

Continuum Many Maximal Consistent Normal Bimodal Logics with Inverses

TIMOTHY WILLIAMSON

Abstract The paper considers extensions of a normal bimodal logic **KL** in which the two necessity operators are mutual one-sided inverses. A continuum of maximal consistent normal extensions of **KL** is constructed, each of which has infinitely many quasi-normal Post complete extensions. Some syntactic properties of maximal consistent normal bimodal logics and in particular of such extensions of **KL** are investigated.

There are just two maximal consistent normal monomodal logics; each consistent normal monomodal logic is extended by at least one of them (see Makinson [2]). In bimodal logic, the situation is less simple. This note investigates one aspect of the problem by considering maximal consistent normal bimodal logics in which the two modal operators are one-sided inverses of each other, that is, $\Box\blacksquare\alpha \equiv \alpha$ is a theorem for every formula α . It will be shown that there are 2^{\aleph_0} such logics. For a general account of bimodal logics in which the two modal operators are one-sided inverses of each other see Humberstone and Williamson [1].

Some terminology for bimodal logic will briefly be rehearsed. The language consists of a set of sentence letters p, \dots , the 0-place falsity constant \perp , the material implication \supset , and two 1-place operators \Box and \blacksquare . None of the results in this paper essentially depends on the cardinality of the set of sentence letters (even if it is 0). Other standard logical symbols are used as metalinguistic abbreviations; $\Diamond = \neg\Box\neg$ and $\blacklozenge = \neg\blacksquare\neg$. A *letterless* formula is one not containing sentence letters. A letterless substitution maps all sentence letters to letterless formulas.

A *logic* is a set of formulas containing all truth-functional tautologies and closed under uniform substitution (US) and modus ponens (MP). If Σ is a logic, we say $\vdash_{\Sigma} \alpha$ when $\alpha \in \Sigma$. A *normal* (bimodal) logic is a logic containing all formulas of the forms $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$ and $\blacksquare(\alpha \supset \beta) \supset (\blacksquare\alpha \supset \blacksquare\beta)$ and closed under $RN_{\Box}(\alpha/\Box\alpha)$ and $RN_{\blacksquare}(\alpha/\blacksquare\alpha)$. Σ is inconsistent if and only if $\perp \in \Sigma$. Σ is *Post complete* if and only if Σ is a consistent logic and for every consistent logic Σ^+ , if

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$\Sigma \subseteq \Sigma^+$ then $\Sigma = \Sigma^+$ (Segerberg [3], corrected in Segerberg [4], discusses Post completeness in monomodal logic). Σ is a *maximal consistent normal* logic if and only if Σ is a consistent normal logic and for every consistent normal logic Σ^+ , if $\Sigma \subseteq \Sigma^+$ then $\Sigma = \Sigma^+$. Proposition 5 below illustrates the difference between Post completeness and maximal consistent normality. Following [1], the smallest normal logic containing $\Box\blacksquare\alpha \equiv \alpha$ for all α is **KL**.

A *frame* is a triple $\langle W, R, S \rangle$, where W is a set and R and S are binary relations on W . A *valuation* on a frame $\langle W, R, S \rangle$ is a mapping V from ordered pairs of formulas and members of W to $\{0, 1\}$ such that for all formulas α and β and $w \in W$: $V(\perp, w) = 0$; $V(\alpha \supset \beta, w) = 1$ if and only if $V(\alpha, w) \leq V(\beta, w)$; $V(\Box\alpha, w) = 1$ if and only if $V(\alpha, x) = 1$ whenever wRx ; $V(\blacksquare\alpha, w) = 1$ if and only if $V(\alpha, x) = 1$ whenever wSx . A set of formulas Σ is *valid* on a frame $\langle W, R, S \rangle$ if and only if for every valuation V on $\langle W, R, S \rangle$, $w \in W$ and $\alpha \in \Sigma$, $V(\alpha, w) = 1$. Σ is *satisfiable* on $\langle W, R, S \rangle$ if and only if for some valuation V on $\langle W, R, S \rangle$ and $w \in W$, for every $\alpha \in \Sigma$, $V(\alpha, w) = 1$.

