On Some Nonlinear Models for the Navier–Stokes Equations

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This thesis is dedicated to my parents
for their unwavering support
and for being such amazing models in my life,
and to my stillborn little brother Davis

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Abstract

We consider in this thesis two nonlinear models for the incompressible Navier-Stokes system. Firstly, we lay the basis for the regularity analysis of those models by establishing various results such as $\epsilon-$regularity theorems (e.g. a version of the Caffarelli-Kohn-Nirenberg theorem), dimension analysis of potential singular sets, partial regularity results in critical cases and various Liouville type theorems.

Secondly, by relying on the previous points and by taking advantage of the locality of our models, we were able to establish some new regularity results such as a higher integrability for the gradient of some particular solutions, regularity in the case of spherical symmetry and a boundary regularity result which is known to be false for the Navier-Stokes system.

Thirdly, we construct two candidates for non-uniqueness for the Navier-Stokes system as singular limits of solutions of our models; we put a particular emphasis on the analysis of this convergence.

Finally, we provide at the end of this work a set of tools to tackle some questions related to establishing energy identity and proving Liouville type theorems for equations of fluids mechanics.

Contents

Chapter 1

Introduction

> All the stars, taken all together, tho' innumerable, must like any other set of points, in turn represent some single gigantic Equation, to the mind of God as straightforward as, say the Equation of a Sphere, [...] to us unreadable, incalculable. A lonely, uncompensated, perhaps even impossible Task, [...] yet some of us must ever be seeking, I suppose.
>
> *Thomas Pynchon*
> in "Mason & Dixon"

This thesis is concerned with understanding whether the non-locality introduced by the presence of the pressure is the principal reason why the global well-posedness question for the 3D incompressible Navier-Stokes equations is so difficult. These equations were introduced in the 19th century as a model governing the evolution of a viscous incompressible fluid (for instance milk, oil and to some extent water) in a domain $\Omega \subset \mathbb{R}^3$. According to this model the velocity field of the fluid $v : \Omega \to \mathbb{R}^3$ and the pressure field $q : \Omega \to \mathbb{R}$ satisfy the following system of nonlinear partial differential equations on $\Omega \times]0, +\infty[$ together with the following initial and boundary conditions:

$$\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla q = f, \quad \operatorname{div} v = 0 \ (\nu > 0), \tag{1.1}$$

$$v|_{\partial\Omega}(\cdot, t) = 0, \tag{1.2}$$

$$v(\cdot, 0) = u_0. \tag{1.3}$$

In the case of $\Omega = \mathbb{R}^3$, the boundary condition (1.2) is replaced by

$$v(x,t) \to 0, \ \text{ as } |x| \to \infty. \tag{1.4}$$

An important feature of these equations, that plays an important role in their mathematical analysis, is that they are invariant under the scaling ($\lambda > 0$):

$$\begin{aligned} v(x,t) &\to v_\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \\ q(x,t) &\to q_\lambda(x,t) = \lambda^2 q(\lambda x, \lambda^2 t), \\ f(x,t) &\to f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t), \\ u_0(x) &\to u_{0\lambda}(x) = \lambda u_0(\lambda x). \end{aligned} \tag{1.5}$$

This scaling is the only one available for system (1.1)-(1.3), and as explained below, this makes it supercritical.

Many fundamental mathematical questions regarding the three-dimensional Navier-Stokes equations remain unsolved. Among those and as foreshadowed, the global well-posedness question, which because of its importance from a mathematical and physical point of view, was recognised as one of the seven Millennium problems. The problem, as stated by Fefferman in [15], asks to prove one of the following four statements **(A)-(D)**.

(A) Existence and smoothness of Navier-Stokes solutions on $\mathbb{R}^3$
Take $\nu > 0$ and let u_0 be any smooth divergence-free vector field satisfying

$$|\nabla^m u_0(x)| \leq C_{mK}(1+|x|)^{-K} \tag{1.6}$$

on $\mathbb{R}^3$ for any $m = 0, 1, \ldots$ and $K > 0$. Take $f(x,t)$ to be identically zero. Then there exist functions

$$(v,q) \in C^\infty(\mathbb{R}^3 \times [0,\infty[) \tag{1.7}$$

that satisfy

$$\int_{\mathbb{R}^3} |v(x,t)|^2 dx \leq C \text{ for all } t > 0 \text{ (bounded energy)}, \tag{1.8}$$

and (1.1)-(1.3) on $\mathbb{R}^3 \times [0,+\infty[$.

(C) Breakdown of Navier-Stokes solutions on $\mathbb{R}^3$
Take $\nu > 0$. Then there exists a smooth, divergence-free vector field $u_0(x)$ and a smooth force $f(x,t)$ on $\mathbb{R}^3 \times [0,+\infty[$, satisfying (1.6),

$$|\partial_t^l \nabla^m f(x,t)| \leq C_{mlK}(1+|x|+t)^{-K} \tag{1.9}$$

(on $\mathbb{R}^3 \times [0,+\infty[$ for any $l, m = 0, 1, \ldots$ and $K > 0$), for which there exist no solutions (v,q) satisfying (1.1)-(1.3), (1.7) and (1.8).
The other two statements **(B)** and **(D)** are just the statements **(A)** and **(C)** stated in the setting of spatially periodic functions.

A situation where **(A)** holds true would mean that the incompressible Navier-Stokes equations constitute a realistic deterministic description of the dynamic of viscous incompressible fluids. On the other hand, a situation where **(C)** holds would suggest, to some extent, that those equations are not a valid physical model for a deterministic description of the flow of a viscous incompressible fluid; this would be quite an interesting and unexpected result from the point of view of Physics since these equations were derived thanks to Newton's second law of motion, which is one of the cornerstones for explaining and describing a wide variety of physical phenomena.

