Complex and Riemannian Geometry:

QUATERNIONIC MANIFOLDS

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## Abstract

The starting point of the thesis is the definition of an almost quaternionic manifold as a real 4n-manifold admitting a $GL(n,\mathbb{H})Sp(1)$-structure. It is shown that there exist two obstructions $C_0, C_1$ to the integrability of such a structure, and the condition $C_0 = 0$ is used to define a quaternionic manifold. It is proved that a quaternionic manifold $M$ has a naturally associated complex $(2n+1)$-manifold $Z$ fibring over $M$ with fibre $\mathbb{C}P^1$, and the relationship between the complex structure of $Z$ and the quaternionic structure of $M$ is studied.

The above theory is then applied to the case in which $M$ is a Riemannian manifold with holonomy contained in $Sp(n)Sp(1)$, using properties of $Z$ to gain information about $M$. Special emphasis is given to the situation when $M$ has positive scalar curvature and it is shown that $M$ is then isometric to quaternionic projective space $\mathbb{H}P^n$ provided a certain mod 2 cohomology class vanishes. When $M$ is 8-dimensional, estimates for its Betti numbers and isometry group are obtained.

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## Contents

# INTRODUCTION

There have been various definitions of quaternionic manifolds by different authors. The most general involves adopting an infinitesimal approach, and imposing the structure of the group $GL(n,\mathbb{H})Sp(1) = GL(n,\mathbb{H})\times Sp(1)/_{\pm 1}$ on the tangent space at each point. This has the advantage of including not only quaternionic projective space $\mathbb{H}P^n$, but also a large class of symmetric spaces as examples. Accordingly we begin by defining an almost quaternionic manifold to be a real 4n-dimensional manifold admitting a $GL(n,\mathbb{H})Sp(1)$-structure and discussing some elementary consequences of this definition. In 4 dimensions an almost quaternionic structure is precisely an oriented conformal structure. There has recently been much interest in 4-dimensional conformal structures satisfying a certain curvature condition - namely self-duality. A construction due to Penrose and described in Atiyah, Hitchin and Singer [7] assigns to such a manifold a 'twistor space' consisting of a complex 3-dimensional manifold. It is the main purpose of part one of the thesis to find a class of almost quaternionic manifolds for which a similar twistor space construction may be carried out.

First we turn attention to the local behaviour of an almost quaternionic manifold M and find conditions for the existence of coordinates preserving the $GL(n,\mathbb{H})Sp(1)$-structure. Because $GL(n,\mathbb{H})Sp(1)$ has finite type, it turns out that there exist two tensors $C_0, C_1$, involving torsion and curvature respectively, which are obstructions to integrability. Moreover $C_0 \equiv C_1 \equiv 0$

implies that M is projectively equivalent to $\mathbb{H}P^n$ and so this case is of little interest. Consequently we adopt the definition that M is a quaternionic manifold if it satisfies the weaker condition $C_0 \equiv 0$. We then show that a quaternionic manifold possesses some interesting first order invariant differential operators.

Our definition of a quaternionic manifold is justified by the fact, proved in §4, that a quaternionic manifold M admits a 'twistor space' consisting of a complex (2n + 1)-dimensional manifold Z fibring over M with fibre $\mathbb{C}P^1$. This fact is then the appropriate generalization of the situation in 4 dimensions described in [7]. In §5 we introduce the concept of a quaternionic vector bundle over a quaternionic manifold. Such a bundle is a generalization of a self-dual vector bundle or 'instanton' in 4 dimensions, and has two related properties. Firstly its pullback to the twistor space is holomorphic and secondly (after modification) its total space is a quaternionic manifold. We conclude part one by investigating further the complex structure of the twistor space of a quaternionic manifold M, and showing that its Dolbeault $\bar{\partial}$-complex corresponds to a certain elliptic complex on M.

The group GL(n,$\mathbb{H}$)Sp(1) has maximal compact subgroup Sp(n)Sp(1) and the latter is one of the few groups that can arise as the holonomy of a locally irreducible Riemannian manifold that is not locally symmetric. Part two of the thesis is devoted to a study of Riemannian manifolds with holonomy contained in Sp(n)Sp(1), the so-called quaternionic Kähler manifolds. These are of interest from the Riemannian viewpoint because they are necessarily Einstein. On the other hand they are also quaternionic manifolds in the above sense and provide us with a large source of examples. In §8 we examine the twistor space of a quaternionic Kähler manifold and show that it possesses a complex contact structure. This generalizes work of Wolf [38] who observed a 1 - 1 correspondence between Riemannian symmetric spaces with holonomy in $Sp(n)Sp(1)$ and certain homogeneous complex contact manifolds.

The remainder of the thesis is concerned with the case of positive scalar curvature. We first show that if $M$ is a quaternionic Kähler manifold with scalar curvature $t > 0$, then its twistor space admits a Kähler-Einstein metric. This in turn has consequences for $M$ itself; for example $M$ must be simply-connected with odd Betti numbers zero. We also include a useful characterization of $\mathbb{H}P^n$ involving the vanishing of a certain mod 2 cohomology class. Finally, the implications of positive scalar curvature in low dimensions are considered. In particular, using the Atiyah-Singer index theorem, we prove that any 8-dimensional quaternionic Kähler manifold with $t > 0$ must have an isometry group of dimension at least 6, and Betti numbers $b_1 = b_3 = 0$, $b_4 = 1 + b_2$.

## Notation

Finite-dimensional representations of Lie groups will occur frequently and most of our vector bundles are defined by means of some representation V. In that case $\underline{V}$ denotes the associated vector bundle and $\underline{V}_x$ its fibre over x. For the sake of simplicity, all representations will be considered as complex vector spaces. For example, $\underline{T}$ generally denotes the complexified tangent bundle of a manifold, but thought of with its obvious real structure given by complex conjugation. When we wish to emphasize a real structure we use the subscript 'r', so that if V is a complex n-dimensional vector space, $V_r$ denotes the real n-dimensional vector space determined by an antilinear involution $\sigma : V \longrightarrow V$ (not the underlying real 2n-dimensional space).

The symbol '+' is used for direct sums and tensor products are often indicated by juxtaposition, so that $EH \otimes T^*$ would mean $E \otimes H \otimes T^*$. We consider the exterior and symmetric algebras $\Lambda^* V$, $S^* V$ as embedded in the tensor algebra; for example $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1$ and $v_1 \vee v_2 = v_1 \otimes v_2 + v_2 \otimes v_1$. Finally the summation convention is used for repeated indices, provided one is up and one is down.

# PART ONE

# QUATERNIONIC MANIFOLDS

## 1. Almost Quaternionic Manifolds

We shall regard $\mathbb{R}^{4n}$ as a right module over the skew-field $\mathbb{H}$ of quaternions by identifying $\mathbb{R}^{4n}$ with $\mathbb{H}^n$ and letting $\mathbb{H}$ act by right multiplication. In particular, the group Sp(1) of unit quaternions acts on $\mathbb{R}^{4n}$ on the right giving an inclusion $\lambda: \ Sp(1) \hookrightarrow GL(4n,\mathbb{R})$. Let $GL(n,\mathbb{H})$ denote the group of quaternionic linear transformations, i.e. elements of $GL(4n,\mathbb{R})$ preserving the module structure or equivalently commuting with the action of $\lambda(Sp(1))$. Then $GL(n,\mathbb{H}) \cap \lambda(Sp(1)) = \{\pm 1\}$ and $GL(n,\mathbb{H})Sp(1)$ will denote the group

$$GL(n,\mathbb{H})\ \lambda(Sp(1)) = GL(n,\mathbb{H})\times Sp(1)\big/\{(1,1),(-1,-1)\} \subset GL(4n,\mathbb{R})$$

Our starting point is the definition, following Bonan [12], Ishihara [25], of an almost quaternionic manifold as a real 4n-dimensional manifold M whose principal bundle of frames has structure group reducing from $GL(4n,\mathbb{R})$ to $GL(n,\mathbb{H})Sp(1)$. $GL(n,\mathbb{H})Sp(1)$ is connected and so contained in the identity component $GL^+(4n,\mathbb{R})$ of $GL(4n,\mathbb{R})$; thus M carries a natural orientation. Note also that $GL(1,\mathbb{H})Sp(1) = (\mathbb{R}^+ \times Sp(1))Sp(1)$ $= \mathbb{R}^+ \times SO(4)$ is the identity component of the conformal group CO(4), so that when n=1 an almost quaternionic structure is precisely an oriented conformal structure.

The principal $GL(n,\mathbb{H})Sp(1)$-bundle P defined by the almost quaternionic structure will lift locally to a principal $GL(n,\mathbb{H})\times Sp(1)$-bundle $\tilde{P}$. Consequently, any representation V of $GL(n,\mathbb{H})Sp(1)$ gives rise to a locally defined vector bundle $\underline{V} = P\times_{GL(n,\mathbb{H})\times Sp(1)}V$ associated to $P$. $\underline{V}$ is of course defined globally on $M$ in case either the representation $V$ factors through $GL(n,\mathbb{H})Sp(1)$ or the lifting $\tilde{P}$ exists globally. In particular, let $H$ denote the standard complex representation of $Sp(1) = SU(2)$ on $\mathbb{C}^2$, and $E$ the dual (for technical reasons) of the standard complex representation of $GL(n,\mathbb{H})\subset GL(2n,\mathbb{C})$ on $\mathbb{C}^{2n}$. $H$ and $E$ both come equipped with quaternionic structure maps $v \longmapsto \tilde{v}$ commuting with the action of the respective groups and satisfying $\widetilde{zv} = \bar{z}\tilde{v}$, $\tilde{\tilde{v}} = -v$, $z\in\mathbb{C}$. In addition, $H$ possesses an invariant skew form $\omega_H \in \Lambda^2H^*$ satisfying $\omega_H(\tilde{h},\tilde{k}) = \overline{\omega_H(h,k)}$, $\omega_H(h,\tilde{h}) > 0$ if $h \neq 0$. $h \longmapsto \omega_H(h,\cdot)$ gives an identification $H = H^*$ and if $h\in H$ satisfies $\omega_H(h,\tilde{h}) = 1$, then under this identification $\omega_H = h\wedge\tilde{h}$. By a standard basis of $H$ we shall mean a basis of the form $\{h_1=h,\ h_2=\tilde{h}\}$ such that $\omega_H(h,\tilde{h}) = 1$. All this structure carries over to the fibres of the associated vector bundle $H$; for example $\omega_H$ becomes a canonical element of $\Gamma(\Lambda^2\underline{H}^*) = \Gamma(\Lambda^2\underline{H})$ and by a standard basis of $\underline{H}$ we shall simply mean a pair of sections $\{h_1,h_2\}$ defined over some open set and satisfying the appropriate conditions at each point.

In terms of $E$ and $H$, the complexified tangent space representation is precisely $E^*\otimes H$, with real structure induced from the quaternionic structure maps of $E$ and $H$, so as vector bundles we may write

$$\underline{T} = \underline{E}^*\otimes \underline{H}, \qquad T^* = \underline{E}\otimes\underline{H}$$

$Sp(1)$ consists of automorphisms of $H$, so its Lie algebra is contained in $\mathrm{End}H$. Indeed, writing $\mathrm{End}H \cong H\otimes H^* \cong H\otimes H = \{\omega_H\} + S^2H$, we have $sp(1) \cong (S^2H)_r$. In addition, $\mathfrak{gl}(4n,R) \cong (\mathrm{End}T)_r$, so the inclusion $\lambda\colon Sp(1)\hookrightarrow GL(4n,R)$ induces an inclusion $\lambda_*$: $S^2H \hookrightarrow \mathrm{End}T$. More explicitly, this action of $S^2H$ on T may be described by the contraction

$$S^2H \otimes T \longrightarrow H \otimes \underbrace{H \otimes (E^* \otimes H)}_{\omega_H} \longrightarrow H \otimes E^* = T \qquad \ldots (1)$$

Since the Lie algebra sp(1) has the structure of the imaginary quaternions, the real 3-dimensional vector bundle $(S^2\underline{H})_r \subset \mathrm{End}\underline{T}$ may be viewed as a coefficient bundle of imaginary quaternions acting on the tangent space at each point. To see this directly, first note from (1) that composition $\mathrm{End}T \otimes \mathrm{End}T \longrightarrow \mathrm{End}T$ when restricted to $S^2H$ is essentially the projection

$$S^2H \otimes S^2H = S^4H + S^2H + A \longrightarrow S^2H + A$$

with skew part in $S^2H$, symmetric part in A. Here, as always, A denotes a trivial 1-dimensional representation. In fact if $v, w \in (S^2H)_r \subset \mathrm{End}T$,

$$v \circ w + w \circ v = -\langle v, w\rangle 1$$

where $\langle , \rangle$ is the metric induced by $\omega_H$ and 1 is the identity endomorphism. It follows that in a neighbourhood of each point $(S^2\underline{H})_r$ has a basis {I,J,K} consisting of sections satisfying the familiar quaternionic identities $I^2 = J^2 = -1$, $IJ = -JI = K$. One just has to choose I,J,K to be orthogonal with norm $\sqrt{2}$ with respect to $\langle , \rangle$. For example, if $\{h_1, h_2\}$ is a standard basis of $\underline{H}$, then such a standard basis of $(S^2\underline{H})_r$ is

$$\left.\begin{aligned} I &= h_1{}^2 + h_2{}^2 \\ J &= i(h_1{}^2 - h_2{}^2) \\ K &= i.\, h_1 \vee h_2 \end{aligned}\right\} \qquad \ldots (2)$$

On overlaps two standard bases {I,J,K}, {I',J',K'} will be related by an SO(3) transformation at each point.

Conversely, suppose that $M$ is a manifold whose tangent bundle possesses in a neighbourhood of each point endomorphisms $I,J,K$ satisfying $I^2 = J^2 = -1$, $IJ = -JI = K$ and such that any two sets $\{I,J,K\}$, $\{I',J',K'\}$ are related by an $SO(3)$ transformation at each point. Firstly, if $v$ is a tangent vector then $\{v,Iv,Jv,Kv\}$ will span a 4-dimensional subspace, so the dimension of $M$ must be $4n$, $n \geqslant 1$. Now a frame at $x \in M$ is simply an isomorphism $p : T \longrightarrow \underline{T}_x$ where as usual $T = \mathbb{R}^{4n}$ is thought of as a $GL(n,\mathbb{H})Sp(1)$ module, and $\underline{T}_x$ is the tangent space at $x$. $p$ induces an isomorphism $p' : \mathrm{End}T \longrightarrow (\mathrm{End}\underline{T})_x$ so we may consider the set $P_x$ of frames for which $I,J,K$ span $p'(\lambda_*(S^2H))$. It is easily seen that $P_x$ is the fibre of a principal $GL(n,\mathbb{H})Sp(1)$ bundle defining an almost quaternionic structure on $M$.

Let $M$ be an almost quaternionic manifold and consider the bundle

$$Z = \{v \in (S^2\underline{H})_r \; : \; \|v\| = \langle v,v\rangle^{\frac{1}{2}} = \sqrt{2}\}$$

whose fibres are real 2-spheres. From above, the fibre $Z_x$ over $x \in M$ actually parametrizes a set of almost complex structures on the tangent space $\underline{T}_x$. Moreover these almost complex structures determine the same orientation on $M$ as its almost quaternionic structure, essentially because any almost complex structure on $T = \mathbb{R}^{4n}$ defined by an element of $\lambda_*(S^2H)_r \subset \mathrm{End}T$ has the standard orientation. For future work it is convenient to have an alternative description of the bundle $Z$. Fix $x \in M$; the vector bundles $\underline{E}$ and $\underline{H}$ may be defined in a neighbourhood of $x$, so let $\underline{E}_x, \underline{H}_x$ denote their fibres over $x$. $h \in \underline{H}_x \setminus 0$ defines an almost complex structure $J_h$ on the tangent space $\underline{T}_x$ of M at x by specifying the subspace of (1,0)-forms (we prefer to work on the cotangent space) as follows:

$$(\underline{T}_x^*)^{1,0} = \underline{E}_x \otimes \mathbb{C}h \subset \underline{T}_x^* \qquad \ldots (3)$$

(recall that by convention, $\underline{T}_x^*$ is to be thought of as the complexified cotangent space so (3) makes sense). Note that taking complex conjugates the subspace of (0,1)-forms must then be

$$(\underline{T}_x^*)^{0,1} = \underline{E}_x \otimes \mathbb{C}\tilde{h}$$

Equivalently,

$$J_h = -i\, h \vee \tilde{h} / \omega_H(h, \tilde{h}) \in (S^2 \underline{H}_x)_r \subset \mathrm{End}\,\underline{T}_x$$

$J_h$ is unchanged when h is multiplied by any non-zero complex scalar and the above construction simply identifies the complex projective bundle $P(\underline{H})$ with Z. Note that although $\underline{H}$ itself may not be well-defined globally on M, $Z = P(\underline{H})$ always is. The manifold Z will play an important rôle in the sequel.

Above we have discussed only those properties of an almost quaternionic manifold M that arise directly from the definition. Next we consider imposing a Riemannian structure on M. This will enable us to talk about almost quaternionic manifolds in the context of Riemannian geometry; in fact in part two this viewpoint will predominate. We say that a Riemannian metric g on an almost quaternionic manifold is compatible with the almost quaternionic structure if for all local sections J of Z, $g(Jv, Jw) = g(v, w)$, i.e. g is Hermitian with respect to the almost complex structure J. Such a g can always be found, for given any Riemannian metric h, put

$$g(v,w) = h(v,w) + h(Iv,Iw) + h(Jv,Jw) + h(Kv,Kw)$$

where $\{I,J,K\}$ is a standard basis of $(S^2\underline{H})_r$. g is in fact independent of the standard basis chosen, so is well-defined globally and has the required property. An equivalent definition is that a Riemannian metric g is compatible with the almost quaternionic structure iff g is a section of

$$\Lambda^2\underline{E}\otimes\Lambda^2\underline{H}\subset S^2\underline{T}^*$$

$\Lambda^2\underline{H}$ is trivialized by $\omega_H$ and a compatible metric can then be written $g = \omega_E\otimes\omega_H$, where $\omega_E\in\Gamma(\Lambda^2\underline{E})$ is a real positive definite skew form. Now the subgroup of $GL(n,\mathbb{H})$ preserving $\omega_E$ at each point is $Sp(n)$, so the subgroup of $GL(n,\mathbb{H})Sp(1)$ preserving g is

$$Sp(n)Sp(1) = Sp(n)\times Sp(1)/\{(1,1),(-1,-1)\}$$

and we have an inclusion $\mu: Sp(n)Sp(1) \hookrightarrow SO(4)$. Therefore the choice of a compatible Riemannian metric is equivalent to a reduction of the structure group of the principal bundle P from $GL(n,\mathbb{H})Sp(1)$ to its maximal compact subgroup $Sp(n)Sp(1)$. Consequently a 4n-dimensional manifold is almost quaternionic iff the structure group of its principal frame bundle admits a reduction to $Sp(n)Sp(1)$. However we emphasize that our definition of an almost quaternionic manifold in no way involves the choice of a Riemannian metric.

Finally, we shall see that reducing to the compact group $Sp(n)Sp(1)$ enables us to find certain topological conditions that an almost quaternionic manifold must satisfy. Now $Sp(1)Sp(1) = SO(4)$, but for $n\geqslant 2$, $Sp(n)Sp(1)$ is a maximal Lie subgroup of SO(4n) (see Gray [17]). Consider the diagram

$$\begin{array}{ccc} Sp(n)xSp(1) & \longrightarrow & Sp(n)Sp(1) \\ & & \downarrow \mu \\ Spin(4n) & \longrightarrow & SO(4n) \end{array}$$

in which the two groups on the left are simply-connected and the horizontal projections are double coverings exhibiting $\pi_1(Sp(n)Sp(1)) \cong \mathbb{Z}_2 \cong \pi_1(SO(4n))$. We calculate $\mu_* \pi_1(Sp(n)Sp(1))$. Left multiplication on $T = \mathbb{H}^n$ by unit quaternions determines an inclusion $Sp(1) \subset Sp(n)$. Take a basis $\{1,i,j,k : i^2 = -1, \text{ etc}\}$ of $\mathbb{H}$. Then $\gamma = \{(e^{\pi ix}, e^{\pi ix}) : x \in [0,1]\} \subset Sp(1)xSp(1) \subset Sp(n)xSp(1)$ is a path projecting to a homotopically non-trivial loop in $Sp(n)Sp(1)$. Because

$$e^{ix}1e^{ix} = e^{2ix}1, \quad e^{ix}ie^{ix} = e^{2ix}i, \quad e^{ix}je^{ix} = j, \quad e^{ix}ke^{ix} = k$$

$\mu^*\gamma$ is the loop

$$\left(\begin{array}{cc|cc|cc|cc|c} \cos 2\pi x & -\sin 2\pi x & & & & & & & \\ \sin 2\pi x & \cos 2\pi x & & & & & & & \\ \hline & & 1 & 0 & & & & & \\ & & 0 & 1 & & & & & \\ \hline & & & & \cos 2\pi x & -\sin 2\pi x & & & \\ & & & & \sin 2\pi x & \cos 2\pi x & & & \\ \hline & & & & & & 1 & 0 & \\ & & & & & & 0 & 1 & \\ \hline & & & & & & & & \text{etc} \end{array}\right), x \in [0,1]$$

Using standard coordinates in the maximal torus of SO(4n) we see that $\mu^*\gamma$ is described by the point $(x_1, \ldots, x_{2n}) = (1,0,1,0,..)$ in the integer lattice. According to the description of $\pi_1(SO(4n))$ in terms of the Stiefel diagram (see Adams [1]), $\mu^*\gamma$ is homotopicall trivial iff $x_1 + \ldots + x_{2n} \equiv 0 \bmod 2$, i.e. n is even. Consequently, if n is even $\mu$ lifts to an inclusion $\tilde{\mu} : Sp(n)Sp(1) \hookrightarrow Spin(4n)$, and only if n is odd is there a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & Sp(n)xSp(1) & \longrightarrow & Sp(n)Sp(1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \mu & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & Spin(4n) & \longrightarrow & SO(4n) & \longrightarrow & 0 \end{array}$$

Let M be an almost quaternionic manifold with associated principal bundle $P \in H^1(M; GL(n,\mathbb{H})Sp(1))$. In the sequence of cohomology groups corresponding to

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow GL(n,\mathbb{H})\times Sp(1) \longrightarrow GL(n,\mathbb{H})Sp(1) \longrightarrow 0$$

let $\xi \in H^2(M; \mathbb{Z}_2)$ denote the image of P under the coboundary homomorphism $\partial$. $\xi$ is the obstruction to the lifting of P to a $GL(n,\mathbb{H})\times Sp(1)$-principal bundle on M, or equivalently to the global existence of the vector bundles $\underline{E}$ and $\underline{H}$. Picking a compatible Riemannian metric g defines $P' \in H^1(M; Sp(n)Sp(1))$ and $P'' \in H^1(M; SO(4n))$.

$$\begin{tikzcd}
\arrow[r] & H^1(M;GL(n,\mathbb{H})\times Sp(1)) \arrow[r] & H^1(M;GL(n,\mathbb{H})Sp(1)) \arrow[r, "\partial"] & H^2(M;\mathbb{Z}_2) \\
\arrow[r] & H^1(M;Sp(n)\times Sp(1)) \arrow[u] \arrow[d, dashed] \arrow[r] & H^1(M;Sp(n)Sp(1)) \arrow[u] \arrow[d, "\mu_*"] \arrow[r, "\partial'"] & H^2(M;\mathbb{Z}_2) \\
\arrow[r] & H^1(M;Spin(4n)) \arrow[r] & H^1(M;SO(4n)) \arrow[r, "\partial''"] & H^2(M;\mathbb{Z}_2)
\end{tikzcd}$$

M is a spin manifold iff the principal bundle of oriented orthonormal frames $P''$ is the image of an element in $H^1(M; Spin(4n))$. If n is even, $\mu_*(P')$ is such an element. On the other hand, when n is odd, the obstruction to the existence of a spin structure, $\partial''(P'')$, which is well known to equal the second Stiefel-Whitney class $w_2$ of M, satisfies

$$w_2 = \partial''(P'') = \partial'(P') = \partial(P) = \xi$$

In the special case $n = 1$, the bottom two horizontal sequences in the above diagram actually coincide; indeed the vector bundles $\underline{E} = \underline{V}_+$ and $\underline{H} = \underline{V}_-$ are precisely the spinor bundles of M (with appropriate conformal weights). To summarize,

<u>Proposition 1.1</u> An almost quaternionic manifold M of dimension 4n, $n \geq 1$, is automatically spin if n is even. When n is odd, M is spin iff $\xi = 0$.

This corrects an erroneous statement of Kulkarni [32; theorem 6.3].

## 2. Integrability

In §1 we adopted an infinitesimal approach in the sense that the definition of an almost quaternionic manifold involved imposing an appropriate structure on the tangent space at each point. In this section we shall be more concerned with local behaviour. More precisely, in the context of G-structures, we ask under what conditions the $GL(n,\mathbb{H})Sp(1)$-structure of an almost quaternionic manifold is integrable. By definition, integrability means that about each point of M there is a chart $(U;x^1, . . ,x^{4n})$ such that $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{4n}}\right\}$ constitutes a local section of the principal G-bundle P defining the G-structure (see for example Guillemin [19]). It follows that on the overlap of two such charts $(U;x^i),(V;y^j)$, the transition functions have derivative in G:-

$$\left(\frac{\partial y^j}{\partial x^i}\bigg|_x\right) \in G = GL(n,\mathbb{H})Sp(1), \qquad x \in U \cap V$$

Equivalently, we may say that the G-structure on M is locally isomorphic to the standard G-structure $\mathbb{R}^{4n}\times G$ on $\mathbb{R}^{4n}$ in the sense that any $x \in M$ has a neighbourhood U admitting a diffeomorphism $\varphi: U \longrightarrow \varphi(U) \subset \mathbb{R}^{4n}$ such that the induced mapping $\varphi_*$ of frame bundles maps $P|U$ isomorphically onto $(\mathbb{R}^{4n}\times G)|\,\varphi(U)$.

The definition of an almost quaternionic manifold is really modelled upon quaternionic projective space $\mathbb{H}P^n$ which itself possesses an integrable $GL(n,\mathbb{H})Sp(1)$-structure. $\mathbb{H}P^n$ is defined as the quotient $\mathbb{H}^{n+1}\setminus 0/\sim$, where $\sim$ is the equivalence relation $(q_0,..,q_n)\sim(q_0u,..,q_nu)$, $u \in \mathbb{H}\setminus 0$. $\mathbb{H}P^n$ may be covered by open sets $U_i = \{(q_0, . . ,q_n): q_i \neq 0\}$ and on each $U_i$ the quaternions $x_i^k = q_kq_i^{-1}$, $k \neq i$ define local cordinates. Note that the mappings

$q \longmapsto qu$, $q \longmapsto q^{-1}$ defined on an open set of $\mathbb{H}$ (where u is a fixed quaternion) have differentials which may be written $d(qu) = dq.u$, $d(q^{-1}) = -q^{-1}dq.q^{-1}$. These do not belong to $GL(n,\mathbb{H})$, although both belong to $GL(n,\mathbb{H})Sp(1)$, and it is this fact that explains why $\mathbb{H}P^n$ does not admit a $GL(n,\mathbb{H})$-structure. Indeed on $U_i \cap U_j$,

$$x_j^k = q_k q_j^{-1} = (q_k q_i^{-1})(q_i q_j^{-1}) = x_i^k (x_i^j)^{-1}$$

and hence

$$dx_j^k = (dx_i^k - x_i^k (x_i^j)^{-1} dx_i^j)(x_i^j)^{-1} \quad \ldots (4)$$

This proves that on overlaps the coordinates $x_i^k$ have transition functions with derivative in $GL(n,\mathbb{H})Sp(1)$ and they therefore define an integrable $GL(n,\mathbb{H})Sp(1)$-structure on $\mathbb{H}P^n$. Denote the corresponding principal $GL(n,\mathbb{H})Sp(1)$ bundle by Q. The action of $GL(n+1,\mathbb{H})$ on $\mathbb{H}^{n+1}$ induces the group $PGL(n,\mathbb{H}) = SL(n+1,\mathbb{H})/_{\pm 1}$ of projective linear transformation on $\mathbb{H}P^n$. An important feature of the almost quaternionic structure of $\mathbb{H}P^n$ is that it is preserved by this group, in the sense that any element of $PGL(n,\mathbb{H})$ induces an automorphism of Q.

Returning to the general case, there is a well-developed theory for determining the conditions under which a G-structure is integrable and results for $G = GL(n,\mathbb{H})Sp(1)$ are implicitly contained in Kobayashi & Nagano [27], and Ochiai [34]. However, for our subsequent work it is essential at this point to have a complete description of the obstructions to the integrability of an almost quaternionic structure. By exploiting knowledge of the finite-dimensional representation theory of $GL(n,\mathbb{H})Sp(1)$, we are able to find these obstructions with relative ease.

We adopt the terminology and results of Guillemin [19].

