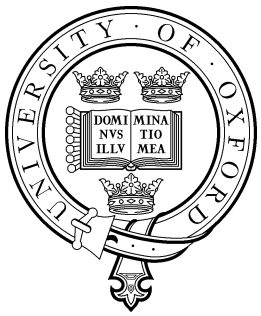
# WELL-POSEDNESS OF DYNAMICS OF MICROSTRUCTURE IN SOLIDS




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*To Özhan*
*for, above all else, supporting me regardless.*

# Abstract

In this thesis, the problem of well-posedness of nonlinear viscoelasticity under the assumptions allowing for phase transformations in solids is considered. In one space dimension we prove existence and uniqueness of the solutions for the quasistatic version of the model using approximating sequences corresponding to the case when initial data takes finitely many values. This special case also provides upper and lower bounds for the solutions which are interesting in their own rights. We also show equivalence of the existence theory we develop with that of gradient flows when the stored-energy function is assumed to be $\lambda$-convex. Asymptotic behaviour of the solutions as time goes to infinity is then investigated and stabilization results are obtained by means of a new argument. Finally, we look at the problem from the viewpoint of curves of maximal slope and follow a time-discretization approach. We introduce a three-dimensional method based on composition of time-increments, as a result of which we are able to deal with the physical requirement of frame-indifference. In order to test this method and distinguish the difficulties for possible generalizations, we look at the problem in a convex setting. At the end we are able to obtain convergence of the minimization scheme as time step goes to zero.

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# Chapter 1

# Introduction

The requirement for a well-posed qualitative mathematical theory for properly formulated dynamics, based on fundamental physical principles, has been recognized for a long time. In order to realize this purpose one needs to have answers to some questions that can be stated generally for any evolution equation associated with a nonelliptic variational integral (see e.g. [12], [46]). This thesis aims to address the following ones within the framework of the evolution of microstructure in solids undergoing phase transformations.

· Does there exist a unique global solution?

· For a given initial state, is there a unique equilibrium state to which the solutions converge as $t \to \infty$?

· On what time scale does the stabilization occur? Do solutions rapidly relax to equilibrium or is the dynamics mainly quasistatic?

In this work we model dynamics of phase transitions using nonlinear viscoelasticity of rate type. In more than one dimension the isothermal case has been studied in various forms by Potier-Ferry [77], [78], Swart & Holmes [92], Rybka [83], [84], Friesecke & McLeod [46], Friesecke & Dolzmann [44], Demoulini [37], Tvedt [95], [96] and Muite [69]. For the one-dimensional case, Andrews [4], Antman & Seidman [8], Ball, Holmes, James, Pego & Swart [18], Dafermos [34], Greenberg, MacCamy & Mizel [48], Kuttler & Hicks [61], MacCamy & Mizel [63], Pego [74] and Yamada [99] have proved existence and uniqueness results for different versions of the model and under different assumptions for the stress. Corresponding results for thermoviscoelasticity can also be found in the work by Zimmer [101] and Racke & Zheng [79]. Differently from all of the work mentioned, we are going to investigate the case of quasistatic nonlinear viscoelasticity, which we believe to be beneficial for the complete understanding of the whole dynamical process. We analyze this case using different methods and under various conditions as described in the sequel.

We start in Chapter 2 by describing the general setting with derivation of the model from the balance law of linear momentum via a constitutive assumption for the stress tensor. We also discuss the approach of phase transformation modelling by considering both the cases of statics and dynamics. In Chapter 3 we prove existence and uniqueness of the solutions in one space dimension and show the relation of the problem with the gradient flow theory. The method we use in this part is a finite-dimensional approximation for our autonomous partial differential equation, which also provides uniform upper and lower bounds for the solutions. In Chapter 5 we look at the three-dimensional problem from the viewpoint of curves of maximal slope following Ambrosio, Gigli and Savaré [3]. We use the method of time-discretization and show that it is possible to ensure the physical requirement of frame-indifference by using the composition of time-increments. In order to get existence of solutions we confine ourselves to the convex case and point out the arguments heavily dependent on this restriction for possible generalizations.

Given a global existence theory, the first point of consideration is the convergence of solutions to equilibrium states as time goes to infinity. This issue becomes even more interesting and challenging especially when the free energy does not attain a minimum as in the case of viscoelastic dynamics. For the one-dimensional setting of our model, the question of stability has been asked by Ball et al. [18], Andrews & Ball [5], Friesecke & McLeod [46], Pego [74], [71], [75], Greenberg, MacCamy & Mizel [48] and Kalies [57], and different answers have been given depending on the assumptions made. Some conclude that no dynamic solution realize global minimizing sequences, whereas some show that all solutions approach the equilibrium set weakly. In Chapter 4 we look at the asymptotic behaviour of the solutions for the one-dimensional quasistatic problem for which we have obtained a global existence theory. We first introduce a new argument and prove stability of solutions as time goes to infinity. Then, using this argument, we reprove the results of Andrews & Ball [5] and Novick-Cohen & Pego [71] for a specially chosen form of the stress, which is compatible with the physical assumptions we make. Consequently, our method sheds new light on the asymptotic behaviour of solutions for the full dynamical model.

# Chapter 2

# Problem setting

## 2.1 Introduction

Before getting to the derivation of our model from different perspectives, we give some general information about nonlinear viscoelasticity and review the description of the deformation of an elastic body and related notions in continuum mechanics.

### 2.1.1 Nonlinear Viscoelasticity

There are some solids in nature which experience deformations that are not elastic. Examples of such materials are metals at certain temperatures and more familiar ones, certainly, are plastics.

We explain a phenomenon observed in some solids by the following experiment suggested by Spencer [90]. Firstly, let us take a solid rod with a certain length. Suppose that we hang a weight on the end of it and wait for a certain period of time. If we measure the length of the rod during this time, we will find out that it gradually extends. How much it extends depends on the material that the rod is made of. If now we remove the weight, we will see that the rod slowly gets shorter again. After a long enough time, it might or might not go back to its original length. This again depends on the particular material we are using. This experiment demonstrates rather strikingly that in some circumstances the way in which a body deforms is determined not only by the size of the forces which are applied to it, but also by the length of time they are allowed to act. This phenomenon is the so-called *viscoelasticity*. In accordance with the effect of time in their mechanical behaviour, viscoelastic materials can also be called time-dependent materials. The experimental study of such materials is more difficult compared to time-independent ones, basically because one cannot keep time constant or eliminate it during an experiment (cf. [42]).

Viscoelasticity combines elasticity and viscosity. Consequently, not only solids but also fluids can possess such a property. However, the way they respond is quite different. In particular, the response of a fluid to a given deformation would be the same starting from any two states, whereas a solid would respond differently, for example, in its initial configuration and after being deformed. More generally, for solids, pure strains might change the behaviour of the material while rotations might not have an influence (cf. [94]) (see also Section 2.3.1).

Finally, it is also worth noting the role of the nonlinear theory as opposed to a linear one. It is indisputable that classical linear theories of solid mechanics can be applied to a larger class of materials simply because many different nonlinear constitutive equations can actually possess the same linear first approximation (see e.g. [94]). However, most natural processes are nonlinear and therefore nonlinear theories are able to provide much more accurate explanations for the behaviour of materials. In our analysis, the time-dependent behaviour of viscoelastic materials are expressed by nonlinear constitutive equations, which include not only the stress and strain variables, but also some information about the history of the motion (see Section 2.2.3).

### 2.1.2 Three-dimensional Continuum Mechanics

In continuum mechanics, in order to formulate problems one can use either material coordinates as independent variables, which corresponds to the "Lagrangian" description, or spatial coordinates, which corresponds to the "Eulerian" description of the problem. In the material description, we fix our attention on a given material particle of the solid and study how it moves. In the spatial description, on the other hand, the focus is on a particular point in space. For fluids, it is common to use the Eulerian description since the governing equations take a relatively simple form. For solids, however, it is more convenient to use Lagrangian description (see e.g. [91], [14]). Even though it is possible to convert a problem described in Lagrangian coordinates into one with an Eulerian description, the former is commonly accepted as a natural choice for nonlinear problems of solids (cf. [6]). Definitions and notations we use in the sequel are mostly those of Antman [7].

For the purpose of the classical mechanics we assume that a three-dimensional body can be informally defined as a set that can occupy regions of $\mathbb{R}^3$, that has volume, that has mass, and that can maintain forces. The elements of a body are called *material points*. We distinguish one configuration of the body, $\Omega \subset \mathbb{R}^3$, and call it the *reference configuration*. This configuration can be a natural stress-free

configuration as well as one which is occupied by the body at a certain instant of time. It might even be some ideal configuration that is unlikely to be occupied by the body. Using the Lagrangian description, we denote the position of a point $x \in \Omega$ at time $t$ in a typical deformed configuration by $y(x,t)$. A *motion* of a body is a one-parameter family $\{y(\cdot,t),\, t \in \mathcal{I}\}$ of its configurations $x \mapsto y(x,t) \in \mathbb{R}^3$, where $\mathcal{I}$ is an interval in $\mathbb{R}$. The gradient of $y$ at $x$ at time $t$ is written as $Dy(x,t)$, and can be identified with the $3 \times 3$ matrix of partial derivatives

$$(Dy)_{i\,\alpha} \;=\; y_{i,\alpha} := \frac{\partial y_i}{\partial x_\alpha}.$$

In the literature, it is called the *deformation gradient*. The elastic energy corresponding to the deformation $y$ is defined as

$$I(y) \;=\; \int_\Omega W(Dy)\,dx, \tag{2.1.1}$$

where $W\colon \mathbb{M}^{3\times 3} \to [0,\infty]$ is the free energy (or stored-energy) function and $\mathbb{M}^{3\times 3}$ denotes the space of real $3 \times 3$ matrices. Unless stated otherwise, we will make the following convention that the initial free energy is finite:

$$\int_\Omega W(Dy(x,0))\,dx \;<\; \infty. \tag{2.1.2}$$

We assume that the body is *homogeneous* so that the mechanical response of the body, which is the stress corresponding to a given strain, is independent of the point $x$. As noted by Ball [11], this is more restrictive than saying that $\Omega$ is occupied by the same material at each point, since it is possible to have some pre-existing stresses.

The first point at issue is related to the choice of functions which we should use as a suitable mathematical model for deformations.

To be physically acceptable we require that for (almost) every $t$, the actual position field $y(\cdot,t)$ is injective, in other words, the deformation $y$ is *invertible* in $\Omega$. We make this assumption to avoid interpenetration of matter so that two distinct material points cannot simultaneously occupy the same position in space. Nevertheless, we can still allow some cases where, for example, self-contact occurs on the boundary (see [11] for more information). Invertibility is a global restriction on $y(\cdot,t)$. A related local requirement is that $y(\cdot,t)$ is *orientation preserving*, i.e. that

$$\det Dy(x,t) \;>\; 0 \tag{2.1.3}$$

for (almost) all $x$ and for (almost) all $t$. If $y \in C^1(\Omega,\mathbb{R})$, by the Inverse Function Theorem, (2.1.3) implies local invertibility. However, local invertibility does not imply global invertibility (see [11], [14] for examples).

We call

$$C(x,t) := Dy^T\, Dy \tag{2.1.4}$$

the *right Cauchy-Green deformation tensor* at $(x,t)$. It is symmetric and is positive-definite where $Dy$ is nonsingular.

The *displacement* vector $u$ of a typical particle $x$ at time $t$ is

$$u(x,t) \,=\, y(x,t) \,-\, x.$$

An obvious advantage of the displacement is that it vanishes in the reference configuration. Nevertheless, the notion of deformation is more commonly used in nonlinear elasticity. If all points in a given body experience the same displacement, then neither the shape not the size of the body is changed. In this case we say that it has been given a *rigid body displacement*. Deformation, on the other hand, occurs if there is a relative displacement between the particles of the body (cf. [51]).

**Rotation**

We now show that we can decompose any deformation gradient tensor into a stretch tensor $U$, which describes distortion, followed by a rotation tensor $R$, which describes the orientation. Our main tool in the analysis of the strain will be the following polar factorization theorem for invertible matrices (see e.g. [14], [28], [49] and also [30, pg. 242] for a version for arbitrary positive operators).

**Theorem 2.1.1** (Polar Decomposition Theorem)**.** *Let* $F \in \mathbb{M}^{3\times 3}$, $\det F > 0$. *Then, there exist positive-definite and symmetric matrices* $U, V$ *and* $R \in \mathrm{SO}(3)$ *such that*

$$F \,=\, R\,U \,=\, V\,R. \tag{2.1.5}$$

*These representations (right and left respectively) are unique.*

Setting $F = Dy$ and using (2.1.5), we can rewrite $C$ in (2.1.4) as

$$C \,=\, Dy^T\, Dy \,=\, U^2.$$

Similarly, we have the *left Cauchy-Green strain tensor* defined as

$$B \,=\, Dy\, Dy^T \,=\, V^2.$$

The matrices $U$ and $V$ are called the *right and left stretch tensors* respectively. We denote the set of rotations as

$$\mathrm{SO}(3) \,=\, \{R \in \mathbb{M}^{3\times 3} \colon R^T R = 1,\, \det R = 1\}.$$

## 2.2 Derivation of the model

### 2.2.1 Modelling phase transformations

In accordance with the above mentioned specifications for deformations, we consider a homogeneous solid body with unit density which occupies a region $\Omega \subset \mathbb{R}^3$ in the reference configuration and is subject to a deformation $y : \Omega \to \mathbb{R}^3$. Our objective is to understand the nature of the microstructure that is observed in materials that can undergo phase transformations, resulting in the coexistence of different phases of the material as seen in Figure 2.1.

Some materials like elastic crystals experience a certain class of phase transformations and as a result they possess a combination of different fine-scale spatial domains. Such transformations might be caused by various mechanical interactions including application of some forces, imposition of electric or magnetic fields, or change in their temperatures. The microstructure observed is due to different or differently oriented atomic lattice structures of the crystal and it develops as a consequence of the multi-well form of the energy density (cf. [24], [46]).

A considerable amount of literature has been published in recent years on the presence of such microstructures and its features, most of which follow the approach of minimization of the free energy in continuum models (see e.g. [19], [20], [16], [58], [23], [21]). Although extensive research has been carried out on variational integrals and their roles in modelling microstructures, very few studies exist which adequately cover the dynamic processes by which such microstructural patterns may be created or evolve (see e.g. [46], [74], [83]). In this thesis we want to present some progress in this direction.

In the theory of nonlinear elasticity, nonconvex stored energy functionals are used to model elastic solids undergoing phase transformations. As a result of this lack of convexity of the free energy, the resulting model becomes a partial differential equation of mixed type, hyperbolic and elliptic. For example in one space dimension, the classical equation of motion for the dynamics of phase transitions in elastic bars is

$$y_{tt} = \sigma(y_x)_x, \tag{2.2.1}$$

where the stress $\sigma$ is a nonmonotone, typically cubic-like, function of the strain $y_x$ (cf. [40], [56]). By results of MacCamy & Mizel [63], we know that global solutions for equation (2.2.1), even for smooth initial data, do not exist in general. In particular, as Pego [74] explains, when the strain $y_x$ is in the ranges where $\sigma$ is decreasing, the equation becomes elliptic and this makes the initial value problem ill-posed. A way

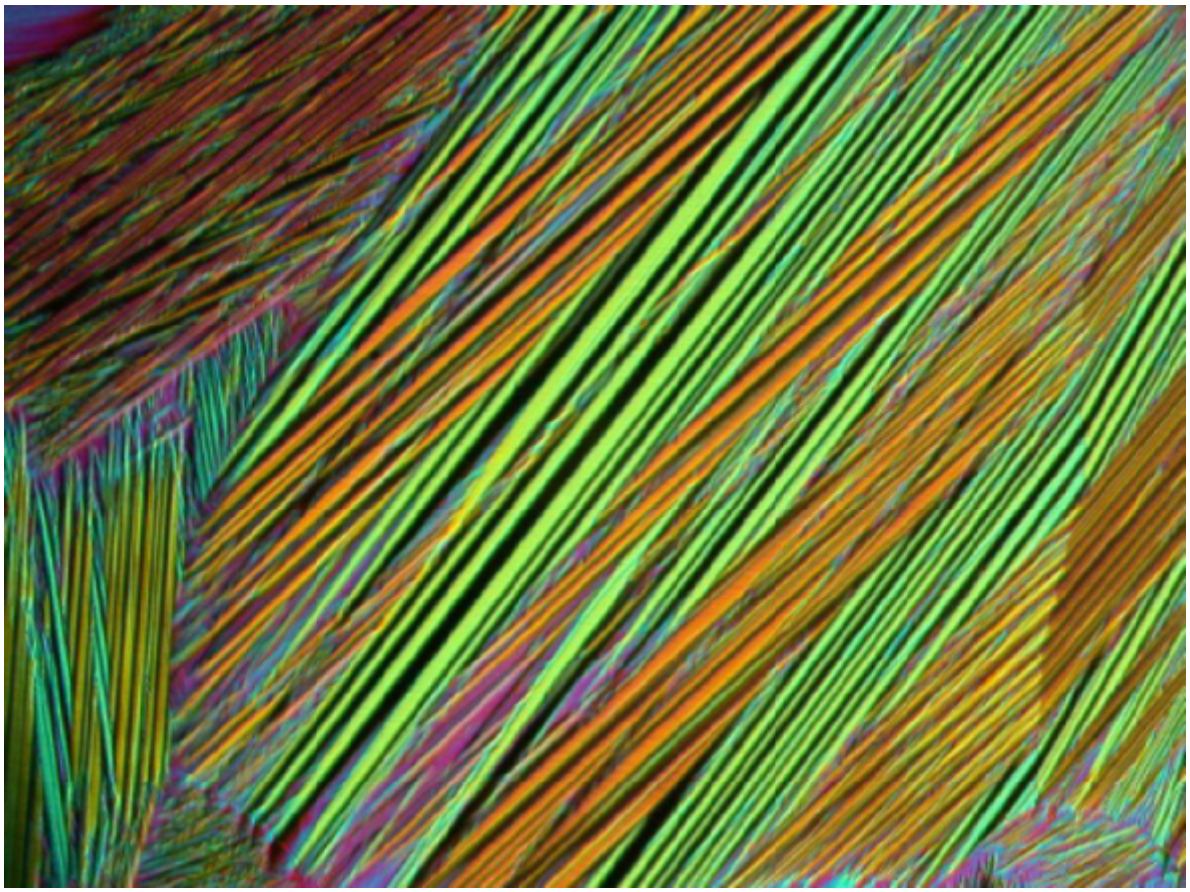


Figure 2.1: Microstructure in CuZnAl (M. Morin)

of overcoming this problem is to consider a physically relevant regularization which can be done by adding capillarity or viscosity effects into the equation.

In this thesis, we focus on the latter method in which the stress includes a viscosity term proportional to the strain rate $y_{xt}$, a general form of which can be written as

$$y_{tt} - \sigma(y_x, y_{xt})_x = 0. \tag{2.2.2}$$

This equation can be thought of as the simplest model of a solid with history dependence, and it has been treated by many authors some of whom are Dafermos [34] and Antman & Seidman [8]. Dafermos [34] proved the existence and uniqueness of the solutions for (2.2.2) under a parabolicity assumption on the stress which ensures that the viscosity is bounded away from zero. He made no assumption on the monotonicity of the stress but the condition on its growth was rather restrictive in the sense that it was suitable for shearing motions of solids but not for longitudinal ones. Moreover, as stated in his article, this growth condition alone was not able to guarantee asymptotic stability of the solutions and a further restriction was necessary. Antman & Seidman

[8], on the other hand, managed to handle the physically natural requirement that in order to produce a total compression an infinite amount of stress is needed. It is difficult to ensure this desired feature basically because it causes the equation to be singular. As a result, they had to impose a number of new restrictions on the constitutive function.

The equation

$$y_{tt} - \sigma(y_x)_x - y_{xxt} = 0, \tag{2.2.3}$$

which is a special case of (2.2.2) was also studied by numerous authors such as Andrews [4], Andrews & Ball [5] and Pego [74]. Andrews [4] considered the problem with two sets of boundary conditions corresponding to the cases when one or both ends of the bar are fixed (see Section 3.1 for more information). He used a fixed point method due to Krasnosel'skii [60] in order to get an existence theory for weak solutions under some modified monotonicity assumptions on the stress $\sigma$. Andrews & Ball [5], on the other hand, mainly focused on the asymptotic behaviour of the solutions as time $t$ goes to infinity. As they explained in their paper, the main purpose of their work was to study the initial boundary-value problem in the case when $\sigma$ is not a monotone increasing function, so that the stored-energy function

$$W(y_x) = \int_0^{y_x} \sigma(z)\, dz$$

is not convex which implies that the equilibrium problem of solving

$$\sigma(y_x(x)) = \text{constant} \tag{2.2.4}$$

has infinitely many roots in general. To see this, one can associate different phases of the material with suitable ranges of the values of the deformation gradient, which in one dimension would be the same as identifying a certain phase with the interval of the values of $y_x$ where $\sigma$ is monotone. If $y_x$ is allowed to have finite discontinuities, then it can jump from one intersection point to another in one equilibrium configuration leading to infinitely many configurations. Ericksen [40] analyzed this problem in the context of one-dimensional equilibrium theory of elastic bars, which he said was an "elementary study" of phase transformations. We discuss the problem of stability for a quasistatic version of (2.2.3) in one space dimension in Chapter 4.

Pego [74] provided a simplified existence theory for (2.2.3) associated with mixed type boundary conditions. His analysis was based on the theory of abstract semilinear parabolic equations as presented by Henry [53]. His main tool was the transformation of the problem into a semilinear system coupling a parabolic partial differential

equation to an ordinary differential equation, leading to new results for the regularity of the weak solutions as well. He proved that each solution tends strongly to a stationary state asymptotically in time and showed that coexistence of phases in stable states might actually be true.

Ball, Holmes, James, Pego and Swart [18] also provided some models in order to investigate the dynamical behaviour of small scale microstructure observed during phase transformations. They were essentially motivated by the mechanical systems that dissipate energy as time $t$ increases and their models were constructed in such a way that the underlying energy functions have minimizing sequences that converge weakly to nonminimizing states rather than attaining a minimum. In one of the models they introduced, the evolution was governed by a nonlinear partial differential equation closely related to (2.2.2) with a cubic-like stress-strain function (see also [47] and [25]). They not only gave existence and uniqueness results, but also study stability of solutions as time $t$ goes to infinity.

In the case of three space dimensions, an equivalent model for (2.2.3) is the equation of viscoelasticity of Kelvin-Voigt type, namely,

$$y_{tt} \;=\; \mathrm{Div}\,\big(DW(Dy) \;+\; Dy_t\big). \tag{2.2.5}$$

This equation models the isothermal case and can be derived from the law of linear momentum by a constitutive assumption for the stress tensor (see Section 2.2.3 for derivation of a more general model) and is therefore coherent with thermomechanics. A theory of existence for (2.2.5) is available by Rybka [83], [84] and Friesecke & Dolzmann [44]. However, the corresponding viscous stress

$$S(Dy, Dy_t) \;=\; Dy_t$$

is not frame-indifferent, which is one of the properties necessary to exclude physically unreasonable effects (see Section 2.3).

The most general form of nonlinear viscoelasticity of strain-rate type can be written as

$$y_{tt} \;-\; \mathrm{Div}\;\, DW(Dy) \;-\; \mathrm{Div}\;\, S(Dy, Dy_t) \;-\; f \;=\; 0 \tag{2.2.6}$$

where the constitutive equation for the stress reads

$$T_R(Dy, Dy_t) \;=\; DW(Dy) \;+\; S(Dy, Dy_t).$$

Equation (2.2.6), complemented with some initial and boundary conditions, is the model we intend to study in this thesis. We show how to derive it from the balance law

of linear momentum in Section 2.2.3 (see (2.2.16)). The only theory for the existence of solutions for this problem with frame indifferent $S(Dy, Dy_t)$ in three dimensions is that of Potier-Ferry [77], [78], who established global existence and uniqueness of solutions for initial data close to a smooth equilibrium for pure displacement boundary conditions. In this work, we study the quasistatic version of (2.2.6) when the applied body force $f$ is assumed to be zero. As explained in more detail in Section 2.4, we introduce a variational method for an existence theory in three dimensions and apply it in a one-dimensional setting. Similarly, in Section 5.3, we introduce another approach in three dimensions using composition of functions in order to deal with the assumption of frame-indifference of the stress and test this method in a convex setting.

A different approach to the existence of solutions in elasticity, as Ball [12] points out, is to change the concept of solution by weakening it to that of a measure-valued solution. The unknown, in this case, is a Young measure $\nu_{x,t}$ in appropriate variables and, roughly speaking, it is obtained by passing to the weak limit in a sequence of approximate solutions. The global existence of such solutions for (2.2.6) has been proved by Demoulini [37] by using a variational time-discretization method. The viscoelastic part of the stress she considered was frame-indifferent and she assumed a uniform dissipation condition, which is much weaker then monotonicity. However, she was unable to handle the constraint $\det Dy > 0$, which is another important physical restriction (see (2.1.3)).

The most recent work on the problem (2.2.6) is by Tvedt [96] (see also [95]), in which existence and uniqueness of weak solutions were obtained with mixed boundary conditions and suitable initial data for a potential energy which was a nonconvex function of the strain. The critical hypothesis he made was that the dependence of the stress function on the strain rate be uniformly strictly monotone. We show in Section 2.3.1 (see also [86]) that this hypothesis by itself is not compatible with frame-indifference.

### 2.2.2 A materials science perspective on microstructure in solids

In this section we explain how one can apply nonlinear viscoelasticity to modelling the microstructure of martensite. We consider the statical problem mostly by following the work of Ball & James [19] and Bhattacharya [24] before we pass to dynamics. By doing so, we plan to point out the importance of the potential energy in the consideration of the dynamical process.

#### 2.2.2.1 Statics

A martensitic phase transformation is a solid to solid phase transformation where the structure of the lattice suddenly changes at a certain temperature. *Austenite* is the phase associated with high temperature and the low-temperature phase is called *martensite*. Various materials can undergo martensitic phase transformations. Some of them are metals, alloys and ceramics. Studies on these transformations have proved the importance of their technological implications, some of which are due to the so-called shape-memory effect (see e.g. [24], [69] for more information).

The reason of interest for us in martensitic phase transformations is the observed microstructure they produce. Let us first understand what microstructure really means in the language of materials science. In a martensitic phase transformation, the symmetry of the austenite phase is more than that of the martensite phase. As an example, one can think of a crystal having a square lattice in the austenite phase and a rectangular lattice in the martensite phase. As a consequence of this property, martensite has multiple variants the number of which depends on the change in the symmetry during transformation. Due to various reasons (nucleation events, etc.) the crystal forces the martensite to make a mixture of different variants which gives rise to some patterns at a very small length scale. These characteristic patterns are called *the microstructure of martensite*.

Let us now try to give an energetic interpretation to these configurations. We only consider materials that are single crystals in the austenite state and suppose that the specimen is subject to a deformation $y$. We assume that the stored-energy function $W$ depends on the local change of the lattice which is measured by the deformation gradient. The total energy of the body is then given by (2.1.1) as defined in Section 2.1.2, namely,

$$I(y) = \int_{\Omega} W(Dy)\, dx.$$

We can think of the equilibrium state of the body as a minimizer of this total free energy which suggests that the behaviour of microstructure is completely determined by $W$. It is important to note that the stored-energy function $W$ is allowed to have several potential wells (see Figure 2.2) and this is the main reason for the equilibrium states to have a mixture of phases. This can be made rigorous as follows.

As explained in the celebrated work of Ball & James [19], the understanding of microstructure should be made in terms of minimizing sequences rather than minimizers of the energy. It is very typical of elasticity models for solids which change phase that the minimizing sequences converge weakly to deformations which are not

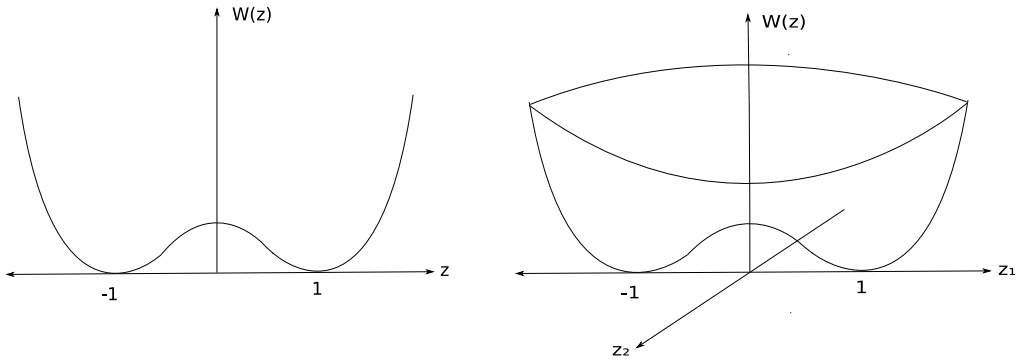



Figure 2.2: A double-well energy density in one and two dimensions

minimizers of the total free energy. In other words, the energy of the weak limit of a sequence of deformations might be greater than the limit of the energy (see [24, Ch. 6], [19] for examples). This implies that the total free energy function is not lower semicontinuous with respect to weak convergence in a suitable Sobolev space $W^{1,p}(\Omega, \mathbb{R}^3)$, $p \geq 1$. That is, the following property does not hold:

$$y_n \rightharpoonup y \text{ in } W^{1,p} \quad \Rightarrow \quad I(y) \leq \lim_{n\to\infty} I(y_n).$$

Such a property is a result of failure of ellipticity of the energy functional which can be associated with the multi-well structure of the energy density. If, on the contrary, the energy of the limit is always smaller than the limit of the energy, then it will not be helpful for the material to make alternating gradients to minimize the energy. We see such a property when the energy density function has a one-well structure and in this case no microstructure is observed.

By well-known results in the calculus of variations, we know that in one-dimensional problems and in problems in higher dimensions where the deformation is scalar, the energy is weakly lower semicontinuous when the energy density is convex (see e.g. [33, Thm. 3.20]). That is,

$$W(\lambda F + (1-\lambda) G) \leq \lambda W(F) + (1-\lambda) W(G)$$

for every $\lambda \in [0,1]$ and every $F$, $G \in \mathbb{R}^3$. In problems in two or higher dimensions where the deformation is vector-valued, the energy functional is weakly lower semicontinuous if the energy density $W$ is *quasiconvex*, which is the following concept introduced by Morrey [68];

$$W(M) \leq \inf_{y|_{\partial\Omega}=Mx} \frac{1}{(\text{volume of } \Omega)} \int_\Omega W(Dy)\, dV \tag{2.2.7}$$

for every matrix $M$. It is very difficult to verify that a given function is quasiconvex because it is not a pointwise condition. One can instead use the notion of *rank-one convexity* which is a weaker condition equivalent to the ellipticity of the Euler-Lagrange equation corresponding to the energy functional (see (2.2.8)).

To sum up, microstructure occurs as a result of coexistence of several phases in a martensitic phase transformation which can be interpreted as $W$ having several local minima. Normally, such $W$ are not convex and thus the functional $I$ is not sequentially weakly lower semicontinuous. As minimizers of the energy might not exist, one is forced to study minimizing sequences instead.

#### 2.2.2.2 Dynamics

As a result of the vanishing of the first variation of the energy functional $I$, any sufficiently smooth minimizer must satisfy the associated Euler-Lagrange equation

$$\text{Div } T_R(Dy) \;=\; 0 \quad \text{in} \quad \Omega, \tag{2.2.8}$$

provided that the energy density $W$ is also smooth enough. As explained in the above section, $W$ is assumed to be nonconvex. Thus, equation (2.2.8), complemented with suitable boundary conditions, typically has a multitude of minimizers corresponding to different phases. This nonuniqueness of the solution can be seen as a result of the fact that the dynamical process which is responsible for selecting from among the many possible equilibrium states, depending on the initial data, the particular one which is preferred by the body, is ignored (see e.g. [47], [56], [92], [1]). Therefore, the corresponding inertial effects should also be included in the model, which can be done by adding the kinetic energy to the energy functional (2.1.1), giving the total energy as

$$I(y, y_t) \;=\; \frac{1}{2}\int_\Omega |y_t|^2\,dx \;+\; \int_\Omega W(Dy)\,dx. \tag{2.2.9}$$

The corresponding equation of motion for (2.2.9) is

$$y_{tt} \;=\; \text{Div } T_R(Dy) \quad \text{in} \quad \Omega, \tag{2.2.10}$$

where the constitutive equation for the first Piola-Kirchhoff stress tensor as a function of the deformation gradient $Dy$ and $x$ is given by

$$T_R(Dy) = DW(Dy).$$

Hence, one should solve (2.2.10) together with suitable boundary conditions. As explained in the previous section in detail, we do not assume $T_R$ to be monotone in

$Dy$ since $W$ is not quasiconvex, and this leads to the loss of ellipticity in the stationary problem (2.2.8). The corresponding feature for the dynamical problem (2.2.10) is the failure of hyperbolicity. As Swart & Holmes [92] explain, because of the hyperbolic nature of the dynamical problem (2.2.10), spatial discontinuities may form in finite time which forces one to study weak solutions that allow for jump discontinuities in the deformation gradient, strain and stress. In a one-dimensional setting, for example, Ericksen [40] discussed deformations of an elastic bar with strain jumps.

The lack of uniqueness for these weak solutions is an indication of incompleteness of the constitutive modelling and there are various possible ways to overcome this problem. The first approach, which is different from the ones we mentioned earlier, is that of constructing more detailed constitutive models which describe thermodynamics of multi-phase materials and the evolution of the microstructure we observe. In the case of nonconvex stored-energy functions $W$, the second law of thermodynamics in the form of the Clausius-Duhem inequality (see (2.2.14)) is not sufficient to provide a unique weak solution, and hence it is necessary to make additional constitutive assumptions, as was done by Abeyaratne & Knowles [1] in the context of one-dimensional isothermal bars (see [92] and references therein for more information).

Another possible method of achieving well-posedness, which is the one we follow, is adding to the stress tensor a higher order regularizing term corresponding to viscosity so that we have

$$\widetilde{T_R}(Dy, Dy_t) \;=\; T_R(Dy, Dy_t) \;+\; Dy_t.$$

As a result, the equation of viscoelasticity becomes (2.2.5), which can be seen as a problem within the theory of dissipative dynamical systems and hence can be analyzed for conditions for long-time existence of solutions, relaxation to equilibrium, structure of the $\omega$-limit set, the existence of a global attractor, etc. A different motivation for studying the regularized problem is the idea of variants of the "viscosity method" in which one can construct weak solutions of (2.2.10) as limits of smooth solutions of the regularized equation. One could also have employed other vanishing higher order quantities such as capillarity or thermal conductivity. Since energy is forced to decrease because of the viscosity, this seems to be a physically natural method for finding local minima of the elastic energy (2.2.9) (cf. [92]).

### 2.2.3 Thermodynamical approach

In this section we derive our model following a thermodynamical approach where we assume that all functions are sufficiently smooth. We mostly refer to the work of Ball [12], [14] (see also [9], [54], [89]).

The fundamental balance laws in thermomechanics are those of the *balance of linear momentum*, which can be written as

$$\frac{d}{dt}\int_E \rho_R\, y_t\, dx \;=\; \int_{\partial E} t_R\, dS \;+\; \int_E \rho_R\, f\, dx, \tag{2.2.11}$$

the *balance of angular momentum*, which is

$$\frac{d}{dt}\int_E \rho_R\, x \wedge y_t\, dx \;=\; \int_{\partial E} x \wedge t_R\, dS \;+\; \int_E x \wedge f\, dx, \tag{2.2.12}$$

and the *balance of energy*, which is

$$\frac{d}{dt}\int_E (\frac{1}{2}\rho_R\, |y_t|^2 + G)\, dx \;=\; \int_{\partial E} t_R \cdot y_t\, dS \;+\; \int_E f \cdot y_t\, dx + \int_E r\, dx - \int_{\partial E} q_R \cdot n\, dS, \tag{2.2.13}$$

where $y = y(x,t)$ is the deformation, $t_R$ the Piola-Kirchhoff stress vector, $\rho_R > 0$ the density in the reference configuration, $f$ the body force, $G$ the internal energy, $q_R$ the heat flux vector and $r$ the heat supply. $E$ denotes an arbitrary open subset of $\Omega$, with sufficiently smooth boundary $\partial E$, and the unit outward normal to $\partial E$ is denoted by $n$. It is worth noting that it is possible to derive the other balance laws from conservation of energy and the physical requirement that its form is the same for different observers (cf. [14]).

In addition to the balance laws, thermomechanical processes are required to obey the Second Law of Thermodynamics, which we assume to hold in the form of the *Clausius-Duhem Inequality*

$$\frac{d}{dt}\int_E \eta\, dx \;\geq\; -\int_{\partial E} \frac{q_R \cdot n}{\theta}\, dS \;+\; \int_E \frac{r}{\theta}\, dx \tag{2.2.14}$$

for all $E$, where $\eta$ is the entropy and $\theta$ the temperature. We have

$$t_R \;=\; T_R\, n,$$

where $T_R$ is the Piola-Kirchhoff stress tensor. The pointwise form of the balance law (2.2.11) is given by

$$\mathrm{Div}\; T_R + \rho_R\, f \;=\; \rho_R\, y_{tt},$$

where we have

$$\big(\text{Div } T_R\big)_i \;=\; \frac{\partial}{\partial x_\alpha}\big(T_R\big)_{i\,\alpha}.$$

The lack of symmetry of $T_R$ suggests introducing another material stress tensor, the *second Piola-Kirchhoff stress tensor* $T$, by

$$T_R \;=\; Dy \cdot T \tag{2.2.15}$$

which is symmetric (see e.g. [7]).

For thermoelastic materials the balance of angular momentum is satisfied as a consequence of the requirement that $T_R$ is frame-indifferent, which can mathematically be expressed as

$$T_R(R\,A,\theta) \;=\; R\,T_R(A,\theta) \quad \text{for all } R \in \text{SO}(3),\;\; A \in \mathbb{M}^{3\times 3}.$$

The equations of isothermal thermoelasticity are obtained by assuming that $\theta(x,t) = \theta_0$ is constant, where $W(A) = T_R(A,\theta_0)$. In the case of thermoviscoelastic materials of strain-rate type (with unit density), on the other hand, we obtain the equation of motion as

$$y_{tt} \;-\; \text{Div } DW(Dy) \;-\; \text{Div } S(Dy, Dy_t) \;-\; f \;=\; 0 \tag{2.2.16}$$

where the constitutive equation for the stress is

$$T_R(Dy, Dy_t) \;=\; DW(Dy) \;+\; S(Dy, Dy_t).$$

The requirement of frame-indifference for $S(Dy, Dy_t)$ takes the form

$$S(Dy, Dy_t) \;=\; Dy\,G(U, U_t), \tag{2.2.17}$$

where $G$ is a symmetric, matrix-valued function of the right stretch tensor $U$ and its time derivative. The function $G$ is closely related to the second Piola-Kirchhoff stress tensor $T$ (see (2.2.15)) as we discuss in more detail in Section 2.3.1.

Equation (2.2.16) is the model we are concerned with in this thesis. We complement it with some initial and boundary values in order to pose our problem. There are some constitutive restrictions we have to impose (one of which is (2.2.17)) in order to exclude physically unrealistic effects. We devote the next section to a detailed explanation of these assumptions.

## 2.3 Constitutive Assumptions

Each material exhibits some characteristic properties by which it is distinguished from other materials. Therefore, not all models are appropriate for every material as physically natural problems. A *constitutive assumption* is a condition which characterizes the material properties of the given body and restricts the possible dynamical processes admissible for it. These assumptions "insure that physically reasonable problems have physically reasonable solutions" (Truesdell & Noll [94]). As Ciarlet [28] points out, constitutive assumptions might reflect natural realities as in the case of material frame-indifference (see Section 2.3.1), they might be purely mathematical conditions as in the case of quasiconvexity (see (2.2.7)), or they might be a combination of the two aspects as in the case of coerciveness condition on the energy density function expressing the behaviour of large strains (see Section 2.3.2). We will express these assumptions in the form of some functional relations for the stress tensor and the energy density function.

### 2.3.1 Frame-indifference

The mechanical behaviour of materials is governed by some general principles one of which is the principle of frame-indifference. As a general axiom in physics, it states that the response of a material must be independent of the observer (see e.g. [94, pg. 36], [28, pg. 100], [64, pg. 194]) In particular, it restricts the form of the constitutive functions and thus plays an important role in nonlinear continuum mechanics. Other than this, as Šilhavý [89] explains, frame-indifference has a theoretical role as well, which is basically forming a link between the general dynamical statements (e.g. the equation of balance of energy) and the specific continuum dynamical concepts (e.g. the equations of balance of linear and angular momentum).
We state it as follows:

> *The Principle of Frame-Indifference (Objectivity)*: Constitutive functions are invariant under rigid motions.

Let us now express this principle as a mathematical statement. First of all, we note that a change of observer (or equivalently the orthogonal basis in which the observable quantity is computed) can be seen as application of rigid-body motions on the current configuration (see [54, Sec. 5.2] for a detailed explanation). Let us first give the definition of a rigid-body motion (or rigid-body deformation).

**Definition 2.3.1** ([7], pg. 453)**.** *If a body undergoes a motion $q$, then a motion differing from $q$ by a rigid motion relative to a different clock is given by*

$$\tilde{q}(x,\tilde{t}) \;=\; c(t) + R(t)\, q(x,t),\;\; \tilde{t} = t + a,\;\; c \in \mathbb{R}^3,\, a \in \mathbb{R},\, R \in \mathrm{SO}(3),$$

*for each point $x \in \Omega$ and time $t$.*

In other words, a rigid-body motion consists of a translation and a rotation. In each of these motions, the relative positions of the points of the material remain the same. As the deformation gradient is not effected by the translations of the origin, if a body undergoes the motion $\tilde{q}$ in Definition 2.3.1, the corresponding expression for the stress becomes

$$\tilde{T}_R(x,\tilde{t}) \;=\; R(t)\, T_R(x,t). \tag{2.3.1}$$

A formal mathematical statement can be given by the following result we prove (see also [70] and [7] for a more general statement regarding the history of motion). We make the abbreviation $F = Dy$ and denote the derivative of $F$ with respect to time as $\dot{F}$.

