Signatures of Gaussian processes and SLE curves

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Dedicated to my parents

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Abstract

This thesis contains three main results.

The first result states that, outside a slim set associated with a Gaussian process with long time memory, paths can be canonically enhanced to geometric rough paths. This allows us to apply the powerful Universal Limit Theorem in rough path theory to study the quasi-sure properties of the solutions of stochastic differential equations driven by Gaussian processes. The key idea is to use a norm, invented by B. Hambly and T.Lyons, which dominates the p-variation distance and the fact that the roughness of a Gaussian sample path is evenly distributed over time.

The second result is the almost-sure uniqueness of the signatures of SLE_κ curves for $\kappa \leq 4$. We prove this by first expressing the Fourier transform of the winding angle of the SLE curve in terms of its signature. This formula also gives us a relation between the expected signature and the n-point functions studied in the SLE and Statistical Physics literature. It is important that the Chordal SLE_κ measure in $\mathbb{D}$ is supported on simple curves from -1 to 1 for $0 \leq \kappa \leq 4$, and hence the image of the curve determines the curve up to reparametrisation.

The third result is a formula for the expected signature of Gaussian processes generated by strictly regular kernels. The idea is to approximate the expected signature of this class of processes by the expected signature of their piecewise linear approximations. This reduces the problem to computing the moments of Gaussian random variables, which can be done using Wick's formula.

Contents

Chapter 1

Introduction

The theory of rough paths began with a desire to define ordinary differential equations (ODEs) driven by paths which do not have bounded total variation. According to T. Lyons [Lyn98], if such a differential equation were to be defined so that the associated Itô map is continuous, it should be considered as one driven by the path together with its "iterated integrals" (see section 1.3). These "iterated integrals" will be a central object of interest in this thesis. In this chapter, we shall recall the basic setting in rough path theory and state the main results of this thesis.

1.1 Paths with finite p-variations

Let $p \geq 1$, $T > 0$ and $d \in \mathbb{N}$. Let $\mathcal{V}^p\left(\mathbb{R}^d\right)$ denote the set of all continuous functions $\gamma : [0,T] \to \mathbb{R}^d$ with finite p-variations, i.e.

$$\|\gamma\|_p := \left(\sup_{\mathcal{P}} \sum_k \left|\gamma_{t_{k+1}} - \gamma_{t_k}\right|^p\right)^{\frac{1}{p}} < \infty \qquad (1.1.1)$$

where the supremum is taken over all finite partitions $\mathcal{P} := (t_0, t_1, .., t_{n-1}, t_n)$, with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$.

It is elementary to show that for $q > p$, $\mathcal{V}^p\left(\mathbb{R}^d\right) \subset \mathcal{V}^q\left(\mathbb{R}^d\right)$. If we define for $\gamma \in \cup_{p\geq 1}\mathcal{V}^p\left(\mathbb{R}^d\right)$ the roughness of γ as

$$\inf\left\{p \geq 1 : \gamma \in \mathcal{V}^p\left(\mathbb{R}^d\right)\right\} \tag{1.1.2}$$

then $\mathcal{V}^p\left(\mathbb{R}^d\right)$ can be considered as the set of paths with roughness less than or equal to p.

Notice thhat in (1.1.2), we have restricted p to be in the range $[1, \infty)$. This is because if γ is continuous and that for some $p < 1$, $\|\gamma\|_p < \infty$, then γ is a constant path.

Remark 1. The notation $\|\gamma\|_{\mathcal{V}^p\left([0,1],\mathbb{R}^d\right)}$ will also be used to denote $\|\gamma\|_p$.

1.2 Tensor algebra

We have briefly mentioned the role of iterated integrals in the study of differential equations. It turns out that if we embed the sequence of iterated integrals in the tensor algebra, then the additivity property of integral will translate into the multiplicative property of the sequence of iterated integrals with respect to the tensor algebra.

Let $T^n\left(\mathbb{R}^d\right)$ and $T\left(\mathbb{R}^d\right)$ denote the graded algebras on $\mathbb{R}^d$ defined by

$$T^n\left(\mathbb{R}^d\right) := \oplus_{k=0}^n\left(\mathbb{R}^d\right)^{\otimes k}$$

and

$$T\left(\mathbb{R}^d\right) := \oplus_{k=0}^{\infty}\left(\mathbb{R}^d\right)^{\otimes k}$$

where $\left(\mathbb{R}^d\right)^{\otimes 0} := \mathbb{R}$.

We shall define two projection maps as follow:

1. $\pi^{(n)}$ will denote the projection map from $T\left(\mathbb{R}^d\right)$ to $\left(\mathbb{R}^d\right)^{\otimes n}$.

2. π_n will denote the projection map from $T\left(\mathbb{R}^d\right)$ onto $T^n\left(\mathbb{R}^d\right)$.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ be the standard basis of $\mathbb{R}^d$. Let $e_1^*, \ldots, e_d^*$ denote the corresponding dual basis of $\mathbb{R}^{d*}$. Note that $\{\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n} : (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n\}$ forms a basis of $\left(\mathbb{R}^d\right)^{\otimes n}$. We now embed, in the algebraic sense, $T\left(\left(\mathbb{R}^d\right)^*\right)$ into $T\left(\left(\mathbb{R}^d\right)\right)^*$ by extending linearly the map $e_{i_1}^* \otimes \ldots \otimes e_{i_n}^*$ in $\left[\left(\mathbb{R}^d\right)^*\right]^{\otimes n}$ defined by

$$\begin{aligned} e_{j_1}^* \otimes \ldots \otimes e_{j_k}^*\left(\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_n}\right) &= 1 \text{ if } n = k \text{ and } j_1 = i_1, \ldots, j_n = i_n. \\ &= 0 \text{ otherwise.} \end{aligned}$$

Let $I := (i_1, \ldots, i_n) \in \{1, \ldots, d\}^{\otimes n}$, we will also denote the functional $e_{i_1}^* \otimes \ldots \otimes e_{i_n}^*$ by e_I^* or π^I. For $\mathbf{w} \in T\left(\mathbb{R}^d\right)$, $e_{i_1}^* \otimes \ldots \otimes e_{i_n}^*$ is called the projection of $\mathbf{w}$ onto the basis $\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_n}$.

We equip $\left(\mathbb{R}^d\right)^{\otimes k}$ with a norm by identifying $\left(\mathbb{R}^d\right)^{\otimes k}$ with $\mathbb{R}^{d^k}$ and we equip $T^n\left(\mathbb{R}^d\right)$ with the norm

$$|\mathbf{w}|_{T^n\left(\mathbb{R}^d\right)} := \max_{0 \leq k \leq n}\left|\pi^{(k)}(\mathbf{w})\right|.$$

Definition 2. Let $\gamma \in \mathcal{V}^1\left(\mathbb{R}^d\right)$and let $\triangle_n(s,t) := \{(t_1, \ldots, t_n) : s < t_1 < \cdots < t_n < t\}$.

The *lift* of γ is a function $S(\gamma) : \{(s,t) : 0 \leq s \leq t\} \to T\left(\mathbb{R}^d\right)$ defined by

$$S(\gamma)_{s,t} = 1 + \sum_{n=1}^{\infty} \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1} \otimes \ldots \otimes \mathrm{d}\gamma_{t_n} \tag{1.2.1}$$

where the sum $+$ is the direct sum operation in $T\left(\mathbb{R}^d\right)$ and the integrals are taken in the Lebesgue-Stieltjes sense.

The signature of a path $\gamma \in \mathcal{V}^1\left(\mathbb{R}^d\right)$ on $[0,T]$ is defined to be $S(\gamma)_{0,T}$.

Remark 3. Note that in particular we have

$$\begin{aligned} e_{j_1}^* \otimes \ldots \otimes e_{j_k}^* \left(S(\gamma)_{0,t}\right) &= \int_{0<t_1<\ldots<t_k<t} e_{j_1}^* \left(\mathrm{d}\gamma_{t_1}\right) \ldots e_{j_k}^* \left(\mathrm{d}\gamma_{t_k}\right). \\ &= \int_{0<t_1<\ldots<t_k<t} \mathrm{d}\gamma_{t_1}^{j_1} \ldots \mathrm{d}\gamma_{t_k}^{j_k} \end{aligned}$$

where γ^{j_i} is the j_ith component of γ with respect to the basis e_{j_i}.

Note in particular that $\pi^{(n)}\left(S(\gamma)_{s,t}\right) = \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1} \otimes \ldots \otimes \mathrm{d}\gamma_{t_n}$ and will be called the n-th grading of the lift of γ. We will denote $\pi_n\left(S(\gamma)_{s,t}\right)$ by $S_n(\gamma)_{s,t}$.

The signature of a path satisfies the Chen's identity:

$$S(\gamma)_{s,u} \otimes S(\gamma)_{u,t} = S(\gamma)_{s,t} \;\; \forall\, 0 \leq s \leq u \leq t \leq T \tag{1.2.2}$$

1.3 Geometric rough paths

For paths which do not have bounded total variations, their iterated integrals cannot be defined in the classical sense. Therefore, we need the notion of geometric rough paths, which are multiplicative functionals which can be approximated by the signatures of smooth functions under some metric.

Let $\triangle_{0,T} := \{(s,t) : 0 \leq s \leq t \leq T\}$.

Definition 4. Let $n \in \mathbb{N}$ and $p \geq 1$. Let $\mathcal{V}^p\left(T^n\left(\mathbb{R}^d\right)\right)$ denote the set of all continuous functions w from $\triangle_{0,T}$ to $T^n\left(\mathbb{R}^d\right)$ such that

1.$\pi_0\left(w_{s,t}\right) \equiv 1$.

2. w satisfies

$$\max_{1 \leq k \leq n} \sup_D \left(\sum_l \left| \pi^{(k)}\left(w_{t_{l-1},t_l}\right)\right|^{\frac{p}{k}} \right)^{\frac{k}{p}} < \infty$$

where $\sup_D$ runs over all partitions $t_0 = 0 < t_1 < t_2 < .. < t_n = T$.

Let w^1, w^2 be elements of $\mathcal{V}^p\left(T^n\left(\mathbb{R}^d\right)\right)$. We define, for each $p \geq 1$, a distance function between w^1 and w^2 by

$$\rho^{(n)}_{p-var}\left(w^1, w^2\right) = \max_{1 \leq k \leq n} \sup_D \left(\sum_l \left| \pi^{(k)}\left(w^1_{t_{l-1},t_l}\right) - \pi^{(k)}\left(w^2_{t_{l-1},t_l}\right)\right|^{\frac{p}{k}} \right)^{\frac{k}{p}}$$

where $\sup_D$ runs over all partitions $t_0 = 0 < t_1 < t_2 < .. < t_n = T$.

For $p \geq 1$, let $\lfloor p \rfloor$ be the integer part of p. The metric $\rho^{(\lfloor p \rfloor)}_{p-var}(\cdot, \cdot)$ defined on $\mathcal{V}^p\left(T^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)\right)$ is known as the *p-variation metric* and will be denoted by $d_p(\cdot, \cdot)$.

Let $G\Omega_p\left(\mathbb{R}^d\right)$ be the completion of the set $\left\{(s,t) \to \pi_{\lfloor p \rfloor}\left(S\left(w\right)_{s,t}\right) : w \in \mathcal{V}^1\left(\mathbb{R}^d\right)\right\}$ under the p-variation metric in $\mathcal{V}^p\left(T^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)\right)$. $G\Omega_p\left(\mathbb{R}^d\right)$ is called the space of *$p-$geometric rough paths.*

It turns out that each geometric rough path can be uniquely extend to a function in $\mathcal{V}^p\left(T^n\left(\mathbb{R}^d\right)\right)$ for all $n \geq \lfloor p \rfloor$.

Theorem 5. *([Lyn98]) Let $p \geq 1$. Let $w \in \mathcal{V}^p\left(T^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)\right)$ and that w is multiplicative, in the sense that*

$$w_{s,u} \otimes w_{u,t} = w_{s,t}$$

for all $0 \leq s \leq u \leq t$. Then for all $N \geq \lfloor p \rfloor$, there exists a unique multiplicative functional $S_N(w) \in \mathcal{V}^p\left(T^N\left(\mathbb{R}^d\right)\right)$ such that

$$\pi_{\lfloor p \rfloor}\left(S_N(w)\right) = w$$

Recall that a continuous function $w : \triangle_{0,T} \to T^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)$ lies in $G\Omega_p\left(\mathbb{R}^d\right)$ if we can approximate w in the p -variation metric by signatures of paths which have bounded total variations.

One candidate for such an approximation is the piecewise linear interpolation of w, defined as below:

Definition 6. Let $w : [0,T] \to \mathbb{R}^d$ be a continuous function.

Let $D = (0 = t_0 < t_1 < .. < t_n = T)$ be a partition of $[0,T]$. We define the piecewise linear interpolation of w with respect to the partition $\mathcal{D}$ by

$$w^D(t) := w(t_i) + \frac{w(t_{i+1}) - w(t_i)}{t_{i+1} - t_i}(t - t_i) \;\; t \in [t_i, t_{i+1}].$$

The piecewise linear interpolation taken with respect to the dyadic partition $\mathcal{P}_m := \left(t_k = \frac{kT}{2^m}, k = 0, \ldots, 2^m\right)$ is known as the dyadic interpolation.

1.4 Range of the signature

Although in Definition 2 we defined $S(\gamma)$ as a function from $\triangle_{0,T}$ to $T\left(\mathbb{R}^d\right)$, the map $S(\gamma)$ in fact takes its value in a much smaller space, which we shall now describe.

Let V be a finite dimensional vector space over a field F. For $a, b \in V$, define $[a,b] = a \otimes b - b \otimes a$. Let A and B be two subsets of $T(V)$ and let $[A,B] := \operatorname{span}\{[a,b] : a \in A,\ b \in B\}$.

Let $\mathcal{L}_1(V) := V$ and

$$\mathcal{L}_{n+1}(V) = [\mathcal{L}_n(V), V].$$

Let $\mathcal{L}(V) = \oplus_{n=1}^{\infty} \mathcal{L}_n(V)$.

Proposition 7. *(K. Chen,[Chen57])Let γ be a continuous path in $\mathbb{R}^d$ with finite total variation. Then* $\log S(\gamma)_{0,1} \in \mathcal{L}\left(\mathbb{R}^d\right)$.

1.4.1 Bibliographical note

The iterated integrals of one-form was first studied by Chen in [Chen54] who proved a version of the identity (1.2.2). While studying the continuity of the Itô map of classical ODEs, T. Lyons[Lyn98] recognised that the key in defining a topology so that the Itô map is continuous, lies in the iterated integrals of the driving signal. In the same paper, he formulated the notion of geometric rough paths, proved Theorem 5 and proved that the Itô map is indeed continuous in the space of geometric rough paths.

His work has since been applied in the context of stochastic analysis, where the object of interest usually do not have bounded total variation. There are a number of stochastic processes whose sample paths could almost surely be lifted to geometric rough paths. (More precisely, this means that all paths on the complement of the null set of the measure in consideration can be lifted to geometric rough paths.). We include some examples as follows:

- Brownian motion (E.M. Sipilainen, [Spl93]).
- Gaussian processes with long time memory (Coutin and Qian[CQ02]).
- Brownian motion on the Sierpinski gasket (Hambly and Lyons [HL98]).
- Symmetric diffusions with generators of elliptic differential operators of second order. (Bass, Hambly and Lyons [BHL02]).
- Planar α-Hölder domains, with $\alpha > \frac{1}{3}$ (Yam [Yam08]).
- The partial sum process of orthogonal expansion (Yang, [DL11]).

In [Wil01], D. Williams used rough path theory to make sense of differential equations driven by Levy processes and proved the existence and uniqueness of solutions. In [FGL13], physical Brownian motions was enhanced to rough paths with a non standard type of area. In [CFV09], Cass, Friz and Victoir proved that the existence of density for the solutions to differential equations driven by non-degenerate Gaussian processes. In [LL05], A. Lejay and T. Lyons showed that it is possible to find two sequences of approximations of Brownian motions in the uniform topology, such that the corresponding sequences of Levy area go to different limits.

1.5 Plan for the rest of this thesis

1.5.1 A Quasi-sure approximation theorem for Gaussian rough paths with long time memory

Lyons' theory has also been used to define and study stochastic differential equations in a pathwise sense. This is possible if the paths can be enhanced into geometric rough paths outside a null set with respect to the probability measure in consideration. In [CQ02], L. Coutin and Z. Qian proved that if $\mathbf{w} := \left(w_t^1, \ldots, w_t^d\right)_{0 \le t \le 1}$ is a centered d-dimensional Gaussian process with independent components satisfying a "long time memory" condition (which we recall in Chapter 2) then there exists a $\mathbb{P}-$null set $\mathcal{N}$ such that on $\mathcal{N}^c$,

$$S_3\left(\mathbf{w}^{\mathcal{P}_m}\right) \text{ is a Cauchy sequence in } G\Omega_p\left(\mathbb{R}^d\right) \tag{1.5.1}$$

This theorem means that stochastic differential equations can be defined for Gaussian processes satisfying the Coutin-Qian condition in a canonical way outside a null set.

In the Chapter 1, we would like to prove a stronger result that the statement (1.5.1) holds on the complement of a slim set, defined in the sense of Malliavin.

After publishing this result online, we were informed of the similar work in [Ihm06] who proved that Brownian sample paths can be quasi-surely and canonically lifted to 2-geometric rough paths.

1.5.2 Signature through Green's theorem

In the introduction, we mentioned the role of the iterated integrals of paths in the study of ODEs. In fact, the study of iterated integrals began long before the birth of rough path theory in Chen's work[Chen54]. In particular, he proved in [Chen58] that for paths which are irreducible and piecewise regular, the sequence of iterated integrals of paths uniquely determine the paths up to translations and reparametrisations.

His result was vastly extended and studied in a new light by Hambly and Lyons [LH02] who proved that two paths $\gamma, \tilde{\gamma} : [0,1] \to \mathbb{R}^d$ with finite total variation has same signature if and only if they can be obtained from each other by a tree-like deformation. In Chapter 3, we prove that if we know *a priori* that a curve is a simple curve in $\overline{\mathbb{D}}$ from -1 to 1 with finite p-variation, where $p < 2$, then we may recover the curve from its signature.

It was first observed by Yam [Yam08] that as tree-like deformation does not change the topological feature of a path, it should be possible to extract topological information of a path from its signature. For paths with bounded total variations, he also gave a formula for the winding number of a path in terms of its signature. In this chapter, we asked if Yam's formula can be simplified if we seek instead an expression for the Fourier transform of a path's winding number around (x, y), considered as a function in (x, y). The answer is positive.

1.5.3 Signatures of SLE curves

Recently the almost sure uniqueness of signature problem for stochastic processes has also been considered. Le Jan and Qian[LQ12] proved that the signatures of Brownian motion sample paths determine the Brownian motion paths almost surely. The result was extended to diffusion processes in [Geng13]. In the fourth chapter, we prove the almost sure uniqueness for SLE_κ curves where $0 < \kappa \leq 4$.

Moreover, we use the formula between winding number and signature provided in Chapter 3 to obtain a relation between the expected signature of the Chordal SLE_κ measure and the n-point functions studied in the SLE literature.

One can also the reverse question of whether the expected signature of the Chordal SLE measure can be expressed in terms of the n-point functions. Here, we are only able to express this relation explicitly up to the fourth grading. The first three grading of the expected signature of SLE curves was computed by B. Werness in [Wer12].

1.5.4 Expected signature for Gaussian processes with strictly regular kernels

The expected signature of a stochastic process can be considered as the Laplace's transform of the law of a process. In the cubature method of numerical integration, it is also used to calculate the cubature measure which approximates the Wiener measure. The expected signature of Brownian motion up to a fixed time were studied independently by T. Fawcett in [Faw03] and N. Victoir in [LV04]. The expected signature of Brownian motion up to an exit time of a domain and of the general diffusion up to a fixed time was studied by Ni and Lyons[LH11].

In the Chapter 5, we calculate the expected signature of Gaussian processes with strictly regular kernels. These are Gaussian processes with regular kernels (see [AMN01]) which do not have Brownian components.

The original motivation for undertaking this study was to find the expected signature of fractional Brownian motions for $H > \frac{1}{2}$. After posting this result on arXiv, we were informed that the expected signature for fractional Brownian motions for Hurst parameter $H > \frac{1}{3}$ had already appeared in [BC07]. Therefore, this chapter should be considered as a generalisation of the result in [BC07].

Chapter 2

A quasi-sure approximation theorem for Gaussian processes with long time memory

Recall that in order to apply the rough path theory to stochastic differential equations, we must prove that the sample paths can be almost surely enhanced into geometric rough paths. Such a result was proved for Gaussian processes with long time memory by L. Coutin and Z. Qian in [CQ02]. In this chapter we shall prove that Coutin-Qian's result holds quasi-surely. In [Ihm06], which we were made aware after publishing this result, Inahama proved that quasi-surely Brownian sample paths can be canonically enhanced into geometric rough paths. In the joint work with X. Geng and Z. Qian, which we shall present in this chapter, we prove that this holds for Gaussian processes with long time memory.

Our strategy is to replace the metric d_p, which is difficult to deal with because of the supremum structure, with the norms $(\rho_j)_{j=1,2,3}$ invented by B. Hambly and T. Lyons in [HL98], which is easier to analyse. Then by observing that the roughness of Gaussian sample paths is evenly distributed over the dyadic partitions (see (2.2.1)), our task is reduced to estimating the Sobelev norm of polynomials of Gaussian random variables. Finally, we use the fact that the Sobelev norm of polynomial functionals in the abstract Wiener space is bounded by their L^2 norm times a constant which depends only on its degree. In the proof for the Brownian motion case in [Ihm06], Inahama got around the same problem by using the norms $(\rho_j)_{j=1,2,3}$ together with the fact that a Borel set A is slim if and only if $\theta(A) = 0$ for any positive Watanabe distribution θ.

First we shall recall some basic notions of Quasi-sure Analysis (for details, please see [Mlv97]).

2.1 Preliminaries

Let B denote the totality of continuous functions $w : [0, 1] \to \mathbb{R}^d$ such that $w_0 = 0$. We equip B with the uniform topology and the corresponding Borel σ-algebra $\mathcal{B}$. We shall put a measure $\mathbb{P}$ on $(B, \mathcal{B})$ that satisfies the following condition.

Condition 1: $\mathbb{P}$ is a probability measure on $(B, \mathcal{B})$ such that the coordinate process $t \to w_t$ is a $d-$dimensional continuous centered Gaussian process with independent components. Moreover, there exists a $H > \frac{1}{4}$ and $C_{Hol}, C_{LTM} > 0$ such that the process $t \to w_t$ satisfies

$$\mathbb{E}\left(w_t^i - w_s^i\right)^2 \leq C_{Hol}\left|t-s\right|^{2H} \ \ \forall s,t \in [0,1]$$
$$\mathbb{E}\left(w_t^i - w_s^i\right)\left(w_{t+\tau}^i - w_{s+\tau}^i\right) \leq C_{LTM}\tau^{2H}\left|\frac{t-s}{\tau}\right|^2 \ \ \forall t,s \in [0,1], \tau > 0, \ \left|\frac{t-s}{\tau}\right| \leq 1 \tag{2.1.1}$$

where w^i are the coordinate components of the coordinate process $t \to w_t$.

Remark 8. It was shown in Remark 4.4.2 in [LQ02] that the fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$ satisfies Condition 1.

Remark 9. The first inequality in (2.1.1) is called the Hölder condition whereas the second inequality in (2.1.1) is called the long time memory condition.

Let $\mathbb{P}$ be a measure satisfying Condition 1. For each i, let $R_i(s,t)$ be the correlation function of the ith coordinate component of the Gaussian process. We shall construct a Hilbert space associated with R_i, which will give rise to a differential structure on the space of functionals on B. Define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_i}$ on the set of indicator functions by

$$\left\langle 1_{[0,s]}, 1_{[0,t]} \right\rangle_{\mathcal{H}_i} = R_i(s_i, t_i)$$

for any $s,t \in [0,1]$. We may extend this inner product by linearity to $\mathbb{R}^d$-valued step functions. Let $\mathcal{H}_i$ be the Hilbert space obtained by taking completion of the set of these step functions with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_i}$. Let $\mathcal{H}$ be the Hilbert space obtained by taking the orthogonal direct sum of the Hilberts spaces $\mathcal{H}_i$. For $q \geq 1$, let $L^q(\mathbb{P})$ denotes the L^q space with respect to the probability measure $\mathbb{P}$. The map $\mathcal{I} : \left(1_{[0,s_1]}, \therefore, 1_{[0,s_d]}\right)^t \to \sum_{i=1}^d w_{s_i}^i$ extends to an isometric embedding from $\mathcal{H}$ to

$L^2(\mathbb{P})$ and $(B, \mathcal{H}, \mathbb{P})$ forms an abstract Wiener space in the sense of Gross. From now on, $\mathcal{H}$ will denote the image of $\mathcal{I}$.

The Cameron-Martin theorem states that for any $h \in \mathcal{H}$, the measure obtained by shifting $\mathbb{P}$ in the direction of h is absolutely continuous with respect to $\mathbb{P}$. This allows us to define the derivative of functionals of B.

Definition 10. A function $F : B \to \mathbb{R}$ is $\mathcal{H}$-differentiable if there exists $DF \in \mathcal{H}^*$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(x + \varepsilon h)|_{\varepsilon=0} = DF(h)$$

for all $h \in \mathcal{H}$.

In general, for each $m \in \mathbb{N}$, F is m times $\mathcal{H}$-differentiable if there exists a bounded multilinear functional $D^m F : \mathcal{H}^m \to \mathbb{R}$ such that

$$\frac{\partial^m F}{\partial t_1 \ldots \partial t_m}(x + t_1 h_1 \ldots + t_m h_m)|_{t_1=\ldots=t_m=0} = D^m F(h_1, \ldots, h_m).$$

A function $F : B \to \mathbb{R}^d$ is m-times $\mathcal{H}$-differentiable if each coordinate component of F is m-times $\mathcal{H}$ differentiable. Moreover, the derivative of $F = (F_1, \ldots, F_d)$ will be defined as

$$D^m F(h_1, \ldots, h_m) = \sum_{i=1}^{d} D^m F_i \otimes \mathbf{e}_i$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ is the standard basis of $\mathbb{R}^d$.

We say a function $F : B \to \mathbb{R}^d$ is smooth if there are $n \in \mathbb{N}$, $\psi_1, \ldots, \psi_n \in B^*$ and infinitely differentiable functions $f_i : \mathbb{R}^n \to \mathbb{R}$ which has polynomial growth such that

$$F(x) = \sum_{i=1}^{d} f_i(\langle x, \psi_1 \rangle, \ldots, \langle x, \psi_n \rangle) \mathbf{e}_i$$

for all $x \in B$.

The set of all smooth functions $B \to \mathbb{R}^d$ will be denoted by $\mathcal{S}$. We also have by a direct calculation that

$$D^k F(x) = \sum_{i=1}^{d} \sum_{j_1, \ldots, j_k = 1}^{n} \partial_{j_1} \ldots \partial_{j_k} f_i(\langle x, \psi_1 \rangle, \ldots, \langle x, \psi_n \rangle) \psi_{j_1} \otimes \ldots \otimes \psi_{j_k} \otimes \mathbf{e}_i$$

where $\psi_{j_1}, \ldots, \psi_{j_k}$ should be regarded as elements of $\mathcal{H}^*$, and for $(h_1, \ldots, h_k) \in \mathcal{H}^k$, $\psi_{j_1} \otimes \ldots \otimes \psi_{j_k}(h_1, \ldots, h_k) = \psi_{j_1}(h_1) \ldots \psi_{j_k}(h_k)$.

Define a norm on $\mathcal{S}$ by

$$\|F\|_{r,q} := \|F\|_{L^q(\mathbb{P})} + \sum_{k=1}^{r} \left\| \left| D^k F \right|_{\mathcal{H}^{\otimes k} \otimes \mathbb{R}^d} \right\|_{L^q(\mathbb{P})}$$

where $\|\cdot\|_{L^q(\mathbb{P})}$ denotes the L^q norm taken with respect to $\mathbb{P}$.

We shall denote the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{r,q}$ by $W^{r,q}$.

One class of subspaces of $\mathcal{S}$ which will be of particular interest to us is the subspaces of polynomial functionals $\mathcal{P}_N$, $N \geq 1$. They are defined as in $\mathcal{S}$ except that the functions f_i are now required to be polynomials of degree no more than N.

The following lemma will imply that the Sobelev norm of a polynomial functional can be estimated in terms of the $L^2(\mathbb{P})$ norm of the polynomial. The first statement was proved as Theorem 3.46 in [Jan97](see Remark 3.45 for the explicit constant) and the second is Proposition 1.2.2 in [Nal95].