Consider any G-structure on a manifold M where $G \subset GL(m,R)$. Denote the standard representation of $GL(m,R)$ on $R^m$ by T. Then the Lie algebra $\mathfrak{g}$ of G is contained in $\operatorname{End}T = T \otimes T^*$. The $r^{th}$ prolongation of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^r = (\mathfrak{g} \otimes S^r T^*) \cap (T \otimes S^{r+1} T^*)$$

We also put $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^{-1} = T$. For $r \geqslant 1$, there is an exact sequence

$$0 \longrightarrow S^r T^* \longrightarrow S^{r-1} T^* \otimes T^* \xrightarrow{\partial} S^{r-2} T^* \otimes \Lambda^2 T^* \longrightarrow \ldots \longrightarrow \Lambda^r T^* \longrightarrow 0$$

where $\partial : S^p T^* \otimes \Lambda^q T^* \longrightarrow S^{p-1} T^* \otimes \Lambda^{q+1} T^*$ is defined in an obvious manner:-

$$\partial(\otimes^p v \otimes v_{i_1} \wedge \ldots \wedge v_{i_q}) = \otimes^{p-1} v \otimes v \wedge v_{i_1} \wedge \ldots \wedge v_{i_q}$$

If we tensor this complex with T, we obtain the following subcomplex :-

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathfrak{g}^{r-1} & \xrightarrow{\partial} & \mathfrak{g}^{r-2} \otimes T^* & \longrightarrow & \mathfrak{g}^{r-3} \otimes \Lambda^2 T^* & \longrightarrow & \ldots & \longrightarrow & \mathfrak{g}^{-1} \otimes \Lambda^r T^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \| & & \\
0 & \longrightarrow & T \cdot S^r T^* & \xrightarrow{1 \otimes \partial} & T \cdot S^{r-1} T^* \otimes T^* & \longrightarrow & T \cdot S^{r-2} T^* \otimes \Lambda^2 T^* & \longrightarrow & \ldots & \longrightarrow & T \cdot \Lambda^r T^* & \longrightarrow & 0
\end{array}$$

Put

$$H^{p,q}(\mathfrak{g}) = \frac{\{v \in \mathfrak{g}^{p-1} \otimes \Lambda^q T^* : \partial v = 0\}}{\partial(\mathfrak{g}^p \otimes \Lambda^{q-1} T^*)}$$

Note that $H^{p,1}(\mathfrak{g}) = 0$ for all $p \geqslant 1$. Then according to [19], the character of the integrability problem is determined by the cohomology groups $H^{p,2}(\mathfrak{g})$.

An important class of G-structures are those for which G has finite type; a group G is said to have finite type k if $\mathfrak{g}^k = 0$, with $\mathfrak{g}^{k-1} \neq 0$. The fundamental result for G-structures of finite type may be stated as follows. There exist obstructions $c_p \in \Gamma(\underline{H^{p,2}(\mathfrak{g})})$, $0 \leqslant p \leqslant k$, all of which vanish if the G-structure is integrable. Here $\mathfrak{g}$ is thought of as the adjoint representation of G, so $H^{p,2}(\mathfrak{g})$ is itself a G-module, and as usual $\underline{H^{p,2}(\mathfrak{g})}$ denotes the associated vector bundle.

In order to calculate the prolongations of the Lie algebra of $GL(n,\mathbb{H})Sp(1)$, we must be able to handle with ease all the finite-dimensional representations of $GL(n,\mathbb{H})Sp(1)$ which occur. Now $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}(2)$, 2x2 complex matrices, and it follows that the real simply-connected Lie group $GL(n,\mathbb{H})$ admits $GL(2n,\mathbb{C})$ as a complexification. Also $GL(2n,\mathbb{C})$ has compact real form $U(2n)$. From the general theory of representations of Lie groups (see for example Zelobenko [40]), there is a one-to-one correspondence between the finite-dimensional irreducible representations of $GL(n,\mathbb{H})$, the irreducible analytic representations of $GL(2n,\mathbb{C})$, and the irreducible (possibly many-valved) representations of $U(2n)$. Therefore for the purpose of calculating representations we may replace $GL(n,\mathbb{H})Sp(1)$ by the group $G = U(2n)\times Sp(1) = U(2n)\times SU(2)$. This enables us to use the machinery for describing irreducible representations of compact Lie groups, and in particular the method of decomposing the tensor product of two irreducible representations into irreducible components outlined in the appendix.

We emphasize that by 'irreducible representation' we shall always mean ' irreducible complex representation'. The representation theory of $SU(2)$ is well-known. Irreducible representations consist precisely of the spaces $S^rH$, $r \geq 0$, and tensor products of these decompose according to the formula

$$S^pH \otimes S^qH = \sum_{r=0}^{\min(p,q)} S^{p+q-2r}H$$

As for $U(2n)$, we shall label irreducible representations by their highest weights. We are supposing that $n \geqslant 1$, but for notational ease we shall often pretend that $n = 2$ and we are dealing with $U(4)$. Thus for the most part we denote weights of $U(2n)$ by 4-tuples; this is justified because in reality most of the representations of $U(2n)$ we need have weights of the form $(a_1, 0, \ldots, 0, a_2, a_3, a_4)$. For example, the standard representation of $U(2n)$ on $\mathbb{C}^{2n}$ has highest weight $(1,0,\ldots,0)$, so will be written as $(1,0,0,0)$. Adopting the practice of identifying corresponding representations of $U(2n)$ and $GL(n,\mathbb{H})$, we shall also denote this representation by $E^*$. The dual representation is $E = (0,0,0,-1)$ and the cotangent space representation takes the familiar form $E \otimes H$. As a simple example of the method of the appendix, the adjoint representation of $U(2n)$ or $GL(n,\mathbb{H})$ is isomorphic to

$$E \otimes E^* = (0,0,0,-1) \otimes (1,0,0,0) = (0,0,0,0) + (1,0,0,-1) = A + B$$

say, where as always $A$ is a trivial 1-dimensional representation, and $B$ is the semisimple part of the Lie algebra. Therefore as G-modules,

$$\begin{aligned} \mathrm{End}T &= T \otimes T^* \\ &= (E^* \otimes H) \otimes (E \otimes H) \\ &= (A + B) \otimes (A + S^2H) \\ &= A + B + S^2H + B \otimes S^2H \end{aligned}$$

Here the space $\mathfrak{g} = A + B + S^2H$ is precisely the adjoint representation of G.

Any irreducible representation of G occurs as a subspace of $(\bigotimes^p E) \otimes (\bigotimes^q E^*) \otimes (\bigotimes^r H)$ for some $p,q,r \geqslant 0$. It will therefore be convenient to express the elements of such a representation in terms of a basis $\{e_i\}$ of $E$, its dual basis $\{e^i\}$ of $E^*$, and a standard basis $\{h_1 = h,\ h_2 = \tilde{h}\}$ of $H$. Furthermore we shall regard the elements $\sum_{i=1}^{2n} e_i e^i$, $\omega_H = h \wedge \tilde{h}$, which trivialize invariant 1-dimensional subspaces of $E\otimes E^*$, $H\otimes H$ respectively, as unit quantities. For example, elements of $\mathcal{Q}$ will be written

$$\left.\begin{aligned} 1 &= e^i h e_i \tilde{h} - e^i \tilde{h} e_i h \in A \\ e^i e_j &= e^i h e_j \tilde{h} - e^i \tilde{h} e_j h \in \mathcal{B} \\ h_i \vee h_j &= e^k h_i e_k h_j + e^k h_j e_k h_i \in S^2 H \end{aligned}\right\} \subset E^* H \otimes EH = T\otimes T^*$$

(summation over repeated indices). Incidentally, the inclusion $S^2H \hookrightarrow \mathrm{End}T$ that this notation defines is consistent with the one discussed in §1.

A more difficult problem is to determine the decomposition under G of the exterior powers $\Lambda^r T^*$ in order to give a quaternionic analogue of the (p,q)-decomposition of $\Lambda^r T^*$ on an almost complex manifold. In fact

<u>Proposition 2.1</u> As a G-module,

$$\Lambda^r T^* = \sum_{k=0}^{[r/2]} (0,..,0,\underbrace{-1,..,-1}_{r-2k},\underbrace{-2,..,-2}_{k}) \otimes S^{r-2k}H$$

<u>Proof</u> We outline a proof. Because $T^* = E\otimes H$, $\Lambda^r T^*$ certainly has the form

$$\Lambda^r T^* = \sum_{k=0}^{[r/2]} V_k \otimes S^{r-2k}H$$

where $V_k \subset \otimes^r E$ is some representation of $U(2n)$. In fact each $V_k$ is non-zero and irreducible (e.g. by passing to general linear groups and using Young diagrams as in Howe [24]). It remains to show that $V_k$ has the required highest weight. Certainly $V_k$ has highest weight of the form $(0,..,0,a_1,..,a_j)$. Suppose that the final digit $a_j \leqslant -3$, for example $a_j = -3$. Then because $S^3E = (0,..,0,-3)$, a careful application of the method of the appendix gives that $V_k$ must occur as a summand in the decomposition of the tensor product

$$S^3E \otimes (0,..,0,0,a_1..,a_{j-1})$$

However $V_k$ would then have to be paired with a representation of $SU(2)$ contained in $\Lambda^3 H \otimes (\otimes^{p-3} H) = 0$. Consequently, $V_k$ must have the form

$$(0,..,0,-1,..,-1,-2,..,-2)$$

In order for $V_k$ to pair with $S^{r-2k}H = \otimes^k(\Lambda^2 H) \otimes S^{r-2k}H$, $V_k$ must occur as a summand in the tensor product $\otimes^k(S^2E) \otimes \Lambda^{r-2k}E$. This forces there to be k -2's and r-2k -1's as required.

<u>Theorem 2.1</u> For $n \geqslant 1$, the group $GL(n,\mathbb{H})Sp(1)$ has finite type 2.

For $n = 1$, we are dealing with the conformal group, and this result is elementary and well-known. For $n \geqslant 2$, it is necessary to use the representation theory outlined above, and the proof is divided into lemmas. $\mathfrak{g}$ denotes the Lie algebra of $GL(n,\mathbb{H})Sp(1)$, or equivalently of $G = U(2n) \times SU(2)$.

<u>Lemma 2.1</u> As a G-module, the first prolongation $\mathfrak{g}^1$ equals $T^* = E \otimes H$.

<u>Proof</u> From the definition of $\mathfrak{g}^1$, there is an exact sequence

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow \mathfrak{g} \otimes T^* \longrightarrow T \otimes \Lambda^2 T^*$$

We decompose $\mathfrak{g} \otimes T^*$, $T \otimes \Lambda^2 T^*$ into irreducible components, using without comment the method of the appendix. First,

$$\begin{aligned} B \otimes E &= (1,0,0,-1) \otimes (0,0,0,-1) \\ &= (1,0,0,-2) + (1,0,-1,-1) + (0,0,0,-1) \\ &= C + D + E \end{aligned}$$

say, and

$$
\begin{aligned}
E^* \otimes S^2E &= (1,0,0,0) \otimes (0,0,0,-2) \\
&= (1,0,0,0,-2) + (0,0,0,-1) \\
&= C + E \\
E^* \otimes \wedge^2 E &= (1,0,0,0) \otimes (0,0,-1,-1) \\
&= (1,0,-1,-1) + (0,0,0,-1) \\
&= D + E
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathfrak{g} \otimes T^* &= (A + B + S^2H) \otimes EH \\
&= (3E + C + D) \otimes H + E \otimes S^3H \\
T \otimes \wedge^2 T^* &= E^*H \otimes \wedge^2(EH) \\
&= E^*H \otimes (\wedge^2E\ S^2H + S^2E) \\
&= (2E + C + D) \otimes H \quad + \quad (E + D) \otimes S^3H
\end{aligned}
$$

In general, if $\alpha : \sum U_i + \sum V_j \longrightarrow \sum U_i + \sum W_k$ is a G-map between G-modules with $U_i$, $V_j$, $W_k$ irreducible and each $V_j$ distinct from each $W_k$, let us say that has maximal rank if $\alpha \big| \sum U_i$ is an isomorphism. By Schur's lemma, to prove that $\alpha$ has maximal rank it suffices to find elements $u_i \in U_i + \sum V_j$ such that $\alpha(u_i) \neq 0$. Here we must show that $\partial : \mathfrak{g} \otimes T^* \longrightarrow T \otimes \wedge^2 T^*$ has maximal rank, but it is readily checked that $\partial$ annihilates none of the elements

$$
\left.
\begin{array}{rcl}
1 \otimes e_1 h & \in & EH \subset A \otimes EH \\
\left.\begin{array}{r} e^1 e_2 \otimes e_2 h \\ e^1 e_2 \otimes e_3 h - e^1 e_3 \otimes e_2 h \end{array}\right. & \begin{array}{c} \in \\ \in \end{array} & \left.\begin{array}{l} CH \\ DH \end{array}\right\} \subset B \otimes EH \\
\left.\begin{array}{r} h^2 \otimes e_1 h \\ 2h^2 \otimes e_1 \tilde{h} - (h \vee \tilde{h}) \otimes e_1 h \end{array}\right. & \begin{array}{c} \in \\ \in \end{array} & \left.\begin{array}{l} ES^3H \\ EH \end{array}\right\} \subset S^2H \otimes EH
\end{array}
\right\} \subset \mathfrak{g} \otimes T^*
$$

This establishes lemma 2.1.

For the sequel, it will be useful to have an explicit representative in $\mathfrak{g}^1 \subset \mathfrak{g} \otimes T^*$. Thus we prove

<u>Lemma 2.2</u> Let

$$\left.\begin{aligned} u_1 &= 1\otimes e_1h &&\in A\otimes EH \\ u_2 &= e^ie_1\otimes e_ih - \tfrac{1}{2n}e^ie_i\otimes e_1h &&\in B\otimes EH \\ u_3 &= 2h^2\otimes e_1\tilde{h} - (h\vee\tilde{h})\otimes e_1h &&\in S^2H\otimes EH \end{aligned}\right\} \subset \mathfrak{g}\otimes T^*$$

Then $u = (n+1)u_1 + 2nu_2 + nu_3 \in \mathfrak{g}'$.

<u>Proof</u> Each $u_i$ is proportional to the image of $e_1h$ under the inclusions $EH \hookrightarrow A\otimes EH,\ B\otimes EH,\ S^2H\otimes EH$. By lemma 2.1, there must exist $a,b,c\in\mathbb{R}$ such that $\partial(au_1 + bu_2 + cu_3) = 0$, i.e. $au_1 + bu_2 + cu_3 \in \mathfrak{g}'$. Recall that as an element of $T\otimes T^*\otimes T^*$, $au_1 + bu_2 + cu_3$ is

$$(a - \tfrac{1}{2n}b)\ (e^ihe_i\tilde{h} - e^i\tilde{h}e_ih)e_2h + b(e^ihe_2\tilde{h} - e^i\tilde{h}e_2h)e_ih$$
$$+c\left[2e^ihe_ihe_2\tilde{h} - (e^ihe_i\tilde{h} + e^i\tilde{h}e_ih)e_2h\right]$$

Applying $\partial$ followed by the contractions $\underbrace{E^*\,\overbrace{H\otimes EH}}_{\omega_H}\otimes EH$, $\underbrace{E^*\,\overbrace{H\otimes EH\otimes EH}}_{\omega_H}$ and equating to zero gives respectively

$$\begin{aligned} 0 &= (a - \tfrac{1}{2n}b)(2n+2n-1) + b(1+1-2n) + c(-2-2n+2n-1) \\ &= (4n-1)a + \frac{1-4n^2}{2n}b - 3c \end{aligned}$$

and

$$\begin{aligned} 0 &= (a - \tfrac{1}{2n}b)(1+1-2n) + b(2n+2n-1) + c(-4n-1+1-2n) \\ &= (2-2n)a + \frac{4n^2-1}{n}b - 6nc \end{aligned}$$

A solution is $a = n+1$, $b = 2n$, $c = n$, proving the lemma.

Finally, to determine the second prolongation $\mathfrak{g}^2$, we use the exact sequence

$$0 \longrightarrow \mathfrak{g}^2 \longrightarrow \mathfrak{g}^1\otimes T^* \longrightarrow \mathfrak{g}\otimes\Lambda^2T^*$$

Theorem 2.1 is equivalent to the statement $\mathfrak{g}^2 = 0$, and this is an immediate consequence of

<u>Lemma 2.3</u> $\partial : \mathfrak{g}^1\otimes T^* \longrightarrow \mathfrak{g}\otimes\Lambda^2T^*$ is injective.

Proof As G-modules, using lemma 2.1,

$$\mathfrak{g}^1 \otimes T^* = EH \otimes EH$$
$$= \Lambda^2 E + S^2 E + (\Lambda^2 E + S^2 E) \otimes S^2 H$$

First observe that in the notation of lemma 2.2,

$$\partial(u_3 \otimes e_2\tilde{h}) = 2h^2 \otimes (e_1\tilde{h} \wedge e_2\tilde{h}) - (h \vee \tilde{h}) \otimes (e_1 h \wedge e_2\tilde{h}) \subset S^2 H \otimes \Lambda^2 T^*$$

has non-zero components in each of the spaces $\Lambda^2 E$, $S^2 E$, $\Lambda^2 E\, S^2 H$, $S^2 E\, S^2 H$. $\partial(u \otimes e_2\tilde{h})$ must also have non-zero components there; because $\partial(u_1 \otimes e_2\tilde{h})$, $\partial(u_2 \otimes e_2\tilde{h})$ together only have components in $(A + B) \otimes \Lambda^2 T^*$. Lemma 2.3 now follows from Schur's lemma.

We now turn attention to the cohomology groups $H^{r,2}(\mathfrak{g})$. The vanishing of $\mathfrak{g}^2$ implies that only the following may be non-zero:

$$H^{0,2}(\mathfrak{g}) = T \otimes \Lambda^2 T^* \big/ \partial(\mathfrak{g} \otimes T^*)$$

$$H^{1,2}(\mathfrak{g}) = (\ker \partial : \mathfrak{g} \otimes \Lambda^2 T^* \longrightarrow T \otimes \Lambda^3 T^*) \big/ \partial(\mathfrak{g}^1 \otimes T^*)$$

$$H^{2,2}(\mathfrak{g}) = \ker \partial : \mathfrak{g}^1 \otimes \Lambda^2 T^* \longrightarrow \mathfrak{g} \otimes \Lambda^3 T^*$$

Theorem 2.2 For $n \geqslant 2$, as G-modules,

$$H^{0,2}(\mathfrak{g}) = D \otimes S^3 H$$
$$H^{1,2}(\mathfrak{g}) = (1,0,0,-3) \; (= U \text{ say})$$
$$H^{2,2}(\mathfrak{g}) = 0$$

Proof The first equality is an immediate corollary of the proof of lemma 2.1. Now

$$B \otimes \Lambda^2 E = (1,0,0,-1) \otimes (0,0,-1,-1)$$
$$= (1,0,-1,-2) + (1,-1,-1,-1) + (0,0,0,-2) + (0,0,-1,-1)$$
$$= M + N + S^2 E + \Lambda^2 E$$

say, and

$$\begin{aligned} B \otimes S^2E &= (1,0,0,-1) \otimes (0,0,0,-2) \\ &= (1,0,0,-3) + (1,0,-1,-2) + (0,0,0,-2) + (0,0,-1,-1) \\ &= U + M + S^2E + \Lambda^2E \end{aligned}$$

Then

$$\begin{aligned} \mathfrak{g} \otimes \Lambda^2T^* &= (A + B + S^2H) \otimes (\Lambda^2E\, S^2H + S^2E) \\ &= 2\Lambda^2E + 2S^2E + U + M + (3\Lambda^2E + 2S^2E + M + N) \otimes S^2H + \Lambda^2E \otimes S^4H \end{aligned}$$

On the other hand, using proposition 2.1,

$$\begin{aligned} T \otimes \Lambda^3T^* &= E^*H \otimes ((0,0,-1,-2)H + \Lambda^3E\, S^3H) \\ &= \Lambda^2E + S^2E + M + (2\Lambda^2E + S^2E + M + N) \otimes S^2H + (\Lambda^2E + N) \otimes S^4H \end{aligned}$$

by the usual methods. Using lemma 2.3, the second equality of the theorem will follow if we can show that $\partial : \mathfrak{g} \otimes \Lambda^2T^* \longrightarrow T \otimes \Lambda^3T^*$ has maximal rank. However this in turn follows from the fact, readily verified, that $\partial$ annihilates none of the elements

$$\left.\begin{array}{r} \left.\begin{array}{rl} 1 \otimes (e_1h \wedge e_1\tilde{h}) \in & S^2E \\ 1 \otimes (e_1h \wedge e_2h) \in & \Lambda^2ES^2H \end{array}\right\} \subset A \otimes \Lambda^2(EH) \\ \left.\begin{array}{rl} e^1e_2 \otimes (e_3h \wedge e_3\tilde{h}) \in & U + M \\ e^1e_2 \otimes (e_3h \wedge e_4h) + e^1e_3 \otimes (e_2h \wedge e_4h) \in & MS^2H \\ e^1e_2(e_3h \wedge e_4h) + e^1e_3(e_4h \wedge e_2h) + e^1e_4(e_2h \wedge e_3h) \in & NS^2H \end{array}\right\} \subset B \otimes \Lambda^2(EH) \\ \left.\begin{array}{rl} h^2 \otimes (e_1h \wedge e_2h) \in & \Lambda^2ES^4H \\ h^2 \otimes (e_1h \wedge e_1\tilde{h}) \in & S^2ES^2H \\ (h \vee \tilde{h}) \otimes (e_1h \wedge e_2h) - h^2 \otimes (e_1h \wedge e_2\tilde{h}) \in & \Lambda^2ES^2H \end{array}\right\} \subset S^2H \otimes \Lambda^2(EH) \end{array}\right\} \subset \mathfrak{g} \otimes \Lambda^2T^*$$

For the final equality, we must show that $\partial : \mathfrak{g}^1 \otimes \Lambda^2T^* \longrightarrow \mathfrak{g} \otimes \Lambda^3T^*$ is injective. There is an inclusion

$$\mathfrak{g}^1 \otimes \Lambda^2T^* \hookrightarrow \mathfrak{g} \otimes T^* \otimes \Lambda^2T^*$$

and by lemma 2.3, an inclusion

$$\mathfrak{g}^1 \otimes \Lambda^2T^* \hookrightarrow \mathfrak{g}^1 \otimes T^* \otimes T^* \xrightarrow{\partial \otimes 1} \mathfrak{g} \otimes \Lambda^2T^* \otimes T^*$$

Thus

$$\partial : \mathfrak{g}^1 \otimes \Lambda^2T^* \hookrightarrow (\mathfrak{g} \otimes \Lambda^2T^* \otimes T^*) \cap (\mathfrak{g} \otimes T^* \otimes \Lambda^2T^*) = \mathfrak{g} \otimes \Lambda^3T^*$$

This completes the proof of theorem 2.2.

The fundamental result for G-structures of finite type, stated earlier implies the

<u>Corollary</u> Let M be an almost quaternionic manifold of dimension $4n$, $n \geqslant 2$. Then there exist tensors $C_0 \in \Gamma(\underline{H^{0,2}(\mathfrak{g})})$, $C_1 \in \Gamma(\underline{H^{1,2}(\mathfrak{g})})$, both of which vanish identically iff the $GL(n,\mathbb{H})Sp(1)$-structure on M is integrable.

$C_0$ and $C_1$ may be realized in terms of classical tensors as follows (see [19]). Choose any $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$ on M (i.e. $\nabla$ defines a connection on the principal $GL(n,\mathbb{H})Sp(1)$-bundle P). Then its torsion $\tau(\nabla) \in \Gamma(\underline{T} \otimes \Lambda^2\underline{T}^*)$, and $C_0$ is the image of $\tau(\nabla)$ under the projection $\underline{T} \otimes \Lambda^2\underline{T}^* \longrightarrow \underline{H^{0,2}(\mathfrak{g})}$. Given this fact, it is easy to see that $C_0$ is precisely the obstruction to the existence of a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection on M. Indeed, given two $GL(n,\mathbb{H})Sp(1)$-connections $\nabla, \nabla'$, their difference $\eta = \nabla - \nabla'$ may be viewed as an element of $\Gamma(\mathfrak{g} \otimes \underline{T}^*)$, and the difference of their torsions satisfies

$$\tau(\nabla) - \tau(\nabla') = \partial\eta \in \Gamma(\underline{T} \otimes \Lambda^2\underline{T}^*)$$

Thus if $C_0 \equiv 0$, take any $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$ and choose $\eta$ such that $\partial\eta = \tau(\nabla)$; then $\nabla - \eta$ is a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection.

Now suppose that M is an almost quaternionic manifold with $C_o \equiv 0$, and choose a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$. By the first Bianchi identity, at each point its curvature satisfies

$$R(\nabla) \in \ker(\partial : \mathfrak{g} \otimes \Lambda^2\underline{T}^* \longrightarrow \underline{T} \otimes \Lambda^3\underline{T}^*)$$

$C_1$ is then the image of $R(\nabla)$ under the projection onto $\underline{H^{1,2}(\mathfrak{g})}$. Moreover $R(\nabla)$ is a section of

$$\partial(\mathfrak{g}^1 \otimes \underline{T}^*) + \underline{U} \subset \mathfrak{g} \otimes \Lambda^2\underline{T}^* \qquad . \; . \; (5)$$

The component in $\underline{U}$ represents $C_1$, whereas the component in $\partial(\underset{\frown}{\underline{g}}^1 \otimes \underline{T}^*)$ represents a form of Ricci curvature in the sense that it may be identified with the image of $R(\nabla)$ under the contraction

$$\underset{\frown}{g} \otimes \Lambda^2 T^* \hookrightarrow \underbrace{(T \otimes T^*) \otimes (T^*}\otimes T^*) \longrightarrow T^* \otimes T^*$$

Remark When n = 1, the spaces D, $\Lambda^3 E$, M, and N all vanish, and the proof of theorem 2.2 can be modified to give

$$\begin{aligned} H^{0,2}(\underset{\frown}{g}) &= 0 \\ H^{1,2}(\underset{\frown}{g}) &= U + \Lambda^2 E\, S^4 H \\ H^{2,2}(\underset{\frown}{g}) &= 0 \end{aligned}$$

Consequently in this case the tensor $C_0$ must be identically zero. However this is well-known because a 4-dimensional almost quaternionic structure is an oriented conformal structure, and the choice of a compatible Riemannian structure determines a torsion-free $GL(1,\mathbb{H})Sp(1)$ -connection. Moreover $C_1$ is precisely the Weyl conformal curvature tensor of M. From above, this has two components $W_- \in \underline{U}$, $W_+ \in \Lambda^2 \underline{E}\, S^4 \underline{H}$, and M is said to be self-dual if $W_- \equiv 0$, anti-self-dual if $W_+ \equiv 0$. Notice that one may compare the situation for $n \geqslant 2$ with that for n = 1 by the statement 'curvature has been traded for torsion'.

Let M be an arbitrary integrable almost quaternionic manifold. Then the principal $GL(n,\mathbb{H})Sp(1)$-bundles P,Q of $M, \mathbb{H}P^n$ are both locally isomorphic to the standard G-structure $\mathbb{R}^{4n} \times GL(n,\mathbb{H})Sp(1)$ on $\mathbb{R}^{4n}$. Consequently, any $x \in M$ has a neighbourhood U admitting a diffeomorphism $\psi : U \longrightarrow \psi(U) \subset \mathbb{H}P^n$ inducing an isomorphism $\psi_* : P|U \longrightarrow Q|U$. If $\phi$ is another such diffeomorphism defined on a neighbourhood of x, then $\psi \circ \phi^{-1}$ induces a local automorphism of Q. It follows from the fact that $GL(n,\mathbb{H})Sp(1)$ has finite type that the group of such local automorphisms is finite-dimensional. This is because its Lie algebra equals $\mathfrak{g}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1}$ (see [19]). Indeed Kulkarni [32] proves that any local automorphism of Q is induced by the restriction of some element of PGL(n,$\mathbb{H}$). Combined with the corollary of theorem 2.2, this implies the following version of his 'uniformization theorem':

Theorem 2.3 Let M be an almost quaternionic manifold of dimension 4n, $n \geqslant 1$, with $C_0 \equiv C_1 \equiv 0$. Then M has an open covering U and diffeomorphisms $\psi_\alpha : U_\alpha \longrightarrow \psi(U_\alpha) \subset \mathbb{H}P^n$ such that if $U_\alpha \cap U_\beta \neq \emptyset$, $\psi_\alpha \circ \psi_\beta^{-1}$ is the restriction of an element of PGL(n,$\mathbb{H}$).

Passing to the universal covering and developing along curves yields the

Corollary If in addition M is compact and simply-connected, then M is diffeomorphic to $\mathbb{H}P^n$.

