{R}^n$. This can be seen by letting $v_0 \in \ker \square(\xi)$ and writing

$$0 = \langle \square(\xi)v_0, v_0 \rangle = \langle \mathcal{B}(\xi)\mathcal{B}^*(\xi)v_0, v_0 \rangle + |\xi|^{2j} \langle \mathcal{A}^*(\xi)\mathcal{A}(\xi)v_0, v_0 \rangle = |\mathcal{B}^*(\xi)v_0|^2 + |\xi|^{2j} |\mathcal{A}(\xi)v_0|^2,$$

so $v_0 \in \ker \mathcal{B}^*(\xi) \cap \ker \mathcal{A}(\xi) = \ker \mathcal{B}^*(\xi) \cap \text{im } \mathcal{B}(\xi) = (\text{im } \mathcal{B}(\xi))^\perp \cap \text{im } \mathcal{B}(\xi) = \{0\}$.

Consequently, we can solve $\square\varphi = v$ for $\varphi \in \dot{W}^{2k,p}(\mathbb{R}^n, \mathbb{V})$ with the elliptic estimate

$$\|D^{2k}\varphi\|_{L^p(\mathbb{R}^n)} \leq C\|v\|_{L^p(\mathbb{R}^n)}. \quad (4.4.3)$$

Now define

$$u \equiv \mathcal{B}^*\varphi, \quad w \equiv \mathcal{A}\Delta^j\varphi;$$

then (4.4.3) already gives the estimate for $\mathcal{B}u$ in the statement, as well as a similar estimate for \mathcal{A}^*w . Note that due to the bounds in (4.4.3), we can assume that $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{V})$, otherwise it can be replaced with an approximating sequence φ_j such that $\square\varphi_j$ converges to v in L^p .

To get the better estimate for \mathcal{A}^*w , we apply \mathcal{A} to the decomposition to get $\mathcal{A}v = \mathcal{A}\mathcal{A}^*w$, so that we can compute in Fourier space, for $\xi \neq 0$,

$$\mathcal{A}^*(\xi)\hat{w}(\xi) = \mathcal{A}^\dagger(\xi)\mathcal{A}(\xi)\mathcal{A}^*(\xi)\hat{w}(\xi) = \mathcal{A}^\dagger\left(\frac{\xi}{|\xi|}\right)\frac{\widehat{\mathcal{A}v}(\xi)}{|\xi|^l},$$

where we used the fact that $\mathcal{A}^\dagger\mathcal{A} = \text{Proj}_{\text{im } \mathcal{A}^*}$. The Hörmander–Mihlin multiplier theorem then implies that

$$\|\mathcal{A}^*w\|_{L^p(\mathbb{R}^n)} \leq C \left\| \mathcal{F}^{-1} \left(\frac{\widehat{\mathcal{A}v}(\xi)}{|\xi|^l} \right) \right\|_{L^p(\mathbb{R}^n)} = C\|\mathcal{A}v\|_{\dot{W}^{-l,p}(\mathbb{R}^n)},$$

which concludes the proof. \square

Proof of Proposition 4.4.2: Let $v \in L^p(\mathbb{R}^n)$ with $\mathcal{A}v = 0$. Using Proposition 4.4.3, we have that $v = \mathcal{B}u + f$, where $f = \mathcal{A}^*w \in L^p(\mathbb{R}^n, \mathbb{V})$ is such that $\mathcal{B}^*f = 0$. This follows since the exactness (4.3.1) can equivalently be written as $\text{im } \mathcal{A}^*(\xi) = \ker \mathcal{B}^*(\xi)$ for $\xi \neq 0$, hence $\mathcal{B}^* \circ \mathcal{A}^* = 0$. On the other hand, since v is \mathcal{A} -free, we also obtain $\mathcal{A}f = 0$. Therefore $\square f = 0$, so that f is analytic by the ellipticity of \square . Since $f \in L^p(\mathbb{R}^n)$, we conclude that $f = 0$, which implies the only non-trivial inclusion in (4.4.2). \square

Through the multiplier $\mathcal{A}^\dagger(\xi/|\xi|)$, the proof of the Helmholtz–Hodge decomposition in Proposition 4.4.3 relies heavily on the Calderón–Zygmund theory to solve an auxiliary partial differential equation in full space. Having a similar decomposition that holds in bounded domains may be a viable tool to tackle other problems in the field. This motivates the following:

Question 4.4.4. Let $\Omega \subset \mathbb{R}^n$ be a sufficiently regular bounded domain and $1 < p < \infty$. Is it the case that each $v \in L^p(\Omega, \mathbb{V})$ has a unique decomposition

$$v = \mathcal{B}u + \mathcal{A}^*w + h,$$

where $u \in \mathcal{W}^{\mathcal{B},p}(\Omega)$, $w \in \mathcal{W}^{\mathcal{A}^*,p}(\Omega)$, and $\mathcal{B}^*h = 0$, $\mathcal{A}h = 0$ in the sense of distributions, with the bounds

$$\|\mathcal{B}u\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)} \leq C_p\|v\|_{L^p(\Omega)}, \quad \|\mathcal{A}^*w\|_{L^p(\Omega)} \leq C\|\mathcal{A}v\|_{\dot{W}^{-l,p}(\Omega)}?$$

It is known that the domain Ω cannot be taken to be an arbitrary open set [186]. The ‘‘harmonic’’ field h is analytic in Ω , since it satisfies $\square h = 0$. It is also known that one cannot hope for a decomposition with $h = 0$, since this is not the case for exterior differentials and codifferentials; in this situation, furthermore, the answer to the question is positive, see for instance [339] for an elementary proof. Question 4.4.4 is also true for $p = 2$:

Answer to Question 4.4.4 for $p = 2$: Note that the orthogonal complement in $L^2(\Omega, \mathbb{V})$ of $X \equiv \{\mathcal{B}u : u \in C_c^\infty(\Omega, \mathbb{U})\}$ is

$$Y \equiv \{v \in L^2(\Omega, \mathbb{V}) : \mathcal{B}^*v = 0 \text{ in the sense of distributions}\}.$$

This follows from the following identity, which holds for all $u \in C_c^\infty(\Omega, \mathbb{U})$ and $f \in L^2(\Omega, \mathbb{V})$:

$$\langle f, \mathcal{B}u \rangle_{L^2} = \int_{\Omega} \langle f, \mathcal{B}u \rangle_{\mathbb{V}} dx = \langle f, \mathcal{B}u \rangle_{\mathcal{D}', \mathcal{D}} = (-1)^k \langle \mathcal{B}^*f, u \rangle_{\mathcal{D}', \mathcal{D}}.$$

The projection theorem yields the orthogonal decomposition $L^2(\Omega, \mathbb{V}) = \overline{X} \oplus Y$. We then note that $Z \equiv \{\mathcal{A}^*w : w \in C_c^\infty(\Omega, \mathbb{W})\}$ is a subspace of Y . An analogous argument shows that the orthogonal complement of Z in Y is $H \equiv \{h \in L^2(\Omega, \mathbb{V}) : \mathcal{A}h = 0, \mathcal{B}^*h = 0\}$. In particular, we obtain the orthogonal decomposition $L^2(\Omega, \mathbb{V}) = \overline{X} \oplus Z \oplus H$, which gives the claim, except for the negative Sobolev bound. To prove this as well, note that we already have a sequence $w_j \in C_c^\infty(\Omega, \mathbb{W})$ such that $\mathcal{A}^*w_j \rightarrow v - \mathcal{B}u - h$ in $L^2(\Omega, \mathbb{V})$, so that $\mathcal{A}\mathcal{A}^*w_j \rightarrow \mathcal{A}v$ in $\dot{W}^{-l,2}(\Omega, \mathbb{V})$. It remains to recall the last estimate from the proof of Proposition 4.4.3, i.e.

$$\|\mathcal{A}^*w_j\|_{L^2(\Omega)} = \|\mathcal{A}^*w_j\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}\mathcal{A}^*w_j\|_{\dot{W}^{-l,2}(\mathbb{R}^n)} = C\|\mathcal{A}\mathcal{A}^*w_j\|_{\dot{W}^{-l,2}(\Omega)},$$

where the equalities follow since w_j are supported inside Ω . □

4.5 Semiconvexity notions

The material from this section is standard and we refer the reader to [97, 98, 295]. We begin with the following definition [149], generalizing the previous notions of MORREY [285] and MEYERS [281]:

Definition 4.5.1. A locally bounded, Borel function $F: \mathbb{V} \rightarrow \mathbb{R}$ is \mathcal{A} -quasiconvex if

$$0 \leq \int_{[0,1]^n} F(z + v(x)) - F(z) dx$$

for all $z \in \mathbb{V}$ and all $v \in C_{\text{per}}^\infty([0,1]^n, \mathbb{V})$ such that $\mathcal{A}v = 0$ and $\int_{[0,1]^n} v = 0$.

An important consequence of Theorem 4.3.1 is that, under a constant rank assumption, the above definition can be changed to resemble more closely the original definition of quasiconvexity in the gradient case (see [319, Corollary 1]):

Corollary 4.5.2. Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty open subset. Given a constant rank operator \mathcal{A} as in the setup of Theorem 4.3.1, a locally bounded Borel function $F: \mathbb{V} \rightarrow \mathbb{R}$ is \mathcal{A} -quasiconvex if and only if

$$0 \leq \int_{\Omega} F(z + \mathcal{B}u(y)) - F(z) dy$$

for all $z \in \mathbb{V}$ and all $u \in C_c^\infty(\Omega, \mathbb{U})$.

Definition 4.5.3. A function $F: \mathbb{V} \rightarrow \mathbb{R}$ is said to be \mathcal{A} -quasiaffine if both F and $-F$ are \mathcal{A} -quasiconvex.

Note that, by Corollary 4.5.2, this is equivalent to saying that, for any $z \in \mathbb{V}$,

$$\int_{\Omega} F(z + \mathcal{B}u) - F(z) dx = 0 \tag{4.5.1}$$

for all $u \in C_c^\infty(\Omega, \mathbb{U})$ and every non-empty open set $\Omega \subset \mathbb{R}^n$.

Besides constant rank, it will be important to assume that the wave cone of \mathcal{A} spans the entire space. This is related to the following well-known lemma [13, Section 2.5]; we give a proof only for the sake of completeness.

Lemma 4.5.4. *We have $\text{span } \Lambda_{\mathcal{A}} = \mathbb{V}$ if and only if all \mathcal{A} -quasiconvex functions are continuous.*

Proof: The direction \Rightarrow is standard and follows from the fact that any such function is Λ -convex and Λ -convex functions are (locally Lipschitz) continuous in $\text{span } \Lambda$, see e.g. [233, Lemma 2.3]. To prove \Leftarrow assume $\text{span } \Lambda \neq \mathbb{V}$. Then we can write $(v_1, v_2) \in \mathbb{V} = \text{span } \Lambda \oplus \tilde{\mathbb{V}}$ where $\tilde{\mathbb{V}} \neq \{0\}$. The function defined by $F(v_1, v_2) = 1_{\{v_2=0\}}(v_1, v_2)$ is a discontinuous \mathcal{A} -quasiconvex function; in fact, it is even \mathcal{A} -quasiaffine. Here we used the fact that periodic \mathcal{A} -free fields take their values in $\text{span } \Lambda$. \square

There is also a purely geometric notion of semi-convexity associated with \mathcal{A} :

Definition 4.5.5. Given an open set $\mathcal{O} \subset \mathbb{V}$, an integrand $F: \mathcal{O} \rightarrow \mathbb{R}$ is Λ -**convex** if, for all v_1, v_2 such that $v_1 - v_2 \in \Lambda$ and $[v_1, v_2] \subset \mathcal{O}$, we have

$$F(tv_1 + (1-t)v_2) \leq tF(v_1) + (1-t)F(v_2) \quad \text{for all } t \in [0, 1].$$

The following result is classical (see e.g. [233] for a quantitative statement):

Lemma 4.5.6. *If $\text{span } \Lambda = \mathbb{V}$ then Λ -convex functions are locally Lipschitz.*

We refer the reader to Section 5.2 for definitions and discussions concerning polyconvexity in this general setting.

Chapter 5

Compensated Compactness under constant rank constraints

In this chapter we develop the general theory of compensated compactness under constant rank constraints. The results of this chapter are joint work with BOGDAN RAIȚĂ [182]. Very general, optimal results in the spirit of the ones presented here can be found in [184], written also with MATTHEW SCHRECKER.

In Section 5.1 we prove a new lower semicontinuity result, c.f. Theorem 3.D. In Section 5.2 we give multiple characterisations of null Lagrangians and their Hardy space integrability is proved in Section 5.3, completing the proof of Theorem 3.B.

5.1 \mathcal{A} -quasiconvexity and weak lower semicontinuity

We will make the standard assumption that $F: \mathbb{V} \rightarrow \mathbb{R}$ satisfies a p -growth condition

$$|F(v)| \leq C(|v|^p + 1) \tag{G_p}$$

The importance of \mathcal{A} -quasiconvexity is its relation to lower semicontinuity, made precise by the following fundamental result by FONSECA–MÜLLER [149] (see also [13, Remark 1.3]):

Theorem 5.1.1. *Let \mathcal{A} have constant rank and let $F: \Omega \times \mathbb{V} \rightarrow \mathbb{R}$ be a Carathéodory integrand. The functional $v \mapsto \int_{\Omega} F(x, v(x)) \, dx$ is sequentially weakly-* lower semicontinuous on $L_{\mathcal{A}}^{\infty}(\Omega)$ if and only if for each fixed $x_0 \in \Omega$ the map $F(x_0, \cdot)$ is \mathcal{A} -quasiconvex.*

Moreover, if (G_p) holds for some $1 < p < \infty$ and we fix $1 < p < q$, then we have

$$\left. \begin{array}{l} v_j \rightharpoonup v \quad \text{in } L^q(\Omega) \\ \mathcal{A}v_j \rightarrow 0 \quad \text{in } W^{-l,q}(\Omega) \end{array} \right\} \implies \liminf_{j \rightarrow \infty} \int_{\Omega} F(x, v_j(x)) \, dx \geq \int_{\Omega} F(x, v(x)) \, dx$$

if and only if for a.e. $x_0 \in \Omega$ the map $F(x_0, \cdot)$ is \mathcal{A} -quasiconvex.

We remark that, in general, the conclusion of the theorem is false in the critical case $p = q$ unless one assumes additional structure on either the integrand, for instance positivity as done in [149], or on the sequence, for instance that it does not concentrate on the boundary nor on the set where $x \mapsto F(x, \cdot)$ is discontinuous. A counterexample illustrating this failure was given for $\mathcal{A} = \text{curl}$ and $F = \det$ in [27, Example 7.1 and 7.3]. We refer the reader to [45] for a detailed discussion of this issue.

The following lemma is well-known and was proved in the $\mathcal{A} = \text{curl}$ case in [1, 271].

Lemma 5.1.2. *Assume Λ spans \mathbb{V} . If $F: \mathbb{V} \rightarrow \mathbb{R}$ is Λ -convex and satisfies (G_p) then*

$$|F(v) - F(w)| \leq C(1 + |v|^{p-1} + |w|^{p-1})|v - w|$$

for all $v, w \in \mathbb{R}^d$.

Proof: By the spanning condition, F is Lipschitz and, for $v, w \in B_r(0) \subset \mathbb{V}$,

$$|F(v) - F(w)| \leq \frac{C}{r} \operatorname{osc}(F, B_{2r}) |v - w|,$$

where C depends only on Λ ; see [233, Lemma 2.3]. Using (G_p) and the triangle inequality, we get

$$|F(v) - F(w)| \leq C \left(1 + \frac{|v|^p}{r} + \frac{|w|^p}{r} \right) |v - w| \leq C(1 + |v|^{p-1} + |w|^{p-1}) |v - w|$$

where we also assumed without loss of generality that $r \geq 1$. \square

We are now ready to begin the proof of the main result of this section. Recall that we always assume (3.1.6). The next proposition, although relatively simple, is a crucial ingredient in the proof of Theorem 5.1.4 below. The point is that when a weakly convergent sequence does not concentrate on the boundary it can be replaced by a sequence of potentials.

Proposition 5.1.3. *Let Ω be a bounded domain. Let $v_j, v \in L^p(\Omega, \mathbb{V})$ be such that*

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{V}), \quad \mathcal{A}v_j \rightarrow \mathcal{A}v \text{ in } W_{\text{loc}}^{-l,p}(\Omega, \mathbb{V})$$

and moreover let λ be such that $|v_j|^p \xrightarrow{*} \lambda$ in $\mathcal{M}(\overline{\Omega})$. Assume that $\lambda(\partial\Omega) = 0$. Up to passing to subsequences in (v_j) , there is a sequence $u_j \in C_c^\infty(\Omega, \mathbb{U})$ such that

$$v_j - v - \mathcal{B}u_j \rightarrow 0 \text{ in } L^p(\Omega, \mathbb{V}).$$

Proof: By linearity we may assume that $v = 0$. Let $U \Subset V \Subset \Omega$ to be determined later and take $\eta \in C_c^\infty(\Omega)$ with $1_U \leq \eta \leq 1_V$ and $|\mathbf{D}^m \eta| \leq 2d^{-m}$ for $m = 1, \dots, k$; here $d \equiv \operatorname{dist}(U, \partial V)$. Write, using the Helmholtz-Hodge decomposition of Proposition 4.4.3,

$$\tilde{v}_j \equiv \eta v_j, \quad \tilde{v}_j = \mathcal{B}u_j + w_j,$$

where we have extended \tilde{v}_j by zero outside Ω so that it is in $L^p(\mathbb{R}^n, \mathbb{V})$. Moreover, we have

$$\|v_j - \mathcal{B}u_j\|_{L^p(\Omega)} \leq \|v_j - \tilde{v}_j\|_{L^p(\Omega)} + \|\tilde{v}_j - \mathcal{B}u_j\|_{L^p(\Omega)} \lesssim \|v_j - \tilde{v}_j\|_{L^p(\Omega)} + \|\mathcal{A}\tilde{v}_j\|_{W^{-l,p}(\Omega)}.$$

Let us estimate the first term: since λ is a positive measure,

$$\lim_{j \rightarrow \infty} \|(1 - \eta)v_j\|_{L^p(\Omega)} = \int_{\overline{\Omega}} (1 - \eta)^p d\lambda \leq \lambda(\overline{\Omega} \setminus U).$$

Taking $U \uparrow \Omega$ the left-hand side goes to zero by the dominated convergence theorem, since $\lambda(\partial\Omega) = 0$. For the second term, we have

$$\|\mathcal{A}(\eta v_j)\|_{W^{-l,p}(\Omega)} \leq \|\eta \mathcal{A}v_j\|_{W^{-l,p}(V)} + \sum_{i=1}^k \|B_i[\mathbf{D}^i \eta, \mathbf{D}^{k-i} v_j]\|_{W^{-l,p}(V)}$$

where the B_i are fixed bilinear pairings given by the chain rule. For the first term note that, up to taking subsequences in v_j if necessary, we can assume that

$$\|\eta \mathcal{A}v_j\|_{W^{-l,p}(V)} \leq \frac{1}{j}$$

by our hypothesis. The second term can be bounded by

$$\|B_i[\mathbf{D}^i \eta, \mathbf{D}^{k-i} v_j]\|_{W^{-l,p}(V)} \lesssim \frac{\|\mathbf{D}^{k-i} v_j\|_{W^{-l,p}(V)}}{d^i} \lesssim \frac{\|v_j\|_{L^p(V)}}{d^i}.$$

Thus, picking $U, V \uparrow \Omega$ such that d approaches zero sufficiently slowly, this term also goes to zero. This finishes the proof: although u_j is only in $\mathcal{W}^{\mathcal{B},p}(\Omega)$, by definition of this Sobolev space there are $\tilde{u}_j \in C_c^\infty(\Omega, \mathbb{U})$ with $\|\mathcal{B}(u_j - \tilde{u}_j)\|_p \rightarrow 0$. \square

We proceed to the proof of the main result of this section; it is inspired by standard lower semicontinuity proofs in the gradient case [1, 82, 245, 271, 281, 285].

Theorem 5.1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $F: \mathbb{V} \rightarrow \mathbb{R}$ is \mathcal{A} -quasiconvex and satisfies (G_p) then, whenever*

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{V}), \quad \mathcal{A}v_j \rightharpoonup \mathcal{A}v \text{ in } W_{\text{loc}}^{-l,p}(\Omega, \mathbb{V}),$$

for all $\rho \in C_c^\infty(\Omega)$ with $\rho \geq 0$ we have

$$\liminf_{j \rightarrow \infty} \int_{\Omega} \rho F(v_j) \, dx \geq \int_{\Omega} \rho F(v) \, dx.$$

Proof: By taking a subsequence, we can assume that $|v_j|^p \overset{*}{\rightharpoonup} \lambda$ in $\mathcal{M}(\Omega)$. Let us also fix $\rho \in C_c^\infty(\Omega)$ with $\rho \geq 0$ and $\varepsilon \in (0, 1)$.

Step 1: We can find $\tilde{v} \in C_c^\infty(\Omega, \mathbb{V})$ such that $\|v - \tilde{v}\|_p < \varepsilon$. Let us also take $\delta \in (0, 1)$ such that, given any triangulation $\tilde{\mathcal{T}}$ of \mathbb{R}^n with $\sup_{T \in \tilde{\mathcal{T}}} \text{diam } T < \delta$, we can find a function a , constant in each $T \in \tilde{\mathcal{T}}$, with the bound $\|\tilde{v} - a\|_{L^p(\Omega)} < \varepsilon$. In particular, a satisfies

$$\|a\|_{L^p(\Omega)} \leq 2\varepsilon + \|v\|_{L^p(\Omega)} < 2 + \|v\|_{L^p(\Omega)}. \quad (5.1.1)$$

We need to wiggle the triangulation slightly so that Proposition 5.1.3 becomes applicable. For this, let $\mathcal{T}_\Omega \equiv \{T \in \tilde{\mathcal{T}} : T \cap B_2(\Omega) \neq \emptyset\}$. Take a direction $e \in \mathbb{S}^{n-1}$ which is not tangent to any face of any simplex $T \in \mathcal{T}_\Omega$. Then, given a face σ of T , the sets $te + \sigma$, for $t \in (0, \delta)$, are disjoint. This shows that the set

$$\{t \in (0, \delta) : \lambda(te + \sigma) > 0\}$$

is at most countable and hence so is the set

$$E \equiv \bigcup_{T \in \mathcal{T}_\Omega} \{t \in (0, \delta) : \lambda(te + \partial T) > 0\}.$$

Select $t \in (0, \delta) \setminus E$ and define the final triangulation $\mathcal{T} \equiv te + \mathcal{T}_\Omega$, which contains $B_1(\Omega)$. Choose a to be constant in each $T \in \mathcal{T}$ and satisfy (5.1.1).

Step 2: Let us write $w_j \equiv a + v_j - v \in L^p(\Omega, \mathbb{V})$; then

$$\begin{aligned} \int_{\Omega} \rho(F(v_j) - F(v)) \, dx &= \int_{\Omega} \rho(F(v_j) - F(w_j)) \, dx + \int_{\Omega} \rho(F(w_j) - F(a)) \, dx \\ &\quad + \int_{\Omega} \rho(F(a) - F(v)) \, dx \equiv \text{I} + \text{II} + \text{III}. \end{aligned}$$

Using the local Lipschitz estimate of Lemma 5.1.2, we get

$$\begin{aligned} |\text{I} + \text{III}| &\lesssim \int_{\Omega} \rho(1 + |v_j|^{p-1} + |w_j|^{p-1})|v_j - w_j| \, dx + \int_{\Omega} \rho(1 + |v|^{p-1} + |a|^{p-1})|v - a| \, dx \\ &\leq \max \rho \int_{\Omega} (1 + |v_j|^{p-1} + 2^p |v_j|^{p-1} + 2^p |v - a|^{p-1})|v - a| \, dx \\ &\quad + \max \rho \left(\int_{\Omega} (1 + |v|^{p-1} + |a|^{p-1})^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |v - a|^p \, dx \right)^{\frac{1}{p}} \end{aligned}$$

Thus, from (5.1.1) and using Hölder again for the first term, we find that

$$|\text{I} + \text{III}| \leq C \left(1 + \|v\|_p^{p-1} + \sup_j \|v_j\|_p^{p-1} \right) \varepsilon = O(\varepsilon)$$

where C now also depends on ρ . To summarize, we have $w_j \rightharpoonup a$ in $L^p(\Omega, \mathbb{V})$ and we have shown that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} \rho(F(v_j) - F(v)) \, dx = O(\varepsilon) + \liminf_{j \rightarrow \infty} \int_{\Omega} \rho(F(w_j) - F(a)) \, dx. \quad (5.1.2)$$

Step 3: Since \mathcal{T} triangulates Ω we have

$$\int_{\Omega} \rho(F(w_j) - F(a)) \, dx = \sum_{T \in \mathcal{T}} \int_{T \cap \Omega} \rho(F(w_j) - F(a)) \, dx. \quad (5.1.3)$$

Using Proposition 5.1.3, take for each $T \in \mathcal{T}$ a sequence $u_{j,T} \equiv u_j \in C_c^\infty(T, \mathbb{V})$ such that $w_j - a - \mathcal{B}u_j \rightarrow 0$ in $L^p(T, \mathbb{V})$. By Lemma 5.1.2,

$$\int_T F(w_j) - F(a + \mathcal{B}u_j) \, dx \rightarrow 0$$

and since F is \mathcal{A} -quasiconvex, from Corollary 4.5.2,

$$\int_T F(a + \mathcal{B}u_j) - F(a) \, dx \geq 0.$$

Putting these together, we have shown that

$$\liminf_{j \rightarrow \infty} \int_T F(w_j) - F(a) \, dx \geq 0. \quad (5.1.4)$$

Take for each $T \in \mathcal{T}$ a point $x_T \in T$ and note that, from (5.1.3),

$$\begin{aligned} & \int_{\Omega} \rho(F(w_j) - F(a)) \, dx = \\ &= \sum_{T \in \mathcal{T}} \rho(x_T) \int_{T \cap \Omega} F(w_j) - F(a) \, dx + \int_{T \cap \Omega} (\rho - \rho(x_T))(F(w_j) - F(a)) \, dx \\ &\geq \sum_{T \in \mathcal{T}} \rho(x_T) \int_{T \cap \Omega} F(w_j) - F(a) \, dx - \max_{T \in \mathcal{T}} \text{diam } \rho(T) \int_{\Omega} C(1 + |w_j|^{p-1} + |a|^{p-1}) |w_j - a| \, dx. \end{aligned}$$

To bound the first term we use (5.1.4) and to bound the second we recall that $w_j - a = v_j - v$ and use the estimate (5.1.1) for a :

$$\liminf_{j \rightarrow \infty} \int_{\Omega} \rho(F(w_j) - F(a)) \, dx \geq -C \max_{T \in \mathcal{T}} \text{diam } \rho(T) \left[\int_{\Omega} 1 + |v|^p \, dx + \sup_j \int_{\Omega} |v_j|^p \, dx \right].$$

Since ρ has compact support it is uniformly continuous and since $\text{diam } T < \delta$ for $T \in \mathcal{T}$ we have that $\max_{T \in \mathcal{T}} \text{diam } \rho(T) \rightarrow 0$ as $\delta \rightarrow 0$. Finally, using (5.1.2) and sending $\varepsilon \rightarrow 0$ the conclusion follows. \square

The above proof can be easily adapted to the case where we do not assume that ρ has compact support, instead assuming that the negative part of the integrand has q -growth for $q < p$, see e.g. the proofs in [98, 271].

5.2 Null Lagrangians and weak continuity

We begin by recording the following definition:

Definition 5.2.1. Given a C^1 integrand $F: \mathbb{V} \rightarrow \mathbb{R}$, we say that it is an \mathcal{A} -null Lagrangian if it satisfies, in the sense of distributions,

$$\mathcal{B}^* (DF(\mathcal{B}u)) = 0, \quad (5.2.1)$$

for all $u \in C^k(\overline{\Omega}, \mathbb{U})$. When the choice of \mathcal{A} is implicit from the context we refer to such integrands simply as null Lagrangians.

We note that Definition 5.2.1 is *independent* of the choice of \mathcal{B} , and only depends on \mathcal{A} , see also Proposition 5.2.2 below. One can also consider null Lagrangians depending on lower order terms, as in [306], but we shall not pursue this here.

Having Theorem 5.1.4 at our disposal, we can give a first abstract characterization of \mathcal{A} -quasiaffine integrands under the main assumption (3.1.6); this will be improved in the next section and is further quantified in [182]. The following proposition is modelled on [26, Theorem 3.4].

Proposition 5.2.2. *Let $F: \mathbb{V} \rightarrow \mathbb{R}$ be locally bounded and Borel and let Ω be a bounded domain. The following are equivalent:*

- (i) F is \mathcal{A} -quasiaffine;
- (ii) F is an \mathcal{A} -null Lagrangian;
- (iii) $F: L^\infty_{\mathcal{A}}(\Omega) \rightarrow \mathbb{R}$ is sequentially weakly-* continuous;
- (iv) F is a polynomial of degree $s \leq \min\{n, \dim \mathbb{V}\}$ and

$$\left. \begin{array}{l} v_j \rightharpoonup v \quad \text{in } L^s(\Omega, \mathbb{V}) \\ \mathcal{A}v_j \rightarrow \mathcal{A}v \quad \text{in } W_{\text{loc}}^{-l,s}(\Omega, \mathbb{V}) \end{array} \right\} \implies F(v_j) \xrightarrow{*} F(v) \text{ in } \mathcal{D}'(\Omega).$$

In light of (ii) above we will sometimes call \mathcal{A} -quasiaffine maps *null Lagrangians*, as it is usual in the Calculus of Variations literature.

Proof: (i) \Leftrightarrow (iii): It is clear that (iii) holds if and only if, for any $\varphi \in C(\Omega)$, the functionals $u \mapsto \pm \int_{\Omega} \varphi(x) F(v(x)) dx$ are sequentially weakly* lower semicontinuous on $L^\infty_{\mathcal{A}}(\Omega)$. By Theorem 5.1.1 this happens if and only if F is \mathcal{A} -quasiaffine.

Clearly (iv) \Rightarrow (iii). We now prove (iii) \Rightarrow (iv), by an argument similar to the one in the first paragraph. It is well-known that F must be Λ -affine (see e.g. [361]), i.e. it is affine along lines parallel to Λ . Since $\text{span } \Lambda = \mathbb{V}$, it must be a polynomial of degree $s \leq \dim \mathbb{V}$ and the inequality $s \leq n$ follows from (5.2.2) below. We apply Theorem 5.1.4 to conclude that if the premise of the implication in (iv) holds then $\int_{\Omega} \varphi(x) F(x, v_j(x)) dx \rightarrow \int_{\Omega} \varphi(x) F(x, v(x)) dx$.

(i) \Rightarrow (ii): We already know that F is a polynomial so in particular it is smooth. Let us take $u_n, \varphi \in C_c^\infty(\Omega, \mathbb{R}^b)$ and $t > 0$. Then, by (4.5.1),

$$0 = \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega} F(\mathcal{B}u_n + t\mathcal{B}\varphi_n) dx = \sum_{i=1}^d \int_{\Omega} \frac{\partial F}{\partial v^i}(\mathcal{B}u_n) (\mathcal{B}\varphi)^i dx.$$

Choosing $u_n \rightarrow u$ in $C^k(\text{supp } \varphi)$, we obtain (ii). The converse direction is identical. \square

Most of the above proposition is essentially contained in the literature, as becomes clear from the proof. The only novelty is (iv), which improves the integrability required for MURAT's result [301] to hold: even in the simplest case where $\mathcal{B} = D^k$, it only follows from his result that a polynomial of degree three is sequentially weakly continuous as a map $W^{k,4}(\Omega) \rightarrow \mathcal{D}'(\Omega)$; this had already been observed and improved in [26], see also [322, 323], but here it is extended to an arbitrary constant rank operator.

While Proposition 5.2.2 gives an abstract characterization of null Lagrangians it is relevant to have an effective way of computing them. For an operator¹ \mathcal{A} not necessarily of constant rank TARTAR [361] showed that (iii) implies the algebraic condition

$$D^r F(v)[\lambda_1, \dots, \lambda_r] = 0 \text{ for all } (\lambda_1, \xi_1), \dots, (\lambda_r, \xi_r) \in V \text{ with } \text{rank}(\xi_1, \dots, \xi_r) < r \quad (5.2.2)$$

for all $v \in \mathbb{V}$ and all $r \geq 2$. MURAT [301] then proved that if moreover \mathcal{A} has constant rank then these conditions are in fact sufficient, i.e. (5.2.2) is equivalent to (iii). Unfortunately, it is in general unclear what are the polynomials, if any, satisfying the above restriction. MURAT [300, page 93] was already aware of this difficulty (emphasis not ours):

¹Strictly speaking, in [301, 361] it is assumed that \mathcal{A} is a first-order operator, but one can easily check that the proof carries through to the case where \mathcal{A} is a general homogeneous l -th order operator.

Encore faut-il, dans chaque cas particulier, *explicitement* quels sont les polynômes homogènes de degré r qui satisfont [(5.2.2)]. Cela conduit à des calculs algébriques qui sont parfois difficiles, voire inextricables.

Even in the case where $\mathcal{B} = \mathcal{D}^k$ it is by no means easy to find all the weakly continuous functions. The following result [26, Theorem 4.1] relies on deep algebraic facts:

Theorem 5.2.3. *Let $F: \odot^k(\mathbb{R}^n, \mathbb{R}^m) \rightarrow \mathbb{R}$ be continuous. Then $F = F(\mathcal{D}^k u)$ is \mathcal{D}^k -quasiaffine if and only if it is an affine combination of Jacobians of $U \equiv \mathcal{D}^{k-1}u$, by which we mean that there exist constants $c_M \in \mathbb{R}$ such that*

$$F = F(0) + \sum_M c_M M(DU)$$

where $M: \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ runs over all $s \times s$ minors of $N \times n$ matrices, for $N = \dim \odot^{k-1}(\mathbb{R}^n, \mathbb{R}^m)$ and $s = 1, \dots, \min\{n, N\}$.

It appears that this result was proved independently around the same time in [10]. We are interested in using the above theorem to make the characterization of Proposition 5.2.2 more explicit. Let us write, following [233, §4],

$$\mathcal{D}(n, k, \mathbb{U}) \equiv \left\{ u \otimes \xi^{\otimes k} : u \in \mathbb{U}, \xi \in \mathbb{R}^n \right\};$$

this cone spans $\odot^k(\mathbb{R}^n, \mathbb{U})$ and when $k = 1$ is the usual cone of rank-one linear transformations. Going back to (4.3.3), we note that it implies that, for $v \in \mathbb{V}$,

$$\mathcal{B}(\xi)v = T(v \otimes \xi^{\otimes k}).$$

Since $\text{im } \mathcal{B}(\xi) = \ker \mathcal{A}(\xi)$, it follows from the definition of Λ that T maps the cone $\mathcal{D}(n, k, \mathbb{U})$ onto Λ . The following straightforward lemma will be helpful:

Lemma 5.2.4. *If $F: \mathbb{V} \rightarrow \mathbb{R}$ is \mathcal{A} -quasiaffine then the composition $F \circ T$ is \mathcal{D}^k -quasiaffine; the converse also holds if $\text{span } \Lambda = \mathbb{V}$.*

Proof: We only prove the converse direction as the other one is absolutely similar, so suppose that F is \mathcal{D}^k -quasiaffine. By assumption, for each $v \in \mathbb{V}$ there is some $z \in \odot^k(\mathbb{R}^n, \mathbb{U})$ such that $Tz = v$. Then for any $u \in C_c^\infty(\Omega, \mathbb{U})$ we have

$$F(v) = F \circ T(z) = \int_{\Omega} F \circ T(z + \mathcal{D}^k u) \, dx = \int_{\Omega} F(v + \mathcal{B}u) \, dx$$

where we used the linearity of T and (4.3.3). This shows that $F \circ T$ is \mathcal{A} -quasiaffine. \square

Remark 5.2.5. An interesting takeaway from this lemma is that there seem to be two competing notions of polyconvexity [56]. We follow the usual definition in the curl-free case [28] and say that $F: \mathbb{V} \rightarrow \mathbb{R}$ is **\mathcal{A} -polyconvex** if it is the pointwise supremum of \mathcal{A} -quasiaffine functions; this is an intrinsic notion. Another possibility is to consider the class of functions F such that $F \circ T$ is \mathcal{D}^k -polyconvex. This class is contained in the class of \mathcal{A} -quasiconvex functions, as one readily checks by a calculation similar to the one in the proof of the lemma. Let us call such functions **extrinsically \mathcal{A} -polyconvex**. We have that

$$\text{convexity} \implies \mathcal{A}\text{-polyconvexity} \implies \text{extrinsic } \mathcal{A}\text{-polyconvexity} \implies \mathcal{A}\text{-quasiconvexity}$$

and in some cases the first two notions coincide, see Example 5.2.9 below, where $\mathcal{B} = \mathcal{E}$. In this case, $F(\mathcal{B}u) = \det \mathcal{E}u$ is extrinsic symmetric polyconvex, but not symmetric polyconvex. It is also clear that the intrinsic and extrinsic classes of polyconvex integrands can be the same, as it is the case when $\mathcal{B} = \mathcal{D}^k$. These notions have been further studied in the particular case where the integrands depend on differential forms [36].

Since we assume that $\text{span } \Lambda = \mathbb{V}$, we have that T is onto \mathbb{V} and the Rank–Nullity Theorem yields the linear isomorphism

$$\odot^k(\mathbb{R}^n, \mathbb{U}) \cong \ker T \oplus \text{im } T = \ker T \oplus \mathbb{V}. \quad (5.2.3)$$

Therefore we think of \mathbb{V} as a subspace of $\odot^k(\mathbb{R}^n, \mathbb{U})$ and of T as a projection onto that subspace. The utility of this viewpoint is illustrated by the previous results: under the assumptions of the lemma, the map $F \circ T$ is an affine combination of Jacobians and under the identification (5.2.3) we can in fact think of F as real-valued map defined on $\mathbb{V} \subseteq \odot^k(\mathbb{R}^n, \mathbb{U})$. Thus, we have shown:

Proposition 5.2.6. *Let $F: \mathbb{V} \rightarrow \mathbb{R}$ be an \mathcal{A} -quasiaffine integrand. Then, under the identification (5.2.3), we can find constants $c_M \in \mathbb{R}$ such that*

$$F \circ T = F(0) + \sum_M c_M M, \quad (5.2.4)$$

where $M: \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$, $N = \dim \odot^{k-1}(\mathbb{R}^n, \mathbb{R}^m)$, runs over all minors of $N \times n$ matrices.

In other words, in the right coordinates, \mathcal{A} -quasiaffine maps are precisely the Jacobians.

It is natural to ponder for a moment whether one can hope for a more invariant statement. The crucial point here is that proper minors, i.e. minors which are not the determinant, have no intrinsic geometric content, in the sense that they are not invariant under changes of coordinates. We make this well-known fact very precise in the following remark.

Remark 5.2.7. Assume that $m \neq n$. A (non-trivial) linear isomorphism $L \in \text{GL}(\mathbb{R}^m \otimes \mathbb{R}^n)$ maps minors into minors, i.e. $M \circ L: \mathbb{R}^{m \times n} \cong \mathbb{R}^m \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ is a minor whenever $M: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is a minor, if and only if

$$L = R \otimes S \quad \text{for some } R \in \text{GL}(m), S \in \text{GL}(n). \quad (5.2.5)$$

This follows from the fact that minors are precisely the rank-one affine functions (and that they are affine only along rank-one lines) and that T maps the rank-one cone into itself if and only if it has the form (5.2.5), see [274, Theorem 1]. This shows the intuitive fact that minors are closely tied with the tensor product structure of the vector space $\mathbb{R}^m \otimes \mathbb{R}^n$ and that, to make sense of them, one should not forget this structure and think of it instead as a generic vector space of dimension $m \times n$.

Remark 5.2.8. ROBBIN–ROGERS–TEMPLE [330, §5.2] asked whether all weakly continuous functions could be obtained in a framework with differential forms. Proposition 5.2.6 gives a positive answer to this question under the main assumption (3.1.6). We refer the reader to the works [208, 340] for further properties of null Lagrangians depending on differential forms.

The above discussion shows that the choice of coordinates (5.2.3) is in some sense very arbitrary. Nonetheless, the identification (5.2.3) also turns out to be computationally effective. The computational problem is to decide which, if any, of the constants c_M that appear in (5.2.4) can be taken to be non-zero. The key to solving this problem is the immediate fact that, if $H: \odot^k(\mathbb{R}^n, \mathbb{U}) \rightarrow \mathbb{R}$ denotes the right-hand side of (5.2.4), then

$$H = H \circ T.$$

We think of both sides of this equality as being polynomials in the algebraically independent variables x_{i_1, \dots, i_k} , $i_j \in \{1, \dots, n\}$, that define an element $X = (x_{i_1, \dots, i_k}) \in \odot^k(\mathbb{R}^n, \mathbb{U})$. Since both sides are equal as polynomials, all the coefficients must be the same. Noting that the coefficients of these polynomials depend linearly on $(c_M)_M$, we find from the equality of coefficients a linear system for the c_M whose solution determines completely the possible null Lagrangians. This system can in turn be solved using symbolic computation software. One can also fix a specific order of the minors in (5.2.4), say s , and solve instead the above system with H replaced by

$$H_s \equiv \sum_{\deg M=s} c_M M$$

since minors of different orders cannot cancel each other out. For the sake of concreteness, we illustrate this method with simple examples.

Example 5.2.9. Let $T = P_{\text{sym}}$, where $P_{\text{sym}}: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n} = \mathbb{V}$ is the orthogonal projection, i.e. $\mathcal{B} = \mathcal{E}$ is the symmetric gradient. The algorithm described above can be very easily implemented; in *Mathematica* a possible implementation is given in Code Listing 5.1.

```

sym[X_] := (X+Transpose[X])/2;
X = Array[Subscript[x, #1, #2]&, {n, n}];
const = Array[Subscript[c, #]&, Binomial[n, s]^2];

Solve[
  DeleteCases[
    CoefficientList[
      const.(Flatten[Minors[sym[X], s] - Minors[X, s]]), Flatten[X]
    ]//Flatten, 0
  ] == 0
]

```

Code Listing 5.1: A possible implementation of the algorithm in the setup of Example 5.2.9

In this case, however, it is relatively easy to verify analytically that there are no non-affine null Lagrangians (when $n = 2, 3$, this was proved in [56] as a consequence of more general statements). For this, it suffices to consider the case where the null Lagrangians are homogeneous polynomials of degree 2. Indeed, if F is an s -homogeneous null Lagrangian then $\partial F/\partial v$ is an $(s-1)$ -homogeneous null Lagrangian, where v is any vector from \mathbb{V} ; this follows straightforwardly from (4.5.1). Thus, if we prove that there are no null Lagrangians with order two then there can be no higher order null Lagrangians.

From the relation $H_2 = H_2 \circ T$ we deduce that, for any $X \in \mathbb{R}^{n \times n}$, $H_2(X) = H_2(X^T)$. Given a 2×2 minor M , let \widetilde{M} be the minor defined by $\widetilde{M}(X) \equiv M(X^T)$; in particular $\widetilde{M} = M$ if M is a principal minor. For the sake of concreteness, let us say that $M(X) = \det[(x_{i,j})_{i \in I, j \in J}]$ where $I = \{i_1, i_2\}$, $J = \{j_1, j_2\} \subset \{1, \dots, n\}$. If we let $X = (x_{i,j})$ be such that

$$x_{i,j} = \begin{cases} 1 & (i,j) = (i_k, j_k) \text{ for } k \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

then

$$c_M = c_M M(X) = H_s(X) = H_s(X^T) = c_{\widetilde{M}} \widetilde{M}(X) = c_{\widetilde{M}}.$$

Now let $Y = X - X^T$ and observe that, since $M(Z) = M(-Z)$ for any $Z \in \mathbb{R}^{n \times n}$,

$$c_M + c_{\widetilde{M}} = c_M M(X) + c_{\widetilde{M}} \widetilde{M}(X) = H_2(Y) = H_2(T(Y)) = 0.$$

The conclusion follows.

Example 5.2.10. Another relevant example is that of solenoidal matrix fields, i.e. $\mathcal{A} = \text{div}$, which can be embedded in the framework of exterior derivatives of differential forms. As above, we are particularly interested in null Lagrangians of degree (at least) two. We will consider divergence-free fields $v: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ for $n = 2, 3$. For $n = 2$, we can set²

$$v = \begin{pmatrix} \partial_2 u_1 & -\partial_1 u_1 \\ \partial_2 u_2 & -\partial_1 u_2 \end{pmatrix} = (Du)J, \quad \text{where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and note that $H_2 = c \det$ for $c \in \mathbb{R}$. To see that this is indeed a null Lagrangian, we need only observe that $\det X = \det(XJ)$ for $X \in \mathbb{R}^{2 \times 2}$.

For $n = 3$, we will show that there are no (homogeneous) quadratic null Lagrangians. First, recall that curl is a potential operator for div in this case, which we write in the form

$$\mathcal{B}u \equiv P_{\text{asym}} Du, \quad \text{for } u: \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

²In this simple case, an example of a potential operator \mathcal{B} is easily chosen by inspection.

where $T \equiv P_{\text{asym}}$ denotes the orthogonal projection onto anti-symmetric matrices. We will test the relation $H_2(X) = H_2(T(X))$ with different matrices $X \in \mathbb{R}^{3 \times 3}$ to show that $H_2 = 0$, since this is enough to show that there are no non-affine null Lagrangians (see also Example 5.2.9). First, note that by taking $X = e_i \otimes e_i + e_j \otimes e_j$, $i \neq j$, the coefficients of the principal minors in H_2 must be zero. The other 2×2 minors touch the main diagonal on exactly one entry, say (i, i) . By taking $X = ae_i \otimes e_i + e_j \otimes e_k$, $j \neq i \neq k$, for $a \in \mathbb{R}$, we see that indeed $H_2 = 0$.

For general dimension $n \geq 3$, it is not too difficult to see that there are no non-affine div-null Lagrangians.

It would be interesting to give a theoretical characterization of the solutions of the computational problem. This is also a relevant question since the linear system described above grows factorially in $\dim \mathbb{V}$, although in applications to continuum mechanics this number is usually relatively small. Unfortunately, even in the special case when \mathcal{B} has order one such a characterization seems difficult. We were unable to give a definitive answer even to the following simple-looking question.

Assume we are given a projection $T: \mathbb{R}^{m \times n} \rightarrow \mathbb{V}$, which can be chosen to be orthogonal, onto some subspace $\mathbb{V} \subseteq \mathbb{R}^{m \times n}$. Consider a function $H_s: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ as above, i.e.

$$H_s(X) = \sum_{\deg M=s} c_M M(X), \quad H_s = H_s \circ T$$

where the sum runs over the set of all minors (not necessarily principal) of order $2 \leq s < \min\{m, n\}$. The second condition can be equivalently rewritten as

$$H_s(X) = H_s(X + Y) \text{ for all } X, \text{ all } Y \text{ such that } T(Y) = 0. \quad (5.2.6)$$

We think of this as saying that the linear combination of minors H_s only depends on the coordinates of \mathbb{V} .

Question 5.2.11. Under which conditions on T can we find non-zero H_s satisfying (5.2.6)? Can we characterize such H_s in terms of \mathbb{V} ?

5.3 Compensated compactness in Hardy spaces

We begin by stating the main theorem of this section; as usual, we assume (3.1.6) holds throughout. Recall that \mathcal{A} -quasiaffine maps are polynomials (c.f. Proposition 5.2.2) and see Definition 4.3.2 for the definition of $\mathbb{I}_{\mathcal{A}}$.

Theorem 5.3.1. *Let $F: \mathbb{V} \rightarrow \mathbb{R}$ be locally bounded and Borel. If the implication*

$$v \in C_{c, \mathcal{A}}^\infty(\mathbb{R}^n) \quad \implies \quad F(v) \in \mathcal{H}^1(\mathbb{R}^n) \quad (5.3.1)$$

holds, F is a sum of homogeneous \mathcal{A} -quasiaffine functions of degree at most $\min\{n, \dim \mathbb{V}\}$.

Conversely, assume that F is an s -homogeneous \mathcal{A} -quasiaffine function. If $s \geq 2$ then (5.3.1) holds and moreover

$$\|F(v)\|_{\mathcal{H}^1} \leq C \|v\|_{L^s}^s \quad \text{for } v \in C_{c, \mathcal{A}}^\infty(\mathbb{R}^n).$$

If $s = 1$, we have that $F(v) = v_0 \cdot v$ for some $v_0 \in \mathbb{V}$ and (5.3.1) holds if and only if $v_0 \perp \mathbb{I}_{\mathcal{A}}$, although nonetheless the above estimate fails.

It will be convenient to prove the homogeneous case first. We will deal with the linear case, which is somewhat degenerate, afterwards.

Proposition 5.3.2. *Let F be a homogeneous polynomial of degree $2 \leq s \leq \min\{n, \dim \mathbb{V}\}$. The following statements are equivalent:*

- (i) $\int_{\mathbb{R}^n} F(v) = 0$ for all $v \in C_{c, \mathcal{A}}^\infty(\mathbb{R}^n)$.

(ii) $\|F(v)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq c\|v\|_{L^s(\mathbb{R}^n)}^s$ whenever $v \in L^s_{\mathcal{A}}(\mathbb{R}^n)$.

Observe that the direction (ii) \Rightarrow (i) is clear, since functions in $\mathcal{H}^1(\mathbb{R}^n)$ have zero mean. To prove the estimate, we follow the original strategy in [87]. In fact, we will use the potential \mathcal{B} and Lemma 5.2.4 to show that the estimate can be inferred from the case $\mathcal{B} = D^k$. The statement for $v = D^k u$ is then known from [264, Theorem 6.2]; here we give a proof by reduction to the div-curl case.

We emphasize the technical fact that the assumption $s \leq n$ will be important in order to apply the Poincaré–Sobolev inequality. Given a ball $B_t(x) \subset \mathbb{R}^n$ we write $(f)_{x,t} \equiv \int_{B_t(x)} f(y) dy$.

Proof of Proposition 5.3.2: From (4.4.2) we see that it is enough to bound $F(\mathcal{B}u)$ for $u \in C_c^\infty(\mathbb{R}^n, \mathbb{U})$. Recalling Lemma 5.2.4, it is natural to first deal with the case $\mathcal{B} = D^k$. This case is already known from [264], but here we give a simpler proof, at least as far as notation is concerned.

We claim that if $\int_{\mathbb{R}^n} F(D^k u) dx = 0$ for $u \in C_c^\infty(\mathbb{R}^n)$ then there is an estimate

$$\|F(D^k u)\|_{\mathcal{H}^1} \leq C\|D^k u\|_{L^s}^s \text{ for } u \in C_c^\infty(\mathbb{R}^n). \quad (5.3.2)$$

The assumption implies that F is D^k -quasiaffine at zero, and hence everywhere, c.f. the proof of Theorem 5.3.1 below. By Theorem 5.2.3 and s -homogeneity, we see that F is a linear combination of minors of order s of DU , where $U \equiv D^{k-1}u$, i.e.

$$F(D^k u) = \sum_{\deg M=s} c_M M(DU).$$

Thus, it is sufficient to prove the estimate in the case $F(D^k u) = M(DU)$. We choose coordinates $x = (x', x'') \in \mathbb{R}^n$ and $T = (T', T'') \in \odot^{k-1}(\mathbb{R}^n, \mathbb{U})$, where x', T' are s -dimensional. Then we can write

$$M(DU(x)) = \det D_{x'} U'(x).$$

Note that $D_{x'}$ can be regarded as a differential operator on \mathbb{R}^n .

To prove the claim, one can use the reasoning used in the proof of [87, Theorem II.1.1]). By looking at the $(1, 1)$ entry of the identity $(\det A)\text{Id} = A(\text{cof } A)^T$ applied to $A = Df$, $f: \mathbb{R}^s \rightarrow \mathbb{R}^s$, we see that $\det Df = Df_1 \cdot \sigma$, where σ is the first row of the matrix $\text{cof } Df$, which is row-wise divergence-free, and moreover we have the pointwise estimate $|\sigma| \lesssim |Df_2||Df_3| \dots |Df_s|$. In our case, it is elementary to adapt these considerations to see that

$$M(DU) = \langle D_{x'} U'_1(x), \Sigma(x) \rangle_{\mathbb{R}^s}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^s}$ is the usual Euclidean inner product and $\Sigma: \mathbb{R}^n \rightarrow \mathbb{R}^s$ is such that

$$\text{div}_{x'} \Sigma = 0 \text{ in } \mathbb{R}^n \quad \text{and} \quad |\Sigma| \lesssim |DU'_2||DU'_3| \dots |DU'_s|.$$

Here $\text{div}_{x'} = D_{x'}^*$ is the adjoint of the differential operator $D_{x'}$.

Let $\psi \in C_c^\infty(B_1(0))$ be a non-negative function with non-zero mean. We have

$$\begin{aligned} |\psi_t * M(DU)|(x) &= \left| \frac{1}{t^n} \int_{\mathbb{R}^n} \psi \left(\frac{x-y}{t} \right) \langle D_{x'} U'_1(y), \Sigma(y) \rangle_{\mathbb{R}^s} dy \right| \\ &= \left| \frac{1}{t^n} \int_{B_t(x)} \left\langle D_{x'} [U'_1(y) - (U'_1)_{x,t}], \psi \left(\frac{x-y}{t} \right) \Sigma(y) \right\rangle_{\mathbb{R}^s} dy \right| \\ &= \left| \frac{1}{t^{n+1}} \int_{B_t(x)} (U'_1(y) - (U'_1)_{x,t}) \left\langle (D_{x'} \psi) \left(\frac{x-y}{t} \right), \Sigma(y) \right\rangle_{\mathbb{R}^s} dy \right| \\ &\lesssim \frac{1}{t^{n+1}} \int_{B_t(x)} |U'_1(y) - (U'_1)_{x,t}| |\Sigma(y)| dy, \end{aligned}$$

where in the third equality we integrated by parts, using the fact that $\operatorname{div}_{x'} \Sigma = 0$. We apply Hölder's inequality with $p = nq/(n+q)$ for some $q \in (1, s)$ to get

$$\begin{aligned} |\psi_t * M(DU)|(x) &\lesssim \frac{1}{t} \left(\int_{B_t(x)} |U'_1(y) - (U'_1)_{x,t}|^p dy \right)^{1/p} \left(\int_{B_t(x)} |\Sigma(y)|^{p'} dy \right)^{1/p'} \\ &\lesssim \left(\int_{B_t(x)} |DU'_1(y)|^q dy \right)^{1/q} \left(\int_{B_t(x)} |\Sigma(y)|^{p'} dy \right)^{1/p'} \end{aligned}$$

where we also used the Poincaré–Sobolev inequality; note that the implicit constant does not depend on t . We further ensure that $p' = p/(p-1) < s/(s-1) = s'$ by requiring $q > ns/(n+s)$. We next note that, writing \mathcal{M} for the Hardy–Littlewood maximal function,

$$\begin{aligned} \sup_{t>0} |\psi_t * M(DU)|(x) &\lesssim \sup_{t>0} \left[\left(\int_{B_t(x)} |DU'_1(y)|^q dy \right)^{1/q} \left(\int_{B_t(x)} |\Sigma(y)|^{p'} dy \right)^{1/p'} \right] \\ &\leq \sup_{t>0} \left(\int_{B_t(x)} |DU'_1(y)|^q dy \right)^{1/q} \sup_{t>0} \left(\int_{B_t(x)} |\Sigma(y)|^{p'} dy \right)^{1/p'} \\ &= \mathcal{M}(|DU'_1|^q)(x)^{1/q} \mathcal{M}(|\Sigma|^{p'})(x)^{1/p'} \end{aligned}$$

Integrating this estimate with respect to x and applying Hölder's inequality twice we obtain

$$\begin{aligned} \|M(DU)\|_{\mathcal{H}^1(\mathbb{R}^n)} &\lesssim \left(\int_{\mathbb{R}^n} \mathcal{M}(|DU'_1|^q)^{s/q} \right)^{1/s} \left(\int_{\mathbb{R}^n} \mathcal{M}(|\Sigma|^{p'})^{s'/p'} \right)^{1/s'} \\ &\lesssim \left(\int_{\mathbb{R}^n} |DU'_1|^s \right)^{1/s} \left(\int_{\mathbb{R}^n} |\Sigma|^{s'} \right)^{1/s'} \\ &\lesssim \left(\int_{\mathbb{R}^n} |DU'_1|^s \right)^{1/s} \left(\int_{\mathbb{R}^n} \prod_{i=2}^s |DU'_i|^{s/(s-1)} \right)^{(s-1)/s} \\ &\leq \prod_{i=1}^s \|DU'_i\|_{L^s(\mathbb{R}^n)} \leq C \|DU'\|_{L^s(\mathbb{R}^n)}^s, \end{aligned}$$

where moreover the second inequality follows by boundedness of the maximal function. This proves the desired claim (5.3.2).

To conclude the proof, we return to the case of a general \mathcal{B} and use Lemma 5.2.4:

$$\|F(\mathcal{B}u)\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|F \circ T(D^k u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|D^k u\|_{L^s(\mathbb{R}^n)}^s \leq C \|\mathcal{B}u\|_{L^s(\mathbb{R}^n)}^s,$$

where the last estimate follows from Theorem 4.2.1, since the left-hand side is kept unchanged by replacing u with $u - P_{\mathcal{B}}u$. \square

Remark 5.3.3. It is possible to give a more abstract proof of the above proposition in the spirit of [264, 346], circumventing the explicit representation of null Lagrangians from [26]. The basic idea is that, since both F and \mathcal{B} are homogeneous, we can write

$$F(\mathcal{B}u) = \sum_{\nu \in \{1, \dots, \dim U\}^s} \sum_{|\beta_1|, \dots, |\beta_s| = k} f_{\beta, \nu} \prod_{i=1}^s \partial^{\beta_i} u^{\nu_i}$$

for some constants $f_{\beta, \nu} \in \mathbb{R}$, where each β_i is an n -multi-index. Using the Leibniz rule together with the cancellation assumption (i) we have, after some elementary calculations,

$$\int_{\mathbb{R}^n} \psi_t(x-y) F(\mathcal{B}u(y)) dy = - \sum_{\beta, \nu} \frac{f_{\beta, \nu}}{t^n} \int_{\mathbb{R}^n} \prod_{i=1}^s \sum_{\gamma < \beta} c_{\beta, \gamma} \partial^{\beta_i - \gamma_i} \phi \left(\frac{x-y}{t} \right) \partial^{\gamma_i} u^{\nu_i}(y) dy$$

where by $\gamma < \beta$ we mean that there is some i such that $\gamma_i < \beta_i$ as multi-indices and $\psi \equiv \phi^s$. The point is that, for each (β, ν) fixed, at least one of the terms on the right has one less derivative than the others. Therefore, subtracting enough moments from u , we see from the Poincaré–Sobolev inequality that this term has higher integrability than the others. One then concludes by suitably applying Hölder’s inequality, similarly to above.

In order to deduce the theorem from the proposition we need to justify the assumption $s \geq 2$. This will be done in the following lemma, which proves a non-inclusion of $L^1_{\mathcal{A}}(\mathbb{R}^n)$ into $\mathcal{H}^1(\mathbb{R}^n)$ and which is somewhat reminiscent of the much deeper Ornstein’s non-inequality [307, 233]. The common theme is, of course, the lack of boundedness of singular integrals on generic subspaces of L^1 , c.f. Proposition 2.1.7. Recall that we assume (3.1.6).

Lemma 5.3.4. *Let $v_0 \in \mathbb{V}$ be a non-zero vector. Then there exists a sequence $v_j \in C^\infty_{c,\mathcal{A}}(\mathbb{R}^n)$ such that $\|v_0 \cdot v_j\|_{\mathcal{H}^1} \geq j$ but $\|v_j\|_{L^1} \leq 1$ for all $j \geq 1$.*

Proof: By the spanning cone condition, there exists non-zero $\tilde{v}_0 \in \mathbb{V}$ and $\xi \in \mathbb{R}^n$ such that $\tilde{v}_0 \in \ker \mathcal{A}(\xi) = \text{im } \mathcal{B}(\xi)$, say $\mathcal{B}(\xi)u_0 = \tilde{v}_0$, and $\tilde{v}_0 \cdot v_0 \neq 0$. Note that if $u(x) = f(x \cdot \xi)u_0$ for some $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\mathcal{B}u(x) = f^{(k)}(x \cdot \xi)\mathcal{B}(\xi)u_0$. In particular, by choosing $f(t) = \max\{t, 0\}^{k-1}$, we obtain that $\mathcal{B}u = (k-1)!\tilde{v}_0(\mathcal{H}^{n-1} \llcorner \{x \cdot \xi = 0\})$.

By defining $\tilde{u} = \rho u$ for some test function ρ that equals one in a neighbourhood of the unit ball, we obtain a compactly supported \mathcal{A} -free measure $\mathcal{B}\tilde{u}$ that is not absolutely continuous.

We now explain how the proof can be concluded easily. Assume for contradiction that the claim of the lemma fails, so that there is a bound

$$\|v \cdot v_0\|_{\mathcal{H}^1} \leq C\|v\|_{L^1} \quad \text{for } v \in C^\infty_{c,\mathcal{A}}(\mathbb{R}^n).$$

Consider a sequence of mollifications \tilde{u}_ε , so that $\tilde{u}_\varepsilon \in C^\infty_c(\mathbb{R}^n)$ and $\mathcal{B}\tilde{u}_\varepsilon \xrightarrow{*} \mathcal{B}\tilde{u}$ as measures. The estimate implies

$$\|\mathcal{B}\tilde{u}_\varepsilon \cdot v_0\|_{\mathcal{H}^1} \leq C \sup_{\varepsilon \in (0,1)} \|\mathcal{B}\tilde{u}_\varepsilon\|_{L^1} < \infty,$$

and so, up to subsequences, $(\mathcal{B}u_\varepsilon \cdot v_0)_\varepsilon$ is convergent in \mathcal{H}^1 . It follows that $\mathcal{B}u \cdot v_0 \in \mathcal{H}^1$, so $\mathcal{B}u \cdot v_0$ is absolutely continuous, which leads to a contradiction since $\tilde{v}_0 \cdot v_0 \neq 0$. \square

We are finally ready to finish the proof.

Proof of Theorem 5.3.1: Note that if (5.3.1) holds then F is \mathcal{A} -quasiaffine at zero, i.e. we have (4.5.1) with $z = 0$: if $u \in C^\infty_c(\mathbb{R}^n, \mathbb{U})$ then $\mathcal{B}u \in C^\infty_{c,\mathcal{A}}$ and therefore $\int_{\mathbb{R}^n} F(\mathcal{B}u) = 0$ since functions in the Hardy space have zero mean. Moreover, if F is \mathcal{A} -quasiaffine at zero then it is quasiaffine everywhere. To see this, fix $z \in \mathbb{V}$ and $u \in C^\infty_c(\mathbb{R}^n, \mathbb{U})$. Let $\phi \in C^\infty_c(\mathbb{R}^n, \mathbb{U})$ be chosen so that $\mathcal{B}\phi = z$ in the support of u ; thus $\int_{\mathbb{R}^n} F(t\mathcal{B}\phi + \mathcal{B}u) = 0$ for any $t \in \mathbb{R}$. Then, since $F(t\mathcal{B}\phi + \mathcal{B}u) = F(t\mathcal{B}\phi)$ outside the support of u ,

$$0 = \frac{d}{dt} \int_{\mathbb{R}^n} F(t\mathcal{B}\phi + \mathcal{B}u) - F(t\mathcal{B}\phi) \, dx = \frac{d}{dt} \int_{\mathbb{R}^n} F(tz + \mathcal{B}u) - F(tz) \, dx$$

so the right-hand side is constant. In particular, comparing the values at $t = 1$ and $t = 0$,

$$\int_{\mathbb{R}^n} F(z + \mathcal{B}u) - F(z) \, dx = \int_{\mathbb{R}^n} F(\mathcal{B}u) \, dx = 0,$$

as wished.

Since F is \mathcal{A} -quasiaffine, it is a polynomial, which we write as a sum of homogeneous terms as $F = \sum_{s=0}^n P_s$. In fact, it is clear that $P_0 = 0$. We note that

$$0 = \int_{\mathbb{R}^n} F(t\mathcal{B}u) \, dx = \sum_{s=1}^n t^s \int_{\mathbb{R}^n} P_s(\mathcal{B}u) \, dx$$

for all $t \in \mathbb{R}$ and u fixed. This implies that each P_s is \mathcal{A} -quasiaffine as well.

Conversely, if F is \mathcal{A} -quasiaffine then it is continuous and, given $v \in C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ we have, from Proposition 4.4.3, a sequence $u_j \in C_c^\infty(\mathbb{R}^n, \mathbb{U})$ such that $\mathcal{B}u_j \rightarrow v$ in $L^p(\mathbb{R}^n, \mathbb{V})$. Therefore

$$0 = \int_{\mathbb{R}^n} F(\mathcal{B}u_j) dx \rightarrow \int_{\mathbb{R}^n} F(v) dx \quad \text{as } j \rightarrow \infty,$$

so we can use Proposition 5.3.2 to see that (5.3.1) and the required estimate for s -homogeneous F , $s \geq 2$, holds.

Finally, let F be linear, say $F(v) = v_0 \cdot v$. By Lemma 5.3.4, there can be no uniform estimate in this case. Moreover, if v_0 is not orthogonal to $\mathbb{I}_{\mathcal{A}}$, we consider $v_1 \in \mathbb{I}_{\mathcal{A}}$ be such that $v_0 \cdot v_1 \neq 0$ and a scalar test field $\rho \in C_c^\infty(\mathbb{R}^n)$ with non-zero integral. Then $\rho v_1 \in C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ but $F(\rho v_1)$ is not in the Hardy space. On the other hand, if v_0 is orthogonal to $\mathbb{I}_{\mathcal{A}}$, we write $v = v_1 + v_2$ for the decomposition of $v \in C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ such that $v_1 \in C_c^\infty(\mathbb{R}^n, \mathbb{I}_{\mathcal{A}})$ and $v_2 \in C_{c,\mathcal{A}}^\infty(\mathbb{R}^n)$ (recall Lemma 4.3.4 and its notation). We then have that $F(v) = v_0 \cdot v_2$, which is a test function with zero integral, as is v_2 by Lemma 4.3.3. It follows that $F(v)$ lies in $\mathcal{H}^1(\mathbb{R}^n)$. The proof is complete. \square

We remark that Theorem 5.3.1 seems to contradict [264, Proposition 6.3], but unfortunately there appears to be a mistake in the calculation presented there. As a simple consequence of the theorem, we have:

Corollary 5.3.5. *If F is an s -homogeneous \mathcal{A} -null Lagrangian, $s \geq 2$, then*

$$F: (L_{\mathcal{A}}^s(\mathbb{R}^n), w) \rightarrow (\mathcal{H}^1(\mathbb{R}^n), w^*) \text{ is sequentially continuous.}$$

Proof: Given a sequence $v_j \in L_{\mathcal{A}}^s(\mathbb{R}^n)$ such that $v_j \rightharpoonup v$ in L^s , we have from (iv) of Proposition 5.2.2 that

$$\int_{\mathbb{R}^n} \varphi F(v_j) dx \rightarrow \int_{\mathbb{R}^n} \varphi F(v) dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

Since $F(v_j), F(v)$ are uniformly bounded in $\mathcal{H}^1(\mathbb{R}^n)$, and by density of test functions in $\text{VMO}(\mathbb{R}^n)$, we can replace C_c^∞ by VMO above; in this case, the integrals should be thought of as shorthand notation for the duality pairing. \square

The utility of Hardy space bounds when dealing with weakly converging sequences is apparent, for instance, from Theorem 2.1.6. To conclude this section we provide some concrete examples which illustrate the way in which Theorem 5.3.1 contains the examples of [87].

Example 5.3.6 (Stationary Maxwell system). Let $E, B \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be such that

$$\operatorname{div} E = 0, \quad \operatorname{curl} B = 0.$$

Then the vector field (E, B) is \mathcal{A} -free, where of course $\mathcal{A} = (\operatorname{div}, \operatorname{curl})$, which is a constant rank operator. The quantity $E \cdot B$ is easily seen to be \mathcal{A} -quasiaffine: indeed, writing $B = Du$ for some smooth u ,

$$\int_{\mathbb{R}^n} E(x) \cdot B(x) dx = - \int_{\mathbb{R}^n} u(x) \operatorname{div} E(x) dx = 0.$$

Therefore, from the theorem,

$$\|E \cdot B\|_{\mathcal{H}^1} \lesssim \|(E, B)\|_2.$$

In particular, and arguing by density, we see that the same holds if $B, E \in L^2(\mathbb{R}^n, \mathbb{R}^n)$.

A generalization of the previous example for quadratic forms was given in [261], even without assuming that \mathcal{A} has constant rank.

Example 5.3.7 (Double cancellation). Let us take vector fields $U, V \in L^2(\mathbb{R}^n, \mathbb{R}^{n \times n})$; again we shall first argue formally as the general case can be recovered by density. We introduce the constant rank operator

$$\mathcal{A} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \text{D}(\text{tr } U) \\ \text{curl } U \\ \text{curl } V \end{bmatrix}.$$

Note that an \mathcal{A} -free test vector field (U, V) can be written as $U = \text{D}u$ and $V = \text{D}v$, where moreover $\text{div } u = 0$ since $\text{div } u = \text{tr } U$ is both constant and zero outside a compact set. The function $F(U, V) = \langle U^T, V \rangle = \sum_{i,j} U^{j,i} V^{i,j}$ is \mathcal{A} -quasiaffine:

$$\int_{\mathbb{R}^n} F(U, V) = \int_{\mathbb{R}^n} \sum_{i,j} \partial^i u^j \partial^j v^i = \int_{\mathbb{R}^n} \text{div } u \text{ div } v = 0.$$

Therefore, from the theorem,

$$\left\| \int_{\mathbb{R}^n} \sum_{i,j} \partial^j u_i \partial^i v_j \right\|_{\mathcal{H}^1} \lesssim \|Du\|_2 \|Dv\|_2$$

whenever u is divergence-free.

Example 5.3.8 (Monge-Ampère). Let \mathcal{A} be an annihilator for D^2 . Given $U, V \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$, the map

$$F(U, V) = U_{11}V_{22} + U_{22}V_{11} - 2U_{12}V_{12}$$

is \mathcal{A} -quasiaffine: writing $U = D^2u, V = D^2v$, we have

$$\int_{\mathbb{R}^n} F(U, V) = \int_{\mathbb{R}^n} \partial_{xx}u \partial_{yy}v + \partial_{yy}u \partial_{xx}v - 2\partial_{xy}u \partial_{xy}v \equiv \int_{\mathbb{R}^n} [u, v] = 0$$

by integration by parts. Thus

$$\|[u, v]\|_{\mathcal{H}^1} \lesssim \|D^2u\|_2 \|D^2v\|_2.$$

Chapter 6

Oscillations and concentrations under constant rank constraints

In this chapter we provide refinements of several results of FONSECA–MÜLLER [149]. In particular, in Section 6.3 we prove an improved lower semi-continuity result and in Section 6.4 we characterize the generalised Young measures arising from sequences satisfying a constant rank constraint by duality with \mathcal{A} -quasiconvex functions. The results of this chapter are joint work with KRISTENSEN and RAIȚĂ [181].

6.1 Statement of the main results

We assume throughout this chapter that \mathcal{A} is an operator satisfying (3.1.6). The first result of this chapter is the following enhanced decomposition lemma:

Lemma 6.1.1. *Let \mathcal{A} as in (3.1.1) be a constant rank operator with potential operator \mathcal{B} such that (4.3.1) holds. Let $1 < p < \infty$ and*

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \rightarrow \mathcal{A}v \text{ in } W^{-\ell, p}(\Omega, \mathbb{W})$$

generate a p -Young measure $\nu = \left((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \bar{\Omega}} \right)$. Then there exist sequences $(u_j), (\tilde{u}_j) \subset C_c^\infty(\Omega, \mathbb{U})$ and $(\tilde{b}_j) \subset L^p(\Omega, \mathbb{V})$ such that

$$\begin{aligned} v_j &= v + \mathcal{B}u_j + \mathcal{B}\tilde{u}_j + \tilde{b}_j, \\ \mathcal{B}u_j, \mathcal{B}\tilde{u}_j, \tilde{b}_j &\rightharpoonup 0 \text{ in } L^p(\Omega, \mathbb{V}), \\ (D^k u_j) &\text{ is } p\text{-uniformly integrable,} \\ D^k \tilde{u}_j &\rightarrow 0 \text{ in } \mathcal{L}^n\text{-measure,} \\ \tilde{b}_j &\rightarrow 0 \text{ in } L_{\text{loc}}^p(\Omega, \mathbb{V}), \end{aligned}$$

and, moreover, in $Y^p(\Omega, \mathbb{V})$,

$$\begin{aligned} (v + \mathcal{B}u_j) &\text{ generates } \left((\nu_x)_{x \in \Omega}, 0, \mathbf{n}/\mathbf{a} \right), \\ (\mathcal{B}\tilde{u}_j) &\text{ generates } \left((\delta_0)_{x \in \Omega}, \lambda \llcorner \Omega, (\nu_x^\infty)_{x \in \Omega} \right), \\ (\tilde{b}_j) &\text{ generates } \left((\delta_0)_{x \in \Omega}, \lambda \llcorner \partial\Omega, (\nu_x^\infty)_{x \in \partial\Omega} \right). \end{aligned}$$

For a complete proof of Lemma 6.1.1 we refer the reader to [181]. In fact, a decomposition of the form $v_j = \mathcal{B}u_j + b_j$, where $(\mathcal{B}u_j)$ captures the oscillation and (b_j) captures the concentration in $\bar{\Omega}$, is also possible under a weaker constraint. In that case, we can improve slightly one of the main results in [149] concerning lower semi-continuity for energy functionals arising from integrands of p -growth, i.e. Borel measurable maps $F: \Omega \times \mathbb{V} \rightarrow \mathbb{R}$ that satisfy

$$|F(x, z)| \leq c(1 + |z|)^p \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega \text{ and all } z \in \mathbb{V}. \tag{6.1.1}$$

Our result, which should be compared to Theorem 5.1.4, is:

Theorem 6.1.2. *Suppose that \mathcal{A} satisfies (3.1.6). Let $1 < p, q < \infty$ and $F: \Omega \times \mathbb{V} \rightarrow [0, \infty)$ be a normal integrand satisfying (6.1.1). Suppose that $z \mapsto F(x, z)$ is \mathcal{A} -quasiconvex for \mathcal{L}^n -a.e. $x \in \Omega$. Then*

$$\left. \begin{array}{l} v_j \rightharpoonup v \quad \text{in } L^p(\Omega, \mathbb{V}) \\ \mathcal{A}v_j \rightarrow \mathcal{A}v \quad \text{in } W^{-\ell, q}(\Omega, \mathbb{W}) \end{array} \right\} \implies \liminf_{j \rightarrow \infty} \int_{\Omega} F(x, v_j(x)) \, dx \geq \int_{\Omega} F(x, v(x)) \, dx. \quad (6.1.2)$$

By normal integrand we mean a jointly Borel measurable function that is lower semi-continuous in the second variable.

For an autonomous integrand $f: \mathbb{V} \rightarrow \mathbb{R}$ satisfying (6.1.1), we define the *upper recession function* by

$$f_p^\infty(z) := \limsup_{(z', t) \rightarrow (z, \infty)} \frac{f(tz')}{t^p} \quad \text{for } (x, z) \in \Omega \times \mathbb{V}. \quad (6.1.3)$$

We have the following characterization result, which shows that autonomous \mathcal{A} -quasiconvex integrands are in duality with a certain class of Young measures:

Theorem 6.1.3. *Suppose that \mathcal{A} satisfies (3.1.6). Let $\nu \in Y^p(\Omega, \mathbb{V})$. If $\nu =: ((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \bar{\Omega}})$ is generated by a sequence $(v_j) \subset L^p(\Omega, \mathbb{V})$ such that $(\mathcal{A}v_j)$ is strongly compact in $W^{-\ell, p}(\Omega, \mathbb{V})$, then*

$$\begin{aligned} \langle f, \nu_x \rangle &\geq f(\bar{\nu}_x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \\ \langle f_p^\infty, \nu_x^\infty \rangle &\geq 0 \quad \text{for } \lambda\text{-a.e. } x \in \Omega, \end{aligned} \quad \text{for all } \mathcal{A}\text{-quasiconvex } f \text{ satisfying (6.1.1)}. \quad (6.1.4)$$

Conversely, suppose that $\lambda(\partial\Omega) = 0$ and write $v(x) := \bar{\nu}_x$. Let \mathcal{B} be a potential operator for \mathcal{A} , i.e. suppose that (4.3.1) holds. Suppose also that the inequalities (6.1.4) hold. Then there exist sequences $(u_j), (\tilde{u}_j) \subset C_c^\infty(\Omega, \mathbb{V})$ such that:

$$\begin{aligned} (v + \mathcal{B}u_j + \mathcal{B}\tilde{u}_j) &\text{ generates } \nu, \\ (D^k u_j) &\text{ is } p\text{-uniformly integrable,} \\ D^k \tilde{u}_j &\rightarrow 0 \text{ in measure.} \end{aligned}$$

The reader can find the notation and definition of Young measures in Section 6.2. The above theorem shows, once again, that \mathcal{A} -quasiconvexity is intrinsic to weak convergence of PDE constrained sequences.