**Lemma 2.3.1.** *Any frame-indifferent stress tensor $T_R(F,\dot{F})$ can be written as*

$$T_R(F,\dot{F}) \;=\; R\, T_R(U,\dot{U}), \tag{2.3.2}$$

*where $R$ and $U$ are as in Theorem 2.1.1.*

*Proof*: By Theorem 2.1.1 we immediately get

$$\begin{aligned}
R^T\, T_R(F,\dot{F}) &= T_R(R^T F, \widehat{R^T F}) \\
&= T_R(R^T R\, U, \dot{R}^T F + R^T \dot{F}) \\
&= T_R(U, \dot{R}^T R\, U + R^T(\dot{R} U + R\dot{U})) \\
&= T_R(U, \dot{R}^T R\, U + R^T \dot{R} U + R^T R\dot{U}) \\
&= T_R(U, (\dot{R}^T R + R^T \dot{R})\, U + \dot{U})) \\
&= T_R(U, \dot{U})
\end{aligned}$$

as required. Q.E.D

We can obtain a more convenient form of (2.3.2) by using the second Piola-Kirchhoff stress tensor which we introduced in (2.2.15) (see [7, pg. 455] for a similar argument):

$$\begin{aligned}
T(F,\dot{F}) &= F^{-1}\, T_R(F,\dot{F}) \\
&= U^{-1}\, R^{-1}\, R\; T_R(U,\dot{U}) \\
&= U^{-1}\, T_R(U,\dot{U}) \\
&=: G(C,\dot{C}).
\end{aligned}$$

Thus we have

$$T_R(F, \dot{F}) \;=\; F\, G(C, \dot{C}). \tag{2.3.3}$$

This result shows that the frame-indifferent constitutive equations for materials of strain-rate type take the form (2.2.17) in Section 2.2.3.

Finally, it is worth clarifying that rotations are involved in both material symmetry and frame-indifference. However, they act differently. More precisely, in material symmetry, the rotation acts in the reference configuration and in frame-indifference, the rotation acts in the deformed configuration. Therefore, it is not possible to obtain one variant by rotating another. In other words, given symmetric matrices $U_1$ and $U_2$, it is not possible to find a rotation $R$ such that $RU_1 = U_2$, since this would be inconsistent with the uniqueness property stated in the Polar Decomposition Theorem (see Section 2.1.2)(cf. [24]).

#### 2.3.1.1 Observations on frame-indifference

This section is devoted to some trivial but crucial observations we make on frame-indifference.

**Lemma 2.3.2.** *Any frame-indifferent stress $T_R(F, \dot{F})$ should satisfy*

$$T_R(F, \dot{F}) : \dot{F} \;=\; \frac{1}{2}\, G(C, \dot{C}) : \dot{C}.$$

*Proof*: We know by (2.3.3) that any frame-indifferent $T_R(F, \dot{F})$ takes the form

$$T_R(F, \dot{F}) \;=\; F\, G(C, \dot{C}).$$

Therefore, using the fact that $G(C, \dot{C})$ is symmetric, we get

$$\begin{aligned} T_R(F, \dot{F}) \colon \dot{F} \;&=\; F\, G(C, \dot{C}) \colon \dot{F} \\ &=\; G(C, \dot{C}) \colon F^T \dot{F} \;=\; G(C, \dot{C}) \colon \dot{F}^T F \\ &=\; G(C, \dot{C}) \colon \frac{1}{2}\left(\dot{F}^T F + F^T \dot{F}\right) \\ &=\; \frac{1}{2}\, G(C, \dot{C}) \colon \dot{C} \end{aligned}$$

as required. Q.E.D

**Lemma 2.3.3.** *The condition*

$$G(C, \dot{C}) : \dot{C} \;\geq\; \gamma\, |\dot{F}|^2, \;\; \gamma > 0 \;\; \textit{a constant}, \tag{2.3.4}$$

*contradicts frame-indifference.*

*Proof*: Assume for contradiction that there exists a frame-indifferent $T_R$ satisfying (2.3.4). By Lemma 2.3.2, we have

$$T_R(F,\dot{F}) : \dot{F} \geq \gamma\,|\dot{F}|^2 \quad \Leftrightarrow \quad G(C,\dot{C}) : \dot{C} \geq 2\,\gamma\,|\dot{F}|^2.$$

Choosing $F = R(t) = \exp(Kt) \in \mathrm{SO}(3)$, where $K$ is skew, we get

$$\dot{F} = K\,\exp(Kt) \;\text{ and }\; |\dot{F}|^2 = |K|^2 \neq 0.$$

However,

$$C = F^T F = R^T R = 1 \;\Rightarrow\; \dot{C} = 0$$

giving a contradiction. Q.E.D

**Remark 2.3.1.** *The condition*

$$G(C,\dot{C}) : \dot{C} \geq \gamma\,|\dot{C}|^2,\;\; \gamma > 0 \;\; \textit{a constant},$$

*does not contradict frame-indifference as can be seen easily by choosing* $G(C,\dot{C}) = \dot{C}$ *in (2.3.4).*

In contradiction to the claim of Tvedt [96, Sec. 1.4] we have that

**Lemma 2.3.4.** *The assumption*

$$(T_R(F,\dot{F}) - T_R(F,\dot{\tilde{F}})) : (\dot{F} - \dot{\tilde{F}}) \geq \gamma\,|\dot{F} - \dot{\tilde{F}}|^2\,,\; \gamma > 0 \tag{2.3.5}$$

*is incompatible with frame-indifference.*

*Proof*: If, for contradiction, the claim was true, then there would exist a frame-indifferent $T_R$ satisfying (2.3.3) so that (2.3.5) would give

$$(F\,G(C,\dot{C}) - F\,G(C,\dot{\tilde{F}}^T F + F^T\dot{\tilde{F}})) : (\dot{F} - \dot{\tilde{F}}) \geq \gamma\,|\dot{F} - \dot{\tilde{F}}|^2. \tag{2.3.6}$$

Let us define

$$A := (G(C,\dot{C}) - G(C,\dot{\tilde{F}}^T F + F^T\dot{\tilde{F}}))$$

so that we get

$$
\begin{aligned}
(F\,G(C,\dot{C}) - F\,G(C,\dot{\tilde{F}}^T F + F^T\dot{\tilde{F}})) : (\dot{F}-\dot{\tilde{F}}) &= A \,:\, F^T(\dot{F}-\dot{\tilde{F}}) = \\
&= \frac{1}{2}\left[A + A^T\right] :\, F^T(\dot{F}-\dot{\tilde{F}}) \\
&= \frac{1}{2}\left[A : F^T(\dot{F}-\dot{\tilde{F}}) + A^T : F^T(\dot{F}-\dot{\tilde{F}})\right] \\
&= \frac{1}{2}\left[A : F^T(\dot{F}-\dot{\tilde{F}}) + A : (\dot{F}-\dot{\tilde{F}})^T F\right] \\
&= \frac{1}{2}\,A \,:\, \left[F^T(\dot{F}-\dot{\tilde{F}}) + (\dot{F}-\dot{\tilde{F}})^T F\right] \\
&= \frac{1}{2}\,A \,:\, \left[F^T\dot{F} - F^T\dot{\tilde{F}} + \dot{F}^T F - \dot{\tilde{F}}^T F\right] \\
&= \frac{1}{2}\,A \,:\, \left[\dot{C} - (F^T\dot{\tilde{F}} + \dot{\tilde{F}}^T F)\right].
\end{aligned}
$$

Therefore, (2.3.6) is now equivalent to

$$
\frac{1}{2}\left(G(C,\dot{C}) - G(C,\dot{\tilde{F}}^T F + F^T\dot{\tilde{F}})\right) : \left(\dot{C} - (F^T\dot{\tilde{F}} + \dot{\tilde{F}}^T F)\right) \,\geq\, \gamma\,|\dot{F}-\dot{\tilde{F}}|^2.
$$

However, for any given $G$, we can choose $F = I$ in this inequality and obtain

$$
\frac{1}{2}\left(G(1,\dot{F}^T + \dot{F}) - G(1,\dot{\tilde{F}}^T + \dot{\tilde{F}})\right) : \left(\dot{F}^T + \dot{F} - \dot{\tilde{F}} - \dot{\tilde{F}}^T)\right) \,\geq\, \gamma\,|\dot{F}-\dot{\tilde{F}}|^2.
$$

Choosing $\dot{F} = 0$ now gives

$$
\frac{1}{2}\left(G(1,0) - G(1,\dot{\tilde{F}}^T + \dot{\tilde{F}})\right) : \left(-\dot{\tilde{F}} - \dot{\tilde{F}}^T)\right) \,\geq\, \gamma\,|-\dot{\tilde{F}}|^2.
$$

Finally, choosing $\dot{\tilde{F}}$ to be a nonzero and skew matrix makes the left-hand side vanish whereas the right hand side remains positive. This gives a contradiction proving the claim. Q.E.D

Contradicting a claim of Antman [7] we have that

**Lemma 2.3.5.** *The following statement is incompatible with frame-indifference.*

$$
\left(T_R(F,\dot{F}+\dot{H}) \,-\, T_R(F,\dot{F})\right) \,:\, \dot{H} \,>\, 0\,, \quad \forall\,\dot{H}\neq 0, \quad \forall\,\dot{F}. \tag{2.3.7}
$$

*Proof*: If, for contradiction, the claim was true, then there would exist a frame-indifferent stress tensor $T_R$ satisfying (2.3.3) and (2.3.7) would be equivalent to

$$
\left\{F[G(F^TF,(\dot{F}+\dot{H})^T F + F^T(\dot{F}+\dot{H})) - G(F^TF,\dot{F}^TF + F^T\dot{F})]\right\} : \dot{H} \,>\, 0,
$$

$\forall\ \dot{H}\neq 0$ and $\forall\ \dot{F}$.

Taking $F = I$ in this inequality gives

$$\Big\{G(1,(\dot{F}+\dot{H})^T+(\dot{F}+\dot{H}))-G(1,\dot{F}^T+\dot{F})\Big\} : \dot{H} \ > \ 0.$$

Letting $\dot{F}$ be skew reduces it further to

$$\Big\{G(1,\dot{H}^T+\dot{H})-G(1,0)\Big\} : \dot{H} \ > \ 0.$$

We can choose $\dot{H}$ to be a nonzero and skew matrix which will make the left-hand side vanish, giving a contradiction. Q.E.D

### 2.3.2 Assumptions on the energy density function

As in (2.1.1), $W$ is the stored-energy function

$$W : \mathbb{M}^{3\times 3} \to [0,\infty]$$

where, as explained by Ball [11], the assumption that $W \geq 0$ is made simply for convenience since it is natural to assume that $W$ is bounded below, and adding a constant to $W$ does not change the problem. There are natural restrictions that a physically admissible energy density $W$ must verify. Among those restrictions (see e.g. [12], [73]), the following are the ones we need:

- **Frame-indifference** : The frame-indifference of $W$ takes the form

$$W(Q\,F) = W(F) \quad \text{for all rotations} \ \ Q \in \mathrm{SO}(3) \ \text{ and } \ F \in \mathbb{M}^{3\times 3}. \tag{2.3.8}$$

  This property states that the energy is not changed as a result of rotations of bodies.

- **Behaviour for extreme deformations** : Another restriction is related to large strains which reflects the idea that infinite stress is associated with extreme strains. We can express this property by the following assumptions on the stored energy function $W$:

$$W(F) \ \to \ \infty \quad \text{if} \quad \det F \ \to \ 0_+, \tag{2.3.9a}$$

$$W(F) \ \geq \ c\,(|F|^p - 1), \quad F \in \mathbb{M}^{3\times 3}, p > 1, c > 0. \tag{2.3.9b}$$

  Assumption (2.3.9a) physically means that an infinite amount of energy is required to compress the material to zero volume. Condition (2.3.9b) is the

*coerciveness inequality* for $W$. In consistency with the orientation preserving condition (2.1.3), we also make the following convention.

$$W(F) = \infty \quad \text{if} \quad \det F \leq 0. \tag{2.3.10}$$

Thus $W$ is continuous with respect to the natural topology on $[0, \infty] = [0, \infty) \cup \{\infty\}$ (cf. [11]).

We immediately have the following remarks (see e.g. [73]):

**Remark 2.3.2.** *Finite upper bounds on $W$ are not compatible with properties (2.3.9a) and (2.3.10).*

**Remark 2.3.3.** *The exponent $p$ in (2.3.9b), in particular its size relative to the dimension, is very important as it determines the Sobolev space to work in.*

## 2.4 Introducing the Quasistatic Case

Quasistatic problems in mechanics arise when the system observed evolves slowly in time. In this case the system is observed over a long time scale and the inertial terms in the equations of motion become negligible. This is never exact in real processes, but in many systems dissipative forces beat the acceleration term and the quasistatic approximation is useful, even though neither mass nor velocity is necessarily small (see e.g. [65], [87], [76]). In this work we investigate the quasistatic case of our model, which we believe is the key step towards the study of the full dynamics.

### 2.4.1 Three-dimensional setting

The quasistatic version of the differential equation (2.2.16) is

$$\text{Div } (DW(F) + S(F, \dot{F})) = 0.$$

Equivalently, putting $T_R(F, \dot{F}) = DW(F) + S(F, \dot{F})$ and $H = \dot{F}$, we can write

$$\text{Div } T_R(F, H) = 0. \tag{2.4.1}$$

We want to develop an existence theory for (2.4.1). One way of doing this is by proving that, for given $F$, the equation (2.4.1) has a unique solution $H$. Once we have $H$, we can write it as a general functional of $F$, given over a time interval $(a, b)$, as

$$\{H(t)\}_{t\in(a,b)} = f\Big(\{F(t)\}_{t\in(a,b)}\Big). \tag{2.4.2}$$

The constitutive assumptions we make on the stress $T_R$ (see Section 2.3) determine the properties of $f$. Using these properties, we can recover given $F$ by, for instance, a fixed point argument.

We adopt a variational version of this general method. In order to obtain (2.4.1), we consider the functional

$$I(H) \,=\, \int_\Omega \big(DW(F)\,H \,+\, \Psi(F,H)\big)\,dx. \tag{2.4.3}$$

The corresponding Euler-Lagrange equation for (2.4.3) is

$$\mathrm{Div}\,\left(DW(F) \,+\, \frac{\partial \Psi}{\partial H}(F,H)\right) \,=\, 0, \tag{2.4.4}$$

and setting

$$S(F,\dot F) = \frac{\partial \Psi}{\partial H}(F,H)$$

gives (2.4.1). Assuming that $\Psi$ is strictly convex in $H$, the integral (2.4.3) has a unique minimizer $H$, which depends on the given $F$. Then, by (2.4.4) one can write this $H$ as a functional of $F$ as in (2.4.2) and, as mentioned before, using the properties of $f$ resulting from the assumptions made, $F$ can be recovered.

The fundamental difficulty for our problem in three space dimensions, as we explained in Section 2.3, is to have a frame-indifferent $S(F,H) = \Psi_H(F,H)$. In order to get a minimizer for (2.4.3), we also need a convexity assumption on $\Psi$. In the following section we give two possible choices of the dissipation potential $\Psi$ which satisfy the necessary convexity condition and at the same time provide frame-indifferent $S$. As these potentials are interesting not only for the quasistatic problem but also for many other applications including the full dynamical model (see e.g. [37]), we devote a separate section to them.

## 2.5 Dissipation Potentials

**Lemma 2.5.1.** *Let*

$$\Psi_1(F,H) \,=\, \frac{1}{4}\,|F^T\,H + H^T F\,|^2. \tag{2.5.1}$$

*Then, $\Psi_1(F,H)$ is convex in $H$, and*

$$S_1(F,\dot F) \,:=\, \frac{\partial \Psi_1}{\partial \dot F}\,(F,\dot F) \,=\, F(\dot F^T\,F \,+\, F^T\,\dot F) \,=\, F\,\dot C,$$

*which is frame-indifferent.*

*Proof*: The convexity of $\Psi_1(F,H)$ follows immediately from the fact that it is a nonnegative quadratic form in $H$. We now show that $S_1(F,\dot{F}) = F\,\dot{C}$. We have

$$\begin{aligned}
\frac{\partial \Psi_1}{\partial H}\,(F,H) \;&=\; \frac{1}{4}\,\frac{\partial}{\partial H_{k\beta}}\,|F^T\,H + H^T F\,|^2 \;= \\
&=\; \frac{1}{2}\,(F_{\alpha i}\,H_{\alpha j} + H_{\alpha i}\,F_{\alpha j})\,(F_{\alpha i}\,\delta_{k\alpha}\,\delta_{j\beta} + \delta_{k\alpha}\,\delta_{i\beta}\,F_{\alpha j}) \\
&=\; \frac{1}{2}\,\big(F^T H + H^T F\big)_{ij}\,\big(F_{ki}\,\delta_{j\beta} + \delta_{i\beta}\,F_{kj}\big) \\
&=\; \frac{1}{2}\,\big((F^T H + H^T F)_{i\beta}\,F_{ki} + (F^T H + H^T F)_{\beta j}\,F_{kj}\big) \\
&=\; \frac{1}{2}\,\big((F^T H + H^T F)^T_{\beta i}\,F^T_{ik} + (F^T H + H^T F)_{\beta j} F^T_{jk}\big) \\
&=\; \frac{1}{2}\,\big(((F^T H + H^T F)^T\,F^T)_{\beta k} \;+\; ((F^T H + H^T F)\,F^T)_{\beta k}\big) \\
&=\; \frac{1}{2}\,\Big[\big((F^T H + H^T F)^T + (F^T H + H^T F)\big)\,F^T\Big]^T_{k\beta} \\
&=\; \frac{1}{2}\,\Big(\big(H^T F + F^T H + F^T H + H^T F\big)F^T\Big)^T \\
&=\; \big((F^T H + H^T F)\,F^T\big)^T \;=\; F\,\big(F^T H + H^T F\big)^T \\
&=\; F\,(H^T F + F^T H).
\end{aligned}$$

As $\dot{C} \;=\; \dot{F}^T F + F^T \dot{F}$, putting $H = \dot{F}$ above proves the claim. Frame-indifference of $S_1(F,\dot{F})$ follows immediately from (2.3.3). Q.E.D

**Lemma 2.5.2.** *Let*

$$\Psi_2(F,H) \;=\; \frac{1}{4}\,|H\,F^{-1} \;+\; F^{-T}\,H^T\,|^2. \tag{2.5.2}$$

*Then, $\Psi_2(F,H)$ is convex in $H$, and*

$$S_2(F,\dot{F}) \;:=\; \frac{\partial \Psi_2}{\partial \dot{F}}\,(F,\dot{F}) \;=\; (\dot{F}\,F^{-1} \;+ (\dot{F}\,F^{-1})^T)\,F^{-T} \;=\; F\,C^{-1}\,\dot{C}\,C^{-1},$$

*which is frame-indifferent.*

*Proof*: Following the proof in Lemma 2.5.1, one can easily show that $\Psi_2(F,H)$ is convex with respect to $H$. Therefore we skip it here and prove only that $S_2(F,\dot{F}) = F\,C^{-1}\,\dot{C}\,C^{-1}$, whose frame-indifference follows from (2.3.3) again. In order to calculate $S_2$ we adopt a different approach from that of the previous result[1]. We have

$$\Psi_2(F, A+\varepsilon B) = \frac{1}{4}\big|(A+\varepsilon B)\,F^{-1} + F^{-T}\,(A+\varepsilon B)^T\big|^2.$$

[1]I thank Gero Friesecke for this proof, which is much simpler than the original one.

Therefore, we obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi_2(F, A+\varepsilon B) = \\
&\quad = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{4}\big|(AF^{-1}+F^{-T}A^T)+\varepsilon(BF^{-1}+F^{-T}B^T)\big|^2 \\
&\quad = \frac{1}{2}\big(AF^{-1}+F^{-T}A^T : BF^{-1}+F^{-T}B^T\big) \\
&\quad = \frac{1}{2}\big(AF^{-1}+F^{-T}A^T : BF^{-1}\big) + \frac{1}{2}\big(AF^{-1}+F^{-T}A^T : F^{-T}B^T\big) \\
&\quad = \frac{1}{2}\big(AF^{-1}F^{-T}+F^{-T}A^TF^{-T} : B\big) + \frac{1}{2}\big(F^{-1}AF^{-1}+F^{-1}F^{-T}A^T : B^T\big) \\
&\quad = \frac{1}{2}\big(AF^{-1}F^{-T}+F^{-T}A^TF^{-T} : B\big) + \frac{1}{2}\big(F^{-T}A^TF^{-T}+AF^{-1}F^{-T} : B\big) \\
&\quad = \big(AF^{-1}F^{-T}+F^{-T}A^TF^{-T} : B\big).
\end{aligned}$$

This gives

$$\frac{\partial \Psi_2}{\partial A}(F, A) = F^{-T}\, A^T\, F^{-T} + A\, F^{-1}\, F^{-T}$$

and hence

$$\begin{aligned}
\frac{\partial \Psi_2}{\partial \dot{F}}(F, \dot{F}) &= F^{-T}\, \dot{F}^T\, F^{-T} + \dot{F}\, F^{-1}\, F^{-T} \\
&= (FF^{-1})\, F^{-T}\, \dot{F}^T\, (FF^{-1})\, F^{-T} + (FF^{-1})\, (F^{-T}F^T)\, \dot{F}\, F^{-1}\, F^{-T} \\
&= F\, (F^{-1}F^{-T})\, (\dot{F}^T F)\, (F^{-1}F^{-T}) + F\, (F^{-1}F^{-T})\, (F^T\dot{F})\, (F^{-1}F^{-T}) \\
&= F\, C^{-1}\, \dot{C}\, C^{-1}
\end{aligned}$$

as required. Q.E.D

**Remark 2.5.1.** *Strict convexity of $\Psi(F, \cdot)$ is incompatible with frame-indifference (see [37, pg. 332]).*

As a result, we conclude that there exist dissipation potentials $\Psi$ which not only satisfy the convexity condition necessary for the existence of a minimizer of (2.4.3), but also give frame-indifferent $S$ in (2.4.4). This proves that the method we introduced in Section 2.4 is consistent with the constitutive assumptions we make and it should, at least theoretically, give rise to well-posedness of the model we study.

# Chapter 3

# One-dimensional Quasistatic Nonlinear Viscoelasticity

## 3.1 Introduction

In this chapter we investigate well-posedness of one-dimensional quasistatic version of the equation (2.2.16). In one space dimension and when no body force is applied it takes the form

$$y_{tt} = \big(\sigma(y_x) + S(y_x, y_{xt})\big)_x, \quad x \in (0,1),\ t \in [0,T], \tag{3.1.1}$$

where, as defined earlier, $y(x,t)$ is the deformation at time $t$ of a material point at position $x$ in the reference configuration, $W$ is the stored energy density, $\sigma = W'$ is the elastic and $S$ is the viscoelastic part of the Piola-Kirchhoff stress. Equation (3.1.1) also describes the motion of a homogeneous viscoelastic bar.

One might have two sets of boundary conditions for this problem. The first one is

$$y(0,t) = 0 \quad \text{and} \quad (\sigma + S)(1,t) = 0 \tag{3.1.2}$$

in which case the end $x = 0$ of the bar is assumed to be fixed and the end $x = 1$ is stress free. The second set is

$$y(0,t) = 0 \quad \text{and} \quad y(1,t) = \mu > 0 \tag{3.1.3}$$

where $\mu$ is a positive constant, and in this case both ends of the bar are assumed to be fixed with the end $x = 1$ having displacement $\mu$.

For the quasistatic problem, (3.1.1) reduces to

$$\big(\sigma(y_x) + S(y_x, y_{xt})\big)_x = 0. \tag{3.1.4}$$

We are going to analyze (3.1.4) when the viscoelastic part of the stress satisfies $S(y_x, y_{xt}) = y_{xt}$, giving the equation we study as

$$\big(\sigma(y_x) + y_{xt}\big)_x = 0. \tag{3.1.5}$$

In the next section we show that (3.1.5) can also be derived by using the variational approach we introduced in Section 2.4.

## 3.2 The variational approach in one space dimension

In this case, the functional (2.4.3) can be written as

$$I_y(z) = \int_0^1 \left[\sigma(y_x)\, z_x + \Psi(y_x, z_x)\right] dx \tag{3.2.1}$$

and hence (3.1.4) comes out as the Euler-Lagrange equation under sufficient smoothness assumptions where we have $S = \Psi_q(y_x, q)$ and $z = y_t$.

As in the three-dimensional case, if $y$ is given and $\Psi(y_x, \cdot)$ is strictly convex, then the unique minimizer $z$ of (3.2.1) exists in a suitable space. This minimizer will be a functional of the given $y$ and hence can be written as

$$z = f(y).$$

The rest of the analysis is to prove existence of $y$ by solving the initial boundary-value problem

$$\begin{cases} y_t = f(y) \\ y(\cdot, 0) = y_0(\cdot) \\ \text{boundary conditions.} \end{cases}$$

We will consider the special case when

$$\Psi(y_x, z_x) = \frac{1}{2} {z_x}^2$$

which is clearly strictly convex in $z_x$. The Euler-Lagrange equation associated to (3.2.1) in this case is (3.1.5), showing that this method also gives the same equation to study.

In the subsequent sections we will analyze well-posedness of this equation with either of the boundary conditions (3.1.2) or (3.1.3).

## 3.3 The case with one-end-free boundary conditions

In this section we complement the equation (3.1.5) with the boundary conditions (3.1.2). We have

$$\big(\sigma(y_x) + y_{xt}\big)_x = 0,$$

which means that $\sigma(y_x) + y_{xt}$ is constant in $x$. However, by (3.1.2) we also have

$$(\sigma(y_x) + y_{xt})(1,t) = 0$$

meaning that $\sigma(y_x) + y_{xt}$ takes the value 0 at $x = 1$. Therefore, we immediately conclude that

$$\sigma(y_x(x,t)) + y_{xt}(x,t) = 0 \quad \text{for} \quad x \in (0,1).$$

Rewriting this equation in terms of $p := y_x$, the problem we study becomes

$$(\bar{P}) \quad \begin{cases} p_t(x,t) = -\sigma(p(x,t)) & \text{for} \quad x \in (0,1), \\ p(x,0) = p_0(x). \end{cases}$$

By a solution to the initial value problem $(\bar{P})$ we mean a function $p(x,t)$ which is continuous in time satisfying

$$p(x,t) = p_0(x) - \int_0^t \sigma(p(x,s))\, ds$$

for $t \in [0,T]$ pointwise for all $x \in (0,1)$. We have the following result for $(\bar{P})$.

**Theorem 3.3.1.** *Assume that*

(i) $\sigma : (0,\infty) \to \mathbb{R}$ *is locally Lipschitz continuous,*

(ii) $\sigma(p) \to \infty$ *as* $p \to \infty$ *and* $\sigma(p) \to -\infty$ *as* $p \to 0_+$,

(iii) $\displaystyle\int_{p_+ +1}^{\infty} \frac{dz}{\sigma(z)} < \infty$, *where* $p_+$ *is the largest root of* $\sigma$.

*Given any measurable* $p_0(\cdot)$ *with* $p_0(x) > 0$ a.e. *and* $\displaystyle\int_0^1 W(p_0(x))\, dx < \infty$, *there exists a unique solution* $p \in C((0,\infty); L^\infty(0,1))$ *to problem* $(\bar{P})$, *and* $p$ *satisfies*

$$\lim_{t \to 0_+} \int_0^1 W(p(x,t))\, dx = \int_0^1 W(p_0(x))\, dx. \tag{3.3.1}$$

*Furthermore, there exist two continuous functions,* $P_1(t) > 0, P_2(t) < \infty$, *independent of* $p_0$, *such that* $p(x,t) \in [P_1(t), P_2(t)]$ *for almost every* $x \in (0,1)$ *and all* $t > 0$. *As* $t \to \infty$,

$$p(x,t) \to \bar{p}(x) \quad \textit{for a.e. } x \in (0,1),$$

*where* $\bar{p} \in L^\infty(0,1)$ *and* $\sigma(\bar{p}(x)) = 0$ *for a.e.* $x \in (0,1)$.

*Proof*: By fixing $x$, we treat our problem as an autonomous ordinary differential equation in time defined by a vector field on the line. By assumption (iii) we have that the initial data $p_0$ for this equation is well-defined. As $\sigma$ is continuous, by well-known existence results in the theory of ordinary differential equations (see e.g. [52, Ch. 7], [10, pg. 36]), we immediately get the existence of a local solution $p(t) \in C((0,T])$. However, by assumption (i), the nonlinearity $\sigma$ is locally Lipschitz continuous. Therefore, the solution $p(t)$ is unique.

Let us now consider the interval $[\varepsilon, C] \subset (0,\infty)$ where $\varepsilon > 0$ is sufficiently small and $C < \infty$ is sufficiently large. By assumption (ii), it is clear that the direction field associated with $(\bar{P})$ points in the positive direction at $\varepsilon$ and in the negative direction at $C$. This shows that

$$p_0 \in [\varepsilon, C] \quad \Rightarrow \quad p(t) \in [\varepsilon, C] \ \text{ for all } \ t > 0. \tag{3.3.2}$$

Therefore, we can say that $p(t)$ is a global solution. As $x$ is a set of full measure in $(0,1)$, this ordinary differential equation is satisfied classically for $t > 0$. Therefore, with $x$ fixed, we have the solution $p(t) = p(x,t)$ to problem $(\bar{P})$ for a.e. $x \in (0,1)$.

We now show the existence of the universal upper and lower bounds. Let $p_-$ and $p_+$ be the smallest and the largest roots of $\sigma$ respectively, and define

$$h(p) := \int_p^\infty \frac{dz}{\sigma(z)}.$$

Then, by assumptions (i) and (ii), $h$ is continuous, strictly monotonic decreasing on $(p_+, \infty)$ and $h(p) \to 0$ as $p \to \infty$. Define

$$P_2(t) := \max\Big\{p_+ + 1, h^{-1}(t)\Big\}. \tag{3.3.3}$$

Note that by (3.3.2)

$$p_0(x) \le p_+ + 1 \ \Rightarrow \ p(x,t) \le p_+ + 1 \ \text{ for all } t \ge 0.$$

Hence, we have $p(x,t) \le P_2(t)$ if $p_0(x) \le p_+ + 1$. If, on the other hand, $p_0(x) > p_+ + 1$, then we argue as follows. At nonstationary points we can rewrite problem $(\bar{P})$ as

$$\frac{d}{dt}\int_0^t \frac{p_t\,dt}{-\sigma(p)} = 1 \quad \Leftrightarrow \quad \frac{d}{dt}\int_t^0 \frac{p_t\,dt}{\sigma(p)} = 1.$$

By a change of variables we get

$$\frac{d}{dt}\int_{p(x,t)}^{p_0(x)} \frac{dz}{\sigma(z)} = 1 \quad \Rightarrow \quad \int_{p(x,t)}^{p_0(x)} \frac{dz}{\sigma(z)} = t.$$

Using assumption (iii) we obtain

$$t = \int_{p(x,t)}^{p_0(x)} \frac{dz}{\sigma(z)} < \int_{p(x,t)}^{p_0(x)} \frac{dz}{\sigma(z)} + \int_{p_0(x)}^{\infty} \frac{dz}{\sigma(z)} = h(p(x,t)) < \infty$$

so that if $p(x,t) > p_+ + 1$ then $p(x,t) \leq h^{-1}(t)$. This shows that $p(x,t) \leq P_2(t)$ in this case as well. For the lower bound, we define

$$g(p) := \int_0^p \frac{-dz}{\sigma(z)}.$$

Then, by assumptions (i) and (ii) again we have that $g$ is continuous, strictly monotonic increasing on $(0, p_-)$ and $g(p) \to 0$ as $p \to 0_+$. We also define

$$P_1(t) := \min\left\{p_-, g^{-1}(t)\right\}. \tag{3.3.4}$$

If $p_- \leq p_0(x)$, by (3.3.2) we obtain $p_- \leq p(x,t)$ for all $t \geq 0$ so that $P_1(t) \leq p(x,t)$. If, on the other hand, we have $0 < p_0(x) < p_-$, then arguing similarly as before we get

$$t = \int_{p_0(x)}^{p(x,t)} \frac{-dz}{\sigma(z)} < \int_0^{p(x,t)} \frac{-dz}{\sigma(z)} = g(p) > 0.$$

Therefore, if $p(x,t) < p_-$ then $p(x,t) \geq g^{-1}(t)$ proving that $p(x,t) \geq P_1(t)$ in this case as well. As a result we have proved that

$$p(x,t) \in [P_1(t), P_2(t)] \text{ for } t > 0, \text{ a.e. } x \in (0,1), \tag{3.3.5}$$

where $P_1(t)$ and $P_2(t)$ are given by (3.3.4) and (3.3.3) respectively. Besides, we know that $p(\cdot,t)$ is measurable because it is measurable on $\{x \in (0,1)\colon p_0(x) \geq \varepsilon > 0\}$. Therefore $p(\cdot,t) \in L^\infty(0,1)$. Moreover, we have that $p\colon (0,\infty) \to L^\infty(0,1)$ is continuous since on any interval $[T_1, T_2] \subset (0,\infty)$ we have $\sigma(p(x,t))$ bounded which implies that $p_t(x,t)$ is bounded and there exists a constant $C$ such that

$$|p(x,t) - p(x,s)| \leq C\,|t-s| \text{ for all } t, s \in [T_1, T_2].$$

For the behaviour of $p(x,t)$ when $t \to \infty$, we look at the equilibrium points $\bar{p}(x)$ for problem $(\bar{P})$ which clearly satisfy

$$\sigma(\bar{p}(x)) = 0.$$

In other words, the equilibrium points (also called fixed points) for $(\bar{P})$ are the roots of $\sigma$. By the uniform bounds we have just obtained, we know that $p(x,t)$ must stay positive and bounded as $t \to \infty$. Moreover, by a similar argument on the direction

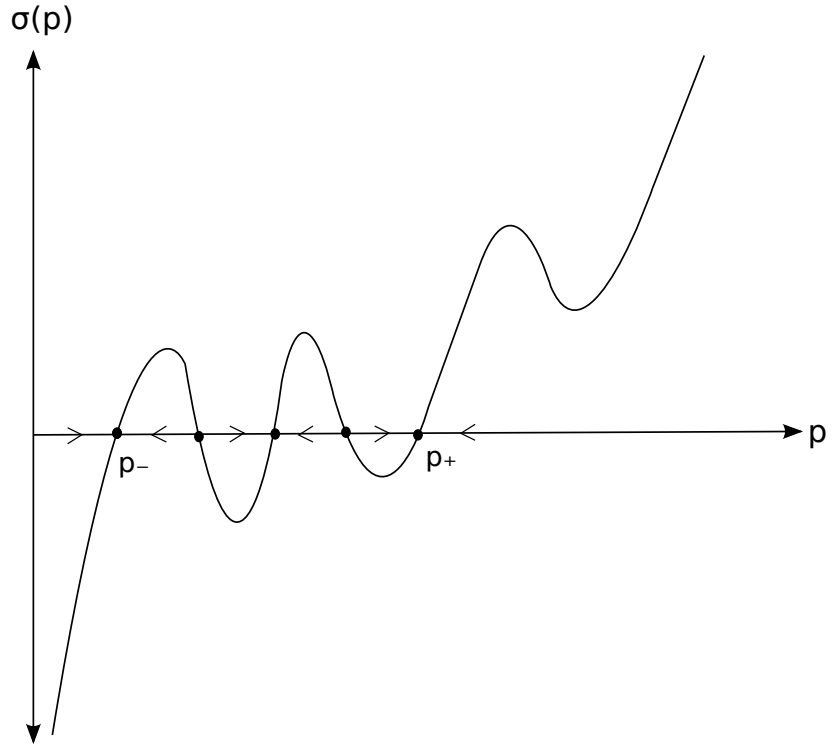



Figure 3.1: One end stress free case

field as in (3.3.2), we can conclude that it converges to a root of $\sigma$ as $t \to \infty$ (see Figure 3.1). We know that any equilibrium point $\bar{p}(\cdot)$ is measurable as it is the almost everywhere limit of a measurable function. Moreover, (3.3.5) holds for $\bar{p}(x)$ proving that $\bar{p}(x) \in L^\infty(0,1)$ as required.

For the convergence of the total energy to its initial value, by continuity of the solution we know that

$$p(x,t) \;\to\; p_0(x) \;\text{ as }\; t \to 0 \;\text{ for a.e. } x \in (0,1).$$

From $(\bar{P})$ we get

$$\frac{\partial}{\partial t} W(p(x,t)) \;=\; -\big(\sigma(p(x,t))\big)^2 \;\leq\; 0.$$

Therefore, $W$ decreases over time and hence

$$W(p(x,t)) \;\leq\; W(p_0(x)) \;\text{ for all }\; t > 0.$$

As the initial energy is assumed to be finite, by Lebesgue's dominated convergence theorem (see e.g. [22]) we get (3.3.1) as required. Q.E.D

## 3.4 The case when both ends are fixed

In this section we consider the boundary conditions (3.1.3) where both ends of the bar are assumed to be fixed. We first show well-posedness under the assumption that the stress $\sigma$ is globally Lipschitz continuous. Then, we replace this assumption with local Lipschitz continuity and make no boundedness assumptions on the initial data. Our aim is to show existence of upper and lower bounds similar to the case of one-end-free boundary conditions and use these bounds to develop a general existence theory associated with the problem.

Let us first derive the problem we want to study. Rewriting (3.1.5) in terms of $z = y_t$ gives

$$z_x + \sigma(y_x) = K(t)$$

where $K(t)$ denotes an arbitrary function of time. Integrating both sides with respect to $x$ and using the boundary conditions (3.1.3) we get

$$\begin{aligned}
&\int_0^1 (z_x + \sigma(y_x))\,dx \;=\; \int_0^1 K(t)\,dx \;=\; K(t) \\
\Rightarrow\quad & z(1,t) \;-\; z(0,t) \;+\; \int_0^1 \sigma(y_x)\,dx \;= z_x + \sigma(y_x) \\
\Rightarrow\quad & z_x + \sigma(y_x) \;=\; \int_0^1 \sigma(y_x)\,dx \\
\Rightarrow\quad & z_x \;=\; -\sigma(y_x) \;+\; \int_0^1 \sigma(y_x)\,dx. \qquad (3.4.1)
\end{aligned}$$

Rewriting (3.4.1) in terms of $p = y_x$ again, gives

$$p_t(x,t) = -\sigma(p(x,t)) \;+\; \int_0^1 \sigma(p(y,t))\,dy.$$

Therefore, the problem we study becomes

$$(P)\quad \begin{cases} p_t(x,t) = -\sigma(p(x,t)) \;+\; \displaystyle\int_0^1 \sigma(p(y,t))\,dy \\ p(x,0) \;=\; p_0(x) \\ \displaystyle\int_0^1 p(x,t)\,dx = \mu > 0 \end{cases}$$

where $\mu$ is as in (3.1.3).

### 3.4.1 The case of globally Lipschitz continuous stress

In this section we assume that the stress is globally Lipschitz continuous.