## 3. Quaternionic Manifolds

According to §2, any almost quaternionic manifold possesses two tensors $C_0, C_1$ which are obstructions to its integrability. Since integrability means the existence in some sense of quaternionic coordinates, it is tempting to define a quaternionic manifold to be an almost quaternionic manifold M having $C_0 \equiv C_1 \equiv 0$, in an attempt to parallel the theory of complex manifolds. However as we have seen (theorem 2.3 and its corollary), this would impose severe restrictions on M. Examining the complex case for the moment, let N be an almost complex manifold, i.e. a manifold with a $GL(n,\mathbb{C})$-structure. Even though $GL(n,\mathbb{C})$ is not of finite type, it is still possible to construct tensors $C_i \in \underline{H^{i,2}(\mathfrak{gl}(n,\mathbb{C}))}$ which can be regarded as formal obstructions to integrability (see Guillemin [19]). In fact $H^{i,2}(\mathfrak{gl}(n,\mathbb{C})) = 0$ unless $i = 0$, so there is only $C_0$ and this is essentially the Newlander-Nirenberg tensor. We know that N is a complex manifold iff $C_0 = 0$. With this motivation, we adopt the definition that <u>a quaternionic manifold is an almost quaternionic manifold of dimension $\geqslant 8$ having $C_0 \equiv 0$</u>. Recall that the condition $C_0 \equiv 0$ is equivalent to the existence of a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection.

It will be our goal to justify the above definition, and to show that the class of quaternionic manifolds possesses important properties. First we remark that there are interesting examples of such manifolds. Recall that a Kähler manifold may be described as a real oriented 2n-dimensional manifold possessing a Riemannian metric whose linear holonomy group is contained in $U(n)$. In analogy, a real oriented 4n-dimensional manifold is said to be quaternionic Kähler if it possesses a Riemannian metric whose linear holonomy group lies in $Sp(n)Sp(1)$ (e.g. Gray [17]). We shall study quaternionic Kähler manifolds in detail in part two, but the important point here is that they are quaternionic in the sense of the above definition. For the holonomy reduction determines a $GL(n,\mathbb{H})Sp(1)$-structure, and the Levi-Civita connection is a torsion-free connection preserving this structure. There are many known examples of quaternionic Kähler manifolds. The most well-known of these are the symmetric spaces discussed by Wolf [38]. More recently, Alekseevskii [5] constructed examples which are homogeneous but not symmetric, and Calabi [15] discovered some non-homogeneous examples with holonomy in $Sp(n)$. We shall see in §7 that most of these are examples of <u>non-integrable</u> quaternionic manifolds, i.e. ones with $C_1 \not\equiv 0$.

We shall now see that the condition $C_0 \equiv 0$ means that a quaternionic manifold possesses a 2nd order G-structure in the sense of Kobayashi [26]. This in turn will enable us to study certain invariant differential operators. Let $M$ be a quaternionic manifold with associated $GL(n,\mathbb{H})Sp(1)$ principal bundle $P$. Since $P$ possesses a torsion-free connection, the set $P^1$ of all those horizontal subspaces at all points $p\in P$ that arise from torsion-free connections on $P$ is non-empty. Let $G$ denote $GL(n,\mathbb{H})Sp(1)$, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^1$ the first prolongation of $\mathfrak{g}$. $\mathfrak{g}^1$ is a vector space, so has an underlying abelian group structure. Given $x\in M$, both $G$ and $\mathfrak{g}^1$ act on the fibre $(P^1)_x$ as follows. Let $V\in(P^1)_x$, $V\subset T_pP$ say. For $g\in G$, put

$$Vg = (R_g)_*V \subset T_{pg}P$$

where $R_g$ denotes right translation by g in P. For $u\in \mathfrak{g}^1 \subset \mathfrak{g}\otimes T^* = \mathrm{Hom}(T,\mathfrak{g})$, we have $u\circ p^{-1} \in \mathrm{Hom}(\underline{T}_x,\mathfrak{g})$ where as usual we regard the frame p as an isomorphism $p : T \longrightarrow \underline{T}_x$ of $T = \mathbb{R}^{4n}$ onto the tangent space at x. Put

$$Vu = \{v + (u\, p^{-1}(\pi_* v))^* \ : \ v\in V\}$$

where $\pi: P \longrightarrow M$ is the projection and $^*$ denotes fundamental vector (see Kobayashi & Nomizu [28]). Now let the Cartesian product $G \times \mathfrak{g}^1$ act on $(P^1)_x$ by

$$V(g,u) = (Vg)u$$

If also $h\in G$ and $t\in \mathfrak{g}^1$, then $(V(g,u))(h,t)$ consists of elements of the form

$$\begin{aligned}(R_h)_*\left[(R_g)_* v + (ug^{-1}p^{-1}(\pi_* v))^*\right] &+ \left[th^{-1}g^{-1}p^{-1}(\pi_* v)\right]^* \\ &= (R_{gh})_* v + \left[(\mathrm{Ad}(h^{-1})uh + t)\ (pgh)^{-1}(\pi_* v)\right]^*\end{aligned}$$

Therefore $G \times \mathfrak{g}^1$ acts as a semidirect product with multiplication given by

$$(g,u)(h,t) = (gh,\ h^{-1}u + t)$$

where $h^{-1}$ here denotes the action of G on its representation $\mathfrak{g}^1 \subset \mathfrak{g}\otimes T^*$. Note that as a vector space, the Lie algebra of this semidirect product $G^1$ is $\mathfrak{g} + \mathfrak{g}^1$. Because two torsion-free G-connections differ by a section of $\mathfrak{g}^1$, $G^1$ acts transitively on $(P^1)_x$ and in fact $P^1$ is a principal of $G^1$-bundle over M.

If $V\subset T_pP$ belongs to $P^1$, it is easy to show that there is a chart $(U;x^i)$ centred at $x = \pi(p)$ such that the local section $s = \left\{\frac{\partial}{\partial x^1}, \cdot\cdot\cdot, \frac{\partial}{\partial x^{4n}}\right\}$ of the frame bundle of M satisfies $s(x) = p$, and has tangent space at p precisely V. V can then be identified with the 2-jet at the origin of the coordinate system $x^i$. This 2-jet is 'structure-preserving to the $1^{st}$ order' in that it defines an isomorphism between the 1-jet at the origin of the standard

G-structure $\mathbb{R}^{4n} \times G$ and the 1-jet at x of P. In general, the 2-jet of a coordinate system at a point is called a 2-frame (see [26]) and $P^1$ is simply a reduction of the principal bundle of 2-frames on M, i.e. a 2nd order G-structure. The group $G^1$ of $P^1$ is the group of automorphisms of the 1-jet at x of P.

Next we investigate the concept of an invariant first order differential operator on a quaternionic manifold. First, consider quaternionic projective space $\mathbb{H}P^n$ with its standard principal G-bundle Q, and fix $x \in \mathbb{H}P^n$. Since the action of $K = PGL(n,\mathbb{H})$ on $\mathbb{H}P^n$ induces automorphisms of Q, the isotropy subgroup $K_0$ at x must lie in $G^1$. Counting dimensions, $K_0 = G^1$, and hence $\mathbb{H}P^n$ has the homogeneous description $\mathbb{H}P^n = K/G^1$. Any representation $G^1 \longrightarrow \mathrm{Aut}\, V$ gives rise to a homogeneous vector bundle $\underline{V} = K \times_{G^1} V$. K acts on $\underline{V}$ and so also on $J_1(\underline{V})$, the bundle of 1-jets of sections of $\underline{V}$. A first order invariant differential operator between homogeneous vector bundles $\underline{V}, \underline{W}$ can be defined as a homomorphism $D : J_1(\underline{V}) \longrightarrow \underline{W}$ which commutes with the action of K. In the general case, we imitate this definition as follows. Let M be a quaternionic manifold and $\underline{V}, \underline{W}$ vector bundles over M associated to the G-structure corresponding to representations V,W of $G = GL(n,\mathbb{H})Sp(1)$. If $x \in M$, the group $G^1$ (more accurately the bundle $(P^1)_x \times_{G^1} G^1$ where $G^1$ acts on itself by conjugation) acts on the fibre $\underline{V}_x$ by means of $G^1 \longrightarrow G \longrightarrow \mathrm{Aut}\, V$ ; similarly for $\underline{W}_x$. Since $G^1$ is the group of automorphisms of the 1-jet of the almost quaternionic structure at x, it also acts on the fibre $J_1(\underline{V})_x$. Then a first order operator $D : J_1(\underline{V}) \longrightarrow \underline{W}$ is said to be quaternionic invariant if the homomorphism D commutes with the action of $G^1$ on each fibre.

Let $D : J_1(\underline{V}) \longrightarrow \underline{W}$ be a quaternionic invariant differential operator. There is an exact sequence

$$0 \longrightarrow \underline{V} \otimes \underline{T}^* \xrightarrow{i} J_1(\underline{V}) \longrightarrow \underline{V} \longrightarrow 0$$

The symbol $\sigma = D \circ i : \underline{V} \otimes \underline{T}^* \longrightarrow \underline{W}$ of D is $G^1$-equivariant, and hence arises from a homomorphism $\sigma : V \otimes T^* \longrightarrow W$ of G-modules. Following Fegan [16], in order to determine which homomorphisms $\sigma : V \otimes T^* \longrightarrow W$ arise as the symbols of invariant operators, we introduce a mapping $\beta$. The representation $\rho : G \longrightarrow \mathrm{Aut}V$ induces a homomorphism of Lie algebras $d\rho : \mathfrak{g} \longrightarrow \mathrm{End}V$, and we have the composition

$$\mathfrak{g}^1 \longrightarrow \mathfrak{g} \otimes T^* \xrightarrow{d\rho \otimes 1} \mathrm{End}V \otimes T^*$$

or equivalently a homomorphism

$$\beta : \mathfrak{g}^1 \otimes V \longrightarrow V \otimes T^*$$

Proposition 3.1 A G-module homomorphism $\sigma : V \otimes T^* \longrightarrow W$ is the symbol of a first order quaternionic invariant differential operator iff $\mathrm{Im}\beta \subset \ker\sigma$.

Proof We omit the details which are an obvious generalization of [16] which actually deals with the case n = 1. The homomorphism $\beta$ although defined differently there is essentially identical to the one above.

Given $\sigma$ satisfying the hypotheses of proposition 3.1, it is easy to construct an invariant operator having it as symbol. First, choose a torsion-free G-connection $\nabla$ on M; $\nabla$ induces a connection on $\underline{V}$ which we can regard as a homomorphism $\nabla : J_1(\underline{V}) \longrightarrow \underline{V} \otimes \underline{T}^*$. $D = \sigma \circ \nabla$ is then the required operator. The condition $\mathrm{Im}\beta \subset \ker\sigma$ ensures that D is independent of the choice of $\nabla$.

Let M be a quaternionic manifold. The existence of invariant differential operators is a local question, so we are free to replace $G = GL(n,\mathbb{H})Sp(1)$ by its double covering $\tilde{G} = GL(n,\mathbb{H})\times Sp(1)$ in the above analysis; in particular we may consider the representations E and H of G. Corresponding to the centre of G are the 1-dimensional representations consisting of powers of $\Lambda^{2n}E$. For reasons of notation, it will be convenient to define

$$\phi = (\Lambda^{2n}E)^{-1/2(n+1)}$$

Equivalently, $\phi$ is the $4(n+1)^{st}$ root of the representation

$$\tilde{G} \longrightarrow G \xrightarrow{\det} \mathbb{R}^+$$

so is a well-defined representation of G. We now construct a series of invariant operators which play an important rôle in the sequel.

<u>Proposition 3.2</u> The homomorphism $\sigma : \phi^r \Lambda^s E\, S^t H\, T^* \longrightarrow \phi^r \Lambda^{s+1} E\, S^{t+1} H$ defined on decomposable elements by

$$\sigma(c^r e_I h_J \otimes eh) = c^r (e_I \wedge e)(h_J \vee h)$$

is the symbol of a quaternionic invariant first order differential operator D iff $r = t - s$. (Here $c \in \phi$. Also $e_I$ is an abbreviation for $e_{i_1} \wedge \dots \wedge e_{i_s}$, and $e_I \wedge e = e_{i_1} \wedge \dots \wedge e_{i_s} \wedge e$ ; similarly for $h_J$)

<u>Proof</u> Put $V = \phi^r \Lambda^s E\, S^t H$, $W = \phi^r \Lambda^{s+1} E\, S^{t+1} H$. Using proposition 3.1, we must prove the composition

$$\mathfrak{g}^1 \otimes V \xrightarrow{\beta} V \otimes T^* \xrightarrow{\sigma} W$$

is zero. As a G-module, by lemma 2.1, $\mathfrak{g}^1 = EH$ is irreducible. It follows from Schur's lemma that it is enough to prove that $\sigma\beta(u \otimes v) = 0$ for some $u \in \mathfrak{g}^1$ and all $v \in V$. For this purpose we use the element $u = (n+1)u_1 + 2nu_2 + nu_3$ of lemma 2.2, where

$$\left.\begin{array}{ll} u_1 = 1 \otimes e_1 h & \in A \otimes EH \\ u_2 = e^i e_1 \otimes e_i h - \frac{1}{2n} e^i e_i \otimes e_1 h & \in B \otimes EH \\ u_3 = 2h^2 \otimes e_1 \tilde{h} - (h \vee \tilde{h}) \otimes e_1 h & \in S^2 H \otimes EH \end{array}\right\} \subset \underset{\sim}{g} \otimes T^*$$

Corresponding to the decomposition $\underset{\sim}{g} = A + B + S^2 H$, we put

$$V_1 = \phi^{r - \frac{s(n+1)}{n}}, \quad V_2 = \phi^{\frac{s(n+1)}{n}} \Lambda^s E, \quad V_3 = s^t H$$

so that the representation $d\rho : \underset{\sim}{g} \longrightarrow \mathrm{End}V$ is simply the sum of the representations

$$A \longrightarrow \mathrm{End}V_1, \quad B \longrightarrow \mathrm{End}V_2, \quad S^2 H \longrightarrow \mathrm{End}V_3 .$$

$V$ is spanned by elements of the form $v_1 \otimes v_2 \otimes v_3$ where

$$v_1 = c^{r - \frac{s(n+1)}{n}}, \quad v_2 = c^{\frac{s(n+1)}{n}} e_I, \quad v_3 = h_J$$

Using the fact that $\underset{\sim}{g}$ acts on tensor products by the derivation law,

$$\beta(u_1 \otimes v) = \frac{-n}{n+1}\left(r - \frac{s(n+1)}{n}\right) c^r e_I h_J \otimes e_1 h$$

$$\beta(u_2 \otimes v) = c^r \left[\sum_p (-1)^{s-p} e_{i_1} \wedge \ldots \overset{i_p}{\wedge} \ldots \wedge e_{i_s} \wedge e_1 . h_J) \otimes e_{i_p} h - \frac{1}{2n}(s\, e_{i_1} \wedge \ldots \wedge e_{i_s} . h_J) \otimes e_1 h\right]$$

$$\beta(u_3 \otimes v) = c^r e_J \left[2 \sum_p h_{i_1} v .. (h, h_{i_p}) h .. v h_{i_t} \otimes e_1 \tilde{h} - h_{i_1} v .. ((h, h_{i_p})\tilde{h} + (\tilde{h}, h_{i_p}) h) .. v h_{i_s} \otimes e_1 h\right]$$

where $(\ ,\ )$ is an abbreviation for $\omega_H(\ ,\ )$. Therefore

$$\sigma\beta(u_1 \otimes v) = -\left(\frac{nr}{n+1} - s\right) c^r \, (e_I \wedge e_1)\, (h_J \vee h)$$

$$\sigma\beta(u_2 \otimes v) = c^r \left[-s\, e_I \wedge e_1 - \frac{1}{2n} s\, e_I \wedge e_1\right] (h_J \vee h)$$

$$= -s\left(1 + \frac{1}{2n}\right) c^r (e_I \wedge e_1)(h_J \vee h)$$

$$\sigma\beta(u_3 \quad v) = c^r (e_I \wedge e_1) \left[\sum_p h_{i_1} v .. ((h, h_{i_p})\tilde{h} - (\tilde{h}, h_{i_p}) h) .. v h_{i_t}\right]$$

$$= t\, c^r\, (e_I \wedge e_1)(h_J \vee h)$$

and

$$\sigma\beta(u \otimes v) = 0 \iff -(n+1)\left(\frac{nr}{n+1} - s\right) - 2ns\left(1 + \frac{1}{2n}\right) + nt = 0$$

$$\iff r = t - s$$

as required.

<u>Remark</u> $V_2 \otimes V_3$ is a representation of the semisimple component of G, and tensoring with a power of $\phi$ is equivalent to extending this representation to one of G. For example, in 4 dimensions tensoring with $\phi^r$ is equivalent to extending a representation of SO(4) to one of CO(4) with conformal weight $-\frac{1}{2}r$. It will be convenient in the future to use $\underline{K}$ as an abbreviation for the 'weighted' vector bundle $\phi \otimes \underline{H}$. This corresponds to a switch of emphasis from the group Sp(1) to GL(1,$\mathbb{H}$).

<u>Proposition 3.3</u> The sequences

$$0 \longrightarrow \phi^{-1}\underline{E} \xrightarrow{D} \phi^{-1}\Lambda^2\underline{E}\,\underline{H} \longrightarrow \cdots \longrightarrow \phi^{-1}\Lambda^{2n}\underline{E}S^{2n-1}\underline{H} \longrightarrow 0 \qquad (r=-1)$$

$$0 \longrightarrow \underline{A} \longrightarrow \underline{EH} \xrightarrow{D} \Lambda^2\underline{E}S^2H \longrightarrow \cdots \longrightarrow \Lambda^{2n}\underline{E}S^{2n}\underline{H} \longrightarrow 0 \qquad (r=0)$$

$$0 \longrightarrow \phi^r S^r\underline{H} \xrightarrow{D} \phi^r\underline{E}S^{r+1}\underline{H} \longrightarrow \cdots \longrightarrow \phi^r\Lambda^{2n}\underline{E}S^{2n+r}\underline{H} \longrightarrow 0$$

are elliptic complexes of quaternionic invariant first order differential operators.

<u>Proof</u> The operator $D^2$ factors through the curvature

$$\Omega : \phi^r \Lambda^s\underline{E}S^tH \longrightarrow \phi^r \Lambda^s\underline{E}S^tH \otimes \Lambda^2\underline{T}^*$$

$\Omega$ is essentially given by (5),§2. The fact that $\Lambda^{s+2}E$ does not occur as a summand in $U \otimes \Lambda^s E$ and the fact that D is quaternionic invariant then imply that $D^2 = 0$.

Let $\sigma$ denote the symbol of D. Ignoring $\phi^r$ because this will not affect exactness, we must show that

$$\Lambda^{s-1}E \otimes S^{t-1}H \xrightarrow{\sigma(v)} \Lambda^s E \otimes S^t H \xrightarrow{\sigma(v)} \Lambda^{s+1}E \otimes S^{t+1}H$$

is exact for any $v \in (T^*)_r$. Assume that $1 \leqslant s \leqslant 2n-1$, $1 \leqslant t$. Let $h \in H \setminus 0$ and put $X^{i,t} = \mathfrak{S}(h^i h^{t-i})$. $\{X^{i,t}\}_{i=0}^{t}$ forms a basis of $S^tH$. A typical element of $\Lambda^s E \otimes S^t H$ has the form

$$Y = \sum_{i=0}^{t} f_i \otimes X^{i,t}$$

where $f_i \in \Lambda^s E$. Also $v = eh + \tilde{e}\tilde{h}$ for some $e \in E$. It is readily verified that $\sigma(\sigma(Y,v),v) = 0$.

On the other hand suppose that $(Y,v) = 0$. Then

$$\begin{aligned} 0 &= \sum_{i=0}^{t} \left[(f_i \wedge e)(X^{i,t}vh) + (f_i \wedge e)(X^{i,t}\tilde{v}\tilde{h})\right] \\ &= \sum_{i=0}^{t} \left[(f_i \wedge e)X^{i+1,t+1} + (f_i \wedge e)X^{i,t+1}\right] \\ &= \sum_{i=0}^{t-1}(f_i \wedge e + f_{i+1} \wedge e)X^{i+1,t+1} + (f_0 \wedge e)X^{0,t+1} + (f_t \wedge e)X^{t+1,t+1} \end{aligned}$$

Hence

$$\begin{cases} f_i \wedge e + f_{i+1} \wedge \tilde{e} = 0, & 0 \leqslant i \leqslant t-1 \\ f_0 \wedge \tilde{e} = 0 \\ f_t \wedge e = 0 \end{cases}$$

and it follows that there exist $a_i, b_i \in \Lambda^{s-1}E$ such that

$$f_i = a_i \wedge \tilde{e} + b_i \wedge e, \quad 0 \leqslant i \leqslant t, \quad a_i = b_{i+1}$$

and $a_t = b_0 = 0$. Then

$$Y = \sigma\left(\sum_{i=0}^{t-1} a_i \otimes X^{i,t-1}, v\right)$$

Similar arguments give exactness at the last step of the symbol sequences, and at the first step provided $r = t-s \geqslant -1$. This completes the proof of proposition 3.3.

The complexes of proposition 3.3 will in general only be globally defined on M when r is even. In particular consider the case $r = 0$ :-

$$0 \longrightarrow \underline{A} \longrightarrow \underline{EH} \xrightarrow{D} \Lambda^2\underline{E}S^2\underline{H} \longrightarrow \cdots \longrightarrow \Lambda^{2n}\underline{E}S^{2n}\underline{H} \longrightarrow 0$$

Because $\Lambda^s ES^s H$ is a subspace of $\Lambda^s T^*$, the operator D factors through exterior differentiation and the above is a subcomplex of the deRham complex on M.

## 4. The Twistor Space

The fundamental result concerning quaternionic manifolds is the following.

Theorem 4.1 Let $M$ be a quaternionic manifold of dimension $4n$, $n \geqslant 2$. Then there is a naturally associated complex $(2n+1)$-dimensional manifold $Z$ fibring over $M$ with fibre $\mathbb{C}P^1$.

Proof Geometrically, the idea is the following. As the notation suggests, the manifold $Z$ of theorem 4.1 is none other than the complex projective bundle $P(\underline{H})$ introduced in §1. However for technical reasons we shall work with the vector bundle $\underline{K}$ and its dual $\underline{K}^*$, where $\underline{K} = \underline{\phi} \otimes \underline{H}$. Note that the weighting is irrelevant when one passes to the projective bundles; in particular $P(\underline{H})$ and $P(\underline{K}^*)$ are naturally isomorphic. First select a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$ on $M$; this induces a connection on $\underline{K}$ and $\underline{K}^*$ which we again denote by $\nabla$. Let $\underline{K}^* \setminus 0$ denote $\underline{K}^*$ with its zero section removed (defined over some open set in $M$) and $p : \underline{K}^* \setminus 0 \longrightarrow M$ the projection. $\nabla$ splits the tangent bundle of $\underline{K}^* \setminus 0$ as

$$T(\underline{K}^* \setminus 0) = \mathcal{V} \oplus \mathcal{H} \qquad \ldots (6)$$

where $\mathcal{V}$ is the bundle of vectors tangent to the fibres, and $\mathcal{H}$ is the bundle of vectors horizontal with respect to $\nabla$. Now $\mathcal{V}$ has a natural almost complex structure because each fibre is isomorphic to $\mathbb{C}^2 \setminus 0$, and as complex vector bundles we may write $\mathcal{V}^{1,0} \cong p^*\underline{K}^*$. In addition, $\mathcal{H}$ has a natural almost complex structure because any $y \in \underline{K}^* \setminus 0$ determines an almost complex structure on $\underline{T}_{p(y)} \cong \mathcal{H}_y$ as in §1 ($[y] \in P(\underline{K}^*) = Z$). The direct sum of these almost complex structures determines an almost complex structure on $T(\underline{K}^* \setminus 0)$ which is in fact independent of the choice of $\nabla$. The proof of the theorem consists of verifying that this is integrable. Since the fibres are then complex submanifolds, the resulting complex structure passes to one on $P(\underline{K}^*) = Z$.

The machinery developed in the preceding sections enables us to use the methods of Atiyah, Hitchin & Singer [7;§3] to prove that the almost complex structure on $\underline{K}^*\setminus 0$ defined above is integrable. First we summarize this theory as it applies to our situation. By our proposition 3.2, there is a quaternionic invariant differential operator

$$D : \underline{\phi H} \longrightarrow \underline{\phi ES^2 H}$$

Thinking of D as a homomorphism $J_1(\underline{K}) \longrightarrow \underline{\phi ES^2 H}$, D is surjective and its kernel $R \subset J_1(\underline{K})$ is a vector bundle. A section $s \in \Gamma(\underline{K})$ defines a complex-valued function $s^v$ on the total space of $\underline{K}^*$ by evaluation :-

$$s^v(k^*) = \langle k^*, s(p(k^*))\rangle$$

Its derivative $ds^v$ belongs to the (complexified) cotangent bundle $T^*(\underline{K}^*)$. Moreover there is a surjective homomorphism

$$V : p^*(J_1(\underline{K})) \longrightarrow T^*(\underline{K}^*\setminus 0)$$

of vector bundles over $\underline{K}^*\setminus 0$ characterized by $V(p^* j_1(s)) = ds^v$. Let $\sigma$ be the symbol of D and put

$$S_1 = \ker\sigma \subset KT^*$$
$$S_2 = S_1 \wedge T^* \subset K\Lambda^2 T^*$$

The choice of the connection $\nabla$ gives a splitting

$$J_1(\underline{K}) \cong \underline{K} \oplus \underline{KT^*}$$
$$R \cong \underline{K} \oplus \underline{S}_1$$

defined by $j_1(s) \longmapsto (s, \nabla s)$. If $s \in \Gamma(\underline{K})$ is covariant constant at $x \in M$, then essentially by definition of the induced connection on the dual $\underline{K}^*$, $ds^v$ annihilates the horizontal subspace $\mathcal{H}$ at any point of $p^{-1}(x)$. On the other hand, any section of $\underline{S}_1$ has the form

$$t = ch_1(eh_2) - ch_2(eh_1)$$

where $c \in \Gamma(\phi)$ and $\{h_1, h_2\}$ is a standard basis of $\underline{H}$. At $k^* \in \underline{K}^* \setminus 0$,

$$V(p^*t) = \langle ch_1, k^*\rangle p^*(eh_2) - \langle ch_2, k^*\rangle p^*(eh_1)$$

If under the isomorphism $\underline{K}^* = \phi^{-1}\underline{H}^* \underset{\omega_H}{\cong} \phi^{-1}\underline{H}$ the element $k^*$ maps to $c^{-1}(z^1h_1 + z^2h_2)$, then

$$V(p^*t) = z^1p^*(eh_1) + z^2p^*(eh_2)$$

In future we shall use the same symbol for both a 1-form on M and its pullback to the total space of any bundle over M, so we write

$$V(p^*t) = e(z^1h_1 + z^2h_2)$$

Using (3) §1, this is a (1,0)-form at $k^*$ with respect to the almost complex structure defined above. We have actually proved that

$$V(p^*R) = V(p^*\underline{K}) \oplus V(p^*\underline{S}_1)$$

is precisely the distribution of (1,0)-forms on $\underline{K}^* \setminus 0$ corresponding to the splitting (6), with $V(p^*\underline{K})$ annihilating $\mathcal{H}$ and $V(p^*\underline{S}_1)$ annihilating $\mathcal{V}$.

According to [7; proposition 3.1], $V(p^*R)$ is involutive iff

(A) $\quad D_1\Gamma(\underline{S}_1) \subset \Gamma(\underline{S}_2)$

(B) $\quad \Omega(\underline{K}) \subset \underline{S}_2$

where $D_1 : \Gamma(\underline{KT}^*) \longrightarrow (\underline{K}\wedge^2\underline{T}^*)$ is the extended covariant derivative and $\Omega \in \Gamma(\mathrm{End}\underline{K} \otimes \wedge^2 T^*)$ is the curvature of $\underline{K}$. It remains to verify these conditions, for then by the Newlander-Nirenberg theorem $\underline{K}^* \setminus 0$, and so Z, is a complex manifold. Notice that the complex structure will indeed only depend upon D and thus on the quaternionic structure of M. This is the force of the word 'naturally' in the statement of theorem 4.1.

To prove (A) and (B), we must first determine $S_1, S_2$ as G-modules. In fact

$$S_1 = \phi E$$
$$S_2 = \phi(S^2E + \wedge^2E)H$$

The first equality is immediate because $H \otimes T^* = H \otimes EH = E + ES^2H$.

For the second, note that

$$H \otimes \Lambda^2 T^* = (S^2E + \Lambda^2 E)H + \Lambda^2 ES^3H$$

$S_2$ is the image of $S_1 \otimes T^*$ under exterior multiplication, so certainly $S_2 \subset \phi(S^2E + \Lambda^2E)H$. Equality follows from Schur's lemma by picking an element such as

$$c(h_1(e_1h_2) - h_2(e_1h_1)) \otimes e_2h \in \phi E \otimes EH = S_1 \otimes T^*$$

whose image in $S_2$ under exterior multiplication has non-zero components in both the irreducible representations $\phi S^2E\,H, \phi\Lambda^2 EH$.