6.2 Generalised Young measures

The presentation here loosely follows the recent lecture notes [251]. Let $1 < p < \infty$. Consider the space of integrands

$$\mathbb{E}_p(\Omega, \mathbb{V}) := \left\{ \Phi \in C(\Omega \times \mathbb{V}) : \Phi_p^\infty(x, z) := \lim_{t \rightarrow \infty, x' \rightarrow x} \frac{\Phi(x', tz)}{t^p} \in \mathbb{R} \text{ uniformly in } \bar{\Omega} \times S_{\mathbb{V}} \right\},$$

which is naturally equipped with the norm

$$\|\Phi\|_{\mathbb{E}_p} := \sup_{(x, z) \in \Omega \times \mathbb{V}} \frac{|\Phi(x, z)|}{(1 + |z|)^p}.$$

When no ambiguity can arise, we may write $\mathbb{E}_p := \mathbb{E}_p(\Omega, \mathbb{V})$. It will thus be convenient to work with the coordinate transformations

$$S: \hat{z} \in B_{\mathbb{V}} \mapsto \frac{\hat{z}}{1 - |\hat{z}|} \in \mathbb{V}, \quad S^{-1}: z \in \mathbb{V} \mapsto \frac{z}{1 + |z|} \in B_{\mathbb{V}},$$

where $B_{\mathbb{V}}$ denotes the open unit ball in \mathbb{V} . With this notation, the space of integrands $\mathbb{E}_p(\Omega, \mathbb{V})$ can be identified with $C(\bar{\Omega} \times B_{\mathbb{V}})$ via the linear isometric isomorphism

$$(T_p \Phi)(x, \hat{z}) := (1 - |\hat{z}|)^p \Phi \left(x, \frac{\hat{z}}{1 - |\hat{z}|} \right), \quad \text{for } x \in \Omega, \hat{z} \in B_{\mathbb{V}}.$$

It follows that its adjoint, $T_p^*: \mathbb{E}_p(\Omega, \mathbb{V})^* \rightarrow C(\overline{\Omega \times B_{\mathbb{V}}})^* \cong \mathcal{M}(\overline{\Omega \times B_{\mathbb{V}}})$ is also a linear isometric isomorphism. We embed $L^p(\Omega, \mathbb{V})$ into \mathbb{E}_p^* via

$$\varepsilon_v(\Phi) := \int_{\Omega} \Phi(x, v(x)) \, dx \leq \|\Phi\|_{\mathbb{E}_p} \int_{\Omega} (1 + |v|)^p \, dx \leq 2^{p-1} \|\Phi\|_{\mathbb{E}_p} (\|\Omega\| + \|v\|_{L^p}^p),$$

so that, by the sequential Banach–Alaoglu theorem, we can conclude that bounded L^p sequences are weakly- $*$ compact in \mathbb{E}_p^* under the above identification. In particular, if (v_j) is bounded in $L^p(\Omega)$, we know that along a subsequence we have $\varepsilon_{v_j} \xrightarrow{*} \nu$ in $\mathbb{E}_p(\Omega, \mathbb{V})^*$. We define $\mu := (T_p^{-1})^* \nu \in \mathcal{M}(\overline{\Omega \times B_{\mathbb{V}}})$ and write for $\Phi \in \mathbb{E}_p$

$$\begin{aligned} \langle \Phi, \nu \rangle &:= \langle \Phi, \nu \rangle_{\mathbb{E}_p, \mathbb{E}_p^*} = \langle T_p \Phi, \mu \rangle \\ &= \int_{\overline{\Omega \times B_{\mathbb{V}}}} (1 - |\hat{z}|)^p \Phi \left(x, \frac{\hat{z}}{1 - |\hat{z}|} \right) \, d\mu(x, \hat{z}) + \int_{\overline{\Omega \times S_{\mathbb{V}}}} \Phi_p^\infty(x, \hat{z}) \, d\mu(x, \hat{z}). \end{aligned}$$

From this formula we derive two necessary conditions for the weakly- $*$ limits of ε_{v_j} , namely that $\mu \geq 0$ in the sense of $\mathcal{M}(\overline{\Omega \times B_{\mathbb{V}}})$ and

$$\int_{\Omega} \varphi(x) \, dx = \int_{\overline{\Omega \times B_{\mathbb{V}}}} \varphi(x) (1 - |\hat{z}|)^p \, d\mu(x, \hat{z}) \text{ for all } \varphi \in C(\overline{\Omega}). \quad (6.2.1)$$

Conversely, these conditions are sufficient to enable us to disintegrate μ into appropriately parametrized (generalized Young) measures that detect both oscillation and concentration behavior of an L^p -weakly convergent sequence (v_j) . We define:

Definition 6.2.1. A parametrized measure $\nu = \left((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \overline{\Omega}} \right)$ is said to be an L^p -Young measure (or p -Young measure) whenever

- (i) $(\nu_x)_{x \in \Omega} \subset \mathcal{M}_1^+(\mathbb{V})$ is weakly- $*$ \mathcal{L}^n -measurable (the *oscillation measure*).
- (ii) $\lambda \in \mathcal{M}^+(\overline{\Omega})$ (the *concentration measure*).
- (iii) $(\nu_x^\infty)_{x \in \overline{\Omega}} \subset \mathcal{M}_1^+(\mathbb{V})$ is weakly- $*$ λ -measurable (the *concentration-angle measure*).
- (iv) $\int_{\Omega} \int_{\mathbb{V}} |z|^p \, d\nu_x(z) \, dx < \infty$ (the *moment condition* holds).

Then ν acts linearly on $\mathbb{E}_p(\Omega, \mathbb{V})$ via

$$\langle \Phi, \nu \rangle := \int_{\Omega} \int_{\mathbb{V}} \Phi(x, \cdot) \, d\nu_x \, dx + \int_{\overline{\Omega}} \int_{S_{\mathbb{V}}} \Phi_p^\infty(x, \cdot) \, d\nu_x^\infty \, d\lambda(x) \text{ for } \Phi \in \mathbb{E}_p(\Omega, \mathbb{V}).$$

We write $Y^p(\Omega, \mathbb{V})$ (or simply Y^p) for the set of all such ν .

It is then easy to check that a Young measure ν actually lies in \mathbb{E}_p^* and, moreover, that the inclusion $Y^p \subset \mathbb{E}_p^*$ is strict. We have the disintegration theorem:

Theorem 6.2.2. $Y^p(\Omega, \mathbb{V}) = T_p^* \{ \mu \in \mathcal{M}^+(\overline{\Omega \times B_{\mathbb{V}}}) : \text{equation (6.2.1) holds} \}$.

The description of Y^p can be pushed further: as a consequence of Theorem 6.1.3 with $\mathcal{A} \equiv 0$ and $\mathcal{B} = \text{Id}$, we can prove that any $\nu \in Y^p$ can be obtained as a weakly- $*$ limit in \mathbb{E}_p^* of *elementary Young measures* $\varepsilon_v = \left((\delta_{v(x)})_{x \in \Omega}, 0, \mathbf{n/a} \right) \in Y^p$.

Coming back to Theorem 6.2.2, it implies that Y^p is weakly- $*$ closed in \mathbb{E}_p^* and convex. Collecting, we obtain the fundamental weak compactness result that we will use:

Theorem 6.2.3 (Fundamental Theorem of Young Measures). *Let $(v_j)_j$ be bounded in $L^p(\Omega, \mathbb{V})$. Then there exists $\nu \in Y^p(\Omega, \mathbb{V})$ such that, along a subsequence, $\varepsilon_{v_j} \xrightarrow{*} \nu$ in $\mathbb{E}_p(\Omega, \mathbb{V})^*$, i.e.,*

$$\lim_{j \rightarrow \infty} \int_{\Omega} \Phi(x, v_j(x)) \, dx = \int_{\Omega} \int_{\mathbb{V}} \Phi(x, z) \, d\nu_x(z) \, dx + \int_{\overline{\Omega}} \int_{S_{\mathbb{V}}} \Phi_p^\infty(x, z) \, d\nu_x^\infty(z) \, d\lambda(x)$$

for all $\Phi \in \mathbb{E}_p(\Omega, \mathbb{V})$.

By taking $\Phi(x, z) = \varphi(x)z_i$ for $\varphi \in C(\bar{\Omega})$, we see that $v(x) := \bar{\nu}_x := \langle \text{id}, \nu_x \rangle$ for \mathcal{L}^n -a.e. $x \in \Omega$, since $\Phi_p^\infty \equiv 0$. It follows that $v_j \rightarrow v$ in $L^p(\Omega, \mathbb{V})$. We will refer to the map v as the *barycentre* of ν .

One can test for weakly-* convergence in \mathbb{E}_p^* with fewer integrands:

Lemma 6.2.4. *There exists a countable family $\{\varphi \otimes f: \varphi \in \text{Lip}(\Omega), f \in \text{Lip}_{\text{loc}}(\mathbb{V}) \cap \mathbb{E}_p(\Omega, \mathbb{V})\}$ whose span is dense in $\mathbb{E}_p(\Omega, \mathbb{V})$. Moreover,*

$$|f(z_1) - f(z_2)| \leq c \|T_p f\|_{\text{Lip}(B_{\mathbb{V}})} |z_1 - z_2| (1 + |z_1| + |z_2|)^{p-1} \quad \text{for } z_1, z_2 \in \mathbb{V},$$

where $\|g\|_{\text{Lip}} := \|g\|_{L^\infty} + \|Dg\|_{L^\infty}$.

The main use of p -Young measures is that they separate the oscillation and L^p -concentration effects:

Theorem 6.2.5. *Let $(v_j) \subset L^p(\Omega, \mathbb{V})$ generate $\nu \in Y^p(\Omega, \mathbb{V})$. Then $v_j \rightarrow v$ in \mathcal{L}^n -measure if and only if $v(x) = \bar{\nu}_x$ and $\nu_x = \delta_{\bar{\nu}_x}$ for \mathcal{L}^n -a.e. $x \in \Omega$.*

Moreover, let $\tilde{v}_j: \Omega \rightarrow \mathbb{V}$ be measurable, such that $v_j - \tilde{v}_j \rightarrow 0$ in \mathcal{L}^n -measure. Then

$$\int_{\Omega} \varphi(x) F(\tilde{v}_j(x)) \, dx \rightarrow \int_{\Omega} \varphi(x) \langle \nu_x, F \rangle \, dx \quad \text{for } \varphi \in C(\bar{\Omega}), F \in C_c(\mathbb{V}).$$

Theorem 6.2.6. *Let $(v_j) \subset L^p(\Omega, \mathbb{V})$ generate $\nu \in Y^p(\Omega, \mathbb{V})$ and $\Phi \in \mathbb{E}_p(\Omega, \mathbb{V})$. Then $(\Phi(\cdot, v_j))$ is uniformly integrable if and only if*

$$\langle |\Phi_p^\infty(x, \cdot)|, \nu_x^\infty \rangle = 0 \quad \text{for } \lambda\text{-a.e. } x \in \bar{\Omega}.$$

In particular, (v_j) is p -uniformly integrable if and only if $\lambda \equiv 0$.

Furthermore, if $(\tilde{v}_j) \subset L^p(\Omega, \mathbb{V})$ generate $\tilde{\nu} \in Y^p(\Omega, \mathbb{V})$ is such that $(v_j - \tilde{v}_j)$ is p -uniformly integrable, then $\lambda_\nu = \lambda_{\tilde{\nu}} (=:\lambda)$ and $\nu_x^\infty = \tilde{\nu}_x^\infty$ for λ -a.e. $x \in \bar{\Omega}$.

Finally, we have the following classical lower semi-continuity result concerning (rough) integrands that are bounded from below and their interaction with the oscillation measure:

Proposition 6.2.7. *Let $1 < p < \infty$ and $F: \Omega \times \mathbb{V} \rightarrow [0, \infty)$ be a normal integrand of p -growth (6.1.1). Let $(v_j) \subset L^p(\Omega, \mathbb{V})$ generate $\nu \in Y^p(\Omega, \mathbb{V})$. Then*

$$\liminf_{j \rightarrow \infty} \int_{\Omega} F(x, v_j(x)) \, dx \geq \int_{\Omega} \langle \nu_x, F(x, \cdot) \rangle \, dx.$$

To conclude this section we note that, under the spanning assumption in (3.1.6), an \mathcal{A} -quasiconvex $f: \mathbb{V} \rightarrow \mathbb{R}$ is locally Lipschitz. In particular, for such integrands the definition (6.1.3) of the p -upper recession function can be shown to equal

$$f_p^\infty(z) = \limsup_{t \rightarrow \infty} \frac{f(tz)}{t^p} \quad \text{for } z \in \mathbb{V}.$$

This fact will be used without mention. Nevertheless, the limit superior in the formula cannot be replaced with the limit, see [293]. It is no surprise, then, that we will need the following approximation from above of quasiconvex integrands by quasiconvex integrands that have regular recession function.

Lemma 6.2.8. *Let $f: \mathbb{V} \rightarrow \mathbb{R}$ be \mathcal{A} -quasiconvex and satisfying (6.1.1). Then there exist \mathcal{A} -quasiconvex $\Phi_i \in \mathbb{E}_p$ such that*

$$\Phi_i \downarrow f \quad \text{and} \quad \Phi_{i,p}^\infty \downarrow f_p^\infty,$$

as $i \rightarrow \infty$, pointwise in \mathbb{V} .

A proof of this fact can be given by modifying the argument in [233, Lem. 6.3]. Moreover, the Φ_i can be chosen to be p -homogeneous outside a large ball that increases with i .

6.3 The Jensen inequalities and lower semi-continuity

We begin this section by proving the Jensen inequalities (6.1.4). To this end, we state without proof a so-called localization result:

Proposition 6.3.1. *Let \mathcal{A} as in (3.1.1) be a constant rank operator with potential operator \mathcal{B} such that (4.3.1) holds. Let $1 < p, q < \infty$ and*

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \rightarrow \mathcal{A}v \text{ in } W^{-\ell, q}(\Omega, \mathbb{W})$$

generate a p -Young measure $\nu = ((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \bar{\Omega}})$. We write $\lambda = \lambda^a(x) \mathcal{L}^n \llcorner \Omega + \lambda^s$ for the Radon–Nykodim decomposition of λ . We have that

(i) For \mathcal{L}^n -a.e. $x_0 \in \Omega$, we have that the homogeneous p -Young measure

$$\left((\nu_{x_0})_{y \in Q_1(0)}, 0, \mathbf{n/a} \right) \text{ is generated by } (v(x_0) + \mathcal{B}u_j),$$

where $u_j \in C_c^\infty(Q_1(0), \mathbb{U})$.

Suppose that $p = q$.

(ii) For \mathcal{L}^n -a.e. $x_0 \in \Omega$, we have that the homogeneous p -Young measure

$$\left((\delta_0)_{x_0 \in Q_1(0)}, \lambda^a(x_0) \mathcal{L}^n \llcorner Q_1(0), (\nu_{x_0}^\infty)_{y \in Q_1(0)} \right) \text{ is generated by } (\mathcal{B}\tilde{u}_j),$$

where $\tilde{u}_j \in C_c^\infty(Q_1(0), \mathbb{U})$.

(iii) For λ^s -a.e. $x_0 \in \Omega$, there exists a (tangent measure of λ^s at x_0) $\tau \in \mathcal{M}_1^+(\bar{Q}_1(0))$, such that we have that the homogeneous p -Young measure

$$\left((\delta_0)_{x_0 \in Q_1(0)}, \tau, (\nu_{x_0}^\infty)_{y \in \bar{Q}_1(0)} \right) \text{ is generated by } (\mathcal{B}U_j),$$

where $U_j \in C_c^\infty(Q_1(0), \mathbb{U})$.

A proof can be found in [251, Sec. 5], see also [13]. Using this result, we can proceed to prove the Jensen inequalities (6.1.4). Since the oscillation inequality holds for anisotropic constraints, we will split the proof in two lemmas.

Lemma 6.3.2. *Suppose that \mathcal{A} satisfies (3.1.6). Let $1 < p, q < \infty$ and*

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \rightarrow \mathcal{A}v \text{ in } W^{-\ell, q}(\Omega, \mathbb{W})$$

generate a p -Young measure $\nu = ((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \bar{\Omega}})$. Then for all \mathcal{A} -quasiconvex $f: \mathbb{V} \rightarrow \mathbb{R}$ satisfying (6.1.1), we have that

$$\langle f, \nu_x \rangle \geq f(\bar{\nu}_x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega.$$

Lemma 6.3.3. *Suppose that \mathcal{A} satisfies (3.1.6). Let $1 < p < \infty$ and*

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \rightarrow \mathcal{A}v \text{ in } W^{-\ell, p}(\Omega, \mathbb{W})$$

generate a p -Young measure $\nu = ((\nu_x)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \bar{\Omega}})$. Then for all \mathcal{A} -quasiconvex $f: \mathbb{V} \rightarrow \mathbb{R}$ satisfying (6.1.1), we have that

$$\langle f_p^\infty, \nu_x^\infty \rangle \geq 0 \quad \text{for } \lambda\text{-a.e. } x \in \Omega.$$

Proof of Lemma 6.3.2: Let $x_0 \in \Omega$ be in a set of full Lebesgue measure where $\bar{\nu}_{x_0} = \nu(x_0)$ (a Lebesgue point of ν) and Proposition 6.3.1(i) applies. We have that, for autonomous and \mathcal{A} -quasiconvex $\Phi \in \mathbb{E}_p$,

$$\langle \Phi, \nu_{x_0} \rangle = \lim_{j \rightarrow \infty} \int_{Q_1(0)} \Phi(v(x_0) + \mathcal{B}u_j(y)) \, dy \geq \Phi(v(x_0)).$$

If f is autonomous and \mathcal{A} -quasiconvex but does not necessarily possess a strong recession function, we employ Lemma 6.2.8 to approximate f from above with well behaved Φ . The conclusion follows from the monotone convergence theorem. \square

Proof of Lemma 6.3.3: Let now $x_0 \in \Omega$ be \mathcal{L}^n -significant such that Proposition 6.3.1(ii) applies. Letting a well behaved \mathcal{A} -quasiconvex integrand Φ approximate f as in the proof of Lemma 6.3.2, we note that $\Phi_p^\infty \in \mathbb{E}_p$ is also \mathcal{A} -quasiconvex, so we can write

$$\lambda^a(x_0) \langle \Phi_p^\infty, \nu_x^\infty \rangle = \lim_{j \rightarrow \infty} \int_{Q_1(0)} \Phi_p^\infty(\mathcal{B}\tilde{u}_j(y)) \, dy \geq \Phi_p^\infty(0) = 0,$$

which implies the required inequality since $\lambda \in \mathcal{M}^+(\bar{\Omega})$.

If we now look at a λ^s -significant $x_0 \in \Omega$ such that Proposition 6.3.1(iii) applies, we have, in a similar fashion,

$$\tau(\bar{Q}_1(0)) \langle \Phi_p^\infty, \nu_x^\infty \rangle = \lim_{j \rightarrow \infty} \int_{Q_1(0)} \Phi_p^\infty(\mathcal{B}U_j(y)) \, dy \geq \Phi_p^\infty(0) = 0,$$

and we can conclude as before since τ is a probability measure. \square

We conclude this section with a proof of the lower semi-continuity theorem:

Proof of Theorem 6.1.2: By Proposition 6.2.7 we have that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} F(x, v_j(x)) \, dx \geq \int_{\Omega} \langle \nu_x, F(x, \cdot) \rangle \, dx \geq \int_{\Omega} F(x, \bar{\nu}_x) \, dx = \int_{\Omega} F(x, v(x)) \, dx$$

where the second inequality follows from Lemma 6.3.2. \square

6.4 Proof of the characterization result

It remains to prove Theorem 6.1.3. The Jensen inequalities (6.1.4) were already proved in Lemmas 6.3.2 and 6.3.3. To establish the converse, we proceed with adapting the strategy from [250, Sec. 3]. We begin with the case of homogeneous Young measures.

Let $Q \subset \mathbb{R}^n$ be a cube and $z \in \mathbb{V}$ and define

$$\begin{aligned} Y_h^p(z) &:= \text{Big}\{(\nu^0, \nu^\infty) \in \mathcal{M}_1^+(\mathbb{V}) \times \mathcal{M}^+(S_{\mathbb{V}}) : \text{there exist } u_j \in C_c^\infty(Q, \mathbb{U}) \text{ such that} \\ &\text{for all } \Phi \in \mathbb{E}_{p,a}, \lim_{j \rightarrow \infty} \int_Q \Phi(z + \mathcal{B}u_j(x)) \, dx = \langle \nu^0, \Phi \rangle + \langle \nu^\infty, \Phi_p^\infty \rangle \text{Big}\}, \end{aligned}$$

where $\mathbb{E}_{p,a}(\mathbb{V})$ denotes the set of autonomous integrands in $\mathbb{E}_p(\Omega, \mathbb{V})$, i.e.,

$$\mathbb{E}_{p,a}(\mathbb{V}) = \left\{ \Phi \in C(\mathbb{V}) : \Phi_p^\infty(x, z) := \lim_{t \rightarrow \infty} \frac{\Phi(tz)}{t^p} \in \mathbb{R} \text{ locally uniformly for } z \in \mathbb{V} \right\}.$$

It is easy to see that, with the norm induced from $\mathbb{E}_p(\Omega, \mathbb{V})$, we have that $Y_h^p(z) \subset \mathbb{E}_{p,a}(\mathbb{V})^* \simeq \mathcal{M}(\mathbb{V}) \times \mathcal{M}(S_{\mathbb{V}})$, where the isomorphism is given by the map T_p . We record that, since $p > 1$, we have that $\bar{\nu}^0 = z$ for elements of $Y_h^p(z)$. Finally, let us mention that in the ‘‘inhomogenization’’ argument we will only look at measures $(\nu^0, \nu^\infty) \in Y_h^p$ that have $\nu^0 = \delta_0$ or $\nu^\infty \equiv 0$, which is completely unlike in [250].

We can now formulate the homogeneous step of the converse of Theorem 6.1.3:

Proposition 6.4.1. *Suppose that \mathcal{A} satisfies (3.1.6). Let \mathcal{B} be a potential operator for \mathcal{A} such that (4.3.1) holds. Let $\nu := (\nu^0, \nu^\infty) \in \mathcal{M}_1^+(\mathbb{V}) \times \mathcal{M}^+(S_{\mathbb{V}})$ and $z \in \mathbb{V}$.*

Then $\nu \in Y_h^p(z)$ if and only if $\bar{\nu}^0 = z$ and

$$\langle \nu^0, f \rangle + \langle \nu^\infty, f_p^\infty \rangle \geq f(z) \quad \text{for all } \mathcal{A}\text{-quasiconvex } f: \mathbb{V} \rightarrow \mathbb{R} \text{ satisfying (6.1.1).}$$

The proof follows the lines of the argument in [250, Sec. 3.3] exactly. We can now proceed with the proof of the main result, which follows closely the construction in [250, Sec. 3.4]:

Proof of Theorem 6.1.3: We already explained that we need only prove the converse. To this end, let $\nu \in Y^p(\Omega, \mathbb{V})$ be such that $\lambda(\partial\Omega) = 0$. By Theorem 6.2.5 and 6.2.6, we have that it suffices to show that there exist sequences $(u_j), (\tilde{u}_j) \subset C_c^\infty(\Omega, \mathbb{U})$ such that

$$(v + \mathcal{B}u_j) \text{ generates } ((\nu_x)_{x \in \Omega}, 0, \mathbf{n}/\mathbf{a}) \quad \text{and} \quad (\mathcal{B}\tilde{u}_j) \text{ generates } ((\delta_0)_{x \in \Omega}, \lambda, (\nu_x^\infty)_{x \in \Omega}).$$

Indeed, this is enough since we would have that $(\mathcal{B}u_j)$ is p -uniformly integrable and $\mathcal{B}\tilde{u}_j \rightarrow 0$ in measure, while both sequences converge weakly to 0 in $L^p(\Omega, \mathbb{V})$. In this case, one can apply the Decomposition Lemma 6.1.1 to refine the two sequences.

We will test with integrands $\varphi \otimes \Phi \in C(\bar{\Omega}) \times \mathbb{E}_{p,a}(\mathbb{V})$ as given by Lemma 6.2.4. In particular, we can assume that $\|\varphi\|_{\text{Lip}}, \|T_p\Phi\|_{\text{Lip}} \leq 1$, so that

$$|\Phi(z) - \Phi(z')| \leq c|z - z'|(1 + |z| + |z'|)^{p-1} \quad \text{for } z, z' \in \mathbb{V}.$$

We are working with $\|\cdot\|_{\text{Lip}} := \|\cdot\|_{L^\infty} + \|D\cdot\|_{L^\infty}$. Let $\varepsilon > 0$.

We write $g(x) := \varphi(x)\langle \nu_x, \Phi \rangle$ and $g_0(x) := \langle \nu_x, \Phi_0 \rangle$ where $\Phi_0 = (1 + |\cdot|)^p$, so $g, g_0 \in L^1(\Omega)$ by the moment condition. We apply Lusin's theorem in the following way: there exists a compact set $C \subset \Omega$ such that, with $G = (g, g_0)$,

$$\mathcal{L}^n(\Omega \setminus C) < \varepsilon|\Omega|, \quad \int_{\Omega \setminus C} |G| dx < \varepsilon|\Omega|, \quad \text{and } G|_C \text{ is continuous.}$$

Using Tietze's extension theorem, we can find $\tilde{G} =: (\tilde{g}, \tilde{g}_0) \in C(\bar{\Omega})$ such that $\tilde{G} = G$ in C and $\|\tilde{G}\|_{L^\infty(\Omega)} = \|G\|_{L^\infty(C)}$. Moreover, \tilde{G} is uniformly continuous, so we can find $\delta \in (0, \varepsilon)$ such that $|\tilde{G}(x) - \tilde{G}(x')| < \varepsilon$ whenever $|x - x'| < \delta$. Finally, consider a regular grid of cubes in \mathbb{R}^n of side length $\delta/2$; we write \mathcal{F}_δ for the family of such cubes that are contained in Ω . Since $\mathcal{L}^n(\partial\Omega) = 0$, it is clear that $|\bigcup \mathcal{F}_\delta| \uparrow |\Omega|$ as $\delta \downarrow 0$. We write

$$\mathcal{F}_\delta^\circ := \{Q \in \mathcal{F}_\delta : Q \cap C \neq \emptyset\}.$$

Then \mathcal{F}_δ° covers $C \cap \bigcup_{Q \in \mathcal{F}_\delta} Q$, so that we can assume by taking δ smaller that

$$\int_{\Omega \setminus \bigcup_{Q \in \mathcal{F}_\delta^\circ} Q} |G| dx < \varepsilon|\Omega|.$$

For each cube $Q \in \mathcal{F}_\delta^\circ$, we choose $x_Q \in Q \cap C$ to be a Lebesgue point of the barycentre $v(x) = \bar{\nu}_x$ and such that the oscillation Jensen inequality holds at x_Q . We also record that

$$|G(x_Q) - \tilde{G}(x)| < \varepsilon \quad \text{for all } x \in Q.$$

We can also assume that we have a piecewise constant approximation of the barycentre in $L^p(\Omega, \mathbb{V})$:

$$\int_{\Omega} |v - v^\varepsilon|^p dx \leq \varepsilon|\Omega|, \quad \text{where } v^\varepsilon := \sum_{Q \in \mathcal{F}_\delta^\circ} v(x_Q)\mathbf{1}_Q.$$

As a consequence of Proposition 6.4.1 with $\nu^0 = \nu_{x_Q}, \nu^\infty = 0$ we can find $u_Q^\varepsilon \in C_c^\infty(Q, \mathbb{U})$ such that

$$\left| \langle \nu_{x_Q}, \Phi \rangle - \int_Q \Phi(\bar{\nu}_{x_Q} + \mathcal{B}u_Q^\varepsilon(x)) dx \right| + \left| \langle \nu_{x_Q}, \Phi_0 \rangle - \int_Q \Phi_0(\bar{\nu}_{x_Q} + \mathcal{B}u_Q^\varepsilon(x)) dx \right| < \varepsilon.$$

Recall here that $\Phi_0 = (1 + |\cdot|)^p$. We can begin to estimate

$$\left| \int_{\Omega} g \, dx - \int_{\bigcup_{Q \in \mathcal{F}_\delta^o} Q} \tilde{g} \, dx \right| \leq \int_{\Omega \setminus \bigcup_{Q \in \mathcal{F}_\delta^o} Q} |g| \, dx + \int_{\bigcup_{Q \in \mathcal{F}_\delta^o} Q} |g - \tilde{g}| \, dx \leq 2\varepsilon|\Omega|,$$

so that

$$\left| \int_{\bigcup_{Q \in \mathcal{F}_\delta^o} Q} \tilde{g} \, dx - \sum_{Q \in \mathcal{F}_\delta^o} |Q|g(x_Q) \right| \leq \varepsilon|\Omega|.$$

We can estimate further

$$\left| \sum_{Q \in \mathcal{F}_\delta^o} \left(|Q|g(x_Q) - \varphi(x_Q) \int_Q \Phi(\bar{\nu}_{x_Q} + \mathcal{B}u_Q^\varepsilon(x)) \, dx \right) \right| \leq \varepsilon|\Omega|.$$

We then have that

$$\sum_{Q \in \mathcal{F}_\delta^o} \varphi(x_Q) \int_Q \Phi(\bar{\nu}_{x_Q} + \mathcal{B}u_Q^\varepsilon(x)) \, dx = \sum_{Q \in \mathcal{F}_\delta^o} \int_Q \varphi(x) \Phi(\bar{\nu}_{x_Q} + \mathcal{B}u_Q^\varepsilon(x)) \, dx + \mathcal{E}_1,$$

where

$$\begin{aligned} |\mathcal{E}_1| &\leq c \sum_{Q \in \mathcal{F}_\delta^o} \int_Q |\varphi(x_Q) - \varphi(x)| \Phi_0(\bar{\nu}_{x_Q} + \mathcal{B}u_Q^\varepsilon(x)) \, dx \leq c\delta \sum_{Q \in \mathcal{F}_\delta^o} |Q|(\langle \nu_{x_Q}, \Phi_0 \rangle + \varepsilon) \\ &\leq c\delta \sum_{Q \in \mathcal{F}_\delta^o} \left(\int_Q \langle \nu_x, \Phi_0 \rangle \, dx + 2\varepsilon|Q| \right) \leq c\delta \left(\int_{\Omega} \langle \nu_x, \Phi_0 \rangle \, dx + 2\varepsilon|\Omega| \right), \end{aligned} \tag{6.4.1}$$

where the integral is finite by the moment condition. We make and recall the abbreviations

$$u^\varepsilon := \sum_{Q \in \mathcal{F}_\delta^o} u_Q^\varepsilon \in C_c^\infty(\Omega, \mathbb{U}) \quad \text{and} \quad v^\varepsilon = \sum_{Q \in \mathcal{F}_\delta^o} v(x_Q) \mathbf{1}_Q \in L^p(\Omega, \mathbb{V}).$$

We next look at

$$\sum_{Q \in \mathcal{F}_\delta^o} \int_Q \varphi(x) \Phi(v(x_Q) + \mathcal{B}u_Q^\varepsilon(x)) \, dx = \sum_{Q \in \mathcal{F}_\delta^o} \int_Q \varphi(x) \Phi(v(x) + \mathcal{B}u_Q^\varepsilon(x)) \, dx + \mathcal{E}_2,$$

where, by using $\|\varphi\|_{L^\infty} \leq 1$,

$$\begin{aligned} |\mathcal{E}_2| &\leq c \int_{\Omega} |v - v^\varepsilon| (1 + |v + \mathcal{B}u^\varepsilon| + |v^\varepsilon + \mathcal{B}u^\varepsilon|)^{p-1} \, dx \\ &\leq c \|v - v^\varepsilon\|_{L^p(\Omega)} \|1 + |v + \mathcal{B}u^\varepsilon| + |v^\varepsilon + \mathcal{B}u^\varepsilon|\|_{L^p(\Omega)}^{p-1} \\ &\leq c \|v - v^\varepsilon\|_{L^p(\Omega)} \left(\left(\int_{\Omega} \Phi_0(v^\varepsilon + \mathcal{B}u^\varepsilon) \, dx \right)^{(p-1)/p} + \|v - v^\varepsilon\|_{L^p(\Omega)}^{p-1} \right) \end{aligned}$$

Since $\|v - v^\varepsilon\|_{L^p(\Omega)} \leq (\varepsilon|\Omega|)^{1/p}$ and the estimation

$$\int_{\Omega} \Phi_0(v^\varepsilon + \mathcal{B}u^\varepsilon) \, dx \leq c \left(\int_{\Omega} \langle \nu_x, \Phi_0 \rangle \, dx + \varepsilon|\Omega| \right)$$

from (6.4.1), we are very close to conclude. Writing

$$\int_{\Omega \setminus \bigcup_{Q \in \mathcal{F}_\delta^c} Q} \varphi(x) \Phi(v(x) + \mathcal{B}u^\varepsilon(x)) \, dx \leq c \int_{\Omega \setminus \bigcup_{Q \in \mathcal{F}_\delta^c} Q} (1 + |v|)^p \, dx \leq c\varepsilon |\Omega|$$

and collecting the estimates, we have that

$$\left| \int_{\Omega} \varphi(x) \langle \nu_x, \Phi \rangle \, dx - \int_{\Omega} \varphi(x) \Phi(v(x) + \mathcal{B}u^\varepsilon(x)) \, dx \right| \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

We have thus showed that the oscillation measure has the right differential structure.

We carry on with the concentration part. We now consider the functions $h(x) := \varphi(x) \langle \nu_x^\infty, \Phi_p^\infty \rangle$ and $h_0(x) := \langle \nu_x^\infty, \Phi_{0,p}^\infty \rangle$, where we recall that $\Phi_0 = (1 + |\cdot|)^p$. We have that $H := (h, h_0) \in L^1(\Omega; d\lambda)$. As in the previous case, we will apply Lusin's theorem and Tietze's extension theorem to find a compact set $K \subset \Omega$ and a continuous extension $\tilde{H} =: (\tilde{h}, \tilde{h}_0) \in C(\bar{\Omega})$, such that

$$\lambda(\Omega \setminus K) < \varepsilon \lambda(\Omega), \quad \int_{\Omega \setminus K} |H| \, d\lambda < \varepsilon \lambda(\Omega), \quad \tilde{H}|_K = H, \quad \|\tilde{H}\|_{L^\infty(\Omega)} = \|H\|_{L^\infty(K)}.$$

We then choose $\delta \in (0, \varepsilon)$ and a collection \mathcal{F}_δ^c of cubes in *exact* analogy with the case of the oscillation measures, by replacing \mathcal{L}^n with λ . At this stage we also use the assumption $\lambda(\partial\Omega) = 0$. Finally, for each $Q \in \mathcal{F}_\delta^c$, we choose $x_Q \in Q \cap K$ such that the concentration Jensen inequality holds at x_Q .

By Proposition 6.4.1 with $\nu^0 = \delta_0$ and $\nu^\infty = |Q|^{-1} \lambda(Q) \nu_{x_Q}^\infty$ we can find $\tilde{u}_Q^\varepsilon \in C_c^\infty(Q, \mathbb{U})$ such that

$$\left| \Phi(0) + \frac{\lambda(Q)}{|Q|} \langle \nu_{x_Q}^\infty, \Phi_p^\infty \rangle - \int_Q \Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x)) \, dx \right| < \varepsilon, \quad \left| \Phi_0(0) + \frac{\lambda(Q)}{|Q|} \langle \nu_{x_Q}^\infty, \Phi_{0,p}^\infty \rangle - \int_Q \Phi_0(\mathcal{B}\tilde{u}_Q^\varepsilon(x)) \, dx \right| < \varepsilon.$$

We can then estimate

$$\int_{\Omega} \varphi \, dx \Phi(0) + \int_{\Omega} h \, d\lambda = \int_{\Omega \setminus \bigcup_{Q \in \mathcal{F}_\delta^c} Q} \varphi \, dx \Phi(0) + \sum_{Q \in \mathcal{F}_\delta^c} |Q| \varphi(x_Q) \left(\Phi(0) + \frac{\lambda(Q)}{|Q|} \langle \nu_{x_Q}^\infty, \Phi_p^\infty \rangle \right) + \mathcal{E}_3,$$

where

$$|\mathcal{E}_3| \leq \sum_{Q \in \mathcal{F}_\delta^c} \int_Q |\varphi - \varphi(x_Q)| \, dx |\Phi(0)| + 2\varepsilon \lambda(\Omega) + \int_Q |\varphi| |\tilde{h} - h(x_Q)| \, d\lambda \leq \delta |\Omega| + 3\varepsilon \lambda(\Omega).$$

We next focus on

$$\left| \sum_{Q \in \mathcal{F}_\delta^c} \left(|Q| \varphi(x_Q) \left(\Phi(0) + \frac{\lambda(Q)}{|Q|} \langle \nu_{x_Q}^\infty, \Phi_p^\infty \rangle \right) - \varphi(x_Q) \int_Q \Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x)) \, dx \right) \right| \leq \varepsilon |\Omega|.$$

Defining $\tilde{u}^\varepsilon := \tilde{u}_Q^\varepsilon$ on each $Q \in \mathcal{F}_\delta^c$ and extending by zero to the rest of Ω , we obtain $\tilde{u}^\varepsilon \in C_c^\infty(\Omega, \mathbb{U})$. Further, we have

$$\sum_{Q \in \mathcal{F}_\delta^c} \varphi(x_Q) \int_Q \Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x)) \, dx = \int_{\bigcup_{Q \in \mathcal{F}_\delta^c} Q} \varphi(x) \Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x)) \, dx + \mathcal{E}_4,$$

where

$$\begin{aligned} |\mathcal{E}_4| &\leq c\delta \sum_{Q \in \mathcal{F}_\delta^c} \int_Q \Phi_0(\mathcal{B}\tilde{u}_Q^\varepsilon(x)) \, dx \leq c\delta \left(\varepsilon |\Omega| + \sum_{Q \in \mathcal{F}_\delta^c} |Q| \Phi_0(0) + \lambda(Q) h_0(x_Q) \right) \\ &\leq c\delta \left(\varepsilon |\Omega| + \int_{\bigcup_{Q \in \mathcal{F}_\delta^c} Q} |h_0| \, d\lambda \right) \leq c\delta (\varepsilon |\Omega| + \lambda(\Omega)) \end{aligned}$$

where the last integral can be computed explicitly. Collecting, we proved that

$$\left| \int_{\Omega} \varphi \, dx \Phi(0) + \int_{\Omega} \langle \nu_x^{\infty}, \Phi_p^{\infty} \rangle \, d\lambda(x) - \int_{\Omega} \varphi(x) \Phi(\mathcal{B}\tilde{u}_Q^{\varepsilon}(x)) \, dx \right| \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

The proof is complete. □

Chapter 7

Quadratic homogenization under constant rank constraints

The purpose of this chapter is to give a short, self-contained overview of the simplest possible setting for homogenization theory: that is, periodic quadratic homogenization. We focus in particular on the G-closure problem, which remains to date an outstanding challenge in the area. We will describe in some detail the methods and results developed by GRABOVSKY, MILTON and coauthors to tackle the G-closure problem in a variety of physical settings. The contents of this chapter are known to the experts and the only novelty here is in the presentation; the reader can find many more details in the recent book [167].

7.1 Postulates and definitions

Let $(\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ be a finite-dimensional inner product space; we will omit the subscript from the inner product whenever no confusion arises. The vector space \mathbb{V} denotes the space where the physical fields describing our system take values.

Consider two vector fields $e, j \in L^2(\mathbb{T}^n, \mathbb{V})$; we think of e as being a field that is applied to the material (say, an electric field) and j as being the resulting flux field (say, the electric current). We also assume that these fields satisfy a linear constitutive relation of the form

$$j(x) = L(x)e(x) \tag{7.1.1}$$

for some $L \in L^\infty(\mathbb{T}^n, \text{End}(\mathbb{V}))$. The operator L is called the *local tensor* of the material, as it describes its physical properties.

In the models of continuum mechanics we shall consider the fields always satisfy a set of differential constraints. We shall assume that these equations take the form

$$\mathcal{A}e = 0, \quad \mathcal{B}^*j = 0 \tag{7.1.2}$$

where \mathcal{A}, \mathcal{B} are linear differential operators, acting on vector fields defined over \mathbb{R}^n , of the form

$$\mathcal{A} = \sum_{|\alpha|=l} A_\alpha \partial^\alpha, \quad A_\alpha \in \text{Lin}(\mathbb{V}, \mathbb{V}'), \tag{7.1.3}$$

$$\mathcal{B} = \sum_{|\beta|=k} B_\beta \partial^\beta, \quad B_\beta \in \text{Lin}(\mathbb{U}, \mathbb{V}). \tag{7.1.4}$$

Here \mathbb{U}, \mathbb{V}' are also finite-dimensional inner product spaces and \mathcal{B}^* is the formal adjoint of \mathcal{B} . We shall moreover assume that, with $\mathcal{A}(\xi)$ and $\mathcal{B}(\xi)$ denoting the symbols of \mathcal{A} and \mathcal{B} ,

$$\mathbb{U} \xrightarrow{\mathcal{B}(\xi)} \mathbb{V} \xrightarrow{\mathcal{A}(\xi)} \mathbb{V}' \quad \text{is an exact sequence,} \tag{7.1.5}$$

i.e. $\text{im } \mathcal{B}(\xi) = \ker \mathcal{A}(\xi)$ for all $\xi \in \mathbb{R}^n$. Due to Theorem 4.3.1, this is equivalent to saying that either \mathcal{A} or \mathcal{B} has constant rank.

A fundamental property of the system (7.1.2) is that it possesses a certain kind of ellipticity; this observation is at the heart of the theory of Compensated Compactness [300, 301, 361, 367] and was exploited in Chapters 4 to 6. In the latter, we proved a more general version of the following result:

Theorem 7.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let \mathcal{A}, \mathcal{B} be as in (7.1.3)–(7.1.5). Given sequences $e^\varepsilon, j^\varepsilon \in L^2(\Omega, \mathbb{V})$ we have*

$$\left. \begin{array}{l} (e^\varepsilon, j^\varepsilon) \rightharpoonup (e, j) \quad \text{in } L^2(\Omega, \mathbb{V} \times \mathbb{V}) \\ \mathcal{A}e^\varepsilon \rightarrow \mathcal{A}e \quad \text{in } H_{\text{loc}}^{-l}(\Omega, \mathbb{V}') \\ \mathcal{B}^*j^\varepsilon \rightarrow \mathcal{B}^*j \quad \text{in } H_{\text{loc}}^{-k}(\Omega, \mathbb{U}) \end{array} \right\} \implies \langle e^\varepsilon, j^\varepsilon \rangle_{\mathbb{V}} \xrightarrow{*} \langle e, j \rangle_{\mathbb{V}} \text{ in } \mathcal{M}(\Omega).$$

We also have a Helmholtz–Hodge decomposition associated with (7.1.2); here we state it in a L^2 periodic setting, but see Chapter 4 for the case of \mathbb{R}^n or of a bounded domain.

Proposition 7.1.2. *Let $v \in L^2(\mathbb{T}^n, \mathbb{V})$ be such that $\int_{\mathbb{T}^n} v \, dx = 0$. There are $u \in H^k(\mathbb{T}^n, \mathbb{U})$, $w \in H^l(\mathbb{T}^n, \mathbb{V}')$ such that*

$$v = \mathcal{B}u + \mathcal{A}^*w.$$

Moreover, this decomposition is continuous:

$$\|\mathcal{B}u\|_{L^2(\mathbb{T}^n)} \lesssim \|v\|_{L^2(\mathbb{T}^n)}, \quad \|\mathcal{A}^*w\|_{L^2(\mathbb{T}^n)} \lesssim \|\mathcal{A}v\|_{H^{-l}(\mathbb{T}^n)}. \quad (7.1.6)$$

7.1.1 The Hilbert space formalism

In order to describe the homogenization theory of composite materials, our first step is to abstract the above operator formalism in order to be able to define concepts that are explicitly independent of the operators, depending only on the bundle $V_{\mathcal{A}}$. This is important because the differential operators are not canonical, c.f. the results of the previous chapter. One can replace, for instance, \mathcal{A} with $\Delta^a \mathcal{A}$ for any $a \in \mathbb{N}$, where Δ is the Laplacian, without any consequences. More disconcertingly, even if \mathcal{A} is fixed there does not seem to be a way of choosing a preferred \mathcal{B} such that (7.1.5) holds.

The abstraction of the operator framework we shall consider is usually known as MILTON’s Hilbert space formalism [282], see also [283, §12]. By (7.1.5) we have an orthogonal decomposition

$$\mathbb{V} = \ker \mathcal{A}(\xi) \oplus \ker \mathcal{B}^*(\xi) \equiv \mathbb{E}_\xi \oplus \mathbb{J}_\xi \quad (7.1.7)$$

and the differential equations (7.1.2) imply immediately that

$$\widehat{e}(\xi) \in \mathbb{E}_\xi \text{ and } \widehat{j}(\xi) \in \mathbb{J}_\xi \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}. \quad (7.1.8)$$

Example 7.1.3. For different choices of the operator \mathcal{A} we obtain different physical models:

- (i) Conductivity: with $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{A} = \text{curl}$, $\mathcal{B} = \text{D}$, the electric field e and the electric current j satisfy (7.1.2), which in this case reads

$$\text{curl } e = 0, \quad \text{div } j = 0.$$

Furthermore, $\mathbb{E}_\xi = \text{span}(\xi)$ and $\mathbb{J}_\xi = \xi^\perp$. The constitutive relation (7.1.1) is usually known as Ohm’s law.

- (ii) Linear elasticity: with $\mathbb{V} = \text{Sym}(\mathbb{R}^n)$, we define the Saint–Venant compatibility operator

$$(\mathcal{A}v)_{ijkl} = (\text{curl curl } v)_{ijkl} \equiv \partial_{kl}^2 v_{ij} + \partial_{ij}^2 v_{kl} - \partial_{jk}^2 v_{il} - \partial_{il}^2 v_{jk}$$

and $\mathcal{B}u = \varepsilon(u) \equiv \frac{1}{2}(Du + Du^T)$. Then (7.1.2), when applied to the strain ε and to the stress σ reads

$$\text{curl curl } \varepsilon = 0 \Leftrightarrow \varepsilon = \varepsilon(u), \text{ for some } u, \quad \text{div } \sigma = 0$$

where the divergence is taken row-wise. The constraint spaces are

$$\mathbb{E}_\xi = \{a \odot \xi \equiv a \otimes \xi + \xi \otimes a : a \in \mathbb{R}^n\}, \quad \mathbb{J}_\xi = \{\sigma \in \text{Sym}(\mathbb{V}) : \sigma \xi = 0\}.$$

In this case, the constitutive relation (7.1.1) is referred to as Hooke’s law.

In general we can couple several admissible models to obtain new ones; we illustrate this simple observation with two physically-relevant situations.

- (iii) Piezoelectricity: when a strain is applied to a piezoelectric material, electric fields or electric displacement fields are generated, and conversely; in this case the strain ε , the stress σ , the electric field e and the electric displacement d satisfy

$$\operatorname{curl} \operatorname{curl} \varepsilon = 0, \quad \operatorname{div} \sigma = 0, \quad \operatorname{curl} e = 0, \quad \operatorname{div} d = 0.$$

We can take $\mathbb{V} = \operatorname{Sym}(\mathbb{R}^n) \oplus \mathbb{R}^n$, so that $(\varepsilon, e), (\sigma, d) \in \mathbb{V}$.

- (iv) Multifield theories: in this case $\mathbb{V} = \bigoplus_{i=1}^m \mathbb{R}^n \cong \mathbb{R}^m \otimes \mathbb{R}^n$ and we have m coupled curl-free fields which induce m divergence-free applied fields. The constraint spaces are

$$\mathbb{E}_\xi = \{a \otimes \xi : a \in \mathbb{R}^m\}, \quad \mathbb{J}_\xi = \{j = (j_1, \dots, j_m) : j_1, \dots, j_m \perp \xi\}.$$

When $m = 2$ this models, for instance, thermoelectric materials. We shall return to this example in greater detail in Section 7.4.

Another example of a coupled theory is the one describing thermoelastic materials, but we refer the reader to [283] for these and other examples, as well as for further discussion of the properties of these models.

We shall forget the differential constraints (7.1.2) and instead *postulate* a decomposition of the form (7.1.7); moreover, the fields e, j must take values in the appropriate constraint spaces, as in (7.1.8). We define the spaces of periodic functions

$$\begin{aligned} \mathcal{E}_{\text{per}} &\equiv \{e \in L^2(\mathbb{T}^n, \mathbb{V}) : \widehat{e}(0) = 0 \text{ and } \widehat{e}(k) \in \mathbb{E}_{k/|k|} \otimes \mathbb{C} \text{ for } k \neq 0\} \\ \mathcal{J}_{\text{per}} &\equiv \{j \in L^2(\mathbb{T}^n, \mathbb{V}) : \widehat{j}(0) = 0 \text{ and } \widehat{j}(k) \in \mathbb{J}_{k/|k|} \otimes \mathbb{C} \text{ for } k \neq 0\} \end{aligned}$$

where $\widehat{f}(k)$ are, for $k \in \mathbb{Z}^n$, the Fourier coefficients of $f \in L^2(\mathbb{T}^n, \mathbb{V})$. There is an orthogonal decomposition

$$L^2(\mathbb{T}^n, \mathbb{V}) = \mathcal{E}_{\text{per}} \oplus \mathcal{J}_{\text{per}} \oplus \mathbb{V}$$

where we identify constant fields with \mathbb{V} . Associated with this decomposition we have the orthogonal projections $\Gamma_{\text{per}} : L^2(\mathbb{T}^n, \mathbb{V}) \rightarrow \mathcal{E}_{\text{per}}$ and $\Sigma_{\text{per}} : L^2(\mathbb{T}^n, \mathbb{V}) \rightarrow \mathcal{J}_{\text{per}}$.

In order to formalize the compensated compactness properties of (7.1.2) in the abstract Hilbert space formalism we introduce the following:

Definition 7.1.4. We say that the decomposition $\mathbb{V} = \mathbb{E}_\xi \oplus \mathbb{J}_\xi$ has the *compensated compactness property* if, for any sequences $e^\varepsilon \in \mathcal{E}_{\text{per}}, j^\varepsilon \in \mathcal{J}_{\text{per}}$,

$$e^\varepsilon \rightharpoonup e, j^\varepsilon \rightharpoonup j \text{ in } L^2(\mathbb{T}^n, \mathbb{V}) \quad \implies \quad \langle e^\varepsilon, j^\varepsilon \rangle_{\mathbb{V}} \xrightarrow{*} \langle e, j \rangle_{\mathbb{V}} \text{ in } \mathcal{M}(\mathbb{T}^n). \quad (7.1.9)$$

We note the following immediate consequence of the compensated compactness property:

Proposition 7.1.5. *If the decomposition $\mathbb{V} = \mathbb{E}_\xi \oplus \mathbb{J}_\xi$ has the compensated compactness property then, for sequences $e^\varepsilon, j^\varepsilon \in L^2(\mathbb{T}^n, \mathbb{V})$ with zero mean,*

$$\left. \begin{aligned} e^\varepsilon \rightharpoonup e, j^\varepsilon \rightharpoonup j \\ \Sigma_{\text{per}} e^\varepsilon \rightarrow \Sigma_{\text{per}} e \text{ in } L^2(\mathbb{T}^n, \mathbb{V}) \\ \Gamma_{\text{per}} j^\varepsilon \rightarrow \Gamma_{\text{per}} j \end{aligned} \right\} \implies \langle e^\varepsilon, j^\varepsilon \rangle_{\mathbb{V}} \xrightarrow{*} \langle e, j \rangle_{\mathbb{V}} \text{ in } \mathcal{M}(\mathbb{T}^n).$$

Proof: Note that it suffices to consider the case $e = j = 0$. By replacing e^ε with $e^\varepsilon - \Sigma_{\text{per}} e^\varepsilon$ and j^ε with $j^\varepsilon - \Gamma_{\text{per}} j^\varepsilon$ we can also assume that $e^\varepsilon \in \mathcal{E}_{\text{per}}$ and $j^\varepsilon \in \mathcal{J}_{\text{per}}$. The conclusion now follows from the compensated compactness property. \square

If $\mathbb{V} = \ker \mathcal{A}(\xi) \oplus \ker \mathcal{B}^*(\xi)$ we see from Proposition 7.1.2 that for $e \in L^2(\mathbb{T}^n, \mathbb{V})$ with $\hat{e}(0) = 0$,

$$e^\varepsilon = \mathcal{B}u^\varepsilon + \mathcal{A}^*w^\varepsilon \quad \implies \quad \Sigma_{\text{per}}e^\varepsilon = \mathcal{A}^*w^\varepsilon$$

and so, from (7.1.6), $\Gamma_{\text{per}}e^\varepsilon \rightarrow \Gamma_{\text{per}}e$ in $L^2(\mathbb{T}^n, \mathbb{V})$ if $\mathcal{A}e^\varepsilon \rightarrow \mathcal{A}e$ in $H^{-l}(\mathbb{T}^n, \mathbb{V}')$. Hence the assumptions of Proposition 7.1.5 agree, in the operator formalism, with those of Theorem 7.1.1.

The relevant properties of the operator formalism, i.e. the constant rank assumption and Theorem 7.1.1 can be summarized as follows:

Assumptions 7.1.6. We shall assume that there is a decomposition

$$\mathbb{V} = \mathbb{E}_\xi \oplus \mathbb{J}_\xi, \quad \text{for } \xi \in \mathbb{S}^{n-1},$$

and we denote by $\Gamma(\xi): \mathbb{E}_\xi \oplus \mathbb{J}_\xi \rightarrow \mathbb{E}_\xi$ the orthogonal projection onto \mathbb{E}_ξ , for $\xi \in \mathbb{S}^{n-1}$. Furthermore, this decomposition has the following properties:

- (i) $\Gamma: \mathbb{S}^{n-1} \rightarrow \text{Lin}(\mathbb{V})$ is continuous and even, i.e. $\mathbb{E}_\xi = \mathbb{E}_{-\xi}$;
- (ii) the decomposition has the compensated compactness property.

Condition (i) is seemingly stronger than the constant rank assumption on \mathcal{A} , but this is not the case: indeed, if $\mathcal{A}^\dagger(\xi)$ denotes the Moore–Penrose pseudoinverse of $\mathcal{A}(\xi)$ and \mathcal{A} has constant rank then the map $\xi \mapsto \mathcal{A}^\dagger(\xi)\mathcal{A}(\xi)$ is continuous, see Chapter 4. We recall that the Moore–Penrose inverse $\mathcal{A}^\dagger(\xi)$ is an operator such that $\mathcal{A}^\dagger(\xi)\mathcal{A}(\xi)$ is the orthogonal projection onto $(\ker \mathcal{A}(\xi))^\perp$ and hence, under a constant rank assumption, Γ is continuous.

7.1.2 H-convergence and G-closure

In the sequel we always work under convexity (or strong ellipticity) assumptions:

Definition 7.1.7. For $\alpha, \beta > 0$, we define the class of coercive linear operators

$$\mathcal{M}_{\alpha, \beta}(\mathbb{V}) \equiv \left\{ L \in \text{GL}(\mathbb{V}) : \langle Lv, v \rangle \geq \alpha|v|^2, \langle L^{-1}v, v \rangle \geq \beta|v|^2 \right\}.$$

We shall also write $\mathcal{M}_{\alpha, \beta}^{\text{sym}}(\mathbb{V}) \equiv \mathcal{M}_{\alpha, \beta}(\mathbb{V}) \cap \text{Sym}(\mathbb{V})$.

We are interested in large-scale properties of the system described at the beginning of this section; these are, to some extent, captured by the homogenized tensor L^* , which is the central object of study of the theory.

Definition 7.1.8. A sequence $L^\varepsilon \in L^\infty(\mathbb{T}^n, \mathcal{M}_{\alpha, \beta}(\mathbb{V}))$ is said to *H-converge* to L^* , denoted $L_\varepsilon \xrightarrow{\text{H}} L^*$, if for every $f \in \mathcal{E}_{\text{per}}$ we have

$$\begin{cases} e^\varepsilon \rightharpoonup e \\ L^\varepsilon e^\varepsilon \rightharpoonup L^*e \end{cases} \quad \text{in } L^2(\mathbb{R}^n, \mathbb{V})$$

where $e^\varepsilon, e \in \mathcal{E}_{\text{per}}$ denote, respectively, the unique solutions to

$$\Gamma_{\text{per}}L^\varepsilon e^\varepsilon = f, \quad \Gamma_{\text{per}}L^*e = f.$$

The tensor L^* is called the *homogenized tensor* and it is also referred to as the *effective tensor* in the literature.

We refer the reader to [7] for a detailed treatment of H-convergence in the conductivity and linear elasticity settings. Definition 7.1.8 requires justification. To begin with, it follows easily from the Lax–Milgram lemma that, for $L \in L^\infty(\mathbb{T}^n, \mathcal{M}_{\alpha, \beta}(\mathbb{V}))$, the map

$$T_L \equiv \Gamma_{\text{per}}L: \mathcal{E}_{\text{per}} \rightarrow \mathcal{E}_{\text{per}}$$

is an isomorphism: thus the problems $T_{L^\varepsilon}e^\varepsilon = f$ are uniquely solvable in \mathcal{E}_{per} . The problem for the homogenized tensor L^* is also well-posed:

Theorem 7.1.9. *Assuming 7.1.6, let $L \in L^\infty(\mathbb{T}^n, \mathcal{M}_{\alpha, \beta}(\mathbb{V}))$ and define $L^\varepsilon \equiv L(\cdot/\varepsilon)$. Then $L^\varepsilon \xrightarrow{H} L^*$ where, for any $v \in \mathbb{V}$,*

$$L^*v = \int_{\mathbb{T}^n} L(x)[v + e(x)] dx$$

and $e \in \mathcal{E}(\mathbb{T}^n)$ is the solution of the periodic cell problem

$$L(x)[v + e(x)] \in \mathcal{J}_{\text{per}} \oplus \mathbb{V} \quad (7.1.10)$$

We remark that the whole sequence L^ε H-converges to L^* and not just a subsequence. The proof of the theorem is similar to the one in the classical settings and we use TARTAR's oscillating test function method [302]. For normed vector spaces E, F we denote by $\mathcal{L}(E, F)$ the space of bounded linear operators $E \rightarrow F$. We begin by stating a basic lemma concerning compactness of the weak operator topology.

Lemma 7.1.10. *Let E, F be Banach spaces such that E is separable and F is reflexive. If $S^\varepsilon \in \mathcal{L}(E, F)$ is norm-bounded, there is $S \in \mathcal{L}(E, F)$ such that, up to a subsequence,*

$$S^\varepsilon f \rightharpoonup Sf \text{ in } F \quad \text{for all } f \in E.$$

The proof of the lemma is elementary and can be found, for instance, in [8].

Proof of Theorem 7.1.9: Let $d = \dim \mathbb{V}$ and take a basis v_1, \dots, v_d of \mathbb{V} , so that we write $\mathbb{V} \cong \mathbb{R}^d$. Fix $f \in \mathcal{E}_{\text{per}}$ and let $e^\varepsilon \equiv T_{L^\varepsilon}^{-1}f$. We denote by e the weak limit of e^ε in $L^2(\mathbb{T}^n, \mathbb{V})$.

We begin by defining auxiliary test functions as follows. For $i = 1, \dots, d$, let $e_i \in \mathcal{E}_{\text{per}}$ be the solution of

$$L(x)[v_i + e_i(x)] \in \mathcal{J}_{\text{per}} \oplus \mathbb{V} \quad \iff \quad \Gamma_{\text{per}}L(v_i + e_i) = 0$$

and set $w_i^\varepsilon(x) \equiv v_i + e_i(x/\varepsilon)$. Note that $w_i^\varepsilon \rightharpoonup v_i$ in $L^2(\mathbb{T}^n, \mathbb{V})$. Consider the operators $S^\varepsilon \equiv L^\varepsilon T_{L^\varepsilon}^{-1}: \mathcal{E}_{\text{per}} \rightarrow L^2(\mathbb{T}^n, \mathbb{V})$. By Lemma 7.1.10 there is an operator $S \in \mathcal{L}(\mathcal{E}_{\text{per}}, L^2(\mathbb{T}^n, \mathbb{V}))$ such that, up to a subsequence,

$$L^\varepsilon e^\varepsilon = S^\varepsilon f \rightharpoonup Sf \quad \text{in } L^2(\mathbb{T}^n, \mathbb{V}).$$

We define $L^* \in L^2(\Omega, \text{End}(\mathbb{V}))$ by its action on the basis vector v_i , so $L^*v_i \equiv \text{w-lim}_{\varepsilon \rightarrow 0} L^\varepsilon w_i^\varepsilon$.

We now prove that $L^\varepsilon \xrightarrow{H} L^*$. To see this note that, for a.e. $x \in \mathbb{T}^n$, $L(x) \in \mathcal{M}_{\alpha, \beta}$ and so

$$0 \leq \langle L^\varepsilon e^\varepsilon - L^\varepsilon w^\varepsilon, e^\varepsilon - w^\varepsilon \rangle$$

where $w^\varepsilon \equiv \sum_{i=1}^d \lambda_i w_i^\varepsilon$ and $\lambda_i \in \mathbb{R}$. We apply the compensated compactness property: since

$$e^\varepsilon, w^\varepsilon \in \mathcal{E}_{\text{per}}, \quad \Gamma_{\text{per}}(L^\varepsilon e^\varepsilon) = f, \quad \Gamma_{\text{per}}(L^\varepsilon w^\varepsilon) = 0,$$

and, writing $\lambda = \sum_{i=1}^d \lambda_i v_i \in \mathbb{V}$, it follows that for a.e. $x \in \mathbb{T}^n$

$$0 \leq \langle Sf - L^*\lambda, e - \lambda \rangle.$$

Let y be a Lebesgue point for all the functions involved, let $\mu \in \mathbb{R}^d$ be arbitrary and set $\lambda = e(y) - t\mu$ where $t > 0$; then

$$0 \leq \langle Sf(y) - L^*(y)e(y) + tL^*(y)\mu, t\mu \rangle.$$

Diving by t and sending $t \rightarrow 0$,

$$0 \leq \langle Sf(y) - L^*(y)e(y), \mu \rangle.$$

Since $\mu \in \mathbb{R}^d$ is arbitrary, we conclude that $Sf(y) = L^*(y)e(y)$ and this holds for a.e. $y \in \mathbb{T}^n$.

It remains to check that in fact $L^* \in \mathcal{M}_{\alpha, \beta}$ a.e. in \mathbb{T}^n . This follows straightforwardly by the compensated compactness property and we leave the details to the reader. \square

Remark 7.1.11. From Theorem 7.1.9 we see that if $L^\varepsilon \in \mathcal{M}_{\alpha, \beta}^{\text{sym}}(\mathbb{V})$ then also $L^* \in \mathcal{M}_{\alpha, \beta}^{\text{sym}}(\mathbb{V})$.

Besides Theorem 7.1.9, there are other representations for the homogenized tensor in the periodic setting; another one will be encountered in Proposition 7.3.1. It is also possible to define L^* through the notion of Γ -limit: for results in the generality we consider here, but without requiring convexity assumptions, see [60, Theorem 1.7]. We remark that, even without convexity assumptions, the homogenized tensor is described in terms of solutions to periodic cell problems, although now one needs to solve an infinite number of such problems instead of just one [288].

Proposition 7.1.12. *Under assumptions 7.1.6, the topology of H -convergence is metrizable.*

Proof: This follows from the fact that the map

$$L^\infty(\mathbb{T}^n, \mathcal{M}_{\alpha,\beta}(\mathbb{V})) \rightarrow \mathcal{L}(\mathcal{E}_{\text{per}}, \mathcal{E}_{\text{per}} \times L^2(\mathbb{T}^n, \mathbb{V})), \quad L \mapsto (T_L^{-1}, LT_L^{-1})$$

is injective and has a bounded image, on which the weak operator topology is metrizable. \square

From Theorem 7.1.9 we are led to introduce the following classical notion:

Definition 7.1.13. The G -closure $\mathsf{G}(\mathbb{M})$ of a closed set $\mathbb{M} \subseteq \mathcal{M}_{\alpha,\beta}(\mathbb{V})$ is the closure in $\mathcal{M}_{\alpha,\beta}(\mathbb{V})$ of the set

$$\{L^* : L^\varepsilon \xrightarrow{H} L^* \text{ where } L^\varepsilon = L(\cdot/\varepsilon) \text{ and } L \in L^\infty(\mathbb{T}^n, \mathbb{M})\}.$$

The set \mathbb{M} is G -closed if $\mathbb{M} = \mathsf{G}(\mathbb{M})$.

Strictly speaking, the above concept should be referred to as H -closure, standing for homogenization closure, but we use the more common terminology of G -closure. We also note for later use the following immediate consequence of Theorem 7.1.9:

Corollary 7.1.14. *The sets $\mathcal{M}_{\alpha,\beta}(\mathbb{V})$ are G -closed.*

A basic condition that a G -closed set must satisfy is closure under lamination.

Definition 7.1.15. The L -closure $\mathsf{L}(\mathbb{M})$ of a set $\mathbb{M} \subset \text{Sym}^+(\mathbb{V})$ is defined as

$$\mathsf{L}(\mathbb{M}) = \bigcup_{i=0}^{\infty} \mathsf{L}_i(\mathbb{M}),$$

where $\mathsf{L}_0(\mathbb{M}) = \mathbb{M}$ and $\mathsf{L}_{i+1}(\mathbb{M})$ is the set of effective tensors of all simple laminates made with any two tensors $L_1, L_2 \in \mathsf{L}_i(\mathbb{M})$, taken in any volume fraction and any orientation of layers. More formally,

$$\mathsf{L}_{i+1}(\mathbb{M}) \equiv \left\{ L(x) = \chi_\theta(x \cdot \xi)L_1 + (1 - \chi_\theta(x \cdot \xi))L_2 : \theta \in [0, 1], \xi \in \mathbb{S}^{n-1}, L_1, L_2 \in \mathsf{L}_i(\mathbb{M}) \right\}$$

where $\chi_\theta : \mathbb{R} \rightarrow \{0, 1\}$ is the periodic extension of the characteristic function of $[0, \theta]$.

A set $\mathbb{M} \subset \text{Sym}^+(\mathbb{V})$ is called L -closed if it is closed in $\text{Sym}^+(\mathbb{V})$ and $\mathbb{M} = \mathsf{L}(\mathbb{M})$.

While we have been working throughout with the torus \mathbb{T}^n coming from the standard lattice \mathbb{Z}^n , one can also work with a rotated lattice $Q\mathbb{Z}^n$ for $Q \in \text{GL}(n)$; in this case all the integrals should be replaced by averaged integrals. Using Theorem 7.1.9 it is easy to check that L^* does not depend on the chosen orientation. This shows, in particular, that Definition 7.1.15 makes sense in the setting of periodic homogenization, since any simple laminate can be regarded as a periodic composite by choosing a cell aligned with the direction of lamination.

Remark 7.1.16. It is well-known that, for the problem of determining the G -closure of a given set, there is no loss of generality in working under periodicity assumptions. This result is usually attributed to an unpublished work of DAL MASO–KÖHN; it was generalized in [320] to nonlinear monotone constitutive relations in the conductivity setting, see also [20], and see [167] for a statement in the general setup we consider here.

7.2 The general theory of exact relations

In this section we describe, for the reader's convenience, the algebraic characterizations of the notions of G- and L-closure from the previous section in the favorable case where \mathbb{M} has empty interior; this is a special, atypical situation [267, 363], see also [282]. We work throughout under the implicit Assumption 7.1.6. The results we present here are well-known and can be found, together with their proofs, in [167, 170, 282, 283]. We denote by $\text{Sym}^+(\mathbb{V})$ the class of symmetric positive-definite operators on \mathbb{V} .

Definition 7.2.1. A submanifold $\mathbb{M} \subset \text{Sym}^+(\mathbb{V})$ with positive codimension is said to be:

- (i) an *exact relation* if it contains the G-closure of all of its compact subsets;
- (ii) an *L-relation* if it is L-closed.

We will, for simplicity, restrict our attention to symmetric operators. The local tensor of the material L is very often symmetric in applications, although this is not always the case as, for instance, in the Hall effect [257].

7.2.1 The W-transform

We begin by recalling the definition of the W-transform, introduced by MILTON in [282].

Definition 7.2.2. For each vector $\xi \in \mathbb{S}^{n-1}$, define the maps $W_\xi: \text{Sym}^+(\mathbb{V}) \rightarrow \text{Sym}(\mathbb{V})$ by

$$W_\xi(L) = (L - I)[I + (L - I)\Gamma(\xi)]^{-1} = [I + (L - I)\Gamma(\xi)]^{-1}(L - I).$$

The following lemma assures that this definition makes sense, see [168, Lemma A.1].

Lemma 7.2.3. For any $\xi \in \mathbb{S}^d$ and any $L \in \text{Sym}^+(\mathbb{V})$ the operator $I + (L - I)\Gamma(\xi)$ is invertible.

Consider the bounded, linear operators $\Pi_\xi: L^2(\mathbb{T}^n, \mathbb{V}) \rightarrow L^2(\mathbb{T}^n, \mathbb{V})$ corresponding to the Fourier multiplier A , where $A_\xi(0) = 0$ and

$$A_\xi(k) = \Gamma(\xi) - \Gamma(k/|k|), \quad k \in \mathbb{Z}^n \setminus \{0\}. \quad (7.2.1)$$

Note that Assumption 7.1.6(i) assures that Π_ξ is a well-defined operator.

The relevance of the W-transform for the study of the homogenized tensor L^* will become apparent from the following result [168, Lemma A.3]:

Proposition 7.2.4. Let $K \subset \text{Sym}^+(\mathbb{V})$ be a compact set and suppose that for a.e. $x \in \mathbb{T}^n$ we have $L(x) \in K$. The operator $P \equiv I - \Pi_\xi W_\xi$ is invertible in $L^2(\mathbb{T}^n, \mathbb{V})$ and, for any $v \in \mathbb{V}$,

$$W_\xi(L^*)v = \int_{\mathbb{T}^n} W_\xi(L(x))P^{-1}v \, dx. \quad (7.2.2)$$

We remark that, on the right-hand side of the equation, we identified the vector $v \in \mathbb{V}$ with the associated constant field $v \in L^2(\mathbb{T}^n, \mathbb{V})$.

In the particular case where L corresponds to a simple laminate, say with lamination direction ξ and phases L_1, L_2 in proportions $\theta, 1 - \theta$ respectively, for $\theta \in (0, 1)$, Proposition 7.2.4 yields the *lamination formula*

$$W_\xi(L^*) = \theta W_\xi(L_1) + (1 - \theta)W_\xi(L_2). \quad (7.2.3)$$

Indeed, since $L(x)$ depends only on $x \cdot \xi$, we have $A_\xi(\xi) = 0$ and so $\Pi_\xi W_\xi(L(x)) = 0$ as well.

Remark 7.2.5. One can fix a tensor $L_0 \in \text{Sym}^+(\mathbb{V})$ and replace the terms $(L - I)$ by $(L - L_0)$ and $\Gamma(\xi)$ by $L_0^{-1}\Gamma'(\xi)$ in Definition 7.2.2, where $\Gamma'(\xi)$ is the projection onto $L_0\mathbb{E}_\xi$ along \mathbb{J}_ξ ; note that this projection is not necessarily orthogonal. The tensor L_0 corresponds to a reference medium, but it can also be thought of as playing the role of a pre-conditioner. Our main interest in the W-transform, however, comes from Proposition 7.2.4 and it turns out that the computation of the homogenized tensor L^* via the W-transform, as in (7.2.2), does not depend on L_0 . Furthermore, as we will be working with polycrystalline exact relations in the sequel, c.f. Section 7.2.3, we have taken $L_0 = I$ for the sake of simplicity, since there is no loss of generality.

We conclude the study of the properties of the W-transform with the following result:

Lemma 7.2.6. *The map W_ξ is a diffeomorphism from $\text{Sym}^+(\mathbb{V})$ onto its image.*

Proof: Let us first show that W_ξ is injective: if there are $L_1, L_2 \in \text{Sym}^+(\mathbb{V})$ such that $W_\xi(L_1) = W_\xi(L_2)$ then, from the two representations for W_ξ ,

$$[I + (L_1 - I)\Gamma(\xi)]^{-1}(L_1 - I) = (L_2 - I)[I + \Gamma(\xi)(L_2 - I)]^{-1}.$$

Multiplying this identity on the right by $I + \Gamma(\xi)(L_2 - I)$ and on the left by $I + (L_1 - I)\Gamma(\xi)$ and expanding we get $L_1 = L_2$.

Moreover, W_ξ is a local diffeomorphism: this follows from the Inverse Function Theorem, the calculation, for $S \in \text{Sym}(\mathbb{V})$,

$$\begin{aligned} DW_\xi(L)S &= [I + (L - I)\Gamma(\xi)]^{-1}S[I - \Gamma(\xi)W_\xi(L)] \\ &= [I + (L - I)\Gamma(\xi)]^{-1}S[I + \Gamma(\xi)(L - I)]^{-1}, \end{aligned}$$

and Lemma 7.2.3. □

From this lemma, together with the lamination formula (7.2.3), we have:

Corollary 7.2.7. *L-closed sets, and hence also G-closed sets, are diffeomorphic images of convex sets under the inverse of the W-transform.*

This corollary shows the remarkable fact that the G-closure is a smoothing operator.

7.2.2 Jordan algebras and the chain conditions

In this section we fix $\xi_0 \in \mathbb{S}^{d-1}$ and we let \mathbb{W} be a subspace in $\text{Sym}(\mathbb{V})$. We will denote by $\mathbb{W}_\mathbb{C}$ the usual complexification of \mathbb{W} . We define a subspace

$$\mathbb{A} \equiv \text{span}\{\Gamma(\xi) - \Gamma(\xi_0) : \xi \in \mathbb{S}^{d-1}\} \quad (7.2.4)$$

and, using Lemma 7.2.6, a submanifold

$$\mathbb{M} \equiv W_{\xi_0}^{-1}(\mathbb{W}) \subset \text{Sym}^+(\mathbb{V}). \quad (7.2.5)$$

We remark that \mathbb{A} is independent of ξ_0 and can be defined in a more invariant fashion as the vector space parallel to the affine hull of $\Gamma(\mathbb{S}^{n-1})$.

Remark 7.2.8. There is no loss of generality in our definition of \mathbb{M} . Indeed, if we translate \mathbb{M} so that $I \in \mathbb{M}$ then, by Corollary 7.2.7, $W_\xi(\mathbb{M})$ must be a convex subset of an affine subspace $\mathbb{W}_\xi = \text{span}(W_\xi(\mathbb{M}))$ of $\text{Sym}(\mathbb{V})$. Finally, the computations in Lemma 7.2.6 show that the differential of W_ξ at I is the identity and hence the subspaces \mathbb{W}_ξ do not depend on ξ , since they all coincide with the tangent space to \mathbb{M} at I . This shows that, indeed, there is no loss of generality in our definition (7.2.5) of \mathbb{M} .

We are interested in deriving algebraic conditions which allow us to decide whether \mathbb{M} is L- or G-closed. For this, it will be helpful to begin by considering the following concept:

Definition 7.2.9. \mathbb{W} is a *Jordan \mathbb{A} -multialgebra*, i.e.

$$K_1 *_A K_2 \equiv \frac{1}{2}(K_1 A K_2 + K_2 A K_1) \in \mathbb{W} \quad \text{for all } K_1, K_2 \in \mathbb{W}, A \in \mathbb{A}. \quad (7.2.6)$$

For each fixed $A \in \mathbb{A}$, the product $*_A$ in (7.2.6) is an example of a Jordan product, i.e. it is a commutative, non-associative product which satisfies the *Jordan identity*

$$(K_1 * K_2) * (K_1 * K_1) = K_1 * (K_2 * (K_1 * K_1)).$$

An algebra equipped with such a product is called a Jordan algebra; these were introduced as algebras of observables in quantum mechanics [221]. We refer the reader to [217, 278] for the classical theory of Jordan algebras.

It turns out that it is useful to consider more general expressions than (7.2.6). We will refer to expressions such as $K_1 A_1 K_2 A_2 \dots K_{j-1} A_{j-1} K_j$ as *j-chains*.

Definition 7.2.10. The subspace \mathbb{W} has the *j-chain property* if, for every $K_1, \dots, K_j \in \mathbb{W}$ and every $A_1, \dots, A_{j-1} \in \mathbb{A}$, the symmetrized *j-chain* lies in \mathbb{W} , i.e.

$$K_1 A_1 K_2 A_2 \dots K_{j-1} A_{j-1} K_j + K_j A_{j-1} K_{j-1} \dots A_2 K_2 A_1 K_1 \in \mathbb{W}.$$

The relevance of *j-chains* for the general theory of exact relations becomes apparent from the following theorem:

Theorem 7.2.11 (Theory of exact relations). *Let \mathbb{M} be a submanifold as in (7.2.5).*

We have the following characterizations:

- (i) \mathbb{M} is an *L-relation* if and only if \mathbb{W} has the *2-chain property*;
- (ii) \mathbb{M} is an *exact relation* if and only if

$$\int_{\mathbb{T}^n} [K(x) \Pi_{\xi_0}]^k K(x) dx \in \mathbb{W}_{\mathbb{C}} \quad \text{for all } k \geq 0 \text{ and all } K \in L^\infty(\mathbb{T}^n, \mathbb{W}_{\mathbb{C}}).$$

Concerning the chain properties, we have:

- (iii) *If \mathbb{W} has the 2-, 3- and 4-chain properties, then it has the j-chain property for any $j \in \mathbb{N}$;*
- (iv) *If \mathbb{W} has the j-chain property for any $j \in \mathbb{N}$ then \mathbb{M} is an exact relation;*
- (v) *If \mathbb{M} is an exact relation then \mathbb{W} has the 2-chain property and, if $n = 2$, it also has the 3-chain property.*

We do not give a full proof of this result here, referring the interested reader instead to [167, §4].