**Theorem 3.4.1.** *Suppose that $\sigma : (0,\infty) \to \mathbb{R}$ is globally Lipschitz continuous and assumption* (ii) *in Theorem 3.3.1 holds. Given any $p_0 \in L^1(0,1)$ with $p_0(x) > 0$ for almost every $x \in (0,1)$, there exists a unique solution to problem $(P)$ in*

$$X \,=\, \left\{ p \,\in\, C([0,\,T]; L^1(0,1)) \,:\, \int_0^1 p(x,t)\,dx = \,\mu\,,\; p(x,0) \,=\, p_0(x) \right\}$$

*where $T > 0$ is arbitrary.*

*Proof*: Define the mapping

$$(\mathcal{T}p)(x,t) \,=\, p(x,0) \,-\, \int_0^t \left( \sigma(p(x,s)) - \int_0^1 \sigma(p(y,s))\,dy \right) ds. \tag{3.4.2}$$

First let us show that $\mathcal{T}$ is a mapping from $X$ to $X$. Clearly,

$$(\mathcal{T}p)(x,0) \,=\, p_0(x) \,-\, 0 \,=\, p_0(x)$$

and

$$\begin{aligned}
\int_0^1 (\mathcal{T}p)(x,t)dx \;&=\; \int_0^1 p_0(x)dx - \int_0^1 \int_0^t \left( \sigma(p(x,s)) - \int_0^1 \sigma(p(y,s))\,dy \right) ds\,dx \\
&=\; \int_0^1 p_0(x)dx \,-\, \int_0^1 \int_0^t \sigma(p(x,s))ds\,dx \,+ \\
&\qquad\qquad + \int_0^1 \int_0^t \left( \int_0^1 \sigma(p(y,s))dy \right) ds\,dx \\
&=\; \int_0^1 p_0(x)\,dx \,-\, \int_0^1 \int_0^t \sigma(p)ds\,dx \,+\, \int_0^1 \int_0^t \sigma(p)ds\,dx \\
&=\; \int_0^1 p_0(x)\,dx \,=\, \mu.
\end{aligned}$$

Moreover, by (3.4.2) we have

$$\begin{aligned}
\|(\mathcal{T}p)(\cdot,t)\|_{L^1(0,1)} \;&\leq\; \|p_0\|_{L^1(0,1)} \,+\, \int_0^t \int_0^1 |\sigma(p(x,s))|\,dx\,ds \,+ \\
&\qquad\qquad + \int_0^t \int_0^1 \int_0^1 |\sigma(p(y,s))|\,dy\,dx\,ds \\
&=\; \|p_0(\cdot)\|_{L^1(0,1)} \,+\, 2\int_0^t \int_0^1 |\sigma(p(x,s)|\,dx\,ds.
\end{aligned}$$

Let $q \in \mathbb{R}$ be a root of $\sigma$. Then, $\sigma(q) = 0$ and by global Lipschitz continuity of $\sigma$, denoting the Lipschitz constant by $k$, we can continue as

$$\begin{aligned}
\|(\mathcal{T}p)(\cdot,t)\|_{L^1(0,1)} \;&\leq\; \|p_0(\cdot)\|_{L^1(0,1)} \,+\, 2\,k \int_0^t \int_0^1 (|p(x,s)| \,+\, |q|\,)\,dx\,ds \\
&\leq\; \|p_0(\cdot)\|_{L^1(0,1)} \,+\, 2\,k\,T\,(\|p\|_{C([0,\,T];\,L^1(0,1))} + |q|) < \infty.
\end{aligned}$$

Hence $(\mathcal{T}p)(\cdot,t) \in L^1(0,1)$. We also need to show that $(\mathcal{T}p)(x,\cdot) \in C([0,T])$. For this we assume $|s-t| \le \delta$. Then, by Lipschitz continuity of $\sigma$ again, we get

$$\begin{aligned}
\|\mathcal{T}p(\cdot,s) - \mathcal{T}p(\cdot,t)\|_{L^1(0,1)} &\le \int_0^1 \int_s^t \left|\sigma(p(x,\tau)) - \int_0^1 \sigma(p(y,\tau))\,dy\right| d\tau\,dx \\
&\le \int_0^1 \int_s^t \int_0^1 |\sigma(p(x,\tau)) - \sigma(p(y,\tau))|\,dy\,d\tau\,dx \\
&\le k \int_0^1 \int_s^t \int_0^1 |p(x,\tau) - p(y,\tau)|\,dy\,d\tau\,dx \\
&\le k\,2\,C\,|t-s|
\end{aligned}$$

where $C \ge \|p(\cdot,t)\|_{L^1(0,1)}$. Therefore, for any given $\varepsilon$, choosing $\delta$ such that $2\,C\,k\,\delta < \varepsilon$ gives $\mathcal{T}p \in C([0,T];L^1(0,1))$. As a result we can conclude that $\mathcal{T} : X \to X$ holds. We now show that $\mathcal{T}$ is a contraction in $X$. For this, we take arbitrary $p, q \in X$. Then, by (3.4.2) we have

$$\begin{aligned}
\|\mathcal{T}p - \mathcal{T}q\|_{C([0,T];L^1(0,1))} &\le \left\| \int_0^t \left( \sigma(p) - \sigma(q) - \int_0^1 (\sigma(p) - \sigma(q))dy \right) ds \right\|_{C([0,T];L^1)} \\
&\le \left\| \int_0^1 \int_0^t |\sigma(p) - \sigma(q)|\,dx\,ds \; + \right. \\
&\qquad \left. + \int_0^1 \int_0^t \int_0^1 |\sigma(p) - \sigma(q)|\,dy\,dx\,ds \right\|_{C([0,T])} \\
&\le \| \,2\,k \int_0^t \int_0^1 |p-q|\,dx\,ds\,\|_{C([0,T])} \\
&= \| \,2\,k \int_0^t \|p-q\,\|_{L^1(0,1)}\,ds\,\|_{C([0,T])} \\
&\le 2\,k\,T\,\|p\,-\,q\|_{C([0,T];\,L^1(0,1))}.
\end{aligned}$$

This shows that for sufficiently small $T$, the mapping $\mathcal{T}$ is a strict contraction in $X$. Then by Banach's Fixed Point Theorem (see e.g. [100, Thm. 1.A]), we have a unique solution to problem $(P)$ on the time interval $[0,T]$. However, $T$ can be chosen independently of the initial data so that we can repeat this process on the intervals $[T,2T]$, $[2T,3T]$, and so forth. Therefore we can conclude that we have a global solution for any finite time interval.

Alternatively, in order to avoid a separate argument to piece the local solutions together into a global one, we can use the following method of Edwards [38, pg. 154], where a new norm depending on the Lipschitz constant $k$ is used. We define

$$|||p||| \;=\; \sup_{t\in[0,T]} \exp(-c\,k\,t)\,||p(t)||_{L^1(0,1)},$$

where $c$ is a fixed number satisfying $c > 2$. Trivially we have that $||\cdot||$ and $|||\cdot|||$ are equivalent and $(C([0,T];L^1(0,1)), |||\cdot|||)$ is a Banach space. Moreover,

$$\begin{aligned}
& |||\mathcal{T}\,p - \mathcal{T}\,q||| \leq \\
\leq\ & \sup_{t\in[0,T]} \exp(-c\,k\,t) \left\| \int_0^t \left( \sigma(p) - \sigma(q) \ - \ \int_0^1 (\sigma(p) - \sigma(q))\,dy \right) ds \right\|_{L^1(0,1)} \\
\leq\ & \sup_{t\in[0,T]} \exp(-c\,k\,t) \left( 2\,k \int_0^t ||p-q||_{L^1(0,1)}\,ds \right) \\
=\ & 2\,k \sup_{t\in[0,T]} \exp(-c\,k\,t) \left( \int_0^t \exp(c\,k\,s)\,\exp(-c\,k\,s)\,||p-q||_{L^1(0,1)}\,ds \right).
\end{aligned}$$

Since

$$\exp(-c\,k\,s)\,||p-q||_{L^1(0,1)} \ \leq \ |||p-q|||$$

we can continue as

$$\begin{aligned}
& |||\mathcal{T}\,p - \mathcal{T}\,q||| \leq \\
\leq\ & 2\,k\,|||p-q||| \sup_{t\in[0,T]} \int_0^t \exp(c\,k\,s)\,\exp(-c\,k\,t)\,ds \\
=\ & 2\,k\,|||p-q||| \sup_{t\in[0,T]} (c\,k)^{-1} \left( \exp(c\,k\,(s-t)) \Big|_{s=0}^{s=t} \right) \\
=\ & 2\,c^{-1}\,|||p-q||| \sup_{t\in[0,T]} (1 - \exp(-c\,k\,t) \\
\leq\ & 2\,c^{-1}\,(1 - \exp(-c\,k\,T))\,|||p-q|||.
\end{aligned}$$

As $c > 2$, $\mathcal{T}$ is a strict contraction for arbitrary $T > 0$. Q.E.D

### 3.4.2 The case of locally Lipschitz continuous stress

Let us state all the assumptions on the stress $\sigma$ in the one-dimensional setting corresponding to the general physical requirements we mentioned in Section 2.3.2.

- **Continuity** : We assume that the stress $\sigma$ is $C^1(0,\infty)$ from which it follows that it is locally Lipschitz continuous. That is, there exists a positive constant $L$, depending on $C \geq 1$, such that

$$|\sigma(p) - \sigma(q)| \leq L(C)\,|p-q| \quad \text{whenever} \quad \frac{1}{C} \leq |p|, |q| \leq C. \tag{3.4.3}$$

- **Behaviour at infinity** : We assume that $\sigma$ is convex and grows superlinearly for large $p$. In other words, for any sufficiently small $\varepsilon > 0$ we have

$$\text{(i)} \ \ \sigma(p) \quad \text{is convex for} \quad p > 1/\varepsilon \tag{3.4.4a}$$

$$\text{(ii)} \ \ \frac{\sigma(p)}{p} \to \infty \quad \text{as} \quad p \to \infty. \tag{3.4.4b}$$

- **Behaviour near zero** : We assume that $\sigma$ is concave and goes to negative infinity as $p$ goes to zero. That is, for any sufficiently small $\varepsilon > 0$ we have

$$\text{(i) } \sigma(p) \quad \text{is concave for} \quad p < \varepsilon \tag{3.4.5a}$$

$$\text{(ii) } \sigma(p) \to -\infty \quad \text{as} \quad p \to 0_+. \tag{3.4.5b}$$

The method we follow in order to prove existence of solutions for $(P)$ is a finite-dimensional approximation for the autonomous differential equation we consider which we discuss in the subsequent section.

### 3.4.3 Initial data taking finitely many values

In this section we study problem $(P)$ when the initial data is positive and takes finitely many values. In other words, we assume that $p_0(x) \in \{p_{01}, p_{02}, \cdots, p_{0N}\}$ where each $p_{0i} > 0$. In this case we can write

$$p_0(x) = \sum_{i=1}^{N} p_{0i}\, \chi_{E_i}(x), \tag{3.4.6}$$

where $\text{meas}(E_i) = \mu_i$, $\sum_i \mu_i = 1$, and $E_i$ are mutually disjoint subsets of $(0, 1)$. We have the following existence result.

**Theorem 3.4.2.** *Assume that the initial data $p_0(x)$ for problem $(P)$ is of the form (3.4.6). Then there exists a unique local solution $p_N(x,t) \in C([0,T]; L^1(0,1))$ satisfying $(P)$ for all $x \in (0,1)$ and $t \in [0,T]$. Moreover,*

$$p_N(x,t) = \sum_{i=1}^{N} p_i(t)\, \chi_{E_i}(x), \quad p_i(0) = p_{0i} > 0. \tag{3.4.7}$$

*Proof*: Let us first prove the last part of the claim. For this suppose that $p(x,t)$ satisfies

$$p_t(x,t) = -\sigma(p(x,t)) + \int_0^1 \sigma(p(y,t))\, dy \tag{3.4.8}$$

with initial data as in (3.4.6). We note that

$$p_0(x_1) = p_0(x_2),\ x_1, x_2 \in E_i \quad \Rightarrow \quad p(x_1,t) = p(x_2,t) \text{ for all } t. \tag{3.4.9}$$

This is because for fixed $x$, (3.4.8) becomes an ordinary differential equation and if $p(x_1, \tilde{t}) \neq p(x_2, \tilde{t})$ at time $t = \tilde{t}$, then we can solve this equation backwards in time

and get $p_0(x_1) \neq p_0(x_2)$ which contradicts the assumption. Therefore, by (3.4.9) we immediately conclude that

$$p(x,t) = \sum_{i=1}^{N} p_i(t)\,\chi_{E_i}(x).$$

That is to say, if a solution to $(P)$ with $p_0(x)$ as in (3.4.6) exists, then it must be of the form $p_N(x,t)$ as in (3.4.7). We now show that $p_N(x,t)$ is actually a solution to (3.4.8). Substituting $p_N(x,t)$ into problem $(P)$ gives

$$\dot{p_i}(t) = -\sigma(p_i(t)) + \sum_{j=1}^{N} \mu_j\,\sigma(p_j(t)), \quad 1 \leq i \leq N \tag{3.4.10a}$$

$$p_i(0) = p_{0i} > 0, \quad \sum_{j=1}^{N} \mu_j p_j(t) = \mu. \tag{3.4.10b}$$

Note that (3.4.10a) is a finite system of ordinary differential equations with locally Lipschitz right-hand sides for $p_i > 0$. Hence, by the Picard-Lindelöf Theorem (see e.g. [52, Thm. 1.1]), they possess unique solutions $p_i(t) \in C([0,T])$, where $T$ is sufficiently small. This proves that $p_N(x,t)$ is well-defined and is a local solution to problem $(P)$ with initial data satisfying (3.4.6) as required. Q.E.D

We next prove that, independently of the initial data, $p_N(x,t)$ is instantaneously bounded and bounded away from zero. This will also show that $p_N(x,t)$ is actually a global solution.

#### 3.4.3.1 The Lower Bound

In this part, we show the existence of a lower bound for the solution $p_N(x,t)$. We first prove a rather general result for $\sigma$, purely based on the assumptions we make.

**Proposition 3.4.3.** *Assume that (3.4.3), (3.4.4b) and (3.4.5) hold. Then, there exist constants $C > 0$ and $\varepsilon_0$ such that*

$$\frac{\sigma(p) - \sigma(\delta)}{p - \delta} > C \quad \textit{for all} \;\; 0 < p \neq \delta,\; \delta \leq \varepsilon_0,$$

*where $\varepsilon_0$ is sufficiently small.*

*Proof*: If the assertion was false, then it would particularly be false for $C = \varepsilon_0 = \frac{1}{j}$ for all $j$, and there would exist sequences $p_j \neq \varepsilon_j \leq \frac{1}{j}$ such that

$$\frac{\sigma(p_j) - \sigma(\varepsilon_j)}{p_j - \varepsilon_j} \leq \frac{1}{j} \quad \text{for all} \;\; j. \tag{3.4.11}$$

We can suppose that $p_j \to p_\infty \in [0,\infty]$ as $j \to \infty$. Then, we need to check three cases separately.

(i) $p_\infty = 0$ : In this case let us assume that $p_j > \varepsilon_j$. By (3.4.5a) we have

$$\frac{\sigma(p_j) - \sigma(\varepsilon_j)}{p_j - \varepsilon_j} \geq \sigma'(p_j).$$

By (3.4.5b) we know that for sufficiently small $p$, the slope of $\sigma(p)$ is positive and increases as $p$ gets closer to zero. Therefore, we have

$$\frac{\sigma(p_j) - \sigma(\varepsilon_j)}{p_j - \varepsilon_j} \geq \sigma'(p_j) \to \infty \text{ as } j \to \infty,$$

contradicting (3.4.11).

(ii) $0 < p_\infty < \infty$ : In this case $\sigma(p_j)$ also stays finite by assumption (3.4.3). Therefore, by (3.4.5b) we get

$$\frac{\sigma(p_j) - \sigma(\varepsilon_j)}{p_j - \varepsilon_j} \sim \frac{\sigma(p_\infty) - \sigma(\varepsilon_j)}{p_\infty - \varepsilon_j} \to \infty \text{ as } j \to \infty,$$

contradicting (3.4.11) again.

(iii) $p_\infty = \infty$ : In this case by (3.4.4b) and (3.4.5b) we immediately obtain

$$\frac{\sigma(p_j) - \sigma(\varepsilon_j)}{p_j - \varepsilon_j} \geq \frac{\sigma(p_j)}{p_j\left(1 - \dfrac{\varepsilon_j}{p_j}\right)} \to \infty \text{ as } j \to \infty,$$

which contradicts (3.4.11). Q.E.D

We now prove some lemmas which gradually lead to the main result of this section.

**Lemma 3.4.4.** *Let $\varepsilon(t)$ be such that*

$$\varepsilon(0) = 0 \quad \textit{and} \quad 0 < \varepsilon(t) < \mu \ \textit{ for } \ t > 0. \tag{3.4.12}$$

*Assume that $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Then, there must exist at least one $i \in \{1, \ldots, N\}$ such that $p_i(t) > \varepsilon(t)$ for $0 \leq t \leq t_0$, where $t_0$ is sufficiently small.*

*Proof*: First, recall by (3.4.10b) that $p_i(0) = p_{0i} > 0$. Therefore, initially we have

$$\varepsilon(t) < p_i(t) \text{ for all } i \in \{1, \ldots, N\}.$$

If, for contradiction, at time $0 < t \leq t_0$ we have $p_j(t) \leq \varepsilon(t)$ for all $j \in \{1, \ldots, N\}$, then by (3.4.10b) we would have

$$\mu = \sum_{j=1}^{N} \mu_j \, p_j(t) \leq \varepsilon(t) \sum_{j=1}^{N} \mu_j = \varepsilon(t)$$

contradicting (3.4.12). Hence, there must exist at least one such $i$ as required. Q.E.D

**Lemma 3.4.5.** *Let $\varepsilon(t)$ be a continuous function satisfying*

$$0 \leq \varepsilon(t) \leq \mu \quad \textit{for} \quad 0 \leq t \leq t_0, \tag{3.4.13}$$

*and assume that $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). If at a certain time $0 \leq t \leq t_0$ we have $p_i(t) = \varepsilon(t)$ for some $i \in \{1, \ldots, N\}$, then there must exist a $j \neq i$ such that $p_j(t) \geq \varepsilon(t)$.*

*Proof*: Assume for contradiction that $p_j(t) < \varepsilon(t)$ for all $j \neq i$. Then we would have

$$\begin{aligned} \mu &= \sum_{j=1}^{N} \mu_j \, p_j(t) = \mu_i \, p_i(t) + \sum_{1 \leq j \leq N,\, j \neq i} \mu_j \, p_j(t) = \mu_i \, \varepsilon(t) + \sum_{1 \leq j \leq N,\, j \neq i} \mu_j \, p_j(t) \\ &< \mu_i \, \varepsilon(t) + \varepsilon(t) \sum_{1 \leq j \leq N,\, j \neq i} \mu_j = \mu_i \, \varepsilon(t) + \varepsilon(t) \, (1 - \mu_i) \\ &= \varepsilon(t) \end{aligned}$$

which contradicts (3.4.13). Q.E.D

**Lemma 3.4.6.** *Assume that (3.4.5) hold, $p_i(t), i = 1, \ldots, N$, and $\varepsilon(t)$ satisfy (3.4.10) and (3.4.12) respectively. Then at time $t \leq t_0$ we have*

$$\left.\begin{array}{l} p_i(t) = \varepsilon(t) \;\; \textit{and} \\ p_j(t) \geq \varepsilon(t) \;\; \textit{for all} \;\; j \neq i \end{array}\right\} \;\Rightarrow\; \dot{p}_i(t) > 0,$$

*where $t_0$ is as in Lemma 3.4.4.*

*Proof*: By Lemma 3.4.4 we know that there must exist at least one $j \in \{1, \ldots, N\}$ such that $j \neq i$ and $p_j(t) > \varepsilon(t)$ for $0 \leq t \leq t_0$. If there is only one such $j$, then without loss of generality letting $j = N$ and putting $p_j(t) = \varepsilon(t)$ for $j \neq N$ into (3.4.10), we obtain

$$\begin{aligned} \dot{p}_i(t) &= -\sigma(p_i(t)) + \sum_{j=1}^{N} \mu_j \, \sigma(p_j(t)) \\ &= -\sigma(\varepsilon(t)) + \sum_{j=1}^{N-1} \mu_j \, \sigma(\varepsilon(t)) + \mu_N \, \sigma(p_N(t)) \\ &= \left( \sum_{j=1}^{N-1} \mu_j - 1 \right) \sigma(\varepsilon(t)) + \mu_N \, \sigma(p_N(t)) \\ &= -\mu_N \, \sigma(\varepsilon(t)) + \mu_N \, \sigma(p_N(t)). \end{aligned}$$

By (3.4.5b) we know that $-\mu_N\sigma(\varepsilon(t))$ is positive and very large for small $t$. Moreover, if $\sigma(p_N) > 0$, then we have

$$0 \;<\; \mu_N\,\sigma(p_N(t)),$$

since $0 < \mu_j < 1$ for all $j \in \{1,\ldots,N\}$. Hence we get $\dot{p_i}(t) > 0$. If, on the other hand, $\sigma(p_N) \leq 0$ then by (3.4.5a) we have

$$p_N(t) \;>\; \varepsilon(t) \quad \Rightarrow \quad \sigma(p_N(t)) \;>\; \sigma(\varepsilon(t)).$$

Therefore

$$\begin{aligned}
\dot{p_i}(t) &= -\mu_N\,\sigma(\varepsilon(t)) + \mu_N\,\sigma(p_N(t)) \\
&> -\mu_N\,\sigma(\varepsilon(t)) + \mu_N\,\sigma(\varepsilon(t)) \\
&= 0
\end{aligned}$$

as required. If there are more than one such $j$, then arguing similarly as above we obtain

$$\begin{aligned}
\dot{p_i}(t) &= -\sigma(p_i(t)) \;+ \sum_{\substack{1\leq j\leq N:\\ p_j(t)=\varepsilon(t)}} \mu_j\,\sigma(p_j(t)) \;+ \sum_{\substack{1\leq j\leq N:\\ p_j(t)>\varepsilon(t)}} \mu_j\,\sigma(p_j(t)) \\
&= \left(\sum_{\substack{1\leq j\leq N:\\ p_j(t)=\varepsilon(t)}} \mu_j \;-\; 1\right)\sigma(\varepsilon(t)) + \sum_{\substack{1\leq j\leq N:\\ p_j(t)>\varepsilon(t)}} \mu_j\,\sigma(p_j(t)) \\
&= -\left(\sum_{\substack{1\leq j\leq N:\\ p_j(t)>\varepsilon(t)}} \mu_j\right)\sigma(\varepsilon(t)) + \sum_{\substack{1\leq j\leq N:\\ p_j(t)>\varepsilon(t)}} \mu_j\,\sigma(p_j(t)) \\
&= \sum_{\substack{1\leq j\leq N:\\ p_j(t)>\varepsilon(t)}} \mu_j\,\big(\sigma(p_j(t)) - \sigma(\varepsilon(t))\big) > 0
\end{aligned}$$

as required. Q.E.D

**Remark 3.4.1.** *Note that as a result of the finite-dimensional reduction for problem $(P)$, we work on an $N$-dimensional hypersurface, which in the case of $N = 3$ becomes a hyperplane. In this case, what Lemma 3.4.6 states is that if at a certain time $t$ we are on $p_i(t) = \varepsilon(t)$ for an $i$, then we must move towards the center of the hyperplane as time increases. See Figure 3.2 for an illustration.*

We now state and prove the main result of this section, namely, existence of a lower bound. Afterwards we will get more information when $t > t_0$ and state the whole analysis as a theorem at the end of the section.

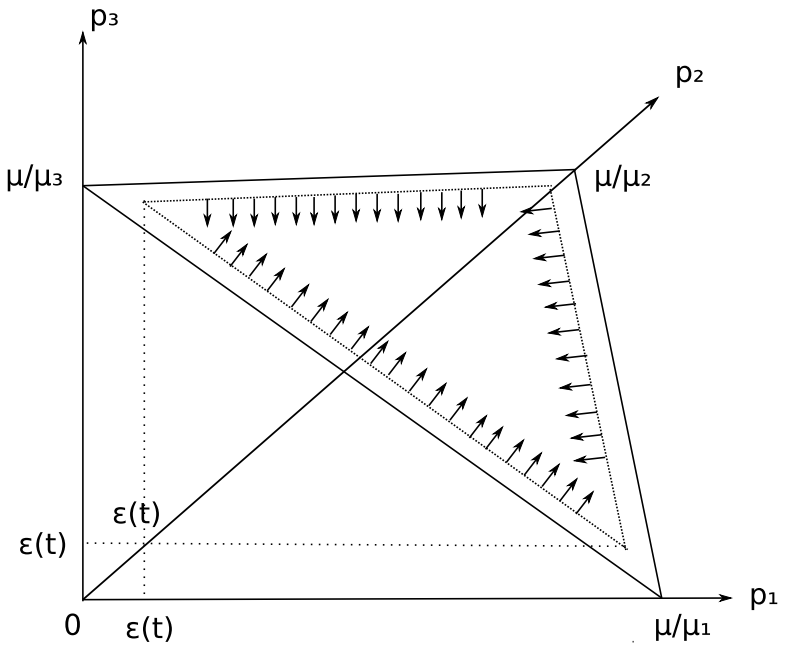



Figure 3.2: Illustration of Lemma 3.4.6 for $N = 3$

**Proposition 3.4.7.** *Assume that (3.4.3), (3.4.4b), (3.4.5) hold and* $p_i(t), i = 1, \ldots, N$, *satisfy (3.4.10). Then there exists a continuous function* $\varepsilon(t)$ *satisfying (3.4.12) such that at time* $t \leq t_0$ *we have*

$$\left.\begin{array}{l} p_i(t) = \varepsilon(t) \quad \textit{and} \\ p_j(t) \geq \varepsilon(t) \quad \textit{for all} \quad j \neq i \end{array}\right\} \;\Rightarrow\; \dot{p_i}(t) > \dot{\varepsilon}(t).$$

*In particular, we may choose*

$$\varepsilon(t) = \mu\,(1 - \exp(-C\,t)) \tag{3.4.14}$$

*where* $C$ *is as in Proposition 3.4.3.*

*Proof*: By (3.4.10), for each $i \in \{1, \ldots, N\}$, we have

$$\begin{aligned}
\dot{p_i}(t) &= -\sigma(p_i(t)) + \sum_{1\leq j\leq N,\, j\neq i} \mu_j\,\sigma(p_j(t)) + \mu_i\,\sigma(p_i(t)) \\
&= -\sigma(\varepsilon(t)) + \sum_{1\leq j\leq N,\, j\neq i} \mu_j\,\sigma(p_j(t)) + \mu_i\,\sigma(\varepsilon(t)) \\
&= (-1+\mu_i)\sigma(\varepsilon(t)) + \sum_{1\leq j\leq N,\, j\neq i} \mu_j\sigma(p_j(t)) \\
&= \left(-\sum_{1\leq j\leq N,\, j\neq i} \mu_j\right)\sigma(\varepsilon(t)) + \sum_{1\leq j\leq N,\, j\neq i} \mu_j\sigma(p_j(t)) \\
&= \sum_{1\leq j\leq N,\, j\neq i} \mu_j\,(\sigma(p_j(t)) - \sigma(\varepsilon(t))).
\end{aligned}$$

By Proposition 3.4.3 with $\delta = \varepsilon(t)$, we obtain

$$\begin{aligned}
\dot{p_i}(t) &= \sum_{1\le j\le N,\, j\neq i} \mu_j \left(\sigma(p_j(t)) - \sigma(\varepsilon(t))\right) \\
&> \sum_{1\le j\le N,\, j\neq i} \mu_j\, C\, (p_j(t) - \varepsilon(t)) \\
&= C \sum_{1\le j\le N,\, j\neq i} \mu_j\, p_j(t) - C\,\varepsilon(t) \sum_{1\le j\le N,\, j\neq i} \mu_j.
\end{aligned}$$

Using $\sum_{j=1}^{N} \mu_j = 1$ again we get

$$\begin{aligned}
\dot{p_i}(t) &> C \sum_{1\le j\le N,\, j\neq i} \mu_j\, p_j(t) - C\,\varepsilon(t)\,(1-\mu_i) \\
&= \sum_{1\le j\le N,\, j\neq i} \mu_j\, C\, p_j(t) - C\,\varepsilon(t) \; - \mu_i\, C\, p_i(t) \\
&= C \left( \sum_{j=1}^{N} \mu_j\, p_j(t) - \varepsilon(t) \right).
\end{aligned}$$

Putting $\sum_{j=1}^{N} \mu_j p_j(t) = \mu$ and using the given $\varepsilon(t)$ we obtain

$$\begin{aligned}
\dot{p_i}(t) &> C\,(\mu - \mu(1 - \exp(-Ct))) = C\,\mu\, \exp(-Ct) \\
&= \dot{\varepsilon}(t)
\end{aligned}$$

as required. Q.E.D

**Lemma 3.4.8.** *Assume that (3.4.3), (3.4.4b), (3.4.5) hold and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Then, $\dot{p_i}(t) > \dot{\varepsilon}(t)$ at time $t \le t_0$ implies*

$$p_i(t) > \varepsilon(t) \;\; \textit{for all } t \le t_0.$$

*Proof*: Assume, for contradiction, that there exists a time $\tilde{t} \le t_0$ such that $p_i(\tilde{t}) = \varepsilon(\tilde{t})$ with $p_j(\tilde{t}) \ge \varepsilon(\tilde{t})$ for all $j \neq i$. Then we would have

$$p_i(t) > \varepsilon(t) \quad \text{for} \quad 0 \le t < \tilde{t} \quad \text{and} \quad p_i(\tilde{t}) = \varepsilon(\tilde{t}). \tag{3.4.15}$$

On the other hand, by Proposition 3.4.7 we have

$$\dot{p_i}(\tilde{t}) > \dot{\varepsilon}(\tilde{t}). \tag{3.4.16}$$

Let us define

$$q_i(t) := p_i(t) - \varepsilon(t).$$

Then (3.4.15) would be equivalent to

$$q_i(t) > 0 \quad \text{for} \quad 0 \leq t < \tilde{t} \quad \text{and} \quad q_i(\tilde{t}) = 0,$$

implying $q_i(t)$ is decreasing in time for $0 \leq t \leq \tilde{t}$. Therefore we would have

$$\dot{q}_i(t) \leq 0 \quad \text{for} \quad 0 \leq t \leq \tilde{t},$$

which contradicts (3.4.16). Q.E.D

**Lemma 3.4.9.** *Assume that (3.4.3), (3.4.4b), (3.4.5) hold and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Then we have*

$$p_i(t) > \varepsilon_0 \quad \textit{for all} \quad t > t_0,$$

*where $\varepsilon_0 = \varepsilon(t_0)$ with $\varepsilon(t)$ as in Proposition 3.4.7.*

*Proof*: If, for contradiction, there exists a time $\tilde{t} > t_0$ such that $p_i(\tilde{t}) = \varepsilon_0$ and $p_j(\tilde{t}) \geq \varepsilon_0$ for all $j \neq i$, then by Lemma 3.4.6 we would have $\dot{p}_i(\tilde{t}) > 0$. This gives a contradiction since by Lemma 3.4.8 we have $p_i(t) > \varepsilon_0$ for all $t \leq \tilde{t}$. Q.E.D

We can now conclude our analysis as follows:

**Theorem 3.4.10.** *Assume (3.4.3), (3.4.4b), (3.4.5) hold and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Then there exists a continuous $\varepsilon(t)$ satisfying (3.4.12) such that if at time $t$ with $0 < t \leq t_0$ we have $p_i(t) = \varepsilon(t)$ and $p_j(t) \geq \varepsilon(t)$ for all $j \neq i$, then*

- $p_i(t) > \varepsilon(t) \quad \textit{for } t \leq t_0$
- $p_i(t) > \varepsilon_0 = \varepsilon(t_0) \quad \textit{for } t > t_0.$

*In particular, we may choose*

$$\varepsilon(t) = \mu\,(1 - \exp(-C\,t))$$

*where $C$ is as in Proposition 3.4.3.*

*Proof*: This result is a direct consequence of Proposition 3.4.7, Lemma 3.4.8 and Lemma 3.4.9. Q.E.D

#### 3.4.3.2 The Upper Bound

In this section we show that $p_N(x,t)$ is instantaneously bounded and stays bounded for all times. We first make the following crucial remark.

**Remark 3.4.2.** *Using precisely the same arguments in Lemmas 3.4.4 and 3.4.5 we can prove that if $E(t)$ is a continuous function with $E(t) \geq \mu$ for $0 \leq t \leq t_0$, where $t_0$ is sufficiently small, and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10), then $p_i(t) = E(t)$ for some $i \in \{i, \ldots, N\}$ implies that there must exist a $j \neq i$ such that $p_j(t) \leq E(t)$.*

Similar to the lower bound case, we have the following general result purely based on the assumptions we make.

**Proposition 3.4.11.** *Assume (3.4.3) and (3.4.4) hold. Then,*

$$\frac{\sigma(\gamma) - \sigma(p)}{\gamma - p} > \frac{\sigma(\gamma)}{\gamma} \quad \text{for all } 0 < p < \gamma < \infty, \tag{3.4.17}$$

*where $\gamma$ is sufficiently large.*

*Proof*: Note that (3.4.17) is equivalent to

$$\begin{aligned} \frac{\sigma(\gamma) - \sigma(p)}{\gamma - p} > \frac{\sigma(\gamma)}{\gamma} \quad &\Leftrightarrow \quad \gamma\,\sigma(\gamma) - \gamma\,\sigma(p) > \gamma\,\sigma(\gamma) - p\,\sigma(\gamma) \\ &\Leftrightarrow \quad \gamma\,\sigma(p) < p\,\sigma(\gamma) \\ &\Leftrightarrow \quad \frac{\sigma(p)}{p} < \frac{\sigma(\gamma)}{\gamma}. \end{aligned}$$

Hence, it is enough to prove that

$$\frac{\sigma(p)}{p} \quad \text{is nondecreasing in } p \text{ for large } p. \tag{3.4.18}$$

Let $q$ be large and $p \geq q$. By the assumptions we have, the minimum of $\dfrac{\sigma(p)}{p}$ for $p \geq q$ is attained. If the minimum is at $q$ then we have

$$\frac{\sigma(p)}{p} \geq \frac{\sigma(q)}{q}.$$

If not, then the minimum must be at $\bar{p} > q$ where

$$\sigma'(\bar{p}) = \frac{\sigma(\bar{p})}{\bar{p}}. \tag{3.4.19}$$

Therefore, the result holds as long as there is no other large $p$ satisfying (3.4.19). However, if there exists such a $\bar{q}$, then for $p \geq \bar{q}$ we would have

$$\begin{aligned} \sigma'(p) &\geq \frac{\sigma'(\bar{p})}{\bar{p}} \geq \frac{1}{\bar{p}}\Big(\sigma(p) + (\bar{p}-p)\,\sigma'(p)\Big) \\ &= \frac{\sigma(p)}{\bar{p}} + \sigma'(p) - \frac{p\,\sigma'(p)}{\bar{p}} \end{aligned}$$

so that

$$\frac{\sigma(p)}{\bar{p}} \leq \frac{p\,\sigma'(p)}{\bar{p}} \quad \Rightarrow \quad \frac{\sigma(p)}{p} \leq \sigma'(p)$$

giving

$$\frac{d}{dp}\frac{\sigma(p)}{p} = \frac{\sigma'(p)\,p - \sigma(p)}{p^2} \geq 0$$

as required. Q.E.D

We now prove the main result of this section which is the existence of a uniform upper bound. We also give an explicit form of the bound for the extreme cases of $\sigma$.

**Proposition 3.4.12.** *Assume that (3.4.3), (3.4.4) hold and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Suppose also that the assumption* (iii) *in Theorem 3.3.1, namely,*

$$\int_{p_+ +1}^{\infty} \frac{dz}{\sigma(z)} < \infty \tag{3.4.20}$$

*holds, where $p_+$ denotes the largest root of $\sigma$. Then, there exists a continuous $E(t)$ satisfying*

$$E(t) \geq 0 \;\; \textit{for} \;\; t \geq 0 \quad \textit{and} \quad E(t) \to \infty \quad \textit{as} \;\; t \to 0 \tag{3.4.21}$$

*such that at time $0 \leq t \leq t_0$, where $t_0$ is sufficiently small, we have*

$$\left.\begin{array}{l} p_i(t) = E(t) \;\; \textit{and} \\ p_j(t) \leq E(t) \;\; \textit{for all} \;\; j \neq i \end{array}\right\} \;\Rightarrow\; \dot{p}_i(t) < \dot{E}(t).$$

*In particular, we may choose*

$$E(t) = \frac{\mu\,\exp(\mu t)}{\exp(\mu t) - 1},$$

*provided $\sigma(p) \sim p^2$ for large $p$.*

*Proof*: First let us note that if $p_i(t) = E(t)$ and $p_j(t) \leq E(t)$ for all $j \neq i$, then by (3.4.10) we get

$$\begin{aligned} \dot{p}_i(t) &= -\sigma(p_i) + \sum_{j=1}^{N} \mu_j \sigma(p_j) \\ &= -\sigma(E(t)) + \sum_{1 \leq j \leq N,\, j \neq i} \mu_j\,\sigma(p_j(t)) + \mu_i \sigma(E(t)) \\ &= (\mu_i - 1)\sigma(E(t)) + \sum_{1 \leq j \leq N,\, j \neq i} \mu_j\,\sigma(p_j(t)). \end{aligned}$$

Using the fact that $\sum_{j=1}^{N} \mu_j = 1$ we obtain

$$\begin{aligned} \dot{p_i}(t) &= \sum_{1\le j\le N,\, j\neq i} \mu_j \left(\sigma(p_j(t)) - \sigma(E(t))\right) \\ &= -\sum_{1\le j\le N,\, j\neq i} \mu_j \left(\sigma(E(t)) - \sigma(p_j(t))\right). \end{aligned}$$

By Proposition 3.4.11 and the boundary conditions $\sum_{j=1}^{N} \mu_j p_j(t) = \mu$, we get

$$\begin{aligned} \dot{p_i}(t) &= -\sum_{1\le j\le N,\, j\neq i} \mu_j \left(\sigma(E(t)) - \sigma(p_j(t))\right) \\ &< \sum_{1\le j\le N,\, j\neq i} \mu_j \left(p_j(t) - E(t)\right) \frac{\sigma(E(t))}{E(t)} \\ &= \frac{\sigma(E(t))}{E(t)} \sum_{j=1}^{N} \mu_j \left(p_j(t) - E(t)\right) \\ &= \frac{\sigma(E(t))}{E(t)} \left(\mu - E(t)\right). \end{aligned}$$

Therefore, in order to prove the claim, it is enough to show existence of $E(t)$ satisfying

$$\frac{\sigma(E(t))}{E(t)} \left(\mu - E(t)\right) \;=\; \dot{E}(t). \tag{3.4.22}$$

Let us denote

$$K(E) = \frac{\sigma(E)}{E}$$

and define

$$g(E) := \int_{Q+1}^{E} \frac{dz}{K(z)(z-\mu)}, \tag{3.4.23}$$

where $Q$ is the largest root of $K(E)(E-\mu)$. Since there exists a constant $C > 0$ such that for $z \geq 2\mu$ we have

$$\left|\frac{1}{K(z)(z-\mu)}\right| \leq C \left|\frac{1}{\sigma(z)}\right|,$$

by assumption (3.4.20) we have that

$$\int_{Q+1}^{\infty} \frac{dz}{K(z)(z-\mu)} \;<\; \infty.$$

Hence, $g(E)$ is well-defined for large $E$ as well. Moreover, by Proposition 3.4.11 we know that $K(E)$ is nondecreasing in $E$ for large $E$. Therefore $g(E)$ is continuous, strictly monotonic decreasing on $(Q+1, \infty)$ and $g(E) \to 0$ as $E \to \infty$. Similar to the both-ends-fixed case, we claim that

$$E(t) = g^{-1}(-t + t_0). \tag{3.4.24}$$

First of all, we need to check that $E(t)$ satisfies (3.4.22). From (3.4.24) we immediately get

$$g(E(t)) = -t + t_0 \;\; \Rightarrow \;\; \dot{g}(E) = -1. \tag{3.4.25}$$

Also, from (3.4.23) we get

$$\dot{g}(E) = \frac{\dot{E}}{K(E)(E-\mu)}. \tag{3.4.26}$$

It is easy to see that (3.4.25) and (3.4.26) gives (3.4.22). By properties of $g(E)$ we immediately have $E(t) \geq 0$ for all $0 \leq t \leq t_0$. Moreover, when $t \to 0$ we have $g(E(t)) \to t_0$ and as $t_0$ is sufficiently small we obtain $E(t) \to \infty$ as required.

For the last part of the claim, it is enough to show that (3.4.22) holds for given $E(t)$ with the corresponding behaviour of $\sigma(p)$.

Assuming that $\sigma(p) \sim p^2$ for large $p$, which is consistent with assumptions (3.4.4), equality (3.4.22) takes the form

$$E(t)\,(\mu - E(t)) \;=\; \dot{E}(t). \tag{3.4.27}$$

One can easily show that if

$$E(t) = \frac{\mu\, \exp(\mu t)}{\exp(\mu t) - 1},$$

then (3.4.27) is satisfied. Moreover, this particular $E(t)$ has the necessary properties stated in (3.4.21) as required. Q.E.D

**Lemma 3.4.13.** *Assume (3.4.3), (3.4.4), (3.4.20) hold and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Then*

$$p_i(t) \;<\; E(t) \;\; \textit{for all} \;\; 0 \leq t \leq t_0,$$

*where $E(t)$ is as in Proposition 3.4.12.*

*Proof*: Under the given assumptions, if there exists a time $\tilde{t} \geq 0$ such that $p_i(\tilde{t}) = E(\tilde{t})$ with $p_j(\tilde{t}) \leq E(\tilde{t})$ for all $j \neq i$, then by Proposition 3.4.12 we have

$$\dot{p_i}(\tilde{t}) < \dot{E}(\tilde{t}). \tag{3.4.28}$$

Therefore we get

$$p_i(t) < E(t) \quad \text{for} \quad 0 \leq t < \tilde{t} \quad \text{and} \quad p_i(\tilde{t}) = E(\tilde{t}).$$

Defining

$$q_i(t) := p_i(t) - E(t),$$

we can equivalently say

$$q_i(t) < 0 \quad \text{for} \quad 0 \le t < \tilde{t} \quad \text{and} \quad q_i(\tilde{t}) = 0.$$

This implies that $q_i(t)$ is increasing in time for $0 \le t \le \tilde{t}$. Thus, we must have

$$\dot{q}_i(t) \ge 0 \text{ for } 0 \le t \le \tilde{t}.$$

However, this contradicts (3.4.28). Q.E.D

We conclude our analysis for this section as follows:

**Theorem 3.4.14.** *Assume (3.4.3), (3.4.4), (3.4.20) hold and $p_i(t), i = 1, \ldots, N$, satisfy (3.4.10). Then, there exists a continuous $E(t)$ satisfying (3.4.21) such that if at time $t$ with $0 \le t \le t_0$ we have $p_i(t) = E(t)$ and $p_j(t) \le E(t)$ for all $j \ne i$, then*

$$p_i(t) < E(t) \quad \text{for all} \quad 0 \le t \le t_0,$$

*where $t_0$ is sufficiently small. In particular, we may choose*

$$E(t) = \frac{\mu \exp(\mu t)}{\exp(\mu t) - 1},$$

*provided $\sigma(p) \sim p^2$ for large $p$.*

*Proof*: This result is a direct consequence of Proposition 3.4.12 and Lemma 3.4.13. Q.E.D

We can now state the global existence result for the solution of $(P)$ together with the bounds we have obtained which are illustrated in Figure 3.3.