For $X \subseteq \omega$, define relations R_X and S on ω :

$$\begin{aligned} iR_Xj &\iff \text{either} && \text{(a) } j = i + 1 \\ &\text{or} && \text{(b) } j = 0 \text{ and } i \text{ is even} \\ &\text{or} && \text{(c) } j = 0 \text{ and for some } k \in X, i = 2k + 1. \end{aligned}$$

$$iSj \iff i = j + 1.$$

Let **KL**[X] be the normal bimodal logic consisting of all formulas α such that $\{\alpha\}$ is valid on the frame $\langle \omega, R_X, S \rangle$. Note that for any formula α and $i, j, k \in \omega$, if iR_XjSk then $j = k + 1$, so $j = i + 1$, so $i = k$, so $V(\alpha \supset \Box\blacksquare\alpha, i) = 1$ for any valuation V on $\langle \omega, R_X, S \rangle$. Conversely, $iR_Xi + 1Si$, so $V(\Box\blacksquare\alpha \supset \alpha, i) = 1$. Thus $\vdash_{\mathbf{KL}[X]} \alpha \equiv \Box\blacksquare\alpha$; **KL** \subseteq **KL**[X] for any $X \subseteq \omega$. Since the argument does not depend on which numbers have R_X to 0, $\{i : iR_X0\}$ can be used to encode X so that **KL**[X] is inconsistent with **KL**[Y] whenever $X \neq Y$. Thus there are uncountably many such logics. Moreover, **KL**[X] and any consistent extension of it is undecidable whenever X is undecidable.

Proposition 1 *If X and Y are distinct subsets of ω , then for some formula α $\vdash_{\mathbf{KL}[X]} \alpha$ and $\vdash_{\mathbf{KL}[Y]} \neg\alpha$.*

Proof: It suffices to show that for any $X \subseteq \omega$ and $i \in \omega$,

- (a) if $i \in X$ then $\vdash_{\mathbf{KL}[X]} \Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp)$;
- (b) if not $i \in X$ then $\vdash_{\mathbf{KL}[X]} \neg\Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp)$.

Let V be any valuation on $\langle \omega, R_X, S \rangle$.

(a) Suppose that $i \in X$. Then $2i + 1R_X0$, so $V(\Diamond\blacksquare\perp, 2i + 1) = 1$. Note that $V(\Diamond^{2i+1}\blacksquare\perp, 2i + 1) = 1$. Hence $V(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp, 2i + 1) = 1$. Now for any $j \in \omega$, either j or $j + 1$ is even, so jR_X0R_X0 or jR_Xj+1R_X0 ; either way, jR_X^20 . Since $0R_X^{2i+1}2i + 1$, $jR_X^{2i+3}2i + 1$. Hence $V(\Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp), j) = 1$ for all $j \in \omega$. Since V was arbitrary, $\vdash_{\mathbf{KL}[X]} \Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp)$.

(b) Suppose that $V(\Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp), j) = 1$ for some $j \in \omega$. Hence for some k , $jR_X^{2i+3}k$ and $V(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp, k) = 1$. Thus $V(\Diamond^{2i+1}\blacksquare\perp, k) = 1$, so $2i+1 = k$ by the structure of the frame, so $V(\Diamond\blacksquare\perp, 2i+1) = 1$, so $2i+1R_X0$, so $i \in X$. Thus if not $i \in X$, $V(\Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp), j) = 0$ for all $j \in \omega$. Since V was arbitrary, $\vdash_{\mathbf{KL}[X]} \neg\Diamond^{2i+3}(\Diamond^{2i+1}\blacksquare\perp \wedge \Diamond\blacksquare\perp)$. \square

Proposition 2 For $X \subseteq \omega$, $\mathbf{KL}[X]$ is a maximal consistent normal logic.

Proof: Let Σ be a consistent normal logic such that $\mathbf{KL}[X] \subseteq \Sigma$. We must show that $\mathbf{KL}[X] = \Sigma$. Let $\#\alpha$ be the degree of embedding of \square (not \blacksquare) in α : $\#p = \#\perp = 0$; $\#\blacksquare\alpha = \#\alpha$; $\#(\alpha \supset \beta) = \max\{\#\alpha, \#\beta\}$; $\#\square\alpha = 1 + \#\alpha$.

(i) Note the following fact.