Proof of (i): Given $\xi_0, \xi \in \mathbb{S}^{n-1}$, the discussion in Remark 7.2.8 shows that \mathbb{M} is an *L-relation* if and only if $W_\xi \circ W_{\xi_0}^{-1}(\mathbb{W}) = \mathbb{W}$. By Lemma 7.2.6, this amounts to checking whether the map $W_\xi \circ W_{\xi_0}^{-1}$ maps a small neighborhood $\mathcal{O} \subset \mathbb{W}$ of 0 to another neighborhood of 0 in \mathbb{W} . We note that, for $K \in \mathcal{O}$, the tensors $[I - K\Gamma(\xi)]^{-1}$ and $K\Gamma(\xi)$ commute; using this, and from the definition of W ,

$$W_{\xi_0}^{-1}(K) = I + [I - K\Gamma(\xi_0)]^{-1}, \quad W_\xi(L) = [I + (L - I)\Gamma(\xi)]^{-1}(L - I).$$

We can then compute

$$W_\xi \circ W_{\xi_0}^{-1}(K) = [I - KA_{\xi_0}(\xi)]^{-1}K \quad \text{for all } K \in \mathcal{O} \subset \mathbb{W}$$

where, similarly to (7.2.1), $A_{\xi_0}(\xi) = \Gamma(\xi_0) - \Gamma(\xi)$. For K sufficiently close to 0 we can expand the right-hand side in a Neumann series:

$$W_\xi \circ W_{\xi_0}^{-1}(\varepsilon K) = \sum_{j=0}^{\infty} \varepsilon^{j+1} (KA_{\xi_0}(\xi))^j K.$$

The first term in the expansion shows that $W_\xi \circ W_{\xi_0}^{-1}(\mathbb{W}) = \mathbb{W}$, i.e. \mathbb{W} does not depend on ξ_0 . The second term shows that we must have $KA_{\xi_0}(\xi)K \in \mathbb{W}$ and, replacing K by $K_1 + K_2$ for $K_i \in \mathbb{W}$, we get (7.2.6). Moreover, (7.2.6) is sufficient to ensure that all higher order terms are in \mathbb{W} : indeed, simply replace K_1 by K and K_2 by $(KA_{\xi_0}(\xi))^{j-1}K$ in (7.2.6) to see that the *j*-th term of the series is in \mathbb{W} . \square

Definition 7.2.12. A subspace $\mathbb{W}' \subset \text{End}(\mathbb{V})$ is said to be an *associative \mathbb{A} -multialgebra* if

$$K_1 A K_2 \in \mathbb{W}' \quad \text{for all } K_1, K_2 \in \mathbb{W}' \text{ and all } A \in \mathbb{A}.$$

Theorem 7.2.13. *Suppose that $\mathbb{W}' \subset \text{End}(\mathbb{V})$ is an associative \mathbb{A} -multialgebra. If we set $\mathbb{W} = \mathbb{W}' \cap \text{Sym}(\mathbb{V})$ then the submanifold \mathbb{M} defined by (7.2.5) is an exact relation.*

Proof: Assume first that $K_1, K_2 \in L^\infty(\mathbb{T}^n, \mathbb{W}'_{\mathbb{C}})$. Then $K_1 \Pi_{\xi_0} K_2 \in L^2(\mathbb{T}^n, \mathbb{W}'_{\mathbb{C}})$: indeed, $\widehat{K}_2(k) \in \mathbb{W}'_{\mathbb{C}}$ for all $k \in \mathbb{Z}^n$ and

$$K_1 \Pi_{\xi_0} K_2 = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} e^{2\pi i x \cdot k} K_1 A_{\xi_0}(k) \widehat{K}_2(k).$$

Since $A_{\xi_0}(k) \in \mathbb{A}$, every term in the series is in $\mathbb{W}'_{\mathbb{C}}$ and hence so is $K_1 \Pi_{\xi_0} K_2$. By induction this implies that, given $K \in L^\infty(\mathbb{T}^n, \mathbb{W}'_{\mathbb{C}})$, $T_k \equiv (K \Pi_{\xi_0})^k K \in L^2(\mathbb{T}^n, \mathbb{W}'_{\mathbb{C}})$ for all $k \geq 1$, since $T_{k+1} = K \Pi_{\xi_0} T_k$.

Finally, we just note that if $K \in L^\infty(\mathbb{T}^n, \mathbb{W}'_{\mathbb{C}})$ then the previous paragraph implies that $T_k \in L^2(\mathbb{T}^n, \mathbb{W}'_{\mathbb{C}})$ since it is a composition of symmetric operators. Hence Theorem 7.2.11(ii) yields the desired conclusion. \square

If we define the space $\mathbb{W}' \equiv \sum_{j=0}^{\infty} (\mathbb{W}\mathbb{A})^j \mathbb{W}$, which is the smallest associative \mathbb{A} -multialgebra containing \mathbb{W} , we see that the j -chain conditions hold for all j if and only if $\mathbb{W} = \mathbb{W}' \cap \text{Sym}(\mathbb{V})$. Thus, in particular, Theorem 7.2.11(iv) follows from Theorem 7.2.13.

7.2.3 Polycrystalline exact relations

In this subsection, we shall make the following assumption:

Rotational equivariance 7.2.14. We assume that there is an orthogonal action of $\text{SO}(n)$ on \mathbb{V} and that $\Gamma(\xi)$ is equivariant with respect to this action, in the sense that

$$R \cdot (\Gamma(\xi)v) = \Gamma(R\xi)(R \cdot v) \quad \text{for all } R \in \text{SO}(n), v \in \mathbb{V} \text{ and } \xi \in \mathbb{S}^{n-1}.$$

We remark that there is an induced action of $\text{SO}(n)$ on $\text{End}(\mathbb{V})$, denoted also by $R \cdot L$: for $L \in \text{End}(\mathbb{V})$, $R \in \text{SO}(n)$ and $v \in \mathbb{V}$,

$$R \cdot L(v) = R \cdot (L(R^{-1} \cdot v)).$$

In practice the action depends on the specific setup one is considering. In conductivity $\mathbb{V} = \mathbb{R}^n$ and $\text{SO}(n)$ acts by multiplication; in linear elasticity, $\mathbb{V} = \text{Sym}(\mathbb{R}^n)$ and $\text{SO}(n)$ acts by conjugation; in multifield theories $\mathbb{V} \cong \mathbb{R}^m \otimes \mathbb{R}^n$ and $\text{SO}(n)$ acts by multiplication on the second component of the tensor product.

Definition 7.2.15. An exact relation $\mathbb{M} \subset \text{Sym}^+(\mathbb{V})$ is said to be a *polycrystal* if $R \cdot L \in \mathbb{M}$ a.e. for all $R \in \text{SO}(n)$ and all $L \in L^\infty(\mathbb{T}^n, \mathbb{M})$.

As a consequence of rotational equivariance we see that

$$R \cdot \Gamma(\xi) = \Gamma(R\xi) \tag{7.2.7}$$

Therefore, from the definition of the W-transform,

$$R \cdot W_\xi(L) = W_{R\xi}(R \cdot L).$$

If \mathbb{M} is given by (7.2.5) then it follows that \mathbb{W} must be rotationally invariant and furthermore so must \mathbb{A} , since

$$R \cdot (\Gamma(\xi) - \Gamma(\xi_0)) = \Gamma(R\xi) - \Gamma(R\xi_0) = (\Gamma(R\xi) - \Gamma(\xi_0)) - (\Gamma(R\xi_0) - \Gamma(\xi_0)) \in \mathbb{A}.$$

In this case there is, in fact, a simpler description of \mathbb{A} than the one in (7.2.4):

Lemma 7.2.16. *Assuming 7.2.14, there is $\tilde{\Gamma} \in \text{Sym}(\mathbb{V})$ such that*

$$\mathbb{A} = \text{span}\{R \cdot \tilde{\Gamma} : R \in \text{SO}(n)\}. \tag{7.2.8}$$

In fact, $\tilde{\Gamma} = \Gamma(\xi_0) - \bar{\Gamma}$ where $\xi_0 \in \mathbb{S}^{n-1}$ is some fixed vector and

$$\bar{\Gamma} \equiv \int_{\text{SO}(n)} Q \cdot \Gamma(\xi_0) d\mu(Q) \quad (7.2.9)$$

where μ denotes the normalized Haar measure on $\text{SO}(n)$.

Proof: Let $\tilde{\mathbb{A}}$ denote the right-hand side of (7.2.8). For $\xi \in \mathbb{S}^{n-1}$ there is $R \in \text{SO}(n)$ such that $R\xi_0 = \xi$; then, from (7.2.7),

$$\Gamma(\xi) - \Gamma(\xi_0) = R \cdot \Gamma(\xi_0) - \Gamma(\xi_0) = R \cdot \tilde{\Gamma} - \tilde{\Gamma}, \quad (7.2.10)$$

and so $\mathbb{A} \subseteq \tilde{\mathbb{A}}$. Conversely, from the definitions of $\tilde{\Gamma}$ and $\bar{\Gamma}$, $\int_{\text{SO}(n)} R \cdot \tilde{\Gamma} d\mu(R) = 0$. Therefore, integrating (7.2.10) over $\text{SO}(n)$, we see that $-\tilde{\Gamma} \in \mathbb{A}$, which proves the other inclusion. \square

In particular examples it is much easier to just consider polycrystalline exact relations, since the number of subspaces of $\text{Sym}(\mathbb{V})$ to be considered decreases dramatically. We remark, however, that rotational invariance cannot always be assumed: for instance piezoelectric materials, c.f. Example 7.1.3(iii), are necessarily anisotropic, since in isotropic materials there is no piezoelectric coupling [283, page 29].

7.2.4 Calculating exact relations

In the previous subsections we described the general theory of exact relations, including necessary and sufficient conditions for a manifold \mathbb{M} to be an exact relation, but we are yet to explain how to find exact relations in practice. As in the last subsection, we focus on the $\text{SO}(n)$ -invariant case, as this simplifies the calculations tremendously. In general, one follows a 4-step strategy:

1. determine all the $\text{SO}(n)$ -invariant Jordan \mathbb{A} -multialgebras;
2. eliminate redundancies from the list obtained in the previous step;
3. check if the Jordan \mathbb{A} -multialgebras obtained satisfy the 3- and 4-chain properties;
4. convert the Jordan \mathbb{A} -multialgebras back to the original coordinates, i.e. apply the inverse of the W-transform.

We shall now describe each of the above steps in greater detail.

Step 1 is essentially a problem in the representation theory of the special orthogonal groups. One has to determine the $\text{SO}(n)$ -invariant subspaces of $\text{Sym}^+(\mathbb{V})$ and verify which of these satisfy the Jordan \mathbb{A} -multialgebra property. In order to do this, it is helpful to identify the symmetries of the problem at hand: these are invertible linear maps $\Phi: \text{Sym}(\mathbb{V}) \rightarrow \text{Sym}(\mathbb{V})$ such that

$$\Phi(KAK) = \Phi(K)A\Phi(K), \quad \text{for all } K \in \text{Sym}(\mathbb{V}) \text{ and } A \in \mathbb{A}; \quad (7.2.11)$$

such maps are called Jordan \mathbb{A} -multialgebra automorphisms. If we are looking for $\text{SO}(n)$ -invariant exact relations then we should also require Φ to be $\text{SO}(n)$ -invariant. By calculating $\Phi[(K_1 + K_2)A(K_1 + K_2)]$ for $K_1, K_2 \in \text{Sym}(\mathbb{V})$ we see that in fact Jordan \mathbb{A} -multialgebra automorphisms satisfy

$$\Phi(K_1AK_2 + K_1AK_2) = \Phi(K_1)A\Phi(K_2) + \Phi(K_2)A\Phi(K_1).$$

There is a general result on the structure of Jordan algebra automorphisms:

Theorem 7.2.17. *Assume that $\dim \mathbb{V} \geq 3$. All Jordan \mathbb{A} -multialgebra automorphisms, if they exist, are given by $\Phi(K) = C^T K C$ where $C \in \text{GL}(\mathbb{V})$ is such that $\mathbb{A} = C\mathbb{A}C^T$. If we are looking for $\text{SO}(n)$ -invariant automorphisms then C must in addition be isotropic, in the sense that $\text{SO}(n) \cdot C = C$.*

The proof is essentially contained in the proof of [170, Theorem 4.9] and so we omit it.

Once Step 1 is performed we have a list of (equivalence classes) of $\text{SO}(n)$ -invariant Jordan \mathbb{A} -multialgebras, say $\mathbb{W}_1, \mathbb{W}_2, \dots$. Typically this list will admit simplifications, which are performed in Step 2: it is possible that $\mathbb{W}_i = \mathbb{W}_{i_1} \cap \mathbb{W}_{i_2}$ for some i, i_1, i_2 , in which case one can recover the exact

relation corresponding to \mathbb{W}_i from the knowledge of \mathbb{W}_{i_1} and \mathbb{W}_{i_2} . Furthermore, not all the Jordan algebras are necessarily physically relevant; in coupled problems, for instance, it can happen that some algebras correspond to the case where the fields are decoupled.

Step 3 can be carried in a few different ways. One possibility is to use a symbolic computing software to check the 3- and 4-chain properties of the simplified list obtained in Step 2: if $\mathbb{W}_{i_1}, \mathbb{W}_{i_2}$ satisfy the chain properties then so does their intersection. An alternative to this brute-force approach is to use the representation theory of $\mathrm{SO}(n)$ to simplify the calculations, as done in [170]; curiously, it seems that this theory is much nicer for $n = 3$ than $n = 2$ for the purposes of these calculations.

Concerning Step 4, in many cases the representation (7.2.5) of \mathbb{M} is suboptimal, since it involves a vector ξ_0 on which the manifold \mathbb{M} does not depend. Often one can use a more simple method:

Definition 7.2.18. A tensor $M \in \mathrm{Sym}^+(\mathbb{V})$ is said to be an *inversion key* for the Jordan \mathbb{A} -multialgebra \mathbb{W} if, for some $\bar{\Gamma}$ in the affine hull of $\Gamma_0(\mathbb{S}^{n-1})$,

$$K_1(\bar{\Gamma} - M)K_2 \in \mathbb{W}, \quad \text{for all } K_1, K_2 \in \mathbb{W}. \quad (7.2.12)$$

The definition of an inversion key M simply states that if \mathbb{W} is a Jordan \mathbb{A} -multialgebra then it is also a Jordan $\mathrm{span}\{\mathbb{A}, \bar{\Gamma} - M\}$ -multialgebra.

It is readily checked that if an inversion key M satisfies (7.2.12) with one choice of $\bar{\Gamma}$ as above then it also satisfies (7.2.12) with any other such choice; in particular, we can take $\bar{\Gamma}$ to be as in (7.2.9) or, for instance, $\Gamma_0(e_1)$. The simplest possible inversion key is $M = 0$, although it needn't satisfy (7.2.12), and the most appropriate choice depends on the specific calculations that need to be performed. Regardless of the choice of M we can determine \mathbb{M} , see [167, §6.2]:

Theorem 7.2.19. *Let M be an inversion key for the Jordan \mathbb{A} -multialgebra \mathbb{W} . Then the set*

$$\mathbb{M}' = \{L_0 + (I - KM)^{-1}K : K \in \mathbb{W}\}$$

is open and dense in \mathbb{M} .

7.3 Variational principles and relaxation

In this section we shall relate the homogenization theory described in Sections 7.1 and 7.2 with the general \mathbb{A} -free Calculus of Variations of Chapter 5, see also [97, 98, 295]. The main point connecting the two theories is the following result, see [283, §13 and §31]:

Proposition 7.3.1. *Assuming 7.1.6, let $L \in L^\infty(\mathbb{T}^n, \mathcal{M}_{\alpha,\beta}^{\mathrm{sym}}(\mathbb{V}))$ and define $L^\varepsilon \equiv L(\cdot/\varepsilon)$. Then $L^\varepsilon \xrightarrow{H} L^*$ where, for any $v \in \mathbb{V}$,*

$$\langle L^*v, v \rangle = \min_{e \in \mathcal{E}(\mathbb{T}^n)} \int_{\mathbb{T}^n} \langle L(x)[v + e(x)], v + e(x) \rangle dx. \quad (7.3.1)$$

Note that by Remark 7.1.11 L^* is symmetric and so the quadratic form $e_0 \mapsto \langle L^*e_0, e_0 \rangle_{\mathbb{V}}$ indeed uniquely determines L^* .

Proof: Let $e \in \mathcal{E}(\mathbb{T}^n)$ be the solution of (7.1.10). Since $L(x) \in \mathcal{M}_{\alpha,\beta}(\mathbb{V})$ for a.e. $x \in \mathbb{T}^n$, we see that for any $\psi \in L^2(\mathbb{T}^n, \mathbb{V})$ we have a.e. the pointwise inequality

$$0 \leq \langle L(v + e - \psi), v + e - \psi \rangle_{\mathbb{V}}$$

which, by the symmetry of L , reduces to

$$\langle L(v + e), 2\psi - (v + e) \rangle_{\mathbb{V}} \leq \langle L\psi, \psi \rangle_{\mathbb{V}}.$$

Take $\phi \in \mathcal{E}(\mathbb{T}^n)$, $w \in \mathbb{V}$ and let $\psi = w + \phi$. Taking averages over \mathbb{T}^n , by Theorem 7.1.9,

$$\langle L^*v, 2w - v \rangle_{\mathbb{V}} \leq \int_{\mathbb{T}^n} \langle L(x)[w + \phi(x)], w + \phi(x) \rangle_{\mathbb{V}}.$$

In order to maximize the left-hand side we take $w = v$ and we get

$$\langle L^*v, v \rangle_{\mathbb{V}} \leq \int_{\mathbb{T}^n} \langle L^*(x)[v + \varphi(x)], v + \varphi(x) \rangle_{\mathbb{V}} dx.$$

Since L is positive definite equality holds in this inequality if and only if $\varphi = e$, and so indeed (7.3.1) holds. \square

Before proceeding further, we recall for the reader's convenience the classical notions of semi-convexity from the vectorial Calculus of Variations, see also Chapter 4

Definition 7.3.2. Given a linear operator \mathcal{A} as in (7.1.3) and a locally bounded, Borel measurable integrand $f: \mathbb{V} \rightarrow \mathbb{R}$, we say that f is:

- (i) \mathcal{A} -quasiconvex if, for all $v \in \mathbb{V}$ and all $\varphi \in C_{\text{per}}^\infty([0, 1]^d, \mathbb{V})$ such that $\mathcal{A}\varphi = 0$ and $\int_Q \varphi(x) dx = 0$, we have

$$f(v) \leq \int_Q f(v + \varphi(x)) dx;$$

- (ii) $\Lambda_{\mathcal{A}}$ -convex if, for all v_1, v_2 such that $v_1 - v_2 \in \Lambda$ and all $\theta \in [0, 1]$, we have

$$f(\theta v_1 + (1 - \theta)v_2) \leq \theta f(v_1) + (1 - \theta)f(v_2).$$

We will omit the subscript \mathcal{A} when talking about $\Lambda_{\mathcal{A}}$ convexity since this is a notion of *directional convexity* and, in particular, it is independent of \mathcal{A} : it only depends on its wave cone. As in the convex case, we can define the associated semi-convex envelopes of f :

Theorem 7.3.3. For $f: \mathbb{V} \rightarrow \mathbb{R}$, the envelopes $\text{env}_{\mathcal{A}}^{\text{qc}}[f], \text{env}_{\Lambda}^{\text{dc}}[f]$ satisfy, for all $v \in \mathbb{V}$,

$$\text{env}_{\mathcal{A}}^{\text{qc}}[f](v) \equiv \sup\{g(v) : g \leq f, g \text{ is } \mathcal{A}\text{-quasiconvex}\} = \inf_{\substack{\varphi \in C_{\text{per}}^\infty(\mathbb{T}^n, \mathbb{V}) \\ \int_{\mathbb{T}^n} \varphi = 0, \mathcal{A}\varphi = 0}} \int_{\mathbb{T}^n} f(v + \varphi(x)) dx, \quad (7.3.2)$$

$$\text{env}_{\Lambda}^{\text{dc}}[f](v) \equiv \sup\{g(v) : g \leq f, g \text{ is } \Lambda\text{-convex}\} = \lim_{i \rightarrow \infty} \text{env}_{\Lambda}^{\text{dc}, i}[f](v), \quad (7.3.3)$$

where $\text{env}_{\Lambda}^{\text{dc}, 0}[f] \equiv f$ and $\text{env}_{\Lambda}^{\text{dc}, i+1}[f]$ is defined inductively by

$$\text{env}_{\Lambda}^{\text{dc}, i+1}[f] \equiv \inf_{\substack{\theta v_1 + (1-\theta)v_2 = v \\ \theta \in [0, 1], v_1 - v_2 \in \Lambda}} \left[\theta \text{env}_{\Lambda}^{\text{dc}, i}[f](v_1) + (1 - \theta) \text{env}_{\Lambda}^{\text{dc}, i}[f](v_2) \right].$$

We remark that $\text{env}_{\mathcal{A}}^{\text{qc}}[f]$ is indeed an \mathcal{A} -quasiconvex function, and likewise for the Λ -convex envelope, and (7.3.2) is was proved in [149]. The inductive definition of the directional Λ -convex envelope should be compared with Definition 7.1.15, see also the proof below, and is usually called the Kohn–Strang [240, 239] approximation.

Remark 7.3.4. In the spirit of Sections 7.1 and 7.2 one can rewrite (7.3.2) and (7.3.3) in a way which is independent of \mathcal{A} , using (7.1.7). Indeed, by a density argument,

$$\text{env}_{\mathcal{A}}^{\text{qc}}[f](v) = \inf_{e \in \mathcal{E}_{\text{per}}} \int_{\mathbb{T}^n} f(v + e(x)) dx;$$

note that the density argument is possible since, for any $e \in \mathcal{E}_{\text{per}}$, there is $u \in H^k(\mathbb{T}^n, \mathbb{U})$ such that $\mathcal{B}u = e$, c.f. Proposition 7.1.2. Moreover, recall that $\Lambda = \bigcup_{\xi \in \mathbb{S}^{n-1}} \mathbb{E}_{\xi}$.

We can now relate the above notions of semi-convexity with Proposition 7.3.1. The following result is well-known [283]:

Theorem 7.3.5. *Assuming 7.1.6, let $\mathbb{M} \subset \text{Sym}^+(\mathbb{V})$ be compact and let $f: \mathbb{V} \rightarrow \mathbb{R}$ be defined by*

$$f(v) \equiv \min_{S \in \mathbb{M}} \langle Sv, v \rangle.$$

Then we have

$$\text{env}_{\Lambda}^{\text{dc}}[f](v) = \min_{S \in L(\mathbb{M})} \langle Sv, v \rangle, \quad \text{env}_{\mathcal{A}}^{\text{qc}}[f](v) = \min_{S \in G(\mathbb{M})} \langle Sv, v \rangle.$$

Proof: Let us begin with the expression for the Λ -convex envelope. We shall prove, by induction, the relation

$$\text{env}_{\Lambda}^{\text{dc},i} f(v) = \min_{S \in L_i K} \langle Sv, v \rangle. \quad (7.3.4)$$

Given $v \in \mathbb{V}$, there are $w_1, w_2 \in \mathbb{V}$ and $\theta \in [0, 1]$ such that $w_1 - w_2 \in \Lambda$, $v = \theta w_1 + (1 - \theta)w_2$ and

$$\text{env}_{\Lambda}^{\text{dc},i+1} f(v) = \theta \text{env}_{\Lambda}^{\text{dc},i} f(w_1) + (1 - \theta) \text{env}_{\Lambda}^{\text{dc},i} f(w_2); \quad (7.3.5)$$

indeed, since there is a constant $c = c(\mathbb{M})$ such that $c|\cdot|^2 \leq f$, we see that the infimum in (7.3.3) is attained. By the inductive hypothesis, there are also $S_j \in L_i(\mathbb{M})$ such that

$$\text{env}_{\Lambda}^{\text{dc},i} f(w_j) = \langle S_j w_j, w_j \rangle, \quad j = 1, 2. \quad (7.3.6)$$

Let $S \in L^\infty(\mathbb{T}^n, \{0, 1\})$ be such that $\int_{\mathbb{T}^n} S(x) dx = \theta S_1 + (1 - \theta)S_2$ and consider a field $\psi \in L^2(\mathbb{T}^n, \mathbb{V})$ such that $\mathcal{A}\psi = 0$ and

$$\psi(x) = w_j \text{ when } S(x) = S_j;$$

note that this is possible by a simple lamination since $w_1 - w_2 \in \Lambda$. Then, from (7.3.1),

$$\begin{aligned} \text{env}_{\Lambda}^{\text{dc},i+1} f(v) &\leq \langle S^* v, v \rangle = \inf_{\varphi} \int_{\mathbb{T}^n} \langle S(x)(v + \varphi), v + \varphi \rangle dx \\ &\leq \int_{\mathbb{T}^n} \langle S(x)(v + \psi), v + \psi \rangle dx \\ &= \theta \langle S_1 w_1, w_1 \rangle + (1 - \theta) \langle S_2 w_2, w_2 \rangle \\ &= \text{env}_{\Lambda}^{\text{dc},i+1} f(v) \end{aligned}$$

where the last equality follows from (7.3.5) and (7.3.6). The other direction is similar.

To prove the identity for the quasiconvex envelope, note that the infimum in (7.3.2) is, in this case, a minimum:

$$\text{env}_{\mathcal{A}}^{\text{qc}} f(v) = \min_{\substack{\varphi \in C^\infty(\mathbb{T}^n, \mathbb{V}) \\ \int_{\mathbb{T}^n} \varphi = 0, \mathcal{A}\varphi = 0}} \min_{S \in \mathbb{M}} \int_{\mathbb{T}^n} \langle S(v + \varphi(x)), v + \varphi(x) \rangle dx. \quad (7.3.7)$$

Defining $T \in L^\infty(Q, \mathbb{M})$ by

$$\langle T(x)(v + \varphi(x)), v + \varphi(x) \rangle \equiv \min_{S \in \mathbb{M}} \langle S(v + \varphi(x)), v + \varphi(x) \rangle,$$

we have, from (7.3.1),

$$\text{env}_{\mathcal{A}}^{\text{qc}} f(v) = \min_{\substack{\varphi \in C^\infty(\mathbb{T}^n, \mathbb{V}) \\ \int_{\mathbb{T}^n} \varphi = 0, \mathcal{A}\varphi = 0}} \int_{\mathbb{T}^n} \langle T(x)(v + \varphi(x)), v + \varphi(x) \rangle dx = \langle T^* v, v \rangle \geq \min_{S \in G(\mathbb{M})} \langle Sv, v \rangle.$$

The other direction is similar: taking the minimum out of the integral in (7.3.7) and swapping the order of the minimums, we find

$$\text{env}_{\mathcal{A}}^{\text{qc}} f(v) \leq \min_{S \in \mathbb{M}} \min_{\substack{\varphi \in C^\infty(\mathbb{T}^n, \mathbb{V}) \\ \int_{\mathbb{T}^n} \varphi = 0, \mathcal{A}\varphi = 0}} \int_{\mathbb{T}^n} \langle S(v + \varphi(x)), v + \varphi(x) \rangle dx = \min_{S \in \mathbb{M}} \langle S^* v, v \rangle = \min_{S \in G(\mathbb{M})} \langle Sv, v \rangle$$

as wished. \square

7.4 Multifold theories and the classical Calculus of Variations

In this section we discuss multifold theories, which we have already encountered briefly in Example 7.1.3(iv). Multifold theories are related to the classical Calculus of Variations, since

$$\mathbb{V} = \oplus_{i=1}^m \mathbb{R}^n \cong \mathbb{R}^m \otimes \mathbb{R}^n = \mathbb{E}_\xi \oplus \mathbb{J}_\xi$$

where, for $\xi \in \mathbb{S}^{n-1}$,

$$\mathbb{E}_\xi = \{a \otimes \xi : a \in \mathbb{R}^m\}, \quad \mathbb{J}_\xi = \{(j_1, \dots, j_m) \in \mathbb{V} : j_1, \dots, j_m \perp \xi\}.$$

Let us note that $\text{Sym}(\mathbb{R}^m \otimes \mathbb{R}^n) \cong \text{Sym}(\mathbb{R}^m) \otimes \text{Sym}(\mathbb{R}^n)$. Under this identification we can write $\Gamma(\xi) = \xi \otimes \xi$, where as usual $\Gamma(\xi)$ denotes the orthogonal projection onto \mathbb{E}_ξ .

As discussed in Section 7.2.3, elements $R \in \text{SO}(n)$ act on \mathbb{V} by $R \cdot (a \otimes b) = a \otimes Rb$ and, moreover,

$$R \cdot [I_m \otimes (\xi \otimes \xi)] = I_m \otimes (R\xi \otimes R\xi).$$

Then, in the notation of Lemma 7.2.16, we see that

$$\text{tr}(\bar{\Gamma}) = \text{tr} \left(\int_{\text{SO}(n)} Qe_1 \otimes Qe_1 \, d\mu(Q) \right) = \int_{\text{SO}(n)} |Qe_1|^2 \, d\mu(Q) = 1.$$

This shows that $\tilde{\Gamma} = \Gamma(\xi_0) - \bar{\Gamma}$ is a symmetric, traceless matrix, denoted $\tilde{\Gamma} \in \text{Sym}_0(\mathbb{R}^n)$, and hence

$$\mathbb{A} = I_m \otimes \text{span}\{R\tilde{\Gamma}R^{-1} : R \in \text{SO}(n)\} = I_m \otimes \text{Sym}_0(\mathbb{R}^n).$$

We want to study polycrystalline L-relations. When $n = 3$, the corresponding Jordan \mathbb{A} -algebras have been calculated in [170] and it turns out that all $\text{SO}(3)$ -invariant L-relations are in fact exact relations. When $n = 2$ the situation is much more complicated and this is what we will discuss here.

7.4.1 Exact relations for two-dimensional composites

We identify $\mathbb{R}^2 \cong \mathbb{C}$ so that, in particular, $\mathbb{V} \cong \mathbb{R}^{m \times 2} \cong \mathbb{C}^m$. There is a convenient parametrization of $\text{End}(\mathbb{R}^{m \times 2})$, whereby we write, for $K \in \text{End}(\mathbb{R}^{m \times 2})$,

$$K(X, Y)v = Xv + Y\bar{v}$$

for $X, Y \in \mathbb{C}^{m \times m}$. We remark that in the left-hand side of this identity v is a vector in $\mathbb{R}^{m \times 2}$ while in the right-hand side it is a vector in \mathbb{C}^m . As a side note, this parametrization is similar to the conformal-anticonformal decomposition of $A \in \mathbb{R}^{2 \times 2}$, where we can write $A \cdot (x, y) = A^+z + A^-\bar{z}$, where $z = x + iy$, for some unique $A^+, A^- \in \mathbb{C}$.

A straightforward calculation shows that¹

$$(K(X, Y))^T = K(X^H, Y^T)$$

and hence $K \in \text{Sym}(\mathbb{V})$ if and only if $X = X^H$ and $Y = Y^T$. We also record the identities²

$$\begin{aligned} K(X_1, Y_1)K(X_2, Y_2) &= K(X_1X_2 + Y_1Y_2^H, X_1Y_2 + Y_1X_2^T), \\ R_\theta \cdot K(X, Y) &= K(X, e^{2i\theta}Y), \end{aligned}$$

¹Here we use the notation A^H for the hermitian conjugate of a matrix A in order to avoid confusion with the effective tensor.

²It is interesting to observe that the multiplication rule resembles, at a formal level, the product in the so-called *spin factors*:

$$(x_1, t_1) * (x_2, t_2) = (t_1t_2 + x_1 \cdot x_2, t_1x_2 + t_2x_1).$$

where R_θ denotes the usual counter-clockwise rotation by an angle θ . We thus see that if \mathbb{W} is an $\text{SO}(2)$ -invariant subspace of $\text{Sym}(\mathbb{V})$ then there are subspaces $\mathbb{X} \subseteq \text{H}(\mathbb{C}^m)$ and $\mathbb{Y} \subseteq \text{Sym}(\mathbb{C}^m)$ such that

$$\mathbb{W} = \mathbb{W}_{\mathbb{X}, \mathbb{Y}} \equiv \mathbb{X} \oplus \mathbb{Y};$$

here $\text{H}(\mathbb{C}^m)$ is the space of Hermitian matrices, seen as a real vector space. In this notation,

$$\mathbb{A} = \mathbb{W}_{0, \mathbb{C}I_m} = \{K(0, zI_m) : z \in \mathbb{C}\},$$

and we easily see that $\mathbb{W}_{\mathbb{X}, \mathbb{Y}}$ is a Jordan \mathbb{A} -multialgebra if and only if

$$\begin{cases} Y^2 + XX^T \in \mathbb{Y} \\ YX + XY^H \in \mathbb{X} \end{cases} \quad \text{for all } X \in \mathbb{X}, Y \in \mathbb{Y}. \quad (7.4.1)$$

When (7.4.1) holds, the 3-chain property is equivalent to

$$iX_1X_2^T X_3 + (iX_1X_2^T X_3)^H \in \mathbb{V}; \quad (7.4.2)$$

recall that this condition is necessary for \mathbb{M} to be an exact relation, c.f. Theorem 7.2.11(v).

7.4.2 High dimensions versus low dimensions

In this section we begin by giving an example, when $m = 4$, of a subspace $\mathbb{W}_{\mathbb{X}, \mathbb{Y}}$ which satisfies (7.4.1) but fails (7.4.2), following [168]. In this case, we identify $\mathbb{R}^4 \cong \mathbb{H}$ through $x \mapsto x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, so that $\mathbb{V} = \mathbb{R}^4 \otimes \mathbb{R}^2 \cong \mathbb{H}^2$.

Let us write $\underline{h} \in \mathbb{R}^4$ for the vector corresponding to $h \in \mathbb{H}$. Consider the \mathbb{R} -linear map

$$Q: \mathbb{H} \rightarrow \text{End}_{\mathbb{R}}(\mathbb{R}^4), \quad Q(g)\underline{h} = g\underline{h}.$$

Note that $Q(g)Q(h) = Q(gh)$ and that $Q(g)^T = Q(\bar{g})$. We can embed naturally

$$\text{End}_{\mathbb{R}}(\mathbb{R}^4) \subset \text{End}_{\mathbb{C}}(\mathbb{C}^4) \subset \text{End}_{\mathbb{R}}(\mathbb{R}^4 \otimes \mathbb{R}^2)$$

by regarding 4×4 real matrices as 4×4 complex matrices, and using $\mathbb{R}^4 \otimes \mathbb{R}^2 \cong \mathbb{C}^4$; we also write $Q: \mathbb{H} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}^4)$. We now take $\mathbb{W} \equiv \mathbb{W}_{\mathbb{X}, \mathbb{Y}}$, where

$$\mathbb{X} \equiv \{iQ(h) : h \in \mathbb{H}, \text{Re}(h) = 0\}, \quad \mathbb{Y} \equiv \{aI_4 : a \in \mathbb{C}\}.$$

It is easy to check that (7.4.1) holds: indeed,

$$\begin{aligned} Y^2 + XX^T &= a^2I_4 + iQ(h)iQ(\bar{h}) = a^2I_4 - Q(|h|^2) = (a^2 - |h|^2)I_4 \in \mathbb{Y}, \\ YX + XY^H &= aiQ(h) + \bar{a}iQ(h) = 2i\text{Re}(a)Q(h) \in \mathbb{X}. \end{aligned}$$

However, (7.4.2) fails: take

$$X_1 \equiv iQ(\mathbf{i}), \quad X_2 \equiv iQ(\mathbf{j}), \quad X_3 \equiv iQ(\mathbf{k})$$

and compute

$$iX_1X_2^T X_3 = i^4Q(-\mathbf{ij}\mathbf{k}) = Q(1) \implies iX_1X_2^T X_3 + (iX_1X_2^T X_3)^H = Q(2) \notin \mathbb{X}.$$

Being more explicit, we can write, for $X = iQ(h)$ and $Y = (a_1 + ia_2)I_4$,

$$K(X, Y) = \begin{bmatrix} a_1 & a_2 - h \\ a_2 + h & -a_1 \end{bmatrix}$$

under the identification $\mathbb{V} \cong \mathbb{H}^2$. Recalling that $\Gamma_0(\xi) = \xi \otimes \xi$, $\xi \in \mathbb{R}^2$, one finds after some calculations that the corresponding manifold $\mathbb{M} = W_{\xi_0}^{-1}(\mathbb{W})$ is given by

$$\mathbb{M} = \left\{ \begin{bmatrix} \lambda & h \\ \bar{h} & \mu \end{bmatrix} : \lambda, \mu > 0, h \in \mathbb{H} \text{ and } \lambda\mu - |h|^2 = 1 \right\}. \quad (7.4.3)$$

By the above argument, \mathbb{M} is L-closed but not G-closed. However, it is also non-compact: thus, to obtain a proper counter-example through Theorem 7.3.5, we intersect \mathbb{M} with the sets from Definition 7.1.7:

$$\mathbb{M}_\gamma \equiv \mathbb{M} \cap \mathcal{M}_{\gamma,\gamma}^{\text{sym}}(\mathbb{H}^2).$$

Let us write $A = (a_1, a_2) \in \mathbb{H}^2$ for a matrix with columns $a_1, a_2 \in \mathbb{H}$. Consider, for $\gamma \in [0, 1)$, the functions

$$f_\gamma(a_1, a_2) \equiv \frac{1}{2} \min_{S \in \mathbb{M}_\gamma} \langle (Sa_1, Sa_2), (a_1, a_2) \rangle_{\mathbb{H}^2} = \frac{1}{2} \min_{S \in \mathbb{M}_\gamma} \text{Re Tr}_{\mathbb{H}}(SAA^{\text{H}});$$

here $f_0 \equiv \lim_{\gamma \rightarrow 0} f_\gamma$ admits the particularly simple expression

$$f_0(A) = \sqrt{\det_{\mathbb{H}}(AA^{\text{H}})},$$

see [168, Lemma 4.6]. Using Theorem 7.3.5 it can be shown that, for $\gamma \in [0, 1)$, the functions f_γ are rank-one convex but non-quasiconvex. We refer the reader to [168] for further details.

We conclude this section by discussing the results of [169] for the low-dimensional case $m = 2$; physically, we may think of this setting as modeling a thermoelectric material. Following the broad strategy discussed in Section 7.2.4, one begins by finding all solutions of (7.4.1); this is a rather intricate exercise in the representation theory of $\text{SO}(2)$. The first step is to identify the symmetries of (7.4.1): the $\text{SO}(2)$ -invariant \mathbb{A} -multialgebra automorphisms $\Phi: \text{Sym}(\mathbb{V}) \rightarrow \text{Sym}(\mathbb{V})$ are of the form

$$\Phi(K(X, Y)) = K(\pm CXC^{\text{H}}, CYC^{\text{T}}) \quad \text{for some } C \in \text{O}(2, \mathbb{C});$$

c.f. Theorem 7.2.17. Once this is done, one can identify all subspaces $\mathbb{Y} \subseteq \text{Sym}(\mathbb{C}^2)$ such that $Y^2 \in \mathbb{W}$ for all $Y \in \mathbb{W}$ and then, for each such \mathbb{W} , look for subspaces $\mathbb{X} \subseteq \text{H}(\mathbb{C}^2)$ such that (7.4.1) holds. These calculations are detailed in [169] and here we summarise them as follows:

Theorem 7.4.1. *All non-trivial solutions (\mathbb{X}, \mathbb{Y}) of (7.4.1) can be obtained, through intersections, from a list of 6 solutions of (7.4.1). Moreover, all solutions satisfy the 3- and 4-chain properties.*

Here we consider only *non-trivial* solutions, i.e. there is coupling between the two fields. In particular, through Theorem 7.2.11(iv), we see from Theorem 7.4.1 that a counter-example as in the higher-dimensional case does not exist.

Chapter 8

Homogenization for Einstein's vacuum equations under symmetry

The purpose of this chapter is to prove a homogenization result for the Einstein vacuum equations: we will prove, in particular, Theorem 3.F. Under elliptic gauge assumptions, Theorem 3.F implies Burnett's conjecture in $U(1)$ -symmetry, Theorem 3.E, in a straightforward way. The results of this chapter are joint work with RITA TEIXEIRA DA COSTA [177].

After giving precise statements of our main hypotheses and results in Section 8.1, in Section 8.2 we briefly summarize the theory of H-measures, which will be used throughout this chapter. In Section 8.3 we completely characterize the H-measure associated to solutions of the scalar covariant wave equation. In Section 8.4 we deal with the case where the background metric also oscillates and we show that, under appropriate conditions, these oscillations do not contribute to the limit. Finally, in Section 8.5 we use classical Compensated Compactness results to prove the non-linear statement of Theorem 3.F.

8.1 Hypotheses and statements of the main results

We consider a wave map from a Lorentzian manifold $(\mathcal{M}, g_\varepsilon)$ to a Riemannian manifold $(\mathcal{N}, \mathbf{g})$:

$$\square_{g_\varepsilon} u_\varepsilon^I + \Gamma_{JK}^I(u_\varepsilon) g^{-1}(du_\varepsilon^J, du_\varepsilon^K) = f_\varepsilon^I, \quad u_\varepsilon^I, f_\varepsilon^I: \mathcal{M} \rightarrow \mathbb{R}, \quad I, J, K \in \{1, \dots, N\}. \quad (8.1.1)$$

Here, $\Gamma_{JK}^I: \mathbb{R} \rightarrow \mathbb{R}$ are the Christoffel symbols of the Riemannian metric \mathbf{g} and depend continuously on u^I . For simplicity, we take $\mathcal{M} \subset \mathbb{R}^{1+n}$ and $\mathcal{N} \subset \mathbb{R}^N$ to be domains; however, in light of the assumptions ensuing, this restriction is without loss of generality. Indeed, we will assume:

Hypotheses 8.1.1. Let $\mathcal{U}_\varepsilon \equiv (g_\varepsilon, (u_\varepsilon^I)_{I=1}^N, (f_\varepsilon^I)_{I=1}^N)$ and $\mathcal{U} \equiv (g, (u^I)_{I=1}^N, (f^I)_{I=1}^N)$ satisfy:

- (a) the eigenvalues of g_ε are uniformly bounded from above and away from zero and g is a smooth metric such that $g_\varepsilon \rightarrow g$ entrywise in C_{loc}^0 ; furthermore, one of the following two conditions holds:
 - (a₁) $g_\varepsilon \rightarrow g$ in $W_{\text{loc}}^{1,\infty}$, or
 - (a₂) g_ε is bounded in $W_{\text{loc}}^{1,\infty}$, with $\partial_0(g_\varepsilon)_{ij} \rightarrow \partial_0(g_\varepsilon)_{ij}$ strongly in L_{loc}^4 , $\delta^{ij} \partial_{ij} g_\varepsilon^{\alpha\beta}$ bounded in L_{loc}^2 ;
- (b) $u_\varepsilon^I \rightharpoonup u^I$ in $W_{\text{loc}}^{1,p}$ and $\|g_\varepsilon^{\alpha\beta} - g^{\alpha\beta}\|_{L^\infty(K)} \|\partial^2(u_\varepsilon^I - u^I)\|_{L^p(K)} \lesssim_K 1$ for every compact $K \subset \mathcal{M}$;
- (c) $f_\varepsilon^I \rightharpoonup f^I$ in L_{loc}^p .

Any sequences satisfying Hypotheses 8.1.1(b) and (c) induce an H-measure through

$$\left((\partial_0 u_\varepsilon^I, \partial_1 u_\varepsilon^I, \dots, \partial_n u_\varepsilon^I, f_\varepsilon^I)_{I=1}^N \right)_\varepsilon \xrightarrow{\text{H}} \left(\left[\begin{array}{cc} \tilde{\nu}^{IJ} & \tilde{\lambda}^{IJ} \\ (\tilde{\lambda}^{IJ})^* & \mu^{IJ} \end{array} \right]_{I,J=1}^N \right), \quad (8.1.2)$$

and since $\mathbb{L}_{\alpha\beta}$ is quadratic, c.f. (3.3.8), H-measures provide a way of calculating the discrepancy between $\mathbb{L}_{\alpha\beta}[u]$ and $\lim_{\varepsilon \rightarrow 0} \mathbb{L}_{\alpha\beta}[u_\varepsilon]$. We note that, in (8.1.2), the H-measure is valued in $N \times N$ block-matrices, where the (I, J) -block is an $(n+2) \times (n+2)$ matrix. See also Section 8.2.

The core of this chapter concerns the case of flat targets, where (8.1.1) reduces to a single linear wave equation. In this case, for simplicity we drop the superscripts. Our main result is:

Theorem 8.1.2. *Let $\mathcal{U}_\varepsilon = (g_\varepsilon, u_\varepsilon, f_\varepsilon)$ be a sequence satisfying Hypotheses 8.1.1 with $p = 2$ if Hypothesis 8.1.1(a₁) holds or $p = 4$ otherwise. Suppose $\mathcal{U}_\varepsilon = (g_\varepsilon, u_\varepsilon, f_\varepsilon)$ satisfies*

$$\square_{g_\varepsilon} u_\varepsilon = f_\varepsilon. \quad (8.1.3)$$

- (a) **Limit equation.** *The triple $\mathcal{U} = (g, u, f)$ is a distributional solution of (8.1.3).*
- (b) **Energy density.** *There are Radon measures ν, λ such that $\tilde{\nu}_{\alpha\beta} = \xi_\alpha \xi_\beta \nu, \tilde{\lambda}_\gamma = \xi_\gamma \lambda$. Moreover, ν is a (radially averaged) measure-valued solution of an inhomogeneous massless Vlasov equation:*
- (b₁) Support property: *ν is supported on the zero mass shell of g , i.e. for all $\varphi \in C_0^\infty(\mathcal{M})$*

$$\int_{S^*\mathcal{M}} \varphi(x) g^{\alpha\beta} \xi_\alpha \xi_\beta d\nu = 0.$$

- (b₂) Propagation property: *for all $\tilde{a} \in C_0^\infty(S^*\mathcal{M})$, extended as a positively 1-homogeneous function to $T^*\mathcal{M} \setminus \{0\}$, the measure ν satisfies*

$$\int_{S^*\mathcal{M}} \left[g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\nu} g^{\alpha\beta} \xi_\alpha \xi_\beta \partial_{\xi_\mu} \tilde{a} \right] d\nu = - \int_{S^*\mathcal{M}} \tilde{a} d(\operatorname{Re} \lambda). \quad (8.1.4)$$

Using classical compensated compactness results, we deduce Theorem 3.F, which we restate here for the reader's convenience:

Theorem 8.1.3. *Let Γ_{JK}^I be continuous Christoffel symbols arising from a Riemannian metric*

$$\mathbf{g} = \mathbf{g}_{IJ}(y) dy^I \otimes dy^J.$$

Let \mathcal{U}_ε be a sequence of solutions to (8.1.1) satisfying Hypotheses 8.1.1 for $p = 4$, such that $u_\varepsilon^I \rightarrow u^I$ in C_{loc}^0 . There exists a Radon measure ν on $S^(\mathcal{M})$ such that:*

- (a) **Limit equation.** *\mathcal{U} is a distributional solution of (8.1.1) and its Lagrangian energy density satisfies*

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} \mathbb{L}_{\alpha\beta}[u_\varepsilon] Y^\alpha Y^\beta d\operatorname{Vol}_{g_\varepsilon} = \int_{\mathcal{M}} \mathbb{L}_{\alpha\beta}[u] Y^\alpha Y^\beta d\operatorname{Vol}_g + \int_{S^*\mathcal{M}} \xi_\alpha \xi_\beta Y^\alpha Y^\beta \sqrt{|g|} d\nu, \quad \forall Y \in C_0^\infty(\mathcal{M}).$$

- (b) **Energy density.** *The measure ν is a (radially averaged) measure-valued solution of an inhomogeneous massless Vlasov equation with respect to g , in the sense that properties (b₁) and (b₂) of Theorem 8.1.2 hold for a measure λ which is defined via $\mathbf{g}_{IJ} \tilde{\lambda}_{\gamma 0}^{IJ} = \xi_\gamma \lambda$.*

8.2 Preliminaries on H-measures and compensated compactness

8.2.1 Symbols and pseudo-differential operators

In this section we gather some basic results about pseudo-differential operators. They can be found, for instance, in the books [196] and [175]. We take $\Omega \subset \mathbb{R}^N$ to be a fixed open set throughout.

Definition 8.2.1. For $m \in \mathbb{R}$, a function a is called a *symbol of order m* , $a \in S^m \equiv S^m(\Omega, \mathbb{C}^{d \times d})$, if $a \in C^\infty(\Omega \times \mathbb{R}^N, \mathbb{C}^{d \times d})$ and, for each compact set $K \subset \Omega$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim_{\alpha, \beta, K} (1 + |\xi|)^{m - |\beta|}.$$

We write $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$.

The following basic lemma gives meaning to asymptotic expansions of symbols:

Lemma 8.2.2. *For $j \in \mathbb{N}_0$ let $a_j \in S^{m_j}$ and $m_j \searrow -\infty$. There is $a \in S^{m_0}$ such that, for every k , $a - \sum_{j < k} a_j \in S^{m_k}$. The symbol a is unique modulo $S^{-\infty}$ and we write $a \sim \sum_{j=0}^{\infty} a_j$ in S^m .*

Each symbol $a \in S^m$ induces an operator A acting on $v \in C_c^\infty(\mathbb{R}^N, \mathbb{C}^d)$ by

$$Av(x) \equiv \int_{\mathbb{R}^N} a(x, \xi) e^{2\pi i x \cdot \xi} \widehat{v}(\xi) d\xi.$$

We say that A is a *pseudo-differential operator* of order m . We write $\sigma(A) \equiv a$ and note that, for any pseudo-differential operator, the symbol $\sigma(A)$ is uniquely determined modulo $S^{-\infty}$.

Lemma 8.2.3. *If $a \in S^m$ then A extends a continuous operator $A: H^s(\mathbb{R}^N, \mathbb{C}^d) \rightarrow H_{\text{loc}}^{s-m}(\Omega, \mathbb{C}^d)$. In particular, if $m < 0$ then $A: L^2(\mathbb{R}^n, \mathbb{C}^d) \rightarrow L_{\text{loc}}^2(\Omega, \mathbb{C}^d)$ is compact.*

We will work with a more restricted class of pseudo-differential operators, the so-called polyhomogeneous operators. To motivate the next definition, observe that if $a \in C^\infty(\Omega \times \mathbb{R}^N)$ satisfies

$$a(x, t\xi) = t^m a(x, \xi) \quad \text{for all } t, |\xi| \geq 1,$$

then $a \in S^m$. Such functions are said to be *positively m -homogeneous in ξ* for $|\xi| \geq 1$.

Definition 8.2.4. A symbol $a \in S^m$ is called *polyhomogeneous* if

$$a \sim \sum_{j=0}^{\infty} a_{m-j} \quad \text{in } S^m$$

where $a_{m-j} \in C^\infty(\Omega \times \mathbb{R}^N)$ is positively $(m-j)$ -homogeneous in ξ for $|\xi| \geq 1$. The term a_m is called the *principal symbol* and is denoted by $\sigma^m(A)$.

The space of pseudo-differential operators with polyhomogeneous symbols in $S^m(\Omega, \mathbb{C}^{d \times d})$ is denoted by $\Psi_d^m(\Omega)$; if their symbols are compactly supported in x , we write $\Psi_{d,c}^m(\Omega)$.

Lemma 8.2.5. *Take $P \in \Psi_d^l(\Omega)$ and $Q \in \Psi_d^m(\Omega)$. Writing $D \equiv \frac{1}{i} \partial$, we have the formulae*

$$\sigma(P^*) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_x^\alpha D_\xi^\alpha \sigma(P)^* \quad \text{in } S^m, \quad \sigma(PQ) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} D_\xi^\alpha \sigma(P) \partial_x^\alpha \sigma(Q) \quad \text{in } S^m.$$

Thus, if $[\sigma(P), \sigma(Q)] = 0$, then $[P, Q] \in \Psi_d^{l+m-1}(\Omega)$ with $\sigma^{l+m-1}([P, Q]) = \frac{1}{i} \{\sigma^l(P), \sigma^m(Q)\}$.

Here, and in the sequel, $[p, q] \equiv pq - qp$ and $\{p, q\}$ denotes the *Poisson bracket*, that is,

$$\{p, q\} \equiv \frac{\partial p}{\partial \xi_j} \frac{\partial q}{\partial x^j} - \frac{\partial p}{\partial x^j} \frac{\partial q}{\partial \xi_j}.$$

Theorem 8.2.6 (Calderón Commutator). *Let $P \in \Psi_1^1(\mathbb{R}^N)$ and let $a(x)$ be a Lipschitz function. Then, for any $1 < p < \infty$, $[P, a]: L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is bounded and*

$$\|[P, a]f\|_{L^p} \leq C_p \|\nabla a\|_{L^\infty} \|f\|_{L^p}.$$

Conversely, if $[P, a]: L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is bounded for $P = \partial_{x_j}$, $j = 1, \dots, N$, then a is Lipschitz.

We refer the reader to [280] for a proof of Theorem 8.2.6.

8.2.2 Existence and properties of H-measures

In this subsection we recall the definition of H-measures, as well as a few useful properties they possess. H-measures were introduced independently by Tartar [364, 367] and Gérard [159], who called them *microlocal defect measures*. Here we adopt Tartar's terminology and refer the reader to [367] for further details.

Theorem 8.2.7 (Existence of H-measures). *Let $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, \mathbb{C}^d)$. Up to a subsequence, there are Radon measures $\mu_{\alpha\beta}$, $\alpha, \beta = 1, \dots, d$, such that*

$$\mu_{\alpha\beta} = \overline{\mu_{\beta\alpha}}, \quad \mu_{\alpha\beta} \xi^\alpha \bar{\xi}^\beta \geq 0 \text{ for all } \xi \in \mathbb{C}^d$$

and, for any $A \in \Psi_{d,c}^0(\Omega)$, we have

$$\lim_{\varepsilon \rightarrow 0} \langle A(v^\varepsilon - v), v^\varepsilon - v \rangle \equiv \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A(v_\varepsilon - v) \cdot \overline{v_\varepsilon - v} \, dx = \int_{S^*\Omega} \sigma^0(A)^{\alpha\beta} \, d\mu_{\alpha\beta} \equiv \langle \mu, \sigma^0(A) \rangle. \quad (8.2.1)$$

The matrix-valued measure $\mu = (\mu_{\alpha\beta})_{\alpha,\beta}$ is called the H-measure associated with (v_ε) .

In Theorem 8.2.7, as usual, $S^*\Omega \equiv \Omega \times S^{N-1}$ denotes the cosphere bundle over Ω . Here, and in the rest of the chapter, we will always write $\langle f, g \rangle \equiv \int_{\Omega} f \bar{g} \, dx$ whenever this integral is meaningful.

Remark 8.2.8. The Stone–Weierstrass Theorem and a standard density argument show that it suffices to test (8.2.1) with symbols of the form $\sigma^0(A)(x, \xi) = b(x)m(\xi)$, see also [153, Remark 2.7].

The following lemma, although simple, describes a very important property of H-measures.

Lemma 8.2.9 (Localization property). *Let (v_ε) be a sequence such that $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, \mathbb{C}^d)$ and let μ be its H-measure. Given $P \in \Psi_d^m(\Omega)$, we have*

$$(Pv_\varepsilon) \text{ is compact in } H_{\text{loc}}^{-m} \iff \sigma^m(P)\mu = 0.$$

To conclude this subsection we define a way of generating, in a *non-canonical* fashion, an H-measure for a sequence that converges only locally in L^2 :

Definition 8.2.10. By passing to a subsequence, $v_\varepsilon \rightharpoonup v$ in $L_{\text{loc}}^2(\Omega, \mathbb{C}^d)$ generates an H-measure μ ,

$$v_\varepsilon \xrightarrow{\text{H}} \mu,$$

as follows. Let $(K_i)_{i=1}^\infty$ be a compact exhaustion of Ω and let $\chi_i \in C_c^\infty(K_{i+1}, [0, 1])$ be such that $\chi_i = 1$ on K_i . Consider a sequence of Radon measures (μ_i) constructed as follows: μ_1 is the H-measure generated by a subsequence $(\chi_1 v_{\varepsilon'})_{\varepsilon'}$ of $(\chi_1 v_\varepsilon)_\varepsilon$, μ_2 is the H-measure generated by a subsequence of $(\chi_2 v_{\varepsilon'})_{\varepsilon'}$, and so on. We define μ through its action on $\varphi \in C_c(S^*\Omega)$: let i be such that $\text{supp } \varphi \subset S^*K_i$ and set $\langle \mu, \varphi \rangle \equiv \langle \mu_i, \varphi \rangle$. It is easy to see that μ is well-defined.

8.2.3 Compensated Compactness

The next theorem, which is due to ROBBIN–ROGERS–TEMPLE [330] and generalizes an earlier result of Murat and Tartar [300], is the main compensated compactness result that we will use in this chapter:

Theorem 8.2.11 (Generalized div-curl lemma). *Let $p_1, p_2 \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. For differential forms $\omega_{i,\varepsilon}$ over Ω of degree k_i , $i = 1, 2$, such that $k_1 + k_2 \leq N$,*

$$\left. \begin{array}{l} \omega_{i,\varepsilon} \rightharpoonup \omega_i \text{ in } L_{\text{loc}}^{p_i}(\Omega) \\ d\omega_{i,\varepsilon} \text{ is compact in } W_{\text{loc}}^{-1,p_i}(\Omega) \end{array} \right\} \implies \omega_{1,\varepsilon} \wedge \omega_{2,\varepsilon} \xrightarrow{*} \omega_1 \wedge \omega_2 \text{ in } \mathcal{D}'(\Omega).$$

8.3 The linear covariant wave equation

This section is concerned with a linear covariant wave equation

$$\square_g u = f, \quad u, f: \mathcal{M} \rightarrow \mathbb{R}, \quad (8.3.1)$$

where g is a smooth Lorentzian metric on an open domain $\mathcal{M} \subset \mathbb{R}^{1+n}$. Recall that

$$\square_g u \equiv \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} g^{\alpha\beta} \partial_\beta u \right) = \nabla^\alpha \nabla_\alpha u, \quad (8.3.2)$$

where $g^{\alpha\beta} \equiv (g^{-1})^{\alpha\beta}$, $|g| \equiv |\det g|$ and ∇^α is the covariant derivative with respect to g . We will also write $d\text{Vol}_g \equiv \sqrt{|g|} dx$ for the volume form induced by g .

It will be convenient to work with a diagonalized form of the wave operator. To this end, define

$$\beta^i \equiv -\frac{g^{0i}}{g^{00}}, \quad e_0 \equiv \partial_0 - \beta^i \partial_i, \quad \tilde{g}^{ij} \equiv g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}, \quad (8.3.3)$$

The symbol of the timelike vector field e_0 appears naturally in relation to the zero mass shell of g : indeed,

$$g^{\alpha\beta} \xi_\alpha \xi_\beta = g^{00} (\xi_0 - \beta^k \xi_k)^2 + \tilde{g}^{ij} \xi_i \xi_j. \quad (8.3.4)$$

In order to use Stokes' theorem, we define some useful geometric quantities associated with the covariant wave operator. Given functions $u_1, u_2: \mathcal{M} \rightarrow \mathbb{R}$ and a smooth vector field X on \mathcal{M} , let us write

$$\begin{aligned} T_{\alpha\beta}[u^1, u^2] &\equiv \partial_\alpha u^1 \partial_\beta u^2 - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu u^1 \partial_\nu u^2, \\ J_\alpha^X[u^1, u^2] &\equiv \frac{1}{2} \left[X u^1 \partial_\alpha u^2 + X u^2 \partial_\alpha u^1 - X_\alpha g^{-1}(du^1, du^2) \right]. \end{aligned} \quad (8.3.5)$$

The energy-momentum tensor T and the associated current J^X are related by the *energy identity*:

$$\nabla^\alpha J_\alpha^X[u^1, u^2] = \frac{1}{2} (X u^1 \square_g u^2 + X u^2 \square_g u^1) + T_{\alpha\beta}[u^1, u^2] \nabla^\alpha X^\beta. \quad (8.3.6)$$

When $u^1 = u^2 = u$ we recover the standard energy identity, see e.g. [6, 104] for further details.

In this section we study the limiting behavior of sequences of solutions to (8.3.1). For the convenience of the reader, we state here a simplified form of Hypotheses 8.1.1:

Hypotheses 8.3.1. Let $u_\varepsilon, f_\varepsilon: \mathcal{M} \rightarrow \mathbb{R}$ be sequences such that $(u_\varepsilon, f_\varepsilon)$ satisfy, for each $\varepsilon > 0$, the linear wave equation (8.3.1). We consider the following regularity conditions:

- (a) g is smooth;
- (b) $u_\varepsilon \rightharpoonup u$ in $W_{\text{loc}}^{1,2}(\mathcal{M})$;
- (c) $f_\varepsilon \rightharpoonup f$ in $L_{\text{loc}}^2(\mathcal{M})$.

According to Definition 8.2.10 and Hypotheses 8.3.1, we may pass to a subsequence so that

$$(\partial_0 u_\varepsilon, \partial_1 u_\varepsilon, \dots, \partial_n u_\varepsilon, f_\varepsilon)_\varepsilon \xrightarrow{\mathbb{H}} \begin{bmatrix} \tilde{\nu} & \tilde{\lambda} \\ \tilde{\lambda}^* & \mu \end{bmatrix}, \quad (8.3.7)$$

where $\tilde{\nu}$ is a $\mathbb{C}^{(n+1) \times (n+1)}$ -valued measure, generated by $(\partial_0 u_\varepsilon, \dots, \partial_n u_\varepsilon)$, and $\tilde{\lambda}$ is \mathbb{C}^{n+1} -valued.

8.3.1 The H-measure and its properties

We are now ready to state the main result of this section, which describes the structure, support and propagation properties of the H-measure defined in (8.3.7).

Theorem 8.3.2. *Let $(u_\varepsilon, f_\varepsilon)$ satisfy Hypotheses 8.3.1 and define $\tilde{\nu}$ and $\tilde{\lambda}$ as in (8.3.7). Then:*

- (a) **Limit equation.** (u, f) satisfy (8.3.1) in the dense of distributions.
- (b) **Energy density.** *There are Radon measures ν and λ on $S^*\mathcal{M}$ such that $\tilde{\nu}_{\alpha\beta} = \xi_\alpha \xi_\beta \nu$ and $\tilde{\lambda}_\gamma = \xi_\gamma \lambda$. Furthermore, ν and λ satisfy the following conditions:*
 - (b₀) *Parity: ν is even and λ is odd, i.e. $\langle \nu, \tilde{a} \rangle = 0$ for any $\tilde{a} \in C_c^\infty(S^*\mathcal{M})$ which is odd in ξ , and likewise for λ .*
 - (b₁) *Support property: for all $\varphi \in C_c^\infty(\mathcal{M})$, ν and λ satisfy*

$$\langle \nu, \varphi(x) g^{\alpha\beta}(x) \xi_\alpha \xi_\beta \rangle = 0, \quad \langle \lambda, \varphi(x) g^{\alpha\beta}(x) \xi_\alpha \xi_\beta \rangle = 0.$$
 - (b₂) *Propagation property: for all $\tilde{a}(x, \xi) \in C_c^\infty(S^*\mathcal{M})$, though of as positively 1-homogeneous functions in ξ , the measure ν satisfies*

$$\langle \nu, \{g^{\alpha\beta}(x) \xi_\alpha \xi_\beta, \tilde{a}\} \rangle = -\langle \text{Re } \lambda, \tilde{a} \rangle.$$

Theorem 8.3.2 follows by standard methods, and similar statements have appeared in [364, Theorem 3.12] and [12, 153]. The main novelty here is that our proof holds for a general *covariant wave operator* where, unlike in these references, the coefficients of the operator are allowed to depend both on $x^0 = t$ and (x^1, \dots, x^n) .

Before proceeding with the core of the proof, we show that we may assume that the convergence in Hypotheses 8.3.1 is *global* and not just local:

Reduction to compact supports: Let $\chi \in C_c^\infty(\mathcal{M})$ satisfy $\chi = 1$ on a compact set K . Then

$$\square_g(\tilde{u}_\varepsilon) \equiv \square_g(\chi u_\varepsilon) = f_\varepsilon \chi + 2g^{-1}(du_\varepsilon, d\chi) + u_\varepsilon \square_g \chi \equiv \tilde{f}_\varepsilon.$$

Suppose that, for every such χ , the conclusion of Theorem 8.3.2 holds, with $\tilde{\nu}$ and $\tilde{\lambda}$ being now the H-measures generated according to (8.3.7), but with u_ε replaced with \tilde{u}_ε and f_ε replaced with \tilde{f}_ε . Since $(u_\varepsilon, f_\varepsilon) = (\tilde{u}_\varepsilon, \tilde{f}_\varepsilon)$ on K , it is then clear, recalling Definition 8.2.10, that the original H-measure generated by $(u_\varepsilon, f_\varepsilon)$ also satisfies the conclusion of Theorem 8.3.2. \square

Thus, from now onwards, we assume that the sequence $(u_\varepsilon, f_\varepsilon)_\varepsilon$ has uniformly bounded support.

Proof of Theorem 8.3.2(a,b₀,b₁): Part (a) follows from the divergence structure of \square_g , see Proposition 8.4.1 for a more general statement.

Noting that $D_\alpha \partial_\beta u_\varepsilon = D_\beta \partial_\alpha u_\varepsilon$, Lemma 8.2.9 yields $\xi_\alpha \tilde{\nu}_{\beta\gamma} = \xi_\beta \tilde{\nu}_{\alpha\gamma}$. It follows that $\tilde{\nu}_{\alpha\beta} = \xi_\alpha \rho_\beta$ for some \mathbb{C}^d -valued Radon measure ρ . Since μ is Hermitian and non-negative, we must have $\rho = \xi \nu$ for another non-negative Radon measure ν . Likewise, $\tilde{\lambda}_\gamma = \xi_\gamma \lambda$ for some Radon measure λ .

The support property of ν in (b₁) follows by applying again Lemma 8.2.9: since $\square_g u_\varepsilon = f_\varepsilon$, by Hypotheses 8.3.1(c) we see that the sequence of vector fields $(\sqrt{|g|} g^{\alpha\beta} v_{\beta,\varepsilon})_\alpha$ has a divergence which is compact in H_{loc}^{-1} and so $g^{\alpha\beta} \xi_\alpha \xi_\beta \nu = 0$. In turn, the support of λ is contained in the support of ν . Indeed, from (8.3.7) and the basic properties of H-measures, for any measurable set $E \subset S^*\mathcal{M}$,

$$M \equiv \begin{bmatrix} \tilde{\nu}(E) & \tilde{\lambda}(E) \\ \tilde{\lambda}^*(E) & \mu(E) \end{bmatrix}$$

is a positive semi-definite matrix and $\mu(E) \geq 0$, hence $\mu(E) \geq 0 \implies \tilde{\lambda}(E) = 0$.

To prove part (b₀) we consider a real symbol $a(x, \xi) = b(x)m(\xi)$; the general case follows according to Remark 8.2.8. Suppose that m is odd: then, using Plancherel's identity,

$$\begin{aligned} \langle Ae_0u_\varepsilon, e_0u_\varepsilon \rangle &= \iint \hat{b}(\xi - \eta)m(\eta)\widehat{e_0u_\varepsilon}(\eta)\overline{\widehat{e_0u_\varepsilon}(\xi)} \, d\xi \, d\eta \\ &= - \iint \hat{b}(\eta - \xi)m(\eta)\overline{\widehat{e_0u_\varepsilon}(\eta)}\widehat{e_0u_\varepsilon}(\xi) \, d\xi \, d\eta = -\langle Ae_0u_\varepsilon, e_0u_\varepsilon \rangle, \end{aligned}$$

where in the last line we made the change of variables $(\xi, \eta) \mapsto -(\xi, \eta)$, used the fact that m is odd and that all functions are real. Hence

$$\langle \nu, (\xi_0 - \beta^i \xi_i)^2 a \rangle = \lim_{\varepsilon \rightarrow 0} \langle Ae_0u_\varepsilon, e_0u_\varepsilon \rangle = 0.$$

Note that, by (8.3.4), $\xi_0 - \beta^i \xi_i$ never vanishes on the zero mass shell $\{g^{\alpha\beta}\xi_\alpha\xi_\beta = 0\}$ where, according to (b₁), ν is supported. Hence we have shown that $\langle \nu, a \rangle = 0$ whenever a is odd in ξ . An identical argument for λ , which is also supported in the zero mass shell, concludes the proof. \square

The proof of part (b₂) is more involved but follows essentially the outline of [364, Theorem 3.12]. The crucial technical ingredient is contained in the following lemma:

Lemma 8.3.3. *Let g be a smooth Lorentzian metric and take $A \in \Psi_{1,c}^0$. Then $[\square_g, A] \in \Psi_1^1$ and*

$$\sigma^1([\square_g, A]) = \sigma^1(iP^\alpha D_\alpha) = \sigma^1(P^\alpha \partial_\alpha),$$

where $P^\alpha \in \Psi_1^0$ is such that

$$\sigma^0(P^\alpha) \equiv 2g^{\alpha\beta} \partial_{x^\beta} a - \partial_{x^\mu} g^{\alpha\beta} \xi_\beta \partial_{\xi^\mu} a.$$

Since g is assumed to be smooth, Lemma 8.3.3 follows at once from the last part of Lemma 8.2.5. Nonetheless, the result still holds if $g \in C^1$, although this is much more difficult:

Remark 8.3.4. The Calderón Commutator (Theorem 8.2.6) shows that $[\square_g, A]: H^1 \rightarrow L^2$ is bounded, even when g is just C^1 , but this assumption cannot be substantially weakened, c.f. [367, pages 336-337] and [88, 368].

Proof of Theorem 8.3.2(b₂): Let us take $A \in \Psi_1^0$ to be a multiplier, so $a(x, \xi) \equiv m(\xi)$. We begin by applying A and \overline{A} to (8.3.1) to get, respectively,

$$\square_g(Au_\varepsilon) = Af_\varepsilon + [\square_g, A]u_\varepsilon, \quad \square_g(\overline{Au_\varepsilon}) = \overline{Af_\varepsilon} + \overline{[\square_g, A]u_\varepsilon}. \quad (8.3.8)$$

Given a smooth vector field X , we multiply the first equation by $X(\overline{Au_\varepsilon})$, the second equation by $X(Au_\varepsilon)$, and sum the two. Using the energy identity (8.3.6) we get

$$\begin{aligned} &\nabla^\alpha J_\alpha^X[Au_\varepsilon, \overline{Au_\varepsilon}] - T_{\alpha\beta}[Au_\varepsilon, \overline{Au_\varepsilon}]\nabla^\alpha X^\beta \\ &= \frac{1}{2} \left[X(\overline{Au_\varepsilon})Af_\varepsilon + X(Au_\varepsilon)\overline{Af_\varepsilon} \right] + \frac{1}{2} \left[X(\overline{Au_\varepsilon})[\square_g, A]u_\varepsilon + X(Au_\varepsilon)\overline{[\square_g, A]u_\varepsilon} \right]. \end{aligned} \quad (8.3.9)$$

Now let $\varphi \in C_c^\infty(\mathbb{R}^{1+n})$ and integrate (8.3.9) against φ with respect to $d\text{Vol}_g$. We deal with each of the corresponding terms separately.

Step 1: the left hand side of (8.3.9). For the first term, we integrate by parts and recall (8.3.5):

$$\begin{aligned} &\int \nabla^\alpha J_\alpha^X[Au_\varepsilon, \overline{Au_\varepsilon}]\varphi \, d\text{Vol}_g \\ &= -\frac{1}{2} \int \left(X(Au_\varepsilon)\partial_\alpha(\overline{Au_\varepsilon}) + X(\overline{Au_\varepsilon})\partial_\alpha(Au_\varepsilon) - X_\alpha g^{\beta\gamma} \partial_\beta(Au_\varepsilon)\partial_\gamma(\overline{Au_\varepsilon}) \right) \nabla^\alpha \varphi \, d\text{Vol}_g. \end{aligned} \quad (8.3.10)$$

Using the fact that A is a multiplier and that $\sigma^0(A^*) = \sigma^0(A)^*$, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int X(Au_\varepsilon) \partial_\alpha (\overline{Au_\varepsilon}) \nabla^\alpha \varphi \, d\text{Vol}_g &= \lim_{\varepsilon \rightarrow 0} \int A^* \left(A(\partial_\beta u_\varepsilon) X^\beta \nabla^\alpha \varphi \sqrt{|g|} \right) \partial_\alpha u_\varepsilon \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \int A^* A(\partial_\beta u_\varepsilon) \partial_\alpha u_\varepsilon X^\beta \nabla^\alpha \varphi \sqrt{|g|} \, dx \\ &= \langle \tilde{\nu}_{\alpha\beta}, |m(\xi)|^2 X^\beta \nabla^\alpha \varphi \sqrt{|g|} \rangle = \langle \nu, g^{\alpha\gamma} \xi_\alpha \xi_\beta |m(\xi)|^2 X^\beta \nabla_\gamma \varphi \sqrt{|g|} \rangle, \end{aligned}$$

where we also used the fact that $[A^*, X^\beta \nabla^\alpha \varphi \sqrt{|g|}] : L^2 \rightarrow L^2$ is compact, c.f. Lemma 8.2.3. The second term on the right-hand side of (8.3.10) is treated identically and has the same limit. Finally, the last term on the right-hand side of (8.3.10) vanishes in the limit: indeed, arguing as before,

$$\lim_{\varepsilon \rightarrow 0} \int X_\alpha g^{\beta\gamma} \partial_\beta (Au_\varepsilon) \partial_\gamma (\overline{Au_\varepsilon}) \nabla^\alpha \varphi \, d\text{Vol}_g = \langle \nu, X_\alpha g^{\beta\gamma} \xi_\beta \xi_\gamma |m(\xi)|^2 \nabla^\alpha \varphi \sqrt{|g|} \rangle = 0,$$

using the support condition on ν .

For the second term in (8.3.9), similar arguments yield

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int T_{\alpha\beta} [Au_\varepsilon, \overline{Au_\varepsilon}] \nabla^\alpha X^\beta \varphi \, d\text{Vol}_g &= \langle \nu, (\xi_\alpha \xi_\beta \nabla^\alpha X^\beta - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \xi_\mu \xi_\nu) |m(\xi)|^2 \varphi \sqrt{|g|} \rangle \\ &= \langle \nu, g^{\alpha\gamma} \xi_\alpha \xi_\beta \nabla_\gamma X^\beta |m(\xi)|^2 \varphi \sqrt{|g|} \rangle. \end{aligned}$$

Setting $\Phi^\beta(x, \xi) \equiv X^\beta |m(\xi)|^2 \varphi(x) \sqrt{|g|}$, $\Phi \equiv \xi_\beta \Phi^\beta$, and using the fact that $\nabla_\gamma g = 0$, we have calculated the limit of the left-hand side of (8.3.9):

$$\lim_{\varepsilon \rightarrow 0} \int \left(\nabla^\alpha J_\alpha^X [Au_\varepsilon, \overline{Au_\varepsilon}] - T_{\alpha\beta} [Au_\varepsilon, \overline{Au_\varepsilon}] \nabla^\alpha X^\beta \right) \varphi \, d\text{Vol}_g = -\langle \nu, g^{\alpha\gamma} \xi_\alpha \xi_\beta \nabla_\gamma \Phi^\beta \rangle.$$

Step 2: the right hand side of (8.3.9). For the first term we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \left(X(\overline{Au_\varepsilon}) A f_\varepsilon + X(Au_\varepsilon) \overline{A f_\varepsilon} \right) \varphi \, d\text{Vol}_g &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \left(X u_\varepsilon A^* A f_\varepsilon + X(A^* A u_\varepsilon) f_\varepsilon \right) \varphi \, d\text{Vol}_g \\ &= \frac{1}{2} \langle X^\beta (\tilde{\lambda}_\beta + \tilde{\lambda}_\beta^*), |m(\xi)|^2 \varphi \sqrt{|g|} \rangle = \langle \text{Re } \lambda, \Phi \rangle. \end{aligned}$$

According to Lemma 8.3.3, the last term yields

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \left(X(\overline{Au_\varepsilon}) [\square_g, A] u_\varepsilon + X(Au_\varepsilon) \overline{[\square_g, A] u_\varepsilon} \right) \varphi \, d\text{Vol}_g \\ = \lim_{\varepsilon \rightarrow 0} -\langle \nu, \partial_{x^\mu} g^{\alpha\gamma} \xi_\alpha \xi_\gamma \xi_\beta X^\beta (m \partial_{\xi_\mu} \overline{m} + \overline{m} \partial_{\xi_\mu} m) \varphi \sqrt{|g|} \rangle = -\frac{1}{2} \langle \nu, \partial_{x^\mu} g^{\alpha\gamma} \xi_\alpha \xi_\gamma \xi_\beta \partial_{\xi_\mu} \Phi^\beta \rangle. \end{aligned}$$

Step 3: putting everything together. Combining the last three computations we find that

$$\langle \nu, -g^{\alpha\gamma} \xi_\alpha (\xi_\beta \nabla_{x^\gamma} \Phi^\beta) + \frac{1}{2} \partial_{x^\mu} g^{\alpha\gamma} \xi_\alpha \xi_\gamma (\xi_\beta \partial_{\xi_\mu} \Phi^\beta) \rangle = \langle \text{Re } \lambda, \xi_0 \Phi \rangle.$$

The left-hand side can be simplified further: note that, as ∇_μ is the Levi-Civita connection,

$$0 = \nabla_\mu g^{\alpha\gamma} = \partial_{x^\mu} g^{\alpha\gamma} + \delta_\beta^\alpha \Gamma_{\mu\delta}^\beta g^{\delta\gamma} + \delta_\beta^\gamma \Gamma_{\mu\delta}^\beta g^{\alpha\delta} \implies \frac{1}{2} \partial_{x^\mu} g^{\alpha\gamma} \xi_\alpha \xi_\gamma + g^{\alpha\gamma} \xi_\alpha \xi_\beta \Gamma_{\gamma\mu}^\beta = 0.$$

Combining this identity with the two equations

$$\partial_{\xi_\mu} \Phi = \xi_\beta \partial_{\xi_\mu} \Phi^\beta + \Phi^\mu, \quad \partial_{x^\gamma} \Phi = \xi_\beta \nabla_{x^\gamma} \Phi^\beta - \Gamma_{\gamma\mu}^\beta \Phi^\mu \xi_\beta$$

we find that

$$-g^{\alpha\gamma}\xi_\alpha(\xi_\beta\nabla_{x^\gamma}\Phi^\beta) + \frac{1}{2}\partial_{x^\mu}g^{\alpha\gamma}\xi_\alpha\xi_\gamma(\xi_\beta\partial_{\xi_\mu}\Phi^\beta) = -g^{\alpha\gamma}\xi_\alpha\partial_{x^\gamma}\Phi + \frac{1}{2}\partial_{x^\mu}g^{\alpha\gamma}\xi_\alpha\xi_\gamma\partial_{\xi_\mu}\Phi = -\{g^{\alpha\beta}\xi_\alpha\xi_\beta, \Phi\}.$$

While the previous calculations hold for an arbitrary vector field X , we now take $X = e_0$, so that $X^\beta\xi_\beta = \xi_0 - \beta^i\xi_i$. As before we note that $\xi_0 - \beta^i\xi_i$ never vanishes on the zero mass shell, where ν is supported. Hence, we have shown that part (b₂) of the theorem holds whenever \tilde{a} is of the form $\tilde{a}(x, \xi) = \Phi(x, \xi) = b(x)q(\xi)$ with q real and positively 1-homogeneous. The case of a general test function follows by considerations analogous to the ones in Remark 8.2.8. \square

8.3.2 Two compensated compactness lemmas

This subsection contains two compensated compactness results for solutions of the wave system (8.3.1) which follow readily from the very classical Theorem 8.2.11. We begin with a *bilinear* result:

Lemma 8.3.5. *Null forms are weakly continuous, i.e.*

$$\left. \begin{array}{l} \text{for } I = 1, 2, \quad u_\varepsilon^I \rightharpoonup u^I \text{ in } W_{\text{loc}}^{1,2} \\ (\square_g u_\varepsilon^I)_\varepsilon \text{ is compact in } W_{\text{loc}}^{-1,2} \end{array} \right\} \implies g^{-1}(du_\varepsilon^1, du_\varepsilon^2) \xrightarrow{*} g^{-1}(du^1, du^2) \text{ in } \mathcal{D}'.$$

Proof: It suffices to consider the case $u_\varepsilon^1 = u_\varepsilon^2$: indeed, one can use the polarization identity

$$g^{-1}(du_\varepsilon^1, du_\varepsilon^1) + 2g^{-1}(du_\varepsilon^1, du_\varepsilon^2) + g^{-1}(du_\varepsilon^2, du_\varepsilon^2) = g^{-1}(du_\varepsilon^1 + du_\varepsilon^2, du_\varepsilon^1 + du_\varepsilon^2)$$

and pass to the limit on both sides to see that $g^{-1}(du_\varepsilon^1, du_\varepsilon^2) \xrightarrow{*} g^{-1}(du^1, du^2)$ in the sense of distributions. We thus drop all superscripts from the sequences.

