Commutative K-theory

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Abstract

The bar construction BG of a topological group G has a subcomplex $B_{\mathrm{com}}G \subset BG$ assembled from spaces of commuting elements in G. If $G = U, O$ (the infinite unitary / orthogonal groups) then $B_{\mathrm{com}}U$ and $B_{\mathrm{com}}O$ are E_∞-ring spaces. The corresponding cohomology theory is called commutative K-theory.

In this work we study properties of the spaces $B_{\mathrm{com}}G$ and of infinite loop spaces built from them, with an emphasis on the cases $G = U, O$. The content of this thesis is organised as follows:

In Chapter 1 we consider a family of self-maps of $B_{\mathrm{com}}G$ and apply these to study the question when the inclusion map $B_{\mathrm{com}}G \subset BG$ admits a section up to homotopy.

In Chapter 2 we show that $B_{\mathrm{com}}U$ is a model for the E_∞-ring space underlying the ku-group ring of $\mathbb{C}P^\infty$. Thus we provide a complete description of complex commutative K-theory. We also study the space $B_{\mathrm{com}}O$. Our results include a computation of the torsionfree part of the homotopy groups of $B_{\mathrm{com}}O$ and a long exact sequence relating real commutative K-theory to singular mod-2 homology.

Chapter 3 is self-contained. We prove a result about the acyclicity of the "comparison map" $M_\infty \to \Omega BM$ in the group-completion theorem and apply this to compare the infinite loop space associated to a commutative $\mathbb{I}$-monoid with the Quillen plus-construction.

Chapter 4 is concerned with a previously known filtration of $\Omega_0^\infty S^\infty$ by certain infinite loop spaces $\{\mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_-)\}_{q\geq 2}$. For each term in this filtration we construct another filtration on the spectrum level, whose subquotients we describe. Our set-up is more general, but the space $\mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_-)$ will serve as our main example.

Appendix A is an excerpt from the author's Oxford transfer thesis. There we gave a construction of an infinite loop space associated to certain subspaces $B(q, \Gamma_{g,1}) \subset B\Gamma_{g,1}$, where $\Gamma_{g,1}$ is the mapping class group of a genus g surface with one boundary component.

Statement of originality

I declare that the work in this thesis, is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known. This thesis has not been submitted for a degree at another university.

Parts of Chapter 2 have been made available on the `arXiv`. The content of Chapter 3 has been submitted for publication.

Simon Philipp Gritschacher
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Introduction

The main topic of this dissertation is commutative K-theory which is a form of classical topological K-theory. The definition is due to Adem and Gómez [5] and it is based on a variation of the classical infinite loop spaces BU and BO which represent complex and real topological K-theory. The relevant construction has been introduced by Adem, Cohen and Torres-Giese [4] and works more generally for any group G.

In order to explain their idea, recall that a model for the classifying space BG is the geometric realisation of the simplicial bar complex B_*G. A k-simplex in B_*G is precisely a k-tuple of elements of G. A natural subcomplex $B_{\mathrm{com}}G \subset BG$ is obtained by restricting to those simplices which are tuples of *pairwise commuting* elements in G. This definition is *ad hoc*, but it was motivated by the study of spaces of homomorphisms from a free abelian group into G, which had been started earlier by Adem and Cohen [2]. Indeed, the space of k-simplices of $B_{\mathrm{com}}G$ is precisely the space $\mathrm{Hom}(\mathbb{Z}^k, G)$.

In [5] it was proved that the space $B_{\mathrm{com}}G$ has the following property: The classifying map of a principal G-bundle factors through it, up to homotopy, precisely if the bundle can be described by a cocycle which at every point of the base space takes values in a commuting subset of G. Adem and Gómez call such a bundle *transitionally commutative.*

The representing spaces for the complex and real variants of commutative K-theory are obtained by choosing for G the infinite unitary and orthogonal groups U respectively O. It was shown by Adem, Gómez, Lind and Tillmann [6] that $B_{\mathrm{com}}U$ and $B_{\mathrm{com}}O$ are E_∞-ring spaces in a way compatible with the E_∞-ring structures on BU and BO. In particular, they represent multiplicative generalised cohomology theories and the inclusion maps $B_{\mathrm{com}}U \subset BU$ and $B_{\mathrm{com}}O \subset BO$ describe natural transformations from the new K-theories to the classical K-theories.

Topological properties of $B_{\mathrm{com}}U$ were first described in [5]. Its rational cohomology ring was identified explicitly as a polynomial algebra. Then in [6] it was proved that the infinite loop space BU is a retract up to homotopy of the infinite loop space $B_{\mathrm{com}}U$, whence it is a direct factor of the latter. A different way of saying this is that commutative complex K-theory K^*_{com} contains ordinary K-theory ku^* as a direct summand, i.e.

$$K^*_{\text{com}} \cong ku^* \oplus \, ? \, .$$

In particular, every complex vector bundle is *stably* transitionally commutative, and the cohomology theory "?" captures the possibly different ways a stable vector bundle can become a transitionally commutative one. It was shown that the analogous results hold for $B_{\text{com}}O$. The homotopy types of $B_{\text{com}}U$ and $B_{\text{com}}O$, however, remained undetermined.

As a central result in this thesis we obtain a complete description of the complex variant of commutative K-theory. For this we determine the homotopy type of a commutative symmetric ring spectrum whose zero space is the infinite loop space $B_{\text{com}}U$. The main input we shall use is a theorem of Lawson in stable representation theory [30]. In fact, after everything has been set up, we will see that the structure of commutative K-theory is a rather straightforward consequence of his work. We also offer a partial description of the classifying space $B_{\text{com}}O$ for the real variant of commutative K-theory. In addition to the work on commutative K-theory we prove several other results relating to [5] and [6] which we outline now.

This thesis has four chapters and a short appendix.

In Chapter 1 we consider $B_{\text{com}}G$ for a compact Lie group G. We first summarize from [5] what will be relevant to our work. We recall the construction of $B_{\text{com}}G$ as a simplicial space, its role as a classifying space when G is a Lie group, and the description of its rational cohomology ring. We then consider self-maps of $B_{\text{com}}G$ induced by the power maps in the group G. They give rise to cohomology operations in commutative K-theory. We use these self-maps to study the natural question if the space BG is a retract up to homotopy of $B_{\text{com}}G$. By the results in [4] this is always true after looping once, and by [6] it holds for U and O. In Theorem 1.2.2 we show that for compact G with $\text{Hom}(\mathbb{Z}^k, G)$ path-connected for all k, BG is a retract up to homotopy of $B_{\text{com}}G$ if and only if G is abelian.

In Chapter 2 we consider K-theory. The chapter consists of three sections. Section 2.1 is mainly recollections from [6], Section 2.2 deals with complex commutative K-theory, and Section 2.3 contains results about the real variant.

In the complex case, we begin by reviewing known results in deformation K-theory. Based on this we define a commutative symmetric ring spectrum E which is a model for complex commutative K-theory. In Theorem 2.2.11 we identify E as the ku-group ring spectrum of $\mathbb{C}P^\infty$. This final form of the statement was obtained after Graeme Segal had made a comment in our talk in the Oxford Topology seminar. We are grateful for his remarks and the insight we gained from them. Previously, we had computed the homotopy ring of E using the cohomology operations introduced in Chapter 1. In retrospect this is a computation of the K-Pontrjagin ring of $\mathbb{C}P^\infty$. Nevertheless, we shall include a summary of our computation, as it may provide a useful perspective in future work. As a corollary, we obtain the homotopy ring of $B_{\mathrm{com}}U$, which is stated as Theorem 2.2.13. As another consequence we obtain a splitting of infinite loop spaces,

$$B_{\mathrm{com}}U \simeq BU \times BU\langle 4\rangle \times BU\langle 6\rangle \times \dots$$

where $BU\langle 2n\rangle$ is the $2n-1$ connected cover of BU. This generalises the aforementioned splitting of Adem, Gómez, Lind and Tillmann. A similar result holds for $B_{\mathrm{com}}SU$ and these results are stated as Corollary 2.2.20. Finally, we compute the rational Hurewicz homomorphism for the space $B_{\mathrm{com}}U$. This has the purpose of relating the new basis of $\pi_*(B_{\mathrm{com}}U)$ given by K-homology theory to the old basis of $H^*(B_{\mathrm{com}}U,\mathbb{Q})$ described in terms of multisymmetric functions [5]. Our result is Theorem 2.2.25.

In the real case, we begin by describing the complexification and realification maps relating complex and real commutative K-theory. In Theorem 2.3.2 they are used to compute the torsionfree part of the homotopy groups $\pi_*(B_{\mathrm{com}}O)$. As a consequence, we obtain in Corollary 2.3.4 the rational cohomology rings of $B_{\mathrm{com}}O$ and $B_{\mathrm{com}}SO$. After that we establish in Theorem 2.3.7 a homotopy fibre sequence relating $B_{\mathrm{com}}O$ to the $\mathbb{Z}_2$-group ring of $\mathbb{R}P^\infty$. This identifies a part of the 2-torsion in $\pi_*(B_{\mathrm{com}}O)$. We use this to compute the real commutative K-theory of S^n for $n \leq 3$. We end Chapter 2 with an outlook on how one may continue the study of $B_{\mathrm{com}}O$ and determine its homotopy type.

Chapter 3 is a self-contained chapter about the group-completion theorem [43]. Suppose that M is a topological monoid satisfying $\pi_0 M \cong \mathbb{N}$ to which the group-completion theorem applies. We prove in Theorem 3.1.3 that if left- and right-stabilisation commute on $H_1(M)$, then the "McDuff-Segal comparison map" $M_\infty \to \Omega_0 BM$ is acyclic. Our considerations in this chapter were motivated by the work in [6, §3], where the infinite loop space associated to a commutative $\mathbb{I}$-monoid (Def. 3.3.1) is identified with a Quillen plus-construction under certain assumptions. As an application of our result, we improve their theorem in certain situations. This is Theorem 3.3.2.

Chapter 4 is largely self-contained and is concerned with a filtration of the infinite loop space $Q_0S^0 := \Omega^\infty_0 S^\infty$ constructed in [6]. For any group G, there is a natural sequence of spaces $B(q,G)$ sitting between $B_{\mathrm{com}}G$ and BG and indexed by the lower central series of the free groups [4]. Choosing for G the symmetric groups Σ_n and applying the formalism of commutative $\mathbb{I}$-monoids Adem-Gómez-Lind-Tillmann construct a sequence of infinite loop spaces

$$\mathrm{hocolim}_{\mathbb{I}} B_{\mathrm{com}}\Sigma_{-} \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B(q,\Sigma_{-}) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B\Sigma_{-} \simeq Q_0 S^0 \,. \qquad (1)$$

We shall not attempt to motivate the study of these spaces, but the purpose of this chapter is to propose a way of analysing them. The construction we use relies on an idea of Lesh [31], but can also be seen as an analogue of the filtration that Lawson considers on the deformation K-theory of a group [29]. In any case, the idea is that of a rank filtration, which leads to a filtration of $\mathrm{hocolim}_{\mathbb{I}} B(q,\Sigma_{-})$ by infinite loop spaces. Our main result, Theorem 4.3.4, explicitly identifies the difference between two successive stages in the filtration.

In Appendix A we construct the analogue of the sequence (1) where the role of the symmetric groups is replaced by the mapping class groups of orientable surfaces, and QS^0 is replaced by $B\Gamma^{+}_{\infty,1}$. This material is taken from our Transfer thesis [19].

Contents

Chapter 1

A classifying space

In this first chapter we discuss general properties of the classifying space for commutativity $B_{\mathrm{com}}G$ and its reduced variant $B_{\mathrm{com}}G_{\mathbb{1}}$. Section 1.1 is mainly a recollection of definitions and results from [5]. This includes the definition of $B_{\mathrm{com}}G$ in Section 1.1.1, the notion of transitionally commutative principal bundles in Section 1.1.2, and a description of the rational cohomology of $B_{\mathrm{com}}G_{\mathbb{1}}$ for Lie groups in Section 1.1.3. A contribution we offer in this section is a description of the canonical map $B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ on rational cohomology. In Section 1.2 we define power operations on $B_{\mathrm{com}}G$, which we used in our original computation of the coefficient ring for complex commutative K-theory. In Section 1.2.1 we consider Adams operations to study the question when the canonical map $B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ admits a section up to homotopy. The main result is proved in Section 1.2.2.

1.1 Definition and elementary properties

1.1.1 The bar construction and commuting elements

Let G be a topological group. Recall that the bar construction for G is a simplicial model for the classifying space BG. The bar construction is the simplicial space $k \mapsto B_kG$, where $B_kG = G^k$ is the k-fold Cartesian product of G, with face maps $d_i : B_kG \to B_{k-1}G$ given by

$$d_i(g_1, \dots, g_k) = \begin{cases} (g_2, \dots, g_k) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_k) & 0 < i < k \\ (g_1, \dots, g_{k-1}) & i = k \end{cases}$$

and degeneracy maps $s_i : B_kG \to B_{k+1}G$ given by

$$s_i(g_1, \dots, g_k) = \begin{cases} (e, g_1, \dots, g_k) & i = 0 \\ (g_1, \dots, g_i, e, g_{i+1}, \dots, g_k) & 0 < i \leq k, \end{cases}$$

where $e \in G$ is the neutral element.

In [2] Adem and Cohen initiated the systematic study of the natural subspace $C_k(G) \subseteq B_kG$ consisting of k-tuples of pairwise commuting elements in G. Explicitly,

$$C_k(G) = \{(g_1, \dots, g_k) \in B_kG \,|\, g_ig_j = g_jg_i \text{ for all } 1 \leq i, j \leq k\} ,$$

topologised as a subset of G^k. The spaces of commuting elements $C_k(G)$ have the following alternative description in terms of spaces of homomorphisms. Let $\mathbb{Z}^k$ be the free abelian group of rank k and let $e_i \in \mathbb{Z}^k$ denote the i-th standard generator. If $f : \mathbb{Z}^k \to G$ is a homomorphism then the assignment $f \mapsto (f(e_1), \dots, f(e_k))$ gives a natural identification

$$\mathrm{Hom}(\mathbb{Z}^k, G) \cong C_k(G)$$

and this induces a topology on $\mathrm{Hom}(\mathbb{Z}^k, G)$. We shall use these two descriptions interchangeably.

In later work by Adem, Cohen and Torres-Giese [4] it was observed that the simplicial operators in the bar construction $k \mapsto B_kG$ respect the commutativity condition, so that the assignment $k \mapsto C_k(G) \cong \mathrm{Hom}(\mathbb{Z}^k, G)$ defines a simplicial space in its own right.

Definition 1.1.1 (Adem-Cohen-Torres-Giese, [4])**.** The space $B_{\mathrm{com}}G$ is defined to be geometric realisation

$$B_{\mathrm{com}}G := |k \mapsto \mathrm{Hom}(\mathbb{Z}^k, G)| .$$

The construction $G \mapsto B_{\mathrm{com}}G$ is natural with respect to homomorphisms of groups and there is a natural inclusion map $i : B_{\mathrm{com}}G \to BG$.

In general the spaces of homomorphisms $\mathrm{Hom}(\mathbb{Z}^k, G)$ are not path-connected. There is a variant of Definition 1.1.1 for which one considers only the distinguished path-component of $\mathrm{Hom}(\mathbb{Z}^k, G)$ containing the trivial homomorphism, see [4, p. 104].

Definition 1.1.2. Let $\mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}$ denote the path-component of $\mathrm{Hom}(\mathbb{Z}^k, G)$ which contains the trivial homomorphism. Then a simplicial space is defined by $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}} \subseteq \mathrm{Hom}(\mathbb{Z}^k, G)$ and its geometric realisation is denoted by

$$B_{\mathrm{com}}G_{\mathbb{1}} := |k \mapsto \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}| .$$

Remark 1.1.3. For the case of a compact Lie group G it is stated in [5, p. 493] that $\mathrm{Hom}(\mathbb{Z}^k, G)$ is path-connected if and only if the maximal abelian subgroups of G are precisely the maximal tori. In particular, this holds for $U(n)$, $SU(n)$ and $Sp(n)$, and thus $B_{\mathrm{com}}G = B_{\mathrm{com}}G_{\mathbb{1}}$ for these groups, see [2, 5]. On the other hand, $G = SO(3)$ is an example where $\mathrm{Hom}(\mathbb{Z}^k, SO(3))$ is not path-connected if $k \geq 2$, see [52].

1.1.2 Bundles with commuting cocycles

From now on we let G be a Lie group. The space $B_{\text{com}}G$ is of interest because of its relationship with bundle theory.

Let us recall that BG is a classifying space for principal G-bundles. There is a principal G-bundle $\pi : EG \to BG$, where EG is a contractible space and a free G-space, with the following universal property. If X is a CW complex and $q : P \to X$ is a principal G-bundle, then there exists a map $f : X \to BG$ which is uniquely determined up to homotopy and an isomorphism of principal G-bundles $P \xrightarrow{\sim} f^*(EG)$ over X. In order to establish the role of a classifying space for $B_{\text{com}}G$ one makes the following definition.

Definition 1.1.4 (Adem-Gómez, [5])**.** Let X be a CW complex. We say that a principal G-bundle $q : P \to X$ is *transitionally commutative* if there exists an open trivialising cover $\{U_i \,|\, i \in I\}$ of X and a representing cocylce $\{g_{ij} : U_i \cap U_j \to G \,|\, i, j \in I\}$ for $q : P \to X$ so that for all $x \in X$ the set $\{g_{ij}(x) \,|\, i, j \in I : \ g_{ij} \text{ is defined at } x\} \subset G$ is a subset of commuting elements in G.

With this definition in place one has the following theorem.

Theorem 1.1.5 (Adem-Gómez, [5, Thm. 2.2])**.** *Suppose that G is a Lie group and let $f : X \to BG$ be the classifying map of a principal G-bundle $q : P \to X$ over a finite CW complex X. Then $q : P \to X$ is transitionally commutative if and only if, up to homotopy, f factors through the inclusion map $i : B_{\text{com}}G \to BG$.*

Note that if a classifying map $f : X \to BG$ lifts up to homotopy under i to a map $f' : X \to B_{\text{com}}G$, then the homotopy class of f' need not be unique. In other words, the map $i_* : [X, B_{\text{com}}G] \to [X, BG]$ of sets of unbased homotopy classes of maps is neither surjective nor injective in general. We shall therefore include the choice of a lift f' into the definition of a transitionally commutative bundle.

Definition 1.1.6. Let X be a CW complex and $q : P \to X$ a principal G-bundle with classifying map $f : X \to BG$. A *transitionally commutative structure* on $q : P \to X$ is the homotopy class of a lift to $B_{\text{com}}G$ of the classifying map $f : X \to BG$.

Example 1.1.7 ([5]). Suppose that X is a based CW-complex. It was shown in [4, 5] that if G is connected every principal G-bundle over the suspension ΣX admits a transitionally commutative structure. In fact, there is a functorial choice of such a structure. This follows because by [4, Thm. 6.3] the looped map $\Omega i : \Omega B_{\text{com}}G \to \Omega BG$ admits a section up to homotopy, that is a map $s : \Omega BG \to \Omega B_{\mathrm{com}}G$ satisfying $(\Omega i)\circ s \simeq id$. The section is natural in G, so that

$$[\Sigma X, BG]_0 \cong [X, \Omega BG]_0 \xrightarrow{s_*} [X, \Omega B_{\mathrm{com}}G]_0 \cong [\Sigma X, B_{\mathrm{com}}G]_0$$

is natural in X and G, where $[-,-]_0$ means based homotopy classes of based maps (note that if G is connected, then BG as well as $B_{\mathrm{com}}G$ are simply connected spaces and therefore the based and unbased homotopy sets coincide). In general, this transitionally commutative structure is not unique. It follows from [5, Ex. 6.4] that for $G = SU(2)$ and $\Sigma X = S^4$ the transitionally commutative structures form the group

$$[\Sigma X, B_{\mathrm{com}}G]_0 = \pi_4(B_{\mathrm{com}}SU(2)) \cong \mathbb{Z}^2,$$

while $\pi_4(BSU(2)) \cong \mathbb{Z}$. The group $\pi_4(BSU(2))$ is generated by the Hopf bundle $S^3 \to S^7 \to S^4$, and so we see that the Hopf bundle admits distinct transitionally commutative structures. We will describe these in Section 2.2.3.6.

1.1.3 The conjugation map and cohomology

In [4] it was shown that for a compact connected Lie group G the rational cohomology ring of $B_{\mathrm{com}}G_{\mathbb{1}}$ has a nice description in terms of Weyl group invariants. Let $T \subset G$ be a maximal torus and let $W = N_G(T)/T$ be its Weyl group. The description of the cohomology of $B_{\mathrm{com}}G_{\mathbb{1}}$ is based on a formula of Baird [8] for the cohomology of $\mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}$. This formula is a generalisation of the classical result in Lie group theory, that the conjugation map $G/T \times T \to G$ sending $(gT, t) \mapsto gtg^{-1}$ induces an isomorphism between $H^*(G, \mathbb{Q})$ and the Weyl group invariant part of $H^*(G/T, \mathbb{Q}) \otimes H^*(T, \mathbb{Q})$ (see e.g. [51]). Here $w \in W$ acts on the flag manifold G/T as $w.gT = gw^{-1}T$, on the torus T as $w.t = wtw^{-1}$, and diagonally on the tensor product.

For every $k \geq 1$, the conjugation map $G/T \times T \to G$ can be extended to a map $\varphi_k : G/T \times T^k \to \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}$ sending $(gT, t_1 \dots t_k) \mapsto (gt_1g^{-1}, \dots, gt_kg^{-1})$. If G/T is regarded as a constant simplicial space, it is easy to see that the collection of maps $\{\varphi_k\}_{k\geq 0}$ defines a simplicial map which realises to a map

$$\varphi : G/T \times BT \to B_{\mathrm{com}}G_{\mathbb{1}}\,. \tag{1.1}$$

The Weyl group acts on G/T as usual and on BT simplicially by conjugation. The map φ factors through the orbit space $G/T \times_W BT$.

Theorem 1.1.8 (Adem-Cohen-Torres-Giese, [4, Thm. 6.1]). *The conjugation map $\varphi : G/T \times BT \to B_{\mathrm{com}}G_{\mathbb{1}}$ induces an isomorphism*

$$H^*(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Q}) \cong \left(H^*(G/T, \mathbb{Q}) \otimes H^*(BT, \mathbb{Q})\right)^W.$$

We next describe the map $i : B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ in rational cohomology. This corrects a mistake in the second part of [4, Thm. 6.1]. Recall the standard fibre bundle $G/T \xrightarrow{l} BT \xrightarrow{j} BG$, where j is the map induced by the inclusion $T \subset G$. Note that since T is abelian there is a multiplication map $\mu : BT \times BT \to BT$.

Lemma 1.1.9. *The composite map $i \circ \varphi : G/T \times BT \to BG$ is homotopic to the following composition*

$$G/T \times BT \xrightarrow{l \times id} BT \times BT \xrightarrow{\mu} BT \xrightarrow{j} BG\,.$$

Proof. It suffices to show that the two principal G-bundles over $G/T \times BT$ classified by $i \circ \varphi$ respectively $j \circ \mu \circ (l \times id)$ are isomorphic. Let $\tau : T \times G \to G$ be the action given by $\tau(t, g) = gt^{-1}$, so that we can identify $ET \times_\tau G \simeq G/T$ via $[x, g] \mapsto gT$. One can explicitly verify that the following diagram is a pullback

$$\begin{array}{ccc} ET^2 \times_{\rho_1} G^2 & \xrightarrow{\quad p \quad} & EG \\ \downarrow q & & \downarrow \pi \\ (ET \times_\tau G) \times BT & \xrightarrow{j \circ \mu \circ (l \times id)} & BG \end{array}$$

where $\rho_1 : T^2 \times G^2 \to G^2$ is the action given by $\rho_1(t, t', g, g') = (gt^{-1}, tt'g')$, p maps $[x, g, g'] \mapsto [E(j \circ \mu)(x), g'] \in EG \times_G G \cong EG$, and q sends $[x, g, g'] \mapsto ([\pi_1(x), g], [\pi_2(x)])$, where $\pi_i : ET^2 \to ET$, $i = 1, 2$, are the two projections. On the other hand, pulling back the universal G-bundle along the map $i \circ \varphi$ produces the following diagram

$$\begin{array}{ccc} ET^2 \times_{\rho_2} G^2 & \xrightarrow{\quad p' \quad} & EG \\ \downarrow q & & \downarrow \pi \\ (ET \times_\tau G) \times BT & \xrightarrow{i \circ \varphi} & BG \end{array}$$

in which $\rho_2 : T^2 \times G^2 \to G^2$ is the action $\rho_2(t, t', g, g') = (gt^{-1}, gt'g^{-1}g')$. The map p' sends $[x, g, g'] \mapsto [E(c_g \circ \pi_2)(x), g']$, where $c_g : T \to G$ is conjugation by $g \in G$. It remains to compare the resulting principal G-bundles over $(ET \times_\tau G) \times BT$. One can check that the isomorphism $G^2 \to G^2$ given by $(g, g') \mapsto (g, g^{-1}g')$ is T^2-equivariant for the actions ρ_1 and ρ_2 and induces an isomorphism of the two principal bundles. □

It is well known that the map $l^* : H^*(BT, \mathbb{Q}) \to H^*(G/T, \mathbb{Q})$ is surjective with kernel the ideal $\mathscr{I}$ generated by the image under j^* of the positive degree elements $H^+(BG, \mathbb{Q}) \subset H^*(BG, \mathbb{Q})$ (see [51]). Therefore, as observed in [5], the isomorphism of Theorem 1.1.8 yields $H^*(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Q}) \cong (H^*(BT, \mathbb{Q})/\mathscr{I} \otimes H^*(BT, \mathbb{Q}))^W$. The inclusion $j : BT \to BG$ factors through $B_{\mathrm{com}}G_{\mathbb{1}}$ and induces an embedding of $H^*(BG, \mathbb{Q})$ onto the Weyl group invariants in $H^*(BT, \mathbb{Q})$. Hence, the map

$$i^* : H^*(BG, \mathbb{Q}) \to H^*(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Q})$$

is injective. The algebra $H^*(BT,\mathbb{Q})$ is a Hopf algebra with comultiplication $\Delta = \mu^*$, where $\mu : BT \times BT \to BT$ comes from multiplication in T.

Corollary 1.1.10. *The image under i^* of $H^*(BG,\mathbb{Q})$ in $H^*(B_{\mathrm{com}}G_{\mathbb{1}},\mathbb{Q})$ agrees with the image of the composition*

$$H^*(BT,\mathbb{Q})^W \xrightarrow{\Delta} (H^*(BT,\mathbb{Q}) \otimes H^*(BT,\mathbb{Q}))^W \xrightarrow{\text{proj.}} (H^*(BT,\mathbb{Q})/\mathscr{I} \otimes H^*(BT,\mathbb{Q}))^W .$$

Proof. This is immediate from Lemma 1.1.9. Note that the image of the composite map is stable under the action of W, because the map $i \circ \varphi$ factors through the W-orbits. □

1.2 Power operations

The classifying space $B_{\mathrm{com}}G$ admits a family of interesting self-maps induced by the k-th power map in the group G. We shall call these maps *power operations* because in those cases where $B_{\mathrm{com}}G$ represents a cohomology theory (such as $G = U$ or $G = O$) they give rise to cohomology operations.

1.2.1 Definition and an application

Let $-^k : G \to G$ denote the map sending $g \mapsto g^k$. In general this does not give a welldefined map on BG, but it always induces a map

$$\phi^k := B_{\mathrm{com}}(-^k) : B_{\mathrm{com}}G \to B_{\mathrm{com}}G .$$

In terms of transitionally commutative bundles this represents the operation which raises a cocycle to its k-th power.

If $T \subset G$ is a maximal torus, then ϕ^k extends the k-th power map in BT. In fact, there is a commutative diagram

$$\begin{array}{ccc} G/T \times BT & \xrightarrow{\varphi} & B_{\mathrm{com}}G_{\mathbb{1}} \\ \downarrow{\scriptstyle id\times -^k} & & \downarrow{\scriptstyle \phi^k} \\ G/T \times BT & \xrightarrow{\varphi} & B_{\mathrm{com}}G_{\mathbb{1}} \end{array}$$

where we abusively write $-^k$ for the map induced on BT by the k-th power map in T. It is an easy standard computation that $-^k$ acts as multiplication by k^n on $H^{2n}(BT,\mathbb{Q})$. This shows that the cohomology isomorphism of Theorem 1.1.8

$$H^{2n}(B_{\mathrm{com}}G_{\mathbb{1}},\mathbb{Q}) \cong \bigoplus_{j=0}^{n} (H^{2n-2j}(G/T,\mathbb{Q}) \otimes H^{2j}(BT,\mathbb{Q}))^W \qquad (1.2)$$

is a decomposition of $H^{2n}(B_{\mathrm{com}}G_{\mathbb{1}},\mathbb{Q})$ into simultaneous eigenspaces for the ϕ^k, $k \in \mathbb{Z}$. The j-th summand in (1.2) is an eigenspace for ϕ^k with eigenvalue k^j.

Remark 1.2.1. Recall that an Adams operation on BG of type $k \in \mathbb{Z}$ is a self-map $\psi^k : BG \to BG$ extending the k-th power map in BT. Thus it seems there is a relationship with Adams operations, but unlike Adams operations the power maps are always defined (making them in a sense less interesting, but they are still quite useful). In general, Adams operations only exist for certain values of $k \in \mathbb{Z}$. For example, for a compact connected semi-simple Lie group G an Adams operation ψ^k exists on BG if and only if k is coprime to the order of the Weyl group of G, see [24] and the references therein.

A natural problem is to study the obstructions for lifting classifying maps of principal bundles under the inclusion map $i : B_{\text{com}}G \to BG$. The universal case asks for the existence of a section up to homotopy of this map. For example, it is an interesting result by Adem-Gómez-Lind-Tillmann [6, Thm. 4.2] that for $G = U$, O (the infinite unitary / orthogonal group) the map $B_{\text{com}}G \to BG$ does admit a homotopy section. On the other hand, we can use the maps ϕ^k and the fact that certain Adams operations do not exist on BG to prove that the same does normally not hold for a compact Lie group G. The existence of a section in the case $G = U$, O is a stable phenomenon.

Theorem 1.2.2. *Let G be a compact connected Lie group. Then the inclusion map $i : B_{\text{com}}G_{\mathbb{1}} \to BG$ admits a section up to homotopy if and only if G is abelian.*

Note that in the theorem we are using the reduced classifying space $B_{\text{com}}G_{\mathbb{1}}$. We will prove the theorem in Section 1.2.2.

Remark 1.2.3. (i) The theorem generalises a result of Adem-Gómez, namely [5, Cor. 7.5], which asserts that the following are equivalent for a compact connected Lie group G,

(1) i is a homotopy equivalence

(2) hofib(i) is rationally acyclic

(3) G is abelian.

Theorem 1.2.2 shows that (1) can be replaced by the a-priori weaker statement

(1') i has a section up to homotopy.

(ii) Our proof of Theorem 1.2.2 will use the fact that BG does in general not admit an Adams operation ψ^k for every $k \geq 0$. If one is willing to invert the order of the Weyl group, then all Adams operations do exist on $BG[1/|W|]$ if G is, say, semi-simple compact and connected [66, Thm. I]. One can then ask if the map $B_{\text{com}}G[1/|W|] \to BG[1/|W|]$ has a section up to homotopy. This is indeed the case, but we are planning to address this question elsewhere.

Corollary 1.2.4. *If G is one of $U(n)$, $SU(n)$ or $Sp(n)$ for some $n \geq 2$, then the universal bundle $\pi : EG \to BG$ is not transitionally commutative.*

Proof. Assume for contradiction that $\pi : EG \to BG$ is transitionally commutative in the sense of Definition 1.1.4. For a compact Lie group G we can approximate $u : B \xrightarrow{\sim} BG$ by a homotopy equivalent paracompact space B, so that the induced principal G-bundle over B is transitionally commutative with respect to a numerable open cover of B. Using the nerve theorem in the form of [56, Prop. 4.1], the proof of [5, Thm. 2.2] then shows that u factors up to homotopy through $B_{\text{com}}G$, and this can be used to construct a section up to homotopy of $i : B_{\text{com}}G \to BG$. If G is $U(n)$, $SU(n)$ or $Sp(n)$ then $B_{\text{com}}G_{\mathbb{1}} = B_{\text{com}}G$ (see Remark 1.1.3) and for $n \geq 2$ we obtain a contradiction to Theorem 1.2.2. □

There are similar statements when $G = \Gamma$ is a discrete group.

Lemma 1.2.5. *Let Γ be a discrete group and let X be a based space. Suppose that $f : X \to B\Gamma$ induces a surjective map on fundamental groups. Then, up to homotopy, f factors through $B_{\text{com}}\Gamma$ if and only if Γ is abelian.*

Proof. The "if"-statement is clear. Suppose now that $f \simeq i \circ f'$ for some $f' : X \to B_{\text{com}}\Gamma$, where $i : B_{\text{com}}\Gamma \to B\Gamma$ is the inclusion map. As Γ is discrete,

$$\pi_1 B_{\text{com}}\Gamma \cong \operatorname{colim}_{A \in \mathcal{N}_2(\Gamma)} A\,,$$

where $\mathcal{N}_2(\Gamma)$ denotes the poset under inclusion of all abelian subgroups of Γ and the colimit is formed in the category of groups [4]. If $[a]$ and $[b]$ are two classes in the colimit, then $[a][b] = [ab]$ whenever a and b commute in Γ. The n-th power map $a \mapsto a^n$, $a \in A$, induces an endomorphism ϕ^n of $\operatorname{colim}_{A \in \mathcal{N}_2(\Gamma)} A$. Since any $a \in \Gamma$ commutes with its inverse a^{-1}, we have $\phi^{-1}([a]) = [a^{-1}] = [a]^{-1}$. The composite map

$$\pi_1 X \xrightarrow{f'_*} \operatorname{colim}_{A \in \mathcal{N}_2(\Gamma)} A \xrightarrow{\phi^{-1}} \operatorname{colim}_{A \in \mathcal{N}_2(\Gamma)} A \xrightarrow{i_*} \Gamma$$

is a homomorphism, where f'_* and i_* denote the maps induced by f' respectively i on fundamental groups. For $h \in \pi_1 X$,

$$(i_* \circ \phi^{-1} \circ f'_*)(h) = i_*(\phi^{-1}(f'_*(h))) = i_*(f'_*(h)^{-1}) = (i_*(f'_*(h)))^{-1} = f_*(h)^{-1}\,.$$

It follows that the inversion map is a homomorphism on $\operatorname{im}(f_*)$, whence $\operatorname{im}(f_*)$ is abelian. By assumption, f_* is surjective, hence Γ is abelian. □

Corollary 1.2.6. *Let X be a finite CW complex. A connected regular covering $q : X' \to X$ is transitionally commutative if and only if the group of covering transformations is abelian.*

Proof. This follows from Theorem 1.1.5 and Lemma 1.2.5, since the automorphism group Γ of a connected regular covering $q : X' \to X$ is a quotient of $\pi_1(X)$ and the covering is classified by a map $X \to B\Gamma$ which induces the quotient map on fundamental groups. □

1.2.2 Proof of Theorem 1.2.2

Suppose that $s : BG \to B_{\mathrm{com}}G_{\mathbb{1}}$ is a homotopy section of the natural map $i : B_{\mathrm{com}}G_{\mathbb{1}} \to BG$. Then we can consider the composite map

$$BG \xrightarrow{s} B_{\mathrm{com}}G_{\mathbb{1}} \xrightarrow{\phi^k} B_{\mathrm{com}}G_{\mathbb{1}} \xrightarrow{i} BG\,.$$

The idea for the proof of Theorem 1.2.2 is to show that if G is non-abelian the existence of this map for all $k \in \mathbb{Z}$ contradicts the following classification result.

Theorem 1.2.7 (Jackowski-McClure-Oliver, [25]). *For any compact connected simple Lie group G with Weyl group W, the correspondence $f \leftrightarrow \mathrm{type}(f)$ defines a bijection*

$$[BG, BG] \xleftrightarrow{\cong} \{0\} \bigsqcup \mathrm{Out}(G) \times \{k \in \mathbb{Z}_{>0} \,|\, (k, |W|) = 1\}\,.$$

Let us very briefly explain this statement. Suppose that G satisfies the assumptions of the theorem and let $f : BG \to BG$ be any self map. The authors show there is a homotopy $f \simeq B\alpha \circ \psi^k$ where $B\alpha$ is induced by a homomorphism $\alpha : G \to G$ and ψ^k is an Adams operation of type $k \in \mathbb{Z}_{>0}$. Since G is simple, the endomorphism α is either trivial or an automorphism. If α is trivial, then $\mathrm{type}(f) := 0$. Otherwise, α defines an element in $\mathrm{Out}(G)$ and the type of f is defined to be the pair $\mathrm{type}(f) := (\alpha, k) \in \mathrm{Out}(G) \times \mathbb{Z}_{>0}$. Conversely, an outer automorphism of G gives rise to a self-map of BG which is welldefined up to homotopy, since any self-map of BG induced by an inner automorphism of G is homotopic to the identity.

Reduction to the case where G is simple. In order to use Theorem 1.2.7 we reduce to the case of a simple Lie group by means of the following lemma.

Lemma 1.2.8. *Suppose that $r : \tilde{G} \to G$ is a covering homomorphism of compact connected Lie groups. Then the diagram*

$$\begin{array}{ccc} B_{\mathrm{com}}\tilde{G}_{\mathbb{1}} & \xrightarrow{B_{\mathrm{com}}(r)} & B_{\mathrm{com}}G_{\mathbb{1}} \\ \downarrow & & \downarrow \\ B\tilde{G} & \xrightarrow{Br} & BG \end{array}$$

is a homotopy pullback diagram.

Proof. We will show that the horizontal homotopy fibres are homotopy equivalent. As all spaces are path-connected, this is enough by a well known characterization of homotopy pullback. Let $K := \ker(r)$ be the covering group of $r : \tilde{G} \to G$. Then K is a discrete normal subgroup of $\tilde{G}$, hence central. The functor $\mathrm{Hom}(\mathbb{Z}^k, -)$ takes the covering $K \to \tilde{G} \to G$ to the sequence of spaces

$$K^k \to \mathrm{Hom}(\mathbb{Z}^k, \tilde{G}) \xrightarrow{r_*} \mathrm{Hom}(\mathbb{Z}^k, G)\,.$$

We next follow an argument in [18]. By a result of Goldman [17, Lem. 2.2] the restricted map

$$r_*^{-1}(\mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}) \xrightarrow{r_*} \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}$$

is a covering projection with covering group K^k. We claim that $r_*^{-1}(\mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}) = \mathrm{Hom}(\mathbb{Z}^k, \tilde{G})_{\mathbb{1}}$. One direction is clear, namely the inclusion $\supset$. Conversely, suppose that $(\tilde{g}_1, \dots, \tilde{g}_k) \in r_*^{-1}(\mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}})$. By the path lifting property there is a path in $\mathrm{Hom}(\mathbb{Z}^k, \tilde{G})$ from $(\tilde{g}_1, \dots, \tilde{g}_k)$ to an element $(h_1, \dots, h_k) \in K^k$. Since K is central and $\tilde{G}$ is compact and connected, K is contained in a maximal torus $\tilde{T} \subset \tilde{G}$, see e.g. [27, Cor. IV.4.47]. Therefore $(h_1, \dots, h_k) \in \tilde{T}^k \subset \mathrm{Hom}(\mathbb{Z}^k, \tilde{G})_{\mathbb{1}}$ and so $(\tilde{g}_1, \dots, \tilde{g}_k) \in \mathrm{Hom}(\mathbb{Z}^k, \tilde{G})_{\mathbb{1}}$, which proves the converse inclusion $\subset$. Hence, the sequence

$$K^k \to \mathrm{Hom}(\mathbb{Z}^k, \tilde{G})_{\mathbb{1}} \xrightarrow{r_*} \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}$$

is a covering sequence.