Lemma 11. *Let $N \geq 1$. Then*

1. For all $q > 1$ and $X \in \mathcal{P}_N$,

$$\|X\|_{L^q(\mathbb{P})} \leq 3^{\frac{N(q+2)}{2}} \|X\|_{L^2(\mathbb{P})}$$

2. Let $F : B \to \mathbb{R}$ be a L^2 random variable with the Wiener chaos expansion $F = \sum_{n=0}^{\infty} J_n F$, where J_n is the projection of $L^2(\mathbb{P})$ onto the nth Wiener chaos. Then $F \in W^{k,2}$ if and only if

$$\sum_{n=1}^{\infty} n^k \|J_n F\|^2_{L^2(\mathbb{P})} < \infty. \tag{2.1.2}$$

Moreover, in this case,

$$\mathbb{E}\left(\left\|D^k F\right\|^2_{H^{\otimes k}}\right) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \|J_n F\|^2_{L^2(\mathbb{P})}.$$

This gives immediately the following.

Corollary 12. *Let $N \in \mathbb{N}$ and $q > 1$. Then*

$$\|F\|_{r,q} \leq 2 \cdot 3^{\frac{N(q+2)}{2}} N^{\frac{r}{2}} \|F\|_{L^2(\mathbb{P})}$$

for all $F \in \mathcal{P}_N$ and all $r \in \mathbb{N} \cup \{0\}$.

Proof. As F is a polynomial functional, the sum in 2.1.2 is trivially finite. Hence

$$\begin{aligned}\mathbb{E}\left(\left\|D^k F\right\|_{H^{\otimes k}}^2\right) &= \sum_{n=k}^{N} \frac{n!}{(n-k)!} \left\|J_n F\right\|_{L^2(\mathbb{P})}^2 \\ &\leq \frac{N!}{(N-k)!} \sum_{n=k}^{N} \left\|J_n F\right\|_{L^2(\mathbb{P})}^2 \\ &\leq \frac{N!}{(N-k)!} \left\|F\right\|_{L^2(\mathbb{P})}^2 .\end{aligned}$$

By the first statement of Lemma 11,

$$\begin{aligned}\left\|\left\|D^k F\right\|_{H^{\otimes k}}\right\|_{L^q(\mathbb{P})} &\leq 3^{\frac{(N-k)(q+2)}{2}} \left\|\left\|D^k F\right\|_{H^{\otimes k}}\right\|_{L^2(\mathbb{P})} \\ &\leq 3^{\frac{(N-k)(q+2)}{2}} \left(\frac{N!}{(N-k)!}\right)^{\frac{1}{2}} \left\|F\right\|_{L^2(\mathbb{P})} .\end{aligned}$$

Therefore

$$\begin{aligned}\left\|F\right\|_{r,q} &= \sum_{k=0}^{r} \left\|\left\|D^k F\right\|_{H^{\otimes k}}\right\|_{L^q(\mathbb{P})} \\ &\leq \sum_{k=0}^{r} 3^{\frac{(N-k)(q+2)}{2}} \left(\frac{N!}{(N-k)!}\right)^{\frac{1}{2}} \left\|F\right\|_{L^2(\mathbb{P})} \\ &\leq 2 \cdot 3^{\frac{N(q+2)}{2}} N^{\frac{r}{2}} \left\|F\right\|_{L^2(\mathbb{P})}\end{aligned}$$

□

We now recall the notion of capacity.

Definition 13. The (r, q)-capacity is defined for open sets O in B as

$$\operatorname{Cap}_{r,q}(O) := \inf\left\{\left\|F\right\|_{r,q} : F \in W^{r,q},\ F \geq 1 \text{ on } O,\ F \geq 0 \text{ on } B\ \mathbb{P}\text{-a.s.}\right\}$$

and for arbitrary sets A in B as

$$\operatorname{Cap}_{r,q}(A) := \inf\left\{\operatorname{Cap}_{r,q}(O) : O \text{ open in }, \ A \subset O\right\}$$

We now recall some basic properties of (r,q)-capacity from the potential theory of Abstract Wiener space.

Proposition 14. *For all $r \in \mathbb{N}$ and $q \geq 1$, the following statements hold*

1.If $X \subset Y$, then $Cap_{r,q}X \leq Cap_{r,q}Y$.

2.Let $(A_n)_{n\geq 0}$ be a countable sequence of sets in B. Then $Cap_{r,q}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} Cap_{r,q}(A_n)$.

3.For all subsets A of B,$Cap_{r,p}A \leq Cap_{s,q}A$ for all $1 \leq r \leq s$ and $1 \leq p \leq q$.

Proof. See [Mlv97]. □

Furthermore, we also have a Tchebychev-type inequality and a Borel-Cantelli lemma for $\operatorname{Cap}_{r,p}$.

Proposition 15. *1. Let $r \in \mathbb{N}$ and $q \geq 1$. Then for all $\lambda > 0$, and all continuous functions $f : B \to \mathbb{R}$ such that $f \in W^{r,p}$,*

$$Cap_{r,q}\{w : f(w) > \lambda\} \leq \frac{1}{\lambda}\|f\|_{r,q}$$

2. Let $\{A_n\}_{n\geq 1}$ be a sequence of subsets of B. If

$$\sum_{n=1}^{\infty} Cap_{r,q}(A_n) < \infty$$

then

$$Cap_{r,q}\left(\limsup_{n} A_n\right) = 0$$

Proof. See [Mlv97]. □

From this point onwards, we will fix $H > \frac{1}{4}$, $p \in (2,4)$ and $Hp > 1$.

We wish to apply the first statement of Proposition 15 with f being the functional $\boldsymbol{w} \to d_p\left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)}\right)$, where $\boldsymbol{w}^{(m)}$ denotes the dyadic interpolation of $\boldsymbol{w}$ and d_p is the p-variation metric defined in Definition 4 of Chapter 1. The key problem is that it is not known (indeed we do not believe) that the functional $\boldsymbol{w} \to d_p\left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)}\right)$ is in $W^{r,p}$ for all $r, p \geq 1$. We get around this problem by using a norm ρ invented in [HL98] which dominates the p-variation norm.

Definition 16. Let $\boldsymbol{w} = (1, w^1, w^2, w^3)$ and $\tilde{\boldsymbol{w}} = (1, \tilde{w}^1, \tilde{w}^2, \tilde{w}^3)$ be two functions from $\triangle := \{(s,t) : 0 \leq s \leq t \leq 1\}$ to $T^3\left(\mathbb{R}^d\right)$. Let $t_k^n := \frac{k}{2^n}$ for $k = 1, \ldots, 2^n$. For $\gamma > p-1$, $j = 1,2,3$, define

$$\rho_j\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right) := \left(\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| w^j_{t^n_{k-1}, t^n_k} - \tilde{w}^j_{t^n_{k-1}, t^n_k} \right|^{\frac{p}{j}}\right)^{\frac{j}{p}}$$

and

$$\rho_j\left(\boldsymbol{w}\right) := \rho_j\left(\boldsymbol{w}, \mathbf{0}\right)$$

where $\mathbf{0} := (1,0,0,0)$.

Remark 17. Note that ρ_j depends also on p and γ. As mentioned, we have fixed p. We will now also fix γ to be a real number greater than 4.

Proposition 18. *1.For $p \in [3,4)$, there exists a $C > 0$ such that for all $\boldsymbol{w}, \tilde{\boldsymbol{w}} : \triangle \to T^3\left(\mathbb{R}^d\right)$,*

$$\begin{aligned} d_p\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right) \leq & C. \max\left\{\rho_1\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right), \rho_2\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right), \rho_3\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\right. \\ & \rho_1\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\left(\rho_1\left(\boldsymbol{w}\right) + \rho_1\left(\tilde{\boldsymbol{w}}\right)\right)^2 + \left(\rho_2\left(\boldsymbol{w}\right) + \rho_2\left(\tilde{\boldsymbol{w}}\right)\right)\right), \\ & \left.\rho_2\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\rho_1\left(\boldsymbol{w}\right) + \rho_1\left(\tilde{\boldsymbol{w}}\right)\right), \rho_1\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\rho_1\left(\boldsymbol{w}\right) + \rho_1\left(\tilde{\boldsymbol{w}}\right)\right)\right\} \\ = & C \max_{\{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \cup \{0\} : i + jk \leq 3\}} \rho_i\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\rho_j\left(\boldsymbol{w}\right) + \rho_j\left(\tilde{\boldsymbol{w}}\right)\right)^k. \end{aligned}$$

2. For $p \in [2,3)$, there exists a $C > 0$ such that for all $\boldsymbol{w}, \tilde{\boldsymbol{w}} : \triangle \to T^2\left(\mathbb{R}^d\right)$,

$$\begin{aligned} d_p\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right) \leq & C. \max\left\{\rho_1\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right), \rho_2\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right), \rho_1\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\rho_2\left(\boldsymbol{w}\right) + \rho_2\left(\tilde{\boldsymbol{w}}\right)\right)\right\} \\ = & C \max_{\{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \cup \{0\} : i + jk \leq 2\}} \rho_i\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\rho_j\left(\boldsymbol{w}\right) + \rho_j\left(\tilde{\boldsymbol{w}}\right)\right)^k. \end{aligned}$$

3. For $p \in [1,2)$, there exists a $C > 0$ such that for all $\boldsymbol{w}, \tilde{\boldsymbol{w}} : \triangle \to \mathbb{R}^d$,

$$\begin{aligned} d_p\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right) \leq & C. \rho_1\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right) \\ = & C \max_{\{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \cup \{0\} : i + jk \leq 1\}} \rho_i\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}\right)\left(\rho_j\left(\boldsymbol{w}\right) + \rho_j\left(\tilde{\boldsymbol{w}}\right)\right)^k. \end{aligned}$$

Proof. See [LQ02], Chapter 4. □

Finally, we will recall the following $L^2(\mathbb{P})$ estimates from [LQ02] on Gaussian processes with long time memory.

For each $w \in B$, let $w^{(m)}$ denote the dyadic interpolation of w as defined in Definition 6.

Lemma 19. *Let $\mathbb{P}$ be a measure satisfying Condition 1 and H be the parameter specified in Condition 1. Let $C^\star$ be defined as $C^\star := \max\left(C_{Hol}, C_{LTM}\right)$ where C_{Hol} and C_{LTM} is defined in Condition 1. Then for $j = 1, \ldots, \left\lfloor \frac{1}{H} \right\rfloor$ and $1 \le k \le 2^n$,*

$$\left\| w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \le \begin{cases} C_j.2^{-nHj}, & \forall n \le m, \\ \frac{C^{\star\frac{j}{2}}}{j!} 3^{j^2(j+1)}.2^{m(1-H)j-nj}, & \forall n \ge m, \end{cases}$$

where

$$\begin{aligned} C_1 &= C^{\frac{1}{2}}_{Hol} 3^4 \\ C_2 &= \max\left(C_{LTM}, C_{Hol}\right)\left(\frac{2d\zeta\left(4-4H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)}} + \frac{3^{24}}{2}\right) \\ C_3 &= \max\left(C^{3/2}_{LTM}, C^{3/2}_{Hol}\right)\left(7d^{3/2}\frac{\zeta\left(2-2H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)k}} + \frac{3^{108}}{6}\right) \end{aligned}$$

and ζ denotes the classical zeta function.

Moreover,

$$\begin{aligned} \left\| w^{(m+1),1}_{t^n_{k-1},t^n_k} - w^{(m),1}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} &\le \begin{cases} 0, & \forall n \le m \\ 2C^{\star\frac{1}{2}} 3^2.2^{m(1-H)-n}, & \forall n \ge m \end{cases}, \\ \left\| w^{(m+1),2}_{t^n_{k-1},t^n_k} - w^{(m),2}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} &\le \begin{cases} 3dC^\star\zeta\left(4-4H\right)^{\frac{1}{2}} 2^{-\frac{1}{2}(4H-1)m-\frac{1}{2}n}, & \forall n \le m \\ \frac{1}{2}C^\star 3^{12}.2^{2m(1-H)-2n}, & \forall n \ge m \end{cases} \\ \left\| w^{(m+1),3}_{t^n_{k-1},t^n_k} - w^{(m),3}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} &\le \begin{cases} 7d^{\frac{3}{2}}C^{\star\frac{3}{2}}\zeta\left(2-2H\right)^{\frac{1}{2}}.2^{-\frac{1}{2}(4H-1)m-\frac{1+2H}{2}n}, & \forall n \le m \\ \frac{1}{6}C^{\star\frac{3}{2}} 3^{36}.2^{3m(1-H)-3n}. & \forall n \ge m \end{cases} \end{aligned}$$

Proof. See Appendix or [LQ02] Proposition 4.4.1 and 4.5.1 and (4.2.4). $\square$

Remark 20. Note that we only have the estimates for $\left\|w^{(m),j}_{t^n_{k-1},t^n_k}\right\|_{L^2(\mathbb{P})}$ and $\left\|w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k}\right\|_{L^2(\mathbb{P})}$ where $1 \leq j \leq \left\lfloor \frac{1}{H} \right\rfloor$. That means for $H \in (\frac{1}{2}, 1]$, we can only prove that, outside a slim set, $\boldsymbol{w}^{(m)}$ is a Cauchy sequence in $G\Omega_p$ for all $\frac{1}{H} < p < 2$. Likewise we can only prove that for $H \in [\frac{1}{2}, \frac{1}{3})$, outside a slim set, $\boldsymbol{w}^{(m)}$ is a Cauchy sequence in $G\Omega_p$ for $\frac{1}{H} < p < 3$. This is not a problem because by the continuity of the extension of multiplicative functional (see Theorem 2.2.2 in [Lyn98]), if $\boldsymbol{w}^{(m)}$ is a Cauchy sequence in $G\Omega_p$, then the extension of $\boldsymbol{w}^{(m)}$ to a multiplicative functional on $T^3\left(\mathbb{R}^d\right)$ is a Cauchy sequence in $G\Omega_q$ for all $q > p$.

2.2 Main result and proof

We now state the main result of this chapter.

Theorem 21. *Let $\mathbb{P}$ be a measure satisfying Condition 1. Let H be the parameter specified in Condition 1. Let $p \in \left(\frac{1}{H}, \left\lfloor \frac{1}{H} \right\rfloor + 1\right)$. Let $\boldsymbol{w}^{(m)} = \left(1, w^{(m),1}, \ldots, w^{(m),\lfloor p \rfloor}\right)$ be the first three levels of the lift of the dyadic interpolation of w, then*

$$Cap_{r,q}\left\{w : \boldsymbol{w}^{(m)} \text{ is not a Cauchy sequence in } G\Omega_p\right\} = 0$$

for any $r \in \mathbb{N}$ and $q \geq 1$.

To prove this, we need an intermediate lemma.

Lemma 22. *Let $\mathbb{P}$ and H satisfy the condition of Theorem21. Let $p \in \left(\frac{1}{H}, 4\right)$, $\theta \in (0, Hp - 1)$, $q \geq 1$, $r \in \mathbb{N}$, $\tilde{N} > \max\left\{\frac{r}{2}, 2^{-1}\left(H - \frac{\theta+1}{p}\right)^{-1}\right\}$ be an integer. For* *all* $m \geq 1$, $\lambda > 0$ *and* $j = 1, \ldots, \left\lfloor \frac{1}{H} \right\rfloor$,

$$Cap_{r,q} \left\{ w : \rho_j \left(\boldsymbol{w}^{(m+1)}, \boldsymbol{w}^{(m)} \right) > \lambda \right\} \leq C_j' . \lambda^{-2\tilde{N}} \left(2^{-m\left(2j\tilde{N}\left(H - \frac{\theta+1}{p}\right) - 1\right)} + 2^{-\tilde{N}m(4H-1)} \right),$$

and

$$Cap_{r,q} \left\{ w : \rho_j \left(\boldsymbol{w}^{(m)} \right) > \lambda \right\} \leq C_j'' . \lambda^{-2\tilde{N}}$$

where

$$C_1' = 3^{8\tilde{N} + q\tilde{N} + 1} 2^{1 + \frac{r}{2} + 2\tilde{N}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{1}{p}} C^{\star\tilde{N}}$$

$$C_2' = 3^{2\left(16\tilde{N} + q\tilde{N} + 1\right)} 2^{3+r} d^{2\tilde{N}} C^{\star 2\tilde{N}} \zeta \left(4 - 4H\right)^{\tilde{N}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{2}{p}} \frac{1}{1 - 2^{4\tilde{N}\left(\frac{\theta+1}{p} - 1\right) + 1}}$$

$$C_3' = 3^{3\left(28\tilde{N} + q\tilde{N} + 1 + \frac{r}{6}\right)} 2^{1 + \frac{r}{2}} d^{3\tilde{N}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{3}{p}} C^{\star 3\tilde{N}} \zeta \left(2 - 2H\right)^{\tilde{N}} \frac{1}{1 - 2^{6\tilde{N}\left(\frac{\theta+1}{p} - 1\right) + 1}}$$

$$C_1'' = 3^{\left(4\tilde{N} + q\tilde{N} + 1\right)} 2^{3 + \frac{r}{2}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{1}{p}} C^{\star\tilde{N}} 3^{8\tilde{N}} \frac{1}{1 - 2^{2\tilde{N}n\left(\frac{\theta+1}{p} - 1\right) + 1}}$$

$$C_2'' = 3^{2\left(4\tilde{N} + q\tilde{N} + 1\right)} 2^{3+r} \tilde{N}^{\frac{r}{2}} C_\theta^{-\tilde{N}\frac{4}{p}} C^{\star 2\tilde{N}} \left(\frac{2d\zeta \left(4 - 4H\right)^{\frac{1}{2}}}{1 - 2^{-\frac{1}{2}(4H-1)}} + \frac{3^{24}}{2} \right)^{2\tilde{N}} \frac{1}{1 - 2^{4\tilde{N}n\left(\frac{\theta+1}{p} - 1\right) + 1}}$$

$$C_3'' = 3^{3\left(4\tilde{N} + q\tilde{N} + 1 + \frac{r}{6}\right)} 2^{3 + \frac{r}{2}} \tilde{N}^{\frac{r}{2}} C_\theta^{-\tilde{N}\frac{6}{p}} C^{\star 3\tilde{N}} \left(7d^{3/2} \frac{\zeta \left(2 - 2H\right)^{\frac{1}{2}}}{1 - 2^{-\frac{1}{2}(4H-1)k}} + \frac{3^{108}}{6} \right)^{2\tilde{N}} \times$$
$$\frac{1}{1 - 2^{4\tilde{N}n\left(\frac{\theta+1}{p} - 1\right) + 1}}.$$

Proof. For $j = 1, \ldots, \left\lfloor \frac{1}{H} \right\rfloor$, define

$$I_j \left(m, \lambda\right) := \mathrm{Cap}_{r,q} \left\{ w : \rho_j \left(\boldsymbol{w}^{(m+1)}, \boldsymbol{w}^{(m)} \right) > \lambda \right\}$$

Recall that

$$\rho_j\left(\boldsymbol{w}^{(m+1)}, \boldsymbol{w}^{(m)}\right) = \left(\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{\frac{p}{j}}\right)^{\frac{j}{p}}$$

Observe that for $j = 1, \ldots, \left\lfloor \frac{1}{H} \right\rfloor$,

$$\begin{aligned}
&\left\{w : \rho_j\left(\boldsymbol{w}^{(m+1)}, \boldsymbol{w}^{(m)}\right) > \lambda\right\} \\
&= \left\{w : \sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{\frac{p}{j}} > \lambda^{\frac{p}{j}} \right\} \\
&\subset \cup_{n=1}^{\infty} \left\{w : \sum_{k=1}^{2^n} \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{\frac{p}{j}} > \frac{\lambda^{\frac{p}{j}} 2^{-n\theta}}{\sum_{n=1}^{\infty} n^{\gamma} 2^{-n\theta}} \right\} \text{ for all } \theta > 0 \\
&\subset \cup_{n=1}^{\infty} \cup_{k=1}^{2^n} \left\{w : \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{\frac{p}{j}} > \frac{\lambda^{\frac{p}{j}} 2^{-n(\theta+1)}}{\sum_{n=1}^{\infty} n^{\gamma} 2^{-n\theta}} \right\} \text{ for all } \theta > 0 \qquad (2.2.1) \\
&= \cup_{n=1}^{\infty} \cup_{k=1}^{2^n} \left\{w : \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{2\tilde{N}} > \frac{\lambda^{2\tilde{N}} 2^{-2\tilde{N}n(\theta+1)\frac{j}{p}}}{\left(\sum_{n=1}^{\infty} n^{\gamma} 2^{-n\theta}\right)^{2\tilde{N}\frac{j}{p}}} \right\} \text{ for all } \theta > 0, \tilde{N} \in \mathbb{N}
\end{aligned}$$

Let $C_\theta = \left(\sum_{n=1}^{\infty} n^{\gamma} 2^{-n\theta}\right)^{-1}$. Then by the countable additivity of $\mathrm{Cap}_{r,q}$,

$$I_j(m; \lambda) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \mathrm{Cap}_{r,q} \left\{w : \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{2\tilde{N}} > C_\theta^{2\tilde{N}\frac{j}{p}} \lambda^{2\tilde{N}} 2^{-2\tilde{N}n(\theta+1)\frac{j}{p}} \right\}$$

for all $\theta > 0$, $q \geq 1$, $r \in \mathbb{N}$ and $\tilde{N} \geq r$.

Observe that the map $w \to \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{2\tilde{N}}$ is a polynomial in finitely many functionals of the form $w \to w_t - w_s$, and thus is continuous with respect to the uniform topology in B. This allows us to apply the Tchebychev's inequality to obtain

$$I_j(m;\lambda) \le \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} \frac{2^{2\tilde{N}n(\theta+1)\frac{j}{p}}}{C_\theta^{2\tilde{N}\frac{j}{p}}\lambda^{2\tilde{N}}} \left\| \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{2\tilde{N}} \right\|_{r,q}. \tag{2.2.2}$$

By Lemma 11 and Corollary 12, for all $q \ge 1$ and $r, \tilde{N} \in \mathbb{N}$,

$$\begin{aligned}
\left\| \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{2\tilde{N}} \right\|_{r,q} &\le 2^{1+\frac{r}{2}} \cdot 3^{\tilde{N}j(q+2)} \left(\tilde{N}j\right)^{\frac{r}{2}} \left\| \left| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right|^{2\tilde{N}} \right\|_{L^2(\mathbb{P})} \\
&= 2^{1+\frac{r}{2}} \cdot 3^{\tilde{N}j(q+2)} \left(\tilde{N}j\right)^{\frac{r}{2}} \left\| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right\|^{2\tilde{N}}_{L^{4\tilde{N}}(\mathbb{P})} \\
&\le 3^{j\left(4\tilde{N}+q\tilde{N}+1\right)} 2^{1+\frac{r}{2}} \left(\tilde{N}j\right)^{\frac{r}{2}} \left\| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right\|^{2\tilde{N}}_{L^2(\mathbb{P})}
\end{aligned}$$

Putting this back into (2.2.2) we have that

$$I_j(m;\lambda) \le C\left(q,r,\tilde{N},\theta,p\right)\lambda^{-2\tilde{N}} \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{j}{p}} \left\| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right\|^{2\tilde{N}}_{L^2(\mathbb{P})} \tag{2.2.3}$$

where

$$C\left(q,r,\tilde{N},\theta,p\right) = 3^{j\left(4\tilde{N}+q\tilde{N}+1\right)} 2^{1+\frac{r}{2}} \left(\tilde{N}j\right)^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{j}{p}}$$

Repeating the entire argument on $\rho_j\left(\boldsymbol{w}^{(m)}\right)$ instead of $\rho_j\left(\boldsymbol{w}^{(m+1)},\boldsymbol{w}^{(m)}\right)$ and on $w^{(m),j}_{t^n_{k-1},t^n_k}$ instead of $w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k}$ we also have

$$\mathrm{Cap}_{r,q}\left(\rho_j\left(\boldsymbol{w}^{(m)}\right) > \lambda\right) \le C\left(q,r,\tilde{N},\theta,p\right)\lambda^{-2\tilde{N}} \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{j}{p}} \left\| w^{(m),j}_{t^n_{k-1},t^n_k} \right\|^{2\tilde{N}}_{L^2(\mathbb{P})}.$$

We now substitute the estimates from Lemma 19 into (2.2.3) to produce our final estimates for $I_j(m;\lambda)$, $j = 1,\ldots,\left\lfloor\frac{1}{H}\right\rfloor$. The following inequalities shall hold for all

$\theta \in (0, Hp-1)$ and $\tilde{N} > \max\left\{\frac{r}{2}, 2^{-1}\left(H - \frac{\theta+1}{p}\right)^{-1}\right\}$, and the constant C may vary from line to line but will depend only on $q, r, \tilde{N}, \theta$ and p.

For $j = 1$, using the formula for the sum of geometric series we have,

$$\begin{aligned} I_1(m,\lambda) &\le c_1'.\lambda^{-2\tilde{N}} \sum_{n=m+1}^{\infty} \sum_{k=1}^{2^n} 2^{2\tilde{N}n\frac{(\theta+1)}{p}} 2^{2\tilde{N}(m(1-H)-n)} \\ &\le c_1'.\lambda^{-2\tilde{N}} 2^{2\tilde{N}m(1-H)} \sum_{n=m+1}^{\infty} 2^{n\left(2\tilde{N}\left[\frac{\theta+1}{p}-1\right]+1\right)} \\ &= c_1'.\lambda^{-2\tilde{N}} 2^{m\left[2\tilde{N}\left(\frac{\theta+1}{p}-H\right)+1\right]}. \end{aligned}$$

where

$$c_1' = 3^{8\tilde{N}+q\tilde{N}+1} 2^{1+\frac{r}{2}+2\tilde{N}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{1}{p}} C^{\star\tilde{N}}$$

and the sum in the second inequality converges because our choice of $\tilde{N}$ means that $2\tilde{N} > \left(H - \frac{\theta+1}{p}\right)^{-1} > \left(1 - \frac{\theta+1}{p}\right)^{-1}$.

For $j = 2$, similarly,

$$\begin{aligned} I_2(m,\lambda) \le c_2'.\lambda^{-2\tilde{N}} &\left(\sum_{n=1}^{m} \sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{2}{p}} 2^{-\tilde{N}(m(4H-1)+n)} \right. \\ &\left. + \sum_{n=m+1}^{\infty} \sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{2}{p}} 2^{2\tilde{N}(2m(1-H)-2n)} \right) \\ &\le \frac{4c_2'}{1 - 2^{4\tilde{N}\left(\frac{\theta+1}{p}-1\right)+1}}.\lambda^{-2\tilde{N}} \left(2^{-\tilde{N}m(4H-1)} + 2^{m\left[4\tilde{N}\left(\frac{\theta+1}{p}-H\right)+1\right]}\right). \end{aligned}$$

where

$$c_2' := 3^{2\left(16\tilde{N}+q\tilde{N}+1\right)} 2^{1+r} d^{2\tilde{N}} C^{\star 2\tilde{N}} \zeta(4-4H)^{\tilde{N}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{2}{p}}.$$

For $j = 3$,

$$
\begin{aligned}
I_3(m,\lambda) \leq c_3'.\lambda^{-2\tilde{N}} & \left(\sum_{n=1}^{m} \sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{3}{p}} 2^{-2\tilde{N}\left[\frac{1}{2}(4H-1)m+\frac{1}{2}(1+2H)n\right]} \right. \\
& \left. + \sum_{n=m+1}^{\infty} \sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{3}{p}} 2^{2\tilde{N}[3m(1-H)-3n]} \right) \\
& \leq \frac{4c_3'}{1-2^{6\tilde{N}\left(\frac{\theta+1}{p}-1\right)+1}}.\lambda^{-2\tilde{N}} \left(2^{m\left[6\tilde{N}\left(\frac{\theta+1}{p}-H\right)+1\right]} + 2^{-\tilde{N}m(4H-1)} \right)
\end{aligned}
$$

where

$$
c_3' = 3^{3\left(28\tilde{N}+q\tilde{N}+1+\frac{r}{6}\right)} 2^{1+\frac{r}{2}} d^{3\tilde{N}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{3}{p}} C^{\star 3\tilde{N}} \zeta(2-2H)^{\tilde{N}}.
$$

Similarly, we have the following estimate for $\mathrm{Cap}_{r,q}\left(\rho_j\left(\boldsymbol{w}^{(m)}\right) > \lambda\right)$,

$$
\begin{aligned}
\mathrm{Cap}_{r,q}\left(\rho_j\left(\boldsymbol{w}^{(m)}\right) > \lambda\right) \leq c_j''.\lambda^{-2\tilde{N}} & \left(\sum_{n=1}^{m} \sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{j}{p}} 2^{-2\tilde{N}jnH} \right. \\
& \left. \sum_{n=m+1}^{\infty} \sum_{k=1}^{2^n} 2^{2\tilde{N}n(\theta+1)\frac{j}{p}} 2^{2\tilde{N}(m(1-H)j-nj)} \right) \\
& \leq c_j''.\lambda^{-2\tilde{N}} \left(\sum_{n=1}^{\infty} 2^{n\left[2\tilde{N}j\left(\frac{\theta+1}{p}-H\right)+1\right]} + \frac{2^{m\left[2\tilde{N}j\left(\frac{\theta+1}{p}-H\right)+1\right]}}{1-2^{2\tilde{N}jn\left(\frac{\theta+1}{p}-1\right)+1}} \right) \\
& \leq \frac{2c_j''}{1-2^{2\tilde{N}jn\left(\frac{\theta+1}{p}-1\right)+1}}.\lambda^{-2\tilde{N}}
\end{aligned}
$$

where the series in the second inequality converges again due to our choice of θ and

$\tilde{N}$ and

$$
\begin{aligned}
c_1'' &= 3^{\left(4\tilde{N}+q\tilde{N}+1\right)} 2^{1+\frac{r}{2}} \tilde{N}^{\frac{r}{2}} C_\theta^{-2\tilde{N}\frac{1}{p}} C^{\star\tilde{N}} 3^{8\tilde{N}} \\
c_2'' &= 3^{2\left(4\tilde{N}+q\tilde{N}+1\right)} 2^{1+r} \tilde{N}^{\frac{r}{2}} C_\theta^{-\tilde{N}\frac{4}{p}} C^{\star 2\tilde{N}} \left(\frac{2d\zeta\left(4-4H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)}} + \frac{3^{24}}{2} \right)^{2\tilde{N}} \\
c_3'' &= 3^{3\left(4\tilde{N}+q\tilde{N}+1+\frac{r}{6}\right)} 2^{1+\frac{r}{2}} \tilde{N}^{\frac{r}{2}} C_\theta^{-\tilde{N}\frac{6}{p}} C^{\star 3\tilde{N}} \left(7d^{3/2} \frac{\zeta\left(2-2H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)k}} + \frac{3^{108}}{6} \right)^{2\tilde{N}} .
\end{aligned}
$$

□

We now prove Theorem 21.