Another interesting perspective on the Navier-Stokes problem was given by Ladyzhenskaya; she asked in [40], the following question: *"Do the Navier-Stokes equations together with the initial and boundary conditions actually give a deterministic description of the dynamics of an incompressible fluid?"*. In contrast to Fefferman, she suggested instead that one should focus on finding the right spaces of initial data that make this problem globally well-posed; that way we won't be too far from the physical observations.

But, whether one would like to answer Fefferman's problem or Ladyzhenskaya's question, one cannot escape the necessity of understanding the regular or singular behaviour of this system (especially the regularity question for the so-called Leray-Hopf solutions to this system). And this will be one of the focal points of this thesis.

Although this regularity problem still remains unsolved, a lot of progress has been made, which allows us to better understand the regular or singular behaviour of this system. Our current understanding of the regularity analysis of Partial Differential Equations (PDEs) allows us to reduce the difficulty of this problem to two main reasons. The first one being the supercriticality of the Navier-Stokes equations in 3D; and we have in general a very poor understanding of supercritical equations. By supercritical, we mean that the globally controlled quantities available for the system are very weak or do not control at all the solution when we move down to smaller scales, in other words when we *zoom-in* on the solution. For the Navier-Stokes equations, those globally controlled quantities are the kinetic energy and the dissipation. To see this, take v to be a smooth solution to (1.1), (1.3) and (1.4) with sufficiently good decay at infinity then test (1.1) with v and integrate by part to get

$$\frac{1}{2}\int_{\mathbb{R}^3}|v(y,t)|^2dy+\int_0^t\int_{\mathbb{R}^3}|\nabla v(y,s)|^2dyds=\frac{1}{2}\int_{\mathbb{R}^3}|u_0(y)|^2dy,\ \text{for all } t>0. \tag{1.10}$$

Next, by taking into account the scaling symmetry (1.5), we observe that

$$\int_{\mathbb{R}^3}|v^\lambda(x,T)|^2dx,\int_0^T\int_{\mathbb{R}^3}|\nabla v^\lambda(x,s)|^2dxds=O(1/\lambda)\ \text{as } \lambda\to 0, \tag{1.11}$$

which supports the lack of control in the smaller scales we mentioned above. The other reason is the non-locality introduced by the incompressibility condition which is characterised by the presence of the pressure. In order to tackle the latter difficulty, one idea would be to find an approximate system to the Navier-Stokes equations which is completely local, study the regularity of solution(s) of this approximation system, and hope to conserve this regularity (if there is) in the limit. Two candidates for these approximation systems are:

$$\mathcal{L}_\kappa u := \partial_t u - \Delta u - \kappa \nabla \operatorname{div} u + u \cdot \nabla u + \frac{1}{2} u \operatorname{div} u = 0 \tag{1.12}$$

and

$$\mathcal{DL}_\kappa u := \partial_t u - \Delta u - \kappa \nabla \operatorname{div} u + \operatorname{div} \left(u \otimes u + \frac{|u|^2}{2} I_3 \right)^1 = 0. \tag{1.13}$$

where $\kappa \geq 0$ and I_3 stands for the identity matrix in 3D. Similarly to the Navier-Stokes system, the above two systems are invariant under the following scaling ($\lambda > 0$):

$$u(x,t) \to u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t); \tag{1.14}$$

and this appears to be the only available scaling symmetry for these systems. Moreover, thanks to the discussion above for the Navier-Stokes system, we also have that the systems $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$ are supercritical in dimension 3. Before moving forward, let us mention the following two papers [55] and [54] where these two systems were studied. In particular, in [54], the authors proved existence of a global smooth (and globally bounded) solution to the systems $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$ with radially symmetric initial data $u_0(x) = -v_0(|x|)x$, such that

$$-C \leq v_0(|x|) \leq C(1 + |x|)^{-\frac{5}{3}} \quad \text{for } x \in \mathbb{R}^3,$$

and $C > 0$.

1.1 The matter of regularity for our toy-model(s)

One of our goals in this thesis was to study the regularity behaviour of the systems (1.12) and (1.13); this was mainly done in Chapter 4, Chapter 6 and Chapter 7. We were able to establish several partial regularity results; for instance, we were able to establish $\epsilon-$regularity results for a much more general class of nonlinear PDEs (see e.g.

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[1] Here $\operatorname{div}(A \otimes B)_i := \partial_1(A_i B_1) + \partial_2(A_i B_2) + \partial_3(A_i B_3)$, with $i = 1, 2, 3$.

Theorem 4.2.1 and Theorem 4.2.5). For our more refined regularity results, we restrict ourselves, most of the time, to the system

$$\mathcal{L}_0 u := \partial_t u - \Delta u + u \cdot \nabla u + \frac{1}{2} u \operatorname{div} u = 0; \tag{1.15}$$

and

$$\mathcal{D}\mathcal{L}_0 u := \partial_t u - \Delta u + \operatorname{div}\left(u \otimes u + \frac{|u|^2}{2} I_3\right) = 0; \tag{1.16}$$

the reasons for such restriction, are on the one hand, that the important results (see for instance Theorem 4.3.1, Theorem 4.4.3, Proposition 4.4.3, Theorem 6.2.1, Theorem 6.3.1) we established for the case $\kappa = 0$ can be extended to the general case $\kappa \neq 0$ just by adding a few lines of computation. On the other hand, because of some technical limitations at this point of time (e.g. the unavailability of a backward uniqueness and unique continuation result for the Lamé operator[2] which we suspect, however, to be true) some important results, such as Theorem 4.4.6 and Theorem 6.4.1, are not completely understood for the general case $\kappa \neq 0$; ultimately, since the term $\kappa \nabla \operatorname{div} u$ introduces some extra coercivity, one can hope to keep for equation (1.12) and (1.13), any extra regularity results obtained for equation (1.15) and (1.16). Some of our results are shared with the Navier-Stokes system and some not, and we achieved that by making use of the absence of the pressure.