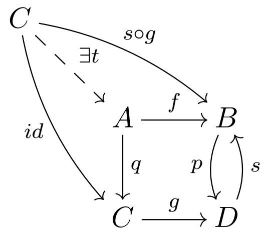
The maps in the covering sequence are compatible with the simplicial structure, and we would like to show that the sequence remains a homotopy fibration sequence after geometric realisation. By [5, Prop. A.1], the simplicial spaces $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, \tilde{G})_{\mathbb{1}}$ and $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}}$ are proper, i.e. the inclusion of the subspace of degenerate simplices in every simplicial level is a cofibration. Also, both simplicial spaces are levelwise path-connected. In this situation we can apply the Bousfield-Friedlander Theorem [10, Thm. B.4] which implies that

$$BK \xrightarrow{\text{incl.}} B_{\mathrm{com}}\tilde{G}_{\mathbb{1}} \xrightarrow{B_{\mathrm{com}}(r)} B_{\mathrm{com}}G_{\mathbb{1}}$$

is a homotopy fibre sequence. Since we also have $BK \simeq \mathrm{hofib}(Br)$ and the natural map $\mathrm{hofib}(B_{\mathrm{com}}(r)) \to \mathrm{hofib}(Br)$ induces a homotopy equivalence, this identifies the displayed diagram as a homotopy pullback. □

Lemma 1.2.9. *Suppose that the map $B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ has a section up to homotopy and that $\tilde{G}$ is a compact connected covering group of G. Then the map $B_{\mathrm{com}}\tilde{G}_{\mathbb{1}} \to B\tilde{G}$ has a section up to homotopy.*

Proof. This follows from Lemma 1.2.8 and the following property of a homotopy pullback. Suppose that the square labelled by A, B, C, D in the diagram of spaces below is a homotopy pullback square and that $s : D \to B$ is a section up to homotopy of $p : B \to D$. Then also $q : A \to C$ has a section up to homotopy. Namely, the outer square in the diagram



commutes up to homotopy, and the universal property of a homotopy pullback implies that there exists a map $t : C \to A$ making the two triangles commute up to homotopy. This map t is then a section up to homotopy of $q : A \to C$. $\square$

The preceding lemma allows us to reduce the proof of the theorem to the case of a simple Lie group, because of the standard fact that a compact connected Lie group G has a covering

$$r : \tilde{G} := H \times G_1 \times \cdots \times G_s \to G\,,$$

where H is a torus and $G_1, \ldots, G_s$ are compact connected simple Lie groups, see for example [27, Ch. IV]. Thus if $\tilde{s} : B\tilde{G} \to B_{\mathrm{com}}\tilde{G}_{\mathbb{1}}$ is a section up to homotopy, then for all $i = 1, \ldots, s$ the composite map

$$BG_i \xrightarrow{\text{incl.}} B\tilde{G} \xrightarrow{\tilde{s}} B_{\mathrm{com}}\tilde{G}_{\mathbb{1}} \xrightarrow{\text{proj.}} (B_{\mathrm{com}}G_i)_{\mathbb{1}}$$

is a section up to homotopy of the inclusion map $(B_{\mathrm{com}}G_i)_{\mathbb{1}} \to BG_i$. Assuming the theorem holds for simple Lie groups, we conclude that G_i is abelian for all $i = 1, \ldots, s$, so $\tilde{G}$ and hence G are abelian.

In fact, by passage to the universal covering, we could use the same argument to reduce to the case of a simple and simply connected Lie group, but we shall not use this fact.

The case when G is simple. If G is simple and $f : BG \to BG$ is any self-map, then the part of f which comes from the Adams operation (see Theorem 1.2.7) can be read off from the induced map on $H^4(BG, \mathbb{Q})$. Thus we will have to understand the corresponding cohomology group of $B_{\mathrm{com}}G_{\mathbb{1}}$. It would suffice to work with rational coefficients, but out of independent interest we will prove the following lemma at the end of this section.

Lemma 1.2.10. *For compact connected G the conjugation map $\varphi : G/T \times BT \to B_{\mathrm{com}}G_{\mathbb{1}}$ induces a monomorphism*

$$\varphi^* : H^4(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Z}) \hookrightarrow H^4(G/T \times BT, \mathbb{Z})^W$$

whose image contains the submodule $2H^4(BT, \mathbb{Z})^W \subset H^4(G/T \times BT, \mathbb{Z})^W$.

Let us finish the proof of Theorem 1.2.2 assuming that Lemma 1.2.10 holds. In the following we write $H^*(-) := H^*(-, \mathbb{Z})$.

Suppose that G is simple and that $s : BG \to B_{\mathrm{com}}G_{\mathbb{1}}$ is a section up to homotopy of $i : B_{\mathrm{com}}G_{\mathbb{1}} \to BG$. We consider the family of self-maps of BG defined by

$$\theta^k := i \circ \phi^k \circ s\,, \quad k \in \mathbb{Z}\,.$$

By Theorem 1.2.7, θ^k is either homotopic to the constant map or $\theta^k \simeq B\alpha(k) \circ \psi^{l(k)}$, where $\alpha(k)$ is an automorphism of G and $l(k)$ is a positive integer, both depending on k. We aim to show that $l(k) = |k|$.

If G is simple then $H^4(BG) \cong \mathbb{Z}$, by [20, Thm. 6], hence $(\theta^k)^* : H^4(BG) \to H^4(BG)$ corresponds to multiplication by an integer which we also denote by $(\theta^k)^*$. Since $B\alpha(k)$ acts on $H^4(BG)$ as multiplication by ± 1, we have $(\theta^k)^* = \pm l(k)^2$, where we set $l(k)$ be zero if θ^k is homotopic to the constant map. The Kunneth splitting

$$H^4(G/T \times BT)^W \cong (H^2(G/T) \otimes H^2(BT))^W \oplus H^4(BT)^W$$

is a decomposition of $H^4(G/T \times BT)^W$ into the sum of two eigenspaces for the k-th power map in BT with eigenvalues k respectively k^2. Let $x \in H^4(BG) \cong \mathbb{Z}$ be a generator. Then $\varphi^*(i^*(x)) = y_1 + y_2$, where $y_1 \in (H^2(G/T) \otimes H^2(BT))^W$ and $y_2 \in H^4(BT)^W$. By Lemma 1.2.10 there are elements $z_i \in H^4(B_{\mathrm{com}}G_{\mathbb{1}})$ so that $\varphi^*(z_i) = 2y_i$ for $i = 1, 2$. Since φ^* is injective, we see that z_i is an eigenvector of $(\phi^k)^*$ with eigenvalue k^i, $i = 1, 2$. Consequently, $(\phi^k)^* i^*(2x) = kz_1 + k^2 z_2$. As $H^4(BG) \cong \mathbb{Z}$, there are integers $c, d \in \mathbb{Z}$, only depending on the splitting map s, so that $s^*(z_1) = cx$ and $s^*(z_2) = dx$. Since s is a homotopy splitting of i, we must have $s^*(z_1 + z_2) = 2x$, hence $d = 2 - c$. Altogether this shows that $(\theta^k)^* = kc + k^2(2 - c)$ and therefore

$$kc + k^2(2 - c) = \pm 2l(k)^2\,. \tag{1.3}$$

The relation (1.3) holds for any $k \in \mathbb{Z}$, in particular for any odd prime $k = p$. But for this choice of k the relation (1.3) implies that p divides c. Then c is divisible by all odd primes, so it must be zero and $k^2 = l(k)^2$. Thus if $k \neq 0$, then $\mathrm{type}(\theta^k) = (\alpha(k), |k|)$.

If G is non-abelian, then $|W| > 1$ and $\mathrm{type}(\theta^{|W|}) = (\alpha(|W|), |W|)$ contradicting Theorem 1.2.7. This finishes the proof of Theorem 1.2.2.

Proof of Lemma 1.2.10

We close this chapter by proving Lemma 1.2.10. Note that in view of Theorem 1.1.8 the content of this lemma is really the assertion that $H^4(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Z})$ is a free abelian group. We show this in the following two lemmas. The Lie group G is always assumed to be compact and connected.

Lemma 1.2.11. *The inclusion $i : B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ is 3-connected.*

Remark 1.2.12. This result may be implicit in the work of [5], but it also appears slightly more general than their Proposition 3.3 which only treats the case where G is one of $SU(n)$, $U(n)$, $Sp(n)$, or a finite product of these.

Proof. We must show that the homotopy fibre $\mathrm{hofib}(i)$ is a 3-connected space. It is well known that G admits a covering $H \times G' \to G$, where H is a torus and G' is a simply connected compact Lie group. By Lemma 1.2.8 and because of the natural homeomorphism $B_{\mathrm{com}}(H \times G')_{\mathbb{1}} \cong BH \times B_{\mathrm{com}}G'_{\mathbb{1}}$, the connectivity of i is the same as the connectivity of the map $B_{\mathrm{com}}G'_{\mathbb{1}} \to BG'$. Thus we may assume, without loss of generality, that our Lie group G is simply connected.

The result follows now from [5], but let us spell out the argument. It is well known that if G is simply connected, then BG is 3-connected, and also $B_{\mathrm{com}}G_{\mathbb{1}}$ is 3-connected by [5, Prop. 3.2]. Therefore, $\pi_j(\mathrm{hofib}(i)) = 0$ for $j \leq 2$. Moreover, the exact same argument as in [4, Thm. 6.3] shows that the looped homotopy fibre sequence

$$\Omega\,\mathrm{hofib}(i) \to \Omega B_{\mathrm{com}}G_{\mathbb{1}} \to \Omega BG \simeq G$$

has a section up to homotopy, i.e. there is a map $s : G \to \Omega B_{\mathrm{com}}G_{\mathbb{1}}$ with $(\Omega i) \circ s \simeq id_G$. As a consequence, the map $i : B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ is surjective on homotopy groups, cf. [5, Cor. 2.3]. This implies $\pi_3(\mathrm{hofib}(i)) = 0$. □

Lemma 1.2.13. *The inclusion $i : B_{\mathrm{com}}G_{\mathbb{1}} \to BG$ induces isomorphisms $H^j(BG, \mathbb{Z}) \cong H^j(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Z})$ for $0 \leq j \leq 3$. Moreover, the group $H^4(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Z})$ is free abelian and there is an isomorphism*

$$H^4(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Z}) \cong H^4(BG, \mathbb{Z}) \oplus \ker(\tau)\,, \tag{1.4}$$

where $\ker(\tau)$ is the kernel of the transgression $\tau : H^4(\mathrm{hofib}(i), \mathbb{Z}) \to H^5(BG, \mathbb{Z})$.

Proof. The group $H^4(BG, \mathbb{Z})$ is free abelian by [20, Thm. 6] and so the statement that $H^4(B_{\mathrm{com}}G, \mathbb{Z})$ is free abelian follows from the isomorphism (1.4) if we show that $\ker(\tau)$ is free.

Consider the homotopy fibration $\mathrm{hofib}(i) \to B_{\mathrm{com}}G_{\mathbb{1}} \to BG$. As G is connected, the base space BG is simply connected. The homotopy fibre $\mathrm{hofib}(i)$ is 3-connected, by Lemma 1.2.11. Thus the Serre exact sequence yields isomorphisms

$$H^j(BG, \mathbb{Z}) \cong H^j(B_{\mathrm{com}}G_{\mathbb{1}}, \mathbb{Z}) \quad \text{for all } 0 \leq j \leq 3$$

and an exact sequence

$$0 \to H^4(BG, \mathbb{Z}) \to H^4(B_{\mathrm{com}} G_{\mathbb{1}}, \mathbb{Z}) \to \ker(\tau) \to 0\,. \tag{1.5}$$

The homology groups of $\mathrm{hofib}(i)$ are finitely generated in every degree. For this we recall from [5, p. 497] that a model for $\mathrm{hofib}(i)$ is the geometric realisation of the simplicial space $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, G)_{\mathbb{1}} \times G$ regarded as a simplcial subspace of the bar construction E_*G. In the literature the realisation is denoted by $E_{\mathrm{com}} G_{\mathbb{1}}$. But every simplicial level of $E_{\mathrm{com}} G_{\mathbb{1}}$ has the homotopy type of a compact CW-complex, see e.g. [3, Rem. p.469], so it has finitely generated homology in every degree. The spectral sequence for the homology of a (semi-)simplicial space, cf. [56, §5], then shows that the homology of $E_{\mathrm{com}} G_{\mathbb{1}}$ is also finitely generated in every degree. Now the universal coefficient theorem shows that $H^4(\mathrm{hofib}(i), \mathbb{Z})$ is the $\mathbb{Z}$-dual of $H_4(\mathrm{hofib}(i), \mathbb{Z})$, hence free. In particular, $\ker(\tau)$ is free and the short exact sequence (1.5) splits. □

To finish the proof of Lemma 1.2.10 note that by Theorem 1.1.8 the map φ^* is a rational isomorphism, so the kernel of φ^* has trivial rank. On the other hand, φ^* is a homomorphism of free abelian groups, by Lemma 1.2.13, so φ^* is injective. It remains to show the second statement of the lemma, which asserts that all even classes $2\,H^4(BT, \mathbb{Z})^W \subset H^4(G/T \times BT, \mathbb{Z})^W$ are in the image of φ^*. Suppose that $x \in H^4(BT, \mathbb{Z})^W$, then it is easy to see that $i^*(x) + \phi^{-1}(i^*(x)) \in H^4(B_{\mathrm{com}} G_{\mathbb{1}}, \mathbb{Z})$ maps to $2x$ under φ^*. This completes the proof of Lemma 1.2.10.

Remark 1.2.14. In general, the homomorphism φ^* in Lemma 1.2.10 is not surjective. For example, in [5, Ex. 6.4] it is shown that $H^4(B_{\mathrm{com}} SU(2), \mathbb{Z}) \cong \mathbb{Z}^2$. However, we will see in Lemma 2.2.22 that $\mathrm{im}(\varphi^*)$ is the span of $(2, 0)$ and $(1, 2)$ in $\mathbb{Z}^2$.

Chapter 2

Commutative K-theory

In this chapter we describe commutative K-theory. This theory was introduced by Adem and Gómez in [5] as a Grothendieck group of transitionally commutative complex vector bundles. In [6] Adem, Gómez, Lind and Tillmann show that this group extends to a generalised cohomology theory.

We begin in Section 2.1.1 by giving a rather detailed exposition of the definition in [6] of commutative K-theory as a generalised cohomology theory. In Section 2.1.2 we describe a categorical model for it, which is a variation of the construction in [6] and which will be used later in Section 2.3.1. In Section 2.2 we offer a complete description of the classifying space $B_{\mathrm{com}}U$. We first construct a characteristic map for commutative K-theory. This was our first attempt to describe the space $B_{\mathrm{com}}U$ and has an interpretation as a refinement of the determinant map (Section 2.2.1). In Section 2.2.2 we review the construction of the deformation K-theory spectrum and the results of Lawson that we will need. Our main results are then contained in Section 2.2.3. This includes the description of the homotopy type of $B_{\mathrm{com}}U$. As a concrete example of a commutative K-theory group, we have included a discussion of the commutative K-theory of S^4. Finally, in Section 2.2.4 we determine the rational Hurewicz homomorphism for $B_{\mathrm{com}}U$ with respect to a particular choice of basis. The idea is to relate our new description of the homotopy groups of $B_{\mathrm{com}}U$ to the rational cohomology calculations carried out previously in [5].

Section 2.3 deals with the classifying space $B_{\mathrm{com}}O$ for real commutative K-theory. In Section 2.3.1 we use the basic operations of complexification and realification to show that all torsion in the homotopy groups of $B_{\mathrm{com}}O$ is 2-primary, and we deduce the torsionfree part from our previous results in complex K-theory. As a corollary, we obtain a description of the rational cohomology ring for $B_{\mathrm{com}}O$ and $B_{\mathrm{com}}SO$, thus extending the list of computations in [5, §8]. In Section 2.3.2 we identify a small part of the 2-torsion which originates from the fact that the spaces of commuting elements $C_k(O)$ are not path-connected. This uses the work of Rojo [52], who counted the number of connected components of the space

$\mathrm{Hom}(\mathbb{Z}^k, O(n))$. Our result will have the form of a homotopy fibre sequence relating $B_{\mathrm{com}}O$ to the $\mathbb{Z}_2$-group ring of $\mathbb{R}P^\infty$. The proof of this result is in Section 2.3.3. As an application we compute the commutative real K-theory of the sphere S^n for $n \leq 3$. For example, we will find that $\widetilde{KO}_{\mathrm{com}}(S^3)$ detects a class in the stable homotopy of $\mathbb{R}P^\infty$. We finish in Section 2.3.4 with a comment on the $\mathbb{Z}_2$-equivariant theory and on another approach that may allow us to identify the homotopy type of $B_{\mathrm{com}}O$.

2.1 Definition and general cohomology properties

Most of this section will be formulated for complex K-theory, but the analogous statements for real K-theory are also true and we refer the reader to [6] or to Section 2.3 for more details. We begin by defining commutative K-theory as a ring, and then as a generalised cohomology theory. This was originally done in [5] and [6] and Section 2.1.1 is a summary of their definitions. As there is a subtlety involved in the multiplicative structure of the theory, it makes sense to spell out these definitions in detail.

2.1.1 Classifying space, monoids and group-completion

We first explain which classifying space is used in the definition of the new K-theory. By a transitionally commutative structure on a vector bundle $q : E \to X$ we understand one on its associated frame bundle [5]. If E is a complex vector bundle of rank n the frame bundle is a principal $GL_n(\mathbb{C})$-bundle, thus classified by a map $X \to BGL_n(\mathbb{C})$.

Recall that every complex vector bundle over a compact CW-complex admits a hermitian metric which is unique up to isomorphism. On the level of classifying spaces this is reflected by the fact that the inclusion $BU(n) \subset BGL_n(\mathbb{C})$ is a homotopy equivalence. This in turn is a consequence of the classical result that a connected Lie group deformation retracts onto a maximal compact Lie subgroup. Now it is a non-trivial fact that this remains true for spaces of commuting elements and for the classifying spaces built from them.

Based on a general result of Pettet-Souto [47], Adem-Gómez show in [5, Thm. 3.1] that both inclusions $U(n) \hookrightarrow GL_n(\mathbb{C})$ and $O(n) \hookrightarrow GL_n(\mathbb{R})$ induce homotopy equivalences

$$B_{\mathrm{com}}U(n) \simeq B_{\mathrm{com}}GL_n(\mathbb{C})$$
$$B_{\mathrm{com}}O(n) \simeq B_{\mathrm{com}}GL_n(\mathbb{R})\,.$$

Thus we shall from now on take the classifying spaces $B_{\mathrm{com}}U(n)$ and $B_{\mathrm{com}}O(n)$ as the basis for our discussion.

Now consider the union

$$M := \coprod_{n\geq 0} B_{\mathrm{com}}U(n)\,.$$

We regard $U(0)$ as the trivial group and give M the basepoint $0 := B_{\mathrm{com}}U(0)$. The set of homotopy classes of unbased maps $X \to M$ is then, by definition, the set of transitionally commutative structures on finite dimensional complex vector bundles over X.

As explained in [6, Sec. 2.5] the direct sum $\oplus : U(m) \times U(n) \to U(m+n)$ and the Kronecker product $\otimes : U(m) \times U(n) \to U(mn)$ induce continuous classifying maps

$$\oplus : M \times M \to M$$
$$\otimes : M \wedge M \to M\,,$$

respectively. Let $1 \in M$ be the unique 0-simplex of $B_{\mathrm{com}}U(1) \subset M$. It is then easy to see that the space M together with the operations $\oplus$ and $\otimes$ and the distinguished points $0, 1 \in M$ satisfies the axioms of an abelian semi-ring up to homotopy. To see distributivity of $\otimes$ over $\oplus$ and commutativity of both operations up to homotopy, one uses the natural action of the symmetric group Σ_n on $B_{\mathrm{com}}U(n)$ induced by conjugation within the unitary groups, together with the fact that the action of any fixed $\sigma \in \Sigma_n$ on $B_{\mathrm{com}}U(n)$ is homotopic to the identity, because the group $U(n)$ is path-connected.

A consequence of this is that for every X the set $[X, M]$ has the structure of an abelian semi-ring, and one defines $K_{\mathrm{com}}(X)$ to be the Grothendieck ring of $[X, M]$ [5].

On compact spaces the functor K_{com} is itself representable by the group-completion of M. This is not automatic, but was proved in [6, Thm. 5.5] using the group-completion theorem [43] (see also [5, Thm. 4.1]). The group-completion theorem implies that $\Omega BM \simeq \mathbb{Z} \times B_{\mathrm{com}}U$ is a ring space up to homotopy, and one can show that for every compact CW-complex X the natural map of semi-rings $[X, M] \to [X, \mathbb{Z} \times B_{\mathrm{com}}U]$ induces an isomorphism of rings $K_{\mathrm{com}}(X) \cong [X, \mathbb{Z} \times B_{\mathrm{com}}U]$. Thus we arrive at the following definition.

Definition 2.1.1 (Adem-Gómez, [5])**.** For a space X of the homotopy type of a CW-complex define the commutative K-theory $K_{\mathrm{com}}(X) = [X, \mathbb{Z} \times B_{\mathrm{com}}U]$.

The functor $X \mapsto K(X)$ extends to a generalised cohomology theory. This is one of the consequences of Bott periodicity or, more abstractly, of the fact that the representing space for K, the Bott space $\mathbb{Z} \times BU$, is an infinite loop space. In fact, it is not only an infinite loop space but an E_∞-ring space, which implies that K-theory is a cohomology theory with cup products. The standard way to see that $\mathbb{Z} \times BU$ is an E_∞-ring space is to identify it as the group-completion of $\coprod_{n \geq 0} BU(n)$, which is the classifying space of a suitable symmetric bimonoidal category. Unfortunately, we cannot argue in the same way for commutative K-theory, as there is no natural candidate for a bimonoidal category whose classifying space is M. This was noted in [6], and there the following precise statement was formulated.

Theorem 2.1.2 (Adem-Gómez-Lind-Tillmann, cf. [6, Thm. 4.1])**.** *The structure of the basepoint component* $\{0\} \times B_{\text{com}}U \subset \mathbb{Z} \times B_{\text{com}}U$ *as a non-unital ring space up to homotopy lifts to that of a non-unital* E_∞*-ring space, and the natural map* $i : B_{\text{com}}U \to BU$ *is an* E_∞*-ring map.*

Let K^*_{com} be the generalised cohomology theory represented by the infinite loop space $\mathbb{Z} \times B_{\text{com}}U$, i.e. $K^*_{\text{com}}(X) = [X, \Omega^{-*}(\mathbb{Z} \times B_{\text{com}}U)]$. The coefficient group $K^{-*}_{\text{com}}(\text{pt})$ of the new theory has then the structure of a graded commutative ring, namely the unitalisation of $\pi_*(B_{\text{com}}U)$ regarded as a $\mathbb{Z}$-algebra, i.e. $K^{-*}_{\text{com}}(\text{pt}) \cong \mathbb{Z} \oplus \pi_*(B_{\text{com}}U)$ as graded abelian groups, where $\mathbb{Z}$ is placed in degree zero. The product in $\mathbb{Z} \oplus \pi_*(B_{\text{com}}U)$ uses the product in $\mathbb{Z}$ and in $\pi_*(B_{\text{com}}U)$ and the structure of $\pi_*(B_{\text{com}}U)$ as a $\mathbb{Z}$-module. However, this ring may not come from an E_∞-ring structure on $\mathbb{Z} \times B_{\text{com}}U$. In fact, it would be an interesting question to decide whether or not the ring $\mathbb{Z} \oplus \pi_*(B_{\text{com}}U)$ can be the coefficient ring of a multiplicative cohomology theory - perhaps using an argument along the lines of [61].

In any case, it is possible and it turns out to be natural to regard commutative K-theory as a module theory over connective K-theory, in fact, as a ku-algebra. The unitalisation of $\pi_*(B_{\text{com}}U)$ regraded as a $\mathbb{Z}[u]$- rather than $\mathbb{Z}$-algebra (where $u \in \pi_2(BU)$ is the Bott element) is then a graded commutative ring which lifts to the spectrum level, i.e. it is the coefficient ring of a multiplicative cohomology theory.

2.1.2 Commutative K-theory via bipermutative categories

There are different ways of how one can endow $B_{\text{com}}U$ with an E_∞-ring space structure or associate a commutative ring spectrum to it. In [6] a bipermutative category is constructed by considering the action of the symmetric groups Σ_n on the spaces $B_{\text{com}}U(n)$ and this category is then used as an input for the machine of Elmendorf-Mandell [14]. When we determine the homotopy type of $B_{\text{com}}U$ in Section 2.2.3 we will build on known results in deformation K-theory. These are formulated using a symmetric ring spectrum which is also an output of the machine of Elmendorf-Mandell, but with a multiplicative Γ-space as an input. In the present section we shall describe a slight variation of the construction in [6] using the unitary groups instead of the symmetric groups in the definition of the bipermutative category. Certainly, this construction is not conceptually new, but it seems more natural from the point of view of what will follow. It is also necessary to easily establish the properties of the complexification and realification maps which we consider in Section 2.3.1.

By a *topological category* we mean a category with an object and a morphism space, so that the domain, co-domain, identity and composition maps are all continuous.

By functoriality, conjugation in the unitary group induces a left action of $U(n)$ on $B_{\rm com}U(n)$.

Definition 2.1.3. Associated to the continuous action of $U(n)$ on $B_{\rm com}U(n)$ for all $n \geq 0$ we have the topological translation category

$$\mathscr{C} := \coprod_{n\geq 0} U(n) \ltimes B_{\rm com}U(n)$$

with spaces of objects and morphisms

$$\mathrm{Ob}(\mathscr{C}) = \coprod_{n\geq 0} B_{\rm com}U(n)\,, \qquad \mathrm{Mor}(\mathscr{C}) = \coprod_{n\geq 0} U(n)\times B_{\rm com}U(n)\,.$$

Let $x \in B_{\rm com}U(m)$ and $y \in B_{\rm com}U(n)$. There are no morphisms from x to y if $n \neq m$, and if $n = m$ a morphism $A : x \to y$ is a pair $(A, x) \in \mathrm{Mor}(\mathscr{C})$ so that $y = Ax$ (A acting on x). The composition of morphisms $A : x \to y$ and $B : y \to z$ is defined by $(B, y) \circ (A, x) = (BA, x)$.

The category $\mathscr{C}$ has *two* permutative structures (see [41] for the definition of a permutative category) denoted by $(\oplus, c_\oplus, 0)$ and $(\otimes, c_\otimes, 1)$ arising from the direct sum and the Kronecker product of matrices, respectively. Thus, the map $\oplus : M \times M \to M$ defined previously lifts to a continuous functor $\oplus : \mathscr{C}\times\mathscr{C} \to \mathscr{C}$ for which the object $0 := B_{\rm com}U(0)$ is a strict two-sided identity. The natural commutativity isomorphism

$$c_\oplus : x \oplus y \xrightarrow{\cong} y \oplus x\,, \quad x, y \in \mathrm{Ob}(\mathscr{C})$$

is defined using the action by a suitable permutation matrix. Similarly, $\otimes : M \wedge M \to M$ lifts to a continuous functor $\otimes : \mathscr{C}\times\mathscr{C} \to \mathscr{C}$. (For definiteness, we let the Kronecker product of $A \in U(m)$ and $B \in U(n)$ be given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix},$$

where the $a_{ij} \in \mathbb{C}$ are the coordinates of A. This operation is strictly associative and, in this convention, satisfies strict *right*-distributivity $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$ over the direct sum.) The basepoint $1 \in B_{\rm com}U(1)$ is a strict two-sided identity for the product $\otimes$ on $\mathscr{C}$, while $0 \in \mathrm{Ob}(\mathscr{C})$ is defined to be a strict two-sided zero object. There is a natural commutativity isomorphism

$$c_\otimes : x \otimes y \xrightarrow{\cong} y \otimes x\,, \quad x, y \in \mathrm{Ob}(\mathscr{C})$$

again defined using a suitable permutation matrix. In fact, the permutations defining $c_\oplus$ and $c_\otimes$ are unique.

The strict right-distributivity of the Kronecker product implies strict right-distributivity of $\otimes$ over $\oplus$ in $\mathscr{C}$,

$$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$$

for all $x, y, z \in \mathrm{Ob}(\mathscr{C})$. Left-distributivity only holds up to natural isomorphism and is determined by going from left to right in the following diagram,

$$x \otimes (y \oplus z) \xrightarrow{c_\otimes} (y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x) \xrightarrow{c_\otimes \oplus c_\otimes} (x \otimes y) \oplus (x \otimes z)\,.$$

A category with two permutative structures and distributivity morphisms as above which furthermore satisfy the coherence diagram displayed in [41, Sec. 12] is called a *bipermutative category*. It is straightforward to check the coherence condition for the category $\mathscr{C}$. This shows

Lemma 2.1.4. *The direct sum and Kronecker product of matrices make the topological category $\mathscr{C}$ into a bipermutative category in the sense of [41].*

The classifying space of a bipermutative category is an E_∞-ring space [41]. May's machine associates to $\mathscr{C}$ an E_∞-ring spectrum $\mathbb{E}\mathscr{C}$ together with a natural ring completion map $B\mathscr{C} \to \mathbb{E}_0\mathscr{C}$ from the classifying space of $\mathscr{C}$ into the zero space of $\mathbb{E}\mathscr{C}$. The nerve of $\mathscr{C}$ is the disjoint union over all $n \geq 0$ of the bar construction $B_*(*, U(n), B_{\mathrm{com}}U(n))$. Its geometric realisation is a model for the homotopy orbit space

$$B\mathscr{C} = \coprod_{n \geq 0} B_{\mathrm{com}}U(n) /\!\!/ U(n)\,,$$

which is therefore an E_∞-ring space. The zero space $\mathbb{E}_0\mathscr{C}$ can be identified using the group completion theorem. Before we do this, we include a proof of the following lemma, which is used implicitly in the literature, e.g. in [5, 6]. However, we were unable to find it stated explicitly in this form. The arguments we use are certainly standard and wellknown, see e.g. [3, Rem. p.469].

Lemma 2.1.5. *For all $n \geq 0$ the stabilisation map $-\oplus 1 : B_{\mathrm{com}}U(n) \to B_{\mathrm{com}}U(n+1)$ is a cofibration.*

Proof. Recall that a simplicial space is good (in the sense of Segal [57, App. A]) if each of the degeneracy maps is a closed cofibration. A map of good simplicial spaces which is a cofibration in every simplicial degree induces a cofibration on geometric realisations (see for example [65, 14-5]). Thus it suffices to show that for all $k, n \geq 0$ the obvious map

$$\mathrm{Hom}(\mathbb{Z}^k, U(n)) \to \mathrm{Hom}(\mathbb{Z}^k, U(n+1))$$

is a cofibration and the simplicial space $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, U(n))$ is good. Both these facts follow from the semi-algebraic triangulation theorem [9, Thm. 9.2.1]. For all $k, n \geq 0$ the space $\mathrm{Hom}(\mathbb{Z}^k, U(n))$ is a real algebraic variety since both, the relations which identify the unitary group as a subspace $U(n) \subset \mathbb{R}^{2n \times 2n}$ as well as the commutator relations are polynomials in the coordinates of Euclidean space. The space $\mathrm{Hom}(\mathbb{Z}^k, U(n))$ is a closed subspace of the compact space $U(n)^k$, hence compact. Moreover, the displayed map is a closed embedding (it is injective and the domain is compact), whence we can identify $\mathrm{Hom}(\mathbb{Z}^k, U(n))$ with a closed subspace of $\mathrm{Hom}(\mathbb{Z}^k, U(n+1))$. In fact, it is a closed subvariety. In this situation, the triangulation theorem says that $\mathrm{Hom}(\mathbb{Z}^k, U(n+1))$ is homeomorphic to a finite simplicial complex in which $\mathrm{Hom}(\mathbb{Z}^k, U(n))$ appears as a closed subcomplex. From this we conclude that the displayed map is a cofibration. In a similar manner one shows that each degeneracy map is a closed cofibration, by identifying $\mathrm{Hom}(\mathbb{Z}^{k-1}, U(n))$ with a closed subvariety of $\mathrm{Hom}(\mathbb{Z}^k, U(n))$. □

The lemma also holds if we replace the unitary groups by the orthogonal groups. This allows us to replace the homotopy colimit over the $B_{\mathrm{com}}U(n)$ or $B_{\mathrm{com}}O(n)$ as n goes to infinity by a strict colimit, i.e. by $B_{\mathrm{com}}U$ respectively $B_{\mathrm{com}}O$. For the rest of this work we will use this fact without further comment.

Lemma 2.1.6. *The zero space $\mathbb{E}_0\mathscr{C}$ has the homotopy type of the homotopy orbit space $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$.*

Proof. This follows from the group-completion theorem [43] applied to the monoid $B\mathscr{C}$, using Lemma 2.1.5 and the fact that $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$ is simply connected. The latter follows from the homotopy fibre sequence $B_{\mathrm{com}}U \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \to \mathbb{Z} \times BU$ and the fact that both, the base space and the fibre, are simply connected. The fibre $B_{\mathrm{com}}U$ is simply connected, because U is connected. □

If we replace in the definition of $\mathscr{C}$ each of the spaces $B_{\mathrm{com}}U(n)$ by a single point, we obtain a bipermutative category $\mathscr{U}$ whose associated E_∞-ring spectrum $\mathbb{E}\mathscr{U}$ is a model for connective K-theory ku. In particular, the zero space $\mathbb{E}_0\mathscr{U}$ is homotopy equivalent to $\mathbb{Z} \times BU$. Since the basepoints $* \in B_{\mathrm{com}}U(n)$ are fixed under the $U(n)$-action, the inclusion maps $* \to B_{\mathrm{com}}U(n)$ define a bipermutative functor $\mathscr{U} \to \mathscr{C}$. This gives rise to a map of E_∞-ring spectra $ku \to \mathbb{E}\mathscr{C}$, which allows us to regard $\mathbb{E}\mathscr{C}$ as a ku-algebra. Notice that on zero spaces this map corresponds to a map of E_∞-ring spaces $\mathbb{Z} \times BU \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$ and that this map is a section of the projection map $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \to \mathbb{Z} \times BU$. This implies that there is a splitting of infinite loop spaces

$$\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \simeq (\mathbb{Z} \times BU) \times B_{\mathrm{com}}U\,. \tag{2.1}$$

Similar arguments apply if we replace U by O.

Remark 2.1.7. (i) We obtain an infinite loop space structure on $B_{\mathrm{com}}U$ by identifying it with the homotopy fibre of the map $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \to \mathbb{Z} \times BU$. The results in [38] and [37, I. 2.2] together with [32] can be used to show that this infinite loop space structure on $B_{\mathrm{com}}U$ agrees with the one obtained in [6] by modeling $B_{\mathrm{com}}U$ as a commutative $\mathbb{I}$-monoid, but we shall not spell out the details.

(ii) We should point out that May's machine [41] and the machine of Elmendorf-Mandell [14], both of which can take a bipermutative category as an input, produce equivalent spectra. To our knowledge, however, there is no comparison result in the literature which guarantees that the spectra are equivalent as commutative ring spectra (e.g. in the category of commutative $\mathbb{S}$-algebras, where the two outputs can be compared). Thus, whenever we refer to commutative K-theory as a ring spectrum, we should have a particular model in mind. In our work, this will always be the model based on deformation K-theory, which we describe in the following sections.

2.2 Complex commutative K-theory

The goal of this section is to describe the complex variant of commutative K-theory. We aim at describing the homotopy type of $B_{\mathrm{com}}U$ and the graded ring structure on its homotopy groups.

2.2.1 The eigenvalue map

Recall that for a space X and an integer $q \geq 1$ the q-th symmetric power is the orbit space $SP^qX = X^q/\Sigma_q$ where the symmetric group Σ_q acts on the cartesian product X^q by permuting the q factors. If X has a basepoint $0 \in X$, there is a natural inclusion $SP^{q-1}X \subset SP^qX$ given by $\{x_1, \dots, x_{q-1}\} \mapsto \{x_1, \dots, x_{q-1}, 0\}$. The infinite symmetric product $SP^\infty X$ is defined to be the union of the sequence of spaces

$$SP^1X \subset SP^2X \subset \cdots \subset SP^qX \subset \dots$$

with the colimit topology. We think of $SP^\infty X$ as the free abelian monoid generated by the points of X with the basepoint $0 \in X$ as the additive unit.

An important role in our description of $B_{\mathrm{com}}U$ is played by a map into the infinite symmetric product of $\mathbb{C}P^\infty$,

$$\lambda : B_{\mathrm{com}}U \to SP^\infty \mathbb{C}P^\infty \,, \tag{2.2}$$

which can be defined as follows. Suppose that $\rho \in \mathrm{Hom}(\mathbb{Z}^k, U(n))$ is a representation. By the spectral theorem ρ is isomorphic to a sum of one-dimensional representations, say $\rho \cong \rho_1 \oplus \cdots \oplus \rho_n$. A representation $\rho_i : \mathbb{Z}^k \to U(1)$ is precisely a point in the k-dimensional torus $(S^1)^k$, so we can define a map

$$\lambda_{k,n} : \mathrm{Hom}(\mathbb{Z}^k, U(n)) \to SP^n((S^1)^k)$$

taking $\rho \mapsto \{\rho_1, \ldots, \rho_n\}$. It is not so difficult to see that this map is welldefined and continuous, cf. [3, Thm. 6.1]. Moreover, if $(S^1)^k$ is given its natural basepoint, the maps $\lambda_{k,n}$ extend to the colimit as n goes to infinity and yield a map

$$\lambda_k : \mathrm{Hom}(\mathbb{Z}^k, U) \to SP^\infty(S^1)^k \,.$$

The collection of maps $\{\lambda_k\}_{k\geq 0}$ defines a map of simplicial spaces, where the simplicial structure on the target space $k \mapsto SP^\infty(S^1)^k$ comes from the bar construction for S^1. The map (2.2) is induced upon geometric realisation.

The map λ can be seen as a refinement of the determinant map $\det : BU \to \mathbb{C}P^\infty$ in complex K-theory. More precisely, there is a commutative diagram

$$\begin{array}{ccc} B_{\mathrm{com}}U & \xrightarrow{\lambda} & SP^\infty\mathbb{C}P^\infty \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{fusion}} \\ BU & \xrightarrow{\det} & \mathbb{C}P^\infty \end{array} \tag{2.3}$$

where $i : B_{\mathrm{com}}U \to BU$ is the inclusion, and "fusion" is the canonical map $SP^\infty\mathbb{C}P^\infty \to \mathbb{C}P^\infty$ extending the identity homomorphism of the abelian group $\mathbb{C}P^\infty$ over the free abelian monoid.

Bott periodicity implies that there is a homotopy fibre sequence

$$\Omega^{-2}BU \longrightarrow BU \xrightarrow{\det} \mathbb{C}P^\infty \tag{2.4}$$

which identifies the connective delooping $\Omega^{-2}BU$ with the homotopy fibre of the first Chern class $c_1 : BU \to K(\mathbb{Z}, 2)$. Using a theorem of Lawson in deformation K-theory [30] we will lift (2.4) to a homotopy fibre sequence

$$\Omega^{-2}B_{\mathrm{com}}U \longrightarrow B_{\mathrm{com}}U \xrightarrow{\lambda} SP^\infty\mathbb{C}P^\infty \,. \tag{2.5}$$

The long exact sequence of homotopy groups resulting from this homotopy fibre sequence completely determines the homotopy groups of $B_{\mathrm{com}}U$. This is because the homotopy groups of $SP^\infty\mathbb{C}P^\infty$ are the homology groups of $\mathbb{C}P^\infty$, which are free abelian and concentrated in even degrees. We will not derive (2.5) directly, but rather derive a homotopy cofibre sequence on the spectrum level. This is Lemma 2.2.10. Working on the spectrum level will then allow us to determine the homotopy type of $B_{\mathrm{com}}U$ as the zero space of a commutative symmetric ring spectrum. This spectrum is identified in Theorem 2.2.11.

Remark 2.2.1. There seems to be another, more direct approach to identify the homotopy type of $B_{\mathrm{com}}U$, which goes via Segal's configuration space model for connective K-homology [58] (see Section 2.3.4). However, the approach using deformation K-theory has the two advantages that (1) the ring structures are explicit, i.e. we will work with commutative symmetric ring spectra at all times and (2) the same approach should also apply to the nilpotent K-theories introduced in [6], even though in this work we shall not concern ourselves with this more general notion.

2.2.2 Deformation K-theory

Suppose G is a finitely generated discrete group. The *deformation K-theory* of G is the K-theory, with respect to direct sum, of the topological category of finite dimensional linear representations of G. The study of this K-theory was first suggested by Carlsson [12]. In [29] Lawson proved important structure theorems for its unitary variant. In this section we briefly outline Lawson's construction of the unitary deformation K-theory spectrum for G and recall a theorem from [30] which we shall use in describing commutative complex K-theory. The spectrum for deformation K-theory will be a symmetric spectrum in the sense of [22].

Remark 2.2.2. In symmetric spectra one must deal with the subtlety of two inequivalent notions of homotopy groups, called the "true" homotopy groups and the "naive" homotopy groups. The latter are the homotopy groups in the usual sense of the classical spectrum underlying the symmetric spectrum, but the former detect the stable equivalences between symmetric spectra. In the following we will never emphasise a difference between the two for the following reason. It is well known that for a symmetric spectrum resulting from a Γ-space (see Section 2.2.2.1) the two notions of homotopy groups coincide (such spectra are called *semistable*). Whenever it seems important to know which notion of homotopy groups we use, we will have identified the corresponding spectrum as one resulting from a Γ-space, so that no confusion will result from this.

2.2.2.1 Γ-spaces

Recall that a Γ-space [57] is a functor

$$A : \Gamma^{\mathrm{op}} \to \mathbf{Top}_* ,$$

where Γ^{op} is the category of finite pointed sets and pointed maps, and $\mathbf{Top}_*$ is the category of pointed spaces. The category Γ^{op} has a skeletal subcategory whose objects are the finite ordinals $\mathbf{n}^+ = \{0, 1, \ldots, n\}$ based at $0 \in \mathbf{n}^+$. In the definition of a Γ-space we require $A(\mathbf{0}^+) = \mathrm{pt}$.

Γ-spaces give rise to connective spectra [57, §3]. A Γ-space A can be prolonged from a functor on Γ^{op} to a continuous functor defined on all CW-complexes [35] and one obtains a spectrum $A(\mathbb{S})$ by letting the n-th space be $A(\mathbb{S})_n := A(S^n)$. The structure map $A(\mathbb{S})_n \wedge S^1 \to A(\mathbb{S})_{n+1}$ can be obtained as the adjoint of the map $S^1 \to F(A(\mathbb{S})_n, A(\mathbb{S})_{n+1})$ given by $x \mapsto A(y \mapsto x \wedge y)$ for $x \in S^1$ and $y \in S^n$ and where F is the based mapping space. In fact, $A(\mathbb{S})$ has the structure of a symmetric spectrum [22] where the action of the symmetric group Σ_n on $A(\mathbb{S})_n$ is induced from the natural action on S^n, thinking of S^n as the one-point compactification of $\mathbb{R}^n$.