Proof. Let $p \in (2,4)$, $\theta \in (0, Hp-1)$, $q \geq 1$, $r \in \mathbb{N}$ and $\tilde{N} > \max\left\{ \frac{r}{2}, 2^{-1}\left(H - \frac{\theta+1}{p}\right)^{-1} \right\}$ be an integer.

Let $0 < \varepsilon < \frac{1}{4\tilde{N}} \min\left\{ \left(2\tilde{N}\left(H-\frac{\theta+1}{p}\right)-1\right), \tilde{N}\left(4H-1\right) \right\}$.

Note first that by the triangle inequality in d_p,

$$
\begin{aligned}
&\left\{ w : \boldsymbol{w}^{(m)} \text{ is not a Cauchy sequence in } G\Omega_p \right\} \\
&\subset \left\{ w : \sum_{m=1}^{\infty} d_p\left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)}\right) = \infty \right\} \\
&\subset \cup_{i+jk \leq \lfloor \frac{1}{H} \rfloor} \left\{ w : \sum_{m=1}^{\infty} \rho_i\left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)}\right) \left(\rho_j\left(\boldsymbol{w}^{(m)}\right) + \rho_j\left(\boldsymbol{w}^{(m)}\right)\right)^k = \infty \right\} \\
&\subset \cup_{i+jk \leq \lfloor \frac{1}{H} \rfloor} \limsup_{m\to\infty} \left\{ w : \rho_i\left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)}\right) \left(\rho_j\left(\boldsymbol{w}^{(m)}\right) + \rho_j\left(\boldsymbol{w}^{(m)}\right)\right)^k > \frac{1}{2^{m\varepsilon}} \right\}
\end{aligned}
$$

Therefore, by the subadditivity of $\mathrm{Cap}_{r,q}$ and by the Borel Cantelli Lemma, it is sufficient to prove that for all $r \in \mathbb{N}$ and $q \geq 1$ and $i + jk \leq \left\lfloor \frac{1}{H} \right\rfloor$.

$$\sum_{m=1}^{\infty} \mathrm{Cap}_{r,q} \left\{ w : \rho_i \left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)} \right) \left(\rho_j \left(\boldsymbol{w}^{(m)} \right) + \rho_j \left(\boldsymbol{w}^{(m)} \right) \right)^k > \frac{1}{2^{m\varepsilon}} \right\} < \infty \qquad (2.2.4)$$

We first consider the case $k > 0$, where

$$\begin{aligned} &\left\{ w : \rho_i \left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)} \right) \left(\rho_i \left(\boldsymbol{w}^{(m)} \right) + \rho \left(\boldsymbol{w}^{(m)} \right) \right)^k > \frac{1}{2^{m\varepsilon}} \right\} \\ \subset &\left\{ w : \rho_i \left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)} \right) > \frac{1}{2^{2\varepsilon m}} \right\} \cup \left\{ w : \left(\rho_i \left(\boldsymbol{w}^{(m)} \right) + \rho \left(\boldsymbol{w}^{(m+1)} \right) \right)^k > 2^{m\varepsilon} \right\} \\ \subset &\left\{ w : \rho_i \left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)} \right) > \frac{1}{2^{2\varepsilon m}} \right\} \cup \left\{ w : \rho_i \left(\boldsymbol{w}^{(m)} \right) > 2^{m\frac{\varepsilon}{k} - 1} \right\} \\ &\cup \left\{ w : \rho \left(\boldsymbol{w}^{(m)} \right) > 2^{m\frac{\varepsilon}{k} - 1} \right\} \end{aligned}$$

Therefore it is sufficient to prove that for all $r \in \mathbb{N}$ and $q \geq 1$,

$$\sum_{m=1}^{\infty} \mathrm{Cap}_{r,q} \left\{ w : \rho_i \left(\boldsymbol{w}^{(m)}, \boldsymbol{w}^{(m+1)} \right) > \frac{1}{2^{2m\varepsilon}} \right\} < \infty \qquad (2.2.5)$$

and

$$\sum_{m=1}^{\infty} \mathrm{Cap}_{r,q} \left\{ w : \rho_i \left(\boldsymbol{w}^{(m)} \right) > 2^{m\frac{\varepsilon}{k} - 1} \right\} < \infty. \qquad (2.2.6)$$

By the first inequality in Lemma 22, there is a constant C which is independent of m, such that

$$\mathrm{Cap}_{r,q}\left\{w:\rho_i\left(\boldsymbol{w}^{(m+1)},\boldsymbol{w}^{(m)}\right)>\frac{1}{2^{2m\varepsilon}}\right\}\le C.2^{4\tilde{N}m\varepsilon}\left(2^{-m\left(2i\tilde{N}\left(H-\frac{\theta+1}{p}\right)-1\right)}+2^{-\tilde{N}m(4H-1)}\right)$$

But by our choice of ε, $\varepsilon<\frac{1}{4\tilde{N}}\left(2\tilde{N}\left(H-\frac{\theta+1}{p}\right)-1\right)$ and $\varepsilon<\frac{4H-1}{4}$, hence (2.2.5) follows.

To prove (2.2.6), note that from the second inequality in Lemma 22, there is a constant C which is independent of m such that

$$\mathrm{Cap}\left\{w:\rho_i\left(\boldsymbol{w}^{(m)}\right)>2^{m\frac{\varepsilon}{k}-1}\right\}\le C.2^{-2\tilde{N}m\frac{\varepsilon}{k}}.$$

From which (2.2.6) follows directly.

To prove the $k=0$ case in (2.2.4) means proving

$$\sum_{m=1}^{\infty}\mathrm{Cap}\left\{w:\rho_i\left(\boldsymbol{w}^{(m)},\boldsymbol{w}^{(m+1)}\right)>\frac{1}{2^{m\varepsilon}}\right\}<\infty.$$

which of course follows from (2.2.5) as

$$\left\{w:\rho_i\left(\boldsymbol{w}^{(m)},\boldsymbol{w}^{(m+1)}\right)>\frac{1}{2^{m\varepsilon}}\right\}\subset\left\{w:\rho_i\left(\boldsymbol{w}^{(m)},\boldsymbol{w}^{(m+1)}\right)>\frac{1}{2^{2m\varepsilon}}\right\}.$$

This completes the proof of Theorem 21. $\square$

Chapter 3

Signature through Green's theorem

As introduced in Chapter 1, the signature of a path is a formal series of its iterated integrals. In [Chen54], K.T. Chen observed that the map that sends a path to its signature forms a homomorphism from the concatenation algebra to the space of formal non-commutative power sereis and used it to study the cohomology of loop spaces. Recent interests in the study of signature has been sparked by its role in rough path theory. In particular, it was shown by Hambly and Lyons in [LH02] that for ODEs driven by paths with bounded total variations, the signature is a fundamental representation of the effect of the driving signal on the solution.

If we consider the signature as a map from the path space to the tensor algebra, then an interesting question is whether this map is injective. This was first considered by Chen himself [Chen58], who proved that irreducible, piecewise regular continuous paths have the same signature if and only if they are equal up to a translation and a reparametrisation.

His result was generalised with a new, quantitative approach by Hambly and Lyons in [LH02] who showed that two paths γ and $\tilde{\gamma}$ with finite total variations have the same signature if and only if γ can be expressed as the concatenation of $\tilde{\gamma}$ with a tree-like path σ. A continuous function $\sigma : [0,1] \to \mathbb{R}^d$ is tree-like if there exists a continuous function $h : [0,1] \to [0,\infty)$ such that $h(0) = h(1) = 0$ and

$$|\sigma_t - \sigma_s| \leq h(t) + h(s) - 2 \inf_{s \leq u \leq t} h(u).$$

In this chapter we establish uniqueness amongst paths γ satisfying the following conditions:

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$.

1. γ can be reparametrised as a continuous curve $\tilde{\gamma} : [0,1] \to \overline{\mathbb{D}}$, such that $\tilde{\gamma}(0) = -1$, $\tilde{\gamma}(1) = 1$ and $\tilde{\gamma}(0,1) \subset \mathbb{D}$.

2. γ has finite p-variation for some $1 \leq p < 2$.

3. γ is a simple curve.

Let $\mathcal{C}_2(-1,1,\mathbb{D})$ denote the set of paths γ satisfying these three conditions. Let $S(\gamma)$ denote the signature (see Definition 26) of a path γ.

Theorem 23. *Let γ, $\gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$. Then $S(\gamma)_{0,1} = S(\gamma')_{0,1}$ if and only if γ and γ' are equal up to a reparametrisation.*

An interesting, but difficult extension is to prove that if the signatures of two curves with finite $p > 1$ -variations are equal, then the paths are equal up to the tree-like path equivalence. The restriction $1 \leq p < 2$ gives us the existence of signature for free, thanks to Young's integration theory.

3.0.1 Young's integrals, signature and approximation theorems

While the Lebesgue-Stieltjes integration theory is sufficient to define integrals against functions with finite 1-variations, we need Young's integration theory to define integrals against functions with finite p variations, where $1 \leq p < 2$. A proof of the following criterion for the existence of Young's integral can be found in [LCL04]:

Theorem 24. *(L. Young[Yng36])Let $T > 0$, $t \in [0,T]$ and $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$, $T > 0$. Then for $\gamma_{\cdot} \in \mathcal{V}^p([0,t], \mathbb{R})$ and $\gamma'_{\cdot} \in \mathcal{V}^q([0,t], \mathbb{R})$, the following limit exists:*

$$\int_0^t \gamma \mathrm{d}\gamma' := \lim_{\|\mathcal{P}\| \to 0} \sum \gamma_{t_i} \left(\gamma'_{t_{i+1}} - \gamma'_{t_i} \right) \tag{3.0.1}$$

where the sum is over partition points $0 = t_0 < t_1 < .. < t_n = t$ in $\mathcal{P}$, and $\|\mathcal{P}\| := \sup_i |t_{i+1} - t_i|$.

Furthermore, as a function of t, $t \to \int_0^t \gamma \mathrm{d}\gamma' \in \mathcal{V}^q([0,T], \mathbb{R})$, and there is a constant $C_{p,q} > 0$ depending only on p and q such that

$$\left\| \int_0^{\cdot} (\gamma_s - \gamma_0) \, \mathrm{d}\gamma'_s \right\|_{\mathcal{V}^q([0,T],\mathbb{R})} \leq C_{p,q} \|\gamma\|_{\mathcal{V}^p([0,T],\mathbb{R})} \|\gamma'\|_{\mathcal{V}^q([0,T],\mathbb{R})} . \tag{3.0.2}$$

The limit in (3.0.1) is called the *Young's integral of γ with respect to γ'*. In the case when γ' has finite 1-variation, the integral coincides with the Lebesgue-Stieltjes integral.

For convenience, we make explicit the following immediate corollary of (3.0.2) :

Corollary 25. *Let $1 \leq p < 2$. Let $\sigma^{(m)}, \gamma^{(m)} : [0,T] \to \mathbb{R}$ be two convergent sequences,* *with limits σ and γ such that $\sigma_0 = \sigma_0^{(m)}$, $\sigma_1 = \sigma_1^{(m)}$, $\gamma_0 = \gamma_0^{(m)}$ and $\gamma_1 = \gamma_1^{(m)}$ for all $m \in \mathbb{N}$, in $\mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$ and $\mathcal{V}^q([0,T], \mathbb{R})$ respectively. Then*

$$\lim_{m\to\infty} \left\| \int_0^{\cdot} \gamma_s^{(m)} \mathrm{d}\sigma_s^{(m)} - \int_0^{\cdot} \gamma_s \mathrm{d}\sigma_s \right\|_{\mathcal{V}^q([0,T],\mathbb{R})} = 0.$$

In particular,

$$\lim_{m\to\infty} \int_0^1 \sigma_s^{(m)} \mathrm{d}\gamma_s^{(m)} = \int_0^1 \sigma_s \mathrm{d}\gamma_s.$$

Theorem 24 allows us to define the signature of $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$ for $1 \le p < 2$. We will use $\triangle_n(s,t)$ to denote the set $\{(t_1, .., t_n) : s < t_1 < ... < t_n < t\}$ and $\triangle_{0,T}$ to denote the set $\{(s,t) : 0 \le s \le t \le T\}$.

Definition 26. Let $1 \le p < 2$ and $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$. The *lift* of γ is a function $S(\cdot) : \triangle_{0,T} \to T\left(\mathbb{R}^d\right)$, defined by

$$S(\gamma)_{s,t} := 1 + \sum_{n=1}^{\infty} \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1} \otimes ... \otimes \mathrm{d}\gamma_{t_n} \tag{3.0.3}$$

where the sum $+$ is the direct sum operation in $T\left(\mathbb{R}^d\right)$.

The signature of γ on $S(\gamma)_{0,T}$ is defined as $S(\gamma)_{0,T}$.

Remark 27. $S(\gamma)_{\cdot,\cdot}$ is well-defined for $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$, with $p < 2$ because, for instance, $\int \mathrm{d}\gamma \otimes \mathrm{d}\gamma$ exists because $\frac{1}{p} + \frac{1}{p} > 1$. Moreover, the resulting integral remains in $\mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$, so the resulting integral can be integrated again with respect to γ.

Remark 28. Let $\mathbf{e}_1, .., \mathbf{e}_d$ be the standard basis in $\mathbb{R}^d$. If $\gamma_\cdot = \sum_{i=1}^d \gamma_\cdot^{(i)} \mathbf{e}_i$, then we have, using the notation from Chapter 1,

$$e_{i_1}^* \otimes \ldots \otimes e_{i_n}^* \left(S(\gamma)_{s,t} \right) = \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1}^{(i_1)} ... \mathrm{d}\gamma_{t_n}^{(i_n)}.$$

The term $\int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1}^{(i_1)} ... \mathrm{d}\gamma_{t_n}^{(i_n)}$ will be denoted by $\gamma^{i_1 i_2 .. i_n}$ if no confusion is possible.

Definition 29. Let $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P}\right)$ be a filtered probability space. Let γ be a stochastic process taking value in $\mathbb{R}^d$ such that, with probability one, its sample paths have finite p-variations, for some $1 \leq p < 2$. Then the expected signature of $\mathbb{P}$ is the expected value of the signature of γ.

We shall use the following properties of signature, whose proofs can be found in [LCL04].

1. (Invariance under reparametrisation)For any $t \in [0, \infty)$, $S(\gamma)_{0,t}$ is invariant under any reparametrisation of γ on $[0, t]$.

2. (Inverse) $S(\gamma)_{0,T} \otimes S(\overleftarrow{\gamma})_{0,T} = \mathbf{1}$, where $\overleftarrow{\gamma}(t) := \gamma(T - t)$ is the reversal of γ and $\mathbf{1}$ is the identity element in $T\left(\mathbb{R}^d\right)$.

3. (Chen's Identity)$S(\gamma)_{s,u} \otimes S(\gamma)_{u,t} = S(\gamma)_{0,t}$ for any $0 \leq s < u < t \leq T$

4.(Shuffle product formula) If we define a *shuffle* of $1, ..., r$ and $r+1, ..., r+s$ to be a permutation of $\{1, 2, ..., r+s\}$ such that $\sigma(1) < \sigma(2) < .. < \sigma(r)$ and $\sigma(r+1) < ... < \sigma(r+s)$.

Proposition 30. *([LCL04],Theorem 2.15)Let* $1 \leq p < 2$ *and* $\gamma \in \mathcal{V}^p\left([0, T], \mathbb{R}^d\right)$, *then using the notation of Remark 28,*

$$\gamma^{k_1 \dots k_r} \cdot \gamma^{k_{r+1} \dots k_{r+s}} = \sum_{shuffles\ \sigma} \gamma^{k_{\sigma^{-1}(1)} \dots k_{\sigma^{-1}(r+s)}}$$

where $\cdot$ is the multiplication operation in $\mathbb{R}$.

In another words,

$$\begin{aligned} & e^*_{k_1} \otimes \dots \otimes e^*_{k_r} \left(S\left(\gamma\right)\right) e^*_{k_{r+1}} \otimes \dots \otimes e^*_{k_{r+s}} \left(S\left(\gamma\right)\right) \\ = & \sum_{shuffles\ \sigma} e^*_{k_{\sigma^{-1}(1)}} \otimes \dots \otimes e^*_{k_{\sigma^{-1}(r+s)}} \left(S\left(\gamma\right)\right). \end{aligned}$$

The sum $\sum_{shuffles\ \sigma} \gamma^{k_{\sigma^{-1}(1)} \dots k_{\sigma^{-1}(r+s)}}$ is denoted by $\gamma^{k_1 \dots k_r \sqcup k_{r+1} \dots k_{r+s}}$. The sum

$$\sum_{shuffles\ \sigma} e^*_{k_{\sigma^{-1}(1)}} \otimes \dots \otimes e^*_{k_{\sigma^{-1}(r+s)}}$$

is denoted by $e^*_{k_1,\dots,k_r} \sqcup e^*_{k_{r+1},\dots,k_{r+s}}$.

Recall that for a continuous function γ and a partition $\mathcal{P} := t_0 = 0 < t_1 < .. < t_n = T$, the piecewise linear interpolation of γ with respect to $\mathcal{P}$ is defined as the following function on $[0, T]$:

$$\gamma_t^{\mathcal{P}} := \gamma_{t_i} + \left(\frac{\gamma_{t_{i+1}} - \gamma_{t_i}}{t_{i+1} - t_i}\right)(t - t_i) \text{ for } t \in [t_i, t_{i+1}]$$

Then the following approximation theorem holds:

Lemma 31. *(Lemma 1.12 and Proposition 1.14, [LCL04])Let p and q be such that*

$1 \leq p < q$. *Let* $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$. *Then for all finite partitions* $\mathcal{P}$,

$$\left\|\gamma^{\mathcal{P}}\right\|_{\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)} \leq \|\gamma\|_{\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)}$$

Furthermore for all $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for all partitions* $\mathcal{P}$ *of* $[0,T]$ *satisfying* $\|\mathcal{P}\| < \delta$ *we have*

$$\begin{aligned} \left\|\gamma - \gamma^{\mathcal{P}}\right\|_{\mathcal{V}^q\left([0,T],\mathbb{R}^d\right)} &< \varepsilon, \text{ and} \\ \sup_{t\in[0,T]} \left\|\gamma_t - \gamma_t^{\mathcal{P}}\right\| &< \varepsilon. \end{aligned}$$

We shall need to use Green's theorem to evaluate the signature. In order to ensure that the domain we are integrating over is a Jordan domain, we will approximate our curves by simple curves. This is made possible by the following lemma:

Lemma 32. *Let* $\gamma : [0,1] \to \mathbb{C}$ *be a continuous simple curve. Then for all* $\varepsilon > 0$, *there exists a partition* $\mathcal{P}$ *of* $[0,1]$ *such that* $\|\mathcal{P}\| < \varepsilon$ *and* $\gamma^{\mathcal{P}}$ *is simple.*

Proof. See[Wer12], Lemma 4.3. ☐

The following corollary follows immediately.

Corollary 33. *Let* $\gamma : [0,1] \to \overline{\mathbb{D}}$ *be a continuous simple curve such that* $\gamma_0 = -1$, $\gamma_1 = 1$ *and* $\gamma_t \subset \mathbb{D}$ *for all* $t \in (0,1)$. *Then for all* $\varepsilon > 0$, *there exists a partition* $\mathcal{P}$ *of* $[0,1]$ *such that* $\|\mathcal{P}\| < \varepsilon$ *and* $\gamma^{\mathcal{P}}$ *is simple. Furthermore,* $\gamma_t^{\mathcal{P}} \subset \mathbb{D}$ *for* $t \in (0,1)$ *and* $\gamma_0^{\mathcal{P}} = -1$, $\gamma_1^{\mathcal{P}} = 1$.

Proof. The only new assertion here is that $\gamma^{\mathcal{P}}(0,1) \subset \mathbb{D}$. Since $\gamma_{t_i}, \gamma_{t_{i+1}} \subset \mathbb{D}$ for $i \neq 0$

or $n-1$, thus by the convexity of $\mathbb{D}$, the line segment between γ_{t_i} and $\gamma_{t_{i+1}}$ lies in $\mathbb{D}$. Note also that the line segment strictly between γ_0 and γ_1 and the segment between γ_{n-1} and γ_n also lies in $\mathbb{D}$ by convexity. □

The following lemma is extremely useful in proving the properties of Young's integral.

Lemma 34. *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuous curve with finite p-variation, where $p < 2$. Let $\mathcal{P}_m$ be a sequence of partitions such that $\mathcal{P}_m$ contains both 0 and 1 for all m and $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$. For any $I \in \{1, \ldots, d\}^n$,*

$$e_I^* \left[S(\gamma)_{0,1} \right] = \lim_{m \to \infty} e_I^* \left[S\left(\gamma_s^{\mathcal{P}_m}\right)_{0,1} \right]. \tag{3.0.4}$$

where e_I is defined in Chapter 1 section 4.

Proof. See Corollary 2.11 in [LCL04]. □

3.1 Uniqueness of signature

The term $e_1^{*\otimes n+1} \otimes e_2^{*\otimes k+1} \left(S(\eta)_{0,1} \right)$ is an important term in the signature of η for our purpose and will be given the special notation $\eta_{n,k}$. These particular iterated integrals can be reduced to just one line integral, which can be evaluated using Green's theorem. This idea is the content of the following lemma.

Lemma 35. *Let $\eta : [0,1] \to \mathbb{R}^2$ be a positively oriented, continuous, simple closed* *curve with bounded total variation and D be its interior, then*

$$e_1^{*\otimes(n+1)} \otimes e_2^{*\otimes(k+1)} \left(S(\eta)_{0,1}\right) \tag{3.1.1}$$

$$= \frac{1}{n!} \frac{(-1)^k}{k!} \int_D \left(x - \gamma_0^{(1)}\right)^n \left(\gamma_0^{(2)} - y\right)^k \mathrm{d}x\mathrm{d}y \tag{3.1.2}$$

Let $\eta_{n,k}$ denote $e_1^{*\otimes(n+1)} \otimes e_2^{*\otimes(k+1)} \left(S(\eta)_{0,1}\right)$.

In general if $(k_1, l_1), ..., (k_n, l_n) \in \mathbb{N}^2$, we have

$$\eta_{k_1,l_1} ... \eta_{k_n,l_n} \tag{3.1.3}$$

$$= C_{\mathbf{k},\mathbf{l}} \int_{\mathbb{R}^{2n}} \left[\Pi_{j=1}^n \left(x_j - \eta_0^{(1)}\right)^{k_j} \left(\eta_0^{(2)} - y_j\right)^{l_j}\right] 1_{D^n} \mathrm{d}x_1 \mathrm{d}y_1 ... \mathrm{d}x_n \mathrm{d}y_n \tag{3.1.4}$$

where $C_{\mathbf{k},\mathbf{l}} = \Pi_{j=1}^n \left[\frac{1}{k_j! l_j!}\right]$ and 1_{D^n} denotes the indicator function of the set

$$D^n := \cap_{j=1}^n \left\{(x_1, y_1, .., x_n, y_n) : (x_j, y_j) \in D\right\}.$$

Proof. Let $\eta^{(1)}$and $\eta^{(2)}$ be the first and second coordinate components of η respectively.

Recall that

$$\eta_{n,k} = \int_0^1 \mathrm{d}\eta_{s_1}^{(1)} \ldots \mathrm{d}\eta_{s_{n+1}}^{(1)} \mathrm{d}\eta_{s_{n+2}}^{(2)} \ldots \mathrm{d}\eta_{s_{n+k+2}}^{(2)}.$$

The key idea here is to integrate with respect to $\eta^{(1)}$s first and then integrate the

$\eta^{(2)}$s.

$$\begin{aligned}
\eta_{n,k} &= \int ... \int_{0<t_1<..<t_{n+1}<s_1<...<s_{k+1}<1} \mathrm{d}\eta^{(1)}_{t_1}...\mathrm{d}\eta^{(1)}_{t_{n+1}}\mathrm{d}\eta^{(2)}_{s_1}...\mathrm{d}\eta^{(2)}_{s_{k+1}} \\
&= \int_{0<s_1<...<s_{k+1}<1} \frac{1}{(n+1)!}\left(\eta^{(1)}_{s_1}-\eta^{(1)}_0\right)^{n+1}\mathrm{d}\eta^{(2)}_{s_1}...\mathrm{d}\eta^{(2)}_{s_{k+1}} \\
&= \int_0^1\int_{s_1}^1 ... \int_{s_{k-1}}^1\int_{s_k}^1 \frac{1}{(n+1)!}\left(\eta^{(1)}_{s_1}-\eta^{(1)}_0\right)^{n+1}\mathrm{d}\eta^{(2)}_{s_{k+1}}...\mathrm{d}\eta^{(2)}_{s_1} \text{ by Fubini's theorem} \\
&= \frac{1}{(n+1)!}\frac{1}{k!}\int_0^1\left(\eta^{(1)}_{s_1}-\eta^{(1)}_0\right)^{n+1}\left(\eta^{(2)}_1-\eta^{(2)}_{s_1}\right)^k\mathrm{d}\eta^{(2)}_{s_1} \\
&= \frac{1}{n!}\frac{1}{k!}\int_D\left(x-\eta^{(1)}_0\right)^n\left(\eta^{(2)}_1-y\right)^k\mathrm{d}x\mathrm{d}y \text{ by Green's Theorem}
\end{aligned}$$

where D is the interior of η.

Let $(k_1,l_1),...,(k_n,l_n)\in\mathbb{N}^2$, then by Fubini's theorem,

$$\begin{aligned}
&\eta_{k_1,l_1}...\eta_{k_n,l_n} \\
=\ & \frac{1}{k_1!}\frac{1}{l_1!}\int_{\mathbb{R}^2}\left(x_1-\eta^{(1)}_0\right)^{k_1}\left(\eta^{(2)}_1-y_1\right)^{l_1}1_D(x_1,y_1)\,\mathrm{d}x_1\mathrm{d}y_1\times... \\
& ...\times\frac{1}{k_n!}\frac{1}{l_n!}\int_{\mathbb{R}^2}\left(x_n-\eta^{(1)}_0\right)^{k_n}\left(\eta^{(2)}_1-y_n\right)^{l_n}1_D(x_n,y_n)\,\mathrm{d}x_n\mathrm{d}y_n \\
=\ & \Pi^n_{j=1}\frac{1}{k_j!}\frac{1}{l_j!}\int_{\mathbb{R}^{2n}}\left[\Pi^n_{j=1}\left(x_j-\eta^{(1)}_0\right)^{k_j}\left(\eta^{(2)}_1-y_j\right)^{l_j}\right] \\
& \cdot 1_{D^n}(x_1,y_1..,x_n,y_n)\,\mathrm{d}x_1..\mathrm{d}y_n
\end{aligned}$$

where $1_{D^n}(x_1,y_1,...,x_n,y_n)$ denotes the indicator function of the set

$$D^n := \cap^n_{j=1}\{(x_1,y_1,..,x_n,y_n) : (x_j,y_j)\in D\}.$$

□

Remark 36. B. Werness [Wer12] is the first to realise that the Green's theorem can be used to compute some terms in the signature of a curve. He used it to prove the $n = 2, k = 1$ case of Lemma 35 in order to compute the first three terms of the expected signature of SLE curve.

Note that Lemma 35 cannot be applied directly to elements of $\mathcal{C}_2(-1, 1, \mathbb{D})$ as they are not closed curves and are "too rough". For the first problem, we need to "close off" the curves in $\mathcal{C}_2(-1, 1, \mathbb{D})$ by concatenating it with the upper semi-circular boundary of $\mathbb{D}$ from 1 to -1. Let us describe this more precisely.