**Corollary 3.4.15.** *Assume that (3.4.3), (3.4.4), (3.4.5) and (3.4.20) hold. Assume also that initial datum $p_0(x)$ is given by (3.4.6). Then, there exists a unique global solution $p_N(x,t) \in C([0,T]; L^1(0,1))$ to problem $(P)$ such that*

$$p_N(x,t) = \sum_{i=1}^{N} p_i(t)\, \chi_{E_i(x)}, \quad p_i(0) = p_{0i} > 0.$$

*Moreover, there exist functions $\varepsilon(t)$ and $E(t)$, independent of $N$, such that*

- $\varepsilon(0) = 0$ *and* $\varepsilon(t) > 0$ *for* $t > 0$,
- $E(t) \to \infty$ *as* $t \to 0$ *and* $E(t) \ge 0$ *for* $t \ge 0$

*which satisfy*

$$\varepsilon(t) < p_N(x,t) < E(t) \quad \text{for all} \quad t > 0.$$

*In particular, there exists a constant $M \ge 1$ and a time $\bar{T} > 0$ such that*

$$\frac{1}{M} \le p_N(x,t) \le M \quad \text{for all} \quad t \ge \bar{T}. \tag{3.4.29}$$

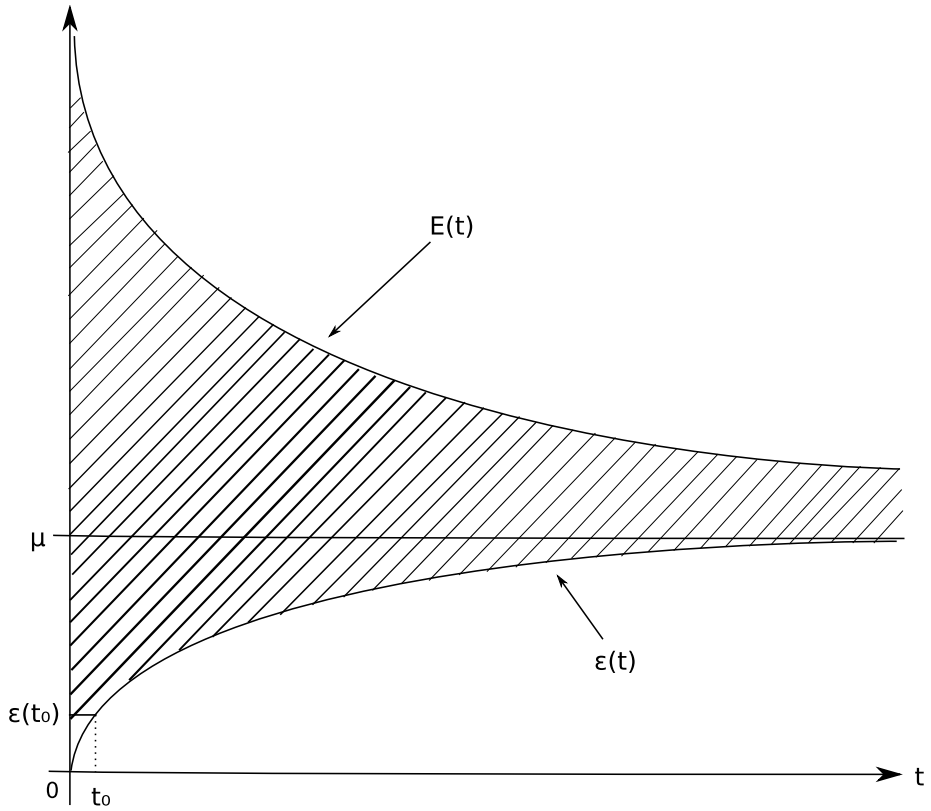



Figure 3.3: Bounds for $p_N(x,t)$

*Proof*: By Theorem 3.4.2 we know that for the initial datum in the given form, the local solution of the problem $(P)$ exists and is

$$p_N(x,t) = \sum_{i=1}^{N} p_i(t)\,\chi_{E_i(x)}.$$

Moreover, by Theorems 3.4.10 and 3.4.14, for any $t > 0$ we have

$$\begin{aligned} & \sum_{i=1}^{N} \varepsilon(t)\chi_{E_i}(x) \;<\; \sum_{i=1}^{N} p_i(t)\chi_{E_i}(x) \;<\; \sum_{i=1}^{N} E(t)\chi_{E_i}(t) \\ \Leftrightarrow\quad & \varepsilon(t) \sum_{i=1}^{N} \chi_{E_i}(x) \;<\; p_N(x,t) \;<\; E(t) \sum_{i=1}^{N} \chi_{E_i}(x) \\ \Leftrightarrow\quad & \varepsilon(t)\,\chi_{(0,1)}(x) \;<\; p_N(x,t) \;<\; E(t)\,\chi_{(0,1)}(x) \end{aligned}$$

which proves that $p_N(x,t)$ is a global solution satisfying the necessary bounds. In particular, defining

$$M := \max\left\{\frac{1}{\varepsilon(\bar{T})}, E(\bar{T})\right\},$$

where $\varepsilon(t)$ and $E(t)$ are as in Propositions 3.4.7 and 3.4.12 respectively, and choosing $\bar{T} > 0$ such that $M \geq 1$, we obtain (3.4.29) and this completes the proof. Q.E.D

### 3.4.4 The General Case

In this section we consider the problem $(P)$ with general initial data $p_0(x)$. We will follow an approximation method and make use of the upper and lower bounds we obtained in the previous section. Before stating the main result, we prove the following technical lemma.

**Lemma 3.4.16.** *Assume that (3.4.3), (3.4.4) and (3.4.5) hold. Then, there exists a positive constant $K > 0$ such that for any $p \neq q$,*

$$\frac{\sigma(p) - \sigma(q)}{p - q} \geq -K.$$

*Proof*: If the claim was not true, then there would exist $p_j$ and $q_j$ with $p_j \neq q_j$ such that

$$\frac{\sigma(p_j) - \sigma(q_j)}{p_j - q_j} \to -\infty.$$

Without loss of generality we can assume that $p_j \to p$, $q_j \to q$, where $p, q \in [0, \infty]$.

*Case 1:* $0 < p < \infty$, $0 < q < \infty$.
In this case, by (3.4.3) we immediately get that

$$\left| \frac{\sigma(p_j) - \sigma(q_j)}{p_j - q_j} \right| \leq C$$

for some constant $C$, giving a contradiction.

*Case 2:* $p = q = 0$ or $p = q = \infty$.
In both cases, by (3.4.4a) and (3.4.5a), for any sufficiently small $\varepsilon > 0$ we have that $\sigma'(p) > 0$ for $p < \varepsilon$ and $p > 1/\varepsilon$. This gives

$$\frac{\sigma(p_j) - \sigma(q_j)}{p_j - q_j} \geq 0$$

for large enough $j$ which is a contradiction.

*Case 3:* $p = 0$, $q \in (0, \infty)$ or $p = \infty$, $q \in (0, \infty)$ (or vice versa).
In these cases, by (3.4.4b) and (3.4.5b) we have either

$$\sigma(p_j) - \sigma(q_j) < 0 \quad \text{and} \quad p_j - q_j < 0$$

or

$$\sigma(p_j) - \sigma(q_j) > 0 \quad \text{and} \quad p_j - q_j > 0$$

for large enough $j$, giving a contradiction.

*Case 4:* $p = 0$, $q = \infty$ (or vice versa).
In this case, similar to the previous case, by (3.4.4b) and (3.4.5b) we obtain

$$\sigma(p_j) - \sigma(q_j) < 0 \quad \text{and} \quad p_j - q_j < 0$$

for large enough $j$, giving a contradiction again. Q.E.D

We now prove a proposition which is fundamental for our main result in this section.

**Proposition 3.4.17.** *Assume (3.4.3), (3.4.4), (3.4.5) hold and we have*

$$p_{0N}(x) \to p_0(x) \quad in \quad L^2(0,1), \tag{3.4.30}$$

*where $p_{0N}(x)$ is the initial datum as in Corollary 3.4.15, having the corresponding solution $p_N(x,t)$. Then, there exists a $p(x,t)$ such that*

$$p_N(x,t) \to p(x,t) \quad in \quad C([0,T]; L^2(0,1)) \quad as \quad N \to \infty.$$

*Proof*: Let us take any two solutions $p_N(x,t)$ and $p_M(x,t)$ of problem $(P)$ as in Corollary 3.4.15. Then we get

$$\begin{aligned} \big(p_N(x,t) - p_M(x,t)\big)_t &= -\big(\sigma(p_N(x,t)) - \sigma(p_M(x,t))\big) + \\ &\quad + \int_0^1 \big(\sigma(p_N(y,t)) - \sigma(p_M(y,t))\big)\, dy. \end{aligned}$$

Together with the boundary conditions, this implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 |\, p_N(x,t) - p_M(x,t)\,|^2\, dx = \\ &= -\int_0^1 (\sigma(p_N) - \sigma(p_M))(p_N - p_M)\, dx + \\ &\qquad + \left(\int_0^1 (p_N - p_M)\, dx\right)\left(\int_0^1 (\sigma(p_N) - \sigma(p_M))\, dy\right) \\ &= -\int_0^1 (\sigma(p_N) - \sigma(p_M))(p_N - p_M)\, dx. \end{aligned}$$

We can have either

$$(\sigma(p_N) - \sigma(p_M))(p_N - p_M) = 0$$

or

$$(\sigma(p_N) - \sigma(p_M))(p_N - p_M) = \frac{(\sigma(p_N) - \sigma(p_M))}{p_N - p_M}(p_N - p_M)^2.$$

Therefore, by Lemma 3.4.16 we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 |p_N(x,t)-p_M(x,t)|^2\,dx = \\
&\quad = -\int_{(\sigma(p_N)-\sigma(p_M))(p_N-p_M)=0} (\sigma(p_N)-\sigma(p_M))(p_N-p_M)\,dx \\
&\qquad -\int_{\frac{(\sigma(p_N)-\sigma(p_M))}{p_N-p_M}>0} (\sigma(p_N)-\sigma(p_M))(p_N-p_M)\,dx \\
&\qquad -\int_{\frac{(\sigma(p_N)-\sigma(p_M))}{p_N-p_M}<0} (\sigma(p_N)-\sigma(p_M))(p_N-p_M)\,dx \\
&\quad \le -\int_{\frac{(\sigma(p_N)-\sigma(p_M))}{p_N-p_M}<0} \frac{(\sigma(p_N)-\sigma(p_M))}{p_N-p_M}(p_N-p_M)^2\,dx \\
&\quad \le K\int_{\left\{x\in(0,1):\ \frac{(\sigma(p_N(x,t))-\sigma(p_M(x,t)))}{p_N(x,t)-p_M(x,t)}<0\right\}} (p_N-p_M)^2\,dx \\
&\quad \le K\int_0^1 (p_N-p_M)^2\,dx.
\end{aligned}$$

Denoting $\displaystyle\int_0^1 |p_N(x,t)-p_M(x,t)|^2\,dx = f(t)$, we have

$$f(t) \le f(0) + 2K\int_0^t f(s)\,ds.$$

By Grönwall's inequality (see e.g. Sec. 1.1 in [97]) we get

$$f(t) \le f(0)\exp(2Kt).$$

Rewriting this in terms of $p_N$ and $p_M$ we obtain

$$\int_0^1 |p_N(x,t)-p_M(x,t)|^2\,dx \le \exp(2Kt)\int_0^1 |p_{0N}(x)-p_{0M}(x)|^2\,dx\,. \tag{3.4.31}$$

By (3.4.30) this shows that $p_N(x,t)\in C([0,T];L^2(0,1))$ and is a Cauchy sequence. Hence there exists a limit $p(x,t)$ such that as $N\to\infty$ we get

$$p_N(x,t) \to p(x,t) \text{ in } C([0,T];L^2(0,1))$$

as required. Q.E.D

We give the definition of a solution to problem $(P)$ with arbitrary positive initial data before we state and prove the main result of this section.

**Definition 3.4.1.** *We say that $p(x,t)$ is a solution of the initial boundary-value problem $(P)$ on $(0,1)\times(0,T)$ if:*

(i) $p(x,t)\in L^1(0,T;L^2(0,1))$ and $\sigma(p(x,t))\in L^1(\tau,T;L^1(0,1))$ where $0<\tau<T$,

(ii) $p(x,t)>0$ for a.e. $x\in(0,1)$ and for all $t\in[0,T]$,

(iii) $p(\cdot,t)\to p_0(\cdot)$ in $L^2(0,1)$ as $t\to 0$,

(iv) for a.e. $x\in(0,1)$ the identity

$$p(x,t)-p(x,s)=-\int_s^t\sigma(p(x,\tau))\,d\tau+\int_s^t\int_0^1\sigma(p(y,\tau))\,dy\,d\tau$$

holds for all $s,t$ with $0<s<t<T$.

**Theorem 3.4.18.** *Assume that (3.4.3), (3.4.4), (3.4.5) hold and*

$$p_0\in L^2(0,1)\ \textit{and}\ p_0(x)>0\ \textit{for a.e.}\ x\in(0,1). \tag{3.4.32}$$

*Then there exists a unique solution $p(x,t)$ to problem $(P)$.*

*Proof*: (*Existence*) Let us take an arbitrary measurable initial data $p_0\in L^2(0,1)$ such that $p_0(x)>0$ a.e. $x\in(0,1)$. We know that for any nonnegative measurable function, there exists a nonnegative increasing sequence of functions converging pointwise to it such that each element of the sequence has only a finite number of values (cf. [22]). Since one can make this sequence positive by adding $1/n$ to each element, we have the existence of (increasing) $p_{0N}(x)$ as in Proposition 3.4.17 such that

$$p_{0N}(x)\to p_0(x)\ \text{ in }\ L^2(0,1). \tag{3.4.33}$$

By Proposition 3.4.17 we know that (3.4.33) implies the existence of a $p(x,t)$ such that

$$p_N(x,t)\to p(x,t)\ \text{ in }\ C([0,T];L^2(0,1)). \tag{3.4.34}$$

It is therefore enough to show that $p(x,t)$ satisfies the conditions in Definition 3.4.1.

We note that (3.4.34) implies $p(x,t)\in L^1(0,T;L^2(0,1))$ as required in (i) of the Definition 3.4.1. Also it immediately gives (iii). Moreover, by Proposition 3.4.17, for any $t\in[\tau,T]$ we have

$$p_N(\cdot,t)\to p(\cdot,t)\text{ in }L^2(0,1).$$

Thus, there exists a subsequence $\{N_j\}_{j\ge1}$ such that

$$p_{N_j}(\cdot,t)\to p(\cdot,t)\ \text{ for a.e.}\,x\in(0,1). \tag{3.4.35}$$

Note that the subsequence $N_j$ might depend on $t$. In this case, for $t_1 \in [\tau, T]$ we have

$$\exists\ N_{j_1} : p_{N_{j_1}}(\cdot, t_1) \to p(\cdot, t_1)\ \text{ a.e. } x \in (0,1).$$

However, $p_{N_{j_1}}(x, t_2)$ is a subsequence of $p_N(x,t)$ for $t_2 \in [\tau, T]$. Hence we get

$$p_{N_{j_1}}(\cdot, t_2) \to p(\cdot, t_2)\ \text{ in } L^2(0,1).$$

Then, there exists a further subsequence of $p_{N_{j_1}}(x, t_2)$, say $p_{N_{j_2}}(x, t_2)$, such that

$$p_{N_{j_2}}(x, t_2) \to p(x, t_2)\ \text{ a.e. } x \in (0,1).$$

By an inductive argument, for $t_i \in [\tau, T]$, we have

$$p_{N_{j_i}}(\cdot, t_i) \to p(\cdot, t_i)\ \text{ a.e. } x \in (0,1). \tag{3.4.36}$$

We then consider the sequence $\{p_{N_{j_j}}(x,t)\}_{j=1}^{\infty}$, which clearly is a subsequence of the sequence $\{p_{N_{j_i}}(x,t)\}_{j=1}^{\infty}$. Therefore, by (3.4.36) we get

$$p_{N_{j_j}}(\cdot, t_i) \to p(\cdot, t_i)\ \text{ a.e. } x \in (0,1)\ \text{ for all } i.$$

Hence we can find a subsequence, denoted by $p_{N_j}(x,t)$, such that

$$p_{N_j}(\cdot, t_k)\ \to\ p(\cdot, t_k)\ \text{a.e. } x \in (0,1)$$

where $\{t_k\}_{k\geq 1}$ is dense in $[\tau, T]$ so that (3.4.35) holds for all $t \in [\tau, T]$ except on a set of measure zero. We also know by Corollary 3.4.15 that

$$\varepsilon(\tau) < p_N(x,t) < E(\tau)\ \text{ for all } t \in [\tau, T]. \tag{3.4.37}$$

Therefore, by assumption (3.4.3) we get

$$|\sigma(p_N(x,t))| \leq C\ \text{ for all }\ x \in (0,1)\ \text{ and }\ t \in [\tau, T], \tag{3.4.38}$$

where $C$ denotes a generic constant depending on $\varepsilon(\tau), E(\tau)$ and the Lipschitz constant for $\sigma$. As $p_N(x,t)$ is a solution of $(P)$, by (3.4.38) and Definition 3.4.1 we obtain

$$\begin{aligned} |p_{N_j}(x,t) - p_{N_j}(x,t_k)| &= \left| \int_{t_k}^{t} \sigma(p_{N_j}(x,\tau) d\tau - \int_{t_k}^{t}\int_0^1 \sigma(p_{N_j}(y,\tau) dy d\tau \right| \\ &\leq \int_{t_k}^{t} |\sigma(p_{N_j}(x,\tau)| d\tau + \int_{t_k}^{t}\int_0^1 |\sigma(p_{N_j}(y,\tau)| dy d\tau \\ &\leq C\,|t - t_k|. \end{aligned}$$

Thus, writing

$$p_{N_j}(x,t) \;=\; p_{N_j}(x,t_k) + p_{N_j}(x,t) - p_{N_j}(x,t_k)$$

and letting $t_k \to t$ and $j \to \infty$, we again get the necessary convergence.

Continuing from (3.4.35), by assumption (3.4.3), for any $t \in [\tau, T]$ we have

$$\sigma(p_{N_j}(\cdot,t)) \;\to\; \sigma(p(\cdot,t)) \;\text{ for a.e. } x \in (0,1). \tag{3.4.39}$$

By (3.4.38) and (3.4.39), we immediately get

$$\int_\tau^t \sigma(p_{N_j}(x,s))ds \to \int_\tau^t \sigma(p(x,s))\,ds \;\text{ for a.e. } x \in (0,1) \tag{3.4.40}$$

and by Lebesgue's dominated convergence theorem we obtain

$$\int_0^1 \sigma(p_{N_j}(y,t))\,dy \;\to\; \int_0^1 \sigma(p(y,t))\,dy \;\text{ for } t \in [\tau,T]. \tag{3.4.41}$$

Let us denote

$$\int_0^1 \sigma(p_{N_j}(x,t))dx =: f_j(t) \quad\text{and}\quad \int_0^1 \sigma(p(x,t))dx =: f(t).$$

Then, (3.4.41) is equivalent to

$$f_j(t) \to f(t) \;\text{ for } t \in [\tau,T].$$

Therefore, by (3.4.38) and (3.4.41) we obtain

$$\int_\tau^t f_j(s)ds \to \int_\tau^t f(s)ds, \quad t \in [\tau,T].$$

Rewriting this information in terms of $\sigma$, for any $t \in [\tau,T]$ we obtain

$$\int_\tau^t \int_0^1 \sigma(p_{N_j}(x,s))dxds \to \int_\tau^t \int_0^1 \sigma(p(x,s))dxds. \tag{3.4.42}$$

Finally, by (3.4.34), (3.4.40) and (3.4.42) we can pass to the limit in the equation

$$p_N(x,t) \;=\; p_N(x,\tau) \;-\; \int_\tau^t \sigma(p_N(x,s))ds \;+\; \int_\tau^t \int_0^1 \sigma(p_N(y,s))\,dy\,ds,$$

which holds by Definition 3.4.1 as $p_N(x,t)$ is a solution of $(P)$. The two integrals converge up to subsequences but we know by (3.4.34) that $p_N(x,t)$ converges to $p(x,t)$ in $C([0,T];L^2(0,1))$. Therefore, we get the convergence for the whole sequence as $N \to \infty$. We obtain

$$p(x,t) \;=\; p(x,\tau) \;-\; \int_\tau^t \sigma(p(x,s))ds \;+\; \int_\tau^t \int_0^1 \sigma(p(y,s))\,dy\,ds \tag{3.4.43}$$

as required for (iv) in Definition 3.4.1. Clearly, (3.4.43) also gives $\sigma(p(x,t)) \in L^1(\tau, T; L^1(0,1))$ which completes the proof of (i). Finally, (3.4.37) and assumption (3.4.32) prove (ii). Therefore we conclude that $p(x,t)$ is a solution of $(P)$.

(*Uniqueness*) By (i), (iv) in Definition 3.4.1 and a result due to Wilcox [98, Thm. 2.1] we ensure, possibly after redefinition on a set of measure zero, that $p(\cdot,t)$ is continuous from $[0,T] \to L^2(0,1)$ and for almost every $x \in (0,1)$, $(P)$ holds. Let us take two different solutions $p_1(x,t)$ and $p_2(x,t)$ of problem $(P)$ with $p_1(x,0) = p_2(x,0)$. Then we get

$$(p_1(x,t))_t = -\sigma(p_1(x,\tau))\,d\tau + \int_0^1 \sigma(p_1(y,\tau))\,dy\,d\tau$$

and

$$(p_2(x,t))_t = -\sigma(p_2(x,\tau))\,d\tau + \int_0^1 \sigma(p_2(y,\tau))\,dy\,d\tau,$$

for almost every $x \in (0,1)$. After subtracting these two equalities and arguing as in the proof of Proposition 3.4.17 we get a similar inequality to (3.4.31) which is

$$\|p_1(x,t) - p_2(x,t)\|_2^2 \leq \exp(2Kt)\,\|p_1(x,0) - p_2(x,0)\|_2^2.$$

As $p_1(x,t)$ and $p_2(x,t)$ have the same initial data we get $p_1(x,t) = p_2(x,t)$ as required. Q.E.D

Having existence of a solution $p(x,t)$ for general initial data, we have the following remark related to the bounds we obtained for $p_N(x,t)$ in Section 3.4.3.

**Remark 3.4.3.** *As stated in Corollary 3.4.15, the uniform upper and lower bounds for $p_N(x,t)$ are independent of $N$. Hence, they remain valid for the solution $p(x,t)$ corresponding to general initial data.*

## 3.5 Relation with the Theory of Gradient Flows

In this section, we obtain an existence theory for problem $(P)$ using the theory of gradient flows. We first give a short introduction to the classical theory appealing mostly to the work of Mielke & Stefanelli [82] and Vrabie [97]. One can also refer to [26], [59] and [31] for the general well-posedness, to [3] and [72] for a metric space approach, to [82], [81] and [36] for generalized existence results, and to [67] and [66] for a variational approach, the former being particularly for $\lambda$-convex functions.

### 3.5.1 Classical Theory of Gradient Flows

For given $T > 0$ and $f : (0,T) \to H$, the gradient flow equation is

$$(GF) \quad \begin{cases} u'(t) + \partial\phi(u(t)) \ni f(t) \text{ for a.e. } t \in (0,T) \\ u(0) = u_0 \end{cases}$$

where $\partial\phi\colon D(\partial\phi) \subset H \to 2^H$ is the Fréchet subdifferential of the proper and lower semicontinuous functional $\phi : H \to (-\infty, \infty]$ with effective domain

$$D(\phi) = \{u \in H : \phi(u) < \infty\}.$$

We denote by $H$ an arbitrary Hilbert space with inner product $(\cdot,\cdot)$ and norm $||\cdot||$ and $D(\partial\phi)$ denotes the subset of $H$ given by all the elements $u \in D(\phi)$ such that $\partial\phi(u) \neq \emptyset$.

Recall that the functional $\phi$ is said to be *proper* if $D(\phi) \neq \emptyset$ and the *Fréchet subdifferential* $\partial\phi$ of $\phi$ at a point $u \in D(\phi)$ is defined as

$$v \in \partial\phi(u) \quad \Leftrightarrow \quad \liminf_{\omega \to u} \frac{\phi(\omega) - \phi(u) - (v, \omega - u)}{||\omega - u||} \geq 0. \tag{3.5.1}$$

We define the solution of $(GF)$ following Vrabie [97] where it is given for any $m$-accretive operator (see e.g. [88] for the definition) rather than $\partial\phi$.

**Definition 3.5.1.** *Let $f \in L^1(0,T;H)$. A function $u : [0,T] \to H$ is called a solution of $(GF)$ if*

(i) $u(t) \in D(\phi)$ for a.e. $t \in (0,T)$,

(ii) $u(t) \in W^{1,1}_{\text{loc}}(0,T;H)$,

(iii) there exists $g \in L^1_{\text{loc}}(0,T;H)$, $g(t) \in \partial\phi(u(t))$ for a.e. $t \in (0,T)$ such that

$$u'(t) + g(t) = f(t) \text{ for a.e. } t \in (0,T).$$

**Remark 3.5.1** ([97], pg. 23)**.** *Note that in the inclusion $(GF)$, $u'(t)$ is the strong derivative of $u$ which, since $u(t) \in W^{1,1}_{\text{loc}}(0,T;H)$, is almost everywhere defined on $(0,T)$.*

It is well-known that $\partial\phi$ is a maximal monotone operator (see e.g. [80] for the definition) and even in the nonconvex case, for any $u \in D(\partial\phi)$, the Fréchet subdifferential $\partial\phi(u)$ is a convex and closed subset of $H$ (see e.g. [67], [81]). When $\phi$ is convex, the Fréchet subdifferential can be characterized by

$$v \in \partial\phi(u) \quad \Leftrightarrow \quad \begin{cases} u \in D(\phi) \text{ and} \\ \phi(\omega) - \phi(u) - (v, \omega - u) \geq 0 \text{ for all } \omega \in H. \end{cases}$$

In this case, existence, uniqueness and regularity of solutions for problem $(GF)$ follow from the renowned theory of nonlinear semigroups in Hilbert spaces developed by Brézis [26], Crandall & Pazy [31] and Komura [59]. In particular, it is well-known that if $u_0 \in D(\phi)$ and $f \in L^2(0,T;H)$ then the Cauchy problem $(GF)$ admits a unique solution $u \in H^1(0,T;H)$ which satisfies the energy identity (cf. [82])

$$\phi(u(t)) + \int_s^t |u'(r)|^2\, dr \;=\; \phi(u(s)) + \int_s^t (f(r), u(r))\, dr, \;\; \forall\, 0 \le s \le t \le T.$$

It is also worth noting the following result due to Minty and Moreau which shows the relation between subdifferential of certain functions and $m$-accretive operators.

**Remark 3.5.2** ([97], Thm. 1.6.2)**.** *If $\phi\colon H \to (-\infty, \infty]$ is a proper, lower semi-continuous, convex function, then its subdifferential $\partial\phi\colon D(\partial\phi) \subset H \to 2^H$ is an $m$-accretive operator.*

### 3.5.2 $\lambda$-convexity

We are particularly interested in $\lambda$-convex functionals, that is, quadratic perturbations of convex functionals. In order to express this mathematically, we give the following definition.

**Definition 3.5.2.** *We say that $\phi$ is $\lambda$-convex if*

$$v \;\mapsto\; \psi(v) := \phi(v) + \frac{\lambda}{2}|v|^2 \text{ is convex for some } \lambda \in \mathbb{R}. \tag{3.5.2}$$

It is also known that $(GF)$ is well-posed when $\phi$ is assumed to be $\lambda$-convex (see e.g. [67], [3] and references therein). Using Definition 3.5.2, we can prove the following trivial result for smooth $\phi$.

**Lemma 3.5.1.** *For any nonconvex $\phi \in C^2(0,\infty)$ which is $\lambda$-convex, $\lambda$ must be non-negative.*

*Proof*: If $\phi \in C^2(0,\infty)$, then by Definition 3.5.2 we have

$$\begin{aligned}
\phi \text{ is } \lambda\text{-convex} \quad &\Leftrightarrow \quad \psi(v) := \phi(v) + \frac{\lambda}{2}|v|^2 \text{ is convex} \\
&\Leftrightarrow \quad \psi''(v) \;\ge\; 0 \;\; \text{for all } v \\
&\Leftrightarrow \quad \phi''(v) + \lambda \;\ge\; 0 \;\; \text{for all } v \\
&\Leftrightarrow \quad \phi''(v) \;\ge\; -\lambda \;\; \text{for all } v.
\end{aligned}$$

Hence if $\lambda \le 0$, then we would have $\phi''(v) \ge 0$ for all $v$. This implies that $\phi$ is convex and gives a contradiction. Q.E.D

As we will put the assumption of $\lambda$-convexity on the stored energy function $W$, we give the above result in a form that we will refer to later.

**Corollary 3.5.2.** *Assume $W \in C^2(0,\infty)$ is $\lambda$-convex for $\lambda \geq 0$. Then*

$$W''(v) \geq -\lambda \quad \textit{for all } v \in (0,\infty).$$

We now show, by reproving Lemma 3.4.16, that the assumption of $\lambda$-convexity makes some of our previous calculations much easier.

**Lemma 3.5.3.** *Assume $W \in C^2(0,\infty)$ is $\lambda$-convex for some $\lambda \geq 0$. Then, for any $p \neq q$ we have*

$$\frac{W'(p) - W'(q)}{p - q} \geq -\lambda.$$

*Proof*: By Corollary 3.5.2 we know that

$$W''(p) \geq -\lambda \text{ for all } p.$$

Suppose without loss of generality that $p < q$. Then, by the Mean Value Theorem, we immediately get

$$\begin{aligned} \frac{W'(p) - W'(q)}{p - q} &= W''(\tilde{p}) \text{ for some } \tilde{p} \in (p,q) \\ &\geq -\lambda \end{aligned}$$

as claimed. Q.E.D

### 3.5.3 Equivalence of the theories

In this section we show that the existence theory we have developed in Section 3.4.2 can also be obtained by the theory of gradient flows for $\lambda$-convex functionals, and vice versa. First, we show that the assumptions we made for the stress $\sigma = W'$ in Section 3.4.2 imply $\lambda$-convexity.

**Proposition 3.5.4.** *Assume $W \in C^2(0,\infty)$ and $W'$ satisfies (3.4.3), (3.4.4a) and (3.4.5a). Then, $W$ is $\lambda$-convex for some real $\lambda \geq 0$.*

*Proof*: First note that by the assumptions we make, we know that $W''(p)$ exists for all $p \in (0,\infty)$. Moreover, by Definition 3.5.2 we have

$$\begin{aligned} W \text{ is } \lambda\text{-convex} &\Leftrightarrow W(p) + \frac{\lambda}{2}|p|^2 \text{ is convex} \\ &\Leftrightarrow W'(p) + \lambda\, p \text{ is nondecreasing.} \end{aligned}$$

Therefore it is enough to show that $W'(p) + \lambda\, p$ is nondecreasing for $\lambda \geq 0$.

For any sufficiently small $\varepsilon > 0$, we know by assumptions (3.4.4a) and (3.4.5a) that if $p \leq \varepsilon$ or $p \geq 1/\varepsilon$, then $W'(p)$ is (strictly) monotonic increasing. This immediately gives $W'(p) + \lambda\, p$ is nondecreasing for any $\lambda \geq 0$.

If, on the other hand, $p \in (\varepsilon, 1/\varepsilon)$, then by assumption (3.4.3) we have

$$|W'(p) - W'(q)\,| \;\leq\; L\,|p - q\,|$$

where $L > 0$ stands for the Lipschitz constant for $\sigma$. Moreover, we have

$$\begin{aligned} |W'(p) - W'(q)\,| \;\leq\; L\,|p - q\,| \quad &\Leftrightarrow \quad \left|\frac{W'(p) - W'(q)}{p - q}\right| \;\leq\; L \\ &\Rightarrow \quad -L \;\leq\; \frac{W'(p) - W'(q)}{p - q}. \end{aligned} \tag{3.5.3}$$

In order to show that $W'(p) + \lambda\, p$ is nondecreasing we take arbitrary $p$ and $q$ in $(\varepsilon, 1/\varepsilon)$ such that $p > q$ and by (3.5.3) we get

$$\frac{W'(p) + L\,p - W'(q) - L\,q}{p - q} \;\geq\; 0$$

which implies

$$W'(p) + L\,p - W'(q) - L\,q \;\geq\; 0.$$

Choosing $\lambda = L$ gives the result. Q.E.D

In order for our notation to be consistent with Definition 3.5.1, let us denote any $p(x,t) \in L^1(0,T;L^2(0,1))$ by $p(t)$. For simplicity, we will only write $p(t) \in L^2(0,1)$, which will mean that it is defined as a function of $x$ for almost every $t \in (0,T)$. We have the following definitions:

**Definition 3.5.3.** *We define the functional $\phi$ as*

$$\phi(p) = \begin{cases} \displaystyle\int_0^1 W(p)\,dx, & \textit{if } \displaystyle\int_0^1 p\,dx = \mu \\ +\infty, & \textit{otherwise} \end{cases}$$

*and its effective domain as*

$$D(\phi(p)) \;=\; \left\{ p \in L^2(0,1) :\; p > 0,\; \int_0^1 W(p)\,dx \;< \infty,\; \int_0^1 p\,dx = \mu \right\}.$$

We now prove the fundamental result necessary for the proof of the equivalence of the theories.

**Proposition 3.5.5.** *Assume $W \in C^2(0,\infty)$ is $\lambda$-convex and $W'$ satisfies (3.4.3). Then we have*

$$v \in \partial\,\phi(p) \;\Leftrightarrow\; \begin{cases} p > 0 \;\; \textit{and} \;\; \displaystyle\int_0^1 p\,dx = \mu, \\ W'(p) \in L^2(0,1) \\ v = W'(p) - c \;\; \textit{for a constant } c. \end{cases}$$

*Proof*: First recall that by (3.5.1) we have $v \in \partial\phi(p)$ if and only if $p \in D(\phi)$ and

$$\liminf_{q\to p} \frac{\phi(q) - \phi(p) - (v, q-p)}{||q-p||_2} \;\geq\; 0.$$

For necessity, as $p > 0$ and $\displaystyle\int_0^1 p\,dx = \mu$, by definition of $\phi$ we immediately get that $p \in D(\phi)$. Moreover, as $v = W'(p) - c$ for a constant $c$, taking any $q^{(j)}(t) \in D(\phi)$ such that $q^{(j)} \to p$ in $L^2(0,1)$ gives

$$\begin{aligned} &\liminf_{q\to p} \frac{\phi(q) - \phi(p) - (v, q-p)}{||q-p||_2} = \liminf_{j\to\infty} \frac{\phi(q^{(j)}) - \phi(p) - (v, q^{(j)}-p)}{||q^{(j)}-p||_2} \\ &= \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \Big( W(q^{(j)}) - W(p) - (W'(p) - c)(q^{(j)} - p) \Big)\, dx. \end{aligned}$$

As $q^{(j)} \in D(\phi)$, it also satisfies the condition $\displaystyle\int_0^1 q^{(j)}\,dx = \mu$. Therefore we obtain

$$\begin{aligned} &\liminf_{q\to p} \frac{\phi(q) - \phi(p) - (v, q-p)}{||q-p||_2} = \\ &\quad = \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \Big( W(q^{(j)}) - W(p) - W'(p)(q^{(j)} - p) \Big)\, dx. \end{aligned}$$

Let us define $\psi(v) := W(v) + \dfrac{\lambda}{2}|v|^2$ so that we can continue as

$$\begin{aligned} &\liminf_{q\to p} \frac{\phi(q) - \phi(p) - (v, q-p)}{||q-p||_2} = \\ &\quad = \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \Big( \psi(q^{(j)}) - \frac{\lambda}{2}|q^{(j)}|^2 - \psi(p) + \\ &\qquad\qquad + \frac{\lambda}{2}|p|^2 - (\psi'(p) - \lambda\,p)(q^{(j)} - p) \Big)\, dx \\ &\quad = \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \Big( \psi(q^{(j)}) - \psi(p) - \psi'(p)(q^{(j)} - p) \Big)\, dx - \\ &\qquad\qquad - \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \frac{\lambda}{2}|q^{(j)} - p|^2\, dx. \end{aligned}$$

By Definition 3.5.2 and the assumption of $\lambda$-convexity of $W$, we know that $\psi$ is convex. Therefore, we obtain

$$\begin{aligned}
&\liminf_{q\to p} \frac{\phi(q)-\phi(p)-(v,q-p)}{||q-p||_2} = \\
&= \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \Big(\psi(q^{(j)})-\psi(p)-\psi'(p)(q^{(j)}-p)\Big)\,dx - \\
&\qquad\qquad - \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \int_0^1 \frac{\lambda}{2}|q^{(j)}-p|^2\,dx \\
&\geq \liminf_{j\to\infty} \frac{1}{||q^{(j)}-p||_2} \left(-\frac{\lambda}{2}\right) \int_0^1 |q^{(j)}-p|^2\,dx \\
&= \liminf_{j\to\infty} \left(-\frac{\lambda}{2}\right) ||q^{(j)}-p||_2 = 0
\end{aligned}$$

as required.

For sufficiency, take $v \in \partial\phi(p)$. Then we know that $p \in D(\phi)$ and hence $p > 0$ and $\int_0^1 p\,dx = \mu$. Moreover, for any $q \in D(\phi)$ such that $q \to p$ in $L^2(0,1)$ we must have

$$\liminf_{q\to p} \frac{\phi(q)-\phi(p)-(v,q-p)}{||q-p||_2} \geq 0. \tag{3.5.4}$$

In particular, let us take $q^{(j)} = p + \dfrac{z}{j}$ where $z$ satisfies

$$\begin{cases}
z \in L^\infty(0,1), \\
z = 0 \text{ outside } E_\tau := \left\{x \in (0,1) :\, \tau < p(x) < \dfrac{1}{\tau},\, \tau \geq 0\right\}, \\
\displaystyle\int_{E_\tau} z\,dx = 0.
\end{cases}$$

Then, clearly, $q^{(j)} \to p$ in $L^2(0,1)$ as $j \to \infty$. Moreover, as

$$\int_0^1 q^{(j)}\,dx = \int_0^1 \left(p + \frac{z}{j}\right) dx = \mu,$$

we have $q^{(j)} \in D(\phi)$. For this special $q^{(j)}$, (3.5.4) implies

$$\liminf_{j\to\infty} \Big(\phi(q^{(j)})-\phi(p)-(v,q^{(j)}-p)\Big) \geq 0$$

which, by Definition 3.5.3, gives

$$\liminf_{j\to\infty} \int_0^1 \left[W\Big(p+\frac{z}{j}\Big) - W(p) - v\,(q^{(j)}-p)\right] dx \geq 0$$

and hence

$$\liminf_{j\to\infty} \int_0^1 \left[W\Big(p+\frac{z}{j}\Big) - W(p) - v\,\frac{z}{j}\right] dx \geq 0. \tag{3.5.5}$$

By assumption (3.4.3) we know that $W''$ is well-defined on $(0,\infty)$. Therefore, we can apply Taylor's theorem to $W$ in (3.5.5) and obtain

$$\liminf_{j\to\infty} \int_0^1 \left[ W(p) + W'(p)\frac{z}{j} + o\left(\frac{1}{j}\right) - W(p) - v\,\frac{z}{j} \right] dx \;\geq\; 0.$$

This gives

$$\begin{aligned}
&\liminf_{j\to\infty} \left[ \int_0^1 (W'(p) - v)\; \frac{z}{j}\, dx \;+\; o\left(\frac{1}{j}\right) \right] \;\geq\; 0 \\
&\Leftrightarrow\; \liminf_{j\to\infty} \left[ \frac{1}{j} \int_0^1 (W'(p) - v)\, z\, dx \;+\; o\left(\frac{1}{j}\right) \right] \;\geq\; 0 \\
&\Rightarrow\; \left( \liminf_{j\to\infty} \frac{1}{j} \right) \int_0^1 (W'(p) - v)\, z\, dx \;\geq\; 0 \\
&\Rightarrow\; \int_0^1 (W'(p) - v)\, z\, dx \;\geq\; 0. \qquad (3.5.6)
\end{aligned}$$

However, by its definition, $z = 0$ outside $E_\tau$, hence (3.5.6) is equivalent to saying that

$$\int_{E_\tau} (W'(p) - v)\, z\, dx \;\geq\; 0.$$

Repeating the above calculations with $-z$ instead of $z$, one gets the same inequality for $-z$. Therefore, we must have

$$\int_{E_\tau} (W'(p) - v)\, z\, dx \;=\; 0.$$

This equation is true in particular for $z = \varphi'$ where $\varphi \in C_0^\infty(E_\tau)$. Hence we have

$$\int_{E_\tau} (W'(p) - v)\, \varphi'\, dx \;=\; \int_{E_\tau} \frac{\partial}{\partial x}(W'(p) - v)\, \varphi\, dx \;=\; 0 \;\text{ for all } \varphi \in C_0^\infty(E_\tau).$$

By well-known results in the theory of Sobolev spaces (see e.g. [2, pg. 74]) we immediately get

$$W'(p) - v \;=\; c(\tau) \;\text{ for a.e. } x \in E_\tau,$$

where $c(\tau)$ is a constant depending on $\tau$. However, by definition, $E_\tau$ is an increasing set which implies that $c(\tau)$ cannot depend on $\tau$. Therefore

$$W'(p) - v \;=\; c$$

must hold for a constant $c$ as required. Q.E.D

We now state and prove the main result of this section which basically tells that the existence theory we developed for one-dimensional nonlinear viscoelasticity is equivalent to that of gradient flows.

**Theorem 3.5.6.** *Assume that $W \in C^2(0,\infty)$ is $\lambda$-convex and $W'$ satisfies (3.4.3), (3.4.4) and (3.4.5). Then, any solution $p(x,t)$ of problem $(P)$ as in Theorem 3.4.18 is a solution of $(GF)$ for almost every $t \in (0,T)$, and vice versa.*

*Proof*: Without loss of generality, we can take $f(t) \equiv 0$ in $(GF)$. First, let us assume that there exists a solution $p(t)$ to $(GF)$. Then, by Definition 3.5.1, we clearly have that $p(t) \in D(\phi(p))$ and also there exists a $g(t) \in \partial\phi(p)$ such that

$$-\frac{\partial p}{\partial t} = g(t) \quad \text{a.e. in } (0,T).$$

By Proposition 3.5.5, we must have

$$g(t) = W'(p(t)) - c(t) \quad \text{and} \quad \int_0^1 p(t)\,dx = \mu.$$

Therefore,

$$c(t) = \int_0^1 \sigma(p(t))\,dx,$$

and we have

$$-\frac{\partial p}{\partial t} = W'(p(t)) - \int_0^1 \sigma(p(t))\,dx.$$

This equation, by Definition 3.4.1, shows that $p(t)$ is a solution of $(P)$ as required.

Conversely, for almost every $t \in (0,T)$, any solution $p(t)$ of problem $(P)$ satisfies

$$p_t(t) = -\sigma(p(t)) + \int_0^1 \sigma(p(t))\,dy$$

where $\displaystyle\int_0^1 p(t)\,dx = \mu$. Equivalently, we can say that

$$-\frac{\partial p}{\partial t} = W'(p(t)) - c(t) \quad \text{for a.e. } t \in (0,T),$$

where we set $\displaystyle c(t) = \int_0^1 \sigma(p(t))\,dx$. By Proposition 3.5.5 and Definition 3.5.1, this immediately implies that $p(t)$ is also a solution of $(GF)$ for almost every $t \in (0,T)$ as required. Q.E.D

# Chapter 4

# Asymptotic Behaviour in One-dimensional Nonlinear Quasistatic Viscoelasticity

## 4.1 Introduction

The study of nonlinear dynamics is very attractive in the sense that it is fundamental for the understanding of many important problems of the natural sciences. Unlike linear systems, the evolution of nonlinear systems generally exhibits complicated behaviour which cannot be understood by pure intuition or by elementary analytical or numerical calculations. Even though they may offer possible solutions, these are often not satisfactory and there is a need for a more geometrical view which could provide the proper procedure for the computations (see [93] for an extensive discussion).

