The QCD Axion, Generally

Arthur Platschorre

Balliol College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

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Abstract

The parameter space to be probed in searches for the axion particle is vast and appears impractical unless more theoretical guidance is provided. This thesis explores the uses of generalised symmetries in identifying promising regions of parameter space for existing and novel experimental axion searches.

We consider the general form of the axion-photon coupling to all orders in the axion. We demonstrate that there exists a model-independent relationship between the axion-photon coupling and the axion mass through a non-invertible generalised symmetry of the axion, thus providing a correlation between the two quantities crucial to current axion searches.

We proceed by exploring the confinement of instantons. Motivated by the absence of global symmetries in quantum gravity, we demonstrate that the vacuum acts as a superconductor when a higher-form generalised symmetry of the particle dual to the axion is absent. In this superconducting phase, instantons are confined by the worldline action of a particle-like soliton travelling between the instantons. We calculate the cost of this additional worldline suppression, comment on the required particle spectrum, and relevance for the strong CP problem.

Finally, we study the minimal requirements to obtain axions of exponentially high quality while also being compatible with a post-inflationary scenario, thereby motivating axion searches in higher-mass regions. These ingredients occur in theories where an axion protected by a higher-form generalised symmetry mixes with the phase of a complex scalar field. We explore how these scenarios arise in extra-dimensional models, including heterotic string theory.

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To my happiness, Felicia.

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Contents

1 Introduction

Despite the many successes of the Standard Model of particle physics in describing the natural world, it is incomplete and leaves several crucial questions unresolved.

On a microscopic level, the Standard Model fails to explain why one of its fundamental pillars, the strong force, behaves in a strikingly uniform way. The strong force is responsible for binding together all quarks into the neutrons and protons that make up the nuclei inside our atoms. Current experiments on the neutron's electric dipole moment [1] demonstrate that the strong force distributes the electric charges of the quarks inside the neutron with unexpected, extraordinary precision, remaining homogeneous up to at least one part in ten billion.

On a macroscopic level, observations on the rotation curves of spiral galaxies [2], gravitational attraction in galactic clusters [3, 4], lensing on intergalactic scales [5, 6], and anisotropies in the light reaching us from the Big Bang [7] indicate that more than 85% of the matter in our universe is an exotic type of 'dark' matter that is not present in the Standard Model and interacts only very weakly with light.

The aforementioned shortcomings strongly suggest that there is physics beyond the Standard Model. One possibility is the existence of new particles, and one well-motivated candidate particle that features prominently in many experimental searches is the axion [8–10]. This particle would explain the homogeneity of the charge distribution inside the neutron [8], would exist in such large numbers that it could account for all of the observed dark matter [11–13], provides a simple test for grand-unified extension of the Standard Model [14, 15], has strong interplay with ideas in quantum gravity [16, 17], and is ubiquitous in extra-dimensional theories [18], including our most promising theory of quantum gravity: string theory [19, 20].

These compelling motivations for axions have spawned an ever-growing experimental program [21, 22] dedicated to finding the axion and/or constraining its properties. As this particle would be feebly coupled to us, a large fraction of the experimental efforts aim to detect the axion through smaller-scale resonant searches, making the axion mass and coupling to photons one of the most important quantities. The axion parameter space these experiments have to cover is vast, spanning more than 10 orders of magnitude in photon-couplings and masses, and seems impractical unless more theoretical guidance is provided.

An important theoretical guide in identifying promising regions of parameter spaces for experimental searches is the study of symmetries. Symmetries *organise* the spectrum of particles and provide powerful constraints on their interactions. In the last decade, the study of symmetries has been revolutionised by the introduction of generalised symmetries [23]. Generalised symmetries extend the notion of ordinary symmetries in a manner that retains the organisational powers of symmetries. This extension allows generalised symmetries to capture a much broader class of the rich, complex phenomena that occur in the natural world.

Generalised symmetries have been successfully applied in many theoretical works. In accordance with ordinary symmetries, generalised symmetries and their anomalies are invariant under renormalisation group flow, making them powerful tools in studying the vacuum structure of strongly-interacting gauge theories [24–35]. Generalised symmetries have also been applied to study holography at finite temperature [36, 37], to identify the Δ^{++} baryon [38], and to understand the complicated scattering of fermions off magnetic monopoles [39, 40]. Moreover, it is widely conjectured [41–43] that consistent theories of quantum gravity have to be free of global symmetries, and the absence of generalised symmetries can be used to provide lower bounds on symmetry breaking effects [44, 45].

In accordance with ordinary global symmetries, continuous generalised symmetries can be spontaneously broken [23, 46–49] resulting in massless Goldstone bosons, providing powerful insights into the low energy particle spectrum. Small explicit breakings of the generalised symmetries can then be used to constrain the interactions between the Goldstone bosons [50, 51].

At present, the application of generalised symmetries to phenomenological studies has remained largely unexplored (see however [38, 52–60]). In this thesis, we aim to constrain the axion parameter space by studying the axion's generalised symmetries, thereby solving modern issues with modern tools. The relevant generalised symmetries to constrain the axion and its crucial interactions with the photon are *non-invertible*, *higher-form* and *higher-group* symmetries [58, 61, 62].

The axion is the Goldstone boson of a spontaneously broken non-invertible symmetry [50, 51]. In chapter 4, we consider the general coupling of this Goldstone boson to the photon. We will demonstrate that, in general, the axion-photon coupling is a non-linear monodromic function of the axion. The non-linearities in the axion-photon coupling are correlated with the axion's mass through the axion's non-invertible shift symmetry. This provides a model-independent relationship between the axion's mass and the form of the axion-photon coupling, quantities key to the many axion searches.

We derive the explicit form of the axion-photon coupling for several examples, including the QCD axion, and show that there is a uniform general prototypical form for this monodromic function. The full non-linear profile of this coupling is phenomenologically relevant to the dynamics induced on axion domain walls/strings and other extended axionic objects. This has been implemented in the recent work [63].

For generic axion models, the quality of the non-invertible shift symmetry can be extremely sensitive to ultraviolet (UV) physics, which has been dubbed the (QCD) axion quality problem [64, 65]. Many of the desired features of axions rely on a high-quality non-invertible shift symmetry, and a great deal of theoretical effort has been dedicated to alleviating the axion quality problem [66–80].

In chapter 5, we study the confinement or suppression of instantons, contributions that worsen the quality of the axion's non-invertible shift symmetry. Recent investigations had already re-examined the axion quality problem from the perspective of the particle dual to the axion [42, 81–85], the associated two-form Kalb-Ramond field [86]. Motivated by these investigations and the absence of generalised global symmetries in quantum gravity, we demonstrate that the vacuum confines instanton contributions when a higher-form shift symmetry of the dual axion is absent.

In this superconducting phase, we show that both instantons and magnetic monopoles are confined. This confinement of instantons corresponds to the worldline action of a particle-like soliton travelling between the instantons analogous to Abrikosov/Nielsen-Oleson vortex solitons that stretch between confined magnetic monopoles in a superconductor. We calculate the cost of this additional worldline suppression, provide several models in which both the confined instantons and confining worldline are dynamical, and comment on the required particle spectrum and relevance for the most prevalent axion, the QCD axion.

Finally, in chapter 6, we are motivated by the fact that, in theory, there exists a unique inference for the mass of the axion from the observed late-time dark matter abundance if the axion is to saturate this abundance. A large number of theoretical and numerical efforts are being dedicated to reducing the uncertainties in this inference [87–95], and generically predict axion masses in regions higher than searched for in current experiments. This inference requires axion models to be both of high quality and to be compatible with a so-called post-inflationary scenario – that is, the scenario where the initial axion misalignment angle is produced after the inflation epoch and randomly distributed on small spatial scales.

In accordance with the dictum that extra-dimensional geometry can provide an alternative for ordinary symmetries, high quality axions can be obtained from extra-dimensional axion models [20, 96], which largely circumvent the axion quality problem [58]. However, generically, extra-dimensional axion models correspond to pre-inflationary scenarios where the initial axion misalignment angle is produced before the inflation epoch and unknown, complicating the inference for the axion's mass.

In this thesis, we present a category of axions that are compatible with a post-inflationary scenario and possess an exponentially high quality. The required ingredients appear in theories where an extra-dimensional axion mixes with an ordinary axion coming from the phase of a complex scalar field. The resulting axion inherits a high-quality non-invertible symmetry from an extra-dimensional higher-form symmetry. As this scenario resembles higher-groups [97], we refer to such axions as higher axions. We proceed to study the realisation of higher axions in several extra-dimensional and string theory models, providing further motivation for higher-mass axion searches.

This thesis is structured as follows. Chapter 2 is devoted entirely to introducing the relevant background on axions, their phenomenological motivations, and current axion searches. This is followed by a formal review of generalised symmetries and effective field theory in chapter 3. In chapter 4, the monodromic axion-photon coupling is introduced, and the correlation with the axion mass is studied. Chapter 5 is devoted to the confinement of instantons. Higher axions, their potential cosmology, and origins in string theory are studied in chapter 6. Our findings are summarised in chapter 7.

This thesis contains only the work of the author except where stated. No part of this thesis has been submitted for any other qualification. This thesis is based on the following works completed during the DPhil,

- Chapter 4

  Prateek Agrawal and Arthur Platschorre. "The monodromic axion-photon coupling". In: JHEP 01 (2024), p. 169. arXiv: 2309.03934 [hep-th] [98]

- Chapter 5

  Arthur Platschorre. "A mass for the dual axion". In: JHEP 10 (2024), p. 253. arXiv: 2405.14931 [hep-th] [99]

- Chapter 6

  Vazha Loladze, Arthur Platschorre, and Mario Reig. "Higher Axion Strings". (Mar. 2025). arXiv: 2503.18707 [hep-ph] [100]

This thesis does not include the following works finished during the DPhil,

- Lakshya Bhardwaj et al. "Lectures on generalized symmetries". In: Phys. Rept. 1051 (2024), pp. 1–87. arXiv: 2307.07547 [hep-th] [101]
- Antonio Pich, Arthur Platschorre, and Mario Reig. "Electroweak mass difference of mesons". In: Phys. Rev. D 108.9 (2023), p. 094044. arXiv: 2308.00030 [hep-ph] [102]

2
A Primer on Axions

This chapter serves as a modern introduction to axions and motivates our efforts in the subsequent chapters. The point of view is taken that a suitable discussion of axions involves the axion's symmetries and redundancies. The axion is introduced in section 2.3 including a review of its phenomenological motivations and standard cosmological history. A key quantity in axion searches is the axion-photon interaction, and this naturally leads us to introducing photons and axion experiments in section 2.4. Finally, gluons and their contribution to the axion mass in both the weak and strong gauge coupling regimes are discussed in sections 2.5.

2.1 Quantum Field Theory

To the best of our knowledge and experimental abilities, the principles that underlie our microscopic world are those of quantum mechanics and relativity. That is, the modern mathematical formalism to describe particles is as physical vectors in a Hilbert space, which can be decomposed into irreducible unitary representations of the Poincaré group. In order to describe the interactions between particles in a Lorentz invariant framework, one is naturally motivated to consider fields ϕ, functions on space-time that sit inside finite representations of the Lorentz group.

Thus, quantum field theory (QFT) emerges naturally as a framework to describe interacting particles obeying the laws of quantum mechanics and relativity.

Being inherently non-deterministic in nature, observables in a quantum field theory can be expressed in terms of a partition or moment-generating function Z of the fields ϕ as,

$$Z = \int D\phi \; e^{iS[\phi]} \,. \tag{2.1}$$

The exponential weight $S[\phi]$ is the action, and $\int D\phi$ is a functional integral over all fields constrained by appropriate boundary conditions. By locality, the action can be expressed as a space-time integral of a local Lagrangian density $\mathcal{L}$,

$$S[\phi] = \int d^d x \; \mathcal{L}[\phi(x)] \,. \tag{2.2}$$

The fields $\phi(x)$ are associated with operators $\hat{\phi}(x)$ on the Hilbert space. Physical observables correspond to time-ordered correlation functions of these observables, which in turn can be expressed as moment functions of the partition function,

$$\langle \Omega | T \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) | \Omega \rangle = \int D\phi \; \phi(x_1) \ldots \phi(x_n) \; e^{iS[\phi]} \,, \tag{2.3}$$

where T indicates an increasing time-ordering of the operators $\hat{\phi}(x)$ and $|\Omega\rangle$ is the vacuum state of the Hilbert space. Throughout this work, the time-ordering, vacuum state and operator notation will often be taken to be implicit, and we will write,

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle = \int D\phi \; \phi(x_1) \ldots \phi(x_n) \; e^{iS[\phi]} \,. \tag{2.4}$$

Moments in the fields ϕ can be obtained from the partition function (2.1) by inserting a source J into the action,

$$Z[J] = \int D\phi \; e^{iS[\phi] + i\int d^d x \;\, J(x)\phi(x)} \,, \tag{2.5}$$

and taking appropriate derivatives with respect to this source.

2.2 Symmetries

In practice, the partition function (2.1) can only be solved for a handful of actions corresponding to simple non-interacting theories and a small set of interacting systems. For a general interacting system with coupling constant g, the standard approach is to express the observables as a perturbative series in the coupling parameter g and the observables of simple non-interacting systems.

This perturbative approach to obtaining physical observables is inherently computationally challenging and intense. The required computational power grows sharply on an order-by-order basis in the coupling g and the number of external particles. Moreover, the rate of convergence of the series explicitly relies on the choice of fields ϕ and coupling constants g.

In this regard, symmetries are indispensable to regain control over the Hilbert space. Symmetries imply constraints between the time-ordered operator products and can therefore effectively cut the number of terms in a perturbative series by a large degree. Furthermore, symmetries can aid in identifying the appropriate degrees of freedom and coupling constants g to use in the perturbative series as explained in section 3.1.

In one sentence, symmetries *organise* the spectrum of the Hilbert space. This organisation goes beyond the perturbative approach. A large class of theories ubiquitous in nature are non-Abelian gauge theories, which admit no perturbative expansion for processes at low energy scales. As such, these theories are shrouded in the mysteries of strong coupling, and much of modern physics has gone into developing tools to understand these phases of QFTs. These tools, by and large, involve the matching of symmetries (and anomalies) of the theory between the weakly and strongly coupled phases [23, 24, 26, 28–35, 41, 51, 103–107], but also strong-weak dualities [25, 108–110], lattice studies [111–113] and large N expansions [114–118].

In the language of classical mechanics, a field redefinition is an assignment $\phi'(x) = \mathcal{F}\left(\phi(x)\right)$. Traditionally, a symmetry is a field redefinition that leaves the action S invariant. Consecutive symmetry operations are often taken to form an algebra associated with a group G. Noether's theorem guarantees that classically, any symmetry is associated with a conserved charge Q. In the case of a continuous symmetry, this charge can be constructed as the integral over a time slice of a conserved current j_μ that satisfies $\partial_\mu j^\mu = 0$,

$$Q(t_0) = \int_{t=t_0} d^{D-1}x \; j^0 \,. \tag{2.6}$$

The *power* of symmetries in classical mechanics is the identification of conserved quantities, thereby reducing the complexity in solving for the system's evolution.

In the absence of anomalies (reviewed in section 3.5.1), a classical symmetry can be lifted to a symmetry of the Hilbert space of the quantum field theory. The symmetry is generated by a unitary operator $U(t)$, given as the exponent of the hermitian charge $Q(t)$. The charges act on charged operators $\phi(x,t)$ by commutation,

$$U(t) = \exp(iQ(t))\,, \qquad U^\dagger(t)\phi(x,t)U(t) = \phi'(x,t)\,, \tag{2.7}$$

and on states $|\psi\rangle$ of the Hilbert space as,

$$U(t)\,|\psi\rangle = |\psi'\rangle \,. \tag{2.8}$$

In the time-ordered language of partition functions (2.1), the commutation action is generated by an insertion of U at times $t+\epsilon$ and $t-\epsilon$ in the time-ordered product,

$$\left\langle e^{-iQ(t+\epsilon)}\phi(x,t)e^{iQ(t-\epsilon)}\right\rangle = \langle \phi'(x,t)\rangle \,. \tag{2.9}$$

Notice that this operator product is independent of the time-displacement parameter ϵ by the time conservation of the symmetry operator. It is these identities (2.9) that make symmetries indispensable in understanding the Hilbert space of a QFT.

Whilst extremely powerful, the previous symmetries are insufficient to effectively constrain the spectrum of the Hilbert space in the presence of axions and the axion's properties, such as mass or coupling to photons. Instead, after properly introducing the axion theory, we will naturally find that we have to expand our concept of symmetries to *generalised symmetries* in order to properly describe the axion's dynamics.

2.3 Axions

Axions are one of the most compelling new physics candidates. Axions could explain the homogeneity of the charge distribution inside the neutron [8–10] as reviewed in section 2.5.1, would exist in such large numbers that they could account for all of the observed dark matter [11–13] as seen in section 2.3.2, and are ubiquitous in extra-dimensional theories [18], including our most promising theory of quantum gravity: string theory [19, 20, 119, 120] as used in chapter 6. Moreover, axions provide simple tests for the grand-unified (GUT) extension of the Standard Model [14, 15] and have strong interplay with ideas in quantum gravity [16, 17]. In this section, we review the axion particle, the dual axion, and the axion's standard cosmological history. This section and the next sections on photons and gluons constitute a more formal introduction to axions and gauge theories than is common in the literature in order to facilitate the transition to generalised symmetries in chapter 3, and the uses thereof in subsequent chapters.

An axion a, is a periodic field with a redundancy $a \equiv a+2\pi$. The axion represents an angular coordinate and is therefore often thought of as a particle on an extra-dimensional circle, and the redundancy makes manifest the topology of this circle.

The action for an axion that is invariant under the redundancy[1] is

$$S = \int d^4x \, \frac{f^2}{2} \left(\partial a\right)^2 - V(a) \,. \tag{2.10}$$

The constant f multiplying the axion kinetic term is known as the axion decay constant, a meaning that will become clear in section 2.4, and $V(a) \equiv V(a+2\pi)$ is the axion potential. The axion can arise after the spontaneous symmetry breaking (SSB) of a complex scalar, in which case the $a \equiv a+2\pi$ identification is an emergent topology below the spontaneous breaking scale.

In general, a path integral or Hilbert space formulation of a quantum field theory is often described in terms of a larger set of fields or Hilbert space, together

[1] To adequately describe a field with a redundancy $a \equiv a+2\pi$ on a general space-time X, one needs to cover the manifold with coordinate patches (open sets) U_i with field values a_i. On the intersection between two open sets $U_i \cap U_j \neq 0$, one additionally has to specify transition functions t_{ij} with $a_i = a_j + t_{ij}$ with $t_{ij} \in 2\pi\mathbb{Z}$. Gauge-invariant observables are those that do not depend on the choice of patches.

with an identification of the fields or equivalence relation among the physical vectors. These redundancies in the Hilbert space follow from a natural desire to write our formulation in a way that makes either topology or Lorentz invariance manifest. The resulting theories are called gauge theories, and the axion is the simplest non-trivial example.

The action (2.10) has a well-known non-linear $U(1)$ shift symmetry under which the axion transforms as,

$$a \to a + c, \qquad \partial_\mu c = 0\,. \tag{2.11}$$

This shift symmetry is generated by the associated current involving the axion's field strength,

$$j_\mu = f^2 \partial_\mu a\,, \tag{2.12}$$

and is conserved by the equations of motion (2.10) of the axion.

In four space-time dimensions, the axion also has an additional trivially conserved three-index anti-symmetric current,

$$j_{\mu\nu\rho} = \frac{1}{2\pi}\epsilon_{\mu\nu\rho\lambda}\partial^\lambda a\,, \qquad \partial^\mu j_{\mu\nu\rho} = 0\,. \tag{2.13}$$

In the frame of the axion, the conservation of this current is a topological condition ($dda = 0$) [121], and the lower form analogue of the Bianchi identity in electromagnetism. In due time, we will associate this conserved current with another symmetry of the axion.

On any closed 1-dimensional curve C, the topology of the axion implies that the integral over a closed curve of the field strength da is always integer,

$$\frac{1}{2\pi}\int da \in \mathbb{Z}\,. \tag{2.14}$$

Physical observables are associated with redundancy-invariant operator insertions into the partition function (2.1). The theory admits three such operators, the field strength da, and the operators,

$$W[x] = e^{ia(x)}\,, \qquad T[S]\,. \tag{2.15}$$

The operator $e^{ia(x)}$ is a Wilson 'point' operator and amounts to an ordinary source for the axion. Insertions of this operator modify the divergence of the shift symmetry current (2.12) to,

$$f^2\partial_\mu\partial^\mu a(x)\; e^{ia(y)} = \delta(x-y)e^{ia(y)}\,. \tag{2.16}$$

As da is the axion's field strength, the conservation (2.12) is the analogue of Gauss' law for axions. Wilson point insertions are said to be electrically charged under the axion, following identical conventions in electromagnetism.

2.3.1 Dual Axions

The operator $T[S]$ describes the insertion of a non-dynamical axion string. An axion string is defined as a 2-dimensional string worldsheet S around which the axion winds,

$$\int da \neq 0\,. \tag{2.17}$$

The operator $T[S]$ is constructed by excising a tube around the worldsheet S of the string and requiring that the integral $\int da \neq 0$ along a closed curve is non-zero if the curve links with S. It is clear by Stokes' law that around such a configuration $dda \neq 0$. From the same Stokes' law, it follows that axion strings are sources for the three-index current (2.13) as,

$$\partial_\mu j^{\mu\nu\rho}(x)T[S] = \delta(x-S)n_1^{[\nu}n_2^{\rho]}T[S] \tag{2.18}$$

where n_1^ν and n_2^ν indicate the unit vectors orthogonal to the surface S. Strings therefore modify the axion Bianchi identity and are magnetically charged under the axion, following similar conventions for the photon. In time, we shall consider these objects as charged under the symmetry generated by the current (2.13).

The operator $T[S]$ can be associated with the insertion of a field in the path integral similar to the Wilson points. This requires going to the so-called dual axion frame [19], where the operator $T[S]$ is the insertion of the operator,

$$e^{i\int_S B}\,, \tag{2.19}$$

where S is the worldsheet of the string and B is a 2-index anti-symmetric form referred to as the dual axion.

The dual axion frame is a change of basis of the partition function of the axion, akin to the Fourier or Legendre duality transforms. The axion partition function,

$$Z = \int Da \ \exp\left[i \int d^4x \ \frac{f^2}{2} (\partial a)^2 \right] , \tag{2.20}$$

is equivalent to the partition function,

$$Z = \int DF \ \exp\left[i \int d^4x \ \frac{f^2}{2} F_\mu F^\mu \right] , \tag{2.21}$$

supplemented by the Bianchi identity $dF = 0$ or $\epsilon^{\mu\nu\rho\sigma}\partial_\mu F = 0$, following from the original $F = da$.

This condition can be implemented by the introduction of an auxiliary 2-index anti-symmetric field $B_{\mu\nu}$,

$$Z = \int DF \ DB \ \exp\left[i \int d^4x \ \frac{f^2}{2} F_\mu F^\mu + \frac{i}{4\pi} \int d^4x \ \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} \partial_\alpha F_\beta \right] . \tag{2.22}$$

Upon using the equations of motion of B,

$$\epsilon^{\mu\nu\alpha\beta}\partial_\alpha F_\beta = 0 \implies F_\mu = \partial_\mu a \, , \tag{2.23}$$

one returns to the axion partition function (2.20).

Integrating out the field strength F_μ, one obtains the equations of motion,

$$\frac{1}{4\pi} \epsilon_{\mu\nu\rho\lambda} \partial^\nu B^{\rho\lambda} = f^2 F_\mu \sim f^2 \partial_\mu a \, , \tag{2.24}$$

where in the last identity we have used the equations of motion (2.23). This relates the axion a and dual axion $B_{\mu\nu}$.

Inserting the equations of motion (2.24) of F_μ into the partition function, we get,

$$Z = \int DB \ \exp\left[i \int d^4x \ \frac{1}{48\pi^2 f^2} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] , \qquad H_{\mu\nu\rho} = \frac{1}{2} \partial_{[\mu} B_{\nu\rho]} \, . \tag{2.25}$$

This is the frame of the dual axion B, but we emphasise that the two partition functions (2.20) and (2.25) are equivalent. The normalisation of the dual axion is chosen [122] such that the worldsheet S of axion strings with unit winding number $\frac{1}{2\pi}\int da = 1$ couples to B as $\int_S B$. The inversion of the coupling strength $f \leftrightarrow \frac{1}{f}$ in the kinetic term of (5.1) is sometimes called a 'weak/strong' duality and is a general feature of dualisations [82].

The Lagrangian should be complemented by the redundancy,

$$B \to B + d\Lambda. \tag{2.26}$$

This guarantees that the integral

$$\frac{1}{2\pi}\int dB \in \mathbb{Z} \tag{2.27}$$

is always integer valued. A massless 2-form with this gauge identification has 1 degree of freedom (d.o.f.), which matches that of the original massless axion.

The equations of motion of the dual axion are,

$$\frac{1}{4\pi^2 f^2}\partial^\mu H_{\mu\nu\rho} = 0\,. \tag{2.28}$$

Under the identification of the axion and dual axion (2.24), it is observed that this is the conservation of the three-index current (2.13). In the frame of the axion, this was a topological condition, $dda = 0$. Similarly, the conserved current (2.12) associated with the axion shift symmetry is a topological condition in the dual axion frame,

$$\frac{1}{4\pi}\epsilon_{\mu\nu\rho\lambda}\partial^\mu\partial^\nu B^{\rho\lambda} = 0\,. \tag{2.29}$$

The inversion of what is kinetic (follows from the equations of motion) and topological between frames is another general feature of dualisations and another hint that the three-index current (2.13) is also associated with a symmetry.

There are three gauge invariant quantities: the field strength $H = dB$, the Wilson point, and axion string insertions,

$$W[x]\,, \qquad T[S] = e^{i\int_S B}\,. \tag{2.30}$$

In this frame, the Wilson point insertion is done by excising a small volume around the space-time point x and demanding that $\int_M dB \neq 0$ for any three-sphere M that links with this point x. The conventions for the dual axion frame are the inverse of those of the axion frame; Wilson points are magnetically charged under the dual axion, and axion strings are electrically charged.

2.3.2 Axion Dark Matter

Phenomenologically, the original motivation for axions was rooted in the strong CP problem (reviewed in section 2.5.1), but axions have found a broader place in particle physics as solutions to several outstanding problems of the Standard Model.

One prominent motivation for axions is that they provide a natural candidate for cold, collisionless dark matter, given their weak self-interactions and interactions with ordinary matter. In fact, axions could account for all of the observed dark matter [11–13]. As axions are only feebly coupled to us, their gravitational interactions and potential dark matter background provide one of the main probes in axion searches. This section provides a review of the standard cosmological history of axion-like dark matter, which will motivate our undertakings in chapter 6.

In contrast to other dark matter candidates, axions are generically not thermally produced[2], interacting too weakly with the Standard Model bath to do so. Nonetheless, they can provide the dominant source of dark matter. If the axion saturates the late-time dark matter abundance, then there exists a unique prediction for the axion's mass in the post-inflationary scenario (see below) [123].

The cosmological history of axions uniquely depends on three quantities: the axion decay constant f, its temperature-dependent mass $m_a(T)$, and the Hubble parameter $H(T)$. In terms of these quantities, the cosmological history of the axion is dictated by the equation of motion of the axion in an expanding universe,

$$\ddot{a} + 3H(T)\dot{a} - \frac{1}{R(t)^2}\nabla^2 a + m_a^2(T)a = 0 \tag{2.31}$$

That is, the axion field obeys the equation of a simple harmonic oscillator with a damping friction term set by the expansion of the universe $H = \frac{\dot{R}(t)}{R(t)}$.

The cosmological history of standard axions arising from the spontaneous symmetry breaking of a complex scalar can be split into two distinct scenarios, respectively referred to as the pre-inflationary and post-inflationary scenarios, characterised by whether f is larger or smaller than Hubble during inflation H_I.

[2]Axions can be produced thermally from the Standard Model bath for low decay constants, for instance through pion interactions, but the thermal population is negligible for generic values of f.

In the pre-inflationary scenario, the axion decay constant f is larger than both Hubble during inflation H_I and the subsequent reheating temperature $T_{\rm RH}$, in which case the vacuum expectation value of the complex scalar field is in a spontaneously broken phase during and after inflation. The axion field will take random uncorrelated values in each causally disconnected patch of space-time. As the universe expands exponentially, the Hubble patch we find ourselves in today corresponds to a tiny patch in this initial space-time, and we therefore observe a uniform axion field[3]. This initial angle is called the axion misalignment angle.

At the start of the radiation-dominated era, the Hubble parameter is initially much smaller than the Compton wavelength of the axion $H \gg m_a(T)$ and the harmonic oscillator (2.31) is overdamped, and the axion is fixed to its initial angle due to the rapid expansion of the universe (Hubble friction). At late times, the axion mass becomes larger than the Hubble constant $H \ll m_a(T)$, and the axion rapidly rolls down to the minimum of the potential and starts oscillating. These oscillations behave like scalar particles and provide a natural source for the energy density in dark matter. For axions whose main mass contribution comes from gluons (see section 2.5), this transition occurs right before the QCD crossover and yields a late-time dark matter energy fraction density of,

$$\Omega_{\rm axion} h^2 = 2 \times 10^4 \left(\frac{f}{10^{16}\ {\rm GeV}}\right)^{\frac{7}{6}} \langle a^2_{\rm initial} \rangle \ , \tag{2.32}$$

where h is the dimensionless Hubble parameter [123]. Comparing this to the observed dark matter density $\Omega_c h^2 \sim 0.12$ [7], an order one misalignment angle would yield an axion decay constant $f \sim 10^{12}$ GeV. Combining this with astrophysical constraints [124, 125] (see section 2.4.2), which favour an axion decay constant $f \geq 10^8$ GeV, yields a QCD axion mass (see (2.65)) in the μeV to meV range.

If the axion decay constant f is smaller than either the inflationary Hubble parameter H_I or the reheating temperature $T_{\rm RH}$, then one is in the so-called post-inflationary scenario. If the axion emerges from the spontaneous symmetry

[3]Moreover, any topological defects associated with a potential SSB in the early universe will be inflated away.

breaking of a $U(1)$ symmetry, then the inflationary temperature $T_I = H_I/2\pi$ or reheating temperature were large enough to restore the symmetry. The symmetry is spontaneously broken post-inflation when $T \sim f$ and the average misalignment angle is random on small spatial scales and drawn from a uniform distribution $\langle a^2 \rangle = \frac{\pi}{3}$. This provides a unique prediction for the axion decay constant in (2.32) (and mass as explained in section 2.5.2) if the axion saturates the dark matter abundance.

Furthermore, topological defects, axion strings, appear at the symmetry-breaking scale by the Kibble-Zurek mechanism [126]. Axion strings are string-like objects that are topologically protected by the property that when a non-contractible loop in physical space is traversed, the field configurations traverse a non-contractible loop on the $U(1)$ vacuum manifold. The number of axion strings is conserved, up to singular collision events or curvatures on the string of energy densities above f.

The energy density in this string network can be transferred to the axion dark matter by string collisions and decay and contributes to the late-time dark matter energy fraction. This lowers the predicted value of the axion decay constant in eq. (2.32), thereby predicting a larger axion mass. A large number of theoretical and numerical efforts are being dedicated to calculating this late-time axion abundance in the post-inflationary scenario [87–95]. Recent simulations point to a (QCD) axion decay constant around the scale $f \sim 10^{10} - 10^{11}$ GeV [93–95], although the exact power spectrum of axions is heavily debated.

Numerical simulations show that string collisions and interactions within a given Hubble patch are extremely efficient, and the string network is expected to quickly enter a scaling regime after a few Hubble times, where the number of long strings within one Hubble patch is of order $\mathcal{O}(1)$ with a length of order the inverse Hubble parameter H^{-1} [89, 91]. In order to maintain this scaling solution in an expanding universe, string length has to be destroyed as new strings enter the Hubble patch and the energy density in the network is released in the form of axion radiation.

This scenario persists until domain walls form when the size of domain walls $1/m_a$ becomes smaller than the Hubble radius H^{-1}. The string network will either collapse or form a stable network, depending on the number of domain walls attached to each string. If each string is attached to only one domain wall, then the network decays and more energy is released into the axion abundance. The string tension is largest right before collapse, and it is expected that the majority of the dark matter is emitted shortly before the network is destroyed [95]. If the network has more than one domain wall per string, then the network is unable to contract and becomes stable, a scenario that is phenomenologically ruled out, unless the degeneracy between the domain walls attached to the string is subsequently broken.

2.4 Axions & Photons

This thesis is entirely devoted to understanding and constraining the axion's properties and interactions in order to guide the numerous experimental axion searches. Axions are generically very light, and we expect them to survive at low-energy scales and interact with other light or massless degrees of freedom of the Standard Model. At energies below the electroweak symmetry-breaking scale ~ 246 GeV, the Standard Model admits one Abelian massless gauge field, the photon. The axion-photon interactions provide one of the main probes for detecting axions, and this section provides an introduction to the axion-photon coupling and current experimental efforts. Our inability to constrain these interactions will naturally lead us to generalised symmetries in chapters 3 and 4.