Let \star be the Hodge star with respect to the metric g^{-1} . We have

$$g^{-1}(du_\varepsilon, du_\varepsilon) d\text{Vol}_{g^{-1}} = du_\varepsilon \wedge (\star du_\varepsilon)$$

and, since u_ε is scalar, $\square_g u_\varepsilon = \star d \star du_\varepsilon$. The conclusion follows from Theorem 8.2.11. \square

The next result is *trilinear* and was essentially known to Tartar: see [366, Lemma I.5], where it is proved when g is the Minkowski metric. The proof given below is the natural adaptation of Tartar's proof, now in the language of geometric wave equations introduced at the beginning of the section. See also [200, Proposition 12.2] for an alternative proof.

Lemma 8.3.6. *Let X be a smooth vector field. Then*

$$\left. \begin{array}{l} \text{for } I = 1, 2, 3, \quad u_\varepsilon^I \rightharpoonup 0 \text{ in } W_{\text{loc}}^{1,3} \\ (\square_g u_\varepsilon^I)_\varepsilon \text{ is bounded in } L_{\text{loc}}^3 \end{array} \right\} \implies Xu_\varepsilon^1 g^{-1}(du_\varepsilon^2, du_\varepsilon^3) \xrightarrow{*} 0 \quad \text{in } \mathcal{D}'.$$

Proof: The assumptions imply that the sequence $J_\alpha^X[u_\varepsilon^1, u_\varepsilon^2]$ is bounded in $L_{\text{loc}}^{3/2}$ and, recalling (8.3.6), that $\nabla^\alpha J_\alpha^X[u_\varepsilon^1, u_\varepsilon^2]$ is compact in $W_{\text{loc}}^{-1,3/2}$. We note that

$$2J_\alpha^X[u_\varepsilon^1, u_\varepsilon^2]\partial_\beta u_\varepsilon^3 g^{\alpha\beta} = Xu^1 g^{-1}(du^2, du^3) + Xu^2 g^{-1}(du^1, du^3) - Xu^3 g^{-1}(du^1, du^2),$$

where the left-hand side is a div-curl product. Using the polarization identity, as in Lemma 8.3.5, to prove the conclusion we can take $u_\varepsilon^2 = u_\varepsilon^3$ without loss of generality. Thus

$$2J_\alpha^X[u_\varepsilon^1, u_\varepsilon^2]\partial_\beta u_\varepsilon^2 g^{\alpha\beta} = Xu_\varepsilon^1 g^{-1}(du_\varepsilon^2, du_\varepsilon^2)$$

or, equivalently, writing again \star for the Hodge star with respect to g^{-1} ,

$$Xu_\varepsilon^1 g^{-1}(du_\varepsilon^2, du_\varepsilon^2) d\text{Vol}_{g^{-1}} = g^{-1}(2J^X[u_\varepsilon^1, u_\varepsilon^2], du_\varepsilon^2) d\text{Vol}_{g^{-1}} = 2J^X[u_\varepsilon^1, u_\varepsilon^2] \wedge \star du_\varepsilon^2.$$

Since $du_\varepsilon^2 \rightharpoonup 0$ in L_{loc}^3 , we can again use Theorem 8.2.11 to pass to the limit. \square

Remark 8.3.7. Taking $u_\varepsilon^1 = u_\varepsilon^2 = u_\varepsilon^3$ in Lemma 8.3.6, we note that the trilinear quantity is weakly continuous *solely at zero*. That this happens is only possible because \square_g , thought of as a first-order operator acting on du , does not have constant rank. Indeed, it follows from the arguments in Chapter 5 that, under constant rank constraints, nonlinearities which are weakly continuous at a point are necessarily weakly continuous *everywhere*. Furthermore, regardless of rank conditions, nonlinearities which are weakly continuous everywhere are polynomials with degree not exceeding the dimension of the domain, i.e. $n + 1$, see also [301]. In contrast, Lemma 8.3.6 is of course valid even when $n = 1$.

8.4 The linear covariant wave equation with oscillating coefficients

This section is devoted to the proof of Theorem 8.1.2. Our strategy is to reduce the analysis of the limiting behavior of sequences of solutions to

$$\square_{g_\varepsilon} u_\varepsilon = f_\varepsilon, \quad u_\varepsilon, f_\varepsilon: (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (8.4.1)$$

to the case where $(u_\varepsilon, f_\varepsilon)$ are solutions of a fixed wave equation, as in the previous section. Hence, we will frequently recast (8.4.1) in the form of (8.3.1), i.e.

$$\square_g u_\varepsilon = (\square_g - \square_{g_\varepsilon}) u_\varepsilon + f_\varepsilon, \quad (8.4.2)$$

but where now the right-hand side also depends on u_ε . Note that, by Hypothesis 8.1.1(b),

$$\square_g u_\varepsilon \text{ is uniformly bounded in } L_{\text{loc}}^p. \quad (8.4.3)$$

We begin by noting that part (a) of Theorem 8.1.2 poses no difficulty, as the covariant wave operator is an operator in divergence form. For later use, we state the result explicitly:

Proposition 8.4.1 (Limit equation). *Let $(g_\varepsilon, u_\varepsilon, f_\varepsilon)_\varepsilon$ be a sequence satisfying Hypotheses 8.1.1 and solving (8.4.1). Then $\square_g u = f$ in the sense of distributions.*

Proof: Note that $\square_{g_\varepsilon} u_\varepsilon \xrightarrow{*} \square_g u$ in \mathcal{D}' , and hence by Hypotheses 8.1.1(c) also weakly in L_{loc}^2 . Indeed, take a test function φ ; then, using the local uniform convergence of g_ε and integrating by parts, we find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int \varphi \square_{g_\varepsilon} u_\varepsilon \, d\text{Vol}_g &= \lim_{\varepsilon \rightarrow 0} \int \varphi \square_{g_\varepsilon} u_\varepsilon \, d\text{Vol}_{g_\varepsilon} \\ &= - \lim_{\varepsilon \rightarrow 0} \int \partial_\alpha \varphi \partial_\beta u_\varepsilon g_\varepsilon^{\alpha\beta} \, d\text{Vol}_{g_\varepsilon} = - \int \partial_\alpha \varphi \partial_\beta u g^{\alpha\beta} \, d\text{Vol}_g = \int \varphi \square_g u \, d\text{Vol}_g, \end{aligned}$$

since we have the product of weakly convergent terms with strongly convergent ones. Recall that $d\text{Vol}_g \equiv \sqrt{|\det g|} \, dx$ and (8.3.2). As $\square_{g_\varepsilon} u_\varepsilon = f_\varepsilon \rightharpoonup f$ in L_{loc}^2 , by uniqueness of limits we see that $\square_g u = f$. \square

For part (b), our starting point is identity (8.4.2). We set

$$h_\varepsilon^{\alpha\beta} \equiv g_\varepsilon^{\alpha\beta} - g^{\alpha\beta}, \quad H_\varepsilon \equiv (\square_{g_\varepsilon} - \square_g) u_\varepsilon;$$

by (8.4.3) and Proposition 8.4.1, H_ε converges weakly in L_{loc}^2 to zero. Besides the H-measures defined in (8.1.2), we will need the H-measure generated when du_ε is combined with the right-hand side in (8.4.2):

$$(\partial_0 u_\varepsilon, \partial_1 u_\varepsilon, \dots, \partial_n u_\varepsilon, H_\varepsilon + f_\varepsilon)_\varepsilon \xrightarrow{\text{H}} \begin{bmatrix} \tilde{\nu} & \tilde{\sigma} \\ \tilde{\sigma}^* & \star \end{bmatrix}.$$

8.4.1 Elementary reductions

Before proceeding with the core of the proof, we make a few basic observations. Firstly, both the structure of the H-measure and part (b₁) of Theorem 8.1.2 follow as in Section 8.3 since, by (8.4.3), $\square_g u_\varepsilon$ is bounded in L^2_{loc} . Likewise, $\tilde{\sigma}_\gamma = \xi_\gamma \sigma$ for some Radon measure σ . Moreover, arguing once more as in Section 8.3, we can and will assume that the sequence u_ε is supported on a fixed bounded set Ω . Hence we can and will also assume that $g_\varepsilon = g$ for all ε , outside a neighborhood of Ω .

The final remark that we make here concerns the parity in ξ of equation (8.1.4): according to the parity of ν and λ , established in Theorem 8.3.2, we only need to test (8.1.4) against 1-homogeneous functions \tilde{a} which are odd in ξ , which corresponds to testing against symbols a which are 0-homogeneous and even in ξ . In particular, in the rest of the proof we will use implicitly the following straightforward lemma:

Lemma 8.4.2. *For $A \in \Psi^0$ such that*

$$\sigma^0(A)(x, \xi) \text{ is real and even in } \xi, \quad (8.4.4)$$

$A\varphi$ and A^φ are real whenever $\varphi \in L^2$ is real.*

In light of Theorem 8.3.2, our task is to show that, in the limit, H_ε does not contribute to the transport equation. This is intuitive if the convergence of the metrics is sufficiently strong and the remainder of this section is dedicated to showing that the notions of convergence given in Hypotheses 8.1.1 are indeed sufficient.

8.4.2 The case of strong convergence of the metrics

In this section, we essentially prove Theorem 8.1.2, assuming Hypothesis (a₁) holds. The first step is a reduction to estimating some commutators. The basic idea is to integrate by parts in order to try to distribute the derivatives in such a way that two derivatives do not land on the same term; this cannot be achieved completely, but the remaining terms have a commutator structure.

Lemma 8.4.3. *Let $A \in \Psi^0_{1,c}$ satisfy (8.4.4). Under Hypotheses 8.1.1, assuming (a₁) holds,*

$$2 \lim_{\varepsilon \rightarrow 0} \langle H_\varepsilon, A \partial_\gamma (u_\varepsilon - u) \rangle = \lim_{\varepsilon \rightarrow 0} \int \partial_\gamma (u_\varepsilon - u) \left[A, h_\varepsilon^{\alpha\beta} \right] \partial_{\alpha\beta}^2 (u_\varepsilon - u) \, dx.$$

Proof: Since derivatives of the metric coefficients converge strongly, $H_\varepsilon = h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 u_\varepsilon + o_{L^2}(1)$, where $o_{L^2}(1)$ denotes a remainder which is compact in L^2 . We begin by noting that

$$\begin{aligned} \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 u_\varepsilon, A \partial_\gamma (u_\varepsilon - u) \rangle &= \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 (u_\varepsilon - u), A \partial_\gamma (u_\varepsilon - u) \rangle + \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 u, A \partial_\gamma (u_\varepsilon - u) \rangle \\ &= \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 (u_\varepsilon - u), A \partial_\gamma (u_\varepsilon - u) \rangle + o(1). \end{aligned}$$

Now, we evaluate the remaining term, setting $w_\varepsilon \equiv u_\varepsilon - u$. First, we integrate by parts in ∂_α :

$$\begin{aligned} \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon, A \partial_\gamma w_\varepsilon \rangle &= -\langle \partial_\alpha h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A \partial_\gamma w_\varepsilon \rangle - \langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, [\partial_\alpha, A] \partial_\gamma w_\varepsilon \rangle - \langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A (\partial_\alpha^2 w_\varepsilon) \rangle \\ &= -\langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A (\partial_\alpha^2 w_\varepsilon) \rangle + o(1). \end{aligned}$$

Then, we integrate the remaining term by parts along ∂_γ , and obtain

$$\begin{aligned} -\langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A (\partial_\alpha \partial_\gamma w_\varepsilon) \rangle &= -\langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, [A, \partial_\gamma] \partial_\alpha w_\varepsilon \rangle + \langle \partial_\gamma h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle + \langle h_\varepsilon^{\alpha\beta} \partial_\beta \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle \\ &= \langle h_\varepsilon^{\alpha\beta} \partial_\beta \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle + o(1). \end{aligned}$$

Finally, integrating the remaining term along ∂_β ,

$$\langle h_\varepsilon^{\alpha\beta} \partial_\beta \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle = -\langle \partial_\beta h_\varepsilon^{\alpha\beta} \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle - \langle h_\varepsilon^{\alpha\beta} \partial_\gamma w_\varepsilon, [\partial_\beta, A] \partial_\alpha w_\varepsilon \rangle - \langle h_\varepsilon^{\alpha\beta} \partial_\gamma w_\varepsilon, A \partial_{\alpha\beta}^2 w_\varepsilon \rangle$$

$$= \langle \partial_\gamma w_\varepsilon, [A, h_\varepsilon^{\alpha\beta}] \partial_{\alpha\beta}^2 w_\varepsilon \rangle - \langle \partial_\gamma w_\varepsilon, A \left(h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon \right) \rangle + o(1).$$

Combining the expressions above yields the identity:

$$\langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon, A \partial_\gamma w_\varepsilon \rangle + \langle \partial_\gamma w_\varepsilon, A \left(h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon \right) \rangle = \langle \partial_\gamma w_\varepsilon, [A, h_\varepsilon^{\alpha\beta}] \partial_{\alpha\beta}^2 w_\varepsilon \rangle + o(1).$$

Since A has real symbol and hence is self-adjoint, up to a compact operator, we conclude the proof. \square

Due to our strong-convergence assumptions, the Calderón commutator immediately yields:

Proposition 8.4.4. *Under Hypotheses 8.1.1, assuming (a₁) holds, for all $A \in \Psi_{1,c}^0$ satisfying (8.4.4),*

$$\lim_{\varepsilon \rightarrow 0} \langle H_\varepsilon, A \partial_\gamma (u_\varepsilon - u) \rangle = 0.$$

Proof: We need only observe that

$$[A, h_\varepsilon^{\alpha\beta}] \partial_{\alpha\beta}^2 (u_\varepsilon - u) = [A \partial_a, h_\varepsilon^{\alpha\beta}] \partial_\beta (u_\varepsilon - u) + o_{L^2}(1).$$

By Theorem 8.2.6 and the fact that $\|\partial_\alpha h_\varepsilon^{\alpha\beta}\|_{L^\infty} \rightarrow 0$, the L^2 -norm of the commutator on the right-hand side goes to zero. It now suffices to appeal to Lemma 8.4.3 and use Hölder's inequality. \square

8.4.3 The general case

The remainder of this section deals with the more complicated case where Hypothesis 8.1.1(a₁) does not hold, and instead we have Hypothesis 8.1.1(a₂). We assume throughout that $p = 4$. Similarly to the previous subsection, we are required to establish the following:

Proposition 8.4.5. *Under Hypotheses 8.1.1, assuming (a₂) holds, for all $A \in \Psi_{1,c}^0$ satisfying (8.4.4),*

$$\lim_{\varepsilon \rightarrow 0} \langle H_\varepsilon, A e_0 (u_\varepsilon - u) \rangle = 0.$$

In Proposition 8.4.5 and in what follows, we use the frame introduced in (8.3.3), in which we have

$$\square_g = g^{00} e_0^2 + \tilde{g}^{ij} \partial_{ij}^2 + \frac{1}{2} \partial_0 g^{00} e_0 + \partial_i g^{i\beta} \partial_\beta + \frac{g^{i\beta} \partial_i \sqrt{|g|}}{\sqrt{|g|}} \partial_\beta - g^{0i} \partial_i \left(\frac{g^{0j}}{g^{00}} \right) \partial_j - \frac{(g^{00})^2}{2q(\tilde{g}^{ij})} \partial_0 \left(q(\tilde{g}^{ij}) \right) e_0, \quad (8.4.5)$$

where $q = q(\tilde{g}^{ij})$ denotes the polynomial in \tilde{g}^{ij} determined implicitly by $|g^{-1}| = -g^{00} q(\tilde{g}^{ij})$ (the existence of such a polynomial is readily verified by considering the LDU decomposition of the matrix-field g).

If Hypotheses 8.1.1(a₁) do not hold, then general first derivatives of the metric coefficients do not converge strongly; however, *spatial* first derivatives of the metric coefficients do: since we assume that $g_\varepsilon = g$ outside a neighborhood of Ω , by integration by parts and our hypotheses,

$$\|\partial_k h_\varepsilon^{\alpha\beta}\|_{L^2} \lesssim \|\partial_{kk}^2 h_\varepsilon^{\alpha\beta}\|_{L^2}^{\frac{1}{2}} \|h_\varepsilon^{\alpha\beta}\|_{L^2}^{\frac{1}{2}} \lesssim \|h_\varepsilon^{\alpha\beta}\|_{L^2}^{\frac{1}{2}}, \quad (8.4.6)$$

which converges to zero. We recall that $\partial_0 (g_\varepsilon)_{ij}$, and hence, $\partial_0 \tilde{g}_\varepsilon^{ij}$ (one may check that $\tilde{g}_\varepsilon^{ij}$ is the inverse of the Riemannian metric $(g_\varepsilon)_{ij}$) also converge strongly. It is now easy to see that, under our assumptions, the last four terms in (8.4.5) only involve strongly converging derivatives of the metric coefficients.

As in the last section, we start by reducing Proposition 8.4.5 to dealing with commutator terms. The proof of the next lemma follows the strategy used for Lemma 8.4.3, but it is much more involved:

Lemma 8.4.6 (Reduction to commutators). *Under Hypotheses 8.1.1, let $A \in \Psi_{1,c}^0$ satisfy (8.4.4). Then,*

$$2 \lim_{\varepsilon \rightarrow 0} \langle H_\varepsilon, A e_0 (u_\varepsilon - u) \rangle = \lim_{\varepsilon \rightarrow 0} \int \partial_\alpha (u_\varepsilon - u) [A, h_\varepsilon^{\alpha\beta}] \partial_\beta e_0 (u_\varepsilon - u) dx.$$

Proof: Let us denote

$$\tilde{h}_\varepsilon^{00} \equiv g_\varepsilon^{00} - g^{00}, \quad \tilde{h}_\varepsilon^{0i} \equiv -g_\varepsilon^{00}(\beta_\varepsilon^i - \beta^i), \quad \tilde{h}_\varepsilon^{ij} \equiv g_\varepsilon^{00}(\beta_\varepsilon^i - \beta^i)(\beta_\varepsilon^j - \beta^j) + \tilde{g}_\varepsilon^{ij} - \tilde{g}^{ij}. \quad (8.4.7)$$

From (8.4.5), we compute

$$\begin{aligned} H_\varepsilon &= (\square_{g_\varepsilon} - \square_g) u_\varepsilon = \tilde{h}_\varepsilon^{ij} \partial_{ij}^2 u_\varepsilon + \tilde{h}_\varepsilon^{00} e_0^2 u_\varepsilon + \frac{1}{2} e_0 \tilde{h}_\varepsilon^{00} e_0 u_\varepsilon + \tilde{h}_\varepsilon^{0i} e_0 \partial_i u_\varepsilon + e_0 (\tilde{h}_\varepsilon^{0i} \partial_i u_\varepsilon) + o_{L^2}(1) \\ &= \tilde{h}_\varepsilon^{ij} \partial_{ij}^2 (u_\varepsilon - u) + \left[\tilde{h}_\varepsilon^{00} e_0 + \frac{1}{2} e_0 \tilde{h}_\varepsilon^{00} \right] e_0 (u_\varepsilon - u) + \tilde{h}_\varepsilon^{0i} e_0 \partial_i (u_\varepsilon - u) + e_0 (\tilde{h}_\varepsilon^{0i} \partial_i (u_\varepsilon - u)) \end{aligned} \quad (8.4.8)$$

$$+ e_0 \tilde{h}_\varepsilon^{00} e_0 u + e_0 \tilde{h}_\varepsilon^{0i} \partial_i u + o_{L^2}(1), \quad (8.4.9)$$

where $o_{L^2}(1)$ denotes a remainder which is strongly converging in L^2 . The proof now proceeds in several steps. Step 5 deals with (8.4.9). In steps 1 through 4, we deal with (8.4.8) and we set $w_\varepsilon \equiv u_\varepsilon - u$ to simplify the notation. We will also find it convenient to note the following identities for $\varepsilon \rightarrow 0$: letting \mathfrak{h}_ε be a suitably regular function with $\mathfrak{h}_\varepsilon \rightarrow 0$ in L^∞ and $\mathfrak{h}_\varepsilon \partial_{\alpha\beta}^2 u_\varepsilon$ uniformly bounded in L^2 , and using Lemma 8.2.3,

$$\begin{aligned} \langle \partial_\gamma u_\varepsilon, [A, \mathfrak{h}_\varepsilon] (\beta^k \partial_k e_0 u_\varepsilon) \rangle &= \langle \partial_\gamma u_\varepsilon, [A, \beta^k \mathfrak{h}_\varepsilon] \partial_k e_0 u_\varepsilon \rangle - \langle \partial_\gamma u_\varepsilon \mathfrak{h}_\varepsilon, [\beta^k, A \partial_k] e_0 u_\varepsilon + A (\partial_k \beta^k e_0 u_\varepsilon) \rangle \\ &= \langle \partial_\gamma u_\varepsilon, [A, \beta^k \mathfrak{h}_\varepsilon] \partial_k e_0 u_\varepsilon \rangle + o(1), \end{aligned} \quad (8.4.10)$$

$$\begin{aligned} \langle \beta^k \partial_k u_\varepsilon, [A, \mathfrak{h}_\varepsilon] \partial_\gamma e_0 u_\varepsilon \rangle &= \langle \partial_k u_\varepsilon, [A, \beta^k \mathfrak{h}_\varepsilon] \partial_\gamma e_0 u_\varepsilon \rangle + \langle \partial_k u_\varepsilon, [\beta^k, A] (\mathfrak{h}_\varepsilon \partial_\gamma e_0 u_\varepsilon) \rangle \\ &= \langle \partial_k u_\varepsilon, [A, \beta^k \mathfrak{h}_\varepsilon] \partial_\gamma e_0 u_\varepsilon \rangle + o(1). \end{aligned} \quad (8.4.11)$$

Step 1: first term in (8.4.8). By its special structure, both time and spatial derivatives of $\tilde{h}_\varepsilon^{ij}$ converge strongly. Hence, by a straightforward adaptation of the proof of Lemma 8.4.3, we find that

$$\lim_{\varepsilon \rightarrow 0} \left\langle \tilde{h}_\varepsilon^{ij} \partial_{ij}^2 w_\varepsilon, A e_0 w_\varepsilon \right\rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{ij}] \partial_{ij}^2 w_\varepsilon dx.$$

Integrating by parts in e_0 and in ∂_j and using the compactness of derivatives of $\tilde{h}_\varepsilon^{ij}$, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{ij}] \partial_{ij}^2 w_\varepsilon dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{ij}] \partial_j e_0 w_\varepsilon dx.$$

Step 2: second term in (8.4.8). An integration by parts in e_0 (which requires both an integration by parts in ∂_t and in a spatial direction) leads to

$$\begin{aligned} \langle \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon, A e_0 w_\varepsilon \rangle &= \langle \partial_k \beta^k \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, [e_0, A] e_0 w_\varepsilon \rangle \\ &\quad - \langle e_0 \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0^2 w_\varepsilon \rangle. \end{aligned}$$

Thus, we have the identity:

$$\langle \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon, A e_0 w_\varepsilon \rangle + \langle e_0 w_\varepsilon, A (\tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon) \rangle + \langle e_0 \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle = \langle e_0 w_\varepsilon, [A, \tilde{h}_\varepsilon^{00}] e_0^2 w_\varepsilon \rangle + o(1).$$

Using the self-adjointness of A on the second term on the left hand side, we conclude that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle 2 \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon + e_0 \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle &= \lim_{\varepsilon \rightarrow 0} \int e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00}] e_0^2 w_\varepsilon dx \\ &= \lim_{\varepsilon \rightarrow 0} \int \left(\partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00}] \partial_0 e_0 w_\varepsilon + \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{00} \beta^i \beta^j] \partial_j e_0 w_\varepsilon \right) dx \\ &\quad - \lim_{\varepsilon \rightarrow 0} \int \left(\partial_i w_\varepsilon [A, \beta^i \tilde{h}_\varepsilon^{00}] \partial_0 e_0 w_\varepsilon + \partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00} \beta^i] \partial_i e_0 w_\varepsilon \right) dx, \end{aligned}$$

where we have applied both (8.4.10) and (8.4.11) to conclude.

Step 3: third term in (8.4.8). An integration by parts in ∂_i leads to

$$\begin{aligned} \langle \tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle &= \langle \tilde{h}_\varepsilon^{0i} [e_0, \partial_i] w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \partial_i \tilde{h}_\varepsilon^{0i} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle h_\varepsilon^{0i} e_0 w_\varepsilon, [\partial_i, A] e_0 w_\varepsilon \rangle \\ &\quad - \langle \tilde{h}_\varepsilon^{0i} e_0 w_\varepsilon, A [\partial_i, e_0] w_\varepsilon \rangle + \langle e_0 w_\varepsilon, [A, \tilde{h}_\varepsilon^{0i}] e_0 \partial_i w_\varepsilon \rangle - \langle e_0 w_\varepsilon, A (\tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon) \rangle. \end{aligned}$$

Thus, we have the identity

$$\langle \tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle + \langle e_0 w_\varepsilon, A (\tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon) \rangle = \langle e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] e_0 \partial_i w_\varepsilon \rangle + o(1).$$

Using the self-adjointness of A , modulo a compact operator, (8.4.11), and interchanging e_0 with ∂_i , we conclude

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle \tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] \partial_i e_0 w_\varepsilon \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] \partial_i e_0 w_\varepsilon \, dx - \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \beta^i \tilde{h}_\varepsilon^{0j}] \partial_j e_0 w_\varepsilon \, dx. \end{aligned}$$

Step 4: fourth term in (8.4.8). To begin, recall that, for any $\mathfrak{h}_\varepsilon \rightarrow 0$ in L^∞ , by (8.4.5),

$$\mathfrak{h}_\varepsilon e_0^2 w_\varepsilon + \mathfrak{h}_\varepsilon \frac{\tilde{g}^{jk}}{g^{00}} \partial_{jk}^2 w_\varepsilon = \mathfrak{h}_\varepsilon \frac{\square_g w_\varepsilon}{g^{00}} + o_{L^2}(1) = o_{L^2}(1), \quad (8.4.12)$$

since $\square_g w_\varepsilon$ is bounded in L^2 by (8.4.3). Consider the term $e_0 (\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon)$; an integration by parts in e_0 yields

$$\begin{aligned} \langle e_0 (\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon), A e_0 w_\varepsilon \rangle &= -\langle \partial_k \beta^k \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, [e_0, A] e_0 w_\varepsilon \rangle - \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle \\ &= -\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle + o(1). \end{aligned} \quad (8.4.13)$$

In the remaining term, we apply (8.4.12) to replace $e_0^2 w_\varepsilon$ with $\partial_{jk}^2 w_\varepsilon$ and we commute A with \tilde{g}^{jk}/g^{00} :

$$-\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle = \langle \tilde{h}_\varepsilon^{0i} \frac{\tilde{g}^{jk}}{g^{00}} \partial_i w_\varepsilon, A \partial_{jk}^2 w_\varepsilon \rangle + o(1).$$

Integrating by parts in j , then i and then k , as in Step 1, we obtain

$$\langle \tilde{h}_\varepsilon^{0i} \frac{\tilde{g}^{jk}}{g^{00}} \partial_i w_\varepsilon, A \partial_{jk}^2 w_\varepsilon \rangle = -\langle \tilde{h}_\varepsilon^{0i} \frac{\tilde{g}^{jk}}{g^{00}} \partial_{jk}^2 w_\varepsilon, A \partial_i w_\varepsilon \rangle + o(1) = \langle \tilde{h}_\varepsilon^{0i} e_0^2 w_\varepsilon, A \partial_i w_\varepsilon \rangle + o(1),$$

where the last step follows from another application of (8.4.12). Combining the previous results, we finally arrive at the identity:

$$-\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle - \langle e_0^2 w_\varepsilon, A (\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon) \rangle = \langle \tilde{h}_\varepsilon^{0i} e_0^2 w_\varepsilon, A \partial_i w_\varepsilon \rangle - \langle e_0^2 w_\varepsilon, A (\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon) \rangle + o(1).$$

Now we use the self-adjointness of A , modulo a compact operator, on all of the terms of the last expression, excluding the first term:

$$-2 \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle = \langle \partial_i w_\varepsilon, A (\tilde{h}_\varepsilon^{0i} e_0^2 w_\varepsilon) \rangle - \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle + o(1) = \langle \partial_i w_\varepsilon, [A, \tilde{h}_\varepsilon^{0i}] e_0^2 w_\varepsilon \rangle + o(1).$$

Recalling (8.4.13), we arrive at

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle e_0 (\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon), A e_0 w_\varepsilon \rangle &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] e_0^2 w_\varepsilon \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] \partial_0 e_0 w_\varepsilon \, dx - \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{0i} \beta^j] \partial_j e_0 w_\varepsilon \, dx, \end{aligned}$$

where we use (8.4.10) in the last equality.

Step 5: the two terms in (8.4.9). We integrate by parts in e_0 :

$$\begin{aligned} & \langle e_0 \tilde{h}_\varepsilon^{00} e_0 u + e_0 \tilde{h}_\varepsilon^{0k} \partial_k u, A e_0 (u_\varepsilon - u) \rangle \\ &= -\langle \partial_j \beta^j (\tilde{h}_\varepsilon^{00} e_0 u + h_\varepsilon^{0k} \partial_k u), A e_0 (u_\varepsilon - u) \rangle - \langle \tilde{h}_\varepsilon^{00} e_0 u + \tilde{h}_\varepsilon^{0k} \partial_k u, [e_0, A] e_0 (u_\varepsilon - u) \rangle \\ & \quad - \langle \tilde{h}_\varepsilon^{00} e_0^2 u + \tilde{h}_\varepsilon^{0k} e_0 \partial_k u, A e_0 (u_\varepsilon - u) \rangle - \langle \tilde{h}_\varepsilon^{00} e_0 u + \tilde{h}_\varepsilon^{0k} \partial_k u, A e_0^2 (u_\varepsilon - u) \rangle \\ &= -\langle \tilde{h}_\varepsilon^{00} e_0 u + \tilde{h}_\varepsilon^{0k} \partial_k u, A e_0^2 (u_\varepsilon - u) \rangle + o(1), \end{aligned}$$

where the last line follows by the uniform convergence of $\tilde{h}_\varepsilon^{\alpha\beta}$. For the remaining term, we may apply the same reasoning as in the previous step: from (8.4.12), we have

$$\begin{aligned} -\langle \tilde{h}_\varepsilon^{00} e_0 u + \tilde{h}_\varepsilon^{0k} \partial_k u, A e_0^2 (u_\varepsilon - u) \rangle &= \langle \tilde{h}_\varepsilon^{00} e_0 u + \tilde{h}_\varepsilon^{0k} \partial_k u, A \frac{\tilde{g}^{jj}}{g_{00}^2} \partial_{ij}^2 (u_\varepsilon - u) \rangle + o(1) \\ &= -\langle \partial_i \tilde{h}_\varepsilon^{00} e_0 u + \partial_i \tilde{h}_\varepsilon^{0k} \partial_k u, A \frac{\tilde{g}^{ij}}{g_{00}^2} \partial_j (u_\varepsilon - u) \rangle + o(1) = o(1), \end{aligned}$$

with the second line following from an integration by parts. Thus, (8.4.9) does not contribute to the limit.

Step 6: conclusion. Combining the previous steps yields

$$\begin{aligned} \langle H_\varepsilon, A e_0 w_\varepsilon \rangle &= \int \partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00}] \partial_0 e_0 w_\varepsilon \, dx + \int \left(\partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i} - \tilde{h}_\varepsilon^{00} \beta^i] \partial_i + \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{i0} - \tilde{h}_\varepsilon^{00} \beta^i] \partial_0 \right) e_0 w_\varepsilon \, dx \\ & \quad + \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{ij} - \tilde{h}_\varepsilon^{0i} \beta^j - \tilde{h}_\varepsilon^{0j} \beta^i + \tilde{h}_\varepsilon^{00} \beta^i \beta^j] \partial_j e_0 w_\varepsilon \, dx + o(1), \end{aligned}$$

and using the definitions in (8.4.7) the conclusion follows. \square

By passing to subsequences if need be, by Hypothesis 8.1.1(b) we may find a sequence $\omega_\varepsilon \searrow 0$ such that

$$\sup_{\alpha, \beta} \|\partial_{\alpha\beta}^2 u_\varepsilon\|_{L^4(\Omega)} \lesssim \omega_\varepsilon^{-1}, \quad \sup_{\alpha, \beta} \|h_\varepsilon^{\alpha\beta}\|_{L^\infty(\Omega)} \lesssim \omega_\varepsilon.$$

In order to prove Proposition 8.4.5, we move to Fourier space. Let $\zeta: \mathbb{R}_0^+ \rightarrow [0, 1]$ be a smooth function such that $\zeta(x) = 1$ for $x \leq 1$ and $\zeta = 0$ for $x \geq 2$. We consider a partition of frequency space into high and low frequencies for each ε , associated to the smooth functions $0 \leq \Theta_{\text{low}, \varepsilon}, \Theta_{\text{spa}, \varepsilon}, \Theta_{\text{time}, \varepsilon} \leq 1$ defined by

$$\begin{aligned} \Theta_{\text{low}, \varepsilon}(\rho) &\equiv \zeta \left(\omega_\varepsilon^{\delta_1} |\rho_{\text{tot}}| \right) \implies \text{supp } \Theta_{\text{low}, \varepsilon} \subseteq \left\{ |\rho_{\text{tot}}| \leq 2\omega_\varepsilon^{-\delta_1} \right\}, \\ \Theta_{\text{spa}, \varepsilon}(\rho) &\equiv (1 - \Theta_{\text{low}, \varepsilon})(\rho) \left[1 - \zeta \left(\frac{|\rho_{\text{spa}}|}{|\rho_{\text{tot}}|^{\delta_2}} \right) \right] \implies \text{supp } \Theta_{\text{spa}, \varepsilon} \subseteq \left\{ |\rho_{\text{spa}}| \geq |\rho_{\text{tot}}|^{\delta_2} \geq \omega_\varepsilon^{-\delta_1 \delta_2} \right\}, \\ \Theta_{\text{time}, \varepsilon}(\rho) &\equiv (1 - \Theta_{\text{low}, \varepsilon})(\rho) \zeta \left(\frac{|\rho_{\text{spa}}|}{|\rho_{\text{tot}}|^{\delta_2}} \right) \implies \text{supp } \Theta_{\text{time}, \varepsilon} \subseteq \left\{ \begin{array}{l} \rho_0^2 \geq |\rho_{\text{tot}}|^2 - 4|\rho_{\text{tot}}|^{2\delta_2} \\ \text{and } |\rho_{\text{tot}}| \geq \omega_\varepsilon^{-\delta_1} \end{array} \right\}, \end{aligned}$$

where $|\rho_{\text{tot}}|^2 = \sum_\alpha \rho_\alpha^2$ and $|\rho_{\text{spa}}|^2 = \sum_i \rho_i^2$. Here, $\delta_1, \delta_2 > 0$ are parameters to be fixed. Clearly

$$\Theta_{\text{low}, \varepsilon} + \Theta_{\text{spa}, \varepsilon} + \Theta_{\text{time}, \varepsilon} = 1.$$

For an L^2 function \mathfrak{h}_ε , we define its projections on a range of frequencies according to

$$\mathcal{P}_{\text{range}, \varepsilon}[\mathfrak{h}_\varepsilon] \equiv \mathcal{F}^{-1} \left(\Theta_{\text{range}, \varepsilon} \mathcal{F}(\mathfrak{h}_\varepsilon) \right), \quad \text{range} \in \{\text{low}, \text{spa}, \text{time}\},$$

where \mathcal{F} denotes the Fourier transform. These projections are linear and commute with derivatives.

We focus on the low frequencies first. Note that, if the frequency parameter is capped, then derivatives, which in frequency space correspond to multiplication by the frequency variable, are not complete. Thus, the strategy of Section 8.4.2 still works under the current convergence assumptions on g_ε as long as one restricts to low frequencies:

Lemma 8.4.7. *Under Hypotheses 8.1.1, as long as $\delta_1 < 1$,*

$$\lim_{\varepsilon \rightarrow 0} \int \partial_\alpha (u_\varepsilon - u) [A, \mathcal{P}_{\text{low}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 (u_\varepsilon - u) \, dx = 0.$$

Proof: Without loss of generality, set $u \equiv 0$. Consider the identity

$$\partial_\alpha u_\varepsilon [A, \mathcal{P}_{\text{low}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 u_\varepsilon = \partial_\alpha u_\varepsilon [A \partial_\beta, \mathcal{P}_{\text{low}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] e_0 u_\varepsilon - \partial_\alpha u_\varepsilon A (\partial_\beta \mathcal{P}_{\text{low}, \varepsilon} [h_\varepsilon^{\alpha\beta}] e_0 u_\varepsilon).$$

We estimate the second term directly and, for the first term, apply the Theorem 8.2.6: for small ε ,

$$\left| \int \partial_\alpha u_\varepsilon [A, \mathcal{P}_{\text{low}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 u_\varepsilon \, dx \right| \lesssim \sup_{\alpha, \beta} \|\mathcal{P}_{\text{low}, \varepsilon} [h_\varepsilon^{\alpha\beta}]\|_{W^{1, \infty}} \|\partial u_\varepsilon\|_{L^2}^2 \lesssim \omega_\varepsilon^{-\delta_1} \sup_{\alpha, \beta} \|h_\varepsilon^{\alpha\beta}\|_{L^\infty} \lesssim \omega_\varepsilon^{1-\delta_1},$$

where we use Bernstein's inequality $\hat{f} \subset B_R(0) \implies \|Df\|_\infty \lesssim R\|f\|_\infty$ in the second inequality. \square

For high frequencies this method fails, as we do not have sufficient control over h_ε^{00} and h_ε^{0i} . If the spatial frequencies dominate, however, we can compensate for this issue by appealing to control on higher order spatial derivatives of $h_\varepsilon^{\alpha\beta}$. This is independent of the commutator structure.

Lemma 8.4.8. *Under Hypotheses 8.1.1, as long as $\delta_1 \delta_2 > \frac{1}{2}$,*

$$\lim_{\varepsilon \rightarrow 0} \int \partial_\alpha (u_\varepsilon - u) [A, \mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 (u_\varepsilon - u) \, dx = 0.$$

Proof: Without loss of generality, set $u \equiv 0$. We have

$$\begin{aligned} & \left| \int \partial_\alpha u_\varepsilon [A, \mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 u_\varepsilon \, dx \right| \\ & \leq \left| \int \partial_\alpha u_\varepsilon A (\mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon) \, dx \right| + \left| \int \mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}] \partial_\alpha u_\varepsilon A \partial_\beta e_0 u_\varepsilon \, dx \right| \\ & \lesssim \sup_{\alpha, \beta, \gamma} \|\partial_\beta e_0 u_\varepsilon\|_{L^4} \|\partial u_\varepsilon\|_{L^4} \|\mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}]\|_{L^2} \lesssim \sup_{\alpha, \beta} \omega_\varepsilon^{-1} \|\mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}]\|_{L^2}. \end{aligned}$$

Using the assumptions directly would imply that the term above is bounded, but not necessarily converging to zero. However, by Plancherel's theorem,

$$\|\mathcal{P}_{\text{spa}, \varepsilon} [h_\varepsilon^{\alpha\beta}]\|_{L^2}^2 = \int \left| \Theta_{\text{spa}, \varepsilon} \widehat{h_\varepsilon^{\alpha\beta}} \right|^2 \, d\xi = \int \frac{|\Theta_{\text{spa}, \varepsilon}(\xi)|^2}{|\xi_{\text{spa}}|^4} \left| \delta^{ij} \xi_i \xi_j \widehat{h_\varepsilon^{\alpha\beta}}(\xi) \right|^2 \, d\xi \lesssim \omega_\varepsilon^{2\delta_1 \delta_2} \|\delta^{ij} \partial_{ij}^2 h_\varepsilon^{\alpha\beta}\|_{L^2}^2,$$

since $|\xi_{\text{spa}}|^2 \gtrsim |\xi|^{2\delta_2} \gtrsim \omega_\varepsilon^{-2\delta_1 \delta_2}$ in the support of $\Theta_{\text{spa}, \varepsilon}(\xi)$. By the boundedness of the spatial laplacian of the metric coefficients, we obtain our result. \square

Finally, we are left with the regime of high frequencies where it is the time frequency which dominates. Here, the lack of control over $\partial h_\varepsilon^{\alpha\beta}$ is compensated by control over $\square_g u_\varepsilon$, see (8.4.3). Crucial to the argument is the commutator structure yielded by Lemma 8.4.6 and the invertibility of e_0 in this frequency regime.

Lemma 8.4.9. *Under Hypotheses 8.1.1 and assuming that A satisfies (8.4.4), if $\delta_1 > \frac{1}{2}$ and $\delta_2 < 1$,*

$$\int \partial_\alpha (u_\varepsilon - u) [A, \mathcal{P}_{\text{time}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 (u_\varepsilon - u) \, dx = 0.$$

Proof: Without loss of generality, set $u \equiv 0$. Throughout the proof, we let $\mathfrak{h}_\varepsilon^{\alpha\beta} \equiv \mathcal{P}_{\text{time}}[h_\varepsilon^{\alpha\beta}]$ and we assume that $\delta_1 > 0$ and $\delta_2 < 1$. We also note that, for sufficiently small ε_0 ,

$$|\beta^j(x)\xi_j/\xi_0| \ll 1, \quad \left| \xi_0 + \beta^j(x)\xi_j \right| \gtrsim \omega_\varepsilon^{-\delta_1} \gg 1 \quad \text{when } \xi \in \text{supp } \Theta_{\text{time}, \varepsilon},$$

whenever $\varepsilon \leq \varepsilon_0$. Hence we may find an operator $Q \in \Psi_{1,c}^{-1}$ and $R \in \Psi_{1,c}^0$ such that

$$\sigma(Q)(x, \xi) = q(x, \xi) \equiv \left[g^{00}(x)(\xi_0 + \beta^j(x)\xi_j) \right]^{-1} = \frac{1 + \sigma(R)(x, \xi)}{g^{00}(x)\xi_0}, \quad \sigma(R)(x, \xi) \equiv \sum_{\ell=1}^{\infty} \left(-\beta^j(x) \frac{\xi_j}{\xi_0} \right)^\ell,$$

whenever $(x, \xi) \in \Omega \times \text{supp } \Theta_{\text{time}, \varepsilon_0}$.

It is now easy to see that we have the estimates

$$\|Q\mathfrak{h}_\varepsilon^{\alpha\beta}\|_{L^2(\Omega)} \lesssim \omega_\varepsilon^{1+\delta_1}, \quad (8.4.14)$$

$$\|\partial_0(Q\mathfrak{h}_\varepsilon^{\alpha\beta})\|_{L^2(\Omega)} \lesssim \omega_\varepsilon^{\delta_1}, \quad \sup_j \|\partial_j(Q\mathfrak{h}_\varepsilon^{\alpha\beta})\|_{L^2(\Omega)} \lesssim \omega_\varepsilon^{\frac{1}{2}+\delta_1}. \quad (8.4.15)$$

Indeed, for (8.4.14), we compute

$$\begin{aligned} \|Q\mathfrak{h}_\varepsilon^{\alpha\beta}\|_{L^2(\Omega)} &\lesssim \|(g^{00})^{-1}\|_{L^\infty(\Omega)} \|\xi_0^{-1}\|_{L^\infty(\text{supp } \Theta_{\varepsilon, \text{time}})} \|h_\varepsilon^{\alpha\beta}\|_{L^2} \sum_{\ell=0}^{\infty} \left[\|\beta^i\|_{L^\infty(\Omega)} \|\xi_j \xi_0^{-1}\|_{L^\infty(\text{supp } \Theta_{\varepsilon, \text{time}})} \right]^\ell \\ &\lesssim \|\xi_0^{-1}\|_{L^\infty(\text{supp } \Theta_{\varepsilon, \text{time}})} \|h_\varepsilon^{\alpha\beta}\|_{L^2} \lesssim \omega_\varepsilon^{1+\delta_1}. \end{aligned}$$

The estimates in (8.4.15) follow similarly. Note that we only require L^2 norms in Ω in what follows as we will always be testing against u_ε and its derivatives, which have compact support in Ω .

Using Q , we may rewrite our commutator as

$$\langle \partial_\alpha u_\varepsilon, [A, \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle = \langle \partial_\alpha u_\varepsilon, A \left(g^{00} e_0 Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon \right) \rangle - \langle \partial_\alpha u_\varepsilon g^{00} e_0 Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta e_0 u_\varepsilon \rangle. \quad (8.4.16)$$

Step 1: integration by parts in e_0 . In this step, we show:

$$\lim_{\varepsilon \rightarrow 0} \langle \partial_\alpha u_\varepsilon, [A, \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle = \lim_{\varepsilon \rightarrow 0} \langle \partial_{\alpha\beta}^2 u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] (g^{00} e_0^2 u_\varepsilon) \rangle + \lim_{\varepsilon \rightarrow 0} \langle \partial_\alpha u_\varepsilon, [A, \partial_\beta (Q \mathfrak{h}_\varepsilon^{\alpha\beta})] (g^{00} e_0^2 u_\varepsilon) \rangle. \quad (8.4.17)$$

To begin, we seek to move the e_0 derivative on $Q \mathfrak{h}_\varepsilon^{\alpha\beta}$ in (8.4.16) onto u_ε through integration by parts in e_0 . Note that, whenever a derivative hits a coefficient of the limit metric g , that term is $o(1)$: using (8.4.14),

$$\langle \partial g Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial^2 u_\varepsilon, A \partial u_\varepsilon \rangle \lesssim \|\partial^2 u_\varepsilon\|_{L^4} \|\partial u_\varepsilon\|_{L^4} \|Q \mathfrak{h}_\varepsilon^{\alpha\beta}\|_{L^2} \lesssim \omega_\varepsilon^{\delta_1} \rightarrow 0, \quad (8.4.18)$$

where ∂ denotes an arbitrary partial derivative and g an arbitrary metric coefficient. We note that the order of the terms in the left-hand side is unimportant. We will use (8.4.18) and its variants implicitly in the sequel.

The first term of (8.4.16) becomes

$$\begin{aligned} \langle \partial_\alpha u_\varepsilon, A \left(g^{00} e_0 Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon \right) \rangle &= \langle \partial_\alpha u_\varepsilon, [A, g^{00} e_0] \left(Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon \right) \rangle + \langle \partial_\alpha u_\varepsilon g^{00}, e_0 A \left(Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon \right) \rangle \\ &\quad - \langle \partial_\alpha u_\varepsilon, A \left(Q \mathfrak{h}_\varepsilon^{\alpha\beta} [g^{00} e_0, \partial_\beta] e_0 u_\varepsilon \right) \rangle - \langle \partial_\alpha u_\varepsilon, A \left(Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta (g^{00} e_0^2 u_\varepsilon) \right) \rangle \\ &= -\langle \partial_\alpha u_\varepsilon, A(Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta (g^{00} e_0^2 u_\varepsilon)) \rangle - \langle g^{00} e_0 \partial_\alpha u_\varepsilon, A(Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon) \rangle + o(1), \end{aligned}$$

and the second term yields

$$-\langle \partial_\alpha u_\varepsilon g^{00} e_0 Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta e_0 u_\varepsilon \rangle = \langle \partial_\alpha u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}, [g^{00} e_0, A \partial_\beta] e_0 u_\varepsilon \rangle + \langle \partial_\alpha u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta (g^{00} e_0^2 u_\varepsilon) \rangle$$

$$\begin{aligned}
& + \langle g^{00} e_0 \partial_\alpha u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta e_0 u_\varepsilon \rangle + \langle (e_0 g^{00} + \partial_k \beta^k g^{00}) \partial_\alpha u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta e_0 u_\varepsilon \rangle \\
& = \langle \partial_\alpha u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta (g^{00} e_0^2 u_\varepsilon) \rangle + \langle g^{00} e_0 \partial_\alpha u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A \partial_\beta e_0 u_\varepsilon \rangle + o(1).
\end{aligned}$$

Combining the previous computations gives

$$\begin{aligned}
\langle \partial_\alpha u_\varepsilon, [A, \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle & = \langle \partial_\alpha u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_\beta (g^{00} e_0^2 u_\varepsilon) \rangle + \langle g^{00} e_0 \partial_\alpha u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_\beta e_0 u_\varepsilon \rangle + o(1) \\
& = \langle \partial_\alpha u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_\beta (g^{00} e_0^2 u_\varepsilon) \rangle + o(1),
\end{aligned} \tag{8.4.19}$$

because, by the symmetry of $\mathfrak{h}_\varepsilon^{\alpha\beta}$, the self-adjointness of A (up to a compact operator), and (8.4.18), we have

$$\begin{aligned}
& \langle g^{00} e_0 \partial_\alpha u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_\beta e_0 u_\varepsilon \rangle \\
& = \langle g^{00} e_0 \partial_\beta u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_\alpha e_0 u_\varepsilon \rangle = \langle A(e_0 \partial_\beta u_\varepsilon g^{00} Q \mathfrak{h}_\varepsilon^{\alpha\beta}), \partial_\alpha e_0 u_\varepsilon \rangle - \langle A(e_0 \partial_\beta u_\varepsilon g^{00}), Q \mathfrak{h}_\varepsilon^{\alpha\beta} \partial_\alpha e_0 u_\varepsilon \rangle \\
& = \langle A(\partial_\beta e_0 u_\varepsilon Q \mathfrak{h}_\varepsilon^{\alpha\beta}), g^{00} e_0 \partial_\alpha u_\varepsilon \rangle - \langle A(\partial_\beta e_0 u_\varepsilon), Q \mathfrak{h}_\varepsilon^{\alpha\beta} g^{00} e_0 \partial_\alpha u_\varepsilon \rangle + o(1) \\
& = -\langle g^{00} e_0 \partial_\alpha u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_\beta e_0 u_\varepsilon \rangle + o(1).
\end{aligned} \tag{8.4.20}$$

To obtain (8.4.17), we need only integrate (8.4.19) by parts in ∂_β .

Step 2: introducing \square_g . In this step, we show:

$$\begin{aligned}
\langle \partial_\alpha u_\varepsilon, [A, \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle & = \langle \partial_{\alpha\beta}^2 u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] \square_g u_\varepsilon \rangle + \langle \partial_\alpha u_\varepsilon, [A, \partial_\beta (Q \mathfrak{h}_\varepsilon^{\alpha\beta})] \square_g u_\varepsilon \rangle \\
& \quad + \langle \partial_\alpha u_\varepsilon, [\partial_i (Q \mathfrak{h}_\varepsilon^{\alpha\beta}), A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon) \rangle + o(1).
\end{aligned} \tag{8.4.21}$$

From (8.4.5), it is clear that terms $g^{00} e_0^2$ in (8.4.17) may be replaced by $\square_g - \tilde{g}^{ij} \partial_{ij}$, as the remaining terms in (8.4.5), which involve derivatives of g , do not contribute, c.f. (8.4.18). Thus, we have

$$\begin{aligned}
& \langle \partial_\alpha u_\varepsilon, [A, \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle \\
& = -\langle \partial_{\alpha\beta}^2 u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] (\tilde{g}^{ij} \partial_{ij}^2 u_\varepsilon) \rangle - \langle \partial_\alpha u_\varepsilon, [A, \partial_\beta (Q \mathfrak{h}_\varepsilon^{\alpha\beta})] (\tilde{g}^{ij} \partial_{ij}^2 u_\varepsilon) \rangle \\
& \quad + \langle \partial_{\alpha\beta}^2 u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] \square_g u_\varepsilon \rangle + \langle \partial_\alpha u_\varepsilon, [A, \partial_\beta (Q \mathfrak{h}_\varepsilon^{\alpha\beta})] \square_g u_\varepsilon \rangle + o(1) \\
& = \langle \partial_\alpha u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta (\tilde{g}^{ij} \partial_{ij}^2 u_\varepsilon) \rangle + \langle \partial_{\alpha\beta}^2 u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] \square_g u_\varepsilon \rangle + \langle \partial_\alpha u_\varepsilon, [A, \partial_\beta (Q \mathfrak{h}_\varepsilon^{\alpha\beta})] \square_g u_\varepsilon \rangle + o(1),
\end{aligned}$$

integrating by parts in ∂_β to arrive at the final equality. Now, we integrate the first term in ∂_i :

$$\langle \partial_\alpha u_\varepsilon, [A, Q \mathfrak{h}_\varepsilon^{\alpha\beta}] \partial_\beta (\tilde{g}^{ij} \partial_{ij}^2 u_\varepsilon) \rangle = \langle \partial_{i\alpha}^2 u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon) \rangle + \langle \partial_\alpha u_\varepsilon, [\partial_i (Q \mathfrak{h}_\varepsilon^{\alpha\beta}), A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon) \rangle + o(1),$$

where we use (8.4.18) as needed. To obtain our claim, it only remains to show that the first term in the above formula vanishes in the limit. To see this, we argue as before, invoking the symmetry of $\mathfrak{h}_\varepsilon^{\alpha\beta}$ and \tilde{g}^{ij} in their indices and self-adjointness of A (up to a compact operator):

$$\begin{aligned}
\langle \partial_{i\alpha}^2 u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon) \rangle & = \langle \partial_{j\beta}^2 u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{i\alpha}^2 u_\varepsilon) \rangle = -\langle \tilde{g}^{ij} \partial_{i\alpha}^2 u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] \partial_{j\beta}^2 u_\varepsilon \rangle \\
& = -\langle \partial_{i\alpha}^2 u_\varepsilon, [Q \mathfrak{h}_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon) \rangle + o(1).
\end{aligned} \tag{8.4.22}$$

Step 3: conclusion. From (8.4.21), and estimates (8.4.3), (8.4.14), (8.4.15), we obtain

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int \partial_\alpha u_\varepsilon [A, \mathcal{P}_{\text{time}, \varepsilon} [h_\varepsilon^{\alpha\beta}]] \partial_\beta e_0 u_\varepsilon \, dx \\
& \lesssim \|Q h_\varepsilon\|_{L^2(\Omega)} \|\partial^2 u_\varepsilon\|_{L^4} \|\square_g u_\varepsilon\|_{L^4} + \|\partial(Q h_\varepsilon)\|_{L^2(\Omega)} \|\partial u_\varepsilon\|_{L^4} \|\square_g u_\varepsilon\|_{L^4} + \|\partial_j(Q h_\varepsilon)\|_{L^2(\Omega)} \|\partial^2 u_\varepsilon\|_{L^4} \|\partial u_\varepsilon\|_{L^4} \\
& \lesssim \omega_\varepsilon^{\delta_1} + \omega_\varepsilon^{\delta_1 - \frac{1}{2}} \rightarrow 0,
\end{aligned}$$

as long as $\delta_1 > \frac{1}{2}$. □

Proof of Proposition 8.4.5: It suffices to pick $\delta_1 \in (\frac{1}{2}, 1)$ and $\delta_2 \in (\frac{1}{2\delta_1}, 1)$; for instance, $\delta_1 = \frac{5}{6}$ and $\delta_2 = \frac{4}{5}$. Now combine Lemma 8.4.6 with Lemmas 8.4.7, 8.4.8 and 8.4.9. □

8.4.4 Conclusion of the proof

Combining the results from the previous subsections we finish the proof of the main theorem.

Proof of Theorem 8.1.2(b₂): Depending on whether (a₁) or (a₂) holds in Hypotheses 8.1.1, we may appeal to either Proposition 8.4.4 or Proposition 8.4.5 to see that, whenever A satisfies (8.4.4),

$$0 = \lim_{\varepsilon \rightarrow 0} \langle A e_0(u_\varepsilon - u), H_\varepsilon \rangle = \langle (\xi_0 - \beta^i \xi_i) \sigma, a \rangle - \langle (\xi_0 - \beta^i \xi_i) \lambda, a \rangle.$$

Since ν is supported on the zero mass shell $\{g^{\alpha\beta} \xi_\alpha \xi_\beta = 0\}$, and since $\xi_0 - \beta^i \xi_i$ never vanishes on that set, see (8.3.4), it follows that $\langle \lambda, \tilde{a} \rangle = \langle \sigma, \tilde{a} \rangle$ for any $\tilde{a} \in C_c^\infty(S^* \mathcal{M})$ which is *odd and 1-homogeneous in ξ* . Thus, according to Theorem 8.3.2(b₂), for any such \tilde{a} ,

$$\int_{S^* \mathcal{M}} \left[g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\mu} g^{\alpha\beta} \xi_\alpha \xi_\beta \partial_{\xi^\mu} \tilde{a} \right] d\nu = - \int_{S^* \mathcal{M}} \tilde{a} d(\operatorname{Re} \sigma) = - \int_{S^* \mathcal{M}} \tilde{a} d(\operatorname{Re} \lambda). \quad (8.4.23)$$

However, ν is even and λ is odd, c.f. Theorem 8.3.2(b₀): thus, whenever \tilde{a} is even in ξ , $\langle \operatorname{Re} \lambda, \tilde{a} \rangle = 0$, and likewise the right-hand side of (8.4.23) vanishes as well in that case. Hence we see that (8.4.23) actually holds for *any* $\tilde{a} \in C_c^\infty(S^* \mathcal{M})$, as wished. \square

8.5 Nonlinear wave map systems with oscillating coefficients

We recall that Theorem 8.1.3 is concerned with sequences of solutions to

$$\square_{g_\varepsilon} u_\varepsilon^I = -\Gamma_{JK}^I(u_\varepsilon) g_\varepsilon^{-1} (du_\varepsilon^J, du_\varepsilon^K) + f_\varepsilon^I, \quad u_\varepsilon^I, f_\varepsilon^I: \mathbb{R}^{1+n} \rightarrow \mathbb{R}, \quad I, J, K \in \{1, \dots, N\}. \quad (8.5.1)$$

We will reduce the study of the wave map system (8.5.1) to the case of wave maps into a flat target, as studied in Section 8.3 and 8.4. By repeating the arguments detailed in Section 8.3 we see that, by replacing u_ε with χu_ε for an arbitrary smooth cut-off function χ , there is no loss of generality in assuming that the sequence (u_ε) has uniformly bounded support.

Before proceeding with the proof, let us introduce the notation

$$H_\varepsilon^I \equiv (\square_{g_\varepsilon} - \square_g) u_\varepsilon^I, \quad Q_\varepsilon^I \equiv -\Gamma_{JK}^I(u_\varepsilon) g_\varepsilon^{-1} (du_\varepsilon^J, du_\varepsilon^K), \quad Q^I \equiv \text{w-} \lim_{\varepsilon \rightarrow 0} Q_\varepsilon^I.$$

Hence we may rewrite (8.5.1) as

$$\square_g u_\varepsilon^I = F_\varepsilon^I, \quad \text{where } F_\varepsilon^I \equiv -H_\varepsilon^I + Q_\varepsilon^I + f_\varepsilon^I.$$

In addition to the H-measures defined in (8.1.2), we will need the H-measure

$$\left((\partial_0 u_\varepsilon^I, \partial_1 u_\varepsilon^I, \dots, \partial_n u_\varepsilon^I, F_\varepsilon^I)_{I=1}^N \right)_\varepsilon \xrightarrow{\text{H}} \left(\left[\begin{array}{cc} \tilde{\nu}^{IJ} & \tilde{\sigma}^{IJ} \\ (\tilde{\sigma}^{IJ})^* & \star \end{array} \right] \right)_{I, J=1}^N.$$

We deal with the terms H_ε^I and Q_ε^I separately. For the former, it suffices to apply, with minor modifications, the arguments in Section 8.4:

Lemma 8.5.1. *Under Hypotheses 8.1.1 and assuming that $u_\varepsilon \rightarrow u$ in C_{loc}^0 , for any $A \in \Psi_{1,c}^0$ satisfying (8.4.4),*

$$\lim_{\varepsilon \rightarrow 0} \langle A e_0(u_\varepsilon^I - u^I), \mathfrak{g}_{IL}(u_\varepsilon) H_\varepsilon^L \rangle = 0.$$

Proof: The proof consists of a small modification of the arguments used to prove Propositions 8.4.4 and 8.4.5. Here we only point out the modifications needed in the proof of Proposition 8.4.5, as the former is much simpler. Note that, by the local uniform convergence of u_ε , it is enough to show that

$$\lim_{\varepsilon \rightarrow 0} \langle A e_0(u_\varepsilon^I - u^I), \mathfrak{g}_{IL}(u) H_\varepsilon^L \rangle = 0.$$

For simplicity of notation we suppress the dependence of \mathbf{g}_{IL} on u .

Similarly to Lemmas 8.4.3 and 8.4.6, we have

$$2 \lim_{\varepsilon \rightarrow 0} \langle Ae_0(u_\varepsilon^I - u^I), g_{IL} H_\varepsilon^L \rangle = \lim_{\varepsilon \rightarrow 0} \int g_{IL} \partial_\alpha (u_\varepsilon^I - u^I) [A, h_\varepsilon^{\alpha\beta}] \partial_\beta e_0 (u_\varepsilon^L - u^L) dx. \quad (8.5.2)$$

Indeed, the proofs of these lemmas consists of integrating by parts using the self-adjointness of A to produce commutators. With \mathbf{g}_{IL} now in the bracket, the integration by parts generates terms with derivatives of \mathbf{g}_{IL} , which however are compact, as they have one fewer derivative on u_ε^J . Using the self-adjointness also yields the same conclusion: e.g. in Step 2 of Lemma 8.4.6, again writing $w_\varepsilon^J \equiv u_\varepsilon^J - u_\varepsilon^J$, we find the commutator

$$\begin{aligned} \langle \mathbf{g}_{IL} e_0 w_\varepsilon^L, [A, \tilde{h}_\varepsilon^{00}] e_0^2 w_\varepsilon^I \rangle &= \langle \mathbf{g}_{IL} \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon^L, Ae_0 w_\varepsilon^I \rangle + \langle e_0 w_\varepsilon^L, A(\mathbf{g}_{IL} \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon^I) \rangle + o(1) \\ &= \langle \mathbf{g}_{IL} \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon^L, Ae_0 w_\varepsilon^I \rangle + \langle \mathbf{g}_{IL} \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon^I, Ae_0 w_\varepsilon^L \rangle + o(1) \\ &= 2 \langle \mathbf{g}_{IL} \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon^L, Ae_0 w_\varepsilon^I \rangle + o(1), \end{aligned}$$

due to the symmetry of \mathbf{g}_{IL} in I, L . Arguing similarly in the other steps, (8.5.2) is established.

The proofs of Proposition 8.4.4 and Lemmas 8.4.7 and 8.4.8 only require cosmetic modifications. In the proof of Lemma 8.4.9, the fact that $I = L$ is used in a non-trivial way in the arguments involving the symmetry in α, β of $h_\varepsilon^{\alpha\beta}$ in (8.4.20) and (8.4.22). However, since we now sum over all I, L and $\mathbf{g}_{IL} = \mathbf{g}_{LI}$, these arguments still apply: for instance, the analogue of (8.4.22) is now

$$\begin{aligned} \langle \mathbf{g}_{IL} \partial_{i\alpha}^2 u_\varepsilon^I, [Qh_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon^L) \rangle &= \langle \mathbf{g}_{IL} \partial_{j\beta}^2 u_\varepsilon^L, [Qh_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{i\alpha}^2 u_\varepsilon^I) \rangle \\ &= - \langle \tilde{g}^{ij} \partial_{i\alpha}^2 u_\varepsilon^I, [Qh_\varepsilon^{\alpha\beta}, A] (\mathbf{g}_{IL} \partial_{j\beta}^2 u_\varepsilon^L) \rangle \\ &= - \langle \mathbf{g}_{IL} \partial_{i\alpha}^2 u_\varepsilon^I, [Qh_\varepsilon^{\alpha\beta}, A] (\tilde{g}^{ij} \partial_{j\beta}^2 u_\varepsilon^L) \rangle + o(1), \end{aligned}$$

where we exchanged α with β , I with L and i with j in the first equality. Here, we have also commuted through \mathbf{g}_{IL} to place it on the right hand side; this follows similarly as for the commutation of \tilde{g}^{ij} , since \mathbf{g}_{IL} is independent of ε . \square

In light of Theorem 8.1.2, the main remaining point in the proof of Theorem 8.1.3 is to characterize the contribution of Q_ε^I to the transport equation. This is done in the next lemma.

Lemma 8.5.2. *Assuming that Hypotheses 8.1.1 hold and that $u_\varepsilon^I \rightarrow u^I$ in C_{loc}^0 , then for any $A \in \Psi_{1,c}^0$*

$$\lim_{\varepsilon \rightarrow 0} \langle A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), Q_\varepsilon^L - Q^L \rangle = \langle g^{\alpha\beta} [\Gamma_{JK}^L(u) + \Gamma_{KJ}^L(u)] \partial_\beta u^K \tilde{\nu}_{\gamma\alpha}^{IJ}, \sigma^0(A) \rangle,$$

Additionally, if Γ_{JK}^I are Christoffel symbols with respect to a Riemannian metric $\mathbf{g} = \mathbf{g}_{IJ}(u) du^I \otimes du^J$, then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), \mathbf{g}_{IL}(Q_\varepsilon^L - Q^L) \rangle &= \\ &= - \langle g^{\alpha\beta} \partial_{x^\beta} \mathbf{g}_{IJ} \tilde{\nu}_{\gamma\alpha}^{IJ}, \sigma^0(A) \rangle + 2i \langle g^{\alpha\beta} \partial_{u^J} \mathbf{g}_{IK} \partial_{x^\beta} u^K \text{Im} \tilde{\nu}_{\gamma\alpha}^{IJ}, \sigma^0(A) \rangle. \end{aligned} \quad (8.5.3)$$

Proof: We have that $w\text{-}\lim_{\varepsilon \rightarrow 0} Q_\varepsilon^L = w\text{-}\lim_{\varepsilon \rightarrow 0} -\Gamma_{JK}^L(u) g^{-1}(du_\varepsilon^J, du_\varepsilon^K)$, by continuity of Γ_{JK}^L and uniform convergence of g_ε and u_ε . Hence, by Lemma 8.3.5,

$$Q_\varepsilon^L \rightharpoonup Q^L = -\Gamma_{JK}^L(u) g^{-1}(du^J, du^K) \quad \text{in } L^2. \quad (8.5.4)$$

The first part of the lemma is now a direct consequence of the trilinear compensated compactness of Lemma 8.3.6. Indeed, for any $A \in \Psi_{1,c}^0$, we have

$$\lim_{\varepsilon \rightarrow 0} \langle A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), Q_\varepsilon^L - Q^L \rangle$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0} \langle \Gamma_{JK}^L(u) A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), g^{-1}(du^J, du^K) - g^{-1}(du_\varepsilon^J, du_\varepsilon^K) \rangle \\
&= \lim_{\varepsilon \rightarrow 0} - \langle \Gamma_{JK}^L(u) A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), g^{-1}(d(u^J - u_\varepsilon^J), d(u^K - u_\varepsilon^K)) \rangle \\
&\quad + \lim_{\varepsilon \rightarrow 0} \langle \Gamma_{JK}^L(u) A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), g^{-1}(d(u^J - u_\varepsilon^J), du^K) + g^{-1}(du^J, d(u^K - u_\varepsilon^K)) \rangle
\end{aligned}$$

again by continuity of Γ_{JK}^L and uniform convergence of u_ε and g_ε . By Lemma 8.3.6, the first limit on the right-hand side vanishes, hence we arrive at

$$\lim_{\varepsilon \rightarrow 0} \langle A(\partial_\gamma u_\varepsilon^I - \partial_\gamma u^I), Q_\varepsilon^L - Q^L \rangle = \langle \Gamma_{JK}^L(u) \tilde{\nu}_{\gamma\alpha}^{IJ}, g^{\alpha\beta} \partial_\beta u^K \sigma^0(A) \rangle + \langle \Gamma_{JK}^L(u) \tilde{\nu}_{\gamma\alpha}^{IK}, g^{\alpha\beta} \partial_\beta u^J \sigma^0(A) \rangle.$$

For the second part, we begin by recalling the formula for the Christoffel symbols:

$$\mathfrak{g}_{IL} \Gamma_{JK}^L = \frac{1}{2} (\mathfrak{g}_{KI,J} + \mathfrak{g}_{JI,K} - \mathfrak{g}_{JK,I}), \quad (8.5.5)$$

where $\mathfrak{g}_{JK,I} \equiv \partial_{u^I} \mathfrak{g}_{JK}$, and likewise for the other terms. Then

$$\begin{aligned}
2\mathfrak{g}_{IL} \Gamma_{JK}^L \partial_\beta u^K \tilde{\nu}^{IJ} &= \mathfrak{g}_{IJ,K} \partial_\beta u^K \tilde{\nu}^{IJ} + (\mathfrak{g}_{IK,J} - \mathfrak{g}_{JK,I}) \partial_\beta u^K \tilde{\nu}^{IJ} \\
&= \mathfrak{g}_{IJ,K} \partial_\beta u^K \tilde{\nu}^{IJ} + \mathfrak{g}_{IK,J} \partial_\beta u^K \tilde{\nu}^{IJ} - \mathfrak{g}_{IK,J} \partial_\beta u^K \tilde{\nu}^{JI} \\
&= \partial_{x^\beta} \mathfrak{g}_{IJ} \tilde{\nu}^{IJ} + 2i \mathfrak{g}_{IK,J} \partial_{x^\beta} u^K \operatorname{Im} \tilde{\nu}^{IJ}
\end{aligned}$$

where in the last line we used the fact that $\tilde{\nu}^{JI} = (\tilde{\nu}^{IJ})^*$. To conclude, it now suffices to use the first part of the lemma, recalling that $\Gamma_{JK}^I = \Gamma_{KJ}^I$. \square

Proof of Theorem 8.1.3: We first note that $(g, (u^I)_{I=1}^N, (f^I)_{I=1}^N)$ is a distributional solution of (8.5.1): this follows at once from Proposition 8.4.1 and (8.5.4).

Using the Localization Lemma, just as in the proof of Theorem 8.3.2, we find that

$$\tilde{\nu}_{\alpha\beta}^{IJ} = \xi_\alpha \xi_\beta \nu^{IJ}, \quad \tilde{\lambda}_{\gamma 0}^{IJ} = \xi_0 \lambda^{IJ}, \quad \tilde{\sigma}_{\gamma 0}^{IJ} = \xi_0 \sigma^{IJ}, \quad (8.5.6)$$

for some Radon measures $\nu^{IJ}, \lambda^{IJ}, \sigma^{IJ}$ and for each I and J . Likewise, the measures $\nu^{IJ}, \lambda^{IJ}, \sigma^{IJ}$ are supported on the zero mass shell of g , and hence the measures $\nu \equiv \mathfrak{g}_{IJ} \nu^{IJ}, \lambda \equiv \mathfrak{g}_{IJ} \lambda^{IJ}$ and $\sigma \equiv \mathfrak{g}_{IJ} \sigma^{IJ}$ are also supported on the same set. Furthermore, ν^{IJ} and hence also ν are even, whereas σ and λ are odd. By the uniform convergence of both u_ε and g_ε , and using the polarization identity,

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} \int \mathbb{L}_{\alpha\beta}[u_\varepsilon] Y^\alpha Y^\beta \, d\operatorname{Vol}_{g_\varepsilon} - \int \mathbb{L}_{\alpha\beta}[u] Y^\alpha Y^\beta \, d\operatorname{Vol}_g \\
&= \lim_{\varepsilon \rightarrow 0} \int \mathfrak{g}_{IJ}(u) \left[\partial_\alpha u_\varepsilon^I \partial_\beta u_\varepsilon^J - \partial_\alpha u^I \partial_\beta u^J \right] Y^\alpha Y^\beta \, d\operatorname{Vol}_g \\
&= \lim_{\varepsilon \rightarrow 0} \int \mathfrak{g}_{IJ}(u) \partial_\alpha (u_\varepsilon^I - u^I) \partial_\beta (u_\varepsilon^J - u^J) Y^\alpha Y^\beta \, d\operatorname{Vol}_g \\
&= \langle \nu^{IJ}, \sqrt{|g|} \mathfrak{g}_{IJ} \xi_\alpha \xi_\beta Y^\alpha Y^\beta \rangle = \langle \nu, \sqrt{|g|} \xi_\alpha \xi_\beta Y^\alpha Y^\beta \rangle,
\end{aligned}$$

for any test vector field Y . This proves part (a).

It remains to prove the propagation property of ν . We first note that

$$\int_{S^* \mathcal{M}} \left[g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\gamma} g^{\alpha\beta} \xi_\alpha \xi_\beta (\partial_{\xi_\gamma} \tilde{a}) \right] d\nu^{IJ} = - \int_{S^* \mathcal{M}} \tilde{a} \, d(\operatorname{Re} \sigma^{IJ}). \quad (8.5.7)$$

This is proved by repeating verbatim the arguments in the proof of Theorem 8.3.2(b₂): the only difference is that we multiply the equation for $\square_g(Au_\varepsilon^1)$ with $X(\overline{Au_\varepsilon^2})$ and the one for $\square_g(Au_\varepsilon^2)$ with $X(Au_\varepsilon^1)$, c.f. (8.3.8). It follows that ν^{IJ} satisfies the equation

$$\int_{S^* \mathcal{M}} \left[g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} (\tilde{a} \mathfrak{g}_{IJ}) - \frac{1}{2} \partial_{x^\gamma} g^{\alpha\beta} \xi_\alpha \xi_\beta (\partial_{\xi_\gamma} \tilde{a}) \mathfrak{g}_{IJ} \right] d\nu^{IJ} = - \int_{S^* \mathcal{M}} \tilde{a} \mathfrak{g}_{IJ} \, d(\operatorname{Re} \sigma^{IJ}), \quad (8.5.8)$$

which is obtained from (8.5.7) by replacing a with $a \mathfrak{g}_{IJ}$. Setting $\sigma \equiv \mathfrak{g}_{IJ} \sigma^{IJ}$, we have

$$\int_{S^* \mathcal{M}} \left[g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\gamma} g^{\alpha\beta} \xi_\alpha \xi_\beta \partial_{\xi^\gamma} \tilde{a} \right] d\nu = -\langle g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \mathfrak{g}_{IJ} \nu^{IJ}, \tilde{a} \rangle - \langle \text{Re } \sigma, \tilde{a} \rangle.$$

Here, repeating the arguments in Section 8.4.1, by the parity of the measures involved, it is clear that we need only consider $\tilde{a}(x, \xi)$ to be odd in ξ , or equivalently, to let A in Lemmas 8.5.1 and 8.5.2 satisfy (8.4.4). Then, Lemma 8.5.1 shows that no contribution to $\text{Re } \sigma$ is made by the metric oscillations, H_ε . The contributions from Q_ε are non-trivial, as shown in the last part of Lemma 8.5.2. Since the second term in the right-hand side of (8.5.3) is imaginary, it follows from (8.5.6) that

$$\langle \text{Re } \sigma, a \rangle = -\langle g^{\alpha\beta} \xi_\alpha \partial_{x^\beta} \mathfrak{g}_{IJ} \nu^{IJ}, a \rangle + \langle \text{Re } \lambda, a \rangle.$$

Combining the previous two computations yields the result. □

Part II

Quasiconvexity and the curl-free Calculus of Variations

Chapter 9

Introduction

9.1 The non-convex Calculus of Variations

An old and important problem in the vectorial Calculus of Variations is to characterize the integrands $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ for which the functional

$$\mathcal{F}[u] \equiv \int_{\Omega} f(Du(x)) \, dx, \quad \text{where } u: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ and } m, n \geq 2,$$

is lower semicontinuous with respect to the weak topology in an appropriate Sobolev space; this is the natural condition for existence of minimizers through the Direct Method.

In his seminal work [285], MORREY recognized that the weak lower semicontinuity of \mathcal{F} is essentially equivalent to *quasiconvexity* of f , see Definition 9.1.2 below. Indeed, we have:

Theorem 9.1.1 (Lower semicontinuity). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, Lipschitz domain and $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$. Then \mathcal{F} is weak* sequentially lower semicontinuous in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ if and only if f is quasiconvex.*

We note that Theorem 9.1.1 has been extended to the case $p < \infty$, as well as to a variety of other settings, see for instance [1, 27, 45, 53, 147, 148, 245, 248, 271, 272, 281].