A quite useful way one can also go about studying the regular or singular behaviour of solutions of PDEs is through Liouville type theorems; we demonstrate this in Chapter 7. Furthermore, in 'Appendix B', by looking at this Liouville question in the particular setting of the Magneto-Hydrodynamic system, we provide some additional tools for how to tackle this type of problems.

It is also worth mentioning that, if one casts aside the relative link between equations (1.12) (or (1.13)) and the Navier-Stokes system (1.1), the regular or singular behaviour of the system (1.15) in itself is of importance; since this might provide us with a toolbox for the regularity analysis of the wider class of supercritical parabolic systems for which very little is known. Let us also point out that, the regularity results we establish in this thesis hold also true for equations of the form

$$\partial_t u - \Delta u + S(u, \nabla u) = 0,$$

where $S : \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ is bilinear and is such that the energy identity (1.10) holds. However when one tries to perform some analysis in the radial symmetric case, an

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[2] See for instance the book "Distributions, Partial Differential Equations, and Harmonic Analysis" by Dorina Mitrea for discussions and properties of this Operator.

explicit knowledge of the structure of the nonlinearity is necessary (and this structure should be also adequate) in order to conclude, but the methodology to do so is similar to the one we present here.

Lastly, when it comes to studying toy-models for the incompressible Navier-Stokes system, we are not the first ones to do so. In fact, this question has been extensively examined (see for instance [66, 52, 17, 46]). Among those works done on this subject, it's worth mentioning the one by Tao in [66] (see Theorem 1.5 therein) who successfully proved existence of a finite time blowup for a model that satisfies, unlike in the other papers, the energy identity (1.10); to be more specific, he considered the following model:

$$\partial_t u = \Delta u + \tilde{B}(u,u),$$

where $\tilde{B}(\cdot,\cdot)$ is an averaged version (involving rotations and Fourier multipliers of zero order) of the Euler bilinear form (this is just the the bilinear form associated to the nonlinearity in the Navier-Stokes system) that satisfies the following cancellation property $\langle \tilde{B}(u,u), u\rangle = 0$; see the section 'Introduction' in [66] for discussions around this operator. Let us point out that this new nonlinearity loses the algebraic structure of the nonlinearity in the Navier-Stokes system, and in particular any hope of obtaining a good equation for the vorticity "$\operatorname{curl} u$"; therefore, proving some fine properties such as the backward uniqueness theorem (see [13]) or even the Caffarelli-Kohn-Nirenberg results (see [6]) for his model does not seem to be possible. In contrast, we consider a local nonlinearity, with an algebraic structure as close as possible to the one in the Navier-Stokes system; and even though we also don't have any good equation for the vorticity in our case, we were still able to prove for our models (1.15) and (1.16) all the known good properties we have for the Navier-Stokes system. Therefore when it comes to gaining a better understanding of the regular or singular behaviour of the incompressible Navier-Stokes equations, our toy-model appears to be a suitable next step following the work of Tao.

Let us give now more context to our toy-model(s) by discussing some notions of solutions.

1.2 Basic notions of solution for our model

1.2.1 Energy solutions

When the initial data u_0 belongs to $L_2(\Omega)$ (with Ω a bounded Lipschitz domain of $\mathbb{R}^3$ or $\mathbb{R}^3$), we have the existence, for all time, of what we call in this work a weak *energy* *solution* to the systems $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$; they are (with their local analogue defined below) a natural starting point for all our regularity analysis. Later on, we will add more and more information on the initial data in order to get finer properties and additional information from these solutions. An explicit definition of these solutions in our setting is as follows:

Definition 1.2.1. Given $u_0 \in L_2(\Omega)$ (with Ω a bounded Lipschitz domain of $\mathbb{R}^3$ or $\mathbb{R}^3$), a function u is called weak *energy solution* to the Cauchy problem for the system $\mathcal{L}_\kappa = 0$ with initial data u_0 if it has the following properties:

1. $u \in L_{2,\infty}(\Omega \times]0,\infty[)$ and $\nabla u \in L_2(\Omega \times]0,\infty[)$;
2. $\mathcal{L}_\kappa u = 0$ holds in the sense of distributions in $\Omega \times]0,\infty[$;
3. for any test function $w \in C_0^\infty(\Omega)$, the function $t \mapsto \int_{\mathbb{R}^3} u(x,t) \cdot w(x)dx$ is continuous at any time $t \in [0,\infty[$;
4. the initial condition is satisfied in the following sense:
$$\|u(\cdot,t) - u_0\|_{L_2(\Omega)} \xrightarrow[t\to 0^+]{} 0;$$
5. the following global energy inequality holds
$$\frac{1}{2}\int_\Omega |u(x,t)|^2 dx + \int_0^t \int_\Omega (|\nabla u(x,s)|^2 + \kappa |\operatorname{div} u(x,s)|^2) dxds \leq \frac{1}{2}\int_\Omega |u_0(x)|^2 dx,$$
for all $t \in [0,\infty[$.