For example, connective topological K-theory can be described by a Γ-space $\mathcal{K}$ as follows. Let $\mathbb{C}^\infty$ be equipped with the standard hermitian inner product. The Γ-space $\mathcal{K}$ associates to a finite set $S \in \Gamma^{\mathrm{op}}$ the space of configurations of finite dimensional, pairwise orthogonal planes in $\mathbb{C}^\infty$ indexed by the elements of S. The basepoint of S is assigned the trivial vector space. For a morphism $\alpha : S \to T$ in Γ^{op} the map $\mathcal{K}(\alpha) : \mathcal{K}(S) \to \mathcal{K}(T)$ is defined in an evident way by forming the direct sum of orthogonal planes. The Γ-space $\mathcal{K}$ has an underlying H-space given by

$$\mathcal{K}(\mathbf{1}^+) = \coprod_{n \geq 0} \mathrm{Gr}_n(\mathbb{C}^\infty)\,,$$

where $\mathrm{Gr}_n(\mathbb{C}^\infty)$ is the Grassmannian of n-dimensional planes in $\mathbb{C}^\infty$. The space $\mathcal{K}(\mathbb{S})_1$ is then the classifying space of this H-space in the sense of [57] and the adjoint structure map $\mathcal{K}(\mathbf{1}^+) \to \Omega\mathcal{K}(\mathbb{S})_1$ is a homotopy group-completion. The spectrum $\mathcal{K}(\mathbb{S})$ is a model for connective complex K-theory ku, in particular $\Omega\mathcal{K}(\mathbb{S})_1 \simeq \mathbb{Z} \times BU$.

That $\mathcal{K}(\mathbf{1}^+)$ is an H-space uses the fact that $\mathcal{K}$ is special. Recall that a Γ-space A is *special* if for every $n \geq 1$ the map

$$A(\mathbf{n}^+) \to A(\mathbf{1}^+) \times \cdots \times A(\mathbf{1}^+) \quad (n \text{ factors})$$

whose j-th component is induced by $\pi_j : \mathbf{n}^+ \to \mathbf{1}^+$, $\pi_j(i) = \delta_{ij}$ is a weak equivalence. If these maps are honest homotopy equivalences (for example, if the Γ-space is special and the spaces in question have the homotopy type of a CW-complex) one obtains an H-space multiplication on $A(\mathbf{1}^+)$ by going from left to right in the diagram

$$A(\mathbf{1}^+) \times A(\mathbf{1}^+) \xleftarrow{\pi_1 \times \pi_2} A(\mathbf{2}^+) \to A(\mathbf{1}^+)\,.$$

The second arrow is induced from the map $\mathbf{2}^+ \to \mathbf{1}^+$ sending $1, 2 \mapsto 1$. If A is a special Γ-space, then $A(\mathbb{S})$ is an Ω-spectrum above the zero space, that is the adjoints of the structure maps $A(\mathbb{S})_n \wedge S^1 \to A(\mathbb{S})_{n+1}$ are weak homotopy equivalences for all $n \geq 1$, see for example [10, Thm. 4.4]. The homotopy groups of the spectrum $A(\mathbb{S})$ are then determined by the unstable homotopy groups of the space $A(\mathbb{S})_1$.

2.2.2.2 Unitary deformation K-theory

In analogy with the construction of ku above, Lawson defines in [29] a Γ-space

$$\mathcal{K}G : \Gamma^{\mathrm{op}} \to \mathbf{Top}_*$$

for the unitary deformation K-theory of G by considering configurations of orthogonal planes in $\mathbb{C}^\infty$ labelled by unitary representations of G. We denote by $\mathscr{L}(\mathbb{C}^n, \mathbb{C}^\infty)$ the space of linear isometric embeddings $\mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$ equipped with the compact open topology. If $S \in \Gamma^{\mathrm{op}}$ is a finite set, then $\mathcal{K}G(S)$ is topologised as a subspace of

$$X_{G,S} := \prod_{a \in S} \coprod_{n \geq 0} \mathscr{L}(\mathbb{C}^n, \mathbb{C}^\infty) \times_{U(n)} \mathrm{Hom}(G, U(n)) \,.$$

The unitary group $U(n)$ acts on $\mathscr{L}(\mathbb{C}^n, \mathbb{C}^\infty)$ by precomposition and it acts on the space of homomorphisms $\mathrm{Hom}(G, U(n))$ by conjugation. A point $(i_a, \rho_a)_{a \in S} \in X_{G,S}$ lies in the subspace $\mathcal{K}G(S) \subset X_{G,S}$ precisely if the images of i_a and $i_{a'}$ in $\mathbb{C}^\infty$ are orthogonal whenever $a \neq a'$ and if ρ_a is the unique zero dimensional representation whenever $a \in S$ is the basepoint.

Two linear isometric embeddings $i : \mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$ and $j : \mathbb{C}^m \hookrightarrow \mathbb{C}^\infty$ whose images are orthogonal subspaces of $\mathbb{C}^\infty$ define a direct sum embedding

$$i \oplus j : \mathbb{C}^{m+n} \cong \mathbb{C}^m \oplus \mathbb{C}^n \hookrightarrow \mathbb{C}^\infty \,.$$

This is used to define $\mathcal{K}G$ on the morphisms of Γ^{op}. Given a map of finite pointed sets $\alpha : S \to T$ we can define a map $\mathcal{K}G(\alpha) : \mathcal{K}G(S) \to \mathcal{K}G(T)$ by

$$\mathcal{K}G(\alpha)((i_a, \rho_a)_{a \in S}) := \Big(\bigoplus_{a \in \alpha^{-1}(b)} (i_a, \rho_a) \Big)_{b \in T} \,, \tag{2.6}$$

where $(i_a, \rho_a) \oplus (i_{a'}, \rho_{a'}) := (i_a \oplus i_{a'}, \rho_a \oplus \rho_{a'})$. This defines a special Γ-space $\mathcal{K}G$. In fact, this is precisely the Γ-space associated - in the manner described in [57, §2] - to the permutative category whose objects are finite dimensional unitary representations of G and whose morphisms are the isomorphisms. If G is the trivial group, then the construction specializes to that of ku described above, since $\mathscr{L}(\mathbb{C}^n, \mathbb{C}^\infty)/U(n) = \mathrm{Gr}_n(\mathbb{C}^\infty)$. Also note that the Γ-space $\mathcal{K}G$ is natural in G with respect to homomorphisms of groups.

It was observed by Lawson that for every $k \geq 0$ the tensor product of representations yields a continuous and natural multiplication map

$$\mathscr{L}((\mathbb{C}^\infty)^{\otimes k}, \mathbb{C}^\infty)_+ \wedge \mathcal{K}G \wedge \cdots \wedge \mathcal{K}G \xrightarrow{\otimes} \mathcal{K}G\,, \tag{2.7}$$

where the smash product is formed in the topologically enriched permutative category of Γ-spaces, cf. [34]. These multiplication maps, which are parametrized by contractible spaces, can be fed into the machine of Elmendorf-Mandell [14] which associates to $\mathcal{K}G$ an E_∞-ring object in the category of symmetric spectra. The notion of an E_∞-ring object in symmetric spectra was introduced in [14]. Loosely speaking, this is a symmetric spectrum which satisfies the axioms of a commutative symmetric ring spectrum up to coherent homotopy.

The general theory of [14] allows one to rigidify this spectrum. Thus it was shown in [29, §7] that there is a commutative symmetric ring spectrum $\mathrm{k}^{\mathrm{def}}\, G$ and a natural zig-zag of stable equivalences between $\mathrm{k}^{\mathrm{def}}\, G$ and $\mathcal{K}G(\mathbb{S})$. Moreover, the map $\mathrm{k}^{\mathrm{def}}\, 1 \to \mathrm{k}^{\mathrm{def}}\, G$ induced by the trivial homomorphism $G \to 1$ makes $\mathrm{k}^{\mathrm{def}}\, G$ into a commutative algebra spectrum over connective complex K-theory, because $\mathrm{k}^{\mathrm{def}}\, 1$ is a model for ku. Thus we can summarize by saying that deformation K-theory yields a functor

$$\mathrm{k}^{\mathrm{def}} : \begin{Bmatrix} \text{finitely generated} \\ \text{discrete groups} \end{Bmatrix}^{\mathrm{op}} \longrightarrow ku\text{-Alg}\,, \tag{2.8}$$

where ku-Alg denotes the category of ku-algebra symmetric spectra.

In order to state Lawson's theorem we need another gadget. Consider the commutative semi-ring space

$$\mathrm{Rep}(G) := \coprod_{n \geq 0} \mathrm{Hom}(G, U(n))/U(n)\,. \tag{2.9}$$

The structure of $\mathrm{Rep}(G)$ as an abelian monoid gives rise, by [57, §1], to a special Γ-space which we denote by $\mathcal{R}G$. The ring multiplication on $\mathrm{Rep}(G)$, which is induced by the tensor product of representations, equips $\mathcal{R}G$ with a multiplicative structure. The symmetric spectrum associated to $\mathcal{R}G$ was introduced in [28]. It is called the *deformation representation ring spectrum* of G and it is denoted by $R[G]$. It can be seen as a topological analogue of the discrete representation ring for G. Thus we have a functor $R[-]$ which takes a finitely generated discrete group G to a commutative symmetric $H\mathbb{Z}$-algebra $R[G]$.

There is a natural map $\mathcal{K}G \to \mathcal{R}G$ which is induced by taking a pair (f, ρ) consisting of an isometric embedding $f : \mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$ and a representation $\rho : G \to U(n)$ to the isomorphism class of the representation. This gives rise to a map of commutative ku-algebras $\mathrm{k}^{\mathrm{def}}\, G \to R[G]$, where $R[G]$ is regarded as a ku-algebra via the linearization map $ku \to H\mathbb{Z}$.

The adjunction between restriction and extension of scalars,

$$\mathrm{map}_{ku\text{-Mod}}(\mathrm{k}^{\mathrm{def}}\, G, R[G]) \cong \mathrm{map}_{H\mathbb{Z}\text{-Mod}}(H\mathbb{Z} \wedge_{ku} \mathrm{k}^{\mathrm{def}}\, G, R[G])\,,$$

defines a map $H\mathbb{Z} \wedge_{ku} \mathrm{k}^{\mathrm{def}}\, G \to R[G]$. Let $(-)_\flat$ denote a cofibrant replacement in the category of ku-module symmetric spectra.

Theorem 2.2.3 (Lawson, [30, Thm. 3])**.** *The adjoint of the ku-module map* $\mathrm{k}^{\mathrm{def}}\, G \to R[G]$ *induces a stable equivalence of symmetric spectra*

$$H\mathbb{Z} \wedge_{ku} \mathrm{k}^{\mathrm{def}}\, G_\flat \xrightarrow{\sim} R[G]\,.$$

Let $u \in \pi_2(ku)$ be the Bott element. By smashing the Bott periodicity cofibre sequence $\Sigma^2 ku \to ku \to H\mathbb{Z}$ over ku with $\mathrm{k}^{\mathrm{def}}\, G$ one obtains the following corollary.

Corollary 2.2.4 (Lawson, [30, Cor. 4])**.** *There is a homotopy cofibre sequence of ku-modules*

$$\Sigma^2\, \mathrm{k}^{\mathrm{def}}\, G \xrightarrow{u\cdot} \mathrm{k}^{\mathrm{def}}\, G \longrightarrow R[G]\,,$$

where the first map is "multiplication by the Bott element".

2.2.3 The commutative K-theory spectrum

We next define a spectrum for commutative complex K-theory and identify its homotopy type. It will be defined as a simplicial deformation K-theory spectrum whose underlying infinite loop space is homotopy equivalent to the homotopy orbit space $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$. We apply Theorem 2.2.3 to identify the spectrum as the ku-group ring spectrum of $\mathbb{C}P^\infty$. We present a computation of the (well known) homotopy ring, the connective K-Pontrjagin ring of $\mathbb{C}P^\infty$, and after that we identify the homotopy type of $B_{\mathrm{com}}U$ and $B_{\mathrm{com}}SU$ as infinite loop spaces. At the end of this section we describe generators for the commutative K-theory group of S^4.

2.2.3.1 Definition

Regarding notation it will be easiest to co-represent the construction $B_{\mathrm{com}}U$ by a cosimplicial group as follows. Let F_k denote the free group on k generators $x_1, \ldots, x_k$.

Definition 2.2.5. Define a cosimplicial group $k \mapsto F_k$ with co-face maps $d^i : F_{k-1} \to F_k$ given by

$$d^i x_j = \begin{cases} x_j\,, & j < i \\ x_j x_{j+1}\,, & j = i \\ x_{j+1}\,, & j > i \end{cases}$$

and co-degeneracy maps $s^i : F_{k+1} \to F_k$ given by

$$s^i x_j = \begin{cases} x_j\,, & j < i+1 \\ 1\,, & j = i+1 \\ x_{j-1}\,, & j > i+1 \end{cases}$$

for $0 \le i \le k$. We also define the cosimplicial abelian group $k \mapsto \mathbb{Z}^k$ as the levelwise abelianisation of F_*. It comes with a natural transformation $F_* \to \mathbb{Z}^*$.

The cosimplicial identities are easily verified, and the simplicial object $\mathrm{Hom}(\mathbb{Z}^*, U)$ is precisely the simplicial space $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, U)$ defining $B_{\mathrm{com}}U$.

Definition 2.2.6. Define a commutative ku-algebra by $E := |\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*|$.

At this point we should be a bit more precise about the meaning of $|\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*|$. Recall that the spectrum $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^k$ was obtained from a special Γ-space $\mathcal{K}\mathbb{Z}^k$. Now the Γ-space itself takes values in topological spaces, but the construction of the deformation K-theory spectrum $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^k$ in [29] involves the passage from spaces to simplicial sets. Namely, it goes via the Γ-space $\mathrm{Sing}(\mathcal{K}\mathbb{Z}^k)$ taking values in simplicial sets, which is obtained from $\mathcal{K}\mathbb{Z}^k$ by composing with the singular complex functor. In particular, the functor $\mathrm{k}^{\mathrm{def}}$ from (2.8) takes values in symmetric spectra of simplicial sets, i.e. the n-th level of $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^k$ is a simplicial set. The n-th level in the simplicial spectrum $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*$ is then a bisimplicial set, and E is the symmetric spectrum of spaces obtained by levelwise geometric realisation.

Now replacing a topological space by the geometric realisation of its singular complex results in a weakly equivalent space. However, if we do this for every level of a simplicial space, we should check that the homotopy type of the simplicial space is wellbehaved under geometric realisation. This is the reason why in the next lemma we will have to check a simple cofibrancy condition.

The assignment $k \mapsto \mathcal{K}\mathbb{Z}^k$ defines a simplicial Γ-space $\mathcal{K}\mathbb{Z}^*$. By the geometric realisation $|\mathcal{K}\mathbb{Z}^*|$ we mean the Γ-space taking $S \mapsto |k \mapsto \mathcal{K}\mathbb{Z}^k(S)|$, $S \in \Gamma^{\mathrm{op}}$. This is again a special Γ-space, namely the one associated to the permutative category $\mathscr{C}$ of Definition 2.1.3. Thus the infinite loop space associated to the Γ-space $|\mathcal{K}\mathbb{Z}^*|$, that is the space $\Omega|\mathcal{K}\mathbb{Z}^*|(S^1)$, is homotopy equivalent to $\mathbb{Z} \times B_{\mathrm{com}}U \,/\!\!/\, U$, see Lemma 2.1.6.

Let $\Omega^\infty E = \mathrm{tel}_n\,\Omega^n E_n$ be the zero space of a symmetric Ω-spectrum stably equivalent to E, i.e. an infinite loop space for E.

Lemma 2.2.7. *There is a zig-zag of stable equivalences between E and the symmetric spectrum associated to the Γ-space $|\mathcal{K}\mathbb{Z}^*|$. In particular, there is a homotopy equivalence $\Omega^\infty E \simeq \mathbb{Z} \times B_{\mathrm{com}}U \,/\!\!/\, U$.*

Proof. Let $\mathrm{Sing}(\mathcal{K}\mathbb{Z}^*)$ denote the simplicial Γ-space $k \mapsto \mathrm{Sing}(\mathcal{K}\mathbb{Z}^k)$ which takes values in bisimplicial sets. Let K_* be the simplicial symmetric spectrum associated to this simplicial Γ-space. The rigidification in [29, §7] which relates the symmetric spectrum K_k to the symmetric spectrum $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^k$ is natural with respect to homomorphisms of groups. Thus there is a zig-zag of maps of simplicial symmetric spectra $K_* \leftarrow \cdots \rightarrow \mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*$ which are stable equivalences in each simplicial degree. By [59, Cor. 4.1.6] these induce a zig-zag of stable equivalences on geometric realisations, i.e. between E and $|K_*|$.

The Γ-space $|\mathcal{K}\mathbb{Z}^*|$ takes values in topological spaces. Let K denote the symmetric spectrum of spaces associated to it. Then K receives a map $|K_*| \to K$ which we need to show is a stable equivalence. Since both spectra are Ω-spectra above the zero level, it suffices to check that the map on level one spaces $|\mathrm{Sing}(\mathcal{K}\mathbb{Z}^*)(S^1)| \to |\mathcal{K}\mathbb{Z}^*(S^1)|$ is a weak homotopy equivalence. By [36, Thm. A.4] this is the case if the simplicial space $k \mapsto \mathcal{K}\mathbb{Z}^k(S^1)$ is *proper*, i.e. the inclusion of the subspace of degenerate simplices in each simplicial level is a closed cofibration. We show this in Lemma 2.2.8 and this finishes the proof. □

Lemma 2.2.8. *The simplicial space $k \mapsto \mathcal{K}\mathbb{Z}^k(S^1)$ is proper.*

Proof. We first show that for every fixed $m \geq 0$ the simplicial space $k \mapsto \mathcal{K}\mathbb{Z}^k(\mathbf{m}^+)$ is proper. To simplify the notation, let us write $Z_k := \mathcal{K}\mathbb{Z}^k(\mathbf{m}^+)$. An element of Z_k is an m-tuple of equivalence classes $([f_1, \underline{A}_1], \ldots, [f_m, \underline{A}_m])$, where each $f_i : \mathbb{C}^{n_i} \hookrightarrow \mathbb{C}^\infty$ is a linear isometric embedding for some integer $n_i \geq 0$, so that the images of all f_i are mutually orthogonal subspaces of $\mathbb{C}^\infty$. Each $\underline{A}_i := (A_{i1}, \ldots, A_{ik}) \in U(n_i)^k$ is a k-tuple of pairwise commuting unitary matrices of the appropriate dimension. The equivalence relation identifies two pairs $(f, \underline{A})$ and $(f', \underline{A}')$ if and only if the images of f and f' have the same dimension, say $n \geq 0$, and there exists $B \in U(n)$ with $\underline{A}' = B\underline{A}B^{-1}$ and $f' = f \circ B$. A simplex in Z_k is degenerate precisely if $A_{ij} = 1 \in U(n_i)$ for all $i \in \{1, \ldots, m\}$ and some $j \in \{1, \ldots, k\}$. We write $S_k \subset Z_k$ for the subspace of all degenerate simplices. We will show that (Z_k, S_k) is a *strong NDR-pair*, see [3, Def. 4.3].

For a given tuple $\underline{n} := (n_1, \ldots, n_m)$ with each $n_i \geq 0$ we write $U(\underline{n})$ for the product group $U(n_1) \times \cdots \times U(n_m)$. Consider the simplicial space $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, U(\underline{n}))$ and denote the subspace of degenerate simplices in degree k by $S(\underline{n})_k$. It follows from [3, Thm. 4.8] applied to the case $G = U(\underline{n})$ and $K = 1$ that $(\mathrm{Hom}(\mathbb{Z}^k, U(\underline{n})), S(\underline{n})_k)$ is a strong $U(\underline{n})$-equivariant NDR-pair. Let $(h_{\underline{n}}, u_{\underline{n}})$ be a $U(\underline{n})$-NDR representation for this pair. We now define a representation (h, u) of (Z_k, S_k) as a strong NDR-pair. For an element $([f_i, \underline{A}_i])_{1 \leq i \leq m} \in Z_k$ we write $\underline{n}$ for the array whose i-th component is the dimension of f_i. There is a natural identification

$$\mathrm{Hom}(\mathbb{Z}^k, U(\underline{n})) \cong \mathrm{Hom}(\mathbb{Z}^k, U(n_1)) \times \cdots \times \mathrm{Hom}(\mathbb{Z}^k, U(n_m))\,. \tag{2.10}$$

For $1 \leq i \leq m$ let pr_i be the projection onto the i-th factor in (2.10). We define the homotopy $h : Z_k \times I \to Z_k$ by

$$h(([f_i, \underline{A}_i])_{1\leq i\leq m}, t) = ([f_i, \mathrm{pr}_i(h_{\underline{n}}((\underline{A}_j)_{1\leq j\leq m}, t))])_{1\leq i\leq m}\,.$$

Here we regard $(\underline{A}_i)_{1\leq i\leq m}$ as an element of $\mathrm{Hom}(\mathbb{Z}^k, U(\underline{n}))$ using (2.10). The level function $u : Z_k \to I$ is defined by

$$u(([f_i, \underline{A}_i])_{1\leq i\leq m}) = u_{\underline{n}}((\underline{A}_j)_{1\leq j\leq m})\,.$$

Both functions are welldefined, since $h_{\underline{n}}$ and $u_{\underline{n}}$ are $U(\underline{n})$-equivariant. It remains to check the conditions (i)-(iv) in [3, Def. 4.1] of an NDR-pair and [3, Def. 4.3] of a strong NDR-pair. For this we note, that under the identification (2.10) degenerate simplices in $\mathrm{Hom}(\mathbb{Z}^k, U(\underline{n}))$ correspond precisely to those tuples $(\underline{A}_i)_{1\leq i\leq m}$ for which there exists $j \in \{1, \ldots, k\}$ such that $A_{ij} = 1$ for all $i \in \{1, \ldots, m\}$. Then all the conditions for an NDR-pair are easy consequences of the definitions and the fact that $(h_{\underline{n}}, u_{\underline{n}})$ is a $U(\underline{n})$-equivariant NDR-representation for $(\mathrm{Hom}(\mathbb{Z}^k, U(\underline{n})), S(\underline{n})_k)$. We conclude that $k \mapsto \mathcal{K}\mathbb{Z}^k(\mathbf{m}^+)$ is proper for every fixed $m \geq 0$.

Now let $\Delta[1]/\partial\Delta[1]$ be the simplicial circle with exactly m non-basepoint simplices in degree m, i.e. the set of m-simplices is $\mathbf{m}^+$. Evaluating $\mathcal{K}\mathbb{Z}^*$ levelwise we obtain the bisimplicial space

$$(k, m) \mapsto \mathcal{K}\mathbb{Z}^k(\mathbf{m}^+) \subset \prod_{a\in\mathbf{m}^+} \coprod_{n\geq 0} \mathscr{L}(\mathbb{C}^n, \mathbb{C}^\infty) \times_{U(n)} \mathrm{Hom}(\mathbb{Z}^k, U(n))\,.$$

The basepoint of the coproduct on the right hand side is disjoint. This implies that every degeneracy map $\mathcal{K}\mathbb{Z}^k(\mathbf{m}^+) \to \mathcal{K}\mathbb{Z}^k(\mathbf{m}+\mathbf{1}^+)$ is the inclusion of a connected component and thus in particular a closed cofibration. This means that for fixed $k \geq 0$ the simplicial space $\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])$ is good in the sense of Segal. Likewise $s\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])$ is a good simplicial space, where we write $s\mathcal{K}\mathbb{Z}^k(\mathbf{m}^+) \subset \mathcal{K}\mathbb{Z}^k(\mathbf{m}^+)$ for the subspace of simplices which are degenerate in the k-direction. The map of simplicial spaces $s\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1]) \to \mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])$, is levelwise a closed cofibration, since $\mathcal{K}\mathbb{Z}^*(\mathbf{m}^+)$ is proper for every $m \geq 0$, as we have previously shown. It follows from [65, 14-5] that the map induced on geometric realisations $|s\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])| \to |\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])|$ is a closed cofibration. Commuting iterated coends we have $|\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])| \cong \mathcal{K}\mathbb{Z}^k(S^1)$ and similarly for $|s\mathcal{K}\mathbb{Z}^k(\Delta[1]/\partial\Delta[1])|$. Thus, the simplicial space $\mathcal{K}\mathbb{Z}^*(S^1)$ is proper, as claimed. $\square$

2.2.3.2 The Bott cofibre sequence

In this section we identify the spectrum E as a ku-algebra.

Lemma 2.2.9. *There is a stable equivalence of $H\mathbb{Z}$-algebras $|R[\mathbb{Z}^*]| \simeq H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$.*

Proof. Fix $k \geq 0$. The symmetric spectrum $R[\mathbb{Z}^k]$ is determined by the Γ-space associated to the abelian semi-ring $\mathrm{Rep}(\mathbb{Z}^k)$ of (2.9). The spectral theorem implies that there is a natural homeomorphism

$$\mathrm{Rep}(\mathbb{Z}^k) \cong SP^\infty((S^1)^k_+)\,, \tag{2.11}$$

where $SP^\infty((S^1)^k_+)$ is the free abelian monoid generated by the space $(S^1)^k$ with a disjoint basepoint, cf. [3, Thm. 6.1]. The basepoint serves as the additive unit. There is a multiplication map $(S^1)^k_+ \wedge (S^1)^k_+ \cong ((S^1)^k \times (S^1)^k)_+ \to (S^1)^k_+$ using the group structure of $(S^1)^k$. This defines a multiplication map

$$SP^\infty((S^1)^k_+) \wedge SP^\infty((S^1)^k_+) \to SP^\infty((S^1)^k_+)$$

via $(\sum_i n_i x_i) \wedge (\sum_k m_k y_k) \mapsto \sum_{i,k} n_i m_k \, x_i y_k$ which makes $SP^\infty((S^1)^k_+)$ into an abelian semi-ring in such a way that (2.11) is an isomorphism of semi-rings. By the Dold-Thom theorem $SP^\infty((S^1)^k_+)$ represents the reduced integral Pontrjagin ring of $(S^1)^k_+$. An explicit identification in the context of symmetric spectra can be found, for example, in [55, Def. 6.24]. The argument there shows that the commutative symmetric ring spectrum associated to $SP^\infty((S^1)^k_+)$ is stably equivalent to $H\mathbb{Z}\wedge(S^1)^k_+$ as an $H\mathbb{Z}$-algebra. The stable equivalence $|R[\mathbb{Z}^*]| \simeq H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$ follows from this upon geometric realisation. □

The following lemma simply describes the geometric realisation of Lawson's Bott cofibre sequence applied to the cosimplicial group $\mathbb{Z}^*$, cf. Corollary 2.2.4. We include a proof for completeness.

Lemma 2.2.10. *There is a homotopy cofibre sequence of ku-modules*

$$\Sigma^2 E \xrightarrow{u\cdot} E \xrightarrow{\lambda} H\mathbb{Z} \wedge \mathbb{C}P^\infty_+\,,$$

in which the first map is multiplication by the Bott element and the second map corresponds to (2.2).

Proof. Models for ku and $H\mathbb{Z}$ are given, respectively, by $\mathrm{k}^{\mathrm{def}}\,1$ and $R[1]$, where 1 is the trivial group. Let $p : ku \to H\mathbb{Z}$ be the linearisation map and let $\beta : \Sigma^2 ku \to ku$ be multiplication by the Bott element. By Bott periodicity the sequence $\Sigma^2 ku \xrightarrow{\beta} ku \xrightarrow{p} H\mathbb{Z}$ is a homotopy cofibre sequence. The choice of a null homotopy determines a factorization of $p \circ \beta$ through the mapping cone $C\beta$ of β. This null homotopy can be chosen through maps of ku-modules. Let $(-)_\flat$ denote a functorial cofibrant replacement in the category of ku-modules and let $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat \to \mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*$ be a levelwise cofibrant replacement. There is a stable equivalence $|\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat| \simeq E$. Smashing the defining pushout diagram for the mapping cone $C\beta$ over ku with $\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat$ gives the pushout diagram

$$\begin{array}{ccc} \Sigma^2\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat & \longrightarrow & \mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat \\ \downarrow & & \downarrow \\ I \wedge \Sigma^2\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat & \longrightarrow & C\beta \wedge_{ku} \mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat \end{array}$$

which, after geometric realisation, exhibits $C\beta \wedge_{ku} |\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat|$ as the homotopy cofibre of the Bott map $\Sigma^2 E \to E$. The stable equivalence of ku-modules $C\beta \simeq H\mathbb{Z}$ together with Theorem 2.2.3 yields stable equivalences

$$C\beta \wedge_{ku} |\,\mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat| \simeq |H\mathbb{Z} \wedge_{ku} \mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^*_\flat| \simeq |R[\mathbb{Z}^*]|\,.$$

This finishes the proof in view of Lemma 2.2.9. □

The following result is the key to understanding the homotopy type of $B_{\mathrm{com}}U$.

Theorem 2.2.11. *There is a stable equivalence of commutative ku-algebras*

$$E \simeq ku \wedge \mathbb{C}P^\infty_+\,.$$

Proof. There is a map of commutative symmetric ring spectra $j : \Sigma^\infty \mathbb{C}P^\infty_+ \to E$ which is induced by the canonical map $\mathbb{C}P^\infty \to B_{\mathrm{com}}U$, i.e. by the inclusion $\{\text{line bundles}\} \subset \{\text{commutative } K\text{-theory}\}$. Recall from Section 2.2.2.2 that if $S \in \Gamma^{\mathrm{op}}$, then a point in the space $\mathcal{K}\mathbb{Z}^k(S)$ is an S-indexed tuple $(V_a, \rho_a)_{a\in S}$ of finite dimensional mutually orthogonal inner product spaces $V_a \subset \mathbb{C}^\infty$ with representations $\rho_a : \mathbb{Z}^k \to U(V_a)$ on them. Now let $\Gamma_{(S^1)^k}$ be the Γ-space which is obtained from $\mathcal{K}\mathbb{Z}^k$ by specifying in addition the data of an unordered orthonormal frame $\{v_{a,1}, \ldots, v_{a,n_a}\} \subset V_a$, $n_a = \dim V_a$, for each $a \in S$ so that the representation ρ_a is diagonal with respect to this frame. The Γ-space $\Gamma_{(S^1)^k}$ has as underlying H-space

$$\Gamma_{(S^1)^k}(\mathbf{1}^+) = \coprod_{n\geq 0} \mathscr{L}(\mathbb{C}^n, \mathbb{C}^\infty) \times_{\Sigma_n} ((S^1)^k)^n\,,$$

the free E_∞-algebra on the space $(S^1)^k$. Moreover, it has multiplication maps similar to (2.7) and thus leads to the group ring spectrum $\Gamma_{(S^1)^k}(\mathbb{S}) = \Sigma^\infty (S^1)^k_+$. There is now an obvious forgetful map of Γ-spaces $\Gamma_{(S^1)^k} \to \mathcal{K}\mathbb{Z}^k$ yielding a map of commutative symmetric ring spectra $\Sigma^\infty (S^1)^k_+ \to \mathrm{k}^{\mathrm{def}}\,\mathbb{Z}^k$. The map $j : \Sigma^\infty \mathbb{C}P^\infty_+ \to E$ is then induced on geometric realisations.

The composite map $\Sigma^\infty \mathbb{C}P^\infty_+ \xrightarrow{j} E \xrightarrow{\lambda} H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$ is easily seen to be the natural one, representing the stable Hurewicz map for $\mathbb{C}P^\infty$. Since E is a ku-algebra, j extends over the free ku-algebra $ku \wedge \mathbb{C}P^\infty_+$. Thus we obtain a sequence

$$ku \wedge \mathbb{C}P^\infty_+ \xrightarrow{j'} E \xrightarrow{\lambda} H\mathbb{Z} \wedge \mathbb{C}P^\infty_+ \,,$$

whose composite is the smash product of the linearization map $ku \to H\mathbb{Z}$ with $\mathbb{C}P^\infty_+$. If we combine this with Lemma 2.2.10, we obtain a map of homotopy cofibre sequences

$$\begin{array}{ccccc} \Sigma^2 ku \wedge \mathbb{C}P^\infty_+ & \longrightarrow & ku \wedge \mathbb{C}P^\infty_+ & \longrightarrow & H\mathbb{Z} \wedge \mathbb{C}P^\infty_+ \\ \downarrow{\scriptstyle \Sigma^2 j'} & & \downarrow{\scriptstyle j'} & & \| \\ \Sigma^2 E & \longrightarrow & E & \xrightarrow{\lambda} & H\mathbb{Z} \wedge \mathbb{C}P^\infty_+ \end{array}$$

where the top cofibering is obtained from the Bott periodicity sequence by smashing (in the derived sense) with $\mathbb{C}P^\infty_+$. Inductively, using the five lemma, we see that j' induces an isomorphism of homotopy groups, hence is a stable equivalence. □

Remark 2.2.12. It is clear that one can make Definition 2.2.6 for other cosimplicial groups which are finitely generated in every degree. Spaces of homomorphisms from a cosimplicial group into a compact Lie group are considered in [63]. We expect that the analogue of Lemma 2.2.7 holds in this more general setting. This suggests that the nilpotent K-theories introduced in [6] as well as the infinite loop spaces constructed in [63, §3] could be studied by the same methods. In particular, one should obtain analogous homotopy cofibre sequences from Lawson's theorem.

2.2.3.3 The homotopy ring of $B_{\mathrm{com}}U$

Because of Lemma 2.2.7, the homotopy ring of the spectrum E computes the homotopy ring of the space $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$. Moreover, by (2.1) the homotopy fibre sequence

$$B_{\mathrm{com}}U \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \to \mathbb{Z} \times BU$$

is split, and the inclusion of the fibre is a map of additive and mutliplicative H-spaces. Thus the homotopy ring $\pi_*(B_{\mathrm{com}}U)$ is contained in π_*E as a subring.

By Theorem 2.2.11, the ring π_*E is the connective K-Pontrjagin ring of $\mathbb{C}P^\infty$,

$$\pi_*E \cong ku_*(\mathbb{C}P^\infty) \,.$$

The structure of this ring is well known. If we think of BU as the second term in the spectrum ku, then the canonical map $\mathbb{C}P^\infty \to BU$ determines a class $x \in ku^2(\mathbb{C}P^\infty)$. Let $y_i \in ku_{2i}(\mathbb{C}P^\infty)$ be dual to $x^i \in ku^{2i}(\mathbb{C}P^\infty)$. It follows from the Atiyah-Hirzebruch spectral sequence that $ku_*(\mathbb{C}P^\infty)$ is a free $\pi_*(ku)$-module on the generators y_i for $i \geq 0$ (where $y_0 \in ku_0(\mathbb{C}P^\infty)$ is the unit), see [1]. We see from [50, Thm. 3.4] that the y_i satisfy the following relation

$$y_1 y_i = (i+1)y_{i+1} + iuy_i \,, \tag{2.12}$$

where $u \in \pi_2(ku)$ is the Bott element. Let $P = \mathbb{Z}[u, y_i \,|\, i \geq 1]$ be the polynomial algebra. Considering dimensions then shows that

$$ku_*(\mathbb{C}P^\infty) \cong P/(\mathscr{I} \otimes \mathbb{Q}) \cap P \,,$$

where $\mathscr{I} \subset P$ is the ideal generated by all relations of the form (2.12).

The map $ku_*(\mathbb{C}P^\infty) \to ku_*(\mathrm{pt}) = \pi_*(ku)$ sends $y_i \mapsto 0$. This shows:

Theorem 2.2.13. *The homotopy ring $\pi_*(B_{\mathrm{com}}U)$ is the ideal $(y_i \,|\, i \geq 1) \subset ku_*(\mathbb{C}P^\infty)$. In particular, the homotopy groups of the space $B_{\mathrm{com}}U$ are as follows,*

$$\pi_{2i}(B_{\mathrm{com}}U) = \mathbb{Z}^i,$$
$$\pi_{2i+1}(B_{\mathrm{com}}U) = 0$$

for all $i \geq 0$.

A computation of π_*E

Before we had identified the homotopy type of E as in Theorem 2.2.11, we computed the ring π_*E only using the cofibering from Lemma 2.2.10 and the power operations introduced in Section 1.2. In retrospect this computation seems far too long-winded. Nevertheless we shall include here a summary of it, because the essential idea of formulating it as an extension problem may be interesting.

Recall from Section 1.2 that for $k \in \mathbb{Z}$ the map $\phi^k : B_{\mathrm{com}}U \to B_{\mathrm{com}}U$ is the self-map induced by the k-th power map in U. In terms of vector bundles it represents the operation which takes a commuting cocycle to its k-th power. Below Definition 2.2.5 we co-represented the simplicial space $B_{\mathrm{com}}U$ by the cosimplicial group $\mathbb{Z}^*$, i.e. $B_{\mathrm{com}}U = |\mathrm{Hom}(\mathbb{Z}^*, U)|$. The map ϕ^k is then the map induced by the endomorphism $kI_* : \mathbb{Z}^* \to \mathbb{Z}^*$ which is multiplication by k in every degree. It is now obvious from Definition 2.2.6 that the map ϕ^k extends to a ku-algebra operation on E. By abuse of notation, we also write $\phi^k : E \to E$ for this operation. Under the map $\lambda : E \to H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$ the operation ϕ^k descends to an honest Adams operation on $\mathbb{C}P^\infty$, i.e. to the self map of $H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$ induced by the k-th power map in the abelian group $\mathbb{C}P^\infty$. This can be seen most easily from the explicit description of the space level map $\lambda : B_{\mathrm{com}}U \to SP^\infty\mathbb{C}P^\infty$ in (2.2). Again abusing the notation, we denote this induced operation on $H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$ by the same symbol ϕ^k. An easy standard computation shows:

Lemma 2.2.14. *The operation ϕ^k acts on $H_{2n}(\mathbb{C}P^\infty, \mathbb{Z})$ as multiplication by k^n.*

Remark 2.2.15. In view of Theorem 2.2.11 we know, of course, that $\phi^k : E \to E$ just corresponds to the self map of $ku \wedge \mathbb{C}P^\infty_+$ induced by the k-th power map in $\mathbb{C}P^\infty$.

Let us explain the basic idea for the computation of $\pi_* E$. The long exact sequence of homotopy groups associated to the homotopy cofibre sequence of Lemma 2.2.10 breaks up into short exact sequences of even degree homotopy groups,

$$0 \to \pi_{2n-2}E \xrightarrow{u\cdot} \pi_{2n}E \xrightarrow{\lambda} H_{2n}(\mathbb{C}P^\infty, \mathbb{Z}) \to 0\,, \tag{2.13}$$

because the homology of $\mathbb{C}P^\infty$ is concentrated in even degrees. The map $\lambda : \pi_* E \to H_*(\mathbb{C}P^\infty, \mathbb{Z})$ is a map of graded rings. The Pontrjagin ring $H_*(\mathbb{C}P^\infty, \mathbb{Z})$ is well known, so the computation of $\pi_* E$ amounts to solving the multiplicative extensions in (2.13). Clearly, as short exact sequences of abelian groups the sequences in (2.13) are all split, because $H_{2n}(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}$ is free. However, we may as well regard (2.13) as a short exact sequence of modules over the ring of operations generated by the ϕ^k, $k \in \mathbb{Z}$. In this category, the extensions (2.13) are in general non-trivial, which therefore introduces a second extension problem. As the power operations are multiplicative, the two extension problems are closely related. We will obtain their solution by simultaneous induction.

To do this we want to understand the inclusion map $B_{\mathrm{com}}U \to BU$ as a map of modules over connective K-theory. We claim that this is a module map if BU is regarded as a module over itself in the natural way. To make this precise, we define the auxiliary spectrum $F := |\,\mathrm{k}^{\mathrm{def}} F_*|$, where F_* is the cosimplicial group of Definition 2.2.5. In degree k it is the free group F_k on k generators. For this spectrum we have the infinite loop space

$$\Omega^\infty F \simeq \mathbb{Z} \times BU /\!\!/ U \simeq (\mathbb{Z} \times BU) \times BU\,. \tag{2.14}$$

The splitting is obtained in the same way as in (2.1). There is an obvious ku-algebra map $i : E \to F$ induced by abelianisation $F_* \to \mathbb{Z}^*$ which on the space level corresponds to the map extending the inclusion $B_{\mathrm{com}}U \to BU$ over the homotopy orbit. Let $u, x \in \pi_2 F$ be the Bott elements corresponding to the two factors in (2.14).