Given a dynamical system starting from an initial state, it is difficult to predict how the system will evolve as time increases. It might converge to an equilibrium state or there might exist some periodic states. Even though the dissipative character of the system may lead to the existence of absorbing sets, there are various difficulties one can encounter especially in infinite dimensions. The mathematical problem here is the study of the long-time behaviour of the system to determine which permanent state will be observed after a certain period of time.

In this section we ask this question for system $(P)$, for which we have obtained a global existence theory in Chapter 3, namely,

$$(P)\quad \begin{cases} p_t(x,t) = -\sigma(p(x,t)) + \displaystyle\int_0^1 \sigma(p(y,t))\,dy \\ p(x,0) = p_0(x) \\ \displaystyle\int_0^1 p(x,t)\,dx = \mu. \end{cases}$$

When the initial data takes only a finite number of values as in Section 3.4.3, the system reduces to the following finite system of ordinary differential equations

$$\dot{p}_i \;=\; \sigma(p_i) \;+\; \sum_j \mu_j \sigma(p_j), \qquad 1 \le i \le N,$$

where $\sum \mu_j = 1$ and $\sum \mu_j p_j = \mu$. Pego [75] proved that for this system every bounded solution stabilizes to some equilibrium as $t \to \infty$. He made use of a theorem due to Hale & Massatt [50] on stabilization of hyperbolic trajectories of systems of ordinary differential equations, which is only valid in the finite dimensional case.

Andrews & Ball [5] studied the asymptotic behaviour as $t \to \infty$ of solutions to the one-dimensional nonlinear viscoelasticity equation (3.1.1) when $S(y_x, y_{xt}) = y_{xt}$ as in our case. They introduced an assumption which they called a "nondegeneracy condition". In Pego's [75] words, assuming that $\sigma$ is piecewise monotone and, in particular, that

> for $z$ in any bounded set of $\mathbb{R}$, the equation $\sigma(z) = s$ has a finite number $M = M(s)$ of roots $z_1(s) < z_2(s) < \cdots < z_M(s)$, where $M(s)$ is piecewise continuous, jumping a finite number of times,

the nondegeneracy condition asserts that

> **Nondegeneracy Condition (NC)** : The derivatives $z_j'(s),\ j = 1, 2, \ldots, M$ are linearly independent on any common interval of definition.

This is equivalent to saying that $1, z_1(s), \cdots, z_M(s)$ are linearly independent functions of $s$ on any interval where $M(s)$ is constant (see [75, pg. 1019]). Under the assumption $(NC)$, Andrews & Ball [5] proved that $\sigma(p(\cdot, t))$ converges to a constant in $L^2(0, 1)$ as $t \to \infty$.

For the regularized nonlinear diffusion equation

$$u_t = \Delta(f(u) + \nu u_t), \quad x \in \Omega,\ t \in \mathbb{R},\ \nu > 0 \text{ constant},$$

motivated by the problem of phase separation in a viscous binary mixture, Novick-Cohen & Pego [71] proved that under the assumption of either $(NC)$ or other certain technical hypotheses, each solution approaches some steady state depending on the initial data. One such hypothesis was that $f$ is cubic with

$$f(u) \;=\; c_1(u - c_2)^3 + c_3(u - c_2) + c_4 \tag{4.1.1}$$

where $c_1 > 0$ and $c_3 < 0$. They also assumed that the mean concentration of the initial data is not equal to $c_2$, i.e.,

$$\int_\Omega u_0(x)\,dx \neq c_2. \tag{4.1.2}$$

In this chapter, we introduce a new argument to study the asymptotic properties of $(P)$. We start in Section 4.7.1 by proving with brute force, and under no technical assumptions, that any solution of problem $(P)$ converges to an equilibrium state as $t \to \infty$ when

$$\sigma(p) = p^3 - p, \tag{4.1.3}$$

which can be thought of as a shifted version of $\sigma$ satisfying the assumptions we made in Chapter 3. Then in Section 4.7.3, we introduce a new approach and apply it to the case when (4.1.3) holds. Moreover, using this new method, we reprove the stability results of Andrews & Ball [5] and Novick-Cohen & Pego [71] in this case.

## 4.2 Stationary solutions

Any stationary solution $\tilde{p} \in L^2(0,1)$ for problem $(P)$ satisfies

$$\begin{cases} 0 = -\sigma(\tilde{p}(x)) + \displaystyle\int_0^1 \sigma(\tilde{p}(y))\,dy \\ \displaystyle\int_0^1 \tilde{p}(x)\,dx = \mu. \end{cases}$$

This implies

$$\int_0^1 \sigma(\tilde{p}(y))\,dy = \sigma(\tilde{p}(x))$$

giving

$$\sigma(\tilde{p}(x)) = \text{constant}.$$

Therefore, we can define the set of equilibrium solutions for problem $(P)$ as

$$\mathcal{E}_P := \left\{ \tilde{p} \in L^2(0,1) : \sigma(\tilde{p}(x)) = C, \text{ for some } C \in \mathbb{R}, \int_0^1 \tilde{p}(x)\,dx = \mu \right\}. \tag{4.2.1}$$

**Remark 4.2.1.** *There might be a continuum of equilibrium points for the following reasons:*

- *By Remark 3.4.3 we know that any solution $p(x,t)$ is bounded for sufficiently large $t$. This implies that $\sigma(p(x,t))$ also stays bounded as $t \to \infty$. However, there might still exist continuum of $C$ such that $\sigma(p(x)) = C$.*

- *It is also possible that for fixed $C$, there might exist a continuum of roots for*

$$\sigma(p(x)) - C \,=\, 0,$$

*such that* $\displaystyle\int_0^1 p(x)\,dx = \mu$. *An example of this is when* $\sigma$ *is constant on a subset of* $(0,\infty)$.

The following result, however, shows that these possibilities are excluded for a particular choice of $\sigma$.

**Lemma 4.2.1.** *If* $\sigma$ *is assumed to be strictly monotone, then there exists a unique stationary solution* $\tilde{p}$. *Moreover,* $\tilde{p}(x) = \mu$ *for all* $x \in (0,1)$.

*Proof*: We first note that $\tilde{p}(x) \equiv \mu$ is clearly an equilibrium solution. On the other hand, by the assumption of strict monotonicity of $\sigma$ we have

$$(\sigma(p) - \sigma(\mu))\,(p - \mu) \,>\, 0 \quad \text{for all} \quad p \neq \mu. \tag{4.2.2}$$

If, for contradiction, there exists an equilibrium point $\tilde{p} \in \mathcal{E}_P$ such that $\tilde{p} \neq \mu$, then by (4.2.1) and (4.2.2) we would have

$$\int_0^1 (C - \sigma(\mu))\,(\tilde{p} - \mu)\,dx \,>\, 0. \tag{4.2.3}$$

However, $\tilde{p}$ satisfies the conditions $\displaystyle\int_0^1 \tilde{p}\,dx = \mu$. Therefore we get

$$\int_0^1 (C - \sigma(\mu))\,(\tilde{p} - \mu)\,dx \,= (C - \sigma(\mu)) \int_0^1 (\tilde{p} - \mu)\,dx \,=\, 0,$$

which contradicts (4.2.3). Q.E.D

## 4.3 The energy

In this section we show that the energy is a Lyapunov function for problem $(P)$. Before doing so, however, we recall from (2.1.2) that the initial energy is assumed to be bounded, i.e.,

$$\int_0^1 W(p_0(x))\,dx \,<\, \infty. \tag{4.3.1}$$

We also assume throughout this chapter that $p_0 \in L^2(0,1)$ with $p_0 > 0$ a.e. on $(0,1)$, so that Theorem 3.4.18 still holds.

**Lemma 4.3.1.** *Let $p(x,t)$ be a solution to problem $(P)$. Then the following dissipation identity holds*

$$\int_0^1 W(p(x,t))\,dx \;=\; \int_0^1 W(p_0(x))\,dx \;-\; \int_0^t \int_0^1 p_s^2(x,s)\,dx\,ds \tag{4.3.2}$$

*for any $t \geq 0$.*

*Proof*: Multiplying the differential equation in $(P)$ by $p_t(x,t)$ and integrating over $x$ we get

$$\begin{aligned}
& p_t(x,t) = -\sigma(p(x,t)) \;+\; \int_0^1 \sigma(p(y,t))\,dy \\
\Rightarrow\;\; & p_t^2(x,t) \;=\; -\sigma(p(x,t))\,p_t(x,t) + p_t(x,t) \int_0^1 \sigma(p(y,t))\,dy \\
\Rightarrow\;\; & \int_0^1 p_t^2(x,t)\,dx = -\int_0^1 \sigma(p(x,t))\,p_t(x,t)\,dx \;+ \\
& \qquad\qquad + \left(\int_0^1 p_t(x,t)\,dx\right)\left(\int_0^1 \sigma(p(y,t))\,dy\right).
\end{aligned}$$

Using the boundary conditions $\displaystyle\int_0^1 p(x,t)\,dx = \mu$ and the fact that $\sigma = W'$ we obtain

$$\begin{aligned}
\int_0^1 p_t^2(x,t)\,dx \;&=\; -\int_0^1 \sigma(p(x,t))\,p_t(x,t)\,dx \\
&=\; -\frac{d}{dt}\int_0^1 W(p(x,t))\,dx.
\end{aligned}$$

Integrating this equality with respect to time from 0 to $t$ for any $t \geq 0$, we get (4.3.2) as required. Q.E.D

**Corollary 4.3.2.** *The energy*

$$\int_0^1 W(p(x,t))\,dx$$

*is a Lyapunov function for problem $(P)$.*

*Proof*: Equality (4.3.2) can be rewritten in the form

$$\frac{d}{dt}\int_0^1 W(p(x,t))\,dx \;=\; -\|p_t(\cdot,t)\|_2^2. \tag{4.3.3}$$

This shows that the energy is nonincreasing along solutions and strictly decreasing provided $\|p_t\|_2$ is nonzero. Therefore, it is a Lyapunov function (see e.g. [15] for the definition) for $(P)$. Q.E.D

## 4.4 Convergence of the time derivative

In this section we obtain results on the long time behaviour of $p_t$.

**Lemma 4.4.1.** *Let $p(x,t)$ be a solution for problem $(P)$. Then there exists a time $T > 0$ and a constant $C > 0$ such that*

$$|p_t(x,t)| \leq C \text{ for all } t \geq T. \tag{4.4.1}$$

*Proof*: By Corollary 3.4.15, Remark 3.4.3 and assumption (3.4.3) we immediately get existence of a constant $K > 0$ such that

$$|\sigma(p(\cdot,t))| \leq K \text{ for all } t \geq T.$$

From $(P)$ we have that

$$p_t(x,t) = -\sigma(p(x,t)) + \int_0^1 \sigma(p(y,t))\,dy.$$

Therefore, we obtain

$$|p_t(x,t)| \leq 2\,K \text{ for all } t \geq T.$$

Setting $C := 2\,K$ gives (4.4.1) as required. Q.E.D

**Lemma 4.4.2.** *Let $p(x,t)$ be a solution of problem $(P)$. Then we have*

$$\int_0^\infty \|p_t(\cdot,t)\|_2^2\,dt < \infty.$$

*Proof*: By Lemma 4.4.1 we easily obtain that

$$\left|\int_0^t \|p_t(\cdot,t)\|_2^2\,dt\right| \leq C \text{ for all } t.$$

Therefore

$$\int_0^t \|p_t(\cdot,t)\|_2^2\,dt$$

is a bounded monotone increasing function of $t$ and hence has a limit as $t \to \infty$ as required. Q.E.D

We now state and prove the main result of this section.

**Proposition 4.4.3.** *Let $p(x,t)$ be a solution of problem $(P)$. Then,*

$$\|p_t(\cdot,t)\|_2^2 \to 0 \quad \text{as} \quad t \to \infty.$$

*Proof*: For any $t \geq T$, where $T$ is as in Corollary 3.4.15, and $\tau > 0$, we can deduce from $(P)$ that

$$\begin{aligned}
&\Big| \|p_t(\cdot, t+\tau)\|_2^2 - \|p_t(\cdot, t)\|_2^2 \Big| = \\
&\quad = \left| \int_0^1 p_t^2(x, t+\tau) - p_t^2(x, t)\, dx \right| \\
&\quad = \left| \int_0^1 \left[ \left( -\sigma(p(x, t+\tau)) + \int_0^1 \sigma(p(y, t+\tau))\, dy \right)^2 - \right. \right. \\
&\qquad\qquad \left. \left. - \left( -\sigma(p(x,t)) + \int_0^1 \sigma(p(y,t))\, dy \right)^2 \right] dx \right| \\
&\quad \leq \int_0^1 \big| \sigma^2(p(x, t+\tau)) - \sigma^2(p(x,t)) \big|\, dx + \\
&\qquad\qquad + \left| \left( \int_0^1 \sigma(p(x, t+\tau))\, dx \right)^2 - \left( \int_0^1 \sigma(p(x,t))\, dx \right)^2 \right|.
\end{aligned}$$

For the first integral on the right hand side, by Remark 3.4.3 and (3.4.3), we get

$$\begin{aligned}
&\int_0^1 \big| \sigma^2(p(x, t+\tau)) - \sigma^2(p(x,t)) \big|\, dx = \\
&\quad = \int_0^1 \big| \sigma(p(x, t+\tau)) - \sigma(p(x,t)) \big| \, \big| \sigma(p(x, t+\tau)) + \sigma(p(x,t)) \big|\, dx \\
&\quad \leq 2\, K\, L \int_0^1 \big| p(x, t+\tau) - p(x,t) \big|\, dx
\end{aligned}$$

where $K$ is such that $K \geq \big|\sigma(p)\big|$. Similarly, for the second integral we obtain

$$\begin{aligned}
&\left| \left( \int_0^1 \sigma(p(x, t+\tau))\, dx \right)^2 - \left( \int_0^1 \sigma(p(x,t))\, dx \right)^2 \right| = \\
&\quad = \left| \int_0^1 \big( \sigma(p(x, t+\tau)) - \sigma(p(x,t)) \big)\, dx \right| \times \\
&\qquad\qquad \times \left| \int_0^1 \big( \sigma(p(x, t+\tau)) + \sigma(p(x,t)) \big)\, dx \right| \\
&\quad \leq \left( L \int_0^1 \big| p(x, t+\tau) - p(x,t) \big|\, dx \right) 2\, K \\
&\quad = 2\, K\, L \int_0^1 \big| p(x, t+\tau) - p(x,t) \big|\, dx.
\end{aligned}$$

Therefore, by Lemma 4.4.1, we obtain

$$\begin{aligned}\Big|\|p_t(\cdot,t+\tau)\|_2^2 - \|p_t(\cdot,t)\|_2^2\Big| &\le 4\,K\,L\int_0^1 \big|p(x,t+\tau)-p(x,t)\big|\,dx\\ &= 4\,K\,L\int_0^1\left|\int_t^{t+\tau}\frac{d}{ds}p(x,s)\,ds\right|dx\\ &\le 4\,K\,L\int_0^1\int_t^{t+\tau}|p_s(x,s)|\,ds\,dx\\ &\le 4\,K\,L\,C\,\tau.\end{aligned}$$

As a result we have that $\|p_t(\cdot,t)\|_2^2$ is uniformly continuous for $t \ge T$. By Lemma 4.4.2 this implies the claim. Q.E.D

## 4.5 $\omega$-limit set

**Definition 4.5.1.** *The $\omega$-limit set for problem $(P)$ can be defined as*

$$\begin{aligned}\omega(p) &:= \bigcap_{t\ge T}\overline{\{p(\cdot,s):\ s\ge t\}}\\ &= \{q\in L^2(0,1):\ \exists\ t_n\to\infty,\ \lim_{n\to\infty}\|p(\cdot,t_n)-q(\cdot)\|_2=0\}\end{aligned}$$

We can prove the following result by using this definition.

**Lemma 4.5.1.** *The $\omega$-limit set for problem $(P)$ is a subset of the set of stationary solutions $\mathcal{E}_P$.*

*Proof*: Take any $q\in\omega(p)$ such that

$$p(\cdot,t_k)\ \to\ q(\cdot)\ \text{ in }\ L^2(0,1)\ \text{ as }\ k\to\infty.$$

Then, up to a subsequence denoted by $t_k$ again, we get

$$p(x,t_k)\ \to\ q(x)\ \text{ for a.e. }\ x\in(0,1).$$

By assumption (3.4.3), we immediately obtain

$$\sigma(p(x,t_k))\ \to\ \sigma(q(x))\ \text{ for a.e. }\ x\in(0,1). \tag{4.5.1}$$

By Corollary 3.4.15 and assumption (3.4.3) again, for sufficiently large $k$ we have

$$\big|\sigma(p(\cdot,t_k))\big|\le K, \tag{4.5.2}$$

for some constant $K$. Hence, by Lebesgue's dominated convergence theorem, we obtain

$$\int_0^1 \sigma(p(x,t_k))\,dx \;\to\; \int_0^1 \sigma(q(x))\,dx \;\text{ as } k\to\infty. \tag{4.5.3}$$

Therefore, by (4.5.1) and (4.5.3), for almost every $x\in(0,1)$ we have

$$\left(\sigma(p(x,t_k)) - \int_0^1 \sigma(p(x,t_k))\,dx\right) \longrightarrow \left(\sigma(q(x)) - \int_0^1 \sigma(q(x))\,dx\right).$$

By (4.5.2) we can say that this convergence also holds in $L^2(0,1)$. On the other hand, by Proposition 4.4.3 we already know that

$$\sigma(p(x,t)) - \int_0^1 \sigma(p(x,t))\,dx \;\to\; 0 \quad \text{in } L^2(0,1).$$

Hence, we can conclude that

$$\sigma(q(x)) \;=\; \int_0^1 \sigma(q(x))\,dx$$

which, by (4.2.1), shows that $q\in\mathcal{E}_P$ as required. Q.E.D

## 4.6 Stability up to a subsequence

In this section we show that $p(x,t)$ converges to a stationary solution up to a subsequence in time. First, we prove a monotonicity condition which is a generalized version of (3.4.9).

**Lemma 4.6.1.** *Let $p(x,t)$ be the unique solution to problem $(P)$ with initial data $p_0(x)$. Then, the following monotonicity condition holds:*

$$p_0(x)\le p_0(y),\;\; x,y\in(0,1) \quad\Rightarrow\quad p(x,t)\le p(y,t) \;\; \textit{for all} \;\; t\ge 0.$$

*Proof*: Let us rewrite the partial differential equation in $(P)$ as

$$p_t(x,t) = -\sigma(p(x,t)) + f(t),$$

where $f(t)$ is a continuous function of time. Then, we can say that for every $x\in(0,1)$, $p(x,t)$ satisfies

$$p_t(\cdot,t) = -\sigma(p(\cdot,t)) + f(t)$$

which is an ordinary differential equation for $t\ge 0$. By uniqueness of $p(x,t)$ we immediately have

$$p_0(x)=p_0(y),\;\; x,y\in(0,1) \quad\Rightarrow\quad p(x,t)=p(y,t) \;\text{ for all }\; t\ge 0.$$

Moreover, by an argument similar to the proof of (3.4.9), we must have

$$p_0(x) < p_0(y),\ x, y \in (0,1) \quad \Rightarrow \quad p(x,t) < p(y,t) \ \text{ for all } \ t \geq 0$$

because if there exists a time $\tilde{t}$ such that $p_0(x) < p_0(y)$ and $p(x,\tilde{t}) = p(y,\tilde{t})$, then by solving the ordinary differential equation backwards in time we would get $p_0(x) = p_0(y)$, giving a contradiction. Q.E.D

**Proposition 4.6.2.** *Let $p(x,t)$ be the unique solution for problem $(P)$ with initial datum $p_0(x)$. Then, there exists a sequence $\{t_j\}_{j\geq 1}$ and a stationary solution $p \in \mathcal{E}_P$ such that*

$$p(\cdot, t_j) \ \rightarrow \ p(\cdot) \ \text{ in } \ L^2(0,1) \ \text{ as } \ j \rightarrow \infty.$$

*Proof*: First let us recall that by Lemma 4.6.1 we have

$$p_0(x) \leq p_0(y),\ x, y \in (0,1) \quad \Rightarrow \quad p(x,t) \leq p(y,t) \ \text{ for all } \ t \geq 0. \tag{4.6.1}$$

On the other hand, by a result due to Ryff [85] (see also [27], [41]) we know that for any real integrable function $f$ defined on the interval $(0,1)$, there exists a measure-preserving map $\delta\colon [0,1] \rightarrow [0,1]$ such that $f$ can be written in terms of its decreasing rearrangement $f^*$ as

$$f = f^* \circ \delta \ \text{ for a.e. } x \in (0,1).$$

Similarly, one can have an increasing rearrangement as well. Moreover, for any measure-preserving map $\delta\colon [0,1] \rightarrow [0,1]$ and integrable function $f \in L^1(0,1)$ we have

$$\int_0^1 f(x)\,dx = \int_0^1 f(\delta(y))\,dy. \tag{4.6.2}$$

Let us apply these results to $p(x,t)$. Let $z(x)\colon [0,1] \rightarrow [0,1]$ be a measure-preserving map such that we have the increasing rearrangement $q_0(x)$ for $p_0(x)$ such that $q_0(x) = p_0(z(x))$. Then by property (4.6.1) we immediately get that the corresponding rearrangement for $p(x,t)$, denoted by $q(x,t)$, is increasing as well. However, we need to check that $q(x,t)$ is a solution for $(P)$. If

$$p_0(z(x)) = q_0(x) \ \text{ and } \ p(z(x),t) = q(x,t),$$

then from $(P)$ we get

$$q(x,t) = q_0(x) - \int_0^t \sigma(q(x,s))\,ds + \int_0^t \int_0^1 \sigma(p(y,s))\,dy\,ds.$$

By (4.6.2) we have

$$\int_0^1 \sigma(p(y,s))\,ds \;=\; \int_0^1 \sigma(q(y,s))\,ds$$

which immediately implies that $q(x,t)$ is also a solution to $(P)$. Therefore, we can rearrange $p_0(x)$ and $p(x,t)$ and make them increasing in $x$ without changing the problem. By Corollary 3.4.15 we also know that $p(\cdot,t) \in L^\infty(0,1)$ after a certain time. As a result, we can say that $p(x,t)$ is a function of uniform bounded variation which is uniformly bounded. Applying Helly's Selection Theorem (see e.g. [3, pg. 70]) we get existence of a subsequence $\{t_j\}_{j\geq 1}$ and an increasing $p \in L^2(0,1)$ such that

$$p(\cdot,t_j) \;\to\; p(\cdot) \;\text{ in }\; L^2(0,1) \;\text{ as }\; j \to \infty.$$

By Remark 3.4.3 and assumption (3.4.3) we also have

$$\sigma(p(x,t_j)) \;\to\; \sigma(p(x)) \;\text{ in }\; L^2(0,1) \;\text{ as }\; j \to \infty.$$

By Proposition 4.4.3 we know that

$$\sigma(p(x,t)) - \int_0^1 \sigma(p(y,t))\,dy \;\to\; 0 \;\text{ in }\; L^2(0,1) \;\text{ as }\; t \to \infty,$$

hence we get

$$\int_0^1 \sigma(p(y,t_j))\,dy \;\to\; \sigma(p(x)) \;\text{ in }\; L^2(0,1) \;\text{ as }\; j \to \infty.$$

However, $\displaystyle\int_0^1 \sigma(p(x,t))\,dx$ is a function of $t$ only. Therefore, we must have

$$\sigma(p(x)) \;=\; \text{constant}.$$

By (4.2.1) we can conclude that $p \in \mathcal{E}_P$ as required. Q.E.D

**Remark 4.6.1.** *Using Proposition 4.6.2 we can prove Lemma 4.5.1 in a much more straightforward way, since we now know that*

$$\bigcup_{t\geq T} \{p(\cdot,t)\} \;\text{ is precompact in }\; L^2(0,1).$$

*That is, for any sequence $\{p(\cdot,t)\}_{t\geq T} \subset L^2(0,1)$, there exists a subsequence $\{t_n\}_{n\geq 1}$ such that*

$$p(\cdot,t_n) \;\to\; l(\cdot) \quad \text{in} \;\; L^2(0,1) \;\; \text{as} \;\; n \to \infty.$$

*Since $\omega(p)$ is invariant and the energy is constant along the stationary solutions, this implies that $\omega(p)$ consists only of stationary solutions $\tilde{p} \in L^2(0,1)$. Therefore,*

$$l(x) \;=\; \tilde{p}(x)$$

*holds.*

## 4.7 Convergence to equilibrium

In this section we give the main result of this chapter after introducing a new method which is valid for the study of more complex stability problems. Our method is heavily dependent upon the following general result.

**Lemma 4.7.1.** *For any $C^1$ function $F : \mathbb{R} \to \mathbb{R}$ we have*

$$\lim_{t\to\infty} \int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx \quad \textit{exists,}$$

*where $M$ is the positive constant as in Corollary 3.4.15.*

*Proof*: We follow the approach of Novick-Cohen & Pego [71]. Let $F \in C^1$. Then we have

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx &= \int_0^1 F(\sigma(p(x,t)))\, p_t(x,t)\, dx \\
&= \int_0^1 F(\sigma(p(x,t)) \left(-\sigma(p(x,t)) + \int_0^1 \sigma(p(y,t))\, dy\right) dx \\
&= -\int_0^1 \int_0^1 F(\sigma(p(x,t))\, (\sigma(p(x,t)) - \sigma(p(y,t)))\, dy\, dx \\
&= -\frac{1}{2}\int_0^1 \int_0^1 \big(F(\sigma(p(x,t))) - F(\sigma(p(y,t)))\big)\big(\sigma(p(x,t)) - \sigma(p(y,t))\big)\, dy\, dx.
\end{aligned}$$

If $F'(z) \geq 0$, then the result immediately follows as we would get from above that

$$\frac{d}{dt}\int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx \;\leq\; 0,$$

which implies by boundedness of $p(x,t)$ that the function

$$\int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx$$

is decreasing and bounded from below. If $F$ is not monotone, then we define

$$h(z) = z + \varepsilon\, F(z).$$

For sufficiently small $|\,\varepsilon|$, $r + \varepsilon\, F(r)$ is monotone increasing for $r$ in a bounded set. Hence, $h'(z) \geq 0$. This implies

$$\begin{aligned}
\varepsilon \int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx \;&=\; \int_0^1 \int_{-M}^{p(x,t)} h(\sigma(z))\, dz\, dx - \int_0^1 \int_{-M}^{p(x,t)} \sigma(z)\, dz\, dx \\
&= \int_0^1 \int_{-M}^{p(x,t)} h(\sigma(z))\, dz\, dx - \int_0^1 W(p(x,t))\, dx + C.
\end{aligned}$$

Therefore the claim follows. Q.E.D

### 4.7.1 An example

In this section, we study a specific case, namely, when $\sigma$ is chosen as a particular cubic function of the strain. Without any technical assumption like $(NC)$ or (4.1.1), we show that stabilization follows.

**Proposition 4.7.2.** *Let $p(x,t)$ be the continuous in time solution of problem $(P)$. Let*

$$W(p) = \frac{1}{4}(p^2-1)^2, \;\; \sigma(p) = p^3 - p, \;\; F(s) = s^2. \tag{4.7.1}$$

*Then,*

$$p(x,t) \to \tilde{p}(x) \;\; \textit{for a.e. } x \in (0,1) \;\; \textit{as} \;\; t \to \infty$$

*where $\tilde{p} \in L^2(0,1)$ is a stationary solution.*

*Proof*: Using the given choices of $\sigma$ and $F$ in (4.7.1) we get that

$$\begin{aligned}
\int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx &= \int_0^1 \int_{-M}^{p(x,t)} (z^3 - z)^2\, dz\, dx \\
&= \int_0^1 \int_{-M}^{p(x,t)} (z^6 - 2\,z^4 + z^2)\, dz\, dx \\
&= \int_0^1 \left( \frac{z^7}{7} - \frac{2}{5} z^5 + \frac{z^3}{3} \right) \Big|_{-M}^{p} dx \\
&= \int_0^1 \left( \frac{p^7}{7} - \frac{2}{5} p^5 + \frac{p^3}{3} + \frac{M^7}{7} - \frac{2}{5} M^5 + \frac{M^3}{3} \right) dx.
\end{aligned}$$

Denoting

$$\left( \frac{1}{7} M^7 - \frac{2}{5} M^5 + \frac{1}{3} M^3 \right) =: K(M),$$

we obtain

$$\int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx = \int_0^1 \left( \frac{p^7}{7} - \frac{2}{5} p^5 + \frac{p^3}{3} \right) dx + K(M).$$

Therefore, by Lemma 4.7.1 we deduce that

$$\begin{aligned}
&\lim_{t\to\infty} \int_0^1 \int_{-M}^{p(x,t)} F(\sigma(z))\, dz\, dx = \\
&\qquad = \lim_{t\to\infty} \int_0^1 \left( \frac{p^7}{7} - \frac{2}{5} p^5 + \frac{p^3}{3} \right) dx + K(M) \quad \text{exists.}
\end{aligned} \tag{4.7.2}$$

We are given that $\sigma(p) = p^3 - p$. Using this we can rewrite $p^3, p^5$ and $p^7$ in terms of $\sigma(p)$ and $p$ as

$$\begin{cases} p^3 = \sigma(p) + p, \\ p^5 = \sigma(p)\, p^2 + \sigma(p) + p, \\ p^7 = \sigma^2(p)\, p + 2\,\sigma(p)\, p^2 + \sigma(p) + p. \end{cases} \tag{4.7.3}$$

Hence,

$$\begin{aligned} \int_0^1 \left( \frac{p^7}{7} - \frac{2}{5}p^5 + \frac{p^3}{3} \right) dx = \\ = \left( \frac{2}{7} - \frac{2}{5} \right) \int_0^1 \sigma(p)\, p^2\, dx + \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) \int_0^1 p\, dx + \frac{1}{7} \int_0^1 \sigma^2(p)\, p\, dx + \\ + \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) \int_0^1 \sigma(p)\, dx \\ = \left( \frac{2}{7} - \frac{2}{5} \right) \int_0^1 \sigma(p)\, p^2\, dx + \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) \mu + \frac{1}{7} \int_0^1 \sigma^2(p)\, p\, dx + \\ + \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) \int_0^1 \sigma(p)\, dx. \end{aligned}$$

By (4.7.2) we get

$$\begin{aligned} \lim_{t\to\infty} \Bigg\{ \left( \frac{2}{7} - \frac{2}{5} \right) \int_0^1 \sigma(p)\, p^2\, dx + \\ + \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) \mu + \frac{1}{7} \int_0^1 \sigma^2(p)\, p\, dx + \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) \int_0^1 \sigma(p)\, dx \Bigg\} \quad \text{exists.} \end{aligned} \tag{4.7.4}$$

On the other hand, we have

$$\begin{aligned} \int_0^1 W(p)\, dx &= \int_0^1 \frac{1}{4}(p^2 - 1)^2\, dx \\ &= \int_0^1 \left( \frac{1}{4}p^4 - \frac{1}{2}p^2 + \frac{1}{4} \right) dx \\ &= \int_0^1 \left( \frac{1}{4}(\sigma(p)\, p + p^2) - \frac{1}{2}p^2 + \frac{1}{4} \right) dx \\ &= \frac{1}{4} \int_0^1 \sigma(p)\, p\, dx - \frac{1}{4} \int_0^1 p^2\, dx + \frac{1}{4}. \end{aligned}$$

Therefore, by (4.3.2) we deduce that

$$\lim_{t\to\infty} \left( \int_0^1 \sigma(p)\, p\, dx - \int_0^1 p^2\, dx \right) \quad \text{exists.} \tag{4.7.5}$$

By Proposition 4.6.2 we know that for a subsequence $\{t_j\}_{j\geq 1}$ there exists a stationary solution $\tilde{p}(x)$ such that

$$\lim_{j\to\infty} p(x, t_j) = \tilde{p}(x).$$

Let us denote $\sigma(\tilde{p}(x)) = \bar{\sigma}$. Then, letting $t_j \to \infty$ in (4.7.4) and (4.7.5), we get

$$\begin{cases} \left(\frac{2}{7} - \frac{2}{5}\right) \bar{\sigma} \int_0^1 \tilde{p}^2(x)\, dx + \frac{\mu}{7} \bar{\sigma}^2 + \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) \bar{\sigma} + \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) \mu = K_1, \\ \bar{\sigma}\, \mu - \int_0^1 \tilde{p}^2(x)\, dx = K_2, \end{cases}$$

where $K_1$ and $K_2$ are constants depending on $F$ and $M$. Substituting the second equation into the first one we obtain

$$\left(\frac{2}{7} - \frac{2}{5}\right) \bar{\sigma}\, (\bar{\sigma}\, \mu - K_2) + \frac{\mu}{7} \bar{\sigma}^2 + \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) \bar{\sigma} + \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) \mu = K_1 \quad (4.7.6)$$

which is a second order polynomial in $\bar{\sigma}$. Solving (4.7.6) gives at most two different values for $\bar{\sigma}$. If we have only one solution, then the claim follows immediately. Assume we have two solutions, $\bar{\sigma}_1$ and $\bar{\sigma}_2$, and assume without loss of generality that $\bar{\sigma}_1 < \bar{\sigma}_2$. Then, we have

$$\begin{cases} p(t_j) \to \tilde{p}_1 \quad \text{where} \quad \sigma(\tilde{p}_1) = \bar{\sigma}_1 \\ p(s_j) \to \tilde{p}_2 \quad \text{where} \quad \sigma(\tilde{p}_2) = \bar{\sigma}_2 \end{cases}$$

for two subsequences $\{t_j\}$ and $\{s_j\}$ of time. By the form of $\sigma$ in (4.7.1) we know that it is not constant on any interval. This would imply that there exists another subsequence $\{r_j\}$ and a stationary solution $\tilde{p}_3$ such that

$$p(r_j) \to \tilde{p}_3 \quad \text{where} \quad \bar{\sigma}_1 < \sigma(\tilde{p}_3) < \bar{\sigma}_2.$$

This contradicts the fact that $\bar{\sigma}$ takes only two different values. As a result we can have only one value for $\bar{\sigma}$ which proves that

$$\sigma(p(x,t)) \to \bar{\sigma} \ \text{ as } \ t \to \infty.$$

By Remark 3.4.3, assumption (3.4.3) and Lebesgue's dominated convergence theorem, this gives

$$\int_0^1 \sigma(p(x,t))\, dx \ \to \ \bar{\sigma} \ \text{ as } \ t \to \infty. \quad (4.7.7)$$

We now quote the following result of Novick-Cohen & Pego ([71, Lemma 3.4]) as it is stated in the article.

> Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous and not constant on any open interval. Assume that $z(t) \in C^1(0, \infty)$ is a bounded solution of $z'(t) = f(z(t)) + e(t)$, where $e(t)$ is continuous with $\lim_{t\to\infty} e(t) = 0$. Then $\lim_{t\to\infty} z(t) = z_\infty$ exists, and $f(z_\infty) = 0$.

Applying this result to

$$\begin{aligned} p_t(x,t) &= -\sigma(p(x,t)) + \bar{\sigma} - \bar{\sigma} + \int_0^1 \sigma(p(y,t))\,dy \\ &= -(\sigma(p(x,t)) - \bar{\sigma}) + e(t), \end{aligned}$$

where we clearly have, by (4.7.7), that $e(t) \to 0$ as $t \to \infty$, we immediately obtain the claim. Q.E.D

### 4.7.2 Generalization of the example

Our aim in this part is to generalize the example in the previous section so that we have a complete understanding of the long time behaviour of solutions of problem $(P)$ when $\sigma(p) = p^3 - p$. Before we prove the main result, we give some definitions and their notations, mostly by appealing to the work of Andrews & Ball [5].

Assume that the stress function $\sigma$ looks like the one in Figure 4.1. We define

$$c(t) := \int_0^1 \sigma(p(x,t))\,dx. \tag{4.7.8}$$

If $c(t)$ does not tend to a limit as $t \to \infty$, then there would exist numbers $p, q$ with $p < q$ such that the bounded continuous function $c(t)$ takes the values $p$ and $q$ for arbitrarily large values of $t$. Clearly, the function $\sigma(z)$ intersects the strip $p \le \sigma(z) \le q$ in exactly three strictly monotonic increasing or decreasing segments of curves, which will be denoted by $\sigma_i,\ i = 1, 2, 3$. If $t$ is such that $c(t) \in [p, q]$, then we denote the points of intersection of $\sigma_i$ with $c(t)$ by $p_i(c)$ (see Figure 4.1).

Following Andrews & Ball [5], for $\varepsilon > 0$ sufficiently small and $t$ such that $c(t) \in [p, q]$, we define the set

$$S_i(t) = \{x \in (0,1) \,:\, |p(x,t) - p_i(c)| < \varepsilon\,\},$$

and

$$\mu_i(c) \,=\, \text{meas}\,(S_i(t)), \tag{4.7.9}$$

which gives the corresponding measure for each $p_i(c)$. Clearly, the sets $S_i(t)$ are disjoint and by (4.4.3) and (4.7.8) we get

$$\lim_{\substack{t\to\infty \\ c(t)\in[p,q]}} \sum_{i=1}^{3} \mu_i(c) \,=\, 1. \tag{4.7.10}$$

Let us denote

$$\vec{\mu} \,=\, (\mu_1(c), \mu_2(c), \mu_3(c)) \ \text{ and } \ \vec{p} \,=\, (p_1(c), p_2(c), p_3(c)). \tag{4.7.11}$$

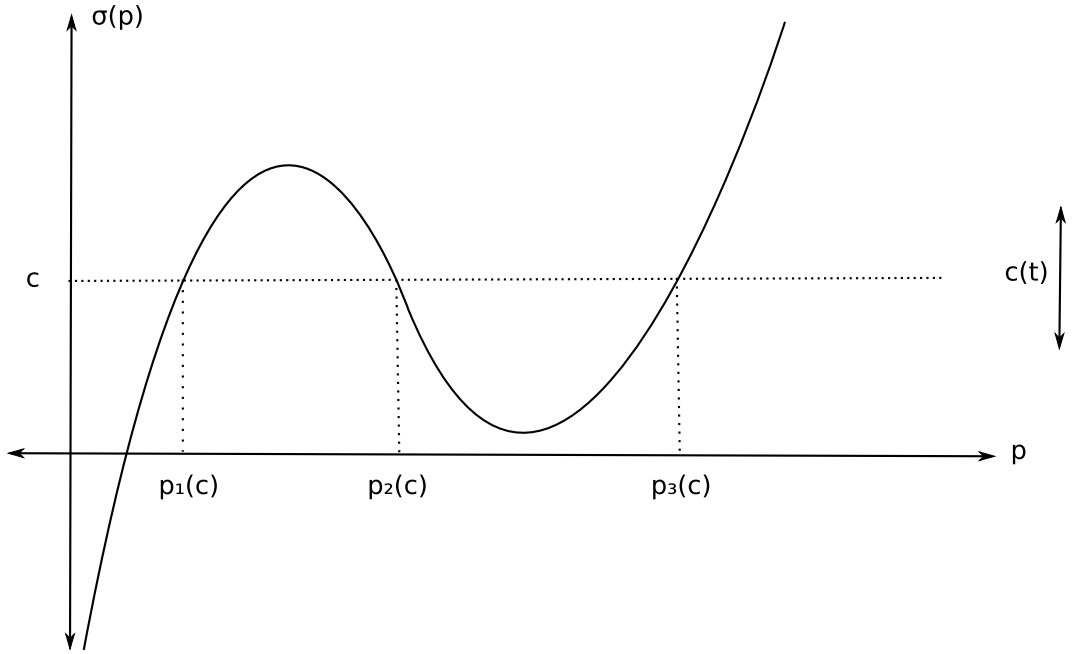



Figure 4.1: A general form of $\sigma$

By (4.7.10) we get

$$\vec{\mu} \cdot \vec{e} = \vec{\mu} \cdot (1,1,1) = 1.$$

By the boundary conditions $\int_0^1 p(x,t)\,dx = \mu$, we get

$$\vec{\mu} \cdot \vec{p} = \vec{\mu} \cdot (p_1(c), p_2(c), p_3(c)) = \mu. \tag{4.7.12}$$

The function $F$ in Lemma 4.7.1 is an arbitrary $C^1$ function. Let us first consider the case when $F(z) \equiv z$. Then, we get

$$\lim_{t\to\infty} \int_0^1 \int_{-M}^{p(x,t)} \sigma(z)\,dz\,dx \quad \text{exists.}$$

Writing $\sigma(z) = W'(z)$ we obtain

$$\begin{aligned} \int_0^1 \int_{-M}^{p(x,t)} \sigma(z)\,dz\,dx &= \int_0^1 \int_{-M}^{p(x,t)} W'(z)\,dz\,dx \\ &= \int_0^1 (W(p(x,t)) - W(-M))\,dx \\ &= \int_0^1 W(p(x,t))\,dx + \text{ constant.} \end{aligned}$$

This clearly implies that

$$\lim_{t\to\infty} \int_0^1 W(p(x,t))\,dx \quad \text{exists.} \tag{4.7.13}$$

Let us denote

$$\lim_{t\to\infty} \int_0^1 W(p(x,t))\,dx \;=\; a_1 \tag{4.7.14}$$

and

$$\begin{aligned} \vec{F}_1(\vec{p}) \;&=\; \big(F_1(p_1(c)), F_1(p_2(c)), F_1(p_3(c))\big) \\ &=\; \big(W(p_1(c)), W(p_2(c)), W(p_3(c))\big). \end{aligned} \tag{4.7.15}$$

Then, by (4.7.11), (4.7.14) and (4.7.15) we obtain

$$\vec{\mu}\cdot\vec{F}_1(\vec{p}) \;=\; \vec{\mu}\cdot\big(W(p_1(c)), W(p_2(c)), W(p_3(c))\big) \;=\; a_1. \tag{4.7.16}$$

Let us now take $F(z) = z^2$. Then Lemma 4.7.1 implies that

$$\lim_{t\to\infty} \int_0^1 \int_{-M}^{p(x,t)} \sigma^2(z)\,dz\,dx \quad \text{exists.}$$

Denoting

$$\lim_{t\to\infty} \int_0^1 \int_{-M}^{p(x,t)} \sigma^2(z)\,dz\,dx \;=\; a_2 \tag{4.7.17}$$

and

$$\begin{aligned} \vec{F}_2(\vec{p}) \;&=\; \big(F_2(p_1(c)), F_2(p_2(c)), F_2(p_3(c))\big) \\ &=\; \left(\int_{-M}^{p_1(c)} \sigma^2(z)\,dz, \int_{-M}^{p_2(c)} \sigma^2(z)\,dz, \int_{-M}^{p_3(c)} \sigma^2(z)\,dz\right) \end{aligned} \tag{4.7.18}$$

we get

$$\vec{\mu}\cdot\vec{F}_2(\vec{p}) \;=\; \vec{\mu}\cdot\big(F_2(p_1(c)), F_2(p_2(c)), F_2(p_3(c))\big) \;=\; a_2. \tag{4.7.19}$$

We rewrite (4.7.12), (4.7.16) and (4.7.19) in the forms

$$\begin{cases} \vec{\mu}\cdot(\,\vec{p} - \mu\,\vec{e}\,) \;=\; 0, \\ \vec{\mu}\cdot(\vec{F}_1(\,\vec{p}) - a_1\,\vec{e}\,) \;=\; 0, \\ \vec{\mu}\cdot(\vec{F}_2(\,\vec{p}) - a_2\,\vec{e}\,) \;=\; 0. \end{cases}$$

These equations tell us that the vectors $(\,\vec{p} - \mu\,\vec{e}\,), (F_1(\,\vec{p}) - a_1\,\vec{e}\,)$ and $(F_2(\,\vec{p}) - a_2\,\vec{e}\,)$ are all orthogonal to the same vector $\vec{\mu}$. This clearly implies that they must be linearly dependent. In other words, we must have that

$$\det\begin{pmatrix} p_1(c)-\mu & p_2(c)-\mu & p_3(c)-\mu \\ F_1(p_1(c))-a_1 & F_1(p_2(c))-a_1 & F_1(p_3(c))-a_1 \\ F_2(p_1(c))-a_2 & F_2(p_2(c))-a_2 & F_2(p_3(c))-a_2 \end{pmatrix} \;=\; 0. \tag{4.7.20}$$

We now reprove Proposition 4.7.2 using this information.