Theorem 9.1.1 shows that quasiconvexity is the fundamental convexity notion in the vectorial Calculus of Variations; interestingly, it is also closely related to the coercivity of integral functionals [82]. Moreover, the motivation for going beyond convex energies is also based on physical grounds. As pointed out by BALL in [28], convexity is an unacceptably strong assumption in many instances of physical interest arising in nonlinear elasticity, as it leads to uniqueness of minimisers and, more generally, of critical points. As soon as one relaxes convexity to polyconvexity, however, uniqueness of minimisers is completely lost [342], although see [360] and the references therein for some uniqueness results for critical points. In fact, there is an important difference between minimisers and critical points in the general quasiconvex setting: while strong local minimisers are regular outside a small set [127, 249, 253], general critical points may be very irregular [299, 353].

Nonetheless, and despite its prominence, quasiconvexity remains an elusive concept, and Part II of this thesis is dedicated to its study. We note that, in Part I, we considered quasiconvexity in the general \mathcal{A} -free setting, whilst here we will take $\mathcal{A} = \text{curl}$; in other words, the energy $\mathcal{F}[u]$ depends on the deformation $u: \Omega \rightarrow \mathbb{R}^m$ only through its gradient.

In this chapter we will give an extensive account of the quasiconvex Calculus of Variations, detailing in particular the relationship between quasiconvexity and rank-one convexity and the deep connections between quasiconvexity and other areas of analysis. Indeed, if one had an easy way of verifying whether a given function is quasiconvex then one would immediately solve a series of outstanding problems in Quasiconformal Analysis [207] and Homogenization Theory [283], see Sections 9.1.5 and 9.1.6 respectively. In Sections 9.2 and 9.3 we will also discuss the original results of Part II.

The results of Chapter 10 concern the geometry of the class of quasiconvex functions, in particular its extremal points, and are mostly based on my own work [176]. Nonetheless, some of the results in that chapter were also obtained jointly with FARACO, either unpublished work or [133].

In Chapter 11 we report on some numerical experiments done with RITA TEIXEIRA DA COSTA [185]. These experiments aimed to numerically find rank-one convex functions which are not quasiconvex.

9.1.1 Polyconvexity, quasiconvexity, rank-one convexity and duality

In this subsection we recall very standard definitions and results concerning semi-convexity notions, the associated envelopes, and the corresponding classes of probability measures. The reader may find a more detailed discussion in [98, 295, 328].

Definition 9.1.2. Let $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$. We say that f is:

- (i) *polyconvex*: if there is a convex function $g: \mathbb{R}^{\tau(m,n)} \rightarrow \mathbb{R}$ such that

$$f(A) = g(\mathbf{M}(A)),$$

where $\mathbf{M}(A)$ is the vector of minors of A and $\tau(m, n) \equiv \sum_{s=1}^{\min\{m,n\}} \binom{m}{s} \binom{n}{s}$ is its length.

- (ii) *quasiconvex* if for all $A \in \mathbb{R}^{m \times n}$ and all $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$ we have

$$f(A) \leq \int_{\Omega} f(A + D\varphi) \, dx,$$

where $\Omega \subset \mathbb{R}^n$ is a fixed bounded domain with $\mathcal{L}^n(\partial\Omega) = 0$.

- (iii) *rank-one convex* if, for all $A, X \in \mathbb{R}^{m \times n}$ such that $\text{rank } X = 1$,

$$t \mapsto f(A + tX) \text{ is convex over } \mathbb{R}.$$

When f is C^2 , rank-one convexity is equivalent to the *Legendre–Hadamard inequality*

$$D^2 f(A)[X, X] \geq 0 \quad \text{for all } A, X \in \mathbb{R}^{m \times n} \text{ such that } \text{rank } X = 1.$$

It is not difficult to see that

$$\text{convexity} \implies \text{polyconvexity} \implies \text{quasiconvexity} \implies \text{rank-one convexity}.$$

In general, none of the reverse implications hold, although the problem of whether rank-one convexity implies quasiconvexity, particularly in low dimensions, will occupy most of this section.

For each of the above classes of semi-convex functions, it is useful to consider the space of probability measures which satisfy Jensen's inequality with respect to all functions in that class:

Definition 9.1.3. Let $K \subset \mathbb{R}^{m \times n}$ be a compact set and recall that we write $\mathcal{M}_1^+(K)$ for the set of probability measures supported on K . For $\nu \in \mathcal{M}_1^+(K)$ we write $\bar{\nu} \equiv \langle \nu, \text{id} \rangle$ for its *barycentre*. Then

$$\begin{aligned} \mathcal{M}^{\text{pc}}(K) &\equiv \{\nu \in \mathcal{M}_1^+(K) : f(\bar{\nu}) \leq \langle \nu, f \rangle \text{ for all polyconvex } f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}\}, \\ \mathcal{M}^{\text{qc}}(K) &\equiv \{\nu \in \mathcal{M}_1^+(K) : f(\bar{\nu}) \leq \langle \nu, f \rangle \text{ for all quasiconvex } f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}\}, \\ \mathcal{M}^{\text{rc}}(K) &\equiv \{\nu \in \mathcal{M}_1^+(K) : f(\bar{\nu}) \leq \langle \nu, f \rangle \text{ for all rank-one convex } f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}\}. \end{aligned}$$

Similar notions are possible in the non-compact case, although one must impose growth conditions on f in that case, see [376] and also [132] and the references therein for related results. Since the above spaces of measures are in duality with the corresponding classes of semi-convex functions, we have

$$\mathcal{M}^{\text{rc}}(K) \subseteq \mathcal{M}^{\text{qc}}(K) \subseteq \mathcal{M}^{\text{pc}}(K).$$

There are more explicit characterisations of the measures in the first two classes. We begin with the characterisation of those measures in $\mathcal{M}^{\text{qc}}(K)$, which is due to KINDERLEHRER and PEDREGAL [227]. This characterisation relies on the notion of *gradient Young measures*, which are simply Young measures [373] which are generated by a sequence of gradients. For the definition of Young measures, as well as an introduction to their properties, we refer the reader to [251, 295], but see also Chapter 6. We then have:

Theorem 9.1.4. *A weak*-measurable map $\nu: \Omega \rightarrow \mathcal{M}(\mathbb{R}^{m \times n})$ is a $(W^{1,\infty})$ gradient Young measure if and only if $\nu_x \geq 0$ a.e. and there is a compact set $K \subset \mathbb{R}^{m \times n}$ and a map $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that:*

- (i) $\text{supp } \nu_x \subset K$ for a.e. x ;
- (ii) $\bar{\nu}_x = Du$ for a.e. x ;
- (iii) $f(\bar{\nu}_x) \leq \langle \nu_x, f \rangle$ for a.e. x and all quasiconvex functions $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$.

Theorem 9.1.4 shows that $\mathcal{M}^{\text{qc}}(K)$ is precisely the set of *homogeneous* gradient Young measures supported in K , i.e. of those gradient Young measures for which ν_x is constant a.e., see also [293, 374]. In the rest of this chapter all gradient Young measures will implicitly be assumed to be homogeneous. We also note that there is a version of Theorem 9.1.4 in the case $p < \infty$ [228, 244].

We now turn our attention to the measures in $\mathcal{M}^{\text{rc}}(K)$, which were characterised by PEDREGAL [311, 313]. To state his result, we briefly recall the notion of *laminates of finite order*: this is the class of probability measures which can be obtained through a finite iterative process in which, at each step, a Dirac measure is split into two Dirac measures along a rank-one line. The class of *laminates* is the weak*-closure of the class of laminates of finite order in the space of probability measures and we refer the reader to [295] for further details. We then have:

Theorem 9.1.5. *Let $K \subset \mathbb{R}^{m \times n}$ be a compact set. A probability measure $\nu \in \mathcal{M}(K)$ is a laminate if and only if $\nu \in \mathcal{M}^{\text{rc}}(K)$.*

To conclude this preliminary subsection we define the semiconvex envelope of a given function.

Definition 9.1.6. Let $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$. Then we set

$$\begin{aligned} f^{\text{pc}}(A) &\equiv \inf \{ g(A) : g \leq f \text{ is polyconvex} \}, \\ f^{\text{qc}}(A) &\equiv \inf \{ g(A) : g \leq f \text{ is quasiconvex} \}, \\ f^{\text{rc}}(A) &\equiv \inf \{ g(A) : g \leq f \text{ is rank-one convex} \}. \end{aligned}$$

Several representation formulas for the envelopes are available and can be found in [98]. We refer the reader to [33, 231] for sharp results concerning the regularity of the envelopes.

9.1.2 Morrey's problem

The rest of this chapter is devoted to a discussion of the following fundamental problem:

Question 9.1.7 (Morrey's problem). Does rank-one convexity imply quasiconvexity? Equivalently, are all compactly supported homogeneous gradient Young measures laminates?

It seems that MORREY himself was not sure about what the answer to Question 9.1.7 should be [285, 286]. Nonetheless, in general, the answer to Question 9.1.7 is *negative*, as attested by a fundamental example due to ŠVERÁK [351]:

Theorem 9.1.8. *Whenever $m \geq 3, n \geq 2$, there is a function $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ which is rank-one convex but not quasiconvex.*

A completely new example was recently found by GRABOVSKY in [168] when $m = 8, n = 2$; we will return to his example in Section 9.1.6 below. As ŠVERÁK's example plays such a fundamental role in the field, we sketch it here.

Proof sketch: It is easy to see that it suffices to prove the theorem when $m = 3, n = 2$. To obtain a counterexample we use a three-step strategy:

- (i) select a subspace L of $\mathbb{R}^{3 \times 2}$ which has few rank-one directions, so that it is easy to build rank-one convex functions over L ;
- (ii) find a solution $\varphi \in W_{\text{per}}^{1, \infty}((0, 1)^2, \mathbb{R}^3)$ of the differential inclusion $D\varphi \in L$;
- (iii) find a rank-one convex function $g: L \rightarrow \mathbb{R}$ which does not satisfy the quasiconvexity inequality with respect to φ .

Once the last step is done, it is relatively straightforward to use the methods from [348] to extend g to a rank-one convex, non-quasiconvex function $f: \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$.

The example can be described as follows: we take the subspace

$$L = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \\ z & z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}, \quad (9.1.1)$$

which only contains three rank-one directions, and we define $g: L \rightarrow \mathbb{R}$ by

$$g: \begin{bmatrix} x & 0 \\ 0 & y \\ z & z \end{bmatrix} \mapsto -xyz, \quad \varphi(x_1, x_2) \equiv \frac{1}{2\pi} \begin{bmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \\ \sin 2\pi(x_1 + x_2) \end{bmatrix}. \quad (9.1.2)$$

Then g is clearly rank-one convex on L and one checks that $\varphi \in W_{\text{per}}^{1,\infty}((0,1)^2, \mathbb{R}^3)$ is such that $D\varphi \in L$. Moreover, $\int_{(0,1)^2} g(D\varphi) = -\frac{1}{4} < 0$. \square

It is interesting to note that the function $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ obtained by ŠVERÁK is a polynomial of degree four and that no counterexample is possible for polynomials of smaller degree. An example related to ŠVERÁK's had appeared previously in the work of TARTAR [361].

ŠVERÁK's counterexample has deep consequences for the study of quasiconvexity. The first such consequence that we want to discuss here is due to KRISTENSEN, who proved the following [246, 247]:

Theorem 9.1.9 (Nonlocality). *A property P is said to be local if there is a local operator \mathcal{P} such that, for an arbitrary smooth function $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$, we have that*

$$f \text{ has property } P \iff \mathcal{P}(f) = 0;$$

moreover, the operator \mathcal{P} is local if

$$f = g \text{ in a neighbourhood of } A \implies \mathcal{P}(f) = \mathcal{P}(g) \text{ in a neighbourhood of } A.$$

Then:

- (i) *Polyconvexity is not a local property when $n \geq 2, m \geq 2$;*
- (ii) *Quasiconvexity is not a local property when $m \geq 3, n \geq 2$;*
- (iii) *Rank-one convexity is always a local property.*

Note that (iii) follows immediately from the definition. The deepest part of Theorem 9.1.9 is (ii), which is proved by a careful modification of ŠVERÁK's example. As polyconvexity has a very explicit characterisation, (i) shows that the non-locality of quasiconvexity is not per se enough to exclude the existence of a more explicit description of quasiconvex functions.

Another consequence of ŠVERÁK's example is that quasiconvexity is not invariant under transposition [296], see also [254]:

Theorem 9.1.10. *There is a quasiconvex function $f: \mathbb{R}^{2 \times 3} \rightarrow \mathbb{R}$ such that $\tilde{f}: \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$, defined by $\tilde{f}: A \mapsto f(A^T)$, is not quasiconvex.*

It is interesting to note that the proof in [296] is based on a weak continuity result for a trilinear quantity under a *non-constant rank* constraint, compare with the results and discussion from Part I.

Despite the above results, the case $n \geq m = 2$ remains open. Over the last decades, evidence towards a *positive answer* to Question 9.1.7, at least when $m = n = 2$, has been mounting, see e.g. [17, 18, 137, 187, 235, 294, 312, 314, 335]. The rest of this section consists of an overview of the existing literature. In Section 9.3 new numerical evidence will be discussed.

9.1.3 Morrey's problem in subspaces

In the previous subsection we saw how ŠVERÁK showed the existence of proper subspaces of $\mathbb{R}^{3 \times 2}$ which admit rank-one convex non-quasiconvex functions. Thus, if one wants to adapt his strategy to the lower-dimensional case, it is natural to consider the following problem:

Question 9.1.11 (Morrey's problem in subspaces). Let $\mathbb{V} \subset \mathbb{R}^{2 \times 2}$ be a proper subspace. Is there a gradient Young measure compactly supported on \mathbb{V} which is not a laminate?

We refer the reader to [265] for higher-dimensional results related to Question 9.1.11. Note that, in Question 9.1.11, the interesting cases are $\dim \mathbb{V} \in \{2, 3\}$. To tackle this question, we first need to have a classification of proper subspaces of $\mathbb{R}^{2 \times 2}$ up to subspaces with the same geometry of rank-one lines. More precisely:

Proposition 9.1.12. *Let \mathbb{V} be a subspace of $\mathbb{R}^{2 \times 2}$. We have the following possibilities:*

1. $\dim \mathbb{V} = 2$: (1.1) all directions in \mathbb{V} are rank-one directions; (1.2) only two are; (1.3) only one or less are.
2. $\dim \mathbb{V} = 3$: (2.1) \mathbb{V} is orthogonal to a rank-one line; (2.2) \mathbb{V} is not orthogonal to a rank-one line.

Then, up to a linear isomorphism preserving rank-one lines, the rank-one convex functions in \mathbb{V} are in one-to-one correspondence with:

- (1.1) convex functions in \mathbb{R}^2 ;
- (1.2) separately convex functions in \mathbb{R}^2 ;
- (1.3) degenerate case;
- (2.1) rank-one convex functions in the upper triangular matrices $\mathbb{R}_{\text{tri}}^{2 \times 2}$;
- (2.2) rank-one convex functions in $\mathbb{R}_{\text{sym}}^{2 \times 2}$.

We refer the reader to [92] for a proof of Proposition 9.1.12

Returning to Question 9.1.11, we begin by discarding the easier cases of the proposition. Standard results on differential inclusions show that, in cases (1.1) and (1.3), no counter-example is possible. Case (1.2) is much deeper and was addressed by MÜLLER in [294], see also [258]:

Theorem 9.1.13. *Every gradient Young measure compactly supported on $\mathbb{R}_{\text{diag}}^{n \times n}$ is a laminate.*

Theorem 9.1.13 is a direct consequence of a remarkable lower semicontinuity result for separately convex functions; in particular, this is a lower semicontinuity for \mathcal{A} -quasiconvex functions, where \mathcal{A} is an operator *without constant rank*, compare with the results and discussion from Part I. Combining Theorem 9.1.13 with a transformation introduced in [128], it was shown in [81] that certain nonlinear hypersurfaces do not support gradient Young measures which are not laminates. As another consequence of Theorem 9.1.13, in [187] the authors were able to tackle case (2.1):

Theorem 9.1.14. *Every gradient Young measure compactly supported on $\mathbb{R}_{\text{tri}}^{2 \times 2}$ is a laminate.*

Thus, apart from the case of symmetric matrices, Question 9.1.11 was solved. Nonetheless, the problem in symmetric matrices remains as an outstanding (and perhaps easier) version of Morrey's problem, so we state it here explicitly:

Question 9.1.15 (Morrey's problem on symmetric matrices). Is there a gradient Young measure supported in $\mathbb{R}_{\text{sym}}^{2 \times 2}$ which is not a laminate?

Gradient Young measures supported in the space of symmetric matrices are generated by sequences of *hessians*; this observation is due to ŠVERÁK [350] and can be proved using the Helmholtz decomposition:

Lemma 9.1.16. *Every homogeneous gradient Young measure supported in $\mathbb{R}_{\text{sym}}^{n \times n}$ is generated by a sequence $\nabla^2 u_j$ where, for any $p \in (1, \infty)$, the sequence $u_j \in W^{2,p}(\Omega)$ is bounded.*

We will return to quasiconvex functions in the symmetric matrices in the next section, but for now we just note that Lemma 9.1.16 motivates the following terminology: we say that a function $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ is *quasiconvex* if

$$f(A) \leq \int_{\Omega} f(A + \nabla^2 \varphi) dx$$

for all $\varphi \in W_0^{1,\infty}(\Omega)$. Note that this definition is consistent with the general definition of \mathcal{A} -quasiconvexity from Chapter 4. The general case of quasiconvexity for higher order derivatives was first considered by MEYERS [281] and, in this case, it has been known for a long time that the corresponding notion of rank-one convexity does not imply quasiconvexity [26].

We conclude this subsection by mentioning a problem which is related to Question 9.1.15:

Question 9.1.17 ([107]). Is every quasiconvex function $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ the restriction of a quasiconvex function $\tilde{f} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$?

In general, the problem of traces of quasiconvex or rank-one convex functions is not very well understood, although the reader may find some partial results in [107, 256].

9.1.4 Morrey's problem for sets

In this subsection we would like to address a variant of Morrey's problem for sets. In order to do so, we need to introduce the appropriate notions of semi-convex hulls:

Definition 9.1.18. Let $K \subset \mathbb{R}^{m \times n}$ be compact. For $\square \in \{\text{pc}, \text{qc}, \text{rc}\}$, set

$$K^{\square} \equiv \{\bar{\nu} : \nu \in \mathcal{M}^{\square}(K)\}.$$

K is *polyconvex* if $K^{\text{pc}} = K$ and likewise for quasiconvexity and rank-one convexity.

The above definition of the semi-convex hull of a set is *functional*, being based on separation of points: thus, for instance, $A \in K^{\text{rc}}$ if and only if $f(A) \leq \sup_K f$ for all rank-one convex functions. A consequence of our definition is that

$$K^{\square} = \{A : \text{dist}^{\square}(A, K) = 0\} \quad \text{for } \square \in \{\text{pc}, \text{qc}, \text{rc}\}, \quad (9.1.3)$$

recall also Definition 9.1.6. This functional definition should be contrasted with other *directional* notions of the hull, like the lamination convex hull, which are less robust, see [230, 277] for a detailed comparison.

We can now state the analogue of Question 9.1.7 for sets:

Question 9.1.19 (Morrey's problem for sets). Are all compact, rank-one convex sets quasiconvex?

We note that, due to (9.1.3), Question 9.1.19 can be rephrased as asking whether the zero sets of the functions $\text{dist}^{\text{qc}}(\cdot, K)$ and $\text{dist}^{\text{rc}}(\cdot, K)$ agree.

In general, the answer to Question 9.1.19 is negative: a complex version of ŠVERÁK's counterexample yields a rank-one convex non-quasiconvex set in $\mathbb{R}^{6 \times 2}$, see Example 9.1.22 below. Actually, this example can be improved: MILTON realised that ŠVERÁK's counterexample could be adapted to yield an example of a rank-one convex non-quasiconvex set in $\mathbb{R}^{3 \times 2}$, see [283, §31.8]. Since MILTON's example does not appear to be very well-known, we sketch it here.

Example 9.1.20. As described in [295], JAMES noticed that the deformation (9.1.2) can be replaced by

$$\tilde{\varphi}(x_1, x_2) \equiv \begin{bmatrix} s(x_1) \\ s(x_2) \\ s(x_1 + x_2) \end{bmatrix}, \quad \text{where } s \text{ is 1-periodic and } s(x) = \begin{cases} -\frac{1}{4} + x & x \in [0, \frac{1}{2}], \\ \frac{1}{4} - x & x \in [\frac{1}{2}, 1]. \end{cases}$$

For a Lipschitz map u we denote by $[Du]$ the essential range of Du . The deformation $\tilde{\varphi}$ is somewhat simpler than the one in (9.1.2) since $\#[D\tilde{\varphi}] = 8$. Moreover, it satisfies $D\tilde{\varphi} \in L$, where L is the subspace in

(9.1.1), and one checks that $\int_{[0,1]^2} g(D\tilde{\varphi}) = -\frac{1}{2} < 0$, where g is as in (9.1.2). Thus we recover ŠVERÁK's result.

Another example is obtained by replacing s with a different saw-tooth function: following [283], take

$$\psi(x_1, x_2) = \begin{bmatrix} h(x_1) \\ h(x_2) \\ h(x_1 + x_2 + 2) \end{bmatrix}, \quad \text{where } h \text{ is 4-periodic and } h(x) = \begin{cases} 3x & x \in [0, 1], \\ 4 - x & x \in [1, 4]. \end{cases}$$

The function ψ was chosen so that $\#[D\psi] = 7$. To see this, it is helpful to introduce the notation

$$X_1 \equiv e_1 \otimes e_2, \quad X_2 \equiv e_2 \otimes e_1, \quad X_3 \equiv e_3 \otimes (e_1 + e_2),$$

where e_i is the canonical basis in \mathbb{R}^n . Then it is easy to check that

$$[D\psi] = \bigcup_{\substack{\varepsilon_i \in \{-1, 3\} \\ (\varepsilon_1, \varepsilon_2, \varepsilon_3) \neq (3, 3, 3)}} \{\varepsilon_1 X_1 + \varepsilon_2 X_2 + \varepsilon_3 X_3\}.$$

Let ν be the gradient Young measure generated by $D\psi$. The analogue of the function g from (9.1.2) is the function

$$l: \begin{bmatrix} 0 & t_1 \\ t_2 & 0 \\ t_3 & t_3 \end{bmatrix} \mapsto (t_1 + 1)^2 (t_2 + 1)^2 (t_3 + 1)^2,$$

which is clearly rank-one convex. The choice of this function is easily explained: the point $3(X_1 + X_2 + X_3)$ is not in K and so we have that $l = 0$ on K . Thus $\langle \nu, l \rangle = 0 < 1 = l(0) = l(\bar{\nu})$, and so l is not quasiconvex.

The set $K \equiv [D\psi]$ is such that $K^{\text{rc}} \neq K^{\text{qc}}$. To see this, note that X_1, X_2, X_3 are the only rank-one directions in the space where l is defined and so

$$K^{\text{rc}} = \bigcup_{i=1}^3 \{t_1 X_1 + t_2 X_2 + t_3 X_3 : t_j = 1 \text{ and } t_j \in [-1, 3] \text{ for all } j\};$$

one can visualise K^{rc} as three faces of the cube with vertices $K \cup \{3X_1 + 3X_2 + 3X_3\}$. The gradient Young measure ν is supported in K but its barycentre $\bar{\nu} = 0 \in K^{\text{qc}}$ is not in K^{rc} .

The previous example settles the general case of Question 9.1.19. However, and in the spirit of Theorem 9.1.9, we may ask a weaker version of Question 9.1.19: are the semi-convexity properties of distance functions local? More precisely, let $K \subset \mathbb{R}^{n \times n}$ be compact and $(U_i)_{i=1}^l$ be pairwise-disjoint open sets; we would like to know whether the following holds:

$$\text{if } K \subset \bigcup_{i=1}^l U_i \text{ then } (K \cap U_i)^\square = K^\square \cap U_i \quad \text{for } \square \in \{\text{pc}, \text{qc}, \text{rc}\}. \quad (9.1.4)$$

For rank-one convexity, as expected, the answer is positive. In fact, KIRCHHEIM [230] proved the following strong form of (9.1.4), see also [277]:

Theorem 9.1.21 (Structure theorem). *For compact sets $B, K \subset \mathbb{R}^{m \times n}$,*

$$K^{\text{rc}} \cap B = [(B \cap K) \cup (\partial B \cap K^{\text{rc}})]^{\text{rc}} \cap B.$$

In particular, when $\square = \text{rc}$, (9.1.4) holds.

For polyconvexity and quasiconvexity, however, the situation is different, as one expects a negative answer in light of Theorem 9.1.9. For polyconvexity, when $n = 2$, an $\text{SO}(2)$ -invariant counterexample can be found in [90]. For quasiconvexity, the same holds in high-enough dimensions, and we have the following example, due to ŠVERÁK, which can be found in the lecture notes [295]:

Example 9.1.22. Consider the complex version of the subspace L from (9.1.1): let $L^{\mathbb{C}}$ be the span of the set

$$K = \left\{ \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \\ z_3 & z_3 \end{bmatrix} : z_i \in \mathbb{C}, |z_i| = 1, z_3 = z_1 z_2 \right\}$$

and take $\varphi(x_1, x_2) = (e^{ix_1}, e^{ix_2}, e^{i(x_1+x_2)})$, which satisfies $\nabla\varphi \in K$. Thus $\text{dist}_K^{\text{qc}}(0) = 0$ and so $0 \in K^{\text{qc}}$. In fact,

$$K^{\text{pc}} = K^{\text{qc}} = K \cup \{0\}.$$

To see this, let $g : L^{\mathbb{C}} \rightarrow \mathbb{R}$, be defined by

$$g(z_1, z_2, z_3) = |z_1 z_2 - z_3|^2 + |\bar{z}_2 z_3 - z_1|^2 + |z_3 \bar{z}_1 - z_2|^2$$

which vanishes precisely on $K \cup \{0\}$ and, under the identification $\mathbb{C}^{3 \times 2} \cong \mathbb{R}^{6 \times 2}$, is polyconvex. It is not hard to extend g to a function f which is still polyconvex but is strictly positive outside L , proving the claimed equality. Note, however, that $K = K^{\text{rc}}$ by Theorem 9.1.21, see also [354].

To summarize, (9.1.4) fails in high enough dimensions for $\square \in \{\text{pc}, \text{qc}\}$.

Let us now restrict our attention to the localisation problem for quasiconvexity in low dimensions, in particular when $n = 2$. Building on earlier work in [355, 356], see also [352, 375], FARACO and SZÉKELYHIDI proved in [137] the following groundbreaking result:

Theorem 9.1.23 (Locality of quasiconvex hulls). *When $\square = \text{qc}$ and $n = 2$, (9.1.4) holds.*

It is not known whether the analogue of the structure theorem also holds for quasiconvexity in the planar case, but other results about planar semi-convex hulls can be found in [90, 189].

Combining the proof of Theorem 9.1.23 with a stability result for the branch set of quasiregular maps, KIRCHHEIM and SZÉKELYHIDI proved in [235] the following:

Theorem 9.1.24. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $u : \Omega \rightarrow \mathbb{R}^2$ be a Lipschitz map. The rank-one convex envelope of the essential range of Du , denoted $[Du]^{\text{rc}}$, is connected. Thus, if $[Du]$ contains an isolated matrix A , then A is rank-one connected to $[Du]^{\text{rc}} \setminus \{A\}$.*

It is important to note that Theorem 9.1.24 fails completely in the higher-dimensional case: in [216] the authors build, for $m > 3, n \geq 2$, a Lipschitz map $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ which vanishes on the space $\mathbb{R}_-^n \equiv \{x_n < 0\}$ and such that $[Du|_{\mathbb{R}_+^n}]^{\text{pc}}$ contains no rank-one matrix.

To conclude this section, we mention that the numerical calculation of semiconvex hulls remains a remarkably difficult problem. Recent results in this direction, as well as a detailed bibliography on the subject, can be found in [11], but see also the foundational papers [155, 276, 277].

9.1.5 Morrey's problem and its relation to Quasiconformal Analysis

In the previous subsections we surveyed the existing literature on general versions of Morrey's problem. In this subsection we will instead focus on some particular rank-one convex functions, which establish deep connections between Quasiconformal Analysis and the Calculus of Variations. It is worth mentioning that the methods from Quasiconformal Analysis, and Geometric Function Theory more broadly, have already proved to be an effective tool for tackling problems concerning quasiconvexity, as the proofs of Theorem 9.1.23 and 9.1.24 exemplify.

Given a matrix $A \in \mathbb{R}^{2 \times 2}$ we will often identify it with the pair $(a^+, a^-) \in \mathbb{C}^2$, where a^\pm denote the conformal/anti-conformal parts of A . In particular, we have the linear isomorphism

$$A \equiv \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto ((x+w) + i(z-y), (x-w) + i(y+z)) \equiv (a^+, a^-),$$

which is particularly useful in light of the formulas $|A| = |a^+| + |a^-|$ and $\det A = |a^+|^2 - |a^-|^2$.

Before proceeding further, it is helpful to recall the definition of quasiregular maps:

Definition 9.1.25. A map $u \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ is K -quasiregular, for $K \in [1, \infty)$, if

$$|Du(x)|^n \leq K \det Du(x) \quad \text{a.e. in } \Omega;$$

here $|\cdot|$ denotes the operator norm. A K -quasiconformal map is a K -quasiregular homeomorphism.

For an introduction to quasiconformal mappings we refer the reader to the classical book [2], but see also [16, 210, 326] for more recent developments. The reader may also find a discussion of aspects of the theory of mappings of finite distortion, which are a generalisation of quasiconformal mappings, in Chapter 13.

An important problem in Quasiconformal Analysis is to determine the sharp integrability exponents of the derivative of a quasiconformal map. The following is a well-known conjecture, see e.g. [207]:

Conjecture 9.1.26. Let $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be K -quasiregular. Then $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ for all $p < \frac{nK}{K-1}$.

While Conjecture 9.1.26 remains open in general, in the planar case it was proved as a consequence of the celebrated area-distortion theorem of ASTALA [14]:

Theorem 9.1.27. Let $u: \mathbb{C} \rightarrow \mathbb{C}$ be K -quasiregular. Then $u \in W_{\text{loc}}^{1,p}(\mathbb{C})$ for all $p < \frac{2K}{K-1}$.

The exponent in Theorem 9.1.27 is the best possible, as the simple example $u(z) = z|z|^{1/K-1}$ shows. ASTALA's original proof is very involved but since then the main ideas have been distilled and the reader can find a relatively elementary proof in [16]. Nonetheless, establishing higher-dimensional analogues of Theorem 9.1.27 remains one of the main problems in the field. To explain the connection between Conjecture 9.1.26 and Morrey's problem we introduce the following functions:

Definition 9.1.28. For $p \in [\frac{n}{2}, \infty)$, the Burkholder function $B_p: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is the integrand

$$B_p(A) \equiv \left(\left| 1 - \frac{n}{p} \right| |A|^n - \det A \right) |A|^{p-n}.$$

Note that $B_n = -\det$. The Burkholder function was introduced by BURKHOLDER [71, 72] in the seemingly-unrelated context of martingale theory and the above generalisation is due to IWANIEC [207], who proved:

Theorem 9.1.29. Take an arbitrary $p \in [\frac{n}{2}, \infty)$.

- (i) B_p is rank-one convex. Moreover, when $n = 2$, B_p is convex in the direction of any matrix with non-positive determinant.
- (ii) If B_p is quasiconvex at zero then Conjecture 9.1.26 holds.

Theorem 9.1.29 shows the close relation between Morrey's problem and the sharp integrability exponent of quasiregular mappings. Despite the remarkable progress in [17], proving that B_p is quasiconvex at zero, even when $n = 2$, remains an outstanding open problem.

Let us comment in more detail on the proof of (ii), in particular when $n = 2$. In the planar theory of quasiconformal maps, a key role is played by the *Beurling–Ahlfors transform*, which is a singular integral operator that can be formally written as

$$\mathcal{S}u(z) = \frac{i}{2\pi} \iint_{\mathbb{C}} \frac{u(\zeta)}{(z - \zeta)^2} d\zeta \wedge d\bar{\zeta},$$

for $u \in L^p(\mathbb{C}, \mathbb{C})$. The importance of \mathcal{S} in the theory is readily justified by the formula

$$\frac{\partial}{\partial z} = \mathcal{S} \circ \frac{\partial}{\partial \bar{z}}. \quad (9.1.5)$$

Since \mathcal{S} is a Calderón–Zygmund operator, let us write $S_p \equiv \|\mathcal{S}\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})}$. The following is a well-known conjecture [205]:

Conjecture 9.1.30 (Iwaniec). $S_p = p^* - 1 \equiv \max\{p - 1, (p - 1)^{-1}\}$.

There are analogues of Conjecture 9.1.30 in higher dimensions, see e.g. [209], and see [214] for related conjectures. These analogues are related to the higher dimensional versions of Conjecture 9.1.26, but to state them precisely we would need to extend \mathcal{S} to act on differential forms. Since these extensions are not crucial for the point that we want to make here, we focus for simplicity on the case $n = 2$.

The next result yields the best available bounds towards Conjecture 9.1.30:

Theorem 9.1.31. $p^* - 1 \leq S_p \leq 1.575(p^* - 1)$.

The lower bound $p^* - 1 \leq S_p$ is very classical and goes back to the work of LEHTO [259], while the upper bound was proved in [38] through martingale techniques, see also the very interesting survey [37] and the references therein.

Note that Conjecture 9.1.30 is equivalent, by (9.1.5), to the beautiful inequality

$$\int_{\mathbb{C}} |u_z|^p dz \leq (p^* - 1)^p \int_{\mathbb{C}} |u_{\bar{z}}|^p dz.$$

Let us define the *Beurling function* $F_p: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ by $F_p(A) \equiv (p^* - 1)^p |a^-|^p - |a^+|^p$. We thus see that Conjecture 9.1.30 is equivalent to the quasiconvexity of F_p at zero. When $n = 2$, it is easy to see that

$$B_p(A) = \frac{2}{p^*} ((p^* - 1)|a^-| - |a^+|) |A|^{p-2} \quad (9.1.6)$$

and an elementary argument, c.f. [207, Lemma 8.1], shows that

$$\alpha_p B_p \leq F_p, \quad \alpha_p \equiv \frac{p p^*}{2} (1 - 1/p^*)^{p-1}. \quad (9.1.7)$$

Thus the quasiconvexity of B_p at zero implies Conjecture 9.1.30. This conjecture, in turn, implies Conjecture 9.1.26: this follows, for instance, from the role played by the constants S_p in Caccioppoli estimates, see [207] and the recent work [125]. This completes the proof sketch of Theorem 9.1.29(ii).

It is worth mentioning that the connection between B_p and F_p is deeper than the pointwise inequality (9.1.7) reveals. It is not difficult to see that the function F_p is not quasiconvex everywhere, since it is not even rank-one convex everywhere. However, a pleasant surprise happens when we compute its rank-one convex envelope: the Burkholder function appears!

Theorem 9.1.32. *For $p \in (1, 2]$, we have*

$$F_p^{\text{rc}}(A) = \begin{cases} F_p(A) & \text{if } (p^* - 1)|a^+| \leq |a^-|, \\ \alpha_p B_p(A) & \text{if } (p^* - 1)|a^+| \geq |a^-|. \end{cases}$$

A similar expression holds for $p \in [2, \infty)$, but the two branches are switched.

Theorem 9.1.32 follows from [370], but see also [18, 37].

We conclude this section by mentioning the relation between Burkholder's function and a seemingly innocent-looking function found by ŠVERÁK in [348]. He observed that, given two rank-one convex functions, their local maxima is also rank-one convex; thus, for instance, the function

$$L(A) \equiv \begin{cases} |a^+|^2 - |a^-|^2 & \text{if } |a^+| + |a^-| \leq 1, \\ 2|a^+| - 1 & \text{if } |a^+| + |a^-| \geq 1, \end{cases} \quad (9.1.8)$$

is easily seen to be rank-one convex. Moreover:

Proposition 9.1.33. *If L is quasiconvex at zero, then B_p is quasiconvex at zero.*

Proof: One can check that, for $p \in (1, 2)$,

$$\int_0^\infty t^{p-1} L\left(\frac{A}{t}\right) dt = c_p B_p(A), \quad c_p \equiv \frac{p(2-p)}{2} \alpha_p,$$

with a similar expression for $p \in (2, \infty)$; see [21] for details. The conclusion follows. \square

The function L also plays an important role in the proof of Theorem 9.1.23, see Example 10.3.3.

9.1.6 Morrey's problem and its relation to Homogenization Theory

In this section we explore the connection between the quasiconvex Calculus of Variations and the theory of homogenization. The basic aim of homogenization theory is to describe the macroscopic behaviour of a system which is heterogeneous at a small scale. A concise overview of periodic, quadratic homogenization under general constant rank constraints can be found in Chapter 7, but we also refer the reader to the books [7, 59].

Before proceeding further, and to fix ideas, we state a fundamental result on the homogenization of a certain class of integral functionals. The next theorem was proved independently by BRAIDES and MÜLLER [57, 288]:

Theorem 9.1.34. *Consider a continuous energy density $f: \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ which satisfies, for some $p \in (1, \infty)$, the conditions*

$$c|A|^p \leq f(y, A) \leq C(1 + |A|^p), \quad (9.1.9)$$

$$|f(y, A) - f(y, B)| \leq L(1 + |A|^{p-1} + |B|^{p-1})|A - B|. \quad (9.1.10)$$

Moreover, define the homogenized density

$$f_{\text{hom}}(A) \equiv \inf_{k \in \mathbb{N}} \inf_{\varphi \in W_0^{1,p}(k[0,1]^n)} \int_{k[0,1]^n} f(y, A + D\varphi) \, dy. \quad (9.1.11)$$

Then we have

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon = \mathcal{F}_{\text{hom}},$$

where $\mathcal{F}_\varepsilon[u] \equiv \int_\Omega f(\frac{x}{\varepsilon}, Du(x)) \, dx$ and $\mathcal{F}_{\text{hom}}[u] \equiv \int_\Omega f_{\text{hom}}(Du(x)) \, dx$.

Note that (9.1.10) follows from (9.1.9) whenever f is separately convex, c.f. Lemma 5.1.2. The precise definition of Γ -convergence does not play an important role in our discussion and so we do not give it here, instead referring the reader to the monograph [106] for a detailed treatment. Whenever f is convex in the last variable the representation formula (9.1.11) for the homogenized integrand is simpler as, in this case, it suffices to solve a single cell problem:

$$f_{\text{hom}}(A) = \inf_{\varphi \in W_{\text{per}}^{1,p}([0,1]^n)} \int_{[0,1]^n} f(y, A + D\varphi) \, dy, \quad (9.1.12)$$

see e.g. [270]. In the general non-convex case the situation is much more complicated and the formula (9.1.11) is optimal, in the sense that one really needs to solve an infinite ensemble of cell problems to determine f_{hom} , see [40, 288].

The homogenized integrand, as defined in (9.1.11), is quasiconvex. An interesting result, due to BRAIDES [58], is that when $f(x, \cdot)$ is polyconvex for every x it is not necessarily the case that f_{hom} is itself polyconvex. In order to explain BRAIDES' example, we note that a polyconvex function with subquadratic growth must be convex. However, this is not the case for quasiconvex functions, as noticed by ŠVERÁK in [349]. In particular, he gave an example of a quasiconvex non-polyconvex function $f: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$, and other examples can be found in [5, 291, 374]. The idea of the example in [58] is precisely to force the (non-convex) integrand f_{hom} to have subquadratic growth, thus making it non-polyconvex. Nonetheless, we note that a different example was given more recently in [39].

The previous paragraph shows that there is a class of integrands which lies somewhere between the polyconvex and quasiconvex classes; thus we arrive at the following problem, c.f. [59, Remark 18.2]:

Question 9.1.35. What is the class of quasiconvex integrands that can be obtained from polyconvex integrands through homogenization, as in Theorem 9.1.34? Can all quasiconvex integrands be obtained in this way?

It is worthwhile mentioning that it is not just polyconvexity that can be lost in the process of homogenization and that the same can happen to the strong ellipticity of the integrands [161].

Theorem 9.1.34 deals with very general non-convex integrands, but the particular case where $\Omega = \mathbb{T}^n$ and the integrands are convex and quadratic is already extremely rich. To see this, let us take

$$f(x, A) \equiv \langle S(x)A, A \rangle$$

for some $S \in L^\infty(\mathbb{T}^n, \text{Sym}^+(\mathbb{R}^{m \times n}))$; thus $S(x)$ is, for a.e. x , a symmetric, positive definite operator on $\mathbb{R}^{m \times n}$. In this case, formula (9.1.12) applies and

$$f_{\text{hom}}(A) \equiv \langle S^*A, A \rangle$$

for some $S^* \in \text{Sym}^+(\mathbb{R}^{m \times n})$; we call S^* the *effective tensor* associated to S . Suppose now that we constrain S to take values in some compact set $\mathbb{M} \subset \text{Sym}^+(\mathbb{R}^{m \times n})$. We can then define the *G-closure* of \mathbb{M} , denoted $\mathbb{G}(\mathbb{M})$, as the set of all effective tensors S^* obtained from a tensor-field $S \in L^\infty(\mathbb{T}^n, \mathbb{M})$ as above. There is a corresponding notion of *lamination-closure*, abbreviated as *L-closure* and denoted by $\mathbb{L}(\mathbb{M})$, where we additionally require S to be a laminate, see Chapter 7 for the precise definition. The standard representation formulas for the quasiconvex and rank-one convex envelopes then yield the following result [283]:

Theorem 9.1.36. *Take a compact subset $\mathbb{M} \subset \text{Sym}^+(\mathbb{R}^{m \times n})$ and define $f_{\mathbb{M}}: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ by*

$$f_{\mathbb{M}}(A) \equiv \min_{S \in \mathbb{M}} \langle SA, A \rangle.$$

Then $f_{\mathbb{M}}^{\text{qc}} = f_{\mathbb{G}(\mathbb{M})}$ and $f_{\mathbb{M}}^{\text{rc}} = f_{\mathbb{L}(\mathbb{M})}$.

For a proof of Theorem 9.1.36, as well as further discussion, see Chapter 7. Theorem 9.1.36 shows that calculating the G-closure of a set corresponds to determining the quasiconvex envelope of an associated function. Calculating the G-closure is remarkably difficult, even for simple sets, see for instance the detailed discussion in [7], and the G-closure problem remains an outstanding problem in Homogenization.

A non-generic yet relevant case where a theory for calculating the G-closure exists is when $\mathbb{G}(\mathbb{M})$ is a manifold of positive codimension; such a manifold is called an *exact relation*. The theory of exact relations is by now very rich (see [168]) and it is outlined in Chapter 7. In a recent breakthrough, GRABOVSKY [167] used this theory to produce a completely new counterexample to Morrey's problem:

Theorem 9.1.37. *When $m = 8$ and $n = 2$ there is a compact, $\text{SO}(2)$ -invariant manifold of positive codimension $\mathbb{G} \subset \text{Sym}^+(\mathbb{R}^{8 \times 2})$ which is L-closed but not G-closed. The function $f_{\mathbb{G}}: \mathbb{R}^{8 \times 2} \rightarrow \mathbb{R}$ is $\text{SO}(2)$ -invariant, 2-homogeneous, rank-one convex, and non-quasiconvex.*

The example of Theorem 9.1.37 is analytically much more involved than ŠVERÁK's example, but has the advantage of having additional structure. We also note, in relation to the results of [233], that $f_{\mathbb{G}}$ is non-negative and vanishes on the rank-one cone.

More recently, in collaboration with some of his students, GRABOVSKY showed in [169] that the strategy used to prove Theorem 9.1.37 does not yield a counterexample to Morrey's problem in the planar case:

Theorem 9.1.38. *When $m = n = 2$ there is no counterexample of the type of Theorem 9.1.37.*

In fact, Theorem 9.1.38 follows from the classification of all poly-crystalline (that is, rotationally-invariant) exact relations for $m = n = 2$, which was obtained in [169].

9.2 Extremal rank-one convex integrands

In the previous section we discussed many interesting properties of the class of quasiconvex integrands, which as we saw remains poorly understood. In this section we present some original results concerning the geometry of this class, in particular its extremal points. These results are proved in Chapter 10 and are mostly contained in [176].

In [350], ŠVERÁK introduced new quasiconvex integrands (see also [138] for generalisations) which, for any $n \times n$ symmetric matrix A , are defined by

$$F_k(A) \equiv \begin{cases} |\det A| & A \text{ has index } k, \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 0, \dots, n$; we recall that the *index* of a matrix is the number of its negative eigenvalues. We also note that the integrand F_0 is sometimes called \det^{++} in the literature, since its support is the set of positive definite matrices. These integrands have played an important role in studying other problems related to the Calculus of Variations, for instance in building counterexamples to the regularity of elliptic systems [299] or in the computation of rank-one convex hulls of compact sets [355].

In order to understand ŠVERÁK's motivation for considering these integrands it is worth making a small excursion into classical convex analysis. Given a real vector space \mathbb{V} and a convex set $K \subset \mathbb{V}$, one can define the set of *extreme points* of K as the set of points which are not contained in any open line segment contained in K . In general the set of extreme points might be very small: this is what happens, for instance, when the set is a convex cone $C \subset \mathbb{V}$, since in this situation all non-zero vectors are contained in a ray through zero. However, if we can find a convex base B for C , then we note that such a ray corresponds to a unique point in B . If this is an extreme point of B then we say that we have an *extremal ray*.

We are interested in the extremal rays of the cone C of rank-one convex integrands. This cone has the inconvenient feature that it is not *line-free*: there is a set of elements $v \in \mathbb{V}$ such that, for any $c \in C$ and any $t \in \mathbb{R}$, the point $c + tv$ is in C ; this set is precisely $C \cap (-C)$. In turn, it is quite clear that this is the set of rank-one affine integrands. A reasonable way of disposing of rank-one affine integrands is by demanding non-negativity from all integrands from C . This leads us to the definition of extremality considered by ŠVERÁK: we say that a non-negative rank-one convex integrand F is *extremal* if, whenever we have $F = E_1 + E_2$ for E_1, E_2 non-negative rank-one convex integrands, then each E_i is a non-negative multiple of F . A weaker notion of extremality was introduced by MILTON in [282] for the case of quadratic forms (see also [188]) but we shall not discuss it further here.

Let us now explain the relation between ŠVERÁK's integrands and extremality. In [350] ŠVERÁK observed that the polyconvex integrand $|\det|: \mathbb{R}^{n \times n} \rightarrow [0, \infty)$ is not extremal, since $|\det| = \det^+ + \det^-$, where as usual

$$\det^+ \equiv \max(0, \det), \quad \det^- \equiv \max(0, -\det),$$

which are also polyconvex. He also observed that \det^\pm are not extremal in $\mathbb{R}_{\text{sym}}^{n \times n}$, since

$$\det^+ = \sum_{k \text{ is even}} F_k \text{ and } \det^- = \sum_{k \text{ is odd}} F_k \quad \text{in } \mathbb{R}_{\text{sym}}^{n \times n}.$$

However, he conjectured that \det^\pm are extremal in $\mathbb{R}^{n \times n}$ and also that each F_k is extremal in $\mathbb{R}_{\text{sym}}^{n \times n}$. The first original result of Chapter 10 gives an affirmative answer to both conjectures¹:

Theorem 9.A. *Given a minor $M: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, let M^\pm be its positive and negative parts. Then M^\pm are extremal non-negative rank-one convex integrands in $\mathbb{R}^{n \times n}$. Moreover, for $k = 0, \dots, n$, the integrands $F_k: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$, are extremal non-negative rank-one convex integrands in $\mathbb{R}_{\text{sym}}^{n \times n}$.*

As a main tool we use the fact that, on a connected open set, a rank-one affine integrand is an affine combination of minors; this is proved by localizing the arguments from BALL–CURRIE–OLVER [26] concerning the classification of Null Lagrangians.

The integrand L , introduced by ŠVERÁK in [348], is another example of an extremal integrand. This example was found jointly with FARACO.

Theorem 9.B. *The integrand $L + 1: \mathbb{R}^{2 \times 2} \rightarrow [0, \infty)$, where L is as in (9.1.8), is extremal.*

¹In fact, ŠVERÁK only conjectures extremality in the cone of quasiconvex integrands, so our results are in this sense slightly stronger than his conjecture.

The importance of extreme points in convex analysis has to do with the Krein–Milman theorem, which states that the closed convex hull of the set of extreme points of a compact, convex subset K of a locally convex vector space is the whole set K —informally, this means that the set of extreme points is a set of “minimal information” needed to recover K . However, KLEE [236] showed that the Krein–Milman theorem is generically true for trivial reasons: if we fix an infinite-dimensional Banach space and we consider the space of its compact, convex subsets (which we can equip with the Hausdorff distance so that it becomes a complete metric space), then for almost every compact convex set K its extreme points are dense in K . Here we mean “almost every” in the sense that the previous statement is false only in a meagre set. Despite this disconcerting result, the situation can be somewhat remedied with the help of Choquet theory, which roughly states that, under reasonable assumptions, an arbitrary point in K can be represented by a measure carried in the set of its extreme points. For precise statements and much more information concerning Choquet theory we refer the reader to the lecture notes [315] or to the monograph [266] and for Krein–Milman-type theorems for semi-convexity notions see [230, 255, 277, 377].

Theorems 9.A and 9.B can be interpreted in light of Choquet theory. As we saw in Section 9.1, Morrey’s problem is equivalent to a dual problem, which is the problem of deciding whether the class of homogeneous gradient Young measures and the class of laminates are the same. Fix a compactly supported Radon probability measure ν on $\mathbb{R}^{n \times n}$; for simplicity we assume that ν has support contained in the interior of the cube $Q \equiv \prod_{i=1}^n [0, 1] \subset \mathbb{R}^{n^2} \cong \mathbb{R}^{n \times n}$. In order to decide whether ν is a laminate, we can resort to PEDREGAL [311] theorem, Theorem 9.1.5, which states that ν is a laminate if and only if Jensen’s inequality holds for any rank-one convex integrand. Since the class of rank-one convex integrands is rather large, the problem of deciding whether a measure is a laminate is in general very hard. However, it follows from Choquet theory that one does not need to test ν against all rank-one convex integrands, it being sufficient to test it in a strictly smaller class:

Theorem 9.C (Choquet theory for rank-one convex integrands). *Let $A_1, \dots, A_{2^{n^2}}$ be the vertices of the cube Q . A Radon probability measure ν supported on the interior of Q is a laminate if and only if*

$$g(\bar{\nu}) \leq \langle \nu, g \rangle$$

for all integrands g which are extreme points of the convex set

$$\left\{ f: Q \rightarrow [0, \infty) : \sum_{i=1}^{2^{n \times n}} f(A_i) = 1, f \text{ is rank-one convex} \right\}.$$

We note that the summation condition is simply a normalization which corresponds to fixing a base of the cone of non-negative rank-one convex integrands on Q .

Our interest in extremal integrands was ignited by the work [18] of ASTALA–IWANIEC–PRAUSE–SAKSMAN, which is based on an observation of SIVALOGANATHAN [341], see also [34]. In [18] it was shown that Burkholder’s function is extreme in the class of homogeneous, isotropic, rank-one convex integrands; in fact, this integrand is also the least integrand in this class, in the sense that no other element of the class is below it at all points. The relevance of this fact is readily seen: from standard results about quasiconvex envelopes it follows immediately that the Burkholder function is either quasiconvex everywhere or quasiconvex nowhere. As discussed in Section 9.1.5, this remarkable function is a bridge between Morrey’s problem and important problems in Geometric Function Theory.

9.3 Numerical evidence towards a positive answer to Morrey’s problem

Recall that Morrey’s problem remains open in the case of two-dimensional targets. In Section 9.1 we discussed partial evidence which suggests that the answer to Morrey’s problem might be positive in this case. However, and despite the remarkable progress in the last decades, it is by no means clear that the answer should be positive even in low dimensions.

Since the analytic study of quasiconvexity remains incredibly challenging, it is natural to look for numerical evidence instead. In earlier attempts to do so in [99, 100, 173], the strategy is to fix a rank-one convex function f and look for deformations $u: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that f does not satisfy Jensen's inequality with respect to Du . A major shortfall in this approach is that explicit rank-one convex non-polyconvex functions are rare and the available examples are relatively simple and have many symmetries.

In Chapter 11, which is based on joint work with TEIXEIRA DA COSTA [185], we take a far more general approach: our candidate rank-one functions are the rank-one envelopes of random functions. This approach enables us to cover a significantly larger portion of the space of rank-one convex functions. To be concrete, in line with the ideas from [351], we fix a Lipschitz deformation $u: \mathbb{T}^n \rightarrow \mathbb{R}^m$ whose gradient has finite image. We check if Jensen's inequality with respect to Du is falsified by the rank-one convex envelopes of functions of the form

$$f(A) = \begin{cases} g(A) & A \in [Du]. \\ 2 & \text{otherwise,} \end{cases}$$

where $[Du]$ denotes the essential range of $Du \in L^\infty(\Omega, \mathbb{R}^{m \times n})$ and $g: [Du] \rightarrow [-1, 1]$ is any function. Note that for a deformation such that $[Du]$ is finite the task of looking for counterexamples is a finite-dimensional problem. As a small technical remark we note that it is important that f only takes finite values; it is easy to build examples of rank-one convex non-quasiconvex functions if the value $+\infty$ is allowed [25], see also [152, 229].

We consider random deformations given by the sum of N plane waves for $N \in \{3, 4, 5\}$, see Section 11.2 for further details. The cases $N = 1, 2$ are not interesting and for $N \geq 6$ we already have that $\#[Du] \geq 64$, so the space of functions $g: [Du] \rightarrow [-1, 1]$ becomes very high-dimensional. Considering such deformations is not very restrictive: in fact, arbitrary deformations can be approximated by sums of plane waves. Moreover, JAMES's interpretation of ŠVERÁK's example shows that, when $m = 3$, there is already a counterexample for $N = 3$, see also Example 9.1.20.

Our findings can be summarised as follows. When $m \geq 3$, our approach finds many potential counterexamples, similar to the ones in [351]. When $m = 2$, and despite sampling thousands of different deformations, none were found. This suggests that, when $m = 2$, rank-one convexity and quasiconvexity may be equivalent. We also observe that on average it is easier to check that a given deformation does not yield a counterexample to Question 9.1.7 as N increases.

We provide a basic heuristic explanation of our findings: for plane wave expansions, the rank-one geometry of the set $[Du]$ is drastically different in the cases $m = 3$ and $m = 2$ and, in the latter, the geometry becomes much richer as N increases. Our considerations are inspired by the very interesting results of SEBESTYÉN-SZÉKELYHIDI [335], where the authors tap into this structure to prove that no counterexamples arise when $m = 2$ and $N = 3$, see also [314].

We conclude this introduction by discussing the algorithm we use to look for counterexamples to Question 9.1.7. By homogenization, the gradient of a Lipschitz deformation $u: \mathbb{T}^n \rightarrow \mathbb{R}^m$ generates a gradient Young measure, which has finite support if $[Du]$ is finite; thus our goal is to determine whether this measure is a laminate. Hence, we are naturally led to consider:

Question 9.3.1 ([234]). Is there an effective algorithm to decide whether a given probability measure supported on a finite subset of $\mathbb{R}^{m \times n}$ is a laminate?

Deciding whether a given measure is a laminate is difficult, as in principle one has to test Jensen's inequality with all rank-one convex functions, as per Theorem 9.1.5. One possible way of circumventing this issue is to consider just the extremal rank-one convex functions, as discussed in Section 9.2; however, and despite the results presented there, the general structure of these functions remains unclear. A different approach is to use a discretized version of the KOHN-STRANG algorithm [239] and in Section 11.1 we show that this yields a partially satisfying answer to Question 9.3.1. We rely on the convergence of approximations to the rank-one convex envelope, which were proved in [42, 123, 124, 303], see also [378] for particular examples. We remark that the related problem of calculating the rank-one convex hull of a set still remains poorly understood, see [11] and the references therein.

Chapter 10

Extremal rank-one convex integrands

In this chapter we study extremal rank-one convex integrands in relation to Morrey's problem. Most of the results presented here can be found in [176], with two notable exceptions: Section 10.3 is based on unpublished work carried jointly with DANIEL FARACO and Section 10.5 is based on the paper [133], written also with DANIEL FARACO.

Section 10.1 contains rigidity results for homogeneous rank-one convex functions which vanish on isotropic cones. In Sections 10.2 and 10.3 we give multiple examples of extremal rank-one convex integrands. Finally, in Section 10.4, we show that, in order to decide whether a given probability measure is a laminate, it is enough to verify Jensen's inequality in the class of extremal non-negative rank-one convex integrands. Finally, Section 10.5 contains a short proof of Ornstein's non-inequality in $\mathbb{R}^{2 \times 2}$.

10.1 Homogeneity properties of rank-one convex integrands

In this section we discuss homogeneity properties of rank-one convex integrands which vanish on cones, both in two and in higher dimensions. We are interested in the family of isotropic cones of aperture $a \geq 1$,

$$\mathfrak{C}_a \equiv \{(z, w) \in \mathbb{C}^2 : a|z| = |w|\},$$

motivated by the fact that the Burkholder function B_p vanishes on \mathfrak{C}_{p^*-1} , c.f. Definition 9.1.28 and (9.1.6). When $a = 1$ we have $\mathfrak{C}_1 = \{\det = 0\}$, which of course can be defined in any dimension.

Lemma 10.1.1. *Let $E: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ be rank-one convex and assume that, for some $a \geq 1$, E is non-positive on \mathfrak{C}_a . Define*

$$h_a(t, k) = \frac{1(a-k)[t(k-1)(a-1) - (a+1)(k+1)]}{t(a+k)[t(k-1)(a+1) - (a-1)(k+1)]}$$

and let $A = (z, w)$ be such that $k \equiv |w|/|z| \leq 1$. Then, for $t \geq 1$,

$$E(A) \leq h_a(t, k)E(tA).$$

Proof: Let us write, for real numbers $x, y \in \mathbb{R}$, $(x, y) \equiv (xz/|z|, yw/|w|) \in \mathbb{C} \times \mathbb{C}$, so $A = (|z|, k|z|)$. We fix $t > 1$ since when $t = 1$ there is nothing to prove. Let us define the auxiliary points

$$A_1 = \frac{|z|}{2}(1+k+t-kt, 1+k-t+kt),$$

$$B_1 = \frac{|z|(1+k)}{a+1}(1, a), \quad B_2 = \frac{t|z|(1-k)}{a+1}(1, -a).$$

Simple calculations show that

$$A = \lambda_1 A_1 + (1 - \lambda_1) B_1, \quad \lambda_1 = \frac{2(a-k)}{(a-1)(k+1) - t(a+1)(k-1)}$$

$$A_1 = \lambda_2(tA) + (1 - \lambda_2)B_2, \quad \lambda_2 = \frac{1 + k + tk - t + a(1 + k + t - kt)}{2(a + k)t}$$

and it is easy to verify that $B_1 - A_1$ and $B_2 - tA$ are rank-one directions. One also needs to verify that $\lambda_1, \lambda_2 \in [0, 1]$, which is a lengthy but elementary calculation using the fact that $0 \leq k \leq 1 \leq a$.

Observe that $B_1, B_2 \in \mathfrak{C}_a$ and so $E(B_1) = E(B_2) \leq 0$. Therefore, from rank-one convexity, we have

$$E(A) \leq (1 - \lambda_1)E(B_1) + \lambda_1(1 - \lambda_2)E(B_2) + \lambda_1\lambda_2E(tA) \leq \lambda_1\lambda_2E(tA)$$

and a simple calculation shows that $\lambda_1\lambda_2 = h_a(t, k)$. \square

We now specialise the lemma to two important situations, in which one can say more. Let us first assume that $k = 0$.

Proposition 10.1.2. *Let $E: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ be rank-one convex, positively p -homogeneous for some $p \geq 1$ and not identically zero. If there is some $a \geq 1$ such that $E = 0$ on \mathfrak{C}_a then $p \geq \frac{1}{a} + 1$.*

We note that this inequality is sharp: indeed, the zero set of the Burkholder function B_p is \mathfrak{C}_{p^*-1} and so, when $1 < p < 2$, we have

$$a = p^* - 1 = \frac{1}{p-1} \Leftrightarrow p = \frac{1}{a} + 1.$$

Thus we can reinterpret this proposition as saying that, for $1 < p < 2$, the Burkholder function has the least possible order of homogeneity of rank-one convex integrands which vanish on \mathfrak{C}_{p^*-1} .

Proof: In this proof we assume that $a > 1$, since the case $a = 1$ follows from Lemma 10.1.3 below. We claim that there is some $z \in \mathbb{C}$ such that $E(z, 0) > 0$. Once this is shown, the conclusion follows easily: take $k = 0$ in Lemma 10.1.1 to find that $E(A) \leq F_a(t)E(A)$ where

$$F_a(t) = t^{p-1} \frac{at + a - t + 1}{at + a + t - 1}$$

and $A = (z, 0)$. Since $E(A) > 0$, we must have $F_a(t) \geq 1$ for all $t \geq 1$. Moreover, $F_a(1) = 1$ and an elementary computation reveals that

$$\left. \frac{dF_a}{dt} \right|_{t=1} = p - 1 - \frac{1}{a}$$

which is non-negative precisely when $p \geq \frac{1}{a} + 1$.

To finish the proof it suffices to prove the claim. Take an arbitrary $z \in S^1$ and take any rank-one line segment starting at $(z, 0)$ and having the other end-point in \mathfrak{C}_a ; such a line must intersect \mathfrak{C}_1 , since $a > 1$, say at P_z . Note that $E(P_z) \geq 0$, since the function $t \mapsto E(tP_z) = t^p E(P_z)$ is convex. We conclude that $E(z, 0) \geq 0$ with equality if and only if $E(P_z) = 0$, in which case E is identically zero along the entire rank-one line segment.

To prove the claim we want to show that if $E(z, 0) = 0$ for all $z \in \mathbb{C}$ then E is identically zero, so let us make this assumption. Then, from the previous discussion, we see that E is identically zero in the “outside” of \mathfrak{C}_a , i.e. in

$$\mathfrak{C}_a^+ \equiv \{(z, w) \in \mathbb{C} \times \mathbb{C} : a|z| > |w|\}.$$

Moreover, given any point P in the interior of \mathfrak{C}_a , there is a rank-one line segment through P with both endpoints, say P_1, P_2 , in \mathfrak{C}_a^+ ; this is the case because we assume $a > 1$. But E is zero in a neighbourhood of P_i and since it is convex along the rank-one line segment $[P_1, P_2]$ we conclude that it is also zero at P . \square

We remark that, in one dimension, the only homogeneous extreme convex integrands are linear (c.f. Proposition 10.4.3) while, from the results of Section 10.2, for $n > 1$ there are extremal rank-one convex integrands in $\mathbb{R}^{n \times n}$ which are positively k -homogeneous for any $k \in \{1, \dots, n\}$. It would be interesting to

know whether there are extremal homogeneous integrands with other degrees of homogeneity, or whether there is an upper bound for the order of homogeneity of such integrands.

If we set $a = 1$ in Lemma 10.1.1, so $\mathfrak{C}_1 = \{\det = 0\}$, we see that $h_1(t, k) = t^{-2}$ and we find the estimate $t^2 E(A) \leq E(tA)$ for $t \geq 1$. In fact, this holds in any dimension, and the proof is a simple variant of the proof of Lemma 10.1.1.

Lemma 10.1.3. *Let $E: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be a rank-one convex integrand which is non-positive on the cone $\{\det = 0\}$. Then for all $A \in \mathbb{R}^{n \times n}$, E satisfies*

$$\begin{aligned} t^n E(A) &\leq E(tA) && \text{if } 1 \leq t. \\ t^n E(A) &\geq E(tA) && \text{if } 0 < t \leq 1. \end{aligned}$$

Proof: Let us begin by observing that the second inequality follows from the first. Indeed, given a matrix $A \in \mathbb{R}^{n \times n}$ and $0 < t < 1$, let $B \equiv tA$ and apply the first inequality to B to get

$$E(tA) = E(B) = t^n \frac{1}{t^n} E(B) \leq t^n E\left(\frac{1}{t}B\right) = t^n E(A);$$

this can be done since $1 < \frac{1}{t}$. Hence we shall prove only the first inequality.

We begin by proving the statement in the case where A is diagonal, so there are real numbers σ_j such that $A = \text{diag}(\sigma_1, \dots, \sigma_n)$. Let $A_0 \equiv A$ and define, for $1 \leq j \leq n$,

$$A_j = \text{diag}(t\sigma_1, \dots, t\sigma_j, \sigma_{j+1}, \dots, \sigma_n) \quad B_j = \text{diag}(t\sigma_1, \dots, t\sigma_{j-1}, 0, \sigma_{j+1}, \dots, \sigma_n).$$

Hence, for any $1 \leq j \leq n$,

$$A_{j-1} = \frac{1}{t}A_j + \frac{t-1}{t}B_j. \quad (10.1.1)$$

This is a splitting of A_{j-1} since we are assuming that $t > 1$ and also

$$A_j - B_j = \text{diag}(0, \dots, 0, t\sigma_j, 0, \dots, 0),$$

which is a rank-one matrix. Iterating (10.1.1) we find

$$A_0 = \frac{1}{t^n}A_n + \sum_{j=1}^n \frac{t-1}{t^j}B_j \quad (10.1.2)$$

and by construction this is a prelaminate. Thus rank-one convexity of E yields

$$E(A) = E(A_0) \leq \frac{1}{t^n}E(A_n) + \sum_{j=1}^n \frac{t-1}{t^j}E(B_j) \leq \frac{1}{t^n}E(tA) \quad (10.1.3)$$

since $\det(B_j) = 0$ for all $1 \leq j \leq n$, hence $E(B_j) \leq 0$, and also $A_n = tA$ by definition.

In the case where A is not diagonal, we consider the singular value decomposition, i.e. $A = Q\Sigma R$ where $Q, R \in \text{SO}(n)$ and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$. We see that (10.1.2) can be rewritten as

$$\Sigma = \frac{1}{t^n}(t\Sigma) + \sum_{j=1}^n \frac{t-1}{t^j}B_j$$

and so, multiplying this by Q and R , we get

$$A = Q\Sigma R = \frac{1}{t^n}(tA) + \sum_{j=1}^n \frac{t-1}{t^j}QB_jR.$$

We still have $\det(QB_jR) = \det(B_j) = 0$ and hence to finish the proof it suffices to show that this decomposition of A is still a prelaminate. For this, we use the following elementary fact:

$$\text{for all } A \in \mathbb{R}^{n \times n} \text{ and } M \in \text{GL}(n), \text{ rank}(AM) = \text{rank}(MA) = \text{rank}(A).$$

The splittings used to obtain this prelaminate are rotated versions of (10.1.1), i.e.

$$QA_{j-1}R = \frac{1}{t}QA_jR + \frac{t-1}{t}QB_jR$$

and this is still a legitimate splitting since

$$\text{rank}(QB_jR - QA_jR) = \text{rank}(B_j - A_j) = 1. \quad \square$$

As a simple consequence of the lemma, we find a rigidity result for decompositions of positively n -homogeneous integrands.

Proposition 10.1.4. *Let $E_1, E_2: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be rank-one convex integrands which are non-positive on $\{\det = 0\}$ and assume there is some positively n -homogeneous integrand F such that $F = E_1 + E_2$. Then each E_i is positively n -homogeneous.*

Proof: Define the ‘‘homogenized’’ integrands $E_i^h: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ by

$$E_i^h(A) \equiv \begin{cases} |A|^n E_i\left(\frac{A}{|A|}\right) & A \neq 0 \\ 0 & A = 0 \end{cases}$$

so that the lemma yields

$$\begin{aligned} \text{if } |A| < 1 \text{ then } E_i(A) &\leq E_i^h(A), \\ \text{if } |A| > 1 \text{ then } E_i(A) &\geq E_i^h(A). \end{aligned}$$

Our claim is that $E_i = E_i^h$. Since $F = E_1 + E_2$ it follows that

$$F \leq E_1^h + E_2^h \quad \text{in } U \equiv \{A \in \mathbb{R}^{n \times n} : |A| \leq 1\}.$$

and we have equality on the sphere $\{A \in \mathbb{R}^{n \times n} : |A| = 1\}$, where $E_i = E_i^h$. As both sides of the inequality are positively n -homogeneous they must be equal in the whole set U and so $E_i = E_i^h$ in U . An identical argument establishes equality in the complement of U . \square

Remark 10.1.5. The proofs of Lemma 10.1.3 and Proposition 10.1.4 are fairly robust. In particular, a similar statement holds if the integrands E_i are defined in $\mathbb{R}_{\text{sym}}^{n \times n}$ instead of $\mathbb{R}^{n \times n}$. Indeed, $\mathbb{R}_{\text{sym}}^{n \times n}$ is the set of (real) matrices that can be diagonalized by rotations. Thus, the prelaminate built in the proof of Lemma 10.1.3 has support in $\mathbb{R}_{\text{sym}}^{n \times n}$ if A is symmetric: for the nondiagonal case, one can take $R = Q^{-1}$.

Returning to the case $n = 2$, it would be pleasant to have a result analogous to Proposition 10.1.4 for integrands vanishing on cones of aperture greater than one. However, this is not possible, since it would yield the extremality of the Burkholder function in the class of isotropic rank-one convex integrands. In order to see that this cannot be the case, we recall that ŠVERÁK introduced in [348] the rank-one convex function

$$L(z, w) \equiv \begin{cases} |z|^2 - |w|^2 & |z| + |w| \leq 1 \\ 2|z| - 1 & |z| + |w| \geq 1 \end{cases},$$

which is related to the Burkholder function B_p , when $1 < p < 2$, by

$$\int_0^\infty t^{p-1} L\left(\frac{z}{t}, \frac{w}{t}\right) dt = \frac{2}{p(2-p)} B_p(z, w);$$

see [21]. In particular, this shows that one cannot drop the homogeneity assumption from the results of [17].

10.2 Proof of extremality for truncated minors

This section is dedicated to the proof of Theorem 9.A. Although truncated minors are not linear along rank-one lines, they are piecewise linear along such lines. For this reason, it will be useful to have at our disposal the classification of rank-one affine integrands, which is due to BALL [28] in dimensions three or lower, DACOROGNA [98] in higher dimensions and also BALL-CURRIE-OLVER [26] in the case of higher order quasiconvexity. Given an open set $\mathcal{O} \subset \mathbb{R}^{n \times n}$ and an integrand $E: \mathcal{O} \rightarrow \mathbb{R}$ we say that E is *rank-one affine* if both E and $-E$ are rank-one convex; such integrands are also often called *Null Lagrangians* or *quasiaffine*.

Theorem 10.2.1. *Let $\mathcal{O} \subset \mathbb{R}^{n \times n}$ be a connected open set and consider a rank-one affine integrand $E: \mathcal{O} \rightarrow \mathbb{R}$. Then $E(A)$ is an affine combination of the minors of A .*

More precisely, let $\mathbf{M}(A)$ be the matrix consisting of the minors of A and let $\tau(n) \equiv (2n)!/(n!)^2$ be its length. There is a constant $c \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{\tau(n)}$ such that

$$E(A) = c + v \cdot \mathbf{M}(A) \quad \text{for all } A \in \mathcal{O}.$$

This theorem is essentially a particular case of [26, Theorem 4.1], the only difference being that in this paper the authors deal only with integrands defined on the whole space. We briefly sketch how to adapt their proof to our case. The first result needed is the following:

Lemma 10.2.2. *Let $\mathcal{O} \subset \mathbb{R}^{n \times n}$ be open. A smooth integrand $E: \mathcal{O} \rightarrow \mathbb{R}$ is rank-one affine if and only if, for any $k \geq 2$,*

$$D^k E(A)[v_1 \otimes w_1, \dots, v_k \otimes w_k] = 0$$

for all $A \in \mathcal{O}$ and all $v_i, w_i \in \mathbb{R}^n$ with w_1, \dots, w_k linearly dependent.

In particular, when \mathcal{O} is connected, any continuous rank-one affine integrand E is a polynomial of degree at most n .

We remark that our proof is very similar to the one in [28, Theorem 4.1].

Proof: We recall that a smooth integrand E is rank-one affine if and only if

$$D^2 E(A)[v \otimes w, v \otimes w] = 0 \quad \text{for all } v, w \in \mathbb{R}^n \tag{10.2.1}$$

and so clearly one of the directions of the lemma holds. Hence let us assume that E is rank-one affine and fix some point $A \in \mathcal{O}$. Define the $2k$ -tensor $T: (\mathbb{R}^n)^{2k} \rightarrow \mathbb{R}$ by

$$T[v_1, \dots, v_k, w_1, \dots, w_k] \equiv D^k E(A)[v_1 \otimes w_1, \dots, v_k \otimes w_k].$$

Moreover, since E is rank-one affine, T is alternating. This follows from the following claim: if $w_j = w_l$ for some $j \neq l$, then

$$T[v_1, \dots, v_k, w_1, \dots, w_k] = 0.$$

To see why this claim is true, we note that it certainly holds when $k = 2$, since (10.2.1) implies that

$$\begin{aligned} 0 &= D^2 E[(v_1 + v_2) \otimes w, (v_1 + v_2) \otimes w] \\ &= D^2 E(A)[v_1 \otimes w, v_2 \otimes w] + D^2 E(A)[v_2 \otimes w, v_1 \otimes w] \end{aligned}$$

since $D^2 E(A)[v_1 \otimes w, v_1 \otimes w] = 0$ and the same with v_2 in the place of v_1 . For a general $k \geq 2$, we use implicit summation to see that

$$\begin{aligned} D^k E(A)[v_1 \otimes w_1, \dots, v_k \otimes w_k] &= \frac{\partial^k E(A)}{\partial A_{i_1}^{\alpha_1} \dots \partial A_{i_k}^{\alpha_k}} v_1^{i_1} \dots v_k^{i_k} w_1^{\alpha_1} \dots w_k^{\alpha_k} \\ &= \frac{\partial^{k-2}}{\partial A_{i_1}^{\alpha_1} \dots \widehat{\partial A_{i_j}^{\alpha_j}} \dots \widehat{\partial A_{i_l}^{\alpha_l}} \dots \partial A_{i_k}^{\alpha_k}} \left[\frac{\partial^2 E(A)}{\partial A_{i_j}^{\alpha_j} \partial A_{i_l}^{\alpha_l}} v_j^{i_j} v_l^{i_l} w_j^{\alpha_j} w_l^{\alpha_l} \right] \times \end{aligned}$$

$$\times v_1^{i_1} \dots \widehat{v_j^{i_j}} \dots \widehat{v_l^{i_l}} \dots v_k^{i_k} w_1^{\alpha_1} \dots \widehat{w_j^{\alpha_j}} \dots \widehat{w_l^{\alpha_l}} \dots w_k^{\alpha_k}$$

where $\widehat{}$ represents an omitted term. Now we can apply the $k = 2$ case to the term in square brackets to see that

$$T[v_1, \dots, v_k, w_1, \dots, w_k] = 0$$

as wished.

To prove the lemma under the assumption that E is smooth, let us take w_1, \dots, w_k linearly dependent, so we can suppose for simplicity that $w_k = w_1 + \dots + w_{k-1}$. Then

$$T[v_1, \dots, v_k, w_1, \dots, w_{k-1}, w_1 + \dots + w_{k-1}] = 0$$

since T is linear and alternating. The last statement of the lemma follows by observing that the first part implies that $D^{n+1}E(A) = 0$ for all $A \in \mathcal{O}$.

When E is merely continuous, let ρ be the standard mollifier and let $\rho_\varepsilon(A) = \varepsilon^{-n^2} \rho(A/\varepsilon)$ for $\varepsilon > 0$. Fix $A \in \mathcal{O}$ and find an $\varepsilon > 0$ such that $\text{dist}(A, \partial\mathcal{O}) > \varepsilon$. Then $E_\varepsilon \equiv \rho_\varepsilon * E$ is smooth and rank-one affine and hence

$$D^k E_\varepsilon(A)[v_1 \otimes w_1, \dots, v_k \otimes w_k] = 0$$

whenever w_1, \dots, w_k are linearly dependent. Since $D^k E_\varepsilon$ converges to $D^k E$ locally uniformly, the conclusion of the lemma follows. \square

Using the lemma, we see that in order to prove the theorem it suffices to consider rank-one affine integrands which are homogeneous polynomials, so let us take such an integrand E which is a homogeneous polynomial of some degree k . Given any $A \in \mathcal{O}$, the total derivative $D^k E(A)$ is a symmetric k -linear function $D^k E(A): (\mathbb{R}^{n \times n})^k \rightarrow \mathbb{R}$; we remark that this operator has as domain the whole matrix space and not just a subset of it. There is an isomorphism between the space of k -homogeneous rank-one affine integrands and the space of symmetric k -linear functions $(\mathbb{R}^{n \times n})^k \rightarrow \mathbb{R}$ and the proof in [26] is unchanged in our case.

We recall that a generic symmetric rank-one matrix is of the form $cv \otimes v$ for some $v \in \mathbb{R}^n$ with $|v| = 1$ and some $c \in \mathbb{R}$. Hence, we have the following analogue of Lemma 10.2.2:

Lemma 10.2.3. *Let $\mathcal{O} \subset \mathbb{R}_{\text{sym}}^{n \times n}$ be open. A smooth integrand $E: \mathcal{O} \rightarrow \mathbb{R}$ is rank-one affine if and only if, for any $k \geq 2$,*

$$D^k E(A)[v_1 \otimes v_1, \dots, v_k \otimes v_k] = 0$$

for all $A \in \mathcal{O}$ and all $v_i, w_i \in \mathbb{R}^n$ with v_1, \dots, v_k linearly dependent.