(A) Let $k \otimes v \in \Gamma(\underline{K} \otimes \underline{T}^*)$. By definition,

$$D_1(k \otimes v) = \nabla k \wedge v + k \otimes dv$$

On the other hand, if $\partial$ denotes the obvious skewing map,

$$\partial\nabla(k \otimes v) = \nabla k \wedge v + k \otimes \partial\nabla v$$

and so

$$D_1 - \partial\nabla = 1 \otimes \tau \;:\; \underline{K} \otimes \underline{T}^* \longrightarrow \underline{K} \otimes \Lambda^2 \underline{T}^* \qquad \ldots (7)$$

where $\tau$ is the torsion of $\nabla$, which by assumption is zero. Thus

$$D_1(\Gamma(\underline{S}_1)) = \partial\nabla(\Gamma(\underline{\phi E})) \subset \Gamma(\partial(\underline{\phi E} \otimes \underline{T}^*)) \subset \Gamma(\underline{S}_2)$$

(B) $\mathcal{R}$ is the image under the mapping

$$d\rho \otimes 1 \;:\; \mathfrak{g} \otimes \Lambda^2 T^* \longrightarrow \mathrm{End}K \otimes \Lambda^2 T^*$$

of the curvature $R \in \Gamma(\underline{\mathfrak{g}} \otimes \Lambda^2 \underline{T}^*)$ of $\nabla$. By (5), §2,

$$R = \partial(u_i \otimes v^i) + R_U$$

where $u_i \in \Gamma(\underline{\mathfrak{g}}^1)$, $v^i \in \Gamma(\underline{T}^*)$ and $R_U \in \Gamma(\underline{U})$. Since U is not an irreducible component of $\mathrm{End}K \otimes \Lambda^2 T^*$, $(d\rho \otimes 1)(R_U) = 0$. Therefore if $k \in \underline{K}$,

$$\begin{aligned} \mathcal{R}(k) &= (d\rho \otimes 1)\partial(u_i \otimes v^i)(k) \\ &= \beta(u_i \otimes k) \wedge v^i \end{aligned}$$

in the notation of §3. But quaternionic invariance implies that $\beta(u_i \otimes k) \in \ker\sigma = \underline{S}_1$, and so $\mathcal{R}(k) \in \underline{S}_2$. The proof of theorem 4.1 is now complete.

Remarks 1. A consequence of theorem 4.1 is that a quaternionic manifold is real analytic.

2. Theorem 2.2 was an essential ingredient in the proof of condition (2) above. For an almost quaternionic manifold M of 4 dimensions, theorem 2.2 has to be modified as explained in §2. In that case $C_0 \equiv 0$ always and condition (1) is automatically satisfied, but condition (2) will remain valid only if M is anti-self-dual. We therefore obtain the statement that the complex manifold Z may be constructed for an oriented 4-dimensional manifold with an anti-self-dual conformal structure. Reversing orientation, this is exactly theorem 4.1 of Atiyah, Hitchin & Singer [7] so that our theorem 4.1 is a natural generalization of this result. Indeed, from the start, our treatment of quaternionic structures, with the vector bundles $\underline{E},\underline{H}$ generalizing the spinor bundles $\underline{V}_-,\underline{V}_+$ and the study of invariant differential operators, was designed so as to parallel [7].

In 4 dimensions the construction of Z has become known as the Penrose twistor construction. Accordingly we shall refer to the complex manifold Z of theorem 4.1 as the twistor space of the quaternionic manifold M. In §3 we defined a quaternionic manifold to be an almost quaternionic manifold of dimension $\geqslant 8$ with $C_0 \equiv 0$. Many of the subsequent results for such manifolds will depend upon the existence of the twistor space Z and will therefore remain valid in the case of a 4-dimensional almost quaternionic manifold which is anti-self-dual.

3. If M is an almost quaternionic manifold with an arbitrary $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$, the above construction still equips $Z = P(\underline{K}^*)$ with an almost complex structure, although this now depends upon the choice of $\nabla$. It is natural to ask whether the integrability of this almost complex structure might imply that M is a quaternionic manifold. Because condition (A) is still a necessary condition for this integrability, the following lemma gives an affirmative answer.

Lemma 4.1 Let M be an almost quaternionic manifold. Then the vector bundle $\underline{K}$ possesses a $GL(1,\mathbb{H})$-connection satisfying $D_1(\Gamma(\underline{S}_1)) \subset \Gamma(\underline{S}_2)$ iff $C_0 \equiv 0$.

Proof It remains to prove the 'only if' statement. Working locally suppose that $\underline{K}$ has a $GL(1,\mathbb{H})$-connection $\nabla'$ satisfying $D_1(\Gamma(\underline{S}_1)) \subset \Gamma(\underline{S}_2)$. Pick any $GL(n,\mathbb{H})$-connection on $\underline{E} \otimes \underline{\phi}^{-1}$; together with $\nabla'$ this yields a $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$ on $\underline{E} \otimes \underline{H}$, i.e. on M. It actually follows from (7) that the torsion of $\nabla$ has components only in the representation

$$E^*(S^2E + \Lambda^2E)H$$

In particular, there is no $D \otimes S^3H$ component. Theorem 2.2 then implies that $C_0 \equiv 0$.

Let M be a quaternionic manifold with twistor space Z; we shall always use q to denote the projection $Z \longrightarrow M$. Before looking at some examples we must describe some elementary properties of Z. Firstly from the definition of the complex structure of Z, each fibre $q^{-1}(x) = \mathbb{C}P^1$ is a complex submanifold of Z. The quaternionic structure map $h \longmapsto \tilde{h}$ on the fibres of $\underline{H}$ induces an involution $\sigma : [h] \longmapsto [\tilde{h}]$ of $Z = P(\underline{H})$. Equivalently if $[h]$ corresponds to the almost complex structure $J_h$ on $\underline{T}_x$ then using (3) §1, $\sigma(J_h) = -J_h$. Thus $\sigma$ has no fixed points on $Z$, but preserves each line $q^{-1}(x) \cong S^2$ where it acts as the antipodal map. If $J$ denotes the almost complex structure on $Z$, then from the definition of $J$ it follows that $\sigma^* J = -J$, so that $\sigma$ acts antilinearly on the tangent bundle of $Z$. $\sigma$ is a real structure in the sense of Atiyah [6], and $M$ parametrizes lines in $Z$ that are real with respect to $\sigma$.

For each point of $M$ it is possible to define the vector bundle $\underline{H}$ over some neighbourhood $U$ of that point. Then over $U$ we have unambiguously $Z = P(\underline{H})$. Consequently over $q^{-1}(U) \subset Z$ there is defined a tautologous* complex line bundle $L^{-1}$ whose fibre at $z = [h] \in P(\underline{H})$ is simply the subspace $\mathbb{C}h$ of $(q^*\underline{H})_z$. (*We use Bott's terminology to avoid the use of the word 'canonical'.) The restriction of its dual $L$ to each line $q^{-1}(x)$ is isomorphic to the hyperplane bundle (with Chern class 1) over $\mathbb{C}P^1$. Notice that because we have used the description $Z = P(\underline{H})$ rather than $P(\underline{K}^*)$, we cannot assume that $L$ is holomorphic over $Z$. However since $\pi : \underline{K}^* \setminus 0 \longrightarrow Z$ is holomorphic, it <u>is</u> true that the line bundle $q^*\phi \otimes L$ is holomorphic.

From the proof of theorem 4.1, the subspace of (1,0)-forms in $q^*\underline{T}^* \subset T^*Z$ at $z = [h]$ is $q^*(\underline{E} \otimes \mathbb{C}h)$. But this is precisely the fibre $(q^*\underline{E} \otimes L^{-1})_z$, so

$$(q^*\underline{T}^*)^{1,0} \cong q^*\underline{E} \otimes L^{-1}$$

and there is a (globally defined) exact sequence

$$0 \longrightarrow q^{*}\underline{E} \otimes L^{-1} \longrightarrow \mathcal{A}^{1,0} \longrightarrow \mathcal{F}^{1,0} \longrightarrow 0 \qquad \text{. . (8)}$$

of complex vector bundles over Z, where $\mathcal{A}^{1,0}$ is the bundle of (1,0)-forms on Z and $\mathcal{F}^{1,0}$ is the bundle of (1,0)-forms on the fibres of $Z \longrightarrow M$. The fact that the holomorphic tangent bundle of $\mathbb{C}P^1$ is isomorphic to the square of the hyperplane bundle implies that $\mathcal{F}^{1,0} \cong L^{-2}$. In particular, $L^2$ is a globally defined line bundle over Z. An immediate consequence of (8) is that the holomorphic normal bundle of each line $q^{-1}(x)$ in Z is isomorphic to $L \otimes \mathbb{C}^{2n}$, where here L means the hyperplane bundle over $\mathbb{C}P^1$. (8) has other consequences, already known in the case $n = 1$. For example Z has no holomorphic p-forms for $p \geqslant 1$, and if $\mathcal{K}$ denotes the canonical bundle of Z then $H^0(Z, \mathcal{O}(\mathcal{K}^m)) = 0$ for $m \geqslant 1$. For proofs see Hitchin [23].

<u>Examples</u> For the moment we describe only a few obvious examples.

1. The space $\mathbb{R}^{4n} = \mathbb{H}^n$ in which the tangent space at each point is endowed with its standard quaternionic structure given by right multiplication by $\mathbb{H}$. The choice of a standard basis of $\mathbb{H}$ determines global almost complex structures I,J,K satisfying $IJ = -JI = K$ and furnishing $\mathbb{R}^{4n}$ with a $GL(n,\mathbb{H})Sp(1)$-structure (in fact a $GL(n,\mathbb{H})$-structure) which is obviously integrable. A point of the twistor space Z is simply a point of $\mathbb{R}^{4n}$ together with an almost complex structure $aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$, so topologically Z is the product $\mathbb{R}^{4n} \times S^2 = \mathbb{R}^{4n} \times \mathbb{C}P^1$. However the holomorphic structure of Z is more subtle. Note first that the projection $p : Z \longrightarrow \mathbb{C}P^1$ is holomorphic; moreover each fibre $p^{-1}(y)$

is isomorphic to $\mathbb{C}^{2n}$, or more accurately $\mathbb{R}^{4n}$ endowed with the complex structure y. Therefore Z can be viewed as a complex vector bundle F over $\mathbb{C}P^1$. The zero section of F may be identified with a line $q^{-1}(x)$ for an arbitrary $x \in \mathbb{R}^{4n}$. Because $q^{-1}(x)$ is a complex submanifold of Z, F is then a holomorphic vector bundle isomorphic to the normal bundle of $q^{-1}(x)$ in Z. Therefore as a complex manifold, Z is the total space of the vector bundle $L \otimes \mathbb{C}^{2n}$ over $\mathbb{C}P^1$.

2. More generally, suppose that M is a manifold possessing a $GL(n,\mathbb{H})$-structure, i.e. global almost complex structures I,J,K satisfying $IJ = -JI = K$. Then if $c_0 \equiv 0$ with respect to the induced $GL(n,\mathbb{H})Sp(1)$-structure, its twistor space Z fibres holomorphically over $\mathbb{C}P^1$. In the special case in which the $GL(n,\mathbb{H})$-structure is integrable, the existence of Z was known to Sommese [37]. For example let $q \in \mathbb{H}$ be a quaternionic with $|q| \neq 1$ and consider the cyclic group $\Gamma = \{q^n\}$ generated by multiplication on the left of $\mathbb{H}^n$ by q. Then $M = \mathbb{H}^n \setminus 0 /_{\Gamma} \cong S^{4n-1} \times S^1$ has an integrable $GL(n,\mathbb{H})$-structure. In the fibration of its twistor space over $\mathbb{C}P^1$ each fibre $S^{4n-1} \times S^1$ is endowed with the structure of a Hopf manifold which incidentally is non-Kähler.

3. We know from §2 that $\mathbb{H}P^n$ has an integrable $GL(n,\mathbb{H})Sp(1)$-structure; hence it is a quaternionic manifold. We shall show that the twistor space of $\mathbb{H}P^n$ is $\mathbb{C}P^{2n+1}$. First consider the tautologous vector bundle F whose fibre at $x \in \mathbb{H}P^n$ is the quaternionic line in $\mathbb{H}^{n+1}$ defined by x. As usual, let $U_i = \{(q_0,..,q_n : q_i \neq 0\}$, $x_i^k = q_k q_i^{-1}$, $k \neq i$ in terms of homogeneous coordinates $(q_0,..,q_n)$. Then $t_i = (x_i^0,..,1,..,x_i^n)$ defines a section of F over $U_i$ such that on $U_i \cap U_j$,

$$t_j = t_i (x_i^j)^{-1}$$

Comparing this with (4), §2, we see that F and H have the same transition functions and are therefore isomorphic. (This gives a direct proof that for $\mathbb{H}P^n$ the cohomology class $\xi = 0$). The twistor space $Z = P_{\mathbb{C}}(F)$ consists of all complex lines through the origin in $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$; i.e. $Z = \mathbb{C}P^{2n+1}$. M parametrizes the projective lines in Z that are real with respect to the action induced by multiplication by j on $\mathbb{H}^{n+1}$.

## 5. Quaternionic Vector Bundles

We have seen that the (locally defined) vector bundle $\underline{H}$ on a quaternionic manifold M can be used to construct its twistor space. The latter is simply the complex projective bundle $P(\underline{H})$. If $\underline{K} = \phi \otimes \underline{H}$ denotes the weighted vector bundle, we also saw that the total space $\underline{K}^* \setminus 0$ is a complex manifold. We now describe a second construction which endows the total space of $\underline{K}$ not with a complex structure, but with a quaternionic structure. More generally

Theorem 5.1 Let M be a quaternionic manifold and F a complex vector bundle over M of dimension 2p over $\mathbb{C}$ which admits an action of $GL(p,\mathbb{H})$ on each fibre. Suppose in addition that F possesses a $GL(p,\mathbb{H})$-connection $\nabla_F$ whose curvature satisfies

$$\Omega_F \in \Gamma(\mathrm{End}F \otimes S^2\underline{E}) \subset \Gamma(\mathrm{End}F \otimes \Lambda^2\underline{T}^*)$$

Then the total space $\tilde{M}$ of the real vector bundle $(F \otimes_{\mathbb{C}} \underline{K})_r$, defined over some open set of M, is a quaternionic manifold.

Proof Each fibre $F_x \otimes \underline{K}_x$, $x \in M$, is a complex vector space, but has a real structure induced from the quaternionic structure maps of $F_x, \underline{K}_x$. Thus $(F \otimes \underline{K})_r$ is a real vector bundle of dimension 4p over $\mathbb{R}$, and its total space $\tilde{M}$ is a real $4(p+n)$-dimensional manifold.

Choose a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection $\nabla$ on M, inducing a connection on $\underline{K}$. $\nabla_F$ and $\nabla$ then define a connection $\nabla'$ on $F \otimes \underline{K}$. Let $p : \tilde{M} \longrightarrow M$ be the projection. $\nabla'$ gives a splitting of the tangent bundle of $\tilde{M}$:

$$\begin{aligned} T(\tilde{M}) &= \mathcal{V} \oplus \mathcal{H} \\ &\cong p^*(F \otimes \underline{K}) \oplus p^*(\underline{E}^* \otimes \underline{H}) \qquad \qquad . . \quad (9) \\ &\cong p^*(\underline{E}^* \oplus F\phi) \otimes p^*\underline{H} \end{aligned}$$

Then $p^*(S^2\underline{H})_r$ acts on $T(M)$ as a coefficient bundle of imaginary quaternions and from §1, $\tilde{M}$ admits a $GL(n+p,\mathbb{H})Sp(1)$-structure.

It remains to show that the first integrability tensor $\tilde{C}_0$ of $\tilde{M}$ is zero. Now $\tilde{M}$ can be given a $GL(n+p,\mathbb{H})Sp(1)$-connection $\tilde{\nabla}$ using (9). Namely take $\tilde{\nabla}$ to be the direct sum of the (pullbacks of) the connections $\nabla', \nabla$. To calculate the torsion $\tilde{\tau}$ of $\tilde{\nabla}$, it is more convenient to work on the cotangent bundle $T^*(\tilde{M})$. From (9),

$$T^*(\tilde{M}) \cong p^*(F\underline{K})^* \oplus p^*\underline{T}^*$$

Take a local basis $\{a_I\}$ of $F\underline{K}$, with corresponding coordinate functions $\lambda^I$ and connection 1-forms given by $\nabla' a_I = a_J \otimes \omega^J_I$. Then the inclusion $p^*(F\underline{K})^* \hookrightarrow T^*(\tilde{M})$ is given by

$$a_I^* \longmapsto d\lambda^I + \lambda^J \omega^I_J \quad .$$

Fix $x \in M$ and suppose without loss of generality that $\nabla' a_I|_x = 0$. Then at any $y \in p^{-1}(x)$,

$$\begin{aligned} \tilde{\tau}(a_I^*) &= (d - \partial\nabla)(a_I^*) \\ &= d(d\lambda^I + \lambda^J \omega^I_I) \\ &= \lambda^J d\omega^I_J \\ &= \lambda^J \Omega^I_J \end{aligned}$$

where $\Omega^I_J$ are the curvature 2-forms of $\nabla'$. Moreover $\tilde{\tau}(u) = 0$ for any $u \in p^*\underline{T}^*$ so that

$$\tilde{\tau} = \lambda^I \Omega'(a_I) \qquad . \quad . \quad (10)$$

Each $a_I$ is a linear combination of elements $f \otimes k$, $f \in F$, $k \in \underline{K}$. Now

$$\Omega'(f \otimes k) = f\,\Omega_K(k) + \Omega_F(f)k$$

By the proof of theorem 4.1, $\Omega_K(k) \in \Gamma(\underline{S}_2)$, whereas by hypothesis $\Omega_F(f) \in \Gamma(F \otimes S^2\underline{E})$. It follows that $\tilde{\tau}$ has no components involving $p^*(S^3\underline{H})$ and therefore by theorem 2.2, $\tilde{C}_0 \equiv 0$. This completes the proof of theorem 5.1.

We shall call a vector bundle $F$ satisfying the hypotheses of theorem 5.1 a quaternionic vector bundle. The curvature condition is equivalent to the statement that the bundle of imaginary quaternions $(S^2\underline{H})_r$ annihilates the curvature 2-forms. This is because $Sp(1)$ acts trivially on the space $S^2E$, so passing to Lie algebras, $S^2H$ annihilates $S^2E$. This is analogous to the case of a complex vector bundle over a complex manifold whose curvature 2-forms are actually (1,1)-forms. For in that case $\Lambda^{1,1}T^*$ is precisely the subspace of (complexified) 2-forms which is annihilated by the action of the complex structure $J \in \mathrm{End}T^*$. It is known (e.g. [7; theorem 5.1]) that such a vector bundle is actually holomorphic; in particular its total space is a complex manifold. Theorem 5.1 tells us that the situation for a quaternionic bundle is similar - after tensoring with $\underline{K}$ the total space of a quaternionic bundle is a quaternionic manifold.

In view of theorem 4.1, an obvious problem is to identify the twistor space of $\tilde{M}$. Suppose that $M$ itself has twistor space $Z$; $q : Z \longrightarrow M$. Let $\phi L = q^*\phi \otimes L$ denote the holomorphic line bundle over $Z$ introduced in §4.

<u>Proposition 5.1</u> With the same hypotheses as theorem 5.1, the twistor space $\tilde{Z}$ of $\tilde{M}$ is the total space of the complex vector bundle $q^*F \otimes \phi L$ over $Z$.

<u>Proof</u> Fix $x \in M$. First we identify the twistor space of the fibre $p^{-1}(x) = (F \otimes \underline{K})_x \subset \tilde{M}$. This fibre is a quaternionic manifold in its

$$\begin{tikzcd} \tilde{M} \arrow[d] & q^*F\otimes\phi L \arrow[l] \arrow[d] \\ M \arrow[loop above right, dashed, "F"] & Z \arrow[l] \arrow[loop above right, dashed, "q^*F"] \end{tikzcd}$$

own right isomorphic to $\mathbb{H}^p$ with its standard quaternionic structure. From example 1, §4, its twistor space is the total space of 2n copies of the hyperplane bundle over $\mathbb{C}P^1$. More invariantly, its twistor space is the total space of the restriction of $q^*(F\phi)\otimes L$ to $q^{-1}(x)$. Therefore Z may be identified with the total space of $q^*\underline{F}\otimes\phi L$.

An immediate consequence of proposition 5.1 is that the total space of the complex vector bundle $q^*F\otimes\phi L$ over Z is actually a complex manifold. Moreover the zero section embeds Z as a complex submanifold of $\tilde{Z}$. Hence the normal bundle of Z in $\tilde{Z}$ is holomorphic, but this normal bundle is isomorphic to $q^*F\otimes\phi L$, so the latter is holomorphic. Thus $q^*F = (q^*F\otimes\phi L)\otimes(\phi L)^{-1}$ is holomorphic. To summarize we have the

<u>Corollary</u> If F is a quaternionic vector bundle over a quaternionic manifold M, its pullback $q^*F$ is a holomorphic bundle over Z.

<u>Examples</u> 1. Take F to be the trivial vector bundle $\mathbb{C}^2$ endowed with its standard quaternionic structure. In that case $\tilde{M}$ is the total space of $(\underline{H}\otimes\mathbb{C}^2)_r$, the underlying real vector bundle of $\underline{H}$. For example over $\mathbb{H}P^n$, $\underline{H}$ is the tautologous quaternionic line bundle and $\tilde{M}$ is $\mathbb{H}P^{n+1}$ with a point removed and we have the diagram

$$\begin{array}{ccc} \tilde{M} = \mathbb{H}P^{n+1} \setminus \{x\} & \longleftarrow & \mathbb{C}P^{2n+3} \setminus \mathbb{C}P^{1} \\ \downarrow & & \downarrow \\ \mathbb{H}P^{n} & \longleftarrow & \mathbb{C}P^{2n+1} \end{array}$$

2. Let M be a 4-dimensional almost quaternionic manifold which is anti-self-dual. As remarked in §4, for all intents and purposes, M is a quaternionic manifold. Certainly theorem 5.1 will apply. A quaternionic vector bundle over M is then anti-self-dual in the sense of Atiyah, Hitchin & Singer [7] and the corollary above is also a consequence of [7; theorem 5.2]. However the proof there is based on the observation that the pullback

$$q^*(S^2\underline{E}) \subset q^*(\Lambda^2\underline{T}^*)$$

actually lies in the bundle $\mathcal{A}^{1,1}$ of (1,1)-forms on Z. This means that $q^*\underline{F}$ is a complex vector bundle over Z with a connection whose curvature forms lie in $\mathcal{A}^{1,1}$, which in turn implies that $q^*F$ is holomorphic. So not only is the definition of a quaternionic vector bundle analogous to that of a vector bundle over a complex manifold with $\mathcal{A}^{1,1}$-curvature, but the pullback of a quaternionic bundle to the twistor space is actually of this type.

3. As a special case of example 2, take $M = \mathbb{H}P^1 = S^4$. The study of (anti-)self-dual vector bundles or 'instantons' over $S^4$ has recently attracted much attention. For example it is possible to construct explicitly anti-self-dual SU(2)-bundles on $S^4$ with arbitrary 2nd Chern class $c_2 > 0$ [8]. If F is one of these vector bundles we shall show that the associated quaternionic manifold $\tilde{M}$ has $\tilde{c}_1 \neq 0$, i.e. is non-integrable.

In the proof of theorem 5.1 we described a connection $\tilde{\nabla}$ on $\tilde{M}$ with tors ion given by (10). With the same notation, there exist

$b_I \in \Gamma(\tilde{M}, \tilde{\underset{\sim}{q}} \otimes \underline{T}^*)$ such that $\partial(b_I) = \Omega'(a_I)$, and then

$$\tilde{\nabla} - \lambda^I b_I$$

defines a torsion-free $GL(n+p,\mathbb{H})Sp(1)$-connection on $\tilde{M}$. We compute its curvature $\tilde{R}$ at $x \in M \subset \tilde{M}$, i.e. $\lambda^I = 0$. The curvature of $\tilde{\nabla}$ may be written $\Omega' + R = \Omega_F + \Omega_K + R$, where $R$ is the curvature of $S^4$. Hence

$$\tilde{R}_x = \Omega_F + \Omega_K + R + b_I \wedge d\lambda^I$$

Now the $\tilde{U}$-component (see (5) §2) of $\tilde{R}_x$ must include $\Omega_F \neq 0$. This forces $\tilde{C}_1 \neq 0$.

## 6. Cohomology

In §4 we introduced the twistor space $Z$ of a quaternionic manifold $M$ and described some properties of $Z$. In this section we shall further investigate the relationship between the quaternionic structure of $M$ and the holomorphic structure of $Z$. In particular we show that there is a correspondence between certain sheaf cohomology groups of $Z$ and the cohomology of the elliptic complexes on $M$ of proposition 3.3.

Let $M$ be a quaternionic manifold with twistor space $Z$; $q : Z \longrightarrow M$. If $L^{-1}$ denotes the tautologous line bundle defined as in §4, the space $H^0(q^{-1}(x),\mathcal{O}(L))$ of holomorphic sections of the restriction of $L$ to a line $q^{-1}(x)$ is naturally isomorphic to the space $\underline{H}_x^*$ of linear forms on $\underline{H}_x$. Indeed, let $\{h_1,h_2\}$ be a standard basis of $\underline{H}$ with corresponding coordinate functions $z^1,z^2$ on its total space. By construction $L^{-1}$ is a subbundle of $q^*\underline{H}$ and

$$t_1 = h_1 + u^{-1}h_2, \quad t_2 = uh_1 + h_2,$$

where $u = z^1/z^2$, define sections of $L^{-1}$ away from $z^1 = 0$, $z^2 = 0$ respectively. Let $s_1,s_2$ be the sections of $L$ defined by $\langle t_i,s_i\rangle = 0$. Then $s_1,s_2$ are well-defined and holomorphic on each line $q^{-1}(x)$ and satisfy $s_1 = us_2$ wherever $u$ is finite.

Because $\underline{H} \cong \underline{H}^*$ by means of $\omega_H$, we also have an isomorphism $H^0(q^{-1}(x),\mathcal{O}(L)) \cong \underline{H}_x$ (natural with respect to the $SU(2)$ structure) given by

$$s_1 \longmapsto h_2, \quad s_2 \longmapsto -h_1$$

The quotient of the inclusion $L^{-1} \xhookrightarrow{i} q^*\underline{H}$ is isomorphic to $L$ because there is an exact sequence

$$0 \longrightarrow L^{-1} \xrightarrow{i} q^*\underline{H} \xrightarrow{\alpha} L \longrightarrow 0 \qquad . . \quad (11)$$

where $\alpha(z,h_1) = -s_2(z)$, $\alpha(z,h_2) = s_1(z)$. (11) enables us to write down an explicit isomorphism $L^{-1} \cong \bar{L}$ between the dual of L and its complex conjugate; namely

$$t \longmapsto \bar{t} = \alpha(\widetilde{i(t)})$$

For example,

$$\bar{t}_2 = \alpha(\bar{u}h_2 - h_1) = \bar{u}s_1 + s_2 = (|u|^2 + 1)s_2 \quad . . (12)$$

From the definition of the complex structure of Z, there is an exact sequence ((8) §4)

$$0 \longrightarrow \underline{E}L^{-1} \longrightarrow \mathcal{A}^{1,0} \longrightarrow \mathcal{J}^{1,0} \longrightarrow 0$$

globally defined over Z, where $\underline{E}$ is an abbreviation for the vector bundle $q^*\underline{E}$. Taking $r^{th}$ exterior powers yields an exact sequence

$$0 \longrightarrow L^{-r}\Lambda^r\underline{E} \longrightarrow \mathcal{A}^{r,0} \longrightarrow L^{-(r-1)}\Lambda^{r-1}\underline{E}\,\mathcal{J}^{1,0} \longrightarrow 0$$

where $\mathcal{A}^{r,0}$ is the bundle of (r,0)-forms over Z. The first mapping here is given by

$$t^r \otimes f \longmapsto f \otimes i(t)^r \in q^*(\Lambda^r\underline{E} \otimes S^r\underline{H}) \subset q^*(\Lambda^r\underline{T}^*) \quad . . (13)$$

A consequence of this is that at $z = [h] \in P(\underline{H}) = \mathcal{Z}$,

$$(q^*\Lambda^r\underline{T}^*)^{r,0} = q^*(\Lambda^r\underline{E} \otimes \mathbb{C}h^r) \subset q^*(\Lambda^r\underline{T}^*)$$

Taking complex conjugates, we have an exact sequence

$$0 \longrightarrow L^r\Lambda^r\underline{E} \xrightarrow{\beta} \mathcal{A}^{0,r} \xrightarrow{\gamma} \bar{L}^{r-1}\Lambda^{r-1}\underline{E}\,\mathcal{J}^{0,1} \longrightarrow 0$$

Using (12), (13), if f is a section of $\Lambda^r\underline{E}$ over M,

$$\begin{aligned} \beta((s_2)^r \otimes f) &= f \otimes (|u|^2 + 1)^{-r}(\widetilde{i(t_2)})^r \\ &= f \otimes (|u|^2 + 1)^{-r}(\bar{u}h_2 - h_1)^r \\ &= (z^2)^r f \otimes \tilde{B}^r \end{aligned} \quad . . (14)$$

where

$$B = \frac{z^1h_1 + z^2h_2}{|z^1|^2 + |z^2|^2}.$$

Let $\underline{L^r\Lambda^r E}$, $\underline{L^r\Lambda^r E\,\mathcal{J}^{0,1}}$, $\underline{\mathcal{A}}^{0,r}$ denote the direct images under $q$ of the sheaves of germs of smooth sections of the respective vector bundles. For example, a section of the sheaf $\underline{L^r\Lambda^r E}$ over an open set $U$ of $M$ is a smooth section of the vector bundle $L^r\Lambda^r\underline{E}$ over $q^{-1}(U)$.