The Lorentz group does not admit a four-vector representation for a photon operator A_μ on a Hilbert space as a linear combination of creation and annihilation operators of helicity ± 1 particles, unless that Hilbert space has a non-trivial redundancy. In the field theory formalism, this becomes a redundancy,

$$A_\mu \to A_\mu + ie^{i\lambda}\partial_\mu e^{-i\lambda}\,. \tag{2.33}$$

where $e^{i\lambda}$ is an element of the gauge group[4] $U(1)$.

[4]The fact that such redundancies can be constructed from ordinary symmetries has led to the confusing notion of 'gauge symmetries', but no symmetry or conserved quantity is associated with the redundancy. In the presence of a boundary, it is sometimes stated that the 'gauge symmetry' acquires a global part, acting as a true symmetry of the theory. It is the view of this author that this notion should also be replaced with that of the higher-form symmetries of the gauge theory (see section 3.4) reducing to an ordinary symmetry on the boundary, as discussed in [47]

The action for a massless vector field that is invariant under the redundancy (2.33) is the Maxwell action,

$$S = -\frac{1}{4e^2}\int d^4x\ F_{\mu\nu}F^{\mu\nu}\,, \tag{2.34}$$

where we have defined the field strength $F = dA$ and introduced the electric coupling constant e, which gives the fine structure constant $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$ [127].

The equations of motion of the Maxwell theory imply the conservation of,

$$j_{\mu\nu} = F_{\mu\nu}\,, \qquad \partial_\mu F^{\mu\nu} = 0\,, \quad \text{or} \quad d\star F = 0\,. \tag{2.35}$$

This is Gauss' law, the statement that electric field lines cannot end in the absence of charges. In topological terms, if Σ is any closed co-dimension 2 manifold (usually taken to be a sphere), Gauss' law states that the electric flux measured through the surface is conserved under topological deformations of Σ to another surface Σ',

$$\int_\Sigma \star F - \int_{\Sigma'} \star F = \int_S d\star F = 0\,. \tag{2.36}$$

where S is a surface bounded by Σ and Σ'. If the surfaces Σ are taken to be spheres on differing time slices, then this implies the conservation of electric field lines.

The second conserved quantity,

$$j_{\mu\nu} = \frac{1}{4\pi}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\,, \tag{2.37}$$

is trivially conserved by the Bianchi identity,

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\nu F^{\alpha\beta} = 0\,, \quad \text{or} \quad dF = 0\,. \tag{2.38}$$

This is equivalent to the fact that the number of magnetic field lines that pierce a 2-dimensional surface is preserved under topological deformations of that surface.

In line with the axion theory, the integral of the field strength $F = dA$ over closed manifolds of appropriate dimension is integral,

$$\frac{1}{2\pi}\int F \in \mathbb{Z}\,, \qquad \frac{1}{8\pi^2}\int F\wedge F \in \mathbb{Z}\,, \quad \text{or} \quad \frac{1}{16\pi^2}\int d^4x\ F_{\mu\nu}\widetilde{F}^{\mu\nu} \in \mathbb{Z}\,, \tag{2.39}$$

where we have introduced the dual field strength,

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,. \tag{2.40}$$

The physical observables[5] in this QFT are constructed out of operators invariant under the redundancy, called gauge invariant operators. The gauge-invariant operators are the field strength $F = dA$ and Wilson and 't Hooft loops associated with closed curves C on the manifold,

$$W[C] = \exp\left[i\int_C A\right]\,, \qquad H[C] = \exp\left[i\int_C \hat{A}\right]\,. \tag{2.41}$$

The insertion of a Wilson loop $W[C]$ in the partition function (2.1) can be seen to be equivalent to the insertion of a background current j coupled to the gauge field as,

$$S \supset \int d^4x\; A_\mu j^\mu\,. \tag{2.42}$$

The current is a generalisation of the delta function that is only non-zero on the curve C and points along the curve. This current can be constructed using the so-called Poincaré dual to the curve C. Poincaré duality states that any closed manifold can be associated with a conserved current and vice versa. More precisely, Poincaré duality is the statement that to any $D-p$ dimensional closed manifold called C, there exists a closed p-form $\widetilde{C}$ such that,

$$\int_C A = \int_X A \wedge \widetilde{C}\,. \tag{2.43}$$

The current can then be constructed as $j = \star\widetilde{C}$ and is conserved by the closure of $\widetilde{C}$. If the closed curve C is in the time direction, the interpretation of the Wilson line is the insertion of an infinitely massive charged particle.

The 't Hooft $H[C]$ loop (a $D-3$ dimensional surface) corresponds to the insertion of an infinitely massive magnetic particle if oriented along the time direction. It is tentatively written in terms of the dual gauge field $\hat{A}$ in (2.41), but the appropriate definition in the electric frame is as the excision of a tube along the particle worldline C, with boundary conditions that specify a unit magnetic flux through any surface Σ linking the loop. We will soon (section 3.4) associate both conserved quantities (2.35) and (2.38) with symmetries of the Wilson and 't Hooft loops.

[5]To adequately describe a gauge field with a redundancy (2.51) on a general space-time X, one needs to cover this space-time with coordinate patches (open sets) U_i with field values a_i. On the intersection between two open sets $U_i \cap U_j \neq 0$, one additionally has to specify transition functions t_{ij} with $A_{\mu,i} = A_{\mu,j} + it^\dagger_{ij}\partial_\mu t_{ij}$ with $t_{ij}(x) \in U(1)$. Gauge-invariant observables are those that do not depend on the choice of patches.

2.4.1 Axion-Photon Interactions

The interactions between axions and photons are highly constrained by the redundancies of the QFTs, and a large part of the axion's compelling features derives from the special topological nature of the axion-photon coupling.

To second order in the photon field, the most general axion-photon Lagrangian is of the form,

$$\mathcal{L} = -\frac{1}{4e^2(a)} F_{\mu\nu}F^{\mu\nu} + \frac{f^2}{2}(\partial a)^2 - V(a) + \frac{g(a)}{16\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu} \,. \tag{2.44}$$

The functions $e^2(a)$, $V(a)$ and $g(a)$ are respectively the axion-dependent electric coupling constant, the axion potential, and the axion-photon coupling.

We will not have much to say about the axion-dependent electric coupling constant $e^2(a)$, which modifies Gauss' law in an axion-dependent way. By CP symmetry, it is of even order in the axion, and any axion-dependent corrections are suppressed by $\left(\frac{1}{f^2}\right)^n$ compared to the mean value of the gauge coupling. Moreover, it is periodic in the axion $e^2(a + 2\pi) = e^2(a)$ (see however [121]), making it less interesting than its monodromic counterpart $g(a)$. In supersymmetric (SUSY) completions of (2.44), both the gauge coupling constant and axion-photon coupling become part of a single constant $\tau(a) = \frac{g(a)}{2\pi} + \frac{4\pi i}{e^2(a)}$ [109, 110, 128].

The potential of the axion, $V(a)$, whilst also periodic in the axion, is of greater interest. It determines the mass of the physical axion,

$$m_a^2 = \frac{1}{f^2}\frac{d^2V(a)}{da^2} \,, \tag{2.45}$$

and is therefore one of the main interests in axion searches, as reviewed in section 2.3.2 and 2.4.2 . The presence of a potential modifies the equations of motion of the axion,

$$f^2\partial^2 a = -\frac{dV(a)}{da} \tag{2.46}$$

and explicitly breaks the axion shift symmetry $a \to a + c$. In this light, the quality of the axion's shift symmetry can be used to constrain the axion's potential.

The third coupling, $g(a)$, is referred to as the monodromic axion-photon coupling.

In contrast to the potential $V(a)$, this coupling is allowed to transform up to a monodromic charge n as,

$$g(a+2\pi) = g(a) + 2\pi n \,, \qquad n \in \mathbb{Z} \,. \tag{2.47}$$

Upon such a transformation, the action (2.44) changes by,

$$S \to S + \frac{2\pi n}{8\pi^2} \int \mathrm{Tr}\,(F \wedge F) \;. \tag{2.48}$$

This integral is integer valued by equation (2.39) and therefore the partition function weight e^{iS} is left-invariant. To be more precise, the quantisation of the monodromic charge depends on the global structure of the gauge group [35, 129], which we will briefly return to in chapter 4.

The quantisation of the monodromic charge n can also be shown by topological arguments similar to Dirac's argument for quantisation of electric charge in $U(1)$ gauge theory [130]. This leads to a more physical interpretation as to why n has to be integer valued. Consider studying the change of the electric charge of a magnetic monopole as it traverses a path along which the axion winds by $a \to a + 2\pi$. In this case, the axion-photon coupling traverses $\int dg(a) = g(a+2\pi) - g(a) = 2\pi n$. The worldline of the monopole represents a non-trivial background flux $\int F = 2\pi$. In this background, the action on the monopole worldline changes as,

$$\exp\left(-\frac{i}{8\pi^2}\int dg(a) \wedge F \wedge A\right) \to \exp\left(-in\int_{\text{Monpole}} A\right) \tag{2.49}$$

The monopole worldline is therefore a source for a Wilson line. The Wilson line needs to have integer charges to be gauge-invariant, thus $n \in \mathbb{Z}$.

The fact that magnetic monopoles acquire an electric charge in the presence of an axion background is referred to as the Witten effect [131]. This effect shows that the axion shift symmetry $a \to a + c$ is absent in the presence of a non-zero axion-photon coupling $g(a)$, and will eventually motivate the introduction of non-invertible symmetries in section 3.3.

2.4.2 Axion Experiments

There is an ever-growing experimental program [21, 22] dedicated to finding the axion on the basis of its gravitational or electromagnetic interactions.

As the axion is only feebly coupled to us, given that $f \geq 10^8$ GeV as favoured by astrophysical constraints [124, 125], a large fraction of the experimental efforts aim to detect axions through smaller-scale resonant searches [132] (for a review see [133]). Resonant searches often involve a microwave cavity with conducting boundaries. Due to the conducting walls, the natural frequency of any electric fields inside the cavity will be set by the inverse size of the cavity. A strong magnetic field is applied inside the cavity, which allows an axion background, such as axion-like dark matter, to convert into photons/electric fields by the axion photon coupling (2.44),

$$\mathcal{L} \supset \frac{a}{16\pi^2} F\widetilde{F} = -\frac{a}{4\pi^2}\ \vec{E}\cdot\vec{B}\,. \tag{2.50}$$

This conversion of dark matter axions into electric fields is resonantly enhanced if the frequency of the created electric fields, set by the mass of the axion, is near the size of the natural frequency of the cavity. These experiments are referred to as axion *haloscope* experiments and include ADMX [134], MADMAX [135], and DALI [136].

Axions can also be created in astrophysical sources (for a recent review see [124]). This includes Primakoff processes in the sun when photons interact with the electric fields of the plasma to create axions. These solar axions are then detected by aiming a 'telescope' at the sun with the lens cap on so that no photons enter the telescope. Inside the telescope, a large magnetic field converts the axions back into photons in the X-ray range, which in turn can be detected. These experiments are called axion *helioscopes* [132] and include the CAST/IAXO collaboration [137, 138]. The production of axions in stars also places a lower bound on the coupling $\sim \frac{1}{f}$ of $f \gtrsim 10^7$ GeV [124] in light of known stellar lifetimes. Further astrophysical constraints on the axion parameter space can be obtained from axion production in supernovea [139, 140], generally favouring $f \gtrsim 10^8$ GeV.

Finally, light axions can extract angular momentum from black holes in a phenomenon called superradiance [141–143]. This process is most efficient for axions with a Compton wavelength of the size of the black hole, or $m_a \sim \frac{1}{GM_{\rm BH}}$. The observation of both spinning stellar and supermassive black holes constrains the axion parameter space for light axion masses of respectively 10^{-13} eV $< m_a < 10^{-11}$ eV and 10^{-19} eV $< m_a < 10^{-17}$ eV [143].

The parameter space axion experiments have to cover spans more than 10 orders of magnitude in photon-couplings and masses. The vast size of this parameter space appears impractical, especially when the most sensitive experiments are resonant searches, which have to be carefully tuned the microwave cavity sizes to the specific mass. An important theoretical guide in identifying promising search regions for experiments is symmetries. In the next chapter, we will identify the relevant symmetries of the axion as *generalised symmetries*.

2.5 Axions & Gluons

One of the main probes in experimental searches for the axion is the axion mass. In general, the axion obtains a mass when its shift symmetry is explicitly broken. In order to increase our theoretical control on the axion parameter space, we review standard scenarios for non-perturbative axion shift symmetry breaking by non-Abelian gauge theories, such as the Standard Model gluons, in this section. This will naturally lead to a discussion on the strong CP problem in section 2.5.1, the introduction of the QCD axion in section 2.5.2, the axion quality problem in section 2.5.3, and instantons in section 2.5.4.

Non-Abelian gauge theories are a natural generalisation of the Abelian gauge fields of the previous section transforming as,

$$A_\mu(x) \equiv U A_\mu(x) U^\dagger + iU\partial_\mu U^\dagger \,, \tag{2.51}$$

where U is an element of a gauge group G. The gauge fields A^μ are matrices valued in the Lie algebra of G. If G has Lie algebra generators T^a (normalised as $\mathrm{Tr}(T^aT^b) = \frac{\delta^{ab}}{2}$), then the gauge field can be decomposed as $A^\mu = A^\mu_a T^a$. .

The action compatible with this redundancy is called the Yang-Mills (YM) action,

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\,. \tag{2.52}$$

where the trace is in the adjoint representation and the field strength $F_{\mu\nu}$ transforms in the adjoint of G. The constant g is the so-called coupling constant.

The equations of motion of the non-Abelian gauge theory and Bianchi identity are,

$$D_\mu F^{\mu\nu} = 0\,, \qquad \frac{1}{4\pi}\epsilon^{\mu\nu\alpha\beta} D_\nu F_{\alpha\beta} = 0\,, \qquad D_\mu = \partial_\mu - i A^a_\mu T^a_{\text{adj}}\,. \tag{2.53}$$

We see that Gauss' law and the Bianchi identity include higher-order terms in the gauge field in order to account for the transformations of $F_{\mu\nu}$ under the gauge group G. The gauge fields are charged under themselves, producing a highly non-trivial classical and quantum field theory and significantly modifying the symmetry structure compared to Maxwell. The Standard Model admits one 'massless' non-Abelian gauge theory below the electroweak symmetry-breaking scale ~ 246 GeV. The gauge bosons are referred to as gluons and the gauge group as the colour group $G = SU(3)_C$. The Standard Model fermions that carry colour charge are called quarks, and the Standard Model sector describing these quarks and the gluons is referred to as the strong sector or quantum chromodynamics (QCD).

One non-trivial consequence of the interacting structure of the non-Abelian gauge theory is that the effective coupling constant g depends on the energy scale μ of the process even in the absence of charged matter. Between two energy scales μ and Λ, the gauge coupling differs as,

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda)} - \frac{11}{3}\frac{C(\text{adj})}{(4\pi)^2}\log\frac{\Lambda^2}{\mu^2}\,. \tag{2.54}$$

We observe that even if we have excellent perturbative control over g at some high-energy scale Λ, then there exists a scale $\mu \sim \Lambda_{\text{QCD}}$ where the gauge coupling becomes order 1. It is said that the gauge coupling runs *strong*. In contrast to Abelian gauge theories, the coupling becomes weaker at higher-energy scales, referred to as asymptotic freedom. At high enough energy scales, the Yang-Mills theory is therefore well described by a set of massless interacting gauge fields.

At low energies, or large distance scales, the coupling g runs strong, and the theory is well described by massive glueball states, composite states formed out of the highly interacting gauge fields. This phase is called the confining phase, and much of modern physics has been devoted to understanding the mechanism by which theories confine and their resulting spectrum.

Powerful probes, in both the free and confining phases, are the Wilson loops and 't Hooft loops,

$$W[C] = \mathrm{Tr}\, \mathcal{P} \exp\left[i \int A\right], \qquad H[C]. \tag{2.55}$$

where $\mathcal{P}$ is a path-ordering on the non-commuting matrices. The trace shows that a charged particle is no longer a gauge-invariant state; instead a Wilson loop should be thought of as a traced density matrix over all colour states of an infinitely massive charged particle. The 't Hooft loop is associated with charges in the dual lattice of the gauge group [35, 104, 144], but we will not require its formal introduction.

In line with the Abelian gauge theory, the integral of the field strength over closed manifolds of appropriate dimension is integral,

$$\frac{1}{2\pi}\int \mathrm{Tr}\,(F) \in \mathbb{Z}, \qquad \frac{1}{8\pi^2}\int \mathrm{Tr}\,(F\wedge F) \in \mathbb{Z} \quad \text{or} \quad \frac{1}{16\pi^2}\int d^4x\ \mathrm{Tr}\left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right) \in \mathbb{Z}, \tag{2.56}$$

The non-linear structure of the field strengths means that these integrals are very different from their Abelian counterparts.

2.5.1 The Strong CP Problem

Historically, the motivation for considering axions and their couplings to the Standard Model gluons is rooted in the strong CP problem [8–10]. At its heart, the strong CP problem challenges us to explain why the strong force distributes the electric charges of the quarks inside the neutron with extraordinary precision, remaining homogeneous up to at least one part in ten billion. This homogeneity is measured by the electric dipole moment (EDM) d_n of the neutron, which is currently observed [1] to be less than,

$$|d_n| \leq 10^{-26}\ \mathrm{e}\cdot\mathrm{cm}. \tag{2.57}$$

This extraordinary uniformity suggests that a symmetry is at play. One candidate symmetry is parity P, a reversal of the spatial coordinates $x \to -x$, under which the EDM vector flips sign compared to the neutron spin vector. A second candidate symmetry is CP (charge-parity), under which the spin vector changes sign but the EDM remains unchanged. As we have not measured two distinct neutron parity or CP states, parity or CP symmetry forbids a non-zero EDM.

In the Standard Model, the strong sector, responsible for the distribution of quarks inside the neutron, has only one term that is variant under both P and CP,

$$S \supset \int d^4x \, \frac{\theta}{16\pi^2} \mathrm{Tr}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)_{\mathrm{QCD}} \tag{2.58}$$

The angle $\theta \sim \theta + 2\pi$ is a Standard Model parameter and can be thought of as a non-dynamical axion. The existence of an EDM is therefore directly related to the angle[6] θ. A calculation of the resulting EDM can be found in [125, 145] as,

$$|d_n| \approx 10^{-16} \, \theta \; \mathrm{e} \cdot \mathrm{cm} \, . \tag{2.59}$$

The strong CP problem is then the question: *why is θ so small?*,

$$\theta \pmod{2\pi} \leq 10^{-10} \, . \tag{2.60}$$

This question turns into a problem when one appreciates that neither parity nor CP are symmetries of the electroweak sector of the Standard Model. The field content and representations of the Standard Model maximally break parity symmetry [146, 147] by assigning weak charges to so-called 'left-handed' quarks and leptons but assigning none to their would be parity partners so-called 'right-handed' quarks and leptons.

The field content of the Standard Model is compatible with a CP symmetry as 'left-handed' quarks and leptons are paired with 'left-handed' anti-quarks and anti-leptons. It is, however, an experimental fact that CP symmetry is violated in the weak sector [148, 149]. This violation is measured by the CKM phase δ_{CKM},

$$\sin(\delta_{\mathrm{CKM}}) \propto \mathrm{Im} \; \mathrm{Det}[y_u y_u^\dagger, y_d y_d^\dagger] \, , \tag{2.61}$$

[6]Here θ should really be replaced by the field redefinition invariant combination $\overline{\theta} = \theta + \mathrm{Arg\ Det}\ y_u^\dagger y_d^\dagger$ in the rest of the discussion, where y_u and y_d are the up and down quark Yukawa matrices.

where y_u and y_d are the Standard Model up and down quark Yukawa matrices.

At this point, nothing prevents us from setting $\theta = 0$ at some scale. The breaking of CP in the electroweak sector yields a non-zero contribution to θ at low scales, but this is extremely suppressed in the Standard Model due to the approximate flavour symmetries (the first contribution above the electroweak symmetry-breaking scale is at 7-loop order) [150, 151]. Below the electroweak symmetry breaking scale, this corresponds to the cheburashka diagram and implies a lower bound [152] on θ of $\sim 10^{-19}$. In this sense, a zero value for the θ angle is technically natural. However, this is unique to the Standard Model field content; GUT or SUSY extensions of the Standard Model can induce running effects at 1-loop order [125].

In the absence of a minimal solution, we refer to the apparent absence of a possible CP violation in the strong sector as the strong CP problem.

There exist several non-axionic solutions to the strong CP problem. The simplest solution is a massless up quark. The neutron EDM depends on the quark mass combination,

$$|d_n| \propto \frac{m_u m_d}{m_u + m_d}\,. \tag{2.62}$$

This vanish when the up quark is massless even when $\theta \neq 0$.

The massless up quark solution requires an additional $U(1)$ chiral symmetry $u \to e^{i\alpha\gamma_5}u$ of the quark, preventing a mass term. In many ways, this solution can be thought of as a proto-axion solution. Current lattice simulations rule out a massless up quark solution compatible with the low energy mass of the up quark [153, 154]. Other prominent solutions to the strong CP problem involve a spontaneous breaking of either parity [155, 156] or CP at a high-energy scale Λ in extension of the Standard Model, in such a way that $\delta_{\text{CKM}} \neq 0$ and $\theta = 0$. In this class, Nelson-Barr [157, 158] has historically been well-regarded, but most models require either additional symmetries and/or an extremely tight coincidence of scales [159].

2.5.2 The QCD Axion

The potential existence of axions motivates us to take a closer look at how the spectrum of Yang-Mills changes when θ is made dynamical. One property of the YM spectrum is the θ-dependent energy of the vacuum state $|\Omega\rangle$ of QCD, which can be calculated in terms of an *Euclidean*[7] path integral,

$$e^{-V(\theta)} = \langle\Omega|e^{-TH}|\Omega\rangle = \int DA\, \exp\left[-S_{\mathrm{YM}}\right] \exp\left[i\int d^4x\, \frac{\theta}{16\pi^2} \mathrm{Tr}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)_{\mathrm{QCD}}\right] \tag{2.63}$$

Importantly, the parity odd topological term $\theta F\widetilde{F}$ has a complex factor of i in Euclidean space-times. The gluonic measure DA and weight $\exp\left[-S_{YM}\right]$ are positive definite and therefore the addition of a phase θ can only serve to increase the vacuum energy $V(\theta) \geq V(0)$. This holds for a general particle spectrum as long as that spectrum does not contain any fermions in chiral representations and is known as the Vafa-Witten theorem [160, 161].

In this context, the Vafa-Witten theorem tells us that, if made dynamical, the angle θ would naturally relax to zero. This is the origin [8–10] of the axion solution to the strong CP problem, in which QCD is complemented by an axion a with a coupling,

$$\exp\left[i\int d^4x\, \frac{(a+\theta)}{16\pi^2} \mathrm{Tr}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)_{\mathrm{QCD}}\right] \tag{2.64}$$

In the absence of any other contributions to the axion potential, this theory will naturally relax to a zero EDM for the neutron.

If the axion is to solve the strong CP problem, then we will refer to it as the QCD axion. The dependence of the strong vacuum (eq. (2.63)) on θ implies a (temperature dependent) mass for the axion, which at low temperatures is [123],

$$m_a = 5.7(1)\ \mu\mathrm{eV}\ \left(\frac{10^{12}\ \mathrm{GeV}}{f}\right). \tag{2.65}$$

This mass implies that the axion is extremely light for generic values of $f \geq 10^8$ GeV favoured by astrophysical constraints [124, 125].

[7]Euclidean path integrals are generally used for state preparation and have a deep connection to non-perturbative physics.

Intriguingly, the parameter space of QCD axion experiments can be constrained if the electromagnetic and colour gauge groups are unified at a large scale [14, 15] such as $SU(5)$ grand unified theories (GUT) or simple heterotic $E_8 \times E_8$ string theory models. In this case, the ratio of the monodromic integers n in the axion's coupling to photons and gluons is known, similar to how the allowed representations of low-energy particles intimately depend on the high-energy completion of these gauge groups. The gluon coupling is the main contributor to the QCD axion's mass, and therefore, GUT theories imply a tight relation between the axion's mass and coupling to photons. Axions can therefore be used as probes of the high-energy gauge group, and test or rule out simple models of Grand Unification and String Theory.

2.5.3 Axion Quality

The QCD axion will generically receive additional contributions to its potential from sectors decoupled from QCD, which in turn can *misalign* the minimum of the potential such that in the vacuum $a+\theta \neq 0$. The ability of a QCD axion to solve the strong CP problem therefore relies on the quality of its continuous shift symmetry. For generic axions, this quality can be plagued by several extreme sensitivities to UV physics. This has been dubbed the QCD axion quality problem [64, 65]. This problem challenges us to explain the absence of local operators breaking the axion shift symmetry, which are expected to be present due to the absence of an axion shift symmetry at low energies and due to quantum gravitational effects [42].

The axion effective field theory (EFT) does not admit a shift symmetry in the presence of the strong sector due to the axion dependence of the vacuum energy. In the EFT, this implies that any operator consistent with the particle content and symmetries of the theory should generically be present and could generate a potential for the axion $V_{\text{break}}(a)$ (we will review these arguments in section 3.1). The requirement that the QCD axion solves the strong CP problem for an order $\mathcal{O}(1)$ angle θ compatible with a combined angle $a+\theta$ that has to be smaller than 10^{-10} means that any misaligned sources have to be extremely suppressed,

$$V_{\text{break}} \lesssim 10^{-10} f^2 m_a^2 \,, \tag{2.66}$$

For a generic axion, including the QCD axion, quantum gravity is believed to break all global symmetries [42]. In this case the breaking potential is expected to have a lower bound of [64],

$$V_{\text{break}} \gtrsim \frac{f^n}{M_{Pl}^{n-4}} e^{ina} + \text{h.c.} \tag{2.67}$$

where n is the number of axion insertions involved in the breaking. Astrophysical constraints favour axion decay constants $f \gtrsim 10^8$ GeV [124, 125], which in turn requires such gravitational sources to be suppressed so that the dimension of the leading operator is $n \geq 9$ in order to be consistent with (2.66).

Many theoretical efforts have been dedicated to alleviating the axion quality problem. Broadly speaking, efforts in four-dimensional field theory have mainly focused on recovering the axion shift symmetry as an accidental symmetry of the particle content of the theory, such that the gauge charges forbid dangerous operators. Examples include models with additional gauge redundancies like $\mathbb{Z}_N$ [66, 67] or $U(1)$ (see for example [68]), and composite axion models from the confinement of additional gauge groups [69–78]. While the latter models are able to explain the large separation between the axion decay constant f and electroweak symmetry breaking scale (~ 246 GeV) as dynamically generated, these models require involved model building. Extra-dimensional efforts are discussed in chapters 3, 4 and 6.

2.5.4 Instantons

The requirement that additional shift symmetry breaking contributions to the axion potential should be suppressed compared to naive EFT estimates is a general feature that plagues axion models beyond just those that aim to solve the strong CP problem. It will therefore be helpful to get a handle on how gauge theories contribute to the potential of a general axion $V(a)$ in both the weakly and strongly coupled regime. An incredibly useful lower-dimensional analogue of equation (2.63) that makes calculations tractable in the strongly coupled regime is an axion a coupled to electromagnetism in $D = 1 + 1$ as [162, 163],

$$\mathcal{L} \supset -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{a}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu} \, , \tag{2.68}$$

The electromagnetic potential $V(r)$ between two charged particles in $D = 1+1$ scales linearly with distance $V(r) \sim r$ and therefore particles are said to be confined, in analogue with QCD at strong coupling. The electromagnetic coupling strength e^2 has energy dimensions 2.

In the absence of charged particles, electromagnetism in $D = 1+1$ admits no propagating waves, as there are no transverse directions, and the theory seems empty. Instead, the theory becomes interesting when compactified on a spatial circle of radius R. In this case, the Aharonov-Bohm phase of the gauge field A around the circle becomes a degree of freedom,

$$q(t) = \int_{S^1} A\,, \tag{2.69}$$

which behaves as another axion $q \sim q + 2\pi$. In fact, the $D = 1+1$ Abelian gauge theory on a spatial circle is equivalent to that of an axion q in $D = 1$. In the presence of the axion-photon coupling (2.68), the action is

$$S = \int dt\, \frac{f^2}{2}\dot{q}^2 + \frac{a}{2\pi}\dot{q}\,, \qquad f^2 = \frac{1}{e^2 2\pi R}\,. \tag{2.70}$$

The potential of the axion a is the dependence of the lowest energy eigenstate of q on the axion a, which for a particle on a circle with action (2.70) yields,

$$V(a) = \frac{e^2}{2}\min_{n\in\mathbb{Z}}\left(n - \frac{a}{2\pi}\right)^2\,. \tag{2.71}$$

This potential is rather interesting. It has an infinite number of branches parametrised by the integer n. Under a shift $a \to a + 2\pi$, the potential remains invariant but only because we move from one branch to the next branch. This kind of behaviour is called monodromic, and the potential is called a monodromic potential for the axion a. It can be shown that this potential is an extremely good approximation for the QCD potential for an axion at strong coupling for large gauge groups in the absence of light quarks [117, 118]. Additional properties of the strongly coupled spectrum that depend on the axion a are the masses of mesons and the number of meson states below a given energy [163].

We wish to redo this calculation at weak-coupling e^2 by adding a mass $m^2 \gg e^2$ to the particle q to account for generic instanton contributions to the axion potential,

$$\mathcal{L} \supset \frac{f^2}{2}\dot{q}^2 - m(1-\cos q) + \frac{a}{2\pi}\dot{q}\,. \tag{2.72}$$

This is the classical problem of a pendulum with angle q in a gravitational field.

We wish to calculate how the vacuum energy depends on a. Classically, a slow-moving pendulum will never wind around the full circle due to the potential. Quantum mechanically, however, there is a small non-zero contribution to the vacuum energy from paths that tunnel through the potential barrier, which in turn contribute to the potential of the aixon. Quantum tunnelling is another feature of QFT that cannot be captured in any perturbative approach to the partition function at finite order [164]. A general observable, such as the energy of the vacuum, would in a perturbative series be expressed as a Taylor series around $g = 0$,

$$\langle 0|H|0\rangle = \sum_{n=0}^{\infty} c_n g^{2n}\,. \tag{2.73}$$

We will find that in the presence of quantum tunnelling, the vacuum energy scales as,

$$\exp\left(-\frac{1}{g^2}\right)\,, \tag{2.74}$$

which admits no such perturbative series. For our purposes, $\frac{1}{g^2} \sim f\sqrt{m}$.

To show this, we calculate the vacuum energy by an *Euclidean* path integral to evolve from the semi-classical state $|0\rangle$ defined as $a = 0 \bmod 2\pi$ to the same state as done in [164, 165]. This state will have overlap with the true vacuum state $|\Omega\rangle$ and for long Euclidean times T the path integral reduces to,

$$\langle 0|e^{-HT}|0\rangle = |\langle 0|\Omega\rangle|^2 e^{-TE_{\text{vac}}}\,, \tag{2.75}$$

where

$$\langle 0|e^{-TH}|0\rangle = \int Dq\,\exp\left[-\int dt\,\left(\frac{f^2\dot{q}^2}{2} + V(q) - i\frac{a}{2\pi}\dot{q}\right)\right]\,. \tag{2.76}$$

The different paths satisfying $q(T) = 0 \bmod 2\pi$ can be broken up into distinct sectors corresponding to paths starting at $q = 0$ and winding a certain number of times n around the circle until $q(T) = 2\pi n$. Each sector has a different topological charge,

$$n = \frac{1}{2\pi}\int_0^T dt\,\dot{q}\,, \tag{2.77}$$

and contributes to the partition function as,

$$\sum_n \int_{q(0)=0}^{q(T)=2\pi n} Dq \, \exp\left[-\int dt \, \left(\frac{f^2}{2}\dot{q}^2 + V(q)\right)\right] e^{ina} \,. \tag{2.78}$$

Importantly, in order to describe a sector with winding number n, we need n transition functions of 2π from $t = 0$ to $t = T$.