Fact 3 For any valuation V on $\langle \omega, R_X, S \rangle$ and $k \in \omega$, there is a substitution σ such that for every formula α , $i \in \omega$ and valuation V^* on $\langle \omega, R_X, S \rangle$, if $i + \#\alpha \leq k$ then $V^*(\sigma\alpha, i) = V(\alpha, i)$.

To see this, let σ be the substitution such that for each sentence letter p , $\sigma p = \bigvee\{\Diamond^j\blacksquare\perp : 0 \leq j \leq k \text{ and } V(p, j) = 1\}$. We can now prove Fact 3 by induction on the complexity of α . The basis for this is as follows: $\alpha = p$. By the structure of the frame, $V(\Diamond^j\blacksquare\perp, i) = 1$ just in case $i = j$. Thus if $i \leq k$, $V^*(\sigma p, i) = 1 \iff V(p, i) = 1$. The cases of \supset and \perp are trivial. For the induction step for \square , note that if $i + \#\square\alpha \leq k$ and iR_Xj then $i + 1 + \#\alpha \leq k$ and $j \leq i + 1$, so $j + \#\alpha \leq k$. For the induction step for \blacksquare , note that if $i + \#\blacksquare\alpha \leq k$ and iSj then $i + \#\alpha \leq k$ and $i \leq j + 1$, so $j + \#\alpha < k$.

(ii) Now suppose that not $\vdash_{\mathbf{KL}[X]} \alpha$. Then for some $i \in \omega$ and valuation V on $\langle \omega, R_X, S \rangle$, $V(\alpha, i) = 0$. By Fact 3 for $k = i + \#\alpha$, there is a substitution σ such that $V^*(\sigma\alpha, i) = V(\alpha, i) = 0$ for every valuation V^* . By an argument such as (a) in the proof of Proposition 1, $jR_X^{i+2}i$ for all $j \in \omega$. Thus $V^*(\Diamond^{i+2}\neg\sigma\alpha, j) = 1$ for every valuation V^* and $j \in \omega$. Thus $\vdash_{\mathbf{KL}[X]} \Diamond^{i+2}\neg\sigma\alpha$, so $\vdash_{\Sigma} \Diamond^{i+2}\neg\sigma\alpha$. But if $\vdash_{\Sigma} \alpha$ then $\vdash_{\Sigma} \square^{i+2}\sigma\alpha$ because Σ is normal, contradicting the consistency of Σ ; hence not $\vdash_{\Sigma} \alpha$. Thus if $\vdash_{\Sigma} \alpha$ then $\vdash_{\mathbf{KL}[X]} \alpha$, as required. \square

Corollary 4 The number of maximal consistent normal extensions of \mathbf{KL} is 2^{\aleph_0} , as is the number of Post complete extensions of \mathbf{KL} .

Proof: By Propositions 1 and 2, \mathbf{KL} has at least 2^{\aleph_0} mutually inconsistent maximal consistent normal extensions. By an argument from Zorn's lemma, every consistent logic is extended by a Post complete logic, so \mathbf{KL} has at least 2^{\aleph_0} Post complete extensions. In both cases the number is at most 2^{\aleph_0} , however many sentence letters there are, for if Σ is a Post complete or maximal consistent normal extension of \mathbf{KL} , then $\Sigma = \{\alpha : \vdash_{\Sigma} \sigma\alpha \text{ for every letterless substitution } \sigma\}$ by an argument such as that for Proposition 6 (\implies (b)) below, so each such extension is determined by its letterless fragment. \square

Proposition 5 $\mathbf{KL}[X]$ has infinitely many Post complete extensions, all of which are nonnormal.

Proof: For $i \in \omega$, let $\mathbf{KL}[X][i] = \{\alpha : V(\alpha, i) = 1 \text{ for every valuation } V \text{ on } \langle \omega, R_X, S \rangle\}$. By standard reasoning, $\mathbf{KL}[X][i]$ is a consistent logic extending $\mathbf{KL}[X]$. By the structure of the frame, $V(\Diamond^j\blacksquare\perp, i) = 1$ just in case $i = j$. Thus $\vdash_{\mathbf{KL}[X][i]}$

$\Diamond^i \blacksquare \perp$ and $\vdash_{\mathbf{KL}[X][i]} \neg \Diamond^j \blacksquare \perp$ whenever $i \neq j$. Hence if $i \neq j$, for some formula α , $\vdash_{\mathbf{KL}[X][i]} \alpha$ and $\vdash_{\mathbf{KL}[X][j]} \neg \alpha$. Since every consistent logic has a Post complete extension, $\mathbf{KL}[X]$ has infinitely many Post complete extensions, all of which are non-normal by Proposition 2. \square

Having established the existence of a nontrivial set of maximal consistent normal extensions of \mathbf{KL} , we proceed to investigate their properties, beginning with a general characterization of maximal consistent normal bimodal logics. A *necessitation* is a sequence (possibly null) of occurrences of \square and \blacksquare in any order.