Recall that $\mathcal{C}_2(-1, 1, \mathbb{D})$ is defined just before the statement of Theorem 23.

Let ϕ denote the anti-clockwise semi-circular boundary of $\mathbb{D}$, or more precisely, $\phi(t) := (\cos t, \sin t), 0 \le t \le \pi$.

Let $p \ge 1$. For elements γ and $\tilde{\gamma}$ in $\mathcal{V}^p\left([0, T_2], \mathbb{R}^d\right)$ and $\mathcal{V}^p\left([0, T_1], \mathbb{R}^d\right)$, define a concatenation product $\star$:$\mathcal{V}^p\left([0, T_2], \mathbb{R}^d\right) \times \mathcal{V}^p\left([0, T_1], \mathbb{R}^d\right) \to \mathcal{V}^p\left([0, T_1 + T_2], \mathbb{R}^d\right)$ by

$$\begin{aligned}
\gamma \star \tilde{\gamma}(u) &:= \gamma(u), \quad u \in [0, T_1], \\
\gamma \star \tilde{\gamma}(u) &:= \tilde{\gamma}(u - T_1) + \gamma(T_1) - \tilde{\gamma}(0), \quad u \in [T_1, T_1 + T_2]
\end{aligned}$$

Then for $\gamma \in \mathcal{C}_2(-1, 1, \mathbb{D})$, $\eta = \gamma \star \phi$ is a simple closed curve.

As η does not in general has bounded total variation, we will prove a version of Lemma 35 that works for η.

Lemma 37. *Let $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$. If $\eta = \gamma \star \phi$ and D is the interior of η, then*

$$e_1^{*\otimes n+1} e_2^{*\otimes k+1}\left(S(\eta)_{0,1}\right) \tag{3.1.5}$$

$$= \frac{1}{n!}\frac{(-1)^k}{k!}\int_D (x+1)^n y^k \mathrm{d}x\mathrm{d}y. \tag{3.1.6}$$

Let $\eta_{n,k}$ denote $e_1^{*\otimes n+1} e_2^{*\otimes k+1}\left(S(\eta)_{0,1}\right)$.

If $(k_1,l_1),...,(k_n,l_n) \in \mathbb{N}^2$, we have

$$\eta_{k_1,l_1}...\eta_{k_n,l_n} \tag{3.1.7}$$

$$= C_{\mathbf{k},\mathbf{l}}\int_{\mathbb{R}^{2n}}\left[\Pi_{j=1}^n (x_j+1)^{k_j} y_j^{l_j}\right] 1_{D^n}\mathrm{d}x_1\mathrm{d}y_1...\mathrm{d}x_n\mathrm{d}y_n \tag{3.1.8}$$

where $C_{\mathbf{k},\mathbf{l}} = \Pi_{j=1}^n\left[\frac{(-1)^{l_j}}{k_j!l_j!}\right]$.

Proof. Let $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$. By Corollary 33, there exists a sequence of partitions $(\mathcal{P}_m)_{m=1}^\infty$ such that $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$, $\gamma_0^{\mathcal{P}_m} = -1$, $\gamma_1^{\mathcal{P}_m} = 1$ and $\gamma^{\mathcal{P}_m} \star \phi$ is a simple closed curve. Let D_m be the interior of $\gamma^{\mathcal{P}_m} \star \phi$, which we shall denote as $\eta^{(m)}$.

Since $\gamma^{\mathcal{P}_m} \to \gamma$ in $\|\cdot\|_\infty$, we have for each $(x,y) \in \mathbb{R}^2 \backslash \eta \cup_{m=1}^\infty \eta^{(m)}$, $1_{D_m}(x,y) \to 1_D(x,y)$, where D is the interior of η. As γ has finite p-variation, where $p < 2$, γ can be reparametrised to be a $\frac{1}{p}$-Hölder continuous path ([LCL04], Section 1.2.2) and hence $\gamma[0,1]$ has Hausdorff dimension strictly less than 2. Therefore, the set $\eta \cup_{m=1}^\infty \eta^{(m)}$ has two dimensional Lebesgue measure is zero. Thus $1_{D_m} \to 1_D$ almost everywhere on $\mathbb{R}^2$ in Lebesgue measure.

By the bounded convergence theorem,

$$\begin{aligned} & \lim_{m\to\infty}\int_{\mathbb{R}^2}(x+1)^{n-1}(-y)^{k-1}1_{D_m}\mathrm{d}x\mathrm{d}y \\ = & \int_{\mathbb{R}^2}(x+1)^{n-1}(-y)^{k-1}1_{D}\mathrm{d}x\mathrm{d}y \end{aligned}$$

By Lemma 34,

$$\lim_{m\to\infty}\gamma_{n,k}^{\mathcal{P}_m}=\gamma_{n,k}.$$

(3.1.5) then follows from Lemma 35.

(3.1.7) follows from (3.1.5) directly. □

Before proving our main result, we need just one more technical lemma.

Lemma 38. *Let $\gamma,\gamma'\in\mathcal{C}_2(-1,1,\mathbb{D})$. If $\gamma[0,1]=\gamma'[0,1]$, then there exists a continuous strictly increasing function $r(t)$ such that*

$$\gamma_{r(t)}=\gamma'_t$$

for all $t\in[0,1]$.

Proof. Let γ^{-1} denote the inverse of the function $t\to\gamma_t$, which exists as γ is a simple curve.

Define a function $r:[0,1]\to[0,1]$ by $r(t)=\gamma^{-1}\circ\gamma'(t)$.

As both γ and γ' are injective continuous functions and $\gamma[0,1]=\gamma'[0,1]$, thus r is a bijective continuous function from $[0,1]$ to $[0,1]$. Hence it is monotone.

But $\gamma_0 = \gamma'_0 = -1$, $\gamma_1 = \gamma'_1 = 1$, so $r(0) = 0$ and $r(1) = 1$. Hence r is an increasing function and the result follows. $\square$

We now prove the main result of this chapter.

Proof. (of Theorem 23)The only if direction follows from the invariance of signature under reparametrisation.

Let $\gamma, \gamma' \in \mathcal{C}_2(-1, 1, \mathbb{D})$ be such that $S(\gamma) = S(\gamma')$.

Let $\eta := \gamma \star \phi$ and $\eta' := \gamma' \star \phi$. By Chen's identity, $S(\gamma) = S(\gamma')$ implies

$$S(\eta)_{0,1} = S(\eta')_{0,1} .$$

Let D and D' be the interior of η and η' respectively.

Since $\gamma, \gamma' \in \mathcal{C}_2(-1, 1, \mathbb{D})$, we have by Lemma 37 that for $\phi = \eta, \eta'$, $A = D, D'$,

$$e_1^{*\otimes n} \otimes e_2^{*\otimes k}(S(\phi)) = \frac{1}{(n-1)!} \frac{(-1)^{k-1}}{(k-1)!} \int_A (x+1)^{n-1} y^{k-1} \mathrm{d}x\mathrm{d}y \tag{3.1.9}$$

Then $S(\eta)_{0,1} = S(\eta')_{0,1}$ implies that

$$\int_{\mathbb{R}^2} (x+1)^{n-1} y^{k-1} 1_D(x, y) \, \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} (x+1)^{n-1} y^{k-1} 1_{D'}(x, y) \, \mathrm{d}x\mathrm{d}y$$

for all n and k.

Thus

$$e^{i\lambda_1}\int_{\mathbb{R}^2} e^{i\lambda_1 x+i\lambda_2 y}1_D(x,y)\,\mathrm{d}x\mathrm{d}y = e^{i\lambda_1}\int_{\mathbb{R}^2} e^{i\lambda_1 x+i\lambda_2 y}1_{D'}(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all $(\lambda_1,\lambda_2)\in\mathbb{R}^2$.

By the fact that Fourier transform is injective on $L^1(\mathbb{R}^2)$,

$$1_D(x,y) = 1_{D'}(x,y) \tag{3.1.10}$$

for almost all $(x,y)\in\mathbb{R}^2$.

Therefore, both $D\backslash\overline{D'}$ and $D'\backslash\overline{D}$ are null sets with respect to the Lebesgue measure.

As both $D\backslash\overline{D'}$ and $D'\backslash\overline{D}$ are open, so they must both be empty. This means $D\subset\overline{D'}$ and $D'\subset\overline{D}$. Thus $\overline{D}=\overline{D'}$.

Note that as γ,γ' are simple curves and $\gamma,\gamma'\subset\mathbb{D}$ except at the endpoints, the domains D and D' are Jordan domains. Using the Jordan curve theorem, we can prove that $\overline{\mathbb{R}^2\backslash\overline{D}}=\mathbb{R}^2\backslash D$ for any Jordan domain D.

As $\overline{D}=\overline{D'}$, we have $\mathbb{R}^2\backslash D=\mathbb{R}^2\backslash D'$, implying that $\gamma[0,1]=\gamma'[0,1]$. The result follows from Lemma 38. $\square$

3.2 Signature and Winding Number

In fact, the Green's theorem calculation from the last section can be generalised to give a relationship between a path's signature and its winding number. It was first pointed out by P. Yam[Yam08] that as any tree-like deformation of a path would not change the path's homotopy class, it should be possible to extract topological information about a path from its signature. Indeed, such a relationship was apparent as far back as 1936, in a paper by Rado[Rad36], who observed that the Levy area of a smooth path is equal to integral of its winding number around (x, y), considered as a function of (x, y). In [Yam08], Yam expressed the winding number of a path in terms of its signature. He starts with the formula

$$\text{Winding number around } z = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} \mathrm{d}w.$$

and smoothened the kernel $w \to \frac{1}{w-z}$ around the singularity at $w = z$. He then expanded $\frac{1}{w-z}$ into a power series of w and used the fact that the line integrals along γ of polynomials in w can be expressed in terms of the signature of γ.

Here we prove that infinitely many terms in a path's signature can be expressed in terms of the moments of its winding number. The proof of the following result, and indeed all results in this section, will appear in the subsection 2.2.2 and 2.2.3.

Theorem 39. *Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuous closed curve with bounded total variation. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of $\mathbb{R}^2$.*

Then

$$\begin{aligned} & \mathbf{e}_1^{*\otimes n+1} \otimes \mathbf{e}_2^{*\otimes k+1} (\log S(\gamma)) \\ = \; & (-1)^k \int_{\mathbb{R}^2} \frac{x^n y^k}{n!k!} \eta(\gamma - \gamma_0, (x,y)) \, \mathrm{d}x \mathrm{d}y. \end{aligned} \tag{3.2.1}$$

where $\eta(\gamma - \gamma_0, (x,y))$ is the winding number of the curve $\gamma - \gamma_0$ around the points (x,y).

As the winding number of a path does not contain information about the order at which it passes through points, whereas signature does, we cannot expect that the signature of a path can be expressed in terms of just winding numbers. In particular, let a and b be two closed curves in $\mathbb{R}^2$, both starting at 0 and let $\star$ denote the concatenation operation between two paths. Then $a \star b$ and $b \star a$ have the same winding number around any point, but in general do not have the same signature. Nevertheless, it is natural to ask how many terms in the signature series of a path can be represented in terms of its winding numbers. The answer is that the first four terms of a closed curve's signature can be expressed in terms of its winding number.

Corollary 40. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve with bounded total variation. The first four terms of $\log(S(\gamma))$ can be expressed in terms of the function $(x,y) \to \eta(\gamma - \gamma_0, (x,y))$ alone.*

In the final section we will prove that the number "four" is sharp. In other words, there are two paths γ, γ' which has the same winding number around every point, but $\pi^{(5)}(\log(S(\gamma)))$ and $\pi^{(5)}(\log(S(\gamma)))$ differs. Let us discuss the reason for that.

Consider the following Hall basis (see, for example, p90 of [Reu93]) for the free Lie algebra generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ up to degree five:

$$[[\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]], \mathbf{e_1}], [[[\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]], \mathbf{e}_1], \mathbf{e}_1], [[\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]], [\mathbf{e}_1, \mathbf{e}_2]], [\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]],$$

$$\mathbf{e}_1, [\mathbf{e}_1, [\mathbf{e}_1, [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2]]], [\mathbf{e}_1, [[[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2]], [[\mathbf{e}_1, \mathbf{e}_2], [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2]], [\mathbf{e}_1, \mathbf{e}_2],$$

$$[[[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2], [[[[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2], [\mathbf{e}_1, [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2]], [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2.$$

Note that for each element h in the Hall basis of degree four or less, there is exactly one pair (n_h, k_h) such that $n_h + k_h \leq 4$ such that

$$e_1^{*\otimes n_h} \otimes e_2^{*\otimes k_h}(h) = 1$$

and for all others $n' \neq n_h$ or $k' \neq k_h$ we have

$$e_1^{*\otimes n'} \otimes e_2^{*\otimes k'}(h) = 0.$$

This changes in the fifth degree, because, for example,

$$e_1^{*n} \otimes e_2^{*k}([[\mathbf{e_1}, \mathbf{e_2}], [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]]]) = 0$$

for all n, k. As the functional $e_1^{*n} \otimes e_2^{*k}$ picks out the elements in the log signature which are the moments of the winding number, this means that we are unable to express $[[\mathbf{e_1}, \mathbf{e_2}], [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]]]^* \left(\log S(\gamma)_{0,1}\right)$ in terms of just the winding number of

γ. This corresponds to the impossibility of expressing the iterated integral

$$\int_{0<s<t<1} \left[\left[\gamma_s, \mathrm{d}\gamma_s\right], \left[\gamma_t, \left[\gamma_t, \mathrm{d}\gamma_t\right]\right]\right]$$

in terms of a single line integral.

A key step in the proof of Theorem 39 is the following lemma, which is a direct consequence of Green's theorem:

Proposition 41. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a closed curve with bounded total variation. Let $\eta\left(\gamma, (x,y)\right)$ denote the winding number of γ around $(x,y) \in \mathbb{R}^2$.*

Then

$$e_1^{*\otimes(n+1)} \otimes e_2^{*\otimes(k+1)} \left(S\left(\gamma\right)_{0,1}\right) = \frac{\left(-1\right)^k}{n!k!} \int_{\mathbb{R}^2} x^n y^k \eta\left(\gamma - \gamma_0; (x,y)\right) \mathrm{d}x\mathrm{d}y. \tag{3.2.2}$$

and consequently,

$$\int_{\mathbb{R}^2} e^{\lambda_1 x + \lambda_2 y} \eta\left(\gamma - \gamma_0, (x,y)\right) \mathrm{d}x\mathrm{d}y \tag{3.2.3}$$

$$= \sum_{n\geq 0, k\geq 0}^{\infty} \lambda_1^n \left(-\lambda_2\right)^k e_1^{*\otimes(n+1)} \otimes e_2^{*\otimes(k+1)} \left(S\left(\gamma\right)_{0,1}\right). \tag{3.2.4}$$

Unfortunately, computing the inverse Fourier transform of the left hand side of (3.2.3) is highly non-trivial, making it difficult in practice to obtain a closed curve's winding number from its signature.

Theorem 41 also means that the information contained in the values of $e_1^{*\otimes n+1} \otimes$

$e_2^{*\otimes k+1}\left(S(\gamma)_{0,1}\right)$, also denoted as $\gamma_{n,k}$, are exactly the increment and the winding angle of a curve γ. More precisely,

Proposition 42. *Let $\gamma, \gamma' : [0,1] \to \mathbb{R}^2$ be two continuous curves with bounded total variations such that $\gamma_0 = \gamma'_0$.*

Then $\gamma_{n,k} = \gamma'_{n,k}$ for all $n, k \geq 0$ and $\gamma_1 = \gamma'_1$ if and only if $\gamma_1 = \gamma'_1$ and the winding angle of γ and γ' around (x,y) are equal for all $(x,y) \in \mathbb{R}^2 \backslash \gamma[0,1] \cup \gamma'[0,1]$.

3.2.1 Winding number

In this section, we shall recall the definition and topological properties of winding number.

Definition 43. A continuous function $\gamma : [0,1] \to \mathbb{R}^2$ is a closed curve if $\gamma_0 = \gamma_1$.

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous function. Let $z \in \mathbb{R}^2 \backslash \gamma[0,1]$. Then

$$g_z^\gamma(s) := \frac{\gamma_s - z}{\|\gamma_s - z\|}$$

defines a function $[0,1] \to \mathbb{S}^1$.

Let $p : \mathbb{R} \to \mathbb{S}^1$, $p(x) = e^{ix}$ be a covering map for $\mathbb{S}^1$. Then there exists a lift $\tilde{g}_z^\gamma : [0,1] \to \mathbb{R}$ such that $p \circ \tilde{g}_z^\gamma = g_z^\gamma$. The winding number of γ will be defined in terms of $\tilde{g}_s(z)$ by the following lemma:

Lemma 44. *([Nar85], Chapter 3 Lemma 1 and 2)Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous* *curve, and* $z \in \gamma[0,1]$*. Then the number*

$$A(\gamma, z) := \tilde{g}_z^{\gamma}(1) - \tilde{g}_z^{\gamma}(0) \tag{3.2.5}$$

depends only on γ *and* z *but not on the lift* $\tilde{g}_z^{\gamma}$*.* $A(\gamma, z)$ *is called the winding angle of* γ *around the point* z*.*

If γ *is a closed curve, then* $\eta(\gamma, z) := \frac{1}{2\pi} A(\gamma, z)$ *is an integer and will be called the winding number of* γ *around* z*.*

Remark 45. We may define the winding angle for any $\gamma : [a, b] \to \mathbb{R}^2$ by simply replacing 0 by a, 1 by b in the above definition.

Remark 46. The winding number is also known as the topological index, degree, or the topological quantum number.

Define the reversal operation $\overleftarrow{\cdot}$ by

$$\overleftarrow{\gamma}(u) := \gamma(1-u). \tag{3.2.6}$$

It follows from (3.2.5) that for all continuous functions γ, γ',

$$\begin{aligned} A(\gamma \star \gamma', (x, y)) &= A(\gamma, (x, y)) + A(\gamma', (x, y)) \\ A(\overleftarrow{\gamma}, (x, y)) &= -A(\gamma, (x, y)) \end{aligned} \tag{3.2.7}$$

The following theorem, which we shall need, is intuitively clear but is highly non-trivial:

Theorem 47. *([Munkres], p404)Let* $\gamma : [0, 1] \to \mathbb{R}^2$ *be a simple closed curve. Let*

Int(γ) and Ext(γ) be its interior and exterior respectively. Then $\eta(\gamma, z) = 0$ for all $z \in$Ext(γ). Moreover, either $\eta(\gamma, z) = 1$ for all $z \in$ Int(γ) or $\eta(\gamma, z) = -1$ for all $z \in Int(\gamma)$. γ is called positively oriented if $\eta(\gamma, z) = 1$ and negatively oriented otherwise.

We shall need a few more properties of $\eta(\gamma, z)$. First we need the notion of freely homotopic, or in another word, continuous deformation.

Definition 48. Let γ and γ' be closed curves in $X \subset \mathbb{R}^2$. Let $\Gamma : [0,1]^2 \to X$ be a continuous function such that

1. $\Gamma(0, t) = \gamma_t$ and $\Gamma(1, t) = \gamma'_t$,
2. $\Gamma(s, 0) = \Gamma(s, 1)$ for all $s \in [0, 1]$.

Then we say that γ and γ' are freely homotopic in X.

Proposition 49. *([Munkres], p403)Let γ,γ' be closed curves in $\mathbb{R}^2$. Then*

1. If γ and γ' are freely homotopic in $\mathbb{R}^2 \backslash \{z\}$, then $\eta(\gamma, z) = \eta(\gamma', z)$.

2.If z, w lies in the same connected component of $\mathbb{R}^2 \backslash \gamma[0, 1]$, then $\eta(\gamma, z) = \eta(\gamma, w)$.

Remark 50. 2. immediately implies that for all $z \in \mathbb{R}^2 \backslash \gamma[0, 1]$, there exists a ball $B_\varepsilon(z)$ around z such that $\eta(\gamma, z) = \eta(\gamma, w)$ for all $w \in B_\varepsilon(z)$.

The following result says that $z \to \eta(\gamma, z)$ is compactly supported.

Lemma 51. *Let $B_R(0)$ be the unit disc centered at 0 with radius R. If $\gamma[0, 1] \subset B_R(0)$, then $\eta(\gamma, z) = 0$ for all $z \in B_R(0)^c$.*

Proof. For each $z \in B_R(0)^c$, the constant path mapping t to 0 for all $t \in [0, 1]$ is freely homotopic to γ in $\mathbb{R}^2 \backslash \{z\}$ via the free homotopy $(s, t) \to s\gamma_t$. $\square$

3.2.2 Proof of Proposition 41 and 42

A key tool in our proof of Proposition 41 and 42 is the following Green's theorem for paths with bounded total variations.

Theorem 52. *([Rad36] and [BP71])Let* $\gamma = \left(\gamma^{(1)}, \gamma^{(2)}\right) : [0,T] \to \mathbb{R}^2$ *be a closed curve with bounded total variation. Let* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *have continuous partial derivatives in both variables. Then*

$$\int_{\mathbb{C}} \left(\partial_x f\left(x,y\right) + \partial_y g\left(x,y\right)\right) \eta\left(\gamma, \left(x,y\right)\right) \mathrm{d}x\mathrm{d}y = \int_{\gamma} f \mathrm{d}\gamma_s^{(2)} - g \mathrm{d}\gamma_s^{(1)}. \tag{3.2.8}$$

and

$$\left\|\eta\left(\gamma,\cdot\right)\right\|_{L^2} \leq \frac{1}{\sqrt{4\pi}} \left\|\gamma\right\|_{\mathcal{V}^1([0,T],\mathbb{R}^2)} \tag{3.2.9}$$

where the equality in (3.2.9) holds if and only if there exists $(x,y) \in \mathbb{R}^2$*,* $n \in \mathbb{N}$ *and* $R > 0$ *such that* $\gamma_t = (x + R\cos 2\pi nt, x + R\sin 2\pi nt)$*.*

The $f(x,y) = x$, $g(x,y) = y$ case of (3.2.8) was proved in [Rad36] and the proof for the general case is essentially the same. New, complete proofs for (3.2.8) were subsequently given by [Vog81] and [Yam08].

The second inequality is the well-known Banchoff-Pohl isoperimetric inequality[BP71].

Recall that we shall denote $e_1^{*\otimes n+1} \otimes e_2^{*\otimes k+1}\left(S\left(\gamma\right)_{0,1}\right)$ by $\gamma_{n,k}$.

The following trick of reducing $\gamma_{n,k}$ to a single line integral was part of the proof in Lemma 35 but for the convenience of later use we state it explicitly here again.

Lemma 53. *Let* $\gamma : [0,1] \to \mathbb{R}^2$ *be a continuous path with bounded total variation.*

Then for all $n \geq 0$ and $k \geq 0$,

$$\gamma_{n,k} = \frac{1}{(n+1)!k!} \int_0^1 \left(\gamma_s^{(1)} - \gamma_0^{(1)}\right)^{n+1} \left(\gamma_1^{(2)} - \gamma_s^{(2)}\right)^k \mathrm{d}\gamma_s^{(2)}.$$

Proof. See the proof of Lemma 35. □

We will now prove Lemma 41.

Proof. (of Lemma 41)By Lemma 53, γ satisfies

$$\gamma_{n,k} = \frac{1}{(n+1)!k!} \int_0^1 \left(\gamma_s^{(1)} - \gamma_0^{(1)}\right)^{n+1} \left(\gamma_1^{(2)} - \gamma_s^{(2)}\right)^k \mathrm{d}\gamma_s^{(2)}. \tag{3.2.10}$$

Using Green's theorem, we have

$$\gamma_{n,k} = \frac{1}{n!}\frac{1}{k!} \int_{\mathbb{R}^2} \left(x - \gamma_0^{(1)}\right)^n \left(\gamma_1^{(2)} - y\right)^k \eta\left(\gamma, (x,y)\right) \mathrm{d}x\mathrm{d}y \tag{3.2.11}$$

where $\eta\left(\gamma, (x,y)\right)$ is the winding number of γ around the point (x,y).

Note that from (3.2.5), $\eta\left(\gamma, (x+x_0, y+y_0)\right) = \eta\left(\gamma - \gamma_0, (x,y)\right)$.

Hence by a change of variable, we obtain (3.2.2).

Multiplying (3.2.11) by λ_1^n and $(-\lambda_2)^k$ and summing over all n and k in we obtain (3.2.3). □

We now prove Proposition 42.

Proof. (of Proposition 42)We shall first prove the only if direction.

By (3.2.10), $\gamma_{n,k} = \gamma'_{n,k}$ for all $n, k \geq 0$ and $\gamma_1 = \gamma'_1$ if and only if

$$\begin{aligned} & \frac{1}{(n+1)!k!}\left[\int_0^1 \left(\gamma_s^{(1)} - \gamma_0^{(1)}\right)^{n+1}\left(\gamma_1^{(2)} - \gamma_s^{(2)}\right)^k \mathrm{d}\gamma_s^{(2)}\right] \\ = & \frac{1}{(n+1)!k!}\left[\int_0^1 \left(\gamma_s'^{(1)} - \gamma_0^{(1)}\right)^{n+1}\left(\gamma_1^{(2)} - \gamma_s'^{(2)}\right)^k \mathrm{d}\gamma_s'^{(2)}\right] \end{aligned} \tag{3.2.12}$$

for all $n, k \geq 0$ and $\gamma_1 = \gamma'_1$.

Let $\sigma := \gamma \star \overleftarrow{\gamma'}$. Then by the additivity of Lebesgue-Stieltjes integral, (3.2.12) holds if and only if,

$$\int_0^1 \left(\sigma_s^{(1)} - \gamma_0^{(1)}\right)^{n+1}\left(\gamma_1^{(2)} - \sigma_s^{(2)}\right)^k \mathrm{d}\sigma_s^{(2)} = 0 \tag{3.2.13}$$

for all $n, k \geq 0$ and $\gamma_1 = \gamma'_1$.

By Green's theorem, (3.2.13) holds and $\gamma_1 = \gamma'_1$ if and only if

$$\int_0^1 \left(x - \gamma_0^{(1)}\right)^n \left(\gamma_1^{(2)} - y\right)^k \eta\left(\sigma, (x, y)\right) \mathrm{d}x\mathrm{d}y = 0 \tag{3.2.14}$$

for all $n, k \geq 0$ and $\gamma_1 = \gamma'_1$.

As $(x, y) \to \eta\left(\sigma, (x, y)\right)$ has compact support on $\mathbb{R}^2$, we may multiply both sides by $\frac{\lambda_1^n}{n!}$ and $\frac{\lambda_2^k}{k!}$, for any λ_1 and λ_2 in $\mathbb{C}$, and sum over n and k to obtain,

$$\int_0^1 e^{\lambda_1\left(x - \gamma_0^{(1)}\right) + \lambda_2\left(\gamma_1^{(2)} - y\right)} \eta\left(\sigma, (x, y)\right) \mathrm{d}x\mathrm{d}y = 0$$

As $(x, y) \to \eta\left(\sigma, (x, y)\right)$ has compact support and is in $L^2\left(\mathbb{R}^2\right)$ by (3.2.9), it is in

$L^1(\mathbb{R}^2)$. The Fourier transform is injective on L^1, so

$$\eta(\sigma, (x, y)) = 0 \tag{3.2.15}$$

almost everywhere in (x, y) with respect to the Lebesgue measure.

According to Remark 50, if there exists a $(x, y) \in \mathbb{R}^2 \backslash \sigma [0, 1]$ such that $\eta(\sigma, (x, y)) \neq 0$, then there exists an open ball $B_\varepsilon((x, y)) \subset \mathbb{R}^2$ such that $\eta(\sigma, (x, y)) \neq 0$ on $B_\varepsilon((x, y))$. This is a contradiction to (3.2.15). Therefore, $\eta(\sigma, (x, y)) = 0$ for all $(x, y) \in \mathbb{R}^2 \backslash \sigma [0, 1]$.

Hence in particular, as

$$\eta(\sigma, (x, y)) = \frac{1}{2\pi} \left(A(\gamma, (x, y)) - A(\gamma', (x, y)) \right)$$

we have $A(\gamma, (x, y)) = A(\gamma', (x, y))$ for all $(x, y) \in \mathbb{R}^2 \backslash (\gamma [0, 1] \cup \gamma' [0, 1])$.

For the "if" direction, note that $A(\gamma, (x, y)) = A(\gamma', (x, y))$ on $(x, y) \in \mathbb{R}^2 \backslash (\gamma [0, 1] \cup \gamma' [0, 1])$ and $\gamma_1 = \gamma_1'$ implies

$$\eta(\sigma, (x, y)) = 0$$

almost everywhere in (x, y). Therefore, equation (3.2.14) holds. As the logic from (3.2.12) to (3.2.14) are "if and only if" and hence if (3.2.14) holds, (3.2.12) also holds, and we have $\gamma_{n,k} = \gamma_{n,k}'$ for all $n, k \geq 0$. $\square$

Our goal now is to prove Theorem 39. We will first recall some basic Lie algebra.