The path integral will be dominated by the classical solution $q_{\text{cl}}(t)$ in any given sector, and we therefore split our path as,

$$q(t) = q_{\text{cl}}(t) + \delta q(t) \,, \qquad \delta q(0) = \delta q(T) = 0 \,. \tag{2.79}$$

In order to find the classical solution, notice that the cosine potential $V(q)$ admits a BPS form,

$$V(q) = \frac{1}{2}\left(\frac{dW(q)}{dq}\right)^2 , \qquad W(q) = -4\sqrt{m}\cos\left(\frac{q}{2}\right) . \tag{2.80}$$

This allows us to complete the square on the action as,

$$\int dt \, \frac{f^2}{2}\dot{q}^2 + \frac{1}{2}\left(\frac{dW}{dq}\right)^2 = \frac{1}{2}\int dt \left(f\dot{q} \pm \frac{dW}{dq}\right)^2 \mp f\int dW \,. \tag{2.81}$$

In a given winding sector, $\int dW$ is fixed, and therefore the classical action in this sector can be obtained by minimising the positive term $\left(\dot{q} \pm \frac{dW}{dq}\right)^2$. The classical action is then provided by,

$$f\dot{q} = \mp\frac{dW}{dq} = \mp 2\sqrt{m}\sin\left(\frac{q}{2}\right) . \tag{2.82}$$

This solution describes an instanton, a classical solution that interpolates between two states (or the same state) of the QFT in Euclidean space-time. The action cost for a single instanton is,

$$e^{-f|\int dW|} = e^{-8f\sqrt{m}} \,. \tag{2.83}$$

Let us calculate the instanton solution for unit winding $n = 1$,

$$f\dot{q} = 2\sqrt{m}\sin\left(\frac{q}{2}\right) \implies q(t) = 4\arctan\left(\exp\left[t\frac{\sqrt{m}}{f} + c_1(T)\right]\right) . \tag{2.84}$$

where $c_1(T)$ is fixed to satisfy $q(T) = 2\pi$.

The contribution of this instanton to the partition function can be calculated by expanding the action S around the classical solution to second order in $(\delta q)^2$,

$$e^{ia}e^{-8f\sqrt{m}}\int_{\delta q(0)=0}^{\delta q(T)=0} Dq\,\exp\left[-\int dt\,\left(\frac{f^2\,(\delta\dot{q})^2}{2}+\frac{1}{2}\frac{d^2V(q_{\rm cl})}{d^2q}\,(\delta q)^2\right)\right]\,. \qquad (2.85)$$

Since the classical action only deviates from the vacuum value for a short time $t \sim \frac{f}{\sqrt{m}}$, we can approximate the remaining partition function in δq as the partition function of a harmonic oscillator of frequency $\omega^2 \sim \frac{m}{f^2}$. Adding in a corresponding fudge factor K close to unity yields our final result for a single instanton,

$$e^{-8f\sqrt{m}}\sqrt{\frac{\omega}{\pi}}e^{-\omega T/2}K\,,\qquad K^2=\frac{\det(f^2\frac{d^2}{dt^2}+m)}{\det\left(f^2\frac{d^2}{dt^2}+\frac{d^2V(q_{\rm cl})}{d^2q}\right)}\,. \qquad (2.86)$$

A general path from $q(0) = 0$ to $q(T) = 2\pi n$ can contain any number of instantons ℓ and anti-instantons $\bar{\ell}$, as long as their sum, $\ell+\bar{\ell}=n$. Each instanton has a width in time of $\frac{f}{\sqrt{m}}$ and the solutions are well separated for large $m^2 \gg e^2$. Each instanton is weighted by the action cost (2.86) where T corresponds to the time between successive instantons.

A general partition function includes the sum over all such instantons for a given winding sector, often referred to as an instanton gas calculation. Any instanton can be located at any time between $t = 0$ and $t = T$. As instantons are indistinguishable, the integral over all such instanton positions introduces the geometric counting factor $\frac{T^{\ell+\bar{\ell}}}{\ell!\bar{\ell}!}$ [165]. The true *Euclidean* path integral is therefore,

$$e^{-TE_{\rm vac}}=\sqrt{\frac{\omega}{\pi}}e^{-\omega T/2}\sum_n\sum_\ell\sum_{\bar{\ell}}e^{ina}\frac{T^{\ell+\bar{\ell}}}{\ell!\;\bar{\ell}!}\delta_{\ell+\bar{\ell},n}\left(e^{-8f\sqrt{m}}K\right)^{\ell+\bar{\ell}}\,. \qquad (2.87)$$

Performing the triple sum yields the vacuum expectation value,

$$V(a)=\frac{\omega}{2}+2Ke^{-S_{\rm cl}}\cos a\,,\qquad S_{\rm cl}=8f\sqrt{m}\,. \qquad (2.88)$$

We see that the vacuum energy at weak coupling is that contribution from the mass of q, $\omega \sim \frac{\sqrt{m}}{f}$ in its local minimum, complemented by the instanton contribution. It is clear that this instanton approximation breaks down as $f \to 0$ or $e^2 \to \infty$, in which case we return to the strongly coupled regime (2.71).

Non-Abelian gauge theories admit a similar expansion in terms of instanton solutions at weak coupling. A standard argument relies on completing the square on the Yang-Mills action [166, 167],

$$S_{\text{YM}} = \frac{1}{4g^2}\int d^4x \ \text{Tr}\left(F_{\mu\nu} \mp \widetilde{F}^{\mu\nu}\right)^2 \pm \frac{1}{2g^2}\int d^4x \ \text{Tr} \ F_{\mu\nu}\widetilde{F}^{\mu\nu} \tag{2.89}$$

We note that the second term is always an integer by equation (2.56), and that the space of solutions again splits into those with different values of n for this integer,

$$\frac{1}{16\pi^2}\int d^4x \ \text{Tr} \ F_{\mu\nu}\widetilde{F}^{\mu\nu} = n\,. \tag{2.90}$$

This integer is referred to as the instanton number.

The first term in the action (2.89) is always positive, and can be minimised if the instanton solution satisfies,

$$F_{\mu\nu} = \pm\widetilde{F}_{\mu\nu}\,. \tag{2.91}$$

It can be checked that in this case the equations of motion (2.53) are automatically satisfied. If this bound is saturated, then the instanton action cost is,

$$\exp\left[-S_{\text{inst}}\right] = \exp\left[-\frac{8\pi^2|n|}{g^2}\right]\,. \tag{2.92}$$

The integer n is once again related to a transition function. Consider this time the space-time $[0,T]\times\ S^3$. The semi-classical ground states are gauge fields A_i. In the same vein that the instanton for a particle on a circle is described by a transition of $2\pi n$ between the same classical state $a = 0 \text{ mod } 2\pi$, the non-Abelian instanton can be related to a non-zero change in the Chern-Simons functional,

$$\text{CS}[A] = \frac{1}{8\pi}\int d^3x \ \epsilon^{ijk} \ \text{Tr}\left(F_{ij}A_k + \frac{2i}{3}A_iA_jA_k\right)\,. \tag{2.93}$$

The instanton number n, is the difference between the Chern-Simons functionals on the temporal boundaries,

$$n = \frac{1}{16\pi^2}\int \text{Tr}\left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right) = \frac{1}{2\pi}\text{CS}[A(T)] - \frac{1}{2\pi}\text{CS}[A(0)]\,. \tag{2.94}$$

Similar to the particle on a circle, the instanton with winding n is completely specified by a non-zero transition function at some time. This transition function is a map from the group G to S^3. These maps are classified by the fundamental group[8] $\pi_3(G)$, and have winding number[9],

$$n = \frac{1}{24\pi^2}\int_{S^3} d^3x\, \epsilon^{ijk}\ \mathrm{Tr}\left(U^\dagger(x)\partial_i U(x) U^\dagger(x)\partial_j U(x) U^\dagger(x)\partial_k U(x)\right) \in \mathbb{Z}\,, \qquad (2.95)$$

called the Wess-Zumino number.

A standard example is the gauge group $G = SU(2)$ with Lie algebra generators $\frac{\sigma^i}{2}$, which is topologically a three-sphere S^3, and therefore has winding numbers $\pi_3(G) = \mathbb{Z}$. We will consider S^3 as a compactified version of R^3 and supply it with coordinates x^i. An instanton with $n = 1$ in $SU(2)$ is associated with the transition function,

$$U = \exp\left(if(r)\frac{\hat{x}^i\sigma_i}{2}\right)\,, \qquad f(r) = \begin{cases} 0 & r = 0 \\ 4\pi n & r = \infty \end{cases} \qquad (2.96)$$

The associated choice of field strength on R^4 turns out to be,

$$F_{\mu\nu} = -\frac{2\rho^2}{(t^2 + x^2 + \rho^2)^2}\eta^i_{\mu\nu}\sigma^i\,. \qquad (2.97)$$

Where $\eta^i_{\mu\nu}$ are the self-dual 't Hooft matrices $\eta^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^i_{\rho\sigma}$. The instanton has, next to 4 translational degrees of freedom, an unfixed size ρ, whose indeterminacy is a consequence of the classical scale invariance of the Yang-Mills action. A sum over instantons therefore consists of both a sum over its location and its size [167, 168]. In the case that the coupling g is under control, this sum yields an axion potential,

$$V(a) \sim e^{-\frac{8\pi^2}{g^2}} \cos a \qquad (2.98)$$

For a general gauge group G, instantons can be embedded in any $SU(2)$ subgroup.

Abelian gauge groups do not have the topology to admit instantons in four flat dimensions of space-time as $\pi_3(U_1)) = 0$. Abelian instantons exist when the Abelian gauge group is the remnant of a spontaneously broken UV gauge group which does admit instantons, such as in $SU(2) \to U(1)$ [167, 169] *or* if the topology of space-time is altered. For instance, on a four-torus $T^2 \times T^2$, one can have monopoles on both 2-cycles $\int F \neq 0$, such that $\int F \wedge F \neq 0$ [170].

[8] In general, the fundamental group $\pi_i(G)$ are the maps from the sphere S^i to the group G modded by the equivalence relation that two maps are the same if they can be continuously deformed into each other. For the particle on a circle, this number was counted by $\pi_0(\mathbb{Z}) = \mathbb{Z}$.

[9] A simple way to recover this is to start with the configuration $A(0) = 0$ and have $A(T)$ be related by a transition function $A_i(T, x) = 0 + U^\dagger(x)\partial_i U(x)$. In this case, the difference between the Chern-Simons functionals (2.94) reduces to equation (2.95).

3

Generalised Symmetries of Axions

The parameter space axion experiments have to cover is vast and seems impractical. An important theoretical guide in identifying promising search regions for experiments is symmetries. This chapter covers how symmetries can be used to constrain axion experiments in section 3.1. The relevant symmetries for axion searches are introduced in section 3.2 as generalised symmetries and require an extension of the ordinary notion of symmetries. This extension includes non-invertible symmetries in section 3.3, higher-form symmetries in section 3.4, and higher-group symmetries in section 3.5.

3.1 Axion Effective Field Theory

As mentioned before, in general, QFT is hard. Perturbative expansions for low-energy scattering processes commonly involve contributions from high-energy virtual states, and the number of virtual states to consider grows starkly with the number of scattered particles and perturbative order under consideration. In this section, we review how symmetries allow for greater theoretical control in determining the low-energy interactions and properties of the axion, thus aiding the numerous experimental axion searches.

Symmetries organise the spectrum of the Hilbert space and thus increase our control on perturbative expansions. Symmetries allow for the appropriate identification of the relevant degrees of freedom to use in the perturbative expansion and constrain the allowed interactions between them. Even if a symmetry is only approximately conserved up to small breaking effects, then the symmetry further subdivides our perturbative series into a double series in the coupling constant and the small breaking effects.

Determining and constraining the axion's parameter space starts with an appropriate identification of the relevant degrees of freedom that interact with the axion below a certain energy scale. This can be accomplished if the system exhibits scale separation, which implies that the Hilbert space can effectively be factorised into the degrees of freedom above and below this energy scale. In this scenario, one can obtain a description in terms of just the low-energy degrees of freedom by averaging over degrees of freedom at shorter distance scales, called an effective field theory (EFT). In this sense, the EFT has a natural cut-off scale Λ above which this averaging procedure is no longer valid.

We will refer to the appropriate degrees of freedom as a field ϕ with mass m. The effective Lagrangian for the low-energy degrees of freedom is an expansion,

$$\mathcal{L} \supset \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - c_3\Lambda\phi^3 - c_4\phi^4 - \frac{c_5}{\Lambda}\phi^5 - \dots\,, \tag{3.1}$$

where we have excluded derivative terms in ϕ for brevity. The coefficients c_i can be obtained by matching scattering amplitudes to those of the full theory and are nominally assumed to be order one and given by dimensionless combinations of the high-energy parameters. The contributions of higher-order operators to physical processes of energy E will scale as $(E/\Lambda)^n$ and are therefore increasingly irrelevant at lower energies. Inversely, if we want to probe the value of higher-order operators, we need to go to higher-energy scattering.

The EFT expansion (3.1) in terms of the low-energy degrees of freedom allows for a large number of possible operators even at relatively low orders in operators. Symmetries are extremely powerful in this regard, forbidding operators that explicitly break the symmetry in the low-energy EFT expansion. For example, the dangerous operator c_3 in the expansion (3.1) is not present in theories obeying a $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. In this way, symmetries can imply powerful constraints on the expansion. Turning this statement on its head, if operator coefficients c_i are measured to be much smaller than the naive $\mathcal{O}(1)$, then this hints at a larger symmetry group.

Symmetries can be accidental on an order-by-order basis in the EFT. In this case, lower-order operators that would violate the symmetry are absent due to the particle content and representations. A standard example in the Standard Model is the baryon number B or lepton number symmetry L, which counts the number N_q of quarks and assigns them baryon number $B = \frac{N_q}{3}$ whilst the lepton number symmetry L of leptons ℓ, counts the number of leptons, such as electrons or neutrinos. The gauge representations of the Standard Model do not allow B or L number-breaking operators at order less than 6 in the EFT, although both symmetries are explicitly broken by non-perturbative effects corresponding to electroweak instantons in the early universe. The lowest-order operator that breaks baryon and lepton number occurs at order 6,

$$\mathcal{L} \supset \frac{c_{B/L}}{\Lambda^2} qqq\ell \,. \tag{3.2}$$

Famously, this operator contributes to proton decay and is generated by loops of massive gauge bosons in the simplest unification models of the Standard Model, in which the SM gauge groups $SU(3)_C \times SU(2)_L \times U(1)_Y$ unify at a scale $\Lambda \sim 10^{16}$ GeV.

3.1.1 Chiral Lagrangian

Symmetries aid in the identification of the appropriate low-energy degrees of freedom ϕ in the EFT that interact with the axion. For instance, suppose that a symmetry G is spontaneously broken to a smaller symmetry group H, which is to say that the vacuum changes under general transformations of G, but is left invariant under a subgroup H,

$$G \left|\Omega\right\rangle \neq \left|\Omega\right\rangle \,, \qquad H \left|\Omega\right\rangle = \left|\Omega\right\rangle \,. \tag{3.3}$$

In this scenario, space-time dependent transformations $g(x)$ valued in G change the vacuum and can thus be used to identify low energy excitations $\phi(x)$, called Goldstone bosons [171, 172]. As transformations in H do not change the vacuum, these excitations lie in the coset manifold,

$$\phi(x) : X \to \frac{G}{H} \,. \tag{3.4}$$

Space-time-independent transformations are a symmetry, and thus the Goldstone bosons are massless. In fact, the EFT should vanish at any order for a constant field ϕ and therefore the EFT expansion is of the form,

$$\mathcal{L} \supset \frac{1}{2}(\partial\phi)^2 - \frac{c_2}{\Lambda^2}\phi^2(\partial\phi)^2 - \frac{c_6}{\Lambda^6}\phi^2(\partial\phi)^4 \dots \,, \tag{3.5}$$

where we have assumed a $\mathbb{Z}_2$ symmetry, and compatibility with the symmetry G.

The proper identification of the low-energy degrees of freedom in an EFT and derivative expansion based on a spontaneously broken symmetry was carried out in the seminal papers by Callan, Coleman, Wess and Zumino [173, 174].

We will not require such generalities here and instead focus on the relevant low-energy degrees of freedom of the Standard Model, the hadrons and photon, and their interactions with the axion. The relevant effective field theory is a non-linear sigma model [174, 175] referred to as the chiral Lagrangian [176].

We restrict ourselves to a Standard Model approximation that only involves two quarks, the up u and down quark d with masses m_u and m_d. The relevant Lagrangian at dimension 4 in quark operators and energies above the confinement scale is,

$$\mathcal{L} \supset \overline{u}\left(i\not{D} - m_u\right)u + \overline{d}\left(i\not{D} - m_d\right)d \,, \tag{3.6}$$

where the covariant derivatives $\not{D}$ include both the gluons and photon.

In the absence of any quark masses, $m_u = m_d = 0$, the quarks have a symmetry group,

$$U(2)_L \times U(2)_R \,, \tag{3.7}$$

which respectively act only on the left or right handed components of the quarks,

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to L \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to R \begin{pmatrix} u_R \\ d_R \end{pmatrix} . \tag{3.8}$$

It is well-known that not all left or right transformations are symmetries of the partition function due to a change of the fermionic measure under the transformation. Considering just the gluonic sector for now, under a general infinitesimal transformation $L = 1+i\ell^\mu/2\,\sigma^\mu$ and $R = 1+ir^\mu/2\,\sigma^\mu$ with $\sigma^\mu = (1, \sigma^i)$, the fermionic measure changes as,

$$\int DuDd \to \int DuDd \exp\left[\int \frac{1}{16\pi^2}\mathrm{Tr}\left((\ell^\mu - r^\mu)\frac{\sigma^\mu}{2}\right)\mathrm{Tr}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)_{\mathrm{QCD}}\right] . \tag{3.9}$$

All symmetries leave this measure invariant expect $U(1)_A$, which transforms $u \to e^{i\alpha\gamma_5}u$ and $d \to e^{i\alpha\gamma_5}d$. Combining this with our QCD calculations around equation 2.66, we observe that the vacuum is sensitive to a $U(1)_A$ phase change, and therefore $U(1)_A$ is not a symmetry of the low-energy theory.

It is known experimentally that in the strongly coupled regime of QCD, a quark condensate forms, breaking the remaining symmetry group spontaneously to the vector subgroup,

$$\frac{U(1)_V \times SU(2)_L \times SU(2)_R}{\mathbb{Z}_2} \to \frac{U(1)_V \times SU(2)_V}{\mathbb{Z}_2} , \tag{3.10}$$

which satisfies $L = R$. The spontaneously broken subgroup acts as $L^\dagger = R$ and is referred to as the axial subgroup $SU(2)_A$.

The coset space of the spontaneous symmetry breaking,

$$\frac{SU(2)_L \times SU(2)_R}{SU(2)_V} \equiv SU(2)_A \tag{3.11}$$

is parametrised by $2^2 - 1 = 3$ Goldstone bosons. We describe this coset space by a 2-dimensional unitary matrix U,

$$U = \exp\left[i\pi_i\sigma_i\right] . \tag{3.12}$$

Here the generators $\sigma_i/2$ are $SU(2)_A$ generators and their coefficients, the Goldstone bosons, are called the pions and were first contemplated by Nambu as carriers of the nuclear force [177]. Due to their electromagnetic interactions, the pions are usually split into the neutral pion π^0 multiplying the σ_3 generator and the charged pions $\pi^\pm$ multiplying $\frac{1}{\sqrt{2}}(\sigma_1 \mp i\sigma_2)$.

The low-energy effective field theory involving the pions needs to be invariant under a general transformation of $SU(2)_L \times SU(2)_R$, which acts on the coset space variables as,

$$U \to LUR^\dagger \,. \tag{3.13}$$

At leading order in U and derivatives, the only non-zero term in the EFT invariant under the transformations is,

$$\mathcal{L} \supset \frac{f_\pi^2}{4} \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) \tag{3.14}$$

This is known as the chiral Lagrangian and f_π as the pion decay constant. Expanding in terms of the pions π_i reveals interactions of the form (3.5).

The change in the fermionic measure in equation (3.9) also receives additional contributions from photons, as the up and down quark, respectively, have electric charge $\frac{2}{3}$ and $-\frac{1}{3}$. Under a transformation $L = 1 + i\ell/2\ \sigma_3$ and $R = 1 - i\ell/2\ \sigma_3$, the measure changes as,

$$\int DuDd \to \int DuDd \exp\left[\int \frac{\ell}{16\pi^2}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)_{\mathrm{EM}}\right] . \tag{3.15}$$

In the low-energy effective field theory, this can be reproduced by adding a pion-photon coupling,

$$\mathcal{L} \supset \frac{\pi^0}{16\pi^2}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)_{\mathrm{EM}} \tag{3.16}$$

A proper derivation of the pion-photon coupling in terms of the variables U requires the introduction of Wess-Zumino-Witten terms [178, 179], but will not be necessary here. The pion-photon coupling provides the dominant decay mode of the neutral pion π^0 to two photons and gives a value of $f_\pi = 92.1 \pm 0.8$ MeV [127].

3.1.2 Axion-Chiral Lagrangian

The low-energy interactions of the (QCD) axion with pions are determined by its coupling to the strong and electromagnetic sector of the Standard Model,

$$\mathcal{L} \supset \frac{Ea}{16\pi^2} F\tilde{F} + \frac{Na}{8\pi^2} \mathrm{Tr}\left(G_{\mu\nu}\widetilde{G}^{\mu\nu}\right) . \tag{3.17}$$

The numbers E and N are referred to as the primordial coupling of axions to photons and gluons. The interactions with gluons explicitly breaks the axion shift symmetry $a \to a + c$ (in the presence of quark masses) and by our EFT arguments, we expect a potential $V(a)$ and axion mass.

The quark masses explicitly breaks the symmetry group (3.10), and allow for a double perturbative series in the small parameters E/Λ and m/Λ in the EFT expansion (3.1). The terms of order $(m/\Lambda)^n$ can be recovered by a simple trick referred to as spurion analysis. In this case, the breaking parameter m is assumed to be an auxiliary field stuck in its minimum. The symmetry can be recovered by having m transform under the symmetry. This restores the organisational power of the symmetry in the EFT and provides the axion-pion interaction.

In order to illustrate this, let us collect the quark masses in the quark mass matrix M as,

$$\mathcal{L} \supset \begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix} \left(i\not{D} - M\right) \begin{pmatrix} u \\ d \end{pmatrix} , \qquad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} . \tag{3.18}$$

The left- and right-handed symmetries in equation (3.7) can be restored if the mass matrix transform as,

$$M \to RML^\dagger . \tag{3.19}$$

At the leading order, the chiral Lagrangian (3.14) is modified to,

$$\mathcal{L} \supset \frac{f_\pi^2}{4} \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{B_0 f_\pi^2}{2} \mathrm{Tr}\left(MU + M^\dagger U^\dagger\right) \tag{3.20}$$

The pion Goldstone bosons obtain a mass $m_{\pi^0} = 134.9768 \pm 0.0005$ MeV [127]. The left and right symmetries (3.7) are broken by the electroweak sector, resulting in additional mass contributions for the charged pions $m_{\pi^\pm}$ due to electromagnetism [180, 181], which current lattice simulations estimate to be $\delta m_\pi = m_{\pi^\pm} - m_{\pi^0} = 4.534 \pm 0.042 \pm 0.043$ MeV [182], and due to Z-bosons $\delta m_\pi = -0.00201(7)(2)(10)$ MeV [102].

The interaction of the axion with the pions can be obtained by rotating the quarks $u \to e^{-i\frac{Na}{2}\gamma_5} u$ and $d \to e^{-i\frac{Na}{2}\gamma_5} d$, which rotates the axion from the $G\widetilde{G}$ term in equation (3.17) into the quark mass matrix,

$$M \to M e^{-i\frac{2Na}{2}} \,. \tag{3.21}$$

As the quarks are charged under electromagnetism, this rotation also shifts the axion-photon coupling. The low-energy effective field theory involving the axion and pions is of the form,

$$\mathcal{L} \supset \frac{f_\pi^2}{4} \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{f^2}{2}\left(\partial a\right)^2 + \frac{B_0 f_\pi^2}{2}\,\mathrm{Tr}\left(MUe^{-i\frac{2Na}{2}} + M^\dagger U^\dagger e^{i\frac{2Na}{2}}\right) \,. \tag{3.22}$$

supplemented by an axion and pion-photon coupling,

$$\left(E - \frac{5}{3}N\right)\frac{a}{16\pi^2} F\tilde{F} + \frac{\pi^0}{16\pi^2} F\tilde{F} \,. \tag{3.23}$$

We will study the QCD axion mass and axion-photon coupling resulting from this axion-pion mixing in chapter 4.

3.2 Generalised Symmetries

Symmetries proved vital in understanding and constraining the interactions between the axion and Standard Model mesons, but several open questions remain.

For one, the axion shift symmetry EFT analysis constrained the axion potential and mass, but left the axion-photon coupling unconstrained and unprotected. Moreover, in the presence of an axion-photon coupling, it is even mysterious which symmetry is formally responsible for protecting the mass of the axion (3.23), as the coupling of axions to photons naively breaks the axion's shift symmetry.

Given the crucial importance to current axion searches of determining both the axion mass and the axion-photon coupling, it is paramount to identify the appropriate symmetry protecting and constraining both quantities. The relevant symmetry constraining both the axion mass and axion-photon coupling will turn out to be a generalised, non-invertible symmetry, linking the two vital quantities through one generalised symmetry, as will be explained in section 3.3 and chapter 4.

Even after the identification of the appropriate non-invertible shift symmetry, generic axion models are still plagued by a quality problem and extreme sensitivities to UV physics. The EFT and ordinary symmetry approach advocated in the previous section give no hint at how to suppress or control these sensitivities and therefore leaves the axion mass parameter space unconstrained. It is well understood that in extra-dimensional axion models, such as those present in the stringy axiverse [20], unwanted contributions to the axion potential are exponentially suppressed [58]. In section 3.4, we will identify the responsible symmetry as a generalised, higher-form symmetry, which we utilise in chapter 4 and 6.

The absence of symmetries constraining the axion parameter space motivates us to consider *generalised* symmetries in this section, which were introduced in the seminal work [23] (although many ideas were already present in the existing literature on non-Abelian gauge theories [105, 183–186], topological field theories [187–189], conformal field theories [190–192], condensed matter [193, 194] and elsewhere [46]).

Generalised symmetries extend the notion of ordinary symmetries in a manner that retains the organisational powers of symmetries. In accordance with ordinary symmetries, generalised symmetries and their anomalies are invariant under renormalisation group (RG) flow, making them powerful tools in studying the vacuum structure of strongly interacting gauge theories. This has been successfully applied to study the strongly-coupled phases of non-Abelian gauge theories [24–35]. Generalised symmetries have also been studied at finite temperatures in holography [36, 37], to identify the Δ^{++} baryon [38], and recently, to understand the scattering of fermions off magnetic monopoles [39, 40]. Moreover, it is widely conjectured [41–43] that consistent theories of quantum gravity have to be free of global symmetries, and the absence of generalised symmetries can be used to provide lower bounds on symmetry breaking effects [44, 45].

More important for our purposes, generalised symmetries can be used to constrain the low-energy spectrum. Akin to ordinary global symmetries, generalised symmetries can be spontaneously broken [23, 46–49] resulting in massless Goldstone bosons in the low energy spectrum if the generalised symmetry is continuous. Small explicit breakings of the generalised symmetry can then be used to constrain the interactions between the Goldstone bosons [50, 51].

The modern mathematical description of generalised symmetries is in terms of category theory [195–200] and symmetry topological field theory [201–203], although we will not require this mathematical machinery here. Excellent pedagogical reviews of generalised symmetries can be found in [101, 204–208]. Generalised symmetries and axions have been considered in [50–52, 56, 58, 60–62].

3.2.1 Symmetries, Encore

In the next sections, we will naturally be led to generalised symmetries by considering the axion-photon/gluon system. The starting point is a reformulation of our common notion of symmetries that preserves the manifest Lorentz invariance of the classical action and the partition function formulation, in contrast to the conserved charges of equation (2.6) on time-slices. The advantage of a reformulation is that it opens up a new perspective that might be easily generalised.

Consider an ordinary continuous symmetry, which by Noether's theorem can be associated with a conserved current j_μ or, equivalently, a 1-form j, whose dual is closed $d \star j = 0$. The existence of a closed form motivates a more topological interpretation of conserved charges. We will associate a charge $Q = \int_\Sigma \star j$ with any closed co-dimension 1 manifold Σ of space-time, not necessarily a time-slice. Note that we integrate the dual of the current over the manifold in order to obtain the flux through the manifold. The notion of conservation now becomes the invariance of this charge under topological deformations of the manifold Σ.

Consider a topological deformation Σ' of Σ obtained by moving and stretching points on Σ. The two manifolds are connected by an interpolating manifold S such that its boundary $\partial S = \Sigma - \Sigma'$. Stokes' theorem and the current conservation tell us that the difference between the charges is,

$$Q(\Sigma) - Q(\Sigma') = \int_\Sigma \star j - \int_{\Sigma'} \star j = \int_S d \star j = 0\,. \tag{3.24}$$

The proper, Lorentz invariant statement is that the value of the charge $Q(\Sigma)$ on any manifold Σ is invariant under topological deformations of the surface Σ. We call such charges in classical mechanics *topological*, which is a simple Lorentz invariant extension of the notion of conservation and extends to discrete symmetries.

The quantum version of topological charges requires the quantum version of Noether's theorem, which is Ward identities. Ward identities can be derived by considering space-time-dependent field redefinitions. Suppose that the product of the action and measure in the partition function is invariant under the field redefinition $\phi \to \phi + \delta\phi$ and consider the insertions,

$$\langle \prod_i \phi(x_i) \rangle = \int D\phi \, \prod_i \phi(x_i) \, e^{iS[\phi]} \,. \tag{3.25}$$

We perform a space-time-dependent transformation of the dummy variables ϕ as $\phi' = \phi + \epsilon 1_R \delta\phi$, where $1_R(x)$ is a characteristic function that is unity in a D-dimensional submanifold of space-time R and zero outside, and ϵ is an arbitrarily small number. As the transformation is a symmetry for space-time independent transformations, the product of the action and measure can at most change by,

$$D\phi' \, \exp\left[iS[\phi']\right] = D\phi \, \exp\left[iS[\phi] + i \int \star j \wedge d\left(\epsilon 1_R\right)\right] \,. \tag{3.26}$$

Therefore, keeping only the terms first order in ϵ, gives,

$$\langle \prod_i \phi(x_i) \rangle = \int D\phi' \, \prod_i \phi'(x_i) \, e^{iS[\phi']} \tag{3.27}$$

$$= \langle \prod_i \phi(x_i) \, i \int \star j \wedge d\left(\epsilon 1_R\right) \rangle + \sum_j \epsilon 1_R \, \langle \delta\phi(x_j) \, \prod_{i \neq j} \phi(x_i) \rangle \,. \tag{3.28}$$

As this should be equal for any choice of manifold R and small number ϵ, we obtain the Ward identity,

$$\left\langle \left(d \star j\right)(x) \, \prod_i \phi(x_i) \right\rangle = -i \sum_i \delta\phi(x_i) \star \delta(x - x_i) \, \langle \prod_{j \neq i} \phi(x_j) \rangle \,. \tag{3.29}$$

To ease the later transition to generalised symmetries, one can think of the 4-form $\star\delta(x - x_i)$ as a bump function Poincaré dual to the point x_i.

The quantum version of a topological charge becomes that of a *topological* operator, a term that also extends to finite symmetry. Topological deformations of the manifold Σ over which the operator is defined leave correlations functions invariant as long as the deformations do not cross a point where charged operators live.

The ability to rewrite the notion of a symmetry in terms of a topological operation motivates us to think about recovering the action of a symmetry operator on a charged particle in a similar topological light. Consider a time-ordered correlation function of the charge of an ordinary continuous symmetry with a charged operator insertion,

$$\langle \exp\left(iQ(\Sigma)\right)\ \phi(x)\rangle \tag{3.30}$$

In this case, we take the manifold Σ to be a D-dimensional closed surface, such as a sphere S^D, that surrounds the point x, such that the point and manifold are linked. We take the surface to be exact and thus the boundary of another manifold S with $\partial S = \Sigma$, which implies that x lies in S. A combination of Stokes' theorem and the exponentiated version of the Ward identity (3.29) then allows us to rewrite eq. (3.30) as,

$$\langle \exp\left(i\int_S d\star j\right)\ \phi(x)\rangle = \langle \phi'(x)\rangle\ . \tag{3.31}$$

The general statement is that the action of any topological operator $U(\Sigma)$ can be found by linking the operator with a charged operator as,

$$\langle U(\Sigma)\phi(x)\rangle = \begin{cases} \langle \phi'(x)\rangle & \mathrm{link}(\Sigma, x) = 1 \\ \langle \phi(x)\rangle & \mathrm{link}(\Sigma, x) = 0 \end{cases} \tag{3.32}$$

The ordinary commutation relation (2.7) can be recovered by linking a field at time t with the closed manifold $\Sigma = \Sigma_{t+\epsilon} - \Sigma_{t-\epsilon}$ defined as two time-slices $\Sigma_{t+\epsilon}$ and $\Sigma_{t-\epsilon}$ at $t+\epsilon$ and $t-\epsilon$ with opposite orientation, which links the point t.