There is a differential operator

$$\bar{\partial}_F : \underline{L^r\Lambda^r E} \longrightarrow \underline{L^r\Lambda^r E\,\mathcal{J}^{0,1}}$$

corresponding to differentiation along the fibres of $q : Z \longrightarrow M$. $\bar{\partial}_F$ may be defined by first restricting a section of $L^r\Lambda^r\underline{E}$ over $q^{-1}(U)$ to each line $q^{-1}(x)$ and then applying the Dolbeault $\bar{\partial}$-operator on $q^{-1}(x)$. The kernel of $\bar{\partial}_F$ consists of germs of sections over $q^{-1}(U)$ holomorphic on each line $q^{-1}(x)$. From above $H^0(q^{-1}(x),\mathcal{O}(L)) \cong \underline{H}_x$; more generally there is an isomorphism $H^0(q^{-1}(x),\mathcal{O}(L^r)) \cong S^r\underline{H}_x$ given by $s_1{}^p s_2{}^{p-r} \longmapsto \frac{1}{p!}\mathfrak{S}(h_2{}^p \otimes (-h_1)^{r-p})$. It follows that there is an exact sequence

$$0 \longrightarrow \underline{\Lambda^r ES^r H} \xrightarrow{\eta} \underline{L^r\Lambda^r E} \xrightarrow{\bar{\partial}_F} \underline{L^r\Lambda^r E\,\mathcal{J}^{0,1}} \longrightarrow 0 \qquad \text{. . (15)}$$

where $\underline{\Lambda^r ES^r H}$ simply denotes the sheaf of germs of smooth sections of the vector bundle $\Lambda^r\underline{E}\otimes S^r\underline{H} \subset \Lambda^r\underline{T}^*$ over $M$. $\bar{\partial}_F$ is surjective essentially because $H^1(\mathbb{C}P^1,\mathcal{O}(r)) = 0$ for $r \geqslant -1$. We now have a diagram

$$\begin{tikzcd}
 & & \underline{\mathcal{J}^{0,1}} & \underline{LE\mathcal{J}^{0,1}} & \underline{L^2\Lambda^2E\mathcal{J}^{0,1}} & & & {} & \underline{L^{2n}\Lambda^{2n}E\mathcal{J}^{0,1}} & \\
0 \arrow[r] & \underline{\mathcal{A}}^{0,0} \arrow[r] & \underline{\mathcal{A}}^{0,1} \arrow[r] \arrow[u, "\gamma"] & \underline{\mathcal{A}}^{0,2} \arrow[r] \arrow[u] & \underline{\mathcal{A}}^{0,3} \arrow[r, "\bar{\partial}"] \arrow[u] & \cdots & \cdots \arrow[r] & \underline{\mathcal{A}}^{0,2n} \arrow[r] \arrow[u] & \underline{\mathcal{A}}^{0,2n+1} \arrow[r] \arrow[u, "\parallel"] & 0 \\
 & \underline{\mathcal{A}}^{0,0} \arrow[u, "\parallel"] \arrow[uur, "\bar{\partial}_F"'] & \underline{LE} \arrow[u, "\beta"] \arrow[uur] & \underline{L^2\Lambda^2E} \arrow[u] \arrow[uur] & {} \arrow[u] & & & \underline{L^{2n}\Lambda^{2n}E} \arrow[u] \arrow[uur] & & \\
0 \arrow[r] & \underline{A} \arrow[r] \arrow[u, "\eta"] & \underline{EH} \arrow[r] \arrow[u] & \underline{\Lambda^2ES^2H} \arrow[r, "D"] \arrow[u] & \cdots & & \cdots \arrow[r] & \underline{\Lambda^{2n}ES^{2n}H} \arrow[r] \arrow[u] & 0 &
\end{tikzcd}$$

of sheaves over M. The following series of lemmas establishes properties of this diagram.

Lemma 6.1 The diagonal sequences given by (15) commute with the remainder of the diagram; i.e. $\bar\partial_F = \gamma\circ\bar\partial\circ\beta$ where $\bar\partial$ is the usual operator on $(0,r)$-forms on $Z$.

Proof Fix $z \in Z$ and choose a standard basis of $\underline{H}$ so that $z$ has coordinate $u = {}^{z^1}/_{z^2} = 0$ and $\nabla h_i|_x = 0$ where $x = q(z)$. It is sufficient to evaluate $\bar\partial_F$ and $\gamma\circ\bar\partial\circ\beta$ on a local section $s$ of $L^r\Lambda^r\underline{E}$ at $z$, where $s$ has the form

$$s = \lambda(s_2)^r \otimes f$$

Here $\lambda \in C^\infty(Z)$ and $f$ is (the pullback of) a section of $\Lambda^r\underline{E}$ over M. Using (14),

$$\beta(s) = \lambda(z^2)^r f \otimes \tilde{B}^r$$

Now $(z^2\tilde{B})_z = -h_1$ and because the connection coefficients of $\underline{H}$ vanish at $x$, $du$ is a (1,0)-form at $z$ (see the proof of theorem 4.1). Hence

$$\bar\partial\beta(s)|_z = (d\lambda)^{0,1}\wedge(f\otimes(-h_1)^r) + \lambda(z)\,d(f\otimes(z^2\tilde{B})^r)^{0,r+1}$$

Because $(q^*\Lambda^{r+1}\underline{T}^*)^{0,r+1}_z = q^*(\Lambda^r\underline{E}\otimes \mathbb{C}h_1{}^{r+1})$, it follows that

$$\begin{aligned} d(f\otimes(z^2\tilde{B})^r)^{0,r+1}_z &= -rd(|u|^2+1)^{0,1}\wedge(f\otimes(-h_1)^r) - d\bar{u}\wedge(f\otimes h_2 r(-h_1)^{r-1}) + d(f\otimes(-h_1)^r)^{0,r+1} \\ &= d(f\otimes(-h_1)^r)^{0,r+1}_z \end{aligned}$$

Hence

$$\bar\partial\beta(s)|_z = (-1)^r(d\lambda)^{0,1}\wedge(f\otimes h_1{}^r) + (-1)^r d(f\otimes h_1{}^r)^{0,r+1}_z \quad \ldots (16)$$

and

$$\gamma\bar\partial\beta(s)|_z = (\bar\partial_F\lambda)\otimes(f\otimes s_2{}^r)$$

However $(s_2)^r\otimes f$ is a holomorphic section of $L^r\Lambda^r\underline{E}$ over each line $q^{-1}(x)$, so this is precisely $\bar\partial_F(s)|_z$. This completes the proof of the lemma.

<u>Lemma 6.2</u> The mapping $\beta\circ\eta$ is also given by the composition

$$\Lambda^r\underline{E}\otimes S^r\underline{H} \hookrightarrow \Lambda^r\underline{T}^* \xhookrightarrow{q^*} \Lambda^r(T^*Z) \longrightarrow \mathcal{A}^{0,r}$$

where the last mapping is simply the projection of a complexified r-form onto its (0,r)-component.

<u>Proof</u> It is sufficient to evaluate both compositions on an element of the form $f\otimes h_1{}^r \in \Lambda^r\underline{E}\otimes S^r\underline{H}$, because such elements (for different choices of a standard basis) span $\Lambda^r\underline{E}\otimes S^r\underline{H}$. Now

$$\eta(f\otimes h_1{}^r) = (-s_2)^r\otimes f$$

and by (14),

$$\beta\eta(f\otimes h_1{}^r) = (-1)^r(z^2)^r f\otimes\tilde{B}^r$$

On the other hand, again by (14), $(q^*\Lambda^r\underline{T}^*)^{0,r}$ is spanned by elements of the form $f\otimes\tilde{B}^r$. Consequently, viewing $f\otimes h_1{}^r$ as a pulled-back r-form on Z,

$$(f\otimes h_1{}^r)^{0,r} = \frac{\omega_H(h_1{}^r, B^r)}{\omega_H(\tilde{B}^r, B^r)}\, f\otimes\tilde{B}^r$$

$$= (-1)^r(z^2)^r f\otimes\tilde{B}^r$$

where $\omega_H$ has its usual meaning, but extended to tensor products in the obvious way. The lemma follows.

<u>Lemma 6.3</u> The operator $\bar\partial : \mathcal{A}^{0,r} \longrightarrow \mathcal{A}^{0,r+1}$ on Z induces a differential operator $D : \underline{\Lambda^r E S^r H} \longrightarrow \underline{\Lambda^{r+1}ES^{r+1}H}$ on M which is identical to the quaternionic invariant operator defined by proposition 3.2.

<u>Proof</u> Let $a\in\underline{\Lambda^r ES^r H}$. $\gamma\bar\partial\beta\eta(a) = \bar\partial_F\eta(a) = 0$ by lemma 6.1, so

$$\bar\partial\beta\eta(a) = \beta(a')$$

for some $a'\in \underline{L^{r+1}\Lambda^{r+1}E}$. But $\bar\partial_F(a') = \gamma\bar\partial\beta(a') = \gamma\bar\partial\bar\partial\beta\eta(a) = 0$, so

$$a' = \eta(a'')$$

for some $a''\in\underline{\Lambda^{r+1}ES^{r+1}H}$. Define $Da = a''$; since $\beta\circ\eta$ is injective, a" is unique.

As in the proof of lemma 6.1, fix $z \in Z$ and choose a standard basis of $\underline{H}$ so that $u = 0$ at $z$ and $\nabla h_i|_x = 0$, $x = q(z)$. It is sufficient to evaluate D on an element $a \in \underline{\Lambda^r ES^r H}$ of the form

$$a = f \otimes h_1^r$$

Using (16),

$$\bar{\partial}\beta\eta(a)|_z = d(f \otimes h_1^r)_z^{0,r+1}$$

Because this equals $\beta\eta(D(a))|_z$, it follows from lemma 6.2 that $D(a)$ is the component of $d(f \otimes h_1^r)$ in $\Lambda^{r+1}\underline{E}S^{r+1}\underline{H}$. Therefore D is indeed the operator defined by proposition 3.2.

Lemma 6.3 tells us that the complex

$$0 \longrightarrow \underline{A} \longrightarrow \underline{EH} \longrightarrow \underline{\Lambda^2 ES^2 H} \longrightarrow \cdots \longrightarrow \underline{\Lambda^{2n} ES^{2n} H} \longrightarrow 0 \qquad \ldots (17)$$

of the diagram is precisely the elliptic complex of proposition 3.3 naturally associated to the quaternionic structure of M. The diagram shows that it is intimately related to the complex

$$0 \longrightarrow \underline{\mathcal{A}}^{0,0} \longrightarrow \underline{\mathcal{A}}^{0,1} \longrightarrow \underline{\mathcal{A}}^{0,2} \longrightarrow \cdots \longrightarrow \underline{\mathcal{A}}^{0,2n+1} \longrightarrow 0 \qquad \ldots (18)$$

defined by the $\bar{\partial}$-operator on Z.

Theorem 6.1 The complexes (17),(18) of sheaves on M have isomorphic cohomology groups with $H^{2n+1}(\underline{\mathcal{A}}^{0,*}) = 0$.

Proof From the proof of lemma 6.3, $\beta\circ\eta$ is a cochain mapping between the complexes (17) and (18); i.e. if $a \in \underline{\Lambda^r ES^r H}$,

$$\bar{\partial}(\beta\circ\eta)(a) = (\beta\circ\eta)(Da)$$

Consequently $\beta\circ\eta$ induces a homomorphism

$$\Theta : H^r(\underline{\Lambda^* ES^* H}) \longrightarrow H^r(\underline{\mathcal{A}}^{0,*})$$

of complex vector spaces for each r. We shall show that $\Theta$ is an isomorphism. Note that because the vector spaces $\Lambda^r ES^r H$, and so

$H^r(\Lambda^* ES^* H)$, have real structures, it follows that each $H^r(\underline{\mathcal{A}}^{0,*})$ also has the structure of a real vector space.

Let $\beta\eta(a)$ be a coboundary in $\underline{\mathcal{A}}^{0,r}$, so that $\beta\eta(a) = \bar{\partial} b$ say. Then $\gamma(b) = \bar{\partial}_F(b')$ for some b'.

$$\gamma(b - \bar{\partial}\beta(b')) = \gamma(b) - \bar{\partial}_F(b') = 0$$

so

$$b - \bar{\partial}\beta(b') = \beta(b'')$$

say. Then

$$\bar{\partial}_F(b'') = \gamma\bar{\partial}(b - \bar{\partial}\beta(b')) = \gamma\bar{\partial}(b) = 0$$

Hence b" = $\eta(a')$ say and

$$\beta\eta(a') = b - \bar{\partial}\beta(b')$$

Thus

$$\beta\eta(Da') = \bar{\partial}\beta\eta(a') = \bar{\partial} b$$

and a = Da' because $\beta\circ\eta$ is injective. Hence $\theta$ is injective.

Let $b \in \underline{\mathcal{A}}^{0,r}$ be a cocycle, so $\bar{\partial} b = 0$. As above, there exist b',b" so that $\gamma(b) = \bar{\partial}_F(b')$ and

$$b - \bar{\partial}\beta(b') = \beta(b'')$$

Moreover

$$\bar{\partial}_F(b'') = \gamma\bar{\partial}(b - \bar{\partial}\beta(b')) = 0$$

and b" = $\eta(a)$ say. Then

$$\beta\eta(a) = b - \bar{\partial}\beta(b')$$

proving that $\theta$ is surjective.

Finally, applying the above arguments at the $(2n+1)^{st}$ step gives $H^{2n+1}(\underline{\mathcal{A}}^{0,*}) = 0$. This completes the proof of theorem 6.1.

All our sheaves are fine, so theorem 6.1 remains valid if we replace (17),(18) by the complexes

$$\cdots \longrightarrow \Gamma(U, \Lambda^r \underline{ES^r H}) \xrightarrow{D} \Gamma(U, \Lambda^{r+1}\underline{ES^{r+1}H}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Gamma(q^{-1}(U), \mathcal{A}^{0,r}) \xrightarrow{\bar{\partial}} \Gamma(q^{-1}(U), \mathcal{A}^{0,r+1}) \longrightarrow \cdots$$

of their sections over any open set $U \subset M$. Because the Dolbeault cohomology groups are then isomorphic to the sheaf cohomology groups $H^r(q^{-1}(U),\mathcal{O})$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions, we obtain the

Corollary The cohomology groups $H^r(U,\Lambda^* \underline{E} S^* \underline{H}), H^r(q^{-1}(U),\mathcal{O})$ are isomorphic, with $H^{2n+1}(q^{-1}(U),\mathcal{O}) = 0$.

Theorem 6.1 may be generalized in two respects. Firstly it may be modified so as to involve the elliptic complexes defined by proposition 3.3 for which $r = t - s$ is not necessarily zero. Secondly it is possible to introduce a quaternionic vector bundle into the picture. Indeed, if F is a quaternionic vector bundle over M with connection $\nabla_F$ and $r = t - s \geqslant -1$, there are elliptic complexes

$$0 \longrightarrow \underline{F\phi^{-1}E} \longrightarrow \underline{F\phi^{-1}\Lambda^2 EH} \longrightarrow \cdots \longrightarrow \underline{F\phi^{-1}\Lambda^{2n}ES^{2n-1}H} \longrightarrow 0 \quad (r=-1)$$
$$0 \longrightarrow \underline{F\phi^r S^r H} \longrightarrow \underline{F\phi^r ES^{r+1}H} \longrightarrow \cdots \longrightarrow \underline{F\phi^r \Lambda^{2n}ES^{r+2n}H} \longrightarrow 0 \quad (r \geqslant 0) \qquad (19)$$

where $\underline{F\phi^r\Lambda^s ES^t H}$ denotes the sheaf of germs of smooth sections of the vector bundle $F\phi^r\Lambda^s \underline{E}S^t\underline{H}$ over M. The operators are $1 \otimes D + p\circ(\nabla_F \otimes 1)$, where D is given by proposition 3.2 and p is the appropriate projection. The curvature condition on $\nabla_F$ (see theorem 5.1) is essential for the sequences (19) to be complexes.

On the other hand, by the corollary to proposition 5.1, $F(\phi L)^r = q^*(F\otimes\phi^r)\otimes L^r$ is a holomorphic vector bundle over some $q^{-1}(U) \subset Z$. Taking the direct image under q of the sheaves of germs of smooth sections of the associated Dolbeault $\bar\partial$-complex gives a complex

$$0 \longrightarrow \underline{F(\phi L)^r \mathcal{A}^{0,0}} \longrightarrow \underline{F(\phi L)^r \mathcal{A}^{0,1}} \longrightarrow \cdots \longrightarrow \underline{F(\phi L)^r \mathcal{A}^{0,2n+1}} \longrightarrow 0 \qquad (20)$$

of sheaves over M.

Theorem 6.2 Let F be a quaternionic vector bundle over a quaternionic manifold M. Then the complexes (19),(20) have isomorphic cohomology groups for $r \geqslant -1$, with $H^{2n+1}(\underline{F(\phi L)^r \mathcal{A}^{0,*}}) = 0$.

Proof Of course in the special case F trivial, r = 0, we have theorem 6.1. In general the proof is almost identical and we omit the details. For example in the case r=-1 there is a diagram

$$\begin{tikzcd}
 & & & \underline{F(\phi L)^{-1}\mathcal{J}^{0,1}} & & \underline{F(\phi L)^{-1}LE\mathcal{J}^{0,1}} & & \\
0 \arrow[r] & \underline{F(\phi L)^{-1}\mathcal{A}^{0,0}} \arrow[rr] & & \underline{F(\phi L)^{-1}\mathcal{A}^{0,1}} \arrow[rr] \arrow[u] & & \underline{F(\phi L)^{-1}\mathcal{A}^{0,2}} \arrow[rr] \arrow[u] & & \cdot\ \cdot \\
 & \underline{F(\phi L)^{-1}} \arrow[u] \arrow[uurr] & & \underline{F(\phi L)^{-1}LE} \arrow[u] \arrow[uurr] & & \underline{F(\phi L)^{-1}L^2\Lambda^2E} \arrow[u] \arrow[uurr] & & \\
0 \arrow[ur] \arrow[rr] & & \underline{F\phi^{-1}E} \arrow[ur] \arrow[rr] & & \underline{F\phi^{-1}\Lambda^2E\,H} \arrow[ur] \arrow[rr] & & \cdot\ \cdot &
\end{tikzcd}$$

The important point is that the diagonal sequences are still exact because $q^*F$ is holomorphically trivial on each line $q^{-1}(x)$ and $H^1(\mathbb{C}P^1, \mathcal{O}(r)) = 0$ if $r \geqslant -1$. Notice that the cohomology vanishes at the first step essentially because $F(\phi L)^{-1}$ cannot admit holomorphic sections on any line $q^{-1}(x)$.

Remarks 1. Theorem 6.2 has a similar corollary to that given for theorem 6.1. For example if F is defined globally over M and r is even, then $H^r(M, \Lambda^*\underline{E}S^*\underline{H}) \cong H^r(Z, \mathcal{O}(F(\phi L)^r))$, with $H^{2n+1}(Z, \mathcal{O}(F(\phi L)^r)) = 0$. If Z is compact, the last statement also follows from Serre duality because the canonical bundle $\mathcal{K}$ of Z may be identified with the line bundle $(\phi L)^{-2(n+1)}$.

2. The reason that theorem 6.2 fails for $r \leqslant -2$ is that $H^1(\mathbb{C}P^1, \mathcal{O}(r))$ no longer vanishes, so that $\bar{\partial}_F$ is no longer surjective and our diagonal sequences break down on the left-hand side of the diagram. For example, the complex corresponding to F trivial, $r = -2$ on an 8-dimensional quaternionic manifold is

$$0 \longrightarrow \underline{\Lambda^2 E} \longrightarrow \underline{\Lambda^3 E\, H} \longrightarrow \underline{\Lambda^4 E\, S^2 H} \longrightarrow 0$$

In fact this is not elliptic; the symbol sequence is not exact at the first step (see the proof of proposition 3.3). However if we choose a Riemannian metric $g = \omega_E \otimes \omega_H$ on M compatible with the quaternionic structure and replace $\Lambda^2 E$ by its subspace $\Lambda^2_0 E$ orthogonal to $\omega_E$, the resulting complex

$$0 \longrightarrow \underline{\Lambda^2_0 E} \longrightarrow \underline{\Lambda^3 E\, H} \longrightarrow \underline{\Lambda^4 E\, S^2 H} \longrightarrow 0$$

<u>is</u> exact. We shall see in §10 that it is precisely the Dirac complex of M (M is spin by proposition 1.1).

3. Extensive investigation of the relationship between certain cohomology groups on M and Z has already been carried out in the case in which M is 4-dimensional; see e.g. Hitchin [22]. However the methods used have depended heavily on the fact that the fibres of Z have complex dimension 1, and have only dealt with cohomology groups of the form $H^0, H^1$.

# PART TWO

## QUATERNIONIC KÄHLER MANIFOLDS

### 7. Curvature

Recall from §3 that a quaternionic Kähler manifold is by definition an oriented 4n-dimensional Riemannian manifold M whose linear holonomy group is contained in Sp(n)Sp(1). We shall also suppose that $n \geqslant 2$. These manifolds are of special interest because Sp(n)Sp(1) occurs in the list

SO(n), U(n), SU(n), Sp(n), Sp(n)Sp(1), Spin 7, $G_2$, of possible holonomy groups of locally irreducible Riemannian manifolds which are not locally symmetric [9], [3]. It follows from the Reduction theorem (see Kobayashi & Nomizu [28]) that the frame bundle of M can be reduced to a principal Sp(n)Sp(1)-bundle P'. Since $Sp(n)Sp(1) \subset GL(n,\mathbb{H})Sp(1)$, M has an almost quaternionic structure with which the metric g is compatible in the sense of §1. Moreover the torsion-free Levi-Civita connection $\nabla$ determined by g reduces to a connection on P' and is therefore a $GL(n,\mathbb{H})Sp(1)$-connection. This in turn implies that M is a quaternionic manifold. We can say that the quaternionic structure of M possesses a quaternionic Kähler metric. For equivalent definitions and elementary properties of quaternionic Kähler manifolds we refer the reader to Gray [17], Ishihara [25] and Kraines [31].

The existence of the principal Sp(n)Sp(1)-bundle P' with its connection $\nabla$ enables us, given a representation V of Sp(n)xSp(1), to construct locally an associated vector bundle $\underline{V}$. Namely we choose locally a lifting of P' to a principal Sp(n)xSp(1)-bundle $\tilde{P}'$ and define $\underline{V} = \tilde{P}' \times_{Sp(n)\times Sp(1)} V$. Given a representation V of

$G = GL(n,\mathbb{H})\times Sp(1)$ we use the same symbol V for the induced representation $Sp(n)\times Sp(1) \longrightarrow G \longrightarrow \mathrm{Aut}V$. For example, E now stands for the standard representation of $Sp(n)$ on $\mathbb{C}^{2n}$, and is equipped with a real positive definite skew form $\omega_E \in \Lambda^2 E$ satisfying

$$g = \omega_E \otimes \omega_H \in \Lambda^2 E \otimes \Lambda^2 H \subset S^2 T^*$$

The identifications $E^* = E$, $H^* = H$, $T = T^*$ given by $e^* \longmapsto \omega_E(\cdot,e^*)$, $h^* \longmapsto \omega_H(\cdot,h^*)$, $v \longmapsto g(v,\cdot)$ are consistent, and we may write without ambiguity

$$T = T^* = E \otimes H$$

Any basis of E of the form $\{e_1,\ldots,e_n,e_{n+1}=\tilde{e}_1,\ldots,e_{2n}=\tilde{e}_n\}$ and satisfying $\omega_E = \sum_{i=1}^{n} e_i \wedge \tilde{e}_i$ will be called a standard basis of E. Using P', the above structure carries over to the associated vector bundles $\underline{E}$, $\underline{H}$ and $\underline{T}$.

In general, an irreducible representation of G will decompose into several irreducible representations of $Sp(n)\times Sp(1)$; for example if $\Lambda_0^2 E$ denotes the subspace of $\Lambda^2 E$ orthogonal to $\omega_E$, then $\Lambda^2 E = \{\omega_E\} + \Lambda_0^2 E$ is an irreducible decomposition under $Sp(n)$. Any 1-dimensional representation of $Sp(n)\times Sp(n)$ must be trivial, so the problem of weighting no longer exists; in particular the vector bundles $\underline{H}$ and $\underline{K}$ are now identical. Any irreducible representation of $Sp(n)\times Sp(1)$ may be realized as a subspace of $(\bigotimes^p E)\otimes(\bigotimes^q H)$. Following the convention of §2, whenever it is necessary to write down the elements of such a subspace, we do so in terms of standard bases of E,H, regarding $\omega_E, \omega_H$, which trivialize invariant 1-dimensional subspaces of $E\otimes E$, $H\otimes H$, as unit quantities. For example, the space of 2-forms on M decomposes under $Sp(n)Sp(1)$ as

$$\Lambda^2 T^* = \Lambda^2 E \otimes S^2 H + S^2 E \otimes \Lambda^2 H$$
$$= S^2 H + S^2 E + \Lambda_0^2 E \otimes S^2 H$$

and we write

$$\left.\begin{array}{l} S^2H \ni h\vee k = (h\vee k)\omega_E = \sum_{i=1}^{2n} (e_i h\tilde{e}_i k + e_i k\tilde{e}_i h) \\ S^2E \ni e\vee f = (e\vee f)\omega_H = \sum_{i=1,2} (eh_i f\tilde{h}_i + fh_i e\tilde{h}_i) \end{array}\right\} \subset \Lambda^2(EH) = \Lambda^2T^*$$

where $\{e_1,\ldots,e_{2n}\}$ and $\{h_1,h_2\}$ are standard bases of E and H. The inclusion $S^2H \subset \Lambda^2T^*$ reminds us that the metric enables us to view the bundle $(S^2\underline{H})_r$ of imaginary quaternions as part of the bundle of 2-forms.

Let M be a quaternionic Kähler manifold. The curvature of an arbitrary connection on its tangent bundle $\underline{T}$ is a section of $\mathrm{End}\underline{T} \otimes \Lambda^2\underline{T}^* \cong \underline{T} \otimes \underline{T}^* \otimes \Lambda^2\underline{T}^*$. The metric enables us to 'lower the first index' and the Riemannian curvature R is then a section of

$$S^2(\Lambda^2\underline{T}^*) \subset \underline{T}^* \otimes \underline{T}^* \otimes \Lambda^2\underline{T}^*$$

The adjoint representation of Sp(n)Sp(1) may be identified with $\mathfrak{a} = S^2E + S^2H \subset \Lambda^2T^*$. Because the holonomy algebra is $\mathfrak{a}$, R is actually a section of

$$S^2\underline{\mathfrak{a}} = S^2(S^2\underline{E}) + S^2\underline{E} \otimes S^2\underline{H} + S^2(S^2\underline{H})$$

Indeed, the holonomy algebra is generated by the curvature in the sense of [28; II, §8]. As a first step to describing the curvature of a quaternionic Kähler manifold, we prove

Lemma 7.1 $\mathbb{H}P^n$ is a quaternionic Kähler manifold with Riemannian curvature

$$R_0 = c(I \otimes I + J \otimes J + K \otimes K) + R' \in \Gamma(S^2\underline{\mathfrak{a}})$$

where $\{I,J,K\}$ is any standard basis of $(S^2\underline{H})_r$, $R' \in \Gamma(S^2(S^2\underline{E}))$ and c is a positive constant.

Proof The fact that $\mathbb{H}P^n$ is a quaternionic Kähler manifold follows from its description as the Riemannian symmetric space

$$Sp(n+1)/Sp(n)\times Sp(1)$$

with linear holonomy group Sp(n)Sp(1) (e.g. Besse [10]). From the general theory of Riemannian symmetric spaces, its curvature $R_0$ is essentially the Killing form form B of Sp(n)xSp(1); in fact (see Kobayashi & Nomizu [28; XI]

$$R_0 = -2cB \in \Gamma(S^2(\underline{a}))$$

where c is a positive constant which we choose so as to make the scalar curvature equal to 1. Now $B = B_1 + B_2$ where $B_1, B_2$ are the Killing forms of Sp(1), Sp(n) respectively. Under the identification $sp(1) = (S^2H)_r$, $B_1$ is minus the metric induced by $\omega_H$. The elements I,J,K of any standard basis of $(S^2H)_r$ are orthogonal and each have norm $\sqrt{2}$ with respect to this metric, so

$$B_1 = -\tfrac{1}{2}(I \otimes I + J \otimes J + K \otimes K)$$

The lemma follows.

<u>Theorem 7.1</u> The Riemannian curvature of an arbitrary quaternionic Kähler manifold M has the form

$$R = tR_0 + R_U$$

where $R_U \in \Gamma(\underline{U})$ (U is the representation of $GL(n,\mathbb{H})Sp(1)$ introduced in §2) and t is the scalar curvature. M is an Einstein manifold.