3.2.3 Proof of Theorem 39

Hall basis

Let A be a set. Let K be a field. The Hall basis, introduced in [Hall50], is a simple way of assigning a basis to a free Lie algebra $\mathcal{L}(A)$ generated by A through the operation of concatenation. Let us describe the construction of the Hall basis.

Let $M(A)$ denote the set of planar rooted complete binary trees with leaves labelled as elements of A. A Hall set H is a subset of $M(A)$ satisfying the following conditions:

- $A \subseteq H$.
- We may assign a total order $\leq$ on H.
- $h = (h_1, h_2)$ in $M(A) \backslash A$ belongs to H if and only if

  $$h_1, h_2 \in H \text{ and } h < h_2,\ h_1 < h_2$$

  and

  $$\text{either } h_1 \in A \text{ or } h_1 = (h_1', h_2') \text{ where } h_2' \geq h_2.$$

There is in general infinitely many Hall sets for each set A, unless $|A| = 1$ or 0.

Let A^* denote the set of all words formed by a finite concatenation of elements of A. Let $f : M(A) \to A^*$ be the map defined by $f(t) = t$ if $t \in A$, and if $t = (t_1, t_2)$, then $f(t) = f(t_1) f(t_2)$.

Let H be a fixed Hall set. For each $h \in f(H)$, define a polynomial P_h by the rule that:

- $P_h = h$ if $h \in A$.
- $P_h = [P_{h_1}, P_{h_2}]$, if $h = h_1 h_2$, where $h_1, h_2 \in f(H)$.

It is a theorem of Hall (for a proof see [Reu93], Theorem 4.9) that for each Hall set H, the set $\{P_h : h \in f(H)\}$ forms a basis for the free Lie algebra generated by A over a field K.

Proof of Theorem 39

Proof. (of Theorem 39)By Proposition 41, it suffices to prove that

$$e_1^{*\otimes n} \otimes e_2^{*\otimes k}(S(\gamma)) = e_1^{*\otimes n} \otimes e_2^{*\otimes k}(\log S(\gamma)).$$

By definition, we have

$$S(\gamma) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (\log S(\gamma))^{\otimes k}.$$

As γ is closed, we have

$$\pi^{(1)}(\log S(\gamma)) = 0.$$

Hence the first term (ie. the one with the least degree) in the Lie series $\log S(\gamma)$ has degree at least two. However, if w is a Lie series generated by the alphabet

$\{e_1^*, e_2^*\}$ and the first term of w has degree at least two, then

$$e_1^{*\otimes n} \otimes e_2^{*\otimes k} \left(w^{\otimes r}\right) = 0$$

if $r \geq 2$.

Therefore,

$$e_1^{*\otimes n} \otimes e_2^{*\otimes k} \left(S\left(\gamma\right)\right) = e_1^{*\otimes n} \otimes e_2^{*\otimes k} \left(\log S\left(\gamma\right)\right).$$

$\square$

We now prove Corollary 40.

Proof. (of Corollary 40)

According to [Reu93] (Example 4.8), a Hall basis for the Lie polynomials up to degree 4 generated by the set $\{\mathbf{e_1}, \mathbf{e_2}\}$ is (a reordering of)

$$\mathbf{e_1}, \mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}], [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]], [\mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}]], \tag{3.2.16}$$

$$[\mathbf{e_1}, [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]]], [\mathbf{e_1}, [\mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}]]], [\mathbf{e_2}, [\mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}]]]. \tag{3.2.17}$$

Let $\mathcal{B}_4^*$ be the dual basis corresponding to the basis (3.2.16). To prove Corollary 40, it is sufficient to express, for each $f \in \mathcal{B}_4^*$, the value $f\left(\log S\left(\gamma\right)\right)$ in terms of the winding number of γ.

As γ is a closed curve, $e_i^*\left(\log\left(S\left(\gamma\right)_{0,1}\right)\right) = 0$ for $i = 1, 2$.

Since

$$
\begin{aligned}
[e_1, e_2]^* &= e_1^* \otimes e_2^* \\
[e_1, [e_1, e_2]]^* &= e_1^{*\otimes 2} \otimes e_2 \\
[e_2, [e_1, e_2]]^* &= -e_1^* \otimes e_2^{*\otimes 2} \\
[e_1, [e_1, [e_1, e_2]]]^* &= e_1^{*\otimes 3} \otimes e_2 \\
[e_1, [e_2, [e_1, e_2]]]^* &= -e_1^{*\otimes 2} \otimes e_2^{\otimes 2} \\
[e_2, [e_2, [e_1, e_2]]]^* &= e_1^* \otimes e_2^{\otimes 3},
\end{aligned}
$$

we have by Theorem 39,

$$
\begin{aligned}
[\mathbf{e_1}, \mathbf{e_2}]^* (\log (S(\gamma))) &= \int \eta (\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_1, [\mathbf{e_1}, \mathbf{e_2}]]^* (\log (S(\gamma))) &= \int x\eta (\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_2, [\mathbf{e_1}, \mathbf{e_2}]]^* (\log (S(\gamma))) &= \int y\eta (\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_1, [\mathbf{e}_1, [\mathbf{e_1}, \mathbf{e_2}]]]^* (\log (S(\gamma))) &= \frac{1}{2} \int x^2\eta (\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_1, [\mathbf{e}_2, [\mathbf{e_1}, \mathbf{e_2}]]]^* (\log (S(\gamma))) &= \int xy\eta (\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_2, [\mathbf{e}_2, [\mathbf{e_1}, \mathbf{e_2}]]]^* (\log (S(\gamma))) &= \frac{1}{2} \int y^2\eta (\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

□

Sharpness of Corollary 40

The purpose of this section is to prove the following sharpness compliment to Corollary 40.

Proposition 54. *There exists two paths γ, γ' such that the winding number of γ and γ' around every point is equal , but the fifth term of their signature differs.*

Proof. With an abuse of notation, let $\mathbf{e}_i$ denote the path $t \to t\mathbf{e}_i$, $t \in [0,1]$ and let

$$\gamma = \mathbf{e_1} \star \mathbf{e_2} \star -\mathbf{e_1} \star -\mathbf{e_2} \star -\mathbf{e_1} \star -\mathbf{e_2} \star \mathbf{e_1} \star \mathbf{e_2}$$

and

$$\gamma' = -\mathbf{e_1} \star -\mathbf{e_2} \star \mathbf{e_1} \star \mathbf{e_2} \star \mathbf{e}_1 \star \mathbf{e}_2 \star -\mathbf{e}_1 \star -\mathbf{e}_2.$$

where $\star$ denote the concatenation product defined just before Lemma 37.

By (3.2.7) and Theorem 47,

$$\eta\left(\gamma, (x,y)\right) = 1_{[0,1]\times[0,1]\cup[-1,0]\times[-1,0]}\left(x,y\right) = \eta\left(\gamma'(x,\dot{;} y)\right).$$

By Lemma 73 (in Chapter 5),

$$S\left(\gamma\right) = e^{\mathbf{e}_1} e^{\mathbf{e}_2} e^{-\mathbf{e}_1} e^{-\mathbf{e}_2} e^{-\mathbf{e}_1} e^{-\mathbf{e}_2} e^{\mathbf{e}_1} e^{\mathbf{e}_2} \tag{3.2.18}$$

and

$$S\left(\gamma'\right) = e^{-\mathbf{e}_1} e^{-\mathbf{e}_2} e^{\mathbf{e}_1} e^{\mathbf{e}_2} e^{\mathbf{e}_1} e^{\mathbf{e}_2} e^{-\mathbf{e}_1} e^{-\mathbf{e}_2}. \tag{3.2.19}$$

We claim that

$$\langle \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^*, S(\gamma) \rangle = 1$$

and

$$\langle \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^*, S(\gamma') \rangle = -1.$$

Note that the word $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1$ is "square-free", i.e. none of the alphabet in the word is identical to the alphabet on its immediate left or right. This means the contribution to the value of both

$$\langle \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^*, S(\gamma) \rangle$$

and

$$\langle \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^*, S(\gamma) \rangle$$

only comes from the first order term in exponentials in (3.2.18) and (3.2.19). For both, the contribution can only comes in one of the following five combinations:

Combination 1. $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}$ exponentials.

Combination 2. $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 7^{th}$ exponentials.

Combination 3. $1^{st}, 2^{nd}, 3^{rd}, 6^{th}, 7^{th}$ exponentials.

Combination 4. $1^{st}, 2^{nd}, 5^{rd}, 6^{th}, 7^{th}$ exponentials.

Combination 5. $1^{st}, 4^{nd}, 5^{rd}, 6^{th}, 7^{th}$ exponentials.

For $S(\gamma)$, the contributions from Combination 1 and Combination 5 is -1, while the contribution from Combination $2-4$ is 1. Therefore,

$$\begin{aligned}
& \langle \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^*, S(\gamma) \rangle \\
= \; & -1+1+1+1-1 \\
= \; & 1.
\end{aligned}$$

For $S(\gamma')$, the contributions from Combination 1 and Combination 5 is 1, while the contribution from Combination $2-4$ is -1. Therefore,

$$\begin{aligned}
& \langle \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^*, S(\gamma') \rangle \\
= \; & 1-1-1-1+1 \\
= \; & -1.
\end{aligned}$$

□

Chapter 4

Signature of SLE curves

Theorem 23, which we proved in the last chapter, only applies to paths with finite p-variations, where $p < 2$. In particular, our results can only be applied to study stochastic processes whose sample paths are almost surely smoother than the Brownian motion sample paths. One example of such processes is the Chordal SLE_κ measure. The SLE measures were born from the study of lattice models which have conformally invariant scaling limit. There are a number of lattice models whose scaling limit have been proved to be an SLE curve under some boundary conditions, such as the loop erased random walk ($\kappa = 2$, [LSW04]), the Ising model ($\kappa = 3$, [CS12]), the level lines of Gaussian Free Field ($\kappa = 4$, [SS09]), percolation on the triangular lattice ($\kappa = 6$, [CN07] and [Snv01]), and the Peano curve of the uniform spanning tree ($\kappa = 8$, [LSW04]).

The path regularity and, in particular, the roughness of SLE curves, in relation to the speed κ of the driving Brownian motion, is an extremely interesting topic. It is intuitively clear that the SLE curves becomes rougher as the speed of the driving

Brownian motion increases. In [JL10], the optimal Hölder exponent for SLE curves under the capacity parametrisation was proved to be

$$\min\left(\frac{1}{2}, 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}\right).$$

In [Bef08], V. Beffara proved that the Hausdorff dimension of SLE curves is $\min\left(1 + \frac{\kappa}{8}, 2\right)$. Therefore, the optimal Hölder exponent cannot exceed $1 + \frac{\kappa}{8}$. B. Werness[Wer12] proved that for $0 < \kappa \leq 4$, almost surely, there exists a parametrisation under which the SLE curve is α-Hölder for any $\alpha < \frac{1}{1+\frac{\kappa}{8}}$. In another words, the roughness of an SLE curve grows linearly with the speed of the driving Brownian motion. It is strongly believed that this remains true for $4 < \kappa < 8$. However, to the best of our knowledge, this problem remains open.

Werness's result allowed him to define the signatures of SLE curves using Young's integral and used Green's theorem to compute the first three terms of the expected signature of SLE curves in terms of the 1-point function.

This chapter has two purposes. The first is to prove the almost sure uniqueness of signature for SLE_κ curves, for $0 < \kappa \leq 4$. The second is to related the expected signature associated with the Chordal SLE_κ measures to the n-point functions studied in the SLE and Statistical Physics literature.

The almost-sure uniqueness problem was first studied by Le Jan and Qian in [LQ12], where they proved that the signatures of Brownian motion sample paths determine the Brownian motion paths almost surely. The result was extended to diffusion processes in [Geng13]. Both results rely on the Strong Markov property.

Although the Chordal SLE$_\kappa$ measure is not Markov, the inversion problem can be tackled for $\kappa \leq 4$ since the Chordal SLE$_\kappa$ measure is supported on simple curves from -1 and 1. Therefore, it follows from Theorem 23 from the last chapter that:

Theorem 55. *Let $0 < \kappa \leq 4$. Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then there exists a $\mathbb{P}_\kappa$-null set $\mathcal{N}$ such that if $\gamma, \gamma' \in \mathcal{N}^c$ and $S(\gamma)_{0,1} = S(\gamma')_{0,1}$, then γ and γ' are equal up to a reparametrisation.*

A proof of Theorem 55 will appear in section 4.1.The expected signature can be considered as the "Laplace's transform" of a stochastic process and has been extensively studied in [Ni12] and [LV04]. The sequence of n-point functions of the Chordal SLE measure was first studied by O. Schramm. Using a generalised Green's theorem for non-smooth curves, we may prove the following relationship between the expected signature and the sequence of n-point functions.

Theorem 56. *Let $0 \leq \kappa \leq 4$. Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points 1 and -1. For each $\gamma \in \mathcal{C}_2(-1, 1, \mathbb{D})$, let $\Phi(\gamma)$ denote the concatenation of γ with the upper semi-circular boundary of $\mathbb{D}$ from 1 to -1. Let j_{n_j,k_j} denote the word $(1, \ldots, 1, 1, 2, \ldots, 2)$ with $n_j + 1$ 1s followed by $k_j + 1$ 2s and let Γ_n denotes the n-point function associated with $\mathbb{P}_\kappa$, then*

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} e^{\sum_{i=1}^N \lambda_i x_i + \mu_i y_i} \Gamma_N\left(\left(x_1, y_1\right), \ldots, \left(x_N, y_N\right)\right) \mathrm{d}x_1 \cdots \mathrm{d}y_N \\
=\ & \sum_{n_1,\ldots,n_N,k_1\ldots k_N \geq 0} \Pi_{i=1}^N \left(\lambda_i\right)^{n_i} \left(-\mu_i\right)^{k_i} \mathbb{E}\left[e^*_{j_{n_1,k_1}} \sqcup \ldots \sqcup e^*_{j_{n_N,k_N}} S\left(\Phi\left(\gamma\right)\right)_{0,1}\right]
\end{aligned}
$$

where $e^*_{j_{n_i+1,k_i+1}}$ *are linear functionals to be defined in section 2.1 and* $\sqcup$ *denotes the shuffle product defined in Proposition 30.*

We will now recall facts from the SLE theory.

4.1 Schramm-Loewner evolution

Let $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space. Let $(B_t : t \geq 0)$ be a one-dimensional standard Brownian Motion. Let $0 < \kappa$. Let $z \in \overline{\mathbb{H}} \backslash \{0\}$. For each $\omega \in \Omega$, consider the initial value problem:

$$\frac{\mathrm{d}g_t(z,\omega)}{\mathrm{d}t} = \frac{2}{g_t(z,\omega) - \sqrt{\kappa}B_t(\omega)} \quad g_0(z) = z \tag{4.1.1}$$

We shall recall the following facts about g_t from [RS05].

1. For each ω, a unique solution to this equation exists up to time $T_z > 0$, where T_z is the first time such that $g_t - \sqrt{\kappa}B_t \to 0$ as $t \to T_z$.

2. Define

$$H_t = \{z \in \mathbb{H} : t < T_z\} \text{ and } K_t = \mathbb{H} \backslash H_t$$

 Then H_t is open and simply connected.

3. For each time $t > 0$, g_t defines a conformal map from H_t onto $\mathbb{H}$. In particular, g_t is invertible.

4. Let $\hat{f}_t(z) := g_t^{-1}(z + \sqrt{\kappa}B_t)$. There exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for all

$\omega \in \mathcal{N}^c$, the limit

$$\hat{\gamma}(t,\omega) := \lim_{z\to 0, z\in\mathbb{H}} \hat{f}_t(z)$$

exists and $t \to \hat{\gamma}(t)$ is continuous. The two dimensional stochastic process $(\hat{\gamma}_t : t \geq 0)$ is called the *Chordal SLE$_\kappa$ curve.*

The Loewner correspondence from a continuous path $t \to B_t(\omega)$ to $t \to \hat{\gamma}(\cdot,\omega)$ is in fact deterministic and one-to-one. Therefore, the measure on the Brownian paths induces, through this correspondence, a measure on paths in $\overline{\mathbb{H}}$ from 0 to ∞, which we shall call the Chordal SLE$_\kappa$ measure in $\mathbb{H}$.

Proposition 57. *Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$. Then with probability one, the following holds:*

1. ([Wer12], Section 4.1)If $0 < \kappa \leq 4$, then for any $p > 1 + \frac{\kappa}{8}$, γ has finite p-variation.

2.([RS05],Theorem 7.1 and Theorem 6.1)$\gamma : [0,\infty) \to \overline{\mathbb{H}}$ satisfies $\gamma_0 = 0$ and $\liminf_{t\to\infty} |\hat{\gamma}_t| = \infty$.

3.([RS05], Theorem 6.1)For $0 \leq \kappa \leq 4$, $t \to \hat{\gamma}_t$ is a simple curve and $\hat{\gamma}(0,\infty) \subset \mathbb{H}$.

The fact that $\lim_{t\to\infty} \hat{\gamma}_t = \infty$ *a.s.* means that the signature $S(\hat{\gamma})_{0,\infty}$ will not be defined. Therefore, we shall follow [Wer12] and opt to study the Chordal SLE$_\kappa$ curve in the unit disc $\mathbb{D}$, from -1 to 1. The Chordal SLE_κ measure in domain $\mathbb{D}$ with marked points -1 and 1 is defined as follows:

Definition 58. For $\kappa > 0$. Let $\mathbb{P}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$, D be a simply connected subdomain of $\mathbb{C}$, $a, b \in \partial D$ and f be a conformal map from $\mathbb{H}$ to D, with

$f(0) = a$ and $f(\infty) = b$. Then the Chordal SLE$_\kappa$ measure in D with marked points a and b is defined as the measure $\mathbb{P} \circ f^{-1}$.

Remark 59. Although there is a one dimensional family of conformal maps f such that f maps $\mathbb{H}$ to D, 0 to a and ∞ to b, the scale invariance of the Chordal SLE measure in $\mathbb{H}$ means that the measure $\mathbb{P} \circ f^{-1}$ is the same no matter which member f in this one dimensional family we use.

We now prove our almost sure uniqueness theorem concerning the signature of SLE curves.

Proof. (of Theorem 55)Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then by Proposition 57, there exists a $\mathbb{P}_\kappa$-null set $\mathcal{N}$, such that for all $\gamma \in \mathcal{N}^c$,

1. $\gamma(0) = -1$, $\gamma(1) = 1$ and $\gamma(0,1) \subset \mathbb{D}$.
2. γ has with finite $1 \leq p < 2$ variations.
3. γ is simple.

Therefore, in particular, $\mathcal{N}^c \subset \mathcal{C}_2(-1, 1, \mathbb{D})$.

Let $\gamma, \gamma' \in \mathcal{N}^c$ be such that $S(\gamma) = S(\gamma')$, then by Theorem 23, γ and γ' are reparametrisations of each other. □

4.2 Expected signature and n-point functions

4.2.1 n-point functions from expected signature

We will need the following immediate consequence of the shuffle product formula.

Lemma 60. *Let* $(k_1, l_1), ..., (k_n, l_n) \in \mathbb{N}^2$*. Let* j_{k_i,l_i} *denote the word with* $k_i + 1$ *1s followed by* $l_i + 1$ *2s. Then*

$$\gamma_{k_1,l_1} \cdots \gamma_{k_n,l_n} = \gamma^{j_{k_1,l_1} \sqcup \cdots \sqcup j_{k_n,l_n}}$$

where the operation $\sqcup$ *is the shuffle product operation defined in Proposition 30.*

Proof. We shall prove this by induction.

$n = 1$ follows from the definition.

Suppose

$$\gamma_{k_1,l_1} \cdots \gamma_{k_n,l_n} = \gamma^{j_{k_1,l_1} \sqcup \cdots \sqcup j_{k_n,l_n}}$$

then by Proposition 30,

$$\left(\gamma_{k_1,l_1} \cdots \gamma_{k_n,l_n}\right)\left(\gamma_{k_{n+1},l_{n+1}}\right) = \gamma^{\left(j_{k_1,l_1} \sqcup \cdots \sqcup j_{k_n,l_n}\right) \sqcup j_{k_{n+1},l_{n+1}}}$$

the induction is completed through the associativity of the shuffle product. □

A well-known observable in the theory of SLE is the following sequence of n-point functions:

Definition 61. Let $0 < \kappa \leq 4$. Let $\mathbb{P}_\kappa$ denote the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked point -1 and 1. For each $\gamma \in \mathcal{C}_2(-1, 1, \mathbb{D})$, let $\Phi(\gamma)$ denote the concatenation of γ and the upper semi-circular boundary of $\mathbb{D}$, oriented in the anti-clockwise direction. We shall define the n-point function associated with the probability measure $\mathbb{P}_\kappa$

to be:

$$\Gamma_n\left(x_1, y_1, .., x_n, y_n\right) = \mathbb{P}_\kappa\left[\left(x_1, y_1\right), \ldots \left(x_n, y_n\right) \in \mathrm{Int}\Phi\left(\cdot\right)\right].$$

The $n-$point functions for SLE$_\kappa$ curves were first studied by O. Schramm who calculated the 1-point function explicitly in terms of hypergeometric functions (see [Sch01]). Although PDEs can be written down for the n-point functions, the analytic expressions for general n and κ are not known. The only exception is $n = 2$ and $\kappa = \frac{8}{3}$, which was predicted in [SC09] and computed rigorously in [BV11].

Proof. (of Theorem 56)

Let $0 < \kappa \leq 4$. Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1.

As in the proof of Theorem55, there exists a $\mathbb{P}_\kappa$ null set $\mathcal{N}$ such that $\mathcal{N}^c \subset \mathcal{C}_2\left(-1, 1, \mathbb{D}\right)$.

By Lemma 37, we have for each $\gamma \in \mathcal{N}^c$,

$$C_{\mathbf{n},\mathbf{k}} \int_{\mathbb{C}^N} \Pi_{i=1}^N \left(x_i + 1\right)^{n_i} y_i^{k_i} 1_{\left(\mathrm{Int}\Phi(\gamma)\right)^N} \mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_N \mathrm{d}y_N = \Phi\left(\gamma\right)_{n_1,k_1} ...\Phi\left(\gamma\right)_{n_N,n_N}$$

where $\left(\mathrm{Int}\Phi\left(\gamma\right)\right)^n$ is the set

$$\cap_{k=1}^N \left\{\left(x_1, y_1, \ldots, x_N, y_N\right) : \left(x_k, y_k\right) \in \mathrm{Int}\Phi\left(\gamma\right)\right\}$$

and

$$C_{\mathbf{n},\mathbf{k}} := \Pi_{i=1}^N \frac{\left(-1\right)^{k_i}}{n_i! k_i!}.$$

By Lemma 60, $\Phi(\gamma)_{n_1,k_1} ... \Phi(\gamma)_{n_N,n_N} = \Phi(\gamma)^{j_{k_1,l_1} \sqcup \cdots \sqcup j_{k_n,l_n}}$.

By taking linear combinations, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} e^{\sum_{i=1}^{N} \lambda_i (x_i+1) + \mu_i y_i} \mathbb{E}\left[1_{D^N}\right] \mathrm{d}x_1 \cdots \mathrm{d}y_N \\ = &\sum_{n_1,\ldots,n_N,k_1\ldots k_N \geq 0} \Pi_{i=1}^{N} \left(\lambda_i\right)^{n_i} \left(-\mu_i\right)^{k_i} \mathbb{E}\left[\Phi(\gamma)^{j_{n_1+1,k_1+1} \sqcup \cdots \sqcup j_{n_N+1,k_N+1}}\right]. \end{aligned}$$

The result then follows by noting that $\mathbb{E}\left[1_{D^N}(\cdot)\right] = \Gamma_N(\cdot)$. □

As we may determine the signature of $\Phi(\gamma)$ from the signature of γ using Chen's identity, this formula gives a relationship between the expected signature of the Chordal SLE measure and the n-point functions.

4.2.2 Expected signature from n-point functions

We may ask whether it is possible to obtain the expected signature from the n-point functions. Unfortunately, here we can do no better than the deterministic case and are only able to obtain an explicit formula only up to the fourth term. To obtain a simpler formula, we choose to study the Chordal SLE$_\kappa$ measure on $1+\mathbb{D}$ so that all paths start from 0. The n-point functions can be defined for the domain $1+\mathbb{D}$ in the obvious way.

Proposition 62. *Let $0 < \kappa \leq 4$. Let γ denote the Chordal SLE$_\kappa$ curve from 0 to 2 in $1+\mathbb{D}$. Let $\Phi(\gamma)$ denote the concatenation of γ with the upper semi-circle of the unit disc $1+\mathbb{D}$, oriented in the anti-clockwise direction. Then the first four terms of* *the tensor element $\Phi(\gamma)$ is*

$$1+\int_{\mathbb{D}}\left(\left[\mathbf{e}_1,\mathbf{e}_2\right]+\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]+\frac{1}{2}\left[\mathbf{x}_1,\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\right)\Gamma_1\left(\left(x_1,y_1\right)\right)\mathrm{d}x_1\mathrm{d}y_1 \tag{4.2.1}$$

$$+\frac{1}{2}\int_{\mathbb{R}^4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_2\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_2\mathrm{d}x_4 \tag{4.2.2}$$

where $\mathbf{x}_1=x_1\mathbf{e}_1+y_1\mathbf{e}_2$ and $\mathbf{x}_2=x_2\mathbf{e}_1+y_2\mathbf{e}_2$, and Γ_n is the n-point function for the Chordal SLE$_\kappa$ measure.

Proof. Let $\mathcal{N}$ be a Chordal SLE$_\kappa$ null set such that for all $\gamma\in\mathcal{N}^c$, γ is simple, $\gamma(0,1)\subset\mathbb{D}$, starts from 0 and ends at 2 and has finite p variation for some $p<2$.

Let $\gamma\in\mathcal{N}^c$. Using the exactly same computation as in the proof of Corollary 40 and replacing the use of Green's theorem with Lemma 37, we have the following

$$\begin{aligned}
\pi_4\left(\log S\left(\Phi\left(\gamma\right)\right)\right) &= \int_{\mathbb{R}^2}\left[\mathbf{e}_1,\mathbf{e}_2\right]1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y \\
&\quad+\int_{\mathbb{R}^2}x\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y \\
&\quad+\int_{\mathbb{R}^2}y\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y \\
&\quad+\int_{\mathbb{R}^2}\frac{x^2}{2}\left[\mathbf{e}_1,\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y \\
&\quad+\int_{\mathbb{R}^2}xy\left[\mathbf{e}_1,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y \\
&\quad+\int_{\mathbb{R}^2}\frac{y^2}{2}\left[\mathbf{e}_2,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y \\
&=\int_{\mathbb{R}^2}\Big(\left[\mathbf{e}_1,\mathbf{e}_2\right]+\left[x\mathbf{e}_1+y\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right] \\
&\quad+\frac{1}{2}\left[x\mathbf{e}_1+y\mathbf{e}_2,\left[x\mathbf{e}_1+y\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\Big)1_{\mathrm{Int}\Phi(\gamma)}\left(x,y\right)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

By taking the exponential and writing $x\mathbf{e}_1 + y\mathbf{e}_2$ as $\mathbf{x}$,

$$\begin{aligned}
\pi_4\left(S\left(\Phi\left(\gamma\right)\right)\right) = 1 &+ \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} \left[\mathbf{x}, \left[\mathbf{e}_1, \mathbf{e}_2\right]\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} \left[\mathbf{x}, \left[\mathbf{x}, \left[\mathbf{e}_1, \mathbf{e}_2\right]\right]\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \otimes \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \otimes \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\
= &\int_{\mathbb{R}^4} \left[\mathbf{e}_1, \mathbf{e}_2\right] \otimes \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x_1, y_1\right) 1_{\mathrm{Int}\Phi(\gamma)}\left(x_2, y_2\right) \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2
\end{aligned}$$

The proof is completed by taking expectation. $\square$

Chapter 5

Expected signature of Gaussian processes with strictly regular kernels

The expected signature of a stochastic process was first studied by T. Fawcett in [Faw03] and N. Victoire in [LV04], who independently calculated the expected signature of Brownian motion. This is used in [LV04] to calculate the cubature measure which approximates the Wiener measure. Since then, interests in the subject have grown. In [LH11], Ni and Lyons expressed the expected signature of Brownian motion in a disc up to the first exit time in terms of the solution of a PDE. In [Wer12], the first three terms of the expected signature of the Chordal SLE measure were explicitly calculated. In this article we calculate the expected signature of Gaussian processes with strictly regular kernel (defined in next section). These are Gaussian processes with regular kernels (see [AMN01]) which do not have Brownian components.

The original motivation for undertaking this study was to find the expected signature of fractional Brownian motions for $H > \frac{1}{2}$. After posting this result on arXiv, we were informed of the calculation of expected signature for fractional Brownian motions for Hurst parameter $H > \frac{1}{2}$ has already appeared in [BC07]. Therefore, this chapter is a generalisation of the calculation in [BC07].