Proposition 6 *A consistent normal bimodal logic Σ is a maximal consistent normal logic just in case*

- (a) *for every letterless formula α , either $\vdash_{\Sigma} \alpha$ or for some necessitations $L_1, \dots, L_k, \vdash_{\Sigma} \neg(L_1 \alpha \wedge \dots \wedge L_k \alpha)$;*

and

- (b) *for every formula α , if $\vdash_{\Sigma} \sigma \alpha$ for every letterless substitution σ then $\vdash_{\Sigma} \alpha$.*

Proof: (\Leftarrow) Assume (a) and (b). Let Σ^+ be a consistent normal logic such that $\Sigma \subseteq \Sigma^+$. Suppose that $\vdash_{\Sigma^+} \alpha$. We must show that $\vdash_{\Sigma} \alpha$. Let σ be a letterless substitution. By (b), we need only show that $\vdash_{\Sigma} \sigma \alpha$. By (a), if not $\vdash_{\Sigma} \sigma \alpha$, then for some necessitations $L_1, \dots, L_k, \vdash_{\Sigma} \neg(L_1 \alpha \wedge \dots \wedge L_k \alpha)$. Since $\Sigma \subseteq \Sigma^+$, $\vdash_{\Sigma^+} \neg(L_1 \alpha \wedge \dots \wedge L_k \alpha)$. By US for Σ^+ , $\vdash_{\Sigma^+} \sigma \alpha$. Since Σ is normal, $\vdash_{\Sigma^+} L_1 \sigma \alpha \wedge \dots \wedge L_k \sigma \alpha$. Thus Σ^+ is inconsistent, contrary to hypothesis.

(\Rightarrow) Let Σ be a maximal consistent normal bimodal logic.

(a) Let α be letterless. Put $\Sigma^+ = \{\beta : \vdash_{\Sigma} (L_1 \alpha \wedge \dots \wedge L_k \alpha) \supset \beta \text{ for some necessitations } L_1, \dots, L_k\}$. If $\perp \in \Sigma^+$ we are done. Suppose that not $\perp \in \Sigma^+$. We show that Σ^+ is a normal extension of Σ ; by hypothesis, it is consistent. Evidently $\Sigma \subseteq \Sigma^+$ and Σ^+ is closed under MP. Σ^+ is closed under US because Σ is, and $\sigma \alpha = \alpha$ since α is letterless. Σ^+ is closed under RN_{\square} and RN_{\blacksquare} because Σ is normal. Thus Σ^+ is a consistent normal extension of Σ . Since Σ is maximal, $\Sigma = \Sigma^+$. Trivially, $\vdash_{\Sigma^+} \alpha$. Thus $\vdash_{\Sigma} \alpha$.

(b) Let $\Sigma^+ = \{\alpha : \vdash_{\Sigma} \sigma \alpha \text{ for every letterless substitution } \sigma\}$. Since Σ is closed under US, $\Sigma \subseteq \Sigma^+$. We need only show that $\Sigma = \Sigma^+$. Since Σ is maximal, we need only show that Σ^+ is consistent and normal. Σ^+ is consistent, otherwise Σ would be inconsistent. Σ^+ is closed under US, for if σ_0 is a substitution and σ is a letterless substitution, then $\sigma \sigma_0$ is a letterless substitution. Σ^+ is closed under MP, RN_{\square} and RN_{\blacksquare} because Σ is and substitutions commute with \supset , \square , and \blacksquare . \square

Corollary 7 *If Σ is a maximal consistent normal bimodal logic, $\vdash_{\Sigma} \Diamond \top$ and $\vdash_{\Sigma} \blacksquare \top$ then $\vdash_{\Sigma} \square \alpha \equiv \alpha$ and $\vdash_{\Sigma} \blacksquare \alpha \equiv \alpha$ for all α .*