This naturally leads to the definition that symmetries in QFT are associated with unitary topological operators defined on co-dimension 1 slices. This redefinition is insufficient. The power of symmetries lies in the means to organise the spectrum of our Hilbert space. There exists a wide class of operators that are topological but neither associated with a group nor a co-dimension 1 manifold. The proper statement will turn out to be, A symmetry is the existence of a topological operator $O(M)$ defined on closed manifolds M

The operator in question, $O(M)$ generates the symmetry by linking with the charged (potentially extended) operators $W[C]$ as,

$$\langle O(M)W[C]\rangle = \begin{cases} \langle W'[C]\rangle & \mathrm{link}(M,C)=1 \\ \langle W[C]\rangle & \mathrm{link}(M,C)=0 \end{cases} . \tag{3.33}$$

The multiplication of $O(M)$ and $O'(M')$ is defined by their operator product expansion (OPE) by fusing the two manifolds M and M'. The OPE only depends on the topological properties of the manifold M and M', and in particular, is independent of the distance between the manifolds.

3.3 Non-Invertible Symmetries of Axions

The first class of generalised symmetries prominent in constraining the axion-photon coupling and mass are non-invertible symmetries, which we review in this section and use in chapter 4 to link the axion mass and photon-coupling.

Non-invertible symmetries correspond to topological operators $O(M)$ whose inverse does not exist, and their algebra cannot be described by a group algebra, which requires inverses. Topological operators whose fusion rules do not form a group algebra are well-known from two-dimensional conformal field theories such as the Ising model and the associated Kramers-Wannier-duality [191, 192, 209–211]. Non-invertible symmetries in the Standard Model have been discussed in the context of QED and QCD in [51] and in the context of axions in [61]. The breaking of non-invertible symmetries and their uses in effective field theories are discussed in [50].

Symmetries can become non-invertible when they have non-trivial interactions with gauge redundancies. Consider for instance the $\mathbb{Z}_2$ symmetry for the axion, which acts as,

$$a \to -a\,. \tag{3.34}$$

This symmetry does not commute with the $U(1)$ shift symmetry of the axion $a \to a+c$, and the symmetry group is $O(2) = U(1) \rtimes \mathbb{Z}_2$. Gauging the $\mathbb{Z}_2$ symmetry turns the axion shift symmetry into a non-invertible symmetry.

To see this, note that the only $\mathbb{Z}_2$ gauge-invariant Wilson point operators under $a \to -a$ are of the 'twisted' form,

$$e^{ina} + e^{-ina} \tag{3.35}$$

The shift symmetry generating operators D_c are of the same twisted form,

$$D_c = \exp(ic \int_\Sigma \star da) + \exp(-ic \int_\Sigma \star da) \,. \tag{3.36}$$

The index c has the identification $c \equiv -c$, and the eigenvalues of the twisted Wilson points under the shift symmetry are,

$$\langle D_c \left(e^{ina} + e^{-ina}\right)\rangle = 2\cos\left(nc\right) \langle \left(e^{ina} + e^{-ina}\right)\rangle \ . \tag{3.37}$$

The twisted symmetry operators are not unitary. In particular, the operator D_π annihilates half of the spectrum. The twisted symmetry operators organise our spectrum by the Ward identities (3.37) inherited from the untwisted theory. Therefore, our standard definition of symmetries is failing to capture the power that comes with non-invertible operators.

The operator algebra of the twisted operators is no longer that of a group but instead satisfies,

$$D_\alpha D_\beta = D_{\alpha+\beta} + D_{\alpha-\beta} \tag{3.38}$$

The general algebra is that of fusion algebras, and the appropriate language is that of (higher) fusion categories [195–200], although we will not require this machinery here.

Non-invertible symmetries are of particular interest in studying the possible interactions between axions and photons. We wish to discuss the fate of the shift symmetry $a \to a + c$ of the axion in the presence of the coupling of axions to photons as,

$$\mathcal{L} \supset \frac{a}{16\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu} \,. \tag{3.39}$$

The equations of motion of the axion are modified by the coupling to,

$$f^2\partial_\mu\partial^\mu a = \frac{1}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\,. \tag{3.40}$$

The axion shift symmetry current $j_\mu = f^2\partial_\mu a$ is broken, but a conserved current can be constructed,

$$j_\mu = f^2\partial_\mu a - \frac{1}{8\pi^2}A_\nu\widetilde{F}^{\mu\nu}\,. \tag{3.41}$$

This current is conserved and allows one to construct unitary operators,

$$U_c = \exp\left[ic\int f^2\star da\right]\exp\left[i\frac{c}{8\pi^2}\int A\wedge F\right]\,, \tag{3.42}$$

which shifts the axion by,

$$a\to a+c\,. \tag{3.43}$$

It can be seen that for $c\neq 2\pi\mathbb{Z}$, this operator is not gauge invariant due to the Chern-Simons term. It could not have been; the unitary operator causes a shift in $a\to a+c$, and in the presence of a non-zero magnetic field F ('t Hooft line), the Witten effect implies that this shift generates an electric field. But this is supposed to be a conserved symmetry of the theory. The only way this is possible is to construct an alternative operator that accounts for this change in electric field by having electric degrees of freedom living on the operator.

Let us attempt to resolve this for simple angles $c=\frac{2\pi}{N}$. The theory on the operator has to include additional charged degrees of freedom. The topological operator that works is of the following 'twisted' form [51, 61],

$$D_{\frac{2\pi}{N}} = \int Db\,\exp\left[\frac{2\pi}{N}\int f^2\star da\right]\,\exp\left[i\int\left(\frac{N}{4\pi}b\wedge db - \frac{1}{2\pi}b\wedge dA\right)\right] \tag{3.44}$$

This additional auxiliary degree of freedom b plays the role of would-be zero modes on the 'domain wall' $a\to a+c$ that can be excited whenever a magnetic monopole crosses the domain wall.

The operator is Gaussian in the auxiliary field b and this degree of freedom can be integrated out. The auxiliary field satisfies the equations of motion,

$$Ndb = dA\,. \tag{3.45}$$

Naively, inserting this back into the operator (3.44), one obtains the gauge-non-invariant operator (3.42). The key here being that integrating out the field *and* solving $db = \frac{dA}{N}$ is not valid as both db and dA have integral periods $\int db \in \mathbb{Z}$. Instead, the correct operator is the operator (3.44).

In this fashion, one can construct an operator that enacts the symmetry for all angles in $\mathbb{Q}$ [51, 61]. The catch is that this gauge-invariant operator is non-unitary.

This can be appreciated in two ways. Consider an 't Hooft line $H[C]$ that wraps a non-contractible cycle on a given time slice. Under a transformation $a \to a + \frac{2\pi}{N}$, the 't Hooft line picks up an electric charge and becomes a mixed Wilson 't Hooft line,

$$H[C] \to H[C] \exp\left[\frac{i}{N}\int_C A\right] . \tag{3.46}$$

This is a fractional Wilson line with fractional charge $\frac{1}{N}$. This line is not gauge-invariant as large gauge transformations exist on the non-contractible cycle C. The only option is that the operator $D_{\frac{2\pi}{N}}$ annihilates states with 't Hooft lines $|H[C]\rangle$ around non-contractible cycles. The operator is non-invertible and is reminiscent of the action of the disorder operator in the Ising model [51, 192].

Another way to approach this fact is to study the fusion algebra of the operator with its would-be inverse,

$$D_{\frac{2\pi}{N}} D_{-\frac{2\pi}{N}} = \int db\, dc \exp\left[i\int\left(\frac{N}{4\pi} b\wedge db - \frac{1}{2\pi} b\wedge dA\right)\right] \exp\left[-i\int\left(\frac{N}{4\pi} c\wedge dc - \frac{1}{2\pi} c\wedge dA\right)\right] \tag{3.47}$$

The auxiliary combination $b + c$ does not couple to A and can be integrated out, leaving $e = (b - c)$ as an auxiliary field,

$$D_{\frac{2\pi}{N}} D_{-\frac{2\pi}{N}} = \int de \exp\left[i\int\left(\frac{N}{4\pi} e\wedge de - \frac{1}{2\pi} e\wedge dA\right)\right] = \mathcal{C}(A) \tag{3.48}$$

Formally, $\mathcal{C}(A)$ is called the condensation number, and the product of operators is just a Chern-Simons theory in a background gauge field A. This theory is non-trivial and $\mathcal{C}(A) \neq 1$ for general backgrounds. The additional quantum Hall state that was required to make the operator conserved renders the operator non-unitary [51, 61].

The non-invertible shift symmetry is formally responsible for protecting the mass of the axion in the presence of an axion-photon coupling. The shift symmetry only exists if the axion-photon coupling is of the form (3.39). We will exploit this connection between the axion mass, axion-photon coupling and non-invertible symmetry in chapter 4.

On a non-compact, non-finite manifold M, such as $\mathbb{R}^3$, the non-invertible operator $D_{\frac{2\pi}{N}}(M)$ is equivalent to the naive shift symmetry operator $U_{\frac{2\pi}{N}}(M)$. This can be seen by either integrating out the auxiliary field or by the absence of large gauge transformation $\pi_3(U(1)) = 0$ [51]. This suggest that one could work with a unitary invertible symmetry to protect the axion mass.

However, this argument is a bit too quick. We cannot use the same argument for potential UV contributions to the axion mass. The topology of spacetime seen by the Abelian gauge field can change in the UV, both in extra-dimensional theories and 4D theories, a simple example being the 't Hooft-Polyakov monopole. It will be much more useful to find a symmetry and associated spurions that parametrise both UV and IR mass generation effects on general manifolds. This is especially valuable for the case of axions, where we expect at least quantum gravitational effects to generate a mass. On a general manifold, the symmetry responsible for protecting the axion mass is the non-invertible symmetry.

The non-invertible symmetry is explicitly broken when the condition (3.45) can no longer be satisfied due to the presence of dynamical fluxes such as magnetic monopoles or instantons [50] which force b to have $\frac{1}{N}$ integral periods. This is the case when the axion Abelian gauge theory has a UV completion as a non-Abelian gauge theory and 't Hooft-Polyakov monopole exists dynamically. The resulting axion mass from loops of magnetic monopoles was calculated in [169].

The spontaneous breaking of continuous non-invertible symmetries is analogous to the breaking of ordinary symmetries, and we refer interested readers to [48].

3.4 Higher-Form Symmetries of Axions

The non-invertible symmetry of the axion is formally responsible for protecting both the mass of the axion and the axion-photon coupling and can be used to constrain the axion parameter space, but leaves the axion quality problem open. Extra-dimensional axions, in contrast to generic axion models, admit a non-invertible symmetry of exponentially high quality [58], alleviating the axion quality problem. It is paramount to axion searches and model-building to identify the responsible symmetry in extra-dimensional models, and in this section, we find that the ultimately responsible symmetry is a higher-form symmetry. This is a class of generalised symmetries whose charge cannot be defined as the integral of a charge density over a co-dimension 1 manifold such as a time slice.

The motivation for higher-form symmetries can be found in extra-dimensional realisations of axions in any space-time X with dimension D and will be essential to studying the quality of axions in the extra-dimensional models in chapters 4 and 6. Moreover, generalised symmetries are formally responsible for axion string conservation.

The axion Lagrangian (2.10) admits an ordinary shift symmetry,

$$a \to a + c\,, \qquad dc = 0\,, \tag{3.49}$$

associated with the conserved quantity $d \star da = 0$.

In its simplest form, the axion theory is the far IR realisation of a gauge theory in $D+1$-dimensions compactified on a circle S^1 of radius R with appropriate boundary conditions. The Lagrangian describing the extra-dimensional fields is that of the $U(1)$ Abelian gauge theory (2.34),

$$S = -\frac{1}{2e^2}\int_{X\times S^1} dA \wedge \star dA\,, \tag{3.50}$$

In this scenario, the axion is associated with an Aharonov-Bohm phase of the photon around the extra dimension,

$$a = \int_{S^1} A\,, \tag{3.51}$$

and the axion decay constant f is inversely proportional to the volume,

$$f^2 = \frac{1}{e^2 2\pi R}\,. \tag{3.52}$$

The question at hand is: *How does the axion shift symmetry uplift?*.

The answer is that the Lagrangian (2.34) admits a generalised shift symmetry of the gauge field,

$$A \to A + c d\theta \,, \tag{3.53}$$

where $d\theta$ is the 1-form along the extra-dimensional circle S^1 and c a constant. This shift symmetry shifts the axion,

$$a \to a + c \,, \tag{3.54}$$

and is not part of the gauge transformations of the gauge field, as it cannot be written as $ie^{i\lambda}\partial_\mu e^{-i\lambda}$ for a single valued transformation $e^{i\lambda} \in U(1)$ along the circle.

In fact, the theory admits a large class of symmetries generated by closed 1-form shifts of the gauge field [23],

$$A \to A + \lambda \,, \qquad d\lambda = 0 \,, \tag{3.55}$$

where λ is a closed 1-form such that the field strength F remains invariant.

The operators that are linearly charged under the symmetry are Wilson lines (2.41),

$$\exp\left(i\int_C A\right) \to \exp\left(i\int_C \lambda\right)\exp\left(i\int_C A\right) \,. \tag{3.56}$$

The symmetry therefore generates a $U(1)$-phase on 1-dimensional objects and is called a 1-form symmetry under which Wilson lines transform by a $U(1)$ phase. The transformations can be seen to reduce to the shift of axion Wilson points by the identification (3.51).

The associated conserved charge is the uplift of the conserved momentum,

$$f^2 d \star da \implies \frac{1}{e^2} d \star dA = 0 \,, \tag{3.57}$$

which is Gauss' law, stating that the electric flux through any closed manifold Σ of co-dimension 2 is conserved under topological deformations of that manifold.

In the QFT, we associated a topological unitary operator U with the shift symmetry of the axion as,

$$U = \exp(i\alpha \int_\Sigma \star da) \tag{3.58}$$

This operator would link with Wilson points e^{ia} to produce the symmetry,

$$\langle U(\Sigma) e^{ia(x)} \rangle = e^{i\alpha \cdot \mathrm{link}(\Sigma, x)} \langle e^{ia(x)} \rangle \ . \tag{3.59}$$

The associated current operator in the higher dimension is a 2-form current $j = \frac{1}{e^2} dA = \frac{1}{e^2} F$. The dual of the current can be integrated over a co-dimension 2-manifold to yield a charge Q that is conserved under topological deformations by Stokes' theorem,

$$Q = \frac{1}{e^2} \int_\Sigma \star F \, . \tag{3.60}$$

This charge measures the electric flux piercing the surface Σ.

This charge acts not on point operators but instead acts on Wilson lines by linking them. The way to show this is completely analogous to the linking of ordinary symmetries (3.33) upon using Poincaré duality. The Ward identity and Noether current can be recovered by making the transformation (3.55) space-time dependent,

$$A \to A + 1_R \lambda \, . \tag{3.61}$$

where 1_R is a characteristic function that is zero outside of a D-dimensional region R of space-time, where it is unity. Making this transformation in the partition function (2.34) with Wilson loop insertions, yields the Ward identity,

$$\langle \frac{i}{e^2} \left(\int d\,(1_R \lambda) \wedge \star F \right) \ \exp\left(i \int_\mathcal{C} A \right) \rangle = \left(i \int_\mathcal{C} 1_R \lambda \right) \langle \exp \left(i \int_\mathcal{C} A \right) \rangle \ . \tag{3.62}$$

We can once again use Poincaré duality to associate the closed curve $\mathcal{C}$ of the Wilson loop with a closed $D-1$-form $\widetilde{C}$ that is a bump function transverse to the closed curve $\mathcal{C}$. This allows us to rewrite the $U(1)$ phase as an integral over the space-time manifold,

$$\langle \frac{i}{e^2} \left(\int d\,(1_R \lambda) \wedge \star F \right) \ \exp\left(i \int_\mathcal{C} A \right) \rangle = \left(i \int 1_R \lambda \wedge \widetilde{C} \right) \langle \exp \left(i \int_\mathcal{C} A \right) \rangle \ . \tag{3.63}$$

By comparing the two sides for any region R and closed form λ, we identify the Ward identity,

$$\frac{1}{e^2}\left\langle (d\star F)\ \exp\left(i\int_{\mathcal{C}} A\right)\right\rangle = \widetilde{C}\left\langle \exp\left(i\int_{\mathcal{C}} A\right)\right\rangle \tag{3.64}$$

This is the equivalent of the Ward identity (3.29) for 1-forms where $\widetilde{C}$ acts as the delta function on the closed curve. The current is recognised as $j = F$ and can be exponentiated on a $(D-2)$-dimensional manifold Σ to yield a unitary topological operator,

$$U = \exp(i\alpha\int_{\Sigma}\frac{1}{e^2}\star F) \tag{3.65}$$

In this case, the exponentiated the Ward identity (3.64) leads to

$$\left\langle U(\Sigma)\ \exp\left(i\int_{\mathcal{C}} A\right)\right\rangle = e^{i\alpha\cdot\mathrm{Link}(\Sigma,\mathcal{C})}\left\langle \exp\left(i\int_{\mathcal{C}} A\right)\right\rangle \tag{3.66}$$

The operator $U(\Sigma)$ is a topological operator in the sense that it is invariant under deformations unless those deformations cross a charged operator in the correlation function. This conserved operator evidently provides non-trivial identities on our QFT following only from the invariance of the Lagrangian under a shift and therefore deserves to be called a symmetry, referred to as the electric 1-form symmetry.

This program can be extended to gauge theories of form fields of arbitrary form p, but shall not be explored here. Instead, we move to generalise the notion of a 1-form symmetry acting on objects defined on one-dimensional closed surfaces to a p-form symmetry as a symmetry that acts on p-dimensional objects. The symmetry is generated by topological defects U that can link with this p-dimensional objects. The associated operators U are therefore defined over closed $D-p-1$-dimensional or codimension $p+1$ manifolds M_{D-p-1}. If the higher form symmetry is continuous, the charge Q can again be written as the integral of $\star j$ over M_{D-p-1}, which makes j a $p+1$-form. The topological nature of the symmetry is then satisfied if $d\star j = 0$. The modern notation for a p-form symmetry associated with a group G is $G^{(p)}$ [62].

In contrast to 0-form symmetries, higher-form symmetries are generated by Abelian groups. Two symmetry operators that have a co-dimension greater than 1 can always be moved past each other in space-time without touching. Moving two symmetry operators past each other generates the reverse ordering of the operators in time-ordered products. By the topological nature of the operators, the two orderings have to be the same, and therefore higher-form symmetry groups are Abelian.

The axion Lagrangian (2.10) admits another 2-form symmetry that shifts the dual axion B by a closed 2-form,

$$B \to B + \lambda^{(2)}\,, \qquad d\lambda^{(2)} = 0\,, \tag{3.67}$$

as can be seen from the invariance of the dual Lagrangian (5.1) under the $U(1)^{(2)}$ shift. The associated conserved current is,

$$j = \frac{1}{4\pi^2 f^2} dB = \frac{1}{2\pi} \star da\,. \tag{3.68}$$

The charged objects are the axion strings (2.30), which shift by a $U(1)$-phase and the symmetry implies their conservation. This symmetry can also be uplifted to a higher-form symmetry of the Abelian gauge theory (2.34) with current,

$$j = \frac{1}{2\pi} \star F\,. \tag{3.69}$$

The conservation of this current, $dF = 0$, is the Bianchi identity. The associated linearly charged operators are the 't Hooft surfaces corresponding to magnetic charges. The 't Hooft surfaces are $D-3$ dimensional, and this is therefore a $(D-3)$-form symmetry. The topological operator is defined on a dimension 2 manifold Σ as

$$U(\Sigma) = \exp(i\alpha \int_\Sigma \frac{1}{2\pi} F)\,. \tag{3.70}$$

In the dual frame, this symmetry is associated with shifts of the dual photon

$$\hat{A} \to \hat{A} + \lambda\,, \qquad d\lambda = 0\,. \tag{3.71}$$

The associated conserved quantity is the magnetic flux through any closed 2-dimensional surface, and the symmetry is referred to as the $U(1)$ magnetic 1-form symmetry.

3.4.1 Explicit Breaking

Generically, extra-dimensional axion models have a shift symmetry of exponentially good quality assuming heavy bulk fields. This in turn can ameliorate the quality issues that plague its four-dimensional counterparts [58] as we will review in this section. See [18] for a recent, comprehensive review.

In this extra-dimensional scenario, the axion is an Aharonov-Bohm phase around an extra-dimensional circle. Only particle worldlines that wrap this circle are sensitive to this phase, a. These worldlines are suppressed by the exponential of the action for such events, which scales with the size of the extra dimension and provides the exponential protection. We have already seen an example of this in the instanton section and model (2.71), where the axion can be interpreted as the Aharonov-Bohm phase for the particle q around the temporal circle, $q(T) - q(0) = a \bmod 2\pi$. The axion obtained an exponentially small mass exactly from worldlines that wind around the circle. Thus, if the axion shift symmetry admits an uplift to a higher-form symmetry, it is generically exponentially protected.

In order to understand this exponential quality, we have to understand how higher-form symmetries can be broken. In general, symmetries are broken when there exist dynamical (finite action) processes that break the current conservation,

$$d \star j \neq 0 \,, \tag{3.72}$$

e.g. if the expectation value of the right-hand side is non-zero.

For the 1-form symmetries in the Abelian gauge theory, the symmetry is broken in the presence of charged matter,

$$\frac{1}{e^2} d \star F \neq 0 \,. \tag{3.73}$$

Electric flux lines are no longer conserved but can end on charged matter.

This can be converted into a more topological argument. Consider the Abelian gauge theory in the presence of a charged Dirac fermion ψ with charge q. In this case, Wilson lines of charge q are endable as insertions of the form,

$$\langle \overline{\psi}(y) \; e^{iq \int_x^y A} \; \psi(x) \rangle \tag{3.74}$$

are gauge-invariant. In this case the symmetry is broken.

To appreciate this, consider linking the topological operator

$$U = \exp\left[i\alpha \int \frac{1}{e^2} \star F\right] \tag{3.75}$$

with the endable Wilson line. One can either collapse the operator on the Wilson line, which would generate the symmetry transformation of the Wilson line. However, if the symmetry was preserved, then the operator is topological and can be unlinked from the endable Wilson line and collapsed, yielding no transformation. This is a contradiction, and the $U(1)$ 1-form symmetry is explicitly broken. Since charge q Wilson lines are invariant under a $\mathbb{Z}_q$ subgroup of $U(1)$, the electric 1-form symmetry is reduced to $\mathbb{Z}_q$.

In Yang-Mills theory, Wilson lines take the form (2.55). Wilson lines in the adjoint representation can be ended by gluon operator insertions (self-screening). This implies that the largest possible electric 1-form symmetry group G for a gauge group G is reduced to transformations that leave adjoint Wilson lines invariant, which is the centre of G. For $G = SU(N)/\mathbb{Z}_K$ where $K|N$, the higher-form symmetries are a $Z_{\mathrm{N/K}}$-1-form electric symmetry and Z_{K}-1-form magnetic symmetry.

The size of the explicit breaking of the electric 1-form symmetry in Maxwell theory is set by the non-conservation of electric field lines. In flat space-time, electric field lines can be broken by Schwinger pair creation events [212], in which a particle-antiparticle pair is created when the local energy density becomes larger than their creation cost $\sim 2m$ where m is the mass of the particle. Schwinger pair creation events are inherently non-perturbative in nature, suppressed by an exponential action cost in the mass over the electric field density, and present an instability of the vacuum. The shift symmetry of extra-dimensional axions is then protected by the cost of a virtual pair production of particles around the extra-dimensional circle, which annihilate on the opposite end of the circle.

The exponential protection offered by higher-form symmetries can also be appreciated by integrating out the Dirac fermion. If the fermion has a mass m, integrating out the fermion with Lagrangian,

$$\mathcal{L} \supset \overline{\psi}\left(i\not{D} - m\right)\psi \tag{3.76}$$

yields the Euler-Heisenberg Lagrangian [213, 214] in four flat space-time dimensions and can be compactly written as,

$$\mathcal{L} \supset -\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s} e^{-sm^2} \frac{\mathrm{Re}\cosh sX}{\mathrm{Im}\cosh sX} F\widetilde{F}\,, \qquad X \equiv \sqrt{\frac{1}{2}F^2 + \frac{i}{2}F\widetilde{F}}\,. \tag{3.77}$$

By expanding this formula in powers of s, we can find a low energy expansion in terms of the field strength F and cut-off scale m,

$$\mathcal{L} \supset -\frac{1}{4e^2}F^2 + \frac{e^4}{1440\pi^2 m^4}\left[F^4 + \frac{7}{4}\left(F\widetilde{F}\right)^2\right] + \ldots \tag{3.78}$$

The non-conservation of electric field lines can be appreciated by considering the Lagrangian (3.77) in the presence of a constant electric field E,

$$\mathcal{L} \supset \frac{1}{2}E^2 - \frac{e^2 E}{8\pi^2}\int_0^\infty \frac{ds}{s} e^{-sm^2/(eE)}\left(\cot s - \frac{1}{s} + \frac{s}{3}\right)\,. \tag{3.79}$$

The cotangent function has poles along the real axis, and a proper integral requires an $i\epsilon$ description. In turn, this means that the Lagrangian has an imaginary contribution [212, 213],

$$\Gamma = \mathrm{Im}\mathcal{L} = \frac{e^2E^2}{8\pi^3}\sum_{n=1}^{\infty}\exp\left(-\frac{m^2\pi n}{eE}\right)\,. \tag{3.80}$$

This indicates the instability of the constant electric field vacuum and shows that the conservation is only broken by exponentially small contributions. We will calculate the Euler-Heisenberg Lagrangian for a non-flat space-time in chapter 4. In this case, the Aharonov-Bohm phase around the compact dimension can be identified as the axion, which will obtain a mass from winding modes of the fermion around the compact dimension.

An alternative formulation of the non-invertible symmetry of the previous section, equation (3.44), involves the gauging of the magnetic 1-form symmetry of the photon on half of space-time. In this case, the non-invertible symmetry will form as a condensation defect on the boundary M of the half-gauging [215–217]. This gives another interpretation as to why the non-invertible symmetry is explicitly broken in the presence of dynamical magnetic monopoles [50].

3.4.2 Spontaneous Breaking

Spontaneous symmetry breaking (SSB) of higher-form symmetries, in accordance with SSB of ordinary symmetries, allows for the appropriate identification of the degrees of freedom relevant to the axion EFT as we review in this section.

Spontaneous breaking is the variance of the vacuum under the (generalised) symmetry,

$$O(M)\,|\Omega\rangle \neq |\Omega\rangle \ . \tag{3.81}$$

In general, spontaneous symmetry breaking can be detected by using what's called an order parameter $\mathcal{O}(x)$ charged under the symmetry. The theory is spontaneously broken if in a given reference vacuum $|\Omega\rangle$, the expectation value of the order parameter is non-zero,

$$\langle\Omega\,|\,\mathcal{O}\,|\,\Omega\rangle \neq 0\,. \tag{3.82}$$

For a general Wilson line in a broken or unbroken phase, the order parameter is a product operator and therefore requires renormalisation. For a line operator, this corresponds to a renormalisation of the tension T of the loop such that $W[C] \to W[C]e^{-\delta T\int_C}$. If upon renormalisation, the vacuum expectation value is non-zero, then the symmetry is said to be in a spontaneously broken phase,

$$\langle W[C]\rangle \sim 1\,. \tag{3.83}$$

This occurs for the $U(1)$-1 form symmetry in Abelian gauge theories in $D = 3 + 1$ dimensions. We recognise the photon as the Goldstone boson of the spontaneous breaking [23]. The low-energy EFT for the Goldstone boson at energy scales below the mass m of any charged particle is the Euler-Heisenberg action (3.78).

If upon renormalisation, the vacuum expectation value still tends to zero in the large loop limit, then the symmetry is in the unbroken phase,

$$\langle W[C]\rangle \sim 0\,. \tag{3.84}$$

Our archetypical example is the 1-form shift symmetry of the photon in $D = 1 + 1$. The spectrum of the photon in $D = 1+1$ was shown to be gapped in equation (2.71) with states labelled by $n \in \mathbb{Z}$. The vacuum state, $n = 0$, is invariant under the 1-form symmetry, which acts as $|n\rangle \to e^{i\alpha n} |n\rangle$. Relatedly, the theory is confining, and the potential scales as $V(r) \sim e^2 r$. We can compute the expectation value of a Wilson line with a temporal extent of T and a spatial extent of R, which correspond to a particle-antiparticle pair existing for a time T separated by a distance R. The answer should be the Euclidean action cost of this state,

$$\langle W[C]\rangle \sim e^{-TV(r)} \sim e^{-TRe^2} \tag{3.85}$$

As expected, the expectation value goes to zero faster than the perimeter of the loop $2(R + T)$. The symmetry is therefore unbroken, and the photon will not enter the low energy EFT, as we saw in section 2.5.4.

This also occurs for Yang-Mills theory,

$$\langle W[C]\rangle \sim e^{-\Lambda^2_{\text{QCD}}\text{Area}(C)} \tag{3.86}$$

where the expectation value roughly scales with the smallest area inscribed in the loop times Λ_{QCD}. The expectation value of a single Wilson loop falls rapidly as the only interactions between different parts of the loop are short-ranged and facilitated by massive glueballs.

In general, a p-form symmetry with a p-dimensional order parameter with surface area L^p is said to be in the unbroken phase if the expectation value scales with the enclosed volume V or faster $\exp(-T_{p+1}V)$. This implies that in the large surface limit this expectation value will be zero and there are short-range correlations between the various parts of the volume. If the order parameter scales with the surface area L^p by $\exp(-T_p L^p)$ or less fast, the theory is said to be in the spontaneously broken phase. In the large surface limit this can be renormalised to 1.

The Hohenberg-Coleman-Mermin-Wagner (HCMW) theorem [218–220] states that ordinary symmetries cannot be broken in $D \leq 2$. A spontaneously broken p-form symmetry in D-dimensions of space-time would lead to a spontaneously broken ordinary symmetry in $(D-p)$-dimensional space-time upon suitable compactification. Therefore, a p-form symmetry cannot be spontaneously broken in $D \leq 2 + p$ space-time dimensions [23, 47].

3.5 Higher-Group Symmetries & Axions

The last category of symmetries relevant to axion physics, especially axions from string theory, is higher-group symmetries and will eventually lead to *higher axions*.

Ordinary global symmetries interact in non-trivial ways, often rephrased in the language of anomalies. The presence of symmetries of different forms opens up interactions between topological operators of different co-dimensions. In general, symmetries can interact when their associated topological operators of varying co-dimensions *cannot* be topologically moved past each other. The resulting symmetry structures are higher-group symmetries [97], and will be relevant when we study extra-dimensional axions in chapter 6. The modern mathematical description is in terms of higher-categorical symmetries [195–200], and higher-group symmetries have been used to constrain (B)SM physics in [59, 62].

The topological notion of symmetries complicates the study of their interaction. Instead, the effect of a p-form symmetry can be probed by including an appropriate background $p+1$-form gauge field A_{p+1}, akin to the introduction of sources J for fields. For a continuous symmetry that can be described in terms of a $p+1$-form current j, this is done as,

$$Z[A] = \int D\phi \, \exp\left[iS[\phi] + i\int A_{p+1} \wedge \star j_{p+1}\right] \tag{3.87}$$

Derivatives of the background gauge field yield the Ward identities (3.29).

The conservation of the symmetry current j implies that the background gauge field is invariant under transformations $A_{p+1} \to A_{p+1} + d\Lambda_p$. Any symmetry is therefore topological if and only if its background gauge field is gauge invariant. The topological movement of the manifold on which the symmetry is defined is equivalent to the application of an appropriate gauge transformation.

Sources J for fields are equivalent to field insertions if and only if they are bump functions around a point x. The generalisation of this is that, if and only if the background field A is closed, $dA = 0$, then the background gauge field is equivalent to the insertion of a symmetry operator by Poincaré duality (see section 2.4).

For discrete symmetries, all background gauge fields are closed and correspond to the insertion of a network of topological operators. For continuous symmetries, one can also consider background gauge fields with non-vanishing field strength $F = dA$ which have no such interpretations.

One speaks of gauging the symmetry if the background gauge field is made dynamical. In this case, one integrates over all gauge configurations, which for discrete symmetries corresponds to summing over all insertions of topological operators. In the continuous case, this corresponds to adding to the partition function

$$Z = \int DA \, \exp\left[\frac{i}{2e^2}\int dA_{p+1} \wedge \star dA_{p+1} + i\int A_{p+1} \wedge \star j_{p+1}\right] \tag{3.88}$$

and any other operators at higher orders in the background gauge field A required to make the action gauge invariant.

3.5.1 Anomalies

Anomalies define the interactions between symmetries. In the background field language, interactions are specified by non-trivial relations between the gauge transformations of the background gauge fields. Moving one symmetry might be sensitive to a non-vanishing background field strength of another symmetry.