5.1 The Main result

We will first recall some notations.

Let $T > 0$ be fixed throughout this chapter.

Let $\triangle' := \{(t,s) \in \mathbb{R}^2 : 0 \leq s < t \leq T\}$.

Let $K(\cdot,\cdot) : \triangle' \to \mathbb{R}$ be a function such that:

(K1) For each r, $K(\cdot, r)$ has an integrable derivative on $(r, T]$

(K2) $K(r^+, r) = 0 \ \forall r \in [0, T]$.

(K3) Let $|K|((r,T], r)$ denote the total variation of $K(\cdot, r)$ on $(r, T]$. Then

$$\int_0^\infty |K|((r,T],r)^2 \, \mathrm{d}r < \infty.$$

If we remove the condition (K2) and weaken the condition of having an integrable derivative in (K1) to bounded total variation, then we recover the notion of regular kernel in [AMN01]. Our extra conditions mean that our Gaussian processes have to be strictly smoother than Brownian motion.

Let W_t be a Gaussian process of the form

$$W_t := \int_0^t K(t,r) \, \mathrm{d}B_r, \quad t \in [0,T], \tag{5.1.1}$$

where $\mathrm{d}B_r$ denotes the integration in the Itô's sense.

We shall call an one-dimensional Gaussian process W of the form (5.1.1), where K satisfies (K1), (K2) and (K3), a Gaussian process with a strictly regular kernel K.

We now fix a basis of $\mathbb{R}^d$, which shall denote by $\{e_1, \ldots, e_d\}$, throughout this chapter. A d-dimensional Gaussian process with a strictly regular kernel is a process whose coordinate components with respect to this basis are independent and identically distributed copies of a Gaussian process with a strictly regular kernel.

The expected signature of such a process will be expressed in terms of pairings, which we shall recall below:

Definition 63. A set π is a pairing of $\{1, 2, \ldots, 2n-1, 2n\}$ if there exists two injective functions $a : \{1, \ldots, n\} \to \{1, \ldots, 2n\}$ and $b : \{1, \ldots, n\} \to \{1, \ldots, 2n\}$ such that $a(i) < b(i)$ for all i, the images of a and b form a partition of $\{1, 2, \ldots, 2n-1, 2n\}$ and

$$\pi = \{(a(i), b(i)) : i = 1, .., n\}.$$

Let E_{2n} be a subset of $\{1, 2, \ldots, d\}^{2n}$ such that $(i_1, \ldots, i_{2n}) \in E_{2n}$ if and only if for all $k \in \{1, 2, \ldots, d\}$, the set

$$\{j : i_j = k\}$$

has an even number of elements.

Let Π_{2n} denote the set of all possible pairings of $\{1, 2, \ldots, 2n\}$. Given $(i_1, \ldots, i_{2n}) \in \{1, 2, \ldots, d\}^{2n}$, define $\Pi_{i_1,\ldots,i_{2n}}$ to be a subset of Π_{2n} whose elements π satisfy

$$(l, j) \in \pi \implies i_l = i_j$$

For example, $\Pi_{1,1,2,2}$ has only one element, which is $\{(1,2),(3,4)\}$. The elements of $\Pi_{1,1,2,2,1,1}$ are $\{(1,2),(3,4),(5,6)\}$,$\{(1,5),(2,6),(3,4)\}$, $\{(1,6),(2,5),(3,4)\}$.

Let $\partial_1 K$ denote the derivative of K with respect to its first coordinate.

We can now state our main result:

Theorem 64. *Let W be a d-dimensional Gaussian process with a strictly regular kernel K. If $k = 2n$ for some $n \in \mathbb{N}$, and $(i_1, \ldots, i_{2n}) \in E_{2n}$, then the projection to the basis $e_{i_1} \otimes e_{i_2} ... \otimes e_{i_k}$ of the expected signature of W up to time T is*

$$\sum_{\pi \in \Pi_{i_1,\ldots,i_{2n}}} \int_{\triangle_{2n}(T)} \Pi_{(l,j)\in\pi} \left[\int_0^{u_j \wedge u_l} \left[\partial_1 K(u_j, r)\right] \left[\partial_1 K(u_l, r)\right] \mathrm{d}r \right] \mathrm{d}u_1..\mathrm{d}u_{2n}$$

where $\triangle_{2n}(T)$ denotes the simplex $\{(u_1, \ldots, u_{2n}) \in \mathbb{R}^{2n} : 0 \leq u_1 < \ldots < u_{2n} \leq T\}$. The projection to the basis $e_{i_1} \otimes e_{i_2} \ldots \otimes e_{i_k}$ is zero otherwise.

The key idea of the calculation, inspired by [CQ00], is to first calculate the expected signature of the piecewise linear approximation of the Gaussian process. This reduces the problem into computing joint moments of Gaussian random variables, which can be done using Wick's formula.

5.2 Signature of Gaussian process

5.2.1 Gaussian process

Recall that $\triangle' := \{(t,s) \in \mathbb{R}^2 : 0 \leq s < t \leq T\}$.

Let $K(\cdot,\cdot) : \triangle' \to \mathbb{R}$ be a function satisfying the conditions (K1), (K2) and (K3) in Section 1.1, and $W : [0,T] \to \mathbb{R}$ be a Gaussian process satisfying (5.1.1).

Then W_t is a Gaussian process with the covariance function

$$R(t,s) := \int_0^{t\wedge s} K(t,r)\,K(s,r)\,\mathrm{d}r \tag{5.2.1}$$

The integral in (5.2.1) exists by condition (K3).

By (K1), we have $\partial_1 K(\cdot,r) \in L^1(r,T]$ and by (K2),

$$K(t,u) = \int_r^t \partial_1 K(u,r)\,\mathrm{d}u.$$

Thus (5.2.1) becomes

$$R(t,s) = \int_0^{t\wedge s}\int_r^s\int_r^t \partial_1 K(u,r)\,\partial_1 K(v,r)\,\mathrm{d}u\mathrm{d}v\mathrm{d}r.$$

and in particular, $(u,v,r) \to \partial_1 K(u,r)\,\partial_1 K(v,r)$ is integrable on the set

$$\{(u,v,r) : r \in [0, s\wedge t],\, u \in [r,t],\, v \in [r,s]\}.$$

By Fubini's theorem, we have the following equality:

$$R(t,s) = \int_0^s\int_0^t \left[\int_0^{u\wedge v} [\partial_1 K(u,r)]\,[\partial_1 K(v,r)]\,\mathrm{d}r\right]\mathrm{d}u\mathrm{d}v. \tag{5.2.2}$$

We shall denote the integrable function $(u,v) \to \int_0^{u\wedge v} [\partial_1 K(u,r)]\,[\partial_1 K(v,r)]\,\mathrm{d}r$ by $f(u,v)$.

To summarise, we have

$$R(t,s) = \int_0^t \int_0^s f(u,v)\,\mathrm{d}u\mathrm{d}v$$

for some $f \in L^1([0,T]\times[0,T])$.

This means that for $\sigma \le \tau$ and $s \le t$ in $[0,T]$, we have

$$\mathbb{E}\left[(W_t - W_s)(W_\tau - W_\sigma)\right] = \int_\sigma^\tau \int_s^t f(u,v)\,\mathrm{d}u\mathrm{d}v.$$

This expression will be key to our computation.

By the symmetry of $R(\cdot,\cdot)$, we also have $f(u,v) = f(v,u)$.

We summarise our calculations in the following lemma:

Lemma 65. *Let $W : [0,T] \to \mathbb{R}$ be a Gaussian process with a strictly regular kernel K. Then there exists an integrable function $f : [0,T]^2 \to \mathbb{R}$ such that $f(u,v) = f(v,u)$, and*

$$\mathbb{E}\left[(W_t - W_s)(W_\tau - W_\sigma)\right] = \int_\sigma^\tau \int_s^t f(u,v)\,\mathrm{d}u\mathrm{d}v.$$

5.2.2 Signature of Gaussian processes

Recall that $\triangle_{0,T}$ denotes $\{(s,t) : 0 \le s \le t \le T\}$. Let $f : \triangle_{0,T} \times \triangle_{0,T} \to \mathbb{R}$ be a function. We shall follow [FV10] and use the notation $f\begin{pmatrix} s,t \\ u,v \end{pmatrix}$ to denote, for

$s \leq t$ and $u \leq v$,

$$f\begin{pmatrix} s,t \\ \\ u,v \end{pmatrix} := f(s,u) + f(t,v) - f(s,v) - f(t,u)$$

and a function $f : [0,T]^2 \to \mathbb{R}$ is said to have finite ρ-variation if $|f|_{\rho-var;[0,T]^2} < \infty$, where

$$|f|_{p-var;[s,t]\times[u,v]} = \sup_{R \in \mathcal{D}([s,t],[u,v])} \left(\sum_{i,j} |f(R)|^p \right)^{\frac{1}{p}} \tag{5.2.3}$$

and $\mathcal{D}([s,t]\times[u,v])$ is the set of all rectangular partitions of $[s,t]\times[u,v]$, ie. the set of all rectangles of the form

$$[s_1, s_2] \times [t_1, t_2]$$

such that if $R_1, R_2 \in \mathcal{D}([s,t]\times[u,v])$, then

$$R_1 \cap R_2$$

is one dimensional and

$$\cup_{R \in \mathcal{D}([s,t]\times[u,v])} R = [s,t] \times [u,v].$$

For a function $\omega : \triangle_{0,T} \times \triangle_{0,T} \to [0,\infty)$, we shall denote, for $[s,t]\times[u,v] \in [0,T]^2$,

$$\omega([s,t]\times[u,v]) := \omega(s,t,u,v)$$

Definition 66. A 2D *control* is a continuous function $\omega : \triangle_{0,T} \times \triangle_{0,T} \to [0, \infty)$ such that for all rectangles R_1,R_2,R in $[0,T] \times [0,T]$, such that if $R_1 \cup R_2 \subset R$, $R_1 \cap R_2$ is one dimensional, then

$$\omega(R_1) + \omega(R_2) \leq \omega(R)$$

and that for all rectangles R with zero Lebesgue measure in $\mathbb{R}^2$,

$$\omega(R) = 0$$

Definition 67. Let $\omega : \triangle_{0,T} \times \triangle_{0,T} \to [0, \infty)$ be a 2D control and let $f : \triangle_{0,T} \times \triangle_{0,T} \to \mathbb{R}$ be a continuous function. We say that ω *controls the p-variation of* f if for all $[s,t] \times [u,v] \subset [0,T]^2$,

$$\left| f \begin{pmatrix} s,t \\ u,v \end{pmatrix} \right|^p \leq \omega([s,t] \times [u,v])$$

Lemma 68. *([FV10], Lemma 5.56) Let $f : \triangle_{0,T} \times \triangle_{0,T} \to \mathbb{R}$ be a continuous function. Then f has finite ρ-variation if and only if there exists a 2D control ω such that ω controls the ρ-variation of f.*

We will now recall the existence and some properties of the signatures of Gaussian processes with strictly regular kernels.

Let X be a d-dimensional Gaussian process, then its covariance function is defined as $R_X(s,t) := \mathbb{E}[X_s \otimes X_t]$.

The following theorem gives the existence of the signatures of some Gaussian processes and the approximation result that the limit of $\mathbb{E}\left[S\left(X^{D}\right)_{0,T}\right]$ as $\|D\| \to 0$ is $\mathbb{E}\left[S\left(X\right)_{0,T}\right]$:

Theorem 69. *([FR]) Let $X = \left(X^{1}, .., X^{d}\right) : [0,T] \to \mathbb{R}^{d}$ be a centered, continuous Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with X^{i}, X^{j} being independent if $i \neq j$. Assume that the covariance function of X, denoted by R_X, has finite ρ-variation for some $\rho \in [1,2)$ and that there exists a finite constant K such that $|R_X|_{p-var,[0,T]^2} \leq K$ (see (5.2.3)). Then there exists a process $\mathbf{X}$ with sample paths almost surely in $\mathcal{V}^{p}\left(T^{\lfloor p \rfloor}\left(\mathbb{R}^{d}\right)\right)$ for all $p \in (2\rho, 4)$, such that for any $\gamma > \rho$, $\frac{1}{\gamma} + \frac{1}{\rho} > 1$ and any $q > 2\gamma$ and $N \in \mathbb{N}$ there exists a constant $C = C\left(q, \rho, \gamma, K, N, T\right)$ such that for all $r \geq 1$,*

$$\left|\rho_{q-var}^{(N)}\left(S_N\left(X^{(k)}\right), S_N\left(\mathbf{X}\right)\right)\right|_{L^r} \leq C r^{\frac{N}{2}} \sup_{0 \leq t \leq T}\left|X_t^{(k)} - X_t\right|_{L^2}^{1-\frac{\rho}{\gamma}}$$

for any piecewise linear interpolation $X^{(k)}$ of X, where the width of the partition is smaller than $\frac{1}{k}$.

Definition 70. Let X be a process satisfying the conditions in Theorem 69, we shall denote the process $\mathbf{X}$ in Theorem 69 by $S\left(X\right)$ and the expected signature of X on $[0,T]$ is defined to be $\mathbb{E}\left[S\left(X\right)_{0,T}\right]$.

Proposition 71. *Let X be a d-dimensional Gaussian process with a strictly regular kernel. Then the covariance function of X has finite 1-variation.*

Moreover, let D^m be the dyadic partition $D^m = \left(0, \frac{T}{2^m}, \frac{2T}{2^m}, .., \frac{(2^m-1)T}{2^m}, T\right)$. Then *for all* $N \in \mathbb{N}$,

$$\left|\mathbb{E}\left[S_N\left(X^{D^m}\right)\right]-\mathbb{E}\left[S_N\left(X\right)\right]\right|\rightarrow 0$$

as $m\rightarrow\infty$, *where* $|\cdot|$ *is the norm induced by identifying* $\left(\mathbb{R}^d\right)^{\otimes n}$ *with* $\mathbb{R}^{d^n}$.

Proof. By Lemma 65 and its notation, the 1-variation of the covariance function of X, R_X, is controlled by

$$\omega\left([s,t]\times[u,v]\right):=\int_s^t\int_v^u\left|f\left(x,y\right)\right|\mathrm{d}x\mathrm{d}y$$

Thus by Lemma 68, R_X has finite 1-variation. Moreover, the total 1-variation of R_X on $[0,T]^2$ is bounded above by $\int_0^T\int_0^T\left|f\left(x,y\right)\right|\mathrm{d}x\mathrm{d}y$. Therefore, Theorem 69 applies.

Observe that we have, in the pathwise sense, the following inequality for all $N\in\mathbb{N}$,$q>2$:

$$\left|S_N\left(X^{D^m}\right)_{0,T}-S_N\left(\mathbf{X}\right)_{0,T}\right|\leq\left|\rho^{(N)}_{q-var}\left(S_N\left(X^{D^m}\right),S_N\left(\mathbf{X}\right)\right)\right|$$

Thus by Theorem 69, the only thing to prove is

$$\sup_{0\leq t\leq T}\left|X_t^{D^m}-X_t\right|_{L^2}\rightarrow 0$$

Let $t_k^m:=\frac{k}{2^m}T$. Then for $t_k^m\leq t\leq t_{k+1}^m$, we have

$$\left|X_t^{D^m}-X_t\right|\leq\left|X_{t_{k+1}^m}-X_t\right|\left(\frac{t-t_k^m}{t_{k+1}^m-t_k^m}\right)+\left|X_{t_k^m}-X_t\right|\left(\frac{t_{k+1}^m-t}{t_{k+1}^m-t_k^m}\right)$$

Thus by Lemma 65 and its notation,

$$\begin{aligned}
\left|X_t^{D^m} - X_t\right|_{L^2} &\leq \left|X_{t_{k+1}^m} - X_t\right|_{L^2}\left(\frac{t - t_k^m}{t_{k+1}^m - t_k^m}\right) + \left|X_{t_k^m} - X_t\right|_{L^2}\left(\frac{t_{k+1}^m - t}{t_{k+1}^m - t_k^m}\right) \\
&\leq \left(\frac{t - t_k^m}{t_{k+1}^m - t_k^m}\right)\left(\int_{\left[t, t_{k+1}^m\right]^2} |f(u,v)| \,\mathrm{d}u\mathrm{d}v\right)^{\frac{1}{2}} \\
&\quad + \left(\frac{t_{k+1}^m - t}{t_{k+1}^m - t_k^m}\right)\left(\int_{\left[t_k^m, t\right]^2} |f(u,v)| \,\mathrm{d}u\mathrm{d}v\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_{\left[t_k^m, t_{k+1}^m\right]^2} |f(u,v)| \,\mathrm{d}u\mathrm{d}v\right)^{\frac{1}{2}}
\end{aligned}$$

Note

$$\begin{aligned}
\left(\sup_{0\leq t\leq 1}\left|X_t^{D^m} - X_t\right|_{L^2}\right)^2 &\leq 4\max_{1\leq k\leq 2^m}\int_{\left[t_k^m, t_{k+1}^m\right]^2} |f(u,v)| \,\mathrm{d}u\mathrm{d}v \\
&\leq 4\int_{\cup_{k=1}^{2^m}\left[t_k^m, t_{k+1}^m\right]^2} |f(u,v)| \,\mathrm{d}u\mathrm{d}v
\end{aligned}$$

The set $\cup_{k=1}^{2^m}\left[t_k^m, t_{k+1}^m\right]^2$ converge to a set with Lebesgue measure zero as $m \to \infty$, and f is integrable on $\mathbb{R}^2$. Thus

$$\left(\sup_{0\leq t\leq 1}\left|X_t^{D^m} - X_t\right|_{L^2}\right)^2 \to 0$$

as $m \to \infty$. $\square$

5.3 Computation

In this section we shall calculate the expected signatures of Gaussian processes with regular kernels.

A key tool in our calculation is the following Wick's formula (also known as Isserlis' theorem):

Theorem 72. *[Isl18]Let $(W_1, \ldots, W_n)$ be a Gaussian vector. Then*

$$\mathbb{E}\left(\Pi_{i=1}^{n} W_i\right) = \begin{cases} 0 & \textit{if } n \textit{ is odd.} \\ \sum_{\pi \in \Pi_n} \Pi_{(i,j)\in\pi} \mathbb{E}\left(W_i W_j\right) & \textit{if } n = 2k \textit{ is even.} \end{cases}$$

We will now carry out some preliminary calculations. Recall that f is the function defined in Lemma 65.

Let t_k^m denote the dyadic partition point $\frac{k}{2^m}T$.

Define $c_{i,j}$, with $1 \leq i \leq 2^m, 1 \leq j \leq 2^m$, by

$$c_{i,j} = \int_{\mathbb{R}^2} f\left(u, v\right) 1_{\left[t_{i-1}^m, t_i^m\right] \times \left[t_{j-1}^m, t_j^m\right]} \mathrm{d}u\mathrm{d}v$$

Recall that given a continuous path $W : [0, T] \to \mathbb{R}^d$, we define the dyadic approximation of W by the function

$$W\left(m\right)_t = \begin{cases} W_0 & t = 0 \\ \sum_i \left(W_{t_k^m}^i + \frac{2^m}{T}\left(W_{t_k^m}^i - W_{t_{k-1}^m}^i\right)\left(t - t_{k-1}^m\right)\right) e_i, & t \in \left(t_{k-1}^m, t_k^m\right] \end{cases}$$

Let W be a d-dimensional Gaussian process with a strictly regular kernel, and let $W(m)$ be the random process obtained from the dyadic linear interpolation of its sample paths.

Let $(k_1, \ldots, k_{2N})$ be a finite sequence of natural numbers, satisfying $1 \leq k_1 \leq k_2 \leq .. \leq k_{2N} \leq 2^m$. Then the symbol $|\#k = 1|$ will denote the number of elements in the set $\{j : k_j = 1\}$, and in general, let $|\#k = i|$ be the number of elements in the set $\{j : k_j = i\}$.

The following lemma summarises our preliminary calculation:

Lemma 73. *If $k = 2n$ for some n and $i_1, .., i_{2n} \in E_{2n}$, then the projection onto the basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$ of $\mathbb{E}\left(S(W(m))_{0,T}\right)$ is*

$$\sum_{\pi \in \Pi_{i_1,\ldots,i_{2n}}} \sum_{1 \leq k_1 \leq k_2 \leq \ldots \leq k_{2n} \leq 2^m} \frac{1}{|\#k = 1|! \ldots |\#k = 2^m|!} \cdot \Pi_{(j,l)\in\pi} C_{k_l,k_j}$$

otherwise the projection onto $e_{i_1} \otimes \cdots \otimes e_{i_k}$ of $\mathbb{E}\left(S(W(m))_{0,T}\right)$ is zero.

Remark 74. In fact we can prove by induction that

$$\frac{1}{|\#k = 1|! \ldots |\#k = 2^m|!} = 2^{2Nm} \int_{\left[t^m_{k_1-1}, t^m_{k_1}\right] \times \ldots \times \left[t^m_{k_{2N}-1}, t^m_{k_{2N}}\right] \cap \triangle_{2N}} 1 \mathrm{d}u_1 \ldots \mathrm{d}u_{2N}$$

but we shall not need it here.

Proof. Define $X_k : \left[t^m_{k-1}, t^m_k\right] \to \mathbb{R}^d$ by

$$X_k(t) = \sum_i \left(W^i_{t^m_k} + \frac{2^m}{T} \left(W^i_{t^m_k} - W^i_{t^m_{k-1}} \right) \left(t - t^m_{k-1} \right) \right) e_i$$

Then

$$\begin{aligned}
S\left(X_k\right) &= 1+\sum_{n=1}^{\infty}\int_{t_{k-1}^m<s_1<..<s_n<t_k^m} dX_{s_1}\otimes\ldots\otimes dX_{s_n} \\
&= 1+\sum_{n=1}^{\infty}\int_{t_{k-1}^m<s_1<..<s_n<t_k^m}\left[\sum_i \frac{2^m}{T}\left(W_{t_k^m}^i-W_{t_{k-1}^m}^i\right)e_i\right]^{\otimes n} ds_1\ldots ds_n \\
&= 1+\sum_{n=1}^{\infty}\left[\sum_i \frac{2^m}{T}\left(W_{t_k^m}^i-W_{t_{k-1}^m}^i\right)e_i\right]^{\otimes n}\int_{t_{k-1}^m<s_1<..<s_n<t_k^m} ds_1\ldots ds_n \\
&= 1+\sum_{n=1}^{\infty}\left[\sum_i \frac{2^m}{T}\left(W_{t_k^m}^i-W_{t_{k-1}^m}^i\right)e_i\right]^{\otimes n}\frac{1}{n!}\left(t_k^m-t_{k-1}^m\right)^n \\
&= e^{\sum_i\left(W_{t_k^m}^i-W_{t_{k-1}^m}^i\right)e_i}
\end{aligned}$$

By Chen's identity,

$$S\left(W\left(m\right)_t\right) = \mathbb{E}\left[e^{\sum_i\left(W_{t_1^m}^i-W_{t_0^m}^i\right)e_i}\otimes\ldots\otimes e^{\sum_i\left(W_{t_{2^m}^m}^i-W_{t_{2^m-1}^m}^i\right)e_i}\right] \tag{5.3.1}$$

We have

$$e^{\sum_i\left(W_k^i-W_{k-1}^i\right)e_i}=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_i\left(W_{t_k^m}^i-W_{t_{k-1}^m}^i\right)e_i\right)^{\otimes j}$$

The coefficient of $e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_n}$ in the expansion of

$$\otimes_{k=1}^{2^m}e^{\sum_i\left(W_k^i-W_{k-1}^i\right)e_i}=\otimes_{k=1}^{2^m}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_i\left(W_{t_k^m}^i-W_{t_{k-1}^m}^i\right)e_i\right)^j\right]$$

is

$$\sum_{1\leq k_1\leq k_2\leq\ldots\leq k_n\leq 2^m}\frac{\Pi_{j=1}^n\left(W_{t_{k_j}^m}^{i_j}-W_{t_{k_j-1}^m}^{i_j}\right)}{|\#k=1|!\ldots|\#k=2^m|!}$$

If n were odd, then as the expected value of the product of an odd number of Gaussian random variables is zero, we have the coefficient of $e_{i_1}\otimes\ldots\otimes e_{i_n}$ in $\mathbb{E}\left(S\left(W\left(m\right)\right)_{0,t}\right)$ being zero.

If $(i_1, .., i_{2n}) \notin E_{2n}$, then since the process $(W_t : t \geq 0)$ has independent components and that the expected value of the product of an odd number of Gaussian random variables is zero, thus

$$\mathbb{E}\left(\Pi_{j=1}^{n}\left(W^{i_j}_{t^m_{k_j}} - W^{i_j}_{t^m_{k_j-1}}\right)\right) = 0,$$

which in turn implies that

$$\pi^{i_1,..,i_{2n}}\mathbb{E}\left[S\left(W(m)_{0,1}\right)\right] = 0 \ ,$$

when $(i_1, .., i_{2n}) \notin E_{2n}$.

We now calculate the projection to $e_{i_1} \otimes \cdots \otimes e_{i_{2n}}$ of $\mathbb{E}\left(S\left(W(m)\right)_{0,t}\right)$, which by Wick's formula equals

$$\begin{aligned} & \pi_{i_1,\ldots,i_{2n}}\left[\mathbb{E}\left(S\left(W(m)\right)_{0,t}\right)\right] \\ = \quad & \textstyle\sum_{\pi\in\Pi_{i_1,\ldots,i_{2n}}}\sum_{1\leq k_1\leq\ldots\leq k_{2n}\leq 2^m}\frac{1}{|\#k=1|!\ldots|\#k=2^m|!}\Pi_{(l,j)\in\pi} \\ & \mathbb{E}\left[\left(W^{i_j}_{t^m_{k_j}} - W^{i_j}_{t^m_{k_j-1}}\right)\left(W^{i_j}_{t^m_{k_l}} - W^{i_j}_{t^m_{k_l-1}}\right)\right] \\ = \quad & \textstyle\sum_{\pi\in\Pi_{i_1,\ldots,i_{2n}}}\sum_{1\leq k_1\leq\ldots\leq k_{2n}\leq 2^m}\frac{1}{|\#k=1|!\ldots|\#k=2^m|!}\cdot\Pi_{(l,j)\in\pi}c_{k_l,k_j} \end{aligned} \tag{5.3.2}$$

where $\sum_{\pi\in\Pi_{i_1,\ldots,i_{2n}}}$ is the sum over all possible pairings (l,j), $j > l$ from the set $\Pi_{i_1,\ldots,i_{2n}}$. $\square$

The following lemma is crucial to our calculation of the sum in Lemma 73.

Lemma 75. For $i \geq 1$ and any pairing π of $\{1,..,2n\}$,

$$\sum_{1\leq k_1\leq..\leq k_i=k_{i+1}\leq k_{i+2}...\leq k_{2n}\leq 2^m} \Pi_{(l,j)\in\pi}\left|c_{k_l,k_j}\right| \to 0 \text{ as } m\to\infty$$

Proof. If $(i,i+1)\in\pi$, then

$$\begin{aligned} & \textstyle\sum_{1\leq k_1\leq...\leq k_i=k_{i+1}\leq k_{i+2}...\leq k_{2n}\leq 2^m} \Pi_{(j,l)\in\pi}\left|c_{k_l,k_j}\right| \\ \leq\ & \textstyle\sum_{k=1}^{2^m}\left|c_{k,k}\right| \times \Pi_{(l,j)\in\pi\backslash(i,i+1)} \sum_{k_j=1}^{2^m}\sum_{k_l=1}^{k_j}\left|c_{k_l,k_j}\right| \end{aligned} \tag{5.3.3}$$

Note that for $(j,l)\neq(i,i+1)$,

$$\begin{aligned} & \textstyle\sum_{k_j=1}^{2^m}\sum_{k_l=1}^{k_j}\left|c_{k_lk_j}\right| \\ =\ & \textstyle\sum_{k_j=1}^{2^m}\sum_{k_l=1}^{k_j}\int_{\mathbb{R}^2}\left|f\left(u,v\right)\right| 1_{\left[t^m_{k_l-1},t^m_{k_l}\right]\times\left[t^m_{k_j-1},t^m_{k_j}\right]}\mathrm{d}u\mathrm{d}v \\ \leq\ & \textstyle\int_{[0,T]\times[0,T]}\left|f\left(u,v\right)\right|\mathrm{d}u\mathrm{d}v \end{aligned} \tag{5.3.4}$$

and that

$$\begin{aligned} \sum_{k=1}^{2^m}\left|c_{k,k}\right| &\leq \sum_{k=1}^{2^m}\int_{\mathbb{R}^2}\left|f\left(u,v\right)\right| 1_{\left[t^m_k,t^m_{k-1}\right]\times\left[t^m_k,t^m_{k-1}\right]}\mathrm{d}u\mathrm{d}v \\ &= \int_{\mathbb{R}^2}\left|f\left(u,v\right)\right| 1_{\cup_{k=1}^{2^m}\left[t^m_k,t^m_{k-1}\right]\times\left[t^m_k,t^m_{k-1}\right]}\left(u,v\right)\mathrm{d}u\mathrm{d}v. \end{aligned}$$

Since $f\left(\cdot,\cdot\right)\in L^1\left([0,T]^2\right)$ and the set $\cup_{k=1}^{2^m}\left[t^m_k,t^m_{k-1}\right]\times\left[t^m_k,t^m_{k-1}\right]$ converges to a null set in $\mathbb{R}^2$ as $m\to\infty$, we have

$$\sum_{k=1}^{2^m}\left|c_{k,k}\right| \to 0$$

as $m \to \infty$.