From this we deduce, by the same arguments as in the situation above, the following result:

Theorem 10.2.4. *Let $\mathcal{O} \subset \mathbb{R}_{\text{sym}}^{n \times n}$ be a connected open set and consider a rank-one affine integrand $E: \mathcal{O} \rightarrow \mathbb{R}$. Then there is a constant $c \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{\tau(n)}$ such that*

$$E(A) = c + v \cdot \mathbf{M}(A) \quad \text{for all } A \in \mathcal{O}.$$

In order to apply Theorems 10.2.1 and 10.2.4 to the integrands we are interested in, we need to know that its supports are connected (in general, it is clear that any integrand with disconnected support cannot be extremal). Given a minor M , let $\mathcal{O}_M \equiv \{M > 0\}$. Moreover, each F_k has support in the set

$$\mathcal{O}_k \equiv \{A \in \mathbb{R}_{\text{sym}}^{n \times n} : A \text{ has index } k \text{ and is invertible}\}.$$

Lemma 10.2.5. *For any minor M the set \mathcal{O}_M is connected. Moreover, for $k = 0, \dots, n$, the sets \mathcal{O}_k are connected.*

Proof: For the first part, let M be an $s \times s$ minor. Let us make the identification

$$\mathbb{R}^{n \times n} \cong \mathbb{R}^{s \times s} \times \mathbb{R}^{(n-s) \times (n-s)}$$

so that $M(A) = \det(P_s(A))$, where P_s is the projection of $\mathbb{R}^{n \times n}$ onto $\mathbb{R}^{s \times s}$. Hence we see that, under this identification,

$$\mathcal{O}_M = \{A \in \mathbb{R}^{s \times s} : \det(A) > 0\} \times \mathbb{R}^{(n-s) \times (n-s)}.$$

Since both spaces are connected and the product of connected spaces is connected, \mathcal{O}_M is connected as well.

For the second part, note that the set \mathcal{O}_k is the set of matrices A for which there is some $Q \in \text{SO}(n)$ and some diagonal matrix $\Lambda = \text{diag}(a_1, \dots, a_k, b_1, \dots, b_{n-k})$, where $a_i < 0$ and $b_j > 0$, such that $QAQ^T = \Lambda$. Clearly the set of Λ 's with this form can be connected to $\text{diag}(-I_k, I_{n-k})$ by a path in \mathcal{O}_k ; here I_l is an $l \times l$ identity matrix. Hence it suffices to prove that there is a continuous path in \mathcal{O}_k connecting $A = Q\Lambda Q^T$ to Λ . Such a path is given by $A(t) = Q(t)AQ(t)^T$, where $Q: [0, 1] \rightarrow \text{SO}(n)$ is a continuous path with $Q(0) = I, Q(1) = Q$. \square

We are finally ready to prove the extremality of truncated minors and of ŠVERÁK's integrands.

Proof of Theorem 9.A.: Let M be a minor and let $E_1, E_2: \mathbb{R}^{n \times n} \rightarrow [0, \infty)$ be rank-one convex integrands such that $M^+ = E_1 + E_2$. For concreteness, let us say

$$M \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \det \begin{bmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k j_1} & \cdots & a_{i_k j_k} \end{bmatrix}.$$

Each E_i is zero outside \mathcal{O}_M and, in this set, it is rank-one affine, so by Theorem 10.2.1 there are constants c_i and $v_i \in \mathbb{R}^{\tau(n)}$ such that

$$E_i(A) = c_i + v_i \cdot \mathbf{M}(A) \quad \text{for all } A \in \mathcal{O}_M$$

and in fact, by continuity, this holds in the entire set $\overline{\mathcal{O}_M} = \{M \geq 0\}$.

Clearly we must have $c_i = 0$. Let us write, for some vectors $v_i^j \in \mathbb{R}^{\binom{n}{j} \times \binom{n}{j}}$,

$$E_i(A) = \sum_{j=1}^n v_i^j \cdot \mathbf{M}_j(A) \quad \text{in } \overline{\mathcal{O}_M},$$

where $\mathbf{M}_j(A)$ is the matrix of the j -th order minors of A (this is denoted by $\text{adj}_j(A)$ in [98]).

We observe that, given some s and some minor M' of order s , there is a matrix A such that M' is the only minor of order s that does not vanish at A . Indeed, if

$$M' \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \det \begin{bmatrix} a_{i'_1 j'_1} & \cdots & a_{i'_1 j'_s} \\ \vdots & \ddots & \vdots \\ a_{i'_s j'_1} & \cdots & a_{i'_s j'_s} \end{bmatrix}$$

then we can take a matrix A whose only non-zero entries are the entries $a_{i'_\alpha j'_\alpha}$ for $\alpha \in \{1, \dots, s\}$ and set these entries to one, so $M'(A) = 1$. Since all other entries of A are zero we see that all other minors of order s vanish at A . Note, moreover, that A has rank s .

The previous observation, applied with $s = k$ and $M = M'$, shows that for $A \in \overline{\mathcal{O}_M}$ we have $v_i^k \cdot \mathbf{M}_k(A) = \lambda_i M(A)$, where $\lambda_i \in \mathbb{R}$ is the entry of v_i^k corresponding to M . We now prove that all the vectors $v_i^j, j \neq k$, are zero.

Let $j \leq k$ be the lowest integer for which $v_i^j \neq 0$ and suppose that $j < k$. Given any minor M' of order j , say $M' = e_\alpha \cdot \mathbf{M}_j$ for some α , there is an A with $\text{rank}(A) = j$ so that M' is the only minor of order j that does not vanish at A . Since A has rank j all of its $(j+1) \times (j+1)$ minors vanish and, in particular, $M(A) = 0$ and hence $A \in \overline{\mathcal{O}_M}$. Since j is the lowest integer for which $v_i^j \neq 0$ we have

$$M(A) = 0 = E_i(A) = v_i^j \cdot \mathbf{M}_j(A) = (v_i^j)_\alpha M'(A).$$

Since α was chosen arbitrarily, this is a contradiction and hence we have $j = k$.

Let $j \geq k$ be the highest integer for which $v_i^j \neq 0$ and suppose that $j > k$. Given any minor M' of order j , say $M' = e_\alpha \cdot \mathbf{M}_j$ for some α , there is an A such that M' is the only minor of order j that does not vanish at A ; moreover, by flipping the sign of the i_1 -th row of A , if need be, we can assume that $A \in \overline{\mathcal{O}_M}$. Since $j > k$ is the highest integer for which $v_i^j \neq 0$, by computing

$$t^k M(A) = M(tA) = E_1(tA) + E_2(tA) = \sum_{j=1}^n t^j (v_1^j + v_2^j) \cdot \mathbf{M}_j(A)$$

and sending $0 < t \nearrow \infty$ we see that we must have $(v_1^j + v_2^j)_\alpha = 0$ and also that the sign of $E_i(tA)$ is, for large t , the sign of $(v_i^j)_\alpha M'(A)$; hence $(v_i^j)_\alpha = 0$ for $i = 1, 2$. Moreover, since α was chosen arbitrarily we have $v_i^j = 0$ and we find a contradiction; thus $j = k$.

The two previous paragraphs show that, in $\overline{\mathcal{O}_M}$,

$$E_i(A) = v_i^k \cdot \mathbf{M}_k(A),$$

and we already know that $v_i^k \cdot \mathbf{M}_k(A) = \lambda_i M(A)$, so the proof that M^+ is extremal is complete. The fact that M^- is extremal follows from the identity $M^- = M^+(J \cdot)$, where $J = (j_{\alpha\beta})$, $j_{\alpha\beta} = \delta_{\alpha\beta}(1 - 2\delta_{\alpha i_1})$, so J changes the sign of the i_1 -th row.

For the second part of the theorem take some $k \in \{0, \dots, n\}$ and assume that there are rank-one convex integrands $E_1, E_2: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow [0, \infty)$ such that $F_k = E_1 + E_2$. The integrand F_k has support in \mathcal{O}_k , which by Lemma 10.2.5 is connected, and in this set each E_i is rank-one affine, so Theorem 10.2.4 implies that there are $c_i \in \mathbb{R}$, $v_i \in \mathbb{R}^{\tau(n)}$ such that

$$E_i(A) = c_i + v_i \cdot \mathbf{M}(A) \quad \text{for all } A \in \mathcal{O}_k.$$

By continuity this in fact holds in $\overline{\mathcal{O}_k}$. From Remark 10.1.5 we see that each E_i has to be positively n -homogeneous and therefore $E_i = \alpha_i \det$ in \mathcal{O}_k , where α_i is the last entry of v_i . Since $E_i \geq 0$ we must have, by possibly changing the sign of α_i , $E_i = \alpha_i |\det|$ in \mathcal{O}_k . Moreover, $E_i = 0$ outside \mathcal{O}_k , and so indeed $E_i = \alpha_i F_k$ as wished. \square

We note that, for the second part of the theorem, it is helpful to employ the homogeneity from Remark 10.1.5. Indeed, it is easy to see, and it follows in particular from the linear algebraic arguments in the proof of the full space case, that minors of a fixed order are linearly independent as functions on $\mathbb{R}^{n \times n}$. However, this is not the case if instead we think of them as functions defined over $\mathbb{R}_{\text{sym}}^{n \times n}$, since there are non-trivial linear relations between minors. For instance, given a 4×4 matrix $A = (a_{ij})$, we have that

$$-(a_{13}a_{24} - a_{14}a_{23}) + (a_{12}a_{34} - a_{14}a_{23}) - (a_{12}a_{34} - a_{13}a_{24}) = 0.$$

This is classical phenomenon and had already been noted, for instance, in [26, Pages 155-156].

10.3 Separation of sets and other extremal integrands

This section consists of joint unpublished work with DANIEL FARACO. We begin by recalling the following construction, which plays a pivotal role in the proof of Theorem 9.1.23 [137, 355]:

Definition 10.3.1. Given a curve Γ in $\mathbb{R}^{2 \times 2}$ such that

$$U_\Gamma \equiv \{A \in \mathbb{R}^{2 \times 2} : \det(A - Y) > 0 \text{ for all } Y \in \Gamma\}$$

consists of two connected components U_1, U_2 , let $U_3 \equiv \mathbb{R}^{2 \times 2} \setminus U_\Gamma$ and define

$$f_\Gamma(A) \equiv \begin{cases} \sup_{Y \in \Gamma} -\det(A - Y) & A \in U_1 \cup U_3 \\ 0 & A \in U_2 \end{cases}.$$

Note that the functions f_Γ are rank-one convex. We give a couple of important examples: in the first one, Γ is a straight line which corresponds to the anti-symmetric matrices, see also [356], while in the second one Γ is a circle.

Example 10.3.2. When $\Gamma = (e_1 \otimes e_2 - e_2 \otimes e_1)\mathbb{R}$, we have that

$$U_\Gamma = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \frac{1}{4}(a_{12} + a_{21})^2 < a_{11}a_{22} \right\}.$$

This set has two connected components:

$$U_1 \equiv \{A \in U_\Gamma : a_{11} + a_{22} < 0\}, \quad U_2 \equiv \{A \in U_\Gamma : a_{11} + a_{22} > 0\}.$$

We thus obtain

$$f_\Gamma(A) = \begin{cases} -\det A + \frac{1}{4}(a_{12} - a_{21})^2 & \text{if } A \in U_1 \cup U_3, \\ 0 & \text{if } A \in U_2, \end{cases}$$

and, in particular, we find that $(f_\Gamma + \det)|_{\mathbb{R}_{\text{sym}}^{2 \times 2}} = \det^{++}$, since $U_2 \cap \mathbb{R}_{\text{sym}}^{2 \times 2} = \text{PosDef}(2)$.

Example 10.3.3. When $\Gamma = \text{SO}(2)$, we obtain ŠVERÁK's function [348]:

$$(\det + f_{\text{SO}(2)})(A) = L(A) \equiv \begin{cases} |a^+|^2 - |a^-|^2 & \text{if } |a^+| + |a^-| \leq 1, \\ 2|a^+| - 1 & \text{if } |a^+| + |a^-| \geq 1. \end{cases}$$

The function L is closely related to Burkholder's function, see [21]. In particular, if L is quasiconvex at zero then so is B_p , recall Proposition 9.1.33. We have:

Proposition 10.3.4. *The function $L + 1$ is an extremal rank-one convex function.*

Proof: Take functions $E_1, E_2: \mathbb{R}^{2 \times 2} \rightarrow [0, \infty)$ which are rank-one convex and such that

$$E_1 + E_2 = L + 1.$$

Let $V_1 \equiv \{A : |a^+| + |a^-| < 1\}$. Since $L + 1$ is rank-one affine in V_1 , we can write

$$E_i = \alpha_i + \langle \gamma_i, \cdot \rangle + \beta_i^1 \det \quad \text{in } V_1$$

where $\gamma_i = (\gamma_i^+, \gamma_i^-) \in \mathbb{R}^4$. Evaluating E_i at $(0, e^{i\theta})$ and varying θ we find that $\gamma_i^- = 0$ and that $\alpha_i = \beta_i^1$. Then we have

$$\left. \begin{aligned} 1 &= (E_1 + E_2)(0) = \alpha_1 + \alpha_2 \\ 1 &= (E_1 + E_2)(e^{i\theta}/2, 1/2) = \alpha_1 + \alpha_2 + \frac{1}{2} \langle \gamma_1^+ + \gamma_2^+, e^{i\theta} \rangle \\ 2 &= (E_1 + E_2)(e^{i\theta}, 0) = \alpha_1 + \alpha_2 + \langle \gamma_1^+ + \gamma_2^+, e^{i\theta} \rangle + \beta_1 + \beta_2 \end{aligned} \right\} \implies \gamma_1^+ = \gamma_2^+ = 0.$$

Thus, we have shown that

$$E_i = \beta_i^1(1 + \det) \quad \text{in } V_1.$$

Now fix $\theta \in [0, \pi)$ and consider coordinates $(x, a^-) \in \mathbb{R} \times \mathbb{C}$ for the 3-dimensional space $S^\theta \equiv (e^{i\theta}, 0)\mathbb{R} \times \mathbb{C}$ (in other words, we fix the direction of the conformal part of the matrix). We split this space into three connected sets

$$V_1^\theta \equiv V_1 \cap S^\theta, \quad V_2^\theta \equiv \{(xe^{i\theta}, a^-) \notin V_1^\theta : x > 0\}, \quad V_3^\theta \equiv \{(xe^{i\theta}, a^-) \notin V_1^\theta : x < 0\}$$

and, in each of them, the function L is rank-one affine and hence each E_i is an affine combination of a linear function with \det . In V_2^θ, V_3^θ , simple growth considerations show that the coefficients of \det in each E_i have to be zero, so only the affine piece remains. Then, if $|a^-| = 1$,

$$L(0, a^-) + 1 = 0 \implies E_i(0, a^-) = 0$$

and this shows that, for some constants $\beta_i^{j,\theta} \in \mathbb{R}$,

$$E_i(xe^{i\theta}, a^-) = 2\beta_i^{j,\theta}|x| \quad \text{for } (xe^{i\theta}, a^-) \in V_j^\theta, j = 2, 3.$$

Recall that the values of E_i in V_1^θ have already been determined and do not depend on θ ; comparing the values of E_i at $(e^{i\theta}, 0)$ when approaching these points from either the inside or the outside of V_1^θ , we see that $\beta_i^1 = \beta_i^{2,\theta} = \beta_i^{3,\theta}$ by continuity. This shows that $E_i = \beta_i^1(1 + L)$, as wished. \square

For the definition of K -elliptic curve, see [137]. The previous examples suggest the following:

Question 10.3.5. Is it the case that, whenever Γ is a K -elliptic curve, the function f_Γ is extremal?

10.4 Choquet theory and Morrey's problem

In this section we shall see the implications of Choquet theory for Morrey's problem. Let us introduce some notation: given a number $d \in \mathbb{N}$, let $Q_d = [0, 1]^d$, denote by A_1, \dots, A_{2^d} its vertices and consider the cone

$$\mathcal{C}_d^c \equiv \{f: Q_d \rightarrow [0, \infty) : f \text{ is convex}\}.$$

In a similar fashion we define the cone $\mathcal{C}_d^{\text{sc}}$ of non-negative separately convex functions on Q_d and when $d = n \times m$ for some $n, m \in \mathbb{N}$, we have the cones $\mathcal{C}_d^{\text{qc}}$ and $\mathcal{C}_d^{\text{rc}}$ of non-negative quasiconvex and rank-one convex integrands. These are closed convex cones in the locally convex vector space \mathbb{R}^{Q_d} of real-valued functions on Q_d ; the topology on \mathbb{R}^{Q_d} is the product topology, i.e. the topology of pointwise convergence. In particular, for any $x \in Q_d$ the evaluation functionals $\varepsilon_x: f \mapsto f(x)$ are continuous.

We claim that each of the above cones has a compact, convex base:

$$\mathcal{B}_d^\square \equiv \mathcal{C}_d^\square \cap \left\{ f \in \mathbb{R}^{Q_d} : \sum_{i=1}^{2^d} f(A_i) = 1 \right\}, \quad \square \in \{c, \text{qc}, \text{rc}, \text{sc}\};$$

here we only take $\square \in \{\text{qc}, \text{rc}\}$ if $d = n \times m$. Clearly each \mathcal{B}_d^\square is a closed, convex base for \mathcal{C}_d^\square , so it suffices to see that $\mathcal{B}_d^{\text{sc}}$ is compact. For this, note that a separately convex function on Q_d attains its maximum at some A_i and, since all functions $f \in \mathcal{B}_d^{\text{sc}}$ are non-negative, we have $f \leq 1$ in Q_d . This shows that $\mathcal{B}_d^{\text{sc}} \subset [0, 1]^{Q_d}$, which is a compact set by Tychonoff's Theorem, and our claim is proved.

The main tool of this section is the following powerful result:

Theorem 10.4.1 (Choquet). *Let K be a metrizable, compact, convex subset of a locally convex vector space X . For each $f \in K$ there is a regular probability measure μ on K which is supported on the set $\text{Ext}(K)$ of extreme points of K and which represents the point f : for all $\varphi \in X^*$,*

$$\varphi(f) = \int_{\text{Ext}(K)} \varphi \, d\mu.$$

For a proof see, for instance, [315, §3]. We note that in general—and this is also the case in our situation—the representing measure is not unique. In order to apply this theorem to \mathcal{B}_d^\square , we need to show that this set is metrizable and this can be done by using a simple result from point-set topology; for a proof see, for instance, [266, Lemma 10.45].

Lemma 10.4.2. *Let K be a compact Hausdorff space. Then K is metrizable if and only if there is a countable family of continuous functions on K which separates points.*

In our situation, it is easy to see that such a family exists: indeed, let $x_n \in Q_d$ be a countable set of points which is dense in Q_d and consider the evaluation functionals $\varepsilon_{x_n}: f \mapsto f(x_n)$ on \mathcal{B}_d^\square . These functionals are continuous on \mathcal{B}_d^\square and separate points, since all elements of \mathcal{B}_d^\square are continuous real-valued functions on Q_d . Therefore the lemma implies that \mathcal{B}_d^\square is metrizable, and hence Choquet's theorem yields:

Proposition 10.4.3. Fix $\square \in \{c, qc, rc, sc\}$ and let ν be a regular probability measure in Q . The measure ν satisfies $f(\bar{\nu}) \leq \langle \nu, f \rangle$ for all $f \in \mathcal{C}_d^\square$ if and only if $g(\bar{\nu}) \leq \langle \nu, g \rangle$ for all $g \in \text{Ext}(\mathcal{B}_d^\square)$.

Proof: Assume that we have Jensen's inequality for all $g \in \text{Ext}(\mathcal{B}_d^\square)$, take any $f \in \mathcal{B}_d^\square$ and let μ be the measure given by Choquet's theorem. If we take $\varphi = \varepsilon_{\bar{\nu}}$ in the theorem, we can apply Fubini's theorem to see that

$$\begin{aligned} f(\bar{\nu}) &= \varepsilon_{\bar{\nu}}(f) = \int_{\text{Ext}(\mathcal{B}_d^\square)} \varepsilon_{\bar{\nu}}(g) d\mu(g) \\ &\leq \int_{\text{Ext}(\mathcal{B}_d^\square)} \int_Q g d\nu d\mu(g) \\ &= \int_Q \int_{\text{Ext}(\mathcal{B}_d^\square)} g d\mu(g) d\nu = \int_Q f d\nu. \end{aligned}$$

Since any $h \in \mathcal{C}_d^\square$ can be written uniquely as $h = \lambda f$ for some $\lambda > 0, f \in \mathcal{B}_d^\square$, the conclusion follows. \square

Theorem 9.C follows easily from Proposition 10.4.3. For the reader's convenience, we restate the theorem here:

Theorem 10.4.4. Let $d = n^2$ and take a Radon probability measure ν supported in the interior of Q_d . Then ν is a laminate if and only if

$$g(\bar{\nu}) \leq \langle \nu, g \rangle$$

for all integrands $g \in \text{Ext}(\mathcal{B}_d^c)$.

Proof: From Pedregal's Theorem, the measure ν is a laminate if and only if Jensen's inequality holds for any rank-one convex integrand $f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$:

$$f(\bar{\nu}) \leq \langle \nu, f \rangle, \quad \bar{\nu} \equiv \int_{\mathbb{R}^{n \times n}} A d\nu(A).$$

Note that if this inequality holds for all *non-negative* rank-one convex integrands then it holds for any rank-one convex integrand: given any such f , one can consider the new integrand

$$f_k \equiv k + \max(f, -k)$$

which is non-negative and rank-one convex, hence by hypothesis $f_k(\bar{\nu}) \leq \langle \nu, f_k \rangle$. This in turn is equivalent to

$$\max(f(\bar{\nu}), -k) \leq \langle \nu, \max(f, -k) \rangle.$$

Sending $k \rightarrow \infty$ we find that $f(\bar{\nu}) \leq \langle \nu, f \rangle$, as we wished; note that f , being continuous, is bounded by below on Q_d .

Therefore, from Proposition 10.4.3, the theorem follows once we show that any rank-one convex integrand $g: Q_d \rightarrow [0, \infty)$ can be extended to a rank-one convex integrand $f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ with $g = f$ in the support of ν . This is a standard result, see [348]. \square

We end this section with some cautionary comments concerning the previous results. In the one dimensional case, where all the above cones coincide, the extreme points are quite easy to identify; the oldest reference we found where this problem is discussed is [50], but see also [266, §14.1].

Proposition 10.4.5. The set of extreme points of \mathcal{B}_1 is the set $\{\varphi_y, \psi_y : y \in [0, 1]\}$, where the functions φ_y, ψ_y are defined by

$$\begin{aligned} \varphi_y : x &\mapsto \frac{(x-y)^+}{1-y} \text{ for } x \in [0, 1], y \in [0, 1), & \varphi_1 &= 1_{\{1\}}, \\ \psi_y : x &\mapsto \frac{(y-x)^+}{y} \text{ for } x \in [0, 1], y \in (0, 1], & \psi_1 &= 1_{\{0\}}. \end{aligned}$$

In higher dimensions the various cones are different. In the case of convex integrands, the set of extreme points of \mathcal{C}_d^c for $d > 1$ is very different from the one-dimensional case, since it is dense in this cone.

Theorem 10.4.6. *Any finite continuous convex function on a convex domain $U \subset \mathbb{R}^d$ can be approximated uniformly on convex compact subsets of U by extremal convex functions.*

This result was proved by JOHANSEN in [218] for $d = 2$ and then generalized to any $d > 1$ in [67]. In these papers the set of extremal convex functions is not fully identified, but it is shown that there is a sufficiently large class of extremal convex functions which approximate any given convex function well: these are certain *polyhedral functions*, i.e. functions of the form $f = \max_{1 \leq i \leq k} a_i$ for some affine functions a_1, \dots, a_k . This disturbing situation, however, is not too unexpected given the result of KLEE [236] already mentioned in the introduction. I do not know whether a similar statement holds for the cones $\mathcal{C}_{n \times m}^{\text{qc}}$ and $\mathcal{C}_{n \times m}^{\text{rc}}$.

We note that the precise extremal structure is unknown even in the case of the cone of convex homogeneous polynomials, see for instance [325].

10.5 A short proof of Ornstein's non-inequality in $\mathbb{R}^{2 \times 2}$

The main result of this section is in a slightly different direction than the other results of this chapter. There are however two reasons for including it here: firstly, the methods used in this section are similar to the ones from Section 10.1; secondly, Theorem 10.5.4 below shows that the considerations of the rest of this chapter have further implications beyond the Calculus of Variations.

Given two linear constant-coefficient homogeneous k -th order differential operators \mathcal{P}_1 , \mathcal{P}_2 and a number $1 \leq p \leq \infty$, consider the inequality

$$\|\mathcal{P}_1 \varphi\|_{L^p(\mathbb{R}^n)} \leq C_p \|\mathcal{P}_2 \varphi\|_{L^p(\mathbb{R}^n)}, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m), \quad (10.5.1)$$

where $0 < C_p$ is some constant. When does such an estimate hold?

The case $1 < p < \infty$ is very classical: if we take $\mathcal{P}_1 = \mathbf{D}^k$ to be the k -th order gradient then (10.5.1) holds if and only if \mathcal{P}_2 is an elliptic operator (in the sense that it has injective symbol); this is a classical result which goes back to the work of CALDERÓN and ZYGMUND [76]. We refer the reader to [183] for a generalisation of (10.5.1) which allows for operators with non-trivial kernels.

At the end-points $p = 1$ or $p = \infty$, (10.5.1) never holds, except in trivial situations: this was proved, respectively, by ORNSTEIN [307] and MITYAGIN [284], but see also [110] for the $p = \infty$ case. In some circumstances one can deduce the result for $p = \infty$ from the one for $p = 1$, see for instance [54, 279] for the case $\mathcal{P}_2 = \text{div}$, and in fact the result for $p = 1$ is much more difficult. Similar results also hold in the anisotropic setting, see [226, 318].

The failure of (10.5.1) when $p = 1$ can also be deduced from the KIRCHHEIM–KRISTENSEN convexity theorem [232, 233]. Besides providing a concise proof of ORNSTEIN's result, their theorem also has applications to the regularity of Hessians of rank-one convex functions and to the characterization of gradient Young measures [250, 252]. Both the failure of (10.5.1) when $p = 1$, $\mathcal{P}_1 = \mathbf{D}$, $\mathcal{P}_2 = \mathcal{E}$, as well as the existence of rank-one convex functions on 3-dimensional spaces with irregular Hessians, were proved in [91, 92] through constructions with unbounded laminates (the so-called staircase laminates) introduced in [131, 132]. See also [15] and [136, 304] for related problems for $p > 1$. Such laminates can be used to provide a fairly explicit deformation showing Ornstein's non inequality, but their construction is somewhat complicated as it includes an infinite process.

The purpose of this section is to give an alternative proof of Ornstein's result when $n = m = 2$ and $k = 1$. These assumptions encompass the case where $\mathcal{P}_1 = \mathbf{D}$ and \mathcal{P}_2 is any of the operators

$$(\text{div}, \text{curl}), \quad \mathcal{E}u \equiv \frac{1}{2}(\mathbf{D}u + (\mathbf{D}u)^T), \quad \bar{\partial}u \equiv \mathcal{E}u - \frac{\text{div } u}{2} \text{Id}_2,$$

which appear respectively in electromagnetism, linearized elasticity and complex analysis. Our strategy is similar to the one of KIRCHHEIM–KRISTENSEN [232, 233] and in fact we prove a particular case of

their convexity theorem. However, our approach is less elaborate, as the singular value decomposition reduces the problem to building a laminate on the diagonal matrices and, since the integrand of interest is 1-homogeneous, the laminate is very simple.

In [233], the following theorem was proved:

Theorem 10.5.1 (Kirchheim–Kristensen). *Let $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ be positively 1-homogeneous and rank-one convex. Then f is convex at all matrices X with $\text{rank } X \leq 1$.*

The reader may find other results concerning positively 1-homogeneous rank-one convex functions in [102, 291, 349]. In the planar case there is a particularly simple proof of Theorem 10.5.1 for 1-homogeneous functions:

Lemma 10.5.2. *Let $f: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ be 1-homogeneous and rank-one convex. Then $f \geq 0$ and moreover, as $f(0) = 0$, f is convex at zero.*

Proof: For $A \in \mathbb{R}^{2 \times 2}$ the singular value decomposition yields $Q, R \in O(2)$ and $\Lambda \in \mathbb{R}_{\text{diag}}^{2 \times 2}$ such that $A = Q\Lambda R$; moreover, the entries of Λ are non-negative. Let us write $(x, y) \equiv \text{diag}(x, y)$. If $0 \neq A$ then, by homogeneity, we can assume that $\Lambda = (1, y)$, where $y \geq 0$. The measure

$$\nu = \frac{1}{2}\delta_{Q(2, -2y)R} + \frac{1}{3}\delta_{Q(1, y)R} + \frac{1}{6}\delta_{Q(-2, -2y)R}$$

is a laminate with barycentre $Q(1, -y)R$. Indeed, we have the splittings

$$(1, -y) \rightarrow \frac{1}{3}(1, y) + \frac{2}{3}(1, -2y) \rightarrow \frac{1}{3}(1, y) + \frac{1}{6}(-2, -2y) + \frac{1}{2}(2, -2y)$$

and the map $A \mapsto QAR$ is rank-preserving. Since f is rank-one convex and 1-homogeneous,

$$\begin{aligned} f(Q(1, -y)R) &\leq \frac{1}{2}f(Q(2, -2y)R) + \frac{1}{3}f(Q(1, y)R) + \frac{1}{6}f(Q(-2, -2y)R) \\ &= f(Q(1, -y)R) + \frac{1}{3}f(A) + \frac{1}{3}f(-A) \end{aligned}$$

Hence $0 \leq f(A) + f(-A) = 2f(A)$ and the proof is finished. \square

Remark 10.5.3. An identical proof gives the same conclusion if $f: \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}$; in this case one takes $R = Q^T$, since symmetric matrices are diagonalisable by orthogonal matrices.

From Lemma 10.5.2 we get a two-dimensional version of Ornstein's non-inequality:

Theorem 10.5.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let \mathcal{P}_i be first-order differential operators, $i = 1, 2$, acting on $\varphi \in C_c^\infty(\Omega, \mathbb{R}^2)$ by $\mathcal{P}_i\varphi = P_i(D\varphi)$, where $P_i \in \text{Lin}(\mathbb{R}^{2 \times 2}, \mathbb{R}^{d_i})$.*

Suppose that there is a constant C such that

$$\|\mathcal{P}_1\varphi\|_{L^1} \leq C\|\mathcal{P}_2\varphi\|_{L^1} \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^2). \quad (10.5.2)$$

Then there is $T \in \text{Lin}(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ such that $P_1 = T \circ P_2$. Moreover, the same conclusion is true if we require that (10.5.2) holds only for those φ of the form $\varphi = \nabla\phi$ for some $\phi \in C_c^\infty(\Omega, \mathbb{R})$.

Proof: Consider the function $f: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ defined by $f(A) = C\|P_2A\| - \|P_1A\|$. Its quasiconvex envelope $f^{\text{qc}}: \mathbb{R}^{2 \times 2} \rightarrow [-\infty, \infty)$ is given by the Dacorogna formula

$$f^{\text{qc}}(A) = \inf_{\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)} \int_{\mathbb{R}^2} f(A + D\varphi) \, dx;$$

it is easily checked that f^{qc} is 1-homogeneous, since the same holds for f . Note that (10.5.2) is equivalent to $f^{\text{qc}}(0) \geq 0$; thus $f^{\text{qc}} > -\infty$ everywhere and hence f^{qc} is rank-one convex. Applying Lemma 10.5.2

we see that $0 \leq f^{\text{qc}} \leq f$ and so we must have $\ker P_2 \subseteq \ker P_1$. Take $T = P_1 P_2^\dagger$, where P_2^\dagger is the *Moore–Penrose inverse*, defined by

$$P_2^\dagger \equiv \left(P_2|_{(\ker P_2)^\perp} \right)^{-1} \text{Proj}_{\text{im } P_2}.$$

Since $P_2^\dagger P_2$ is the orthogonal projection onto $(\ker P_2)^\perp$, the conclusion follows.

The last part is identical, except that we replace Lemma 10.5.2 with Remark 10.5.3: if (10.5.2) holds for all potential vector fields then $(f|_{\mathbb{R}_{\text{sym}}^{2 \times 2}})^{\text{qc}}(0) \geq 0$, see [26, 350] for quasiconvexity on $\mathbb{R}_{\text{sym}}^{n \times n}$. \square

In particular, from the second part of Theorem 10.5.4 we recover [307, Part 1]:

Corollary 10.5.5. *Given a bounded open set $\Omega \subset \mathbb{R}^2$, there is no constant C such that*

$$\int_{\Omega} |\partial_{x_1 x_2} \phi(x)| \, dx \leq C \int_{\Omega} |\partial_{x_1 x_1} \phi(x)| + |\partial_{x_2 x_2} \phi(x)| \, dx \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

Chapter 11

Numerical evidence towards a positive answer to Morrey's problem

This chapter is based on the paper [185], written jointly with TEIXEIRA DA COSTA.

In Section 11.1 we show that the problem of deciding whether a measure with finite support is a laminate is semidecidable. In Section 11.2 we present the class of deformations that we will consider. Section 11.3 contains a heuristic explanation of the numerical results obtained; these results, in turn, are described in detail in Section 11.4.

11.1 Deciding whether a measure is a laminate

In this section we discuss Question 9.3.1: throughout ν is a fixed probability measure with support on a finite set $K \subset \mathbb{R}^{m \times n}$. We will use a discretized version of the KOHN–STRANG algorithm to show the following:

Proposition 11.1.1. *Let ν be a probability measure supported in a finite set of points in $\mathbb{R}^{m \times n}$. The problem of deciding whether ν is a laminate is semidecidable, i.e. there is an algorithm which terminates in finite time with a positive answer if ν is a laminate.*

To prove this, we will resort to Theorem 9.1.5: ν is a laminate if and only if

$$f^{\text{rc}}(\bar{\nu}) \leq \langle \nu, f \rangle, \quad \bar{\nu} \equiv \langle \nu, \text{id} \rangle \tag{11.1.1}$$

for all continuous $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$, where f^{rc} denotes the rank-one convex envelope of f .

Lemma 11.1.2. *If ν is not a laminate there is $g: K \rightarrow [-1, 1]$ such that, for $0 < \delta < c(K, n, m)$ small enough, the continuous function $f_\delta: \mathbb{R}^{m \times n} \rightarrow [-1, 2]$, defined by*

$$f_\delta(A) \equiv \begin{cases} g(A_0) + \frac{2-g(A_0)}{\delta}|A - A_0| & \text{if } |A - A_0| \leq \delta \text{ for some } A_0 \in K \\ 2 & \text{otherwise,} \end{cases} \tag{11.1.2}$$

satisfies $f_\delta^{\text{rc}}(\bar{\nu}) > \langle \nu, f_\delta^{\text{rc}} \rangle$.

Although this is not needed, note that $\lim_{\delta \rightarrow 0} f_\delta = f_0 \equiv g \mathbb{1}_K + 2 \times \mathbb{1}_{\mathbb{R}^{m \times n} \setminus K}$ pointwise.

Proof: Since ν is not a laminate, there is $\tilde{g}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ rank-one convex and such that $g(\bar{\nu}) > \langle \nu, g \rangle$; by scaling, we can assume that $\tilde{g}([-2, 2]^{mn}) \subseteq [-1, 1]$. Since \tilde{g} is rank-one convex it is locally Lipschitz and, see [33],

$$\text{Lip}(\tilde{g}, [-1, 1]^{mn}) \leq \min\{m, n\} \text{osc}(\tilde{g}, [-2, 2]^{mn}) \leq 2 \min\{m, n\}.$$

Let us take $g = \tilde{g}|_K$, $\delta \leq \frac{1}{2} \min\{|X_1 - X_2| : X_1, X_2 \in K\}$ so that f_δ is well-defined and, in addition, we require that $\delta < (2 \min\{m, n\})^{-1}$. Thus, for $A \in B_\delta(A_0)$ and $A_0 \in K$,

$$\tilde{g}(A) \leq \tilde{g}(A_0) + 2 \min\{m, n\}|A - A_0| \leq \tilde{g}(A_0) + \frac{1}{\delta}|A - A_0| \leq f_\delta(A).$$

This shows that $f_\delta \geq \tilde{g}$; since g is rank-one convex, also $f_\delta^{\text{rc}} \geq \tilde{g}$ and so

$$f_\delta^{\text{rc}}(\bar{\nu}) \geq \tilde{g}(\bar{\nu}) > \langle \nu, g \rangle = \langle \nu, f_\delta^{\text{rc}} \rangle. \quad \square$$

Lemma 11.1.2 shows that in order to decide whether ν is a laminate one has to explore the finite-dimensional space of functions $g: K \rightarrow [-1, 1]$. In order to compute an approximation of f_δ^{rc} we use a discrete version of the KOHN–STRANG algorithm [239].

Algorithm 11.1.3. We fix $\delta > 0$ small enough so that we do not need to worry about it; thus we drop the subscript δ . By translation invariance we can assume that $\bar{\nu} = 0$. Then:

1. Fix an odd integer L , consider the grid $\mathcal{G}_L \equiv \frac{1}{L}\mathbb{Z}^{mn} \cap [-1, 1]^{mn}$, and choose a finite set of directions \mathcal{D} consisting of rank-one matrices which are in \mathcal{G}_L .
2. Set $f_{L,\mathcal{D}}^{\text{rc},0} := f$ and, for $A \in \mathcal{G}_L$,

$$f_{L,\mathcal{D}}^{\text{rc},i+1}(A) = \min_{X \in \mathcal{D}: A \pm X \in \mathcal{G}_L} \left\{ \frac{f_{L,\mathcal{D}}^{\text{rc},i}(A+X) + f_{L,\mathcal{D}}^{\text{rc},i}(A-X)}{2}, f_{L,\mathcal{D}}^{\text{rc},i}(A) \right\}.$$

We terminate the algorithm if either the maximum difference between iterates stabilizes or $f_{L,\mathcal{D}}^{\text{rc},i}$ satisfies Jensen's inequality with respect to ν .

Let $f^{\text{rc},i}$ be the i -th Kohn–Strang iterate, i.e. $f^{\text{rc},i}$ is defined inductively by $f^{\text{rc},0} = f$ and

$$f^{\text{rc},i}(A) = \inf \left\{ \lambda f^{\text{rc},i-1}(X) + (1-\lambda)f^{\text{rc},i-1}(Y) : \lambda X + (1-\lambda)Y = A, \text{rank}(X-Y) = 1 \right\},$$

where λ runs over $(0, 1)$. Clearly, for $A \in \mathcal{G}_L$, $f^{\text{rc},i}(A) \leq f_{L,\mathcal{D}}^{\text{rc},i}(A) \leq f(A)$, and so if $f_{L,\mathcal{D}}^{\text{rc},i}$ satisfies Jensen's inequality with respect to ν , f_δ satisfies (11.1.1). Conversely, we have:

Proposition 11.1.4. *Let $f_{L,\mathcal{D}}^{\text{rc}} \equiv \lim_{i \rightarrow \infty} f_{L,\mathcal{D}}^{\text{rc},i}$. Then $f_{L,\mathcal{D}}^{\text{rc}}$ converges uniformly to f_δ^{rc} as $L \rightarrow \infty$ and as the largest angle between any rank-one matrix and its best approximation in \mathcal{D} goes to zero.*

For a proof see [303]. Note that we take $0 < \delta < c(m, n, K)$ and that f_δ is continuous, so their results apply. It is clear that combining Lemma 11.1.2 with Proposition 11.1.4, we deduce Proposition 11.1.1.

11.2 Gradient Young measures versus laminates

In this section we address Question 9.1.7. Recall that a function $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is said to be *quasi-convex* if, for all $A \in \mathbb{R}^{m \times n}$,

$$f(A) \leq \int_{\mathbb{T}^n} f(A + D\varphi(x)) \, dx \quad \text{for all } \varphi \in C^\infty(\mathbb{T}^n, \mathbb{R}^m). \quad (11.2.1)$$

Equivalently, f is quasiconvex if and only if $f(\bar{\nu}) \leq \langle \nu, f \rangle$, where ν is any compactly supported gradient Young measure, c.f. Theorem 9.1.4.

We want to test the inequality (11.2.1) with deformations of the form

$$\varphi(x) = \sum_{i=1}^N a_i s(x \cdot n_i + c_i), \quad (11.2.2)$$

where $N \in \mathbb{N}$, $a_i \in \mathbb{R}^m$, $n_i \in \mathbb{Z}^n$ are vectors, $c_i \in \mathbb{R}$ are phases and s is the 1-periodic sawtooth function, defined for $t \in [0, 1]$ by $s(t) = t1_{[0,1/2]}(t) + (1-t)1_{[1/2,1]}(t)$. The idea of approximating an arbitrary deformation with a simplified deformation with the form (11.2.2) is known in the Applied Harmonic Analysis literature as a ridgelet expansion [316]. We remark as a somewhat inconvenient fact that

orthonormal ridgelet bases in L^2 , just like Fourier series, are never unconditional bases in L^p for $p \neq 2$, although we do not prove this here.

The advantage of an expansion as in (11.2.2) is that, with $h \equiv s'$ being the Haar wavelet,

$$D\varphi(x) = \sum_{i=1}^N h(x \cdot n_i + c_i) a_i \otimes n_i;$$

hence the gradient $D\varphi$ takes values in a finite set. In our context, considering plane-wave expansions as in (11.2.2) is a classical idea, and we are motivated by JAMES' interpretation of ŠVERÁK's example [295], see also [283, §31] and [314, 335]. Moreover, φ generates a homogeneous gradient Young measure ν , which takes the form

$$\nu = \sum_{\varepsilon \in \{-1,1\}^N} \nu_\varepsilon \delta_{X_\varepsilon}, \tag{11.2.3}$$

where we defined the weights ν_ε and the matrices X_ε as

$$\nu_\varepsilon \equiv |\{x \in \mathbb{T}^2 : h(x \cdot n_i + c_i) = \varepsilon_i, i = 1, \dots, N\}|, \quad X_\varepsilon \equiv \sum_{i=1}^N \varepsilon_i a_i \otimes n_i. \tag{11.2.4}$$

Note that ν_ε depends on n_i but not on a_i . Furthermore, the measure ν has barycentre zero.

For the sake of conciseness, we introduce the following definition:

Definition 11.2.1. For $N \in \mathbb{N}$, we say that $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is N -wave quasiconvex at zero if

$$f(0) \leq \sum_{\varepsilon \in \{-1,1\}^N} \nu_\varepsilon f(X_\varepsilon)$$

for all $(a_i, n_i, c_i) \in \mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{R}$, where ν_ε and X_ε are defined by (11.2.4). Moreover, f is N -wave quasiconvex if, for any $A \in \mathbb{R}^{m \times n}$, the function $f(\cdot - A)$ is N -wave quasiconvex at zero.

It seems that variants of this notion were studied in [222] for $N = 3, 4$. By periodicity, in Definition 11.2.1 we can assume that $c_1 = \dots = c_I = 0$ where $I = \min\{n, N\}$. We have:

Proposition 11.2.2. f is quasiconvex if and only if it is N -wave quasiconvexity for all N .

Proof: We prove that if f is N -wave quasiconvex at zero it is quasiconvex at zero, as the converse is clear. We rely on the following standard fact: for $\varphi \in C^\infty(\mathbb{T}^n, \mathbb{R}^m)$ there is a sequence φ_j of the form (11.2.2) which converges to φ strongly in $W^{1,\infty}(\mathbb{T}^n, \mathbb{R}^m)$. For a quantitative version of this fact when $m = 1$ see e.g. [166], although there the authors take several different functions s_i , for $i = 1, \dots, N$, instead of a fixed sawtooth function; regardless, any s_i can be approximated by scaled and translated copies of s . The general case $m > 1$ follows by straightforward argument and we omit it.

Let ν_j be the gradient Young measure generated by the deformation φ_j ; by assumption,

$$f(0) = f(\bar{\nu}_j) \leq \langle \nu_j, f \rangle = \int_{\mathbb{T}^n} f(D\varphi_j) dx.$$

Since $\varphi_j \rightarrow \varphi$ in $W^{1,\infty}(\mathbb{T}^n, \mathbb{R}^m)$, we see that $f(0) \leq \int_{\mathbb{T}^n} f(D\varphi) dx$. Thus f is quasiconvex at zero. \square

The following theorem gathers several results from the literature.

Theorem 11.2.3. N -wave quasiconvexity has the following properties:

- (i) 1-wave quasiconvexity is equivalent to rank-one convexity;
- (ii) 2-wave quasiconvexity is equivalent to rank-one convexity;
- (iii) if $m \geq 3, n \geq 2$ then 3-wave quasiconvexity is different from rank-one convexity and is a nonlocal property;
- (iv) if $m = n = 2$ then 3-wave quasiconvexity is implied by rank-one convexity.

Proof: (i) follows straightforwardly, (ii) follows by [335, Lemma 2.1], (iii) follows from the example in [351] together with an adaptation of the arguments in [246] and (iv) is the main result of [335]. \square

11.3 Counting rank-one connections

The behaviour of gradients of maps changes dramatically from the higher-dimensional to the planar case [137, 216, 235]. One of the basic explanations for this difference is that the relative size of the cone

$$\Lambda \equiv \{A \in \mathbb{R}^{m \times 2} : \text{rank } A \leq 1\}$$

is much larger when $m = 2$ than when $m \geq 3$: for instance, it separates the matrix space into two components in the former case.

The previous insight is also relevant towards the goal of understanding the behaviour of the particular deformations of Section 11.2. In fact, the proof of Theorem 11.2.3(iv) in [335] also explores the fact that Λ is large: using arguments somewhat in the spirit of [355], the abundance of rank-one connections is used to build complicated laminates supported in the 3-cube $\{X_\varepsilon\}_{\varepsilon \in \{-1,1\}^3}$. In view of Proposition 11.2.2 it is natural to ponder what can be said for a general $N > 3$.

In this section our goal is to roughly quantify the number of rank-one connections between points in the lamination hull of the N -cube. Our observations are merely heuristic, i.e. we do not provide any proofs, and they are the consequence of analysing thousands of computer-generated random configurations.

For a given choice of matrices X_ε as in (11.2.4), let us write

$$K_N \equiv \{X_\varepsilon : \varepsilon \in \{-1,1\}^N\} \subset \mathbb{R}^{m \times 2}, \quad Q_N \equiv [-1,1]^N \subset \mathbb{R}^N.$$

We can visualise K_N as the vertices of the N -cube Q_N by considering the map $X_\varepsilon \mapsto \varepsilon$. Note, however, that for $N > 2m$ the map $\varepsilon \mapsto X_\varepsilon$ cannot be an embedding.

Let us denote by $K_N^{\text{lc},i}$ the usual i -th lamination convex hull of K_N , see [295] for the definition. Since the edges of the cube correspond to rank-one segments, it is clear that, under the above identification, $K_N^{\text{lc},i}$ contains the i -skeleton of Q_N : for instance, $K_N^{\text{lc},1}$ contains the edges of the cube, $K_N^{\text{lc},2}$ contains the faces, and so on.

We say that X_ε and $X_{\varepsilon'}$ are neighbours if ε and ε' are adjacent vertices in Q_N . Generically, each vertex X_ε is rank-one connected only to its N neighbours and thus $K_N^{\text{lc},1}$ is in fact the 1-skeleton of the N -cube, i.e. it consists of the vertices and the edges $E_N \equiv K_N^{\text{lc},1} \setminus K_N$; note that each edge is an open segment parallel to a rank-one line.

We now want to compare $K_N^{\text{lc},2}$ with the 2-skeleton of the N -cube. We call a rank-one connection *trivial* if it exists in the 2-skeleton of the N -cube. A vertex is trivially connected to the N edges that have that vertex as one of their endpoints. An edge, which we write in the form $\{(\varepsilon_1, \dots, \varepsilon_{i-1}, t, \varepsilon_{i+1}, \dots, \varepsilon_N) : t \in [0,1]\}$, is trivially connected to the $N - 1$ edges that arise by flipping the sign of one of the ε_j , for $j \neq i$.

Associated to a fixed deformation, we consider two vectors, one with length 2^N and the other with length $N2^{N-1}$. In each of these vectors, the i -th entry represents the number of non-trivial edges to which the i -th vertex, respectively the i -th edge, is rank-one connected. We calculate the mean deviation of each of these vectors. Finally, sampling randomly thousands of deformations, we get approximate values for the average number of connections, see Tables 11.1 and 11.2.

	$m = 2$	$m = 3$
$N = 3$	0.95 (0.47)	0 (0)
$N = 4$	4.79 (1.57)	0 (0)
$N = 5$	15.59 (3.65)	0 (0)
$N = 6$	41.70 (8.31)	0 (0)

Table 11.1: Average number (and average mean deviation) of the number of non-trivial edges to which a vertex is rank-one connected.

Remark 11.3.1. We would like to make a few points concerning Tables 11.1 and 11.2:

	$m = 2$	$m = 3$
$N = 3$	2.90 (0.63)	0 (0)
$N = 4$	12.50 (2.39)	0 (0)
$N = 5$	36.78 (7.06)	0 (0)
$N = 6$	92.17 (18.28)	0 (0)

Table 11.2: Average number (and average mean deviation) of the number of non-trivial edges to which an edge is rank-one connected.

- (i) The values obtained should be understood in a probabilistic sense: it is not true that, when $m = 3$, there are never non-trivial connections. In fact, if we randomise vectors $a_i \in (\mathbb{Z} \cap [-L, L])^3, n_i \in (\mathbb{Z} \cap [-L, L])^2$ with L a small number, say $L = 5$, then we find non-trivial rank-one connections in many of the corresponding configurations.
- (ii) The low average mean deviations in the tables show that the connections are not concentrated in a few vertices or edges; see also Figure 11.1.
- (iii) When $m = 2$, an increase in N also increases the number of connections dramatically. Thus, although the set K_N becomes exponentially more complicated as N increases, the geometry of its rank-one lines also becomes much richer.

Remark 11.3.2. Rank-one lines are very fragile: even if sometimes rank-one connections exist, they are easily destroyed by small perturbations [230]. It is therefore more appropriate to consider the rank-one convex hull, which is often much larger than the lamination convex hull, albeit it is also much more difficult to calculate.

What we find the most remarkable about Tables 11.1 and 11.2 is not the fact that there are almost no rank-one connections when $m = 3$ but rather that there are so many connections when $m = 2$. Thus, in low-dimensions, simple lamination seems to be a viable option to produce very complex gradients. We believe that Tables 11.1 and 11.2 can be taken as partial evidence towards a positive answer to Question 9.1.7 when $m = n = 2$.

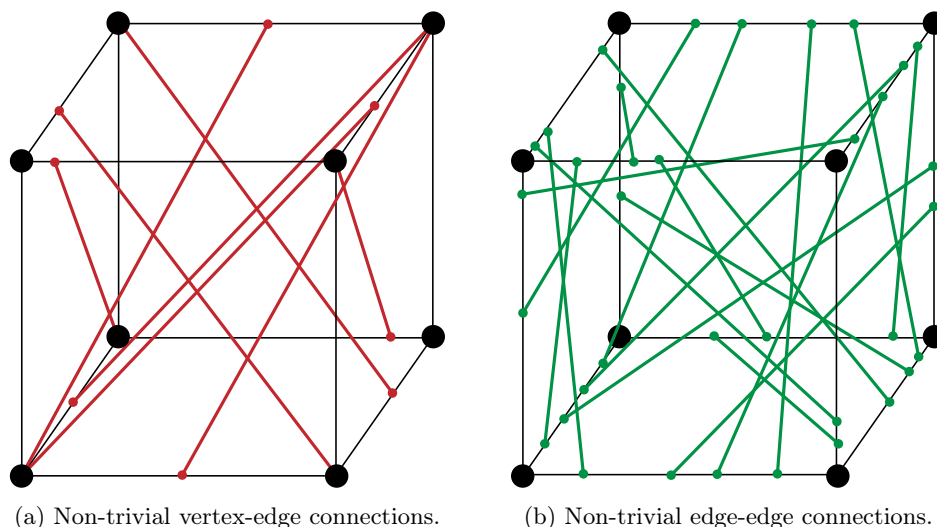


Figure 11.1: Depiction of a “typical” configuration when $N = 3$ and $m = 2$, with an average of 1 vertex-edge connection per vertex and 3 edge-edge connections per edge. Each line denotes the existence of at least one point in the edge which is rank-one connected.

11.4 Numerical search for counterexamples to Morrey's problem

In this section we report on numerical experiments which bring together Sections 11.1 and 11.2. Our goal was to find numerical evidence towards a resolution of Question 9.1.7.

We set $n = 2$ and run the following algorithm:

Algorithm 11.4.1. Fix $L, N \in \mathbb{N}$, with L odd and sufficiently large, and a threshold $\gamma \in [0, 1]$. Set $\mathcal{G}_L \equiv \frac{1}{L}\mathbb{Z}^{2m} \cap [-1, 1]^{2m}$ and choose a finite set of directions \mathcal{D} consisting of rank-one matrices which are in \mathcal{G}_L . Then:

1. Randomly generate a set of directions, $(n_i)_{i=1}^N$ in $([-L, L] \cap \mathbb{Z})^2$. Check that n_i and n_j are linearly independent for $i \neq j$; if not, repeat the previous instruction.
2. For each $(n_i)_{i=1}^N$, randomly generate a set of phases, $(c_i)_{i=1}^N \in \mathbb{R}$. The set $(n_i, c_i)_{i=1}^N$ determines the weights, $(\nu_\varepsilon)_\varepsilon$, at each point in the support of the measure, see (11.2.4).
3. Randomly generate a set of vectors $(a_i)_{i=1}^N$ in $(\frac{1}{L}\mathbb{Z} \cap [-1, 1])^m$. Check that $a_i \neq 0$ for all i and that the matrices X_ε where the measure is supported, defined in (11.2.4) in terms of $(a_i, n_i)_{i=1}^N$, are in \mathcal{G}_L ; if not, repeat the previous instruction.

Repeat Step 1 a number M_n of times; for each of those, repeat Step 2 M_c times; and for each $(n_i, c_i)_{i=1}^N$, generate M_a different sets $(a_i)_{i=1}^N$ by 3. We thus obtain $M_\nu \equiv M_n \times M_c \times M_a$ sets $(a_i, n_i, c_i)_{i=1}^N \in \mathbb{Z}^m \times \mathbb{Z}^2 \times \mathbb{R}$, each of which defines a measure ν supported on \mathcal{G}_L , see (11.2.3). Then, for each such ν , we execute the following:

4. Randomly generate vectors in $g \in [-1, 1]^{2^N}$ and, for each such vector, define a function f as in (11.1.2).
5. Apply the Kohn–Strang algorithm, as described in Algorithm 11.1.3, to calculate the approximation $f_{L, \mathcal{D}}^{\text{rc}}(0)$ of $f^{\text{rc}}(0)$.
6. Check whether $f_{L, \mathcal{D}}^{\text{rc}}(0) - \langle \nu, g \rangle > \gamma$. If so, pick another measure of those generated in Steps 1-3 and go back to Step 4. If not, and if this step hasn't been performed more than M_g times, using the same measure ν , go back to Step 4.

The measure ν is *suspicious* if it seems that Jensen's inequality fails, i.e. if at least one g generated in Step 4 is such that $f_{L, \mathcal{D}}^{\text{rc}}(0) - \langle \nu, g \rangle > 0$. Suspicious measures are further examined:

7. For each suspicious pair (ν, g) , make the changes $(L, \mathcal{D}) \mapsto (L', \mathcal{D}')$, where $L' = 2L - 1$ and $\#\mathcal{D} \leq \#\mathcal{D}'$, and rerun Step 5. Repeat the previous instruction as needed.

Remark 11.4.2. Note that the parameter γ ensures that, in Step 4, one keeps looking for g 's until one finds a sufficiently suspicious measure; we typically took $\gamma = 0.1$ and we note that in ŠVERAK's example $f^{\text{rc}}(0) - \langle \nu, f \rangle \approx \frac{1}{4}$. In fact, suppose that $0 < f_{L, \mathcal{D}}^{\text{rc}}(0) - \langle \nu, g \rangle \ll 1$; when refining the approximation of $f^{\text{rc}}(0)$ as in Step 7 it is likely that one finds $f_{L', \mathcal{D}'}^{\text{rc}}(0) - \langle \nu, g \rangle < 0$ and indeed this has often happened in our calculations.

We implemented Steps 1 and 2 of Algorithm 11.4.1, which determine the weights in the measure (11.2.3), in Mathematica, as it is well suited to computing ν_ε as given by (11.2.4). We note that, due to the complexity of this computation, we were unable to apply our algorithm to look for counterexamples with $N \geq 6$. Moreover, for $N = 3$, it follows from the work of SEBESTYÉN–SZÉKELYHIDI [335] that the admissible weights form a line segment in \mathbb{R}^8 , so it is enough to look for counterexamples at the endpoints. Only for $N = 4, 5$ do we, *a priori*, actually require $N_n \times N_c$ to be large in order to have a good sampling of the parameter space.

The bulk of Algorithm 11.4.1, i.e. Steps 3-7, was implemented in the C programming language. Our implementation is quite fast for $m = 2$: for instance with $L = 25$, $\#\mathcal{D} = 64$ and $M_g = 50$, it typically takes around 3 minutes to perform Steps 4-5, even when Step 5 is performed the maximum number of times. For $m > 2$, the algorithm has a very large computational cost: for instance, with $L = 19$ and $\#\mathcal{D} = 168$, it typically takes around 13 hours to perform Steps 4-5 a number $M_g = 50$ times. We remark that in this case the number of points in the grid is approximately 47×10^6 .

11.4.1 The case $m = 2$

For $m = 2$ we considered deformations given by sums of N plane waves with $N = 3, 4, 5$.

For $N = 3$, we verified numerically the analytical result of [335]. Using a gridsize of $L = 25$, we selected a total of 210 measures and randomised 50 different functions g , which were rank-one convexified using $\#\mathcal{D} = 64$ rank-one directions, see Table 11.3. About 5% of the pairs (ν, g) were found to be suspicious, though none above the threshold $\gamma = 0.1$. Upon rescaling the grid to $L' = 49$ and increasing the set of rank-one directions to $\#\mathcal{D}' = 256$, all but one of these pairs was shown to satisfy Jensen’s inequality; the remaining potential counterexample was ruled out by rescaling the grid again to $L' = 97$ and increasing $\#\mathcal{D}' = 784$.

It is for $N = 4, 5$, where Question 9.1.7 is open, that our results are most interesting. As the structure of the weights in these cases is unknown, we consider a much larger set of measures, around 1000, in our numerical tests; we have also increased the maximum number of functions g to test to 100×2^N , see Table 11.3. We have found that, when compared to a run for $N = 3$ with the same L and \mathcal{D} , in the case $N = 4, 5$ there is a drastic decrease in the percentage of suspicious measures initially flagged by Algorithm 11.4.1: when using a gridsize of $L = 25$ and $\#\mathcal{D} = 64$ rank-one directions, for example, only 0.06% of the pairs (ν, g) are found suspicious when $N = 4$ and none are flagged in this way when $N = 5$. From the point of view of our algorithm, Jensen’s inequality is clearly easier to verify as N increases, at least within the range of N we test, which could be explained by the increase in size of the 2nd lamination convex hull, c.f. Section 11.3. None of the pairs (ν, g) flagged as suspicious was found to be a counterexample after rescaling the grid to $L' = 49$ and increasing the set of rank-one directions to $\#\mathcal{D}' = 256$. We also tested configurations generated randomly in finer grids, having obtained identical results to the case $L = 25$.

To summarize: after testing thousands of randomly generated measures and hundreds of randomly generated functions, we have not found any counterexamples to Question 9.1.7.

N	M_n	M_c	M_a	M_ν	M_g
3	7	1	30	210	50
4	7	7	20	980	160
5	7	7	20	980	320

Table 11.3: Parameter space sampled in numerical experiments with $m = 2$.

11.4.2 The case $m = 3$

For $m = 3$ and $N = 3$, let us consider directions (n_1, n_2, n_3) which are non-degenerate in the sense that, for some choice of phases, there is $\varepsilon \in \{-1, 1\}^3$ with $\nu_\varepsilon \neq \frac{1}{8}$. It follows from the example in [351] that, with probability one, any such measure is a counterexample to Question 9.1.7. Due to the high computational cost of Algorithm 11.4.1 for $m = 3$, which severely limits our ability to explore the parameter space, we decided to focus only on $N = 3$ and attempt to recover these analytic results.

With a grid of size $L = 19$ and $\#\mathcal{D} = 168$, over the course of two weeks, we tested around 30 measures corresponding to 3-wave deformations. All but one measure was found to be suspicious and around 90% of the measures were found to be sufficiently suspicious, in the sense that there was one rank-one convexified function for which Jensen’s inequality failed by a margin superior to the threshold of $\gamma = 0.1$. We were unable to verify how many of our candidate counterexamples would survive after rescaling the grid (Step 7 of Algorithm 11.4.1), as those computations would take around a month per measure. However, these results are in agreement with what is known analytically for $N = 3$, further validating our implementation of Algorithm 11.4.1.

11.4.3 The case $m > 3$

It is interesting to consider GRABOVSKY’s example [168] of a rank-one convex, non quasiconvex function $G: \mathbb{R}^{8 \times 2} \rightarrow \mathbb{R}$. G is quasiconvex at zero, although not at the point $\text{Id}_{\mathbb{H}^2} \equiv e_1 \otimes e_1 + e_5 \otimes e_2$.

However, the paper [168] does not give an explicit deformation falsifying the quasiconvexity inequality (11.2.1); the deformation is only obtained indirectly through the variational principle for the effective tensor in periodic homogenization.

It would be interesting to find an explicit deformation falsifying the quasiconvexity inequality with respect to G and, in particular, to find the smallest value of N for which G is no longer N -wave quasiconvex. In an attempt to do so, we randomly generated deformations according to steps 1–3 of Algorithm 11.4.1. After testing hundreds of such deformations, and finding no counter-example to Jensen's inequality, we are led to suppose that G is N -wave quasiconvex for $N \leq 5$. As mentioned above, the case $N \geq 6$ is very computationally demanding.

We also note that there is a curious similarity between the plane-wave expansions of Section 11.2 and [168, equation (2.18)].

Part III

Nonlinear open mapping principles and the Jacobian equation

Chapter 12

Introduction

In Part III of this thesis we will discuss a circle of related problems concerning *underdetermined* nonlinear PDEs in low-regularity. One of the simplest equations of this type is the Jacobian equation,

$$Ju \equiv \det Du = f, \tag{12.0.1}$$

where f is a given integrable function and the solution is a map $u: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. This equation appears naturally in Optimal Transport [62, 114], Differential Geometry [19, 287] and Continuum Mechanics [96]. In fact, (12.0.1) has a geometric flavour: whenever u is a smooth solution of (12.0.1), we have

$$\int_E |f| dx = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) dy \quad \text{for any measurable set } E \subseteq \Omega, \tag{12.0.2}$$

where $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$.

The following basic question will occupy most of Part III of this thesis:

when can we solve (12.0.1), and what is the optimal regularity of solutions?

The answer to this question turns out to have interesting implications for problems closely related to those addressed in Parts I and II, as we will now briefly explain.

In Part I we characterized the class of functions which map solutions of constant rank PDEs into the real Hardy space $\mathcal{H}^1(\mathbb{R}^n)$; the Jacobian determinant is, of course, in that class. In their seminal paper, COIFMAN–LIONS–MEYER–SEMMES [87] addressed the question of whether the Jacobian determinant is onto $\mathcal{H}^1(\mathbb{R}^n)$; in other words, can one always solve (12.0.1) when $f \in \mathcal{H}^1(\mathbb{R}^n)$? Although they could not solve this problem, they showed, among other things, that $\mathcal{H}^1(\mathbb{R}^n)$ is the smallest Banach space containing the range of the Jacobian.

Part II is mostly concerned with quasiconvexity, and thereby with lower semicontinuity and with the existence theory for variational problems. A typical problem in the Calculus of Variations is to

$$\text{minimise } \mathcal{F}[u] \equiv \int_{\Omega} F(x, Du) dx, \quad \text{among all } u \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^n), \tag{12.0.3}$$

where $u_0: \bar{\Omega} \rightarrow \bar{O}$ is an orientation-preserving diffeomorphism and the integrand $F: \bar{\Omega} \times \text{GL}^+(n) \rightarrow \mathbb{R}$ is continuous [28, 29]. Besides weak lower semicontinuity, in order to apply the Direct Method it is crucial that \mathcal{F} be *coercive*; interestingly, coercivity is also closely related to quasiconvexity [82]. Nonetheless, one is sometimes led to consider *non-coercive* integrands: this is the case, for instance, when F depends on Du only through Ju , see e.g. [83, 142, 151] for examples in the study of elastic crystals and [96] for a different example. In this case, a solution of (12.0.3) is found, at least formally, by solving a Dirichlet problem for (12.0.1):

$$\begin{cases} Ju = f & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \quad \text{where } f \text{ is such that } F(x, f(x)) \equiv \min_{\xi > 0} F(x, \xi).$$

Thus we will also study the existence and regularity of solutions to the Dirichlet problem for (12.0.1).

The original results of Part III of this thesis are based on joint work with LUKAS KOCH and SAULI LINDBERG. We begin by recalling some standard tools from Geometric Function Theory in Chapter 13.

In Chapter 14 we will address the Dirichlet problem for (12.0.1) when f is in an L^p space. We will prove that, generically, one can at best hope for solutions with rather low regularity. The contents of this chapter are essentially contained in [180].

Chapter 15 concerns the possibility of using energy minimization as a selection criterion for (12.0.1); we will see, however, that uniqueness of energy minimizers is completely false and that for spherically symmetric data, even under symmetric boundary conditions, minimizers are not symmetric. The original results of this chapter have appeared in [179].

In Chapter 16 we will prove a nonlinear version of the classical Open Mapping Theorem from Functional Analysis, which applies to the nonlinear operators with $\mathcal{H}^1(\mathbb{R}^n)$ -integrability which we characterised in Part I. As an application of this result, we will prove a partial selection criterion for solutions of (12.0.1). This chapter is based on [178].

12.1 The Dirichlet problem for the Jacobian determinant

In this section we will consider the Dirichlet problem for (12.0.1) over the unit ball $B \subset \mathbb{R}^n$. For now we will also assume that $f \geq 0$, so that we may hope to find injective solutions, i.e. $\mathcal{N}(\cdot, u, E) = 1$ in (12.0.2), for any $E \subset B$.

Note that, for smooth solutions, the PDE formulation (12.0.1) and the geometric formulation (12.0.2) are *equivalent*. In low regularity, it is easier to solve the geometric formulation, as it admits solutions even when f is a measure:

Theorem 12.1.1. *Let μ be a Borel measure on \overline{B} . There is a homeomorphism $u: \overline{B} \rightarrow \overline{B}$ such that $\mu(E) = \mathcal{L}^n(u(E))$ for any Borel set $E \subset \overline{B}$ if and only if:*

- (i) μ is atomless;
- (ii) μ is positive on all non-empty open subsets of B ;
- (iii) μ vanishes on the boundary of B ;
- (iv) $\mu(\overline{B}) = \mathcal{L}^n(\overline{B})$.

In that case, we write $u^ \mathcal{L}^n = \mu$. Moreover, u can be chosen to be the identity on ∂B .*

Theorem 12.1.1 was first proved by VON NEUMANN in an unpublished work and was then reproved independently by OXTOPY and ULAM [309], but see also [9, Appendix 2]. In light of the boundary condition provided by the theorem, from now onward we will consider the Dirichlet problem for (12.0.1) with identity boundary data:

$$\begin{cases} Ju = f & \text{in } B, \\ u = \text{id} & \text{on } \partial B. \end{cases} \quad (12.1.1)$$

If one has an appropriate existence theory for (12.1.1) it is easy to obtain existence results for more general boundary conditions.

12.1.1 The classical theory

Theorem 12.1.1 is a statement about the existence of a pullback, but it does not say anything about the way in which the regularity of the map u is impacted by the regularity of μ . Let us consider the simplest case, which is when $\mu \ll \mathcal{L}^n$. It is easy to see that this is equivalent to u satisfying the Lusin (N) property, c.f. Definition 13.1.1. If additionally μ satisfies the conditions of Theorem 12.1.1, then $d\mu = f dx$ for some $f \in L^1(B)$ such that

$$\int_B f dx = 1, \quad f > 0 \text{ a.e. in } B. \quad (12.1.2)$$

Note that the first condition is simply a *compatibility condition* which follows by integration by parts or, equivalently, from the fact that the Jacobian is a null Lagrangian. In order to develop a regularity theory for the Jacobian equation, it is convenient to strengthen slightly the positivity hypothesis, and in fact we will often assume the following:

$$\int_B f \, dx = 1, \quad \operatorname{ess\,inf}_B f \geq c > 0. \quad (12.1.3)$$

Under these hypotheses, we have the following classical results:

Theorem 12.1.2. *In the next statements, we always assume that f satisfies (12.1.3).*

- (i) *If $f \in C^\infty(\overline{B})$ there is a smooth diffeomorphism u solving (12.1.1).*
- (ii) *If $f \in C^{k,\alpha}(\overline{B})$ there is a solution $u \in C^{k+1,\alpha}(\overline{B}, \overline{B})$ of (12.1.1) and moreover $u^{-1} \in C^{k+1,\alpha}(\overline{B}, \overline{B})$.*
- (iii) *If $f \in C^0(\overline{B})$ is Dini-continuous there is a solution $u \in C^1(\overline{B})$ of (12.1.1).*

In Theorem 12.1.2, (i) is due to MOSER [287], (ii) is due to DACOROGNA–MOSER [103], and (iii) is due to RIVIÈRE–YE [329]. We also note that YE proved in [372] results similar to (ii), but in the scale of Sobolev spaces. Further references can be found in the book [94].

We will now present an abstract framework to prove Theorem 12.1.2, as well as other variants; although the main ideas are well-known [103, 372], they are not systematized anywhere else. The basic idea is to employ a linearization procedure through the classical open mapping theorem:

Theorem 12.1.3. *Let X, Y be Banach spaces and let $T: X \rightarrow Y$ be a nonlinear operator which is Gâteaux-differentiable. Take $u_0 \in X$, let $f_0 \equiv Tu_0$, and suppose that*

- (i) *$DT(u_0): X \rightarrow Y$ is onto;*
- (ii) *T is Lipschitz in a neighbourhood of u_0 .*

There is $\varepsilon > 0$ such that, when $\|f - f_0\|_Y \leq \varepsilon$, there is $u \in X$ with $Tu = f$ and $\|u - u_0\|_X \leq C\|f - f_0\|_Y$.

For a proof we refer the reader to [117]. We are interested in applying Theorem 12.1.3 when T is a multilinear operator. In that case, condition (ii) is equivalent to boundedness:

Lemma 12.1.4. *Fix $m \in \mathbb{N}$. A bounded multilinear operator $T: X^m \rightarrow Y$ is locally Lipschitz continuous.*

Proof: Indeed, we have $DT(x_1, \dots, x_m)(v_1, \dots, v_m) = \sum_{i=1}^m T(x_1, \dots, x_{i-1}, v_i, x_{i+1}, \dots, x_m)$. □

We say that a Banach space Y of integrable functions over B is a *quasi-Banach algebra* if

$$\|fg\|_Y \leq C\|f\|_Y\|g\|_Y. \quad (12.1.4)$$

Combining the previous lemma with Theorem 12.1.3, we arrive at the following result, which gives existence of solutions for (12.1.1) whenever f is close to a constant in a quasi-Banach algebra:

Theorem 12.1.5. *Let X, Y be Banach spaces such that:*

- (i) *$X \subset W_0^{1,n}(B)$;*
- (ii) *$Y \subset L^1(B)$ is a quasi-Banach algebra;*
- (iii) *$\nabla: X \rightarrow Y_0^n$ is bounded and $\operatorname{div}: X^n \rightarrow Y_0$ is onto, where $Y_0 \equiv \{f \in Y : \int_B f \, dx = 0\}$.*

There are $\varepsilon, C > 0$ such that if $\|f - 1\|_Y \leq \varepsilon$ then there is a solution u of (12.1.1) with $\|u - \operatorname{id}\|_{X^n} \leq C$.

Proof: We begin with a simple observation from linear algebra. We have

$$\det(A + \operatorname{Id}) = (a^1 + e^1) \wedge \cdots \wedge (a^n + e^n),$$

where e^i is the canonical basis of \mathbb{R}^n and a^i is the i -th column of A . By expanding the above wedge product, we see that we may write

$$\det(A + \text{Id}) = G(A) + \text{tr } A + 1,$$

where $G = \sum_{k=2}^n G_k$ is a sum of polynomials G_k on the entries of k , each of which has degree k . Set $T(v) \equiv G(Dv) + \text{tr}(Dv) = G(Dv) + \text{div } v$; this is a multilinear operator. With $u = v + \text{id}$ and $f = g + 1$, we have

$$Ju = f \quad \iff \quad Tv = g.$$

Note that $DT(0) = \text{div}: X^n \rightarrow Y_0$, which by assumption is surjective. By Lemma 12.1.4, to apply Theorem 12.1.3 and conclude the proof, it suffices to check that $T: X^n \rightarrow Y_0 \subset Y$ is bounded. For $k = 2, \dots, n$, with $u \equiv (u^1, \dots, u^n) \in X^n$,

$$\|G_k(Du)\|_Y \lesssim \sum_{\sigma} \prod_{j=1}^k \|\partial_{\sigma(j)} u^j\|_Y \lesssim \prod_{j=1}^k \|\nabla u^j\|_{Y^n} \lesssim \prod_{j=1}^k \|u^j\|_X,$$

where σ runs over all possible permutations of the set $\{1, \dots, n\}$. In the first inequality above we used the fact that Y is an algebra, while in the last one we used the boundedness of $\nabla: X \rightarrow Y^n$. This concludes the proof. \square

We remark that the role of condition (i) is simply to ensure that the Jacobian determinant is meaningful, c.f. Definition 12.1.9 below. To remove the smallness condition in Theorem 12.1.5, we use an idea which goes back to [372], but see also [98]:

Corollary 12.1.6. *Consider the assumptions of Theorem 12.1.5 and assume also that $Y \subset C^0(\overline{B})$ and $1 \in Y$. Let Y_* be the set of those functions f in Y which can be connected to 1 through a path γ such that*

$$\gamma: [0, 1] \rightarrow Y \text{ has finite length,} \quad 1/\gamma: [0, 1] \rightarrow Y \text{ has finite length.} \quad (12.1.5)$$

Assume also that X^n is closed under composition and that

$$f \in 1 + Y_* \text{ and } u \in \text{id} + X^n \text{ solves (12.1.1)} \quad \implies \quad u^{-1} \in \text{id} + X^n. \quad (12.1.6)$$

Then, for any $f \in 1 + Y_*$ there exists a solution $u \in \text{id} + X^n$ of (12.1.1).

Proof: Let $\gamma: [0, 1] \rightarrow Y_*$ be a path such that (12.1.5) holds, $\gamma(0) = f$ and $\gamma(1) = 1$. Let $\varepsilon > 0$ be as in Theorem 12.1.5. Take $N \in \mathbb{N}$ to be a large number to be chosen later and consider a partition of $[0, 1]$ into $N + 1$ intervals $[t_i, t_{i+1}]$, where $t_i \equiv i/N$ and $i = 0, \dots, N$. We write

$$f_i \equiv \gamma(t_i), \quad \tilde{f}_i \equiv \lambda_i f_i, \quad \text{where } \lambda_i \equiv \int_B \frac{f(x)}{f_i(x)} dx.$$

Since $\int_B f dx = 1$, we have the simple estimate

$$|\lambda_i - \lambda_{i+1}| \leq \sup_B \left| \frac{1}{f_i} - \frac{1}{f_{i+1}} \right|;$$

the right-hand side goes to zero as $N \rightarrow \infty$, according to (12.1.5), as $Y \subset C^0(B)$ continuously. In fact, (12.1.5) also implies that $f_i > 0$ for any i and that we have $c \leq \lambda_i \leq C$, uniformly in i and N . Hence, if N is chosen large enough, and using (12.1.4) and again (12.1.5), we have

$$\left\| \frac{\tilde{f}_i}{\tilde{f}_{i+1}} - 1 \right\|_Y \lesssim \frac{|\lambda_i \|f_i - f_{i+1}\|_Y + |\lambda_i - \lambda_{i+1}| \|f_{i+1}\|_Y}{|\lambda_{i+1}| \|f_{i+1}\|_Y} \leq \varepsilon.$$

Let $\psi_0 = \text{id}$. By the last estimate, we may use Theorem 12.1.5 to define ψ_{i+1} inductively as a solution of

$$\begin{cases} \mathbf{J}\psi_{i+1} = \frac{\tilde{f}_i}{\tilde{f}_{i+1}} \circ \Psi_i^{-1} & \text{in } B, \\ \psi_{i+1} = \text{id} & \text{on } \partial B, \end{cases} \quad \text{where } \Psi_i \equiv \psi_i \circ \dots \circ \psi_0.$$

Indeed, note that $\mathbf{J}\Psi_i = f/\tilde{f}_i$ and hence the required compatibility condition is satisfied, i.e.

$$\int_B \frac{\tilde{f}_i}{\tilde{f}_{i+1}} \circ \Psi_i^{-1}(x) \, dx = \int_B \frac{\tilde{f}_i(y)}{\tilde{f}_{i+1}(y)} \mathbf{J}\Psi_i(y) \, dy = \int_B \frac{f(y)}{\tilde{f}_{i+1}(y)} \, dy = 1.$$

Moreover, $\psi_i \in \text{id} + X^n$ and so, by assumption, Ψ_i and Ψ_i^{-1} have the same regularity. The map $u \equiv \Psi_N$ is the required solution of (12.1.1). \square

In standard function spaces, one can take γ to be a straight line between 1 and f , and the abstract hypotheses (12.1.5)–(12.1.6) follow from the assumption that f is bounded away from zero, as in (12.1.3). The hypothesis (12.1.5) is self-explanatory, while the other hypotheses essentially ensure that the solution space is high-regularity, in the sense that it is closed under composition and inverses.

It is easy to deduce Theorem 12.1.2 from Corollary 12.1.6: the main thing to be checked is the surjectivity of the divergence between the relevant function spaces. For Theorem 12.1.2(ii), this surjectivity follows from the Schauder theory for the Poisson equation, while for Theorem 12.1.2(iii) it follows from the solvability of the divergence equation for Dini continuous data [48].