The general language to capture these interactions is that of 't Hooft anomalies[1] . A symmetry with background gauge field A is said to be 't Hooft anomalous if the partition function is variant under a gauge transformation,

$$Z[A + d\lambda] \neq Z[A]\,. \tag{3.89}$$

In this case, the partition function is invariant up to terms containing the gauge invariant field strength of $F = dA$. The symmetry is a symmetry of the quantum field theory, as symmetry operators are associated with closed background gauge fields $F = 0$, but the symmetry cannot be gauged.

[1]Often a class of anomalies is cited called ABJ anomalies, in which case the partition function is not invariant under the gauge transformation of a background gauge field, even when the field strength of that background gauge field vanishes. This is not a symmetry of the quantum field theory, and we refrain from using this terminology.

The simplest example to describe the power of these tools is that of a set N of Weyl fermions in $D = 3+1$ and a partition function

$$Z = \int D\psi D\overline{\psi} \exp\left[i \int d^4x \sum_i \overline{\psi}_i i\partial\!\!\!/ \psi_i\right] \tag{3.90}$$

This system has a $U(N)$ symmetry and we will consider background fields for the various $U(1)^N$ subgroups. Suppose the i'th Weyl fermion has a charge q_i^A under a $U(1)$ symmetry A. It is well-known that in the presence of a background gauge field A, the partition function transforms under a transformation $\psi \to e^{i\alpha}\psi$ as

$$Z[A] \to Z[A] \exp\left[i\frac{\alpha}{8\pi^2}\sum_i \left(q_i^A\right)^2 \int F^A \wedge F^A\right], \tag{3.91}$$

where $F^A = dA$. We therefore see that the symmetry suffers an 't Hooft anomaly

$$Z[A+d\lambda] \to Z[A] \exp\left[i\sum_i \left(q_i^A\right)^3 \int \frac{\lambda}{8\pi^2} F^A \wedge F^A\right], \tag{3.92}$$

when $\sum_i \left(q_i^A\right)^3 \neq 0$ and in this case the symmetry cannot be gauged. Often the anomaly is attributed to the non-invariance of the infinitesimal field space element $D\psi$ under the symmetry transformation. This statement has led to erroneously attributing all anomalies to a change in the measure of the massless fermions. This statement is however highly dependent on the choice of dummy variables in the path integral, as can be seen in, for instance, $D = 1+1$ where a bosonisation yields the anomaly as part of the equation of motion. The appropriate statement is that the measure plus action is variant under the symmetry.

If the partition function is not invariant under a gauge transformation of a background gauge field C due to the presence of another background gauge field A,

$$Z[A, C+d\lambda] \neq Z[A, C], \tag{3.93}$$

then the symmetry is said to have a mixed 't Hooft anomaly. This occurs when the Weyl fermions have charges under both the symmetries and is an immediate generalisation of the previous case by replacing $q_i^A F^A \to q_i^A F^A + q_i^C F^C$ in equation (3.91).

As one example, suppose the charges are such that $\sum_i \left(q_i^A\right)^3$, $\sum_i \left(q_i^C\right)^3$ and $\sum_i q_i^C \left(q_i^A\right)^2$ all vanish. In this case, the symmetry C has a mixed 't Hooft anomaly of the form,

$$Z[A, C + d\lambda] = Z[A, C] \exp\left[i \sum_i q_i^A \left(q_i^C\right)^2 \int \frac{\lambda}{8\pi^2} F^A \wedge F^C\right] . \tag{3.94}$$

Sometimes, it is said that mixed 't Hooft anomalies of this kind imply that both symmetries cannot be gauged at the same time. This statement is not entirely true and leads to the last class of symmetries called higher-group symmetries.

3.5.2 Higher-Group Symmetries

Our final new class of generalised symmetries is associated with the mixed 't Hooft anomalies of the previous section and equation (3.94).

Suppose we were to gauge the symmetry A by introducing a dynamical gauge field $F^A \to f^A$. The partition function shifts under gauge transformations of C as,

$$Z[C + d\lambda] = \int DAD\psi \exp\left[iS[\psi, A, C] + i \sum_i q_i^A \left(q_i^C\right)^2 \int \frac{\lambda}{8\pi^2} f^A \wedge F^C\right] \tag{3.95}$$

The symmetry appears to be absent in the quantum field theory, but there is another new symmetry corresponding to the magnetic higher-form symmetry of the newly introduced gauge field f^A. The magnetic 1-form symmetry couples to the current $\frac{1}{2\pi} \star f^A$. The introduction of a 2-form background gauge field B for this symmetry introduces an additional interaction in the partition function,

$$S \supset \int B \wedge \frac{1}{2\pi} f^A . \tag{3.96}$$

The introduction of B allows us to cancel the anomalous background field transformation if B shifts under transformations of C as,

$$B \to B - \frac{\lambda}{4\pi} \sum_i q_i^C \left(q_i^A\right)^2 F^C . \tag{3.97}$$

In this case, the partition function is invariant

$$Z\left[B - \frac{\lambda}{4\pi} \sum_i q_i^C \left(q_i^A\right)^2 F^C, C + d\lambda\right] = Z\left[B, C\right] . \tag{3.98}$$

A symmetry structure where the respective background gauge fields transform under each other's gauge transformations is referred to as a higher-group symmetry [97]. The symmetry in equation (3.98) is a 2-group, referring to the highest form of the background field participating in the higher-group. Structures of this kind imply non-trivial interactions on the intersections between the topological operators associated with B and C [97, 210].

In string theory, anomaly cancellations of this kind occur in the Green-Schwarz mechanism [221]. In this scenario, the background fields B and C are dynamical. The gauged symmetry C is referred to as an 'anomalous' $U(1)$ symmetry, although no anomaly is present. The 2-form field B is the dynamical Kalb-Ramond field present in the supergravity action. The corresponding Lagrangian of B has to be invariant under the gauge transformations of equation (3.98),

$$\mathcal{L} \supset \frac{1}{2f^2}\left(dB - \sum_i q_i^C \left(q_i^A\right)^2 \frac{1}{4\pi} C\wedge F^C\right)\wedge\star\left(dB - \sum_i q_i^C \left(q_i^A\right)^2 \frac{1}{4\pi} C\wedge F^C\right) + B\wedge\frac{1}{2\pi} f^A\,. \tag{3.99}$$

Important for us is that one category of axions in string theory is associated with the reduction of the Kalb-Ramond field B or its dual B_6 on compact extra-dimensional cycles. If the axion a is the reduction,

$$a = \int_{S^2} B\,, \tag{3.100}$$

on an extra-dimensional 2-cycle S^2 and there is a non-zero flux $\int F^C$ on the 2-cycle, then the low-energy axion is eaten,

$$\mathcal{L} \sim \frac{1}{2f^2\mathrm{Vol}(S^2)}(da - C)^2 + \dots\,. \tag{3.101}$$

The higher-form shift symmetry of B or the axion a does not disappear from the spectrum. The 2-group structure in equation (3.98) involves a gauging of the shifts of the axion *and* the fermions. The orthogonal shift survives in the low-energy spectrum, and inherits the exponentially good quality of the higher-form symmetry. We will return to this in chapter 6, when this higher-form shift symmetry will be inherited by the phases of complex scalars, resulting in an axion category we dub *higher axions*.

4

The Monodromic Axion-Photon Coupling

In this chapter, we apply generalised symmetries to constrain the vast axion mass and axion-photon coupling parameter space. We demonstrate that, in general, the axion-photon coupling is a non-linear monodromic function of the axion in section 4.2. The non-linearities in the axion-photon coupling are correlated with the axion mass through a non-invertible symmetry. We derive the general form of the axion-photon coupling for several examples, including the QCD axion, in sections 4.3, 4.4, and 4.5, and show that there is a prototypical form for this monodromic function. We summarise our findings and the phenomenological relevance in the discussion in section 4.6.

4.1 Introduction

The axion-Maxwell Lagrangian describes the low-energy physics of one of the most compelling new physics candidates, the axion, and its experimentally important mass and coupling to photons. In section 2.4.1, we argued that the general low-energy axion-Maxwell Lagrangian takes the form,

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}f^2\partial_\mu a\partial^\mu a - V(a) + \frac{g(a)}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\,. \tag{4.1}$$

The discovery of the axion-photon interaction will not just be a discovery of a new particle but can provide deep insights into the structure of the standard model as reviewed in chapter 2. The axion could elegantly explain the absence of CP violation in the strong sector [8–10] (see section 2.5.1), is a natural solution to the cosmological puzzle of dark matter [11–13] as seen in section 2.3.2, and is ubiquitous in extra-dimensional theories [18], including string theory [19, 20]. Furthermore, the axion parameter space provides a low-energy probe for the grand-unified extension of the Standard Model [14], including heterotic string theory [15], and axions have strong connections with conjectures in quantum gravity [16, 17].

A large part of the wealth of information that we would derive from the discovery of the axion comes from the special topological nature of the axion-photon coupling $g(a)$ and the associated symmetries and redundancies.

The axion-photon coupling $g(a)$ (see section 2.4.1) is a monodromic function, defined by the property,

$$g(a + 2\pi) = g(a) + 2\pi n\,. \tag{4.2}$$

The integer n is the monodromic charge of the function $g(a)$. This property of the monodromic function arises from the discrete gauge symmetry of the axion, $a \to a + 2\pi$, under which the path integral weight, e^{iS}, is required to be invariant.

It is an extremely important fact that the monodromy of $g(a)$ does not imply that the perturbative coupling of the axion to photons around the CP-conserving point $a = 0$ is quantised. Indeed, the coupling for canonically normalised fields,

$$g_{a\gamma\gamma} = \frac{\alpha_{\rm em}}{\pi f} g'(a)|_{a=0}, \tag{4.3}$$

which can be an arbitrary number for a non-linear monodromic function $g(a)$, and provides one of the main experimental probes in axion searches.

This resolves a small puzzle in the QCD axion coupling to photons, as was noted in [222] and further discussed in [223]. On one hand, we usually justify the non-quantised couplings of the axion by invoking the mixing with the pion. On the other hand, for all values of the axion, the pion remains heavy and can stay integrated out, leaving an apparent non-monodromic function. The resolution to the non-quantisation of the coupling therefore should appear in the low-energy axion-Maxwell theory without needing to invoke the pion. Indeed, this is achieved by a monodromic non-linear function $g(a)$.

In this chapter, we will find that a linear integer form $g(a) = na$ is protected by the continuous non-invertible shift symmetry (3.44) of the axion, which also protects the axion from getting a mass. Both a potential $V(a)$ and a non-linear $g(a)$ act as spurions for the breaking of the axion's continuous shift symmetry [222, 223]. This gives a precise sense in which the deviation from quantisation (linear form) of axion couplings and the generation of a mass are linked. Thus, while the monodromy of $g(a)$ follows from the topology of the axion, the special case of $g(a) = na$ additionally requires the presence of a continuous non-invertible symmetry.

A general monodromic $g(a)$ can be split into an integer linear form plus an infinite sum of Fourier modes. In some cases only the first few terms in the expansion dominate. This is simply the expected contribution from axion-dependent perturbative corrections to non-topological quantities like $\alpha_{\rm em}$. However, in many cases, including the case of the QCD axion, the final form of $g(a)$ requires the sum over the entire Fourier tower, and it is interesting that a closed form expression for $g(a)$ can be derived.

The functional coupling $g(a)$ elucidates many interesting physics points. As mentioned above, it captures the correct monodromy in the axion-Maxwell Lagragian when all other fields can be integrated out for all values of the axion. In cases where this is not possible (e.g. when some particles become light at some value of the axion field) $g(a)$ also captures fast rearrangement of degrees of freedom through its singularities at isolated points. This correlates with cusps and singularities in the axion potential at the same point and interesting dynamics induced on an axion domain wall.

Phenomenologically, the full non-linear form of $g(a)$ is most relevant for scenarios where the axion traverses an $O(1)$ fraction of its field range. This is certainly true for axion strings and domain walls (as implemented in the recent paper [63]), and sharp features or jumps in $g(a)$ can affect axion emission from these objects. For instance, in an electromagnetic environment, the axion emission by the Primakoff effect will be dominated by shell regions where the axion field value is close to the jump. Similar effects can also be true for dense axion objects, like axion miniclusters (of nuclear density) or superradiant axion clouds surrounding rotating black holes.

The fact that in many simple models the whole Fourier tower needs to be summed up to get the relevant $g(a)$ highlights another interesting point. For effective field theories involving compact fields, the standard polynomial basis might not be the most convenient basis to work in.

This chapter is organised as follows. In section 4.2, the general properties of the axion-photon coupling $g(a)$ and the symmetries of the axion-Maxwell Lagrangian in the presence of $g(a)$ are discussed, together with the connection between the mass and non-quantisation of $g(a)$. In section 4.3, the important case of perturbative non-invertible symmetry breaking is introduced and shown to share many features of the QCD axion. Section 4.4 discusses the QCD axion, non-perturbative breaking of the non-invertible symmetry and the corresponding axion-photon coupling. The final section 4.5 is entirely devoted to axion potentials and photon couplings in the presence of a tower of states. We finish by summarising our findings and the phenomenological relevance in the discussion in section 4.6.

4.2 General properties of $g(a)$

In the presence of a linear axion-photon coupling $g(a) = na$, a massless axion admits a non-invertible shift symmetry $a \to a + c$ as discussed around equation (3.44). In general, a mass or potential $V(a)$ for the axion breaks this non-invertible shift symmetry. If the axion-photon coupling is a general function $g(a)$ as in (4.1), then the non-invertible symmetry is also broken. The associated current for the ordinary shift symmetry is,

$$\partial_\mu j^\mu = \frac{g'(a)}{16\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} \,. \tag{4.4}$$

If the slope $g'(a)$ is not a constant, then one cannot construct a non-invertible shift symmetry as in equation (3.44), as no conserved current can be constructed whether gauge-invariant or not.

Moreover, if the slope $g'(a)$ is a constant, then the monodromic property (4.2) forces the slope to be integer. This implies that the spurion for non-invertible shift symmetry breaking is the non-integer part of the slope $g'(a)$ mod 1. A non-integer part to the slope at any point means that the axion-photon coupling has to have higher terms in the axion $a^3, a^5, \ldots$ etc to satisfy the monodromy, in turn breaking the non-invertible symmetry.

Both a mass for the axion m and an axion-photon coupling that is not of the pure linear quantised form $g(a) = na$ explicitly break the non-invertible shift symmetry and act as spurions. In a generic EFT, we expect that if the shift symmetry is broken by the presence of one of the spurions, then any spurion breaking that symmetry will be generated. This leads to the simple heuristic rule that a mass for the axion implies a non-linear axion photon-coupling and vice versa,

$$m = 0 \iff g(a) = \mathbb{Z}a\,. \tag{4.5}$$

If the axion-photon coupling is nearly quantised, then we can express the degree of non-quantisation as

$$g(a) - na = zh(a), \tag{4.6}$$

with $0 \leq z \ll 1$ and $h(a)$ an $O(1)$ periodic function. If we expect the size of the two spurions to be commensurate, then an estimate for the mass of the axion is

$$m^2 \sim z\frac{\Lambda^4}{f^2}\,. \tag{4.7}$$

where Λ is the UV cutoff of the effective theory consistent with the coupling $g(a)$ (e.g. for the QCD axion $\Lambda \simeq \Lambda_{\text{QCD}}$).

We emphasise that this is a heuristic estimate, and the actual correlation may be different in specific examples especially if more symmetries are at play (see for instance [128]). However, this correlation highlights the point that if there is an axion that is parametrically lighter than its naively expected mass, that also corresponds to a coupling to photons that is very nearly an integer. Similarly, if an axion coupled to photons picks up a mass, it generically also picks up a non-linear $g(a)$ coupling to photons [223].

Common to all of the examples considered in this chapter will be a positive real parameter $z \in [0, \infty)$ that breaks the non-invertible shift symmetry of the axion. In the same spirit as the spurion analysis of section (3.1.2), the non-invertible shift symmetry $a \to a + c$ can be restored if z transforms as $z \to ze^{-ic}$. The connection between the potential $V(a)$ and axion-photon coupling $g(a)$ is best provided by the repackaging of z together with the axion a into a complex quantity,

$$\mathcal{Z}(a) = ze^{ia}\,, \tag{4.8}$$

that is invariant under the non-invertible symmetry.

The low energy EFT will be a function of $\mathcal{Z}$ and by CP symmetry, the real and imaginary parts of powers of $\mathcal{Z}$ respectively contribute to the CP even potential $V(a)$ and CP odd effective axion-photon coupling $g(a)$, providing the connection between the two. A heuristic estimate for the axion mass for small z was provided in equation (4.7).

A repackaging of the axion of the form (4.8) is well-known from supersymmetry and was already noted in contributions to the superpotential in [224] where $z = e^{-S_{\text{inst}}}$ represents the instanton contribution. More recently, the full axion-photon coupling $g(a)$ and potential $V(a)$ have been calculated for $\mathcal{N} = 2$ SYM in [128].

The low-energy EFT is a function of the massless photon and the shift symmetry invariant quantity $\mathcal{Z}(a)$. The invariance of $\mathcal{Z}(a)$ under the non-invertible symmetry allows the construction of a more general axion-photon coupling,

$$\mathcal{L} \supset \frac{g(a)}{16\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}\,, \qquad g(a) = na + \sum_{k\neq 0} c_k \mathcal{Z}^k\,, \tag{4.9}$$

which returns to the original form when the non-invertible symmetry is restored, which will turn out to be $z \to 0$ (and $z \to \infty$).

The function $g(a)$ can be thought of as a generalisation of the original anomaly with electromagnetism. The original anomaly coefficient can still be recovered from the monodromy of $g(a)$,

$$g(a + 2\pi) - g(a) = 2\pi n\,. \tag{4.10}$$

It is an extremely important fact that the monodromy of $g(a)$ does not imply that the perturbative coupling of the axion to photons around the CP conserving point $a = 0$ is quantised. Indeed, the coupling for canonically normalised fields,

$$g_{a\gamma\gamma} = \frac{\alpha_{\rm em}}{\pi f} g'(a)|_{a=0}, = \frac{\alpha_{em}}{\pi f}\left(n + \sum_{k\neq 0} ikc_k z^k\right) \tag{4.11}$$

which can be an arbitrary number for a non-linear monodromic function $g(a)$.

Common to our examples will be a prototypical axion-photon coupling $g(a)$ that can be expressed as a contour integral,

$$g(a) = \mathrm{Im}\int_C \frac{d\mathcal{Z}}{\mathcal{Z}}\frac{1-\mathcal{Z}}{1+\mathcal{Z}}\,, \tag{4.12}$$

where the contour C is an arc at radius z of angular size a. The monodromic charge n can be extracted from equation (4.12) by the poles of the integrand that are included in the closed contour at radius z. The poles for this particular function are located at $\mathcal{Z} = 0$ and $\mathcal{Z} = -1$ with respective residues 1 and -2, giving a monodromic charge that is $n = \mathrm{sign}(1 - z)$.

The effective axion-photon coupling can be extracted from equation (4.12) by performing the contour integral over the arc C,

$$g(a) = 2\arctan\left(\frac{1-z}{1+z}\tan\frac{a}{2}\right) + 2\pi\mathrm{sign}(1-z)\Theta(a-\pi)\,, \tag{4.13}$$

where Θ is the Heaviside function. The full profile of $g(a)$ is plotted for several relevant parameter values in figure 4.1. The function $g(a)$ can be decomposed into a monodromic part na and a periodic part; the latter captures the explicit breaking of the non-invertible axion shift symmetry.

The feature most relevant to current experiments is the slope of the effective axion-photon coupling around the minimum $a = 0$ of the potential,

$$g'(0) = \frac{1-z}{1+z}\,. \tag{4.14}$$

There are three values of the real parameter z that are interesting. At the points $z = \{0, \infty\}$, $g'(a) \in \mathbb{Z}$ and the non-invertible symmetry is restored. In our examples, the axion potential also vanishes for these values of z. The function

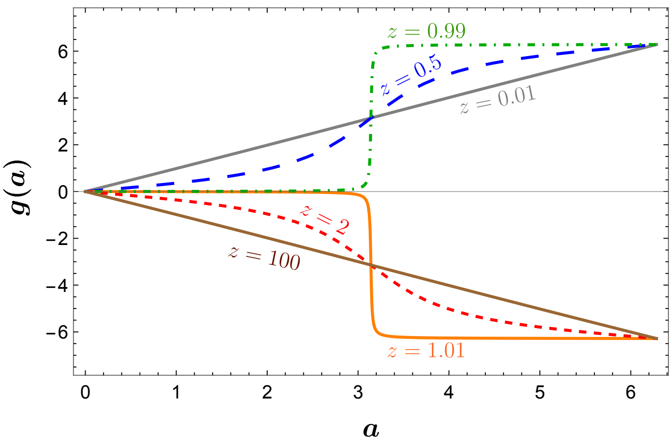

Figure 4.1: The effective axion-photon coupling $g(a)$ for the prototypical example in equation (4.13) at values $z = \{0.01, 0.5, 0.99, 1.01, 2, 100\}$ showing that $g(a)$ jumps across $z = 1$ and further becomes discontinuous as $z \to 1^{\pm}$ at $a = \pi$.

$g(a)$ does not have a well-defined limit as $z \to 1$, it changes discontinuously across $z = 1$. In this limit, $g'(0) = k \in \mathbb{Z}$, but the axion shift symmetry is not restored, and the monodromy is not equal to k.

Common to our examples will be the restoration of a $\mathbb{Z}_2$ discrete symmetry at $z = 1$, which has an anomaly with electromagnetism. The anomaly is captured in the low-energy effective theory by $g(a)$ changing discontinuously across $z = 1$. Furthermore, the different profiles $\lim_{z\to1^-} g(a)$ and $\lim_{z\to1^+} g(a)$ are both discontinuous at the point $a = \pi$, describing a fast rearrangement of degrees of freedom and restoration of a $U(1)$ symmetry. This discontinuity in $g(a)$ at $a = \pi$ is reproduced at the same point by a singularity in the potential $V(a)$, but can be sensitive to higher-order terms.

4.3 Perturbative Breaking

An instructive example is the explicit breaking of the non-invertible shift symmetry of the axion by a massive charged Dirac fermion Ψ coupled to a $U(1)$ gauge field. The axion a is coupled through a chiral mass term, resulting in a Lagrangian of the form,

$$\mathcal{L} = i\overline{\Psi}\not{D}\Psi - f\overline{\Psi}e^{ia\gamma_5}\Psi - m_\Psi\overline{\Psi}\Psi\,. \tag{4.15}$$

In the absence of a mass m_Ψ, this Lagrangian has an axion shift symmetry $a \to a + c$ and $\Psi \to e^{-i\frac{c}{2}\gamma_5}\Psi$, which is non-invertible due to the fermion's electric charge [51].

This non-invertible symmetry is explicitly broken by the mass term m_Ψ, but can be restored by having m_Ψ transform as $m_\Psi \to e^{ic\gamma_5} m_\Psi$. The low-energy theory is therefore an EFT in the independent CP even and odd terms,

$$\mathrm{Tr}\left(m_\Psi e^{ia\gamma_5}\right) , \qquad \mathrm{Tr}\left(\gamma_5 m_\Psi e^{ia\gamma_5}\right) . \tag{4.16}$$

As the non-invertible shift symmetry is broken, we expect to generate both an axion-photon coupling $g(a)$ and potential $V(a)$ and the low-energy effective Lagrangian should be of the form equation (4.1). We shall see that this simple model captures a lot of the features of the QCD axion.

In order to calculate $g(a)$ and $V(a)$ in this model, we rewrite the two complex mass terms for the fermion as one,

$$\overline{\Psi}\left(fe^{ia\gamma_5} + m_\Psi\right)\Psi = \overline{\Psi}\ m(a)e^{ig(a)\gamma_5}\ \Psi . \tag{4.17}$$

The modulus is the effective axion-dependent mass of the fermion,

$$m(a)^2 = |fe^{ia} + m_\Psi|^2 = m_\Psi^2 + 2m_\Psi f\cos a + f^2 . \tag{4.18}$$

The phase $g(a)$ will be the axion-photon coupling,

$$g(a) = \mathrm{Arg}(fe^{ia} + m_\Psi) = \frac{1}{2}a - \arctan\left(\frac{1-z}{1+z}\tan\frac{a}{2}\right) - \mathrm{sign}(1-z)\pi\Theta(a-\pi) . \tag{4.19}$$

where we have defined the parameter

$$z = \frac{f}{m_\Psi} . \tag{4.20}$$

The trick to calculating the axion-photon coupling is to consider the Lagrangian with the single compact term (4.17) and make the chiral transformation,

$$\Psi \to e^{-ig(a)\gamma_5/2}\Psi , \tag{4.21}$$

In this case, the Lagrangian reduces to,

$$\mathcal{L} = i\overline{\Psi}\not{D}\Psi - \overline{\Psi}m(a)\Psi - \frac{g(a)}{16\pi^2}F\widetilde{F} . \tag{4.22}$$

We recognise $g(a)$ in equation (4.19) as the axion-photon coupling. Similar to the potential, $g(a)$ captures many features of the symmetries of the Lagrangian in its fully summed form. For instance, in the shift symmetry-preserving limit $z = \infty$ or $m_\Psi = 0$, $g(a)$ becomes linear, $aF\widetilde{F}$. When $z = 0$, the effective coupling of axions to the fermions vanishes and $g(a) = 0$. Thus in both cases $g(a)$ becomes of the form $\mathbb{Z}a$ when the axion becomes massless as predicted by general symmetry arguments.

In the limit $z \to 1$, there is an apparent restoration of a $\mathbb{Z}_2$ symmetry that acts on the fields as

$$\Psi(t,x) \to \gamma^0\Psi(t,-x) \qquad A_\mu(t,x) \to (-1)^\mu A_\mu(t,-x) \qquad a(t,x) \to a(t,-x)\,, \tag{4.23}$$

where $(-1)^\mu = 1$ if $\mu = t$ and -1 otherwise. This symmetry leaves the Lagrangian invariant in this limit up to the change of the Chern-Pontryagin density $F\widetilde{F}$. Correspondingly, the profiles for $\lim_{z\to 1^-} g(a)$ and $\lim_{z\to 1^+} g(a)$ differ, and the monodromies are respectively 0 and 1. Both profiles of $g(a)$ also have a discontinuous jump at the chiral symmetry restoring point $z = 1,\ a = \pi$ where the fermion becomes massless $m(a = \pi) = 0$ (see (4.18)) and the EFT breaks down.

The effective potential $V(a)$ can be computed by integrating out the fermion,

$$iS_{\text{eff}} = \mathrm{Tr}\ln\left[i\left(i\not{D} - m(a)\right)\right]. \tag{4.24}$$

This yields an effective potential for a constant axion a as

$$V(a) = 2i\mathrm{Tr}\ln\left[\partial^2 + m(a)^2\right]. \tag{4.25}$$

This is simply the Coleman-Weinberg potential following from a particle with an effective mass $m(a)$,

$$V(a) = -c_1 m(a)^2 - \frac{m(a)^4}{16\pi^2}\ln\frac{m(a)^2}{c_2^2}\,, \tag{4.26}$$

where c_1 and c_2 are renormalisation scheme-dependent quantities.

This potential shares many features of the QCD axion. The potential becomes axion independent when $z \to \{0, \infty\}$ as this is when either f or m_Ψ is zero and the shift symmetry is restored. In the limit $z = 1$, the effective mass of the particle $m(a)$ vanishes at the chiral symmetry restoring point $a = \pi$ and should not be integrated out. This is reflected by a singularity or cusp in V'''' at $a = \pi$.

4.4 QCD Axion

We study QCD in the two-flavour approximation $N_f = 2$ coupled to the axion with the Lagrangian as introduced in section 3.1.2,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}f^2(\partial a)^2 - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) \\ &+ \sum_{i=1}^{2}\overline{\Psi}_i\left(i\not{D} - m_i\right)\Psi_i + \frac{Na}{8\pi^2}\mathrm{Tr}\left(G_{\mu\nu}\widetilde{G}^{\mu\nu}\right) + \frac{Ea}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu},\end{aligned} \tag{4.27}$$

where f is the axion decay constant, E is the primordial anomaly with electromagnetism, and $N \in \frac{1}{2}\mathbb{Z}$ is the anomaly coefficient with QCD[1].

The condition on E in order for the axion to have 2π periodicity depends on the chosen subgroup $\Gamma = 1, \mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_6$ of the standard model gauge group $SU(3) \times SU(2) \times U(1)/\Gamma$ [60, 129, 170]. In this chapter we take $\Gamma = \mathbb{Z}_6$, and a sufficient condition for axion 2π periodicity is

$$E - \frac{2N}{3} \in \mathbb{Z}\,. \tag{4.28}$$

The axion-gluon coupling explicitly breaks the non-invertible shift symmetry of the axion non-perturbatively. We therefore expect the low-energy effective axion theory to have a potential $V(a)$ and generate an effective axion-photon coupling $g(a)$. The symmetries, phases, and domain walls of this theory have been well-studied using the chiral Lagrangian [225–228] and anomaly matching [25, 229].

The mass for the axion and its coupling to photons have been calculated at high precision [230],

$$m_a^2 = \frac{m_\pi^2 f_\pi^2 N^2}{f^2}\left(\frac{4}{z + \frac{1}{z} + 2} + \dots\right), \qquad g_{a\gamma\gamma} = \frac{\alpha_{\mathrm{em}}}{\pi f}\left(E - \frac{5}{3}N - \frac{1-z}{1+z}N + \dots\right), \tag{4.29}$$

where $z = \frac{m_u}{m_d}$ measures the isospin breaking of $SU(2)_V$ and $\dots$ denote higher order terms in the chiral Lagrangian. Note that as is conventional, we have written the coupling $g_{a\gamma\gamma}$ in the canonical basis for both axions and photons.

The usual explanation for the non-rational contribution of $\frac{1-z}{1+z}$ to the axion-photon coupling slope is that it arises from the mixing with the pion. This is

[1] Here we have used the unfortunate standard convention making N in general half-integer.

certainly true but raises a minor puzzle. In the effective theory, we can integrate out the pion, and for all values of the axion, the pion degree of freedom is heavy and the EFT is valid. Therefore, the quantisation of the monodromy of the axion-photon coupling should be visible in the effective theory.

The resolution to this puzzle has been discussed in [222] and [223] and arises exactly through the monodromic function $g(a)$. As we will show below, the axion coupling to photons can be packaged in this functional form, such that $g(a)$ has integer monodromy under the axion discrete gauge symmetry, but $g'(0)$ can be irrational. We review the calculation of the axion potential in the chiral Lagrangian and derive the form of $g(a)$ relevant for the QCD axion below.

The effective Lagrangian for the photon, the QCD axion a and the pion π^0, was derived in section 3.1.2,

$$\mathcal{L} = \frac{f^2}{2}\,(\partial a)^2 + \frac{f_\pi^2}{2}\,(\partial \pi^0)^2 - V(a,\pi^0) + \left(E - \frac{5}{3}N\right)\frac{a}{16\pi^2}F\tilde{F} + \frac{\pi^0}{16\pi^2}F\tilde{F}\,, \tag{4.30}$$

with a potential $V(a,\pi^0)$ given by

$$V(a,\pi^0) = f_\pi^2 m_\pi^2\left(1 - \cos\frac{2Na}{2}\cos\pi^0 + \frac{1-z}{1+z}\sin\frac{2Na}{2}\sin\pi^0\right)\,. \tag{4.31}$$

In this basis, the two discrete gauge symmetries involving the axion and the pion are implemented by $(a,\pi^0) \to (a+2\pi, \pi^0+2N\pi)$ and $\pi^0 \to \pi^0 + 2\pi$. The potential has characteristic eigenvector directions which reverse roles when the sign of $1-z$ flips, which will be important to our discussion throughout.

We would like to study the low-energy limit of this theory. In the limit that $f_\pi \ll f$, the pion is much more massive than the axion and can be integrated out. If this can be done consistently at every value of the axion a, then axion domain walls are completely describable within the effective field theory. There can in general be additional domain walls (perhaps metastable or unstable) that also involve rearrangements of heavy degrees of freedom or new massless states appearing on the domain walls. These domain walls are not described completely within the EFT.

An axion domain wall $a \to a + 2\pi$ in this particular basis of the potential (equation (4.31)) requires a pion domain wall $\pi^0 \to \pi^0 + n\pi$ with $n \in 2N\mathbb{Z}$. For any

$z \geq 0$, to first order in $\frac{f_\pi}{f}$, we can integrate out the pion using its equation of motion,

$$\frac{\partial V}{\partial \pi^0} = 0 \implies \pi^0 = -\arctan\left(\frac{1-z}{1+z}\tan\frac{2Na}{2}\right) - \pi \sum_{k=1}^{2N} \mathrm{sign}(1-z)\Theta\left(a-(2k-1)\frac{\pi}{2N}\right). \tag{4.32}$$

The Θ Heaviside function is obtained after inverting trigonometric functions, and its strength is such that the axion domain wall $a \to a + 2\pi$ has a smooth profile. Other choices for the strength of the Heaviside function compatible with the axion-pion discrete gauge symmetry (below equation (4.31)) would lead to discontinuities in the axion EFT and are associated with additional excitations of heavy pion degrees of freedom on the domain wall.