Remark 1. We have an analogous definition for the system $\mathcal{DL}_\kappa = 0$.

We can find in the literature several existence of energy solutions results for the model $\mathcal{L}_\kappa = 0$. We refer, for instance, to [42, 55, 67] for initial data in $L_2(\Omega)$ (where Ω is the whole $\mathbb{R}^2$ or $\mathbb{R}^3$, or a bounded Lipschitz sub-domain); the construction of these solutions can be done in a same way as for the so-called *Leray-Hopf solutions* in the case of the Navier-Stokes equations (see [45, 25]). If we choose Leray's construction, we find that our energy solution satisfies in addition the following local energy inequality:

$$\int_0^\infty \int_\Omega \left(|\nabla u|^2 + \kappa |\operatorname{div} u|^2\right) \phi(x,t) dxdt \leq \int_0^\infty \int_\Omega \frac{|u|^2}{2}(\partial_t \phi + \Delta\phi) dxdt + \int_0^\infty \int_\Omega (\frac{|u|^2}{2} - \kappa \operatorname{div} u) u \cdot \nabla\phi dxdt,$$

for all $0 \leq \phi \in C_0^\infty(\Omega \times]0,\infty[)$. A simple adaptation of the arguments in those papers yields similar results for the model $\mathcal{DL}_\kappa = 0$. Let us point out that it is not clear, even in the case of the Navier-Stokes system, whether the solution obtained by using Hopf's construction satisfies the above energy inequality.

On more general notes, when one allows the initial data to "grow", for instance, if we require $u_0 \in L_{2,loc}(\mathbb{R}^3)$ such that $\sup_{x_0 \in \mathbb{R}^3} \int_{B(x_0)} |u|^2 dx < \infty$, we are still able to prove global in time existence of a solution. The solution constructed for such initial data is what we call in this work *local-energy solution* to our models; a more concrete definition is as follows.

Definition 1.2.2. A vector field $u \in L_{2,loc}(\mathbb{R}^3 \times [0,\infty[)$ is called a local-energy solution to the system $\mathcal{L}_\kappa = 0$ with initial data u_0 if

1. for all $R > 0$, we have

$$\sup_{x_0 \in \mathbb{R}^3} \left(\operatorname*{ess\,sup}_{0 \le t < R^2} \int_{B(x_0,R)} |u(x,t)|^2 dx + \int_0^{R^2} \int_{B(x_0,R)} |\nabla u(x,t)|^2 dxdt \right) < \infty,$$

and

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B(x_0,R)} |u(x,t)|^2 dxdt = 0 \quad \text{for a.e. } t > 0; \tag{1.17}$$

2. $\mathcal{L}_\kappa u = 0$ holds in the sense of distributions in $\mathbb{R}^3 \times]0,\infty[$ and

$$\|u(\cdot,t) - u_0\|_{L_2(K)} \xrightarrow[t \to 0^+]{} 0,$$

for any compact set $K \subset \mathbb{R}^3$;

3. for any $0 \le \phi \in C_0^\infty(\mathbb{R}^3 \times]0,\infty[)$, the following local energy inequality holds

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \kappa |\operatorname{div} u|^2 \right) \phi(x,t) dxdt \le & \int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) dxdt \\ & + \int_0^\infty \int_{\mathbb{R}^3} (\frac{|u|^2}{2} - \kappa \operatorname{div} u) u \cdot \nabla \phi dxdt. \end{aligned}$$

Remark 2. We have an analogous definition for the model $\mathcal{DL}_\kappa = 0$.

Remark 3. A similar notion of solution is also available for the incompressible Navier-Stokes system; for a proof of existence and important discussions concerning these solutions, see for instance [43] Chapters 32 & 33 or [60] Appendix B (which is based on [34]).

For later purpose, we state the following existence result; this will be discussed in Chapter 3.

Theorem 1.2.1. *Given initial data* $u_0 \in L_{2,loc}(\mathbb{R}^3)$ *such that*

$$\sup_{x_0 \in \mathbb{R}^3} \int_{B(x_0)} |u_0(x)|^2 dx < \infty \quad \textit{and} \quad \lim_{|x_0| \to \infty} \int_{B(x_0)} |u_0(x)|^2 dx = 0,$$

there exists at least one local-energy solution to the Cauchy problem for the system $\mathcal{L}_\kappa = 0$ *and* $\mathcal{DL}_\kappa = 0$*, with initial data* u_0*.*

Remark 4. Like in the case of the incompressible Navier-Stokes system, we don't know whether or not we have a uniqueness result for the energy solutions (or for their local counterpart) for our models in 3D; we discuss this at the end of this chapter.

In this work, we put great emphasis on the local regularity behaviour of solutions to the models $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$. The minimal adequate setting for this analysis is what we call *suitable weak solutions*; for the system $\mathcal{L}_\kappa = 0$, a concrete definition is as follows.