**Theorem 4.7.3.** *Assume that functions $W$ and $\sigma(p)$ in problem $(P)$ are of the form*

$$W(p) = \frac{1}{4}(p^2-1)^2, \;\; \sigma(p) \;=\; p^3 - p.$$

*Then,*

$$\lim_{t\to\infty} \int_0^1 \sigma(p(x,t))\, dx \quad \textit{exists} \tag{4.7.21}$$

*and hence the conclusion of Proposition 4.7.2 holds.*

*Proof*: Assume for contradiction that $c(t)$ defined in (4.7.8) does not tend to a limit as $t \to \infty$. Then, we can use the setting described above for the given choice of $\sigma$. First let us find vectors $\vec{F}_1(\vec{p})$ and $\vec{F}_2(\vec{p})$ in this case. By (4.7.15) we get

$$\vec{F}_1(\vec{p}) \;=\; \left(\frac{p_1^4}{4} - \frac{p_1^2}{2} + \frac{1}{4}, \frac{p_2^4}{4} - \frac{p_2^2}{2} + \frac{1}{4}, \frac{p_3^4}{4} - \frac{p_3^2}{2} + \frac{1}{4}\right)$$

and by (4.7.18) we get

$$\begin{aligned} \vec{F}_2(\vec{p}) \;\; = \;\; & \left(\frac{p_1^7}{7} + \frac{p_1^3}{3} - \frac{2p_1^5}{5} + K(M), \right. \\ & \frac{p_2^7}{7} + \frac{p_2^3}{3} - \frac{2p_2^5}{5} + K(M), \\ & \left. \frac{p_3^7}{7} + \frac{p_3^3}{3} - \frac{2p_3^5}{5} + K(M)\right). \end{aligned}$$

where $K(M)$ is as defined in the proof of Proposition 4.7.2. Using the specific form of $\sigma(p)$ as was done in (4.7.3) we deduce that

$$\begin{aligned} \vec{F}_1(\vec{p}) \;\; = \;\; & \left(-\frac{1}{4}\, p_1^2 + \frac{1}{4}\, \sigma(p_1)\, p_1 + \frac{1}{4}, \right. \\ & -\frac{1}{4}\, p_2^2 + \frac{1}{4}\, \sigma(p_2)\, p_2 + \frac{1}{4}, \\ & \left. -\frac{1}{4}\, p_3^2 + \frac{1}{4}\, \sigma(p_3)\, p_3 + \frac{1}{4}\right). \end{aligned} \tag{4.7.22}$$

and

$$\begin{aligned} \vec{F}_2(\vec{p}) \;\; = \;\; & \left(-\frac{4}{35}\, \sigma(p_1)\, p_1^2 + \frac{1}{7}\, \sigma^2(p_1)\, p_1 + \frac{8}{105}\, p_1 + \frac{8}{105}\, \sigma(p_1) + K(M), \right. \\ & -\frac{4}{35}\, \sigma(p_2)\, p_2^2 + \frac{1}{7}\, \sigma^2(p_2)\, p_2 + \frac{8}{105}\, p_2 + \frac{8}{105}\, \sigma(p_2) + K(M), \\ & \left. -\frac{4}{35}\, \sigma(p_3)\, p_3^2 + \frac{1}{7}\, \sigma^2(p_3)\, p_3 + \frac{8}{105}\, p_3 + \frac{8}{105}\, \sigma(p_3) + K(M)\right). \end{aligned} \tag{4.7.23}$$

Therefore determinant (4.7.20) becomes

$$\begin{vmatrix} p_1(c)-\mu & p_2(c)-\mu & p_3(c)-\mu \\ -\frac{1}{4}p_1^2+\frac{1}{4}\sigma(p_1)p_1+\frac{1}{4}-a_1 & \cdots & \cdots \\ -\frac{4}{35}\sigma(p_1)p_1^2+\frac{1}{7}\sigma^2(p_1)p_1+ & & \\ \qquad +\frac{8}{105}p_1+\frac{8}{105}\sigma(p_1)+C & \cdots & \cdots \end{vmatrix} = 0,$$

where $C := K(M) - a_2$.

By Proposition 4.6.2 we know that for a subsequence $\{t_j\}_{j\geq 1}$, $p(x,t)$ converges to an equilibrium point. Letting $t_j \to \infty$ we get $\sigma(\vec{p}(c)) = c$, hence in the above determinant we obtain

$$\begin{vmatrix} p_1(c)-\mu & p_2(c)-\mu & p_3(c)-\mu \\ -\frac{1}{4}p_1(c)^2+\frac{1}{4}cp_1(c)+\frac{1}{4}-a_1 & \cdots & \cdots \\ -\frac{4}{35}c\,p_1(c)^2+\frac{1}{7}c^2\,p_1(c)+\frac{8}{105}p_1(c)+\frac{8}{105}c+C & \cdots & \cdots \end{vmatrix} = 0.$$

Calculating this determinant explicitly gives

$$0 \;=\; (p_1(c)-p_3(c))(p_3(c)-p_2(c))(p_2(c)-p_1(c))(\mu\, c^2\, k \;+\circ(c)) \tag{4.7.24}$$

where $k$ is a positive constant. As $p_1(c) \neq p_2(c) \neq p_3(c)$, (4.7.24) gives

$$0 \;=\; \mu\, c^2\, k \;+o(c) \tag{4.7.25}$$

proving that $c$ is a solution of a quadratic polynomial as in (4.7.6). The rest of the proof is the same as in Proposition 4.7.2 and hence we get the claim as required. Q.E.D

### 4.7.3 Revisiting some results

In this section we will reprove the stability results of Andrews & Ball [5] and Novick-Cohen & Pego [71] using the general method we discussed in Section 4.7.2. We will confine ourselves to the case when the number of the increasing or decreasing segments of curves is exactly three. The setting is shown in Figure 4.2.

**Theorem 4.7.4.** *Assume that $\sigma$ is as shown in Figure 4.2 and $p(x,t)$ is a solution of problem $(P)$. Then,*

$$\lim_{t\to\infty} \int_0^1 \sigma(p(x,t))\,dx \quad \textit{exists.}$$

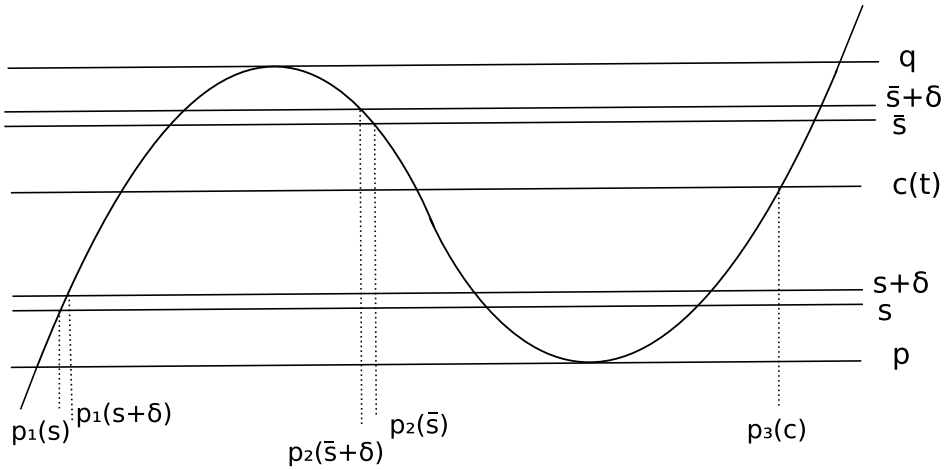



Figure 4.2: A special $\sigma$

*Proof*: Keeping the notation of Section 4.7.2, assume that $c(t)$ does not tend to a limit as $t \to \infty$. We note that using Lemma 4.7.1 and following Andrews & Ball [5], one can show that

$$\lim_{t\to\infty} \int_0^1 \int_{-M}^{p(x,t)} \chi(\sigma(z))\, dz\, dx \text{ exists,}$$

for any characteristic function $\chi$ of a closed interval.

We define

$$E_s(p) = \lim_{\delta\to 0} \frac{1}{\delta} \int_{-M}^{p(x,t)} \chi_{[s,s+\delta]}(\sigma(z))\, dz, \tag{4.7.26}$$

where $p < s < s+\delta < \bar{p} < \bar{q} < q,\ c(t) \in [\bar{p}, \bar{q}]$, and

$$E_{\bar{s}}(p) = \lim_{\delta\to 0} \frac{1}{\delta} \int_{-M}^{p(x,t)} \chi_{[\bar{s},\bar{s}+\delta]}(\sigma(z))\, dz, \tag{4.7.27}$$

where $p < \bar{p} < \bar{q} < \bar{s} < \bar{s}+\delta < q,\ c(t) \in [\bar{p}, \bar{q}]$. By the arguments introduced in Section 4.7.2, we obtain a determinant similar to (4.7.20), namely,

$$\det \begin{pmatrix} p_1(c)-\mu & p_2(c)-\mu & p_3(c)-\mu \\ E_s(p_1(c))-a_1 & E_s(p_2(c))-a_1 & E_s(p_3(c))-a_1 \\ E_{\bar{s}}(p_1(c))-a_2 & E_{\bar{s}}(p_2(c))-a_2 & E_{\bar{s}}(p_3(c))-a_2 \end{pmatrix} = 0 \tag{4.7.28}$$

where

$$a_1(s) = \vec{\mu} \cdot \big(E_s(p_1(c)), E_s(p_2(c)), E_s(p_3(c))\big)$$
$$a_2(\bar{s}) = \vec{\mu} \cdot \big(E_{\bar{s}}(p_1(c)), E_{\bar{s}}(p_2(c)), E_{\bar{s}}(p_3(c))\big).$$

By (4.7.26) we have

$$\begin{aligned} E_s(p_1(c)) &= \lim_{\delta\to 0}\frac{1}{\delta}\int_{-M}^{p_1(c)}\chi_{[s,s+\delta]}(\sigma(z))\,dz \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\sigma_1^{-1}(s)}^{\sigma_1^{-1}(s+\delta)} dz \\ &= \lim_{\delta\to 0}\frac{\sigma_1^{-1}(s+\delta)-\sigma_1^{-1}(s)}{\delta} \\ &= \big(\sigma_1^{-1}(s)\big)' =: p_1'(s) \end{aligned} \tag{4.7.30}$$

and

$$E_s(p_2(c)) \,=\, E_s(p_1(c)) \,=\, p_1'(s). \tag{4.7.31}$$

Moreover, we have

$$\begin{aligned} E_s(p_3(c)) &= \lim_{\delta\to 0}\frac{1}{\delta}\int_{-M}^{p_3(c)}\chi_{[s,s+\delta]}(\sigma(z))\,dz \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\sigma_1^{-1}(s)}^{\sigma_1^{-1}(s+\delta)} dz + \int_{\sigma_2^{-1}(s+\delta)}^{\sigma_2^{-1}(s)} dz + \int_{\sigma_3^{-1}(s)}^{\sigma_3^{-1}(s+\delta)} dz\right) \\ &= \big(\sigma_1^{-1}(s)\big)' - \big(\sigma_2^{-1}(s)\big)' + \big(\sigma_3^{-1}(s)\big)' \\ &= p_1'(s) - p_2'(s) + p_3'(s). \end{aligned} \tag{4.7.32}$$

Similarly, we find

$$\begin{cases} E_{\bar s}(p_1(c)) = 0 \\ E_{\bar s}(p_2(c)) = p_1'(\bar s) - p_2'(\bar s) \\ E_{\bar s}(p_3(c)) = p_1'(\bar s) - p_2'(\bar s). \end{cases} \tag{4.7.33}$$

By (4.7.30), (4.7.31) and (4.7.32), (4.7.28) becomes

$$\det\begin{pmatrix} p_1(c)-\mu & p_2(c)-\mu & p_3(c)-\mu \\ p_1'(s)-a_1 & p_1'(s)-a_1 & p_1'(s)-p_2'(s)+p_3'(s)-a_1 \\ -a_2 & p_1'(\bar s)-p_2'(\bar s)-a_2 & p_1'(\bar s)-p_2'(\bar s)-a_2 \end{pmatrix} \,=\, 0$$

Calculating this determinant explicitly gives

$$\begin{aligned} 0 \;=\;& a_2(\bar s)\,(p_2(c)-p_1(c))\,(p_2'(s)-p_3'(s)) + \\ &+ (p_1'(\bar s)-p_2'(\bar s)\,(p_3(c)-p_2(c))\,(p_1'(s)-a_1(s)) + \\ &+ (p_1'(\bar s)-p_2'(\bar s))\,(p_1(c)-\mu)\,(p_2'(s)-p_3'(s)). \end{aligned} \tag{4.7.34}$$

We can rewrite this equality in the form

$$\big(p_1(c),p_2(c),p_3(c)\big)\cdot\vec{A}(s,\bar s) \,=\, g(s,\bar s) \tag{4.7.35}$$

where

$$g(s,\bar{s}) \;=\; \mu\,(p_1'(s)-p_2'(\bar{s}))\,(p_2'(s)-p_3'(s))$$

and $\vec{A}(s,\bar{s})$ is the rest. By (4.7.34) we can immediately say that provided

$$p_2'(s)\neq p_3'(s) \;\text{ and }\; p_2'(\bar{s})\neq p_1'(\bar{s}),$$

we have

$$\vec{A}(s,\bar{s})\neq 0.$$

However, these conditions are included in $(NC)$. Therefore, taking derivatives with respect to $c$ in (4.7.35) gives

$$\big(p_1'(c),p_2'(c),p_3'(c)\big)\cdot\vec{A} \;=\; 0$$

which, as $\vec{A}(s,\bar{s})$ is nonzero, means that $p_i'(c)$ are linearly dependent, contradicting the nondegeneracy condition and hence proving the claim. Q.E.D

# Chapter 5

# Quasistatic Nonlinear Viscoelasticity as a Curve of Maximal Slope

## 5.1 Introduction

In this chapter we consider the problem of quasistatic nonlinear viscoelasticity in three dimensions, namely,

$$\text{Div } (DW(Dy) + S(Dy, Dy_t)) = 0 \tag{5.1.1}$$

as introduced in Section 2.4. We follow the method of time-discretization for this problem in the context of gradient flows following Ambrosio, Gigli and Savaré [3]. Our aim is to develop a comprehensive theory for the three-dimensional problem while ensuring the requirement of frame-indifference. We achieve this goal by introducing a method using composition of time-increments. However, we confine ourselves to the one-dimensional setting to get an existence theory as a result of the simplifications we obtain by the assumption of the convexity of the stored-energy function. Other motivations for us to study the one-dimensional case are to test the general ideas and see how efficient they are, as well as to point out the arguments heavily dependent on the convexity assumption for possible future generalizations.

## 5.2 Direct time-discretization

This section is devoted to showing that a direct implicit time-discretization scheme for (5.1.1) does not work due to lack of weak continuity of the right Cauchy-Green strain tensor $C$. We follow the approach of Friesecke & Dolzmann [44] and the notations are kept separate from that of the subsequent sections.

We define approximate solutions to (5.1.1) by means of the following implicit time-discretization scheme for functions $y\colon \Omega\times[0,\infty) \to \mathbb{R}^3$. For a fixed time stepsize $h > 0$ and initial data $y_0 \in W^{1,2}(\Omega)$ we inductively define

$$\begin{aligned} y^{h,0} &:= y_0 \\ y^{h,j} &:= \text{ a minimizer of the functional } J^{h,j}[y] \ \ (j \in \mathbb{N}), \end{aligned}$$

where

$$J^{h,j}[y] = \int_\Omega W(Dy) + h\,\Psi\left(Dy^{h,j-1}, \frac{Dy - Dy^{h,j-1}}{h}\right) dx. \tag{5.2.1}$$

Note that for $j \geq 1$, the minimizers $y^{h,j}$ satisfy the Euler-Lagrange equations

$$\int_\Omega \left[DW(Dy^{h,j}) + \Psi_q\left(Dy^{h,j-1}, \frac{Dy^{h,j} - Dy^{h,j-1}}{h}\right)\right] \cdot D\xi\, dx \ = \ 0 \tag{5.2.2}$$

for all $\xi \in C_0^\infty(\Omega)$, which represent a weak, time-discretized version of (5.1.1) for $\Psi_q(p,q) = S(p,q)$.

We define the energy functional as

$$E[y] := \int_\Omega W(Dy)\, dx. \tag{5.2.3}$$

**Lemma 5.2.1.** *Assume $W \in C^1(\mathbb{M}^{3\times 3})$ and $y \in C^1((0,\infty); H^1(\Omega))$. Then, the following dissipation identity is satisfied for any $t \geq 0$ :*

$$E[y(t)] + \int_0^t \int_\Omega S(Dy, Dy_s)\, Dy_s\, dx\, ds = E[y(0)]. \tag{5.2.4}$$

*Proof*: Multiplying (5.1.1) by $y_t$ and integrating over $x$ gives

$$\begin{aligned} &\int_\Omega \text{Div } (DW(Dy))\, y_t\, dx \ + \ \int_\Omega \text{Div } (S(Dy, Dy_t))\, y_t\, dx \ = \ 0 \\ \Rightarrow \ &\int_\Omega DW(Dy)\, Dy_t\, dx \ + \ \int_\Omega S(Dy, Dy_t)\, Dy_t\, dx \ = \ 0. \end{aligned}$$

By (5.2.3) we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega W(Dy)\, dx \ + \ \int_\Omega S(Dy, Dy_t)\, Dy_t\, dx \ = \ 0 \\ \Rightarrow \ &E[y(t)] - E[y(0)] \ + \ \int_0^t \int_\Omega S(Dy, Dy_s)\, Dy_s\, dx\, ds \ = \ 0 \end{aligned}$$

as required. Q.E.D

We now prove that a time-discretized energy inequality corresponding to (5.2.4) is satisfied.

**Proposition 5.2.2.** *Let $y_0$ and $y^{h,j}$ be defined as above and assume that $\Psi(\cdot,0)=0$. Then, for all $h>0$, we have the discrete energy inequality*

$$\sup_{j\in\mathbb{N}} E[y^{h,j}] + \sum_{j=1}^{\infty} h \int_\Omega \Psi\left(Dy^{h,j-1}, \frac{Dy^{h,j}-Dy^{h,j-1}}{h}\right) dx \leq E[y_0]. \tag{5.2.5}$$

*Proof*: As $y^{h,j}$ is a minimizer of the functional $J^{h,j}[y]$, by the assumption $\Psi(\cdot,0)=0$, for each $j$ we must have

$$E[y^{h,j}] + h\int_\Omega \Psi\left(Dy^{h,j-1}, \frac{Dy^{h,j}-Dy^{h,j-1}}{h}\right) dx \;\leq\; E[y^{h,j-1}].$$

Equivalently, we can write

$$\int_\Omega (W(Dy^{h,j}) - W(Dy^{h,j-1}))\, dx + h\int_\Omega \Psi\left(Dy^{h,j-1}, \frac{Dy^{h,j}-Dy^{h,j-1}}{h}\right) dx \leq 0.$$

Applying this estimate successively for all $j\in\mathbb{N}$ gives the result. Q.E.D

We want our model to satisfy the condition of frame-indifference as mentioned before. For this reason, we will consider the potential $\Psi_1(F,\dot{F})$ we introduced in Section 2.4 which not only provides a frame-indifferent $S$, but also is strictly convex in its second argument as desired.

**Lemma 5.2.3.** *Assume $W$ is convex and let $\Psi_1(F,\dot{F})$ be as in Lemma 2.5.1. Then,*

(i) $\Psi_1\left(F, \dfrac{Dy-Dy^{h,j-1}}{h}\right)$ *is convex in* $y$,

(ii) *putting* $\Psi=\Psi_1$ *in* $J^{h,j}[y]$ *gives*

$$y \longmapsto J^{h,j}[y]$$

*is weakly sequentially lower semicontinuous on* $W^{1,2}(\Omega)$.

(iii) *if* $h>0$ *and* $y^{h,j-1}\in W^{1,2}(\Omega)$, *then* $J^{h,j}[y]$ *attains its infimum on* $W^{1,2}(\Omega)$.

*Proof*: Part (i) is a direct consequence of convexity of $\Psi_1(F,H)$ with respect to $H$, which we proved in Lemma 2.5.1. Part (ii) can be proved easily by standard theorems in the calculus of variations (see e.g. [32, Thm. 2.6]) using part (i). Similarly, part (ii) and the assumption on $W$ immediately yield part (iii). Q.E.D

Using (5.2.5), we can now prove the following convergences.

**Lemma 5.2.4.** *Assume that*

(i) $E[y_0] < \infty$,

(ii) *There exist constants* $c > 0$, $C > 0$ *and* $p \geq 2$ *such that*

$$c\,|F|^p - C \leq W(F) \leq C(|F|^p + 1),$$

(iii) $\Psi = \Psi_1 = \dfrac{1}{4}\,|F^T H + H^T F\,|^2$.

*Then after extracting suitable subsequences and defining the piecewise constant interpolant in time as*

$$\tilde{y}^h(x,t) := y^{h,j}(x) \quad \textit{for} \;\; t \in ((j-1)h, jh],$$

*we have the following convergences as* $h \to 0$ :

$$\begin{cases} y_0^h \rightharpoonup y_0 \;\; \textit{in} \;\; W^{1,2}(\Omega), \\ \tilde{y}^h \overset{*}{\rightharpoonup} y \;\; \textit{in} \;\; L^\infty((0,\infty); W^{1,2}(\Omega)), \\ \tilde{C}^h \rightharpoonup C \;\; \textit{in} \;\; H^1((0,\infty); L^2(\Omega)). \end{cases}$$

*Proof*: The first two convergences are direct consequences of the assumptions (i), (ii) and Proposition 5.2.2. For the last assertion, we recall that

$$\dot{C} \;=\; F^T \dot{F} + \dot{F}^T F$$

so that assumption (iii) and Proposition 5.2.2 immediately give the desired convergence. Q.E.D

One way of showing that $y$ is a weak solution to (5.1.1) is to pass to the limit in (5.2.2) as $h \to 0$. In order to do this, roughly speaking, we need to have

$$\left.\begin{array}{l} y^{(j)} \overset{*}{\rightharpoonup} y \;\text{ in }\; L^\infty(W^{1,p}) \\ C^{(j)} \rightharpoonup C \;\text{ in }\; H^1(L^2) \end{array}\right\} \;\Longrightarrow\; C \;=\; Dy^T\, Dy.$$

We recall the following result and the corresponding counterexample from [86], which are worth mentioning even though there is no dynamics involved.

**Theorem 5.2.5.** *Assume that*

(i) $y^{(j)} \overset{*}{\rightharpoonup} y \;\; \textit{in} \;\; W^{1,\infty}(0,1)$,

(ii) $\det Dy,\, \det Dy^{(j)} > 0 \;\; \textit{for}$ a.e. $x \in (0,1) \;\; \textit{and for all} \;\; j$,

(iii) $U^{(j)} = \sqrt{Dy^{(j)^T} Dy^{(j)}} \;\to\; U \;\; \textit{for}$ a.e. $x \in (0,1)$.

*Then,* $Dy^T Dy \;=\; U^2$ *holds.*

*Proof*: Assumption (i) immediately gives, by Theorem 3.4 in [17], that

$$\det Dy^{(j)} \overset{*}{\rightharpoonup} \det Dy \ \text{ in } \ L^\infty(0,1) \tag{5.2.6a}$$

$$\operatorname{cof} Dy^{(j)} \overset{*}{\rightharpoonup} \operatorname{cof} Dy \ \text{ in } \ L^\infty(0,1). \tag{5.2.6b}$$

By assumption (ii) and Theorem 2.1.1 we have $Dy^{(j)} = R^{(j)}\, U^{(j)}$, where $R^{(j)} \in \mathrm{SO}(3)$ and $U^{(j)}$ is the right stretch tensor. This gives

$$\det Dy^{(j)} \ = \ \det R^{(j)} \, \det U^{(j)} \ = \ \det U^{(j)}.$$

Therefore, by assumption (iii) we obtain

$$\det Dy^{(j)} \ \to \ \det U \ \text{ for a.e. } x \in (0,1). \tag{5.2.7}$$

Convergences (5.2.6a) and (5.2.7) immediately give

$$\det Dy \ = \ \det U. \tag{5.2.8}$$

Similarly, we have

$$\operatorname{cof} Dy^{(j)} \ = \ \operatorname{cof} R^{(j)} \operatorname{cof} U^{(j)} \ = \ R^{(j)} \operatorname{cof} U^{(j)}$$

and hence, by (5.2.6b), we obtain

$$R^{(j)} \operatorname{cof} U^{(j)} \ \overset{*}{\rightharpoonup} \ \operatorname{cof} Dy \ \text{ in } \ L^\infty(0,1).$$

This shows that $\operatorname{cof} U^{(j)}$ is uniformly bounded, which together with assumption (iii) gives

$$\operatorname{cof} U^{(j)} \ \to \ \operatorname{cof} U \ \text{ in } \ L^q(0,1),\ 1 \le q < \infty. \tag{5.2.9}$$

Without loss of generality, we can say that $R^{(j)} \overset{*}{\rightharpoonup} R$ in $L^\infty(0,1)$, which implies

$$R^{(j)} \ \rightharpoonup \ R \ \text{ in } \ L^p(0,1),\ 1 \le p < \infty. \tag{5.2.10}$$

Choosing $q = p'$ in (5.2.9) thus gives

$$R^{(j)} \operatorname{cof} U^{(j)} \ \rightharpoonup \ R \operatorname{cof} U \ \text{ in } \ L^1(0,1). \tag{5.2.11}$$

Convergences (5.2.6b) and (5.2.11) imply that

$$\operatorname{cof} Dy \ = \ R \operatorname{cof} U. \tag{5.2.12}$$

By (5.2.8), (5.2.12) and the fact that $\operatorname{cof} F = (\det F)\, F^{-T}$, for any $F \in \mathbb{M}^{3\times 3}$, we obtain

$$(\det U)\, Dy^{-T} \ = \ (\det Dy)\, Dy^{-T} \ = \ \operatorname{cof} Dy \ = \ R \operatorname{cof} U \ = \ R\, (\det U)\, U^{-T},$$

giving

$$Dy^{-T} \;=\; R\,U^{-T}. \tag{5.2.13}$$

As we do not know whether $R$ is a rotation and $U$ is symmetric or not, (5.2.13) is still not enough. However, by boundedness of $Dy^{(j)}$ and $R^{(j)}$, and assumption (iii), we deduce that

$$U^{(j)} \;\to\; U \;\text{ in }\; L^r(0,1),\; 1 \le r < \infty.$$

Choosing $p = r'$ in (5.2.10) thus gives

$$R^{(j)}\,U^{(j)} \;\rightharpoonup\; R\,U \;\text{ in }\; L^1(0,1). \tag{5.2.14}$$

Assumption (i) and (5.2.14) imply that

$$Dy \;=\; R\,U. \tag{5.2.15}$$

Therefore, by (5.2.13) and (5.2.15) we have

$$Dy^T\,Dy \;=\; U^T\,R^{-1}\,R\,U \;=\; U^T\,U. \tag{5.2.16}$$

Equations (5.2.15) and (5.2.16) prove that $R \in \mathrm{SO}(3)$. Hence by assumption (ii) and Theorem 2.1.1 we can conclude that $U$ is symmetric which, by (5.2.16), immediately gives the result. Q.E.D

We now state the following crucial remark.

**Proposition 5.2.6.** *If assumption* (iii) *is not satisfied, then the conclusion of Theorem 5.2.5 does not hold.*

*Proof*: We give the following counterexample in order to prove the claim. Consider the simple laminate in a two-well problem as shown in Figure 5.1 formed from gradients $A, B$ satisfying $A - B = a \otimes n$ with separating interfaces with normal $n$, the $A$ layers having thickness $\lambda/j$ and $B$ layers $(1-\lambda)/j$ for $0 < \lambda < 1$.

Let $A \in \mathrm{SO}(3)\,U_1, B \in \mathrm{SO}(3)\,U_2$ and choose

$$A = U_1 \;\text{ and }\; B = U_1 + a \otimes n.$$

Then, $Dy^{(j)}$ satisfies (see e.g. [11], [13])

$$Dy^{(j)} \;\overset{*}{\rightharpoonup}\; \lambda U_1 + (1-\lambda)(U_1 + a \otimes n) = U_1 + (1-\lambda)\,a \otimes n =: Dy.$$

Therefore,

$$U \;=\; \sqrt{Dy^T Dy} \;=\; \sqrt{(U_1 + (1-\lambda)\,n \otimes a)(U_1 + (1-\lambda)\,a \otimes n)}.$$

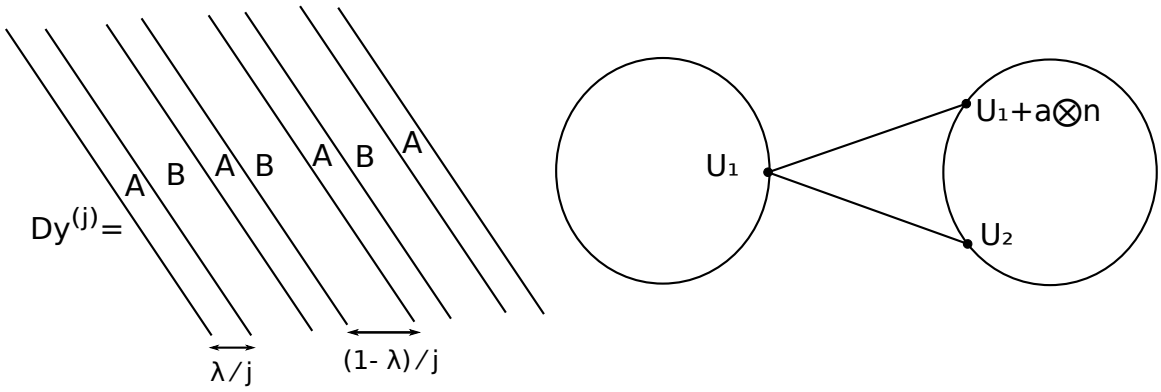



Figure 5.1: Laminate in a two-well problem

On the other hand, we could also choose

$$U_A^{(j)} = U_1 \quad \text{and} \quad U_B^{(j)} = U_2$$

where $U_A^{(j)} = U^{(j)}(x)\big|_{x\in A}$ and similarly for $B$. In this case we would get

$$U^{(j)} \overset{*}{\rightharpoonup} \lambda U_1 + (1-\lambda)U_2 =: U.$$

However,

$$\lambda U_1 + (1-\lambda)U_2$$

and

$$\sqrt{(U_1 + (1-\lambda)\, n \otimes a)(U_1 + (1-\lambda)\, a \otimes n)}$$

are not necessarily equal, contradicting the conclusion of Theorem 5.2.5. Q.E.D

This result shows that we should have stronger convergence results when $h \to 0$. This leads us to look at our problem from a different point of view which we introduce in the next section.

## 5.3 Gradient Flows in the Sense of Curves of Maximal Slope

We would like to fit our problem (5.1.1) into the theory of curves of maximal slope for which we make use of the techniques developed by Ambrosio, Gigli and Savaré [3]. We define the energy functional $\phi$ as

$$\phi(v) := \int_\Omega W(Dv)\, dx \tag{5.3.1}$$

and

$$\Phi(\tau, U_\tau^{n-1}; V) := \frac{1}{2\tau}\, d^2(V, U_\tau^{n-1}) \,+\, \phi(V), \tag{5.3.2}$$

where $d$, in a general metric context, is the distance. For a given time step $\tau > 0$, associated to the partition of the time interval $(0, \infty)$ as

$$\mathcal{P}_\tau := \{0 = t_\tau^0 < t_\tau^1 < \cdots < t_\tau^n < \cdots\},\ I_\tau^n := (t_\tau^{n-1}, t_\tau^n],\ \lim_{n\to\infty} t_\tau^n = \infty, \tag{5.3.3}$$

we want to find an approximate solution $U_\tau^n \approx y(\cdot, t_\tau^n)$, $n = 1, \cdots$, by solving iteratively a suitable time-discretized version of (5.1.1) in the unknown $U_\tau^n$ starting from an initial value $U_\tau^0 \approx y(\cdot, 0)$.

The first question we consider is concerned with the choice of $d$ in (5.3.2) which should be defined suitably so that we get the right Euler-Lagrange equation corresponding to $\Phi$. Having this, we can restrict the set of solutions to the minimum points of (5.3.2) and thus we end up with the recursive scheme

$$\begin{cases} U_\tau^0 \ \text{ is given; whenever } \ U_\tau^1, \cdots, U_\tau^{n-1} \ \text{ are known,} \\ \\ \text{find } \ U_\tau^n : U_\tau^n \,\in\, \operatorname{argmin} \Phi(\tau, U_\tau^{n-1}; \cdot) \ \text{ for all } n \geq 1. \end{cases} \tag{5.3.4}$$

## 5.3.1 Choice of the distance

In this section we discuss the question of defining $d$ in such a way that not only we get the necessary Euler-Lagrange equation for $\Phi$, but also we have a frame-indifferent $S$ and the existence of a minimizer at each time step.

### 5.3.1.1 A first attempt

Taking the analysis of Section 5.2 into account, we define $d$ as

$$d(U_\tau^n, U_\tau^{n-1}) \,:=\, \frac{1}{2}\, \|C(U_\tau^n) - C(U_\tau^{n-1})\|_2, \tag{5.3.5}$$

where $C$ is the right Cauchy-Green strain tensor (see (2.1.4)). In this case $\Phi$ becomes

$$\Phi(\tau, U_\tau^{n-1}; U_\tau^n) = \int_\Omega \frac{1}{4\,\tau} |C(U_\tau^n) - C(U_\tau^{n-1})|^2\, dx \,+\, \int_\Omega W(DU_\tau^n)\, dx$$

and the corresponding Euler-Lagrange equations are calculated as follows.

$$\begin{aligned} \Phi(\tau, U_\tau^{n-1}; U_\tau^n + t\, z) &= \int_\Omega \frac{1}{4\,\tau} |C(U_\tau^n + t\, z) - C(U_\tau^{n-1})|^2 + W(D(U_\tau^n + t\, z))\, dx \\ &= \int_\Omega \frac{1}{4\,\tau} |D(U_\tau^n + t\, z)^T\, D(U_\tau^n + t\, z) - C(U_\tau^{n-1})|^2 \,+\, W(DU_\tau^n + t\, Dz)\, dx. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial \Phi}{\partial t}(\tau, U_\tau^{n-1}; U_\tau^n + t\, z)\Big|_{t=0} = \\
&= \int_\Omega \frac{1}{2\,\tau}\Big((DU_\tau^n)^T DU_\tau^n - C(U_\tau^{n-1})\Big) : \Big(Dz^T DU_\tau^n + (DU_\tau^n)^T Dz\Big)\, dx + \\
&\qquad\qquad + \int_\Omega DW(DU_\tau^n) : Dz\, dx \\
&= \int_\Omega \frac{1}{2}\left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right) : Dz^T DU_\tau^n\, dx + \\
&\quad + \int_\Omega \frac{1}{2}\left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right) : (DU_\tau^n)^T Dz\, dx + \int_\Omega DW(DU_\tau^n) : Dz\, dx.
\end{aligned}$$

Using algebra of tensors (see Ch. 11, Sec. 1 in [7]) we obtain

$$\begin{aligned}
&\frac{\partial \Phi}{\partial t}(\tau, U_\tau^{n-1}; U_\tau^n + t\, z)\Big|_{t=0} = \\
&= \int_\Omega \frac{1}{2}\left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right) (DU_\tau^n)^T : Dz^T\, dx + \\
&\quad + \int_\Omega \frac{1}{2}\left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right)^T (DU_\tau^n)^T : Dz^T\, dx + \int_\Omega DW(DU_\tau^n) : Dz\, dx.
\end{aligned}$$

As $C$ is symmetric, we get

$$\begin{aligned}
&\frac{\partial \Phi}{\partial t}(\tau, U_\tau^{n-1}; U_\tau^n + t\, z)\Big|_{t=0} = \\
&\qquad = \int_\Omega \left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right)^T (DU_\tau^n)^T : Dz^T\, dx + \int_\Omega DW(DU_\tau^n) : Dz\, dx \\
&\qquad = \int_\Omega \left[DU_\tau^n \left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right) + DW(DU_\tau^n)\right] : Dz\, dx.
\end{aligned}$$

Hence the Euler-Lagrange equation can be written as

$$\operatorname{Div}\left[DW(DU_\tau^n) + DU_\tau^n \left(\frac{C(U_\tau^n) - C(U_\tau^{n-1})}{\tau}\right)\right] = 0. \tag{5.3.6}$$

Equation (5.3.6) is a time-discretized version of (5.1.1) with

$$S(F, \dot{F}) = F\, \dot{C},$$

which is frame-indifferent by (2.3.3) (see also Lemma 2.5.1).

Having frame-indifference at hand, we now need to prove existence of a minimizer at each time step. However, the well-known result stated below shows that it is impossible with this choice of $d$.

**Proposition 5.3.1.** *The function*

$$\Psi(F) \;=\; |C - I|^2$$

*is not quasiconvex.*

*Proof*: We will show that $\Psi$ is not rank-one convex which immediately implies that it is not quasiconvex. By definition of rank-one convexity (see e.g. [33], [73]) we know that $\Psi$ is rank-one convex if

$$t \;\mapsto\; \Psi(G + t\, a \otimes b) \;\text{ is convex for all }\; a, b \in \mathbb{R}^3, G \in \mathbb{M}^{3\times 3}.$$

Therefore, it is enough to find a rank-one direction towards which the function $\Psi$ is not convex. We claim that this direction, denoted by $H$, is

$$H := a \otimes b = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.3.7}$$

Let us now take $G \equiv I$ and define

$$F_t := I + t\, H = \begin{pmatrix} 1+t & -t & 0 \\ -t & 1+t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that we have

$$\begin{aligned} C_t = F_t^T F_t \;&=\; \begin{pmatrix} 1+t & -t & 0 \\ -t & 1+t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+t & -t & 0 \\ -t & 1+t & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &=\; \begin{pmatrix} 2t^2+2t+1 & -2t^2-2t & 0 \\ -2t^2-2t & 2t^2+2t+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and hence

$$\Psi(F_t) = \big|C_t - I\big|^2 = 4\,(2t^2+2t) = 16\,(t^4+2t^3+t^2).$$

It is clear to see that

$$\frac{d^2\Psi}{dt^2} \;=\; 32\,(1+6t+6t^2)$$

with real negative roots

$$\frac{-1}{2} + \frac{\sqrt{3}}{6} \;\text{ and }\; \frac{-1}{2} - \frac{\sqrt{3}}{6}.$$

Therefore, $\dfrac{d^2\Psi}{dt^2} < 0$ for some $t$, proving that $\Psi$ is not convex in the chosen direction. Q.E.D

#### 5.3.1.2 Composition of functions

We believe that in order to deal with frame-indifference and the existence of a minimizer at the same time, one should make use of a multiplicative characterization of functions. That is, we need to compose functions rather than adding them up. The argument goes as follows. Let

$$u^n : \Omega \to \Omega^n,\ x \mapsto u^n(x)$$
$$v : \Omega^n \to \Omega^{n+1},\ u^n \mapsto u^{n+1}.$$

Then by composing $u^n$ and $v$ we can define

$$u^{n+1} : \Omega \to \Omega^{n+1},\ x \mapsto u^{n+1}(x)$$

so that we have

$$u^{n+1}(x) = v(u^n(x)) \tag{5.3.9}$$

or equivalently

$$v = u^{n+1} \circ (u^n)^{-1}.$$

From (5.3.9) it easily follows that

$$D_{u^n} v = D_x u^{n+1}\, D_{u^n}(u^n)^{-1}.$$

By applying the chain rule we obtain

$$D_{u^n} v = D_x u^{n+1}\, (D_x u^n)^{-1} = Du^{n+1}\, (Du^n)^{-1}. \tag{5.3.10}$$

It is worth pointing out that the Kirchhoff stress tensor defined as

$$K(F) = D_F W(F)\, F^T \tag{5.3.11}$$

occurs as a multiplicative derivative of $W$ (see e.g. [12], [43]). That is,

$$K(F) : H = D_F W(F) : (H\, F) = \frac{d}{ds} W((I + s\, H)\, F)\bigg|_{s=0}.$$

Therefore, it is particularly suitable for problems in which the multiplicative character of matrices is important. We will see later that it suits our purposes very well in this chapter.