Plugging in the pion profile yields the effective Lagrangian as,

$$\mathcal{L} = \frac{f^2}{2}(\partial a)^2 - V(a) + \frac{g(a)}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4.33}$$

with

$$V(a) = -f_\pi^2 m_\pi^2 \sqrt{1 - \frac{4z}{(1+z)^2}\sin^2\left(\frac{2Na}{2}\right)}, \tag{4.34}$$

$$g(a) = Ea - \frac{5}{3}Na - \arctan\left(\frac{1-z}{1+z}\tan\frac{2Na}{2}\right) - \pi \mathrm{sign}(1-z)\sum_{k=1}^{2N}\Theta\left(a-(2k-1)\frac{\pi}{2N}\right). \tag{4.35}$$

We see the prototypical example of the function $g(a)$ – it has a monodromy under axion shift symmetry given by

$$g(a+2\pi) = g(a) + 2\pi\left(E - \frac{2N}{3} - N\left(1+\mathrm{sign}(1-z)\right)\right). \tag{4.36}$$

The monodromic charge $\left(E - \frac{2N}{3} - N\left(1+\mathrm{sign}(1-z)\right)\right) \in \mathbb{Z}$ by equation (4.28). For the specific choice $\frac{E}{N} = \frac{8}{3}$ relevant for simple grand-unified theories, the monodromy vanishes when $0 \leq z < 1$ and is $2N$ when $z > 1$.

The slope around the axion minimum for a generic z is irrational, and the axion-photon coupling at this point is given by

$$g_{a\gamma\gamma} = \frac{\alpha_{\mathrm{em}}}{\pi f} g'(0) = \frac{\alpha_{\mathrm{em}}}{\pi f}\left(E - \frac{5}{3}N - \frac{1-z}{1+z}N\right). \tag{4.37}$$

Note that both the potential $V(a)$ and $g(a)$ depend on the axion-pion mixing parameter z and that $g(a)$ is quantised exactly in the limit $z \to \{0, \infty\}$ when the mass vanishes. Under the transformation $z \leftrightarrow \frac{1}{z}$, the effective potential is left unaltered, and the monodromy of $g(a)$ changes by $2N$.

In the isospin restoring limit $z \to 1$, the potential (equation (4.31)) has an additional $\mathbb{Z}_2 \subset SU(2)_V$ pion parity $(-1)^{N_\pi}$ symmetry that sends $\pi^0 \to -\pi^0$ and the tension difference between the domain walls $\pi^0 \to \pi^0 \pm 2N\pi$ goes to zero. The profiles and monodromies for $\lim_{z\to 1^-} g(a)$ and $\lim_{z\to 1^+} g(a)$ differ, as this $\mathbb{Z}_2$ is broken by the Wess-Zumino-Witten term. Additionally, in the limit $z \to 1$, the pion shift symmetry is restored at $a = \pi$ and the pion becomes massless. The potential $V(a)$ has a corresponding cusp at this point, and $g(a)$ is discontinuous due to the massless pion jump. This cusp and the discontinuity at $a = \pi$ is an accidental restoration of the pion shift symmetry at $a = \frac{\pi}{2N}$ for $N_f = 2$ at this order in the chiral Lagrangian and can be resolved at higher orders [231].

4.5 Extra-dimensional $g(a)$

A class of interesting axions is those that descend from gauge theories and higher form fields in extra dimensions. In the previous chapter 3.4, we have argued that such axions inherit a high-quality global symmetry that descends from a higher-form symmetry in the bulk. The study of such axions is further motivated by the fact that they arise generically in string compactifications as the Aharonov-Bohm phase of form fields around compact cycles of the internal manifold [18].

In extra-dimensional models, axion potentials arise from charged objects wrapping internal cycles which are sensitive to the Aharonov-Bohm phase and appear as instantons in the 4D theory, see e.g. [224, 232–235] as argued in section (3.4). Alternatively, this potential can be thought of as arising from the axion dependence of a KK tower of states which undergoes spectral flow as axion $a \to a + 2\pi$. A similar effect arises from a tower of dyonic states in axion-Maxwell theory [169]. In this section we bridge the relationship between contributions to the axion potentials from towers of states and winding modes wrapping the extra-dimensional cycle through an instructive example.

We consider a $U(1)$ gauge theory with gauge field A in 5D Euclidean space ($g_{\mu\nu} = \delta_{\mu\nu}$) with a massive charged fermion Ψ, with the fifth dimension y compactified on a circle of radius R,

$$S = \int d^4x \int dy \left[-\frac{1}{4e^2} F_{MN}F^{MN} - \Psi^\dagger \left(\not{D} + m \right) \Psi \right] . \tag{4.38}$$

The axion is identified with a Wilson loop $\int A$ around the compact extra dimension and can be isolated in almost axial gauge as

$$A_5(x, y) = \frac{a(x)}{2\pi R} . \tag{4.39}$$

In any theory with a compact dimension, the modes of fields can be understood in terms of a tower of states (KK modes). In the present theory, this leads to a description of the 5D fermion as a tower of electrically charged massive 4D fermions with axion-dependent masses. In such a formulation, the axion dependence of the theory can be put into a twisted boundary condition for the fermions [236],

$$\Psi(y + 2\pi kR) \simeq e^{ik(\pi - a)}\Psi(y) , \tag{4.40}$$

in which we have also given the fermion additional anti-periodic boundary conditions to align the minimum of the potential with $a = 0$. The fermionic modes then decompose into a twisted tower of states,

$$\Psi(x, y) = \sum_n \exp\left[i \left(n + 1/2 - \frac{a}{2\pi} \right) y/R \right] \Psi_n(x) . \tag{4.41}$$

and the action becomes,

$$S = \int d^4x \int dy \left[-\frac{1}{4e^2} F_{MN}F^{MN} - \sum_n \Psi_n^\dagger \left(\not{D}_4 - i\gamma_5 \left(n + 1/2 - \frac{a}{2\pi} \right) /R + m \right) \Psi_n \right] . \tag{4.42}$$

The axion acts as a chiral mass, and in complete analogy with the previous examples, the two mass terms can be combined into one complex number,

$$m_n(a)e^{ig_n(a)\gamma_5} = m - i\gamma_5 \left(n + 1/2 - \frac{a}{2\pi} \right) /R , \tag{4.43}$$

where the modulus and phase are,

$$m_n(a)^2 = |m^2 + \left(n + 1/2 - \frac{a}{2\pi}\right)^2 /R^2|^2 \,, \qquad g_n(a) = \mathrm{Arg}\left(m - i\left(n + 1/2 - \frac{a}{2\pi}\right)/R\right) . \tag{4.44}$$

After rotating the individual KK modes of the fermion by $\Psi_n \to e^{-i\gamma_5 g_n(a)/2}\Psi_n$, the axion-photon coupling is,

$$g(a) = \sum_n \arctan\left(\frac{Rm}{n + 1/2 - \frac{a}{2\pi}}\right) . \tag{4.45}$$

This sum can be done explicitly[2], resulting in,

$$g(a) = \pm\frac{1}{2}a + \arctan\left(\frac{1-z}{1+z}\tan\left(\frac{a}{2}\right)\right) + \pi\mathrm{sign}(1-z)\Theta(a-\pi) . \tag{4.46}$$

where we recognise the factor $z = e^{-2\pi Rm}$ as the instanton action of the fermion to loop around the extra dimension.

In the limit $R \to \infty$, equation (4.46) reduces to the well-known result for the 'parity anomaly' of a massive fermion,

$$g(a) = \frac{1}{2}\left(\pm 1 + \frac{m}{|m|}\right) a . \tag{4.47}$$

Similar to the previous examples, there is an apparent $\mathbb{Z}_2$ (5D parity) restoration as $m \to 0$ or $z \to 1$ in equation (4.38). The profiles and monodromies of $\lim\limits_{z\to 1^-} g(a)$ and $\lim\limits_{z\to 1^+} g(a)$ differ, however, due to the gauge-parity anomaly. In the same limit $z \to 1$, there is a jump in both profiles of $g(a)$ at $a = \pi$ due to the lightest fermion in the tower becoming massless. In the 5D theory, a domain wall describing the $-m \to m$ transition has a massless chiral fermion on it and describes anomaly inflow consistent with our 4D analysis.

4.5.1 Winding Modes

In order to obtain the potential, one can integrate out the fermion KK modes with mass $m_n(a)$ and obtain a Coleman-Weinberg potential as before. However, the sum over the entire tower is horribly inefficient in this formulation.

[2]On flat space, integrating out $5D$ fermions generates a level $\frac{1}{2}$-Chern-Simons term. In order to still have an axion gauge symmetry, the action therefore implicitly includes a primordial level $\pm\frac{1}{2}$ Chern-Simons term, which we have added to the result.

An alternative and more useful representation for our purposes is in terms of winding modes of the fermion around the compact dimension. A spurion analysis of the KK modes (4.42) suggest that the non-invertible symmetry $a \to a + c$ is restored when we apply the additional shift,

$$2\pi Rm \to 2\pi Rm - ic\,. \tag{4.48}$$

The contributions to the axion effective action can therefore be packaged in the complex quantity,

$$\mathcal{Z} = e^{-2\pi Rm} e^{-ia}\,. \tag{4.49}$$

The real and imaginary parts of powers of $\mathcal{Z}$ respectively contribute to the CP even potential $V(a)$ and CP odd effective axion-photon coupling $g(a)$.

The weight assigned to the axion contributions, $e^{-2\pi mR}$ is the action cost for a fermion to loop around the compact dimension. This is in line with the previous discussion that only loops around the compact dimension are sensitive to the Aharonov-Bohm phase and can hence generate a potential for the axion and an effective axion-photon $g(a)$. This effect is suppressed by the small spacelike propagator for heavy Ψ to loop around the extra dimension $z = e^{-2\pi Rm}$. Non-local loops of the fermion around the compact dimension appear as instanton effects in 4D, giving a mass to the axion.

The equivalence between a tower of states (KK modes) labelled by n and a tower of instantons (winding modes) labelled by k is given by Poisson resummation [169, 237],

$$\sum_{n=-\infty}^{\infty} s\left(n - \frac{a}{2\pi}\right) = \sum_{k=-\infty}^{\infty} e^{-ika} S(k) \qquad , \qquad S(k) = \int_{-\infty}^{\infty} dx \; e^{-i2\pi kx} s(x)\,. \tag{4.50}$$

The root of this equivalence is well-known in thermal field theory and follows from the following set of relations,

$$\sum_{n=-\infty}^{\infty} s\left(n - \frac{a}{2\pi}\right) = \oint_{\text{poles}} \frac{dx}{2} s(x) \coth\left(i\pi\left(x + \frac{a}{2\pi}\right)\right). \tag{4.51}$$

That is, the effect of the entire tower of KK modes can be captured by an appropriate contour integral in the complex plane around the poles of the coth function. We can utilise the exponential expansion of the coth,

$$\coth\left(i\pi x\right) = \frac{1+e^{-2\pi i x}}{1-e^{-2\pi i x}} = 1 + 2\sum_{k=1}^{\infty} e^{-2\pi k i x}\,. \tag{4.52}$$

Plugging this back into equation (4.51) and appropriately deforming the contour, we find the equivalence (4.50).

The Poisson resummation can be used to find an alternative expression for the the sum over the Green's functions of a set of massive KK modes with masses $m_n(a)$ as a sum over twisted flat space Green's functions D_F,

$$G(x,y) = \sum_{k=-\infty}^{\infty} e^{ik(a+\pi)} D_F(x, y+2\pi k R)\,. \tag{4.53}$$

We proceed to calculate the potential $V(a)$ in the alternative winding mode basis by integrating out the fermions,

$$e^{S_{\text{eff}}[a,A]} = \int D\Psi D\overline{\Psi}\; e^{S[a,A,\Psi]}\,. \tag{4.54}$$

This yields an effective action

$$S_{\text{eff}}[a,A] = S[a,A] + \text{Tr}\left(\log\left(-\not{D} - m\right)\right)\,. \tag{4.55}$$

A simple way to calculate $V(a)$ is to take a derivative of the effective action (4.55) with respect to a constant axion a and set the photon field to zero. This yields

$$(2\pi R)\,\frac{\delta S_{\text{eff}}}{\delta a} \supset -\text{Tr}\left(\gamma_5 G\right)\,. \tag{4.56}$$

The compact Green's function G can be expanded in terms of twisted flat space Green's functions by equation (4.53) as,

$$\text{Tr}\left(\gamma_5 G\right) = \sum_{k=-\infty}^{\infty} e^{ik(a+\pi)}\text{Tr}\left(i\gamma_5 \frac{e^{k2\pi R\partial_5}}{\not{D}+m}\right)\,. \tag{4.57}$$

Multiplying top and bottom by the same factor, one arrives at

$$\text{Tr}\left(\gamma_5 G\right) = \sum_{k=-\infty}^{\infty} e^{ik(a+\pi)}\text{Tr}\left(i\gamma_5 \frac{e^{k2\pi R\partial_5}\left(\not{D}-m\right)}{\left(\not{D}\right)^2 - m^2}\right)\,. \tag{4.58}$$

This allows us to calculate the 4D potential by equation (4.56) as

$$\frac{\partial V}{\partial a} = 4 \sum_{k=-\infty}^{\infty} e^{ik(a+\pi)} \int \frac{dp^5}{(2\pi)^5} \frac{p_5 e^{ik2\pi R p_5}}{p^2 + (p_5)^2 + m^2} \,. \tag{4.59}$$

Integrating over the 5D momenta yields,

$$\frac{\partial V}{\partial a} = -4 \sum_{k=1}^{\infty} (-1)^k \sin(ka) \int \frac{dp^4}{(2\pi)^4} e^{-k|2\pi R|\sqrt{p^2+m^2}} \,. \tag{4.60}$$

Integrating over momenta and with respect to the axion a yields the effective potential,

$$V(a) = \frac{m^2}{(2\pi R)^2} \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^3} e^{-k2\pi|Rm|} (-1)^n \cos(ka) \left(1 + \frac{3}{2\pi|Rm|k} + \frac{3}{(2\pi Rmk)^2}\right) \,. \tag{4.61}$$

This potential is generally known as the one generated by a four-dimensional particle with a rotor degree of freedom coupled to the axion and was also discussed in various limits in [169, 236–239].

We see that for the 5D instantons, the spurion is parametrised by the parameter $z = e^{-2\pi Rm}$ with the symmetry $z \leftrightarrow \frac{1}{z}$ leaving the potential invariant and implementing the $-m$ to m domain wall.

At the symmetric point $z = 1$, the 5D fermion becomes massless, and the Lagrangian has an apparent $\mathbb{Z}_2$ (5D parity) symmetry, which is broken by the topological Chern-Simons term. In this limit, at the point $a = \pi$, the lightest 4D fermion in the tower becomes massless, and a 4D $U(1)$-chiral symmetry is restored, meaning that this fermion should not have been integrated out. This is reflected by a singularity of V'' at $a = \pi$.

In [98], we reformulate the results of this section in terms of an effective worldline formalism of a charged massive 4D fermion with additional compact degrees of freedom coupled to the axion. In doing so, we derive the effective axion-Euler-Heisenberg Lagrangian to all orders in the constant axion a and Maxwell field strength F.

4.6 Discussion

We have considered the general properties of the monodromic axion-photon coupling $g(a)$ and the symmetries of the low-energy axion-Mawell Lagrangian in the presence of such a coupling. We argued that the non-quantisation of $g'(a)$ is a spurion for the non-invertible axion shift symmetry. The connection between the axion potential and this coupling has been considered supported by several examples, including the QCD axion, perturbative shift symmetry breaking and fermions with additional compact degrees of freedom. In all such cases, a prototypical monodromic function $g(a)$ was derived and could be expanded in terms of a linear monodromic function with the same monodromic charge as $g(a)$ and a periodic function. In some cases only the first few terms in the expansion of the periodic function dominated. However, in many simple models the whole Fourier tower needs to be summed up to get the relevant $g(a)$. In such cases $g(a)$ captured the rearrangement of heavy degrees of freedom through its singularities at isolated points. This correlated with cusps and singularities in the axion potential.

There are a number of model-building applications of this formulation. Instead of building effective field theories (EFTs) with polynomial axion couplings, more general non-linear couplings can arise naturally through the $g(a)$ portal. This may have interesting avenues for constructing more general natural potentials for axions. Phenomenologically, the full non-linear form of $g(a)$ is most relevant for scenarios where the axion traverses an $O(1)$ fraction of its field range. This is certainly true for axion strings and domain walls [63], and sharp features in $g(a)$ can affect axion emission from these objects. It can also be true for dense axion objects, like axion miniclusters (of nuclear density) or superradiant axion clouds.

It will be interesting to study the effective photon coupling for mesons in the chiral Lagrangian, e.g. for the pion $g(\pi^0)$ (see [240] and references therein) after integrating out η'. It has been shown [38] in the context of a one-flavor QCD N that degrees of freedom rearrange on the $\eta' \to \eta' + 2\pi$ domain wall, leading to a fractional quantum hall droplet and a potential jump in $g(\eta')$. It will be nice to see this physics captured within the effective field theory.

5
A Mass for the Dual Axion

In this chapter, we explore the confinement of instantons. Motivated by the absence of generalised symmetries in quantum gravity, we demonstrate in section 5.3 that the vacuum acts as a superconductor when the 2-form symmetry of the dual axion is explicitly broken. In this superconducting phase, instantons are confined by the worldlines of charged particles, and we consider models for instanton confinement in section 5.4. We comment on the required particle spectrum and relevance for the strong CP problem in section 5.5, and summarise our findings in the discussion in section 5.6.

5.1 Introduction

Among the many beyond the Standard Model (BSM) extensions, the axion, a (pseudo)-Nambu-Goldstone boson, has received ever-growing attention in recent decades. Most of the desired properties of an axion rely on the existence of an (approximate) continuous shift symmetry, making the axion generically very light. In quantum gravity, global symmetries are expected to be absent (see section 2.5.3). The axion's shift symmetry is expected to be either broken or gauged. For parametrically light axions, this provides an axion quality problem.

Recent investigations have re-examined the axion quality problem from the perspective of the dual of the axion [42, 81–85], the associated two-form Kalb-Ramond field [86] introduced in section 2.3.1. From this dual perspective, breaking the axion shift symmetry corresponds to gauging 2-form shift symmetry of the dual axion.

In this chapter, we consider instead the effect of breaking the 2-form shift symmetry of the dual axion (as introduced in section 3.4) by potential quantum gravitational effects. The dual axion acquires a mass, and we demonstrate that this corresponds to the second option allowed by quantum gravity, a gauged shift symmetry of the axion. The axion gets 'eaten' and becomes the longitudinal degree of freedom of a massive vector field. A potential loss of control over the dual axion mass due to sensitivities to UV physics presents a dual axion quality problem and dramatically alters the interactions of the axion with axion strings and instantons. Generically, such axions arise in string theory in anomaly cancellation mechanisms in either compactifications of heterotic string theory [19] or type II intersecting brane models [241–243], resulting in massive vector fields with a stringy mass scale.

The presence of a massive vector field implies that the vacuum will act as a superconductor. Electric field lines can end in the vacuum, and field lines emanating from electrically charged particles are screened at long distances. Magnetic field lines are forced into flux tubes, and magnetic monopoles are confined.

This motivates a general question: *What happens to instantons and strings coupled to the 'eaten' axion in this superconducting phase?*

As the axion's degree of freedom is gauged, it is most instructive to first consider this question from the perspective of the massive dual axion. Strings and instantons are respectively 'electrically' and 'magnetically' charged under the massive dual axion. Analogous to a massive vector field, a massive dual axion implies that the 'electric flux' lines of the dual axion can end in the vacuum and the dual axion profile around 'electrically' coupled strings is screened at large distances. The dual axion's 'magnetic flux' lines surrounding instantons are forced into 'flux tubes' and instantons are confined. This phase in which the dual axion is massive should be contrasted with a phase in which the axion is massive; strings are confined by axion domain walls, and the axion field around instantons is screened at large distances.

The confinement of instantons and screening of the dual axion profile around strings can be seen as natural consequences of gauging the axion shift symmetry. The profile of the 'eaten' axion around axion strings is compensated by that of the vector field at large distances, and the strings become non-interacting. Interactions of the strings with their environment can come from coupling to other long-range fields such as electromagnetism or gravity. Their phenomenology is that of gauge strings [122] and we will refer interested readers to [99].

The gauged axion shift symmetry implies that instantons coupled to the axion exist only in charge-neutral dipole configurations and any far-separated instanton configurations are exponentially suppressed. The absence of isolated instantons in such theories was already noted in [44, 45]. We show that this confinement of instantons corresponds to the worldline action of a particle-like soliton traveling between the instantons. This soliton is analogous to Abrikosov/Nielsen-Oleson vortex solitons that stretch between confined magnetic monopoles in a superconductor. We proceed to calculate the cost of this additional worldline suppression.

The particle-like soliton travelling between instantons is electrically charged under the massive vector field. This completes the duality of the superconductor; magnetic monopoles of the massive vector field are confined by 'electrically' charged strings of the massive dual axion. The 'magnetically' charged instantons of the dual axion are confined by electrically charged particles under the massive vector field.

The confinement of instantons and their contributions to the path integral presents interesting model-building opportunities. We will consider a non-exhaustive list of models in which the confined instantons and confining worldline are dynamical, including adaptations of particle confinement and the Green-Schwarz [221].

In other areas of the literature, axions with gauged shift symmetries have been discussed in field theoretic contexts as dark matter candidates [244, 245], at the LHC [246–250], as mixing with ordinary axions [223, 251–254], in brane-bulk scenarios [80] or as solutions to the strong CP problem [255].

This chapter is structured as follows. In section 5.2, we consider the massive dual axion theory. This is followed by section 5.3 and 5.4 on instantons and possible confinement mechanisms for instantons. We comment on the relevance for the strong CP problem in section 5.5. We summarise our findings in the discussion in section 5.6.

5.2 Massive Dual Axion

The dual axion was introduced in section 2.3.1 as a two-form field B with Lagrangian,

$$Z = \int DB \ \exp\left[i\int d^4x \ \frac{1}{48\pi^2 f^2} H_{\mu\nu\rho}H^{\mu\nu\rho}\right] , \qquad H_{\mu\nu\rho} = \frac{1}{2}\partial_{[\mu}B_{\nu\rho]} , \tag{5.1}$$

and a redundancy,

$$B \to B + d\Lambda. \tag{5.2}$$

As explored in section 3.4, the massless dual axion has a $U(1)$ 2-form shift symmetry that shifts the dual axion as,

$$B \to B + c^{(2)} , \qquad dc^{(2)} = 0 . \tag{5.3}$$

The aim of this section is to study the effect of mass contributions to the dual axion B and breaking of higher-form symmetry by quantum gravitational effects.

To this end, we add a mass m to Lagrangian (5.7) through a generalisation of the Proca/Stückelberg mechanism. In its simplest form, this is accomplished by adding a one-form $U(1)$ vector field $\tilde{A}$ as,

$$\mathcal{L} = \frac{1}{48\pi^2 f^2} H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{m^2}{16\pi^2 f^2}\left(B_{\mu\nu} - \frac{f}{m}\partial_{[\mu}\widetilde{A}_{\nu]}\right)^2 . \tag{5.4}$$

Massive 2-form fields have 3 d.o.f. and their properties have been studied in, for instance, [256–260] and in the context of cosmology in [261].

Several mass-generating mechanisms that lead to a low-energy Lagrangian (5.4) exist for massless two-forms in field/string theories. Any strings coupled to the dual axion with either a finite creation action or which can be wrapped around compact 2-cycles of the manifold give a mass to (components of) the dual axion. Such cycles can be absent in the IR, but the topology of spacetime seen by the dual axion can change in the UV, both in extra-dimensional theories and 4D theories. A four-dimensional mass for the dual axion can also be generated by the dimensional reduction of higher-dimensional theories with couplings of the form B^2F^N where F is the field strength of some p-form field. Activating fluxes $\int F = n$ on the internal/compactified cycles of the manifold will generate a mass $m^2 \sim n^N$ [261]. Such fluxes are standard in KKLT [262] or Large Volume Scenarios [263] in string theory.

5.2.1 Massive Vector Field

Strings are 'electrically' coupled to the dual axion, and understanding their phenomenology in a phase in which the dual axion has a mass can be done from the massive dual axion perspective (5.4). Their phenomenology is that of gauge strings, as their flux is screened by the mass of the dual axion, and we refer interested readers to [122].

Instantons are 'magnetically' coupled to the dual axion and understanding their fate is inevitably tied to understanding the fate of the original massless axion. The Lagrangian (5.4) is a dual description of several well-known models, one of which is a massive vector field, which has 'eaten' the axion.

We first proceed by dualising the vector field $\widetilde{A} \to A$, which yields the BF-theory,

$$\mathcal{L} = \frac{1}{48\pi^2 f^2} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4\pi} B_{\mu\nu} \widetilde{F}^{\mu\nu} \,. \tag{5.5}$$

The coupling, field strength and dual field strength are normalized as

$$e = \frac{m}{f} \,, \quad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \,, \quad \widetilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \,. \tag{5.6}$$

The BF coupling $B_{\mu\nu}\widetilde{F}^{\mu\nu}$ between the dual axion and vector field is a topological term and should be contrasted with the kinetic coupling in (5.4) involving the dual axion and the dual of the vector field. This is another manifestation of duality; dynamic/kinetic couplings in one frame become topological in a dual frame (5.5).

The topological coupling in (5.5) implies that the axion is eaten by the vector field A. This can be seen by also dualising the dual axion $B \to a$, which yields the Proca/Stückelberg action of a massive vector field,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{f^2}{2} \left(\partial_\mu a - A_\mu\right)^2 \,. \tag{5.7}$$

The original axion becomes embedded as the longitudinal mode of a $U(1)$ gauge field A matching the 3 d.o.f. of theory (5.4). The vacuum acts as a superconductor, and the fate of instantons in this phase is discussed in the next section.

5.3 Instantons in Superconductors

Instantons can be electrically coupled to the axion a. In the superconducting phase, instantons no longer couple to just axion insertions but instead couple to the gauge-invariant operator

$$\mathrm{Exp}\left[ia - i\int_C A\right] . \tag{5.8}$$

where C is any curve extending from the instanton to infinity and $e = \frac{m}{f}$. The presence of this additional Wilson line specifies the worldline of a charged (non-dynamical) particle stretching from the instanton (see figure 5.1). The space-time process describes the creation/decay of the particle at the instanton. The additional worldline cost of this charged particle to the partition function Z of any isolated instanton confines this instanton. Only gauge-invariant combinations in which the worldline C stretches between different instantons contribute to the vacuum-to-vacuum partition function. Instantons will still contribute to amplitudes in which the Wilson lines can end on the boundary, such as those that involve the confining particles in the initial or final states.

This confinement of instantons is analogous to the ordinary confinement of magnetic monopoles in a superconductor by Abrikosov/Nielsen-Oleson vortex strings [264, 265] (or electric monopoles in dual superconductors with a massive dual photon [266]). An artistic impression of the case of an instanton-anti-instanton dipole is provided in figure 5.1.

Confinement of instantons is not unique to (5.4) and has a long history. One of the earliest examples of instanton confinement-deconfinement transitions occurs in 2D (space-time dimensions) in the well-known XY model. This model has the coarse-grained description of a simple massless compact scalar. Vortices or strings of this compact scalar are instantons in 2D and are confined by a logarithmic potential,

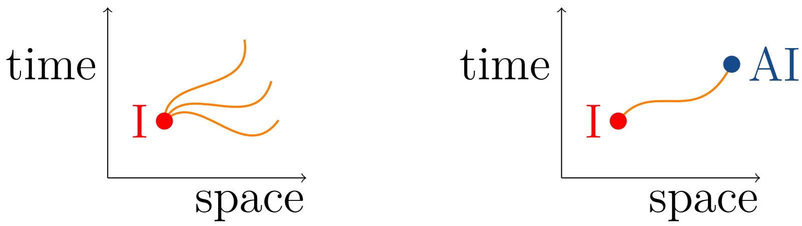

Figure 5.1: *Left*: Space-time diagram of an instanton (I) (red) confined by particle-like soliton worldlines (orange) describing the creation/decay of such particles at the instanton vertex. *Right*: Space-time diagram of an instanton (I) (red) and anti-instanton (AI) (blue) configuration with a particle-like soliton worldline (orange) stretched between the instantons describing the creation and subsequent decay of the particle.

forming bound vortex pairs. Increasing the temperature allows for a transition to a gas of free unbound vortices above the critical temperature described by the infinite order Berezinskii–Kosterlitz–Thouless [267, 268] phase transition.

Confinement of instantons in the presence of massless fermions also played an important role in the early studies of the vacuum structure of QCD [269, 270] in the presence of massless fermions. In his seminal paper, 't Hooft showed [167] that the existence of normalisable zero mode solutions to the massless Dirac equation implied the vanishing of the single instanton partition function in the presence of massless fermions. The exchange of massless fermions gives rise to strong logarithmic interactions between instantons and anti-instantons, leading to logarithmic confinement [270, 271]. Instantons and anti-instantons form closely bound dipole pairs, and vacuum tunnelling effects become negligible in the absence of chiral symmetry-breaking sources. Instantons will instead contribute to correlation functions that include the fermion zero mode, such as a chiral condensate.

Confinement of instantons is crucial for extended particle-like field configurations that enjoy topological protection, like vortices and monopoles. Smooth deformations of their field profile in time cannot change the topological index associated with these solitons, and their number is conserved. However, singular field configurations can describe quantum tunnelling events between these distinct topological sectors. Therefore, an additional prerequisite to the number conservation of such solitons is the confinement of the associated instantons. This is most pronounced for topological defects that lack core singularities, such as Skyrmions. For instance, unconfined instantons exist in 3D in the $\mathbb{C}P^1$ model describing a S^2 valued field ϕ on a compact space S_2 [272], where instantons mediate the sudden decay of the Skymrion and violation of Skyrmion number. However, not all such defects are confined. Hopfions [273, 274], non-trivial maps $\pi_3(S^2)$ from a compact space S^3 to a S^2 valued field do have an associated conserved charge in 4D space-time. This is due to the logarithmic confinement of the associated Hopf instantons. A general discussion on the stability of topological defects due to instanton confinement and the renormalisation flow of instanton interactions can be found in [275].

5.3.1 Instanton Gas

Confinement forces instantons into gauge-invariant dipole configurations such as depicted in figure 5.1. In complete analogy with flux tubes stretching between confined particles generating a linear potential, the worldline of the charged particle stretching between the instanton-anti-instanton dipole generates an additional cost to the instanton-anti-instanton partition function. We proceed by calculating this additional cost.

In a system consisting of a gas of instanton-anti-instanton configurations, in which the particle worldline is dynamical, the worldline would relax to the shortest distance between any instanton-anti-instanton pair. In the massive dual axion theory (5.7), both the worldline and the instantons are non-dynamical probe objects. We will rectify this in the next subsection by introducing additional degrees of freedom to (5.7) in order to make both the instantons and worldlines dynamical, but for now we wish to describe only this steady-state situation in which the worldline has relaxed to the shortest distances between the instantons.

There is an equivalent formulation of the coupling (5.8) first introduced by Zwanziger [276] for the analogous electromagnetic coupling of photons to magnetic monopoles, which will be useful to describe this instanton-anti-instanton dipole. In this formulation, one introduces axion shift symmetry sources by an instanton density J_{inst}. To each source, one attaches a Dirac line described by a unit vector n^μ.

The coupling between the density and the axion is then achieved by the appropriate addition of the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = J_{\text{inst}} \left(a - \frac{n^\mu}{n \cdot \partial} A_\mu \right) . \tag{5.9}$$

In the limit $m \to 0$ keeping f fixed, the coupling e of the photon vanishes, and one returns to a theory of a compact scalar a, the axion, coupled to instantons. When $m \neq 0$, the Dirac line becomes physical and is identified with the worldline of the non-dynamical particle. Since the worldline of the particle is non-dynamical, the interaction (5.9) is Lorentz violating.

In the gauge $a = 0$, the system in the Zwanziger formulation (5.9) is described by an effective non-conserved current

$$J^\mu = \frac{n^\mu}{n \cdot \partial} J_{\text{inst}} , \tag{5.10}$$

coupled to a massive vector field A_μ with corresponding equations of motion in Euclidean space-time

$$A^\mu = \frac{g^{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2}}{-\partial^2 + m^2} e^2 J^\nu . \tag{5.11}$$

The current (5.10) describes the worldline of a charged particle in the direction n^μ that emanates from a non-zero instanton density J_{inst}. This should be compared to the equivalent Wilson line formulation (5.8) which requires the introduction of a non-conserved current J^μ along the curve C coupled to the massive gauge field A^μ that emerges from an insertion of e^{ia}.