Definition 1.2.3. Let Ω be a domain of $\mathbb{R}^3$. We say that u is a *suitable* weak solution to $\mathcal{L}_\kappa = 0$ in $\Omega \times]T_1, T_2[$ if u obeys the following conditions:

1. $u \in L_{2,\infty}(\Omega \times]T_1, T_2[) \cap L_2(T_1, T_2; W^1_2(\Omega))$;

2. $\mathcal{L}_\kappa u = 0$ holds in the sense of distributions in $\Omega \times]T_1, T_2[$;

3. The local energy inequality

$$\int_\Omega \phi |u(x,t)|^2 dx + 2 \int_{T_1}^t \int_\Omega \phi (|\nabla u|^2 + \kappa |\operatorname{div} u|^2) dx ds \leq \int_{T_1}^t \int_\Omega |u|^2 (\partial_t \phi + \Delta \phi) dx ds$$
$$+ \int_{T_1}^t \int_\Omega u.\nabla \phi (|u|^2 - \kappa \operatorname{div} u) dx ds,$$

holds for a.e $t \in]T_1, T_2[$ and all non-negative functions $\phi \in C^\infty_0(\Omega \times]T_1, \infty[)$.

Remark 5. Once again, we have an analogous definition for the model $\mathcal{DL}_\kappa = 0$.

Remark 6. Local-energy solutions to our models are obviously suitable weak solutions; and so are energy solutions constructed by following Leray's method.

Remark 7. The notion of suitable weak solutions appears quite naturally in the literature regarding regularity of weak solutions to PDEs. And, an interesting question about them is whether or not one can prove that any weak solution is actually a suitable one, i.e. whether, by starting with a distributional solution in the energy class associated to the equation, one is able to show that this solution satisfies a local energy inequality. For both of our models $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$, we are unable to prove such result, and this is linked to the fact that we do not have a good theory of uniqueness for these models. However, we provide in Appendix A, a toolbox for tackling this question.

1.2.2 $L_{p,\infty}$-integrability for energy solutions

Rusin proved in [55], for the system $\mathcal{L}_\kappa = 0$, the existence of a mild solution that satisfies an integral equation with an initial data in $L_3(\mathbb{R}^3)$. When the L_3-norm of the initial data is small, this mild solution is global in time, and for initial data with arbitrarily large L_3-norm he could only guarantee short-time existence; although he didn't explicitly state the proof of the last statement, this can be obtained without too much trouble thanks to Lemma 5.2 in his paper (and by using for instance ideas from [16] Theorem 7.1). This mild solution is also smooth away from the initial time; a simpler adaptation of his result yields the same conclusions for the model $\mathcal{D}\mathcal{L}_\kappa = 0$. However, for the case $u_0 \in L_p(\mathbb{R}^3)$ with $p > 3$, we were not able to find in the literature any proof of a similar result; although we cover this case in this manuscript, we are more interested in knowing what happens to our global weak energy solutions when the initial data, in addition of being in $L_2(\mathbb{R}^3)$, is assumed also to be in $L_p(\mathbb{R}^3)$ with $p > 3$. The main result of this section is as follows.

Theorem 1.2.2. *Let $u_0 \in L_2(\mathbb{R}^3) \cap L_p(\mathbb{R}^3)$ with $3 < p < \infty$. Then, for any weak energy solution u to the system $\mathcal{L}_\kappa = 0$ or the system $\mathcal{D}\mathcal{L}_\kappa = 0$, we have that there exists a $T = T(\kappa, u_0) > 0$ such that*

$$\|u\|_{L_\infty(0,T;L_p(\mathbb{R}^3))} + \|u\|_{L_{\frac{5p}{3}}(\mathbb{R}^3\times]0,T[)} + \|\sqrt{t}\nabla u\|_{L_{\frac{5p}{3}}(\mathbb{R}^3\times]0,T[)} \leq C(\kappa, u_0);$$

moreover, we have uniqueness of energy solutions on the time interval $[0, T]$.

Remark 8. We prove an analogue of this result for local-energy weak solutions in Chapter 5 (see Theorem 5.3.1).

The proof of this theorem relies on the following two results.

Theorem 1.2.3. *Given initial data $u_0 \in L_2(\mathbb{R}^3) \cap L_p(\mathbb{R}^3)$ with $3 < p < \infty$, there exist a positive time $T_* = T_*(\kappa, u_0)$, and a weak energy solution u to the system $\mathcal{L}_\kappa = 0$ or $\mathcal{D}\mathcal{L}_\kappa = 0$, with initial data u_0 such that: for all $t \in [0, T_*]$,*

$$\|u(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_0^t (\|\nabla u(\cdot,t')\|^2_{L_2(\mathbb{R}^3)} + \kappa\|\operatorname{div} u(\cdot,t')\|^2_{L_2(\mathbb{R}^3)})dt' = \|u_0\|^2_{L_2(\mathbb{R}^3)},$$

$$\|u(\cdot,t) - u_0\|_{L_2(\mathbb{R}^3)\cap L_p(\mathbb{R}^3)} \xrightarrow[t\to 0^+]{} 0,$$

$$\|u\|_{L_\infty(0,T_*;L_p(\mathbb{R}^3))} + \|u\|_{L_{\frac{5p}{3}}(\mathbb{R}^3\times]0,T_*[)} + \|\sqrt{t}\nabla u\|_{L_{\frac{5p}{3}}(\mathbb{R}^3\times]0,T_*[)} \leq C(\kappa, u_0);$$

moreover, we have the following lower bound on T_*:

$$T_* \geq \frac{c(\kappa,p)}{\|u_0\|^{\frac{2p}{p-3}}_{L_p(\mathbb{R}^3)}}.$$

Remark 9. The case $p = 3$, of the above two results, was considered by Rusin in [55].

Next, we have a version of weak-strong uniqueness for our systems.