<u>Proof</u> The Levi-Civita connection $\nabla$ is a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection, so from (5) §2, its curvature is a section of

$$\partial(\underline{g}^1 \otimes \underline{T}^*) + \underline{U} \subset \mathrm{End}\,\underline{T} \otimes \Lambda^2 \underline{T}^*$$

But from the Riemannian viewpoint, the curvature is also a section of $S^2\underline{a}$. We must show that $\partial(\underline{g}^1 \otimes \underline{T}^*) \cap S^2\underline{a}$ is spanned by $R_0$. Using lemma 2.1, as Sp(n)Sp(1) modules,

$$\partial(g^1 \otimes T^*) \subset EH \otimes EH = S^2H + S^2E + \Lambda_0^2 ES^2H + A + \Lambda_0^2 E + S^2ES^2H$$

Of these, $S^2H$ and $\Lambda_0^2ES^2H$ do not occur as submodules of $S^2\mathfrak{a}$. On the other hand, with the same notation as lemma 2.2, it is immediate that

$$\left.\begin{array}{llll} \partial(u_1\otimes e_2h_2) & \text{has non-zero} & S^2E \text{ component in} & A\otimes S^2E \\ \partial(u_2\otimes e_2h_2) & \quad " \quad " & S^2ES^2H \quad " \quad " & B\otimes\Lambda_0^2ES^2H \\ \partial(u_3\otimes e_2h_2) & \quad " \quad " & \Lambda_0^2E \quad " \quad " & S^2H\otimes\Lambda_0^2ES^2H \end{array}\right\} \subset (S^2\mathfrak{a})^{\perp}$$

where $(S^2\mathfrak{a})^{\perp}$ denotes the orthogonal complement of $S^2\mathfrak{a}$ in $\mathrm{End}T\otimes\Lambda^2T^*$. Thus $\partial(u\otimes e_2h_2)$ has non-zero $S^2E, S^2ES^2H, \Lambda_0^2E$ components in $(S^2\mathfrak{a})^{\perp}$ so none of these spaces occur in $\partial(\mathfrak{g}^1\otimes T^*)\cap S^2\mathfrak{a}$, as required.

The Ricci tensor of M is the image of R under the contraction $\mathrm{End}T\otimes\Lambda^2T^* \longrightarrow T^*\otimes T^*$. $R_U$ maps to zero (see (5), § 2), whereas $tR_0$, being invariant under Sp(n)Sp(1), must map to a multiple of the metric. There M is Einstein.

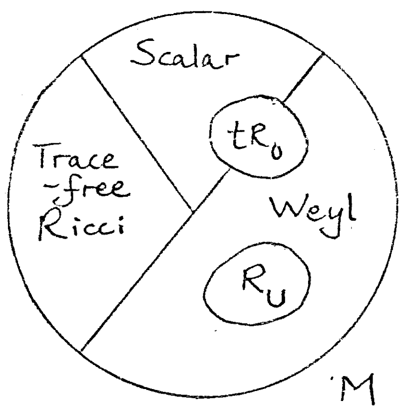
Remarks 1. The fact that a quaternionic Kähler manifold is Einstein is well-known. For the decomposition of R under the action of Sp(n)Sp(1) (without proofs), see Alekseevskii [3]. The space $U$ is reducible under Sp(n)Sp(1) and contains the irreducible component $S^4E$. We know that $U\cap S^2\mathfrak{a}\subset S^2(S^2E)$; in fact

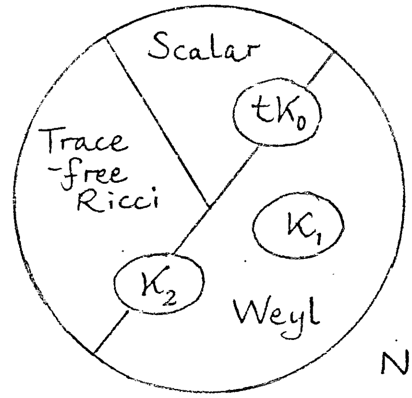
$$U\cap S^2\mathfrak{a} = S^4E$$

so that actually $R_U\in\Gamma(S^4\underline{E})$.

2. It is interesting to compare the Riemannian curvature of a quaternionic Kähler manifold M to that of a Kähler manifold N. Suppose that dimM = dimN = 4n, $n\geqslant 2$. Of course under O(4n), the curvature of M and N decomposes into three irreducible components corresponding to the scalar, trace-free Ricci and Weyl curvature. Moreover it is known (e.g. [3]) that the curvature of N has three

irreducible components under the action of $U(2n)$. Namely if $K_0$ denotes the constant holomorphic curvature (with scalar curvature 1) of $CP^{2n}$, then N has curvature $tK_0 + K_1 + K_2$, where $K_1$ (often called the Bochner component) maps to zero under the Ricci contraction and $K_2$ maps to zero under the scalar curvature contraction. In the following diagrams the components of the curvature of M and N are shown in relation to the $O(4n)$ decomposition:




It will be convenient to have an explicit description of the curvature $\Omega_H \in \Gamma(\mathrm{End}\underline{H} \otimes \Lambda^2 \underline{T}^*)$ of the vector bundle $\underline{H}$. Relative to a standard basis $\{h_1, h_2\}$ of $\underline{H}$, the curvature 2-forms are defined by

$$\Omega_H(h_i) = h_j \otimes \Omega_i^j$$

If the induced connection on $\underline{H}$ is given by $\nabla h_i = h_j \otimes \omega_i^j$, then

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

<u>Proposition 7.1</u> The curvature of $\underline{H}$ is given by

$$\Omega_i^j = \tfrac{1}{2} ct\ h_i \vee \tilde{h}_j$$

<u>Proof</u> Since $\underline{T} = \underline{E} \otimes \underline{H}$, the curvature of M can be written

$$R = \Omega_E + \Omega_H$$

where $\Omega_E \in \Gamma(\mathrm{End}\underline{E} \otimes \Lambda^2 \underline{T}^*)$ is the curvature of $\underline{E}$. By lemma 7.1, theorem 7.1 and (2) §1,

$$\mathcal{R}_H = ct(I \otimes I + J \otimes J + K \otimes K)$$
$$= ct\left[2h_1^2 \otimes h_2^2 + 2h_2^2 \otimes h_1^2 - (h_1 \vee h_2) \otimes (h_1 \vee h_2)\right]$$

in terms of standard bases. The proposition follows.

By theorem 7.1 any quaternionic Kähler manifold is Einstein; in particular its scalar curvature $t$ is a constant. Thus there are three classes of examples, those with $t > 0$, $t = 0$ and $t < 0$, or equivalently with Ricci tensor positive, zero and negative definite. Theorem 7.1 also implies that $t = 0$ iff the holonomy reduces from $Sp(n)Sp(1)$ to $Sp(n)$. By a result of Alekseevskii [3], any quaternionic Kähler manifold with $t \neq 0$ is irreducible as a Riemannian manifold.

Let $M$ be a quaternionic Kähler manifold. By theorem 4.1, $M$ admits a twistor space $Z$ which is locally isomorphic to the complex projective bundle $P(\underline{H})$. To introduce some topology, first suppose that $M$ has $\xi = 0$ (see §1), so that $\underline{H}$ is well-defined globally. The tautologous holomorphic line bundle $L^{-1} = (\phi L)^{-1}$ is then well-defined on $Z$. Put

$$w = -c_2(\underline{H}) \quad (c_1(\underline{H})=0 \text{ because } \underline{H} \cong \underline{H}^*)$$
$$\ell = c_1(L)$$

Because $Z = P(\underline{H})$, the Leray-Hirsch theorem implies that the real cohomology of $Z$ is a free $H^*(M;\mathbb{R})$-module generated by $\ell$ subject to the relation

$$\ell^2 = w \qquad \qquad . . (21$$

In general, if $\xi \neq 0$, $S^2\underline{H}$ and $L^2$ are still well-defined and we can put

$$w = -\tfrac{1}{4}c_2(S^2\underline{H}) = \tfrac{1}{4}p_1((S^2\underline{H})_r) \in H^4(M;\mathbb{R})$$
$$= \tfrac{1}{2}c_1(L^2) \in H^2(Z;\mathbb{R})$$

Moreover (21) remains valid.

Note that w may also be defined as the real cohomology class determined by the closed 4-form

$$-\frac{1}{8\pi^2}\,\mathrm{trace}\,(\Omega_H{}^2) = \frac{1}{4\pi^2}c^2t^2(I\wedge I + J\wedge J + K\wedge K)$$

in terms of a standard basis $\{I,J,K\}$ of $(S^2H)_r \subset \Lambda^2\underline{T}^*$. At each point I,J,K define almost complex structures consistent with the orientation of M, and it follows that

$$\Theta = I\wedge I + J\wedge J + K\wedge K$$

has maximal rank with $\Theta^n = \Theta\wedge\ldots\wedge\Theta$ a positive multiple of the volume element. If M is compact,

$$\varkappa[w^n] = \int_M\left(\frac{1}{4\pi^2}c^2t^2\Theta\right)^n > 0 \qquad \ldots (22)$$

Because $\Omega_H$ is invariant under Sp(n)Sp(1), the 4-form $\Theta$ is in fact covariant constant. $\Theta$ is analogous to the fundamental 2-form on a Kähler manifold and was introduced by Kraines [31] to establish inequalities for the Betti numbers of a quaternionic Kähler manifold. The main point is that wedging with $\Theta$ determines an injection $H^p(M;\mathbb{R}) \hookrightarrow H^{p+4}(M;\mathbb{R})$, $p \leqslant n-3$. As a special case, $H^{4p}(M;\mathbb{R}) \neq 0$.

<u>Examples</u> 1. We saw in §4 that $S^{4n-1}\times S^1$ is a quaternionic manifold $(n \geqslant 2)$, but it cannot admit a quaternionic Kähler metric. For otherwise its $4^{th}$ Betti number would be non-zero.

2. For $t > 0$, the only known examples of quaternionic Kähler manifolds are the symmetric spaces described by Wolf [38]. He

observed that all the compact simply-connected Riemannian symmetric spaces which are quaternionic Kähler arise as follows. Let $G$ be an arbitrary compact simple Lie group with trivial centre. Let $G_1 = Sp(1)$ denote the regular 3-dimensional subgroup of $G$ corresponding to the highest root of its Lie algebra (with respect to some ordering), and let $N(G_1)$ be the connected normalizer of $G_1$ in $G$. Then $G/N(G_1)$ has the structure of a Riemannian symmetric space with linear holonomy $N(G_1) \subset Sp(n)Sp(1)$. For details see [38]. In particular there are three families

$$\frac{Sp(n+1)}{Sp(n)\times Sp(1)} = \mathbb{H}P^n, \qquad \frac{SU(n+2)}{S(U(n)\times U(2))} = X^n, \qquad \frac{SO(n+4)}{S(O(n)\times O(4))} = Y^n$$

all of dimension $4n$, $n \geqslant 2$. Note that $X^2 \cong Y^2$ and if we include $n = 1$ we obtain the two examples

$$\mathbb{H}P^1 \cong X^1 \cong S^4, \qquad Y^1 \cong \mathbb{C}P^2$$

of self-dual manifolds. We may also take $G$ to be an exceptional Lie group, in which case after a careful examination of the isotropy representations we obtain the spaces

$$\frac{G_2}{SO(4)}, \qquad \frac{F_4}{Sp(3)\times_{\mathbb{Z}_2} Sp(1)}, \qquad \frac{E_6}{SU(6)\times_{\mathbb{Z}_2} Sp(1)},$$

$$\frac{E_7}{Spin(12)\times_{\mathbb{Z}_2} Sp(1)}, \qquad \frac{E_8}{E_7\times_{\mathbb{Z}_2} Sp(1)}$$

of dimension 8, 28, 40, 62, 112 respectively.

It follows from the above descriptions that all the Wolf spaces apart from $\mathbb{H}P^n, X^1$ have $\xi \neq 0$. Moreover if $M$ is any one of the exceptional spaces or $Y^n$, $n \geqslant 3$, we obtain from the homotopy exact sequence

$$\mathbb{Z}_2 = \pi_2(M) \cong H^2(M;\mathbb{Z}).$$

Let M be a Wolf space with holonomy strictly contained in $Sp(n)Sp(1)$. Then from theorem 7.1, its curvature must have $R_U \neq 0$, so as a quaternionic manifold $C_1 \neq 0$. Therefore of the Wolf spaces described above, $\mathbb{H}P^n$ is the only one that is an integrable quaternionic manifold.

3. Non-trivial examples of quaternionic Kähler manifolds with $t = 0$ have been found by Calabi [15]. Indeed he has shown that the total space of the holomorphic cotangent bundle of $\mathbb{C}P^n$ admits a complete Riemannian metric with holonomy equal to $Sp(n)$ $(n \geqslant 1)$. Such a manifold M has a set $\{I, J, K\}$ of complex structures, each of which actually defines a Kähler metric on M. The Riemannian curvature $R = R_U \neq 0$, so again the $GL(n,\mathbb{H})Sp(1)$-structure of M is not integrable. A priori, the $GL(n,\mathbb{H})$-structure is non-integrable. This answers in the negative a question of Sommese who, in a study of manifolds with an integrable $GL(n,\mathbb{H})$-structure [37], asked whether the integrability of the almost complex structures I,J,K is sufficient to ensure the integrability of the $GL(n,\mathbb{H})$-structure.

The Calabi examples are not compact. In fact Bogomolov [11] has recently proved that there are no compact locally irreducible Riemannian manifolds with holonomy contained in $Sp(n)$, $n \geqslant 2$.

4. Finally, examples of quaternionic Kähler manifolds with $t < 0$ are furnished by the non-compact duals of the symmetric spaces described above. Alekseevskii [5] has also found examples which are homogeneous, but not symmetric. To date, no non-homogeneous quaternionic Kähler manifolds with $t \neq 0$ are known.

## 8. Properties of the Twistor Space

We know that a quaternionic Kähler manifold M admits a twistor space Z. In this section we shall see how the existence of the quaternionic Kähler metric on M imposes additional holomorphic structure on Z. As usual let $L^{-1}$ be the tautologous holomorphic line bundle over Z, and let $\mathcal{A}^1 \cong \mathcal{A}^{1,0}$ denote the holomorphic cotangent bundle of Z.

Proposition 8.1 The twistor space Z of a quaternionic manifold M has a nowhere-zero holomorphic section $\beta \in H^0(Z, \mathcal{O}(L^2 \otimes \mathcal{A}^1))$.

Proof Working locally over some open set of M, we begin by considering the complex manifold $\underline{H}\setminus 0$, the total space of $\underline{H}$ with its zero section removed. The induced connection on $\underline{H}$ gives rise to splittings

$$\left.\begin{aligned} T(\underline{H}\setminus 0) &= \mathcal{V} \oplus \mathcal{H} \\ T^*(\underline{H}\setminus 0) &= \mathcal{J} \oplus p^*\underline{T}^* \end{aligned}\right\} \qquad ..(24)$$

where $\mathcal{J}$ is the annihilator of $\mathcal{H}$. We first recall how the complex structure was defined on these bundles in §4.

Take a standard basis $\{h_1, h_2\}$ of $\underline{H}$ and let $z^1, z^2$ denote corresponding coordinate functions on its total space.Then from the proof of theorem 4.1, the fibre of $(p^*\underline{T}^*)^{1,0}$ at each point is spanned by elements of the form

$$z^1 p^*(eh_1) + z^2 p^*(eh_2) = e(z^i h_i)$$

If $\nabla h_i = h_j \otimes \omega^j_i$, a local section $s = s^i h_i$ of $\underline{H}$ is covariant constant at $x \in M$ iff

$$0 = ds^i + s^j \omega^i_j, \quad i = 1,2$$

there. It follows that the 1-forms

$$\theta^i = dz^i + z^j \omega^i_j, \quad i = 1,2$$

defined on $\underline{H} \setminus 0$ annihilate vectors in $\mathcal{H}$. Moreover, since the restrictions of the functions $z^1, z^2$ to each fibre $\mathbb{C}^2 \setminus 0$ are holomorphic, $\{\theta^i\}$ spans $\mathcal{J}^{1,0}$ at each point.

Now $\tilde{h}_i \longmapsto \theta^i$ gives an isomorphism $p^*\underline{H} \cong \mathcal{J}^{1,0}$ which is natural with respect to the SU(2) structure on the fibres. As a consequence, the canonical section of $p^*\underline{H}$ can be viewed as an invariant section $\alpha = z^2\theta^1 - z^1\theta^2$ of $\mathcal{J}^{1,0}$. We claim that $\alpha$ is in fact a holomorphic 1-form on $\underline{H} \setminus 0$. Because of the SU(2)-invariance, it is enough to prove that $d\alpha$ is a (2,0)-form at say $y = h_2(x)$ (i.e. $z^1 = 0$, $z^2 = 1$) and we are free to assume that $\nabla h_i|_x = 0$, so that $\omega_i^j\big|_x = 0$. Then at y,

$$
\begin{aligned}
d\alpha &= d(z^2\theta^1 - z^1\theta^2) \\
&= d(dz^1 + z^1\omega_1^1 + z^2\omega_2^1) - 2dz^1 \wedge dz^2 \\
&= d\omega_2^1 - 2dz^1 \wedge dz^2 \\
&= \Omega_2^1 - 2dz^1 \wedge dz^2
\end{aligned}
$$

where by proposition 7.1, $\Omega_2^1 = ct(h_2)^2 \in \Lambda^{2,0}(p^*\underline{T}^*)_y$. Therefore $d\alpha\big|_y$ is a (2,0)-form, as required.

Now pass to $Z = P(\underline{H})$. The functions $z^1, z^2$ define, as in §6, sections $s_1, s_2$ of L satisfying $s_1 = us_2$ wherever $u = z^1/z^2$ is finite. The restrictions of $s_1, s_2$ to each line $q^{-1}(x) \cong \mathbb{CP}^1$ are holomorphic, although $s_1, s_2$ themselves are not holomorphic sections of L. Put

$$\theta = du + \omega_2^1 - u^2\omega_1^2 + u(\omega_1^1 - \omega_2^2) \in \Gamma(T^*Z)$$

$$\beta = (s_2)^2 \otimes \theta \in \Gamma(L^2 \otimes T^*Z)$$

both defined on $Z \setminus \{z^2=0\}$. Regarding $\beta$ as an $L^2$-valued 1-form, its pullback under $\pi : \underline{H} \setminus 0 \longrightarrow Z$ satisfies

$$\pi^*\beta = (z^2)^2\pi^*\theta$$
$$= z^2dz^1 - z^1dz^2 + \omega_2^1 - u^2\omega_1^2 + u(\omega_1^1 - \omega_2^2)$$
$$= z^2\theta^1 - z^1\theta^2$$
$$= \alpha$$

Therefore $\theta$ is actually a (1,0)-form on $Z\setminus\{z^2=0\}$, and the equation $\pi^*\beta = \alpha$ defines a nowhere-zero holomorphic section $\beta\in H^0(Z,\mathcal{O}(L^2\otimes\mathcal{A}^1))$.

The next theorem is really a corollary of proposition 8.1.

Theorem 8.1. Let M be a quaternionic Kähler manifold with twistor space Z. Then the (locally defined) vector bundle $\underline{E}$ on M is quaternionic, and there is a (globally defined) holomorphic short exact sequence

$$0 \longrightarrow L^{-2} \longrightarrow \mathcal{A}^1 \longrightarrow L^{-1}\otimes q^*\underline{E} \longrightarrow 0 \qquad .. \text{ (25)}$$

of vector bundles over Z.

Proof Let R denote the Riemannian curvature of M. The curvature $\Omega_E$ of $\underline{E}$ is the image of R under the mapping

$$S^2\mathfrak{a} \subset \mathfrak{a}\otimes\Lambda^2T^* \xrightarrow{d\rho\otimes 1} \mathrm{End}E\otimes\Lambda^2T^*$$

where $\rho$ is the representation $Sp(n)xSp(1) \longrightarrow \mathrm{Aut}E$. From theorem 7.1 and the proof of proposition 7.1, $\Omega_E$ is a section of $S^2(S^2\underline{E})$, so certainly its 2-forms belong to $S^2\underline{E}$. Hence $\underline{E}$ is a quaternionic vector bundle in the sense of §5.

Corresponding to (24) there are splittings

$$\left.\begin{aligned} TZ &= \mathcal{V}\oplus\mathcal{H} \\ T^*Z &= \mathcal{F}\oplus q^*\underline{T}^* \end{aligned}\right\} \qquad .. \text{ (26)}$$

where by abuse of notation we have used the same symbols for the summands, although now the fibres of $\mathcal{V}$ and $\mathcal{F}$ have only 2 real dimensions instead of 4. Now $\beta$ determines an inclusion $L^{-2} \hookrightarrow \mathcal{A}^1$ and the (1,0)-part of the above splitting of $T^*Z$ can be written

$$\mathcal{A}^1 = \beta(L^{-2}) \oplus L^{-1}q^*\underline{E}$$

because $\beta(L^{-2}) \cong \mathcal{F}^{1,0}$ and $L^{-1}q^*\underline{E} = (q^*\underline{T}^*)^{1,0}$. Refer also to (8) §4. Therefore (25) exists as a $C^\infty$ sequence of vector bundles. But $\beta$ is holomorphic and induces a holomorphic structure on $L^{-1}\otimes q^*\underline{E}$ so that (25) is holomorphic.

On the other hand $q^*\underline{E}$, and so $L^{-1}q^*\underline{E}$, has a holomorphic structure by the corollary to theorem 5.1. Using theorem 6.2, this holomorphic structure satisfies $H^0(Z, \mathcal{O}(q^*\underline{E})) = \ker \nabla : \underline{E} \longrightarrow \underline{E} \otimes \underline{T}^*$. Fix $z \in Z$ with $q(z) = x$. Then the space of 1-jets at $z$ of local holomorphic sections of $q^*\underline{E}$ is spanned by elements of the form $j_1(q^*e)$, where $e$ is a local section of $E$ over $M$ such that $\nabla e|_x = 0$. Fix such an $e$. We shall show that $j_1(q^*e)$ is the image of a 1-jet of a holomorphic section of $L \otimes \mathcal{A}^1$ under the mapping $L \otimes \mathcal{A}^1 \longrightarrow q^*\underline{E}$ arising from (25). This will imply that the two holomorphic structures on $L^{-1}q^*\underline{E}$ are identical, so that there is no ambiguity in the statement of theorem 8.2.

Take a standard basis $\{h_1, h_2\}$ of $\underline{H}$ such that $z = [h_2(x)]$ and $\nabla h_i|_x = 0$. Choose a smooth (necessarily complex-valued) function $\lambda$ on $M$ satisfying

$$d\lambda|_x = (eh_1)_x \in (EH)_x = T^*_x$$

Then

$$\sigma = s_2 \otimes [e(uh_1 + h_2) + \lambda\theta]$$

is a section of $L \otimes \mathcal{A}^1$ whose 1-jet at $z$ is holomorphic. For at $h_2(x)$,

$$\begin{aligned} d(\pi^*\sigma) &= d\left[e(z^1h_1 + z^2h_2) + \frac{1}{z^2}\lambda\alpha\right] \\ &= dz^2 \wedge (eh_2) + \lambda(x)(d\alpha - dz^2 \wedge dz^1) \\ &\in \Lambda^{2,0}T^*(\underline{H} \setminus 0) \end{aligned}$$

But under $L \otimes \mathcal{A}^1 \longrightarrow q^*\underline{E}$, $\sigma$ maps to $q^* e$, as required.

Remark Combining theorems 5.1,8.1 we see that the total space of tangent bundle $\underline{T} = \underline{E} \otimes \underline{H}$ of a quaternionic Kähler manifold M is in fact a quaternionic manifold. However, unless M is flat, it does not admit a compatible quaternionic Kähler metric. More generally, if F is any non-flat quaternionic vector bundle over M, the quaternionic manifold $\tilde{M}$ of theorem 5.1 does not admit a quaternionic Kähler metric.

For definiteness, take an anti-self-dual SU(2) bundle F over $M = S^4$ with $c_2(F) > 0$. In example 3, §5 we computed the curvature of a torsion-free $GL(2,\mathbb{H})Sp(1)$-connection on $\tilde{M}$ at a point $x \in M$. Because the $\tilde{U}$-component of this is an invariant of the quaternionic structure, 'lowering the first index' gives the component $\tilde{R}_U$ of the Riemannian curvature of any compatible quaternionic Kähler metric. In our case, at x,

$$\tilde{R}_U = \Omega_F + \text{terms arising from } b_I \wedge d\lambda^I$$

With respect to the natural splitting $T^*\tilde{M} = \underline{T}^* \oplus F^*\underline{H}$ at x, $\Omega_F$ has non-zero component in $\Lambda^2(F^*\underline{H}) \otimes \Lambda^2\underline{T}^*$. Because $b_I \in \tilde{\underline{g}}^{\mathbb{C}} \otimes \underline{T}^*$, $\tilde{R}_U$ cannot lie in $S^2(\Lambda^2 T^*\tilde{M})$.

To sum up, the existence of anti-self-dual vector bundles over $S^4$ provides us with examples of non-trivial quaternionic structures which do not admit a quaternionic Kähler metric.

The sequence (25) gives a natural expression for the canonical bundle $\mathcal{K}$ of the complex manifold Z :-

$$\begin{aligned}\mathcal{K} &= \Lambda^{2n+1}(\mathcal{A}^1)\\ &= L^{-2}\otimes\Lambda^{2n}(L^{-1}\otimes q^*\underline{E})\\ &= L^{-2(n+1)}\otimes q^*\Lambda^{2n}\underline{E}\\ &= L^{-2(n+1)}\end{aligned} \qquad \ldots (27)$$

where for the last equality $\Lambda^{2n}\underline{E}$ is trivialized by $\omega_E{}^n$. Equivalently, there is a natural section $\gamma\in H^0(Z,\mathcal{O}(L^{2(n+1)}\otimes\mathcal{K}))$ which may be calculated by the formula

$$\gamma = \beta\wedge((\Lambda^{2n}\delta)(\omega_E{}^n)$$

where $\delta$ is any $C^\infty$ splitting of (25) :-

$$0 \longrightarrow L^{-2} \longrightarrow \mathcal{A}^1 \overset{\delta}{\underset{\longrightarrow}{\longleftarrow}} L^{-1}q^*\underline{E} \longrightarrow 0$$

But (8) §4 gives a natural candidate for $\delta$. Namely in terms of a standard basis $\{h_1,h_2\}$ of $\underline{H}$,

$$\delta((s_2)^{-1}\otimes e) = e(uh_1 + h_2)$$

Hence

$$\begin{aligned}\gamma &= \beta\wedge((\Lambda^{2n}\delta)(\omega_E{}^n))\\ &= ((s_2)^2\otimes\theta)\wedge\left[(uh_1 + h_2)^n\omega_E{}^n\right]\\ &= (s_2)^{2(n+1)}\otimes\left[\theta\wedge(uh_1 + h_2)^n\omega_E{}^n\right]\end{aligned}$$

On the other hand, regarding $\beta\in H^0(Z,\mathcal{O}(L^2\otimes\mathcal{A}^1)$ as a $L^2$-valued 1-form on Z, the expression

$$\beta\wedge(d\beta)^n\in H^0(Z,\mathcal{O}(L^{2(n+1)}\otimes\mathcal{K}))$$

makes sense because it is independent of the local section of $L^2$ used to evaluate $d\beta$. Moreover at $z = [h_2(x)]\in Z$, where we assume for simplicity that $\nabla h_i\big|_x = 0$,

$$\begin{aligned}\beta\wedge(d\beta)^n &= ((s_2)^2\otimes\theta)\wedge(d\theta)^n\\ &= (s_2)^{2(n+1)}du\wedge(\Omega^1_2)^n\\ &= (s_2)^{2(n+1)}du\wedge(ct(h_2)^2)^n\omega_E{}^n \quad \text{by proposition 7.1}\\ &= c^nt^n\gamma\end{aligned}$$

Consequently $\beta\wedge(d\beta)^n$ is nowhere-vanishing provided that $t\neq 0$. In this case $\beta$ is precisely a complex contact structure on Z. For an appropriate definition of the latter, see Boothby [13]. To summarize,

Theorem 8.2 Let M be a quaternionic Kähler manifold with non-zero scalar curvature. Then the twistor space of M admits a complex contact structure.

Examples Let $M = G/N(G_1)$ be a Wolf space with G compact, as described in §7. $N(G_1) = K\,Sp(1)$ where $K \subset Sp(n)$ and G has Lie algebra $\mathfrak{g} = \mathfrak{k} + sp(1) + \mathfrak{m}$ with $[\mathfrak{k} + sp(1), \mathfrak{m}] \subset \mathfrak{m}$. Choose a standard basis $\{I,J,K\}$ of $sp(1)$ and let $U(1) \subset Sp(1)$ denote the subgroup generated by I. Then the twistor space of M is the complex homogeneous space $Z = G/K\,U(1)$ with the obvious fibration $q : Z \longrightarrow M$. I defines an almost complex structure on the tangent space $\underline{T}_o \cong \mathfrak{m}$ to M at the origin. An arbitrary $z \in Z$ then corresponds to the almost complex structure $g_* \circ I \circ g_*^{-1}$ on $\underline{T}_{q(z)}$ where g is any representative of the coset z and $g_* : \underline{T}_o \longrightarrow \underline{T}_{q(z)}$ is the action of g on $\underline{T}$. The holomorphic tangent space to Z at the origin may be identified with $\{J-iK\} + \mathfrak{m}^{1,0}$ where $\mathfrak{m}^{1,0} \subset \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$ is the i-eigenspace of adI. The form $\beta$ defining the contact structure is essentially projection onto the $\{J-iK\}$- summand.