Thus by (5.3.3) we have

$$\begin{aligned} & \textstyle\sum_{1 \le k_1 \le \ldots \le k_i = k_{i+1} \le k_{i+2} \cdots \le k_{2n} \le 2^m} \prod_{(j,l) \in \pi} \left| c_{k_l k_j} \right| \\ \le \quad & \left[\textstyle\int_{[0,T] \times [0,T]} \left| f(u,v) \right| \mathrm{d}u \mathrm{d}v \right]^{n-1} \textstyle\int_{\cup_{k=1}^{2m} \left[t_k^m, t_{k-1}^m \right] \times \left[t_k^m, t_{k-1}^m \right]} \left| f(u,v) \right| \mathrm{d}u \mathrm{d}v \\ & \to 0 \ \text{ as } \quad m \to \infty. \end{aligned}$$

Now if $(i, i+1) \notin \pi$, then let $\pi(i), \pi(i+1)$ denote the unique integers satisfying $(i, \pi(i)), (i+1, \pi(i+1)) \in \pi$.

Assume now that $(i, i+1) \notin \pi$, then we have

$$\begin{aligned} & \textstyle\sum_{1 \le k_1 \le \ldots \le k_i = k_{i+1} \le k_{i+2} \ldots \le k_{2n} \le 2^m} \prod_{(l,j) \in \pi} \left| c_{k_l k_j} \right| \\ \le \quad & \textstyle\prod_{(l,j) \in \pi \setminus \{(i,\pi(i)),(i+1,\pi(i+1))\}} \sum_{k_j=1}^{2^m} \sum_{k_l=1}^{k_j} \left| c_{k_l k_j} \right| \\ & \times \textstyle\sum_{k=1}^{2^m} \sum_{k_{\pi(i)}=1}^{2^m} \sum_{k_{\pi(i+1)}=1}^{2^m} \left| c_{k,k_{\pi(i)}} \right| \left| c_{k,k_{\pi(i+1)}} \right|. \end{aligned} \tag{5.3.5}$$

Let $F(v) := \int_0^T |f(u,v)| \mathrm{d}u$. Then as $f(\cdot,\cdot) \in L^1[0,T]^2$, we have $F(\cdot) \in L^1[0,T]$. Thus

$$\sum_{k_{\pi(i)}=1}^{2^m} \int_{\mathbb{R}^2} |f(u,v)| \, 1_{\left[t_{k_{\pi(i)}}^m, t_{k_{\pi(i)}-1}^m \right] \times \left[t_k^m, t_{k-1}^m \right]} \mathrm{d}u \mathrm{d}v = \int_0^T F(v) \, 1_{\left[t_k^m, t_{k-1}^m \right]}(v) \, \mathrm{d}v.$$

Note that since $F \ge 0$, and that $F(\cdot) \in L^1$, the integral

$$\int_0^T \int_0^T F(u) F(v) \, \mathrm{d}u \mathrm{d}v = \int_0^T F(u) \, \mathrm{d}u \int_0^T F(v) \, \mathrm{d}v$$

exist and thus $(u,v) \to F(u) F(v)$ is integrable on $[0,T]^2$.

Hence

$$\begin{aligned}
& \sum_{k=1}^{2^m} \sum_{k_{\pi(i)}=1}^{2^m} \sum_{k_{\pi(i+1)}=1}^{2^m} \left|c_{k,k_{\pi(i)}}\right| \left|c_{k,\pi(i+1)}\right| \\
\leq \; & \sum_{k=1}^{2^m} \left\{ \left[\sum_{k_{\pi(i)}=1}^{2^m} \int_{\mathbb{R}^2} |f(u,v)| \, 1_{\left[t^m_{k_{\pi(i)}}, t^m_{k_{\pi(i)}-1}\right] \times \left[t^m_k, t^m_{k-1}\right]} \mathrm{d}u \mathrm{d}v \right] \right. \\
& \left. \times \left[\sum_{k_{\pi(i+1)}=1}^{2^m} \int_{\mathbb{R}^2} |f(u,v)| \, 1_{\left[t^m_{k_{\pi(i+1)}}, t^m_{k_{\pi(i+1)}-1}\right] \times \left[t^m_k, t^m_{k-1}\right]} \mathrm{d}u \mathrm{d}v \right] \right\} \\
\leq \; & \sum_{k=1}^{2^m} \int_{\mathbb{R}^2} F(v) \, F(u) \, 1_{\left[t^m_k, t^m_{k-1}\right] \times \left[t^m_k, t^m_{k-1}\right]} (u,v) \, \mathrm{d}u \mathrm{d}v.
\end{aligned}$$

As $(u,v) \to F(u)\,F(v)$ is integrable on $L^1[0,T]^2$, and the set $\cup_{k=1}^{2^m} \left[t^m_k, t^m_{k-1}\right] \times \left[t^m_k, t^m_{k-1}\right]$ converges to a null set in $\mathbb{R}^2$, so that

$$\sum_{k=1}^{2^m} \int_{\mathbb{R}^2} F(v) \, F(u) \, 1_{\left[t^m_k, t^m_{k-1}\right] \times \left[t^m_k, t^m_{k-1}\right]} (u,v) \, \mathrm{d}u \mathrm{d}v \to 0,$$

as $m \to \infty$.

Together with (5.3.4) and (5.3.5) we have

$$\begin{aligned}
& \sum_{1 \leq k_1 \leq \ldots \leq k_i = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^m} \Pi_{(l,j)\in\pi} \left|c_{k_l k_j}\right| \\
\leq \; & \left(\sum_{k=1}^{2^m} \int_{\mathbb{R}^2} F(v) \, F(u) \, 1_{\left[t^m_k, t^m_{k-1}\right] \times \left[t^m_k, t^m_{k-1}\right]} (u,v) \, \mathrm{d}u \mathrm{d}v \right) \\
& \times \left(\int_{[0,T]\times[0,T]} |f(u,v)| \, \mathrm{d}u \mathrm{d}v \right)^{n-2} \\
\to \; & 0,
\end{aligned}$$

as $m \to \infty$.

This completes the proof of the lemma. $\square$

We now prove our main result Theorem 64.

Proof. (of Theorem 64) By Proposition 71, the expected signature is given by the limit as $m \to \infty$ of the sum in Lemma 73.

However, by Lemma 75, for any $i \geq 1$, as $m \to \infty$,

$$\sum_{1 \leq k_1 \leq \ldots \leq k_i = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^m} \frac{1}{|\#k = 1|! \ldots |\#k = 2^m|!} \cdot \Pi_{(l,j) \in \pi} \left| c_{k_l, k_j} \right| \to 0$$

Thus

$$\begin{aligned} & \lim_{m \to \infty} \sum_{1 \leq k_1 \leq \ldots \leq k_{2n} \leq 2^m} \frac{1}{|\#k=1|! \ldots |\#k=2^m|!} \cdot \Pi_{(j,l) \in \pi} c_{k_l k_j} \\ = & \lim_{m \to \infty} \sum_{1 < k_1 < \ldots < k_{2n} \leq 2^m} \frac{1}{|\#k=1|! \ldots |\#k=2^m|!} \Pi_{(j,l) \in \pi} c_{k_l k_j} \\ = & \lim_{m \to \infty} \sum_{1 < k_1 < \ldots < k_{2n} \leq 2^m} \Pi_{(j,l) \in \pi} c_{k_l k_j} \end{aligned}$$

where the last equality uses the fact that for $k_1 < k_2 < \ldots < k_{2N}$, we have $|\#k = 1|! \ldots |\#k = 2^m|! = 1$.

Note that

$$\begin{aligned} & \Pi_{(j,l) \in \pi} c_{k_l k_j} \\ = & \Pi_{(l,j) \in \pi} \int_{\mathbb{R}^2} f(u,v) \, 1_{\left[t^m_{k_l}, t^m_{k_l - 1}\right] \times \left[t^m_{k_j}, t^m_{k_j - 1}\right]} \mathrm{d}u \mathrm{d}v \\ = & \int_{\mathbb{R}^2} \Pi_{(j,l) \in \pi} f(u_j, u_l) \, 1_{\left[t^m_{k_1 - 1}, t^m_{k_1}\right] \times \ldots \times \left[t^m_{k_{2n} - 1}, t^m_{k_{2n}}\right]} (u_1, \ldots, u_{2n}) \, du_1 \ldots du_{2n}. \end{aligned}$$

Thus

$$\begin{aligned} & \lim_{m\to\infty} \sum_{1<k_1<k_2<\ldots<k_{2n}\leq 2^m} \Pi_{(j,l)\in\pi} c_{k_l k_j} \\ = & \lim_{m\to\infty} \sum_{1<k_1<\ldots<k_{2n}\leq 2^m} \int_{\left[t^m_{k_1-1},t^m_{k_1}\right]\times\ldots\times\left[t^m_{k_{2n}-1},t^m_{k_{2n}}\right]} \Pi_{(j,l)\in\pi} f\left(u_j,u_l\right) du_1 \ldots du_{2n}. \end{aligned}$$

Let $\triangle_{2n}(0,T)$ denote the simplex $\{(u_1,..,u_{2n})\in\mathbb{R}^{2n} : 0\leq u_1<\ldots<u_{2n}\leq T\}$. Then,

$$\begin{aligned} & \int_{\left[t^m_{k_1-1},t^m_{k_1}\right]\times\ldots\times\left[t^m_{k_{2n}-1},t^m_{k_{2n}}\right]\cap\triangle_{2N}(0,T)} \Pi_{(l,j)\in\pi} \left|f\left(u_j,u_l\right)\right| du_1 \ldots du_{2n} \\ \leq & \int_{\left[t^m_{k_1-1},t^m_{k_1}\right]\times\ldots\times\left[t^m_{k_{2n}-1},t^m_{k_{2n}}\right]} \Pi_{(l,j)\in\pi} \left|f\left(u_j,u_l\right)\right| du_1 \ldots du_{2n} \\ \leq & \Pi_{(j,l)\in\pi} \left|c_{k_l k_j}\right|. \end{aligned}$$

However, by Lemma 75,

$$\sum_{1\leq k_1\leq..\leq k_i=k_{i+1}\leq k_{i+2}\ldots\leq k_{2n}\leq 2^m} \Pi_{(j,l)\in\pi} \left|c_{k_l k_j}\right| \to 0 \text{ as } m\to\infty.$$

Thus for any $1\leq i$, as $m\to\infty$,

$$\begin{aligned} & \sum_{1\leq k_1\leq..\leq k_i=k_{i+1}\leq k_{i+2}\ldots\leq k_{2n}\leq 2^m} \int_{\left[t^m_{k_1-1},t^m_{k_1}\right]\times\ldots\times\left[t^m_{k_{2n}-1},t^m_{k_{2n}}\right]\cap\triangle_{2N}(0,T)} \\ & \Pi_{(l,j)\in\pi} \left|f\left(u_j,u_l\right)\right| du_1 \ldots du_{2n} \end{aligned}$$

converges to zero.

Hence,

$$\begin{aligned}
&\lim_{m\to\infty}\sum_{1<k_1<\ldots<k_{2n}\le 2^m}\int_{\left[t^m_{k_1},t^m_{k_1-1}\right]\times\ldots\times\left[t^m_{k_{2n}},t^m_{k_{2n}-1}\right]}\Pi_{(j,l)\in\pi}f\left(u_j,u_l\right)du_1\ldots du_{2n}\\
=&\lim_{m\to\infty}\sum_{1\le k_1\le\ldots\le k_{2n}\le 2^m}\int_{\left[t^m_{k_1},t^m_{k_1-1}\right]\times\ldots\times\left[t^m_{k_{2n}},t^m_{k_{2n}-1}\right]\cap\triangle_{2N}(0,T)}\Pi_{(j,l)\in\pi}f\left(u_j,u_l\right)du_1\ldots du_{2n}\\
=&\int_{\triangle_{2n}}\Pi_{(l,j)\in\pi}f\left(u_j,u_l\right)du_1\ldots du_{2n}\ .
\end{aligned}$$

Finally by Lemma 73 and Lemma 71,

$$\begin{aligned}
&\pi_{i_1,..,i_{2n}}\left(\mathbb{E}\left(S\left(W\right)\right)_{0,T}\right)\\
=&\lim_{m\to\infty}\pi_{i_1,..,i_{2n}}\left(\mathbb{E}\left(S\left(W\left(m\right)\right)\right)_{0,T}\right)\\
=&\sum_{\pi\in\Pi_{i_1,\ldots,i_{2n}}}\int_{\triangle_{2N}}\Pi_{(l,j)\in\pi}f\left(u_j,u_l\right)du_1\ldots du_{2n}\ .
\end{aligned}$$

□

5.4 The fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$

We will now show that the formula we give coincide with the following formula for the expected signature of fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$ calculated in [BC07].

Proposition 76. *(See [BC07], Theorem 31)Let $H>\frac{1}{2}$. If $k=2n$ for some $n\in\mathbb{N}$, and $(i_1,\ldots,i_{2n})\in E_{2n}$, then the projection to the basis $e_{i_1}\otimes e_{i_2}\ldots\otimes e_{i_k}$ of the expected* *signature of fractional Brownian Motion with Hurst parameter H up to time T is*

$$\sum_{\pi\in\Pi_{i_1,\ldots,i_{2n}}} \left(H\left(2H-1\right)T^{H}\right)^{n}\int_{\triangle_{2n}(1)}\Pi_{(l,j)\in\pi}\left(u_j-u_l\right)^{2H-2}du_1...du_{2n}$$

where $\triangle_{2n}(1)$ denotes the simplex $\{(u_1,\ldots,u_{2n})\in\mathbb{R}^{2n}:0\le u_1<\ldots<u_{2n}\le 1\}$. The projection to the basis $e_{i_1}\otimes e_{i_2}\ldots\otimes e_{i_k}$ is zero otherwise.

Proof. (of Proposition 76)For $H>\frac{1}{2}$, let $K_H(t,s)$ be defined, for $t>s$, as

$$K_H(t,s)=c_H s^{\frac{1}{2}-H}\int_s^t |u-s|^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u$$

where $c_H=\left[\frac{H(2H-1)}{\beta\left(2-2H,H-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$ and β denotes the beta function.

Then by [NVV99], the fractional Brownian motion $B^H_\cdot$ can be represented as

$$B^H_t=\int_0^t K_H(t,s)\,\mathrm{d}B_s$$

where $\mathrm{d}B_s$ denotes integration in the sense of Itô.

Note that $K(s^+,s)=0$ for all s, $K(\cdot,s)$ is differentiable and has a positive, integrable derivative, and thus K_H satisfies (K1) and (K2).

Now note

$$\begin{aligned}K_H(t,s) &\le c_H s^{\frac{1}{2}-H}t^{H-\frac{1}{2}}\int_s^t|u-s|^{H-\frac{3}{2}}\mathrm{d}u\\ &\le \frac{c_H}{H-\frac{1}{2}}s^{\frac{1}{2}-H}t^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}\\ &\le \frac{c_H}{H-\frac{1}{2}}s^{\frac{1}{2}-H}t^{2H-1}\end{aligned}$$

and thus

$$K_H(t,s)^2 \leq \left[\frac{c_H}{H-\frac{1}{2}}\right]^2 s^{1-2H}t^{4H-2}$$

and the right hand side is integrable in s.

As $K(\cdot, s)$ is increasing and $K(s^+, s) = 0$, $|K|((s,t], s) = K(t,s) - K(s^+, s) = K(t,s)$ and (K3) is satisfied.

Therefore, Theorem 64 applies and Proposition 76 follows from a change of variable. $\square$

5.5 Right continuity at $H = \frac{1}{2}$

We shall prove that the formula for the expected signature in Propositon 76 reconciles with the expected signature of Brownian motion when we take limit as $H \to \frac{1}{2}$. By the self-similarity property of fractional Brownian motions, it is sufficient to establish this continuity in the case $T = 1$. First we recall the expected signature of Brownian motion up to time 1:

Proposition 77. *([Faw03],[LV04])The n^{th} level term of the expected signature of Brownian motion up to time* 1 *is*

$$\frac{1}{2^n n!}\left(\sum_{i=1}^{d} e_i \otimes e_i\right)^n$$

Equivalently, the projection to any basis of the form $e_{i_1} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{i_n}$ is equal to $\frac{1}{2^n n!}$.

Note that a term $e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_{2n}}$ would appear in the expansion of $\left(\sum_{i=1}^{d} e_i \otimes e_i\right)^n$

if and only if the pairing $\pi_n := \{(1,2),(3,4),\ldots,(2n-1,2n)\}$ is in $\Pi_{i_1,\ldots,i_{2n}}$. Thus to prove the continuity of the expected signature as $H \to \frac{1}{2}$, it suffices to prove

1. Let π_n denote the pairing $\{(1,2),(3,4),\ldots,(2n-1,2n)\}$. Then the integral

$$I_n := (H(2H-1))^n \int_{\triangle_{2n}} \Pi_{(l,j)\in\pi_n} (u_j - u_l)^{2H-2} du_1 \ldots du_{2n}$$

   converges to $\frac{1}{2^n n!}$ as $H \to \frac{1}{2}$.

2. The integral

$$(H(2H-1))^n \int_{\triangle_{2n}} \Pi_{(l,j)\in\pi} (u_j - u_l)^{2H-2} du_1 \ldots . du_{2n},$$

   converges to 0 as $H \to \frac{1}{2}$ for any $\pi \in E_{2n} \backslash \pi_n$.

We will calculate first the integral I_n, which we may write explicitly as

$$(H(2H-1))^n \int_{\triangle_{2n}} \Pi_{j=1}^n (u_{2j} - u_{2j-1})^{2H-2} du_1 \ldots du_{2n}$$

By Fubini's theorem, and using the notation $u_{2n+1} = 1$, we can "integrate u_{2n}s first" to obtain:

$$\begin{aligned}
& (H(2H-1))^n \int_{\triangle_{2n}} \Pi_{j=1}^n (u_{2j} - u_{2j-1})^{2H-2} du_1 \ldots du_{2n} \\
= & (H(2H-1))^n \int_0^1 \int_0^{u_{2n}} \ldots \int_0^{u_2} \Pi_{j=1}^n (u_{2j} - u_{2j-1})^{2H-2} du_1 \ldots du_{2n} \\
= & (H(2H-1))^n \int_0^1 \int_0^{u_{2n-3}} \ldots \int_0^{u_3} \Pi_{j=1}^n \left[\int_{u_{2j-1}}^{u_{2j+1}} (u_{2j} - u_{2j-1})^{2H-2} du_{2j} \right] du_1 du_3 \ldots du_{2n-1} \\
= & H^n \int_0^1 \int_0^{u_{2n-3}} \ldots \int_0^{u_3} \Pi_{j=1}^n (u_{2j+1} - u_{2j-1})^{2H-1} du_1 du_3 du_5 \ldots du_{2n-1}.
\end{aligned}$$

Taking limit as $H \to \frac{1}{2}$ and using the bounded convergence theorem, we have

$$\begin{aligned}
& \lim_{H\to\frac{1}{2}} H^n \int_0^1 \int_0^{u_{2n-3}} \dots \int_0^{u_3} \Pi_{j=1}^n \left(u_{2j+1}-u_{2j-1}\right)^{2H-1} du_1 du_3 du_5 \dots du_{2n-1} \\
= & \left(\tfrac{1}{2}\right)^n \int_0^1 \int_0^{u_{2n-3}} \dots \int_0^{u_3} du_1 du_3 du_5 \dots du_{2n-1} \\
= & \tfrac{1}{2^n n!}.
\end{aligned}$$

Now we consider other pairings π in $\Pi_{i_1,\dots,i_{2n}}$, that is when there exists a k,i such that $k-i>1$ but$(i,k)\in\pi$. We have

$$\begin{aligned}
& \left(H\left(2H-1\right)\right)^n \int_{\triangle_{2n}} \Pi_{(l,j)\in\pi} \left(u_j-u_l\right)^{2H-2} du_1 \dots du_{2n} \\
\leq & \left(H\left(2H-1\right)\right)^{n-1} \int_{\triangle_{2n-2}} \Pi_{(l,j)\in\pi\backslash(i,k)} \left[\left(u_j-u_l\right)^{2H-2} du_l du_j\right] \\
& \times \left(H\left(2H-1\right)\right) \int \left(u_k-u_i\right)^{2H-2} 1_{[u_{i-1},u_{i+1}]\times[u_{k-1},u_{k+1}]} \left(u_i,u_k\right) du_i du_k.
\end{aligned} \tag{5.5.1}$$

Note that as $k-1\geq i+1$,

$$\begin{aligned}
& H\left(2H-1\right) \int_{\mathbb{R}^2} \left(u_k-u_i\right)^{2H-2} 1_{[u_{i-1},u_{i+1}]\times[u_{k-1},u_{k+1}]} \left(u_i,u_k\right) du_i du_k \\
\leq\ & H\left(2H-1\right) \int_{\mathbb{R}^2} \left(u_k-u_i\right)^{2H-2} 1_{[0,u_{k-1}]\times[u_{k-1},1]} \left(u_i,u_k\right) du_i du_k \\
=\ & \frac{1}{2}\left[1-u_{k-1}^{2H}-\left(1-u_{k-1}\right)^{2H}\right] \\
\leq\ & \frac{1}{2}\left(\frac{1}{2}-\left(\frac{1}{2}\right)^{2H}\right),
\end{aligned}$$

where the final inequality holds because $0\leq 1-x^{2H}-\left(1-x\right)^{2H}\leq 1-\left(\frac{1}{2}\right)^{2H-1}$ for $H>\frac{1}{2}$.

Thus by (5.5.1),

$$
\begin{aligned}
& \left(H\left(2H-1\right)\right)^{n}\int_{\triangle_{2n}}\Pi_{(l,j)\in\pi}\left(u_{j}-u_{l}\right)^{2H-2}du_{1}\ldots du_{2n}\\
\leq\ & \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{2H-1}\right)\left(H\left(2H-1\right)\right)^{n-1}\int_{\triangle_{2n-2}}\Pi_{(l,j)\in\pi\setminus(i,k)}\left[\left(u_{j}-u_{l}\right)^{2H-2}du_{l}du_{j}\right].
\end{aligned}
$$

Note first that

$$
\begin{aligned}
& \left(H\left(2H-1\right)\right)^{n-1}\int_{\triangle_{2n-2}}\Pi_{(l,j)\in\pi\setminus(i,k)}\left[\left(u_{j}-u_{l}\right)^{2H-2}du_{l}du_{j}\right]\\
\leq\ & \left(H\left(2H-1\right)\right)^{n}\int_{[0,1]^{2n}}\Pi_{(l,j)\in\pi}\left|u_{j}-u_{l}\right|^{2H-2}du_{1}\ldots du_{2n}\\
=\ & \left(2H\left(2H-1\right)\right)^{n}\left(\int_{0}^{1}\int_{0}^{1}\left|y-x\right|^{2H-2}dxdy\right)^{n}\\
=\ & \left(2H\left(2H-1\right)\right)^{n}\left(\int_{0}^{1}\int_{0}^{y}\left|y-x\right|^{2H-2}dxdy\right)^{n}\\
=\ & 1,
\end{aligned}
$$

therefore,

$$
\begin{aligned}
& \left(H\left(2H-1\right)\right)^{n}\int_{\triangle_{2n}}\Pi_{(l,j)\in\pi}\left(u_{j}-u_{l}\right)^{2H-2}du_{1}\ldots du_{2n}\\
\leq\ & \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{2H-1}\right)\\
\to\ & 0
\end{aligned}
$$

as $H\to\frac{1}{2}$.

Appendix

Proof of Lemma 19

We now give a proof to Lemma 19 in Chapter 1 which appeared in [LQ02](Proposition 4.4.1 and 4.5.1 and equation (4.24)). The purpose of this Appendix is to make this thesis self-contained and to give a slightly simplified calculation to Lemma 19. Given the assumption on our Gaussian processes, the analytical estimates are necessarily technical, but the algebraic calculation can be greatly simplified using the following explicit version of the Baker-Hausdorff-Campbell formula:

Proposition 78. *([SS99])The first four terms of the Hausdorff series*

$$\log\left(\exp a \otimes \exp b\right)$$

is

$$a + b + \frac{1}{2}[a,b] + \frac{1}{12}\left([a,[a,b]] + [b,[b,a]]\right).$$

The following lemma appears as Lemma 4.2.1 and (4.29) in [LQ02].

Lemma 79. *For $m \geq n$,*

$$\pi^{(2)}\left(w^{(m+1)}_{t^n_{k-1},t^n_k} - w^{(m)}_{t^n_{k-1},t^n_k}\right) = \frac{1}{2}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]$$

$$\begin{aligned}\pi^{(3)}\left(w^{(m+1)}_{t^n_{k-1},t^n_k} - w^{(m)}_{t^n_{k-1},t^n_k}\right) &= \frac{1}{12}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l-1}}, \left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]\right] \\ &+ \left[\triangle w_{t^m_{2l}}, \left[\triangle w_{t^{m+1}_{2l}}, \triangle w_{t^{m+1}_{2l-1}}\right]\right] \\ &+ \frac{1}{4}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\triangle w_{t^m_l}\otimes\left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right] \\ &+ \frac{1}{4}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]\otimes\triangle w_{t^m_l} \\ &+ \frac{1}{2}\sum_{l_1,l_2=2^{m-n}(k-1)+1,l_1<l_2}^{2^{m-n}k}\triangle w_{t^m_{l_1}}\otimes\left[\triangle w_{t^{m+1}_{2l_2-1}}, \triangle w_{t^{m+1}_{2l_2}}\right] \\ &+ \frac{1}{2}\sum_{l_1,l_2=2^{m-n}(k-1)+1,l_1<l_2}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l_1-1}}, \triangle w_{t^{m+1}_{2l_1}}\right]\otimes\triangle w_{t^m_{l_2}}.\end{aligned}$$

Proof. Let $t^n_k := \frac{k}{2^n}$. for each continuous function w, let $\triangle w_{t^n_k}$ denote $w_{t^n_k} - w_{t^n_{k-1}}$. Let $m \geq n$, from the calculation in Lemma 73, we have

$$\begin{aligned}w^{(m+1)}_{t^n_{k-1},t^n_k} &= \exp\triangle w_{t^{m+1}_{2^{m+1-n}(k-1)+1}}\otimes\ldots\otimes\exp\triangle w_{t^{m+1}_{2^{m+1-n}k}} \\ &= \otimes^{2^{m+1-n}k}_{k=2^{m+1-n}(k-1)+1}\exp\triangle w_{t^{m+1}_l} \\ &= \otimes^{2^{m-n}k}_{k=2^{m-n}(k-1)+1}\exp\triangle w_{t^{m+1}_{2l-1}}\otimes\exp\triangle w_{t^{m+1}_{2l}}\end{aligned}$$

where the notation $\otimes^{2^{m+1}}_{l=1}$ is understood to be tensor product multiplication from left to right.

By Proposition 78, we have

$$w^{(m+1)}_{t^n_{k-1},t^n_k} = \otimes^{2^{m-n}k}_{l=2^{m-n}(k-1)+1} \exp\left(\triangle w_{t^m_l} + \frac{1}{2}\left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right] + \frac{1}{12}\left[\triangle w_{t^{m+1}_{2l-1}}, \left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]\right]\right.$$
$$\left. + \frac{1}{12}\left[\triangle w_{t^{m+1}_{2l}}, \left[\triangle w_{t^{m+1}_{2l}}, \triangle w_{t^{m+1}_{2l-1}}\right]\right] + \ldots\right)$$

where $\ldots$ denotes Lie polynomials of degree higher than 3.

By using the series expansion of exp and collecting the terms in $\left(\mathbb{R}^d\right)^{\otimes 2}$ and $\left(\mathbb{R}^d\right)^{\otimes 3}$,

$$\pi^{(2)}\left(w^{(m+1)}_{t^n_{k-1},t^n_k} - w^{(m)}_{t^n_{k-1},t^n_k}\right) = \frac{1}{2}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]$$

$$\begin{aligned}
\pi^{(3)}\left(w^{(m+1)}_{t^n_{k-1},t^n_k} - w^{(m)}_{t^n_{k-1},t^n_k}\right) &= \frac{1}{12}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l-1}}, \left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]\right] \\
&+ \left[\triangle w_{t^m_{2l}}, \left[\triangle w_{t^{m+1}_{2l}}, \triangle w_{t^{m+1}_{2l-1}}\right]\right] \\
&+ \frac{1}{4}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} \triangle w_{t^m_l} \otimes \left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right] \\
&+ \frac{1}{4}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} \left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right] \otimes \triangle w_{t^m_l} \\
&+ \frac{1}{2}\sum_{l_1,l_2=2^{m-n}(k-1)+1,l_1<l_2}^{2^{m-n}k} \triangle w_{t^m_{l_1}} \otimes \left[\triangle w_{t^{m+1}_{2l_2-1}}, \triangle w_{t^{m+1}_{2l_2}}\right] \\
&+ \frac{1}{2}\sum_{l_1,l_2=2^{m-n}(k-1)+1,l_1<l_2}^{2^{m-n}k} \left[\triangle w_{t^{m+1}_{2l_1-1}}, \triangle w_{t^{m+1}_{2l_1}}\right] \otimes \triangle w_{t^m_{l_2}}.
\end{aligned}$$

□

We need one more easy algebraic lemma.