Proposition 1.2.1. *Let* $u_0 \in L_2(\mathbb{R}^3) \cap L_p(\mathbb{R}^3)$ *with* $3 \leq p < \infty$*; we denote by* u_* *the weak energy solution constructed in the previous theorem. Then, for any weak energy solution* u *to the system* $\mathcal{L}_\kappa = 0$ *or* $\mathcal{DL}_\kappa = 0$ *with initial data* u_0*, we have*

$$u = u_* \text{ in } \mathbb{R}^3 \times]0, T_*[.$$

The proof of these results will be presented in Chapter 3.

1.2.3 Strong solutions

Let $u_0 \in \overset{\circ}{L}{}^1_2(\Omega)$, with Ω a bounded Lipschitz subdomain of $\mathbb{R}^3$ or $u_0 \in H^1(\mathbb{R}^3)$; here $\overset{\circ}{L}{}^1_2(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot \|_{L_2(\Omega)}$, and $H^1(\mathbb{R}^3)$ the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\| \cdot \|_{L_2(\mathbb{R}^3)} + \|\nabla \cdot \|_{L_2(\mathbb{R}^3)}$. In this work, a strong solution to our models is defined as follows.

Definition 1.2.4. Let u_0 be as above, with Ω a bounded smooth subdomain of $\mathbb{R}^3$ or $\mathbb{R}^3$; a function u is said to be a strong solution to the model $\mathcal{L}_\kappa = 0$ in $\Omega \times]0, T[$ with initial data u_0 if the following points hold:

1. $u \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; \overset{\circ}{L}{}^1_2(\Omega))$,

$$\|u\|^2_{C([0,T];L_2(\Omega))} + 2\|\nabla u\|^2_{L_2(\Omega \times]0,T[)} + 2\kappa \|\operatorname{div} u\|^2_{L_2(\Omega \times]0,T[)} = \|u_0\|^2_{L_2(\Omega)}$$

   and

$$\|u(\cdot, t) - u_0\|_{L_2(\Omega)} \xrightarrow[t \to 0^+]{} 0;$$

2. $\mathcal{L}_\kappa u = 0$ holds in the sense of distributions in $\Omega \times]0, T[$ and;

3. $\nabla u \in L_\infty(0, T; L_2(\Omega))$.

Remark 10. We have a similar definition for the model $\mathcal{DL}_\kappa = 0$.

We have the following two results concerning existence of strong solutions for our models.

Theorem 1.2.4. *There exists a positive constant* $c_0 = c_0(\Omega)$ *such that if*

$$\arctan(\|\nabla u_0\|^2_{2,\Omega}) + c_0 \|u_0\|^2_{2,\Omega} < \frac{\pi}{2}, \tag{1.18}$$

then there exists a unique strong solution to $\mathcal{L}_\kappa = 0$ *or* $\mathcal{DL}_\kappa = 0$ *in* $\mathbb{R}^3 \times]0, T[$*, for all* $T > 0$*, and with initial data* u_0*.*

Next, we have

Theorem 1.2.5. *Given initial data* $u_0 \in \overset{\circ}{L}{}^1_2(\Omega)$ *or* $H^1(\mathbb{R}^3)$*, there exists a positive time* T_* *such that the models* $\mathcal{L}_\kappa = 0$ *and* $\mathcal{DL}_\kappa = 0$ *each have a unique strong solution in* $\Omega \times]0, T_*[$*, with initial data* u_0*; moreover, we have the following estimation of* T_**:*

$$T_* \geq \frac{1}{\|\nabla u_0\|^4_{2,\Omega}}.$$

We give the proof of the above theorems in Chapter 3.

Remark 11. We have also an analogue of Proposition 1.2.1, for strong solutions, if the solution u_* therein is assumed instead to be a strong solution. The proof of this statement is similar to the one of Proposition 1.2.1.

Remark 12. Unfortunately, we are unable (even for the Navier-Stokes system) to show that the horizon of time we obtained in the previous two sections can be pushed to infinity. If we were to have that ($T_* = +\infty$), this combined with our regularity results would guarantee global smoothness.

1.3 The question of uniqueness for the Navier-Stokes system and for our toy-model(s)

Let us discuss now the question of uniqueness for the Cauchy problem for system (1.1) with divergence-free initial data u_0 in L_2 or $(-1)-$homogeneous divergence-free initial data; these are spaces of initial data for which the existence of a global in time solution is known (see for instance [45, 25] and [31, 4, 9]). An answer to this question will give more insight on Ladyzhenskaya's question. Regarding this matter, Jia and Šverak in [32], upon assuming a certain spectral condition proved non-uniqueness for such a Cauchy problem. Let us give the intuition behind what they did; we focus here only on the $(-1)-$homogeneous case since the L_2 initial data case uses some technical localisation arguments which are not our goal here (see [32] Section 5 where this was done). For a divergence-free $(-1)-$homogeneous vector field $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ and for any $\sigma \in \mathbb{R}$, Jia and Šverak were able to prove existence of a solution to the Cauchy problem for the Navier-Stokes system with initial data σu_0. What they did, in order to achieve that, was to seek a solution of the form:

$$u_\sigma(x,t) = \frac{1}{\sqrt{t}} U_\sigma(\frac{x}{\sqrt{t}}),$$

where $U_\sigma \in C^\infty(\mathbb{R}^3)$ and is such that

$$|\partial^\alpha(U_\sigma - \sigma e^\Delta u_0)(y)| \leq \frac{C(\alpha, \sigma u_0)}{(1+|y|)^{3+|\alpha|}}, \quad (\forall \alpha \in \mathbb{N}^3),$$

and U_σ satisfies the following equation:

$$\begin{cases} -\Delta U_\sigma - U_\sigma \cdot \nabla U_\sigma - \dfrac{x}{2} \cdot \nabla U_\sigma - \dfrac{U_\sigma}{2} + \nabla P = 0 \\ \operatorname{div} U_\sigma = 0 \end{cases} \text{in } \mathbb{R}^3;$$