Wolf [38] observed that

$$q : G/K\,U(1) \longrightarrow G/K\,Sp(1)$$

establishes a 1-1 correspondence between the compact simply-connected homogeneous complex contact manifolds and the compact quaternionic Kähler symmetric spaces. He also observed that a similar correspondence holds in the dual non-compact case. Consequently our construction of the twistor space of a quaternionic manifold may be regarded as a simultaneous generalization of the results of Atiyah, Hitchin & Singer [7] and those of Wolf [38].

## 9. Positive Scalar Curvature

From now on, we shall be primarily concerned with a quaternionic Kähler manifold $M$ having positive scalar curvature. Provided we assume that $M$ is complete and connected, the fact that $M$ has positive definite Ricci tensor with eigenvalues bounded away from zero implies that $M$, and so its twistor space $Z$, is compact (Kobayashi & Nomizu [28 ; VIII, theorem 5.8]).

Theorem 9.1 Let $M$ be a quaternionic Kähler manifold with scalar curvature $t > 0$. Then its twistor space $Z$ admits a Kähler-Einstein metric of positive scalar curvature.

Proof First we establish the existence of a Kähler metric. Recall the splittings (26) :-

$$TZ = \mathcal{V} \oplus \mathcal{H}$$
$$T^*Z = \mathcal{F} \oplus q^*\underline{T}^*$$

Now the fibres of $\mathcal{V}$ possess a Riemannian metric induced from the $SU(2)$-invariant Fubini-Study metric on each fibre $\mathbb{C}P^1$ of $Z$. In addition, the pullback $q^*g$ of the Riemannian metric on $M$ determines a Riemannian metric on the fibres of $\mathcal{H}$. Both of these metrics are Hermitian with respect to the complex structure of $TZ$. Choose a standard basis $\{h_1, h_2\}$ of $\underline{H}$ with corresponding coordinate functions $z^1, z^2$, $u = z^1/z^2$, connection 1-forms $\omega_i^j$ and 1-form

$$\theta = du + \omega_2^1 - u^2\omega_1^2 + u(\omega_1^1 - \omega_2^2)$$

Then on $Z \setminus \{z^2 = 0\}$ the above metrics may be described by the forms

$$\Phi(u) = \frac{-i\theta \wedge \bar{\theta}}{(1 + |u|^2)^2} \in \Lambda^2 \mathcal{F}$$

$$\Psi(u) = \frac{i(uh_1 + h_2) \vee (\widehat{uh_1 + h_2})}{1 + |u|^2} \in q^*(S^2\underline{H}) \subset \Lambda^2(q^*\underline{T}^*)$$

For any $k>0$, $k\Phi + \Psi \in \Gamma(\Lambda^{1,1}T^*Z)$ defines a positive definite Hermitian metric on Z. We shall show that k may be chosen so that $d(k\Phi + \Psi) = 0$. $k\Phi + \Psi$ is then the fundamental 2-form of the desired Kähler metric.

Since $\Phi, \Psi$ are actually independent of the choice of basis, it is sufficient to evaluate $d\Phi$, $d\Psi$ at $z = [h_2(x)] \in Z$ (i.e. $u=0$) and as usual we are free to suppose that $\nabla h_i|_x = 0$. At z,

$$\begin{aligned} d\Phi &= -i(d\theta\wedge\bar\theta - \theta\wedge d\bar\theta) \\ &= -i(\Omega_2^1\wedge d\bar z - dz\wedge\Omega_1^2) \\ &= -ict((h_2)^2\wedge d\bar z - dz\wedge(h_1)^2) \end{aligned}$$

by proposition 7.2, whereas again at z,

$$d\Psi = -2i((h_1)^2\wedge dz - d\bar z\wedge(h_2)^2)$$

Hence $d(k\Phi + \Psi) = 0$ iff $k = {}^2/_{ct} > 0$.

Take the Kähler metric on Z determined by the 2-form $\frac{2}{ct}\Phi + \Psi$ and let G denote the corresponding Riemannian metric. By construction, the isomorphism $q_* : \mathcal{H} \longrightarrow \underline{T}$ mapping horizontal vectors on Z to tangent vectors on M is an isometry at each point. In other words $q : (Z,G) \longrightarrow (M,g)$ is a Riemannian submersion, and in order to verify that G is an Einstein metric we can employ the methods of O'Neill [35]. The fibres of $q : Z \longrightarrow M$ are totally geodesic, so there is one tensor A (analogous to the second fundamental form of an immersion) defined on Z which describes the submersion. Essentially $A \in \mathrm{HOM}(\Lambda^2\mathcal{H}, \mathcal{V})$ and measures the obstruction to integrability of the distribution $\mathcal{H}$, but it is customary to describe A as follows. For any horizontal vector $X \in \mathcal{H}_z$, $A_X$ is a skew-symmetric endomorphism of $T_zZ$ which interchanges the subspaces $\mathcal{H}_z$ and $\mathcal{V}_z$. Moreover if $Y \in \mathcal{H}_z$,

$$A_XY = \tfrac{1}{2} \text{ vertical component of } [X,Y]_z$$

where to compute the Lie bracket, X,Y are extended to horizontal vector fields.

The above is standard and contained in [35], but now we define the tensor $A^2$ by

$$A^2(R,S) = \sum_i G(A_{X_i}R, A_{X_i}S)$$

where $R,S \in T_zZ$ and $\{X_i\}$ is an orthonormal basis of $\mathcal{H}_z$. Note that $A^2(R,S)$ is non-zero only if $R,S$ are both horizontal or both vertical. This definition enables us to state

<u>Lemma 9.1</u> Let $q : Z \longrightarrow M$ be a Riemannian submersion with totally geodesic fibre $F$. Then the Ricci tensors of $Z,M,F$ are related by

$$\mathrm{Ric}^Z(X,V) = \sum_i G((\nabla_{X_i}A)_{X_i}X,V)$$

$$\mathrm{Ric}^Z(X,Y) = \mathrm{Ric}^M(q_*X,q_*Y) - 2A^2(X,Y)$$

$$\mathrm{Ric}^Z(V,W) = \mathrm{Ric}^F(V,W) + A^2(V,W)$$

for any $X,Y \in \mathcal{H}$, $V,W \in \mathcal{V}$, where $\{X_i\}$ is an orthonomal basis of $\mathcal{H}$.

<u>Proof</u> All three formulae follow from the 'fundamental equations' of [35] by putting $T \equiv 0$ and using symmetries of $A$ and $\nabla A$.

Returning to the case in which $Z$ is the twistor space of a quaternionic Kähler manifold, we calculate the tensor $A$. Let $X,Y \in \Gamma(\mathcal{H})$. Choose a standard basis $\{h_1,h_2\}$ of $\underline{H}$ with $\nabla h_i|_x = 0$ and with associated 1-form $\theta$, curvature forms $\Omega_i^j$. At any $z \in q^{-1}(x)$,

$$\begin{aligned}
\langle \theta, A_XY\rangle_z &= \tfrac{1}{2}\langle\theta,[X,Y]\rangle \\
&= -\tfrac{1}{2}d\theta(X,Y) + \tfrac{1}{2}X\langle\theta,Y\rangle - \tfrac{1}{2}Y\langle\theta,X\rangle \\
&= -\tfrac{1}{2}\left[\Omega_2^1 - u^2\,\Omega_1^2 + u(\Omega_1^1 - \Omega_2^2)\right](X,Y) \\
&= -\tfrac{1}{2}ct(uh_1 + h_2)^2(X,Y)
\end{aligned}$$

using proposition 7.2. Thus if $\frac{\partial}{\partial u} \in \mathcal{V}$ denotes the dual to $\theta$, we may write

$$A(u) = -ct\ \mathrm{Re}\ (uh_1 + h_2)^2 \otimes \frac{\partial}{\partial u}$$

The above formula was proved only for points on the line $q^{-1}(x)$, but because the right-hand side is independent of the choice of standard basis, it is valid everywhere. For example,

$$A\big|_{u=0} = -ct\left[(h_2)^2\frac{\partial}{\partial u} + (h_1)^2\frac{\partial}{\partial \bar{u}}\right] \quad . \quad . \qquad (28)$$

It is now an easy matter to compute that

$$A^2\big|_{\Lambda^2\mathcal{H}} = 2ctG, \qquad A^2\big|_{\Lambda^2\mathcal{V}} = 8nctG \quad . \quad . \qquad (29)$$

Finally we show that $Ric^Z(X,V) = 0$ for $X \in \Gamma(\mathcal{H})$, $V \in \Gamma(\mathcal{V})$. Fix $z \in Z$, $q(z) = x$. By the lemma, it suffices to prove that $G((\nabla_Y A)_Y X, V) = 0$ for $Y \in \mathcal{H}_z$. Let $\{h_1, h_2\}$ be a standard basis of $\underline{H}$ with $\nabla h_i\big|_x = 0$. Using without comment the methods of 35 ,

$$q^*\Lambda^2\underline{T}^* \text{ component of } \nabla_Y(h_i^{\,2}) = 0$$

$$\mathcal{V} \text{ component of } \nabla_Y(\tfrac{\partial}{\partial u}) = 0$$

Because the section $\{u = 0\}$ of Z over M is horizontal at z, using (28) we can conclude that $G((\nabla_Y A)_Y X, V) = 0$.

From lemma 9.2 & (29), the Ricci form of Z is $a\Phi + b\Psi$ where a,b are constants, $b > 0$. But the Ricci form of a Kähler manifold is closed, so $d(a\Phi + b\Psi) = 0$ and from the first part of the proof, this forces the Ricci form to be a positive multiple of the fundamental 2-form. Therefore G is a Kähler-Einstein metric of positive scalar curvature.

Because the first Chern class of a Kähler manifold has deRham representative ($^1/_{2\pi}$ times) the Ricci form, theorem 9.1 implies that $c_1(Z)$ is a positive element. On the other hand, from §7 we can express the Chern classes of Z in terms of $\ell$ and $H^*(M;\mathbb{R})$. In particular from (27),

$$c_1(Z) = -c_1(\mathcal{K}) = -c_1(L^{-2(n+1)}) = 2(n+1)\ell$$

Hence

$$\ell \text{ is a positive element of } H^{1,1}(Z;\mathbb{R}) \qquad \ldots (30)$$

If M has $\xi = 0$, then $\ell \in H^2(Z,\mathbb{Z})$ and a result of Kobayashi & Ochiai [29; theorem 2.2] implies that Z is biholomorphically equivalent to $\mathbb{C}P^{2n+1}$. The following result then comes as no surprise.

<u>Theorem 9.2</u> Let M be a quaternionic Kähler manifold with $\xi = 0$ and positive scalar curvature. Then M is isometric to $\mathbb{H}P^n$ with its standard metric.

<u>Proof</u> We shall actually prove this directly, making use of our theorem 6.2. If F is a holomorphic vector bundle over Z we use the notation $h^p(F) = \dim H^p(Z,\mathcal{O}(F))$; also $h^p = \dim H^p(Z,\mathcal{O})$. First we calculate the Euler characteristic

$$P(r) = \chi(Z,\mathcal{O}(L^r)) = \sum_{p=0}^{2n+1} (-1)^p h^p(L^r)$$

By the Riemann-Roch theorem (see Hirzebruch [20]),

$$P(r) = \varkappa[\mathrm{ch}(L^r)\,\mathrm{td}(Z)] = \varkappa[e^{r\ell}\mathrm{td}(Z)]$$

is a polynomial of degree $\leqslant 2n+1$ in r. Here td(Z) denotes the Todd class of the holomorphic tangent bundle of Z. If $r<0$, $L^r$ is a negative line bundle by (30) and Kodaira's vanishing theorem gives

$$h^s(L^r) = 0 \text{ if } r<0 \text{ and } s \leqslant 2n.$$

Moreover by Serre duality,

$$\begin{aligned} h^{2n+1}(L^r) &= h^0(L^{-r}\otimes \kappa) \\ &= h^0(L^{-r-2(n+1)}) \quad \text{by (27)} \\ &= 0 \quad \text{if } r \geqslant -(2n+1) \end{aligned}$$

Thus

$$P(r) = 0 \quad \text{if } r = -1,-2,\ldots,-(2n+1)$$

Also

$$\begin{aligned} h^s &= h^{2n+1-s}(L^{-2(n+1)}) \\ &= 0 \quad \text{if } s \geqslant 1 \end{aligned}$$

and the Todd genus of Z,

$$\varkappa[td(Z)] = P(0) = 1$$

We now know the values of P(r) at 2n+2 distinct points and are able to conclude that

$$P(r) = \binom{r+2n+1}{r}, \quad r \geqslant 0 \quad \quad .. \quad (32)$$

Lemma 9.2 Let M be a compact quaternionic Kähler manifold with scalar curvature $t \neq 0$. Then the space of infinitesimal isometries on M is isomorphic to the sheaf cohomology group $H^0(Z, \mathcal{O}(L^2))$.

Proof By theorem 6.2, $H^0(Z, \mathcal{O}(L^2))$ is isomorphic to the kernel $\mathscr{S}$ of the differential operator

$$D : S^2\underline{H} \longrightarrow \underline{E} \otimes S^3\underline{H}$$

defined by the quaternionic structure. The symbol of D is the symmetrization mapping

$$S^2H \otimes EH \longrightarrow E \otimes S^3H$$

Let $\delta$ be the differential operator acting on $S^2\underline{H}$ with symbol given by the contraction

$$S^2H \otimes EH \hookrightarrow \underbrace{H \otimes H}_{\omega_H} \otimes EH \longrightarrow EH = T^*$$

(If one regards $S^2\underline{H}$ as a subbundle of $\Lambda^2\underline{T}^*$, then $\delta$ is simply the adjoint of exterior differentiation.) We identify the space of infinitesimal isometries on M with the space $\mathcal{K}$ of 1-forms $\eta$ satisfying $\nabla\eta \in \Gamma(\Lambda^2\underline{T}^*)$. If $\sigma \in \mathscr{S}$, then $\delta\sigma \in \Gamma(\underline{T}^*)$ and we shall prove that $\sigma \longmapsto \delta\sigma$ gives an isomorphism $\mathscr{S} \cong \mathcal{K}$. Note that this will be an isomorphism of real vector spaces since $\mathscr{S}$ and $\mathcal{K}$ have obvious real structures.

Let $\sigma \in \mathscr{S}$. $\nabla\nabla(\sigma) \in \Gamma(S^2\underline{H} \otimes \underline{T}^* \otimes \underline{T}^*)$ and the skew part $\partial\nabla\nabla(\sigma) \in \Gamma(S^2\underline{H})$ is essentially $\mathcal{R}_H(\sigma)$ where $\mathcal{R}_H$ is the curvature of

$\underline{H}$. We claim that the composition

$$S^2H \otimes \Lambda^2T^* \hookrightarrow H \otimes \underbrace{H \otimes (EH}_{\omega_H} \otimes EH) \longrightarrow EH \otimes EH$$

of the obvious inclusion with the indicated contraction sends $S^2H \otimes \Lambda^2T^*$ onto the irreducible $GL(n,\mathbb{H})Sp(1)$ modules

$$\Lambda^2ES^2H, \quad \Lambda^2E, \quad S^2ES^2H \qquad \ldots (33)$$

of $EH \otimes EH$ (i.e. all but $S^2E$). By Schur's lemma, it is enough to produce elements of $S^2H \otimes \Lambda^2T^*$ mapping onto non-zero elements in each of these spaces, and it is readily checked that

$$\left.\begin{array}{r} h_1{}^2 \otimes e_1{}^2 \in S^2H \otimes S^2E \\ h_1{}^2 \otimes (e_1 \wedge e_2 . h_2{}^2) \in S^2H \otimes \Lambda^2ES^2H \end{array}\right\} \subset S^2H \otimes \Lambda^2T^*$$

(standard bases) do the job. This observation implies that $\mathcal{R}_H(\sigma)$ determines the components (33) of $\nabla(\delta\sigma)$. From proposition 7.2, $\mathcal{R}_H$ is trivial as an $Sp(n)Sp(1)$ module, so reverting to the action of $Sp(n)Sp(1)$, $\mathcal{R}_H(\sigma)$ is a section of $S^2\underline{H} \subset \Lambda^2\underline{E}S^2\underline{H}$, Therefore $\nabla(\delta\sigma) \in \Gamma(S^2\underline{H} + S^2\underline{E}) = \Gamma(\underline{a})$, and $\delta\sigma \in \mathcal{K}$.

Now suppose that $\sigma \in \mathcal{K}$, $\delta\sigma \equiv 0$. From above, the $S^2H$-component of $\nabla(\delta\sigma)$ is $\mathcal{R}_H(\sigma) = t\sigma$ (up to a universal constant). Therefore $\sigma \equiv 0$ and $\delta : \mathcal{S} \longrightarrow \mathcal{K}$ is injective. Because $\mathcal{S}, \mathcal{K}$ are finite-dimensional, $\mathcal{S} \cong \mathcal{K}$.

Returning to the proof of the theorem,

$$h^2(L^2) = h^{2n+1-s}(L^{-2} \otimes L^{-2(n+1)})$$
$$= 0 \quad \text{if } s \geqslant 1.$$

Thus by (32),

$$h^0(L^2) = \chi(Z, \mathcal{O}(L^2)) = P(0) = (n+1)(2n+3)$$

From the lemma, M has an isometry group of dimension $(n+1)(2n+3)$. The linear isotropy subgroup $G_0$ at each point then has dimension $\geqslant (n+1)(2n+3) - 4n = n(2n+1) + 3$. But $G_0$ must be contained in the holonomy group, so counting dimensions $G_0 = Sp(n)Sp(1)$ and $M$ is homogeneous. Now a Riemannian homogeneous space whose isotropy and holonomy groups coincide is locally symmetric.

Finally, the twistor space $Z$, and so $M$ itself, is simply-connected. This follows from the fact that $c_1(Z) > 0$ and a theorem of Kobayashi (see [28; note 23]). Therefore $M$ is a Riemannian symmetric space with holonomy equal to $Sp(n)Sp(1)$. Wolf's classification [38] implies that $M$ is isometric to $\mathbb{H}P^n$.

Theorem 9.2 means that the vanishing of $\xi$ characterizes $\mathbb{H}P^n$ among the quaternionic Kähler manifolds of positive scalar curvature. Even if $\xi \neq 0$, we can again use the holomorphic structure of the twistor space to gain information about $M$.

Theorem 9.3 Let $M$ be a quaternionic Kähler manifold with positive scalar curvature. Then the Kähler manifold $Z$ is simply-connected and has only $(p,p)$ cohomology.

Proof The fact that $\pi_1(Z) = 0$ was proved above. The second statement will follow after we have proved by induction on $q$ the assertion that for $0 \leqslant q \leqslant n$,

$$h^p(L^{-2r} \otimes L^{-q} \Lambda^q \underline{E}) = \begin{cases} 0 \quad \forall p & \text{if } 1 \leqslant r \leqslant n-q \\ h^{p,q} = 0 & \text{if } r = 0 \text{ and } p \neq q \\ h^{p,p} & \text{if } r = 0 \text{ and } p = q \end{cases}$$

Here $\underline{E}$ is an abbreviation for the holomorphic vector bundle $q^*E$ over $Z$, and $h^{p,q} = \dim H^q(Z, \mathcal{O}(\mathcal{A}^p))$ where $\mathcal{A}^p$ is the bundle of holomorphic q-forms ($h^{p,q} = h^{q,p}$ since $Z$ is Kähler).

First, by Kodaira's vanishing theorem,

$$h^p(L^{-2r}) = 0 \quad \forall p \quad \text{if } 1 \leqslant r \leqslant n$$

and using Serre duality and (27), §8,

$$h^{p,0} = h^p(\mathcal{O}) = h^{2n+1-p}(L^{-2(n+1)}) = \begin{cases} 0 & \text{if } p \geqslant 1 \\ 1 & \text{if } p = 0 \end{cases}$$

Therefore our assertion is valid when $q = 0$. As inductive hypothesis suppose that the assertion is valid when $q \leqslant Q-1$. The $Q^{th}$ exterior power of (25) gives a holomorphic short exact sequence

$$0 \longrightarrow L^{-Q-1}\Lambda^{Q-1}\underline{E} \longrightarrow \mathcal{A}^Q \longrightarrow L^{-Q}\Lambda^Q\underline{E} \longrightarrow 0$$

Tensoring with $L^{-2r}$,

$$0 \longrightarrow L^{-2(r+1)} \otimes L^{-(Q-1)}\Lambda^{Q-1}\underline{E} \longrightarrow L^{-2r} \otimes \mathcal{A}^Q \longrightarrow L^{-2r} \otimes L^{-Q}\Lambda^Q\underline{E} \longrightarrow 0$$

By hypothesis

$$h^p(L^{-2(r+1)} \otimes L^{-(Q-1)}\Lambda^{Q-1}\underline{E}) = \begin{cases} 0 \ \forall p & \text{if } 0 \leqslant r \leqslant n-Q \\ h^{2n+1-p}(L^{-(Q-1)}\Lambda^{Q-1}\underline{E}) = 0 & \\ & \text{if } r = n-Q+1,\ p \neq 2n-Q+2 \end{cases}$$

so that from the associated long exact sequence of cohomology groups,

$$h^p(L^{-2r} \otimes L^{-Q}\Lambda^Q\underline{E}) = h^p(L^{-2r} \otimes \mathcal{A}^Q) = h^{Q,p}(L^{-2r}) \begin{cases} \forall p & \text{if } 0 \leqslant r \leqslant n-Q \\ p \leqslant 2n-Q+1, & r = n-Q+1 \end{cases} \quad (34$$

But if $p+Q \leqslant 2n$ and $r \geqslant 1$, by the Akizuki-Nakano vanishing theorem [2], $h^{Q,p}(L^{-2r}) = 0$. Moreover by hypothesis, $h^{Q,p} = 0$ for $p \leqslant Q-1$. Therefore

$$h^p(L^{-2r} \otimes L^{-Q}\Lambda^Q\underline{E}) = 0 \quad \begin{cases} p \leqslant 2n-Q, & 1 \leqslant r \leqslant n-Q+1 \\ p \quad Q-1, & r = 0 \end{cases} \quad \ldots \quad (35$$

On the other hand,

$$\begin{aligned} h^p(L^{-2r} \otimes L^{-Q}\Lambda^Q\underline{E}) &= h^{2n+1-p}(L^{2r+Q}\Lambda^Q\underline{E} \otimes L^{-2(n+1)}) \\ &= h^{2n+1-p}(L^{-2(n-Q+1-r)} \otimes L^{-Q}\Lambda^Q\underline{E}) \\ &= 0 \quad \text{if } p \geqslant Q+1, \quad 0 \leqslant r \leqslant n-Q \end{aligned} \quad \ldots \quad (36$$

the last line by (35). By assumption $0 \leqslant Q \leqslant n$, so that $Q+1 \leqslant 2n-Q+1$ and (34), (35), (36) imply that our original assertion is valid for $q = Q$.

As a corollary of the above proof, we have

Proposition 9.1 Let $0 \leq q \leq n$. With the same hypotheses as in theorem 9.3,

$$\chi(Z, \mathcal{O}(L^{-2r} \otimes L^{-q} \Lambda^q \underline{E}) = \begin{cases} 0 & \text{if } 1 \leq r \leq n-q \\ (-1)^q h^{q,q} & \text{if } r = 0 \end{cases}$$

Theorem 9.3 has the following immediate consequences for the manifold M itself.

Theorem 9.4 Let M be a quaternionic Kähler manifold with positive scalar curvature. Then M is simply-connected and has odd Betti numbers zero.

The statement that $\pi_1(M) = 0$ generalizes a result of Alekseevskii [4; assertion 6].

## 10. Low Dimensions

We now specialize to the case in which M is an 8-dimensional quaternionic Kähler manifold with scalar curvature $t > 0$. Let $\chi$, $p_1, p_2$ denote the Euler and Pontrjagin classes of M, and $\tau$ the signature. By evaluation on the fundamental cycle we identify $H^8(M;\mathbb{R})$ with $\mathbb{R}$.

Proposition 10.1 $\chi = 3\tau$

Proof The proposition follows immediately from the formulae below.

$$\frac{1}{45}(7p_2 - p_1{}^2) = \tau \qquad \text{. . (37)}$$

$$7p_1{}^2 - 4p_2 = 0 \qquad \text{. . (38)}$$

$$4p_2 - p_1{}^2 = 8\chi \qquad \text{. . (39)}$$

(37) is the usual formula for the signature in terms of the L-genus (see e.g. Hirzebruch [20]).

The left-hand side of (38) equals $45.2^7.\hat{A}_2(M)$. By proposition 1.1, M is a spin manifold, so applying the Atiyah-Singer index theorem to the Dirac operator, the $\hat{A}$-genus $\hat{A}_2(M)$ equals the dimension of the space of harmonic spinors. By the Lichnerowicz vanishing theorem [33], this is zero because $t > 0$.

Finally we prove (39). As usual let Z denote the twistor space of M, $q : Z \longrightarrow M$. From (7) §4, the pullback $q^*(\underline{T})_r$ of the tangent bundle of M is isomorphic to the real vector bundle underlying $Lq^*\underline{E}$. Thus if $c_1, c_2, c_3, c_4$ are the Chern classes of $Lq^*\underline{E}$, then $c_4 = q^*\chi$. Because $q^* : H^*(M;\mathbb{R}) \longrightarrow H^*(Z,\mathbb{R})$ is injective, we shall omit $q^*$ and write $c_4 = \chi$.

Now $\underline{T} = \underline{E} \otimes \underline{H}$, so

$$(Lq^*\underline{E}) \otimes q^*\underline{H} = L \otimes q^*\underline{T}$$

From (11) §6,

$$\mathrm{ch}(q^*\underline{H}) = \mathrm{ch}(L) + \mathrm{ch}(L^{-1})$$
$$= e^{\ell} + e^{-\ell}$$

Hence

$$\mathrm{ch}(Lq^*\underline{E}) = (1 + e^{-2\ell})^{-1}\mathrm{ch}(q^*\underline{T})$$
$$= \tfrac{1}{2}(1+\ell-\tfrac{1}{3}\ell^3+\tfrac{2}{15}\ell^5)(8 + p_1 + \tfrac{1}{12}(p_1^{\,2} - 2p_2))$$

which implies that

$$c_1 = 4\ell \qquad \ldots (40)$$
$$c_1^{\,2} - 2c_2 = p_1 \qquad \ldots (41)$$
$$c_1^{\,3} - 3c_1c_2 + 3c_3 = -8\ell^3 + 3p_1\ell \qquad \ldots (42)$$
$$c_1^{\,4} - 4c_1^{\,2}c_2 + 2c_2^{\,2} + 4c_1c_3 - 4c_4 = p_1^{\,2} - 2p_2 \qquad \ldots (43)$$

(40),(42) give

$$c_1^{\,4} - 3c_1^{\,2}c_2 + 3c_1c_3 = -32\ell^4 + 12p_1\ell^2$$

Using (43),

$$c1^4 - 6c_2^{\,2} + 12c_4 = -128\ell^4 + 48p_1\ell^2 - 3p_1^{\,2} + 6p_2$$

Using (40),(41),

$$c_4 = \tfrac{1}{8}(4p_2 - p_1^{\,2})$$

which implies (39).

Remark The above proof of formula (39) will hold for any 8-dimensional Riemannian manifold with structure group reducing from SO(8) to Sp(2)Sp(1), for in general it is still possible to construct the manifold Z which will have a natural almost complex structure. In fact the same formula holds for any 8-dimensional Riemannian manifold with structure group reducing to Spin7, SU(4) or Sp(2). The case of SU(4) or Sp(2) is easy because $\chi, p_1, p_2$ are then given in terms of the Chern classes of M by $\chi = c_2$, $p_1 = -2c_2$, $p_2 = c_2^{\,2}$ ($c_1(M) = 0$) and (39) is immediate.

For Spin7 see Gray [18].

<u>Proposition 10.2</u> The spinor bundles of M are given by

$$\underline{V}_+ = \Lambda^2_0\underline{E} + S^2H$$

$$\underline{V}_- = \underline{E} \otimes \underline{H} = \underline{T}$$

In particular, tangent vectors can be thought of as spinors.

<u>Proof</u> Both the inclusions $SU(4) \subset SO(8)$, $Sp(2)Sp(1) \subset SO(8)$ lift to spin, the former because $SU(4)$ is simply-connected and the latter by proposition 1.1. Thus there is a commutative diagram

$$\begin{tikzcd}
 & SU(2) \arrow[dr, hook] & \\
Sp(2) \arrow[ur, hook] \arrow[dr, hook] & & \text{Spin } 8 \\
 & Sp(2)Sp(1) \arrow[ur, hook] &
\end{tikzcd}$$

It is known (see e.g. Hitchin [21]) that the spin representations $V_+, V_-$ decompose under the action of $SU(4)$ as

$$V_+ = \sum_{q \text{ even}} \Lambda^{0,q} = \Lambda^{0,0} + \Lambda^{0,2} + \Lambda^{0,4}$$

$$V_- = \sum_{q \text{ odd}} \Lambda^{0,q} = \Lambda^{0,1} + \Lambda^{0,3}$$

where $\Lambda^{0,q}$ is the representation of $SU(4)$ corresponding to the space of $(0,q)$-forms on a manifold with an $SU(4)$ structure. The representations $\Lambda^{0,0}$, $\Lambda^{0,4}$ are both trivial. If we restrict to $Sp(2)$ then the space $\Lambda^{0,1}$ is precisely what we have called E, so in that case,

$$V_+ = A + \Lambda^2 E + A = \Lambda^2_0 E + 3A$$

$$V_- = E + E = 2E$$

However the decomposition of $V_+, V_-$ under the action of $Sp(2)Sp(1)$ must yield the above upon restriction to $Sp(2)$. This restriction corresponds to replacing the representation H by the trivial representation 2A, and because Clifford multiplication implies that $V_+ \subset V_- \otimes EH$, $V_- \subset V_+ \otimes EH$, the only possibility is

$$V_+ = \Lambda_0^2 E + S^2 H$$

$$V_- = E \otimes H$$

as required.