Using the composition idea, let us now take a general function $\psi\colon \mathbb{M}^{3\times 3}_+ \to [0,\infty]$ and define the distance $d$ as

$$d(v,u) = \|\psi(D_u v)\|_2.$$

We want to find the corresponding Euler-Lagrange equation for $\Phi$. In this case, the functional $\Phi$ becomes

$$\begin{aligned}\Phi(\tau, u; v) &= \frac{1}{2\tau}\, d^2(v, u) \,+\, \phi(v) \\ &= \int_\Omega \frac{1}{2\tau}\, \psi^2(Dv\, Du^{-1}) \,+\, W(Dv)\, dx.\end{aligned}$$

Then, for any $z \in W^{1,2}(0,1)$, we get

$$\Phi(\tau, u; v + t\, z) \,=\, \int_\Omega \left( W(Dv + tDz) \,+\, \frac{1}{2\tau}\, \psi^2((Dv + tDz)\, Du^{-1}) \right) dx.$$

Hence we have

$$\begin{aligned}\left.\frac{\partial \Phi}{\partial t}\right|_{t=0} &= \int_\Omega \left( DW(Dv) : Dz + \frac{1}{2\tau}\, D\psi^2(Dv\, Du^{-1}) : Dz\, Du^{-1} \right) dx \\ &= \int_\Omega \left( DW(Dv) : Dz \,+\, \frac{1}{2\tau}\, D\psi^2(Dv\, Du^{-1})\, Du^{-T} : Dz \right) dx \\ &= \int_\Omega \left( DW(Dv) \,+\, \frac{1}{2\tau}\, D\psi^2(Dv\, Du^{-1})\, Du^{-T} \right) : Dz\, dx.\end{aligned}$$

The Euler-Lagrange equation, in the most general form, is therefore

$$\mathrm{Div} \left( DW(Dv) \,+\, \frac{1}{2\tau}\, D\psi^2(Dv\, Du^{-1})\, Du^{-T} \right) \,=\, 0. \tag{5.3.12}$$

Let us investigate the term $\dfrac{1}{2\tau}\, D\psi^2(Dv\, Du^{-1})\, Du^{-T}$ further. Assume that for any $G \in \mathbb{M}^{3\times 3}_+$ we have

$$\psi^2(G) \,\approx\, \mathrm{dist}^2(G, \mathrm{SO}(3)) \tag{5.3.13}$$

for $G$ near $\mathrm{SO}(3)$. Linearizing $\mathrm{dist}(\cdot, \mathrm{SO}(3))$ near the identity (see e.g. [45]) we get

$$\mathrm{dist}(G, \mathrm{SO}(3)) \,=\, \left| \frac{1}{2}(G + G^T) - I \right| \,+\, \mathcal{O}(|G - I|^2).$$

This implies that

$$\psi^2(G) \,\approx\, |G^{\mathrm{sym}} - I|^2 \quad \text{when} \quad G \,\approx\, I. \tag{5.3.14}$$

Thus, in our case we have

$$\psi^2(Dv\, Du^{-1}) \,\approx\, \left| \frac{1}{2} \left( (Dv\, Du^{-1}) + (Dv\, Du^{-1})^T \right) - I \right|^2.$$

For $(v-u)/\tau \sim \dot{u}$ as $\tau \to 0$, this implies

$$\begin{aligned}
&\frac{1}{2\tau}\, D\psi^2(DvDu^{-1})\, Du^{-T} \sim \\
&\quad\sim \frac{1}{\tau}\left[\frac{1}{2}\Big((Dv\, Du^{-1}) + (Dv\, Du^{-1})^T\Big) - I\right] Du^{-T} \\
&\quad= \frac{1}{2}\left[\left(\frac{Dv - Du}{\tau} Du^{-1}\right) + \left(Du^{-T}\frac{Dv^T - Du^T}{\tau}\right)\right] Du^{-T} \\
&\quad\sim \frac{1}{2}\Big((D\dot{u}\, Du^{-1}) + (D\dot{u}\, Du^{-1})^T\Big)\, Du^{-T} \quad \text{as} \quad \tau \to 0.
\end{aligned}$$

In other words,

$$S(F, \dot{F}) \sim \frac{1}{2}\Big((\dot{F}\, F^{-1}) + (\dot{F}\, F^{-1})^T\Big)\, F^{-T}. \tag{5.3.15}$$

In Lemma 2.5.2 we showed that (5.3.15) is a frame-indifferent tensor corresponding to a convex potential. This (at least formally) proves that by using the composition of time-increments one can actually obtain a frame-indifferent stress in the Euler-Lagrange equation corresponding to the minimization problem for $\Phi$.

**Remark 5.3.1.** *The implication of (5.3.14) from (5.3.13) suggests that in order to handle the frame-indifference of the stress, it is enough to use (5.3.14), which might be easier to deal with from an analytical point of view.*

#### 5.3.1.3 Defining the distance

Motivated by the above calculations we define the distance function $d$ as

$$d(v, u) \;=\; \|\psi(D_u v)\|_2 \tag{5.3.16}$$

where $\psi\colon \mathbb{M}^{3\times 3}_{+} \to [0, \infty]$ satisfies the following general properties:

- Assumption (5.3.13) in the above calculation suggests that we have $\psi(I) = 0$.
- In order to get existence of a minimizer at each time step we need that $\psi$ is polyconvex (or at least quasiconvex).
- We also want $d$ to be a real metric since we are using the approach of gradient flows in metric spaces. In other words, it should satisfy the triangle inequality

  $$d(u, w) \;\le\; d(u, v) \;+\; d(v, w)$$

  for any admissible function $u, v$ and $w$.

In order to see how these ideas work, we will look at the one-dimensional case where we can assume that $W$ is convex without contradicting the physical assumptions we mentioned in Section 2.3.2. By doing so, we will also be able to distinguish between the parts heavily dependant on one-dimensional arguments and the ones which can be generalized to a three-dimensional setting. Firstly, let us give some preliminary information about the theory of gradient flows in the sense of curves of maximal slope.

## 5.4 Preliminaries : Gradient Flows in Metric Spaces

There are two main concepts behind the idea of the notion of gradient flows in a metric space $\mathcal{M}$: the metric derivative of an absolutely continuous curve with values in $\mathcal{M}$ and the upper gradients of a functional defined in $\mathcal{M}$. In this section, we give definitions of all the notions we require and state related results without proofs. All the information we present can be found in a more detailed fashion in the book by Ambrosio, Gigli and Savaré [3].

### 5.4.1 Absolutely continuous curves and metric derivative

**Definition 5.4.1** (Absolutely continuous curves)**.** *Let $(\mathcal{M}, d)$ be a complete metric space and let $v : (a, b) \to \mathcal{M}$ be a curve where $(a, b) \subset \mathbb{R}$; we say that $v$ belongs to $AC^p(a, b; \mathcal{M})$ for $1 \leq p \leq \infty$, if there exists a function $A \in L^p(a, b)$ such that*

$$d(v(s), v(t)) \leq \int_s^t A(r)\, dr \quad \forall\, a < s \leq t < b. \tag{5.4.1}$$

*In the case $p = 1$ we are dealing with absolutely continuous curves and we will denote the corresponding space simply by $AC(a, b; \mathcal{M})$.*

**Remark 5.4.1.** *Any curve in $AC^p(a, b; \mathcal{M})$ is uniformly continuous.*

Among all the possible choices of $A$ in (5.4.1) there exists a minimal one, which is provided by the following theorem which we give as a definition.

**Definition 5.4.2** (Metric derivative)**.** *Let $p \in [1, \infty]$. Then for any curve $v$ in $AC^p(a, b; \mathcal{M})$ the limit*

$$|v'|(t) := \lim_{s \to t} \frac{d(v(s), v(t))}{|s - t|} \tag{5.4.2}$$

*exists for $\mathcal{L}^1$-a.e. $t \in (a, b)$. Moreover the function $t \mapsto |v'|(t)$ belongs to $L^p(a, b)$, it is an admissible integrand for the right hand side of* (5.4.1)*, and it is minimal in the following sense:*

$$|v'|(t) \leq A(t) \ \text{ for } \mathcal{L}^1\text{-a.e. } t \in (a, b), \tag{5.4.3}$$

*for each function $A$ satisfying* (5.4.1).

**Remark 5.4.2** (Derivative in Banach Spaces)**.** *Suppose that $\mathcal{M} = \mathcal{B}$ is a reflexive Banach space, then a curve $v$ belongs to $AC^p(a,b;\mathcal{M})$ if and only if it is differentiable at $\mathcal{L}^1$-*a.e. *point $t \in (a,b)$, its derivative $v'$ belongs to $L^p(a,b;\mathcal{B})$ and*

$$v(t) - v(s) \;=\; \int_s^t v'(r)\,dr \quad \forall\, a < s \le t < b. \tag{5.4.4}$$

*In this case,*

$$\|v'(t)\|_{\mathcal{B}} = |v'|(t) \quad \mathcal{L}^1\text{-a.e. } in\ (a,b).$$

### 5.4.2 Upper gradients and slopes

In this section we define a "modulus of the gradient" for real valued functions defined on metric spaces. Let us initially consider the finite dimensional case of the Euclidean space $\mathcal{M} = \mathbb{R}^d$ with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$.

Let $\phi : \mathcal{M} \to \mathbb{R}$ be a functional. We define its domain as

$$D(\phi) := \{v \in \mathcal{M} :\, \phi(v) \,<\, +\infty\} \tag{5.4.5}$$

and say that it has a proper domain provided $D(\phi) \neq \varnothing$. The gradient $\nabla\phi$ of a differentiable $\phi$ can be defined by taking the derivative of $\phi$ along regular curves, i.e.

$$g = \nabla\phi \quad \Leftrightarrow \quad (\phi\circ v)' = (g(v), v') \quad \text{for every regular curve } v : (0,\infty) \to \mathcal{M}.$$

Its modulus $|\nabla\phi|$ has the following natural variational characterization:

$$g \ge |\nabla\phi| \quad \Leftrightarrow \quad |(\phi\circ v)'| \le g(v)|v'| \quad \text{for every regular curve } v : (a,b) \to \mathcal{M}. \tag{5.4.6}$$

We want to define a notion of "upper gradient" $g$ for $\phi$ modelled on (5.4.6). A first possibility is to use an integral formulation along absolutely continuous curves.

**Definition 5.4.3** (Strong upper gradients)**.** *A function $g : \mathcal{M} \to [0,\infty]$ is a strong upper gradient for $\phi$ if for every absolutely continuous curve $v \in AC(a,b;\mathcal{M})$ the function $g\circ v$ is Borel and*

$$|\,\phi(v(t)) \,-\, \phi(v(s))\,| \;\le\; \int_s^t g(v(r))\,|v'|(r)\,dr \quad \forall\, a \,<\, s \,\le\, t \,<\, b. \tag{5.4.7}$$

*In particular, if $g\circ v\,|v'| \in L^1(a,b)$ then $\phi\circ v$ is absolutely continuous and*

$$|(\phi\circ v)'(t)| \;\le\; g(v(t))\,|v'|(t) \quad \textit{for } \mathcal{L}^1\text{-a.e. } t \in (a,b). \tag{5.4.8}$$

We also introduce a weaker notion, based on a pointwise formulation:

**Definition 5.4.4** (Weak upper gradients)**.** *A function $g : \mathcal{M} \to [0, \infty]$ is a weak upper gradient for $\phi$ if for every curve $v \in AC(a, b; \mathcal{M})$ such that*

(i) $g \circ v\, |v'| \in L^1(a, b)$;

(ii) $\phi \circ v$ *is* $\mathcal{L}^1$-a.e. *equal in* $(a, b)$ *to a funtion* $\varphi$ *with finite pointwise variation in* $(a, b)$;

*we have*

$$|\varphi'(t)| \le g(v(t))\, |v'|(t) \text{ for } \mathcal{L}^1\text{-a.e. } t \in (a, b). \tag{5.4.9}$$

*In this case, if $\phi \circ v \in AC(a, b)$ then $\varphi = \phi \circ v$ and (5.4.7) holds.*

Among all the possible choices for an upper gradient of $\phi$, we give the definition of the local and global slopes:

**Definition 5.4.5** (Slopes)**.** *The local and global slopes of $\phi$ at $v \in D(\phi)$ are defined by*

$$|\partial\phi|(v) := \limsup_{\omega \to v} \frac{(\phi(v) - \phi(\omega))^+}{d(v, \omega)}, \quad \Upsilon_\phi(v) := \sup_{\omega \neq v} \frac{(\phi(v) - \phi(\omega))^+}{d(v, \omega)}. \tag{5.4.10}$$

**Remark 5.4.3.** *The function $|\partial\phi|$ is a weak upper gradient for $\phi$. If $\phi$ is d-lower semicontinuous, then $\Upsilon_\phi$ is a strong upper gradient for $\phi$.*

### 5.4.3 Curves of maximal slope

The notion of curves of maximal slope was introduced by De Giorgi, Marino and Tosques [35]. Here we follow an upper gradient point of view as in the book of Ambrosio, Gigli and Savaré [3] and confine ourselves to the quadratic case.

**Definition 5.4.6** (Curves of maximal slope)**.** *We say that a locally absolutely continuous map $u : (a, b) \to \mathcal{M}$ is a curve of maximal slope for the functional $\phi$ with respect to its upper gradient $g$, if $\phi \circ u$ is* $\mathcal{L}^1$-a.e. *equal to a non-increasing map $\varphi$ and*

$$\varphi'(t) \le -\frac{1}{2}\, |u'|^2(t) - \frac{1}{2}\, g^2(u(t)) \text{ for } \mathcal{L}^1\text{-a.e. } t \in (a, b). \tag{5.4.11}$$

**Remark 5.4.4.** *Observe that (5.4.9) and (5.4.11) yield*

$$|u'|^2(t) = g^2(u(t)) = -\varphi'(t) \ \ \mathcal{L}^1\text{-a.e. in } (a, b), \tag{5.4.12}$$

*in particular* $u \in AC^2_{loc}(a,b;\mathcal{M})$ *and* $g \circ u \in L^2_{loc}(a,b)$. *If* $u$ *is a curve of maximal slope for* $\phi$ *with respect to a strong upper gradient* $g$, *then* $\phi(u(t)) \equiv \varphi(t)$ *is a locally absolutely continuous map in* $(a,b)$ *and the energy identity*

$$\frac{1}{2}\int_s^t |u'|^2(r)\,dr \,+\, \frac{1}{2}\int_s^t g^2(r)\,dr \,=\, \phi(u(s)) \,-\, \phi(u(t)) \tag{5.4.13}$$

*holds in each interval* $[s,t] \subset (a,b)$.

#### 5.4.3.1 An illustration in a Hilbert space setting

In this section, we review how one can relate curves of maximal slope and gradient flows when $\mathcal{M} = \mathcal{B} = \mathcal{B}'$ is a Hilbert space (see [3, pg. 2]). The case when $\mathcal{B}$ is a reflexive Banach space and $\phi$ is convex is also studied by Colli & Visintin [29]. We denote by $(\cdot,\cdot)$ the duality between $\mathcal{B}$ and its dual $\mathcal{B}'$. Consider the classical setting of a gradient flow

$$u'(t) \,=\, -\nabla\phi(u(t)) \tag{5.4.14}$$

in a Hilbert space. If we take the modulus in both sides we have the equation

$$|u'(t)|(t) \,=\, |\nabla\phi(u(t))|$$

which makes sense in a metric setting, interpreting the left hand side as the metric derivative and the right hand side as an upper gradient of $\phi$. However, in passing from (5.4.14) to a scalar equation we clearly have a loss of information. This information can be retained by looking at the derivative of the energy as follows. If (5.4.14) holds, then by Remark 5.4.2 we have

$$\begin{aligned}
\frac{d}{dt}\phi(u(t)) &= (u'(t), \nabla\phi(u(t))) = -\|\nabla\phi(u(t))\|^2_{\mathcal{B}} \\
&= -|\nabla\phi(u(t))|^2 = -|u'(t)||\nabla\phi(u(t))| \\
&= -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\nabla\phi(u(t))|^2.
\end{aligned}$$

The second equality holds if and only if $u'$ and $-\nabla\phi(u)$ are parallel and the third equality holds if and only if $|u'|$ and $|\nabla\phi(u)|$ are equal, so that we can write (5.4.14) equivalently as

$$\frac{1}{2}|u'|^2(t) + \frac{1}{2}|\nabla\phi(u(t))|^2 \,=\, -\frac{d}{dt}\phi(u(t)).$$

Passing to an integral formulation and replacing $|\nabla\phi(u)|$ with $g(u)$, where $g$ is a strong upper gradient of $\phi$, for any $s,t \geq 0$ with $s \leq t$, we obtain

$$\frac{1}{2}\int_s^t |u'|^2(r)\,dr \,+\, \frac{1}{2}\int_s^t g^2(r)\,dr \,=\, \phi(u(s)) \,-\, \phi(u(t))$$

which is (5.4.13). Therefore, we can say that the concept of curve of maximal slope is very natural in a Hilbert space setting as well.

## 5.5 The Abstract Convergence Theorem

Let us state the main result of this chapter in a general abstract form following Theorem 2.3.3 in [3] where $\bar{U}_\tau(t)$ will denote the piecewise constant interpolant. For the completeness of the text, we postpone defining it to Section 5.6.3.2.

**Theorem 5.5.1.** *Let $(\mathcal{M}, d)$ be a complete metric space and $\sigma$ a Hausdorff topology on $\mathcal{M}$ which is weaker than the topology $d$. Assume that*

- *$\phi$ is $\sigma$-lower semicontinuous*
- *$d$ is $\sigma$-lower semicontinuous with respect to both its variables*
- *the sublevel sets of $d(\cdot, x)$ are $\sigma$-compact*
- *$|\partial\phi|$ is a strong upper gradient for $\phi$ and is $\sigma$-lower semicontinuous.*

*Then, every curve $u(t)$ for which there exists a sequence of discrete solutions $\{U_{\tau_n}\}_{n\in\mathbb{N}}$ such that*

$$\bar{U}_{\tau_n}(t) \overset{\sigma}{\rightharpoonup} u(t) \text{ for all } t \in [0,\infty),$$

*with $u_0 \in D(\phi)$, is a curve of maximal slope for $\phi$ with respect to $|\partial\phi|$ and in particular $u$ satisfies the identity*

$$\frac{1}{2}\int_0^T |u'|^2(t)\,dt + \frac{1}{2}\int_0^T |\partial\phi|^2(u(t))\,dt + \phi(u(T)) = \phi(u_0) \text{ for all } T > 0.$$

*Moreover, we have*

$$\begin{aligned}
\lim_{n\to\infty} \phi(\bar{U}_{\tau_n}(t)) &= \phi(u(t)) \text{ for all } t \in [0,\infty) \\
\lim_{n\to\infty} |\partial\phi|(\bar{U}_{\tau_n}) &= |\partial\phi|(u) \text{ in } L^2_{\mathrm{loc}}([0,\infty)) \\
\lim_{n\to\infty} |U'_{\tau_n}| &= |u'| \text{ in } L^2_{\mathrm{loc}}([0,\infty)).
\end{aligned}$$

## 5.6 One-dimensional case

### 5.6.1 Logarithmic metric

In one space dimension, the composition (5.3.10) takes the form

$$D_u v = \frac{dv}{du} = \frac{dv}{dx}\frac{dx}{du} = dv\,du^{-1}.$$

Writing the triangle inequality property pointwise for $\psi$ we get

$$\left|\psi\left(\frac{dv}{du}\right)\right| \le \left|\psi\left(\frac{dv}{dw}\right)\right| + \left|\psi\left(\frac{dw}{du}\right)\right| \tag{5.6.1}$$

which immediately suggests that one needs a logarithmic behaviour for $\psi$. Let us therefore define $d$ as the following logarithmic metric

$$d(v,u) := \left\| \log\left(\frac{dv}{du}\right) \right\|_2 \qquad \begin{array}{l} \text{where } u, v \in H^1(0,1) \text{ and} \\ du, dv > 0 \text{ for a.e. } x \in (0,1). \end{array} \tag{5.6.2}$$

**Proposition 5.6.1.** *The function $d$ defined in (5.6.2) is a metric on*

$$\mathcal{S} := \{z \in H^1(0,1) \colon z(0) = 0,\, d(z,x) \le C,\, dz > 0 \text{ a.e.}\} \tag{5.6.3}$$

*where $C > 0$ is a fixed constant.*

*Proof*: It is clear that we have $d(v,u) \ge 0$ for all $v, u \in \mathcal{S}$. Moreover we have

$$\begin{aligned} d(v,u) = 0 \quad &\Leftrightarrow \quad \log\left(\frac{dv}{du}\right) = 0 \text{ for a.e. } x \in (0,1) \\ &\Leftrightarrow \quad \frac{dv}{du} = 1 \text{ for a.e. } x \in (0,1) \\ &\Leftrightarrow \quad v = u \end{aligned}$$

as required. We also need to check that $d$ is symmetric. Indeed, we have

$$\begin{aligned} d(v,u) &= \left\| \log\left(\frac{dv}{du}\right) \right\|_2 = \\ &= \| \log(dv) - \log(du) \|_2 = \| \log(du) - \log(dv) \|_2 = \\ &= \left\| \log\left(\frac{du}{dv}\right) \right\|_2 = d(u,v). \end{aligned}$$

Finally, $d$ also satisfies the triangle inequality since for any $z, v, u \in \mathcal{S}$ we obtain

$$\begin{aligned} d(v,u) &= \left\| \log\left(\frac{dv}{du}\right) \right\|_2 = \left\| \log\left(\frac{dv}{dz}\frac{dz}{du}\right) \right\|_2 \\ &= \left\| \log\left(\frac{dv}{dz}\right) + \log\left(\frac{dz}{du}\right) \right\|_2 \\ &\le \left\| \log\left(\frac{dv}{dz}\right) \right\|_2 + \left\| \log\left(\frac{dz}{du}\right) \right\|_2 \\ &= d(v,z) + d(z,u). \end{aligned}$$

Q.E.D

#### 5.6.1.1 Corresponding Euler-Lagrange equations

Before proving some properties of the logarithmic metric $d$, let us investigate the Euler-Lagrange equation corresponding to it. By (5.3.12) in one space dimension we must have

$$\frac{d}{dx}\left(DW(dv) + \frac{1}{2\tau}\, D\psi^2(dvdu^{-1})\, du^{-1}\right) = 0. \tag{5.6.4}$$

By (5.6.2) we know that

$$\psi(dvdu^{-1}) = \log\left(\frac{dv}{du}\right)$$

and hence (5.6.4) becomes

$$\begin{aligned}
&\frac{d}{dx}\left(DW(dv) + \frac{1}{\tau}\log\left(\frac{dv}{du}\right)\frac{du}{dv}\, du^{-1}\right) = \\
&\quad = \frac{d}{dx}\left(DW(dv) + \frac{1}{\tau}\log\left(\frac{dv}{du}\right) dv^{-1}\right) = \\
&\quad = \frac{d}{dx}\left(DW(dv) + \frac{\log(dv) - \log(du)}{\tau}\, dv^{-1}\right) = 0.
\end{aligned}$$

Passing to the limit as $\tau \to 0$ gives

$$\frac{d}{dx}\left(DW(dv) + (d\dot{v}\, dv^{-1})\, dv^{-1}\right) = 0. \tag{5.6.5}$$

It is important to note that $(d\dot{v}\, dv^{-1})\, dv^{-1}$ in (5.6.5) is a one-dimensional version of $S$ in (5.3.15) which suggests that logarithm was a good choice for $\psi$.

#### 5.6.1.2 Lower semicontinuity of $d$

Even though the square of the logarithmic function is not convex, $d$ is lower semicontinuous with respect to the following notion of weak convergence:

**Definition 5.6.1** ($\sigma$-convergence)**.** *We say that* $u_n \overset{\sigma}{\rightharpoonup} u$ *if*

$$\log(du_n) \rightharpoonup \log(du) \ \text{ in } \ L^2(0,1).$$

We now show that $d$ in (5.6.2) is $\sigma$-lower semicontinuous.

**Proposition 5.6.2.** *For any* $u_n, v_n \in \mathcal{S}$ *we have*

$$\left.\begin{array}{l} u_n \overset{\sigma}{\rightharpoonup} u \\ v_n \overset{\sigma}{\rightharpoonup} v \end{array}\right\} \ \Rightarrow \ d^2(v,u) \le \liminf_{n\to\infty} d^2(v_n, u_n).$$

*Proof*: By Definition 5.6.1 we know that

$$\begin{array}{l} u_n \overset{\sigma}{\rightharpoonup} u \Rightarrow \log(du_n) \rightharpoonup \log(du) \text{ in } L^2(0,1) \\ v_n \overset{\sigma}{\rightharpoonup} v \Rightarrow \log(dv_n) \rightharpoonup \log(dv) \text{ in } L^2(0,1). \end{array} \tag{5.6.6}$$

On the other hand, we can write

$$\begin{aligned} d^2(v_n,u_n) &= \int_0^1 \left(\log\frac{dv_n}{du_n}\right)^2 dx = \int_0^1 \left(\log\frac{dv_n}{dv}\frac{dv}{du}\frac{du}{du_n}\right)^2 dx \\ &= \int_0^1 \left(\log\frac{dv}{du}\right)^2 + \left(\log\frac{dv_n}{dv}+\log\frac{du}{du_n}\right)^2 + 2\log\frac{dv}{du}\left(\log\frac{dv_n}{dv}+\log\frac{du}{du_n}\right) dx \\ &\geq \int_0^1 \left(\log\frac{dv}{du}\right)^2 dx + 2\int_0^1 \log\frac{dv}{du}\big(\log(dv_n)-\log(dv)\big)\,dx + \\ &\qquad + 2\int_0^1 \log\frac{dv}{du}\big(\log(du)-\log(du_n)\big)\,dx. \end{aligned}$$

Since $\log\frac{dv}{du} \in L^2(0,1)$, the last two integrals converge to zero by (5.6.6) as $n \to \infty$. Hence, taking inferior limit of both sides in the above inequality, we obtain

$$\liminf_{n\to\infty} d^2(v_n,u_n) \geq \int_0^1 \left(\log\frac{dv}{du}\right)^2 dx = d^2(v,u)$$

as required. Q.E.D

#### 5.6.1.3 Compactness of sublevels

In this section we prove that $d$-sublevel sets are compact.

**Proposition 5.6.3.** *For any $\{u_n\} \subset \mathcal{S}, v \in \mathcal{S}$, we have*

$$\sup_n d(u_n,v) < \infty \Rightarrow \exists\,\{n_j\}_{j\geq 1} \text{ and } u \in \mathcal{S} \text{ such that } u_{n_j} \overset{\sigma}{\rightharpoonup} u.$$

*Proof*: By Definition 5.6.2 we have

$$d(u_n,v) = \big\|\log du_n - \log dv\big\|_2.$$

Therefore, the condition $\sup_n d(u_n,v) < \infty$ implies

$$\big\|\log du_n\big\|_2 \text{ is bounded.}$$

Hence there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\xi \in L^2(0,1)$ such that

$$\log du_{n_j} \rightharpoonup \xi \text{ in } L^2(0,1).$$

Defining

$$u(x) := \int_0^x \exp(\xi(s))\,ds$$

we get $du = \exp(\xi)$ and hence $\xi = \log du$, proving $u_{n_k} \overset{\sigma}{\rightharpoonup} u$. Moreover, we have $u(0) = 0$ so that $u \in \mathcal{S}$ as desired. Q.E.D

### 5.6.2 Energy and slopes

#### 5.6.2.1 Properties of the energy functional

In addition to the assumptions we made on $W$ in Section 2.3.2, we have:

- **Convexity** : We assume that $W(\exp(\cdot))$ is convex (cf. [39]), i.e. for all $\xi, \eta \in \mathbb{R}$ we have

$$W(\exp(\xi)) \;\geq\; W(\exp(\eta)) \;+\; DW(\exp(\eta))\,(\exp(\xi) - \exp(\eta)). \tag{5.6.7}$$

As a result, the energy functional $\phi$, which was defined in (5.3.1), satisfies the following properties.

- **Lower semicontinuity** : By assumption (5.6.7) above we immediately have that $\phi$ is $\sigma$-lower semicontinuous (see e.g. [33, Thm. 3.23]),

$$u_n \overset{\sigma}{\rightharpoonup} u \quad \Rightarrow \quad \liminf_{n\to\infty} \phi(u_n) \geq \phi(u). \tag{5.6.8}$$

- **Coercivity** : By assumption (2.3.9b) we have that

$$\phi(u) \geq 0 \;\text{ for all }\; u \in \mathcal{S}. \tag{5.6.9}$$

#### 5.6.2.2 The local slope

In this section we prove that the local slope of $\phi$ with respect to the logarithmic distance $d$ is equal to the norm of the Kirchhoff tensor.

**Theorem 5.6.4.** *Let $\phi$ be as in (5.3.1) satisfying (5.6.8) and (5.6.9). Assume also that $W$ satisfies (5.6.7). Then, for any $u \in \mathcal{S}$ we have*

$$|\partial\phi|(u) \;=\; \|K(du)\|_2.$$

*Proof*: Since $u, v \in \mathcal{S}$ we can define $\xi$ and $\eta$ for almost every $x \in (0,1)$ such that

$$du = \exp(\xi) \;\text{ and }\; dv = \exp(\eta).$$

Let us also define

$$\widetilde{W}(z) := W(\exp(z)). \tag{5.6.10}$$

(i) *The upper bound*:
By Definition 5.4.5, (5.3.1) and (5.6.10) we have

$$|\partial\phi|(u) = \limsup_{d(v,u)\to 0} \frac{(\phi(u) - \phi(v))^+}{d(v,u)} = \limsup_{\|\xi-\eta\|_2\to 0} \frac{\left(\int_0^1 \widetilde{W}(\xi) - \widetilde{W}(\eta)\,dx\right)^+}{\|\xi - \eta\|_2}. \tag{5.6.11}$$

Using assumption (5.6.7) and the Cauchy-Schwarz inequality we can continue as

$$\begin{aligned}
|\partial\phi|(u) &\leq \limsup_{\|\xi-\eta\|_2\to 0} \left(\int_0^1 -\widetilde{W}'(\xi)\frac{(\eta-\xi)}{\|\xi-\eta\|_2}\right)^+ \\
&\leq \limsup_{\|\xi-\eta\|_2\to 0} \left\|\widetilde{W}'(\xi)\right\|_2 = \|W'(\exp(\xi))\exp(\xi)\|_2 \\
&= \|W'(du)du\|_2 = \|K(du)\|_2
\end{aligned}$$

as required.

(ii) *The lower bound*:

As we are trying to get a lower bound and we have superior limit in the definition of the slope, it is enough to get a bound for a particular sequence. Let us choose

$$\eta = \xi + \alpha\,\mu \quad \text{where } \alpha \in \mathbb{R},\ \|\mu\|_2 = 1. \tag{5.6.12}$$

Continuing from (5.6.11) we get

$$\begin{aligned}
|\partial\phi|(u) &= \limsup_{\|\xi-\eta\|_2\to 0} \frac{\left(\int_0^1 \widetilde{W}(\xi) - \widetilde{W}(\eta)\,dx\right)^+}{\|\xi-\eta\|_2} \\
&\geq \limsup_{\alpha\to 0} \frac{\left(\int_0^1 \widetilde{W}(\xi) - \widetilde{W}(\xi+\alpha\,\mu)\,dx\right)^+}{\|\alpha\,\mu\|_2} \\
&= \limsup_{\alpha\to 0} \left(\frac{1}{\alpha}\int_0^1 \left(\widetilde{W}(\xi) - \widetilde{W}(\xi+\alpha\,\mu)\right)dx\right)^+.
\end{aligned}$$

Using Fatou's lemma we can continue as

$$\begin{aligned}
|\partial\phi|(u) &\geq \limsup_{\alpha\to 0} \left(\frac{1}{\alpha}\int_0^1 \left(\widetilde{W}(\xi) - \widetilde{W}(\xi+\alpha\,\mu)\right)dx\right)^+ \\
&\geq \liminf_{\alpha\to 0} \left(\frac{1}{\alpha}\int_0^1 \left(\widetilde{W}(\xi) - \widetilde{W}(\xi+\alpha\,\mu)\right)dx\right)^+ \\
&\geq \left(\liminf_{\alpha\to 0} \frac{1}{\alpha}\int_0^1 \left(\widetilde{W}(\xi) - \widetilde{W}(\xi+\alpha\,\mu)\right)dx\right)^+ \\
&\geq \left(\int_0^1 \liminf_{\alpha\to 0} \left(\frac{\widetilde{W}(\xi) - \widetilde{W}(\xi+\alpha\,\mu)}{\alpha}\right)dx\right)^+ \\
&= \left(\int_0^1 \widetilde{W}'(\xi)\,\mu\,dx\right)^+ \quad \text{for a.e. } x \in (0,1).
\end{aligned}$$

However, $\mu \in L^2(0,1)$ was arbitrary with $\|\mu\|_2 = 1$, hence we have

$$\begin{aligned}
|\partial\phi|(u) &\geq \sup_{\|\mu\|_2=1} \int_0^1 \widetilde{W}'(\xi)\,\mu\,dx \\
&= \left\|\widetilde{W}'(\xi)\right\|_2 = \|K(du)\|_2
\end{aligned}$$

as required. Q.E.D

We state the following result due to Ambrosio, Gigli and Savaré [3, Lemma 3.1.3] which we will refer to later.

**Lemma 5.6.5** (Slope estimate)**.** *If* $\bar{u} \in \operatorname{argmin} \Phi(\tau, u; \cdot)$*, then* $\bar{u} \in D(|\partial\phi|)$ *and*

$$|\partial\phi|(\bar{u}) \leq \frac{d(\bar{u}, u)}{\tau}. \tag{5.6.13}$$

#### 5.6.2.3 Strong upper gradient property

In this section we make use of the global slope in order to prove that $\|K\|_2$ is a strong upper gradient for $\phi$.

**Proposition 5.6.6.** *Let* $K$ *and* $\phi$ *be defined in (5.3.11) and (5.3.1) respectively. Then,* $\|K(du)\|_2$ *is a strong upper gradient for* $\phi$*.*

*Proof*: First note that by definitions of local and global slopes in (5.4.10) and Theorem 5.6.4 we immediately have

$$\Upsilon_\phi(u) \geq |\partial\phi|(u) = \|K(du)\|_2 \tag{5.6.14}$$

for all $u \in \mathcal{S}$. On the other hand, we have

$$\begin{aligned} \Upsilon_\phi(u) &= \sup_{v \neq u} \frac{\Big(\phi(u) - \phi(v)\Big)^+}{d(v,u)} \\ &= \sup_{\xi \neq \eta} \frac{1}{\|\xi - \eta\|_2} \left( \int_0^1 \widetilde{W}(\xi) - \widetilde{W}(\eta)\, dx \right)^+. \end{aligned}$$

By assumption (5.6.7) we get

$$\begin{aligned} \Upsilon_\phi(u) &\leq \sup_{\xi \neq \eta} \frac{1}{\|\xi - \eta\|_2} \left( \int_0^1 \widetilde{W}'(\xi)(\xi - \eta)\, dx \right)^+ \\ &\leq \sup_{\xi \neq \eta} \|\widetilde{W}'(\xi)\|_2 = \|K(du)\|_2. \end{aligned} \tag{5.6.15}$$

Therefore, by (5.6.14) and (5.6.15) we obtain

$$\Upsilon_\phi(u) = \|K(du)\|_2.$$

As we already know by (5.6.8) and Remark 5.4.3 that $\Upsilon_\phi(u)$ is a strong upper gradient, we conclude that so is $\|K(du)\|_2$, as required. Q.E.D

#### 5.6.2.4 Lower semicontinuity of the Kirchhoff tensor

In order to get $\sigma$-lower semicontinuity of $\|K\|_2$, which is vital for the proof of convergence of the scheme, we make the following additional assumption on $W$.

- We assume that

$$\mathbb{R} \ni z \mapsto (W'(\exp(z))\exp(z))^2 = K^2(\exp(z)) \text{ is convex.} \tag{5.6.16}$$

**Remark 5.6.1.** *An example of $W$ satisfying (5.6.16) which is also compatible with the earlier constitutive assumptions can be given as*

$$W(z) = z^2 + z^{-3}.$$

*It is clear that we have*

$$W(\exp(z)) = \exp(2z) + \exp(-3z),$$

*which is a convex function (see Figure 5.2). Moreover,*

$$K(z) = W'(z)\,z = 2z^2 - 3z^{-3}$$

*and hence*

$$K^2(\exp(z)) = \big(2\,\exp(2z) - 3\,\exp(-3z)\big)^2.$$

*It can easily be checked that $K^2(\exp(z))$ is also convex (see also Figure 5.2).*

Having assumption (5.6.16), we can prove $\sigma$-lower semicontinuity of $\|K\|_2$.

**Proposition 5.6.7.** *The map*

$$u \mapsto \|K(du)\|_2^2$$

*is $\sigma$-lower semicontinuous. That is,*

$$\|K(du)\|_2^2 \le \liminf_{n\to\infty} \|K(du_n)\|_2^2 \quad \textit{whenever } u_n \overset{\sigma}{\rightharpoonup} u. \tag{5.6.17}$$

*Proof*: We consider

$$\|K(du_n)\|_2^2 = \int_0^1 K^2(\exp(\log(du_n)))\,dx.$$

By Definition 5.6.1 and assumption (5.6.16) we have that the integrand of this integral is lower semicontinuous, positive and convex. Therefore, we can apply Ioffe's theorem on lower semicontinuity of integral functionals (see [55, Thm. 3]) and get (5.6.17) as desired. Q.E.D

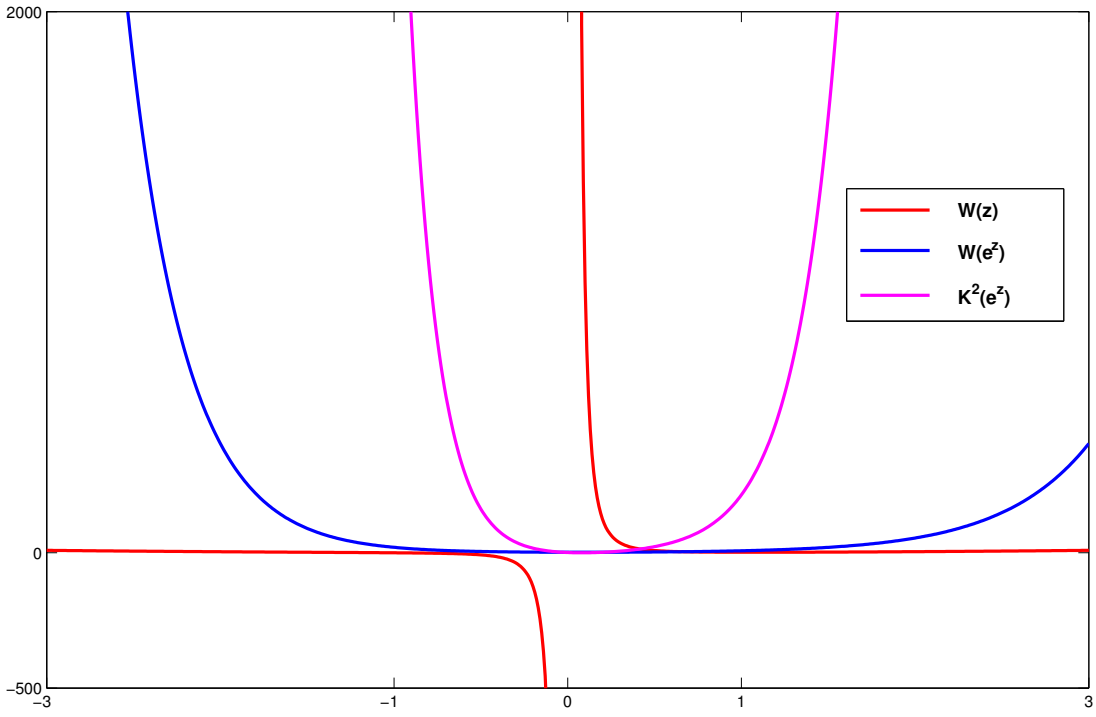



Figure 5.2: Graphs for Remark 5.6.1

### 5.6.3 Existence and convergence

#### 5.6.3.1 Existence of minimizers

In this section, we prove existence of a minimizer at each time step so that our recursive scheme makes sense.

**Theorem 5.6.8.** *There exists a sequence* $\{U_\tau^n\}_{n=1}^\infty \subset \mathcal{S}$ *satisfying* (5.3.4). *That is,*

$$U_\tau^n \in \operatorname{argmin} \Phi(\tau, U_\tau^{n-1}; \cdot) \quad \forall\, n \geq 1.$$

*Proof*: We will show existence of a minimizer by using direct methods in the calculus of variations (cf. [32], [33], [73]). We denote

$$\Phi(\tau, u; v) = \frac{1}{2\tau}\, d^2(v, u) + \phi(v). \tag{5.6.18}$$

We need to show that

(i) $\Phi$ is $\sigma$-lower semicontinuous,

(ii) $\Phi$ is coercive, i.e. sublevels of $\Phi$ are $\sigma$-compact.

By Proposition 5.6.2 and (5.6.8) we already know that (i) holds. Moreover, by (5.6.9) and Proposition 5.6.3 we also get (ii) as required. Q.E.D

#### 5.6.3.2 Interpolants and their derivatives

We define the following interpolants and the corresponding derivatives.