Lemma 80. *Let* $a=\sum_{i=1}^{d}a_i\mathbf{e}_i$ *,* $b=\sum_{j=1}^{d}b_j\mathbf{e}_j$ *and* $c=\sum_{k=1}^{d}c_k\mathbf{e}_k$*. Then*

$$\begin{aligned}
[a,b] &= \sum_{i\neq j}(a_jb_i-a_ib_j)\,\mathbf{e}_i\otimes\mathbf{e}_j \\
[a,[a,b]] &= \sum_{j\neq k}a_i(a_jb_k-b_ja_k)\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k-\sum_{i\neq j}(a_ib_j-b_ia_j)\,a_k\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k \\
a\otimes[b,c] &= \sum a_i(b_jc_k-c_jb_k)\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k.
\end{aligned}$$

We will now proceed with the analytical estimates.

Lemma 81. *Let w be a continuous path. Then for all $m\geq n$,*

$$w^{(m+1),1}_{t^n_{l-1},t^n_l} \quad = \quad w^{(m),1}_{t^n_{l-1},t^n_l}$$

Proof. This follows directly from the definition. □

Lemma 82. *Let $\mathbb{P}$ be a measure satisfying Condition 1 and H be the parameter specified in Condition 1. Assume that $H\leq\frac{1}{2}$. Then for $1\leq k\leq 2^n$ and for all $m\geq n$,*

$$\left\|w^{(m+1),2}_{t^n_{k-1},t^n_k}-w^{(m),2}_{t^n_{k-1},t^n_k}\right\|_{L^2(\mathbb{P})} \quad\leq\quad 2d\max\left(C_{LTM},C_{Hol}\right)\zeta\left(4-4H\right)^{\frac{1}{2}}.2^{-\frac{1}{2}(4H-1)m-\frac{1}{2}n}.$$

Proof. Note that

$$w^{(m+1),2}_{t^n_{k-1},t^n_k}-w^{(m),2}_{t^n_{k-1},t^n_k}=\frac{1}{2}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\triangle w_{t^{m+1}_{2l-1}},\triangle w_{t^{m+1}_{2l}}\right].$$

Thus

$$
\begin{aligned}
&\mathbb{E}\left\|w^{(m+1),2}_{t^n_{k-1},t^n_k}-w^{(m),2}_{t^n_{k-1},t^n_k}\right\|^2_{\left(\mathbb{R}^d\right)^{\otimes 2}}\\
\leq&\frac{1}{4}\sum_{i\neq j}\sum_{r,l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left|\mathbb{E}\left(\triangle w^j_{t^{m+1}_{2l-1}}\triangle w^i_{t^{m+1}_{2l}}-\triangle w^i_{t^{m+1}_{2l-1}}\triangle w^j_{t^{m+1}_{2l}}\right)\right.\\
&\left.\cdot\left(\triangle w^j_{t^{m+1}_{2r-1}}\triangle w^i_{t^{m+1}_{2r}}-\triangle w^i_{t^{m+1}_{2r-1}}\triangle w^j_{t^{m+1}_{2r}}\right)\right|\\
\leq&d^2\sum_{r<l}\left|\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l-1}}\triangle w^1_{t^{m+1}_{2r-1}}\right)\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l}}\triangle w^1_{t^{m+1}_{2r}}\right)\right|\\
&+d^2\sum_{r<l}\left|\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l-1}}\triangle w^1_{t^{m+1}_{2r}}\right)\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l}}\triangle w^1_{t^{m+1}_{2r-1}}\right)\right|\\
&+\frac{1}{2}d^2\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l-1}}\right)^2\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l}}\right)^2\\
&+\frac{1}{2}d^2\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left|\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l}}\triangle w^1_{t^{m+1}_{2l-1}}\right)\mathbb{E}\left(\triangle w^1_{t^{m+1}_{2l}}\triangle w^1_{t^{m+1}_{2l-1}}\right)\right|\\
\leq&d^2\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\sum_{r=1}^{l-1}C^2_{LTM}\frac{1}{2^{4Hm}\left(l-r\right)^{4-4H}}+d^2\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\sum_{r=1}^{l-1}C^2_{LTM}\frac{1}{2^{4Hm}\left(l-r\right)^{4-4H}}\\
&+\frac{1}{2}d^2\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\frac{C^2_{Hol}}{2^{4Hm}}+\frac{1}{2}d^2\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\frac{C^2_{LTM}}{2^{4Hm}}\\
\leq&d^2C^2_{LTM}\zeta\left(4-4H\right)2^{m-n}\frac{1}{2^{4m}}+\frac{d^2}{2}2^{m-n}C^2_{Hol}\frac{1}{2^{4Hm}}+\frac{d^2}{2}2^{m-n}\frac{C^2_{LTM}}{2^{4Hm}}\\
\leq&3d^2\max\left(C^2_{LTM},C^2_{Hol}\right)\zeta\left(4-4H\right)2^{m-n}2^{-4Hm}
\end{aligned}
$$

where the third last inequality comes from Condition 1. □

We now proceed to the estimates for the third term of the signature.

Lemma 83. *Let $\mathbb{P}$ be a measure satisfying Condition 1 and H be the parameter* *specified in Condition 1. Assume that $H \leq \frac{1}{3}$. Then for $1 \leq k \leq 2^n$ and all $m \geq n$,*

$$\left\| w^{(m+1),3}_{t^n_{k-1},t^n_k} - w^{(m),3}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \leq 7d^{3/2} \max\left(C^{3/2}_{LTM}, C^{3/2}_{Hol}\right) \zeta\left(2-2H\right)^{\frac{1}{2}} 2^{-\frac{1}{2}(4H-1)m-\frac{1+2H}{2}n} .$$

Proof. We shall estimate the first term in the formula for $w^{(m+1),3}_{t^n_{k-1},t^n_k} - w^{(m),3}_{t^n_{k-1},t^n_k}$ in Proposition 79.

$$\begin{aligned}
&\left\| \sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} \left[\triangle w_{t^{m+1}_{2l-1}}, \left[\triangle w_{t^{m+1}_{2l-1}}, \triangle w_{t^{m+1}_{2l}}\right]\right] \right\|^2_{L^2(\mathbb{P})} \\
&= \sum_{r,l=2^{m-n}(k-1)+1}^{2^{m-n}k} \sum_{i,j,u} \mathbb{E}\left(\triangle w^i_{t^{m+1}_{2r-1}} \triangle w^j_{t^{m+1}_{2r-1}} \triangle w^u_{t^{m+1}_{2r}} - 2\triangle w^i_{t^{m+1}_{2r-1}} \triangle w^u_{t^{m+1}_{2r-1}} \triangle w^j_{t^{m+1}_{2r}}\right. \\
&\quad \left. + \triangle w^u_{t^{m+1}_{2r-1}} \triangle w^j_{t^{m+1}_{2r-1}} \triangle w^i_{t^{m+1}_{2r}}\right) \times \qquad (5.5.2) \\
&\quad \left(\triangle w^i_{t^{m+1}_{2l-1}} \triangle w^j_{t^{m+1}_{2l-1}} \triangle w^u_{t^{m+1}_{2l}} - 2\triangle w^i_{t^{m+1}_{2l-1}} \triangle w^u_{t^{m+1}_{2l-1}} \triangle w^j_{t^{m+1}_{2l}} + \triangle w^u_{t^{m+1}_{2l-1}} \triangle w^j_{t^{m+1}_{2l-1}} \triangle w^i_{t^{m+1}_{2l}}\right) \qquad (5.5.3)
\end{aligned}$$

After expanding each bracket inside the sum, we see that each term in the sum is of the form

$$\mathbb{E}\left(\triangle w^{i_1}_{t^{m+1}_{2r-1}} \triangle w^{i_2}_{t^{m+1}_{2r-1}} \triangle w^{i_3}_{t^{m+1}_{2r}} \triangle w^{i_4}_{t^{m+1}_{2l-1}} \triangle w^{i_5}_{t^{m+1}_{2l-1}} \triangle w^{i_6}_{t^{m+1}_{2l}}\right)$$

for indices $i_1, \ldots, i_6$.

By Wick's formula (Theorem 72),

$$\begin{aligned}
&\left|\mathbb{E}\left(\triangle w^{i_1}_{t^{m+1}_{2r-1}} \triangle w^{i_2}_{t^{m+1}_{2r-1}} \triangle w^{i_3}_{t^{m+1}_{2r}} \triangle w^{i_4}_{t^{m+1}_{2l-1}} \triangle w^{i_5}_{t^{m+1}_{2l-1}} \triangle w^{i_6}_{t^{m+1}_{2l}}\right)\right| \\
&\leq \sum_{\text{pairings } \pi} \Pi_{(j,l)\in\pi} \left|\mathbb{E}\left(\triangle w^{i_j}_{n_j} \triangle w^{i_l}_{n_l}\right)\right| \qquad (5.5.4)
\end{aligned}$$

where $n_1 = n_2 = 2r-1$, $n_3 = 2r$, $n_4 = n_5 = 2l-1$, $n_6 = 2l$.

At least one term in the product "$\Pi_{(j,l)\in\pi}$" will pair one of $\triangle w^{i_1}_{t^{m+1}_{2r-1}}, \triangle w^{i_2}_{t^{m+1}_{2r-1}}, \triangle w^{i_3}_{t^{m+1}_{2r}}$ to one of $\triangle w^{i_4}_{t^{m+1}_{2l-1}}, \triangle w^{i_5}_{t^{m+1}_{2l-1}}, \triangle w^{i_6}_{t^{m+1}_{2l}}$.If $r \neq l$, then this term will be bounded by

$$\begin{aligned} & \max\left(\frac{C_{LTM}}{2^{2Hm}\left|l-r\right|^{2-2H}}, \frac{C_{LTM}}{2^{2Hm}\left(\left|2l-2r+1\right|\right)^{2-2H}}\right) \\ = \; & \frac{C_{LTM}}{2^{2Hm}\left|l-r\right|^{2-2H}}. \end{aligned}$$

All other terms in the product $\Pi_{(j,l)\in\pi}$ will be bounded by

$$\max\left(C_{LTM}, C_{Hol}\right)\frac{1}{2^{2Hm}}. \tag{5.5.5}$$

As there are $\frac{6!}{2^3 3!} = 15$ terms in the sum $\sum_{\text{pairings } \pi}$, we have

$$\begin{aligned} & \left|\mathbb{E}\left(\triangle w^{i_1}_{t^{m+1}_{2r-1}} \triangle w^{i_2}_{t^{m+1}_{2r-1}} \triangle w^{i_3}_{t^{m+1}_{2r}} \triangle w^{i_4}_{t^{m+1}_{2l-1}} \triangle w^{i_5}_{t^{m+1}_{2l-1}} \triangle w^{i_6}_{t^{m+1}_{2l}}\right)\right| \\ \leq \; & 15\frac{\max\left(C^3_{LTM}, C^3_{Hol}\right)}{2^{6Hm}\left|l-r\right|^{2-2H}}. \end{aligned}$$

If $r = l$, then using (5.5.4) together with (5.5.5), we have

$$\begin{aligned} & \left|\mathbb{E}\left(\triangle w^{i_1}_{t^{m+1}_{2r-1}} \triangle w^{i_2}_{t^{m+1}_{2r-1}} \triangle w^{i_3}_{t^{m+1}_{2r}} \triangle w^{i_4}_{t^{m+1}_{2l-1}} \triangle w^{i_5}_{t^{m+1}_{2l-1}} \triangle w^{i_6}_{t^{m+1}_{2l}}\right)\right| \\ \leq \; & 15\frac{\max\left(C^3_{LTM}, C^3_{Hol}\right)}{2^{6Hm}}. \end{aligned}$$

Therefore (5.5.3) is bounded by

$$\begin{aligned}
& 2\sum_{r<l}\sum_{i,j,u} 15\frac{\max\left(C_{LTM}^3, C_{Hol}^3\right)}{2^{6Hm}\left|l-r\right|^{2-2H}} + 15\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\sum_{i,j,u}\frac{\max\left(C_{LTM}^3, C_{Hol}^3\right)}{2^{6Hm}} \\
\leq\ & \frac{45d^3\max\left(C_{LTM}^3, C_{Hol}^3\right)\zeta\left(2-2H\right)2^{m-n}}{2^{6Hm}}. \qquad (5.5.6)
\end{aligned}$$

The procedure of estimating the second term in (79) is exactly the same and yields the same bound. For the third term and fourth in (79), we note that

$$\triangle w_{t_l^m} = \triangle w_{t_{2l}^{m+1}} + \triangle w_{t_{2l-1}^{m+1}} \qquad (5.5.7)$$

and the logic for estimating the first term in (79) applies. The upper bound for the third and fourth term in (79) will be double that of the the first term due to (5.5.7).

We shall now estimate the final term in (79).

$$\begin{aligned}
& \mathbb{E}\left\|\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left(w_{t_{l-1}^m} - w_{t_{2^{m-n}(k-1)}^m}\right)\otimes\left[\triangle w_{t_{2l-1}^{m+1}}, \triangle w_{t_{2l}^{m+1}}\right]\right\|_{\left(\mathbb{R}^d\right)^{\otimes 3}}^2 \\
=\ & \sum_{j\neq u}\sum_{r,l=2^{m-n}(k-1)+1}^{2^{m-n}k}\mathbb{E}\left(w_{t_{r-1}^m}^i - w_{t_{2^{m-n}(k-1)}^m}^i\right)\left(\triangle w_{t_{2r-1}^{m+1}}^j\triangle w_{t_{2r}^{m+1}}^u - \triangle w_{t_{2r-1}^{m+1}}^u\triangle w_{t_{2r}^{m+1}}^j\right)\times \\
& \left(w_{t_{l-1}^m}^i - w_{t_{2^{m-n}(k-1)}^m}^i\right)\left(\triangle w_{t_{2l-1}^{m+1}}^j\triangle w_{t_{2l}^{m+1}}^u - \triangle w_{t_{2l-1}^{m+1}}^u\triangle w_{t_{2l}^{m+1}}^j\right) \qquad (5.5.8)
\end{aligned}$$

Note that the terms obtained by expanding the brackets in (5.5.8) are either of the form

$$\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}}\triangle w^b_{t^{m+1}_{2r}}\triangle w^a_{t^{m+1}_{2l-1}}\triangle w^b_{t^{m+1}_{2l}}\right) \tag{5.5.9}$$

or

$$\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}}\triangle w^b_{t^{m+1}_{2r}}\triangle w^a_{t^{m+1}_{2l}}\triangle w^b_{t^{m+1}_{2l-1}}\right). \tag{5.5.10}$$

The process of estimating (5.5.9) and (5.5.10) are exactly the same. We will only do the estimates for (5.5.9) here.

We will first consider the case $i \neq b$. The procedure of estimating case $i \neq a$ is identical. Here we have

$$\begin{aligned}
&\left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}}\triangle w^b_{t^{m+1}_{2r}}\triangle w^a_{t^{m+1}_{2l-1}}\triangle w^b_{t^{m+1}_{2l}}\right)\right| \\
= &\left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}}\triangle w^a_{t^{m+1}_{2l-1}}\right)\right|\left|\mathbb{E}\left(\triangle w^b_{t^{m+1}_{2r}}\triangle w^b_{t^{m+1}_{2l}}\right)\right|.
\end{aligned}$$

However, by Condition 1., if $r < l$,

$$\left|\mathbb{E}\left(\triangle w^b_{t^{m+1}_{2r}}\triangle w^b_{t^{m+1}_{2l}}\right)\right| \leq \frac{C_{LTM}}{2^{2Hm}\left|l-r\right|^{2-2H}}$$

and if $r = l$,

$$\left|\mathbb{E}\left(\triangle w^b_{t^{m+1}_{2r}}\triangle w^b_{t^{m+1}_{2l}}\right)\right| \leq \frac{C_{Hol}}{2^{2Hm}}.$$

By Wick's formula,

$$
\begin{aligned}
& \left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}} \triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
\leq\ & \left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\right|\left|\mathbb{E}\left(\triangle w^a_{t^{m+1}_{2r-1}} \triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
& + \left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2r-1}}\right|\left|\mathbb{E}\left(\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
& + \left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2l-1}}\right|\left|\mathbb{E}\left(\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2r-1}}\right)\right| \qquad (5.5.11)
\end{aligned}
$$

We may apply the Cauchy-Schwarz ineqaulity and Condition 1 to the first term in (5.5.11) to obtain

$$
\begin{aligned}
& \left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\right|\left|\mathbb{E}\left(\triangle w^a_{t^{m+1}_{2r-1}} \triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
\leq\ & \frac{\max\left(C^2_{LTM}, C^2_{Hol}\right)}{2^{2Hm}\left|l-r\right|^{2-2H}}\left\|w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right\|_{L^2(\mathbb{P})}\left\|w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right\|_{L^2(\mathbb{P})} \\
\leq\ & \frac{\max\left(C^2_{LTM}, C^2_{Hol}\right)}{2^{2Hm}\left|l-r\right|^{2-2H}}\left[\frac{C_{Hol}}{2^{2nH}}\right] \\
\leq\ & \frac{\max\left(C^3_{LTM}, C^3_{Hol}\right)}{2^{2H(m+n)}\left|l-r\right|^{2-2H}}
\end{aligned}
$$

We will estimate the second term in (5.5.11).

By Cauchy-Schwarz inequality again,

$$
\begin{aligned}
& \left|\mathbb{E}\left(\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
\leq\ & \frac{C_{Hol}}{2^{H(m+n)}}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
&\left|\mathbb{E}\left(\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
\leq\ & \frac{C_{Hol}}{2^{H(m+n)}}.
\end{aligned}$$

Therefore, the second term in (5.5.11) is estimated as

$$\begin{aligned}
&\left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2r-1}}\right|\left|\mathbb{E}\left(\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2l-1}}\right)\right| \\
\leq\ & \frac{C^2_{Hol}}{2^{2H(m+n)}}.
\end{aligned}$$

Exactly the same computation gives the estimates for the third term in (5.5.11) as

$$\begin{aligned}
&\left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2l-1}}\right|\left|\mathbb{E}\left(\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\triangle w^a_{t^{m+1}_{2r-1}}\right)\right| \\
\leq\ & \frac{C^2_{Hol}}{2^{2H(m+n)}}.
\end{aligned}$$

Thus we obtain the estimates for (5.5.9) for the case $r < l$ as

$$\begin{aligned}
&\left|\mathbb{E}\left(w^i_{t^m_{r-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}} - w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}}\triangle w^b_{t^{m+1}_{2r}}\triangle w^a_{t^{m+1}_{2l-1}}\triangle w^b_{t^{m+1}_{2l}}\right)\right| \\
\leq\ & \frac{\max\left(C^3_{Hol}, C^3_{LTM}\right)}{2^{2H(m+n)+2Hm}\left|l-r\right|^{2-2H}}
\end{aligned}$$

while if $r=l$,

$$
\begin{aligned}
& \left|\mathbb{E}\left(w^i_{t^m_{r-1}}-w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(w^i_{t^m_{l-1}}-w^i_{t^m_{2^{m-n}(k-1)}}\right)\left(\triangle w^a_{t^{m+1}_{2r-1}}\triangle w^b_{t^{m+1}_{2r}}\triangle w^a_{t^{m+1}_{2l-1}}\triangle w^b_{t^{m+1}_{2l}}\right)\right| \\
\leq\ & \frac{\max\left(C^3_{Hol},C^3_{LTM}\right)}{2^{4H(m+n)+2Hm}}.
\end{aligned}
$$

Hence by (5.5.8)

$$
\begin{aligned}
& \mathbb{E}\left\|\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left(w_{t^m_{l-1}}-w_{t^m_{2^{m-n}(k-1)}}\right)\otimes\left[\triangle w_{t^{m+1}_{2l-1}},\triangle w_{t^{m+1}_{2l}}\right]\right\|^2_{\left(\mathbb{R}^d\right)^{\otimes 3}} \\
\leq\ & 8d^3\sum_{r<l}\frac{\max\left(C^3_{Hol},C^3_{LTM}\right)}{2^{4H(m+n)+2Hm}\left|l-r\right|^{2-2H}}+4d^3\sum_{r=2^{m-n}(k-1)+1}^{2^{m-n}k}\frac{\max\left(C^3_{Hol},C^3_{LTM}\right)}{2^{4H(m+n)+2Hm}} \\
\leq\ & \frac{12d^3\max\left(C^3_{Hol},C^3_{LTM}\right)2^{m-n}}{2^{2H(m+n)+2Hm}}\zeta\left(2-2H\right).
\end{aligned}
$$

This combining with (5.5.6) and Lemma 79 we have,

$$
\left\|w^{(m+1),3}_{t^n_{k-1},t^n_k}-w^{(m),3}_{t^n_{k-1},t^n_k}\right\|_{L^2(\mathbb{P})}\leq\frac{7d^{3/2}\max\left(C^{3/2}_{LTM},C^{3/2}_{Hol}\right)\zeta\left(2-2H\right)^{\frac{1}{2}}2^{\frac{m-n}{2}}}{2^{2H(m+n)+2Hm}}
$$

□

We now proceed to estimate

$$
\left\|w^{(m+1),j}_{t^n_{k-1},t^n_k}-w^{(m),j}_{t^n_{k-1},t^n_k}\right\|_{L^2(\mathbb{P})}
$$

when $n \geq m$, the easier case.

Lemma 84. *Let $\mathbb{P}$ be a measure satisfying Condition 1 and H be the parameter specified in Condition 1. There exists a constant C depending only on d and H such that for $j = 1, \ldots, 3$ and $1 \leq k \leq 2^n$, $n \geq m$,*

$$\left\| w^{(m),j}_{t^n_{k-1}, t^n_k} \right\|_{L^2(\mathbb{P})} \leq \frac{C_{Hol}^{\frac{j}{2}}}{j!} 3^{j^2(2j+2)} . 2^{m(1-H)j-nj} \tag{5.5.12}$$

and

$$\left\| w^{(m+1),j}_{t^n_{k-1}, t^n_k} - w^{(m),j}_{t^n_{k-1}, t^n_k} \right\|_{L^2(\mathbb{P})} \leq \frac{2C_{Hol}^{\frac{j}{2}}}{j!} 3^{j^2(2j+2)} 2^{m(1-H)-jn}. \tag{5.5.13}$$

Proof. Let l be such that $\left[t^n_{k-1}, t^n_k\right] \subset \left[t^m_{l-1}, t^m_l\right]$. Then by Lemma 73,

$$S\left(w^{(m)}\right)_{t^n_{k-1}, t^n_k} = e^{2^m\left(w_{t^m_l} - w_{t^m_{l-1}}\right)\left(t^n_k - t^n_{k-1}\right)}.$$

Hence in particular, for $j = 1, 2, 3$,

$$w^{(m),j}_{t^n_{k-1}, t^n_k} = \frac{2^{jm}\left(w_{t^m_l} - w_{t^m_{l-1}}\right)^j \left(t^n_k - t^n_{k-1}\right)^j}{j!}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left(w^{(m),j}_{t^n_{k-1}, t^n_k}\right)^2 &\leq \frac{2^{2jm}\left(t^n_k - t^n_{k-1}\right)^{2j}}{(j!)^2} 3^{\frac{j(2j+2)2j}{2}} \left(\mathbb{E}\left(w_{t^m_l} - w_{t^m_{l-1}}\right)^2\right)^j \\
&\leq \frac{2^{2jm}}{(j!)^2} \frac{C_{Hol}^j 3^{j^2(2j+2)}}{2^{2jHm+2jn}} \\
&\leq 3^{j^2(2j+2)} \frac{C_{Hol}^j}{(j!)^2} 2^{2j(1-H)m-2jn}.
\end{aligned}$$

We will now prove (5.5.13). We will first estimate it cruelly as,

$$\left\| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \leq \left\| w^{(m+1),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} + \left\| w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})}.$$

It then follows easily that

$$\left\| w^{(m+1),j}_{t^n_{k-1},t^n_k} - w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \leq \frac{2C^{\frac{1}{2}}_{Hol}}{j!} 3^{j^2(2j+2)} 2^{j(1-H)m-jn}.$$

$\square$

Finally we will give the estimates for $\left\| w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})}$ when $m \geq n$.

Lemma 85. *Let $\mathbb{P}$ be a measure satisfying Condition 1 and H be the parameter specified in Condition 1. Then for $j = 1, \ldots, \left\lfloor \frac{1}{H} \right\rfloor$ and $1 \leq k \leq 2^n$, $m \geq n$,*

$$\left\| w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \leq C_j . 2^{-nHj}.$$

where

$$\begin{aligned}
C_1 &= C^{\frac{1}{2}}_{Hol} 3^4 \\
C_2 &= \max\left(C_{LTM}, C_{Hol}\right) \left(\frac{2d\zeta\left(4-4H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)}} + \frac{3^{24}}{2} \right) \\
C_3 &= \max\left(C^{3/2}_{LTM}, C^{3/2}_{Hol}\right) \left(7d^{3/2} \frac{\zeta\left(2-2H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)k}} + \frac{3^{108}}{6} \right).
\end{aligned}$$

Proof. Observe that

$$\left\| w^{(m),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \leq \sum_{k=n}^{m-1} \left\| w^{(k+1),j}_{t^n_{k-1},t^n_k} - w^{(k),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} + \left\| w^{(n),j}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})}.$$

For $j = 1$, we have

$$\begin{aligned} \left\| w^{(m),1}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} &\leq C_{Hol}^{\frac{1}{2}} 3^4 2^{(1-H)m-n} \\ &\leq C_{Hol}^{\frac{1}{2}} 3^4 2^{-nH}. \end{aligned}$$

For $j = 2$, we have

$$\begin{aligned} \left\| w^{(m),2}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} &\leq \sum_{k=n}^{m-1} \left\| w^{(k+1),2}_{t^n_{k-1},t^n_k} - w^{(k),2}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} + \left\| w^{(n),2}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \\ &\leq 2d \max\left(C_{LTM}, C_{Hol}\right) \zeta\left(4-4H\right)^{\frac{1}{2}} \sum_{k=n}^{m-1} 2^{-\frac{1}{2}(4H-1)k-\frac{1}{2}n} \\ &\quad + \frac{C_{Hol}}{2} 3^{24} 2^{-2nH} \\ &\leq 2d \max\left(C_{LTM}, C_{Hol}\right) \zeta\left(4-4H\right)^{\frac{1}{2}} 2^{-2Hn} \left(\frac{1-2^{-\frac{1}{2}(4H-1)(m-n)}}{1-2^{-\frac{1}{2}(4H-1)}} \right) \\ &\quad + \frac{C_{Hol}}{2} 3^{12} 2^{-2nH} \\ &\leq \max\left(C_{LTM}, C_{Hol}\right) \left(\frac{2d\zeta\left(4-4H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)}} + \frac{3^{12}}{2} \right) 2^{-2Hn}. \end{aligned}$$

For $j=3$, we have

$$\begin{aligned}
\left\| w^{(m),3}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} &\leq \sum_{k=n}^{m-1} \left\| w^{(k+1),3}_{t^n_{k-1},t^n_k} - w^{(k),3}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} + \left\| w^{(n),3}_{t^n_{k-1},t^n_k} \right\|_{L^2(\mathbb{P})} \\
&\leq 7d^{3/2} \max\left(C^{3/2}_{LTM}, C^{3/2}_{Hol} \right) \zeta\left(2-2H\right)^{\frac{1}{2}} \sum_{k=n}^{m-1} 2^{-\frac{1}{2}(4H-1)k - \frac{1+2H}{2}n} \\
&\quad + \frac{C^{\frac{3}{2}}_{Hol}}{6} 3^{108} 2^{-3nH} \\
&\leq 7d^{3/2} \max\left(C^{3/2}_{LTM}, C^{3/2}_{Hol} \right) \zeta\left(2-2H\right)^{\frac{1}{2}} 2^{-3nH} \frac{1}{1-2^{-\frac{1}{2}(4H-1)k}} \\
&\quad + \frac{C^{\frac{3}{2}}_{Hol}}{6} 3^{108} 2^{-3nH} \\
&\leq \max\left(C^{3/2}_{LTM}, C^{3/2}_{Hol} \right) \left(7d^{3/2} \frac{\zeta\left(2-2H\right)^{\frac{1}{2}}}{1-2^{-\frac{1}{2}(4H-1)k}} + \frac{3^{108}}{6} \right) 2^{-3nH}.
\end{aligned}$$

$\square$

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