In general on a spin manifold, tensoring the Dirac complex

$$0 \longrightarrow V_+ \longrightarrow V_- \longrightarrow 0$$

with $V_+$ give half the deRham complex. In our case the Dirac complex looks like

$$0 \longrightarrow \Lambda_0^2 \underline{E} + S^2 \underline{H} \longrightarrow \underline{EH} \longrightarrow 0$$

or alternatively as a 3-step complex

$$0 \longrightarrow \Lambda_0^2 \underline{E} \longrightarrow \Lambda^3 \underline{EH} \longrightarrow \Lambda^4 \underline{E} S^2 \underline{H} \longrightarrow 0$$

since $\Lambda^3 E = E$, $\Lambda^4 E = A$. If we tensor this not with $V_+$, but simply with the vector bundle $S^2 \underline{H}$, we obtain the complex

$$0 \longrightarrow \underline{A} \longrightarrow \underline{EH} \longrightarrow \Lambda^2 \underline{E} S^2 \underline{H} \longrightarrow \Lambda^3 \underline{E} S^3 \underline{H} \longrightarrow \Lambda^4 \underline{E} S^4 \underline{H} \longrightarrow 0$$

where each operator is exterior differentiation followed by the appropriate projection. More generally, tensoring the Dirac complex with $S^{r+2}\underline{H}$ gives the complex

$$0 \longrightarrow S^r \underline{H} \longrightarrow \underline{E} S^{r+1} \underline{H} \longrightarrow \Lambda^2 \underline{E} S^{r+2} \underline{H} \longrightarrow \Lambda^3 \underline{E} S^{r+3} \underline{H} \longrightarrow \Lambda^4 \underline{E} S^{r+4} \underline{H} \longrightarrow 0$$

of proposition 3.3. If F is a quaternionic vector bundle over M we can obtain additional complexes by tensoring with F. Theorem 6.2 then yields

Proposition 10.3 The Dolbeault $\bar{\partial}$-complex $FL^r \otimes \mathcal{A}^{0,*}$ on Z and the complex $FS^{r+2}\underline{H} \otimes \underline{V}_*$ on M have isomorphic cohomology.

We are now in a position to state

<u>Theorem 10.1</u> An 8-dimensional quaternionic Kähler manifold of positive scalar curvature has Betti numbers

$$b_1 = 0, \quad b_2, \quad b_3 = 0, \quad b_4^+ = 1 + b_2, \quad b_4^- = 0$$

<u>Proof</u> By theorem 9.4, $b_1 = b_3 = 0$. Moreover

$$h^{1,1} = \dim H^2(Z;\mathbb{R}) = 1 + b_2$$

But using propositions 9.1, 10.3, and the Atiyah-Singer index theorem,

$$\begin{aligned} h^{1,1} &= -\chi(Z, \mathcal{O}(L^{-1}E)) \\ &= -\chi(M, \underline{H}\,\underline{E} \otimes \underline{V}_*) \\ &= \varkappa_M[\mathrm{ch}(\underline{H}\,\underline{E})\hat{A}(M)] \end{aligned}$$

where $\hat{A}(M)$ is the $\hat{A}$-class of $M$. But $\mathrm{ch}(\underline{H}\,\underline{E}) = \mathrm{ch}(\underline{T})$ $= 8 + p_1 + \frac{1}{12}(p_1{}^2 - 2p_2)$, so

$$\begin{aligned} 1 + b_2 &= \varkappa_M\left[(8 + p_1 + \tfrac{1}{12}(p_1{}^2 - 2p_2)(1 - \tfrac{1}{24}p_1)\right] \\ &= \tfrac{1}{24}p_1{}^2 - \tfrac{1}{6}p_2 \\ &= -\tfrac{1}{3}\chi \end{aligned}$$

by (39). On the other hand by proposition 10.1,

$$\tfrac{1}{3}\chi = \tfrac{1}{2}(\chi - \tau) = 1 + b_2 + b_4^-$$

Therefore $b_4^- = 0$ and $\chi = 3 + 3b_2$, completing the proof.

<u>Examples</u> 1. There are three 8-dimensional Wolf spaces with $t > 0$, namely

$$\mathbb{H}P^n, \quad \frac{SU(4)}{S(U(2) \times U(2))}, \quad \frac{G_2}{SO(4)}$$

Computing $\pi_2(M)$ as in §7, we see that these spaces have second Betti number $b_2 = 0,1,0$ respectively. It is interesting to check independently that $b_2 = 0$ for $G_2/SO(4)$ by considering the action of the holonomy on the space of 2-forms

$$\Lambda^2 T^* = S^2E + S^2H + \Lambda_0^2 E S^2 H$$

Any harmonic 2-form is invariant under the linear holonomy group which has the form KSp(1) where K = Sp(1). If V denotes the standard representation of K on $\mathbb{C}^2$, then under

$$K \hookrightarrow Sp(2) \longrightarrow \mathrm{Aut}E$$

E must decompose into a sum of spaces $S^pV$. From proposition 10.2, there must be no trivial summands in $\Lambda_0^2E$, because there are no harmonic spinors. This forces $E = S^3V$ with $\Lambda_0^2E = S^4V$. Therefore

$$S^2E = S^6V + S^2V$$

has no invariants and $b_2 = 0$.

2. The manifold $\widehat{\mathbb{H}P}^2$, the quaternionic projective plane with a point blown up, cannot admit a quaternionic Kähler metric of positive scalar curvature. In our language $\mathbb{H}P^2$ may be described as the quaternionic projective line bundle $P_{\mathbb{H}}(\underline{H} \oplus 1)$, and has $b_2 = 0$, but $b_4 = 2$.

<u>Theorem 10.2</u> An 8-dimensional quaternionic Kähler manifold of positive scalar curvature admits an isometry group of dimension at least 6.

<u>Proof</u> By lemma 9.2, it suffices to show that

$$h^0(L^2) = \dim H^0(Z, \mathcal{O}(L^2)) \geqslant 6$$

Firstly, using propositions 9.1, 10.3,

$$\begin{aligned} 1 &= \chi(Z,\mathcal{O}) \\ &= \chi(M, S^2\underline{H} \otimes \underline{V}_*) \\ &= \varkappa_M\left[ch(S^2\underline{H})\hat{A}(M)\right] \end{aligned}$$

Now $ch(S^2\underline{H}) = (ch\underline{H})^2 - 1 = (2+w+\frac{1}{12}w^2)-1 = 3+4w+\frac{4}{3}w^2$. So

$$\begin{aligned} 1 &= \varkappa_M\left[(3+4w+\tfrac{4}{3}w^2)(1-\tfrac{1}{24}p_1)\right] \\ &= \tfrac{4}{3}w^2 - \tfrac{1}{6}p_1 w \end{aligned} \qquad \ldots (44)$$

On the other hand from the proof of theorem 9.2,

$$\begin{aligned} h^0(L^2) &= \chi(Z,\mathcal{O}(L^2)) \\ &= \chi(M, S^4\underline{H}\otimes\underline{V}_*) \\ &= \chi_M\left[ch(S^4\underline{H})\hat{A}(M)\right] \end{aligned}$$

Now $ch(S^4\underline{H}) = (ch(S^2\underline{H}))^2 - ch(S^2\underline{H}) - 1 = 5 + 20w + \frac{68}{3}w^2$. So

$$\begin{aligned} h^0(L^2) &= \chi_M\left[(5+20w+\tfrac{68}{3}w^2)(1-\tfrac{1}{24}p_1)\right] \\ &= \tfrac{68}{3}w^2 - \tfrac{5}{6}p_1 w \end{aligned}$$

Combined with (22) §7, and (44),

$$h^0(L^2) = 5 + 16w^2 \geqslant 6$$

<u>Remarks</u> 1. The essential ingredients of theorems 10.1, 10.2 were vanishing theorems for certain sheaf cohomology groups on Z. Proposition 10.3 translates these into Bochner-type vanishing theorems on M. For example, the assumption of positive scalar curvature implies that

$$\begin{aligned} \dim H^p(Z,\mathcal{O}(L^r)) &= h^p(L^r) \\ &= h^{2n+1-p}(L^{-r-2n-1}) \\ &= 0 \quad \text{if } p \geqslant 1,\ r \geqslant 0. \end{aligned}$$

Hence $H^p(S^{r+2}\underline{H}\otimes\underline{V}_*) = 0$ unless $p = 0$. This is a purely differential-geometric statement and is easy to prove independently by arguments involving curvature. The crucial point then is that with hindsight, theorems 10.1, 10.2 could be proved without reference to the twistor space Z.

2. Proposition 10.3 generalizes to the case in which M has arbitrary dimension 4n. In particular the complex $L^r\otimes\mathcal{A}^{0,*}$ on Z and the complex $S^{r+n}\underline{H}\otimes\underline{V}_*$ on M have isomorphic cohomology, where $\underline{V}_*$ again denotes the Dirac complex on M. We omit a proof which would involve decomposing the spin representations under Sp(n)Sp(1).

However it is easy to check the following consequence directly.

Proposition 10.4 Let M be a compact quaternionic Kähler manifold with twistor space Z. For even r,

$$\chi(Z,\mathcal{O}(L^r)) = \chi(M, S^{r+n}\underline{H}\otimes\underline{V}_*)$$

Proof By the Riemannian-Roch theorem,

$$\chi(Z,\mathcal{O}(L^r)) = \varkappa_Z\left[\mathrm{ch}(L^r)\,\mathrm{td}(Z)\right]$$

where td(Z) is the Todd class of the holomorphic tangent bundle of Z. By (25), §8 the latter is isomorphic to $L^2\oplus(p^*\underline{T})^{1,0}$ as a $GL(2n+1,\mathbb{C})$-bundle, so

$$\begin{aligned}\mathrm{td}(Z) &= \mathrm{td}(L^2)\,\mathrm{td}((p^*\underline{T})^{1,0})\\ &= \mathrm{td}(L^2)\; e^{n\ell}\;\hat{A}((p^*\underline{T})_r)\\ &= \frac{2\ell}{1-e^{-2\ell}}\;\hat{A}(M)\end{aligned}$$

(see Hirzebruch [20] for details). Thus

$$\begin{aligned}\chi(Z,\mathcal{O}(L^r)) &= \varkappa_Z\left[\frac{2\ell e^{(n+r)\ell}}{1-e^{-2\ell}}\hat{A}(M)\right]\\ &= \varkappa_M\left[\frac{2e^{(n+r)\ell}}{1-e^{-2\ell}}\hat{A}(M)\right]\end{aligned}$$

On the other hand,

$$\chi(M,S^{r+n}\underline{H}\otimes\underline{V}_*) = \varkappa_M\left[\mathrm{ch}(S^{r+n}\underline{H})\hat{A}(M)\right]$$

To complete the proof we must show that on M, ignoring odd powers of $\ell$,

$$\frac{2e^{(n+r)\ell}}{1-e^{-2\ell}} = \mathrm{ch}(S^{r+n}\underline{H})$$

or equivalently,

$$2e^{(n+r+1)\ell} = (e^{\ell}+e^{-\ell})\,\mathrm{ch}(S^{r+n}\underline{H})$$

But this follows from the formula

$$\mathrm{ch}(S^k\underline{H}) = e^{k\ell}+e^{(k-2)\ell} + e^{(k-4)\ell} + \ldots + e^{-(k-2)\ell} + e^{-k\ell} \quad \ldots \quad (45)$$

The methods of theorems 10.1, 10.2 are of little use in arbitrary dimensions because the characteristic numbers of M provide too many unknowns. However our last result does extend theorem 10.2 to the case of 12 and 16 dimensions.

Theorem 10.3 Let M be a quaternionic Kähler manifold of positive scalar curvature and dimension 12 or 16. Then M admits an isometry group of dimension at least 5 or 8 respectively.

Proof First suppose M is 12-dimensional. Using (45) with $w = -c_2(\underline{H}) = \ell^2$,

$$\mathrm{ch}\,\underline{H} = 2 + w + \frac{1}{12}w^2 + \frac{1}{360}w^3$$

$$\mathrm{ch}S^2\underline{H} = 3 + 4w + \frac{4}{3}w^2 + \frac{8}{45}w^3$$

$$\mathrm{ch}S^3\underline{H} = 4 + 10w + \frac{41}{6}w^2 + \frac{73}{36}w^3$$

$$\mathrm{ch}S^4\underline{H} = 5 + 20w + \frac{68}{3}w^2 + \frac{104}{9}w^3$$

$$\mathrm{ch}S^5\underline{H} = 6 + 35w + \frac{707}{12}w^2 + \frac{3271}{72}w^3$$

By proposition 10.4,

$$P(r) = \chi(Z,\mathcal{O}(L^r)) = \chi_M\left[\mathrm{ch}(S^{r+n}\underline{H})\hat{A}(M)\right]$$

where $\hat{A}(M) = 1 + \hat{A}_1 + \hat{A}_2 + \hat{A}_3$. Using proposition 9.1,

$$0 = P(-2) = \frac{1}{360}w^3 + \frac{1}{12}\hat{A}_1w^2 + \hat{A}_2w + \hat{A}_3$$

$$1 = P(0) = \frac{73}{36}w^3 + \frac{41}{6}\hat{A}_1w^2 + 10\hat{A}_2w + 4\hat{A}_3$$

$$I = P(2) = \frac{3271}{72}w^3 + \frac{707}{12}\hat{A}_1w^3 + 35\hat{A}_2w + 6\hat{A}_3$$

where I is the dimension of the space of infinitesimal isometries.

Eliminating $\hat{A}_2w$, $\hat{A}_3$ and using $\hat{A}_1 = -\frac{1}{24}p_1$,

$$I = 4 - \tfrac{4}{3}p_1w^2 + \tfrac{112}{3}w^3 \qquad \ldots \quad (46)$$

Lemma 10.1 On a compact quaternionic Kähler manifold of dimension 4n,

$$2(n-1)w^n - p_1w^{n-1} \geqslant 0.$$

Proof There exist constants a,b such that

$$p_1w^{n-1} = \int_M (at^2 + b\|R_U\|^2)t^{2(n-1)}\,\Theta^n$$

because from §7 quadratic Sp(n)Sp(1)-invariants of the curvature are generated by $t^2, \|R_U\|^2$. The signs of a and b are determined by the orientation of

$$\sigma \wedge \sigma \wedge \Theta^{n-1}$$

for $\sigma \in S^2\underline{H}$, $S^2\underline{E} \subset \Lambda^2\underline{T}^*$ respectively. In fact $b<0$, so

$$a'w^n - p_1w^{n-1} \geqslant 0$$

for some universal constant $a'$. For $\mathbb{H}P^n$, $w = -c_2(\underline{H})$ is the standard generator of the cohomology so $w^n = 1$. But it is known (e.g. Borel & Hirzebruch [14]) that $p_1 = 2(n-1)w$. Therefore $a' = 2(n-1)$ as required.

Combining the lemma with (46),

$$I \geqslant 4 + 32w^2 \geqslant 5$$

We can repeat the above procedure in 16 dimensions because M is the spin forcing $\hat{A}_4 = 0$. Proposition 9.1 yields

$$(\,0 = P(-4) = \hat{A}_4\,)$$

$$0 = P(-2) = \frac{71}{5040}w^4 + \frac{8}{45}\hat{A}_1w^3 + \frac{4}{3}\hat{A}_2w^2 + 4\hat{A}_3w$$

$$1 = P(0) = \frac{16483}{5040}w^4 + \frac{104}{9}\hat{A}_1w^3 + \frac{68}{3}\hat{A}_2w + 20\hat{A}_3w$$

$$I = P(2) = \frac{6235}{72}w^4 + \frac{6352}{45}\hat{A}_1w^3 + \frac{392}{3}\hat{A}_2w^2 + 56\hat{A}_3w$$

Eliminating $\hat{A}_2 w^2$, $\hat{A}_3 w$,

$$I = 7 - \tfrac{8}{3}p_1 w^3 + 64w^4$$

and using lemma 10.1,

$$I \geqslant 7 + 48w^4 \geqslant 8.$$

We conclude by raising the question 'do there exist non-symmetric quaternionic Kähler manifolds of positive scalar curvature?' First consider the situation in four dimensions. In that case we know from part one that a quaternionic manifold corresponds to an anti-self-dual conformal structure. The nearest there is to a quaternionic Kähler metric is then simply an Einstein metric. In fact the methods of §10 imply that a self-dual Einstein manifold $M$ of positive scalar curvature has an isometry group of dimension at least 4, and Hitchin [23] has proved that $M$ must be $S^4$ or $\mathbb{C}P^2$, i.e. a Wolf space. However even in eight dimensions the problem is significantly harder because theorem 10.1 only guarantees the existence of a group of dimension $\geqslant 6$, compared with 14, 15 and 21 for the Wolf spaces.

## Appendix

### Decomposition of a Tensor Product.

The purpose of this appendix is to explain a method for decomposing the tensor product of two irreducible representations of a compact Lie group into irreducible components. This method was successfully used by Fegan [16], but appears to be relatively unknown to differential geometers and is no doubt more familiar to physicists. The problem is that although the relevant theorem is well-documented (e.g. Zelobenko [40]), it is not immediately obvious that it is of any practical use. In reality it provides a rapid method for tackling many representations that occur in differential geometry, without the aid of a computer. As an example, it is an invaluable supplement to Alekseevskii's paper [3] on exceptional holonomy groups, in which certain decompositions are stated without indication as to proof.

To establish notation, we first summarize relevant facts about compact Lie groups (see Adams [1]). Let G be a compact Lie group, and choose a maximal torus T, Lie algebra $\mathfrak{t}$. Take any inner product on the dual $\mathfrak{t}^*$ invariant under AdG and put $\langle\alpha,\beta\rangle = 2(\alpha,\beta)/(\beta,\beta)$. The action of AdT on $\mathfrak{g}$ defines the set R of roots of G as a subset of $\mathfrak{t}^*$ satisfying $\langle\alpha,\beta\rangle \in \mathbb{Z}$, $\alpha,\beta \in R$. The Weyl group W is the group generated by reflections in the hyperplanes of $\mathfrak{t}^*$ orthogonal to the roots. Any $\sigma \in W$ permutes the roots, so W is contained in an appropriate symmetric group, and it makes sense to define sign $\sigma = \pm 1$. A base for R is a subset $\Delta$ such that any $\alpha \in R$ may be written $\alpha = \sum_{\gamma_i \in \Delta} n_i \gamma_i$ where the $n_i$ are integers, either all positive or all negative. Taking those $\alpha$ with $n_i$ positive defines the set

$R^+$ of positive roots. A choice of $\Delta$ determines a fundamental (dual) Weyl chamber $\mathcal{C}$ in $\mathfrak{t}^*$ by

$$\mathcal{C} = \{\alpha \in \mathfrak{t}^* : (\alpha, \gamma) > 0 \quad \forall \gamma \in \Delta\}$$

If $\exp : \mathfrak{t} \longrightarrow T$ is the exponential mapping, $\alpha \in \mathfrak{t}^*$ is said to be a weight if $\alpha(x) \in \mathbb{Z}$ whenever $\exp(x) = e$. Roots are weights, and if $\alpha$ is a weight $\langle \alpha, \beta \rangle \in \mathbb{Z}$, $\forall \beta \in R$. If G is simply-connected, the last property characterizes the weights. W preserves weights and any weight belonging to $\overline{\mathcal{C}}$ is called dominant. An important quantity is

$$d \doteq \tfrac{1}{2} \sum_{\gamma \in R^+} \gamma$$

which has the property that if $\alpha$ is a weight,

$$\alpha \in \overline{\mathcal{C}} \text{ iff } \alpha + d \in \mathcal{C}$$

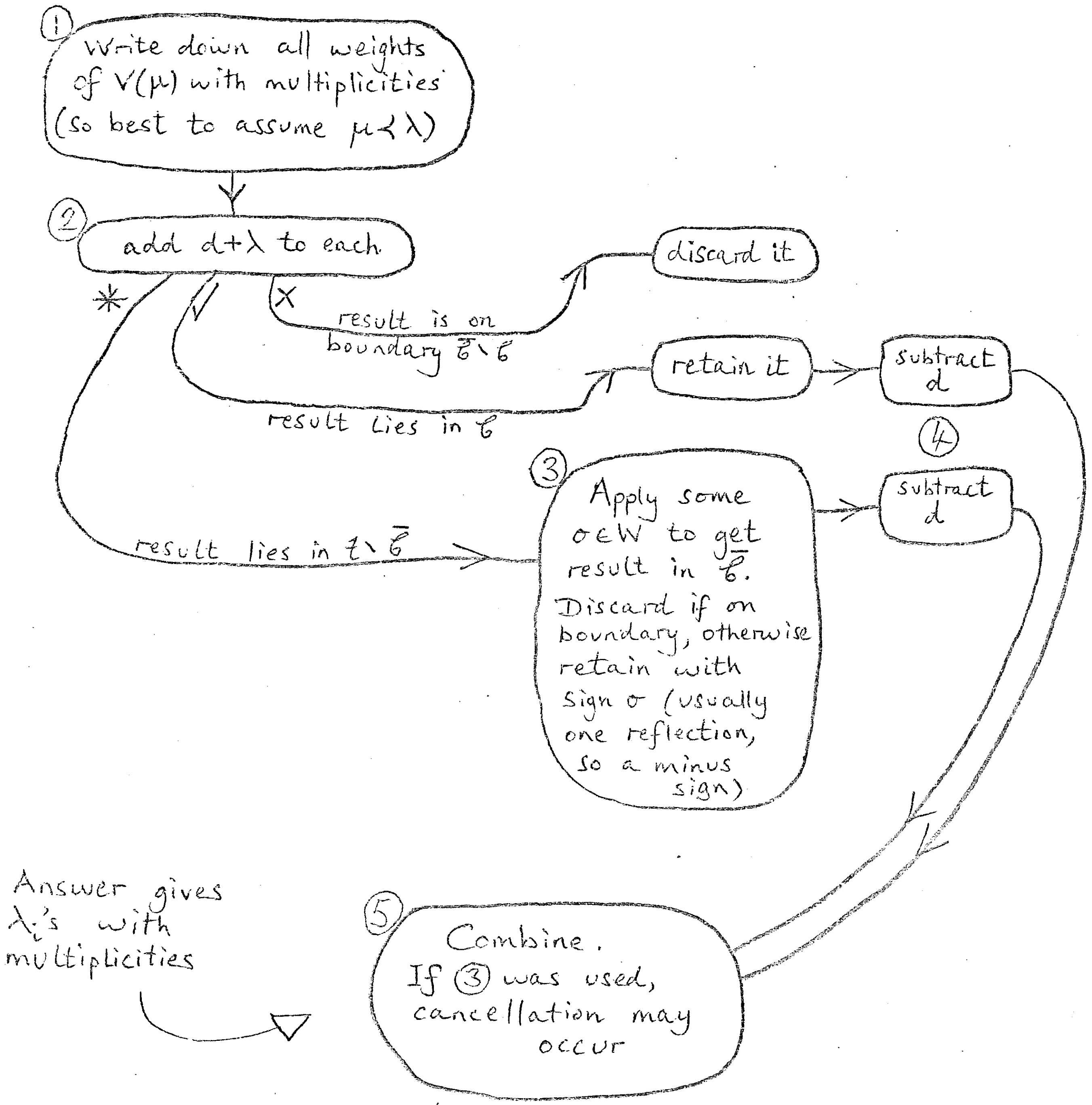
If $\gamma \in R$, $\langle d, \gamma \rangle \in \mathbb{Z}$, but if G is not simply-connected, d may not be a weight. There is a partial ordering defined on the weights by $\alpha \prec \beta$ iff $\alpha - \beta$ is a sum of positive roots or zero.

Any irreducible complex representation of G has the form

$$V = \sum_{\alpha} V_\alpha$$

where each $\alpha$ is a weight such that $\mathfrak{t} \subset \mathfrak{g}$ acts on $V_\alpha$ by $x(v) = \alpha(x)v$, $x \in \mathfrak{t}$, $v \in V_\alpha$. This collection of weights of V is invariant under W. A given $\alpha$ may appear in the sum with multiplicity $> 1$, but there is a unique 'highest' weight $\lambda$, i.e. maximal with respect to $\prec$, which appears with multiplicity one. For example, the weights of the adjoint representation $\mathfrak{g}$ of G are precisely the roots with multiplicity one, and 0 with multiplicity $\dim \mathfrak{t} = \operatorname{rank} G$. In general, $V \longleftrightarrow \lambda$ sets up a 1-1 correspondence between irreducible complex representations and dominant weights. Therefore we may label an irreducible representation $V(\lambda)$ by its highest weight $\lambda \in \overline{\mathcal{C}}$.

Suppose that $V(\lambda)$, $V(\mu)$ are irreducible representations with highest weights $\lambda, \mu$. The weights of $V(\lambda) \otimes V(\mu)$ are of the form $\alpha + \beta$ where $\alpha$ is a weight of $V(\lambda)$ and $\beta$ a weight of $V(\mu)$, but the decomposition of the tensor product into irreducible components is not at all obvious. However it is easy to see that the representation $V(\lambda+\mu)$ must occur with multiplicity one; moreover if $\lambda = \mu$, this belongs to $S^2(V(\lambda))$. The following diagram then gives a method for computing the irreducible components of $V(\lambda) \otimes V(\mu) = \sum V(\lambda_i)$ in terms of their highest weights $\lambda_i$:



Finally we give an example with $G = U(4)$. In suitable coordinates (e.g. see [1]), the weights are given by 4-tuples $(a_1,..,a_4)$ with $a_i \in \mathbb{Z}$, and the Weyl group consists of all permutations of the $a_i$'s. The set

$$(1,-1,0,0),\ (0,1,-1,0),\ (0,0,1,-1)$$

is a base for R giving rise to

$$\mathcal{C} = \{(a_1,a_2,a_3,a_4) : a_1 > a_2 > a_3 > a_4\}$$

$$d = (\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2})$$

We shall decompose $B \otimes S^2E = (1,0,0,-1) \otimes (0,0,0,-2)$, a result used for theorem 2.2. From $E^* \otimes E = A + B$, B has dimension 15. Its weights must consist of the Weyl orbit containing $(1,0,0,-1)$, and 0 with multiplicity 3.

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| $(1,0,0,-1)$ | $(\frac{5}{2},\frac{1}{2},-\frac{1}{2},-\frac{9}{2})\checkmark$ | $\longrightarrow$ | $(1,0,0,-3)$ |
| $(1,0,-1,0)$ | $(\frac{5}{2},\frac{1}{2},-\frac{3}{2},-\frac{7}{2})\checkmark$ | $\longrightarrow$ | $(1,0,-1,-2)$ |
| $(1,-1,0,0)$ | $\times$ | | |
| $(0,1,-1,0)$ | $\times$ | | |
| $(0,1,0,-1)$ | $\times$ | | |
| $(0,0,1,-1)$ | $\times$ | | |
| $(-1,0,0,1)$ | $\times$ | | |
| $(-1,0,1,0)$ | $\times$ | | |
| $(-1,1,0,0)$ | $(\frac{1}{2},\frac{3}{2},-\frac{1}{2},-\frac{7}{2})*$ $\longrightarrow$ | $-(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{7}{2})$ $\longrightarrow$ | $-(0,0,0,-2)$ |
| $(0,-1,1,0)$ | $(\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{7}{2})*$ $\longrightarrow$ | $-(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{7}{2})$ $\longrightarrow$ | $-(0,0,0,-2)$ |
| $(0,-1,0,1)$ | $\times$ | | |
| $(0,0,-1,1)$ | $(\frac{3}{2},\frac{1}{2},-\frac{3}{2},-\frac{5}{2})\checkmark$ | $\longrightarrow$ | $(0,0,1,-1)$ |
| $3(0,0,0,0)$ | $3(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{7}{2})\checkmark$ | $\longrightarrow$ | $3(0,0,0,-2)$ |

5 Answer $= (1,0,0,-3) + (1,0,-1,-2) + (0,0,0,-2) + (0,0,-1,-1)$

$= U + M + S^2E + \Lambda^2E$

<u>Remarks</u> The above was probably the most difficult decomposition required in §2; most were much simpler. It is clear that under U(n) there will still be four components in the corresponding decomposition of

$$(1,0,..,0,-1) \otimes (0,..,0,-2)$$

This is because although more weights will appear in step 1, these will all be discarded in step 2. With practice one can write down only weights in step 1 that do not get discarded; this is the reason for the method's efficiency.

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