As mentioned above, such an existence result was also proved in [4, 9]. Next, for the non-uniqueness they looked for a distinct solution, in the vicinity of u_σ of the form

$$u(x,t) = \frac{1}{\sqrt{t}} U_\sigma(\frac{x}{\sqrt{t}}) + \frac{1}{\sqrt{t}} \phi(\frac{x}{\sqrt{t}}, t);$$

The equation for ϕ is:

$$t\phi_t = \Delta\phi + \frac{x}{2} \cdot \nabla\phi + \frac{\phi}{2} - U_\sigma \cdot \nabla\phi - \phi \cdot \nabla U_\sigma - \phi \cdot \nabla\phi + \nabla\pi \tag{1.19}$$

with $\phi_t := \phi(\cdot, t)$. The linearisation of this equation (around 0) will be written as:

$$\mathcal{K}_\sigma \phi = t\phi_t \tag{1.20}$$

where

$$\mathcal{K}_\sigma \phi = \Delta\phi + \frac{x}{2} \cdot \nabla\phi + \frac{\phi}{2} - U_\sigma \cdot \nabla\phi - \phi \cdot \nabla U_\sigma + \nabla P,$$

and P is chosen so that $\operatorname{div} K_\sigma \phi = 0$. The goal now is to solve equation (1.19) via perturbation methods; this requires a good understanding of the operator K_σ and a fortiori its spectral properties. Heuristically, we can assimilate equation (1.19) into equation (1.20); now for the existence of an adequate solution to (1.20), this is done as soon as we have an eigenfunction, say Φ, to K_σ with an eigenvalue λ which has a positive real part and $\Phi(x) = o(|x|^{-1})$ as $|x| \to \infty$. Indeed, $\phi(x,t) = t^\lambda \Phi(x)$ is a solution to (1.20) with initial value 0; and this will guarantee the desired non-uniqueness. All that is left is to tweak the spectrum of K_σ (by playing with the values of σ) in order to obtain the above spectral property. And this is where the spectral conditions in their paper come from (see assumptions **(A)** and **(B)** in their paper which are quite technical). So far, these conditions can only be verified experimentally; and this was done by Guillod and Šverak in [23].

In this thesis, we will also address the uniqueness question for the Cauchy problem for the system (1.1) for a $(-1)-$homogeneous divergence-free initial data. Our strategy is to construct solutions to the system (1.1) as singular limits of solutions to $\mathcal{L}_\kappa = 0$

and $\mathcal{DL}_\kappa = 0$ (we used ideas from [31] for our construction). Our hope is that, since we have a priori at least two solutions in our hands (one being a limit of solutions to $\mathcal{L}_\kappa = 0$ and the other being a limit of solutions to $\mathcal{DL}_\kappa = 0$), this might lead to a non-uniqueness phenomenon for the Navier-Stokes system. At this point in time, we are unable to prove that these singular limits are distinct. However, in our effort to do so, we were able to prove some new estimates (see Theorem 5.1.3 and Theorem 5.6.1) and global (on the whole of $\mathbb{R}^3$) convergence results (see Theorem 5.1.4 and Theorem 5.6.1).

Let us conclude this section and chapter by making the following remarks. Thanks to numerical experiments performed by Guillod (this will be presented in a forthcoming paper) we noticed similar non-uniqueness behaviour, as in the case of the Navier-Stokes system, for the system $\mathcal{L}_\kappa = 0$. However, for the system $\mathcal{DL}_\kappa = 0$, things behave differently and quite strangely; we observed, in the same experimental settings, a unique solution with a gradient that blows-up, which is inconsistent with the mathematical analysis.

1.4 Outline of this work

Our work is structured as follows.

In the next chapter, Chapter 2, we explain our notation; we also establish and recall some results that we will use often in this work.

In Chapter 3, we discuss and provide proofs of the various existence results enunciated in this chapter.

In Chapter 4, we lay the basis for the regularity analysis we carry out in this work. We prove here several partial regularity results and discuss their implications; most of the results in this chapter are shared with the incompressible Navier-Stokes system. This chapter is based on my paper [27].

In Chapter 5, we present two constructions of forward self-similar solutions to the 3D incompressible Navier-Stokes system, as the singular limits of the systems $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$. We establish several new estimates and convergence results. Our presentation is based on my paper [29].

In Chapter 6, we record some new regularity results for our model; some of these results are not yet known to hold for the Navier-Stokes system and some are simply not true. The main reason why we were able to obtain these results is the absence of the pressure term in our models. This chapter is based on my paper [28].

In Chapter 7, we make the link between Liouville type results and regularity analysis for our models; we give also a couple of Liouville type theorems.

In Chapter 8: Conclusion, we present a conclusion to the work carried out in this thesis and give some interesting directions for further study of the regularity question for our models.

Finally, in the appendices, we study some particular equations from fluid dynamics and provide some tools to tackle questions such as “establishing a local energy identity” and proving “Liouville type theorems”. Part of this chapter is based on my paper [26].

Chapter 2

Preliminary material

2.1 Notations

We introduce in this section the notation we will use throughout this work.

We denote for simplicity $f_{,i} := \partial_i f$ and summation over repeated indices running from 1 to n is adopted.