- *Piecewise constant interpolant*:

$$\bar{U}_\tau(t) := U_\tau^n, \quad t^{n-1} < t \leq t^n. \tag{5.6.19}$$

  Its derivative with respect to $d$ is

$$|U_\tau'|(t) = \frac{d(U_\tau^n, U_\tau^{n-1})}{\tau} \tag{5.6.20}$$

  where $t^{n-1} < t < t^n$.

- *De Giorgi interpolant*:

$$\hat{U}_\tau(t) := \hat{U}_\tau(t^{n-1} + \delta) \in \operatorname{argmin} \Phi(\delta, U_\tau^{n-1}; \cdot) \tag{5.6.21}$$

  where $t = t^{n-1} + \delta \in (t^{n-1}, t^n]$ and its derivative is

$$G_\tau(t) = \frac{d(\hat{U}_\tau(t), U_\tau^{n-1})}{t - t^{n-1}} \tag{5.6.22}$$

  for $t = t^{n-1} + \delta \in (t^{n-1}, t^n]$.

**Remark 5.6.2.** *From definitions (5.6.20) and (5.6.22) we have that*

$$G_\tau(t) = |U_\tau'|(t) \quad \text{at} \quad t = t^n.$$

**Theorem 5.6.9.** *We have the following a priori bounds on the discrete solution and the interpolants:*

$$\begin{aligned} \|U_\tau^n\|_{W^{1,2}} &\leq C \\ \|\bar{U}_\tau(t)\|_{W^{1,2}} &\leq C \\ \|\hat{U}_\tau(t)\|_{W^{1,2}} &\leq C, \end{aligned}$$

*where $C$ is a generic constant independent of $n$ and $\tau$.*

*Proof*: The result easily follows from the assumption (2.3.9b), Theorem 5.6.8, definitions (5.6.19), (5.6.21) and the fact that the energy is finite initially (see (2.1.2)). Q.E.D

**Remark 5.6.3.** *By the slope estimate (5.6.13) and Theorem 5.6.4 we immediately have that*

$$\|K(d\hat{U}_\tau(t))\|_2 = |\partial\phi|(\hat{U}_\tau(t)) \leq G_\tau(t) \quad \text{for all } t \in (0, \infty).$$

#### 5.6.3.3 Discrete energy estimates

All arguments we follow in this section are presented in the book by Ambrosio, Gigli and Savaré [3] where they prove very similar results in a more general setting. The main result we want to obtain is the following discrete energy equality.

**Proposition 5.6.10.** *Let $\{U_\tau^n\}_{n=0}^\infty$ be a sequence solving the variational scheme (5.3.4) with $d$ defined in (5.6.2), and let $|U_\tau'|, G_\tau$ be respectively defined by (5.6.20) and (5.6.22). Then for all $n \geq 1$ we have*

$$\phi(U_\tau^{n-1}) - \phi(U_\tau^n) \;=\; \frac{1}{2}\int_{t^{n-1}}^{t^n} |U_\tau{}'|^2(t)\,dt + \frac{1}{2}\int_{t^{n-1}}^{t^n} G_\tau^2(t)\,dt. \tag{5.6.23}$$

In order to prove this proposition we need the following lemmas.

**Lemma 5.6.11.** *Let $\Phi$ and $\hat{U}_\tau(t)$ be defined in (5.3.2) and (5.6.21) respectively. Then we have that*

$$\lim_{t\to t^{n-1}} \Phi(t - t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t)) \;=\; \phi(U_\tau^{n-1}).$$

*Proof*: By definition (5.6.21), for any $t > 0$, we have that

$$\hat{U}_\tau(t) \in \operatorname*{argmin}_{v\in\mathcal{S}} \Phi(t - t^{n-1}, U_\tau^{n-1}; v).$$

Therefore,

$$\Phi(t - t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t)) \;\leq\; \Phi(t - t^{n-1}, U_\tau^{n-1}; U_\tau^{n-1}) \;=\; \phi(U_\tau^{n-1}).$$

Equivalently,

$$d^2(\hat{U}_\tau(t), U_\tau^{n-1}) \;\leq\; 2\,(t - t^{n-1})\,(\phi(U_\tau^{n-1}) - \phi(\hat{U}_\tau(t))). \tag{5.6.24}$$

Since the energy is assumed to be bounded initially (see (2.1.2)), we have

$$\phi(U_\tau^{n-1}) - \phi(\hat{U}_\tau(t)) \quad \text{is bounded.}$$

Therefore, we get

$$d(\hat{U}_\tau(t), U_\tau^{n-1}) \;\to 0 \;\text{ as }\; t \to t^{n-1}. \tag{5.6.25}$$

By Definition 5.6.2 this means

$$\log(d\hat{U}_\tau(t)) \;\to\; \log(dU_\tau^{n-1}) \;\text{ in }\; L^2(0,1) \;\text{ as }\; t \to t^{n-1}.$$

By Definition 5.6.1, we can conclude that

$$\hat{U}_\tau(t) \;\overset{\sigma}{\rightharpoonup}\; U_\tau^{n-1} \;\text{ as }\; t \to t^{n-1}.$$

Therefore, by (5.6.8) we obtain

$$\begin{aligned}
\phi(U_\tau^{n-1}) &\leq \liminf_{t\to t^{n-1}} \phi(\hat{U}_\tau(t)) \\
&\leq \liminf_{t\to t^{n-1}} \left(\phi(\hat{U}_\tau(t)) + \frac{1}{2(t-t^{n-1})} d^2(\hat{U}_\tau(t), U_\tau^{n-1})\right) \\
&= \liminf_{t\to t^{n-1}} \Phi(t-t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t)) \\
&\leq \phi(U_\tau^{n-1}).
\end{aligned}$$

Hence, all the inequalities above are equalities, and we have

$$\lim_{t\to t^{n-1}} \Phi(t-t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t)) \;=\; \phi(U_\tau^{n-1})$$

as required. Q.E.D

**Lemma 5.6.12.** *Let $t^{n-1} < t < t^n$. Then, $\Phi$ is differentiable for almost every $t \in (0,\infty)$ and*

$$\frac{d}{dt}\, \Phi(t-t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t)) \;=\; -\frac{1}{2}\, G_\tau^2(t).$$

*Proof*: Let us denote

$$u_{\tau_0} = \operatorname*{argmin}_{v\in\mathcal{S}} \Phi(\tau_0, u; v) \;\text{ and }\; u_{\tau_1} = \operatorname*{argmin}_{v\in\mathcal{S}} \Phi(\tau_1, u; v).$$

Then, for any $\tau_0 < \tau_1$ we have that

$$\begin{aligned}
\Phi(\tau_0, u; u_{\tau_0}) - \Phi(\tau_1, u; u_{\tau_1}) &\leq \Phi(\tau_0, u; u_{\tau_1}) - \Phi(\tau_1, u; u_{\tau_1}) \\
&= \frac{1}{2\tau_0}\, d^2(u_{\tau_1}, u) - \frac{1}{2\tau_1}\, d^2(u_{\tau_1}, u) \\
&= \frac{\tau_1-\tau_0}{2\tau_1\tau_0}\, d^2(u_{\tau_1}, u). \qquad (5.6.26)
\end{aligned}$$

Similarly for $u_{\tau_0}$ we have

$$\begin{aligned}
\Phi(\tau_0, u; u_{\tau_0}) - \Phi(\tau_1, u; u_{\tau_1}) &\geq \Phi(\tau_0, u; u_{\tau_0}) - \Phi(\tau_1, u; u_{\tau_0}) \\
&= \frac{1}{2\tau_0}\, d^2(u_{\tau_0}, u) - \frac{1}{2\tau_1}\, d^2(u_{\tau_0}, u) \\
&= \frac{\tau_1-\tau_0}{2\tau_1\tau_0}\, d^2(u_{\tau_0}, u) \geq 0. \qquad (5.6.27)
\end{aligned}$$

Combining (5.6.26) and (5.6.27) we get

$$0 \;\leq\; \frac{d^2(u_{\tau_0}, u)}{2\tau_1\tau_0} \;\leq\; \frac{\Phi(\tau_0, u; u_{\tau_0}) - \Phi(\tau_1, u; u_{\tau_1})}{\tau_1-\tau_0} \;\leq\; \frac{d^2(u_{\tau_1}, u)}{2\tau_1\tau_0}. \qquad (5.6.28)$$

This immediately implies that $\Phi$ is differentiable for almost every $t \in (0, \infty)$. Moreover, letting $\tau_0, \tau_1 \to \tau$ in (5.6.28) we obtain

$$-\frac{d}{d\tau}\,\Phi(\tau, u; u_\tau) \;=\; \lim_{\tau_0,\tau_1\to\tau} \frac{\Phi(\tau_0, u; u_{\tau_0}) - \Phi(\tau_1, u; u_{\tau_1})}{\tau_1 - \tau_0} \;=\; \frac{d^2(u_\tau, u)}{2\tau^2}.$$

Putting

$$\tau = t - t^{n-1}, u = U_\tau^{n-1} \;\text{ and }\; u_\tau = \hat{U}_\tau(t)$$

we get the claim. Q.E.D

**Proof of Proposition 5.6.10 :**

By Lemma 5.6.12 we have that

$$-\frac{1}{2}\int_{t^{n-1}}^{t^n} {G_\tau}^2(t)\,dt \;=\; \int_{t^{n-1}}^{t^n} \frac{d}{dt}\Phi(t - t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t))\,dt.$$

By Lemma 5.6.11 and the fact that $\hat{U}_\tau(t^n) = U_\tau^n$ we can continue as

$$\begin{aligned} -\frac{1}{2}\int_{t^{n-1}}^{t^n} {G_\tau}^2(t)\,dt &= \int_{t^{n-1}}^{t^n} \frac{d}{dt}\Phi(t - t^{n-1}, U_\tau^{n-1}; \hat{U}_\tau(t))\,dt \\ &= \Phi(\tau, U_\tau^{n-1}; U_\tau^n) - \phi(U_\tau^{n-1}) \\ &= \frac{1}{2\,\tau}\,d^2(U_\tau^n, U_\tau^{n-1}) + \phi(U_\tau^n) - \phi(U_\tau^{n-1}). \end{aligned}$$

That is,

$$2\,(\phi(U_\tau^{n-1}) - \phi(U_\tau^n)) = \tau\,\frac{d^2(U_\tau^n, U_\tau^{n-1})}{\tau^2} + \int_{t^{n-1}}^{t^n} G_\tau^2(t)\,dt.$$

By Definition 5.6.20 and the fact that $\tau = t^n - t^{n-1}$, we get

$$\begin{aligned} 2\,(\phi(U_\tau^{n-1}) - \phi(U_\tau^n)) &= |U_\tau'|^2(t)\int_{t^{n-1}}^{t^n} dt \;+\; \int_{t^{n-1}}^{t^n} G_\tau^2(t)\,dt \\ &= \int_{t^{n-1}}^{t^n} |U_\tau'|^2(t)\,dt \;+\; \int_{t^{n-1}}^{t^n} G_\tau^2(t)\,dt, \end{aligned}$$

as required. Q.E.D

**Corollary 5.6.13.** *We have that*

$$\frac{1}{2}\sum_{n=1}^{N} \frac{d^2(U_\tau^n, U_\tau^{n-1})}{\tau} \;\le\; C$$

*where $C$ is independent of $\tau$ and $N$.*

*Proof*: Summing the identity in Proposition 5.6.10 from $n = 1$ to $n = N$, we get

$$\frac{1}{2}\sum_{n=1}^{N}\tau\,\frac{d^2(U_\tau^n, U_\tau^{n-1})}{\tau^2} + \frac{1}{2}\int_0^{t_N} G_\tau^2\,dt = \phi(U_\tau^0) - \phi(U_\tau^N).$$

We also know from a priori estimates on the energy that the right hand side in the above equation is bounded by a constant which is independent of $\tau$ and $N$. This gives the result. Q.E.D

#### 5.6.3.4 Compactness

In this section we prove the following compactness result by following the ideas in [3].

**Proposition 5.6.14.** *Suppose that for every $\tau \in \mathbb{N}$, the given time step associated with the partition of the interval $(0,\infty)$ as in (5.3.3), $\{U_\tau^n\}_{n\in\mathbb{N}}$ is a discrete solution to (5.3.4) and $U_\tau^0 = u_0$. Then there exist a subsequence $\{\tau_n\}_{n=1}^{\infty}$, $\tau_n \downarrow 0$, a limit curve $u$ which belongs to $AC^2_{loc}([0,\infty);\mathcal{S})$, a non-increasing function $\varphi : [0,\infty) \to \mathbb{R}$, and a function $A \in L^2_{loc}([0,\infty))$ such that*

$$\bar{U}_{\tau_n}(t) \overset{\sigma}{\rightharpoonup} u(t) \quad as \quad n \to \infty \quad \forall t \geq 0, \tag{5.6.29}$$

$$\varphi(t) := \lim_{n\to\infty} \phi(\bar{U}_{\tau_n}(t)) \geq \phi(u(t)) \quad \forall t \geq 0, \quad \phi(u(0)) = \phi(u_0), \tag{5.6.30}$$

$$|U'_{\tau_n}| \rightharpoonup A, \quad A(t) \geq |u'|(t) \quad for \quad \mathcal{L}^1\text{-a.e. } t \in (0,\infty). \tag{5.6.31}$$

*Proof*: Equation (5.6.23) and uniform boundedness of the energy shows that $|U'_\tau|$ is uniformly bounded and hence weakly compact in $L^2(0,T)$ for any positive time $T$. This immediately gives existence of a subsequence $\{\tau_n\}$ such that

$$|U'_{\tau_n}| \rightharpoonup A \quad \text{in} \quad L^2_{\text{loc}}([0,\infty))$$

for some function $A \in L^2_{\text{loc}}([0,\infty))$. Moreover, by Helly's theorem, possibly upon extracting a further subsequence, we get a limit function $\varphi$ such that

$$\phi(\bar{U}_{\tau_n}(t)) \to \varphi(t) \quad \text{for every } t \geq 0, \ \varphi(0) = \phi(u_0).$$

Corollary 5.6.13 shows that $d(U_\tau^n, U_\tau^{n-1})$ remains uniformly bounded with respect to $\tau$ for $0 \leq t \leq T$. Therefore by Proposition 5.6.3 we can immediately say that the curves $\bar{U}_\tau(t)$ take their values in a $\sigma$-compact set. For fixed $0 \leq s \leq t$, let us define

$$s(n) = \max\{r \in \mathcal{P}_{\tau_n} : r < s\}, \ \ t(n) = \min\{r \in \mathcal{P}_{\tau_n} : t < r\}$$

so that

$$s(n) \leq s \leq t \leq t(n), \ \lim_{n\to\infty} s(n) = s, \ \lim_{n\to\infty} t(n) = t.$$

By Definition 5.6.19 and the triangle inequality property for $d$ we get

$$\begin{aligned} d(\bar{U}_{\tau_n}(t), \bar{U}_{\tau_n}(s)) \leq\ & d(U_{\tau_n}^{t(n)}, U_{\tau_n}^{t(n)-1}) + d(U_{\tau_n}^{t(n)-1}, U_{\tau_n}^{t(n)-2}) + \cdots \\ & \cdots + d(U_{\tau_n}^{s(n)+1}, U_{\tau_n}^{s(n)}) \\ \leq\ & |U'_{\tau_n}|(t)\,(t(n) - t(n) + 1) + |U'_{\tau_n}|(t)\,(t(n) - 1 - t(n) + 2) + \cdots \\ & \cdots + |U'_{\tau_n}|(t)\,(s(n) + 1 - s(n)) \\ =\ & |U'_{\tau_n}|(t)(t(n) - s(n)) \ + \int_{s(n)}^{t(n)} |U'_{\tau_n}|(r)\,dr. \end{aligned}$$

Taking superior limit of both sides gives

$$\limsup_{n\to\infty}\ d(\bar{U}_{\tau_n}(t), \bar{U}_{\tau_n}(s)) \ = \ \int_s^t A(r)\,dr. \tag{5.6.32}$$

By a refined version of Arzelà-Ascoli (see e.g. [3, pg. 69]), possibly extracting one more subsequence, we can find $u(t) \in AC^2_{\text{loc}}([0,\infty); \mathcal{S})$ such that (5.6.29) holds. The property (5.6.8) of $\phi$ gives (5.6.30). Moreover, from (5.6.32) and Proposition 5.6.2 we get

$$d(u(t), u(s)) \ \leq \ \liminf_{n\to\infty} d(\bar{U}_{\tau_n}(t), \bar{U}_{\tau_n}(s)) \ \leq \ \limsup_{n\to\infty}\ d(\bar{U}_{\tau_n}(t), \bar{U}_{\tau_n}(s)) \ = \int_s^t A(r)\,dr.$$

Using Definition 5.4.2 for $|u'|(t)$ we can write

$$\begin{aligned} |u'|(t) \ &= \ \limsup_{s\to t} \frac{d(u(t), u(s))}{|t-s|} \leq \limsup_{s\to t} \frac{1}{|t-s|} \int_s^t A(r)\,dr \\ &= \ \limsup_{s\to t} \frac{\displaystyle\int_0^t A(r)\,dr - \int_0^s A(r)\,dr}{|t-s|} \\ &= \ \frac{d}{dt}\int_0^t A(r)\,dr \ = \ A(t). \end{aligned}$$

which shows that $u(t) \in AC^2_{\text{loc}}([0,\infty); \mathcal{S})$ and (5.6.31) holds. Q.E.D

#### 5.6.3.5 Convergence of the scheme

We state and prove our main result as follows:

**Theorem 5.6.15.** *Suppose that the assumption (5.6.7) holds, and (5.6.8), (5.6.9) are satisfied by $\phi$. Also assume that $d$ is given by (5.6.2) and $K$ satisfies (5.6.16). Then, $u$ as in Proposition 5.6.14 is a curve of maximal slope for $\phi$ with respect to its strong upper gradient $\|K\|_2$ and in particular $u$ satisfies the energy identity*

$$\frac{1}{2}\int_0^T |u'|^2(t)\,dt + \frac{1}{2}\int_0^T \|K(du(t))\|_2^2\,dt + \phi(u(T)) = \phi(u_0). \tag{5.6.33}$$

*Proof*: By (5.6.31) we have

$$\begin{aligned}\frac{1}{2}\int_0^t |u'|^2(s)\,ds + \frac{1}{2}\int_0^t \|K(du(s))\|_2^2\,ds + \phi(u(t)) \leq \\ \leq \frac{1}{2}\int_0^t A^2(s)\,ds + \frac{1}{2}\int_0^t \|K(du(s))\|_2^2\,ds + \phi(u(t)).\end{aligned}$$

By Remark 5.6.3 and (5.6.30) we can continue as

$$\begin{aligned}\frac{1}{2}\int_0^t |u'|^2(s)\,ds + \frac{1}{2}\int_0^t \|K(du(s))\|_2^2\,ds + \phi(u(t)) \leq \\ \leq \frac{1}{2}\int_0^t A^2(s)\,ds + \frac{1}{2}\int_0^t \liminf_{n\to\infty} G^2_{\tau_n}(t)\,ds + \liminf_{n\to\infty} \phi(\bar{U}_{\tau_n}(t)).\end{aligned}$$

By Fatou's lemma and (5.6.31) again, we get

$$\begin{aligned}\frac{1}{2}\int_0^t |u'|^2\,ds + \frac{1}{2}\int_0^t \|K(du(s))\|_2^2\,ds + \phi(u(t)) \leq \\ \leq \liminf_{n\to\infty} \left(\frac{1}{2}\int_0^t |U'_{\tau_n}|^2(s)\,ds + \frac{1}{2}\int_0^t G^2_{\tau_n}(s)\,ds + \phi(\bar{U}_{\tau_n}(t))\right).\end{aligned}$$

By Proposition 5.6.10 we conclude that

$$\begin{aligned}\frac{1}{2}\int_0^t |u'|^2\,ds + \frac{1}{2}\int_0^t \|K(du(s))\|_2^2\,ds + \phi(u(t)) \leq \\ \leq \liminf_{n\to\infty} \left(\frac{1}{2}\int_0^t |U'_{\tau_n}|^2(s)\,ds + \frac{1}{2}\int_0^t G^2_{\tau_n}(s)\,ds + \phi(\bar{U}_{\tau_n}(t))\right) \\ \leq \liminf_{n\to\infty} \phi(\bar{U}^0_{\tau_n}) \;=\; \phi(u(0)). \qquad (5.6.34)\end{aligned}$$

By Proposition 5.6.6 we know that $\|K(du)\|_2$ is a strong upper gradient for $\phi$. Hence we have

$$\phi(u(0)) \leq \phi(u(t)) + \int_0^t \|K(du)\|_2\,|u'|(s)\,ds.$$

Combining this inequality with (5.6.34) we obtain

$$|u'|(t) \;=\; \|K(du(t))\|_2 \;\text{ for }\; \mathcal{L}^1\text{-a.e. } t \in (0,\infty),$$

and hence (5.6.33) holds. It follows that the map $t \mapsto \phi(u(t))$ is locally absolutely continuous and

$$\frac{d}{dt}\phi(u(t)) = -\|K(du(t))\|_2|u'|(t) \;\text{ for }\; \mathcal{L}^1\text{-a.e. } t \in (0,\infty).$$

By definition (5.4.11) we can conclude that $u$ is a curve of maximal slope for $\phi$ with respect to its strong upper gradient $\|K(du)\|_2$ as required. Q.E.D

### 5.6.4 Curves of maximal slope vs. Quasistatic viscoelasticity

In this section we prove that the curve of maximal slope for which we have proved existence in the above parts satisfies the equation of quasistatic nonlinear viscoelasticity.

**Theorem 5.6.16.** *Assume that $u(t) \in W^{1,2}((0,\infty); W^{1,2}(0,1))$ is a curve of maximal slope for the functional $\phi$ defined by (5.3.1) with respect to its strong upper gradient. Then, $u$ is also a solution of the equation of one-dimensional quasistatic nonlinear viscoelasticity*

$$\frac{d}{dx}\Big(W'(du) + (d\dot{u}du^{-1})\, du^{-1}\Big) \;=\; 0, \tag{5.6.35}$$

*which corresponds to (5.1.1) in three dimensions with a frame-indifferent stress.*

*Proof*: First of all, by Definition 5.4.6 we obtain

$$\begin{aligned}
\frac{d}{dt}\phi(u(t)) &= \int_0^1 \frac{d}{dt}W(du)\,dx = \int_0^1 W'(du)\,d\dot{u}\,dx \\
&= \int_0^1 \Big(W'(du)du\Big)\frac{d\dot{u}}{du}\,dx \\
&\geq -\|K(du)\|_2 \left\|\frac{d\dot{u}}{du}\right\|_2 \qquad (5.6.36)\\
&\geq -\frac{1}{2}\|K(du)\|_2^2 - \frac{1}{2}\left\|\frac{d\dot{u}}{du}\right\|_2^2 \qquad (5.6.37)\\
&= \frac{d}{dt}\phi(u(t)).
\end{aligned}$$

Therefore, all inequalities in the above calculation are equalities and hence by (5.6.36) we have

$$K(du) \;\text{ is parallel to }\; \frac{d\dot{u}}{du},$$

whereas (5.6.37) implies

$$K(du) = \frac{d\dot{u}}{du}.$$

As a result we get

$$\frac{d\dot{u}}{du} \;=\; \pm\, K(du).$$

However, we know that $\phi$ is decreasing over time, so we can conclude that

$$\frac{d\dot{u}}{du} \;=\; -K(du). \tag{5.6.38}$$

Equation (5.6.38) is equivalent to

$$\frac{d\dot{u}}{(du)^2} + W'(du) \;=\; 0,$$

which immediately gives (5.6.35) as required. Let us now recall the equation of quasistatic nonlinear viscoelasticity given in (5.1.1), namely,

$$\text{Div } (DW(Du) + S(Du, Du_t)) = 0.$$

In a one-dimensional setting this equation can be written as

$$\frac{d}{dx}\big(W'(du) + S(du, d\dot{u})\big) = 0. \tag{5.6.39}$$

Also, in Section 5.3.1.2, we proved that it is possible to have

$$S(F, \dot{F}) = \frac{1}{2}\Big((\dot{F}\,F^{-1}) + (\dot{F}\,F^{-1})^T\Big)\,F^{-T}$$

in the Euler-Lagrange equations for $\Phi$, which is defined in (5.3.2). In a one-dimensional setting, $S$ can be written as

$$S(du, d\dot{u}) = \frac{1}{2}\Big(d\dot{u}du^{-1} + d\dot{u}du^{-1}\Big)du^{-1} = (d\dot{u}du^{-1})du^{-1},$$

and (5.6.39) gives (5.6.35) as required. Q.E.D

# Chapter 6

# Conclusions and further work

## 6.1 Conclusions

The following conclusions that can be drawn from this thesis.

- Well-posedness of nonlinear viscoelasticity under the assumptions allowing for phase transformations in solids is still open in three space dimensions. One major technical difficulty is that of ensuring frame-indifference.

- In the quasistatic case there exists a variational method which may lead to the existence of solutions while handling the assumption of frame-indifference.

- There exist explicit examples of potential functions which not only are convex but also give rise to frame-indifferent stresses in the corresponding Euler-Lagrange equations.

- Some assumptions on the stress, which have been believed to be compatible with frame-indifference, are actually inconsistent with it.

- In one-dimensional setting, the problem is well-posed. Moreover, there exist uniform upper and lower bounds for the solutions, which are also very interesting from a dynamical systems point of view.

- There is a strong relation between the theory of gradient flows and quasistatic nonlinear viscoelasticity in the sense that under certain assumptions any solution obtained in one of them is also a solution for the other.

- Stability of solutions in one space dimension can be achieved for some choices of stress functions without having further technical assumptions.

- A direct time-discretization method in three dimensions does not work due to lack of weak continuity of the Cauchy-Green strain tensor $C$. However, by composing the time-increments within a discrete scheme in the sense of curves of maximal slope, one might be able to prove existence of solutions in the convex setting.

## 6.2 Further work

The following are some ideas for future research.

- Showing that the abstract method introduced in Section 2.4 is valid for a specific example in a three-dimensional setting.

- Constructing other dissipation potentials $\psi$ as in Section 2.5 which not only satisfy the necessary convexity condition, but also give frame-indifference stresses in the Euler-Lagrange equations.

- Proving similar results for long-time behaviour of solutions for more general forms of the stress using the argument introduced in Section 4.7. The first attempt might be to take $\sigma$ to be a general cubic. A further step would then be to try to generalize to polynomials of arbitrary degree.

- Obtaining a bound on $Dy_t$ using the results in Section 5.2 in the context of direct time-discretization, and trying to pass to the limit with the help of this bound which is not possible naively. Dissipation potentials introduced in Section 2.5 might be useful and one can consult to some compactness results due to Lions [62, Ch. 1, Sec. 5].

- Constructing other distance functions $d$ such that similar results for Chapter 5 can be obtained.

- Reproving the convergence and existence results in Chapter 5 using convexity along geodesics, which may lead to stronger conclusions under weaker assumptions on $W$. In this case, one can also make use of the results by Ambrosio, Gigli and Savaré [3, Ch. 4].

- Considering what would be the effect of the inertia term on the arguments introduced throughout the thesis so that it might be possible to find ways of generalizations in order to deal with the full dynamical model.

# Bibliography


[1] R. Abeyaratne and J. K. Knowles, *On the dissipative response due to discontinuous strains in bars of unstable elastic material*, Int. J. Solids Structures, 24 (1988), pp. 1021–1044.

[2] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, vol. 140 of Pure and Applied Mathematics Series, Elsevier, 2nd ed., 2007.

[3] L. Ambrosio, N. Gigli, and G. Savare, *Gradient flows : in metric spaces and in the space of probability measures*, Boston : Birkhuser, 2005.

[4] G. Andrews, *On the existence of solutions to the equation* $u_{tt} = u_{xxt} + \sigma(u_x)_x$, Journal of Differential Equations, 35 (1980), pp. 200–231.

[5] G. Andrews and J. M. Ball, *Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity*, Journal of Differential Equations, 44 (1982), pp. 306–341.

[6] S. S. Antman, *Physically unacceptable viscous stress*, Z. angew. Math. Phys., 49 (1998), pp. 980–988.

[7] ——, *Nonlinear Problems of Elasticity*, Springer, 2004.

[8] S. S. Antman and T. I. Seidman, *Quasilinear hyperbolic-parabolic equations of one-dimensional viscoelasticity*, Journal of Differential Equations, 124 (1996), pp. 132–185.

[9] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, Springer, 2nd ed., 1989.

[10] ——, *Ordinary differential equations*, Universitext, Springer, 3rd ed., 2006.

[11] J. M. Ball, *Some recent developments in nonlinear elasticity and its applications to material science*, in Nonlinear Mathematics and Its Applications (Guilford, 1995), P. Aston, ed., Cambridge University Press, Cambridge, 1996, pp. 93–119.

[12] ——, *Some open problems in elasticity*, in Geometry, Mechanics and Dynamics: Volume in Honor of the 60th Birthday of J. E. Marsden, P. Newton, P. Holmes, and A. Weinstein, eds., Springer, 2002, ch. 1, pp. 3–59.

[13] ——, *Mathematical models of martensitic microstructure*, Mat. Sci. Eng. A, 378 (2004), pp. 61–69.

[14] ——, *Lecture notes for the course of 'Solid Mechanics'*, Michaelmas Term 2006. University of Oxford.

[15] ——, *Lecture notes for the course of 'Infinite-dimensional Dynamical Systems'*, Trinity Term 2008. University of Oxford.

[16] J. M. Ball, C. Chu, and R. D. James, *Hysteresis during stress-induced variant rearrangement*, Journal de Physique, 4 (1995), pp. 245–251.

[17] J. M. Ball, J. C. Currie, and P. Olver, *Null lagrangians, weak continuity, and variational problems of arbitrary order*, Journal of Functional Analysis, 41 (1981), pp. 135–174.

[18] J. M. Ball, P. J. Holmes, R. D. James, R. L. Pego, and P. J. Swart, *On the dynamics of fine structure*, J. Nonlinear Sci., 1 (1991), pp. 17–70.

[19] J. M. Ball and R. D. James, *Fine phase mixtures as minimizers of energy*, Arch. Rational Mech. Anal., 100 (1987), pp. 13–52.

[20] ——, *Proposed experimental tests of a theory of fine microstructure and the two-well problem*, Phil. Trans. R. Soc. London A, 338 (1992), pp. 389–450.

[21] J. M. Ball and C. Mora-Corral, *A variational model allowing both smooth and sharp phase boundaries in solids*, Comm. in Pure and Applied Analysis, 8 (2009), pp. 55–81.

[22] R. Bartle, *The Elements of Integration and Lebesgue Measure*, Wiley Classics Library, 1995.

[23] K. Bhattacharya, *Self-accommodation in martensite*, Arch. Rational. Mech. Anal., 120 (1992), pp. 201–244.

[24] ——, *Microstructure of Martensite -Why it forms and how it gives rise to the shape-memory effect*, Oxford University Press, 2003.

[25] D. Brandon, I. Fonseca, and P. Swart, *Oscillations in a dynamical model of phase transitions*, Proc. Roy. Soc. Edinburgh Sect. A, 131 (2001), pp. 59–81.

[26] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland : Amsterdam, 1973.

[27] K. M. Chong, *An induction principle for spectral and rearrangement inequalities*, Transactions of the American Mathematical Society, 196 (1974), pp. 371–383.

[28] P. G. Ciarlet, *Mathematical Elasticity, Volume I: Three-dimensional Elasticity*, Amsterdam : North-Holland, 1987.

[29] P. Colli and A. Visintin, *On a class of doubly nonlinear evolution equations*, Commun. in Partial Differential Equations, 15 (1990), pp. 737–756.

[30] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, Springer, 2nd ed., 1990.

[31] M. G. Crandall and A. Pazy, *Semi-groups of nonlinear contractions and dissipative sets*, J. Funct. Anal., 3 (1969), pp. 376–418.

[32] B. Dacorogna, *Direct Methods in the Calculus of Variations*, vol. 78 of Applied Mathematical Sciences, Springer-Verlag, 1st ed., 1989.

[33] ——, *Direct Methods in the Calculus of Variations*, vol. 78 of Applied Mathematical Sciences, Springer-Verlag, 2nd ed., 2008.

[34] C. M. Dafermos, *The mixed initial-boundary value problem for the equations of nonlinear one-dimensional viscoelasticity*, Journal of Differential Equations, 6 (1969), pp. 71–86.

[35] E. De Giorgi, A. Marino, and M. Tosques, *Problems of evolution in metric spaces and maximal decreasing curve*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 68 (1980), pp. 180–187.

[36] M. Degiovanni, A. Marino, and M. Tosques, *Evolution equations with lack of convexity*, Nonlinear Analysis, Theory, Methods and Applications, 9 (1985), pp. 1401–1443.

[37] S. Demoulini, *Weak solutions for a class of nonlinear systems of viscoelasticity*, Arch. Rational Mech. Anal., 155 (2000), pp. 299–334.

[38] R. E. Edwards, *Functional Analysis*, Holt, Rinehart and Winston, 1965.

[39] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, SIAM Classics in Applied Mathematics, 28, 1999.

[40] J. L. Ericksen, *Equilibrium of bars*, Journal of Elasticity, 5 (1975), pp. 191–201.

[41] A. Ferone and R. Volpicelli, *Polar factorization and pseudo-rearrangements: applications to Pólya - Szegö type inequalities*, Nonlinear Analysis, 53 (2003), pp. 929–949.

[42] W. N. Findley, *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North-Holland, 1976.

[43] G. Francfort and A. Mielke, *Existence results for a class of rate-independent material models with nonconvex elastic energies*, J. reine angew. Math., 595 (2006), pp. 55–91.

[44] G. Friesecke and G. Dolzmann, *Implicit time discretization and global existence for a quasi-linear evolution equation with nonconvex energy*, SIAM J. Math. Anal., 28 (1997), pp. 363–380.

[45] G. Friesecke, R. D. James, and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, Comm. Pure Appl. Maths., 55 (2002), pp. 1461–1506.

[46] G. Friesecke and J. B. McLeod, *Dynamics as a mechanism preventing the formation of finer and finer microstructure*, Arch. Rational Mech. Anal., 133 (1996), pp. 199–247.

[47] ——, *Dynamic stability of non-minimizing phase mixtures*, Proc. R. Soc. Lond. A, 453 (1997), pp. 2427–2436.

[48] J. M. Greenberg, R. C. MacCamy, and V. J. Mizel, *On the existence, uniqueness, and stability of solutions of the equation* $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$, J. Math. Mech., 17 (1968), pp. 707–728.

[49] M. E. Gurtin, *An introduction to continuum mechanics*, Academic Press, California, 1981.

[50] J. K. Hale and P. Massatt, *Asymptotic behaviour of gradient-like systems*, in Dynamical Systems, A. R. Bednarek and L. Cesari, eds., vol. II, Academic Press, 1982, pp. 85–101.

[51] A. S. Hall, *An Introduction to the Mechanics of Solids*, John Wiley and Sons, 1969.

[52] P. Hartman, *Ordinary Differential Equations*, SIAM, 2002.

[53] D. Henry, *Geometric theory of semilinear parabolic equations*, vol. 840 of Lecture Notes in Math., Springer, New York, 1981.

[54] G. A. Holzapfel, *Nonlinear Solid Mechanics*, John Wiley & Sons, 2000.

[55] A. D. Ioffe, *On lower semicontinuity of integral functionals I*, SIAM J. Control and Optimization, 15 (1977), pp. 521–538.

[56] R. D. James, *The propagation of phase boundaries in elastic bars*, Arch. Rational Mech. Anal., 73 (1980), pp. 125–158.

[57] W. D. Kalies, *On the asymptotic behaviour of a phase-field model for elastic phase transitions*, Journal of Dynamics and Differential Euqations, 9 (1997), pp. 289–306.

[58] R. V. Kohn and S. Müller, *Branching of twins near an austenite/twinned martensite interface*, Phil. Mag. A, 66 (1992), pp. 697–715.

[59] Y. Komura, *Nonlinear semi-groups in Hilbert spaces*, J. Math. Soc. Japan, 19 (1967), pp. 493–507.

[60] M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, Elmsford, N.Y., 1964.

[61] K. Kuttler and D. Hicks, *Initial-boundary value problems for the equation* $u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f$, Quarterly of Applied Mathematics, 66 (1988), pp. 393–407.

[62] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod: Gauthier-Villars, Paris, 2002.

[63] R. C. MacCamy and V. J. Mizel, *Existence and nonexistence in the large of solutions of quasilinear wave equations*, Arch. Rational Mech. Anal., 25 (1967), pp. 299–320.

[64] J. E. Marsden and T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Dover Publications, 1993.

[65] J. A. C. Martins, M. D. P. Monteiro Marques, A. Petrov, N. V. Rebrova, V. A. Sobolev, and I. Coelho, *(In)stability of quasi-static paths of some finite dimensional smooth or elastic-plastic systems*, Journal of Physics, 22 (2005), pp. 124–138.

[66] A. Mielke and M. Ortiz, *A class of minimum principles for characterizing the trajectories and the relaxation of dissipative systems*, ESAIM Control Optim. Calculus of Variations, 14 (2008), pp. 494–516.

[67] A. Mielke and U. Stefanelli, *Weighted energy-dissipation functionals for gradient flows*, ESAIM : COCV, (2009). (publication ahead of print journal).

[68] C. B. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math., 2 (1952), pp. 25–53.

[69] B. K. Muite, *Analysis, Modelling and Simulation of Shape Memory Alloys*, PhD thesis, University of Oxford, 2009.

[70] W. Noll, *A mathematical theory of the mechanical behavior of continuous media*, Arch. Rational Mech. Anal., 2 (1958), pp. 197–226.

[71] A. Novick-Cohen and R. L. Pego, *Stable patterns in a viscous diffusion equation*, Transactions of the American Mathematical Society, 324 (1991), pp. 331–351.

[72] C. Ortner, *Two variational techniques for the approximation of curves of maximal slope*, Tech. Report NA05/10, University of Oxford, Computing Laboratory, 2005.

[73] P. Pedregal, *Variational methods in nonlinear elasticity*, SIAM, 2000.

[74] R. L. Pego, *Phase transitions in one-dimensional nonlinear viscoelasticity: Admissibility and stability*, Arch. Rational Mech. Anal., 97 (1987), pp. 353–394.

[75] ——, *Stabilization in a gradient system with a conservation law*, Proceedings of the American Mathematical Society, 114 (1992), pp. 1017–1024.

[76] M. A. Peshkin and A. Sanderson, *A variational principle for quasistatic mechanics*, Tech. Report CMU-RI-TR-86-16, Carnegie Mellon University, 1986.

[77] M. Potier-Ferry, *The linearization principle for stability in quasilinear parabolic equations I*, Arch. Rational Mech. Anal., 77 (1981), pp. 301–320.

[78] ——, *On the mathematical foundations of elastic stability theory I*, Arch. Rational Mech. Anal., 78 (1982), pp. 55–72.

[79] R. Racke and S. Zheng, *Global existence and asymptotic behavior in nonlinear thermoviscolesticity*, J. Differential Equations, 134 (1997), pp. 46–67.

[80] R. T. Rockafellar, *Convexity properties of nonlinear maximal monotone operators*, Bulletin of the American Mathematical Society, 75 (1969), pp. 74–77.

[81] R. Rossi and G. Savaré, *Gradient flows of nonconvex functionals in Hilbert spaces and applications*, ESAIM: COCV, 12 (2006), pp. 564–614.

[82] R. Rossi, A. Segatti, and U. Stefanelli, *Attractors for gradient flows of nonconvex functionals and applications*, Arch. Rational Mech. Anal., 187 (2008), pp. 91–135.

[83] P. Rybka, *Dynamical modelling of phase transitions by means of viscoelasticity in many dimensions*, Proceedings of the Royal Society of Edinburgh, 121A (1992), pp. 101–138.

[84] ——, *A priori estimates for the gradient of the solution to the system of viscoelasticity in several dimensions*, Topological Methods in Nonlinear Analysis, 3 (1994), pp. 235–256.

[85] J. V. Ryff, *Measure preserving transformations and rearrangements*, Journal of Mathematical Analysis and Applications, 31 (1970), pp. 449–458.

[86] Y. Şengül, *Well-posedness of dynamics of microstructure in solids.* Transfer Thesis, Mathematical Institute, University of Oxford, 2008.

[87] M. Shillor, *Quasistatic problems in contact mechanics*, Int. J. Appl. Math. Comput. Sci., 11 (2001), pp. 189–204.

[88] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, vol. 49 of Mathematical Surveys and Monographs, American Mathematical Society, 1997.

[89] M. Šilhavý, *The Mechanics and Thermodynamics of Continuous Media*, Springer, 1997.

[90] A. J. M. Spencer, *Mechanics, mathematics and materials*. Inaugural Lecture, University of Nottingham, 1966.

[91] ——, *Continuum Mechanics*, Longman, 1980.

[92] J. P. Swart and J. P. Holmes, *Energy minimization and the formation of microstructure in dynamic anti-plane shear*, Arch. Rational Mech. Anal., (1992), pp. 37–85.

[93] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer, 1997.

[94] C. Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics*, Springer-Verlag, 2004.

[95] B. Tvedt, *Global existence of solutions and propagation of regularity for quasilinear viscoelastic systems of differential type*, PhD thesis, University of California at Berkeley, 1997.

[96] ——, *Quasilinear equations for viscoelasticity of strain-rate type*, Arch. Rational Mech. Anal., 189 (2008), pp. 237–281.

[97] I. I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 1995.

[98] C. H. Wilcox, *Initial boundary-value problems for linear hyperbolic partial differential equations of the second order*, Arch. Rational Mech. Anal., 10 (1962), pp. 361–400.

[99] Y. Yamada, *Some remarks on the equation* $y_{tt} - \sigma(y_x)_x - y_{xtx} = f$, Osaka J. Math., 17 (1980), pp. 303–323.

[100] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I : Fixed-point Theorems*, Springer, 1985.

[101] J. Zimmer, *Global existence for a nonlinear system in thermoviscoelasticity with nonconvex energy*, J. Math. Anal. Appl., 292 (2002), pp. 589–604.