To conclude this section, we mention an alternative approach to Theorem 12.1.2(ii). In the previous approach, we saw the Jacobian equation as the nonlinear analogue of the divergence equation; however, we can also see it as the underdetermined analogue of the Monge–Ampère equation, c.f. Table 12.1. By doing so, one may use CAFFARELLI’s regularity theory [74, 75] for the Monge–Ampère equation in order to prove Theorem 12.1.2, see [78] for further details.

	Linear	Non-linear
Determined	$\Delta\phi = f$	$\det D^2\phi = f$
Underdetermined	$\text{div } u = f$	$\det Du = f$

Table 12.1: Particular solutions to the underdetermined problem are obtained by taking $u = D\phi$ for ϕ a solution of the corresponding determined problem.

The equations on the top row of Table 12.1 are *elliptic*; thus, it is natural to expect that, to find solutions with optimal regularity for the underdetermined equations, one should take the solution obtained by solving the corresponding determined elliptic problem. In a truly remarkable paper [54], BOURGAIN–BREZIS showed that this is not the case:

Theorem 12.1.7. *For every mean-free $f \in L^n(\mathbb{T}^n)$ there is $u \in C^0 \cap W^{1,n}(\mathbb{T}^n)$ such that $\text{div } u = f$.*

However, for some mean-free $f \in L^n(\mathbb{T}^n)$, the solution of $\text{div } u = f$ obtained by solving the Poisson equation is not bounded! In fact, the same happens for the Jacobian equation: solutions of the Monge–Ampère equation do not always have optimal regularity. We refer the reader to [372, page 293] for an instance where the solution obtained through Monge–Ampère is only $C^{1,1}$, although there are smooth solutions to (12.1.1). The non-optimality of the solution provided by the Monge–Ampère equation is particularly clear whenever the data does not have a sign, i.e. whenever (12.1.3) does not hold, as in that case the Monge–Ampère equation is no longer elliptic. Nonetheless, for the Jacobian equation, we have the following result [95]:

Theorem 12.1.8. *Take $k \geq 1$ and $f \in C^k(\overline{B})$ with $\int_B f \, dx = 1$. There is $u \in C^k(\overline{B})$ solving (12.1.1).*

We will discuss the case of data without a sign in greater detail below.

12.1.2 The Dirichlet problem in low regularity

It follows from Theorem 12.1.1 that there are always solutions to the geometric formulation of (12.1.1), as soon as $f \in L^1$ satisfies (12.1.2). Moreover, if f is Dini continuous or better and satisfies the stronger condition (12.1.3), one also has classical C^1 solutions of (12.1.1). It is thus natural to ask what happens in the low regularity setting between L^1 and C^0 . Prior to the original results presented in Part III of this thesis, almost nothing was known in this regime.

To be precise, let us distinguish between the geometric and the PDE formulations of our problem:

Definition 12.1.9. Let $f \in L^1(B)$ satisfy (12.1.2) and let $u: B \rightarrow \mathbb{R}^n$ be a map. We say that:

- (i) u is a *weak solution* of (12.1.1) if $u \in C^0(\overline{B})$, $u = \text{id}$ on ∂B and

$$\mathcal{L}^n(u(E)) = \int_E f \, dx \quad \text{for all measurable sets } E \subset B;$$

- (ii) u is a *distributional solution* of (12.1.1) if $u \in \text{id} + W_0^{1,n}(\overline{B}, \mathbb{R}^n)$ and $Ju(x) = f(x)$ for a.e. $x \in B$.

At first sight, it may seem strange that we refer to the solutions in Definition 12.1.9(ii) as distributional, since they are really *pointwise solutions*. However, it is well-known that for maps in $W^{1,n}$ the pointwise and the distributional Jacobians agree. One could also extend Definition 12.1.9(ii) to Sobolev spaces below $W^{1,n}$ but, in that case, the distributional Jacobian differs from the pointwise Jacobian and, instead of measuring the distortion of area, it measures the topological singularities of the maps. We will not discuss distributional Jacobians further here, instead referring the reader to [3, 28, 64, 242, 289].

For maps in $W^{1,n}$, the notions of weak and distributional solutions coincide:

Proposition 12.1.10. *Let $f \in L^1(B)$ satisfy (12.1.2) and let $u \in \text{id} + W_0^{1,n}(B, \mathbb{R}^n)$. Then u is a distributional solution if and only if it is a weak solution. If additionally (12.1.3) holds and $u \in W^{1,n(n-1)}(B, \mathbb{R}^n)$ is a distributional solution, then u is also a homeomorphism.*

The proof is a simple application of the theory of mappings of finite distortion and we refer the reader to Chapter 13 for further details. Due to Proposition 12.1.10, we will often talk about solutions of (12.1.1) without specifying which type of solution is meant.

In this thesis we are mostly interested in distributional solutions or, similarly, in weak solutions with Sobolev regularity. One obvious reason for considering solutions with Sobolev regularity is that we want to be able to make sense of the PDE (12.0.1). However, it may also be interesting to study solutions with Hölder regularity. Such solutions will not be considered further here, but we simply note that there are Hölder-continuous weak solutions exhibiting extremely wild behaviour:

Example 12.1.11 ([164, 165]). For any $\alpha \in (0, 1)$, there is a bi- α -Hölder continuous map u such that:

- (i) u is a weak solution of (12.1.1) for some $f \in L^1(B)$ satisfying (12.1.2);
(ii) u is approximately differentiable a.e. and $\det(\text{ap}Du) < 0$ a.e. in B .

For such maps, the *topological* and the *analytic* definition of degree are different.

We now consider (12.1.1) with data which is not Dini continuous. The first result in this direction is an ill-posedness result for data which is merely continuous:

Theorem 12.1.12. *For some $f \in C^0(B)$ satisfying (12.1.3) there are no Lipschitz solutions of (12.1.1).*

Theorem 12.1.12 was proved independently by BURAGO–KLEINER [70] and MCMULLEN [279]. The result is not completely surprising, due to the failure of end-point estimates: for instance, the divergence operator is not bounded in C^0 .

It is natural to ask what happens between L^1 and L^∞ . One of the main difficulties in this setting is that the spaces where the divergence and the Jacobian are bounded are different: in particular, in the

terminology of Section 12.1.1, the space L^p is not a quasi-Banach algebra whenever $p \in [1, \infty)$. In order to formulate the problem precisely, let us introduce the spaces

$$X_p(B) \equiv \begin{cases} \{f \in L^p(B) : f \text{ satisfies (12.1.3)}\} & \text{if } p > 1, \\ \{f \in L \log L(B) : f \text{ satisfies (12.1.3)}\} & \text{if } p = 1. \end{cases} \quad (12.1.7)$$

The particular definition of $X_1(B)$ is due to the improved integrability of the Jacobian determinant, discussed in Part I; one may think of $L \log L$ as the analogue of the Hardy space in our setting. Indeed, it was proved in [194] that, if $u \in \text{id} + W_0^{1,n}(B, \mathbb{R}^n)$ and $Ju \geq 0$ then $Ju \in L \log L(B)$. We then have the following natural question, see [372, Question 3] and [194, page 206]:

Question 12.1.13. Let $1 \leq p < \infty$. For $f \in X_p(B)$, does (12.1.1) have a solution $u \in W^{1,np}(B, \mathbb{R}^n)$?

The first major original result of Part III of this thesis shows that, surprisingly, the answer is negative:

Theorem 12.A (Non-existence of solutions with high regularity). *Fix $0 < c < 1$ and $1 \leq p < \infty$. There is $f \in X_p(B)$ such that (12.1.1) has no solution $u \in W^{1,np}(B, \mathbb{R}^n)$. In fact, for a Baire-generic $f \in X_p(B)$, (12.1.1) has no solution in the space $\bigcup_{n \leq q, p < q} W^{1,q}(B, \mathbb{R}^n)$.*

The restriction $n \leq q$ at the end is simply to ensure that the solutions are distributional. Without this constraint, the proof of Theorem 12.A still provides an ill-posedness statement, although not a non-existence one:

Corollary 12.B (Ill-posedness in spaces with high regularity). *Fix $1 \leq p < q < \infty$. For data in $X_p(B)$, (12.1.1) is ill-posed in $W^{1,q}(B, \mathbb{R}^n)$ in the following sense: one cannot find a function $C: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that, for every $f \in X_p(B)$, there is a weak solution of (12.1.1) with*

$$\|Du\|_{L^q(B)} \leq C(\|f\|_{X_p(B)}).$$

Let us briefly describe the proof of Theorem 12.A. The starting point of our proof is the observation that if $f \in X_p(B)$ is a radially symmetric function then the unique spherically symmetric solution of (12.1.1) is, in general, in $W^{1,p} \setminus W^{1,q}$ for any $q > p$, see Proposition 13.4.3. Hence we first show that any L^p function admits perturbations which are radially symmetric in a small neighbourhood of ∂B and such that the $W^{1,q}$ -norm of the symmetric solutions in that neighbourhood is unbounded. The crucial step of the proof is to show that, for these perturbations, the symmetric solutions have comparable q -Dirichlet energy to that of the energy-minimiser: hence we deduce that the $W^{1,q}$ -norm of q -energy-minimal solutions (see equation (14.1.2)) is unbounded at every $f \in X_p(B)$. The weak continuity of the Jacobian, together with an application of the Baire Category Theorem, then concludes the proof.

An interesting problem, left unanswered by the above results, is whether one can still find solutions with lower regularity. We state explicitly a particularly relevant question:

Question 12.1.14. For $n \leq p$ and $f \in X_p(B)$, is there a solution of (12.1.1) in $W^{1,p}(B, B)$?

A related but more general question is whether one always has ill-posedness whenever the data does not belong to a quasi-Banach algebra.

12.1.3 Energy minimisation as a selection criterion

Apart from the nonlinear character of the Jacobian, the main obstacle in studying existence and regularity of solutions of (12.0.1) is the *underdetermined* nature of the equation. A natural way to circumvent this issue is to look for solutions which minimise an appropriate energy function: it may be that such solutions are better behaved and unique. Since we are interested in determining the optimal Sobolev regularity of solutions to (12.0.1), the natural energy to consider is the np -Dirichlet energy:

Definition 12.1.15. Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $p \in [1, \infty)$. For $f \in L_{\text{loc}}^1(\Omega)$, define the np -energy of f as

$$\mathcal{E}_{np}(f, \Omega) \equiv \inf \left\{ \int_{\Omega} |Dv|^{np} dx : v \in \dot{W}^{1,np}(\Omega, \mathbb{R}^n) \text{ satisfies } Jv = f \text{ a.e. in } \Omega \right\}.$$

Given $f \in L^1_{\text{loc}}(\Omega)$, we say that $u \in \dot{W}^{1,np}(\Omega, \mathbb{R}^n)$ is a *np-energy minimiser* for f if

$$\int_{\Omega} |Du|^{np} dx = \mathcal{E}_{np}(f, \Omega) \quad \text{and} \quad Ju = f \text{ a.e. in } \Omega.$$

Due to the weak continuity of the Jacobian, we can apply the Direct Method to see that if there is a solution of $Ju = f$ in $\dot{W}^{1,np}(\Omega, \mathbb{R}^n)$ then there is an *np-energy minimiser* for f . We again contrast Definition 12.1.15 with that used in the theory of Optimal Transport, where typically one minimises

$$\int_{\Omega} \frac{|u(x) - x|^2}{2} f(x) dx.$$

This procedure selects the unique solution of (12.0.1) obtained from the Monge–Ampère equation, c.f. Table 12.1.

Establishing regularity of energy minimisers for (12.0.1) is a difficult task, even in the incompressible case $f = 1$. There is an extensive literature on the topic, and we refer the reader to [43, 80, 128, 223, 224], as well as the references therein, for further information.

In this subsection we are particularly concerned with the following question:

is energy minimisation an effective selection criterion?

We will also return to this point at the end of Section 12.2.

In order to keep technicalities to a minimum, we will henceforth focus on the case $n = 2$. Nevertheless most of the results discussed below permit straightforward extensions to the higher dimensional setting.

Whenever B is a ball centred at the origin and f is a radially symmetric function, i.e. $f = f(|z|)$, which satisfies (12.1.3), there is a unique radial stretching, denoted ϕ_1 , which solves (12.1.1):

$$\phi_1(z) \equiv \rho(|z|) \frac{z}{|z|}, \quad \text{where } \rho(r) \equiv \sqrt{\int_0^r 2sf(s) ds}.$$

Here the subscript ‘1’ denotes the topological degree of the map, see Definition 13.5.3 for more general solutions. We are thus led to the following very natural problem, c.f. [372, Question 9]:

Question 12.1.16 (Hélein). Let f be radially symmetric. Is the unique radial stretching ϕ_1 solving (12.1.1) a 2-energy minimiser¹ for f in the class $\text{id} + W_0^{1,2}(B, B)$?

In Theorem 12.C, we give a negative answer to Question 12.1.16 and we identify a class of data for which symmetry, and hence uniqueness, holds. Recall that the spaces $X_p(B)$ were defined in (12.1.7).

Theorem 12.C (Symmetry vs symmetry breaking of energy minimisers). *Energy minimisers are symmetric for some data but not for other. Precisely:*

- (i) Take $1 \leq p < \infty$ and let $f \in X_p(B)$ be radially symmetric. If, for a.e. $r \in (0, R)$,

$$f(r) \leq \int_{B_r(0)} f dx, \tag{12.1.8}$$

then the unique radial stretching ϕ_1 solving (12.1.1) is a $2p$ -energy minimiser for f and

$$\|D\phi_1\|_{L^{2p}(B)}^2 \lesssim \|f\|_{X_p(B)}, \tag{12.1.9}$$

Moreover, ϕ_1 is the unique $2p$ -energy minimiser for f in the class $\text{id} + W_0^{1,2p}(B, B)$.

- (ii) Fix $p_0 \in [1, \infty)$. There is $f \in L^\infty(B)$ satisfying (12.1.3) and such that, for any $p \in [1, p_0]$, ϕ_1 is not a $2p$ -energy minimiser for f in the class $\text{id} + W_0^{1,2p}(B, B)$. In this case there are uncountably many energy minimizers for f in $\text{id} + W_0^{1,2p}(B, B)$.

¹In higher dimensions, one should consider the n -harmonic energy, as done in Definition 12.1.15. We believe the original question in [372] has a misprint.

The local $L \log L$ integrability of non-negative Jacobians of $W^{1,n}$ maps was proved by MÜLLER in [290], and a global version, under suitable boundary regularity, was proved in [194], while here (12.1.9) is the reverse inequality, at least when $p = 1$.

Condition (12.1.8) is closely related to the regularity and symmetry properties of minimisers. Concerning the regularity, estimate (12.1.9) is in general false if f does not satisfy (12.1.8): Theorem 12.A shows that, for a Baire-generic f in $X_p(B)$, solutions of (12.1.1) are at best in $W^{1,p}(B)$, and not in $W^{1,2p}(B)$, as in (12.1.9). Concerning the symmetry, part (ii) of Theorem 12.C shows that, in the absence of (12.1.8), energy minimisers are in general not symmetric and not unique.

Let us now explain the role played by condition (12.1.8) in studying energy minimisation. Writing $z = re^{i\theta}$, note that

$$|Du|^2 = |\partial_r u|^2 + \frac{|\partial_\theta u|^2}{r^2}, \quad Ju = \frac{\partial_\theta u}{r} \wedge \partial_r u.$$

Hence we see that energy minimisation favours maps for which:

- (i) $\partial_r u$ is approximately perpendicular to $\partial_\theta u$, so that $|Ju| \approx |\partial_r u| |\partial_\theta u| / r$;
- (ii) $|\partial_r u| \approx \frac{1}{r} |\partial_\theta u|$, so that $|\partial_r u| \frac{|\partial_\theta u|}{r} \approx \frac{1}{2} (|\partial_r u|^2 + |\partial_\theta u|^2 / r^2) = \frac{1}{2} |Du|^2$.

Radial stretchings accomplish (i) perfectly: indeed,

$$D\phi_1(z) = \frac{\rho(r)}{r} \text{Id} + \left(\dot{\rho}(r) - \frac{\rho(r)}{r} \right) \frac{z \otimes z}{r^2} \implies \begin{cases} \partial_r \phi_1 = \dot{\rho}(r) z, \\ \frac{1}{r} \partial_\theta \phi_1 = \frac{\rho(r)}{r} z^\perp. \end{cases}$$

There is, however, no reason for radial stretchings to satisfy (ii), and this is where condition (12.1.8) comes in: it is easy to see that this condition is equivalent to

$$|\partial_r \phi_1| \leq \frac{1}{r} |\partial_\theta \phi_1|. \quad (12.1.10)$$

The isoperimetric inequality shows that ϕ_1 has optimal angular derivatives among all solutions of $Ju = f$, while (12.1.10) ensures that these derivatives control the Dirichlet energy $|D\phi_1|^2$, which enables us to prove that ϕ_1 is an energy minimiser.

A simple sufficient criterion for (12.1.10) to hold is that $r \mapsto f(r)$ is non-increasing. We also note that condition (12.1.10) is not new: a radial stretching satisfying (12.1.10) was called *conformally non-expanding* in [212], as it does not increase the conformal modulus of annuli.

12.2 The range of the Jacobian and a nonlinear open mapping principle

In this section we will study the Jacobian equation in low regularity over \mathbb{R}^n :

$$Ju = f \quad \text{a.e. in } \mathbb{R}^n. \quad (12.2.1)$$

In this setting, the following question, essentially set by COIFMAN, LIONS, MEYER and SEMMES in [87], remains an outstanding open problem:

Question 12.2.1. Fix $p \in [1, \infty)$. Is the Jacobian $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ surjective?

Here $\mathcal{H}^p(\mathbb{R}^n)$ stands for the real Hardy space, c.f. Section 2.1. We remind the reader that, for $p > 1$, $\mathcal{H}^p(\mathbb{R}^n)$ is simply the usual Lebesgue space $L^p(\mathbb{R}^n)$ and that if $f \in \mathcal{H}^1(\mathbb{R}^n)$ then necessarily $\int_{\mathbb{R}^n} f \, dx = 0$. In particular, there are *no* functions in $\mathcal{H}^1(\mathbb{R}^n)$ satisfying the positivity assumptions that we considered in the last section. Nonetheless, recall that we presented in Theorem 12.1.8 an existence result for sufficiently smooth but not necessarily positive data.

Question 12.2.1 is very natural as, in [87, 201], it was shown that $\mathcal{H}^p(\mathbb{R}^n)$ is the smallest Banach space containing the range of the Jacobian. In [206], see also [52], IWANIEC went further than Question 12.2.1 by conjecturing:

Conjecture 12.2.2. For each $p \in [1, \infty)$, the Jacobian has a *continuous right inverse*: there is a continuous map $E: \mathcal{H}^p(\mathbb{R}^n) \rightarrow \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ such that $J \circ E = \text{Id}$.

In [206], IWANIEC proposed the following route towards Conjecture 12.2.2:

Strategy 12.2.3. A possible way of proving Conjecture 12.2.2 is to establish the following claims:

- (i) Every np -energy minimiser satisfies $\|Du\|_{L^{np}(\mathbb{R}^n)}^n \lesssim \|Ju\|_{\mathcal{H}^p(\mathbb{R}^n)}$.
- (ii) For all $f \in \mathcal{H}^p(\mathbb{R}^n)$ there is a unique np -energy minimiser u_f for f , modulo rotations.
- (iii) For all $f \in \mathcal{H}^p(\mathbb{R}^n)$ there is a rotation $Q_f \in \text{SO}(n)$ such that $f \mapsto Q_f u_f$ is continuous.

In this section we will present several original contributions in the direction of Conjecture 12.2.2. The first new result of this section is a nonlinear open mapping theorem which shows that (i) is *equivalent* to a positive answer to Question 12.2.1.

Theorem 12.D (Open map principle for the Jacobian). *Fix $1 \leq p < \infty$. The following are equivalent:*

- (i) $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ is surjective;
- (ii) there is a bounded operator $E: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ such that $J \circ E = \text{Id}$;
- (iii) J is open at zero: for all $f \in \mathcal{H}^p(\mathbb{R}^n)$ there is $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ such that $Ju = f$ and

$$\|Du\|_{L^{np}(\mathbb{R}^n)}^n \lesssim \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}. \quad (12.2.2)$$

Theorem 12.D is surprising since, in general, the open mapping principle is false already for bilinear operators, *even if all the spaces involved are finite dimensional!* Indeed, the hope for a general nonlinear open mapping principle, which seems to go back to RUDIN [331, page 67], was quickly destroyed by counterexamples due to COHEN [86] and HOROWITZ [197], see also [46, 122]. Nonetheless, partial differential operators, and in particular weakly- $*$ continuous ones, are a very special class of operators and, as we will see in Section 12.3 below, for such operators nonlinear versions of the open mapping principle do hold. At this point we just mention that Theorem 12.D extends with only cosmetic changes to any of the weakly continuous operators characterized in Theorem 1.A, see Theorem 16.1.1.

Theorem 12.D may seem to be a purely abstract result, but this is not so, as we will now explain. In Proposition 12.1.10 we have seen that, whenever the data is non-negative, the weak and the distributional formulations of our problem agree; we further note that, when the data changes sign, the notion of weak solution has to be changed in order to account for the multiplicity of the map. Thus, in general, a weak solution of (12.2.1) is simply a continuous map which satisfies the change of variables formula (12.0.2). We then have the following two cases:

- if $p > 1$ then, even without sign assumptions, Proposition 12.1.10 extends to our setting: a map in $\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ solves (12.2.1) distributionally if and only if it is a weak solution;
- if $p = 1$ then distributional solutions are not necessarily weak solutions. Indeed, there are continuous maps in $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ which do not satisfy the change of variables formula (12.0.2)—for instance, the image of a null set under such a map can have positive measure! The existence of such maps is classical and goes back to the work of CESARI [79], but see also [269] for a more refined version. We also remind the reader that the Jacobian of *every* non-constant function in $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ changes sign, and hence positivity assumptions are meaningless in this context.

To summarize: the geometric and the PDE formulations of our problem are different when $p = 1$.

Question 12.2.1 is concerned with distributional solutions, but it is important to know whether such solutions are also weak solutions. In this direction, let us introduce the following definition:

Definition 12.2.4. A distributional solution u of (12.2.1) is *admissible over* $\Omega \subseteq \mathbb{R}^n$ if $u \in C^0(\Omega)$ and

$$\int_E f = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) \, dy \quad \text{for any measurable set } E \subseteq \Omega.$$

As we have just seen, distributional solutions in $\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$, for $p > 1$, are admissible over \mathbb{R}^n .

Using Theorem 12.D, one can show that the existence of rough solutions implies the existence of admissible solutions. This is made precise in the following result:

Theorem 12.E. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Take $f \in \mathcal{H}^1(\mathbb{R}^n)$ with $f \geq 0$ in Ω and assume that $J: \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^1(\mathbb{R}^n)$ is onto. Then there is a distributional solution $u \in \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ of (12.2.1) which is admissible over Ω :*

- (i) u is continuous in Ω ;
- (ii) u has the Lusin (N) property in Ω ;
- (iii) $\int_{\mathbb{R}^n} |Du|^n dx \leq C \|f\|_{\mathcal{H}^1}$ with $C > 0$ independent of f .

If moreover $n = 2$ and there is an open set $\Omega' \subseteq \Omega$ with $f = 0$ a.e. in Ω' , then:

- (iv) for any set $E \subset \Omega'$, we have $u(\partial E) = u(\overline{E})$;
- (v) for $y \in u(\Omega')$, if C denotes a connected component of $u^{-1}(y) \cap \Omega'$ then C intersects $\partial\Omega'$.

Theorem 12.E is proved through a regularisation argument, in particular involving the compactness of mappings of finite distortion.

The previous discussion shows that, to answer Question 12.2.1 in the positive, or even to prove Conjecture 12.2.2, one needs to show that energy minimisers satisfy the required a priori estimate. In this direction, IWANIEC suggested that one should prove (i) by constructing a Lagrange multiplier for every np -energy minimiser, see [262, 263] for results in this direction.

We now resume the discussion of Strategy 12.2.3; we have already addressed part (i). Concerning the uniqueness of energy minimisers proposed in (ii), one can easily adapt the proof of Theorem 12.C to \mathbb{R}^n to find conditions on the data under which both claims (i) and (ii) hold:

Corollary 12.F. *Let $f \in L^p(\mathbb{R}^n)$ be a radially symmetric function such that*

$$|f(r)| \leq \int_{B_r(0)} f dx \quad \text{for a.e. } r \in (0, \infty)$$

and let ϕ_1 be the unique radial stretching solving (12.2.1). Then

$$\|D\phi_1\|_{L^{np}(\mathbb{R}^n)}^n \lesssim \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}$$

and, for $p > 1$, ϕ_1 is the unique np -energy minimiser for f , modulo rotations. For $p = 1$ the same statement holds in the class of admissible solutions.

The reader may find further uniqueness results in [262]. Despite these positive results, the final original contribution of this section is to show that, in general, claim (ii) of Strategy 12.2.3 is false:

Theorem 12.G. *Fix $1 \leq p < \infty$. There is a radially symmetric function $f \in \mathcal{H}^p(\mathbb{R}^2)$ which has uncountably many $2p$ -energy minimisers, modulo rotations.*

Theorem 12.G shows that energy minimisation is not a suitable selection criterion. It is also very difficult to work with energy minimisers directly: when $p = 1$, we cannot decide whether they are admissible, although Theorem 12.E shows that, under natural assumptions, the existence of energy minimisers implies the existence of admissible solutions.

12.3 More general nonlinear open mapping principles, with applications to the equations of incompressible fluid flow

In this section we return to the open mapping principle for the Jacobian determinant, Theorem 12.D. Our aim is to state a general nonlinear open mapping principle that applies to a wide range of nonlinear PDEs. The reader can find precise statements, as well as more detailed examples, in [179].

For the sake of concreteness we will look at time-dependent equations. We consider the equation

$$Tu(x, t) = f(x, t), \quad (x, t) \in \mathbb{R}^n \times [0, \infty)$$

which we assume to be invariant under a one-parameter group of scalings:

$$\tau_\lambda^X[u](x, t) \equiv \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma}\right), \quad \tau_\lambda^Y[f](x, t) \equiv \frac{1}{\lambda^\delta} f\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma}\right). \quad (12.3.1)$$

Here X, Y are the Banach spaces where the solution and the data live, respectively, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are fixed and the group parameter is $\lambda > 0$. We also assume that X and Y are dual Banach spaces, equipped with a certain weak* topology. In our typical applications, f is the initial datum of a Cauchy problem and T is the solution-to-datum map.

Invariance under translations and scalings is an ubiquitous feature of physical processes. It is an example of the *covariance principle* that the solutions of a PDE representing a physical phenomenon should not have a form which depends on the location of the observer or the units that the observer is using to measure the system [69]. For the computation of the symmetry groups of several representative PDEs we refer to [305, §2.4] and for the general role of scaling symmetries in physics and other sciences to [41].

It often happens that a PDE has *several scaling symmetries*. For instance, the positive n -homogeneity of the Jacobian operator can be expressed as symmetry of the equation $Ju = f$ under the scaling $u_\lambda = \lambda u$, $f_\lambda = \lambda^n f$ for all $\lambda > 0$, but the Jacobian equation also has the scaling symmetry $u_\lambda(x) = \lambda u(x/\lambda)$, $f_\lambda(x) = f(x/\lambda)$. An important theme in this section is that, whenever a PDE has several scaling symmetries, these symmetries must be *compatible* in order for the equation to be solvable for all data.

The next result encapsulates the two previous points: many scale-invariant PDEs satisfy a nonlinear open mapping principle and, for the equation to be solvable, the associated scalings need to be compatible.

Theorem 12.H (General nonlinear open map principle, rough version). *Consider a constant-coefficient system of PDEs, posed over $\mathbb{R}^n \times [0, \infty)$, which moreover is preserved under weak* convergence. Let T be the solution-to-datum operator associated with the PDE.*

Suppose that the equation $Tu = f$ is invariant under the scalings (12.3.1) and that the solutions and the data lie in homogeneous function spaces, which satisfy, for some $r, s \in \mathbb{R}$,

$$\|\tau_\lambda^X[u]\|_{X^*} \equiv \lambda^r \|u\|_X, \quad \|\tau_\lambda^{Y^*}[f]\|_Y \equiv \lambda^s \|f\|_{Y^*}, \quad \text{where } rs > 0.$$

The following statements are then equivalent:

- (i) *For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$;*
- (ii) *For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$ and $\|u\|_{X^*}^{s/r} \leq C \|f\|_{Y^*}$.*

Moreover, suppose that T is invariant under another pair of scalings $\tilde{\tau}_\lambda^X, \tilde{\tau}_\lambda^Y$, which satisfy

$$\|\tilde{\tau}_\lambda^{X^*}[u]\|_X \equiv \lambda^{\tilde{r}} \|u\|_X, \quad \|\tilde{\tau}_\lambda^Y[f]\|_Y \equiv \lambda^{\tilde{s}} \|f\|_Y, \quad \text{where } \tilde{r}\tilde{s} > 0.$$

Then solvability of the equation $Tu = f$ requires compatibility of the scalings, i.e.

$$T \text{ is surjective} \quad \implies \quad r/s = \tilde{r}/\tilde{s}.$$

The first part of Theorem 12.H yields a generalisation of Theorem 12.D. We note that the assumption that r and s have the same sign, i.e. $rs > 0$, ensures that the norms in question are either *subcritical* or *supercritical*; the critical case $r = s = 0$ is beyond the scope of this thesis. We also refer the reader to [179] for solvability and a priori estimates in inhomogeneous function spaces.

Concerning the hypothesis of stability under weak* convergence, we note that it is typically satisfied by solutions above a certain regularity threshold: for instance, both the Navier–Stokes equations and the cubic wave equation in $\mathbb{R}^3 \times [0, +\infty)$ are preserved under weak* convergence in the corresponding energy spaces, see [179] for further details. Moreover, by considering a relaxed version of the PDE, the

assumption of stability under weak* convergence can sometimes be bypassed, an idea which we will now discuss.

In order to give a representative application of Theorem 12.H we consider energy-dissipating solutions of the incompressible Euler equations

$$\partial_t u + \operatorname{div}(u \otimes u) - \nabla p = 0, \tag{12.3.2}$$

$$\operatorname{div} u = 0, \tag{12.3.3}$$

$$u(\cdot, 0) = u^0, \tag{12.3.4}$$

in $\mathbb{R}^n \times [0, \infty)$, for $n \geq 2$. Note that equations (12.3.2)–(12.3.4) are invariant under scalings of the form

$$u_\lambda(x, t) \equiv \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right), \quad u_\lambda^0(x, t) \equiv \frac{1}{\lambda^\alpha} u^0\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right), \quad p_\lambda(x, t) \equiv \frac{1}{\lambda^{2\alpha}} p\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right) \tag{12.3.5}$$

for any $\alpha, \beta > 0$. An interesting connection between (12.3.2)–(12.3.3) and the Jacobian equation (12.2.1) is detailed in [298].

Before proceeding further, we note that the incompressibility constraint (12.3.3) will often be codified in the appropriate function spaces through the subscript div ; thus, for instance, we use the notation $L^2_{\operatorname{div}} \equiv \{v \in L^2 : \operatorname{div} v = 0\}$.

Solutions of (12.3.2)–(12.3.4) which fail to conserve energy have been studied extensively in relation to the so-called Onsager conjecture, see [68, 112, 204] and the references therein. By a theorem of SZÉKELYHIDI–WIEDEMANN [357], for a dense set of initial data in L^2_{div} there exist infinitely many *admissible* solutions $u \in L^\infty_t L^2_{\operatorname{div},x}$ of (12.3.2)–(12.3.4), that is, weak solutions which satisfy the energy inequality

$$\int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq \int_{\mathbb{R}^n} |u^0(x)|^2 dx \quad \text{for a.e. } t \geq 0.$$

Moreover, the data u^0 can be chosen to be C^β regular for any given $\beta \in (0, 1/3)$, at least on the torus [108].

For some data $0 \neq u^0 \in L^2_{\operatorname{div}}$ there are even admissible compactly supported solutions. SCHEFFER had already constructed in [333] solutions of the Euler equations which are compactly supported and square integrable in space-time, see also the work of SHNIRELMAN [338], and a systematic study via convex integration was initiated by DE LELLIS and SZÉKELYHIDI in the groundbreaking works [112, 113]. Nevertheless, Theorem 12.H easily implies that, for a Baire-generic datum in $L^2_{\operatorname{div}}(\mathbb{R}^n)$, the kinetic energy of weak solutions cannot decay too quickly. To state the result, let us say that a weak solution is *strongly dissipative* if there is $\varepsilon > 0$ such that

$$\int_{\mathbb{R}^n} |u(x, t)|^2 dx = O(t^{-1/2+\varepsilon}) \quad \text{as } t \rightarrow \infty.$$

We then have:

Corollary 12.I. *For a Baire-generic initial data in $L^2_{\operatorname{div}}(\mathbb{R}^n)$, the Cauchy problem (12.3.2)–(12.3.4) does not admit strongly dissipative solutions in $L^\infty([0, +\infty), L^2_x(\mathbb{R}^n))$.*

We contrast Corollary 12.I with the periodic setting: in [371], it is shown that for *any* initial datum $u^0 \in L^2_{\operatorname{div}}(\mathbb{T}^n)$ there is a (non-admissible) weak solution in $L^\infty_t L^2_x$ with exponential energy decay.

To deduce Corollary 12.I from Theorem 12.H we consider a *linear* relaxation of the equations (12.3.2)–(12.3.4); this is an idea in the spirit of TARTAR’s framework for studying oscillations and concentrations in conservation laws [361, 362], which was discussed at length in Part I of this thesis. Such a relaxation is used here in order to render the associated solution-to-datum operator weak*-to-weak* continuous.

The proof of Corollary 12.I also applies to many other models in fluid dynamics. For instance, concerning the Navier–Stokes equations, we prove in an elementary fashion upper bounds for the generic energy dissipation rate of weak solutions. Another example is given by the equations of ideal magneto-hydrodynamics, for which the analogue of Corollary 12.I holds true. In that context bounded solutions with compact support in space-time were constructed in [135]. On the torus \mathbb{T}^3 , solutions in $L^\infty_t H^\beta_x$, for a small $\beta > 0$, violating magnetic helicity conservation were constructed in [44]. We refer the reader to [179] for further details.

Chapter 13

Tools from Geometric Function Theory

This chapter collects, for the convenience of the reader, several results from Geometric Function Theory. The first two sections contain well-known results about the Lusin (N) property, the change of variables formula and mappings of finite distortion. Section 13.3 compares the weak and distributional formulations of the Dirichlet problem for the Jacobian equation. Section 13.4 contains notation and some basic properties of radial stretchings, while in Section 13.5 we present some technical results which allow us to write a Sobolev map in polar coordinates.

13.1 The change of variables formula

The following notions are very relevant in relation to the change of variables formula:

Definition 13.1.1. Let $u: \Omega \rightarrow \mathbb{R}^n$ be a map which is differentiable a.e. in Ω . Then:

- (i) u has the *Lusin (N) property* if $\mathcal{L}^n(u(E)) = 0$ for any $E \subset \Omega$ such that $\mathcal{L}^n(E) = 0$;
- (ii) u has the *(SA) property* if $\mathcal{L}^n(u(E)) = 0$ for any open set $E \subset \Omega$ with $Ju = 0$ a.e. in E .

In the one-dimensional case, the Lusin (N) property is well understood: for instance, on an interval, a continuous function of bounded variation has the Lusin (N) property if and only if it is absolutely continuous. However, in higher dimensions, the situation is much more complicated, although we have the following important result [269, 273]:

Proposition 13.1.2. *For $p > n$, maps in $W^{1,p}(\Omega, \mathbb{R}^n)$ always satisfy the Lusin (N) property. In fact, any Hölder continuous map in $W^{1,n}(\Omega, \mathbb{R}^n)$ satisfies the Lusin (N) property.*

In the critical case $p = n$ we have the following characterisation, proved in [275]:

Proposition 13.1.3. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map with $Ju \geq 0$ in Ω . Then u has the Lusin (N) property if and only if it has the (SA) property.*

We remark that Proposition 13.1.3 is in general false if $Ju \not\geq 0$, see [324] for a counterexample. The following result, see [269], is also useful for our purposes:

Proposition 13.1.4. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map such that, for some $K \geq 1$,*

$$\text{diam}(u(B_r(x))) \leq K \text{diam}(u(\partial B_r(x))) \quad \text{for all } B_r(x) \Subset \Omega. \quad (13.1.1)$$

Then u has the Lusin (N) property.

The change of variables formula is closely related to the Jacobian determinant and to the topological degree. Here we state some basic properties of the topological degree and we refer the reader to [143] for further details.

Lemma 13.1.5. *For $u \in C^0(\overline{\Omega}, \mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus u(\partial\Omega)$, the topological degree of u at x with respect to Ω , denoted by $\text{deg}(x, u, \Omega)$, has the following properties:*

- (i) $\deg(\cdot, u, \Omega) : \mathbb{R}^n \setminus u(\partial\Omega) \rightarrow \mathbb{Z}$;
- (ii) if $u = v$ on $\partial\Omega$, then $\deg(\cdot, u, \Omega) = \deg(\cdot, v, \Omega)$ on $\mathbb{R}^n \setminus u(\partial\Omega)$;
- (iii) if $x \notin u(\partial\Omega)$ and $\deg(x, u, \Omega) \neq 0$ then $x \in u(\Omega)$.

We then have:

Theorem 13.1.6. *Let $u \in C^0(\Omega, \mathbb{R}^n) \cap W^{1,n}(\Omega, \mathbb{R}^n)$ be a map with the Lusin (N) property. Then*

$$\int_E |Ju| \, dx = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) \, dy \quad \text{for all measurable sets } E \subset \Omega, \quad (13.1.2)$$

where \mathcal{N} is the multiplicity function, defined as $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$. We also have

$$\int_E Ju \, dx = \int_{\mathbb{R}^n} \deg(y, u, E) \, dy \quad \text{for all open sets } E \subset \Omega.$$

The reader may find the proof of Theorem 13.1.6, together with a wealth of information on geometric properties of Sobolev maps, in [143].

13.2 Mappings of finite distortion

In this section we gather a few results from the theory of mappings of finite distortion, which is detailed in [191, 210], see also [16] for the planar case.

Definition 13.2.1. Let $u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$ be such that $0 \leq Ju \in L_{\text{loc}}^1(\Omega)$. We say that u is a *map of finite distortion* if there is a function $K : \Omega \rightarrow [1, \infty]$ such that $K < \infty$ a.e. in Ω and

$$|Du(x)|^n \leq K(x) Ju(x) \quad \text{for a.e. } x \text{ in } \Omega.$$

If u has finite distortion, we can set $Ku(x) = \frac{|Du|^n}{Ju(x)}$ if $Ju(x) \neq 0$ and $Ku(x) = 1$ otherwise; this function is the (optimal) *distortion* of u .

We note that, in Definition 13.2.1, $|\cdot|$ denotes the operator norm of a matrix. We next summarise some of the key analytic and topological properties of mappings of finite distortion:

Theorem 13.2.2. *Let $\Omega \subset \mathbb{R}^n$ and let $u \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ be a map of finite distortion. Then:*

- (i) u has a continuous representative and, whenever $r < R$ and $B_R(x_0) \subset \Omega$,

$$\text{diam}(u(B_r(x_0)))^n \leq \frac{C}{\log(R/r)} \int_{B_R(x_0)} |Du|^n \, dx;$$

- (ii) u has the Lusin (N) property;
- (iii) u is differentiable a.e. in Ω .

Let $p : \mathbb{N} \rightarrow \mathbb{R}$ be such that $p(2) = 1$ and $p(n) > n - 1$ for $n \geq 2$. Then:

- (iv) if $Ku \in L^{p(n)}(\Omega)$ then u is open and discrete;
- (v) if $Ku \in L^{p(n)}(\Omega)$ then for each $\Omega' \Subset \Omega$ there is $m = m(\Omega')$ such that

$$\mathcal{N}(y, u, \Omega') \leq m \quad \text{for all } y \in u(\Omega').$$

We recall that a map is *discrete* if the preimage of every point is locally finite. Whenever u is a map of finite distortion we always implicitly assume that u denotes the continuous representative of the equivalence class in $W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$. If u is such that $Ku \in L^1(\Omega)$, we say that u has *integrable distortion*; in the plane, the theory of such maps was pioneered in [215].

We remark that the first three properties of Theorem 13.2.2 are a consequence of the fact that mappings of finite distortion are *monotone in the sense of Lebesgue*:

Proposition 13.2.3. *Let $u \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ be a map of finite distortion; then (13.1.1) holds. In fact, if we measure the diameter in \mathbb{R}^n with respect to the ℓ^∞ norm, we can take $K = 1$.*

To conclude this section, we note that the exponent $p(n)$ in Theorem 13.2.2(iv) is *optimal* for discreteness [193], but it remains an open problem whether $Ku \in L^{n-1}$ implies that u is open, as conjectured by IWANIEC and ŠVERÁK in [215]. Nonetheless, under mild assumptions on the behaviour of u near the boundary of Ω , the answer is known to be positive [192]:

Theorem 13.2.4. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map with $Ku \in L^{n-1}(\Omega)$. If there is a compact set $K \subset \Omega$ such that $u|_{\Omega \setminus K}$ is discrete then u is open and discrete.*

The statement of Theorem 13.2.4 is again optimal: a counterexample due to BALL [29] shows that the conclusion of Theorem 13.2.4 does not hold if the distortion is in $L^q(\Omega)$ for $q < n - 1$.

13.3 Weak and distributional solutions of the Jacobian equation

The purpose of this section is to show that weak and distributional solutions of the Dirichlet problem for the Jacobian equation agree whenever the solution is in $W^{1,n}$, c.f. Definition 12.1.9. In particular, we will prove Proposition 12.1.10 over a general smooth domain Ω :

Proposition 13.3.1. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a distributional solution of (12.1.1), assuming (12.1.2). Then u is a weak solution of (12.1.1), i.e. $u \in C^0(\bar{\Omega})$ and*

$$\mathcal{L}^n(u(E)) = \int_E f \, dx \quad \text{for all measurable sets } E \subseteq \Omega.$$

Conversely, if $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ is a weak solution of (12.1.1) then u is a distributional solution.

The proof is a simple consequence of the theory of maps of finite distortion.

Proof: The non-degeneracy assumption (12.1.2) implies that u is a map of finite, and even integrable, distortion. Hence, by Theorem 13.2.2, the change of variables formula of Theorem 13.1.6 applies:

$$\int_{u(E)} \mathcal{N}(y, u, E) \, dy = \int_E f \, dx, \quad \text{for all measurable sets } E \subseteq \Omega. \quad (13.3.1)$$

Moreover, u is injective a.e., that is $\mathcal{N}(y, u, \Omega) \leq 1$ for a.e. $y \in \mathbb{R}^n$: it suffices to apply (13.3.1) with $E = \Omega$, recalling that $\int_\Omega f \, dx = |\Omega|$ by (12.1.3). Hence the conclusion follows from (13.3.1).

Similarly, the converse direction follows from the fact that, under (12.1.2), weak solutions of (12.1.1) satisfy the Lusin (N) property, and hence Theorem 13.1.6 is applicable. \square

Moreover, distributional solutions in a sufficiently high Sobolev space are necessarily homeomorphisms:

Corollary 13.3.2. *If $u \in W^{1,n(n-1)}(\Omega, \mathbb{R}^n)$ solves (12.1.1), assuming (12.1.2), then $u \in \text{Hom}(\bar{\Omega}, \bar{\Omega})$.*

Proof: It follows from our assumptions that the distortion of u is in $L^{n-1}(\Omega)$. Consider a sufficiently small neighbourhood Ω_δ of Ω and extend u to be the identity in $\Omega_\delta \setminus \Omega$. Theorem 13.2.4, applied in Ω_δ , shows that $u|_\Omega$ is open and discrete and it follows that $u \in \text{Hom}(\bar{\Omega}, u(\bar{\Omega}))$, see e.g. [191, Theorem 3.27]. Moreover, $u(\bar{\Omega}) = \bar{\Omega}$. Indeed, it suffices to prove that $u(\Omega) \subseteq \bar{\Omega}$, since

$$u(\bar{\Omega}) = u(\Omega) \sqcup u(\partial\Omega) = u(\Omega) \sqcup \partial\Omega \quad (13.3.2)$$

where the unions are disjoint since u is a homeomorphism, and $\Omega \subseteq u(\Omega)$ by Lemma 13.1.5. Suppose that there is $x \in \Omega$ such that $u(x) \in \mathbb{R}^n \setminus \bar{\Omega}$. Take $y \in \Omega$ such that $u(y) \in \Omega$ and consider a continuous path in Ω joining x and y . The image of such a path under u must cross $\partial\Omega$ somewhere, which contradicts (13.3.2). \square

A simple argument using the change of variables formula shows that the inverse map u^{-1} is in $W^{1,n}(\Omega, \Omega)$. Note that the situation in the planar case is particularly pleasant, since then $n - 1 = 1$. Other results in the direction of Corollary 13.3.2 can be found in [29, 347].

13.4 Radial stretchings

A function $f: B_R(0) \rightarrow \mathbb{R}$ is said to be *radially symmetric* if $|x| = |y| \implies f(x) = f(y)$ and we identify any such function with a function $f: [0, +\infty) \rightarrow \mathbb{R}$ in the obvious way. For such a function, it is natural to look for solutions of (12.2.1) possessing some symmetry:

Definition 13.4.1. The class of *radial stretchings* consists of maps of the form $\phi(x) \equiv \rho(|x|) \frac{x}{|x|}$, for some function $\rho: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$.

The following is a useful criterion concerning the Sobolev regularity of radial stretchings:

Lemma 13.4.2. *Let $p \in [1, \infty)$. Given $R \in (0, +\infty)$, $\phi \in \dot{W}^{1,p}(B_R(0), B_R(0))$ if and only if ρ is absolutely continuous on $(0, R)$ and*

$$\|\mathbf{D}\phi\|_{L^p(B_R(0))}^p \approx_n \int_0^R \left(|\dot{\rho}(r)|^p + \left| \frac{\rho(r)}{r} \right|^p \right) r^{n-1} dr < \infty.$$

In this case, for a.e. x in $B_R(0)$, and writing $r = |x|$, we have the formulae

$$\mathbf{J}\phi(x) = \frac{1}{r^{n-1}} \dot{\rho}(r) \rho^{n-1}(r), \quad (13.4.1)$$

$$\mathbf{D}\phi(x) = \frac{\rho(r)}{r} \text{Id} + \left(\dot{\rho}(r) - \frac{\rho(r)}{r} \right) \frac{x \otimes x}{r^2}. \quad (13.4.2)$$

We refer the reader to [30, Lemma 4.1] for a proof. We also record here the standard notation $\partial_r u(x) \equiv (\mathbf{D}u(x)) \cdot \frac{x}{r}$; in particular, for a radial stretching ϕ , we have

$$\partial_r \phi(x) = \dot{\rho}(r) \frac{x}{r}. \quad (13.4.3)$$

From Lemma 13.4.2, we can easily deduce a sharp result concerning the regularity of radial stretchings:

Proposition 13.4.3. *Let $\Omega = B_R(0)$ and $p \in [1, +\infty]$. If $f \in L^p(\Omega)$ is radially symmetric and satisfies (12.1.3) then there is a unique radial stretching $\phi(x) = \rho(r) \frac{x}{r}$ solving (12.2.1). It satisfies*

$$\rho^n(r) = \int_0^r n f(s) s^{n-1} ds. \quad (13.4.4)$$

Further, $\phi \in W^{1,p}(B_R(0))$ and

$$\|\mathbf{D}\phi\|_{L^p(B_R(0))} \lesssim_n \frac{\|f\|_{L^p(B_R(0))}}{(\inf f)^{\frac{n-1}{n}}} + R^{\frac{n-1}{p}} \|f\|_{L^p(B_R(0))}^{\frac{1}{n}}. \quad (13.4.5)$$

Moreover, this inequality is sharp: in general $\phi \notin W^{1,q}$ for any $q > p$.

Proof: The existence and uniqueness of a radial stretching ϕ solving (12.2.1) follows immediately from (13.4.1). We turn to the regularity properties and assume that $1 \leq p < \infty$, as the case $p = \infty$ is similar. Using (13.4.4) and Jensen's inequality, we deduce that

$$\left| \frac{\rho(r)}{r} \right|^p \lesssim \left(\int_{B_r(0)} f(x) dx \right)^{\frac{p}{n}} \leq \frac{1}{r} \left(\int_{B_r(0)} f(x)^p dx \right)^{\frac{1}{n}} \leq \frac{1}{r} \|f\|_{L^p(B_R(0))}^{\frac{p}{n}}$$

and therefore, integrating in r ,

$$\int_0^R \left| \frac{\rho(r)}{r} \right|^p r^{n-1} dr \lesssim R^{n-1} \|f\|_{L^p(B_R(0))}^{\frac{p}{n}}.$$

Since f satisfies (12.1.3), we deduce using (13.4.4) that $(\inf f)^{\frac{1}{n}} \leq \rho(r)/r$. Thus, from (13.4.1), we have

$$\dot{\rho}(r) = \frac{r^{n-1}}{\rho^{n-1}(r)} f(r) \implies \int_0^R |\dot{\rho}(r)|^p r^{n-1} dr \lesssim \frac{\|f\|_{L^p(B_R(0))}^p}{(\inf f)^{\frac{p(n-1)}{n}}}$$

and the desired estimate follows from Lemma 13.4.2. To see that this inequality is sharp, it suffices to note using (13.4.1) that for $\delta > 0$ there is a constant C_δ such that $C_\delta f(r) \leq \dot{\rho}(r)$ for $r \in (\delta, R)$. By choosing $f \in L^p(0, R)$ such that $f \notin L^q(\delta, R)$ for $q > p$ we have $Du \notin L^q(B_R(0))$. \square

13.5 Polar coordinates for Sobolev maps

In this section we set, for simplicity, $n = 2$. Given a planar Sobolev map $u \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$, we consider polar coordinates both in the domain and in the target; that is, we want to write

$$u(re^{i\theta}) = \psi(r, \theta) \exp(i\gamma(r, \theta)) \quad (13.5.1)$$

for some functions $\psi: (0, \infty) \times [0, 2\pi] \rightarrow [0, \infty)$ and $\gamma: (0, \infty) \times [0, 2\pi] \rightarrow \mathbb{R}$, where furthermore we must have the compatibility conditions

$$\psi(r, 0) = \psi(r, 2\pi) \quad \text{and} \quad \gamma(r, 0) - \gamma(r, 2\pi) \in 2\pi\mathbb{Z} \quad \text{for all } r. \quad (13.5.2)$$

We will freely identify $(r, \theta) \equiv re^{i\theta}$, adopting either notation whenever it is more convenient.

The existence of a representation as in (13.5.1) is a standard problem in lifting theory:

Proposition 13.5.1. *Let $0 \leq R_1 < R_2$ and $p \geq 2$. Let $u \in W^{1,p}(\mathbb{A}(R_1, R_2), \mathbb{R}^2)$ and, if $p = 2$, suppose moreover that u is continuous. Assume $u^{-1}(0) \subseteq \{0\}$. Then there are continuous functions*

$$\psi \in W^{1,p}([R_1, R_2] \times [0, 2\pi]), \quad \gamma \in W^{1,p}((\max\{R_1, \varepsilon\}, R_2) \times [0, 2\pi]),$$

where $\varepsilon \in (0, R_2)$ is arbitrary, which satisfy (13.5.2) and such that the representation (13.5.1) holds.

Proof: Let $\varepsilon > 0$ and consider the keyhole domains

$$\begin{aligned} \mathbb{A}_{1,\varepsilon} &\equiv [\max\{R_1, \varepsilon\}, R_2] \times [\varepsilon, 2\pi - \varepsilon], \\ \mathbb{A}_{2,\varepsilon} &\equiv [\max\{R_1, \varepsilon\}, R_2] \times ([0, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi]). \end{aligned}$$

We freely identify $\mathbb{A}_{i,\varepsilon}$ with the respective domains in \mathbb{R}^2 .

We first show the existence of a representation (13.5.1) in each $\mathbb{A}_{i,\varepsilon}$. Note that if $u \in W^{1,p}$ then $\psi = |u|$ is also in $W^{1,p}$ and is continuous whenever u is. Thus, since $0 \notin u(\mathbb{A}_{i,\varepsilon})$, it suffices to prove the existence of a continuous function $\gamma_i \in W^{1,p}(\mathbb{A}_{i,\varepsilon}, \mathbb{R})$ such that $u/|u| = e^{i\gamma_i}$ for $i = 1, 2$. Since u is continuous, $u/|u| \in W^{1,p}(\mathbb{A}_{i,\varepsilon}, \mathbb{S}^1)$, and so the existence of γ_i follows from the results in [49], see also [55].

Thus, for almost every $(r, \theta) \in \mathbb{A}_{1,\varepsilon} \cap \mathbb{A}_{2,\varepsilon}$,

$$\psi(r, \theta) e^{i\gamma_1(r, \theta)} = u(re^{i\theta}) = \psi(r, \theta) e^{i\gamma_2(r, \theta)} \iff \gamma_1(r, \theta) - \gamma_2(r, \theta) = 2\pi k(r, \theta),$$

where $k(r, \theta) \in \mathbb{Z}$. As γ_1, γ_2 are continuous in $\mathbb{A}_{1,\varepsilon} \cap \mathbb{A}_{2,\varepsilon}$, we must have

$$k(r, \theta) = \begin{cases} k_1 & \text{for } \varepsilon < \theta < \pi - \varepsilon, \\ k_2 & \text{for } \pi + \varepsilon < \theta < \varepsilon. \end{cases}$$

Without loss of generality, upon redefining γ_1 we may assume $k_1 = 0$. Hence we may define

$$\gamma_\varepsilon(r, \theta) = \begin{cases} \gamma_1(r, \theta) & \text{if } (r, \theta) \in \mathbb{A}_{1,\varepsilon}, \\ \gamma_2(r, \theta) & \text{if } (r, \theta) \in \mathbb{A}_{2,\varepsilon}. \end{cases}$$

By a similar argument, we see that we may take $\gamma_\varepsilon = \gamma_\delta$ in $\mathbb{A}(R_1, R_2) \setminus (B_\delta \cup B_\varepsilon)$, so that in fact $u = \psi(r, \theta) e^{i\gamma(r, \theta)}$ with $\gamma \in W^{1,p}(\max\{R_1, \varepsilon\}, R_2) \times [0, 2\pi]$ for all $\varepsilon > 0$. The conclusion follows. \square

We remark that the conclusion of Proposition 13.5.1 is false if $p < 2$, see [55, §4].

Corollary 13.5.2. *In the setting of Proposition 13.5.1, we have a.e. the formulae*

$$Ju = \frac{1}{2r} \frac{\partial(\psi^2, \gamma)}{\partial(r, \theta)} = \frac{1}{2r} \left(\partial_r(\psi^2) \partial_\theta \gamma - \partial_\theta(\psi^2) \partial_r \gamma \right), \quad (13.5.3)$$

$$|Du|^2 = |\partial_r \psi|^2 + |\psi \partial_r \gamma|^2 + \frac{|\partial_\theta \psi|^2}{r^2} + \frac{|\psi \partial_\theta \gamma|^2}{r^2}. \quad (13.5.4)$$

Proof: It is not difficult to formally derive the above formulae whenever the representation (13.5.1) holds. To make the argument rigorous it suffices to note that, due to the regularity of ψ and γ , the right-hand sides in (13.5.3)–(13.5.4) define locally integrable functions. Thus the corollary follows by a standard density argument. \square

There is a natural class of maps which admits a representation as in (13.5.1) and which extends the class of radial stretchings:

Definition 13.5.3. The class of *generalised radial stretchings* consists of maps of the form

$$\phi_k(z) \equiv \frac{\rho(r)}{\sqrt{|k|}} e^{ik\theta}$$

where $k \in \mathbb{Z} \setminus \{0\}$ is the topological degree of the map and $\rho \geq 0$.

Note that generalised radial stretchings are spherically symmetric in the sense that they map circles centred at zero to circles centred at zero.

Chapter 14

A non-existence result in the Dirichlet problem for the Jacobian equation

The purpose of this chapter is to prove Theorem 12.A, concerning non-existence of solutions to the Dirichlet problem. This result was originally proved in [180], which is joint work with LUKAS KOCH and SAULI LINDBERG.

14.1 Preliminary notation

Throughout this chapter we fix $c \in (0, 1)$. For $p \in [1, +\infty)$ and $\eta \in [0, 1)$, let

$$\begin{aligned} Z_p^\eta &\equiv \left\{ f \in L^p(B_1) : \int_{B_1} f \, dx = 1, f \geq (1 - \eta)c \right\}, & \text{for } p > 1 \\ Z_1^\eta &\equiv \left\{ f \in L \log L(B_1) : \int_{B_1} f \, dx = 1, f \geq (1 - \eta)c \right\}; \end{aligned} \tag{14.1.1}$$

and note that $X_p \equiv Z_p^0$, c.f. (12.1.7). We make a few immediate remarks about the sets Z_p^η :

- (i) clearly $Z_p^{\eta_1} \subset Z_p^{\eta_2}$ if $\eta_1 < \eta_2$;
- (ii) each set Z_p^η is a complete metric space under the distance

$$\text{dist}_p(f, g) \equiv \begin{cases} \|f - g\|_{L^p(B_1)}, & p > 1; \\ \|f - g\|_{L \log L(B_1)}, & p = 1; \end{cases}$$

- (iii) if $c = 1$ then the only elements of X_p are functions which are 1 a.e. in B_1 . Thus without loss of generality we assume throughout this chapter that $c < 1$.

We write $B_{Z_p^\eta}(f, \varepsilon)$ for a ball of radius ε around f , under this distance. Given $f \in X_p$ and $\max\{p, n\} \leq q$, we recall that in Definition 12.1.15 we introduced the q -energy of f :

$$\mathcal{E}_q(f) \equiv \inf \left\{ \int_{B_1} |Dv|^q \, dx : v \text{ solves (12.1.1)} \right\}. \tag{14.1.2}$$

Thus (12.1.1) admits a $W^{1,q}$ solution if and only if $\mathcal{E}_q(f) < \infty$. We say that w is a q -energy-minimal solution for f if w solves (12.1.1) and moreover

$$\mathcal{E}_q(f) = \int_{B_1(0)} |Dw|^q \, dx.$$

14.2 The proof of Theorem 12.A

We begin with the following lemma:

Lemma 14.2.1. *Fix $1 \leq p < q < \infty$ and let $\varepsilon, \eta > 0$ be arbitrary. For any $f \in X_p$ there are sequences $f_j \in B_{Z_p^n}(f, \varepsilon)$ and $R_j \nearrow 1$ such that*

- (i) $\text{dist}_p(f_j, f) \rightarrow 0$ as $j \rightarrow \infty$;
- (ii) f_j is radially symmetric in $\mathbb{A}(R_j, 1)$;
- (iii) if u_j is the radial stretching such that $Ju_j = f_j$ in $\mathbb{A}(R_j, 1)$ and $u_j = \text{id}$ on \mathbb{S}^{n-1} , then

$$\lim_{j \rightarrow \infty} \int_{\mathbb{A}(R_j, 1)} |\partial_r u_j(x)|^q dx = +\infty.$$

Proof: For $\gamma, R \in (\frac{3}{4}, 1)$, consider the functions

$$f_{\gamma, R}(x) \equiv \begin{cases} \frac{\gamma^n}{\int_{B_R} f dx} f(x), & 0 < |x| < R, \\ \frac{M[\gamma R + M(|x| - R)]^{n-1}}{|x|^{n-1}}, & R \leq |x| < 1, \end{cases} \quad \text{where } M \equiv \frac{1 - \gamma R}{1 - R}. \quad (14.2.1)$$

The choice of M ensures that $\int_{B_1} f_{\gamma, R} dx = 1$, since

$$\int_R^1 f_{\gamma, R}(r) r^{n-1} dr = \frac{1 - (\gamma R)^n}{n}. \quad (14.2.2)$$

Fix $\alpha \in (-\frac{q}{p}, -1)$ and choose $\gamma = \gamma(R)$ in such a way that

$$1 - \gamma R = (1 - R)^{1+\alpha/q}; \quad (14.2.3)$$

in particular, we have $\gamma(R) \rightarrow 1$ as $R \rightarrow 1$. Take any sequence $R_j \nearrow 1$ and consider the associated numbers $\gamma_j = \gamma_j(R_j)$. We will prove that, for j large enough, $f_j \equiv f_{\gamma_j, R_j}$ satisfies the properties above.

Concerning the lower bounds of the sequence, for $r \in (R, 1)$, straightforward algebraic manipulations show that

$$f_{\gamma, R}(r) = M \left(\frac{\gamma R + M(r - R)}{r} \right)^{n-1} \geq 1 \times \gamma^{n-1};$$

thus $f_{\gamma, R}(r) > c$ for γ sufficiently close to 1. For $r \in (0, R)$, clearly $f_{\gamma, R}(r) \geq \gamma^n c / \int_{B_R} f \rightarrow c$ as $\gamma, R \rightarrow 1$. Hence the lower bounds are satisfied. Moreover, (ii) clearly holds.

For (iii), denote by $u_{\gamma, R}(x) = \rho_{\gamma, R}(r) \frac{x}{r}$ the unique radial stretching solving $Ju_{\gamma, R} = f_{\gamma, R}$ in $\mathbb{A}(R, 1)$ and such that $u_{\gamma, R} = \text{id}$ on \mathbb{S}^{n-1} . By Lemma 13.4.2 we find that

$$\rho_{\gamma, R}(r) = \gamma R + M(r - R), \quad r \in (R, 1] \quad (14.2.4)$$

and so, by (13.4.3), since $r^{n-1} dr \approx dr$ for $r \in (\frac{1}{2}, 1)$, and recalling our choice of γ in (14.2.3),

$$\int_{\mathbb{A}(R, 1)} |\partial_r u_{\gamma, R}(x)|^q dx \approx \int_R^1 |\dot{\rho}(r)|^q dr = M^q (1 - R) = \frac{(1 - \gamma R)^q}{(1 - R)^{q-1}} = (1 - R)^{1+\alpha}.$$

Thus, since $\alpha < -1$, we see that (iii) also holds.

Hence it remains to prove (i), and we split the proof into two cases.

Case $p > 1$: Whenever $\gamma, R \nearrow 1$, we have

$$\int_{B_R} |f - f_{\gamma, R}|^p dx = \left| \frac{\gamma^n}{\int_{B_R} f dx} - 1 \right|^p \int_{B_R} |f|^p dx \rightarrow 0.$$

Thus, since $\int_{\mathbb{A}(R,1)} |f(x)|^p dx \rightarrow 0$ as $R \rightarrow 1$, in order to prove that $f_{\gamma,R} \rightarrow f$ in $L^p(B_1)$ it suffices to show that, as $\gamma, R \nearrow 1$,

$$\int_{\mathbb{A}(R,1)} |f_{\gamma,R}(x)|^p dx \rightarrow 0. \quad (14.2.5)$$

For this, note that for $|x| = r \in (R, 1)$, since $\gamma R < 1$ and $R > \frac{1}{2}$,

$$f_{\gamma,R}(x) \leq \frac{M}{r^{n-1}} \leq 2^{n-1} \frac{(1-\gamma R)}{1-R}.$$

Thus, as $r^{n-1} dr \approx dr$ for $r \in (\frac{1}{2}, 1)$, and using (14.2.3),

$$\int_{\mathbb{A}(R,1)} |f_{\gamma,R}(x)|^p dx \approx \int_R^1 |f_{\gamma,R}(r)|^p dr \lesssim \frac{(1-\gamma R)^p}{(1-R)^{p-1}} = (1-R)^{\alpha \frac{p}{q} + 1}.$$

Since $\alpha > -\frac{q}{p}$ we have that $\alpha \frac{p}{q} + 1 > 0$ and so (14.2.5) is proved.

Case $p = 1$: As before, we have that, as $\gamma, R \nearrow 1$,

$$\|f_{\gamma,R} - f\|_{L \log L(B_R)} = \left| 1 - \frac{\gamma^n}{\int_{B_R} f dx} \right| \|f\|_{L \log L(B_R)} \rightarrow 0$$

and also $\|f\|_{L \log L(\mathbb{A}(R,1))} \rightarrow 0$ as $R \rightarrow 1$. Thus it suffices to show $\|f_{\gamma,R}\|_{L \log L(\mathbb{A}(R,1))} \rightarrow 0$ as $R \rightarrow 1$. We make the simple observation that

$$1 - (\gamma R)^n = (1 - \gamma R)(1 + \gamma R + \cdots + (\gamma R)^{n-1}) \approx_n (1 - \gamma R).$$

Then, using the estimates on $f_{\gamma,R}$ from the previous case and (14.2.2), we obtain

$$\begin{aligned} \|f_{\gamma,R}\|_{L \log L(\mathbb{A}(R,1))} &\approx \int_{\mathbb{A}(R,1)} f_{\gamma,R}(x) \log \left(e + \frac{f_{\gamma,R}(x)}{\|f_{\gamma,R}\|_{L^1(\mathbb{A}(R,1))}} \right) dx \\ &\lesssim \int_R^1 \frac{1-\gamma R}{1-R} \log \left(e + \frac{1-\gamma R}{(1-R)(1-\gamma^n R^n)} \right) dr \\ &\leq (1-R)^{\alpha/q+1} \log \left(e + \frac{1}{1-R} \right). \end{aligned}$$

The right-hand side converges to zero as $R \rightarrow 1$, since $\alpha/q + 1 > 0$. \square

Our goal is to show that $\mathcal{E}_q(f_{\gamma,R}) \rightarrow +\infty$ as $\gamma, R \rightarrow 1$. The idea is that energy-minimal solutions are controlled by the radial solution.

Proposition 14.2.2. *Fix $1 \leq p < q$ and suppose that $v \in W^{1,n}(B_1, \mathbb{R}^n)$ is a solution of $Jv = f_{\gamma,R}$ with $v = \text{id}$ on ∂B_1 , where $f_{\gamma,R}$ is as in (14.2.1) and (14.2.3). There is a constant $C = C(n, q) > 0$ such that*

$$\int_{\mathbb{A}(R,1)} |\partial_r u(x)|^q dx \leq C \int_{\mathbb{A}(R,1)} |\partial_r v(x)|^q dx,$$

where $u = u_{\gamma,R}$ is the radial stretching such that $Ju_{\gamma,R} = f_{\gamma,R}$ in $\mathbb{A}(R, 1)$ and $u = \text{id}$ on \mathbb{S}^{n-1} .

Proof: Throughout the proof θ will denote an element of \mathbb{S}^{n-1} and we write $r\theta$ for the corresponding element in a sphere of radius r .

Since $R > \frac{1}{2}$, from Hölder's inequality,

$$2^{n-1} \int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)|^q r^{n-1} dr d\theta \geq \int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)|^q dr d\theta$$

$$\geq \left(\int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)| dr d\theta \right)^q.$$

On the other hand, since $\partial_r u(r\theta) = M\theta$ for all $r \in (R, 1)$, c.f. (14.2.1) and (14.2.4), we can estimate

$$\left(\int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)| dr d\theta \right)^q = \int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)|^q dr d\theta \geq \int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)|^q r^{n-1} dr d\theta.$$

Thus, the proposition will be proved once we show that

$$\int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r v(r\theta)| dr d\theta \geq \frac{1}{4n^2} \int_{\mathbb{S}^{n-1}} \int_R^1 |\partial_r u(r\theta)| dr d\theta$$

for all $R \in (\frac{1}{2}, 1)$. Note that $\int_R^1 |\partial_r u(r\theta)| dr = (1-R)M = 1 - \gamma R$ for all $\theta \in \mathbb{S}^{n-1}$.

Let us take the set

$$\Theta \equiv \{ \theta \in \mathbb{S}^{n-1} : r \mapsto v(r\theta) \text{ is absolutely continuous in } [R, 1] \}.$$

For $\lambda \in (0, 1)$ to be chosen later, let

$$\Theta_1 \equiv \left\{ \theta \in \Theta : \int_R^1 |\partial_r v(s\theta)| ds \leq \lambda \right\}, \quad \Theta_2 \equiv \left\{ \theta \in \Theta : \int_R^1 |\partial_r v(s\theta)| ds > \lambda \right\}.$$

By the ACL property of Sobolev functions, $\mathcal{H}^{n-1}(\Theta) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ and so the set $\mathbb{S}^{n-1} \setminus \Theta$ is an \mathcal{H}^{n-1} -null set. Fubini's theorem and the fact that v satisfies the Lusin (N) property, c.f. Theorem 13.2.2, then implies

$$\mathcal{L}^n(v(\Theta_1 \times [R, 1])) + \mathcal{L}^n(v(\Theta_2 \times [R, 1])) = \mathcal{L}^n(v(\mathbb{A}(R, 1))).$$

By the fundamental theorem of calculus, for $\theta \in \Theta_1$, $r \in [R, 1]$,

$$1 - |v(r\theta)| \leq |v(\theta) - v(r\theta)| \leq \int_r^1 |\partial_r v(s\theta)| ds \leq \lambda$$

and thus $v(\Theta_1 \times [R, 1]) \subset \mathbb{A}[1 - \lambda, 1]$. Combined with the change of variables formula from Corollary 13.3.1, (14.2.2) and Bernoulli's inequality, this estimate yields

$$\begin{aligned} \mathcal{L}^n(v(\Theta_2 \times [R, 1])) &= \int_{\mathbb{A}(R, 1)} f_{\gamma, R}(x) dx - \mathcal{L}^n(v(\Theta_1 \times [R, 1])) \\ &\geq \omega_n [(1 - \gamma^n R^n) - (1 - (1 - \lambda)^n)] \\ &\geq \omega_n (1 - n\lambda - \gamma^n R^n), \end{aligned}$$

since $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$. Moreover, by Markov's inequality and (14.2.2),

$$\begin{aligned} \mathcal{L}^n(v(\Theta_2 \times [R, 1])) &= \int_{\Theta_2 \times [R, 1]} f_{\gamma, R}(x) dx \\ &= \mathcal{H}^{n-1}(\Theta_2) \frac{1 - \gamma^n R^n}{n} \leq \frac{1 - \gamma^n R^n}{n\lambda} \int_{\mathbb{A}(R, 1)} |\partial_r v(r\theta)| dr d\theta. \end{aligned}$$

Combining the last two estimates and choosing $\lambda = \frac{1 - \gamma^n R^n}{2n}$, we find

$$\frac{1}{4} \int_{\mathbb{A}(R, 1)} |\partial_r u(r\theta)| dr d\theta = \frac{n\omega_n(1 - \gamma R)}{4} \leq \frac{n\omega_n(1 - (\gamma R)^n)}{4} \leq \int_{\mathbb{A}(R, 1)} |\partial_r v(r\theta)| dr d\theta$$

since $(\gamma R)^n < \gamma R < 1$. The conclusion follows. \square

Remark 14.2.3. It is clear from the proof that the boundary condition $v = \text{id}$ on $\partial B_1(0)$ can be weakened to the requirement that $v(\theta) \in \mathbb{S}^{n-1}$ for \mathcal{H}^{n-1} -a.e. $\theta \in \mathbb{S}^{n-1}$. Note that this condition is independent of the representative of the equivalence class of $v \in W^{1,q}(B_1(0), \mathbb{R}^n)$. The argument above carries through simply by replacing the set Θ with $\Theta \cap \{v(\theta) \in \mathbb{S}^{n-1}\}$.

Combining Lemma 14.2.1 with Proposition 14.2.2, we immediately obtain:

Corollary 14.2.4. *Let $1 \leq p < q < \infty$ and $n \leq q$. For any $\varepsilon, \eta > 0$ and any $f \in X_p$, there is a sequence $f_j \in B_{Z_p^\eta}(f, \varepsilon)$ such that $\text{dist}_p(f_j, f) \rightarrow 0$ and $\mathcal{E}_q(f_j) \rightarrow \infty$.*

We are ready to prove Theorem 12.A, an equivalent formulation of which we restate here for convenience of the reader.

Theorem 14.2.5. *Fix $0 < c < 1$ and $1 \leq p < \infty$. The set of those $f \in X_p$ such that there is a solution $u \in \bigcup_{n \leq q, p < q} W^{1,q}(\Omega, \mathbb{R}^n)$ of (12.1.1) is meagre in X_p .*

Proof: We focus on the case $\Omega = B_1(0)$ first. Fix $\delta > 0$ sufficiently small; in particular, $\delta < 1 - c$ suffices. We replace c with $c/(1 - \delta) < 1$ in the definition (14.1.1) of Z_p^η , in order to account for the slightly worse lower bound satisfied by the perturbations of Lemma 14.2.1.

Let us first fix q such that $n \leq q$ and $p < q$. For $k \in \mathbb{N}$ let $Y_k \equiv \{f \in Z_p^\delta : \mathcal{E}_q(f) \leq k\}$. Recall that $E_q(f)$ was defined in (14.1.2). Note that each Y_k is closed in Z_p^δ . Indeed, given a sequence $f_j \in Y_k$ such that $\text{dist}_p(f_j, f) \rightarrow 0$ for some $f \in Z_p^\delta$, if v_j are energy-minimal solutions corresponding to f_j , then from weak compactness there is a function $v \in W^{1,q}(B_1, \mathbb{R}^n)$ such that $v_j \rightharpoonup v$ in $W^{1,q}(B_1, \mathbb{R}^n)$. By the weak continuity of the Jacobian and weak lower semicontinuity of the q -Dirichlet energy, it follows that $\mathcal{E}_q(v) \leq \liminf_j \mathcal{E}_q(v_j) \leq k$.

We show that Y_k has empty interior in Z_p^δ . Indeed, take $f_0 \in Y_k$ and consider an arbitrary $\varepsilon > 0$: we claim that $B_{Z_p^\delta}(f_0, \varepsilon) \not\subset Y_k$. For this, it suffices to show that $B_{X_p}(f_0, \varepsilon) \not\subset Y_k$, since $B_{X_p}(f_0, \varepsilon) \subset B_{Z_p^\delta}(f_0, \varepsilon)$. Any such ball $B_{X_p}(f_0, \varepsilon)$ contains an element f_1 which is in X_p ; thus, by replacing f_0 with f_1 and shrinking ε if need be, we can assume without loss of generality that $f_0 \in X_p$. By Corollary 14.2.4, there are $f_j \in B_{Z_p^\delta}(f_0, \varepsilon) \subset Y_k$ such that $\text{dist}_p(f_j, f) \rightarrow 0$ but $j \leq \mathcal{E}_q(f_j)$. This proves the claim.

Since each set Y_k is closed and has empty interior, the Baire Category Theorem implies that $\bigcup_{k \in \mathbb{N}} Y_k$ is meagre in Z_p^δ . Equivalently, $J(\text{id} + W_0^{1,q}(\Omega, \mathbb{R}^n))$ is meagre in Z_p^δ . For $p \geq n$, we have that

$$J\left(\text{id} + \bigcup_{n \leq q, p < q} W_0^{1,q}(\Omega, \mathbb{R}^n)\right) = \bigcup_{j=1}^{\infty} J\left(\text{id} + W_0^{1,q_j}(\Omega, \mathbb{R}^n)\right), \quad q_j \searrow p,$$

and the right-hand side is a countable union of meagre sets, hence meagre, so the conclusion follows. The case $p < n$ is identical. \square

Chapter 15

Energy minimisers with prescribed Jacobian

In this chapter we study energy-minimal solutions of the Dirichlet problem (12.1.1) for the Jacobian equation, c.f. Definition 12.1.15. The results presented here are joint work with LUKAS KOCH and SAULI LINDBERG and can be found in [178].

In Section 15.1 we will prove a more general version of Theorem 12.C(i), from which Corollary 12.F also follows. In Section 15.2 we prove the symmetry-breaking result stated in Theorem 12.C(ii). Finally, Section 15.3 concerns non-uniqueness of energy minimisers, and it is there that we prove Theorem 12.G.

Throughout this chapter we will work in the plane, i.e. $n = 2$ and we write

$$\mathbb{S}_r \equiv \{z \in \mathbb{R}^2 : |z| = r\}, \quad \nu \equiv \frac{z}{|z|}.$$

We will also use the notation ϕ_k for a generalised radial stretching, as in Definition 13.5.3.

15.1 A class of data with symmetric energy minimisers

The key step in establishing Theorem 12.C is the following more general proposition which may be of independent interest.

Proposition 15.1.1. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^p(\mathbb{R}^2)$ be a radially symmetric function such that, for some $\lambda \geq 1$ and a.e. $r \in (0, +\infty)$,*

$$|f(r)| \leq \lambda \int_{B_r(0)} f \, dx. \tag{15.1.1}$$

Let ϕ_1 denote the radial stretching solving $J\phi_1 = f$.

(i) *We have the estimate*

$$\|D\phi_1\|_{L^{2p}(\mathbb{R}^2)}^2 \leq C(\lambda) \|f\|_{\mathcal{H}^p(\mathbb{R}^2)}. \tag{15.1.2}$$

(ii) *Let $u \in W_{\text{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ be a solution of $Ju = f$ such that, for a.e. $r \in (0, +\infty)$,*

$$4\pi \int_{B_r} Ju \, dx \leq \left(\int_{\mathbb{S}_r} |Du \cdot \nu^\perp| \, d\theta \right)^2. \tag{15.1.3}$$

Then, with Z denoting the Zhukovsky function $Z(\lambda) \equiv \frac{1}{2} \left(\frac{1}{\lambda} + \lambda \right)$, we have the estimate

$$\int_{\mathbb{S}_r} |D\phi_1|^{2p} \leq Z(\lambda) \int_{\mathbb{S}_r} |Du|^{2p} \tag{15.1.4}$$

for a.e. $r \in (0, \infty)$.

(iii) *If $\lambda = 1$ and if (15.1.4) holds with equality then (15.1.3) holds with equality.*

In the statement of the theorem, as well as in its proof, u denotes the precise representative of the equivalence class $[u] \in W_{\text{loc}}^{1,2p}$. We refer the reader to [129] for the definition and properties of precise representatives.

We note that condition (15.1.3) is a parametric version of the isoperimetric inequality. In particular, it holds under natural assumptions including the setting of Theorem 12.C, see already Proposition 15.1.4.