Proof: Make the assumptions. By induction on the complexity of β , if β is letterless then either $\vdash_{\Sigma} \beta$ or $\vdash_{\Sigma} \neg \beta$. Thus if σ is a letterless substitution, either $\vdash_{\Sigma} \sigma \alpha$ or $\vdash_{\Sigma} \neg \sigma \alpha$; either way, $\vdash_{\Sigma} \square \sigma \alpha \equiv \sigma \alpha$ and $\vdash_{\Sigma} \blacksquare \sigma \alpha \equiv \sigma \alpha$. By (b) of Proposition 6, $\vdash_{\Sigma} \square \alpha \equiv \alpha$ and $\vdash_{\Sigma} \blacksquare \alpha \equiv \alpha$. \square

Corollary 8 *If Σ is a maximal consistent normal extension of **KL** and $\vdash_{\Sigma} \Diamond \top$ then $\vdash_{\Sigma} \Box \alpha \equiv \alpha$ and $\vdash_{\Sigma} \blacksquare \alpha \equiv \alpha$ for all α .*

Proof: From Corollary 7, for $\vdash_{\Sigma} \Diamond \top$ because $\vdash_{\mathbf{KL}} \Box \blacksquare \perp \equiv \perp$ and **KL** is normal, so $\vdash_{\mathbf{KL}} \Diamond \top$. \square

Proposition 9 *A consistent normal extension Σ of **KL** is a maximal consistent normal logic just in case*

(a') *for every letterless formula α , either $\vdash_{\Sigma} \alpha$ or for some $i(1), \dots, i(k) \in \omega$ $\vdash_{\Sigma} \neg(\Box^{i(1)} \alpha \wedge \dots \wedge \Box^{i(k)} \alpha)$,*

and

(b) *for every formula α , if $\vdash_{\Sigma} \sigma \alpha$ for every letterless substitution σ , then $\vdash_{\Sigma} \alpha$.*

Proof: By Proposition 6, we need only show that if Σ is a normal extension of **KL**, α is letterless and for some necessitations $L_1, \dots, L_k, \vdash_{\Sigma} \neg(L_1 \alpha \wedge \dots \wedge L_k \alpha)$ then for some $i(1), \dots, i(k) \in \omega, \vdash_{\Sigma} \neg(\Box^{i(1)} \alpha \wedge \dots \wedge \Box^{i(k)} \alpha)$. Now for each necessitation L there are $m, n \in \omega$ such that for every formula $\alpha \vdash_{\mathbf{KL}} L \alpha \equiv \blacksquare^m \Box^n \alpha$; this can easily be proved by induction on the length of L . Thus we can assume that for some $m(1), \dots, m(k), n(1), \dots, n(k) \in \omega, \vdash_{\Sigma} \neg(\blacksquare^{m(1)} \Box^{n(1)} \alpha \wedge \dots \wedge \blacksquare^{m(k)} \Box^{n(k)} \alpha)$. Let $m = \max\{m(1), \dots, m(k)\}$. Since Σ is normal, $\vdash_{\Sigma} \Box^m \neg(\blacksquare^{m(1)} \Box^{n(1)} \alpha \wedge \dots \wedge \blacksquare^{m(k)} \Box^{n(k)} \alpha)$. Since $\vdash_{\mathbf{KL}} \Diamond \top$ and Σ is a normal extension of **KL**, $\vdash_{\Sigma} \neg(\Box^m \blacksquare^{m(1)} \Box^{n(1)} \alpha \wedge \dots \wedge \Box^m \blacksquare^{m(k)} \Box^{n(k)} \alpha)$. Now $\vdash_{\mathbf{KL}} \Box^i \blacksquare^i \beta \equiv \beta$ for every formula β and $i \in \omega$ (proof: by induction on i , using the normality of **KL**). Thus $\vdash_{\Sigma} \Box^m \blacksquare^{m(j)} \beta \equiv \Box^{m-m(j)} \beta$ for $1 \leq j \leq k$. Since **KL** $\subseteq \Sigma$, $\vdash_{\Sigma} \neg(\Box^{m-m(1)+n(1)} \alpha \wedge \dots \wedge \Box^{m-m(k)+n(k)} \alpha)$. \square