Lemma 2.2.16. *There is an isomorphism $\pi_* F \cong \mathbb{Z}[u, x]/(x^2 - ux)$.*

Sketch proof. First one shows that $F \simeq ku \vee \Sigma^2 ku$ as ku-modules. For this one may use [30, Prop. 6] which describes $\mathrm{k}^{\mathrm{def}} F_k$ as a ku-module. The equivalence implies that $\pi_* F$ is generated as a free $\mathbb{Z}[u]$-module by $1 \in \pi_0 F$ and the Bott element $x \in \pi_2 F$. Then we use the fact that the inclusion of the fibre $BU \to \mathbb{Z} \times BU /\!\!/ U$ is a map of additive and multiplicative H-spaces, and on homotopy groups in degree 4 we have a short exact sequence of groups $0 \to \mathbb{Z}\langle x^2 \rangle \to \mathbb{Z}\langle u^2, ux \rangle \to \mathbb{Z}\langle u^2 \rangle \to 0$, where the third arrow is the projection $ux \mapsto 0$. This shows that $x^2 = \pm ux$ in $\pi_4 F$. $\square$

Let us now summarize what we shall need for solving the multiplicative extensions in (2.13). For $k \in \mathbb{Z}$ and an integer $m \geq 0$ we write $k_{(m)}$ for the falling factorial, i.e. the integer

$$k_{(m)} = k(k-1)(k-2)\cdots(k-m+1)\,.$$

One may readily check that

$$(k+m-1)_{(2m-1)} = 0 \quad \text{if} \quad -(m-1) \leq k \leq m-1 \tag{2.15}$$

$$(k+m-1)_{(2m)} = 0 \quad \text{if} \quad -(m-1) \leq k \leq m\,. \tag{2.16}$$

Let us write $d_n = [\mathbb{C}P^n]$ for the canonical generator of $H_{2n}(\mathbb{C}P^\infty;\mathbb{Z})$. Recall that the Pontrjagin ring $H_*(\mathbb{C}P^\infty,\mathbb{Z})$ is the divided polynomial algebra $\Gamma_{\mathbb{Z}}[d_1]$. This is the quotient of the free graded commutative algebra on the generators d_n for $n \geq 1$ by the relations $d_n d_m = (n,m)d_{n+m}$ for all $n,m \geq 1$, where $(n,m) = (n+m)!/n!m!$ is the binomial coefficient.

Also recall that we have the following three maps of commutative ring spectra: The map $\lambda : E \to H\mathbb{Z} \wedge \mathbb{C}P^\infty_+$, an augmentation map $\varepsilon : E \to ku$ of the ku-algebra E (on the space level this is the projection map $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \to \mathbb{Z} \times BU$), and the forgetful map $i : E \to F$ from commutative K-theory to ordinary K-theory. They induce maps of graded rings which we denote by the same letters, so

$$\begin{aligned} \lambda &: \pi_*E \to \Gamma_{\mathbb{Z}}[d_1] \\ \varepsilon &: \pi_*E \to \mathbb{Z}[u] \\ i &: \pi_*E \to \mathbb{Z}[u,x]/(x^2-ux)\,. \end{aligned}$$

Proposition 2.2.17. *There exist unique classes $x_n \in \pi_{2n}E$ for all $n \geq 1$, so that*

(i) $\lambda(x_n) = d_n$ *for all* $n \geq 1$

(ii) $\varepsilon(x_n) = 0$ *for all* $n \geq 1$

(iii) $i(x_n) = 0$ *for all* $n \geq 2$

and the multiplicative relations

$$R_n := \begin{cases} x_1 x_{2m-1} = 2m\, x_{2m} + m\, u x_{2m-1}\,, & n = 2m-1 \\ x_1 x_{2m} = (2m+1)\, x_{2m+1} - m\, u x_{2m}\,, & n = 2m \end{cases}$$

hold for all $n \geq 1$. For $1 \leq i \leq n$ and $k \in \mathbb{Z}$ let the integer $c^k_{n,i} \in \mathbb{Z}$ be defined by

$$\phi^k(x_n) = \sum_{i=1}^{n} c^k_{n,i}\, u^{n-i} x_i\,.$$

For all $n \geq 1$ and all $k \in \mathbb{Z}$ they satisfy

$$C_n^k := \begin{cases} c_{2m-1,1}^k = \frac{(k+m-1)_{(2m-1)}}{(2m-1)!}\,, & n = 2m-1 \\ c_{2m,1}^k = \frac{(k+m-1)_{(2m)}}{(2m)!}\,, & n = 2m\,. \end{cases}$$

Proof. We will prove this statement by induction on n. For the base of the induction we will construct classes x_1 and x_2 satisfying (i)-(iii), and we will show that R_1 holds as well as C_n^k for $n = 1, 2$ and $k \in \mathbb{Z}$. The properties (i)-(iii) and R_1 will determine x_1 and x_2 uniquely.

Proof of the base case. For every $n \geq 1$ we have the exact sequence of abelian groups (2.13)

$$0 \longrightarrow \pi_{2n-2}E \xrightarrow{\cdot u} \pi_{2n}E \xrightarrow{\lambda} H_{2n}(\mathbb{C}P^\infty; \mathbb{Z}) \longrightarrow 0\,.$$

As the power operations are ku-module operations and descend under λ, we may regard this as a short exact sequence of modules over the 'monoid ring' of power operations. For $n = 1$ we obtain the following exact sequence of modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot u} \mathbb{Z}\langle u, x_1\rangle \xrightarrow{\lambda} \mathbb{Z}\langle d_1\rangle \longrightarrow 0\,.$$

Here we take $x_1 \in \pi_2 E$ to be the unique lift of d_1 so that $\varepsilon(x_1) = 0$. The action of ϕ^k on d_1 is determined by Lemma 2.2.14, $\phi^k(d_1) = k\,d_1$. Note that if $\eta : ku \to E$ denotes the ku-algebra unit, then $E \simeq ku \vee \mathrm{hcofib}(\eta)$ naturally with respect to ϕ^k (which acts trivially on ku). This implies that the extension is the trivial one with $\phi^k(u) = u$ and $\phi^k(x_1) = k\,x_1$ for all $k \in \mathbb{Z}$. We have thus shown (i)-(iii) for the case $n = 1$, as well as C_1^k. Also note that $i(x_1) = x \in \pi_2 F$, since $(\varepsilon \circ i)(x_1) = \varepsilon(x_1) = 0$.

For $n = 2$ we get an exact sequence

$$0 \longrightarrow \mathbb{Z}\langle u, x_1\rangle \xrightarrow{u\cdot} \mathbb{Z}\langle u^2, ux_1, x_2\rangle \xrightarrow{\lambda} \mathbb{Z}\langle d_2\rangle \longrightarrow 0\,,$$

where we choose $x_2 \in \pi_4 E$ to be a lift of d_2 with $\varepsilon(x_2) = 0$. Note, however, that this does not determine x_2 uniquely, since $\varepsilon(ux_1) = u\varepsilon(x_1) = 0$ by our previous choice of x_1. Again we can split off the part coming from the unit $\eta : ku \to E$ and consider the reduced extension problem

$$0 \longrightarrow \mathbb{Z}\langle x_1\rangle \xrightarrow{u\cdot} \mathbb{Z}\langle ux_1, x_2\rangle \xrightarrow{\lambda} \mathbb{Z}\langle d_2\rangle \longrightarrow 0\,.$$

Since the Pontrjagin ring of $\mathbb{C}P^\infty$ is a divided polynomial algebra, we know that

$$x_1^2 = 2x_2 + a\,ux_1 \tag{2.17}$$

for some $a \in \mathbb{Z}$. Applying i to both sides of this equation yields

$$0 = 2bx^2 + (a-1)x^2\,,$$

where we write $i(x_2) = b\,x^2$ for some $b \in \mathbb{Z}$. This equation holds in the infinite cyclic group $\mathbb{Z}\langle x^2 \rangle$, whence a must be odd. As we can still alter x_2 by any integer multiple of ux_1, we can choose x_2 so that $a = 1$ in (2.17), that is R_1 holds. This choice determines x_2 uniquely and implies $i(x_2) = 0$. We can now apply ϕ^k to R_1 and obtain

$$\phi^k(x_2) = k^2\,x_2 + \frac{k(k-1)}{2}\,ux_1\,.$$

In particular, C_2^k holds for all $k \in \mathbb{Z}$. This proves the base case of the induction.

We may now assume that unique classes $x_1, x_2, \ldots, x_{2m}$ have been constructed, so that (i)-(iii) hold, R_n holds for all $n \leq 2m-1$, and C_n^k holds for all $n \leq 2m$ and $k \in \mathbb{Z}$.

Proof of R_{2m}. Let us choose any class $x_{2m+1} \in \pi_{2m+1}E$ which lifts d_{2m+1} under λ. Because of the exact sequences (2.13) we know that such a class exists and that moreover

$$x_1 x_{2m} = \sum_{i=0}^{2m+1} a_i\,u^{2m+1-i} x_i \tag{2.18}$$

for unique $a_i \in \mathbb{Z}$ (here we take x_0 to be the unit in $\pi_0 E \cong \mathbb{Z}$). Applying λ to both sides of this equation yields

$$(2m+1)\,d_{2m+1} = \lambda(x_1)\lambda(x_{2m}) = a_{2m+1}\,d_{2m+1}\,,$$

hence $a_{2m+1} = (2m+1)$. We may alter our choice of x_{2m+1} by adding any integer multiples of classes $u^{2m+1-i}x_i$ for $0 \leq i \leq 2m$, thereby changing the integers a_i by multiples of $2m+1$. The a_i are well-defined modulo $2m+1$, and the choice of representatives for all a_i mod $(2m+1)$ defines a unique class x_{2m+1}.

To determine the congruence classes of all a_i we first apply ε to (2.18) to see that $a_0 \equiv 0\ (2m+1)$ using the induction hypothesis in (ii). We then apply i and use the induction hypothesis in (iii) to conclude that $a_1 \equiv 0\ (2m+1)$. Next, we let ϕ^k act on (2.18) and look at the coefficient of $u^{2m}x_1$ modulo $(2m+1)$. Here we find

$$k\,c^k_{2m,1} \equiv \sum_{i=1}^{m} a_{2i}\,c^k_{2i,1} + a_{2i-1}\,c^k_{2i-1,1}\ \ (2m+1)\,, \tag{2.19}$$

where on the left hand side we used the induction hypothesis in R_n and the fact that $a_1 \equiv 0\ (2m+1)$. Note that congruence (2.19) holds for any choice of $k \in \mathbb{Z}$.

We now claim that $a_j \equiv 0\ (2m+1)$ for all $1 \leq j \leq 2m-1$. We proceed by induction on j. We have already proved the base case $j = 1$. Now assume that $a_j \equiv 0\ (2m+1)$ for all $j \leq 2l-1$ for some $1 \leq l \leq m-1$. We choose $k = -l$ in (2.19). We use the vanishing of the falling factorial (2.15) and (2.16) and the induction hypothesis in C_n^{-l} to see that $c^{-l}_{2i,1} = c^{-l}_{2i-1,1} = 0$ for all $i \geq l+1$ and $c^{-l}_{2l,1} = 1$. This leaves us with $a_{2l} \equiv 0\ (2m+1)$. Next we choose $k = l+1$ in (2.19) and use again the induction hypotheses to see that $c^{l+1}_{2i,1} = 0$ for all $i \geq l+1$ and $c^{l+1}_{2i-1,1} = 0$ for all $i \geq l+2$. Furthermore, $c^{l+1}_{2l+1,1} = 1$ so that we are left with $a_{2l+1} \equiv 0\ (2m+1)$. We have thus proved the statement for all $j \leq 2l+1$ and this finishes the induction step.

The congruence (2.19) now reads

$$k\, c^k_{2m,1} \equiv a_{2m}\, c^k_{2m,1}\ (2m+1)\,.$$

In this equation we can choose $k = -m$ and observe that $c^{-m}_{2m,1} = 1$ (using C^{-m}_{2m}) to conclude that $a_{2m} \equiv -m\ (2m+1)$. We finally choose representatives for all congruence classes, setting $a_i = 0$ for all $0 \leq i \leq 2m-1$ and $a_{2m} = -m$, and thereby define a unique x_{2m+1} satisfying (i)-(iii) and R_{2m}.

Proof of C^k_{2m+1}. Let $N \subset \pi_* E$ denote the submodule generated as a $\pi_*(ku)$-module by $x_2, x_3, \ldots, x_{2m+1}$. Letting ϕ^k act on the multiplicative relation R_{2m} gives

$$(2m+1)\phi^k(x_{2m+1}) = (k+m)\, c^k_{2m,1}\, u^{2m} x_1 + N\,.$$

But

$$\frac{k+m}{2m+1}\, c^k_{2m,1} = \frac{k+m}{2m+1}\frac{(k+m-1)_{(2m)}}{(2m)!} = \frac{(k+m)_{(2m+1)}}{(2m+1)!}\,,$$

proving C^k_{2m+1}.

One then proceeds with the construction of x_{2m+2} and the proofs of R_{2m+1} and C^k_{2m+2} in an analogous manner. We leave these proofs to the reader. This finishes the induction step and the proof of the proposition. □

The proposition shows that $\pi_* E$ has a basis as a free $\pi_*(ku)$-module given by $1 \in \pi_0 E$ and the $x_n \in \pi_{2n} E$ for $n \geq 1$, and they satisfy the relations

$$x_1 x_n = (n+1)x_{n+1} + (-1)^{n+1}\left\lfloor \frac{n+1}{2} \right\rfloor u x_n \tag{2.20}$$

for all $n \geq 1$. At first sight this looks different from (2.12), but the difference only amounts to a change of basis. The precise relationship between the classes $\{y_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 1}$ is

$$x_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{\lfloor \frac{n-1}{2} \rfloor}{i} u^i y_{n-i} \tag{2.21}$$

for all $n \geq 1$.

2.2.3.4 Involutive operations on $B_{\mathrm{com}}U$

There are three natural involutions on complex commutative K-theory induced by inversion, transposition and complex conjugation within the unitary groups. We next record a lemma which describes their effect on the homotopy groups of $B_{\mathrm{com}}U$. The result for complex conjugation will be needed in Section 2.3.1 on real commutative K-theory.

Thus let ϕ^{-1}, ϕ^t and ψ^{-1} be the self-maps of $B_{\mathrm{com}}U$ induced respectively by $A \mapsto A^{-1}$, $A \mapsto A^t$ (transpose) and $A \mapsto A^*$ (the complex conjugate matrix) for $A \in U$. It is not difficult to see that all three maps extend to self-maps of the commutative symmetric ring spectrum E. Under the stable equivalence $E \simeq ku \wedge \mathbb{C}P^\infty_+$ the operation ϕ^{-1} extends to the map which is complex conjugation on $\mathbb{C}P^\infty$ and the identity on ku, ϕ^t corresponds to complex conjugation on ku and the identity on $\mathbb{C}P^\infty$, and ψ^{-1} is complex conjugation on both factors.

Lemma 2.2.18. *The effect of ϕ^{-1}, ϕ^t and ψ^{-1} on $\pi_*(B_{\mathrm{com}}U)$ is determined by the following table*

	u	x_{2m-1}	x_{2m}
ϕ^{-1}	u	$-x_{2m-1}$	$x_{2m} + ux_{2m-1}$
ϕ^t	$-u$	x_{2m-1}	$x_{2m} + ux_{2m-1}$
ψ^{-1}	$-u$	$-x_{2m-1}$	x_{2m}

Proof. The formulae for ϕ^t follow from the composition of ϕ^{-1} and ψ^{-1}. The proof for ϕ^{-1} is by induction over the multiplicative relations (2.20). The proof for ψ^{-1} is similar.

The operation ϕ^{-1} is a ku-algebra operation, whence $\phi^{-1}(u) = u$. The formulae $\phi^{-1}(x_1) = -x_1$ and $\phi^{-1}(x_2) = x_2 + ux_1$ were derived in the proof of Proposition 2.2.17. Now assume that the formulae for ϕ^{-1} hold for x_{2j-1} and x_{2j} for all $j \le m$. Applying ϕ^{-1} to the relation (2.20) for $n = 2m$ yields

$$(-x_1)(x_{2m} + ux_{2m-1}) = (2m+1)\,\phi^{-1}(x_{2m+1}) - m\,u(x_{2m} + ux_{2m-1})$$

and again using (2.20) one easily obtains $\phi^{-1}(x_{2m+1}) = -x_{2m+1}$. Similarly, applying ϕ^{-1} to (2.20) for $n = 2m+1$ yields

$$x_1 x_{2m+1} = (2m+2)\,\phi^{-1}(x_{2m+2}) - (m+1)\,ux_{2m+1}$$

from which we infer that $\phi^{-1}(x_{2m+2}) = x_{2m+2} + ux_{2m+1}$. This finishes the induction step.

For ψ^{-1} the relation $\psi^{-1}(u) = -u$ is clear, because in K-theory complex conjugation on $\mathbb{Z} \times BU$ has the effect of sending the Bott class to its additive inverse. Also the formula

$\psi^{-1}(x_1) = -x_1$ can be easily obtained from the fact that complex conjugation acts as multiplication by -1 on $H_2(\mathbb{C}P^\infty;\mathbb{Z})$. The rest follows by induction. □

From (2.21) we see that in the basis for $\pi_* E$ given by $\{y_n\}_{n\geq 1}$ we also have that $\psi^{-1}(y_n) = (-1)^n\, y_n$, but the formulae for ϕ^{-1} and ϕ^t look more complicated in this basis.

2.2.3.5 The homotopy type of $B_{\mathrm{com}}U$ and $B_{\mathrm{com}}SU$

In [6, Thm. 4.2] it is shown that the inclusion map $i : B_{\mathrm{com}}U \to BU$ admits a splitting up to homotopy $s : BU \to B_{\mathrm{com}}U$ which is also an infinite loop map. As a consequence, the authors obtain a splitting of infinite loop spaces

$$B_{\mathrm{com}}U \simeq BU \times \mathrm{hofib}(i)\,. \tag{2.22}$$

The main result of the present section is a generalisation of this splitting which completely determines the homotopy type of $B_{\mathrm{com}}U$. In addition, we describe the relationship between $B_{\mathrm{com}}U$ and $B_{\mathrm{com}}SU$ and thus describe the homotopy type of $B_{\mathrm{com}}SU$.

The splitting of $ku \wedge \mathbb{C}P^\infty_+$ as a wedge of suspensions of ku is well known. We state it in the following form. Recall the auxiliary spectrum F from (2.14) which allows us to implement the map $i : B_{\mathrm{com}}U \to BU$ as a map of ku-algebras $i : E \to F$.

Lemma 2.2.19. *There is a diagram of ku-modules*

$$\begin{array}{ccc} \bigvee_{n\geq 0} \Sigma^{2n} ku & \xrightarrow{\simeq} & E \\ \big\downarrow{\scriptstyle \mathrm{proj.}} & & \big\downarrow{\scriptstyle i} \\ ku \vee \Sigma^2 ku & \xrightarrow{\simeq} & F \end{array}$$

commuting up to homotopy, where the two horizontal maps are stable equivalences.

By the infinite wedge $\bigvee_{n\geq 0} \Sigma^{2n} ku$ we mean the union of the finite wedges $\bigvee_{0\leq k\leq n} \Sigma^{2k} ku$ as n goes to infinity.

Proof. Recall the homotopy classes $\{y_n\}_{n\geq 0}$ from Section 2.2.3.3. Using the structure of E as a ku-module, we define for every $n \geq 0$ a ku-module map $f_n : \Sigma^{2n} ku \to E$ as the composite

$$S^{2n} \wedge ku \xrightarrow{y_n \wedge id} E \wedge ku \xrightarrow{\mathrm{mult.}} E\,.$$

The coproduct over the f_n defines the top horizontal map in the diagram,

$$\bigvee_{n\geq 0} f_n : \bigvee_{n\geq 0} \Sigma^{2n} ku \to E\,.$$

It induces an isomorphism of homotopy groups in view of the fact that $\pi_* E$ is a free $\mathbb{Z}[u]$-module on generators $\{y_n\}_{n\geq 0}$. The splitting $ku \vee \Sigma^2 ku \simeq F$ was mentioned in the proof of Lemma 2.2.16. It is defined in a similar way, using the Bott class $x \in \pi_2 F$. The diagram commutes up to homotopy, because $i(y_n) = 0$ for all $n \geq 2$, cf. Proposition 2.2.17 and (2.21). $\square$

Let $\eta : ku \to E$ be the unit of the ku-algebra spectrum E. We define

$$b_{\mathrm{com}}u := \mathrm{hocofib}(\eta)\,.$$

This spectrum has naturally the structure of a ku-module. If $\Omega^\infty b_{\mathrm{com}}u$ denotes the zero space of a stably equivalent Ω-spectrum then

$$\Omega^\infty b_{\mathrm{com}}u \simeq B_{\mathrm{com}}U\,.$$

So $b_{\mathrm{com}}u$ serves as our spectrum model for the infinite loop space $B_{\mathrm{com}}U$. The equivalence follows, because in the homotopy category we have the distinguished triangle $\Sigma^{-1}b_{\mathrm{com}}u \to ku \to E$ which yields a fibration sequence of infinite loop spaces

$$\Omega^\infty\Sigma^{-1}b_{\mathrm{com}}u \to \Omega^\infty ku \to \Omega^\infty E\,.$$

Now $\Omega^\infty\Sigma^{-1}b_{\mathrm{com}}u \simeq \Omega(\Omega^\infty b_{\mathrm{com}}u)$ and this space must be equivalent to the homotopy fibre of the map $\Omega^\infty ku \to \Omega^\infty E \simeq \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$ which is just $\Omega B_{\mathrm{com}}U$. Since both $B_{\mathrm{com}}U$ and $\Omega^\infty b_{\mathrm{com}}u$ are path-connected, the equivalence $\Omega(\Omega^\infty b_{\mathrm{com}}u) \simeq \Omega B_{\mathrm{com}}U$ deloops to give $\Omega^\infty b_{\mathrm{com}}u \simeq B_{\mathrm{com}}U$.

The cofibre sequence $S^0 \to \mathbb{C}P^\infty_+ \to \mathbb{C}P^\infty$ and the stable equivalence $E \simeq ku \wedge \mathbb{C}P^\infty_+$ show that there is a stable equivalence of ku-modules

$$b_{\mathrm{com}}u \simeq ku \wedge \mathbb{C}P^\infty$$

and therefore a splitting of ku-modules $b_{\mathrm{com}}u \simeq \bigvee_{n\geq 1}\Sigma^{2n}ku$. In particular, this gives us a map

$$b_{\mathrm{com}}u \to \Sigma^2 ku$$

by projection. If $ku\langle 2\rangle \to ku$ denotes the simply connected cover of ku then multiplication by the Bott element defines a map $\Sigma^2 ku \xrightarrow{\simeq} ku\langle 2\rangle$ and this map is an equivalence by Bott periodicty. Since $\Omega^\infty ku\langle 2\rangle \simeq BU$, the map $b_{\mathrm{com}}u \to \Sigma^2 ku$ corresponds to a map $B_{\mathrm{com}}U \to BU$ on the space level. In view of Lemma 2.2.19 this is just the standard map.

Let $ku\langle 2n\rangle \to ku$ denote more generally the $2n-1$ connected cover of ku. Multiplication by the n-th power of the Bott element defines a map $\Sigma^{2n}ku \xrightarrow{\simeq} ku\langle 2n\rangle$ which is again an equivalence by Bott periodicity, and $\Omega^\infty ku\langle 2n\rangle \simeq BU\langle 2n\rangle$. Altogether this yields the following generalisation of [6, Thm. 4.2].

Corollary 2.2.20. *Let $BU\langle 2n\rangle \to BU$ denote the $2n-1$ connected cover of BU. There is a homotopy equivalence of infinite loop spaces*

$$B_{\mathrm{com}}U \simeq BU \times \prod_{n\geq 2} BU\langle 2n\rangle\,,$$

so that the inclusion $i : B_{\mathrm{com}}U \to BU$ corresponds to the projection. Similarly, there is a splitting of infinite loop spaces

$$B_{\mathrm{com}}SU \simeq BSU \times \prod_{n\geq 2} BU\langle 2n\rangle\,.$$

Proof. The first part of the corollary follows from the preceding paragraph. The splitting for $B_{\mathrm{com}}SU$ is a direct consequence of the splitting for $B_{\mathrm{com}}U$ in view of the next lemma. □

Lemma 2.2.21. *There is a homotopy fibre sequence of infinite loop spaces*

$$B_{\mathrm{com}}SU \xrightarrow{\text{incl.}} B_{\mathrm{com}}U \xrightarrow{\det} BS^1\,.$$

Proof. The proof is similar to the proof of Lemma 1.2.8. For each $n \geq 1$ we consider the n-sheeted covering map

$$\begin{aligned} q : S^1 \times SU(n) &\longrightarrow U(n)\\ (z, A) &\longmapsto zA \end{aligned}$$

with covering group the cyclic group $\mathbb{Z}_n$. Applying $\mathrm{Hom}(\mathbb{Z}^k, -)$ gives a sequence of maps

$$\mathrm{Hom}(\mathbb{Z}^k, \mathbb{Z}_n) \longrightarrow \mathrm{Hom}(\mathbb{Z}^k, S^1 \times SU(n)) \xrightarrow{q_*} \mathrm{Hom}(\mathbb{Z}^k, U(n))\,.$$

The result Goldman [17, Lem. 2.2] shows that q_* is a covering map with covering group $\mathrm{Hom}(\mathbb{Z}^k, \mathbb{Z}_n) \cong (\mathbb{Z}_n)^k$. Here we make use of the fact that $\mathrm{Hom}(\mathbb{Z}^k, U(n))$ is path-connected for all $k, n \in \mathbb{N}$, see [2, Cor. 2.4]. The resulting covering sequence fits into a commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}_n)^k & \longrightarrow & (S^1)^k \times \mathrm{Hom}(\mathbb{Z}^k, SU(n)) & \xrightarrow{q_*} & \mathrm{Hom}(\mathbb{Z}^k, U(n)) \\ \| & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \det} \\ (\mathbb{Z}_n)^k & \longrightarrow & (S^1)^k & \xrightarrow{(z_1,\ldots,z_k)\mapsto(z_1^n,\ldots,z_k^n)} & (S^1)^k \end{array}$$

The bottom row of the diagram comes from the k-fold cartesian product of the n-sheeted covering map $S^1 \longrightarrow S^1$, $z \mapsto z^n$. Since both rows are homotopy fibre sequences, the right hand square is homotopy cartesian. Taking vertical homotopy fibres yields a homotopy fibre sequence

$$\mathrm{Hom}(\mathbb{Z}^k, SU(n)) \longrightarrow \mathrm{Hom}(\mathbb{Z}^k, U(n)) \xrightarrow{\det} (S^1)^k$$

for every $k \in \mathbb{N}$ and $n \geq 1$. Each term in the sequence forms a levelwise path-connected simplicial space when k varies. We can now apply the theorem of Bousfield-Friedlander [10, Thm. B.4] which implies that

$$B_{\mathrm{com}}SU(n) \xrightarrow{\text{incl.}} B_{\mathrm{com}}U(n) \xrightarrow{\det} BS^1 \tag{2.23}$$

is a homotopy fibre sequence for every $n \geq 1$. Here we used the fact that these are good simplicial spaces, so that we can replace them up to weak equivalence by the diagonals of bisimplicial sets. The maps in (2.23) are natural with respect to the standard maps $B_{\mathrm{com}}SU(n) \to B_{\mathrm{com}}SU(n+1)$ and $B_{\mathrm{com}}U(n) \to B_{\mathrm{com}}U(n+1)$. Passing to homotopy colimits as $n \to \infty$ yields the homotopy fibre sequence in the lemma. □

2.2.3.6 Example: The commutative K-theory of S^4

Consider the forgetful map

$$\pi : \tilde{K}_{\mathrm{com}}(S^n) \to \tilde{K}(S^n)$$

from commutative K-theory to ordinary K-theory. It follows from our computation of the homotopy groups of $B_{\mathrm{com}}U$ in Theorem 2.2.13 that the smallest dimension n in which π is not an isomorphism is $n = 4$. In fact, we see that $\tilde{K}_{\mathrm{com}}(S^4) \cong \mathbb{Z}^2$ and π is a surjection onto $\tilde{K}(S^4) \cong \mathbb{Z}$. In this section we describe the group $\tilde{K}_{\mathrm{com}}(S^4)$ and construct a non-trivial class in the kernel of π.

Let $\mathbb{H}$ denote the quaternions. The group $\tilde{K}(S^4)$ is generated by the reduced class of the complex 2-plane bundle underlying the tautological quaternionic line-bundle

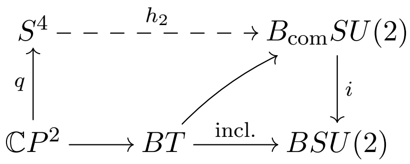
$$\mathbb{H} \hookrightarrow H \to \mathbb{H}P^1 \simeq S^4 .$$

Its structural group is $\mathbb{H}^\times \simeq SU(2)$ and the corresponding principal $SU(2)$-bundle is the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$. Note that $\pi_4(BSU(2)) \cong H^4(S^4) \cong \mathbb{Z}$, the isomorphism is given by the second Chern class, and $c_2(H) = -1$. In particular, the inclusion $BSU(2) \subset BU$ induces an isomorphism $\pi_4(BSU(2)) \cong \pi_4(BU)$. A classifying map for the Hopf bundle is obtained by taking the inclusion $\Sigma SU(2) \to BSU(2)$ of the simplicial 1-skeleton and choosing an identification $\Sigma SU(2) \cong S^4$.

Let us begin by putting two non-isomorphic transitionally commutative structures on the Hopf bundle. First, we can simply factor the inclusion map $\Sigma SU(2) \to BSU(2)$ through $B_{\mathrm{com}}SU(2)$, because the simplicial 1-skeleta of $BSU(2)$ and $B_{\mathrm{com}}SU(2)$ agree; so we obtain a classifying map h_1 as in the following diagram:

$$\begin{array}{ccc} & & B_{\mathrm{com}}SU(2) \\ & \overset{h_1}{\nearrow} & \big\downarrow i \\ S^4 & \xrightarrow{\text{incl.}} & BSU(2) \end{array}$$

In order to construct a second classifying map, we consider the tautological complex line bundle $\mathbb{C} \hookrightarrow L \to \mathbb{C}P^2$. Let L^* denote its complex conjugate bundle. Then $L \oplus L^*$ has a canonical reduction of its structural group to the maximal torus $T \subset SU(2)$ which consists of all the diagonal matrices in $SU(2)$. We can then draw the following diagram:



The arrow $BT \to B_{\text{com}}SU(2)$ is the map induced by the inclusion $T \subset SU(2)$, using the fact that T is abelian so that $B_{\text{com}}T = BT$. Composition with the classifying map for $L \oplus L^*$ gives a map $\mathbb{C}P^2 \to B_{\text{com}}SU(2)$. It follows from [5, Prop. 3.2] that the space $B_{\text{com}}SU(2)$ is 3-connected (this also follows from Lemma 1.2.11), so this classifying map factors up to homotopy through the quotient map $q : \mathbb{C}P^2 \to \mathbb{C}P^2/\mathbb{C}P^1 \cong S^4$. This shows that the dashed arrow h_2 exists, and its homotopy class is uniquely determined. By the Cartan formula we have

$$c_2(L \oplus L^*) = c_2(L) + c_1(L) \cup c_1(L^*) + c_2(L^*) = -1\,.$$

As the quotient map q has degree one, this implies that the map h_2 classifies a bundle with second Chern class -1, hence is a lift of the Hopf bundle.

We claim that the principal $SU(2)$-bundles represented by h_1 and h_2 are not isomorphic as transitionally commutative bundles. In fact, Lemma 2.2.22 will show that h_1 and $(h_2 - h_1)/2$ form a free basis for the group $\pi_4(B_{\text{com}}SU(2))$. We can check this in cohomology: By the Hurewicz theorem there is an isomorphism

$$h_* : \pi_4(B_{\text{com}}SU(2)) \xrightarrow{\cong} H_4(B_{\text{com}}SU(2))\,,$$

because $B_{\text{com}}SU(2)$ is 3-connected. Now $H_4(B_{\text{com}}SU(2))$ is torsionfree: by the universal coefficient theorem, any torsion would be visible in $H^5(B_{\text{com}}SU(2))$ which is the trivial group, by [5, Ex. 6.4]. So we obtain an isomorphism

$$\pi_4(B_{\text{com}}SU(2)) \cong \mathrm{Hom}(H^4(B_{\text{com}}SU(2)), \mathbb{Z}) \qquad (2.24)$$

taking $f \mapsto (x \mapsto \langle x, h_*(f)\rangle)$.

Lemma 2.2.22. *The maps h_1 and $(h_2 - h_1)/2$ freely generate the group $\pi_4(B_{\text{com}}SU(2))$.*

Proof. Recall the conjugation map

$$\varphi : SU(2)/T \times BT \to B_{\text{com}}SU(2)$$

from Section 1.1.3. Let us write $H^*(-) := H^*(-;\mathbb{Z})$. Then

$$H^*(SU(2)/T \times BT) \cong \mathbb{Z}[a,b]/(b^2)\,,$$

where $a \in H^2(BT) \cong H^2(\mathbb{C}P^\infty)$ and $b \in H^2(SU(2)/T) \cong H^2(\mathbb{C}P^1)$ are generators. We first show that the image of $\varphi^* : H^4(B_{\mathrm{com}}SU(2)) \to H^4(SU(2)/T \times BT)$ is the submodule spanned by $2a^2$ and $a^2 + 2ab$. From Lemmas 1.1.9 and 1.2.10 we see that the classes $2a^2$ and $a^2 + 2ab$ lie in the image of φ^*, the latter being the image of $-c_2 \in H^4(BSU(2))$ under the map $i \circ \varphi$. On the other hand, commutativity of the diagram

$$\begin{array}{ccc} SU(2)/T \times \Sigma T & \longrightarrow & SU(2)/T \times BT \\ \big\downarrow {\scriptstyle \times 2} & & \big\downarrow {\scriptstyle \varphi} \\ \Sigma SU(2) & \longrightarrow & B_{\mathrm{com}}SU(2) \end{array}$$

shows that neither ab nor $a^2 + ab$ are possibly contained in the image of φ^*. In the diagram the two horizontal arrows are the inclusions of the simplicial 1-skeleta, and the left hand vertical arrow is a map of degree 2 (the order of the Weyl group of $SU(2)$). The top horizontal arrow maps ab isomorphically and has a^2 in its kernel. Finally, inspection of the Mayer-Vietoris sequence in [5, Ex. 6.4] shows that a^2 is not contained in the image of φ^* either. Thus $\mathrm{im}(\varphi^*)$ is freely generated by $2a^2$ and $a^2 + 2ab$. Let α and β be the classes in $H^4(B_{\mathrm{com}}SU(2))$ corresponding to $2a^2$ respectively a^2+2ab under the identification $H^4(B_{\mathrm{com}}SU(2)) \cong \mathrm{im}(\varphi^*)$. To finish the proof, one can now check from the definition of the classifying maps h_1 and h_2, that h_1 is the dual of β and $h_2 - h_1$ is twice the dual of α under the isomorphism (2.24). □

We are now interested in the question which classes h_1 and h_2 represent in commutative K-theory. In other words, we would like to describe the map

$$\pi_4(B_{\mathrm{com}}SU(2)) \to \pi_4(B_{\mathrm{com}}U) \cong \tilde{K}_{\mathrm{com}}(S^4) \tag{2.25}$$

coming from the inclusion $SU(2) \subset U$. Recall from Theorem 2.2.13 that

$$\tilde{K}_{\mathrm{com}}(S^4) \cong \mathbb{Z}\langle uy_1, y_2\rangle\,.$$

On this group we have defined two natual maps: The first one is the projection map $\pi : \tilde{K}_{\mathrm{com}}(S^4) \to \tilde{K}(S^4)$. Remember that the class $y_1 \in \tilde{K}_{\mathrm{com}}(S^2)$ is the Bott element represented by the canonical map $S^2 \cong \mathbb{C}P^1 \to B_{\mathrm{com}}U$. Since π is a map of $\pi_*(ku)$-modules, it takes the class uy_1 to the square of the Bott element, i.e. to the generator of $\tilde{K}(S^4)$, while it sends the class y_2 to zero. The second map comes from the characterisitc map $\lambda : B_{\mathrm{com}}U \to SP^\infty\mathbb{C}P^\infty$ defined in (2.2). On homotopy groups λ induces a map

$$\lambda_n : \tilde{K}_{\mathrm{com}}(S^{2n}) \to H_{2n}(\mathbb{C}P^\infty, \mathbb{Z})$$

for every $n > 0$. In particular, for $n = 1$ this is an isomorphism given by the first Chern class, as we can see from (2.3). For $n = 2$ we get a map $\lambda_2 : \tilde{K}_{\text{com}}(S^4) \to H_4(\mathbb{C}P^\infty, \mathbb{Z})$. If d_2 denotes the canonical generator of $H_4(\mathbb{C}P^\infty, \mathbb{Z})$, then λ_2 takes $y_2 \mapsto d_2$ (by definition of y_2) and it takes $uy_1 \mapsto 0$ (see e.g. (2.13)).

Now let $\bar{h}_i$ denote the image of h_i, $i = 1, 2$, under the map (2.25). We wish to express $\bar{h}_i$ in terms of uy_1 and y_2. We know that $\bar{h}_i$ is a lift of the generator of $\tilde{K}(S^4)$, so $\bar{h}_i = uy_1 + a_i y_2$ for some $a_i \in \mathbb{Z}$. To determine a_i we compute the characteristic class $\lambda_2(\bar{h}_i)$.

Let us begin with $\bar{h}_1$. By definition, the classifying map h_1 factors through the simplicial 1-skeleton of $B_{\text{com}}U$. On the spectrum level, and under the stable equivalence $b_{\text{com}}u \simeq ku \wedge \mathbb{C}P^\infty$, it is not so difficult to see that the inclusion of the simplicial 1-skeleton into $b_{\text{com}}u$ corresponds to the natural map $ku \wedge \mathbb{C}P^1 \to ku \wedge \mathbb{C}P^\infty$. Thus $\bar{h}_1$ will factor through this map. Computing $\lambda_2(\bar{h}_1)$ then involves the composite

$$ku \wedge \mathbb{C}P^1 \to ku \wedge \mathbb{C}P^\infty \to H\mathbb{Z} \wedge \mathbb{C}P^\infty \simeq \bigvee_{n>0} \Sigma^{2n} H\mathbb{Z} \xrightarrow{\text{pr}_2} \Sigma^4 H\mathbb{Z}\,,$$

where the n-th summand $\Sigma^{2n}H\mathbb{Z}$ in the splitting corresponds to the canonical generator $[\mathbb{C}P^n] \in H_{2n}(\mathbb{C}P^\infty, \mathbb{Z})$. The composite map is then trivial, because it factors through the summand $\Sigma^2 H\mathbb{Z}$. Therefore, $\lambda_2(\bar{h}_1) = 0$.

Now we consider $\bar{h}_2$. This class was constructed using the sum of the tautological line bundle over $\mathbb{C}P^\infty$ and its complex conjugate. Thus, if $\xi : \Sigma^\infty \mathbb{C}P^\infty \to b_{\text{com}}u$ is the standard map, we first want to compute the composite

$$\Sigma^\infty \mathbb{C}P^\infty \xrightarrow{\xi} b_{\text{com}}u \xrightarrow{\lambda} H\mathbb{Z} \wedge \mathbb{C}P^\infty \simeq \bigvee_{n>0} \Sigma^{2n} H\mathbb{Z} \xrightarrow{\text{pr}_2} \Sigma^4 H\mathbb{Z}\,.$$

Since $\lambda \circ \xi$ is just the stable Hurewicz map for $\mathbb{C}P^\infty$, the cohomology class determined by this composite is just the dual of $[\mathbb{C}P^2] \in H_4(\mathbb{C}P^\infty, \mathbb{Z})$, i.e. the square of the first Chern class $c_1(\xi)^2 \in H^4(\mathbb{C}P^\infty, \mathbb{Z})$. If we consider instead the sum of ξ with its complex conjugate bundle, i.e. $\xi \oplus \xi^*$, then we get $c_1(\xi)^2 + c_1(\xi^*)^2 = 2c_1(\xi)^2$. In our construction of h_2 we restrict $\xi \oplus \xi^*$ to $\mathbb{C}P^2 \subset \mathbb{C}P^\infty$ and the resulting class descends under the quotient map $q : \mathbb{C}P^2 \to S^4$. Since the quotient map has degree one, we find that the composite

$$\Sigma^\infty S^4 \xrightarrow{\bar{h}_2} b_{\text{com}}u \xrightarrow{\lambda} H\mathbb{Z} \wedge \mathbb{C}P^\infty \simeq \bigvee_{n>0} \Sigma^{2n} H\mathbb{Z} \xrightarrow{\text{pr}_2} \Sigma^4 H\mathbb{Z}$$

corresponds to twice the generator of $H^4(S^4, \mathbb{Z})$, hence $\lambda_2(\bar{h}_2) = 2d_2$.

Altogether this shows that $\bar{h}_1 = uy_1$ and $\bar{h}_2 = uy_1 + 2y_2 = y_1^2$. We record:

Lemma 2.2.23. *The map $\pi_4(B_{\text{com}}SU(2)) \to \pi_4(B_{\text{com}}U)$ is an isomorphism and is determined by $h_1 \mapsto uy_1$ and $h_2 - h_1 \mapsto 2y_2$.*

Note that we have not explicitly described a representing map $S^4 \to B_{\mathrm{com}}SU(2)$ for the generator $(h_2 - h_1)/2$ of $\pi_4(B_{\mathrm{com}}SU(2))$. For the generator y_2 of $\pi_4(B_{\mathrm{com}}U)$ we can describe a map as follows. Let $\xi : \Sigma^\infty \mathbb{C}P^\infty \to b_{\mathrm{com}}u$ denote the classifying map of the tautological complex line-bundle over $\mathbb{C}P^\infty$ thought of as a transitionally commutative bundle in a natural way. Let $\bar{\xi} : \mathbb{C}P^\infty \to \Sigma^2 ku$ denote the class of the same bundle in ordinary K-theory. We have a map

$$s : \Sigma^2 ku \to b_{\mathrm{com}}u\,,$$

namely the natural map $ku \wedge \mathbb{C}P^1 \to ku \wedge \mathbb{C}P^\infty$ followed by the stable equivalence $ku \wedge \mathbb{C}P^\infty \simeq b_{\mathrm{com}}u$. This map can be seen as a spectrum level version of the splitting map constructed in [6]. We claim that the restriction of the difference class

$$\xi - s(\bar{\xi}) \in \tilde{K}_{\mathrm{com}}(\mathbb{C}P^\infty)$$

along the inclusion $j : \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^\infty$ descends to a unique class on S^4 which represents the generator $y_2 \in \tilde{K}_{\mathrm{com}}(S^4)$.