Before proceeding with the proof, it is useful to note that condition (15.1.1) can be rewritten in terms of the radial stretching ϕ_1 as

$$|\dot{\rho}(r)| \leq \lambda \left| \frac{\rho(r)}{r} \right|, \quad (15.1.5)$$

see also the discussion in Chapter 12. It is worth mentioning that, in (15.1.1), we make implicitly a *choice of orientation*. Indeed, in order to ensure the existence of generalised radial stretchings solving the equation, it must be the case that the map $r \mapsto \int_{B_r} f \, dx$ does not change sign, see Section 13.4. Clearly (15.1.1) implies that this map is non-negative. There is an analogue of Proposition 15.1.1 in the case where $\int_{B_r} f \, dx$ is always non-positive: in that case, we replace ϕ_1 with ϕ_{-1} .

Proof of Proposition 15.1.1(i): For the case $p > 1$, we combine (15.1.5) with Lemma 13.4.2 to get

$$\|D\phi_1\|_{L^{2p}(\mathbb{R}^2)}^{2p} \approx \int_0^\infty \left(|\dot{\rho}(r)|^{2p} + \left| \frac{\rho(r)}{r} \right|^{2p} \right) r \, dr \lesssim_\lambda \int_0^\infty \left| \frac{\rho(r)}{r} \right|^{2p} r \, dr = \int_{\mathbb{R}^2} \left| \int_{\bar{B}_{|x|}(0)} f \, dy \right|^p dx.$$

Denoting by M be the (non-centred) Hardy–Littlewood maximal function, we have

$$\|D\phi_1\|_{L^{2p}(\mathbb{R}^2)}^{2p} \lesssim_\lambda \int_{\mathbb{R}^2} |Mf(x)|^p \, dx \lesssim \int_{\mathbb{R}^2} |f(x)|^p \, dx,$$

as wished.

For the case $p = 1$, we need to argue in a more careful way and we use the fact that

$$\|f(|r|)\|_{\mathcal{H}^1(\mathbb{R}, |r| \, dr)} \approx \|f(|x|)\|_{\mathcal{H}^1(\mathbb{R}^2, dx)},$$

see the proof of [89, Corollary (2.27)]. Recall that an $\mathcal{H}^1(\mathbb{R}, |r| \, dr)$ -atom is simply a function $a: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\text{supp } a \subset [r_1, r_2], \quad \|a\|_\infty \leq \frac{1}{\int_{r_1}^{r_2} |s| \, ds}, \quad \int_{\mathbb{R}} a(r) |r| \, dr = 0,$$

for some real numbers $r_1 < r_2$, and that moreover for any $f \in \mathcal{H}^1(\mathbb{R}, |r| \, dr)$ there exist atoms a_i and real numbers $\lambda_i \in \mathbb{R}$ such that

$$0 = \lim_{N \rightarrow \infty} \left\| f - \sum_{i=1}^N \lambda_i a_i \right\|_{\mathcal{H}^1(\mathbb{R}, |r| \, dr)}, \quad \sum_{i=1}^\infty |\lambda_i| \lesssim \|f\|_{\mathcal{H}^1(\mathbb{R}, |r| \, dr)}. \quad (15.1.6)$$

Arguing as in the case $p > 1$ we see that

$$\|D\phi_1\|_{L^{2p}(\mathbb{R}^2)}^{2p} \lesssim_\lambda \int_0^\infty \frac{1}{r} \int_0^r 2 f(s) s \, ds \, dr = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^{1/\varepsilon} \frac{1}{r} \int_{-r}^r f(s) |s| \, ds \, dr,$$

where we also used $f = f(|r|)$ in the last equality.

Fix $\varepsilon > 0$. By using (15.1.6) and the dominated convergence theorem,

$$\int_\varepsilon^{1/\varepsilon} \frac{1}{r} \int_{-r}^r f(s) |s| \, ds \, dr = \lim_{N \rightarrow \infty} \sum_{j=1}^N \lambda_j \int_\varepsilon^{1/\varepsilon} \frac{1}{r} \int_{-r}^r a_j(s) |s| \, ds \, dr.$$

When $N \in \mathbb{N}$, suppose a is one of the atoms a_1, \dots, a_N and let $0 \leq \tilde{r}_1 < \tilde{r}_2$ be, respectively, the minimum and the maximum of $|\cdot|$ over $[r_1, r_2]$. Then

$$\begin{aligned} \int_{\varepsilon}^{1/\varepsilon} \frac{1}{r} \int_{-r}^r a(s)|s| \, ds \, dr &= \int_{\max\{\varepsilon, \tilde{r}_1\}}^{\min\{1/\varepsilon, \tilde{r}_2\}} \frac{1}{r} \int_{-r}^r a(s)|s| \, ds \, dr \\ &\leq \frac{\int_{\tilde{r}_1}^{\tilde{r}_2} \frac{1}{r} \int_{-r}^r |s| \, ds \, dr}{\int_{r_1}^{r_2} |s| \, ds} = \frac{\tilde{r}_2^2 - \tilde{r}_1^2}{2 \int_{r_1}^{r_2} |s| \, ds} \leq 1. \end{aligned}$$

By letting first $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, the conclusion follows from (15.1.6). \square

Before finishing the proof of Proposition 15.1.1, we record the following elementary lemma:

Lemma 15.1.2. *Define $\psi: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ by $\psi(a, b) \equiv a + b^2/a$. Then*

- (i) *the function ψ is convex;*
- (ii) *for each $b \in \mathbb{R}$, the function $a \mapsto \psi(a, b)$ is increasing in $(0, b)$ and decreasing in $(b, +\infty)$ and it has a global minimum at $a = |b|$;*
- (iii) *for $\lambda > 0$, if $a_2 \leq a_1$ and $|b| \leq \lambda a_2$ then $\psi(a_2, b) \leq Z(\lambda)\psi(a_1, b)$.*

Proof: The first two properties are readily checked. To prove (iii), note that when $|b| \leq a_2$ the conclusion follows from (ii), since $1 \leq Z(\lambda)$. When $a_2 < |b|$ then, by applying (ii) twice,

$$\psi(a_2, b) \leq \psi(b/\lambda, b) = Z(\lambda)\psi(b, b) \leq Z(\lambda)\psi(a_1, b). \quad \square$$

Completion of the proof of Proposition 15.1.1: We first deal with the case $p = 1$. Note that ϕ_1 is continuous and denote also by u the precise representative of the class $[u] \in W_{\text{loc}}^{1,2}$. Consider the set of “good” radii

$$\mathcal{G} \equiv \left\{ r \in (0, \infty) : \begin{array}{l} u|_{\mathbb{S}_r} \text{ is absolutely continuous, (15.1.1) and (15.1.3) hold,} \\ \text{and } Ju(x) = f(x) \text{ for } \mathcal{H}^1\text{-a.e. } x \in \mathbb{S}_r \end{array} \right\}.$$

Since u is a Sobolev function, our hypotheses together with an application of Fubini’s theorem show that the \mathcal{G} has full measure, i.e. $\mathcal{L}^1(\mathbb{R}^+ \setminus \mathcal{G}) = 0$.

Fix $r \in \mathcal{G}$. The crucial observation is that ϕ_1 satisfies the isoperimetric inequality (15.1.3) with equality: this is easily checked, but it can also be seen as a consequence of the fact that ϕ_1 maps circles to circles and has degree one. Hence, as u satisfies (15.1.3) by assumption,

$$\left(\int_{\mathbb{S}_r} |D\phi_1 \cdot \nu^\perp| \, d\theta \right)^2 = 4\pi \int_{B_r} f \, dx \leq \left(\int_{\mathbb{S}_r} |Du \cdot \nu^\perp| \, d\theta \right)^2.$$

Moreover, $D\phi_1(re^{i\theta})$ is constant on \mathbb{S}_r and so, using Jensen’s inequality, we arrive at

$$|D\phi_1 \cdot \nu^\perp|^2 = \left(\int_{\mathbb{S}_r} |D\phi_1 \cdot \nu^\perp| \, d\theta \right)^2 \leq \int_{\mathbb{S}_r} |Du \cdot \nu^\perp|^2 \, d\theta. \quad (15.1.7)$$

Here we implicitly assume that $D\phi_1$ and Du are evaluated at the point $x = re^{i\theta}$, in order to lighten the notation; the same convention is used in the rest of the proof.

We now note the following cofactor identity: if $\nu \in \mathbb{S}^1$ and $A \in \mathbb{R}^{2 \times 2}$,

$$\det A = \det A(\nu, \nu) = \langle \text{cof}(A)^T A\nu, \nu \rangle = \langle A\nu, \text{cof}(A)\nu \rangle.$$

Using the Cauchy–Schwarz inequality and the fact that $|\text{cof}(A)\nu| = |A\nu^\perp|$, we have

$$\det A \leq |A\nu| |\text{cof}(A)\nu| \quad \implies \quad |A\nu^\perp|^2 + \frac{(\det A)^2}{|A\nu^\perp|^2} \leq |A\nu^\perp|^2 + |A\nu|^2 = |A|^2. \quad (15.1.8)$$

We apply (15.1.8) to $A = Du(x)$, choosing $\nu = x/r$: since $Ju = f$,

$$\int_{\mathbb{S}_r} \psi(|Du \cdot \nu^\perp|^2, f(r)) \, d\theta = \int_{\mathbb{S}_r} |Du \cdot \nu^\perp|^2 + \frac{f^2}{|Du \cdot \nu^\perp|^2} \, d\theta \leq \int_{\mathbb{S}_r} |Du|^2 \, d\theta, \quad (15.1.9)$$

where ψ is as in Lemma 15.1.2. By Lemma 15.1.2(i), Jensen's inequality applies to yield

$$\psi\left(\int_{\mathbb{S}_r} |Du \cdot \nu^\perp|^2 \, d\theta, f(r)\right) = \psi\left(\int_{\mathbb{S}_r} |Du \cdot \nu^\perp|^2 \, d\theta, \int_{\mathbb{S}_r} f(r) \, d\theta\right) \leq \int_{\mathbb{S}_r} \psi(|Du \cdot \nu^\perp|^2, f(r)) \, d\theta. \quad (15.1.10)$$

Let us also note that we have equality in (15.1.9) whenever we have equality in (15.1.8), i.e. whenever we have equality in Cauchy–Schwarz. In other words, we have equality in (15.1.9) if and only if $\text{cof}(Du)\nu$ is parallel to $Du \cdot \nu$ (or, equivalently, if and only if $\partial_r u \perp \partial_\theta u$), which is the case if u is a radial stretching¹, see also the discussion in the Introduction.

We now take

$$a_1 = \int_{\mathbb{S}_r} |Du \cdot \nu^\perp|^2 \, d\theta, \quad a_2 = |D\phi_1 \cdot \nu^\perp|^2 = \frac{\rho^2(r)}{r^2}, \quad b = f(r) = \frac{\rho(r)\dot{\rho}(r)}{r}.$$

From (15.1.7) we have that $a_2 \leq a_1$ and from (15.1.5) we have $|b| \leq \lambda a_2$. Hence Lemma 15.1.2(iii), combined with (15.1.9) and (15.1.10), gives

$$\psi\left(|D\phi_1 \cdot \nu^\perp|^2, f(r)\right) \leq Z(\lambda) \psi\left(\int_{\mathbb{S}_r} |Du \cdot \nu^\perp|^2 \, d\theta, f(r)\right) \leq Z(\lambda) \int_{\mathbb{S}_r} |Du|^2 \, d\theta.$$

As noted above, ϕ_1 satisfies (15.1.9) with equality and so

$$\int_{\mathbb{S}_r} |D\phi_1|^2 \, d\theta = |D\phi_1|^2 = \psi\left(|D\phi_1 \cdot \nu^\perp|^2, f(r)\right).$$

This proves (15.1.4) when $p = 1$.

The case $p > 1$ follows from the case $p = 1$: since $x \mapsto x^{2p}$ is a strictly convex, increasing function over \mathbb{R}^+ , we can apply Jensen's inequality to conclude that

$$\int_{\mathbb{S}_r} |D\phi_1|^{2p} \, d\theta = \left(\int_{\mathbb{S}_r} |D\phi_1|^2 \, d\theta\right)^p \leq \left(\int_{\mathbb{S}_r} |Du|^2 \, d\theta\right)^p \leq \int_{\mathbb{S}_r} |Du|^{2p} \, d\theta,$$

where we also used the fact that $D\phi_1$ is constant in \mathbb{S}_r in the first equality.

Finally, (iii) follows by inspection of the proof. Since $\psi(a_2, b) < \psi(a_1, b)$ if $b \leq a_2 < a_1$, to have equality in (15.1.4) we must have $a_2 = a_1$, that is, we must also have equality in (15.1.3). \square

Remark 15.1.3. The dependence on λ in the estimate (15.1.2) is not uniform. That this must be the case is easily seen by considering the family of data $\varepsilon 1_{B_1} + \frac{4-\varepsilon}{3} 1_{\mathbb{A}(1,2)}$, see also Section 15.2.

We next show that (15.1.3) holds under natural assumptions.

Proposition 15.1.4. Fix $p \in [1, \infty)$ and $R > 0$. Let $u \in W^{1,2p}(B_R(0), \mathbb{R}^2)$ be a continuous map such that $Ju = f$ a.e. in $B_R(0)$. Suppose furthermore that for a.e. $r \in (0, R)$ the change of variables formula

$$\int_{B_r} Ju \, dx = \int_{\mathbb{R}^2} \text{deg}(y, u, B_r) \, dy, \quad (15.1.11)$$

holds. Then (15.1.3) holds for a.e. $r \in (0, R)$. Moreover, equality holds in (15.1.3) if and only if $u(\mathbb{S}_r)$ is a circle which is traversed one time.

¹Although this is not important for our purposes, it is also the case if u is conformal.

Here $\deg(y, u, B_r)$ denotes the *topological degree* of u at y with respect to B_r , c.f. Lemma 13.1.5.

Proof: We first note that due to the Sobolev regularity of u , $u(\mathbb{S}_r)$ is a continuous rectifiable curve for almost every $r \in (0, R)$ and hence we may restrict to such r without loss of generality. We now recall the following *generalised isoperimetric inequality*: given a continuous rectifiable curve Γ , let $(E_k)_k$ be the components of $\mathbb{R}^2 \setminus \Gamma$; on each E_k , Γ has a well-defined winding number w_k . Then we have

$$4\pi \sum_k w_k^2 \mathcal{L}^2(E_k) \leq l(\Gamma)^2, \quad (15.1.12)$$

with equality if and only if Γ is a circle traversed a finite number of times in a given direction. Here $l(\Gamma)$ denotes the length of Γ . This inequality was proved implicitly in [139, page 487] and then later in [35], but see also [308] for a comprehensive overview.

We want to apply (15.1.12) when $\Gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is the curve $\Gamma(\theta) = u(re^{i\theta})$. Recall that, at a point y , the winding number of the curve Γ with respect to y is just $\deg(y, u, B_r)$, see for instance [117, §6.6]. Since $l(\Gamma) = \int_{\mathbb{S}^1} |\operatorname{cof}(Du)\nu| d\theta$, we can use (15.1.12) to get

$$\int_{\mathbb{R}^2} \deg(y, u, B_r)^2 dy \leq \frac{1}{4\pi} \left(\int_{\mathbb{S}^1} |\operatorname{cof}(Du)\nu| d\theta \right)^2.$$

As the topological degree is an integer, we deduce from (15.1.11) that

$$\int_{B_r} Ju dx \leq \frac{1}{4\pi} \left(\int_{\mathbb{S}^1} |\operatorname{cof}(Du)\nu| d\theta \right)^2. \quad (15.1.13)$$

This proves (15.1.3), since $|\operatorname{cof}(A)\nu| = |A\nu^\perp|$ for $A \in \mathbb{R}^{2 \times 2}$.

The equality cases follow from the equality cases for (15.1.12) together with the fact that we must have $\deg(y, u, B_r) = \pm 1$ for $y \in u(B_r)$ to get equality in (15.1.13). \square

Remark 15.1.5. For $p > 1$, the continuity assumption in Proposition 15.1.4 is not restrictive and moreover (15.1.11) also holds automatically, as maps in a supercritical Sobolev space always satisfy the Lusin (N) property, c.f. Proposition 13.1.2.

For $p = 1$ it is not in general the case that solutions are continuous and satisfy (15.1.11). However, both properties are satisfied over open sets where $f > 0$ a.e., as in this case solutions have finite distortion and Theorem 13.2.2 applies. Assuming a positive answer to Question 12.2.1, one can always find solutions satisfying both properties over bounded domains where $f \geq 0$ a.e., as per Theorem 12.E.

We conclude this section by showing how Theorem 12.C(i) follows from Proposition 15.1.1.

Proof of Theorem 12.C(i): Fix $p \in [1, \infty)$. Since (12.1.3) holds, Proposition 15.1.4 applies. As f satisfies (15.1.1) with $\lambda = 1$, we conclude from (15.1.4) that ϕ_1 is a $2p$ -energy minimiser.

That ϕ_1 is the unique $2p$ -energy minimiser in $\operatorname{id} + W_0^{1,2p}(B, B)$ follows from Proposition 15.1.1(iii) and the equality case of Proposition 15.1.4. Indeed, for any $2p$ -energy minimiser u we must have that, for a.e. $r \in (0, R)$, $u(\mathbb{S}_r)$ is a circle; that is, using Proposition 13.5.1, we may write

$$u(re^{i\theta}) = \psi(r)e^{i\gamma(r,\theta)}.$$

Since $\deg(y, u, B_r) = 1$ for all $y \in u(B_r)$, as f is positive a.e., we see from (13.5.3) that

$$\psi(r) = \sqrt{\int_0^r 2sf(s) ds} = \rho(r).$$

It is now easy to see from (13.5.4) that, as u is a minimiser, $\partial_r \gamma(r, \theta) = 0$ for a.e. $r \in (0, R)$. It follows from the boundary condition $u = \operatorname{id}$ on ∂B that $u = \phi_1$.

Finally we prove (12.1.9). Arguing as in the proof of the case $p > 1$ of Proposition 15.1.1(i), we have that

$$\int_B |D\phi_1|^{2p} dx \lesssim \int_B |Mf(x)|^p dx,$$

for any $p \in [1, \infty)$. Through the maximal inequality this immediately implies (12.1.9) for $p > 1$. To deal with the endpoint $p = 1$ we recall that, whenever $\text{supp } f \subset \bar{B}$, then

$$\|f\|_{L \log L(B)} \approx \|Mf\|_{L^1(B)},$$

see for instance [344, page 23]. Extending $f \in X_p(B)$ by zero outside B we finish the proof. \square

Inspecting the above proof we also readily obtain Corollary 12.F. We also note that the proof of Theorem 12.C(i) does not use any information about the behaviour of solutions on the boundary of the domain. It would be interesting to know the extent to which the boundary condition impacts the symmetry of energy minimisers. A model problem in this direction is to consider, for $\varepsilon > 0$, the datum $f_\varepsilon(r) \equiv c_\varepsilon r^\varepsilon$, where $c_\varepsilon \equiv \frac{2}{2+\varepsilon}$ is such that $\int_{B_1} f_\varepsilon dx = 1$. It is easy to see that

$$f_\varepsilon(r) = \frac{2+\varepsilon}{2} \int_{B_r(0)} f dx$$

and hence (15.1.4) shows that, for $\varepsilon \ll 1$, the energy of the corresponding radial stretchings is arbitrarily close to that of any other energy minimiser. However, we do not know whether the corresponding radial stretchings are $2p$ -energy minimisers in $\text{id} + W_0^{1,2p}(B, B)$.

15.2 Non-symmetric energy minimisers

In this section we prove part (ii) of Theorem 12.C. For a point $z = (x, y) \in \mathbb{R}^2$, let us write $|z|_1 \equiv |x| + |y|$ for its ℓ^1 -norm and

$$Q_r \equiv \{z \in \mathbb{R}^2 : |z|_1 < r\}, \quad \mathbb{A}_1(r, R) \equiv \{z \in \mathbb{R}^2 : r < |z|_1 < R\}$$

for the corresponding balls and annuli. The following example, although simple, is useful:

Example 15.2.1 (Mapping a ball onto a square). The map

$$\eta(x, y) \equiv \frac{r \operatorname{sgn}(x)}{\sqrt{2}} \begin{cases} (1, 4/\pi \arctan(y/x)) & \text{if } |y| < |x|, \\ (4/\pi \arctan(x/y), 1) & \text{if } |y| \geq |x|, \end{cases}$$

is bi-Lipschitz and satisfies a.e. $\det D\eta = 2/\pi$. For any $r > 0$, we also have $R \circ \eta(B_r) = Q_r$, where R is a rotation by angle $\frac{\pi}{4}$.

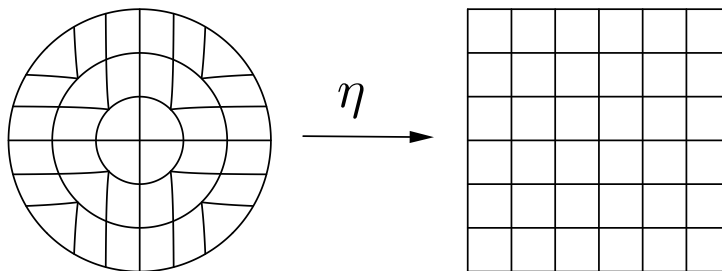


Figure 15.1: The map from Example 15.2.1.

The map in Example 15.2.1 can be found in [174]. In fact, Example 15.2.1 is an explicit particular case of a more general construction, due to FONSECA–PARRY [151, Theorem 5.4]. Their result applies to all domains of the following class:

Definition 15.2.2. A domain $\Omega \subset \mathbb{R}^n$ is of class \mathcal{C} if there are $\varepsilon, \delta > 0$ and $N \in \mathbb{N}$ such that:

- (i) $B_\varepsilon(0) \subset \Omega$ and Ω is bounded and star-shaped with respect to 0, that is, every ray starting at 0 intersects $\partial\Omega$ exactly once;
- (ii) there is a finite partition $\Omega = \bigcup_{i=1}^N \Omega_i$ such that each Ω_i is a cone with vertex at 0, $B_\varepsilon(0) \cap \Omega_i$ is convex, $\partial\Omega_i \cap \partial\Omega$ is C^1 and satisfies $\nu(x) \cdot x \geq \delta$ for all $x \in \partial\Omega_i \cap \partial\Omega$, where ν denotes the outward unit normal.

Given two domains $\Omega, \tilde{\Omega}$ of class \mathcal{C} , as they are star-shaped with respect to 0, there is a unique Lipschitz function $\psi: \partial\Omega \rightarrow (0, +\infty)$ such that $\psi(x)x \in \partial\tilde{\Omega}$ for all $x \in \partial\Omega$. The next theorem was proved in [151], although the statement here is more precise than theirs:

Theorem 15.2.3. *Let $\Omega, \tilde{\Omega}$ be two domains of class \mathcal{C} . Then there is a surjective map $v: \Omega \rightarrow \tilde{\Omega}$ which is L -bi-Lipschitz, i.e.*

$$\frac{1}{L}|x - y| \leq |v(x) - v(y)| \leq L|x - y| \quad \text{for all } x, y \in \bar{\Omega},$$

and which solves, for ψ as above,

$$\begin{cases} Jv = |\tilde{\Omega}|/|\Omega| & \text{in } \Omega, \\ v(x) = \psi(x)x & \text{for } x \in \partial\Omega. \end{cases}$$

Moreover, $L > 0$ is a constant which depends only on $\delta, \varepsilon, n, N, \text{diam}(\Omega)$ and $\text{diam}(\tilde{\Omega})$.

Our goal is to use Theorem 15.2.3 to prove Theorem 12.C(ii). If we do not require f to be bounded away from zero, the following yields a simple example:

Example 15.2.4. Let $f = \frac{4}{3}1_{\mathbb{A}(1,2)}$ and note that the radial stretching ϕ_1 solving $J\phi_1 = f$ is not in $W^{1,2}(B_{1+\delta}(0))$, for any $\delta > 0$, c.f. (15.2.5). Actually, it is a general fact that $W^{1,2}$ solutions of (12.2.1) cannot be constant in open sets where $f = 0$, for otherwise they would have integrable distortion and hence would be open mappings.

We can apply Theorem 15.2.3 to the domains

$$\Omega = \mathbb{A}(1,2) \cap \{x > 0, y > 0\}, \quad \tilde{\Omega} = B_2(0) \cap \{x > 0, y > 0\},$$

which are star-shaped with respect to $(1,1)$, to find a bi-Lipschitz map $u: \Omega \rightarrow \tilde{\Omega}$ which has constant Jacobian in Ω . One can then extend u to $B_1(0) \cap \{x > 0, y > 0\}$ in a trivial way, using the boundary data on the arc $\mathbb{S}_1 \cap \{x > 0, y > 0\}$, and then extend u to $B_2(0)$ through reflections along the axes, i.e. by setting

$$u(x, y) = \begin{cases} (u^1(x, -y), -u^2(x, -y)) & \text{if } x > 0, y < 0, \\ (-u^1(-x, y), u^2(-x, y)) & \text{if } x < 0, y > 0, \\ (-u^1(-x, -y), -u^2(-x, -y)) & \text{if } x < 0, y < 0, \end{cases} \quad (15.2.1)$$

see Figure 15.2. Hence there is a Lipschitz solution $u: B_2(0) \rightarrow B_2(0)$ of (12.1.1).

In order to find an example where f is bounded away from zero we need a substantially more intricate construction. Our goal is to prove the following result:

Theorem 15.2.5. *For $\varepsilon \in [0, 1]$, consider the family of data*

$$f_\varepsilon \equiv \varepsilon 1_{B_1(0)} + 1_{\mathbb{A}(1,2)} + \frac{6-\varepsilon}{5} 1_{\mathbb{A}(2,3)}. \quad (15.2.2)$$

There is a Lipschitz map $u_\varepsilon: B_3(0) \rightarrow B_3(0)$ such that

$$\begin{cases} Ju_\varepsilon = f_\varepsilon & \text{in } B_3(0), \\ Ju_\varepsilon = \text{id} & \text{on } \mathbb{S}_3, \end{cases} \quad (15.2.3)$$

and moreover there is a constant C , independent of ε , such that

$$\|Du_\varepsilon\|_\infty \leq C. \quad (15.2.4)$$

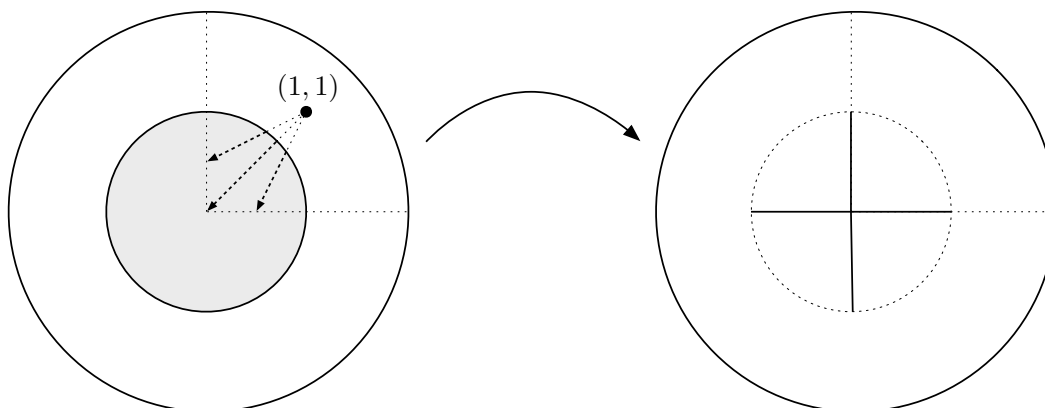


Figure 15.2: A Lipschitz map which is the identity on ∂B_2 and which has $\frac{4}{3}1_{A(1,2)}$ as Jacobian. It maps $\mathbb{S}_1 \cap \{x > 0, y > 0\}$ onto $\{0\} \times [0, 1] \cup [0, 1] \times \{0\}$ according to the dotted arrows.

Let us just note that, once Theorem 15.2.5 is proved, the proof of Theorem 12.C is easily finished:

Proof of Theorem 12.C(ii): Note that $f_\varepsilon \geq \varepsilon$ and that $\int_{B_3(0)} f_\varepsilon dx = 1$, so that indeed f_ε satisfies (12.1.3). Let ϕ_ε be the unique radial stretching solving (15.2.3), where f_ε is as in (15.2.2). Explicitly, $\phi_\varepsilon(z) = \rho_\varepsilon(r)\frac{z}{r}$ where, for $r \in (1, 2)$,

$$\rho_\varepsilon(r) = \sqrt{r^2 - 1 + \varepsilon} \implies |\rho'_\varepsilon(r)|^2 = \frac{r^2}{r^2 - 1 + \varepsilon}. \tag{15.2.5}$$

Using Lemma 13.4.2 we see that, as $\varepsilon \searrow 0$,

$$(9\pi)^{\frac{p-1}{2p}} \|D\phi_\varepsilon\|_{L^{2p}(B_3)} \geq \|D\phi_\varepsilon\|_{L^2(B_3)} \nearrow +\infty,$$

for any $p \in [1, \infty)$. Moreover, by (15.2.4), the maps u_ε satisfy

$$\|Du_\varepsilon\|_{L^{2p}(B_3)} \lesssim 1,$$

uniformly in ε and p . This completes the proof. □

It thus remains to prove Theorem 15.2.5. We begin by constructing an auxiliary map.

Lemma 15.2.6 (Mapping a wedge onto an ‘A’). *For $\varepsilon \in [0, 1]$, consider the sets*

$$\Lambda \equiv \mathbb{A}_1(2, 3) \cap \{y > 0\}, \quad A_\varepsilon \equiv \Lambda \cup \{1 + \varepsilon(1 - |x|) < y \leq 2 - |x|\}.$$

Let us write $\partial\Lambda = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \equiv \partial\Lambda \setminus A_\varepsilon$, and consider boundary data

$$\gamma_\varepsilon(x, y) = \begin{cases} (x, y) & \text{on } \Gamma_1, \\ (x, 1 + \varepsilon(y - 1)) & \text{on } \Gamma_2. \end{cases}$$

There is a surjective Lipschitz map $w_\varepsilon: \Lambda \rightarrow A_\varepsilon$, with $\|Dw_\varepsilon\|_\infty \leq C$, and such that

$$\begin{cases} Jw_\varepsilon = \frac{6-\varepsilon}{5} & \text{in } \Lambda, \\ w_\varepsilon = \gamma_\varepsilon & \text{on } \partial\Lambda. \end{cases}$$

Proof: Take $\Lambda^+ \equiv \Lambda \cap \{x > 0\}$ and $A_\varepsilon^+ \equiv A_\varepsilon \cap \{x > 0\}$. Consider the map $\tau_\varepsilon \equiv (\tau_\varepsilon^1, \tau_\varepsilon^2)$ defined for $(x, y) \in \Lambda^+$ by

$$\tau_\varepsilon^1(x, y) \equiv x, \quad \tau_\varepsilon^2(x, y) \equiv \begin{cases} \frac{1}{2}(2\varepsilon(x-1)(x+y-3) - x^2 + 3x + y^2 - y) & \text{if } x \in [0, 1], \\ \frac{1}{2}(x(2y-5) + x^2 + y^2 - 3y + 6) & \text{if } x \in [1, 2], \\ \frac{1}{2}y(x+y-1) & \text{if } x \in [2, 3]. \end{cases}$$

Since $J\tau_\varepsilon = \partial_y \tau_\varepsilon^2$, it follows that for $(x, y) \in \Lambda^+$ we have

$$J\tau_\varepsilon(x, y) = \begin{cases} \varepsilon(x-1) + y - \frac{1}{2} & \text{if } x \in [0, 1], \\ x + y - \frac{3}{2} & \text{if } x \in [1, 2], \\ \frac{1}{2}(x-1) + y & \text{if } x \in [2, 3]. \end{cases}$$

It is easy to check that $J\tau_\varepsilon \in C^{0,1}(\overline{\Lambda^+})$ and $J\tau_\varepsilon \geq \frac{1}{2}$ in Λ^+ . Note that $\tau_\varepsilon: \Lambda^+ \rightarrow A_\varepsilon^+$ is a bi-Lipschitz homeomorphism such that

$$\tau_\varepsilon|_{\partial\Lambda^+ \cap \partial\Lambda} = \gamma_\varepsilon|_{\partial\Lambda^+ \cap \partial\Lambda} \quad \text{and} \quad \tau_\varepsilon(\partial\Lambda^+ \setminus \partial\Lambda) = \partial\Lambda^+ \setminus \partial\Lambda;$$

in fact, we found τ_ε by looking for maps with these properties such that τ_ε^2 is a piecewise second order polynomial in y . See also Figure 15.3.

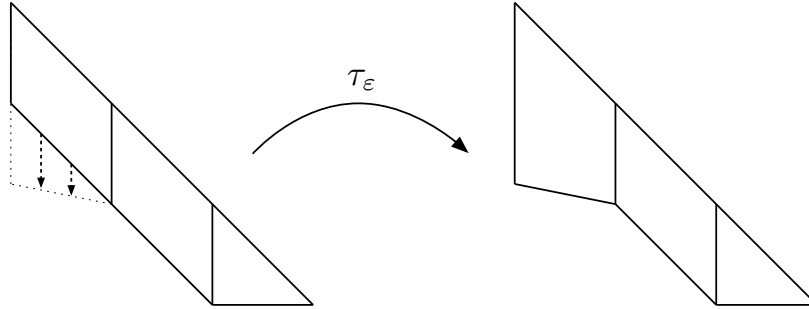


Figure 15.3: The map τ_ε , mapping Λ^+ onto A_ε^+ . Apart from the segment with the two dashed arrows, τ_ε is the identity on $\partial\Lambda^+$.

We now want to apply the Dacorogna–Moser theory to find a map $w_\varepsilon: \Lambda^+ \rightarrow A_\varepsilon^+$ with constant Jacobian. However, since A_ε^+ is just Lipschitz² this cannot be done directly. Instead, we use Theorem 15.2.3 to find a bi-Lipschitz homeomorphism $a_\varepsilon: A_\varepsilon^+ \rightarrow B_1(0)$ with constant Jacobian (explicitly, $J a_\varepsilon = \frac{2\pi}{6-\varepsilon}$), and we take a solution of

$$\begin{cases} J\sigma_\varepsilon = g_\varepsilon & \text{in } B_1(0), \\ \sigma_\varepsilon = \text{id} & \text{on } \mathbb{S}^1, \end{cases} \quad g_\varepsilon \equiv \frac{6-\varepsilon}{5} \frac{1}{J\tau_\varepsilon \circ \tau_\varepsilon^{-1} \circ a_\varepsilon^{-1}}$$

Note that, by the change of variables formula, and writing $\chi_\varepsilon \equiv a_\varepsilon \circ \tau_\varepsilon$,

$$\begin{aligned} \int_{B_1(0)} g_\varepsilon &= \frac{6-\varepsilon}{5} \int_{B_1(0)} \frac{J\chi_\varepsilon^{-1}}{J\tau_\varepsilon \circ \chi_\varepsilon^{-1} J\chi_\varepsilon^{-1}} = \frac{6-\varepsilon}{5} \int_{B_1(0)} \frac{J\chi_\varepsilon \circ \chi_\varepsilon^{-1}}{J\tau_\varepsilon \circ \chi_\varepsilon^{-1}} J\chi_\varepsilon^{-1} \\ &= \frac{6-\varepsilon}{5} \int_{\Lambda^+} \frac{J\chi_\varepsilon}{J\tau_\varepsilon} = \frac{6-\varepsilon}{5} \frac{2\pi}{6-\varepsilon} |\Lambda^+| = |B_1(0)|, \end{aligned}$$

²The Dacorogna–Moser theory [103] requires the domain to be at least of class $C^{3,\alpha}$.

thus g_ε satisfies the required compatibility condition. For any $\alpha \in (0, 1)$, we can additionally suppose that

$$\|\sigma_\varepsilon - \text{id}\|_{C^{1,\alpha}} \leq C(\alpha, \|g_\varepsilon\|_{C^{0,1}}) \|g_\varepsilon - 1\|_{C^{0,\alpha}} \leq C(\alpha),$$

see [329, Theorem 8]. Here the last inequality follows from the fact that the bi-Lipschitz constants of $a_\varepsilon, \tau_\varepsilon$ are uniformly bounded with $\varepsilon \in [0, 1]$, since the geometric parameters of A_ε^+ , according to Definition 15.2.2, are also bounded. We now take $w_\varepsilon: \Lambda^+ \rightarrow A_\varepsilon^+$ to be

$$w_\varepsilon \equiv a_\varepsilon^{-1} \circ \sigma_\varepsilon \circ a_\varepsilon \circ \tau_\varepsilon$$

and then extend w_ε to $\Lambda \setminus \Lambda^+$ through a reflection, similarly to (15.2.1). This yields the required map. \square

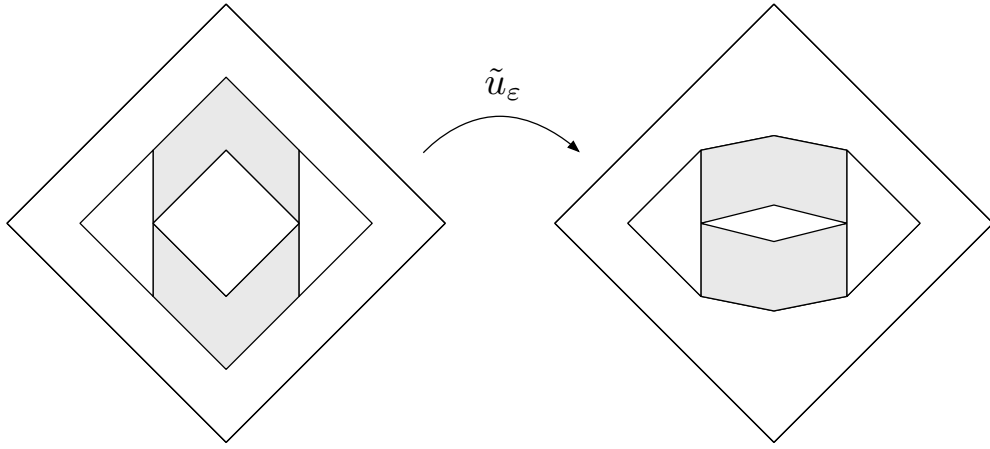


Figure 15.4: The map \tilde{u}_ε constructed in the proof of Theorem 15.2.5.

Proof of Theorem 15.2.5: Consider the map v_ε defined on Q_2 by

$$v_\varepsilon(x, y) = \begin{cases} (x, \varepsilon y) & \text{if } (x, y) \in Q_1, \\ (x, y) & \text{if } (x, y) \in \mathbb{A}_1(1, 2) \text{ and } |x| > 1, \\ (x, y - (1 - \varepsilon)(1 - |x|)) & \text{if } (x, y) \in \mathbb{A}_1(1, 2) \text{ and } |x| < 1, y > 0, \\ (x, y + (1 - \varepsilon)(1 - |x|)) & \text{if } (x, y) \in \mathbb{A}_1(1, 2) \text{ and } |x| < 1, y < 0. \end{cases} \quad (15.2.6)$$

It is easy to check that $Jv_\varepsilon = \varepsilon 1_{Q_1} + 1_{\mathbb{A}_1(1,2)}$. Let w_ε be the map from Lemma 15.2.6 and consider

$$\tilde{u}_\varepsilon \equiv \begin{cases} v_\varepsilon & \text{in } Q_2, \\ w_\varepsilon & \text{in } \Lambda, \\ \bar{w}_\varepsilon & \text{in } \bar{\Lambda}, \end{cases} \quad \text{where } \bar{\Lambda} \equiv \{(x, -y) : (x, y) \in \Lambda\}$$

and $\bar{w}_\varepsilon(x, y) \equiv (w_\varepsilon^1(x, -y), -w_\varepsilon^2(x, -y))$, see Figure 15.4. Thus

$$J\tilde{u}_\varepsilon = \varepsilon 1_{Q_1} + 1_{\mathbb{A}_1(1,2)} + \frac{6 - \varepsilon}{5} 1_{\mathbb{A}_1(2,3)}.$$

Recall the map η from Example 15.2.1 and let R be a rotation by angle $\frac{\pi}{4}$. Taking

$$u_\varepsilon \equiv (R \circ \eta)^{-1} \circ \tilde{u}_\varepsilon \circ (R \circ \eta),$$

the proof is finished. \square

We also note that the claim of non-uniqueness of energy minimisers in Theorem 12.C(ii) follows readily by combining Theorem 15.2.5 with the arguments of the next section.

15.3 Non-uniqueness of energy minimisers

The goal of this section is to prove Theorem 12.G, which we restate here:

Theorem 15.3.1. *Fix $1 \leq p < \infty$. There is a radially symmetric function $f \in \mathcal{H}^p(\mathbb{R}^2)$ which has uncountably many $2p$ -energy minimisers, modulo rotations.*

A more informative statement can be found in Corollary 15.3.6, at the end of the section. The proof of Theorem 15.3.1 relies mostly on elementary tools and the most sophisticated result that we use is the following:

Theorem 15.3.2 (Sierpiński). *Let (X_n) be disjoint closed sets such that $I = \bigcup_{n \in \mathbb{N}} X_n$, where $I = [a, b] \subset \mathbb{R}$. There is at most one $n \in \mathbb{N}$ such that X_n is non-empty.*

Theorem 15.3.2 is only needed to obtain uncountably many distinct minimisers, as non-uniqueness follows already from more elementary means. We also note that Theorem 15.3.2 holds more generally for a compact, connected Hausdorff space, see e.g. [126, Theorem 6.1.27]. In the case of an interval there is a simple proof, which we give here for the sake of completeness:

Proof: Take $Y \equiv \bigcup_n \partial X_n = I \setminus \bigcup_n \text{int}(X_n)$, which is closed, thus a complete metric space.

We observe that the set Y has empty interior in I , i.e. any open interval L contains an open set U disjoint from Y . Indeed, from the Baire Category Theorem we see that there is an open set $U \subseteq L$ and some X_m which is dense in U . Since X_m is closed, we must have $U \subseteq \text{int} X_m$ and thus U is disjoint from Y .

By the Baire Category Theorem there is also some open subinterval J of I and some $n \in \mathbb{N}$ such that ∂X_n is dense in $Y \cap J$. Since ∂X_n is closed we have $\partial X_n \cap J = Y \cap J$. Thus $(Y \setminus \partial X_n) \cap J = \emptyset$.

Suppose now that $X_n \neq I$. It follows that J intersects $Y \setminus \partial X_n$. Indeed, since Y has empty interior in I , J intersects $I \setminus X_n$ and so it intersects $\text{int}(X_k)$ for some k . Actually, J must intersect ∂X_k : otherwise, $\text{int}(X_k) \cap J$ is non-empty, open and closed in J , thus $\text{int} X_k = J$, since J is connected; clearly this is impossible, since X_k is disjoint from X_n . So we proved that J intersects $Y \setminus \partial X_n$, contradicting the previous paragraph. \square

We are now ready to begin the proof of Theorem 15.3.1, whose core idea is contained in the following lemma.

Lemma 15.3.3. *Let u be a $2p$ -energy minimiser for a radially symmetric function $f \in \mathcal{H}^p(\mathbb{R}^2)$. For $\alpha_0 \in [0, 2\pi]$, consider the set*

$$X_{\alpha_0} \equiv \{ \alpha \in [0, 2\pi] : u_\alpha = u_{\alpha_0} \text{ modulo rotations} \}, \quad \text{where } u_\alpha(z) \equiv u(e^{i\alpha}z). \quad (15.3.1)$$

Assume that $f \in C^0(B_R)$ has a sign. If $X_{\alpha_0} = [0, 2\pi]$ then there is $k \in \mathbb{Z} \setminus \{0\}$ such that

$$u(z) = \phi_k(z) \quad \text{in } B_R, \text{ modulo rotations.}$$

Proof: If $X_{\alpha_0} = [0, 2\pi]$ then, for any $\alpha \in [0, 2\pi]$ and $z \in B_R$, we have $|u(e^{i\alpha}z)| = |u(z)|$; that is, circles in B_R , centred at zero, are mapped to circles centred at zero.

For each $r \in (0, R)$, we have $0 \notin u(\mathbb{S}_r)$. Indeed, for each ball $B \Subset B_R$, there is $c = c(B) > 0$ such that $f \geq c$ in B (or $f \leq -c$, but by reversing orientations we can always consider the first case without loss of generality). Thus, in B_r , u is a map of integrable distortion and so it is both continuous and open [215]. Therefore $\partial(u(B_r)) \subseteq u(\partial B_r) = u(\mathbb{S}_r)$ and we see that $u(\mathbb{S}_r) \neq \{0\}$. Since $u(\mathbb{S}_r)$ is a circle, we conclude that $0 \notin u(\mathbb{S}_r)$.

By Proposition 13.5.1 we may write

$$u(r, \theta) = \psi(r, \theta) e^{i\gamma(r, \theta)} \quad (15.3.2)$$

where $\psi \in W^{1,2p}([0, R] \times [0, 2\pi])$ and $\gamma \in W^{1,2p}([\varepsilon, R] \times [0, 2\pi])$ satisfy (13.5.2) and $\varepsilon > 0$ is arbitrary. For $r < R$, $u(\mathbb{S}_r) = \mathbb{S}_{r'}$, that is, $\psi(r, \theta)$ is independent of θ . Thus, by (13.5.3), $Ju = f$ reduces to

$$\partial_r(\psi^2)\partial_\theta\gamma = 2rf(r), \quad (15.3.3)$$

which is valid for almost every $(r, \theta) \in (0, R] \times [0, 2\pi]$. Since both ψ and the right-hand side are independent of θ we must have $\gamma(r, \theta) = k\theta + \beta(r)$ and additionally there is the compatibility constraint (13.5.2) which yields $k \in \mathbb{Z}$. We may assume that $k \neq 0$: otherwise (15.3.3) shows that $f = 0$ a.e., which is impossible. Since u is a $2p$ -energy minimiser, (13.5.4) readily implies that β is constant. We integrate both sides of (15.3.3), using $\psi(0) = 0$, to find

$$\psi(r)^2 = \frac{1}{k} \int_0^r 2sf(s) ds \quad \text{for } r < R.$$

Thus, modulo rotations, $u = \phi_k$ in B_R . □

In fact, the same argument applied in an annulus $\mathbb{A}(R_0, R)$ gives the following variant:

Lemma 15.3.4. *Consider the setup of Lemma 15.3.5, but replace B_R by $\mathbb{A}(R_0, R)$. Then there is $k \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{R}$ such that, in $\mathbb{A}(R_0, R)$,*

$$u(z) = \psi(r)e^{2\pi ik\theta} \text{ modulo rotations,} \quad \text{where } \psi(r)^2 = \frac{1}{k} \int_{R_0}^r 2sf(s) ds + c.$$

We now combine the previous two lemmas.

Lemma 15.3.5. *There is a radially symmetric $f \in \mathcal{H}^p(\mathbb{R}^2)$, admitting a $2p$ -energy minimiser u , for which we have $X_0 \neq [0, 2\pi]$, where X_0 is as in (15.3.1).*

Proof: We take a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} f &\in C^1(\mathbb{R}^2) \text{ is radially symmetric,} \\ \int_{B_2} f dx &= \int_{\mathbb{R}^2} f dx = 0 \\ f(r) &< 0 \text{ if } 0 < r < 1, \quad f(r) > 0 \text{ if } 1 < r < 2, \quad f(r) = (4-r)^+ \text{ if } 3 < r. \end{aligned} \quad (15.3.4)$$

By [237, Theorem 4], there is $v \in C^1(\overline{B_4}, \mathbb{R}^2)$ such that $Jv = f$ and $v = 0$ on \mathbb{S}_4 ; in particular, by extending v by zero outside B_4 , we have $v \in W^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$. Since the $2p$ -Dirichlet energy is convex, the Direct Method, combined with the sequential weak continuity of the Jacobian, shows that f has at least one $2p$ -energy minimiser and we call it u , using it to define the sets in (15.3.1).

Suppose, for the sake of contradiction, that $X_0 = [0, 2\pi]$. Using Lemmas 15.3.3 and 15.3.4, we deduce that there are angles $\alpha, \alpha' \in [0, 2\pi)$, numbers $k, k' \in \mathbb{Z}$ and $c \in \mathbb{R}$ such that

$$u = e^{i\alpha} \phi_k \text{ in } B_1, \quad u = e^{i\alpha'} \left(\psi(r)e^{2\pi ik'\theta} \right) \text{ in } \mathbb{A}(1, 2),$$

where, for $r \in (1, 2)$,

$$\psi(r)^2 = \frac{1}{k'} \int_1^r 2sf(s) ds + c.$$

In the notation of Definition 13.5.3, we must have

$$e^{i\alpha + ik\theta} \frac{\rho(1)}{\sqrt{|k|}} \equiv \text{Tr}_{\mathbb{S}_1} u|_{B_1} = \text{Tr}_{\mathbb{S}_1} u|_{\mathbb{A}(1,2)} \equiv e^{i\alpha' + ik'\theta} c$$

in $L^{2p}(\mathbb{S}_1)$. It is easy to conclude that $\alpha = \alpha'$, $k = k'$ and $c = \rho(1)/\sqrt{|k|}$, and so, modulo rotations, actually $u = \phi_k$ in B_2 . It is now easy to verify directly that, for f as in (15.3.4), we have

$$\int_0^2 |\dot{\rho}(r)|^2 dr = +\infty,$$

and so by Lemma 13.4.2 $u \notin W^{1,2}(B_2, \mathbb{R}^2)$, which is a contradiction. Alternatively, one can infer that $u \notin W^{1,2}(B_2, \mathbb{R}^2)$ from [263, Theorem 3.4]. □

Proof of Theorem 15.3.1: Let f and u be as in Lemma 15.3.5. For each $\alpha \in [0, 2\pi]$, it is easy to check that the set X_α is closed. We may write, for some index set A ,

$$[0, 2\pi] = \bigcup_{\alpha \in A} X_\alpha, \quad \text{where the union is disjoint.}$$

For distinct $\alpha, \alpha' \in A$, X_α and $X_{\alpha'}$ correspond to distinct equivalence classes of $2p$ -energy minimisers, and so by Lemma 15.3.5 we must have $\#A > 1$. But now Theorem 15.3.2 shows that A must be uncountable. \square

We also note that the proof of Lemma 15.3.3 yields the following corollary:

Corollary 15.3.6. *Let $f \in \mathcal{H}^p(\mathbb{R}^2)$ be radially symmetric and suppose u is its unique $2p$ -energy minimiser, modulo rotations. If u is continuous then $u = \phi_k$ for some $k \in \mathbb{Z} \setminus \{0\}$.*

Clearly the continuity assumption is not restrictive if $p > 1$.

Proof: As in the proof of Lemma 15.3.3 we conclude that u maps circles centred at zero to circles and that $(r, \theta) \mapsto |u(re^{i\theta})|$ is independent of θ . Thus we write simply $|u(r)|$.

We show that the set $\{r \in (0, \infty) : |u(r)| > 0\}$ is connected. Suppose, by way of contradiction, that there are $r_1 < r_2 < r_3$ such that $|u(r_1)|, |u(r_3)| > 0$ but $|u(r_2)| = 0$. We get another $2p$ -energy minimiser for f by setting

$$v(z) = \begin{cases} u(z), & |z| \leq r_2, \\ e^{i\pi} u(z), & |z| > r_2, \end{cases}$$

contradicting the assumption that the $2p$ -energy minimiser for f is unique modulo rotations.

Thus we can write, for some $0 \leq R_1 \leq R_2 \leq \infty$,

$$\{r \in (0, \infty) : |u(r)| > 0\} = (R_1, R_2).$$

We can use Lemma 15.3.4 to conclude that $u = \phi_k$ in $\mathbb{A}(R_1, R_2)$, modulo rotations. Moreover, clearly we must have $f(r) = 0$ if $r \notin (R_1, R_2)$. Thus $\phi_k(z) = 0$ if $r \notin (R_1, R_2)$ and so $u = \phi_k$ outside $\mathbb{A}(R_1, R_2)$ as well. \square

Chapter 16

Nonlinear open mapping principles

In this chapter we prove a general nonlinear open mapping principle for positively homogeneous operators, see Section 16.1. We will prove, in particular, Theorem 12.D. In Section 16.2 we discuss some applications of this result to the Jacobian operator and prove Theorem 12.E. The results of this chapter are based on joint work with LUKAS KOCH and SAULI LINDBERG [179].

16.1 A nonlinear open mapping principle for positively homogeneous operators

For the Jacobian, by adapting a standard proof of the classical Open Mapping Theorem, one obtains the following statement: if $J(\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)) = \mathcal{H}^p(\mathbb{R}^n)$, then for every $f \in \mathcal{H}^p(\mathbb{R}^n)$ there exist $u, v \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ with

$$Ju + Jv = f \quad \text{and} \quad \int_{\mathbb{R}^n} (|Du|^{np} + |Dv|^{np}) \, dx \leq C \|f\|_{\mathcal{H}^p}^p. \quad (16.1.1)$$

Thus, quantitative control is gained at the expense of introducing an extra term Jv .

One could attempt to show the non-surjectivity of J by disproving the *a priori* estimate in (16.1.1). However, the extra term Jv makes this a formidable task since the equation $Ju + Jv = f$ admits much more pathological solutions than $Ju = f$. As a prototypical example, there exist Lipschitz maps $u, v: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ vanishing in the lower half-plane and satisfying $Ju + Jv = 1$ in the upper half-plane [207, Lemma 5]. In Theorem 12.D, the extra Jacobian Jv is removed, leading to a genuinely nonlinear version of the Open Mapping Theorem.

16.1.1 The proof of the main result

Theorem 16.1.1. *Let X and Y be Banach spaces such that \mathbb{B}_{X^*} is sequentially weak* compact. We make the following assumptions:*

- (A1) $T: X^* \rightarrow Y^*$ is a weak*-to-weak* sequentially continuous operator.
- (A2) $T(au) = a^s T(u)$ for all $a > 0$ and $u \in X^*$, where $s > 0$.
- (A3) For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}: X^* \rightarrow X^*$, $\sigma_k^{Y^*}: Y^* \rightarrow Y^*$ such that

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \quad \text{for all } k \in \mathbb{N}, \quad \sigma_k^{Y^*} f \xrightarrow{*} 0 \quad \text{for all } f \in Y^*.$$

Then the following conditions are equivalent:

- (i) $T(X^*)$ is non-meagre in Y^* .
- (ii) $T(X^*) = Y^*$.
- (iii) T is open at the origin.

(iv) For every $f \in Y^*$ there exists $u \in X^*$ such that

$$Tu = f, \quad \|u\|_{X^*}^s \leq C\|f\|_{Y^*}. \quad (16.1.2)$$

A sufficient condition for \mathbb{B}_{X^*} to be sequentially weak* compact is that X is a *weak Asplund space* [343, Theorem 3.5]. For instance, reflexive or separable spaces are weak Asplund [118].

Proof of Theorem 16.1.1: We have (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) and so we just prove (i) \Rightarrow (iv).

Assume that (i) holds. We may write $T(X^*)$ as a union $\cup_{\ell=1}^{\infty} K_{\ell}$, where

$$K_{\ell} \equiv \{f \in Y^* : \text{there exists } u \in X^* \text{ with } Tu = f \text{ and } \|u\|_{X^*}^s \leq \ell\|f\|_{Y^*}\}.$$

Since balls in X^* are sequentially weak* compact, by (A1), the sets K_{ℓ} are norm-closed. Now, by the Baire Category Theorem, some K_{ℓ} contains a closed ball $\bar{B}_r(f_0)$.

Our aim is to solve (16.1.2) whenever $\|f\|_{Y^*} = r$; assumption (A2) then implies the claim. Suppose, therefore, that $\|f\|_{Y^*} = r$. For every $k \in \mathbb{N}$ we have $f_0 + (\sigma_k^{Y^*})^{-1}f \in \bar{B}_r(f_0)$. Hence, we may choose $u_k \in X^*$ such that $Tu_k = f_0 + (\sigma_k^{Y^*})^{-1}f$ and

$$\|\sigma_k^{X^*} u_k\|_{X^*}^s = \|u_k\|_{X^*}^s \leq \ell\|f_0 + (\sigma_k^{Y^*})^{-1}f\|_{Y^*} \leq \ell(\|f_0\|_{Y^*} + r).$$

Since balls in X^* are sequentially weak* compact, after passing to a subsequence if need be, $\sigma_k^{X^*} u_k$ converges weakly* to some $u \in X^*$, so that $T(\sigma_k^{X^*} u_k) \xrightarrow{*} Tu$. By the lower semicontinuity of the norm we have

$$\|u\|_{X^*}^s \leq \liminf_{k \rightarrow \infty} \|\sigma_k^{X^*} u_k\|_{X^*}^s \leq \ell(\|f_0\|_{Y^*} + r).$$

On the other hand, (A3) gives

$$T(\sigma_k^{X^*} u_k) = \sigma_k^{Y^*}(Tu_k) = \sigma_k^{Y^*} f_0 + f \xrightarrow{*} f,$$

so that, by (A1), $Tu = f$. Thus u solves (16.1.2) and the proof is complete. \square

The theory of Compensated Compactness, as described in Part I of this thesis, provides many examples of nonlinear operators to which Theorem 16.1.1 applies. Here we give a general formulation in the spirit of Chapter 5, see also [301, 361], which we then illustrate with more concrete examples.

Example 16.1.2. Let \mathcal{A} be an l -th order homogeneous linear operator, which for simplicity we assume to have constant coefficients; that is, for $v \in C^{\infty}(\mathbb{R}^n, \mathbb{V})$,

$$\mathcal{A}v = \sum_{|\alpha|=l} A_{\alpha} \partial^{\alpha} v, \quad A_{\alpha} \in \text{Lin}(\mathbb{V}, \mathbb{W}),$$

where \mathbb{V}, \mathbb{W} are finite-dimensional vector spaces. For $p \in [1, +\infty)$ and $s \in \mathbb{N}$, $s \geq 2$, take

$$X^* = L_{\mathcal{A}}^{ps}(\mathbb{R}^n, \mathbb{V}), \quad Y^* = \mathcal{H}^p(\mathbb{R}^n).$$

Here $L_{\mathcal{A}}^{ps}(\mathbb{R}^n, \mathbb{V})$ is the space of those $v \in L^{ps}(\mathbb{R}^n, \mathbb{V})$ such that $\mathcal{A}v = 0$ in the sense of distributions. We will further need the following standard non-degeneracy assumption:

$$\text{the symbol of } \mathcal{A}, \text{ seen as a matrix-valued polynomial, has constant rank.} \quad (16.1.3)$$

Whenever (16.1.3) holds, we say that \mathcal{A} has *constant rank*. We will not discuss this assumption here but it holds in all of the examples below; the reader may find other a lot information concerning constant rank operators in Chapter 4.

Let $T: X^* \rightarrow Y^*$ be a homogeneous sequentially weakly continuous operator. Under the assumption (16.1.3), such operators were completely characterised in Chapter 5, and they are often called *Compensated Compactness quantities*. They can be realised as certain constant-coefficient partial differential operators and so they necessarily satisfy (A3) if one takes the isometries $\sigma_k^{X^*}, \sigma_k^{Y^*}$ to be translations. The following are standard examples of such operators:

- (i) $\mathcal{A} = \text{curl}$ and $T = \text{J}$. For this example, take $\mathbb{V} = \mathbb{R}^{n \times n}$ and choose \mathcal{A} in such a way that $\mathcal{A}v = 0$ if and only if $v = Du$, for some $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$. For instance, we may take $(\text{curl } v)_{ijk} = \partial_k v_{ij} - \partial_j v_{ik}$. We also choose $s = n$ and so $X^* = \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$. The only positively n -homogeneous sequentially weakly continuous operator $X^* \rightarrow Y^*$ is the Jacobian, and in particular we recover Theorem 12.D.
- (ii) $\mathcal{A} = \text{curl}^2$ and $T = \text{H}$. Here \mathcal{A} is chosen similarly to the previous example, but now $\mathcal{A}v = 0$ if and only if $v = D^2u$, for some $u: \mathbb{R}^n \rightarrow \mathbb{R}$. Again we take $s = n$ and so $X^* = \dot{W}^{2,np}(\mathbb{R}^n, \mathbb{R}^n)$. We may take $T = \text{H}: X^* \rightarrow Y^*$ to be the Hessian, and Theorem 16.1.1 shows that it satisfies the open mapping principle.

The two previous examples admit a straightforward generalisation, where one considers s -th order minors (instead of the determinant) and a j -th order curl (instead of $j = 1, 2$).

- (iii) $\mathcal{A} = (\text{div}, \text{curl})$ and $T = \langle \cdot, \cdot \rangle$. In this example, $s = 2$ and T is the standard inner product acting on a pair $v \equiv (B, E): \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$; here, B is thought of as a “magnetic field” and E as an “electric field”. As before, Theorem 16.1.1 shows that T satisfies the open mapping principle.

We conclude this subsection by comparing the above example with [87]. There, the authors address the problem of deciding whether Compensated Compactness quantities are surjective, particularly when $p = 1$. Thus Theorem 16.1.1 can be read as saying that openness at zero is a necessary condition for a positive answer to this problem.

16.1.2 A nonlinear uniform boundedness principle

We now present a nonlinear version of the Uniform Boundedness Principle in the spirit of Theorem 16.1.1: under certain structural conditions, a family of operators which is pointwise bounded in a neighbourhood of the origin is uniformly bounded in a (possibly smaller) neighbourhood of the origin.

Proposition 16.1.3. *Let X and Z be Banach spaces and let I be an index set. Suppose the following conditions hold:*

- (i) *For every $i \in I$, the mapping $T_i: X \rightarrow Z$ is such that $u \mapsto \|T_i u\|_Z: X \rightarrow \mathbb{R}$ is weakly sequentially lower semicontinuous.*
- (ii) *There is $\varepsilon > 0$ such that $\sup_{i \in I} \|T_i(u)\|_Z < \infty$ whenever $\|u\|_X \leq \varepsilon$.*
- (iii) *For $j \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^X: X \rightarrow X$ and $\sigma_k^Z: Z \rightarrow Z$ such that*

$$\begin{aligned} T_i \circ \sigma_k^X &= \sigma_k^Z \circ T_i && \text{for all } i \in I \text{ and } k \in \mathbb{N}, \\ \sigma_k^X u &\rightarrow 0 && \text{for all } u \in X. \end{aligned}$$

Then there exists $\delta > 0$ such that

$$\sup_{\|u\|_X \leq \delta} \sup_{i \in I} \|T_i u\|_Z < \infty.$$

Proof: By (ii), we may write $\varepsilon \mathbb{B}_X = \cup_{\ell=1}^{\infty} C_\ell$, where $C_\ell \equiv \{u \in \varepsilon \mathbb{B}_X: \sup_{i \in I} \|T_i u\|_Z \leq \ell\}$ and (i) shows that each C_ℓ is norm closed. Thus, by the Baire Category Theorem, some C_ℓ contains a closed ball $\bar{B}_\delta(u_0)$.

Let now $\|u\|_X \leq \delta$ and $i \in I$. By (iii), we have $u + \sigma_k^X u_0 = \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u] \in \bar{B}(u_0, \delta)$ and moreover $u + \sigma_k^X u_0 \rightarrow u$. So by (i) and again (iii), we have

$$\|T_i u\|_Z \leq \liminf_{k \rightarrow \infty} \|T_i \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u]\|_Z = \liminf_{k \rightarrow \infty} \|\sigma_k^Z T [u_0 + (\sigma_k^X)^{-1} u]\|_Z \leq \ell.$$

The proof is complete. □

We note that, in the linear case, it is possible to prove the Banach–Steinhaus Uniform Boundedness Principle without using Baire’s Category Theorem: the proof relies, instead, on the so-called “gliding hump method”. For an extension of the classical Uniform Boundedness Principle using this method, we refer the reader to [158].

16.1.3 Atomic decompositions in terms of T

The main motivation behind this subsection is Theorem 16.1.4. It establishes an analogue of the atomic decomposition of $\mathcal{H}^1(\mathbb{R}^n)$, giving a weak factorization on $\mathcal{H}^p(\mathbb{R}^n)$ in the spirit of the classical work of COIFMAN, ROCHBERG and WEISS [88]:

Theorem 16.1.4. *Let $p \in [1, \infty)$. For every $f \in \mathcal{H}^p(\mathbb{R}^n)$ there are functions $u_i \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ and real numbers c_i such that*

$$f = \sum_{i=1}^{\infty} c_i J u_i, \quad \|u_i\|_{\dot{W}^{1,np}(\mathbb{R}^n)} \leq 1, \quad \sum_{i=1}^{\infty} |c_i| \lesssim \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}. \tag{16.1.4}$$

In particular, $\mathcal{H}^p(\mathbb{R}^n)$ is the smallest Banach space containing the range $J(\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n))$.

Theorem 16.1.4 was proved in [87] for $p = 1$, while the case $p > 1$ is much harder and was established only recently by HYTÖNEN in [201]. It is conceivable that the operator $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ is not surjective but (16.1.4) improves to a *finitary* decomposition of $\mathcal{H}^p(\mathbb{R}^n)$ in terms of Jacobians. In Proposition 16.1.5, we formulate a rather precise classification of infinitary and finitary decompositions in the setting of Theorem 16.1.1.

Take $\omega \in \bar{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\}$. Given T as in Theorem 16.1.1, if every $f \in Y^*$ can be written as

$$f = \sum_{j=1}^{\omega} c_j T u_j, \quad c_j \in \mathbb{R}, u_j \in \mathbb{B}_{X^*}, \tag{16.1.5}$$

then, following [122], T is said to be $1/\omega$ -surjective. If, furthermore,

$$\sum_{j=1}^{\omega} |c_j| \lesssim \|f\|_{Y^*} \tag{16.1.6}$$

for all $f \in Y^*$, then T is said to be $1/\omega$ -open. DIXON [122] generalised HOROWITZ's example [197] by constructing, for every $m \in \mathbb{N}$, a continuous $1/m$ -surjective bilinear map between Banach spaces which is not $1/m$ -open. In fact, in Dixon's notation, the constants c_j are subsumed by the elements u_j . The formalism (16.1.5)–(16.1.6) is, however, more standard in the context of atomic decompositions.

In Proposition 16.1.5 we show that, for $\omega \in \bar{\mathbb{N}}$, and under the assumptions of Theorem 16.1.1, $1/\omega$ -surjectivity implies $1/\omega$ -openness.

Proposition 16.1.5. *Suppose X, Y and T satisfy the assumptions of Theorem 16.1.1. Let us define, for $\omega \in \bar{\mathbb{N}}$, the sets*

$$\Lambda_{\omega} \equiv \left\{ \sum_{j=1}^{\omega} c_j T u_j : u_j \in \mathbb{B}_{X^*}, c_j \in \mathbb{R} \text{ and } \sum_{j=1}^{\omega} |c_j| < \infty \right\}.$$

If Λ_{∞} is not meagre in Y^ , there is $\omega \in \bar{\mathbb{N}}$ such that $\Lambda_{\omega} = Y^*$ and $\bigcup_{m < \omega} \Lambda_m$ is meagre in Y^* ; moreover, T is $1/\omega$ -open.*

Proof: We show that if $\bigcup_{m < \infty} \Lambda_m$ is not meagre in Y^* , then there is $m \in \mathbb{N}$ such that $\Lambda_m = Y^*$ and Λ_{m-1} is meagre in Y^* . Note that, for each $m \in \mathbb{N}$, the set Λ_m is closed; it follows from the Baire Category Theorem that one of the sets Λ_m contains a ball. By using the s -homogeneity of T , we write $\Lambda_m = \{ \sum_{j=1}^m d_j T v_j : d_j \in \mathbb{R}, v_j \in X^* \}$. By applying Theorem 16.1.1 to the $(s+1)$ -homogeneous operator

$$\tilde{T}: \mathbb{R}^m \times (X^*)^m \rightarrow Y^*, \quad \tilde{T}(\{d_j\}_{j=1}^m, \{v_j\}_{j=1}^m) \equiv \sum_{j=1}^m d_j T v_j,$$

we find that for each $f \in Y^*$ there are $d_j \in \mathbb{R}$ and $v_j \in X^*$ such that

$$\sum_{j=1}^m d_j T v_j = f, \quad \sum_{j=1}^m (|d_j|^{s+1} + \|v_j\|_{X^*}^{s+1}) \lesssim \|f\|_{Y^*}. \tag{16.1.7}$$

We set $c_j = d_j \|v_j\|_{X^*}^s$ and denote $u_j = v_j / \|v_j\|_{X^*}$ if $v_j \neq 0$ and $u_j = 0$ if $v_j = 0$. Thus $c_j T u_j = d_j T v_j$ for $j = 1, \dots, m$. Consequently, through Young's inequality, (16.1.7) yields

$$\sum_{j=1}^m c_j T u_j = f, \quad \sum_{j=1}^m |c_j| \lesssim \|f\|_{Y^*}, \quad u_j \in \mathbb{B}_{X^*}. \quad (16.1.8)$$

It now suffices choose the smallest $m \in \mathbb{N}$ such that $T: X^* \rightarrow Y^*$ is $1/m$ -surjective; the $1/m$ -openness of T is given by (16.1.8).

We finally show that if $\bigcup_{m < \infty} \Lambda_m$ is meagre but Λ_∞ is non-meagre, then in fact $\Lambda_\infty = Y^*$ and T is $1/\infty$ -open. We denote $V \equiv \{\varepsilon T u : \varepsilon = \pm 1, u \in \mathbb{B}_{X^*}\} \subset Y^*$. Now V is bounded and symmetric and, by assumption, $\{\sum_{j=1}^\infty c_j f_j : f_j \in V \text{ for all } j \text{ and } \sum_{j=1}^\infty |c_j| < \infty\}$ is non-meagre in Y^* . By [264, Lemma 3.1], $\{\sum_{j=1}^\infty c_j T u_j : \sum_{j=1}^\infty |c_j| = 1, u_j \in \mathbb{B}_{X^*}\} \subset Y^*$ contains a ball centred at the origin. It immediately follows that given $f \in Y^*$, conditions (16.1.5)–(16.1.6) can be satisfied with $\omega = \infty$. \square

Remark 16.1.6. It is tempting to try and prove the last part of Proposition 16.1.5 by defining an auxiliary operator $\tilde{T}: \ell^{s+1}(\mathbb{N}) \times \ell^{s+1}(\mathbb{N}; X^*) \rightarrow Y^*$ via $T(\{d_j\}_{j=1}^\infty, \{v_j\}_{j=1}^\infty) \equiv \sum_{j=1}^\infty d_j T v_j$ and using Theorem 16.1.1 on \tilde{T} , in analogy to the case $\omega < \infty$. However, such an operator is never weak*-to-weak* continuous unless $T \equiv 0$. Indeed, suppose $T u \neq 0$ and set $d_{jk} = \delta_{jk}$ and $v_{jk} = \delta_{jk} u$. Now $\tilde{T}(\{d_{jk}\}_{j=1}^\infty, \{v_{jk}\}_{j=1}^\infty) = T u$ for all $k \in \mathbb{N}$ but $(\{d_{jk}\}_{j=1}^\infty, \{v_{jk}\}_{j=1}^\infty) \xrightarrow{*} 0$.

Example 16.1.7. Let us denote by \mathcal{H} the Hilbert transform and by $T: L^2(\mathbb{R}, \mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ the operator $T(\chi, \eta) \equiv \mathcal{H}\chi \mathcal{H}\eta - \chi\eta$. The strong factorization $\mathcal{H}^1(\mathbb{C}_+) = \mathcal{H}^2(\mathbb{C}_+) \cdot \mathcal{H}^2(\mathbb{C}_+)$ of analytical Hardy spaces, see e.g. [262] for a proof, yields the surjectivity result

$$\mathcal{H}^1(\mathbb{R}) = \left\{ T(\chi, \eta) : \chi, \eta \in L^2(\mathbb{R}) \right\}. \quad (16.1.9)$$

Thus, in this case, $\Lambda_1 = \mathcal{H}^1(\mathbb{R})$.

Another example is obtained by considering the operator $J: W^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$, where $n \geq 2$ and $p \in [1, \infty)$; we emphasise that the Sobolev space is *inhomogeneous*. In this case, Λ_∞ is meagre in $\mathcal{H}^p(\mathbb{R}^n)$, see [263]. However, if we instead consider the Jacobian as defined on $\dot{W}^{1,np}$, then $\Lambda_\infty = \mathcal{H}^p(\mathbb{R}^n)$ by the results of [201], although it is unclear whether this is optimal. We note that for $J: \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2) \rightarrow \mathcal{H}^p(\mathbb{R}^2)$, the statement $\Lambda_1 = \mathcal{H}^p(\mathbb{R}^2)$ is equivalent to

$$\mathcal{H}^p(\mathbb{R}^2) = \left\{ |\mathcal{S}\omega|^2 - |\omega|^2 : \omega \in L^{2p}(\mathbb{R}^2, \mathbb{R}^2) \right\},$$

compare with (16.1.9). Here \mathcal{S} is the Beurling–Ahlfors transform, which one may think of as the square of a complex Hilbert transform [210].

We are not aware of operators satisfying the assumptions of Theorem 16.1.1 and for which there is $1 < m \in \mathbb{N}$ such that $\Lambda_m = Y^*$ but $\bigcup_{m' < m} \Lambda_{m'}$ is meagre in Y^* .

16.2 Applications to the Jacobian equation

In this section we focus on the case $n = 2$ for simplicity and we assume throughout that the operator $J: \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2) \rightarrow \mathcal{H}^p(\mathbb{R}^2)$ is surjective. We are particularly interested in the case $p = 1$. Our goal is to illustrate the way in which Theorem 12.D yields the following principle:

the existence of rough solutions implies the existence of well-behaved solutions.

The following is an example a rough solution, and something that we would like to avoid:

Example 16.2.1 ([269]). There is a map $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$Ju = 0 \text{ a.e. in } \mathbb{R}^2 \quad \text{and} \quad u([0, 1] \times \{0\}) = [0, 1]^2.$$

In particular, u does not have the Lusin (N) property.

The main result of this subsection is the following theorem, which shows that in some sense it suffices to deal with non-pathological solutions.

Theorem 16.2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and take $f \in \mathcal{H}^1(\mathbb{R}^2)$ such that $f \geq 0$ in Ω . Assume that $J: \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \rightarrow \mathcal{H}^1(\mathbb{R}^2)$ is onto. Then there is a solution $u \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ of (12.2.1) such that:*

- (i) u is continuous in Ω ;
- (ii) u has the Lusin (N) property in Ω .
- (iii) $\int_{\mathbb{R}^2} |Du|^2 dx \leq C \|f\|_{\mathcal{H}^1}$ with $C > 0$ independent of f .

In particular, u satisfies the change of variables formula (13.1.2). Moreover, let $\Omega' \subseteq \Omega$ be an open set such that $f = 0$ a.e. in Ω' . Then:

- (iv) for any set $E \subset \Omega'$, we have $u(\partial E) = u(\overline{E})$;
- (v) for $y \in u(\Omega')$, if C denotes a connected component of $u^{-1}(y) \cap \Omega'$ then C intersects $\partial\Omega'$.

Before proceeding with the proof, we note that (iv) is a type of degenerate monotonicity which had already appeared in the study of the hyperbolic Monge–Ampère equation [84, 230].

Proof: The point of the proof is to perturb f appropriately; then the solution u is obtained as a limit of mappings of integrable distortion.

Let B^+ be a ball containing Ω and let B^- be another ball, disjoint from Ω , and with the same volume as B^+ . Consider the perturbations

$$f_\varepsilon \equiv f + \varepsilon a, \quad a \equiv \chi_{B^+} - \chi_{B^-},$$

which satisfy $f_\varepsilon > 0$ a.e. in Ω . Clearly $a \in \mathcal{H}^1(\mathbb{R}^2)$, being bounded, compactly supported and with zero mean. Hence $f_\varepsilon \rightarrow f$ in $\mathcal{H}^1(\mathbb{R}^2)$ and, from Theorem 12.D, we see that we can choose solutions u_ε of $Ju_\varepsilon = f_\varepsilon$ such that $\int_{\mathbb{R}^2} |Du_\varepsilon|^2 \leq C \|f_\varepsilon\|_{\mathcal{H}^1}$ for all $\varepsilon > 0$. Since the maps u_ε have finite distortion, we can apply Theorem 13.2.2(i) to conclude that the family (u_ε) is equicontinuous. Hence, upon normalising the maps so that $u_\varepsilon(x_0) = 0$ for some fixed $x_0 \in \Omega'$, and up to taking subsequences, (u_ε) converges both locally uniformly in Ω and weakly in $\dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ to a limit u . This already proves (i) and (iii).

To prove (ii), we note that each u_ε satisfies (13.1.1), c.f. Proposition 13.2.3. Since u is the uniform limit of the sequence (u_ε) , u also satisfies (13.1.1) and (ii) follows from Proposition 13.1.4.

For (iv), note that $\varepsilon \leq f_\varepsilon$ in Ω and so each map u_ε , having integrable distortion, is open; it follows that $\partial u_\varepsilon(E) \subseteq u_\varepsilon(\partial E)$. Suppose, for the sake of contradiction, that there is $y \in u(\overline{E}) \setminus u(\partial E)$. On the one hand, there is some $\delta > 0$ such that, for all ε small enough,

$$B_\delta(y) \cap \partial u_\varepsilon(\text{int } E) \subset B_\delta(y) \cap u_\varepsilon(\partial E) = \emptyset;$$

on the other hand, since $y \in u(\text{int } E)$, for all ε small enough,

$$B_\delta(y) \cap u_\varepsilon(\text{int } E) \neq \emptyset.$$

It follows that $B_\delta(y) \subseteq u_\varepsilon(\text{int } E)$. We also have that $|u_\varepsilon(\text{int } E)| \rightarrow 0$ as $\varepsilon \rightarrow 0$: by the change of variables formula,

$$|u_\varepsilon(\text{int } E)| \leq \int_{u_\varepsilon(\text{int } E)} \mathcal{N}(y, u_\varepsilon, \text{int } E) dy = \int_E Ju_\varepsilon = \varepsilon |E| \rightarrow 0.$$

Thus, since $|B_\delta(y)| \leq |u_\varepsilon(E)|$, a contradiction is reached by sending $\varepsilon \rightarrow 0$.

Finally, (v) follows from (iv), as shown for instance in [230, Lemma 2.10]. \square

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