Corollary 10 *If Σ is a maximal consistent normal extension of **KL** and $\vdash_{\Sigma} \Box \alpha$ then $\vdash_{\Sigma} \alpha$.*

Proof: Suppose that $\vdash_{\Sigma} \Box \alpha$. Let σ be a letterless substitution. By Proposition 9, either $\vdash_{\Sigma} \sigma \alpha$ or for some $i(1), \dots, i(k) \in \omega \vdash_{\Sigma} \neg(\Box^{i(1)} \sigma \alpha \wedge \dots \wedge \Box^{i(k)} \sigma \alpha)$. Suppose that not $\vdash_{\Sigma} \sigma \alpha$. Since Σ is closed under US, $\vdash_{\Sigma} \Box \sigma \alpha$. Since Σ is normal, $\vdash_{\Sigma} \Box^{i(k)} \sigma \alpha$ whenever $i(k) \geq 1$. Thus $\vdash_{\Sigma} \neg \sigma \alpha$. Since Σ is normal, $\vdash_{\Sigma} \Box \neg \sigma \alpha$, so $\vdash_{\Sigma} \Box \perp$. But $\vdash_{\mathbf{KL}} \Diamond \top$, so Σ is inconsistent, contrary to hypothesis. Thus $\vdash_{\Sigma} \sigma \alpha$. But σ was arbitrary, so $\vdash_{\Sigma} \alpha$ by Proposition 9. \square

The result for \blacksquare is automatic: if Σ is a normal extension of **KL** and $\vdash_{\Sigma} \blacksquare \alpha$ then $\vdash_{\Sigma} \Box \blacksquare \alpha$, so $\vdash_{\Sigma} \alpha$.

Corollary 11 *If Σ is a maximal consistent normal extension of **KL** then $\vdash_{\Sigma} \Diamond \alpha \supset \blacksquare \alpha$.*

Proof: $\vdash_{\mathbf{KL}} \alpha \supset \Box \blacksquare \alpha$ and $\vdash_{\mathbf{KL}} \neg \alpha \supset \Box \blacksquare \neg \alpha$, so $\vdash_{\mathbf{KL}} \Box \blacksquare \alpha \vee \Box \blacksquare \neg \alpha$. Since **KL** is normal, $\vdash_{\mathbf{KL}} \Box(\Diamond \alpha \supset \blacksquare \alpha)$. Since **KL** $\subseteq \Sigma$, $\vdash_{\Sigma} \Box(\Diamond \alpha \supset \blacksquare \alpha)$. By Corollary 10, $\vdash_{\Sigma} \Diamond \alpha \supset \blacksquare \alpha$. \square

Corollary 12 *If Σ is a maximal consistent normal extension of **KL** then either $\vdash_{\Sigma} \Box \alpha \equiv \alpha$ and $\vdash_{\Sigma} \blacksquare \alpha \equiv \alpha$ for all α or for some $i(1), \dots, i(k) \in \omega \vdash_{\Sigma} \Diamond^{i(1)} \blacksquare \perp \vee \dots \vee \Diamond^{i(k)} \blacksquare \perp$.*

Proof: By Proposition 9, either $\vdash_{\Sigma} \blacklozenge \top$ or for some $i(1), \dots, i(k) \in \omega \vdash_{\Sigma} \neg(\Box^{i(1)} \blacklozenge \top \wedge \dots \wedge \Box^{i(k)} \blacklozenge \top)$. The result follows by Corollary 8. \square

Proposition 13 *If a maximal consistent normal extension Σ of **KL** is satisfiable on a finite frame, then $\vdash_{\Sigma} \Box \alpha \equiv \alpha$ and $\vdash_{\Sigma} \blacksquare \alpha \equiv \alpha$.*