The cofibre sequence $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \xrightarrow{q} S^4$ yields an exact sequence

$$0 = \tilde{K}_{\mathrm{com}}(\Sigma\mathbb{C}P^1) \longrightarrow \tilde{K}_{\mathrm{com}}(S^4) \xrightarrow{q^*} \tilde{K}_{\mathrm{com}}(\mathbb{C}P^2) \longrightarrow \tilde{K}_{\mathrm{com}}(\mathbb{C}P^1)\,,$$

which shows that a class in the kernel of the restriction map to $\mathbb{C}P^1$ comes from a unique class in $\tilde{K}_{\mathrm{com}}(S^4)$. Notice now that $\tilde{K}_{\mathrm{com}}(\mathbb{C}P^1) \cong \tilde{K}(\mathbb{C}P^1)$ and the elements in this group are classified by their first Chern class. In particular, this implies that the restriction of $\xi - s(\bar{\xi})$ to $\tilde{K}_{\mathrm{com}}(\mathbb{C}P^1)$ is zero. Thus there exists a unique $y \in \tilde{K}_{\mathrm{com}}(S^4)$ so that $q^*(y) = j^*(\xi - s(\bar{\xi}))$. By construction, y lies in the kernel of the projection map $\pi : \tilde{K}_{\mathrm{com}}(S^4) \to \tilde{K}(S^4)$. Furthermore, we have

$$\lambda_2(j^*(\xi - s(\bar{\xi}))) = j^*(\lambda_2(\xi) - \lambda_2(s(\bar{\xi}))) = c_1(j^*(\xi))^2 \in H^4(\mathbb{C}P^2;\mathbb{Z})\,,$$

showing that $\lambda_2(y) = \lambda_2(y_2)$. This proves that $y = y_2$ as claimed.

Remark 2.2.24. In [5, Ex. 2.5] a non-trivial class $\tilde{h} : S^4 \to B_{\mathrm{com}}SU(2)$ is constructed and it is claimed that its composition with $i : B_{\mathrm{com}}SU(2) \to BSU(2)$ is the trivial map on rational cohomology. However, one can check from Lemma 1.1.9 that $i \circ \tilde{h}$ pulls back the second Chern class non-trivially. Therefore, the transitionally commutative $SU(2)$-bundle described by $\tilde{h}$ is non-trivial even as an ordinary principal bundle. In our notation, $\tilde{h} = 2h_1$.

2.2.4 The rational Hopf ring and the unstable Hurewicz map

In the preceding section we have argued that $B_{\mathrm{com}}U$ is a model for the augmentation ideal in the ku-group ring of $\mathbb{C}P^\infty$. The rational homology of the space $B_{\mathrm{com}}U$ has therefore a description in terms of K-homology theory. On the other hand, we can regard $B_{\mathrm{com}}U$ as the union of the $B_{\mathrm{com}}U(n)$ whose rational homology can be described using Lie group theory. The purpose of this section is to compare these two descriptions.

In [5] Adem and Gómez show that there is an isomorphism of $\mathbb{Q}$-algebras

$$H^*(B_{\mathrm{com}}U, \mathbb{Q}) \cong \mathbb{Q}[z_{a,b} \,|\, (a,b) \in \mathbb{N}^2,\, b \geq 1]\,,$$

where the class $z_{a,b}$ has degree $2(a+b)$. We will recall the definition of $z_{a,b}$ below. Let $\zeta_{a,b} := z_{a,b}^*$ denote the dual homology class. Furthermore, for $1 \leq j \leq n-1$ let us write $e_{j,n} := e_j(-1, \ldots, -(n-1))$ for the value of the j-th elementary symmetric polynomial

$$e_j(x_1, \ldots, x_{n-1}) = \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} x_{i_1} \cdots x_{i_j}$$

at $(x_1, \ldots, x_{n-1}) = (-1, \ldots, -(n-1))$. The main result of this section is the following formula.

Theorem 2.2.25. *The Hurewicz homomorphism*

$$h_* : \pi_*(B_{\mathrm{com}}U) \to \tilde{H}_*(B_{\mathrm{com}}U, \mathbb{Q})$$

is determined by the formula

$$h_*(y_n) = \sum_{j=0}^{n-1} \binom{n}{j} e_{j,n}\, \zeta_{j,n-j}$$

for $n \geq 1$.

Remark 2.2.26. Recall that we have a splitting $B_{\mathrm{com}}U \simeq \prod_{n \geq 1} BU\langle 2n \rangle$. On each factor of the right hand side we have defined a (truncated) Chern character. The theorem may then be used, for example, to express the characteristic classes $z_{a,b}$ defined on the left hand side of this equivalence in terms of the Chern character defined on the right hand side.

In order to derive the formula in Theorem 2.2.25 we shall use the concept of a *Hopf ring*. The original reference for this notion is Ravenel-Wilson [50].

Suppose that X is a space equipped with an additive H-space product $\oplus : X \times X \to X$ and a multiplicative one $\otimes : X \wedge X \to X$ satisfying the axioms of a commutative ring up to homotopy. For example, X could be an E_∞-ring space. The additive structure induces the graded Pontrjagin product on $H_*(X)$. Using the diagonal map $\Delta : X \to X \times X$ the

Pontrjagin algebra $H_*(X)$ becomes a graded Hopf algebra. In the context of Hopf rings the Pontrjagin product is usually denoted by the symbol $*$ and in this section only we shall follow this convention. The multiplicative H-space structure induces an additional graded product on $H_*(X)$ which is denoted by the symbol $\circ$ and which factors through the reduced homology $\tilde{H}_*(X)$. The Hopf algebra $H_*(X)$ together with the $\circ$-product is called a Hopf ring.

Because of the homotopy commutative diagrams which relate the operations $\oplus$, $\otimes$ and Δ, such as the distributivity of $\otimes$ over $\oplus$, the Pontrjagin product, the coproduct and the $\circ$-product satisfy various formulae. For example, there is a left-distributivity law, see [50, Lem. 1.12], for $a, b, c \in H_*(X)$

$$a \circ (b * c) = \sum (-1)^{|a''||b|} (a' \circ b) * (a'' \circ c)\,, \tag{2.26}$$

where $\Delta(a) = \sum a' \otimes a''$ is the coproduct and $*$ denotes the Pontrjagin product. The right-distributivity law is similar.

Let $X_0 \subset X$ denote the path-component of the basepoint. What is important now is the fact that the Hurewicz homomorphism $\pi_*(X_0) \to \tilde{H}_*(X_0)$ is a homomorphism of graded rings with respect to the $\circ$-product on homology. The rational homotopy $\pi_*(B_{\mathrm{com}}U) \otimes \mathbb{Q}$ is generated as a $\mathbb{Q}[u]$-algebra by the Bott class $x_1 \in \pi_2 B_{\mathrm{com}}U$. Thus, in order to derive Theorem 2.2.25, all we need to do is to determine the $\circ$-product on homology classes $\zeta_{a,b}$ in $\tilde{H}_*(\{0\} \times B_{\mathrm{com}}U \,/\!\!/\, U, \mathbb{Q})$, which is done in Proposition 2.2.28, and compute the image of u and x_1 under the Hurewicz map.

Proof of Theorem 2.2.25

From now on H^* and H_* will always mean (co-)homology with rational coefficients. We begin by recalling the cohomology results of [5]. Let $T(n) \subset U(n)$ denote the maximal torus consisting of diagonal matrices with entries in $U(1)$. The associated Weyl group is the symmetric group Σ_n which acts on $T(n)$ by permuting the diagonal. The classifying space $BT(n)$ is a product of n copies of $\mathbb{C}P^\infty$, so its rational cohomology ring is a polynomial algebra $\mathbb{Q}[\mathbf{y}]$ on a set $\mathbf{y} = \{y_1, \ldots, y_n\}$ of n independent variables of degree two. The rational cohomology algebra $H^*(B_{\mathrm{com}}U(n))$ was determined by Adem-Gómez in [5, 8.1] using Theorem 1.1.8. The result is

$$H^*(B_{\mathrm{com}}U(n)) \cong (\mathbb{Q}[\mathbf{x}] \otimes \mathbb{Q}[\mathbf{y}])^{\Sigma_n}/J_n\,, \tag{2.27}$$

where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is another set of variables of degree two, the symmetric group acts diagonally on the tensor product by permuting the variables in $\mathbf{x}$ and $\mathbf{y}$, and J_n is the ideal generated by all $e_n(\mathbf{x}) \otimes 1$ for $n \geq 1$, where $e_n(\mathbf{x})$ denotes the n-th elementary symmetric polynomial in $\mathbf{x}$. For $(a,b) \in \mathbb{N}^2$ with $b \geq 1$ let $z_{a,b,n} \in \mathbb{Q}[\mathbf{x}] \otimes \mathbb{Q}[\mathbf{y}]$ denote the Σ_n-invariant polynomial of degree $2(a+b)$ given by

$$z_{a,b,n} = x_1^a y_1^b + \cdots + x_n^a y_n^b \,. \tag{2.28}$$

It is shown in [5] that $H^*(B_{\mathrm{com}}U(n))$ is generated as an algebra by the $z_{a,b,n}$ for $(a,b) \in \mathbb{N}^2$, $b \geq 1$. The authors also prove that $H^*(B_{\mathrm{com}}U) \cong \lim_n H^*(B_{\mathrm{com}}U(n))$ and it is generated as a *polynomial* algebra by the $z_{a,b} := (z_{a,b,n})_{n\geq 1}$ for $b \geq 1$. The inverse limit is over the projections $H^*(B_{\mathrm{com}}U(n)) \to H^*(B_{\mathrm{com}}U(n-1))$ sending $x_n, y_n \mapsto 0$.

We now consider the ring space $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$. For $n \in \mathbb{Z}$, let us write $X[n]$ for the subspace $\{n\} \times B_{\mathrm{com}}U /\!\!/ U$ so that

$$\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U = \coprod_{n \in \mathbb{Z}} X[n] \,.$$

Let us first describe the homology and cohomology of a component $X[n]$. Using the fact that the basepoint of $B_{\mathrm{com}}U$ is fixed by U the homotopy fibre sequence $B_{\mathrm{com}}U \to B_{\mathrm{com}}U /\!\!/ U \to BU$ can be split by a map of infinite loop spaces $BU \to B_{\mathrm{com}}U /\!\!/ U$ so that for all $n \in \mathbb{Z}$ we have an isomorphism of Hopf algebras

$$H^*(X[n]) \cong H^*(BU) \otimes H^*(B_{\mathrm{com}}U) \,.$$

Recall that $H^*(BU) \cong \mathbb{Q}[z_{a,0} \,|\, a \geq 1, |z_{a,0}| = 2a]$, where $z_{a,0} := a!\,\mathrm{ch}_a$ denotes, up to a constant, the a-th component of the Chern character. Therefore, $H^*(X[n])$ is a polynomial algebra on all classes $z_{a,b}$ of degree $2(a+b)$ for $(a,b) \in \mathbb{N}^2 - \{(0,0)\}$. Let

$$\zeta_{a,b} = z_{a,b}^* \in H_{2(a+b)}(X[n])$$

denote the dual class. We will need the following fact.

Lemma 2.2.27. *The polynomial generators $z_{a,b} \in H^{2(a+b)}(X[n])$ are primitive. Dually, the Pontrjagin algebra $H_*(X[n])$ is the polynomial algebra primitively generated by the classes $\{\zeta_{a,b} \in H_{2(a+b)}(X[n]) \,|\, (a,b) \in \mathbb{N}^2 - \{(0,0)\}\}$.*

We will not prove this lemma, because it is very similar to the proof of Lemma 2.2.29 below, but easier.

The tensor product $\otimes : (\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U) \wedge (\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U) \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$ restricts to component maps

$$\mu_{n,m} : X[n] \times X[m] \to X[nm] \tag{2.29}$$

for all $n, m \in \mathbb{Z}$, and for $n = m = 0$ this factors through the smash product $X[0] \wedge X[0] \to X[0]$. Let $\circ$ denote the product induced on homology groups.

Proposition 2.2.28. *The product* $\circ : \tilde{H}_*(X[0]) \otimes \tilde{H}_*(X[0]) \to \tilde{H}_*(X[0])$ *is given by*

$$\zeta_{a,b} \circ \zeta_{c,d} = \binom{a+c}{c}\binom{b+d}{d}\zeta_{a+c,b+d}$$

on the primitive basis for the Pontrjagin algebra $H_*(X[0])$.

Theorem 2.2.25 follows directly from this proposition.

Proof of Theorem 2.2.25. Working over $\mathbb{Q}$ we can write (using the relations (2.12))

$$y_n = \frac{y_1(y_1-u)(y_1-2u)\cdots(y_1-(n-1)u)}{n!},$$

where $u \in \pi_2(BU)$ and $y_1 \in \pi_2(B_{\mathrm{com}}U)$ are the Bott elements. The formula for $h_*(y_n)$ follows immediately from the fact that $h_*(x_1) = \zeta_{0,1}$ and $h_*(u) = \zeta_{1,0}$ and from the relations in Proposition 2.2.28. □

The rest of this section is devoted to the proof of Proposition 2.2.28. The first lemma describes the effect of the multiplication map (2.29) on cohomology groups.

Lemma 2.2.29. *Let* $z_{a,b} \in H^*(X[nm])$ *with* $n, m \geq |z_{a,b}| = 2(a+b)$*. Then*

$$\mu^*_{n,m}(z_{a,b}) = \sum_{i=0}^{a}\sum_{j=0}^{b}\binom{a}{i}\binom{b}{j} z_{a-i,b-j} \otimes z_{i,j},$$

where $1 \otimes z_{0,0} := m$ *and* $z_{0,0} \otimes 1 := n$.

Proof. The product $\mu_{n,m} : X[n] \times X[m] \to X[nm]$ is induced by the tensor product, so we begin by describing the map induced by the tensor product $\alpha_{n,m} : BT(n) \times BT(m) \to BT(nm)$ on cohomology. It makes the following diagram commute for all $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\begin{array}{ccc} BT(n) \times BT(m) & \xrightarrow{\alpha_{n,m}} & BT(nm) \\ \Big\downarrow{\scriptstyle p_i \times p_j} & & \Big\downarrow{\scriptstyle p_{ij}} \\ BU(1) \times BU(1) & \xrightarrow{\text{mult.}} & BU(1). \end{array}$$

The lower horizontal map is induced by the product in $U(1)$ and the vertical maps denote the projections when we identify $BT(n) = BU(1)^n$ etc. For $BT(nm)$ we index the factors $BU(1)$ lexicographically. Now fix a generator $x \in H^2(BU(1))$ and write $H^*(BT(nm)) \cong \mathbb{Q}[\mathbf{s}]$ where $\mathbf{s} = \{s_{11}, \ldots, s_{nm}\}$ and $s_{ij} = p^*_{ij}(x)$. In a similar way we define the sets of generators $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_n\}$ so that $H^*(BT(n)) \cong \mathbb{Q}[\mathbf{v}]$ and $H^*(BT(m)) \cong \mathbb{Q}[\mathbf{w}]$. It follows from the diagram that the map

$$\alpha^*_{n,m} : \mathbb{Q}[\mathbf{s}] \to \mathbb{Q}[\mathbf{v}] \otimes \mathbb{Q}[\mathbf{w}]$$

satisfies $\alpha^*_{n,m}(s_{ij}) = v_i \otimes 1 + 1 \otimes w_j$ and therefore

$$\alpha^*_{n,m}(s^a_{11} + \cdots + s^a_{nm}) = \sum_{i=0}^{a} \binom{a}{i} (v_1^{a-i} + \cdots + v_n^{a-i}) \otimes (w_1^i + \cdots + w_m^i) \tag{2.30}$$

for all $a \geq 0$.

Next we describe the effect on cohomology of the map

$$\beta_{n,m} : B_{\mathrm{com}}U(n) /\!\!/ U(n) \times B_{\mathrm{com}}U(m) /\!\!/ U(m) \to B_{\mathrm{com}}U(nm) /\!\!/ U(nm)$$

induced by tensor product. Consider the homotopy fibre sequence

$$B_{\mathrm{com}}U(n) \to B_{\mathrm{com}}U(n) /\!\!/ U(n) \to BU(n)\,. \tag{2.31}$$

The E_2-page of the associated rational Leray-Serre spectral sequence is concentrated in even bi-degrees. Hence, all differentials vanish for degree reasons and the spectral sequence collapses at the E_2-page. Let us write

$$\mathcal{A}_n := \mathbb{Q}[\mathbf{t}']^{\Sigma_n} \otimes (\mathbb{Q}[\mathbf{x}'] \otimes \mathbb{Q}[\mathbf{y}'])^{\Sigma_n}/J_n\,, \tag{2.32}$$

where $\mathbf{t}'$, $\mathbf{x}'$ and $\mathbf{y}'$ each denote a set of n polynomial generators of degree two, and J_n is the ideal defined as in (2.27). Then, using the isomorphism (2.27), we have an isomorphism of graded vector spaces

$$H^*(B_{\mathrm{com}}U(n) /\!\!/ U(n)) \cong H^*(BU(n)) \otimes H^*(B_{\mathrm{com}}U(n)) \cong \mathcal{A}_n\,.$$

Let us write $\mathcal{A}_m$ and $\mathcal{A}_{nm}$ as in (2.32) with suitable sets of variables $\mathbf{t}''$, $\mathbf{x}''$, $\mathbf{y}''$ and $\mathbf{t}$, $\mathbf{x}$, $\mathbf{y}$, respectively. Then there is a map

$$\psi_{n,m} : \mathcal{A}_{nm} \to \mathcal{A}_n \otimes \mathcal{A}_m$$

induced by $\alpha^*_{n,m} : \mathbb{Q}[\mathbf{t}] \to \mathbb{Q}[\mathbf{t}'] \otimes \mathbb{Q}[\mathbf{t}'']$ etc. For $b \geq 1$ let $z_{a,b,nm}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ be defined as in (2.28) using the variables $\mathbf{x}$ and $\mathbf{y}$, and for $b = 0$ define $z_{a,0,nm}(\mathbf{t}, \mathbf{x}, \mathbf{y}) := t^a_{11} + \ldots t^a_{nm}$. Let us also write $z_{0,0,nm}(\mathbf{t}, \mathbf{x}, \mathbf{y}) := nm$. Using equation (2.30) it is easily checked that

$$\psi_{n,m}(z_{a,b,nm}(\mathbf{t}, \mathbf{x}, \mathbf{y})) = \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} z_{a-i,b-j,n}(\mathbf{t}', \mathbf{x}', \mathbf{y}') \otimes z_{i,j,m}(\mathbf{t}'', \mathbf{x}'', \mathbf{y}'')$$

for all $(a, b) \in \mathbb{N}^2$. The tensor product $\otimes : U(n) \times U(m) \to U(nm)$ induces a pairing of the corresponding homotopy fibre sequences (2.31). We also observe that it induces a welldefined map $U(n)/T(n) \times U(m)/T(m) \to U(nm)/T(nm)$ compatible with (1.1), i.e compatible with the maps $\varphi : U(k)/T(k) \times BT(k) \to B_{\mathrm{com}}U(k)$ for $k = n, m, nm$, respectively. As a consequence, there is a commutative diagram

$$\begin{array}{ccc} H^*(B_{\mathrm{com}}U(nm) /\!\!/ U(nm)) & \xrightarrow{\beta^*_{n,m}} & H^*(B_{\mathrm{com}}U(n) /\!\!/ U(n)) \otimes H^*(B_{\mathrm{com}}U(m) /\!\!/ U(m)) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{A}_{nm} & \xrightarrow{\psi_{n,m}} & \mathcal{A}_n \otimes \mathcal{A}_m \end{array}$$

Finally, we also need the following homotopy commutative diagram

$$\begin{array}{ccc} B_{\mathrm{com}}U(n) /\!\!/ U(n) \times B_{\mathrm{com}}U(m) /\!\!/ U(m) & \xrightarrow{\beta_{n,m}} & B_{\mathrm{com}}U(nm) /\!\!/ U(nm) \\ \downarrow j_n \times j_m & & \downarrow j_{nm} \\ X[n] \times X[m] & \xrightarrow{\mu_{n,m}} & X[nm] \end{array}$$

where $j_k : B_{\mathrm{com}}U(k) /\!\!/ U(k) \to X[k]$ denotes the inclusion into the direct limit $X[k] = \mathrm{colim}_{l \geq k} B_{\mathrm{com}}U(l) /\!\!/ U(l)$. It was observed in [5, 8.1] that the maps in the inverse system of cohomology groups $\{H^*(B_{\mathrm{com}}U(l))\}_{l \geq 0}$ are all surjective. As a consequence of this, and because of the isomorphism $H^*(B_{\mathrm{com}}U(l) /\!\!/ U(l)) \cong H^*(BU(l)) \otimes H^*(B_{\mathrm{com}}U(l))$, we also have that $H^*(X[k]) \cong \lim_l H^*(B_{\mathrm{com}}U(l) /\!\!/ U(l))$. The map

$$j_k^* : H^*(X[k]) \to H^*(B_{\mathrm{com}}U(k) /\!\!/ U(k))$$

is then the projection onto the k-th component in the inverse limit. By [62, Prop. 3.1 (2)], the map j_k^* is injective in degrees $* \leq k$. The lemma follows now by chasing the class $z_{a,b} \in H^*(X[nm])$ through the two diagrams and using the fact that j_n^*, j_m^* are both injective because of the assumption $n, m \geq |z_{a,b}|$. □

In the next lemma we dualise to describe the corresponding formula on homology. Let

$$\mu_* : H_*(X[n]) \otimes H_*(X[m]) \to H_*(X[nm])$$

be the map induced by the product $\mu_{n,m} : X[n] \times X[m] \to X[nm]$. Recall that $*$ denotes the Pontrjagin product on homology.

Lemma 2.2.30. *Let $\zeta_{a,b} \in H_{2(a+b)}(X[n])$ and $\zeta_{c,d} \in H_{2(c+d)}(X[m])$ and assume that $n, m \geq |\zeta_{a,b}| + |\zeta_{c,d}|$. Then*

$$\mu_*(\zeta_{a,b} \otimes \zeta_{c,d}) = \binom{a+c}{c}\binom{b+d}{d} \zeta_{a+c,b+d} + nm\, \zeta_{a,b} * \zeta_{c,d}\,.$$

Proof. Let $q = a + b + c + d$. By Lemma 2.2.27 we can generally write

$$\mu_*(\zeta_{a,b} \otimes \zeta_{c,d}) = \sum_j \text{primitives } \zeta_{q-j,j} + \sum \text{decomposables of total degree } q\,. \quad (2.33)$$

To determine the right hand side of this equation, we compute the Kronecker pairing with suitable cohomology classes. Suppose that $\zeta_{a_1,b_1}^{c_1} * \cdots * \zeta_{a_r,b_r}^{c_r}$ is a monomial in the Pontrjagin ring $H_*(X[nm])$ written in such a way that $(a_i, b_i) \neq (a_j, b_j)$ whenever $i \neq j$. One can use the fact that the homology classes ζ_{a_i,b_i} and the cohomology classes z_{a_i,b_i} are primitive (cf. Lemma 2.2.27) to show that the dual cohomology class of $\zeta_{a_1,b_1}^{c_1} * \cdots * \zeta_{a_r,b_r}^{c_r}$ is given by

$$(\zeta_{a_1,b_1}^{c_1} * \cdots * \zeta_{a_r,b_r}^{c_r})^* = \frac{z_{a_1,b_1}^{c_1} \cdots z_{a_r,b_r}^{c_r}}{c_1! \cdots c_r!} \,. \tag{2.34}$$

Now assume that $\zeta_{a_1,b_1}^{c_1} * \cdots * \zeta_{a_r,b_r}^{c_r}$ has total degree q. Its coefficient in the expansion (2.33) can be computed via

$$\frac{\langle z_{a_1,b_1}^{c_1} \cdots z_{a_r,b_r}^{c_r}, \mu_*(\zeta_{a,b} \otimes \zeta_{c,d})\rangle}{c_1! \cdots c_r!} = \frac{\langle \mu_{n,m}^*(z_{a_1,b_1})^{c_1} \cdots \mu_{n,m}^*(z_{a_r,b_r})^{c_r}, \zeta_{a,b} \otimes \zeta_{c,d}\rangle}{c_1! \cdots c_r!} \,,$$

which is non-zero only if the term $z_{a,b} \otimes z_{c,d}$ appears in the expansion of the cohomology product on the right hand side. By Lemma 2.2.29 (this uses the assumption on n, m), this happens if either $z_{a_1,b_1}^{c_1} \cdots z_{a_r,b_r}^{c_r} = z_{a+c,b+d}$ in which case the pairing evaluates to $\binom{a+c}{c}\binom{b+d}{d}$, or if $z_{a_1,b_1}^{c_1} \cdots z_{a_r,b_r}^{c_r} = z_{a,b} z_{c,d}$. In the latter case we have

$$\begin{aligned} \mu_{n,m}^*(z_{a,b})\mu_{n,m}^*(z_{c,d}) = (z_{0,0} \otimes z_{c,d})(z_{a,b} \otimes z_{0,0}) + (z_{c,d} \otimes z_{0,0})(z_{0,0} \otimes z_{a,b}) \\ + \text{ terms evaluating to zero on } \zeta_{a,b} \otimes \zeta_{c,d} \,. \end{aligned}$$

If $z_{0,0}$ appears on the left hand side of $\otimes$, it gives multiplicity n while on the right hand side of $\otimes$ it gives multiplicity m. If $(a, b) \neq (c, d)$ then only the first term contributes and evaluates to nm on $\zeta_{a,b} \otimes \zeta_{c,d}$. If $(a, b) = (c, d)$ then the first two terms evaluate to nm, but the result is divided by 2! because of the denominator in (2.34). In any case the pairing evaluates to nm. □

A basic computation in Hopf rings will now prove Proposition 2.2.28.

Proof of Proposition 2.2.28. We have an isomorphism of Pontrjagin algebras

$$H_*(\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U) \cong \mathbb{Q}[\mathbb{Z}] \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_{a,b} \,|\, (a, b) \in \mathbb{N}^2 - \{(0, 0)\}] \,, \tag{2.35}$$

where $\mathbb{Q}[\mathbb{Z}]$ is the rational group ring of the additive group of integers. Denote the coproduct by Δ. If $[n] \in H_0(\mathbb{Z})$ denotes the homology class determined by $n \in \mathbb{Z}$, then the coproduct in $\mathbb{Q}[\mathbb{Z}]$ is given by $\Delta([n]) = [n] \otimes [n]$. Furthermore, because $\mathbb{Z}$ is a ring, the group ring $\mathbb{Q}[\mathbb{Z}]$ is a Hopf ring with $[n] \circ [m] = [nm]$.

Let $i_n : X[n] \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$ be the inclusion of the n-th component. Then i_n sends the class $\zeta_{a,b} \in H_{2(a+b)}(X[n])$ to the class $[n] \otimes \zeta_{a,b}$ under the isomorphism (2.35). Now let $\zeta_{a,b},\, \zeta_{c,d} \in H_*(B_{\mathrm{com}}U /\!\!/ U)$ be given and choose $n, m \geq |\zeta_{a,b}| + |\zeta_{c,d}|$. The commutative diagram

$$\begin{array}{ccc} X[n] \times X[m] & \xrightarrow{\mu_{n,m}} & X[nm] \\ \downarrow{\scriptstyle i_n \times i_m} & & \downarrow{\scriptstyle i_{nm}} \\ (\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U) \times (\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U) & \xrightarrow{\otimes} & \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \end{array}$$

together with Lemma 2.2.30 implies that

$$([n] \otimes \zeta_{a,b}) \circ ([m] \otimes \zeta_{c,d}) = \binom{a+c}{c}\binom{b+d}{d}[nm] \otimes \zeta_{a+c,b+d} + nm\,[nm] \otimes (\zeta_{a,b} * \zeta_{c,d}) \quad (2.36)$$

in $H_*(\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U)$. Recall from Lemma 2.2.27 that $\zeta_{a,b}$ is primitive, hence

$$\Delta([n] \otimes \zeta_{a,b}) = ([n] \otimes \zeta_{a,b}) \otimes ([n] \otimes 1) + ([n] \otimes 1) \otimes ([n] \otimes \zeta_{a,b})\,.$$

We can write $[m] \otimes \zeta_{c,d} = ([0] \otimes \zeta_{c,d}) * ([m] \otimes 1)$ in the Pontrjagin ring and apply left distributivity (2.26) to obtain

$$\begin{aligned} &([n] \otimes \zeta_{a,b}) \circ ([m] \otimes \zeta_{c,d}) \\ &\quad = (([n] \otimes \zeta_{a,b}) \circ ([0] \otimes \zeta_{c,d})) * (([n] \otimes 1) \circ ([m] \otimes 1)) \\ &\qquad + (([n] \otimes 1) \circ ([0] \otimes \zeta_{c,d})) * (([n] \otimes \zeta_{a,b}) \circ ([m] \otimes 1)) \\ &\quad = (([n] \otimes \zeta_{a,b}) \circ ([0] \otimes \zeta_{c,d})) * ([nm] \otimes 1) \\ &\qquad + (([n] \otimes 1) \circ ([0] \otimes \zeta_{c,d})) * (([0] \otimes \zeta_{a,b}) \circ ([m] \otimes 1)) * ([nm] \otimes 1)\,. \end{aligned}$$

To evaluate $([n] \otimes 1) \circ ([0] \otimes \zeta_{c,d})$ we write $[n] \otimes 1 = ([n-1] \otimes 1) * ([1] \otimes 1)$ and apply right distributivity, which yields

$$\begin{aligned} ([n] \otimes 1) \circ ([0] \otimes \zeta_{c,d}) &= (([n-1] \otimes 1) \circ ([0] \otimes \zeta_{c,d})) * (([1] \otimes 1) \circ ([0] \otimes 1)) \\ &\quad + (([n-1] \otimes 1) \circ ([0] \otimes 1)) * (([1] \otimes 1) \circ ([0] \otimes \zeta_{c,d})) \\ &= ([n-1] \otimes 1) \circ ([0] \otimes \zeta_{c,d}) + [0] \otimes \zeta_{c,d}\,. \end{aligned}$$

In the second equation we have used the relation $([k] \otimes 1) \circ ([0] \otimes 1) = [0] \otimes 1$ and the fact that $[1] \otimes 1$ is the unit with respect to the $\circ$ product. Inductively, we see that

$$([n] \otimes 1) \circ ([0] \otimes \zeta_{c,d}) = n\,[0] \otimes \zeta_{c,d}$$

and similarly $([0] \otimes \zeta_{a,b}) \circ ([m] \otimes 1) = m\,[0] \otimes \zeta_{a,b}$.

Writing $[n] \otimes \zeta_{a,b} = ([n] \otimes 1) * ([0] \otimes \zeta_{a,b})$ and applying right distributivity we also find that

$$([n] \otimes \zeta_{a,b}) \circ ([0] \otimes \zeta_{c,d}) = ([0] \otimes \zeta_{a,b}) \circ ([0] \otimes \zeta_{c,d})\,.$$

This uses the relation $([0] \otimes \zeta_{a,b}) \circ ([0] \otimes 1) = 0$.

Putting all this together and multiplying both sides of (2.36) by $[-nm] \otimes 1$ in the Pontrjagin ring we obtain

$$\begin{aligned}([0] \otimes \zeta_{a,b}) \circ ([0] \otimes \zeta_{c,d}) + nm\,[0] \otimes (\zeta_{a,b} * \zeta_{c,d})\\ = \binom{a+c}{c}\binom{b+d}{d}\,[0] \otimes \zeta_{a+c,b+d} + nm\,[0] \otimes (\zeta_{a,b} * \zeta_{c,d})\,.\end{aligned}$$

This can be read as a formula in $\tilde{H}_*(X[0])$. Cancelling the two appearances of $nm\,[0] \otimes (\zeta_{a,b} * \zeta_{c,d})$ yields the desired relation. □

2.3 Real commutative K-theory

This section records some basic results about the classifying space $B_{\mathrm{com}}O$ for the real variant of commutative K-theory. It is shown in [6] that $B_{\mathrm{com}}O$ is a non-unital E_∞-ring space. At this stage we shall not be concerned too much with the multiplicative structure, but shall be mainly interested in the infinite loop space $B_{\mathrm{com}}O$. Similarly to the unitary case, the infinite loop space structure can be obtained by regarding $B_{\mathrm{com}}O$ as the homotopy fibre of an infinite loop map as follows. Analogously to Definition 2.1.3 we form the topological category

$$\mathscr{D} = \coprod_{n\geq 0} O(n) \ltimes B_{\mathrm{com}}O(n) \tag{2.37}$$

using the action of $O(n)$ on $B_{\mathrm{com}}O(n)$ induced by conjugation. The direct sum of matrices can be used to make $\mathscr{D}$ into a permutative category, so that $B\mathscr{D}$ is an E_∞-space and its group-completion $\Omega B(B\mathscr{D})$ is an infinite loop space. For every $n \geq 1$ the space $B_{\mathrm{com}}O(n)$ is path-connected and has fundamental group $\pi_1(B_{\mathrm{com}}O(n)) \cong \mathbb{Z}_2$, by [6, Lem. 4.3]. Using the fact that the basepoint of $B_{\mathrm{com}}O$ is fixed under the action by O we get a section of the projection map $B_{\mathrm{com}}O /\!\!/ O \to BO$. This shows that $B_{\mathrm{com}}O /\!\!/ O$ has fundamental group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The group-completion theorem [43, 49] implies that $\Omega B(B\mathscr{D}) \simeq \mathbb{Z} \times B_{\mathrm{com}}O /\!\!/ O$ and we obtain a (split) homotopy fibre sequence of infinite loop spaces

$$B_{\mathrm{com}}O \to \mathbb{Z} \times B_{\mathrm{com}}O /\!\!/ O \to \mathbb{Z} \times BO\,.$$

Note that the second arrow is an infinite loop map, because it is obtained, upon group-completion, from the permutative functor $\mathscr{D} \to \coprod_{n\geq 0} O(n)$ which collapses each space $B_{\mathrm{com}}O(n)$ to a point. This infinite loop space structure on $B_{\mathrm{com}}O$ coincides with the one described in [6] using commutative $\mathbb{I}$-monoids, cf. Remark 2.1.7.

2.3.1 The torsionfree part of $\pi_*(B_{\mathrm{com}}O)$

In order to describe complexification and realification in commutative K-theory we work with the permutative categories $\mathscr{C}$ from Definition 2.1.3 and $\mathscr{D}$ from (2.37).

By *complexification* we mean the map of infinite loop spaces

$$c : B_{\mathrm{com}}O \to B_{\mathrm{com}}U\,, \tag{2.38}$$

which results from the obvious inclusions $O(n) \subset U(n)$. More precisely, the embedding $O(n) \hookrightarrow U(n)$ induces a permutative functor $C : \mathscr{D} \to \mathscr{C}$ and therefore an infinite loop map after group-completion $\mathbb{Z} \times B_{\mathrm{com}}O \,/\!\!/\, O \to \mathbb{Z} \times B_{\mathrm{com}}U \,/\!\!/\, U$. The map (2.38) is then induced on homotopy fibres, i.e. we take for c the composition

$$B_{\mathrm{com}}O \xrightarrow{\text{incl.}} \mathbb{Z} \times B_{\mathrm{com}}O \,/\!\!/\, O \longrightarrow \mathbb{Z} \times B_{\mathrm{com}}U \,/\!\!/\, U \simeq (\mathbb{Z} \times BU) \times B_{\mathrm{com}}U \xrightarrow{\text{proj.}} B_{\mathrm{com}}U\,,$$

where $\simeq$ is the splitting of infinite loop spaces in (2.1). If we were also taking into account the tensor product, the functor C could be easily enhanced to a bipermutative functor showing that (2.38) is a map of E_∞-ring spaces, but we shall not use this fact.

For *realification* we consider the standard embedding $U(n) \hookrightarrow O(2n)$ defined in the following way. If an element $X \in U(n)$ is written as a sum of its real and imaginary part, i.e. $X = A + iB$, then

$$X \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,. \tag{2.39}$$

This is a group homomorphism, hence induces a functor $R : \mathscr{C} \to \mathscr{D}$. As opposed to C the functor R is not permutative, as the embedding $U(n) \hookrightarrow O(2n)$ is compatible with block sum only up to conjugation in $O(2n)$ by a suitable permutation matrix. However, it is easily checked that R is a strong symmetric monoidal functor, which is enough to produce an infinite loop map upon group-completion, in view of a well known and classical result of Isbell [36, 23]. Thus realification yields an infinite loop map

$$r : B_{\mathrm{com}}U \to B_{\mathrm{com}}O\,. \tag{2.40}$$

Let $\psi^{-1} : B_{\mathrm{com}}U \to B_{\mathrm{com}}U$ denote complex conjugation.

Lemma 2.3.1. *There are homotopies of infinite loop maps $r \circ c \simeq 2$ and $c \circ r \simeq 1 + \psi^{-1}$.*

Proof. We only need adapt the arguments for ordinary K-theory. The composition of the standard maps $O(n) \to U(n) \to O(2n)$ takes $X \in O(n)$ to $X \oplus X$, from which the first homotopy follows. On the other hand, the composite map $U(n) \to O(2n) \to U(2n)$ takes $X = A + iB \in U(n)$ to the matrix displayed in (2.39), which is conjugate in $U(2n)$ to the matrix

$$X \oplus X^* := \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}\,.$$

The element conjugating (2.39) into $X \oplus X^*$ is the matrix $\gamma_n = \begin{pmatrix} iI_n & I_n \\ -iI_n & I_n \end{pmatrix}$ which may be seen as a morphism in the category $\mathscr{C}$. In fact, the γ_n for $n \geq 0$ are the components of a natural isomorphism of functors

$$C \circ R \xrightarrow{\cong} \oplus \circ (id \times \psi^{-1}) \circ \Delta\,, \tag{2.41}$$

where $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ is the diagonal and where we abusively write $\psi^{-1} : \mathscr{C} \to \mathscr{C}$ for complex conjugation in the category $\mathscr{C}$. It is straightforward to check that (2.41) is a symmetric monoidal natural transformation, hence induces a homotopy of E_∞-maps on classifying spaces. This shows that $c \circ r \simeq 1 + \psi^{-1}$ as infinite loop maps.[1] $\square$

Theorem 2.3.2. *All torsion in the homotopy groups of $B_{\mathrm{com}}O$ is two-primary and the free part is*

$$\pi_n(B_{\mathrm{com}}O)/\mathrm{torsion} \cong \begin{cases} \mathbb{Z}^{2k}\,, & n = 4k \\ 0\,, & \text{else.} \end{cases}$$

Proof. We invert 2 and consider the spaces $B_{\mathrm{com}}O[1/2]$ and $B_{\mathrm{com}}U[1/2]$. It follows from Lemma 2.3.1 that the map $r \circ c : B_{\mathrm{com}}O[1/2] \to B_{\mathrm{com}}O[1/2]$ is a homotopy equivalence. In particular, c is injective and r is surjective on homotopy groups after inverting 2. By Lemma 2.2.18 and Lemma 2.3.1, the map $c \circ r : B_{\mathrm{com}}U[1/2] \to B_{\mathrm{com}}U[1/2]$ induces an isomorphism on homotopy groups in degrees $\equiv 0 \bmod 4$ and it induces the trivial map in all other degrees. Thus, by Theorem 2.2.13

$$\pi_{4k}(B_{\mathrm{com}}O)[1/2] \cong \pi_{4k}(B_{\mathrm{com}}U)[1/2] \cong \mathbb{Z}[1/2]^{2k}$$

for all $k \geq 0$. If $n \not\equiv 0 \bmod 4$ the composite map

$$\pi_n(B_{\mathrm{com}}U)[1/2] \xrightarrow{r} \pi_n(B_{\mathrm{com}}O)[1/2] \xrightarrow{c} \pi_n(B_{\mathrm{com}}U)[1/2]$$

is zero, but c is injective and r is surjective, whence $\pi_n(B_{\mathrm{com}}O)[1/2] = 0$. $\square$

As a corollary we can describe the rational cohomology rings of $B_{\mathrm{com}}O$ and $B_{\mathrm{com}}SO$. We first record the following fact.

Lemma 2.3.3. *There is a homotopy fibre sequence of infinite loop spaces*

$$B_{\mathrm{com}}SO \xrightarrow{\text{incl.}} B_{\mathrm{com}}O \xrightarrow{\det} \mathbb{R}P^\infty\,.$$

[1] To be precise, we get a homotopy between maps defined on $\mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U$ rather than $B_{\mathrm{com}}U$. To get the homotopy $c \circ r \simeq 1 + \psi^{-1}$ we consider the composition $B_{\mathrm{com}}U \times [0,1] \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \times [0,1] \to \mathbb{Z} \times B_{\mathrm{com}}U /\!\!/ U \simeq (\mathbb{Z} \times BU) \times B_{\mathrm{com}}U \to B_{\mathrm{com}}U$, where the first map is inclusion, the second map is the homotopy induced by (2.41) and the last map is projection. The equivalence $\simeq$ is the splitting (2.1) which is natural with respect to both maps $c \circ r$ and $1 + \psi^{-1}$.