We are interested in the field configuration in Euclidean space-time describing an instanton-anti-instanton dipole configuration with a separation length R in time. In a spherical coordinate system (t, r, ϕ, θ), such a configuration is described by the instanton density $J_{\text{inst}} = \delta(x^\mu + \frac{R}{2}\hat{t}) - \delta(x^\mu - \frac{R}{2}\hat{t})$ and corresponds to an effective current (5.10) of the form

$$J^0 = \delta(r) \text{Rect}\left(\frac{t}{R}\right) \tag{5.12}$$

in which the Dirac line $n = \hat{t}$ stretches between the instantons and $\mathrm{Rect}\left(\frac{t}{R}\right)$ is a unit box with size R centred at $t = 0$. This current describes the existence of a charged particle that sits at the origin for a time R.

In the limit $R \to \infty$, the profile for J^μ is that of a charged particle sitting at the origin for all time, and one has the ordinary screened solution (in the normalisation of (5.7)) for a charged particle,

$$A^0 = \frac{e^2}{4\pi r} e^{-mr} \,. \tag{5.13}$$

At a finite distance R, we can calculate the contribution to the partition function Z due to this additional charged worldline. Given an effective current J^μ, such an evaluation of the partition function amounts to completing the square of (5.7) as,

$$\ln Z = -\frac{e^2}{2} J_\mu \left(\frac{g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2}}{-\partial^2 + m^2} \right) J_\nu \,. \tag{5.14}$$

In the Wilson line formulation (5.8) and the gauge $a = 0$, this is equivalent to calculating the expectation value of $e^{i\int_C A}$ along the contour C stretching between the instantons.

We proceed by plugging in the explicit expression (5.12) for J^μ,

$$\ln Z = -\frac{e^2}{2} \int_{-\frac{R}{2}}^{\frac{R}{2}} dt \int_{-\frac{R}{2}}^{\frac{R}{2}} dt' \; G(t - t') \,. \tag{5.15}$$

Here, $G(t)$ is the Green's function of the massive vector field. Plugging in the expression for this Green's function in momentum space, introducing the three momentum $\vec{k}$ as $k_\mu = (k_0, \vec{k})$ and performing the temporal integrals gives

$$\ln Z = -2e^2 \int \frac{dk^0}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \frac{\sin^2\left(\frac{k_0 R}{2}\right)}{k_0^2} \frac{1 - \frac{k_0^2}{m^2}}{k^2 + m^2} \,. \tag{5.16}$$

Performing the temporal momentum integrals, keeping only the R-dependent pieces and splitting the partition function for later convenience yields

$$\ln Z = -\frac{e^2}{2} S - \frac{e^2}{(2\pi)^2} \frac{K_1(mR)}{mR} \,, \qquad e = \frac{m}{f} \,, \tag{5.17}$$

where S is dominated by the action of the charged soliton,

$$S = \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{R}{\vec{k}^2 + m^2} + \frac{e^{-R\sqrt{\vec{k}^2 + m^2}}}{\left(\vec{k}^2 + m^2\right)^{\frac{3}{2}}} \right) \sim R\Lambda \,. \tag{5.18}$$

From the partition function (5.17) it is clear what the interpretation of the addition of the current J_{inst} to the system is. In the limit $mR \ll 1$ keeping f fixed, the second term in (5.17) dominates $\ln Z \sim -\frac{1}{f^2R^2}$. This simply describes a massless axion propagating between the instanton and anti-instanton.

In the opposite limit $mR \gg 1$ keeping f fixed, the massive photon is coupled to the current $\frac{n^\mu}{(n\cdot\partial)}J_{\text{inst}}$. This current describes a unit box or, equivalently, the insertion of a charged particle for a time R. The partition function of such a configuration scales as $\ln Z \sim -e^2R\Lambda$ where the UV cutoff Λ is the electric self-energy of the charged particle. The probe instantons are linearly confined in the distance R.

Similar to the confinement of colour, the presence of instantons with a finite action e^{-S_0}, reduces the effective range of the confining contribution. In the presence of such instantons, the worldline of the charged particle can be broken by a dipole instanton-anti-instanton pair. Such a breaking becomes favourable whenever the distance $e^2R \gtrsim \frac{S_0}{\Lambda}$ and therefore the confining contribution is only of finite range. This length is further reduced by virtual instanton fluctuations [275].

5.4 Confinement of Instantons

The discussion in the previous section pertained to the steady-state scenario in which the charged particle worldline has relaxed to the shortest distance between the instantons. The instanton and confining worldline can be made dynamical by introducing additional degrees of freedom to the massive theory (5.4), yielding models that could potentially be relevant for the strong CP problem.

5.4.1 Bosonic Confinement

Simple models of confinement of instantons by worldlines of bosons can be obtained in 3D. We will focus on the confinement of magnetic monopoles, which are instantons in this dimension and are associated with a non-trivial second homotopy index π_2. As $\pi_2(G) = 1$ for any compact, connected Lie group G [277], such instantons only form when a gauge group is spontaneously broken $G \to H$ and are associated with the index $\pi_2(G/H)$. Subsequent breaking of H will always eventually result in confinement of such instantons, as they should be absent from the full breaking $\pi_2(G/1) = 1$.

Consider simple two-step breaking models in which a gauge group G breaks as $G \to H_1 \to H_2$. We will focus on models in which instantons that form in the first breaking step $G \to H_1$ confine due to meta-stable particles that form in the second breaking step $H_1 \to H_2$. Particles formed in the second breaking step are classified by their topological index $\pi_1(H_1/H_2)$. In the larger gauge group G, such configurations are described by the topological index $\pi_1(G/H_2)$ and for each particle there exists a mapping

$$\pi_1(H_1/H_2) \to \pi_1(G/H_2) \tag{5.19}$$

If this mapping has a non-trivial kernel, then two distinct particles p_1 and p_2 associated with distinct non-trivial loops in H_1/H_2 can be continuously connected in G/H_2. In such cases, there exists an instanton associated with an element in $\pi_2(G/H_1)$ describing a surface in G/H_2 that connects the two loops. The winding configurations are said to be able to unwind in the larger gauge group G and the instanton allows for the particle transition/decay $p_1 \to p_2$. If a particle is mapped to the identity in $\pi_1(G/H_2)$, then the instanton describes the decay of the particle to the vacuum. A full account of the decays of metastable topological defects in any dimension can be found in [278].

If the instanton is absent from the total breaking $\pi_2(G/H_2) = 1$, then the instanton only exists as an endpoint or kink or twist on the worldline of meta-stable particle(s). Simple examples of such models are breaking patterns

$$G \to U(1) \to 1 \tag{5.20}$$

for compact, connected and simply connected Lie groups G. Particles form during the second breaking step $\pi_1(U(1)) = \mathbb{Z}$, but can unwind in the larger simply connected gauge group as $\pi_1(G) = 1$. The decays/transitions of these particles are facilitated by instantons. The instantons are absent from the full breaking and instead are confined as the end-points of such particle worldlines. Similar considerations apply to instantons confined on the worldlines of non-Abelian particles, such as in [279, 280].

A well-known example is magnetic monopoles (instantons in this dimension) that form in $SU(2) \to U(1) \to 1$ breaking patterns by subsequently one adjoint and one fundamental scalar. During the second breaking by the fundamental scalar, non-perturbative particles form $\pi_1(U(1)) = \mathbb{Z}$, which are time-independent configurations in which the fundamental scalar winds. In the full breaking pattern, all such winding configurations are trivial $\pi_1(SU(2)) = 1$ and can therefore be unwound by instanton insertions $\pi_2(SU(2)/U(1)) = \mathbb{Z}$. These instantons are 't Hooft-Polyakov monopoles [281, 282] that form during the first breaking of $SU(2) \to U(1)$ by the adjoint scalar. These instantons are absent in the full breaking $\pi_2(SU(2)) = 1$ and only occur at the end-points of the worldlines of the non-perturbative particles. The instantons are therefore confined.

If the second breaking in (5.20) is instead to a discrete subgroup $Z_N \subset U(1)$, then the instantons become kinks or twists on the worldline of particles. A well-known example is magnetic monopoles, which are kinks on particle worldlines in $SU(2) \to U(1) \to \mathbb{Z}_2$ double breaking models [283] by two orthogonal adjoints. During the second breaking $U(1) \to \mathbb{Z}_2$ by an adjoint ϕ, particles form $\pi_1(U(1)/\mathbb{Z}_2) = \mathbb{Z}$ as winding configurations of ϕ. In the full breaking, there are only two topologically inequivalent winding configurations: $\pi_1(SU(2)/\mathbb{Z}_2) = \mathbb{Z}_2$. The magnetic monopole formed during the first breaking connects the worldlines of these configurations. No closed loop can be formed, and the instanton is confined.

The $SU(2)$ double breaking models provide examples of completions of the massive dual axion Lagrangian (5.4) in 3D in which the instanton and confining worldline are dynamical. The axion is identified with the magnetic dual of the $U(1)$ photon. After the first breaking step, magnetic monopoles (instantons) form which are magnetically charged and couple to the axion/dual photon. A gas of these instantons would have given a mass $m \sim e^{-S_{\rm mon}}$ to the axion/dual photon [166].

This is prevented by the second breaking step at scales v, where particles associated with the winding of the second scalar ϕ form attached to the monopole.

These particles come attached to any 't Hooft vertex associated with the monopole, and the monopole becomes confined. The photon (dual axion) obtains a mass from eating the second scalar ϕ. The partition function of any pair of instantons separated by a space-time distance R is suppressed by the action cost of the charged particle travelling between the instantons. As the non-perturbative particle has mass v (which includes the self-energy), this results in a partition function suppression of $\ln Z \sim -vR$.

Extending two-step bosonic models of instanton confinement to 4D, one runs into issues. In this dimension, instantons are associated with the topological index π_3, and already exist; $\pi_3(G) \neq 0$ for a generic non-Abelian gauge group G without spontaneous breaking. In two-step breaking models, we therefore need to follow both the original gauge instantons and possible additional instantons created in the breaking associated with the index $\pi_3(G/H_1)$.

Associated with the first breaking $G \to H_1$, there is a fibration $H_1 \to G \to G/H_1$ and a long exact sequence[1] [284] in homotopy

$$\cdots \to \pi_3(H_1) \to \pi_3(G) \to \pi_3(G/H_1) \to \pi_2(H_1) = 1 \to \cdots \tag{5.21}$$

The exactness of the first three entries of the sequence implies that the instantons of G either descend to an instanton in the unbroken gauge group H_1 or form a subset of the instantons associated with non-trivial elements of the index $\pi_3(G/H)$. In fact, the exactness of the last three entries and the vanishing index $\pi_2(H_1) = 1$ imply that no other instantons are created during the breaking process,

$$\pi_3(G/H_1) \equiv \pi_3(G)/\pi_3(H_1)\,. \tag{5.22}$$

All instantons descend from those of G and split into instantons associated with either the unbroken part of the gauge group $\pi_3(H_1)$ or the broken part $\pi_3(G/H_1)$.

A simple example involves the breaking of $SU(2) \to U(1)$ by an adjoint scalar. The fibration associated with the breaking is the Hopf fibration $S^1 \to S^3 \to S^2$.

[1] A long exact sequence is a series of mappings $\cdots \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to \ldots$ such that the image of each map is the kernel of the next, $\mathrm{Im}(f_i) \equiv \mathrm{Ker}(f_{i+1})$.

The low-energy gauge group $U(1)$ has no instantons; instead $\pi_3(SU(2)/U(1)) = \mathbb{Z} = \pi_3(SU(2))$.

The important question at hand is: *what happens to the instantons associated with $\pi_3(G/H_1)$ during the second breaking $H_1 \to H_2$?* This is where the contrast with 3D occurs.

Particle-like defects can form during the second breaking, this time associated with the topological index $\pi_2(H_1/H_2)$. In the larger gauge group G, the worldlines of particles that can end on instantons are associated with the kernel of the mapping

$$\pi_2(H_1/H_2) \to \pi_2(G/H_2)\,. \tag{5.23}$$

However, the kernel of such a mapping must always be trivial [278] as $\pi_2(H_1) = 1$ for all compact finite-dimensional Lie groups H_1. Any magnetic monopole associated with $\pi_2(H_1/H_2)$ remains absolutely stable when embedded into a larger gauge group G. Even if instantons associated with the index $\pi_3(G/H_1)$ do connect particle worldlines, then these worldlines are topologically equivalent with respect to the index $\pi_2(H_1/H_2)$. There is no topological protection against connecting the particle worldlines and shrinking/removing the resulting loop, which would result in a remaining unconfined instanton. This is in stark contrast with the 3D models.

Even though the topological index of instantons is protected against confinement, there could still be dynamical reasons for the absence or confinement of instantons (see [99]).

5.4.2 Fermionic Confinement

Next to the confinement of instantons by bosons, instantons can become confined by the existence of fermion zero modes. By the Atiyah-Singer index theorem [285], the difference between the number of left and right chiral solutions to the massless Dirac equation on an even-dimensional compact space is given by the instanton number. In his seminal paper, 't Hooft showed [167] that the existence of normalisable zero mode solutions to the massless Dirac equation in the background of an instanton implied the vanishing of the single instanton partition function.

Instead, the instanton will contribute to correlation functions that include the fermion zero mode, such as the chiral condensate.

Instanton-anti-instanton dipoles will receive contributions to their partition function from both the exchange of gluons and massless fermions. The gluonic exchange gives rise to dipole-dipole interactions $\ln Z \sim -\frac{1}{R^4}$, where R is the distance between the instanton and anti-instanton [270, 286]. The exchange of massless fermions gives rise to strong logarithmic interactions $\ln Z \sim -\ln R$ between the instanton and anti-instanton, leading to logarithmic confinement [270, 271].

Coupling logarithmically confined instantons to a gauged axion results in the well-known Green-Schwarz mechanism [221] and another realisation of a superconductor in which both the confined instantons and confining particles are dynamical.

In order to show this, consider coupling the massive dual axion or gauged axion to instantons. The instantons can be those of the massive vector field or possibly an external gauge group G, which we take to have traceless generators for simplicity. The axion-instanton coupling is of the form[2],

$$\mathcal{L} \supset c_a \frac{a}{16\pi^2} F\widetilde{F} + c_G \frac{a}{16\pi^2} \mathrm{tr} G\widetilde{G} \,. \tag{5.24}$$

This coupling is anomalous under the gauged $U(1)^0$ shift symmetry $a \to a + \lambda$. In order to cancel the anomaly, one introduces a set of massless Weyl fermions charged under both the non-Abelian gauge group G and the gauged axion shift symmetry $U(1)^0$. One can choose the anomalous variation of the fermion measure to be invariant under G transformations. Under variations λ of $U(1)^{(0)}$, the total fermion measure[3] then changes as,

$$D\Psi \to D\Psi \, \exp\left[\frac{1}{16\pi^2}\int d^4x \; \lambda \left(c_a F\widetilde{F} + c_G \mathrm{tr} G\widetilde{G}\right)\right] \,. \tag{5.25}$$

The constants c_a and c_G are determined by the exact fermion content. If the i-th Weyl fermion i sits in representations (q_i, r_i) of $U(1)^0 \times G$, then the total change

[2] The allowed values of c_a and c_G depend on the spacetime manifold and the global gauge group; see [60, 170]. For simplicity, we pick a spin manifold and the product group $U(1)^0 \times G$, which implies $(3c_a, c_G) \in \mathbb{Z} \times \mathbb{Z}$

[3] In theories of gravity there would also be a gravitational anomaly $\sim \; \mathrm{tr} R \wedge R$ where R is the Riemann curvature 2-form.

of the measure is (see section 3.5.1),

$$c_a = \frac{1}{3}\sum_i q_i^3 \mathrm{dim}(r_i)\,, \quad c_G = \sum_i q_i I(r_i)\,, \tag{5.26}$$

where $I(r_i)$ is the Dynkin index of representation r_i normalised such that $I(\square) = 1$ for the fundamental representation. If the fermion content can be chosen to cancel the anomalous variations of (5.24), then the gauge symmetry is non-anomalous. The instantons associated with a non-zero total instanton number $I = \frac{c_a}{16\pi^2}F\widetilde{F}+\frac{c_G}{16\pi^2}\mathrm{tr}G\widetilde{G}$ are then confined by the massless fermions and will only contribute to amplitudes involving these massless fermions in the initial/final states.

One of the simplest examples is $c_a = 0$ and the gauge group $G = SU(2)$ with gauge field B_μ and field strength $G_{\mu\nu}$. The massive dual axion is coupled to the instantons as,

$$\mathcal{L} \supset \frac{f^2}{2}\left(\partial_\mu a - A_\mu\right)^2 + c_G\frac{a}{16\pi^2}\mathrm{tr}G\widetilde{G}\,. \tag{5.27}$$

This theory is anomalous and requires the introduction of a set of four massless left-handed Weyl fermions ψ_i in the fundamental representation of G. We pick the charges of these fermions under A^μ to be the Fermat charges $q_i = \{-3,-4,-5,6\}$, such that the constants $c_a = 0$ and $c_G = -6$. The fermionic Lagrangian is

$$\mathcal{L} \supset \sum_i \psi_i^\dagger \overline{\sigma}_\mu \left(i\partial^\mu + B_\mu + q_i A_\mu\right)\psi_i\,. \tag{5.28}$$

The gauge field A^μ is coupled to the chiral current j_5^μ, which satisfies the equations of motion,

$$j_5^\mu = \sum_i q_i \psi^\dagger \overline{\sigma}_\mu \psi\,, \quad \partial_\mu j_5^\mu = \frac{c_G}{16\pi^2}\mathrm{tr}G\widetilde{G}\,. \tag{5.29}$$

The non-zero divergence implies that instantons act as a source for this chiral current. Every instanton vertex comes with associated fermion worldlines with a total non-zero chiral charge (see figure 5.1) and is confined.

Classically, one can calculate the worldline contribution to the partition function of an instanton-anti-instanton dipole pair separated by a distance R in time by studying the equations of motion (5.29), which for such a configuration are

$$\partial_\mu j_5^\mu = \delta\left(x^\mu + \frac{R}{2}\hat{t}\right) - \delta\left(x^\mu - \frac{R}{2}\hat{t}\right)\,. \tag{5.30}$$

In a spherical coordinate system (t, r, ϕ, θ), these equations are solved by

$$j_5^\mu = \delta(r)\mathrm{Rect}\left(\frac{t}{R}\right)\delta_{\mu,0} + \chi_\mu\,, \tag{5.31}$$

where χ_μ has zero divergence $\partial_\mu \chi^\mu = 0$ and represents the contribution of a compact particle worldline.

The first part of this current was already studied in (5.12) and integrating out the gauge field A^μ would lead to linear confinement, this time by chirally charged fermion zero modes. However, the vacuum contains massless fermions; the Wilson line interpretation of the fermion breaks down, and quantum corrections may be large. A full calculation of the partition function would require both the fermion profile in an instanton-anti-instanton background and a control over these quantum corrections. We will not attempt this here.

Instanton can become unconfined if the fermion zero mode Wilson lines attached to the instanton vertex can end in the bulk. For instance, an interaction with strength g between the four fermions can be added of the form

$$\mathcal{L} \supset -g_1 e^{-ic_G a}\left(\psi_1\psi_2\right)\left(\psi_3\psi_4\right) + \text{h.c.}\,, \tag{5.32}$$

or two two-point interactions

$$\mathcal{L} \supset -g_2 e^{-i(q_1+q_2)a}\psi_1\psi_2 - g_3 e^{-i(q_1+q_2)a}\psi_3\psi_4 + \text{h.c.} \tag{5.33}$$

The colour and Lorentz indices are contracted pairwise using Levi-Civita symbols.

The zero modes associated with the gauge-variant instanton vertex $e^{ic_G a}$ can end in the bulk by an insertion of the term above. The instanton no longer couples to the gauged axion and is unconfined.

A second way to unconfine such instantons is the introduction of a second axion b^μ, whose shift symmetry is also gauged,

$$\mathcal{L} \supset \frac{f_b^2}{2}\left(\partial_\mu b - A_\mu\right)^2\,. \tag{5.34}$$

Instantons can now be coupled to the gauge-invariant combination $a - b$ and become unconfined. This can be done either by directly altering (5.24) by $a \to a - b$ or by adding a fermion mass of the form (5.32) with $a \to b$. This model will become important in the next chapter when we discuss higher axions.

5.5 Implications for the Strong CP Problem

In the previous section, a simple toy model (5.28) was presented in which the axion's shift symmetry is gauged, seemingly making any CP-odd angle unobservable. By returning to the gauge $a = 0$, we recognise this simple model as a massive vector field coupled to the anomalous current of a set of massless fermions. In this gauge, the unobservability of the θ-angle is enforced by the axial shift symmetry of the massless fermions. These simple toy models thus pose no new solutions to the strong CP problems beyond the massless up quark solution and will therefore also suffer the same challenges [154] (for a review see [153]). Field theoretical solutions to the strong CP problem that involve a gauged axion and no massless fermions are known [255], but require the introduction of Wess-Zumino terms and additional dimensions.

The simple toy model also involves an additional massive vector field coupled to the massless fermions. This vector field could further suppress instanton contributions by turning the logarithmic confinement into linear confinement, but it will also lead to several additional phenomenological signatures. These signatures include amplitudes involving a longitudinal mode emission of the massive vector field proportional to the ratio (energy of the process/vector mass). For Standard Model currents, such couplings have been extensively studied in the literature (see [249, 250] and references therein) and include currents broken at tree level, such as axial currents broken by fermion masses or tree-level conserved currents broken by a chiral anomaly, such as the SM baryon number or lepton number currents. Both couplings lead to FCNC and rare flavour-changing meson decays and can be constrained by missing energy signatures or visible decays depending on the specific model. By the Goldstone boson equivalence theorem, such couplings are equivalent to axion-like FCNC [287] and constrained in [288]. The non-observation of such processes would imply that such couplings can only exist for small couplings e or large masses m.

5.6 Discussion

We studied a modification of axion physics in which the dual axion acquires a mass. In this superconducting phase, dipole configurations of instantons received an additional exponential suppression proportional to the distance between the individual instantons, corresponding to a charged particle travelling between the instantons. We have considered several models of instanton confinement with dynamical instantons and dynamical worldlines. Much about the confinement of particles in non-supersymmetric theories is still poorly understood, and we do not expect this to be an exhaustive list of instanton confinement models either.

The dynamics confining instantons and their contributions to the path integral presented here were bosonic confinement in 3D and fermionic confinement in 4D, the latter being equivalent to the massless up quark and its solution to the strong CP problem.

It would be interesting to further extend the list of such models to also include confinement of instantons akin to particle colour confinement in QCD in which the confining flux tubes are dynamically generated and do not require the introduction of additional degrees of freedom. Such models could provide alternative solutions to the strong CP problem.

Useful insight in such models of particle confinement has been gained in the past by considering supersymmetric QCD. In these theories, Seiberg and Witten [110] were able to demonstrate that, in a pure SU(2) gauge theory, confinement of electric monopoles could be described by magnetic monopole condensation. Furthermore, by adding matter multiplets, they showed that it is possible to interpolate between a Higgs phase in which quarks are condensed to a phase in which magnetic monopoles condense [109]. It would be interesting to apply similar techniques to instanton confinement. Given the complexity of such an endeavor, we leave this as an open problem for future studies.

6 Higher Axion Strings

In this chapter, we study the minimal requirements to obtain post-inflationary axions with exponentially good quality, thereby motivating axion searches in higher-mass regions. This occurs when an axion coming from a higher-form gauge field mixes with the phase of a complex scalar field. The resulting axion is perturbatively massless and inherits a high-quality shift symmetry from the gauge field's higher-form symmetry while being compatible with a post-inflationary scenario. Axions produced in this manner share features of both extra-dimensional and ordinary axions but are ultimately distinct from either. We refer to such axions as *higher axions*, as the mechanism responsible for the mixing is that of higher-groups. We present a toy model on a 5-dimensional manifold with boundary in section 6.2. The boundary hosts the complex scalar that provides axion strings through standard mechanisms. We proceed by studying the resulting post-inflationary scenario in section 6.3. Finally, we present realisations of the aforementioned scenarios in string theory compactifications in section 6.4, and finalise our findings in section 6.5.

6.1 Introduction

In theory, the mass of the axion can be inferred from the observed dark matter (DM) abundance if the late-time axion density is to saturate this abundance. In practice, this inference has proven difficult in the more predictive post-inflationary scenarios – that is, the scenario where the shift symmetry of the axion is spontaneously broken after the end of inflation (see section 2.3.2) – due to the large hierarchy of scales between the axion decay constant and relevant cosmological parameters. Large theoretical and numerical efforts are being dedicated to reducing the uncertainties in this inference [87–95]. In the post-inflationary scenario, a network of axion strings is expected to form, and simulations have shown that they can dominate the production of DM axions. Because of this, the required decay constant is smaller than in a situation with only a misalignment contribution (see eq. (2.32)) and the axion mass prediction increases. Thus, realising post-inflationary scenarios can motivate higher-mass axion searches. Recent simulations point to a QCD axion decay constant around the scale $f \sim 10^{10} - 10^{11}$ GeV [93–95].

The post-inflationary scenario where the axion comes from the phase of a complex scalar field benefits from theoretical simplicity and predictivity but is plagued by extreme sensitivities to UV physics dubbed the QCD axion quality problem (section 2.5.3). Many theoretical efforts have been dedicated to alleviating the axion quality problem. Efforts in four-dimensional field theories are recounted in section 2.5.3 and have mainly focused on recovering the axion shift symmetry as an accidental symmetry of the particle content of the theory and include models with additional gauge redundancies like $\mathbb{Z}_N$ [66, 67] or $U(1)$ (see for example [68]), and composite axion models from the confinement of additional gauge groups [69–78]. While the latter models are able to explain the large separation between the axion decay constant and weak scale as dynamically generated, these models require involved model building, and the post-inflationary scenario is, in some cases, disfavoured by heavy relic production [69].

As shown in section 3.4, akin to other fine-tuning problems of the Standard Model, an alternative to ordinary symmetries can be played by higher-form symmetries and higher-dimensional constructions [289, 290]. Extra-dimensional field theories or string theory constructions [20] are generically expected to contain a large number of axions depending on the number of closed cycles and gauge fields in the compact dimensions. In these scenarios, axions arise as the zero modes of bulk higher-form gauge fields integrated over closed extra-dimensional cycles and inherit a high-quality shift symmetry from the more robust global higher-form shift symmetries of gauge fields [15, 58], ameliorating several quality issues that plague their four-dimensional counterparts. See [18] for a recent, comprehensive review.

However, this category of axions lacks the predictability of the post-inflationary field theoretic axion. Intuitively, the axion in such scenarios comes from the Goldstone of the spontaneously broken higher-form symmetry (see section 3.4). In contrast to its four-dimensional counterparts, in the standard cosmological history this symmetry is never expected to be restored, and therefore no phase transition or network of solitonic axion strings is expected. In this sense, higher-form axions generically correspond to the pre-inflationary scenario, where the DM abundance depends quadratically on the unknown initial misalignment angle (see eq. (2.32)), reducing the predictability of the theory [291] and inference of the axion mass.

In this work we present a different kind of axion where both the predictivity of the post-inflationary scenario and the exponentially good axion quality, can be present simultaneously. To gain some intuition, let us consider a simple field theoretic model where the axion comes as a linear combination of the zero-mode of the higher-dimensional gauge field and the phase of a complex scalar. An illustrative example is an orbifold scenario with 5D bulk gauge fields and a 4D boundary – that is, a 3−brane localised at the orbifold fixed point – where the complex scalar lives[1]. At low energies, the model has a one-dimensional approximate vacuum manifold on the brane:

$$U(1)_{PQ} \equiv \frac{U(1)^{(0)} \times U(1)^{(1\downarrow 0)}}{U(1)_{\text{gauge}}} \,. \tag{6.1}$$

The $U(1)^{(0)}$ symmetry is an ordinary 0-form symmetry associated with the phase of a complex scalar on the brane. The $U(1)^{(1\downarrow 0)}$ symmetry is a 0-form symmetry that descends from a global 1-form symmetry in the 5D bulk. A diagonal direction on the vacuum manifold is a gauge redundancy, leaving the orthogonal direction $U(1)_{PQ}$ as the axion, θ. Any symmetry breaking or lifting of this vacuum manifold

[1]Another interesting 5D model with a distinct set-up was presented in [80, 292] also offering a high-quality axion.

needs to – by construction – leave the gauge direction invariant and therefore always has to break the more robust $U(1)^{(1\downarrow 0)}$ symmetry, guaranteeing an exponentially good quality. This situation differs qualitatively from field theory as well as higher-dimensional axions and resembles situations that appear in the context of higher-group symmetries [97] (see section 3.5.2). For this reason, we call the 4D axion, θ, a *higher axion*.

The approximate vacuum manifold also allows for string solutions: 1-dimensional field configurations that are topologically protected by the property that when a non-contractible loop in physical space is traversed, the field configuration traverses a non-contractible loop on the vacuum manifold. There are two independent global string solutions, corresponding to the $U(1)^{(1\downarrow 0)}$ and $U(1)^{(0)}$ non-contractible cycles, that can be related by a local string involving both cycles. The $U(1)^{(1\downarrow 0)}$ global strings admit no symmetry restoration and are expected to require a decompactified core [293] and are generically difficult to be produced in the standard cosmological history (see however [291, 294]). On the other hand, the $U(1)^{(0)}$ strings can be produced by the Kibble-Zurek mechanism [126] as the temperature of the universe drops below the spontaneous breaking scale of the symmetry (see section 2.3.2). At this instance, the gauged direction on the vacuum manifold provides mass to a gauge boson, which - in the scenario of a small bulk - is much larger than the other scales associated with the $U(1)^{(0)}$ strings. In this limit, the cosmology is similar to that of ordinary field theory strings.

As noted above, the key ingredient is the mixing between the higher-form and a zero-form symmetry. This mixing is also the crucial ingredient in *pseudo*-anomalous $U(1)$ scenarios where 4D gauge anomalies are cancelled by bulk higher-form fields. This intuition allows us to identify string theory constructions where similar mechanisms operate and provide *higher axions* – that is, a perturbatively massless Goldstone boson that emerges as a linear combination of a higher-form axion and an ordinary 0-form axion (phase of a complex scalar). We will sketch how to obtain these axions in extra-dimensional and heterotic string theory compactifications.

The structure of this chapter is as follows. The set-up and details of the toy model is described in section 6.2 along with the quality of the resulting higher axion. Section 6.3 describes the string solutions and cosmological history. In section 6.4 we describe the minimal ingredients to realise this mechanism in string theory. Finally, in section 6.5 we summarise our findings.

Whilst finalising the work [100], [292] appeared, partially overlapping with the results of [100] and this thesis and reaching similar conclusions.

6.2 Minimal Set-Up

Let us consider a standard orbifold scenario consisting of a 5-dimensional spacetime with four flat dimensions parametrised by coordinates x^μ and one compact dimension of length R, parametrised by an angular coordinate $\phi \in [0, 2\pi)$ or $y = R\phi$. The angular coordinate is subject to the gauge identification $(x^\mu, \phi) \sim (x^\mu, -\phi)$, effectively reducing the compact space to $S^1/\mathbb{Z}_2$ [295]. This orbifold has two fixed points, $\phi = 0, \pi$, which can host $(3+1)$-dimensional field theories on fixed 3-branes[2]. Latin indices will denote 5D indices $M = 0, 1, 2, 3, 5$, whilst greek indices will be used for the four flat dimensions $\mu = 0, 1, 2, 3$.

The bulk of the 5-dimensional spacetime contains a gauge field A_M with coupling strength g_A. Under the orbifold parity identification, the gauge field transforms as $(A_\mu, A_5) \sim (-A_\mu, A_5)$. Additionally, the gauge field has an ordinary gauge identification,

$$A \to A + d\lambda_A\,. \tag{6.2}$$

In the limit where this gauge field is decoupled from all other dynamics of the theory, there exists a global one-form and two-form symmetries, the electric $U(1)^{(1)}$ and magnetic $U(1)^{(2)}$, under which Wilson lines and ‘t Hooft surface operators transform by a $U(1)$ phase, respectively [101] (see section 3.4). These symmetries

[2]We will focus on the physics on one of the branes. In principle, the discussion can be generalised to include fields on both branes, but this would not alter the conclusions regarding the low-energy physics.

are spontaneously broken and act non-linearly on their respective Goldstone bosons, the photon A_M and the dual photon $\widetilde{A}_{MN}$, by shifting by a closed one (two)-form,

$$A \xrightarrow{U(1)^{(1)}} A + c^{(1)}, \quad dc^{(1)} = 0\,, \tag{6.3}$$

$$\tilde{A} \xrightarrow{U(1)^{(2)}} \tilde{A} + \tilde{c}^{(2)}, \quad d\tilde{c}^{(2)} = 0\,. \tag{6.4}$$

The electric (magnetic) symmetry is emergent below the scale of the lowest electrically charged particles (magnetically charged lines), at which point electric (magnetic) field lines (surfaces) cannot end, and their number penetrating any surface is conserved. It is well-established that this conservation is only broken by non-perturbative Schwinger-like pair production processes, providing the exponential protection for these symmetries.