Proof: Suppose that a maximal consistent normal extension Σ of **KL** is satisfiable on a finite frame $\langle W, R, S \rangle$. Thus for some valuation V on $\langle W, R, S \rangle$ and $x \in W$, $V(\alpha, x) = 1$ whenever $\vdash_{\Sigma} \alpha$. Now for all $i \in \omega$, $\{w \in W : V(\blacksquare^i \perp, w)\} \subseteq \{w \in W : V(\blacksquare^{i+1} \perp, w)\}$. Since W is finite, for some i $\{w \in W : V(\blacksquare^i \perp, w)\} = \{w \in W : V(\blacksquare^{i+1} \perp, w)\}$. Thus for all $w \in W$, $V(\blacksquare^{i+1} \perp \supset \blacksquare^i \perp, w) = 1$. Hence for all $w \in W$ and $j \in \omega$, $V(\Box^{i+j}(\blacksquare^{i+1} \perp \supset \blacksquare^i \perp), w) = 1$. Since $\vdash_{\mathbf{KL}} \neg \Box^i \blacksquare^i \perp$ and $\vdash_{\mathbf{KL}} \Box^i \blacksquare^{i+1} \perp \equiv \blacksquare \perp$, $\vdash_{\mathbf{KL}} (\Box^i \blacksquare^{i+1} \perp \supset \Box^i \blacksquare^i \perp) \supset \neg \blacksquare \perp$, so by normality $\vdash_{\mathbf{KL}} \Box^i (\blacksquare^{i+1} \perp \supset \blacksquare^i \perp) \supset \neg \blacksquare \perp$, so $\vdash_{\mathbf{KL}} \Box^{i+j} (\blacksquare^{i+1} \perp \supset \blacksquare^i \perp) \supset \Box^j \neg \blacksquare \perp$. Since $\mathbf{KL} \subseteq \Sigma$, $\vdash_{\Sigma} \Box^{i+j} (\blacksquare^{i+1} \perp \supset \blacksquare^i \perp) \supset \Box^j \neg \blacksquare \perp$. Thus $V(\Box^{i+j} (\blacksquare^{i+1} \perp \supset \blacksquare^i \perp) \supset \Box^j \neg \blacksquare \perp, x) = 1$. But $V(\Box^{i+j} (\blacksquare^{i+1} \perp \supset \blacksquare^i \perp), x) = 1$, so $V(\Box^j \neg \blacksquare \perp, x) = 1$ for all j . Hence for all $i(1), \dots, i(k) \in \omega$, $V(\Diamond^{i(1)} \blacksquare \perp \vee \dots \vee \Diamond^{i(k)} \blacksquare \perp, x) = 0$, so not $\vdash_{\Sigma} \Diamond^{i(1)} \blacksquare \perp \vee \dots \vee \Diamond^{i(k)} \blacksquare \perp$. The result follows by Corollary 12. \square

The \blacksquare -fragment of a bimodal logic Σ is the result of replacing \blacksquare by \Box throughout the set of all formulas in Σ not containing \Box (the replacement is for notational uniformity with standard monomodal logic). If Σ is (bi)normal, its \blacksquare -fragment is a normal monomodal logic. We determine the \blacksquare -fragments of all maximal consistent normal extensions of **KL**. Let *Triv* be the smallest normal monomodal logic containing all formulas of the form $\Box \alpha \equiv \alpha$ and \mathbf{KD}_c the smallest normal monomodal logic containing all formulas of the form $\Diamond \alpha \supset \Box \alpha$.

Proposition 14 *If Σ is a maximal consistent normal extension of **KL**, then the \blacksquare -fragment of Σ is either \mathbf{KD}_c or *Triv*.*

Proof: Let Δ be the \blacksquare -fragment of Σ . By Corollary 11, $\mathbf{KD}_c \subseteq \Delta$. Thus either $\Delta = \mathbf{KD}_c$ or Δ is a proper normal extension of \mathbf{KD}_c . In the latter case, $\vdash_{\Delta} \Box^i \Diamond \top$ for some i (Segerberg [5]). Then $\vdash_{\Sigma} \blacksquare^i \Diamond \top$, so $\vdash_{\Sigma} \Box^i \blacksquare^i \Diamond \top$, so $\vdash_{\Sigma} \Diamond \top$ since Σ extends **KL**, so $\vdash_{\Sigma} \blacksquare \alpha \equiv \alpha$ for all α by Corollary 8, so *Triv* $\subseteq \Delta$. Hence $\Delta = \text{Triv}$, for the only proper normal extension of *Triv* is inconsistent. \square

The more difficult problem of finding the \Box -fragments of all maximal consistent normal extensions of **KL** is left open.

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Department of Philosophy
University of Edinburgh
Edinburgh EH89JX
UNITED KINGDOM
email: Timothy.Williamson@ed.ac.uk