Proof. This follows easily from the isomorphisms

$$\begin{aligned}\{\pm I_{2n+1}\} \times SO(2n+1) &\xrightarrow{\cong} O(2n+1)\\ (\epsilon, A) &\longmapsto \epsilon A\end{aligned}$$

for all $n \geq 0$. □

Corollary 2.3.4. *Let* $\mathfrak{M}_O := \{(a,b) \in \mathbb{N}^2 \,|\, b > 0,\, a+b \equiv 0 \bmod 2\}$. *Then we have isomorphisms of* $\mathbb{Q}$*-algebras*

$$H^*(B_{\mathrm{com}}O, \mathbb{Q}) \cong H^*(B_{\mathrm{com}}SO, \mathbb{Q}) \cong \mathbb{Q}[z_{a,b} \,|\, (a,b) \in \mathfrak{M}_O]\,,$$

where the $z_{a,b}$ *are of degree* $|z_{a,b}| = 2(a+b)$.

Proof. Theorem 2.3.2 implies that $\pi_*(B_{\mathrm{com}}O) \otimes \mathbb{Q}$ is a vector space of dimension $2k$ in degree $4k$ for all $k \geq 0$ and of dimension zero in the remaining degrees. The classical theorem of Milnor-Moore [45, App. A] asserts that the rational Pontrjagin algebra $H_*(B_{\mathrm{com}}O, \mathbb{Q})$ is a free graded commutative algebra with one primitive generator for each basis element of $\pi_*(B_{\mathrm{com}}O) \otimes \mathbb{Q}$. We can then dualise to obtain the displayed result for $H^*(B_{\mathrm{com}}O, \mathbb{Q})$. The isomorphism $H^*(B_{\mathrm{com}}O, \mathbb{Q}) \cong H^*(B_{\mathrm{com}}SO, \mathbb{Q})$ follows from Lemma 2.3.3 which shows that the inclusion $B_{\mathrm{com}}SO \to B_{\mathrm{com}}O$ is a rational homotopy equivalence. □

Remark 2.3.5. Alternatively, the algebra $H^*(B_{\mathrm{com}}O, \mathbb{Q})$ can be computed using the methods in [5, §8] (i.e. Theorem 1.1.8), since the inclusion $B_{\mathrm{com}}SO_{\mathbb{1}} \subset B_{\mathrm{com}}O$ is a rational homotopy equivalence by (2.46) below. This method involves the problem of determining Weyl group invariants, but this problem will be the same as the one arising for the symplectic groups in [5, §8.3].

2.3.2 Orthogonal representations and path-components

In this section we apply the work of Rojo [52] to derive an exact sequence relating real commutative K-theory to singular mod-2 homology. The idea is that of an eigenvalue map similar to the one we considered in Section 2.2.1 for the complex K-theory.

Let $\tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$ denote the augmentation ideal in the $\mathbb{Z}_2$-group ring of $\mathbb{R}P^\infty$. This is the space denoted by $AG(\mathbb{R}P^\infty, 0; 2)$ in the classical paper of Dold-Thom [13, §4.9]. We define a continuous map

$$\theta : B_{\mathrm{com}}O \to \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty] \tag{2.42}$$

in the following way. Let $\rho \in \mathrm{Hom}(\mathbb{Z}^k, O(n))$. The *normal form* in linear algebra tells us that there are representations $\eta_i : \mathbb{Z}^k \to SO(2)$ and $\ell_j : \mathbb{Z}^k \to O(1)$, so that ρ is isomorphic to a direct sum

$$\rho \cong \eta_1 \oplus \cdots \oplus \eta_m \oplus \ell_1 \oplus \cdots \oplus \ell_{n-2m}\,, \tag{2.43}$$

see e.g. [53, App. A]. Note that such ℓ_j is precisely an element in $O(1)^k$. Let $\mathbb{Z}_2[O(1)^k]$ denote the $\mathbb{Z}_2$-group ring. For every $k \geq 0$ and $n \geq 1$ we define a set map

$$\theta_{k,n} : \mathrm{Hom}(\mathbb{Z}^k, O(n)) \to \mathbb{Z}_2[O(1)^k] \tag{2.44}$$

by choosing for $\rho \in \mathrm{Hom}(\mathbb{Z}^k, O(n))$ any splitting of the form (2.43) and sending

$$\rho \mapsto \sum_{i=1}^{n-2m} \ell_i \,.$$

For $k = 0$ we interpret this as the map which sends the unique element in $\mathrm{Hom}(\mathbb{Z}^k, O(n))$ to n mod 2.

Lemma 2.3.6. *$\theta_{k,n}$ is welldefined.*

Proof. Two splittings of the form (2.43) can differ by

(1) a permutation of the summands

(2) a replacement $\eta_i \leftrightarrow \eta_i^{-1}$ (by η_i^{-1} we mean the representation given by $\eta_i^{-1}(a) = \eta_i(a)^{-1}$ for $a \in \mathbb{Z}^k$)

(3) regarding the sum of two $O(1)$-representations as an $SO(2)$-representation or vice versa,

or a combination of (1)-(3). It is clear that the image of ρ under $\theta_{k,n}$ is unaffected by (1) and (2). For (3) we note that an $SO(2)$-representation η splits as a sum of two $O(1)$-representations, i.e. $\eta \cong \ell \oplus \ell'$, if and only if $\ell = \ell'$, for otherwise the determinant of η were not the trivial representation. Since we are taking mod-2 coefficients, this shows that the image of ρ under $\theta_{k,n}$ is invariant under (3). Altogether we see that $\theta_{k,n}$ is welldefined. □

We next argue that the $\theta_{k,n}$ induce a welldefined map in the limit $n \to \infty$. Let $\mathbb{1} \oplus \mathbb{1}$ denote the trivial $SO(2)$-representation. We obtain the stable representation space $\mathrm{Hom}(\mathbb{Z}^k, O)$ as the colimit of the sequence

$$\mathrm{Hom}(\mathbb{Z}^k, O(0)) \xrightarrow{\oplus \mathbb{1} \oplus \mathbb{1}} \mathrm{Hom}(\mathbb{Z}^k, O(2)) \xrightarrow{\oplus \mathbb{1} \oplus \mathbb{1}} \dots \xrightarrow{\oplus \mathbb{1} \oplus \mathbb{1}} \mathrm{Hom}(\mathbb{Z}^k, O(2n)) \xrightarrow{\oplus \mathbb{1} \oplus \mathbb{1}} \dots .$$

Thus to define a map on $\mathrm{Hom}(\mathbb{Z}^k, O)$ it suffices to define compatible maps on spaces of *even dimensional* representations. For an even dimensional representation the number of $O(1)$-representations in a splitting of the form (2.43) is even, so the map $\theta_{k,2n}$ factors through the subspace of $\mathbb{Z}_2[O(1)^k]$ of elements of even augmentation. Thus let $\tilde{\mathbb{Z}}_2[O(1)^k]$ denote the kernel of the augmentation map $\mathbb{Z}_2[O(1)^k] \to \mathbb{Z}_2$. Then $\theta_{k,2n}$ lifts to a map

$\tilde{\theta}_{k,2n} : \mathrm{Hom}(\mathbb{Z}^k, O(2n)) \to \tilde{\mathbb{Z}}_2[O(1)^k]$. Because we are stabilising by adding a trivial *two* dimensional representation and we are taking coefficients *mod*-2, the maps $\tilde{\theta}_{k,2n}$ are invariant under adding $\mathbb{1} \oplus \mathbb{1}$ and therefore induce a welldefined map from the colimit

$$\theta_k : \mathrm{Hom}(\mathbb{Z}^k, O) \to \tilde{\mathbb{Z}}_2[O(1)^k]\,.$$

Finally, we claim that θ_k is continuous. In fact, more is true: In [52] Rojo counts the number of path-components of $\mathrm{Hom}(\mathbb{Z}^k, O(n))$. To do this he sets up a bijection between the set $\pi_0(\mathrm{Hom}(\mathbb{Z}^k, O(n)))$ and a certain set of subsets of $O(1)^k$ depending on n. He also shows that $\pi_0(\mathrm{Hom}(\mathbb{Z}^k, O(n)))$ becomes a $\mathbb{Z}_2$-module once n is large compared to k. The abelian group structure is induced by the direct sum of representations. In the large n limit, Rojo's work can be read so as to say that θ_k is *precisely* the projection map sending a representation to its path-component in $\pi_0(\mathrm{Hom}(\mathbb{Z}^k, O))$. This means that $\pi_0(\mathrm{Hom}(\mathbb{Z}^k, O)) \cong \tilde{\mathbb{Z}}_2[O(1)^k]$ as $\mathbb{Z}_2$-modules and that θ_k is the projection map

$$\theta_k \equiv \pi_0 : \mathrm{Hom}(\mathbb{Z}^k, O) \to \pi_0(\mathrm{Hom}(\mathbb{Z}^k, O))\,. \tag{2.45}$$

Letting k vary we obtain a map of simplicial spaces $\theta_* : \mathrm{Hom}(\mathbb{Z}^*, O) \to \tilde{\mathbb{Z}}_2[B_*O(1)]$ and our desired map (2.42) is the map induced on geometric realisations.

The following theorem is the main result of this section. We prove the theorem in Section 2.3.3.

Theorem 2.3.7. *The following sequence of E_∞-ring spaces and -maps is a homotopy fibre sequence,*

$$B_{\mathrm{com}}SO_{\mathbb{1}} \xrightarrow{\text{incl.}} B_{\mathrm{com}}O \xrightarrow{\theta} \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]. \tag{2.46}$$

The Dold-Thom theorem [13, Thm. 6.10] implies that the homotopy groups of the space $\tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$ are the reduced mod-2 homology groups of $\mathbb{R}P^\infty$. For a choice of a splitting $\tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty] \simeq \prod_{n\geq 1} K(\mathbb{Z}_2, n)$ we see that θ describes a natural transformation of cohomology theories

$$\widetilde{KO}_{\mathrm{com}} \xrightarrow{\theta} \prod_{n\geq 1} \tilde{H}^n(-, \mathbb{Z}_2)\,. \tag{2.47}$$

The definition of θ also shows that the composition

$$B_{\mathrm{com}}O \xrightarrow{\theta} \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty] \xrightarrow{\text{fusion}} \mathbb{R}P^\infty$$

is simply the determinant map $\det : B_{\mathrm{com}}O \to BO(1) \simeq \mathbb{R}P^\infty$. By "fusion" we mean the map extending the identity homomorphism of $\mathbb{R}P^\infty$ over $\tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$. Thus we may interpret (2.47) as a refinement of the first Stiefel-Whitney class $w_1 : \widetilde{KO} \to \tilde{H}^1(-, \mathbb{Z}_2)$.

We end this section with the following application of Theorem 2.3.7.

Lemma 2.3.8. *The real commutative K-theory of S^n in dimensions $n = 0, 1, 2, 3$ is as follows:*

n	0	1	2	3
$\widetilde{KO}_{\mathrm{com}}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
$\widetilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0

Proof. It is clear that $\widetilde{KO}_{\mathrm{com}}(S^0) \cong \mathbb{Z}$, so we concentrate on the remaining cases. The inclusion $B_{\mathrm{com}}SO_{\mathbb{1}} \to BSO$ is 3-connected, by Lemma 1.2.11, thus $\pi_i(B_{\mathrm{com}}SO_{\mathbb{1}}) = 0$ for $i = 1, 3$ and $\pi_2(B_{\mathrm{com}}SO_{\mathbb{1}}) \cong \widetilde{KO}(S^2) \cong \mathbb{Z}_2$. In particular, the map θ induces an isomorphism on fundamental groups, which we identify with the first Stiefel-Whitney class

$$\widetilde{KO}_{\mathrm{com}}(S^1) \xrightarrow{\cong} \widetilde{KO}(S^1) \xrightarrow{\cong} \mathbb{Z}_2 \,.$$

From the homotopy fibre sequence in the theorem we then get the following exact sequence,

$$0 \to \widetilde{KO}_{\mathrm{com}}(S^3) \to \tilde{H}_3(\mathbb{R}P^\infty, \mathbb{Z}_2) \to \widetilde{KO}(S^2) \to \widetilde{KO}_{\mathrm{com}}(S^2) \to \tilde{H}_2(\mathbb{R}P^\infty, \mathbb{Z}_2) \to 0 \,.$$

From exactness and $\tilde{H}_3(\mathbb{R}P^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2$ we see that $\widetilde{KO}_{\mathrm{com}}(S^3)$ must be either trivial or isomorphic to $\mathbb{Z}_2$. There are different ways to see that the group is non-trivial. For example, there is a canonical map $\mathbb{R}P^\infty \to B_{\mathrm{com}}O$ induced from the inclusion $O(1) \subset O$ which extends over the stable homotopy of $\mathbb{R}P^\infty$. It is easy to see that the composite map

$$Q\mathbb{R}P^\infty \longrightarrow B_{\mathrm{com}}O \xrightarrow{\theta} \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$$

represents the stable mod-2 Hurewicz homomorphism $\pi_*^s(\mathbb{R}P^\infty) \to \tilde{H}_*(\mathbb{R}P^\infty, \mathbb{Z}_2)$. It is well known that this map is non-trivial in degree 3. This follows because the double suspension of $\mathbb{R}P^3$ splits off the top cell, so the restriction of the Hurewicz homomorphism to the 3-skeleton $\mathbb{R}P^3 \subset \mathbb{R}P^\infty$ is non-trivial. In particular, we see that $\pi_3^s(\mathbb{R}P^\infty) \to \pi_3(B_{\mathrm{com}}O)$ is non-zero, so $\widetilde{KO}_{\mathrm{com}}(S^3) \cong \mathbb{Z}_2$.[2] Now we are left with an extension

$$0 \to \widetilde{KO}(S^2) \to \widetilde{KO}_{\mathrm{com}}(S^2) \to \tilde{H}_2(\mathbb{R}P^\infty, \mathbb{Z}_2) \to 0 \,.$$

The map $\widetilde{KO}(S^2) \to \widetilde{KO}_{\mathrm{com}}(S^2)$ is split by the projection map from commutative K-theory to ordinary K-theory, so we conclude that $\widetilde{KO}_{\mathrm{com}}(S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

[2] In fact, by [33] the group $\pi_3^s(\mathbb{R}P^\infty)$ is cyclic of order 8, so we can identify the map $\pi_3^s(\mathbb{R}P^\infty) \to \widetilde{KO}_{\mathrm{com}}(S^3)$ with mod-2 reduction $\mathbb{Z}_8 \to \mathbb{Z}_2$.

2.3.3 Proof of Theorem 2.3.7

We first claim that all three terms in (2.46) are E_∞-ring spaces. The group-ring $\tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$ is even a strictly commutative ring, so there is nothing to show. That $B_{\mathrm{com}}O$ is an E_∞-ring space was shown in [6]; the reader may also look at the introduction to this chapter. For $B_{\mathrm{com}}SO_{\mathbb{1}}$ it follows in exactly the same way, by noting that the conjugation action of $O(n)$ on $B_{\mathrm{com}}O(n)$ restricts to the subspace $B_{\mathrm{com}}SO(n)_{\mathbb{1}}$. This last observation is obvious.

Next we argue that both maps in (2.46) are E_∞-ring maps. This is clear for the inclusion $B_{\mathrm{com}}SO_{\mathbb{1}} \to B_{\mathrm{com}}O$. For the characteristic map $\theta : B_{\mathrm{com}}O \to \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$ we must analyse how the maps $\theta_{k,n}$ in (2.44) behave with respect to direct sum and tensor product. One property is clear, namely additivity. If $\rho_n \in \mathrm{Hom}(\mathbb{Z}^k, O(n))$ and $\rho_m \in \mathrm{Hom}(\mathbb{Z}^k, O(m))$, then clearly

$$\theta_{k,n+m}(\rho_n \oplus \rho_m) = \theta_{k,n}(\rho_n) + \theta_{k,m}(\rho_m)\,.$$

From this we infer that θ is an infinite loop map. To see that θ is an E_∞-ring map we must check if

$$\theta_{k,nm}(\rho_n \otimes \rho_m) = \theta_{k,n}(\rho_n)\theta_{k,m}(\rho_m) \tag{2.48}$$

holds in $\mathbb{Z}_2[O(1)^k]$. The definition shows that the maps $\theta_{k,n}$ factor through isomorphism classes of representations. This observation was already made by Rojo [52]. For isomorphism classes the tensor product is strictly distributive over the direct sum. Thus, if we assume that ρ_n and ρ_m are split as in (2.43), then there are three types of products which can occur in $\rho_n \otimes \rho_m$:

(i) Suppose $\rho_n \otimes \rho_m$ contains a summand of the form $\eta_1 \otimes \eta_2$ with $\eta_i : \mathbb{Z}^k \to SO(2)$, $i = 1, 2$. As $\mathrm{Hom}(\mathbb{Z}^k, SO(2))$ is path-connected, the product $\eta_1 \otimes \eta_2$ lies in the component of the trivial representation, that is $\eta_1 \otimes \eta_2 \in \mathrm{Hom}(\mathbb{Z}^k, SO(4))_{\mathbb{1}}$. The description of the path-components of $\mathrm{Hom}(\mathbb{Z}^k, SO(n))$ in [52] implies that every element of $\mathrm{Hom}(\mathbb{Z}^k, SO(4))_{\mathbb{1}}$ splits as a direct sum of $SO(2)$-representations. Therefore, the summand $\eta_1 \otimes \eta_2$ does not contribute to $\theta_{k,nm}(\rho_n \otimes \rho_m)$.

(ii) The product $\rho_n \otimes \rho_m$ may contain a summand of the form $\eta \otimes \ell$ with $\eta : \mathbb{Z}^k \to SO(2)$ and $\ell : \mathbb{Z}^k \to O(1)^k$. Then $\eta \otimes \ell$ is again an $SO(2)$-representation, because of the fact from Linear algebra that $\det(\eta \otimes \ell) = \det(\eta)\det(\ell)^2 = \mathbb{1}$. Thus $\eta \otimes \ell$ does not contribute to $\theta_{k,nm}(\rho_n \otimes \rho_m)$.

(iii) Finally, $\rho_n \otimes \rho_m$ may contain a tensor product of $O(1)$-representations, but on these the maps $\theta_{k,n}$ are clearly multiplicative.

Together (i)-(iii) show that (2.48) holds, from which we conclude that $\theta : B_{\mathrm{com}}O \to \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$ is an E_∞-ring map.

It remains to show that the sequence of spaces $B_{\mathrm{com}}SO_{\mathbb{1}} \xrightarrow{\text{incl.}} B_{\mathrm{com}}O \xrightarrow{\theta} \tilde{\mathbb{Z}}_2[\mathbb{R}P^\infty]$ is a homotopy fibre sequence. For this we prove the following lemma, which is stated in a stronger form than we shall eventually require.

Lemma 2.3.9. *Let $k \geq 0$. The direct sum of representations makes* $\mathrm{Hom}(\mathbb{Z}^k, O)$ *into an infinite loop space.*

Sketch proof. There are different ways to see this. For example, we could show that $\mathrm{Hom}(\mathbb{Z}^k, O(-))$ defines an $\mathscr{I}$-functor in the sense of [37, Def. 1.4]. Thus $\mathrm{Hom}(\mathbb{Z}^k, O)$ admits an action of the linear isometries operad. The following sketch proof is more in the spirit of [6]. We use the action of $O(n)$ on $\mathrm{Hom}(\mathbb{Z}^k, O(n))$ to build a permutative translation category $\mathscr{O}$ whose classifying space is

$$B\mathscr{O} = \coprod_{n \geq 0} \mathrm{Hom}(\mathbb{Z}^k, O(n)) /\!\!/ O(n)\,,$$

cf. Section 2.1.2. We then wish to apply the group-completion theorem to the E_∞-space $B\mathscr{O}$ and identify the space $\mathrm{Hom}(\mathbb{Z}^k, O)$ as the homotopy fibre of the infinite loop map $\Omega B(B\mathscr{O}) \to \mathbb{Z} \times BO$ induced from the projection maps $\mathrm{Hom}(\mathbb{Z}^k, O(n)) \to \mathrm{pt}$. This works indeed, but the group-completion process is now slightly more complicated as we have to deal with the fundamental group as well as the many path-components of $\mathrm{Hom}(\mathbb{Z}^k, O(n))$ for all $n \geq 0$.

One way to argue cleanly is to use a result about the group-completion theorem due to Ramras [48, Thm. 3.6]. He introduces the notion of an *anchored element* in a homotopy abelian monoid. Loosely speaking, if M is a homotopy commutative monoid, then an element $x \in M$ is anchored, if there exists an interchanging homotopy for the product in M which fixes the powers $x^n \in M$ for all $n \geq 0$. In applying the group-completion theorem to M, we form an infinite mapping telescope

$$M_\infty = \mathrm{tel}(M \xrightarrow{\cdot m_0} M \xrightarrow{\cdot m_1} M \xrightarrow{\cdot m_2} \dots)\,.$$

What Ramras shows is that if M_∞ can be formed by stabilising with respect to a single anchored element $x \in M$, then the fundamental group of M_∞ with any choice of base-point is abelian.

The proof of the Lemma now proceeds by showing (a) to group-complete $B\mathscr{O}$ it suffices to stabilise by multiplication with the trivial representation $\mathbb{1} \in \mathrm{Hom}(\mathbb{Z}^k, O(1))$ and (b) there exists a homotopy anchoring $\mathbb{1}^{2N} \in B\mathscr{O}$ for every $N \in \mathbb{N}$. Assertion (a) is proved by using the result of Rojo [52, Prop. 6.5] that $\pi_0(\mathrm{Hom}(\mathbb{Z}^k, O(n)))$ has the structure of an abelian group once n is large compared to k, and (b) follows along the lines of [48, Cor. 4.4]. Using (a) and (b), the group-completion theorem [43, 49] together with [48, Thm. 3.6] implies that $\mathbb{Z} \times \mathrm{Hom}(\mathbb{Z}^k, O) /\!\!/ O$ is an infinite loop space, and the lemma follows. $\square$

Let us go back to the proof of our theorem. For ease of notation, let us write $\Pi_k := \pi_0(\mathrm{Hom}(\mathbb{Z}^k, O))$. For every $k \geq 0$ we have a homotopy fibre sequence

$$\mathrm{Hom}(\mathbb{Z}^k, SO)_{\mathbb{1}} \xrightarrow{\text{incl.}} \mathrm{Hom}(\mathbb{Z}^k, O) \xrightarrow{\theta_k} \tilde{\mathbb{Z}}_2[O(1)^k]\,,$$

because θ_k coincides with the projection $\pi_0 : \mathrm{Hom}(\mathbb{Z}^k, O) \to \Pi_k$, by [52] (cf. (2.45)). We aim to prove that this remains a homotopy fibre sequence after replacing each term by the geometric realisation of the corresponding simplicial space. For this we use the theorem of Bousfield-Friedlander [10, Thm. B.4] which requires us to check that the simplicial spaces $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, O)$ and $k \mapsto \tilde{\mathbb{Z}}_2[O(1)^k]$ satisfy the π_*-Kan condition; but this condition is satisfied for simplicial grouplike E_2-algebras. Let us make this more explicit. We shall use the criterion derived in [10, B.3.1], which applies to simple spaces, in particular to the infinite loop space $\mathrm{Hom}(\mathbb{Z}^k, O)$. For a topological space X and an integer $q \geq 1$ let $\pi_q(X)_{\mathrm{free}}$ denote the set of unbased homotopy classes of maps $S^q \to X$. In order to see that $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, O)$ satisfies the π_*-Kan condition it then suffices to check, by [10, B.3.1], that the canonical map

$$p : \{k \mapsto \pi_q(\mathrm{Hom}(\mathbb{Z}^k, O))_{\mathrm{free}}\} \longrightarrow \{k \mapsto \Pi_k\} \tag{2.49}$$

is a fibration of simplicial sets for all $q \geq 1$. By Lemma 2.3.9, $\mathrm{Hom}(\mathbb{Z}^k, O)$ is a homotopy abelian grouplike H-space, so there is a homotopy equivalence of H-spaces

$$\mathrm{Hom}(\mathbb{Z}^k, O) \simeq \Pi_k \times \mathrm{Hom}(\mathbb{Z}^k, SO)_{\mathbb{1}}\,.$$

Note that this equivalence depends on a choice of representatives for the path-components in Π_k. However, if we take unbased homotopy classes of maps, the isomorphism

$$\pi_q(\mathrm{Hom}(\mathbb{Z}^k, O))_{\mathrm{free}} \cong \Pi_k \times \pi_q(\mathrm{Hom}(\mathbb{Z}^k, SO)_{\mathbb{1}})_{\mathrm{free}} \tag{2.50}$$

is independent of the choice of representatives for the elements of Π_k. Also, the simplicial structure maps for $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, O)$ are maps of H-spaces. Together this implies that the isomorphism (2.50) is an isomorphism of simplicial sets, if we give the right hand side the product simplicial structure. Since $\mathrm{Hom}(\mathbb{Z}^k, SO)_{\mathbb{1}}$ is a path-connected simple space, we also have

$$\pi_q(\mathrm{Hom}(\mathbb{Z}, SO)_{\mathbb{1}})_{\mathrm{free}} \cong \pi_q(\mathrm{Hom}(\mathbb{Z}^k, SO)_{\mathbb{1}}, \mathbb{1})\,. \tag{2.51}$$

This demonstrates that the π_*-Kan condition is satisfied for $k \mapsto \mathrm{Hom}(\mathbb{Z}^k, O)$. Indeed, the map p in (2.49) is isomorphic under (2.50) to the projection onto the first factor and this is a fibration of simplicial sets, because the based homotopy groups (2.51) form a Kan complex. Also $k \mapsto \tilde{\mathbb{Z}}_2[O(1)^k]$ satisfies the π_*-Kan condition, for it can be given the structure of a bisimplicial group, cf. [10, B.3]. This completes the proof of Theorem 2.3.7.

2.3.4 Concluding remark

$\mathbb{Z}_2$-equivariant theory

Instead of treating the complex and real commutative K-theories separately, we could also attempt to describe its $\mathbb{Z}_2$-equivariant refinement, where $\mathbb{Z}_2$ refers to the involution on $B_{\mathrm{com}}U$ given by complex conjugation. The result that $B_{\mathrm{com}}U$ is the infinite loop space underlying the group ring $ku \wedge \mathbb{C}P^\infty$ lets us guess that the $\mathbb{Z}_2$-equivariant theory should be described by $kr \wedge \mathbb{C}P^\infty$, where kr is the connective version of Atiyah's Real K-theory [7] and $\mathbb{C}P^\infty$ is considered as a space with involution by complex conjugation. Real commutative K-theory should then be described by the $\mathbb{Z}_2$-fixed points of this spectrum. From a private conversation that we had with Markus Hausmann it appears that the homotopy groups of this fixed point spectrum indeed agree with our partial computation of $\pi_*(B_{\mathrm{com}}O)$ in Theorem 2.3.2 and Lemma 2.3.8. We are thankful for this enlightening discussion.

K-homology theory

An approach that may lead us to identify the homotopy type of $B_{\mathrm{com}}O$ goes via a model for K-homology theory due to Graeme Segal. We are grateful to him for hinting us at [58] and pointing out an alternative approach to show that $B_{\mathrm{com}}U$ represents the ku-homology of $\mathbb{C}P^\infty$. For a space X with a basepoint $0 \in X$ let $\mathrm{Conf}_0(X, (\mathrm{Vect}_{\mathbb{R}}, \oplus))$ denote the space of finite configurations of unordered points in X each labelled by a finite dimensional real vector space. The topology is so that points can be created at $0 \in X$ and if two points collide their labels are added using direct sum. In [58] Segal shows that the functor $X \mapsto \mathrm{Conf}_0(X, (\mathrm{Vect}_{\mathbb{R}}, \oplus))$ represents the reduced ko-homology of X. Suppose now that we wanted to understand the infinite loop space $B_{\mathrm{com}}SO_{\mathbb{1}}$ which appears as the homotopy fibre in Theorem 2.3.7. Let $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ be the space of linear isometric embeddings of $\mathbb{R}^n$ into $\mathbb{R}^\infty$. Instead of $B_{\mathrm{com}}SO_{\mathbb{1}}$ we would consider the homotopy orbit space

$$B_{\mathrm{com}}SO_{\mathbb{1}} /\!\!/ O \simeq \bigcup_{n\geq 1} \mathscr{L}(\mathbb{R}^{2n}, \mathbb{R}^\infty) \times_{O(2n)} B_{\mathrm{com}}SO(2n)_{\mathbb{1}}\,, \tag{2.52}$$

which tells us just as much about $B_{\mathrm{com}}SO_{\mathbb{1}}$ in view of the infinite loop space splitting $B_{\mathrm{com}}SO_{\mathbb{1}} /\!\!/ O \simeq BO \times B_{\mathrm{com}}SO_{\mathbb{1}}$. Because a representation of $\mathbb{Z}^k$ into $SO(2n)$ can be conjugated into normal form (see (2.43)) and because we only consider representations which are in the path-component of the trivial representation, a point on the right hand side of

(2.52) describes essentially a configuration of finitely many points in $BSO(2) \simeq \mathbb{C}P^\infty$ each labelled by an oriented two dimensional real vector space. However, because the Weyl group of $SO(2n)$ acts by *signed* permutations on the maximal torus, the points in a configuration will not only be unordered, but they will come in pairs in which a point in $BSO(2)$ is identified with its complex conjugate point if at the same time the orientation of its label is reversed. Thus we expect that (2.52) is related to the real oriented K-homology of $BSO(2)$. For example, it may be a quotient construction thereof, which takes into account the involution on $BSO(2)$ by complex conjugation and on kso by reversing the orientation, or something alike. We hope to pursue this approach in future work.

Chapter 3

A remark on the group-completion theorem

This self-contained chapter is concerned with the group-completion theorem of McDuff-Segal [43]. For M a topological monoid, the group-completion theorem gives a sufficient condition as to when a certain map $M_\infty \to \Omega BM$ defined on an infinite mapping telescope M_∞ is a homology equivalence with integer coefficients. For the case $\pi_0 M = \mathbb{N}$ we shall give here a sufficient condition under which this map is *acyclic*, i.e. induces an isomorphism of homology groups for every choice of a local coefficient system on the target space. In [49] Randal-Williams shows that homotopy commutativity of M is a sufficient condition. Our condition will be weaker than this, but the proof assumes that $\pi_0 M = \mathbb{N}$.

This was a natural question motivated by the results in [6, §3]. As an application of our result we give conditions on a commutative $\mathbb{I}$-monoid X which imply that $\mathrm{hocolim}_{\mathbb{I}} X$ is equivalent to a Quillen plus-construction.

3.1 Background and result

Let M be a topological monoid and let BM be its classifying space. The group-completion theorem as stated by McDuff and Segal [43] relates the homology of ΩBM to the localization of the Pontryagin ring $H_*(M)$ at its multiplicative subset $\pi_0 M$.

Theorem 3.1.1 (McDuff-Segal, [43])**.** *Suppose the localization $H_*(M)[(\pi_0 M)^{-1}]$ can be constructed by right-fractions. Then the natural map $M \to \Omega BM$ induces an isomorphism*

$$H_*(M)[(\pi_0 M)^{-1}] \xrightarrow{\cong} H_*(\Omega BM)\,.$$

In this chapter all homology groups are understood to be singular homology groups with constant integer coefficients. For what is meant by "can be constructed by right-fractions" we refer the reader to [43, Rem. 1], where this is explained. Let us assume for simplicity that $\pi_0 M$ is finitely generated. Let $x_1, \ldots, x_k \in M$ represent a set of generators and define $x := x_1 \cdots x_k \in M$. Then the infinite mapping telescope

$$\mathbb{M}_\infty := \mathrm{tel}(M \xrightarrow{\cdot x} M \xrightarrow{\cdot x} M \xrightarrow{\cdot x} \ldots)$$

has $\pi_0 M$-local homology $H_*(\mathbb{M}_\infty) = H_*(M)[(\pi_0 M)^{-1}]$. In general there is no natural map $\mathbb{M}_\infty \to \Omega BM$ inducing the isomorphism of Theorem 3.1.1. However, a natural map into a weakly equivalent space can be constructed, see [43, 49] for details. We shall not take this too precisely, but simply speak of a comparison map

$$f : \mathbb{M}_\infty \to \Omega BM \, .$$

In the process of group-completion it is desirable to know if this map induces an isomorphism on homology for all choices of local coefficients on the target space; for then ΩBM is a model for the homotopy theoretic group-completion of M. A map inducing an isomorphism for all local coefficients is acyclic, i.e. its homotopy fibre is an acyclic space. If the comparison map f is acyclic, it can be converted into a weak homotopy equivalence by means of the Quillen plus-construction. Randal-Williams [49] has proved the following strengthening of Theorem 3.1.1 under the hypothesis that M is homotopy commutative, see also [44].

Theorem 3.1.2 (Randal-Williams, [49])**.** *Suppose M is homotopy commutative. Then the comparison map f is acyclic.*

The objective of this chapter is to study the acyclicity of the comparison map under a weaker condition than homotopy commutativity. This is condition (†) below. We restrict ourselves to a simplified setting by assuming that the monoid of components is the natural numbers $\pi_0 M = \mathbb{N}$. Our main result is:

Theorem 3.1.3. *Let M be a topological monoid satisfying $\pi_0 M = \mathbb{N}$. Suppose that the localization $H_*(M)[(\pi_0 M)^{-1}]$ can be constructed by right-fractions and that (†) holds. Then the comparison map $f : \mathbb{M}_\infty \to \Omega BM$ is acyclic.*

As a consequence of acyclicity we obtain

Corollary 3.1.4. *With the assumptions of Theorem 3.1.3, the fundamental group of $\mathbb{M}_\infty$ with any choice of basepoint has a perfect commutator subgroup and the induced map $\mathbb{M}_\infty^+ \to \Omega BM$ is a weak homotopy equivalence.*

Here the plus-construction is applied to each path-component separately and with respect to the maximal perfect subgroup of the fundamental group.

Our hypothesis (†) is only a mild commutativity condition which should be easy to check in examples. The proof of Theorem 3.1.3 is given in Section 3.2. In view of Theorem 3.1.1 it reduces to showing that if $f : \mathbb{M}_\infty \to \Omega BM$ induces an isomorphism of integral homology then the assumption (†) is enough to conclude that f induces an isomorphism for all local coefficients on ΩBM. This will be proved using covering spaces.

We now set up some notation. We assume that $\pi_0 M = \mathbb{N}$. Thus we study a sequence of basepointed, path-connected spaces $(M_n, m_n)_{n\geq 0}$ with strictly associative, basepoint preserving product maps

$$\mu_{m,n} : M_m \times M_n \to M_{m+n}$$

for all $m, n \geq 0$. The basepoint $m_0 \in M_0$ serves as a two-sided unit. The following condition on the Pontryagin ring $H_*(M)$ expresses commutativity of left- and right-stabilisation in $H_1(M)$. Let us regard the basepoint $m_n \in M_n$ as a class in $H_0(M_n)$ and write $- \cdot -$ for the Pontryagin product.

(†) There is a cofinal sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ so that for all $k \in \mathbb{N}$ and all $a \in H_1(M_{n_k})$ the equality

$$a \cdot m_{n_k} = m_{n_k} \cdot a$$

holds in $H_1(M) \subseteq H_*(M)$.

Example 3.1.5. If we assume that $\pi_0 M$ is central in $H_*(M)$, then (†) obtains. In particular, (†) holds for homotopy commutative M.

We introduce some further notation. We write $G_n := \pi_1(M_n, m_n)$ and $G'_n := [G_n, G_n]$ for its derived subgroup. We avoid the use of a designated symbol for the concatenation product on fundamental groups. For $a, b \in G_n$ we employ the convention where ab means that a is traversed first, followed by b. We let e_n denote the neutral element of G_n, that is the class of the constant loop based at $m_n \in M_n$. Via pointwise multiplication of loops the maps $\mu_{m,n}$ induce homomorphisms

$$\oplus : G_m \times G_n \to G_{m+n}$$

which we commonly denote by the symbol $\oplus$ to keep the notation simple. In particular, multiplication by e_n describes a homomorphism $- \oplus e_n : G_m \to G_{m+n}$, hence is distributive over the concatenation product. Similarly, $e_m \oplus -$ is a homomorphism. We clearly have $e_n = e_1 \oplus \cdots \oplus e_1$ (n times).

Remark 3.1.6. The identity in (†) is equivalent to saying that for all $a \in G_{n_k}$ there exists $c \in G'_{2n_k}$ so that $a \oplus e_{n_k} = (e_{n_k} \oplus a)c$ holds in G_{2n_k}. Indeed, the definition of the Pontryagin product implies that for all $n \geq 0$ the diagram

$$\begin{array}{ccc} G_n & \xrightarrow{e_1 \oplus -} & G_{n+1} \\ \downarrow & & \downarrow \\ H_1(M_n) & \xrightarrow{m_1 \cdot -} & H_1(M_{n+1}) \end{array}$$

commutes, where the vertical maps are abelianization.

3.2 Proof of Theorem 3.1.3

We will assume that all spaces are locally path-connected and semi-locally simply connected, so that universal covering spaces exist. For example, this includes all CW-complexes. For our purposes this is not a restriction, since all the results remain valid by replacing the spaces by the realization of their singular complex.

Let G_∞ be the colimit of the direct system of groups $- \oplus e_1 : G_n \to G_{n+1}$. It is isomorphic to the fundamental group of the infinite mapping telescope

$$M_\infty := \mathrm{tel}(M_0 \xrightarrow{\cdot m_1} M_1 \xrightarrow{\cdot m_1} M_2 \xrightarrow{\cdot m_1} \dots)\,, \tag{3.1}$$

which we assume is based at $m_0 \in M_0 \subset M_\infty$. Note that $\mathbb{M}_\infty \simeq \mathbb{Z} \times M_\infty$. It suffices to prove the theorem for $f : M_\infty \to \Omega_0 BM$, where $\Omega_0 BM$ is the basepoint component of ΩBM. This will be evident from our proof. In view of Theorem 3.1.1, to prove acyclicity of $M_\infty \to \Omega_0 BM$ it is enough to show that

- the commutator subgroup of $G_\infty = \pi_1(M_\infty, m_0)$ is perfect
- the space M_∞^+ is *weakly simple*, i.e. its fundamental group acts trivially on the homology of its universal covering space.

Indeed, Theorem 3.1.1 asserts that f is an integer homology equivalence, and the first item implies that the induced map $M_\infty^+ \to \Omega_0 BM$ is in addition a π_1-isomorphism. An argument with the Serre spectral sequence (compare (3.2)), which uses the fact that M_∞^+ is weakly simple, then shows that the map induced by $M_\infty^+ \to \Omega_0 BM$ on universal coverings is a homology isomorphism, which is to say that $M_\infty^+ \to \Omega_0 BM$, hence f, is acyclic.

However, instead of working with M_∞^+ and its universal covering, we shall work with M_∞ and the covering space corresponding to the commutator subgroup G'_∞. Let Y_∞ denote this covering space. We use the following lemma, which we prove below:

Lemma 3.2.1. *The action of G_∞/G'_∞ on $H_*(Y_\infty)$ through deck translations is trivial.*

Conceptually, our proof of Theorem 3.1.3 is similar to Wagoner [64, Prop. 1.2].

Proof of Theorem 3.1.3. Our assumptions permit us to apply Theorem 3.1.1, which asserts that $M_\infty \to \Omega_0 BM$ is a homology equivalence for constant integer coefficients. In particular, this gives us an isomorphism $G_\infty / G_{\infty'} \cong \pi_1(\Omega_0 BM)$. Consider the map of homotopy fibre sequences

$$\begin{array}{ccccc} Y_\infty & \longrightarrow & M_\infty & \longrightarrow & B(G_\infty/G'_\infty) \\ \downarrow & & \downarrow & & \| \\ W & \longrightarrow & \Omega_0 BM & \longrightarrow & B(G_\infty/G'_\infty) \end{array} \tag{3.2}$$

where the bottom row is the universal covering sequence for $\Omega_0 BM$, i.e. W is the universal covering space of $\Omega_0 BM$. As is true for any H-space, the fundamental group of $\Omega_0 BM$ acts trivially on the homology of its universal covering space. Moreover, by Lemma 3.2.1, the action of G_∞/G'_∞ on $H_*(Y_\infty)$ is trivial. Thus both fibre sequences in (3.2) have a simple system of local coefficients. Consider now the map of Serre spectral sequences associated to the two rows of diagram (3.2). It follows from the Zeeman comparison theorem [42, Thm. 3.26] that the map between the fibres $Y_\infty \to W$ is an integer homology equivalence. In fact, since W is a simply connected space, this map is acyclic and $\pi_1 Y_\infty = G'_\infty$ is perfect. Taking vertical homotopy fibres in (3.2) we see that the homotopy fibre of $M_\infty \to \Omega_0 BM$ has trivial integer homology, i.e. $M_\infty \to \Omega_0 BM$ is acyclic. □

It remains to show Lemma 3.2.1. Consider one component of the monoid M. As a model for the universal covering space $\tilde{M}_n \to M_n$ we may take

$$\tilde{M}_n = \{\text{homotopy classes of paths } \gamma : I \to M_n \text{ rel } \partial I \mid \gamma(0) = m_n\},$$

suitably topologised [40, 3.8]. The projection to M_n is evaluation at the endpoint of a path. To simplify the notation, we shall denote a path and its homotopy class rel ∂I by the same letter. An element $a \in G_n$ acts on $\tilde{M}_n$ via deck translation. In our chosen model this action corresponds to precomposition of paths $\gamma \mapsto a\gamma$. We define $Y_n := G'_n \backslash \tilde{M}_n$ with the quotient topology. By passage to the quotient, the induced projection $Y_n \to M_n$ is the connected regular covering corresponding to the commutator subgroup G'_n of G_n. In the same way we define $Y_\infty \to M_\infty$, the regular covering corresponding to the subgroup G'_∞ of G_∞.