As described around Eq. (6.1), part of the light axion will be identified with the zero mode of this gauge field along the compact dimension, called a, and provided by the Wilson loop around the compact dimension

$$a = \oint A \; dy\,. \tag{6.5}$$

On the compact dimension, the 1-form shift symmetry of the bulk gauge field reduces to a 0-form shift symmetry $U(1)^{(1\downarrow 0)}$ of the zero mode

$$a \to a + c, \qquad dc = 0\,. \tag{6.6}$$

This symmetry inherits the exponentially good quality from the 1-form shift symmetry of the gauge field, A.

In addition to the bulk gauge field, we assume that the brane at the orbifold fixed point $y = 0$ supports another gauge field C_μ with gauge identification

$$C \to C + d\lambda_C\,. \tag{6.7}$$

A crucial ingredient of our mechanism is that the shift symmetry of the zero mode a will transform under this identification. One way to accomplish this that is compatible with locality is,

$$A_5 \to A_5 + \delta(y)\lambda_C + \partial_5\lambda_A\,. \tag{6.8}$$

Generically, a δ-function localisation in shifts of 1-form gauge fields can be inherited from a 1-form analogue of the Green-Schwarz mechanism [221] as in section 3.5.2. For example, the gauge field A can shift as $A \to A + \lambda_C d\alpha$ under $U(1)_C$ gauge transformations, where $\alpha \sim \alpha + 2\pi$ is a 0-form periodic field. In the presence of a background flux, $d\alpha = \delta(x)$, the gauge transformation becomes localised, although we stress that our model is independent of a particular UV realisation. This is a higher-group structure between a (gauged) (-1)-form symmetry [296] and two (gauged) 0-form symmetries. This is analogous to the Green-Schwarz mechanism, where a 2-form field B shifts as $B \to B + \lambda_C d\tilde{A}$ where $\tilde{A}$ is a 1-form gauge field. The latter scenario is generic in string theory compactifications, as discussed in the next section 6.4, where background fluxes $d\tilde{A} = \delta(S_2)$ on a sphere S^2 localise a gauge symmetry on a cycle or D-brane. The modern terminology for a GS mechanism is that of a (gauged) higher group [97] (see section 3.5.2), further motivating the name higher axions.

As the gauge interaction (6.8) is local in position space, the entire KK tower shifts under the gauge symmetry:

$$a \to a + \lambda_C \,, \qquad A_5^{(n)} \to A_5^{(n)} + \lambda_C + n\lambda_A^{(n)} \,, \qquad n = 1, 2, \ldots \tag{6.9}$$

The zero mode is part of a tower of Kaluza-Klein (KK) modes $(A_\mu^{(n)}, A_5^{(n)})$ provided by the Fourier transform of A_M along the compact dimension subject to the orbifold identification

$$A_\mu(x,\phi) = \frac{1}{\pi}\sum_{n=1}^{\infty} A_\mu^{(n)}(x)\sin(n\phi) \,, \qquad A_5(x,\phi) = \frac{a(x)}{2\pi R} + \frac{1}{\pi R}\sum_{n=1}^{\infty} A_5^{(n)}(x)\cos(n\phi) \,. \tag{6.10}$$

The five-dimensional action on the orbifold has to be invariant under the gauge identification. When reduced into the individual KK components, the four-dimensional Lagrangian is:

$$\mathcal{L} \supset -\frac{R}{4g_A^2\pi}\sum_{n=1}^{\infty}\left(\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}\right)^2 + \frac{1}{2g_A^2\pi R}\sum_{n=1}^{\infty}\left(\partial_\mu A_5^{(n)} - nA_\mu^{(n)} - C_\mu\right)^2 + \frac{1}{4g_A^2\pi R}\left(\partial_\mu a - C_\mu\right)^2 \,. \tag{6.11}$$

The physical consequences of having mixed the gauge redundancies between the bulk gauge field and brane gauge field are most apparent in the axial gauge choice $A_5^{(n>0)} = 0$. In this gauge, all gauge bosons obtain a mass apart from one zero eigenvector[3] C_μ of the mass matrix. The additional tower of massive gauge fields is heavy $m_{KK}^2 \sim \frac{1}{R^2}$ and can be integrated out. Upon integrating out these gauge bosons, the electric coupling of C_μ shifts by

$$\frac{1}{e^2} \to \frac{1}{e^2}\left(1 + \frac{e^2 R\pi}{6g_A^2}\right) . \tag{6.12}$$

As seen in Eq. (6.11), the would-be higher-dimensional axion a has been eaten by the 4D gauge boson C_μ. To introduce an axion in the 4D EFT we have to modify the boundary theory at the orbifold fixed point $y = 0$. Let us consider a brane-localised charged complex scalar $\Phi = |\Phi|e^{ib}$, whose phase b shifts under the gauge transformations of C_μ as

$$b \to b + \lambda_C . \tag{6.13}$$

The potential of this scalar forces it to have a minimum at the vev $|\Phi| = F_\Phi$. Below this scale, the low-energy Lagrangian is of the form:

$$\mathcal{L} \supset -\frac{1}{4e^2}\sum_{n=1}^{\infty}(\partial_\mu C_\nu - \partial_\nu C_\mu)^2 + \frac{F_a^2}{2}(\partial_\mu a - C_\mu)^2 + \frac{F_\Phi^2}{2}(\partial_\mu b - C_\mu)^2 - V(a-b) , \tag{6.14}$$

where $F_a^2 = \frac{1}{2\pi R g_A^2}$ is the decay constant of the zero mode a, which is fixed and necessarily tied to the compactification scale, and we have included a potential for the ungauged linear combination $a - b$, whose contributions will be detailed in section 6.2.2.

An alternative construction would be to consider $U(1)_C$ as a bulk gauge symmetry [100]. In this case, one imposes the orbifold boundary condition $(C_\mu, C_5) \to (C_\mu, -C_5)$. The IR limit of this theory will coincide with the one described above. In both cases, we will be interested in coupling the ungauged axion linear combination, $a - b$, to the SM gauge bosons. Due to the gauge transformation of a and b, this will induce boundary-localised anomalies that have to be cancelled by a bulk Chern-Simons (CS) interaction.

[3] The zero eigenvector of the mass matrix is $C_\mu - \sum_{n=1}^{\infty} \frac{e^2 R}{n g_A^2 \pi} A_\mu^{(n)}$ but will henceforth be simply called C_μ

6.2.1 The Axion

The Lagrangian (6.14) below the compactification scale $1/R$ and spontaneous breaking scale F_Φ includes a zero mode a and a phase b. The linear combination $a+b$ is 'eaten' by the gauge field C_μ and does not appear in the low-energy EFT. The orthogonal combination will form a compact scalar $\theta \sim \theta + 2\pi$ – the *higher axion* – with decay constant f. This can be seen by diagonalising [223] to the eaten/uneaten basis $(F_\pi \pi, f\theta)$ by rotating the original basis vector $(F_a a, F_\Phi b)$ as

$$F_\pi \pi = \cos(\alpha) F_a a + \sin(\alpha) F_\Phi b\,, \tag{6.15}$$

$$f\theta = -\sin(\alpha) F_a a + \cos(\alpha) F_\Phi b\,. \tag{6.16}$$

Requiring that θ is uncharged under the gauge transformations fixes the angle as

$$\cos(\alpha) = \frac{F_a}{\sqrt{F_a^2 + F_\Phi^2}}\,, \qquad \sin(\alpha) = \frac{F_\Phi}{\sqrt{F_a^2 + F_\Phi^2}}\,. \tag{6.17}$$

Defining the field π as having charge 1 under $U(1)_C$ and the periodicity of θ as $\theta \sim \theta + 2\pi$ lifts the degeneracy between the new decay constants F_π and f and the fields π and θ and uniquely defines the π field as

$$\pi = \frac{1}{F_\pi^2}\left(F_a^2 a + F_\Phi^2 b\right)\,, \qquad F_\pi^2 = F_a^2 + F_\Phi^2\,, \tag{6.18}$$

and the axion θ as

$$\theta = -a + b\,, \qquad f = \frac{F_a F_\Phi}{F_\pi}\,. \tag{6.19}$$

After performing this rotation, the Lagrangian can be expressed in the rotated basis as

$$\mathcal{L} \supset \frac{1}{2} F_\pi^2 \left(\partial_\mu \pi - C_\mu\right)^2 + \frac{1}{2} f^2 \left(\partial_\mu \theta\right)^2 - V(\theta)\,. \tag{6.20}$$

The field π will be eaten by the gauge boson C_μ, which can be integrated out, leaving behind a perturbatively massless axion θ.

6.2.2 Axion Quality

The axion θ of the previous section is best described as excitations on the vacuum manifold:

$$U(1)_{PQ} \equiv \frac{U(1)^{(0)} \times U(1)^{(1\downarrow 0)}}{U(1)_{\text{gauge}}} \,. \tag{6.21}$$

The ability of this axion to solve the strong CP problem relies on the quality of its anomalous continuous shift symmetry and its coupling to gluons. For generic axions, this quality can be plagued by several extreme sensitivities to UV physics (section 2.5.3).

In section 3.4, we demonstrated that extra-dimensional axion scenarios largely circumvent these extreme sensitivities to UV physics [79] through an exponential protection from a higher-form symmetry. Any effect breaking the shift symmetries of such axions will require an *extension* into the compact dimension and will be suppressed by the exponential of the action, which is proportional to the size of the compact space. In the mechanism described here, any effect lifting the vacuum manifold (6.21) of the axion θ by construction must leave the gauge direction invariant and always has to break the more robust $U(1)^{(1\downarrow 0)}$ symmetry, guaranteeing an exponentially good quality analogous to other extra-dimensional axion scenarios.

In this case, the least suppressed gauge invariant operator that breaks the shift-symmetry of the uneaten linear combination, $\theta = -a + b$ is

$$\mathcal{O} = c\Lambda_{UV}^3 \Phi^* e^{-S} e^{ia} + \text{h.c.} \,, \tag{6.22}$$

with $c \sim O(1)$. This operator leads to the axion potential

$$V(\theta) = -c\, F_\Phi \Lambda_{UV}^3 e^{-S} \cos\theta \,. \tag{6.23}$$

We notice, as expected for higher-dimensional axions, that θ has exponentially good quality despite that it contains, in part, the phase of the complex scalar field, Φ. In typical higher-dimensional and string constructions the instanton action is related to the 4D effective gauge coupling of bulk symmetries as $S \sim \frac{2\pi R}{g_5^2} \sim \frac{2\pi}{\alpha(R^{-1})}$. Provided that the gauge couplings are small in the UV, this guarantees the quality of θ to solve the strong CP problem.

6.2.3 Anomaly Cancellation

In general, the axion has a topological coupling to gluons if required to solve the strong CP problem and to electromagnetism, which provides the main detection strategies in a large class of experiments [22]. In the extra-dimensional axion scenario, this coupling is facilitated by extending the gauge fields of electromagnetism and QCD as zero modes of extra-dimensional gauge fields with field strengths F^{EM}, F^{QCD} [79]. This extension into the five-dimensional bulk allows for CS interactions between the Standard Model gauge fields and bulk gauge field,

$$\mathcal{L} \supset \frac{\kappa_{EM}}{16\pi^2} A_M \left(F_{NL}\widetilde{F}^{MNL}\right)_{EM} + \frac{\kappa_{QCD}}{16\pi^2} A_M \left(F_{NL}\widetilde{F}^{MNL}\right)_{QCD} , \tag{6.24}$$

and the coupling constants $\kappa_{EM}, \kappa_{QCD} \in \mathbb{Z}$ are to be integer valued in the normalisation that the electric coupling constants sit in front of the gauge field kinetic terms.

Due to the gauge transformation in eq. (6.8), these CS terms require additional fermions localised on the brane that cancel the induced anomalies with respect to the gauge redundancies of C_μ. The brane fermions admit couplings to the complex scalar Φ, at which point the fermions become massive below the spontaneous breaking scale and can be integrated out, resulting in a low-energy four-dimensional coupling of the axion θ to electromagnetism and QCD,

$$\mathcal{L} \supset \frac{\kappa_{EM}}{16\pi^2} \theta \left(F\widetilde{F}\right)_{EM} + \frac{\kappa_{QCD}}{16\pi^2} \theta \left(F\widetilde{F}\right)_{QCD} , \tag{6.25}$$

similar to standard four-dimensional QCD axions. Altogether, Eqs. 6.24 and 6.25 result in the cancellation of the mixed anomalies $[SU(3)_{\text{QCD}}]^2 \times U(1)_C$ and $[U(1)_{EM}]^2 \times U(1)_C$.

The fermions on the brane introduce additional cubic anomalies $[U(1)_C]^3$. In the minimal version of the mechanism where $U(1)_C$ is a brane-localised gauge symmetry, the cubic anomaly cannot be directly cancelled by a CS term since C_μ does not propagate into the bulk. These anomalies are cancelled by a subtle mechanism which is a lower-dimensional analogue of M-theory [297]. In order to comply with gauge invariance of C_μ, the Bianchi identity of the field strength F^A of the bulk gauge field A is altered on the brane

$$dF^A = \delta(y) F^C \wedge dy\,. \tag{6.26}$$

Similar to M-theory, the Bianchi identity can be solved by having the field strength along the direction of the brane given by

$$F^A_{\mu\nu} = \frac{1}{2}\epsilon(y) F^C_{\mu\nu} + \ldots \tag{6.27}$$

where $\epsilon(y)$ is the parity-odd step function around the orbifold fixed point $y = 0$. The dots indicate terms that have to vanish on the brane due to the orbifold parity identification. This altered field strength admits a CS self-coupling in the bulk

$$\mathcal{L} \supset \frac{\kappa_A}{24\pi^2} A \wedge F^A \wedge F^A\,. \tag{6.28}$$

Plugging the expansion (6.27) back into this CS coupling cancels the boundary-localised cubic anomaly $[U(1)_C]^3$ for appropriately chosen $\kappa_A \in \mathbb{Z}$, which corresponds to the anomaly coefficient induced by chiral fermions on the boundary.

6.3 Higher Axion Strings

In this section we argue that there exist higher axion string solutions that can be thermally produced through a standard Kibble-Zurek mechanism [126]. In the limit that the mass of the gauge boson, C_μ, is much larger than the VEV of the complex scalar, $|\Phi|$, the string consists mostly of the uneaten axion mode, and its decay will contribute to the relic abundance of axion dark matter.

The vacuum manifold (6.21) is said to admit string solutions; field configurations with 2-dimensional worldsheets that are topologically protected by the non-trivial winding in the $U(1) \times U(1)$ vacuum manifold as one traverses a loop in spacetime enclosing the string. The topological index associated with the winding is best classified in terms of the winding numbers of the original phases a and b around the string.

The phenomenology of strings with non-trivial windings of a is that of Stückelberg strings. There is no symmetry restoration at the core of the string, but instead local effective field theory is expected to break down. In principle, special inflationary paradigms exist to UV complete such a description and non-thermally create these strings including [291, 294], which always require a phase in the early universe where the 4D EFT breaks. If formed, such strings could potentially overproduce the DM abundance in scenarios without strong warping [291].

Strings in which the phase b winds do admit a description of the core captured by the four-dimensional field theory. In the cosmological history of the universe, such strings are expected to form when the temperature drops below the scale $|\Phi| = F_\Phi$. Below this scale, the angle b obtains random values in uncorrelated patches, and a network of strings forms due to the Kibble-Zurek mechanism. Regions in space-time with non-trivial windings of b will enclose string cores. In the absence of any potential for b, the string solution that winds once is axially symmetric, and the complex scalar and massive gauge field at a distance r and angle α in a plane orthogonal to the string is described by:

$$\Phi(r,\alpha) = f(r)e^{i\alpha}\,, \qquad C_\alpha(r,\alpha) = \frac{g(r)}{r}\,, \tag{6.29}$$

where $f(r)$ and $g(r)$ have to vanish sufficiently quickly near the core of the string.

The full string solution, including the core profile, can be found by minimising the string's tension or energy per unit length [298]. The tension of the global string is logarithmically dominated by the IR dynamics and proportional to the scale of symmetry breaking F_Φ^2. A small winding of the massive gauge field around the string reduces this scale to f. The massive gauge field obtains the IR value:

$$C_\alpha = \frac{1}{r}\frac{F_\Phi^2}{F_a^2 + F_\Phi^2}\,, \tag{6.30}$$

implying that these strings carry a small amount (non-integer) of flux. The full tension, including that of the core, can then be estimated as

$$T \sim \frac{\pi F_\Phi^4}{2F_\pi^2} + \pi F_\Phi^2 \ln\frac{m}{eF_\pi} + \pi f^2 \ln\frac{eF_\pi L}{2} + \frac{\pi}{2}F_\Phi^2\,. \tag{6.31}$$

Where m is the mass of the radial mode of Φ and size of the string core. The length scale L provides an IR cut-off of the theory, which could be the diameter of a single closed string or the distance between two extended strings of opposite orientation.

In the limit $F_a \gg F_\Phi$, it is disfavoured to have any heavy gauge field C_μ around the string, and we recover the ordinary tension of a global string

$$F_a \to F_\pi\,, \qquad F_\Phi \to f\,, \qquad T \to \frac{\pi}{2} f^2 + \pi f^2 \ln \frac{mL}{2}\,. \tag{6.32}$$

In this limit, the string profile is a global string, and we recover an ordinary axion string around which the *higher axion* θ winds by 2π. Their evolution and contribution to the relic axion abundance will follow standard axion string simulations [87–95].

6.4 Heterotic Axion Strings

The field theoretic models above resemble anomaly cancellation in different string theories, inspiring us to build heterotic string theory realisations, although we stress that none of the features of the toy model in section 6.2 rely on any specific string theory embedding. Additional type II-B compactifications can be found in [100].

The $D = 10$ dimensional low-energy bosonic excitations common to all superstring theories are captured by the dilaton ϕ, the anti-symmetric two-index Kalb-Ramond field B_{MN} and the graviton g_{MN}, and collected in the $\mathcal{N} = 1$ SUGRA action [299],

$$S = \frac{1}{2\kappa_0^2} \int d^{10}x \, \sqrt{-g} e^{-2\phi} \left(R - \frac{1}{2} H_{MNL} H^{MNL} + 4\partial_M \phi \partial^M \phi \right)\,. \tag{6.33}$$

In addition, the bosonic part of the various types of superstring theories contains additional forms and gauge fields specific to the type of string theory under consideration; see [300] for a comprehensive discussion on the different possibilities.

Typically, the vacuum state of the ten-dimensional geometry is $M_4 \times K$, where K is a Calabi-Yau (CY) if the theory admits $\mathcal{N} = 1$ SUSY in $D = 4$ [301]. The low-energy spectrum is determined by the topological invariants of the Calabi-Yau.

The $D = 4$ dimensional massless spectrum includes the graviton and a set of massless chiral superfields S, T^i, Z^j. The superfield S is the dilaton-axion chiral superfield, whose real part is the dilaton zero mode s and complex part is the model-independent axion that is dual to the zero-mode of $B_{\mu\nu}$. The superfield T^i is the set of Kähler moduli, whose real part t_i sets the size of the 2-cycles (whose number is determined by the Hodge number $h^{1,1}$) and complex part are the model-dependent axions corresponding to the integral of B_{MN} along the cycle. The volume V of the Calabi-Yau can be expressed in terms of the Kähler moduli t_i and the triple intersection numbers d_{ijk} and is proportional to the quantity κ defined as,

$$\kappa = d_{ijk} t^i t^j t^k \,. \tag{6.34}$$

The additional four-dimensional graviton and the complex structure moduli Z^j are not relevant for the rest of our discussion.

In the case of heterotic string theory, there will in addition be complex scalars fields Φ_i descending as zero modes of the $D = 10$ gauge fields in the presence of non-trivial fluxes. The low-energy gauge group will also contain Abelian gauge groups $U(1)_j$ with gauge fields V_j with the complex scalars carrying gauge charges p_{ij}. In type II, fluxes and $D-$branes on the Calabi-Yau may play a relevant role and result in an additional set of chiral superfields Φ_i with charges p_{ij} under a set of (distinct) $U(1)_j$ gauge groups with gauge fields V_j.

If the $U(1)_j$ symmetries are anomalous from the $D = 4$ dimensional perspective, then (part of) the moduli should also transform under the symmetry to cancel the anomaly by a standard Green-Schwarz mechanism [221] or gauged higher-group as given in section 3.5.2. In general, this transformation of the moduli can be a complicated transformation under each of the $U(1)$ symmetries. In line with the toy model, the low-energy axion will be a combination of the phase of the chiral superfields Φ_i and the imaginary parts of the moduli fields.

The $D = 4$, $\mathcal{N} = 1$ supersymmetric Lagrangian contains terms of the form,

$$\mathcal{L} \supset \int d^4\theta \ \left[\Phi_i^\dagger e^{2p_{ij}V_j}\Phi_i + K(S, T^i, V_j)\right] + \int d^2\theta \frac{1}{4} f(S, T^i) W^j W_j \,. \tag{6.35}$$

The Kähler potential K is a function of only the superfields $S + \overline{S}$ and $T^i + \overline{T^i}$ up to non-perturbative corrections [299]. The Kähler potential also includes the gauge fields V_j, in order to compensate for the transformation of S and T^i under the gauge symmetry. These symmetries can be anomalous, and the anomaly is cancelled by the shift of the gauge kinetic function $f(S, T^i)$ under the symmetry, which is coupled to the super field strength W^j of the low energy gauge fields. For the model-independent axion, the Kähler potential in the absence of gauge fields is of the universal form $K(S+\overline{S}) = -M_P^2 \ln\left(S+\overline{S}\right)$.

The effect of the Kähler potential is twofold. It contributes to the mass of the gauge field V_j as,

$$M_j^2 = \frac{\partial^2 K}{\partial V_j^2}\,, \tag{6.36}$$

and to the D-term as a Fayet-Iliopoulos (FI) term,

$$D_j = \frac{\partial K}{\partial V_j} - p_{ij}|\Phi_i|^2\,. \tag{6.37}$$

In order to preserve $N=1$ supersymmetry, these D terms have to vanish [302]. In a general scenario, Eq. (6.37) fixes the VEV of the complex scalar Φ_i to be close to the mass of the gauge boson [302]. However, in principle, the D terms and mass M_j^2 can be scaled independently, and there exist special loci in moduli space along which $\frac{\partial K}{\partial V_j}$ vanishes. In this scenario, the VEV of the complex scalars can be kept suitably small by small deviations from this locus, whilst the mass of the gauge boson M_j^2 is typically proportional to the string scale, M_s.

6.4.1 Heterotic $E_8 \times E_8$

In heterotic string theory [303], the massless 10 dimensional spectrum consists of $\mathcal{N}=1$ supergravity with either an $SO(32)$ or $E_8\times E_8$ supersymmetric Yang-Mills gauge group and contains a universal dilaton and model-independent axion $S = s + i\sigma$ and a number of Kähler moduli $T^i = t^i + 2i\chi^i$. The Kähler potential in the absence of gauge fields is a function of the superfields,

$$K = -M_P^2 \ln\left(S+\overline{S}\right) - M_P^2 \ln\kappa\,. \tag{6.38}$$

where κ is defined in Eq. 6.34. While the field metric on the model-independent axion is universal, the field metric on the Kähler moduli is,

$$G_{ij} = -\frac{3}{2}M_P^2\left(\frac{\kappa_{ij}}{\kappa} - \frac{3}{2}\frac{\kappa_i\kappa_j}{\kappa^2}\right)\,, \qquad \text{with:} \quad \kappa_i = d_{ijk}t^j t^k\,, \qquad \kappa_{ij} = d_{ijk}t^k\,. \tag{6.39}$$

We will be focussing on Calabi-Yau manifolds with fluxes in an $S(U(1)^5)$ subgroup embedded in E_8 as $S(U(1)^5) \supset SU(5) \supset E_8$, corresponding to the line-bundle models considered in [304–306]. This set-up can result in low-energy axions with low decay constants – that is, below the string scale, for certain values of κ_i and $U(1)$ charges q_{ij}, as originally shown in [307]. Other axion setups with low decay constants in heterotic string theory and M-theory can be found in [308].

In certain constructions, the low-energy theory is the GUT gauge group $SU(5)_{\mathrm{GUT}} \times S(U(1)^5)$. The fact that the 5 symmetries are really 4 is encoded by identifying the charge vectors $\vec{q}_i = (q_{ij})$ and $\vec{\tilde{q}}_i = (\tilde{q}_{ij})$ if,

$$\vec{q}_i - \vec{\tilde{q}}_i = \mathbb{Z}(1,1,1,1,1)\,. \tag{6.40}$$

In addition, the charged matter spectrum contains several complex scalar fields $\Phi_i = \hat{\Phi}_i e^{ib_i}$ with charges p_{ij} under the $S(U(1)^5)$ symmetries. When these complex scalars obtain a VEV, the $S(U(1)^5)$ symmetries are broken, and the fluxes occupy a larger non-Abelian subgroup of rank 4 inside $SU(5) \supset E_8$. The general potential for the matter fields requires a full description of this non-Abelian bundle and the Calabi-Yau, which is far beyond the scope of this work, and would require a combination of the methods presented in [309–312].

Instead, our focus will lie on the D-terms and the Kähler potential. In general, several of the $S(U(1)^5) \cong U(1)^4$ symmetries are anomalous, and therefore the Kähler moduli must participate in the anomaly cancellation and transform as,

$$\chi^i \to \chi^i - q_{ij}\lambda_j\,, \qquad \sigma \to \sigma - 2q_{\sigma j}\lambda_j\,. \tag{6.41}$$

The gauge-invariant Kähler potential is therefore of the form,

$$K(S + \overline{S} + q_{\sigma j}V_j,\ T_i + \overline{T}_i + q_{ij}V_j)\,. \tag{6.42}$$

The mass of the j-th gauge boson can be recovered from the Kähler potential of the Kähler moduli by equation (6.36) and transformation (6.41),

$$M_j^2 = \frac{\partial^2 K}{\partial V_j^2} = \sum_{i,k} q_{ij} G_{ik} q_{kj} + \frac{q_{\sigma j}^2}{4s^2} M_P^2\,. \tag{6.43}$$

In this case, the mass is simply proportional to the field space metric on the moduli suitably contracted with the charge vector. Given that the typical 2-cycle volume is smaller than the overall CY volume in string units, $t^i/s \ll 1$, it can be appreciated that the dilaton term constitutes a small correction.

The Kähler potential also gives rise to a contribution to the D-term, which in combination with the contribution of the complex scalar, becomes

$$D_j = M_P^2 \left(\frac{3q_{ij}\kappa_i}{\kappa} + \frac{q_{\sigma j}}{s} \right) - p_{ij}|\Phi_i|^2 \,. \tag{6.44}$$

The Calabi-Yau admits special loci in moduli space, called split loci, along which the first two terms vanish (see [305, 306] for details). Close to the split loci, the VEV of the complex scalar field Φ_i can be made suitably small whilst preserving supersymmetry and a vanishing D-term.

At these points in moduli space, the low-energy theory is that of a gauge field V_j with mass M_j^2, a higher-dimensional axion $a_j = \sum_i c_{ij}\chi^i$, where the coefficients c_{ij} depend on the $U(1)_j$ charge of the χ^i field, and the phase of the complex scalar, b_j. The decay constant of a_j is proportional to the gauge boson mass, $F_a^2 \sim M_j^2$, while the VEV of the complex scalar can be suitably smaller, $|\Phi|^2 \ll M_j^2 = F_a^2$. We remark that the gauge invariant linear combination – the *higher axion* θ – contains b_j as well as a_j in close analogy to the toy model in section 6.2.

The low-energy theory admits axion string solutions with a profile for the radial mode and a corresponding winding of its phase. The core of the higher axion string does not correspond to a decompactification limit but rather a change in the nature of the vector bundle on the Calabi-Yau. The geometric interpretation of the radial profile in the $D = 10$ dimensional theory is that, far away from the core of the string, the complex scalar has a VEV, and the fluxes occupy a rank 4 subgroup of $SU(5) \supset E_8$. At the core of the string, the complex scalar vanishes, the fluxes sit in an $S(U(1)^5)$ and the Kähler moduli lie along the split locus. This results in an apparent enhanced symmetry group $S(U(1)^5)$ at the core of the string, but the gauge fields *eat* the higher-form axion, which shifts under the anomalous $U(1)$s as required by anomaly cancellation.

6.5 Discussion

In this work we presented a new mechanism to obtain field theoretic axion strings for exponentially good quality axions. The minimal version of the mechanism – presented in the form of a 5D theory with boundary – involves the mixing between an axion coming from the fifth component of a bulk gauge field and the phase of a complex scalar that lives on the boundary. This differs qualitatively from standard field theory as well as higher-dimensional axions and resembles situations that appear in the context of higher-group symmetries. For this reason, we named the 4D axion θ as *higher axion*. Crucially, higher axion strings can be produced by the standard Kibble-Zurek mechanism [126] in the early universe and constitute a natural way to reconcile the predictivity of the post-inflationary axion scenario together with the attractive features of the string axiverse.

We have shown that the key ingredients are common in most string theory constructions, providing guidance to obtain higher axions in heterotic string theory compactifications. In field theory, one has in principle freedom to choose the VEV of the complex scalar with respect to the compactification scale. However, in supersymmetric compactifications, the two scales are tied through the $D-$term potential. We demonstrated that it is possible to separate $|\Phi|$ from the string scale, M_s, if multiple moduli fields shift under the anomalous $U(1)$. This occurs through cancellations between different contributions to the FI term in special regions of moduli space.

For the case of the QCD axion, numerical simulations of the post-inflationary scenario seem to point towards a low axion decay constant around $f \sim 10^{10} - 10^{11}$ GeV. In the case of heterotic string theory, where the string scale is typically around $M_s \sim 10^{17}$ GeV, this seems to require a fine cancellation between different FI term contributions (see Eq. (6.44)). We note that moduli stabilisation may have a large impact on the cancellations, and understanding their contributions and finding explicitly string theory constructions of the mechanisms presented here are very interesting directions that we leave for future work.

7 Conclusion & Outlook

In this thesis, we set out to constrain the vast axion parameter space by studying the generalised symmetries of the axion and, in doing so, provide timely theoretical guidance for the numerous existing axion experiments and motivate novel axion searches.

The relevant background on axions, photons and gluons was reviewed in chapter 2, whereas chapter 3 provided a general review of generalised symmetries and effective field theory with an emphasis on applications to axions.

In chapter 4, we studied the axion mass and photon-coupling parameter space. We explored the general properties of the axion-photon coupling and demonstrated that there exists a model-independent correlation between the non-linearities in the axion-photon coupling and axion mass, as both are determined by the quality of the same generalised non-invertible symmetry of the axion. We derived the general form of the axion-photon coupling for several examples, including the QCD, and showed that there is a generic, prototypical form for this monodromic function.

Phenomenologically, the full non-linear form of the axion-photon coupling is most relevant for scenarios with axion strings, domain walls or other extended objects that probe the full axion field range. A natural next step is to consider the influence of the non-linear form on axion-light scattering in Primakoff processes or in axion miniclusters or superradiant axion clouds surrounding rotating black holes. Knowing the full non-linear form is also essential to axion birefringence experiments [313].

Moreover, it could be interesting to use the formalism described in chapter 4 to study the effective photon coupling for mesons in the chiral Lagrangian, e.g. for the pion $g(\pi^0)$ (see [240] and references therein), or to understand the fractional quantum hall droplet of the η' in [38] in this context.

Lastly, several non-invertible symmetry-breaking contributions to the axion-photon coupling could be considered, such as magnetic monopoles (with fermions) (see [128] for a SUSY example) and non-Abelian instantons.

In chapter 5, we considered a superconducting phase of axion physics in which the higher-form shift symmetry of the dual axion is explicitly broken, motivated by the absence of generalised global symmetries in quantum gravity. In this phase, instantons are confined, and we proceeded by calculating the exponential suppression of instanton dipoles and provided several dynamical models of said confinement.

It would be interesting to extend the list of models provided in this thesis to also include confinement of instantons akin to particle colour confinement in QCD, thereby potentially extending the list of solutions to the strong CP problem. Given the complexity of such an endeavour, we leave this to future studies.

Finally, in chapter 6, we presented the minimal ingredients for obtaining an axion with an exponentially high quality shift symmetry compatible with a post-inflationary scenario, thereby motivating higher-mass axion searches. The minimal mechanism involved the mixing between the phase of a complex scalar and an axion protected by a higher-form shift symmetry, and the resulting axion was dubbed the higher axion. Whilst extra-dimensional theories in general yielded higher axions with realistic decay constants, string theoretic constructions required the consideration of special regions of moduli space. In light of this, understanding moduli stabilisation in such models provides an interesting direction for future work.

In conclusion, the central idea of this thesis is that generalised symmetries can be effectively used to constrain the axion parameter space, from the axion's mass and photon coupling to the axion's cosmological history, and thereby aid in the axion's experimental discovery. The next natural step is to apply generalised symmetries to constrain other facets of Standard Model physics and beyond.

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