For $m, n \geq 0$ there is an induced continuous pairing

$$\oplus : Y_m \times Y_n \to Y_{m+n} \tag{3.3}$$

given by pointwise multiplication of paths, i.e. $(\gamma \oplus \eta)(t) = \gamma(t) \cdot \eta(t)$ (product in M). To see that this product is well-defined, let $\gamma \in \tilde{M}_m$ and $\eta \in \tilde{M}_n$ be homotopy classes of paths, and let $c \in G'_m$ and $d \in G'_n$ be commutators. Then (in $\tilde{M}_{m+n}$)

$$\begin{aligned}(c\gamma) \oplus (d\eta) &= ((ce_m)\gamma) \oplus ((e_n d)\eta) = ((ce_m) \oplus (e_n d))(\gamma \oplus \eta) \\ &= (c \oplus e_n)(e_m \oplus d)(\gamma \oplus \eta)\,,\end{aligned}$$

but $(c \oplus e_n)(e_m \oplus d) \in G'_{m+n}$.

In particular, we have the diagram of spaces $- \oplus e_1 : Y_n \to Y_{n+1}$ and we can consider the groups $\operatorname{colim}_n H_*(Y_n)$. The direct limit group G_∞ acts upon these as follows. Let $[a] \in G_\infty$ be represented by some $a \in G_m$ and let $[z] \in \operatorname{colim}_n H_*(Y_n)$ be represented[1] by some $z \in H_*(Y_n)$. Then choose $k \geq \max\{m, n\}$ and define

$$[a][z] := [(a \oplus e_{k-m})(z \oplus e_{k-n})]\,, \tag{3.4}$$

where on the right hand side we use the action of G_k on $H_*(Y_k)$ through deck translations (the action factors through the abelianization G_k/G'_k which is the deck group for Y_k). One may verify that the action (3.4) is well-defined.

Lemma 3.2.2. *There is an isomorphism* $\operatorname{colim}_n H_*(Y_n) \cong H_*(Y_\infty)$ *which respects the action of* G_∞.

Proof. We begin by describing a map $j : \operatorname{tel}_n Y_n \to Y_\infty$. It suffices to give maps $j_n : Y_n \times [n, n+1] \to Y_\infty$ for all $n \geq 0$ which are compatible at the endpoints of the intervals where they are glued together in the telescope. A point in the telescope M_∞ is specified by a pair (x, t) with $t \in \mathbb{R}$ and $x \in M_{\lfloor t \rfloor}$, where $\lfloor t \rfloor \in \mathbb{N}$ denotes the integral part of t. For $t \in \mathbb{R}$ define a path

$$\begin{aligned}\alpha_t : I &\to M_\infty \\ s &\mapsto (m_{\lfloor st \rfloor}, st)\,.\end{aligned}$$

This is the 'straight' path from the basepoint $m_0 \in M_\infty$ to the basepoint $m_{\lfloor t \rfloor} \in M_{\lfloor t \rfloor}$ of the 'slice' at coordinate t in the telescope.

Now suppose $(\gamma, t) \in Y_n \times [n, n+1]$. Then γ represents a homotopy class of a path in $M_{\lfloor t \rfloor}$ based at $m_{\lfloor t \rfloor}$. To this pair we assign the class in Y_∞ of the path

$$j_n(\gamma, t) = \alpha_t \gamma : I \to M_\infty\,.$$

[1]For ease of notation, we shall not distinguish between homology cycles and the homology classes they represent.

One may check that this is well-defined and continuous.[2] Also, the maps j_n for $n \geq 0$ fit nicely together whenever t approaches an integer, and we obtain the desired map j from the homotopy colimit to Y_∞.

We now consider the following diagram

$$\begin{array}{ccccc} \operatorname{tel}_n Y_n & \xrightarrow{\mathrm{ev}_1} & \operatorname{tel}_n M_n & \longrightarrow & \operatorname{tel}_n B(G_n/G'_n) \\ \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle c} \\ Y_\infty & \xrightarrow{\mathrm{ev}_1} & M_\infty & \longrightarrow & B(G_\infty/G'_\infty)\,. \end{array}$$

The map denoted ev_1 is evaluation at the endpoint of a path. By construction of j the left hand square commutes. The vertical arrow labelled c is a weak homotopy equivalence, as one can see by commuting homotopy groups with the telecope. The map $M_\infty \to B(G_\infty/G'_\infty)$ is the classifying map for the covering space Y_∞. The top row is the telescope over the natural homotopy fibre sequences $Y_n \to M_n \to B(G_n/G'_n)$ and thus again a homotopy fibre sequence. Since the right hand square commutes up to homotopy[3], the map j is a weak homotopy equivalence. It therefore induces an isomorphism $H(j) : \operatorname{colim}_n H_*(Y_n) \to H_*(Y_\infty)$.

Finally, we must check that $H(j)$ is compatible with the action of G_∞. Consider the following two diagrams

$$\begin{array}{ccc} Y_k \times \{k\} & \xrightarrow{i_k} & Y_\infty \\ \downarrow{\scriptstyle \text{incl.}} & \nearrow{\scriptstyle j} & \\ \operatorname{tel}_n Y_n & & \end{array} \qquad\qquad \begin{array}{ccc} H_*(Y_k) & \xrightarrow{H(-\oplus e_l)} & H_*(Y_{k+l}) \\ \downarrow{\scriptstyle H(i_k)} & \swarrow{\scriptstyle H(i_{k+l})} & \\ H_*(Y_\infty) & & \end{array} \tag{3.5}$$

The left hand triangle defines the map $i_k : Y_k \to Y_\infty$. With this definition it is readily verified that the right hand triangle commutes. Now suppose we are given equivalence classes $[a] \in G_\infty$ and $[z] \in \operatorname{colim}_n H_*(Y_n)$ represented respectively by $a \in G_m$ and $z \in H_*(Y_n)$. Then, using (3.4)

$$\begin{aligned} H(j)([a][z]) &= H(j)([(a \oplus e_{k-m})(z \oplus e_{k-n})]) = H(i_k)((a \oplus e_{k-m})(z \oplus e_{k-n})) \\ &= \alpha_k (a \oplus e_{k-m})(z \oplus e_{k-n})\,, \end{aligned} \tag{3.6}$$

for some $k \geq \max\{m, n\}$. On the other hand, we have the action of G_∞ on $H_*(Y_\infty)$ through the deck translations. If $[a] \in G_\infty$ is represented by $a \in G_m$, then the isomorphism $G_\infty \cong \pi_1(M_\infty, m_0)$ takes $[a] \mapsto \alpha_m a \bar{\alpha}_m$. Here we denote by $\bar{\alpha}_m$ the inverse path of α_m.

---

[2] In fact, upon passage to quotient spaces, the map j_n arises as a lift of the composite map $\tilde{M}_n \times [n, n+1] \to M_n \times [n, n+1] \hookrightarrow M_\infty$ under the universal covering map $\tilde{M}_\infty \to M_\infty$.

[3] For example, commuting homology with homotopy colimits one may verify that the square commutes on the level of $H_1(-;\mathbb{Z})$; hence, by the universal coefficient theorem, also on $H^1(-; G_\infty/G'_\infty)$. But homotopy classes of maps into $B(G_\infty/G'_\infty)$ are uniquely determined by their effect on $H^1(-; G_\infty/G'_\infty)$.

Thus $[a]$ acts on $\gamma \in Y_\infty$ as $\gamma \mapsto (\alpha_m\, a\, \bar{\alpha}_m)\,\gamma$. Therefore, using commutativity of the right hand diagram in (3.5)

$$\begin{aligned}[a]H(j)([z]) = [a]H(i_n)(z) &= [a \oplus e_{k-m}]H(i_k)(z \oplus e_{k-n}) \\ &= \alpha_k(a \oplus e_{k-m})\bar{\alpha}_k\alpha_k(z \oplus e_{k-n})\,,\end{aligned}$$

which equals (3.6). So $H(j)$ commutes with the action of G_∞ and the assertion of the lemma follows. □

Proof of Lemma 3.2.1. Let $[a] \in G_\infty$ and let z be a cycle in $H_*(Y_\infty)$. By Lemma 3.2.2 and cofinality of the sequence $(n_k)_{k\in\mathbb{N}}$ (cf. (†)) we may assume that there is $n \in \{n_k\}_{k\in\mathbb{N}}$ so that z is represented by a cycle in $H_*(Y_n)$ and that $[a]$ is represented by an element $a \in G_n$. Recall that the colimit system of homology groups is induced by the maps $- \oplus e_1 : Y_n \to Y_{n+1}$. Therefore $z \in H_*(Y_n)$ and $z \oplus e_n \in H_*(Y_{2n})$ represent the same classes in the direct limit. Similarly, $a \in G_n$ and $a \oplus e_n \in G_{2n}$ coincide in G_∞. The commutativity relation (†) implies that there exists $c \in G'_{2n}$ so that $a \oplus e_n = (e_n \oplus a)c$, cf. Remark 3.1.6. Since the action of G_{2n} on Y_{2n} factors through the abelianization, the action of $a \oplus e_n$ on $z \oplus e_n$ can be written

$$(a \oplus e_n)(z \oplus e_n) = (e_n \oplus a)(z \oplus e_n) = (e_n z) \oplus (a e_n) = z \oplus a\,. \tag{3.7}$$

Note that a may be considered as a point in Y_n. Therefore, using (3.7) the action of $[a] \in G_\infty$ on $[z] \in H_*(Y_\infty)$ can be described by choosing representatives $a \in Y_n$ and $z \in H_*(Y_n)$ and computing the image of z under the map

$$H_*(Y_n) \xrightarrow{H(-\oplus a)} H_*(Y_{2n}) \to \operatorname{colim}_n H_*(Y_n) \cong H_*(Y_\infty)\,. \tag{3.8}$$

Here $H(- \oplus a)$ is the map induced on homology by the map of spaces $- \oplus a : Y_n \to Y_{2n}$ (3.3). Since Y_n is path-connected, there is a path from a to e_n which induces a homotopy from $- \oplus a$ to $- \oplus e_n$ (as maps $Y_n \to Y_{2n}$). As a consequence, the first map in (3.8) is the stabilisation map $H(- \oplus e_n)$ for the colimit $\operatorname{colim}_n H_*(Y_n)$, whence we conclude that the action of a on z is trivial. □

3.3 Application to commutative $\mathbb{I}$-monoids

Let $\mathbb{I}$ denote the skeletal category of finite sets $\mathbf{n} = \{1, \ldots, n\}$ (including the empty set $\mathbf{0} := \emptyset$) and injective maps between them. It is a permutative category under the disjoint union of sets, i.e.

$$(\mathbf{m}, \mathbf{n}) \mapsto \mathbf{m} \sqcup \mathbf{n} := \{1, \ldots, m, m+1, \ldots, m+n\}\,. \tag{3.9}$$

The monoidal unit is given by the initial object $\mathbf{0} \in \mathbb{I}$ and the commutativity isomorphism $\mathbf{m} \sqcup \mathbf{n} \cong \mathbf{n} \sqcup \mathbf{m}$ is the evident block permutation.

A functor $\mathbb{I} \to \mathbf{Top}_*$ is called an $\mathbb{I}$-space. By the usual construction, the category of $\mathbb{I}$-spaces inherits a symmetric monoidal structure from $\mathbb{I}$. The following definition is standard in the literature.

Definition 3.3.1 (E.g. [54, §2.2])**.** A commutative $\mathbb{I}$-monoid X is a commutative monoid object in the symmetric monoidal category of $\mathbb{I}$-spaces $\mathbb{I} \to \mathbf{Top}_*$.

It is well known that for a commutative $\mathbb{I}$-monoid X the space $\operatorname{hocolim}_{\mathbb{I}} X$ is an E_∞-space structured by an action of the Barratt-Eccles operad. For details we refer the reader to [54], or to [6] and the references therein, where the basic definitions and results are summarized. Let us write $X_n := X(\mathbf{n})$ for short. Let Σ_n denote the symmetric group on n letters. Unravelling the definition, a commutative $\mathbb{I}$-monoid X consists of a sequence of pointed Σ_n-spaces X_n and equivariant structure maps

$$\oplus : X_m \times X_n \to X_{m+n}$$

for all $m, n \geq 0$ satisfying suitable associativity and unit axioms. Moreover, commutativity of X implies that for all $m, n \geq 0$ the diagram

$$\begin{array}{ccc} X_m \times X_n & \xrightarrow{\oplus} & X_{m+n} \\ \downarrow t & & \downarrow \tau_{m,n} \\ X_n \times X_m & \xrightarrow{\oplus} & X_{n+m} \end{array} \tag{3.10}$$

commutes, where t is the transposition and $\tau_{m,n} \in \Sigma_{m+n}$ is the block permutation $\mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$.

Let us write $X_\infty = \operatorname{tel}_n X_n$ for the infinite mapping telescope which is formed using the maps $X_n \to X_{n+1}$ induced by the standard maps $\mathbf{n} = \mathbf{n} \sqcup \mathbf{0} \to \mathbf{n} \sqcup \mathbf{1}$. Combining our Theorem 3.1.3 with results of [6] we obtain conditions under which the homotopy colimit over $\mathbb{I}$ is equivalent to the Quillen plus-construction on X_∞.

Theorem 3.3.2. *Let X be an objectwise path-connected commutative $\mathbb{I}$-monoid such that*

- *the induced Σ_n-action on $H_*(X_n; \mathbb{Z})$ is trivial for all $n \geq 0$.*

Then the fundamental group of X_∞ is quasiperfect and X_∞^+ is an infinite loop space. If, furthermore

- *all maps $X_m \to X_n$ induced by injections $\mathbf{m} \to \mathbf{n}$ are injective*
- *for all $x \in X_m$ and $y \in X_n$ the element $x \oplus y \in X_{m+n}$ is in the image of a map induced by a non-identity order preserving injection if and only if x or y is,*

then the inclusion $X_\infty \to \mathrm{hocolim}_{\mathbb{I}} X$ *induces a weak homotopy equivalence of infinite loop spaces* $X_\infty^+ \simeq \mathrm{hocolim}_{\mathbb{I}} X$.

Proof. Consider the topological monoid $\mathbf{X} := \coprod_{n\geq 0} X_n$. Commutativity of (3.10) and the assumption that Σ_n acts trivially on $H_*(X_n;\mathbb{Z})$ for all $n \geq 0$ imply that the Pontrjagin ring $H_*(\mathbf{X})$ is abelian. Thus we find ourselves in a situation to which Theorem 3.1.3 applies, cf. Example 3.1.5. It follows that X_∞ has a quasiperfect fundamental group and that X_∞^+ is a simple space[4]. On passage to colimits as $n \to \infty$ we also have that Σ_∞ acts trivially on $H_*(X_\infty) = \mathrm{colim}_n H_*(X_n;\mathbb{Z})$. One can now check that all assumptions in [6, Thm. 3.1] are satisfied, from which the first assertion follows. The second statement follows under the given additional hypotheses from [6, Thm. 3.3]. □

3.4 Perfection of the commutator group

Recall the infinite mapping telescope M_∞ from (3.1). It follows that in the situation of Theorem 3.1.3 the group $G_\infty = \pi_1(M_\infty)$ is quasiperfect, i.e. G_∞ has a perfect commutator subgroup G'_∞. This statement can be proved directly and for a general system of groups $(G_n, \oplus)_{n\geq 0}$ as described in Section 3.1, by repeating an argument of Randal-Williams [49, Prop. 3.1].

Lemma 3.4.1. *Let* $(G_n, \oplus)_{n\geq 0}$ *be a system of groups as in Section 3.1 and suppose that* (†) *holds*[5]*. Then the commutator subgroup of* $G_\infty = \mathrm{colim}_n G_n$ *is perfect.*

Proof. For simplicity let us assume that the cofinal sequence $(n_k)_{k\in\mathbb{N}}$ is all of $\mathbb{N}$. The proof goes through verbatim in the more general case.

It suffices to show that every commutator $[a, b]$ in G_∞ can be written as a commutator $[c, d]$ with $c, d \in G'_\infty$. We may assume that $a, b \in G_\infty$ be represented by $a, b \in G_n$. Using (†) we find

$$[a \oplus e_n, b \oplus e_n] = [a \oplus e_n, (e_n \oplus b)c]$$

for some $c \in G'_{2n}$. Since the product $\oplus$ is a homomorphism, we have that

$$(a \oplus e_n)(e_n \oplus b) = (ae_n) \oplus (e_n b) = (e_n a) \oplus (be_n) = (e_n \oplus b)(a \oplus e_n)\,,$$

that is $a \oplus e_n$ and $e_n \oplus b$ commute with respect to the product in G_{2n}. Thus the commutator can be written as

$$[a \oplus e_n, b \oplus e_n] = (e_n \oplus b)[a \oplus e_n, c](e_n \oplus b)^{-1}\,.$$

[4]Recall that a path-connected space is called *simple*, if its fundamental group is abelian and acts trivially on the higher homotopy groups. Examples of simple spaces include all path-connected H-spaces.

[5]In its equivalent formulation of Remark 3.1.6

Multiplication by e_{2n} from the right defines a homomorphism $G_{2n} \to G_{4n}$. Applied to the previous line it gives

$$[a \oplus e_{3n}, b \oplus e_{3n}] = (e_n \oplus b \oplus e_{2n})[a \oplus e_{3n}, c \oplus e_{2n}](e_n \oplus b \oplus e_{2n})^{-1}\,.$$

There exist $d \in G'_{4n}$ such that

$$a \oplus e_{3n} = (a \oplus e_n) \oplus e_{2n} = (e_{2n} \oplus a \oplus e_n)d\,.$$

Now $e_{2n} \oplus a \oplus e_n$ commutes with $c \oplus e_{2n}$ in G_{4n}, and we can write

$$\begin{aligned}&[a \oplus e_{3n}, b \oplus e_{3n}]\\ &= (e_n \oplus b \oplus e_{2n})(e_{2n} \oplus a \oplus e_n)[d, c \oplus e_{2n}](e_{2n} \oplus a \oplus e_n)^{-1}(e_n \oplus b \oplus e_{2n})^{-1}\,.\end{aligned}$$

Let v be the element in the direct limit represented by $(e_n \oplus b \oplus e_{2n})(e_{2n} \oplus a \oplus e_n) \in G_{4n}$. Then in the direct limit the above equation becomes

$$[a, b] = v[d, c]v^{-1}\,,$$

where $c, d \in G'_\infty$. Thus $[a, b] \in [G'_\infty, G'_\infty]$. □

Chapter 4

Constructions for the symmetric groups

Recall that the descending central series of a discrete group Γ is the normal series

$$\cdots \subset \gamma^{q+1}(\Gamma) \subset \gamma^{q}(\Gamma) \subset \cdots \subset \gamma^{2}(\Gamma) \subset \gamma^{1}(\Gamma) = \Gamma\,,$$

where $\gamma^1(\Gamma) := \Gamma$ and $\gamma^{q+1}(\Gamma) := [\gamma^q(\Gamma), \Gamma]$ for $q \geq 1$. The group Γ has *nilpotency class q* if q is the smallest integer so that $\gamma^{q+1}(\Gamma) = 1$ is the trivial group. In [4] Adem, Cohen and Torres-Giese introduced a sequence of spaces

$$B(2,\Gamma) \subset \cdots \subset B(q,\Gamma) \subset B(q+1,\Gamma) \subset \cdots \subset B\Gamma \tag{4.1}$$

defined as follows. The space $B(q,\Gamma) \subset B\Gamma$ is the subspace of the bar construction in which a k-simplex is a k-tuple $(g_1,\ldots,g_k) \in \Gamma^k$ so that the subset $\{g_1,\ldots,g_k\} \subset \Gamma$ generates a subgroup of Γ of nilpotency class less than q. Such a k-tuple corresponds precisely to a homomorphism from the free q-nilpotent group $F_k/\gamma^q(F_k)$ into Γ. For example, we have $B(2,\Gamma) = B_{\mathrm{com}}\Gamma$ because a group of nilpotency class less than two is precisely an abelian group.

Now we consider (4.1) for the symmetric groups $\Gamma = \Sigma_n$. In this situation, the authors of [6] associate to (4.1) a sequence of infinite loop spaces as follows. The sequence of classifying spaces $\{B\Sigma_n\}_{n\geq 0}$ defines in a natural way a commutative $\mathbb{I}$-monoid (see Section 3.3 for the definition of a commutative $\mathbb{I}$-monoid). It is well known that $\mathrm{hocolim}_{\mathbb{I}} B\Sigma_{-} \simeq B\Sigma_\infty^+$, where the plus-construction is applied with respect to the perfect commutator subgroup $[\Sigma_\infty,\Sigma_\infty] = A_\infty$ (see e.g. [54, Rem. 2.2]). Therefore, by the Barratt-Priddy-Quillen theorem, $\mathrm{hocolim}_{\mathbb{I}} B\Sigma_{-} \simeq Q_0S^0$ is the basepoint component of the infinite loop space QS^0. For all $q \geq 2$ the structure of a commutative $\mathbb{I}$-monoid restricts to $\{B(q,\Sigma_n)\}_{n\geq 0}$ and the inclusion maps $B(q,\Sigma_n) \to B(q+1,\Sigma_n)$ induce morphisms of commutative $\mathbb{I}$-monoids. As a result, one obtains a sequence of infinite loop spaces

$$\mathrm{hocolim}_{\mathbb{I}} B(2,\Sigma_{-}) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B(q,\Sigma_{-}) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B\Sigma_{-} \simeq Q_0S^0\,, \tag{4.2}$$

see [6, Cor. 4.5].

In this chapter we propose a way to study these infinite loop spaces by using ideas from [31]. To this end we shall first generalise the construction of $B(q,\Gamma)$ slightly to allow for a general family of subgroups of Γ as opposed to only nilpotent subgroups of certain nilpotency class. This is described in Section 4.1. In Section 4.2 we associate a spectrum to certain collections of families of subgroups of the symmetric groups. The construction goes via a translation category in a similar manner as described in [6]. In Section 4.3 we use the essential idea of the *filtration by complexity* from [31]. This is a filtration of a collection of families of subgroups of the symmetric groups, which induces a filtration on the spectrum level. Our main result in this section identifies the subquotients of the filtration. We prove the theorem in Section 4.4. As a result, we obtain a filtration by infinite loop spaces of each of the terms in the sequence (4.2). In Example 4.3.7 we identify the first term in this filtration.

4.1 Definitions

In this section the letter Γ stands for a finite discrete group.

Definition 4.1.1. A *family of subgroups of* Γ is a set $\mathcal{F}$ of subgroups which is closed under conjugation in Γ and under taking subgroups.

Notation. For $(g_1,\dots,g_k)\in\Gamma^k$ we let $\langle g_1,\dots,g_k\rangle\subset\Gamma$ denote the subgroup of Γ generated by $\{g_1,\dots,g_k\}$. If the dimension k is clear form the context, we will sometimes abbreviate $\underline{g}:=(g_1,\dots,g_k)$.

Definition 4.1.2. For a family $\mathcal{F}$ of subgroups of Γ a simplicial set $B_*(\mathcal{F},\Gamma)$ is defined by

$$k\mapsto B_k(\mathcal{F},\Gamma):=\{(g_1,\dots,g_k)\in\Gamma^k\mid\langle g_1,\dots,g_k\rangle\in\mathcal{F}\}\,.$$

The degeneracy maps s_i and face maps d_i are the restrictions of the simplicial operators in the bar construction for the group Γ, see Section 1.1.1. This is welldefined; for $\underline{g}\in B_k(\mathcal{F},\Gamma)$ we have $\langle s_i(\underline{g})\rangle=\langle\underline{g}\rangle$ and $\langle d_i\underline{g}\rangle\subset\langle\underline{g}\rangle$ for all $0\le i\le k$, and the family $\mathcal{F}$ is closed under taking subgroups. The fact that $\mathcal{F}$ is closed under conjugation is not needed for this definition, but will be important later on.

Write $B(\mathcal{F},\Gamma):=|B_*(\mathcal{F},\Gamma)|$. It was noted in [4, Thm. 4.3] that $B(q,\Gamma)$ can be expressed as a colimit. The following lemma is the exact analogue for the space $B(\mathcal{F},\Gamma)$.

Lemma 4.1.3. *Regard the family $\mathcal{F}$ as a partially ordered set under inclusion. Then*

$$B(\mathcal{F},\Gamma)\cong\mathrm{colim}_{H\in\mathcal{F}}BH\,.$$

Proof. There are inclusion maps $B_*H \to B_*(\mathcal{F}, \Gamma)$ for all $H \in \mathcal{F}$ inducing a map

$$\bigcup_{H \in \mathcal{F}} B_*H \to B_*(\mathcal{F}, \Gamma)\,,$$

where the union is taken inside $B_*\Gamma$. A map in the other direction is obtained by sending $\underline{g} \in B_k(\mathcal{F}, \Gamma)$ to $\underline{g} \in B_k\langle \underline{g} \rangle \subset \bigcup_{H \in \mathcal{F}} B_k H$. For this to extend to a map of simplicial sets it is important that we map into the union. The two maps just described are mutual inverses. □

Example 4.1.4. (i) If $\mathcal{F}$ is the family consisting of only the trivial subgroup $\{1\} \subset \Gamma$, then $B(\mathcal{F}, \Gamma) = \text{pt}$. On the other hand, if $\mathcal{F}$ consists of all subgroups of Γ, then $B(\mathcal{F}, \Gamma) = B\Gamma$. More generally, if $\mathcal{F}$ is a family containing a unique maximal object $A \in \mathcal{F}$, then $B(\mathcal{F}, \Gamma) = BA$. For example, if $\Gamma = \Sigma_n$ is the symmetric group on n letters and $\mathcal{A}_n$ is the family consisting of all subgroups of the alternating group $A_n \subset \Sigma_n$ then $B(\mathcal{A}_n, \Sigma_n) = BA_n$.

(ii) Let p be a prime and let $\mathcal{S}yl_p$ be the family consisting of all Sylow-p subgroups of the symmetric group Σ_p together with the trivial group. Then $B(\mathcal{S}yl_p, \Sigma_p)$ is a wedge of copies of BC_p one for each of the Sylow-p subgroups of Σ_p. Here C_p denotes the cyclic group of order p.

(iii) For an integer $q \geq 2$ let $\mathcal{N}_q$ be the family of all subgroups of Γ of nilpotency class less than q. Then $B(\mathcal{N}_q, \Gamma) = B(q, \Gamma)$ is the space introduced in [4]. In particular, if $q = 2$ then $\mathcal{N}_2$ is the family of all abelian subgroups of Γ and $B(\mathcal{N}_2, \Gamma) = B_{\text{com}}\Gamma$.

For later reference we record the following lemma, a form of which is proved by Okay in [46]. For p a prime and $\mathcal{F}$ a family of subgroups of a finite group Γ we denote by $\mathcal{F}^p \subset \mathcal{F}$ the sub-family consisting of only the p-subgroups in $\mathcal{F}$.

Lemma 4.1.5. *Suppose that p is a prime and that $\mathcal{F}$ is a family of nilpotent subgroups of the finite group Γ. Then the map*

$$B(\mathcal{F}^p, \Gamma) \to B(\mathcal{F}, \Gamma)$$

induced by $\mathcal{F}^p \subset \mathcal{F}$ is a mod-p homology equivalence.

Proof. For $\mathcal{F} = \mathcal{N}_q$ this follows from [46, Theorem 3.4]. For an arbitrary family of nilpotent subgroups this follows in the same way. We summarize an argument.

Recall that a finite nilpotent group H has a unique Sylow-r subgroup $H_{(r)}$ for every prime $r \,|\, |H|$ and that H is isomorphic to the direct product of these. This also implies that the inclusion $H_{(r)} \hookrightarrow H$ induces an isomorphism

$$H_*(H_{(r)}; \mathbb{F}_r) \xrightarrow{\cong} H_*(H; \mathbb{F}_r)\,. \tag{4.3}$$

We think of a family of groups as a poset under inclusion. Okay notices that the projection of a group $H \in \mathcal{F}$ onto its Sylow-p subgroup if $p \mid |H|$, or onto the trivial group if $p \nmid |H|$, defines a functor $\pi_p : \mathcal{F} \to \mathcal{F}^p$. This functor and the inclusion $\mathcal{F}^p \subset \mathcal{F}$ induce a map

$$B(\mathcal{F}^p, \Gamma) \cong \mathrm{colim}_{H \in \mathcal{F}^p} BH \xrightarrow{\cong} \mathrm{colim}_{H \in \mathcal{F}} B\pi_p(H)\,,$$

which is an isomorphism as one can easily check. The inclusion $\pi_p(H) \hookrightarrow H$ induces another map

$$\mathrm{hocolim}_{H \in \mathcal{F}} B\pi_p(H) \to \mathrm{hocolim}_{H \in \mathcal{F}} BH\,,$$

and a corresponding map on ordinary colimits. Together, these maps form the following commutative diagram

$$\begin{array}{ccccc} & & \mathrm{hocolim}_{H \in \mathcal{F}} B\pi_p(H) & \longrightarrow & \mathrm{hocolim}_{H \in \mathcal{F}} BH \\ & & \downarrow \simeq & & \downarrow \simeq \\ B(\mathcal{F}^p, \Gamma) & \xrightarrow{\cong} & \mathrm{colim}_{H \in \mathcal{F}} B\pi_p(H) & \longrightarrow & B(\mathcal{F}, \Gamma) \end{array}$$

The vertical arrows are the natural maps from the homotopy colimit to the colimit and they are weak equivalences, essentially because $\mathcal{F}$ is a directed Reedy category and the diagrams over which we take the homotopy colimits are cofibrant (see e.g. [21, §5.2]). It remains to show that the map between the homotopy colimits is a mod-p homology equivalence.

Now there is a spectral sequence computing the homology of a homotopy colimit with coefficients in any abelian group, see [11] or [16, Cor. IV.2.12]. For example, for the space $\mathrm{hocolim}_{H \in \mathcal{F}} BH$ and with mod-p coefficients the E^2-page is given by

$$E^2_{s,t} \cong (L^s\mathrm{colim}) H_t(B(-), \mathbb{F}_p)\,,$$

where $L^s\mathrm{colim}$ denotes the s-th left derived colimit and $H_t(B(-), \mathbb{F}_p)$ is the $\mathcal{F}$-indexed diagram of abelian groups given by $H \mapsto H_t(BH, \mathbb{F}_p)$. By (4.3) the natural transformation $H_t(B\pi_p(-), \mathbb{F}_p) \xrightarrow{\cong} H_t(B(-), \mathbb{F}_p)$ is an isomorphism of $\mathcal{F}$-indexed diagrams, so the map $\mathrm{hocolim}_{H \in \mathcal{F}} B\pi_p(H) \to \mathrm{hocolim}_{H \in \mathcal{F}} BH$ induces an isomorphism on E^2-pages of the spectral sequences. Thus it is a mod-p homology equivalence. As a consequence, the composition in the bottom row of the diagram is a mod-p homology equivalence. □

4.2 The case $\Gamma = \Sigma_n$ and the spectrum associated to a collection of families

We now specialize to the symmetric groups. We write Σ_n for the symmetric group on n letters. The letter $\mathcal{F}$ will from now on denote a *collection* of families of groups, i.e. for every $n \geq 0$ we have a family $\mathcal{F}_n$ of subgroups of Σ_n and we write $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$. Suppose that $\mathcal{F}$ satisfies the following condition:

(C1) If $H \in \mathcal{F}_n$ and $K \in \mathcal{F}_m$ then $H \times K \in \mathcal{F}_{n+m}$.

Note that in (C1) we implicitly identify $H \times K$ with a subgroup of Σ_{n+m} via the canonical homomorphism $\oplus : \Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}$ given by direct sum of permutations.

To a collection $\mathcal{F}$ satisfying (C1) we can associate a spectrum using the construction employed in [6]: For every $n \geq 0$ the simplicial set $B_*(\mathcal{F}_n, \Sigma_n)$ admits an action of Σ_n by conjugation, so we can define the translation category

$$\mathscr{C}_*(\mathcal{F}) := \coprod_{n \geq 0} \Sigma_n \ltimes B_*(\mathcal{F}_n, \Sigma_n)\,, \tag{4.4}$$

cf. Def. 2.1.3. The canonical homomorphisms

$$\oplus : \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$$

for $n, m \geq 0$ can be used to endow $\mathscr{C}_*(\mathcal{F})$ with the structure of a simplicial permutative category. This uses condition (C1), which guarantees that we get a welldefined monoidal product on $\mathscr{C}_*(\mathcal{F})$ without leaving the collection $\mathcal{F}$. To the permutative category $\mathscr{C}_*(\mathcal{F})$ we can associate a Γ-space and hence a spectrum.

Definition 4.2.1. We write $X(\mathcal{F})$ for the spectrum associated to $\mathscr{C}_*(\mathcal{F})$.

Relationship with $\mathbb{I}$-monoids. There is a projection to the sphere spectrum $X(\mathcal{F}) \to \mathbb{S}$ induced by the functor of permutative categories $\mathscr{C}_*(\mathcal{F}) \to \coprod_{n \geq 0} \Sigma_n$ which collapses each simplicial set $B(\mathcal{F}_n, \Sigma_n)$ to a point. The homotopy fibre of the induced map of infinite loop spaces $\Omega^\infty X(\mathcal{F}) \to QS^0$ can often be identified with the infinite loop space associated to a commutative $\mathbb{I}$-monoid.

Recall from Section 3.3 the skeletal category $\mathbb{I}$ of finite sets and injective maps and the definition of a commutative $\mathbb{I}$-monoid. For every morphism $i : \mathbf{n} \to \mathbf{m}$ in $\mathbb{I}$ there is a canonical homomorphism $i_* : \Sigma_n \to \Sigma_m$. Suppose that the collection $\mathcal{F}$ satisfies the following additional condition for all $n, m \geq 0$.

(C2) If $i : \mathbf{n} \to \mathbf{m}$ is an order-preserving injection and $H \in \mathcal{F}_n$ then $i_*(H) \in \mathcal{F}_m$.

Then it is easy to check that the assignment

$$\mathbf{n} \mapsto B(\mathcal{F}_n, \Sigma_n)$$

defines a commutative $\mathbb{I}$-monoid. The space $\mathrm{hocolim}_{\mathbb{I}} B(\mathcal{F}_-, \Sigma_-)$ is then an infinite loop space. The following observation is based on a result from [15], which was revisited in [19, §4]. For a proof see [6, Thm. 3.3].

Lemma 4.2.2. *There is a natural homotopy fibre sequence of infinite loop spaces*

$$\mathrm{hocolim}_{\mathbb{I}} B(\mathcal{F}_-, \Sigma_-) \to \Omega^\infty X(\mathcal{F}) \to QS^0\,.$$

4.3 A filtration on the spectrum level

In this section we assume that $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ is a collection of families of subgroups of the symmetric groups satisfying condition (C1), so that we have the spectrum $X(\mathcal{F})$ of Definition 4.2.1. In [31] Lesh considered spectra, which were built from the classifying spaces $\{B\mathcal{F}_n\}_{n\geq 0}$ (the classifying space of a family is a generalisation of the classifying space of a group). She used a natural sequence of sub-collections of $\mathcal{F}$ to construct a filtration of these spectra. In this section we apply the same idea to construct a filtration of the spectrum $X(\mathcal{F})$. The main result is Theorem 4.3.4 which identifies the subquotients of this filtration.

The action of a permutation group $H \in \mathcal{F}_n$ partitions the set $\mathbf{n} = \{1, \dots, n\}$ into orbits whose lengths are invariant under conjugation of the group H.

Definition 4.3.1. For any $n \geq 0$ and any $N > 0$ let $\mathcal{F}_n(N) \subseteq \mathcal{F}_n$ be the sub-family consisting of groups $H \in \mathcal{F}_n$ whose orbits have length at most N.

It is clear that $\mathcal{F}(N) := \{\mathcal{F}_n(N)\}_{n\geq 0}$ is again a collection of families of subgroups satisfying condition (C1). For every $N > 0$ the inclusion $\mathcal{F}(N-1) \subset \mathcal{F}(N)$ induces a functor of permutative categories $\mathscr{C}_*(\mathcal{F}(N-1)) \to \mathscr{C}_*(\mathcal{F}(N))$ and therefore a morphism of spectra $X(\mathcal{F}(N-1)) \to X(\mathcal{F}(N))$. Thus we have a filtration by spectra of $X(\mathcal{F})$,

$$\mathbb{S} = X(\mathcal{F}(1)) \to \cdots \to X(\mathcal{F}(N-1)) \to X(\mathcal{F}(N)) \to \cdots \to X(\mathcal{F})\,. \tag{4.5}$$

Note that $\mathcal{F}_n(N) = \mathcal{F}_n$ whenever $n \leq N$. The following lemma is easy to check:

Lemma 4.3.2. *There is an equivalence* $\mathrm{hocolim}_N X(\mathcal{F}(N)) \simeq X(\mathcal{F})$.

We need another condition on $\mathcal{F}$. In [31] this is called the *product projection condition.*

(P) Suppose that $H \in \mathcal{F}_n$ is a subgroup of a product $\Sigma_{i_1} \times \cdots \times \Sigma_{i_k} \subset \Sigma_n$ so that $i_1 + \cdots + i_k = n$. Then $\pi_j(H) \in \mathcal{F}_{i_j}$ for all $1 \leq j \leq k$, where π_j is the projection onto the j-th factor.

Example 4.3.3 ([31]). Let $\mathcal{A} = \{\mathcal{A}_n\}_{n\geq 0}$ be the collection with $\mathcal{A}_n$ the family of subgroups of the alternating group $A_n \subset \Sigma_n$. In [31] it was noted that (P) fails to hold for the collection $\mathcal{A}$. For instance, $\langle(12)(34)\rangle \subset \Sigma_2 \times \Sigma_2$ is a subgroup of A_4 but $\pi_1(\langle(12)(34)\rangle) = \langle(12)\rangle = \Sigma_2$ is not a subgroup of $A_2 = 1$. Consequently, the theorem below will not apply to the spectrum $X(\mathcal{A})$.

If $\mathcal{F}_n$ is a family of subgroups of Σ_n let us write $\mathcal{F}'_n$ for the sub-family consisting of only those subgroups in $\mathcal{F}_n$ which do *not* act transitively on $\{1, \dots, n\}$.

Theorem 4.3.4. *Let $\mathcal{F}$ be a collection of families of subgroups of the symmetric groups satisfying (C1) and (P). For every $N > 0$ there is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} \Sigma^\infty(B(\mathcal{F}'_N, \Sigma_N) /\!\!/ \Sigma_N)_+ & \longrightarrow & \Sigma^\infty(B(\mathcal{F}_N, \Sigma_N) /\!\!/ \Sigma_N)_+ \\ \downarrow & & \downarrow \\ X(\mathcal{F}(N-1)) & \longrightarrow & X(\mathcal{F}(N)). \end{array} \tag{4.6}$$

Thus we have a homotopy cofibre sequence of spectra

$$X(\mathcal{F}(N-1)) \to X(\mathcal{F}(N)) \to \Sigma^\infty(E\Sigma_N)_+ \wedge_{\Sigma_N} B(\mathcal{F}_N, \Sigma_N)/B(\mathcal{F}'_N, \Sigma_N)\,.$$

We prove the theorem in Section 4.4.

The main example. As an application of the theorem, suppose we want to understand the infinite loop space $\mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_-)$ from (4.2). In our notation this is the space $\mathrm{hocolim}_{\mathbb{I}} B(\mathcal{N}_{q,-}, \Sigma_-)$, where $\mathcal{N}_{q,n}$ is the family of nilpotent subgroups of Σ_n of nilpotency class less than q. We can study the space localized at a fixed prime p.

Recall our notation that if $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ is a collection of families of groups, then $\mathcal{F}^p := \{\mathcal{F}_n^p\}_{n \geq 0}$ is the collection obtained from $\mathcal{F}$ by considering the p-subgroups only. If $\mathcal{F}$ satisfies (C1), (C2) or (P), then so does $\mathcal{F}^p$.

Lemma 4.3.5. *Suppose that p is a prime and that $\mathcal{F}$ is a collection of families of nilpotent subgroups of the symmetric groups satisfying (C1) and (C2). Then the map*

$$j : \mathrm{hocolim}_{\mathbb{I}} B(\mathcal{F}_-^p, \Sigma_-) \to \mathrm{hocolim}_{\mathbb{I}} B(\mathcal{F}_-, \Sigma_-)$$

induced by $\mathcal{F}^p \subset \mathcal{F}$ is a p-local equivalence.

Proof. Both homotopy colimits are infinite loop spaces. It suffices to show that j induces an isomorphism on rational homology and on homology with mod-p coefficients.

To check this we use again the Bousfield-Kan spectral sequence for the homology of a homotopy colimit [16, Cor. IV.2.12]. By Lemma 4.1.5, the map $B(\mathcal{F}_n^p, \Sigma_n) \to B(\mathcal{F}_n, \Sigma_n)$ is a mod-p homology equivalence for all $n \geq 0$. Thus the argument used in the proof of Lemma 4.1.5 shows that j is a mod-p homology equivalence. Moreover, both homotopy colimits have trivial rational homology. For example, we can write $B(\mathcal{F}_n, \Sigma_n) \simeq \mathrm{hocolim}_{H \in \mathcal{F}_n} BH$ as in the proof of Lemma 4.1.5. The rational homology of a finite group vanishes, and the category $\mathcal{F}_n$ is contractible because it has an initial object, so the spectral sequence shows that $B(\mathcal{F}_n, \Sigma_n)$ has trivial rational homology. Also the category $\mathbb{I}$ is contractible, so the rational homology of $\mathrm{hocolim}_{\mathbb{I}} B(\mathcal{F}_-, \Sigma_-)$ is trivial. □

Thus Lemma 4.3.5 leads us to study the space $\mathrm{hocolim}_{\mathbb{I}} B(\mathcal{N}^p_{q,-}, \Sigma_-)$, where $\mathcal{N}^p_{q,n} \subset \mathcal{N}_{q,n}$ is the sub-family consisting of all p-groups in $\mathcal{N}_{q,n}$. By Lemma 4.2.2 we can study this infinite loop space by analysing the spectrum $X(\mathcal{N}^p_q)$. Now we make the following observation.

Lemma 4.3.6. *Let $\mathcal{F}$ be a collection as in the theorem. Then the map $X(\mathcal{F}^p(N-1)) \to X(\mathcal{F}^p(N))$ is an equivalence if N is not a power of p.*

Proof. If N is not a power of p, then the orbit-stabilizer theorem implies that there is no transitive p-subgroup of Σ_N. In other words, the family $\mathcal{F}^p_N$ consists of non-transitive subgroups only. Thus, by Theorem 4.3.4, the homotopy cofibre of the map $X(\mathcal{F}^p(N-1)) \to X(\mathcal{F}^p(N))$ is contractible. □

Now we summarize: Let us define $A(q,p)_k := \mathrm{hocolim}_{\mathbb{I}} B(\mathcal{N}^p_{q,-}(p^k), \Sigma_-)$. Moreover, define

$$E(q,p)_k := (E\Sigma_{p^k})_+ \wedge_{\Sigma_{p^k}} B(\mathcal{N}^p_{q,p^k}, \Sigma_{p^k})/B((\mathcal{N}^p_{q,p^k})', \Sigma_{p^k}) .$$

Then we have constructed a filtration of the p-localization of $\mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_-)$ by infinite loop spaces $\{A(q,p)_k\}_{k\geq 0}$ so that

$$A(q,p)_{k-1} \to A(q,p)_k \to QE(q,p)_k$$

is a homotopy fibre sequence of infinite loop spaces for every $k > 0$.

Example 4.3.7. The first approximation to the p-localization of $\mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_-)$ is the infinite loop space

$$A(q,p)_1 \simeq QE(q,p)_1 \simeq Q(ENC_p)_+ \wedge_{NC_p} BC_p , \tag{4.7}$$

which is independent of q. Here $C_p \subset \Sigma_p$ is the cyclic subgroup generated by the standard p-cycle and $NC_p = C_p \rtimes C_{p-1}$ is the normalizer of C_p in Σ_p. In the semidirect product $n \in C_{p-1} \cong \mathbb{F}_p^\times$ acts on $x \in C_p$ by $x \mapsto x^n$. To check the second equivalence in (4.7) we analyse the quotient space $B(\mathcal{N}^p_{q,p}, \Sigma_p)/B((\mathcal{N}^p_{q,p})', \Sigma_p)$. The non-trivial p-subgroups of Σ_p are precisely the Sylow-p subgroups, which are cyclic groups C_p generated by a single p-cycle. In particular, they are all transitive and abelian. Thus $B((\mathcal{N}^p_{q,p})', \Sigma_p) = \mathrm{pt}$ and the quotient space is simply a wedge of copies of BC_p. The action of Σ_p conjugates all of these copies of BC_p into each other, hence

$$E(q,p)_1 = (E\Sigma_p)_+ \wedge_{\Sigma_p} \bigvee BC_p \simeq (E\Sigma_p)_+ \wedge_{NC_p} BC_p$$

and replacing Σ_p by NC_p proves the equivalence.

4.4 Proof of Theorem 4.3.4

We will construct diagram (4.6), including the horizontal homotopy cofibres, as a diagram of simplicial permutative categories. This will be diagram (4.9).

We first show how every object in the category $\mathscr{C}_*(\mathcal{F})$ can be decomposed into a direct sum of "irreducibles". Let $n \geq 1$ and let $\underline{x} = (x_1, \ldots, x_k)$ denote a k-simplex in the simplicial set $B_*(\mathcal{F}_n, \Sigma_n)$ generating a subgroup $\langle \underline{x} \rangle \in \mathcal{F}_n$. The group $\langle \underline{x} \rangle$ acts on the set $\mathbf{n} = \{1, \ldots, n\}$ and induces a partition

$$\mathbf{n} = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_r$$

into $r \geq 1$ orbits of length $s_j := |\mathcal{O}_j| \geq 1$. Each orbit $\mathcal{O}_j$ inherits an ordering of its elements from the natural ordering on $\mathbf{n}$. We assume that the set of orbits $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$ is ordered as follows. Let $N = \max\{s_1, \ldots, s_r\}$ and let $r_{\max} \in \{0, \ldots, r\}$ be the number of orbits of length N. Then we let $\{\mathcal{O}_1, \ldots, \mathcal{O}_{r_{\max}}\}$ be the set of orbits of length N and $\{\mathcal{O}_{r_{\max}+1}, \ldots, \mathcal{O}_r\}$ the set of orbits of length less than N. Each of the two sets is ordered by the minimal elements of its orbits, that is $\min(\mathcal{O}_1) < \min(\mathcal{O}_2) < \cdots < \min(\mathcal{O}_{r_{\max}})$ and $\min(\mathcal{O}_{r_{\max}+1}) < \cdots < \min(\mathcal{O}_r)$. Let $\mathbf{s}_j = \{1, \ldots, s_j\}$ and decompose $\mathbf{n} = \mathbf{s}_1 \sqcup \cdots \sqcup \mathbf{s}_r$ (we read this disjoint union as a product in the symmetric monoidal category $\mathbb{I}$, see (3.9)). Now there exists a unique permutation $\sigma_{\underline{x}} \in \Sigma_n$ mapping $\mathcal{O}_j$ to $\mathbf{s}_j \subseteq \mathbf{s}_1 \sqcup \cdots \sqcup \mathbf{s}_r = \mathbf{n}$ and so that the restriction $\sigma_{\underline{x}}|_{\mathcal{O}_j}$ is an order preserving injection for all $1 \leq j \leq r$.

We regard $\sigma_{\underline{x}}$ as a morphism in the category $\mathscr{C}_k(\mathcal{F})$, i.e. in the k-th simplicial level of the category $\mathscr{C}_*(\mathcal{F})$ in (4.4). The source of $\sigma_{\underline{x}}$ is the simplex $\underline{x}$ and the target is the conjugate element $\underline{x}^{\sigma_{\underline{x}}} := \sigma_{\underline{x}} \underline{x} \sigma_{\underline{x}}^{-1}$. By construction of $\sigma_{\underline{x}}$ we have

$$\langle \underline{x}^{\sigma_{\underline{x}}} \rangle \subset \Sigma_{s_1} \times \cdots \times \Sigma_{s_r}$$

and if $\pi_j : \Sigma_{s_1} \times \cdots \times \Sigma_{s_r} \to \Sigma_{s_j}$ denotes the projection onto the j-th component, the subgroup $\pi_j(\langle \underline{x}^{\sigma_{\underline{x}}} \rangle) \subset \Sigma_{s_j}$ acts transitively on the subset $\mathbf{s}_j \subset \mathbf{n}$. Note that if we write $\pi_j(\underline{x}^{\sigma_{\underline{x}}}) := (\pi_j(x_1^{\sigma_{\underline{x}}}), \ldots, \pi_j(x_k^{\sigma_{\underline{x}}}))$ then the simplex $\underline{x}^{\sigma_{\underline{x}}}$ splits as a direct sum

$$\underline{x}^{\sigma_{\underline{x}}} = \pi_1(\underline{x}^{\sigma_{\underline{x}}}) \oplus \cdots \oplus \pi_r(\underline{x}^{\sigma_{\underline{x}}}) \tag{4.8}$$

in the monoidal category $\mathscr{C}_k(\mathcal{F})$.

Fix $N \geq 1$. We now show that there is a splitting of the simplicial category $\mathscr{C}_*(\mathcal{F}(N))$ in each simplicial degree, and that this is a splitting of permutative categories.

A generic object of $\mathscr{C}_k(\mathcal{F}(N))$ is a pair $(\mathbf{n}, \underline{x})$ where $n \geq 1$ and $\underline{x} = (x_1, \ldots, x_k)$ is a k-simplex in $B_*(\mathcal{F}_n(N), \Sigma_n)$ generating a subgroup $\langle \underline{x} \rangle \in \mathcal{F}_n$ all of whose orbits have length less than or equal to N. In addition, there is an object $(\mathbf{0}, *) \in \mathscr{C}_k(\mathcal{F}(N))$ acting as a symmetric monoidal unit. Write $\mathbf{n}N = \{1, 2, \ldots, nN\}$. Let $\mathscr{D}_{k,N}$ denote the full subcategory of $\mathscr{C}_k(\mathcal{F}(N))$ on the object $(\mathbf{0}, *)$ and on all objects of the form

$$(\mathbf{n}N, \underline{x}_1 \oplus \cdots \oplus \underline{x}_n)$$

for $n \geq 1$, where each $\langle \underline{x}_i \rangle \subset \Sigma_N$, $1 \leq i \leq n$, acts transitively on the set $\{1, \ldots, N\}$. A morphism $\sigma : (\mathbf{n}N, \underline{x}_1 \oplus \cdots \oplus \underline{x}_n) \to (\mathbf{n}N, \underline{x}'_1 \oplus \cdots \oplus \underline{x}'_n)$ in the category $\mathscr{D}_{k,N}$ corresponds to an element $\sigma \in \Sigma_{nN}$ of the form $\sigma = \alpha \circ (\beta_1 \oplus \cdots \oplus \beta_n)$, where $\beta_i \in \Sigma_N$ acts on $\underline{x}_i$ by conjugation and $\alpha \in \Sigma_n$ permutes the n summands. The category $\mathscr{D}_{k,N}$ inherits the structure of a permutative category from $\mathscr{C}_k(\mathcal{F}(N))$.

Lemma 4.4.1. *For every $k \geq 0$ and every $N \geq 1$ there is an equivalence of permutative categories*

$$\mathscr{C}_k(\mathcal{F}(N)) \simeq \mathscr{D}_{k,N} \times \mathscr{C}_k(\mathcal{F}(N-1))\,.$$

Proof. We define functors

$$R_N : \mathscr{C}_k(\mathcal{F}(N)) \to \mathscr{D}_{k,N}$$

and

$$r_N : \mathscr{C}_k(\mathcal{F}(N)) \to \mathscr{C}_k(\mathcal{F}(N-1))$$

in the following way. We can translate each object $(\mathbf{n}, \underline{x}) \in \mathscr{C}_k(\mathcal{F}(N))$ along the isomorphism $\sigma_{\underline{x}}$ defined in the first paragraph of this section. Recall from (4.8) that $\underline{x}^{\sigma_{\underline{x}}}$ splits as a direct sum. As before, let r denote the number of orbits of the group $\langle \underline{x} \rangle$ and let $r_{\max} \in \{0, \ldots, r\}$ denote the number of orbits of length N. Write $\mathbf{r}_{\max} = \{1, \ldots, r_{\max}\}$. We let

$$R_N((\mathbf{n}, \underline{x})) := (\mathbf{r}_{\max}N, \pi_1(\underline{x}^{\sigma_{\underline{x}}}) \oplus \cdots \oplus \pi_{r_{\max}}(\underline{x}^{\sigma_{\underline{x}}}))$$

be the part of the simplex $\underline{x}^{\sigma_{\underline{x}}}$ generating the $r_{\max}$ maximal orbits, and we let

$$r_N((\mathbf{n}, \underline{x})) := (\mathbf{n} - \mathbf{r}_{\max}N, \pi_{r_{\max}+1}(\underline{x}^{\sigma_{\underline{x}}}) \oplus \cdots \oplus \pi_r(\underline{x}^{\sigma_{\underline{x}}}))$$

be the part of the simplex generating orbits of length less than N. If $r_{\max} = 0$ we set $R_N((\mathbf{n}, \underline{x})) := (\mathbf{0}, *)$ and if $r_{\max} = r$ we set $r_N((\mathbf{n}, \underline{x})) := (\mathbf{0}, *)$. Condition (P) guarantees that the projections R_N and r_N do not leave the collection $\mathcal{F}$.

If we translate along $\sigma_{\underline{x}}$ the domain and target of a morphism $\tau : (\mathbf{n}, \underline{x}) \to (\mathbf{n}, \underline{x}')$, then the morphism τ splits accordingly,

$$\sigma_{\underline{x}'} \circ \tau \circ \sigma_{\underline{x}}^{-1} = R_N(\tau) \oplus r_N(\tau)\,,$$

into a morphism $R_N(\tau) : R_N((\mathbf{n}, \underline{x})) \to R_N((\mathbf{n}, \underline{x}'))$ in $\mathscr{D}_{k,N}$ and a morphism $r_N(\tau) : r_N((\mathbf{n}, \underline{x})) \to r_N((\mathbf{n}, \underline{x}'))$ in $\mathscr{C}_k(\mathcal{F}(N-1))$. It is easy to check that the pair

$$(R_N, r_N) : \mathscr{C}_k(\mathcal{F}(N)) \to \mathscr{D}_{k,N} \times \mathscr{C}_k(\mathcal{F}(N-1))$$

is a permutative functor. It is also an equivalence of categories with inverse equivalence $\mathscr{D}_{k,N} \times \mathscr{C}_k(\mathcal{F}(N-1)) \to \mathscr{C}_k(\mathcal{F}(N))$ given by inclusion of each factor into $\mathscr{C}_k(\mathcal{F}(N))$ and then forming the direct sum using the monoidal structure on $\mathscr{C}_k(\mathcal{F}(N))$. $\square$

We now extend the functor $R_N : \mathscr{C}_k(\mathcal{F}(N)) \to \mathscr{D}_{k,N}$ defined in the proof of Lemma 4.4.1 to a functor of simplicial categories. For a morphism $f : [k] \to [l]$ in the simplex category Δ let $f^* : \mathscr{C}_l(\mathcal{F}(N)) \to \mathscr{C}_k(\mathcal{F}(N))$ be the induced functor. Note that $\mathscr{D}_{k,N}$ was defined as a subcategory of $\mathscr{C}_k(\mathcal{F}(N))$. We can make $k \mapsto \mathscr{D}_{k,N}$ into a simplicial category by sending $f : [k] \to [l]$ to the functor $R_N(f^*) : \mathscr{D}_{l,N} \to \mathscr{D}_{k,N}$ defined so as to make the following diagram commute,

$$\begin{array}{ccc} \mathscr{D}_{l,N} & \xrightarrow{\text{incl.}} & \mathscr{C}_l(\mathcal{F}(N)) \\ \downarrow{\scriptstyle R_N(f^*)} & & \downarrow{\scriptstyle f^*} \\ \mathscr{D}_{k,N} & \xleftarrow[R_N]{} & \mathscr{C}_k(\mathcal{F}(N)) \end{array}$$

We need to check that this assignment is functorial, that is $R_N((g \circ f)^*) = R_N(f^*) \circ R_N(g^*)$ for any two composable morphisms f and g in Δ. To see this, let $(\mathbf{n}N, \underline{x}_1 \oplus \cdots \oplus \underline{x}_n)$ be an object of $\mathscr{D}_{k,N}$. Then $g^*(\underline{x}_1 \oplus \cdots \oplus \underline{x}_n) = g^*\underline{x}_1 \oplus \cdots \oplus g^*\underline{x}_n$ and applying R_N removes all summands $g^*\underline{x}_i$ for which $\langle g^*\underline{x}_i \rangle \subset \Sigma_N$ is a non-transitive subgroup. But if $\langle g^*\underline{x}_i \rangle$ is non-transitive, then so is $\langle f^*(g^*\underline{x}_i) \rangle$. This is enough to see that $R_N((g \circ f)^*) = R_N(f^*) \circ R_N(g^*)$ and we get a simplicial category $\mathscr{D}_{*,N}$.

Note that $R_N(f^*)$ is a permutative functor, since both f^* and R_N are permutative. It follows that $\mathscr{D}_{*,N}$ is a simplicial permutative category. The functor R_N extends to a map of simplicial permutative categories which we denote by the same symbol,

$$R_N : \mathscr{C}_*(\mathcal{F}(N)) \to \mathscr{D}_{*,N} \, .$$

Altogether we obtain a sequence of simplicial permutative categories

$$\mathscr{C}_*(\mathcal{F}(N-1)) \xrightarrow{\text{incl.}} \mathscr{C}_*(\mathcal{F}(N)) \xrightarrow{R_N} \mathscr{D}_{*,N}$$

which, by Lemma 4.4.1, induces a homotopy cofibre sequence on the spectrum level.

Next we consider the category

$$V_{k,N} := \Sigma_N \ltimes B_k(\mathcal{F}_N(N), \Sigma_N)$$

and the full subcategory

$$V'_{k,N} := \Sigma_N \ltimes B_k(\mathcal{F}_N(N-1), \Sigma_N) \, ,$$

whose objects are those simplices $\underline{x} \in B_k(\mathcal{F}_N(N), \Sigma_N)$ which generate a non-transitive subgroup $\langle \underline{x} \rangle \subset \Sigma_N$. Let $(V_{k,N})^n = V_{k,N} \times \cdots \times V_{k,N}$ be the n-fold product category and let Σ_n act on it by permuting the factors. Then we let

$$\mathscr{S}_{k,N} := \coprod_{n \geq 0} \Sigma_n \ltimes (V_{k,N})^n$$

be the free permutative category. Similarly, we define

$$\mathscr{S}'_{k,N} := \coprod_{n \geq 0} \Sigma_n \ltimes (V'_{k,N})^n.$$

Lemma 4.4.2. *For every $k \geq 0$ and every $N \geq 1$ there is an equivalence of permutative categories*

$$\mathscr{S}_{k,N} \simeq \mathscr{D}_{k,N} \times \mathscr{S}'_{k,N}\,.$$

Proof. The idea is the same as in the proof of Lemma 4.4.1, so we do not spell out the details here. We separate each object $(\mathbf{n}, \underline{x}_1, \ldots, \underline{x}_n) \in \mathscr{S}_{k,N}$ into the part consisting of all those $\underline{x}_i$ which generate a transitive subgroup $\langle \underline{x}_i \rangle \subset \Sigma_N$ and those $\underline{x}_i$ which generate a non-transitive subgroup. This will define permutative functors $P_N : \mathscr{S}_{k,N} \to \mathscr{D}_{k,N}$ and $p_N : \mathscr{S}_{k,N} \to \mathscr{S}'_{k,N}$, so that

$$(P_N, p_n) : \mathscr{S}_{k,N} \to \mathscr{D}_{k,N} \times \mathscr{S}'_{k,N}$$

is an equivalence of categories. □

There is an obvious functor of permutative categories $\mu_N : \mathscr{S}_{k,N} \to \mathscr{C}_k(\mathcal{F}(N))$ given on objects by $(\mathbf{n}, \underline{x}_1, \ldots, \underline{x}_n) \mapsto (\mathbf{n}N, \underline{x}_1 \oplus \cdots \oplus \underline{x}_n)$. Similarly, there is a permutative functor $\mu'_N : \mathscr{S}'_{k,N} \to \mathscr{C}(\mathcal{F}(N-1))$. The functors P_N, μ_N and μ'_N extend to functors of simplicial permutative categories. It is easy to check that the following diagram commutes,

$$\begin{array}{ccccc} \mathscr{S}'_{*,N} & \longrightarrow & \mathscr{S}_{*,N} & \xrightarrow{P_N} & \mathscr{D}_{*,N} \\ \downarrow{\scriptstyle \mu'_N} & & \downarrow{\scriptstyle \mu_N} & & \Vert \\ \mathscr{C}_*(\mathcal{F}(N-1)) & \longrightarrow & \mathscr{C}_*(\mathcal{F}(N)) & \xrightarrow{R_N} & \mathscr{D}_{*,N} \end{array} \qquad (4.9)$$

By Lemma 4.4.1 and Lemma 4.4.2, both rows induce homotopy cofibre sequences when we pass from permutative categories to spectra. To finish the proof of the theorem we only need to identify the spectra associated to $\mathscr{S}_{*,N}$ and $\mathscr{S}'_{*,N}$. Since $\mathscr{S}_{*,N}$ is the free permutative category on $\Sigma_N \ltimes B_*(\mathcal{F}_N, \Sigma_N)$, it gives rise to the suspension spectrum $\Sigma^\infty(B(\mathcal{F}_N, \Sigma_N) /\!\!/ \Sigma_N)_+$. Similarly, $\mathscr{S}'_{*,N}$ corresponds to the spectrum $\Sigma^\infty(B(\mathcal{F}'_N, \Sigma_N) /\!\!/ \Sigma_N)_+$. This finishes the proof.

Appendix A

Constructions for the mapping class groups

This is an excerpt from our Transfer thesis [19].

Let $\Gamma_{g,1}$ be the mapping class group rel ∂ of an orientable surface of genus g with one boundary component. By a well known theorem of Tillmann [60], the plus-construction applied to the classifying space of the stable mapping class group $\Gamma_{\infty,1}$ is an infinite loop space $B\Gamma_{\infty,1}^{+}$. For $q \geq 2$ let $B(q, \Gamma_{g,1}) \subset B\Gamma_{g,1}$ be the subspace described in the introduction to Chapter 4. In this appendix we describe a construction that allows one to associate in a natural way an infinite loop space to the spaces $\{B(q, \Gamma_{g,1})\}_{g \geq 0}$ together with an infinite loop map into $B\Gamma_{\infty,1}^{+}$.

To explain the idea recall from [6] (see also Lemma 4.2.2) that under certain conditions a commutative $\mathbb{I}$-monoid $X : \mathbb{I} \to \mathbf{Top}$ gives rise to a homotopy fibre sequence of infinite loop spaces

$$\mathrm{hocolim}_{\mathbb{I}} X \to \Omega B \coprod_{n \geq 0} B(\Sigma_n \ltimes X_n) \xrightarrow{\text{proj.}} \Omega B \coprod_{n \geq 0} B\Sigma_n .$$

In our construction we generalise the term in the middle by replacing the symmetric groups by a categorical operad, the sequence of spaces $\{X_n\}_{n \geq 0}$ by a diagram of spaces defined on this operad, and the translation category $\ltimes$ by a Grothendieck construction. This gives Definition A.0.7 below. To associate an infinite loop space to $\{B(q, \Gamma_{g,1})\}_{g \geq 0}$ we will then choose the operad to be the mapping class group operad $\mathscr{M}$ from [60].

Let G be a discrete group considered as a category with one object. We recall the notion of G-category, quotient category and G-functor. G-functors were introduced in [26]. Since all our categories and functors will be equipped with *right* G-actions we give the relevant definitions of right G-category etc. Later we shall omit the word "right" as long as no confusion will result from this. **Cat** is the category of small categories.

Definition A.0.1. A *right G-category* is a functor $\mathscr{C} : G^{op} \to \mathbf{Cat}$. A *morphism of right G-categories* is a natural transformation of functors $\mathscr{C} \to \mathscr{D}$.

In other words, a right G-category is a small category $\mathscr{C}$ together with endofunctors $\mathbf{g} : \mathscr{C} \to \mathscr{C}$ for all $g \in G$ describing an action of G on $\mathscr{C}$, that is $\mathbf{e} = id_{\mathscr{C}}$ and $\mathbf{g} \circ \mathbf{h} = \mathbf{hg}$ ($e \in G$ is the neutral element and $\mathbf{hg}$ is the functor corresponding to $hg \in G$). In case the category $\mathscr{C}$ is G-free the quotient category $\mathscr{C}/G := \mathrm{colim}_{G^{op}}\mathscr{C}$ can be explicitly described as follows: The object set is the set of orbits $\mathrm{ob}(\mathscr{C})/G$ of objects of $\mathscr{C}$ under the action by endofunctors. Similarly, the morphism set is the quotient set $\mathrm{mor}(\mathscr{C})/G$ and composition of morphisms in $\mathscr{C}$ descends to a composition in $\mathscr{C}/G$.

Definition A.0.2. For a right G-category $\mathscr{C}$ and any category $\mathscr{D}$ a *right G-functor* is a functor $F : \mathscr{C} \to \mathscr{D}$ together with a natural transformation $\alpha(\mathbf{g}) : F \to F \circ \mathbf{g}$ for every $g \in G$, such that $\alpha(\mathbf{e}) = id_F$ and $\alpha(\mathbf{g}) \circ \alpha(\mathbf{h}) = \alpha(\mathbf{hg})$. We denote the G-functor by the pair (F, α). A *morphism of right G-functors* $(F, \alpha) \to (F', \alpha')$ is a natural transformation $t : F \to F'$ such that $t \circ \alpha = \alpha' \circ t$.

Next follows a familiar definition.

Definition A.0.3. Let $\mathscr{C}$ be a small category and $F : \mathscr{C} \to \mathbf{Cat}$ a diagram of small categories. The *Grothendieck construction of F on $\mathscr{C}$* is the category $\mathscr{C} \ltimes F$ defined as follows: An object is a pair (c, x) consisting of an object $c \in \mathscr{C}$ and an object $x \in F(c)$. A morphism $(c, x) \to (c', x')$ is a pair (f, l) where $f : c \to c'$ is a morphism in $\mathscr{C}$ and $l : F(f)(x) \to x'$ is a morphism in $F(c')$. Composition is defined by $(f', l') \circ (f, l) = (f' \circ f, l' \circ F(f')(l))$.

The following lemma follows directly from the definitions, cf. [26, Prop. 2.4].

Lemma A.0.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be G-categories with free G-action and let (F, α), $(H, \beta) : \mathscr{C} \to \mathbf{Cat}$ be G-functors.*

(i) The category $\mathscr{C} \ltimes F$ becomes a G-category with free G-action in a natural way. We denote the quotient category by $\mathscr{C} \ltimes_G F := (\mathscr{C} \ltimes F)/G$.

(ii) A morphism of G-functors $t : (F, \alpha) \to (H, \beta)$ induces a functor $\mathscr{C} \ltimes_G F \to \mathscr{C} \ltimes_G H$.

(iii) A morphism of G-categories $s : \mathscr{D} \to \mathscr{C}$ induces a functor $\mathscr{D} \ltimes_G s^*F \to \mathscr{C} \ltimes_G F$.

We next recall from [39] the definition of a Σ-operad in $\mathbf{Cat}$.

Definition A.0.5. A Σ-*operad* $\mathscr{O}$ *in* **Cat** is a right Σ_n-category $\mathscr{O}_n$ for all $n \geq 0$ together with a distinguished object $1 \in \mathscr{O}_1$ and structure maps

$$\gamma : \mathscr{O}_k \times \mathscr{O}_{j_1} \times \cdots \times \mathscr{O}_{j_k} \to \mathscr{O}_j$$

for all $k \geq 1$ and all $j_1, \ldots, j_k \geq 0$ with $j_1 + \cdots + j_k = j$, satisfying the usual associativity, unit and equivariance axioms (see [39] for details). A Σ-operad $\mathscr{O}$ in **Cat** is called Σ-*free* if $\mathscr{O}_n$ is Σ_n-free for all $n \geq 0$. A *morphism* $s : \mathscr{O} \to \mathscr{P}$ *of* Σ-*operads in* **Cat** is a sequence of morphisms of Σ_n-categories $s_n : \mathscr{O}_n \to \mathscr{P}_n$ commuting with the structure maps in the obvious way.

The following is our main definition.

Definition A.0.6. Let $\mathscr{O}$ be a Σ-operad in **Cat**. A *left* $\mathscr{O}$-*module* in **Top** is a sequence of right Σ_n-functors $(F_n, \alpha) : \mathscr{O}_n \to$ **Top** satisfying the axioms listed below, where the c_i, a_i, b etc. run through objects of $\mathscr{O}$ of the appropriate degree, the σ, τ_i are permutations, and the indices sum appropriately, e.g. $j = j_1 + \cdots + j_k$, so that everything makes sense:

(i) Gluing: There are continuous maps $\varphi(b; c_1, \ldots, c_k)$ such that for every morphism $(f; g_1, \ldots, g_k) : (b; c_1, \ldots, c_k) \to (b'; c'_1, \ldots, c'_k)$ in $\mathscr{O}_k \times \mathscr{O}_{j_1} \times \cdots \times \mathscr{O}_{j_k}$ we have a commutative diagram of spaces:

$$\begin{array}{ccc} F_{j_1}(c_1) \times \cdots \times F_{j_k}(c_k) & \xrightarrow{\varphi(b;c_1,\ldots,c_k)} & F_j(\gamma(b; c_1, \ldots, c_k)) \\ \Big\downarrow{\scriptstyle F_{j_1}(g_1)\times\cdots\times F_{j_k}(g_k)} & & \Big\downarrow{\scriptstyle F_j(\gamma(f;g_1,\ldots,g_k))} \\ F_{j_1}(c'_1) \times \cdots \times F_{j_k}(c'_k) & \xrightarrow{\varphi(b';c'_1,\ldots,c'_k)} & F_j(\gamma(b'; c'_1, \ldots, c'_k)) \end{array}$$

(ii) Associativity: The following diagram of spaces commutes, where $p := p_1 + \cdots + p_k$, $d := \gamma(b; a_1, \ldots, a_k) \in \mathscr{O}_p$, and $s_i := \gamma(a_i; c_{i,1}, \ldots, c_{i,p_i}) \in \mathscr{O}_{j_i}$:

$$\begin{array}{ccc} F_{m_{1,1}}(c_{1,1}) \times \cdots \times F_{m_{k,p_k}}(c_{k,p_k}) & \xrightarrow{\varphi(d;c_{1,1},\ldots,c_{k,p_k})} & F_j(\gamma(d; c_{1,1}, \ldots, c_{k,p_k})) \\ \Big\downarrow{\scriptstyle \varphi(a_1;c_{1,1},\ldots,c_{1,p_1})\times\cdots\times\varphi(a_k;c_{k,1},\ldots,c_{k,p_k})} & & \Big\| \\ F_{j_1}(s_1) \times \cdots \times F_{j_k}(s_k) & \xrightarrow{\varphi(b;s_1,\ldots,s_k)} & F_j(\gamma(b; s_1, \ldots, s_k)) \end{array}$$

(iii) Unit: If $k = 1$ then for all $c \in \mathscr{O}_j$ the map $\varphi(1; c) : F_j(c) \to F_j(c)$ is the identity.

(iv) Equivariance: The following diagram commutes:

$$\begin{array}{ccc} F_{j_1}(c_1) \times \cdots \times F_{j_k}(c_k) & \xrightarrow{\varphi(b;c_1,\ldots,c_k)} & F_j(\gamma(b; c_1, \ldots, c_k)) \\ \Big\downarrow{\scriptstyle \alpha(\tau_1)\times\cdots\times\alpha(\tau_k)} & & \Big\downarrow{\scriptstyle \alpha(\tau_1\oplus\cdots\oplus\tau_k)} \\ F_{j_1}(c_1\tau_1) \times \cdots \times F_{j_k}(c_k\tau_k) & \xrightarrow{\varphi(b;c_1\tau_1,\ldots,c_k\tau_k)} & F_j(\gamma(b; c_1\tau_1, \ldots, c_k\tau_k)) \end{array}$$

(v) Commutativity: The following diagram commutes:

$$\begin{array}{ccc}
F_{j_{\sigma^{-1}1}}(c_{\sigma^{-1}1}) \times \cdots \times F_{j_{\sigma^{-1}k}}(c_{\sigma^{-1}k}) & \xrightarrow{\varphi(b;c_{\sigma^{-1}1},\ldots,c_{\sigma^{-1}k})} & F_j(\gamma(b;c_{\sigma^{-1}1},\ldots,c_{\sigma^{-1}k})) \\
\mathrm{sh}_\sigma \big\uparrow & & \big\downarrow \alpha(\sigma(j_1,\ldots,j_k)) \\
F_{j_1}(c_1) \times \cdots \times F_{j_k}(c_k) & \xrightarrow{\varphi(b\sigma;c_1,\ldots,c_k)} & F_j(\gamma(b\sigma;c_1,\ldots,c_k))
\end{array}$$

where the map sh_σ permutes the factors according to $\sigma \in \Sigma_k$.

From now on we call a left $\mathscr{O}$-module just "$\mathscr{O}$-module".

A *morphism between $\mathscr{O}$-modules* (F,φ) and (H,ψ) is a sequence of morphisms of right Σ_n-functors $t_n : F_n \to H_n$ such that all the following diagrams commute:

$$\begin{array}{ccc}
F_{j_1}(c_1) \times \cdots \times F_{j_k}(c_k) & \xrightarrow{\varphi(b;c_1,\ldots,c_k)} & F_j(\gamma(b;c_1,\ldots,c_k)) \\
t_{j_1}(c_1)\times\cdots\times t_{j_k}(c_k) \big\downarrow & & \big\downarrow t_j(\gamma(b;c_1,\ldots,c_k)) \\
H_{j_1}(c_1) \times \cdots \times H_{j_k}(c_k) & \xrightarrow{\psi(b;c_1,\ldots,c_k)} & H_j(\gamma(b;c_1,\ldots,c_k))
\end{array}$$

For example, the data of a commutative $\mathbb{I}$-monoid gives rise to a $\tilde{\Sigma}$-module, where $\tilde{\Sigma}$ is the categorical Barratt-Eccles operad (i.e. $\tilde{\Sigma}_n$ is the translation category of the symmetric group Σ_n).

Definition A.0.7. Let $\mathscr{O}$ be a Σ-free operad in **Cat**. For an $\mathscr{O}$-module F we define the category

$$\mathscr{O} \ltimes_\Sigma F := \coprod_{n\geq 0} \mathscr{O}_n \ltimes_{\Sigma_n} F_n \,.$$

Definition A.0.6 is formulated precisely so that the following statement is true. By an $\mathscr{O}$-algebra we mean a category with an action by the operad $\mathscr{O}$.

Lemma A.0.8. *Let $\mathscr{O}$ be a Σ-free operad in* **Cat** *and let $F : \mathscr{O} \to$* **Top** *be an $\mathscr{O}$-module. The category $\mathscr{O} \ltimes_\Sigma F$ is naturally an $\mathscr{O}$-algebra and any morphism of $\mathscr{O}$-modules $t : F \to H$ induces a morphism of $\mathscr{O}$-algebras $T : \mathscr{O} \ltimes_\Sigma F \to \mathscr{O} \ltimes_\Sigma H$.*

Proof. An action $\theta_k : \mathscr{O}_k \times (\mathscr{O} \ltimes_\Sigma F)^k \to \mathscr{O} \ltimes_\Sigma F$ is defined by

$$\theta_k(b;[c_1,x_1],\ldots,[c_k,x_k]) = [\gamma(b;c_1,\ldots,c_k),\varphi(b;c_1,\ldots,c_k)(x_1,\ldots,x_k)]$$

on objects and by $\theta_k(f;[g_1],\ldots,[g_k]) = [\gamma(f;g_1,\ldots,g_k)]$ on morphisms. One can easily check that this is welldefined and satisfies [39, Def. 2] for **Cat**-operads. $\square$

For the constant $\mathscr{O}$-module $* : \mathscr{O} \to \mathbf{Top}$, which sends every object to a point, we have

$$B(\mathscr{O} \ltimes_\Sigma *) \cong \coprod_{n \geq 0} B\mathscr{O}_n/\Sigma_n,$$

which is the free $B\mathscr{O}$-algebra associated to the space S^0. Here $B : \mathbf{Cat} \to \mathbf{Top}$ is the classifying space of a category. The unique natural transformation $F \to *$ from any other $\mathscr{O}$-module F is a morphism of $\mathscr{O}$-modules and therefore induces a morphism of $B\mathscr{O}$-spaces $B(\mathscr{O} \ltimes_\Sigma F) \to \coprod_{n\geq 0} B\mathscr{O}_n/\Sigma_n$.

Let $\mathscr{M}$ be the surface operad constructed by Tillmann in [60]. Recall that $\mathscr{M}$ is an operad in **Cat** whose n-th category $\mathscr{M}_n$ has objects the pairs (c, σ) where c is a compact orientable surface (built from atomic surfaces, see below) of genus $g \geq 0$ with one ingoing and n outgoing boundary components, the latter of which are labelled by the element $\sigma \in \Sigma_n$. The set of morphisms $(c, \sigma) \to (c', \sigma')$ is $\pi_0(\mathrm{Diff}^+(c, c'; \partial))$, the set of isotopy classes of orientation preserving diffeomorphisms fixing the ingoing boundary component pointwise and permuting the outgoing boundary components according to the labels σ, σ' but preserving a fixed parametrization of a collar neighbourhood. Thus the endomorphism group of (c, σ) is precisely the mapping class group of c whose elements fix the boundary components pointwise.

The structure maps $\gamma : \mathscr{M}_k \times \mathscr{M}_{j_1} \times \cdots \times \mathscr{M}_{j_k} \to \mathscr{M}_j$ are induced by gluing k surfaces $c_i \in \mathscr{M}_{j_i}$ along their ingoing boundary component onto the outgoing boundaries of a surface $b \in \mathscr{M}_k$ according to the label of b. The objects in $\mathscr{M}$ are obtained from three atomic surfaces by gluing: the disc $D \in \mathscr{M}_0$, the torus with two discs removed $T \in \mathscr{M}_1$ and the pair of pants surface $P \in \mathscr{M}_2$. The operadic unit is the cylinder $S^1 \times [0, 1] \in \mathscr{M}_1$ which in $\mathscr{M}$ is identified with the circle S^1.

Theorem A.0.9 (Tillmann, [60, Theorem 4.4])**.** *The surface operad $\mathscr{M}$ detects infinite loop spaces.*

The theorem says that algebras over the surface operad group-complete to infinite loop spaces.

Let $\Gamma_{g,n+1}$ denote the group of mapping classes of a genus g orientable surface fixing the $n+1$ boundary components pointwise. In [60] an infinite loop space structure on $\mathbb{Z} \times B\Gamma^+_{\infty,1}$ is obtained by observing that the monoid

$$B\mathscr{M}_0 \simeq \coprod_{g \geq 0} B\Gamma_{g,1}$$

is naturally a $B\mathscr{M}$-algebra. In a similar way, we would like to associate an infinite loop space to every subspace $B(q, \Gamma_{\infty,1}) \subset B\Gamma_{\infty,1}$. For this we define an $\mathscr{M}$-module $\hat{F}_q : \mathscr{M} \to \mathbf{Top}$ in the following way.

We define the forgetful functor $\widehat{\ }: \mathscr{M} \to \mathscr{M}_0$ by $\hat{c} := \gamma(c; D, \dots, D)$ on objects and by $\hat{\psi} := \gamma(\psi; 1_D, \dots, 1_D)$ on morphisms (where D is the cap). The hat $\widehat{\ }$ shall remind us of capping off the outgoing boundary components. Let $\Gamma : \mathscr{M} \to \mathbf{Grp}$ be the functor which sends an object $(c, \sigma) \in \mathscr{M}$ to its endomorphism subcategory $\mathscr{M}(c, \sigma)$ and a mapping class $\psi : (c, \sigma) \to (c', \sigma')$ to the homomorphism $\mathrm{conj}_\psi : \Gamma(c, \sigma) \to \Gamma(c', \sigma')$ which is given by conjugation by ψ. We let F_q be the composition of Γ with the functor $B(q, -) : \mathbf{Grp} \to \mathbf{Top}$, and we let $\hat{F}_q$ be the composition of the forgetful functor $\widehat{\ }$ with F_q. Note that if $c \in \mathscr{M}_n$ is represented by a surface of genus g, then $\hat{F}_q(c) \simeq B(q, \Gamma_{g,1})$.

Lemma A.0.10. *The functor $\hat{F}_q : \mathscr{M} \to \mathbf{Top}$ is an $\mathscr{M}$-module.*

Proof. This is a straightforward check of Definition A.0.6. □

Let $MS^0 := \coprod_{n \geq 0} B\mathscr{M}_n/\Sigma_n$ be the free $B\mathscr{M}$-algebra associated to the space S^0. By Lemma A.0.8 and by Theorem A.0.9, we see that the space

$$MC_q := \mathrm{hofib}(\Omega B(B(\mathscr{M} \ltimes_\Sigma \hat{F}_q)) \xrightarrow{\text{proj.}} \Omega B(MS^0))$$

is an infinite loop space, since it is the homotopy fibre of a map of infinite loop spaces. By naturality, the inclusions $B(q, \Gamma_{g,1}) \to B(q+1, \Gamma_{g,1})$ induce maps of infinite loop spaces $MC_q \to MC_{q+1}$ for all q.

If we formally set $B(\infty, \Gamma_{g,1}) := B\Gamma_{g,1}$, then $\hat{F}_\infty$ is also an $\mathscr{M}$-module. In this case, the space MC_∞ is precisely the space $B\Gamma^+_{\infty,1}$, by the same argument as in [6, Thm. 3.1]. Thus for every q we also have an infinite loop map $MC_q \to B\Gamma^+_{\infty,1}$.

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