We define:

$$]a,b[:= \{t : a < t < b\}, \quad [a,b] := \{t : a \leq t \leq b\},$$
$$[a,b[:= \{t : a \leq t < b\}, \quad]a,b] := \{t : a < t \leq b\}.$$

For arbitrary vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ in $\mathbb{R}^n$, and arbitrary matrices $F = (F_{ij})$ and $G = (G_{ij})$ in $\mathbb{R}^{n\times n}$, we have the following operations:

$$a \cdot b = a_i b_i, \quad |a| = \sqrt{a \cdot a}, \quad a \otimes b = (a_i b_j) \in \mathbb{R}^{n\times n}$$
$$F : G = F_{ij} G_{ij}, \quad |F| = \sqrt{F : F}.$$

For spatial and space-time domains, we use in this work the following notations:

$$x_0 = (x_{01}, x_{02}, x_{03}) \in \mathbb{R}^3,\ R > 0,\ B(x_0, R) = \{x \in \mathbb{R}^3 : |x - x_0| < R\}$$
$$\overline{B(x_0, R)} = \{x \in \mathbb{R}^3 : |x - x_0| \leq R\},\ B(R) = B(0, R),\ \overline{B(R)} = \overline{B(0, R)},$$
$$z_0 = (x_0, t_0),\ Q(z_0, R) = B(x_0, R) \times]t_0 - R^2, t_0[$$
$$\overline{Q(z_0, R)} = \overline{B(x_0, R)} \times [t_0 - R^2, t_0],\ Q(R) = Q(0, R),\ \overline{Q(R)} = \overline{Q(0, R)}$$
$$B_+(x_0, R) = \{x = (x_1, x_2, x_3) \in B(x_0, R) : x_3 > x_{03}\}$$
$$\overline{B_+(x_0, R)} = \{x = (x_1, x_2, x_3) \in \overline{B(x_0, R)} : x_3 \geq x_{03}\}$$
$$Q_+(z_0, R) = B_+(x_0, R) \times]t_0 - R^2, t_0[$$
$$\overline{Q_+(z_0, R)} = \overline{B_+(x_0, R)} \times [t_0 - R^2, t_0],\ Q_+(R) = Q_+(0, R),\ \overline{Q_+(R)} = \overline{Q_+(0, R)}$$

For various mean values of functions, we write

$$[f]_{x_0,R} := \fint_{B(x_0,R)} f(x)dx \left(= \frac{1}{|B(R)|}\int_{B(x_0,R)} f(x)dx\right), \quad [f]_{,R} := [f]_{0,R},$$
$$(g)_{z_0,R} := \fint_{Q(z_0,R)} g(z)dz \left(= \frac{1}{|Q(R)|}\int_{Q(z_0,R)} g(z)dz\right) \quad (g)_{,R} = (g)_{0,R}.$$

Here $|\omega|$ and $|\Omega|$ stand for the 3 and 4-dimensional Lebesgue measure of the domains ω and Ω respectively.

Set $Q_{T_1,T_2} = \Omega\times]T_1,T_2[$, where Ω is a domain in $\mathbb{R}^3$. The notation for mixed Lebesgue and Sobolev spaces is as follows: $L_{m,n}(Q_{T_1,T_2}) := L_n(T_1,T_2;L_m(\Omega))$, the Lebesgue space with the norm

$$\|v\|_{m,n,Q_{T_1,T_2}} = \begin{cases} \left(\displaystyle\int_{T_1}^{T_2} \|v(\cdot,t)\|^n_{L_m(\Omega)}dt\right)^{1/n} & \text{if } 1 \leq n < \infty \\ \operatorname*{ess\,sup}_{t\in(T_1,T_2)} \|v(\cdot,t)\|_{L_m(\Omega)} & \text{if } n = \infty, \end{cases}$$

$$L_m(Q_{T_1,T_2}) = L_{m,m}(Q_{T_1,T_2}), \quad \|v\|_{m,m,Q_{T_1,T_2}} = \|v\|_{m,Q_{T_1,T_2}};$$

$W^{1,0}_{m,n}(Q_{T_1,T_2})$, $W^{2,1}_{m,n}(Q_{T_1,T_2})$ are the Sobolev spaces with mixed norm,

$$W^{1,0}_{m,n}(Q_{T_1,T_2}) = \{v, \nabla v \in L_{m,n}(Q_{T_1,T_2})\},$$

$$W^{2,1}_{m,n}(Q_{T_1,T_2}) = \left\{v, \nabla v, \nabla^2 v, \partial_t v \in L_{m,n}(Q_{T_1,T_2})\right\},$$
$$W^{1,0}_{m}(Q_{T_1,T_2}) = W^{1,0}_{m,m}(Q_{T_1,T_2}), \quad W^{2,1}_{m}(Q_{T_1,T_2}) = W^{2,1}_{m,m}(Q_{T_1,T_2}).$$

Now, we introduce the following parabolic distance:

$$|z_1 - z_2|_{par} := |x_1 - x_2| + \sqrt{|t_2 - t_1|},$$

for any $z_1 = (x_1,t_1)$ and $z_2 = (x_2,t_2)$. We say that $u \in C^{\alpha,\frac{\alpha}{2}}(Q_{T_1,T_2})$, i.e. u is Hölder continuous with exponent α if:

$$|u|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_1,T_2})} := \sup_{\substack{z_1,z_2\in Q_{T_1,T_2}\\ z_1\neq z_2}} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|_{par}} < \infty;$$

and the Hölder norm is defined as follows:

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_1,T_2})} := \|u\|_{\infty,Q_{T_1,T_2}} + |u|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_1,T_2})}.$$

In this work, we denote by $S(\nu t)$ the semigroup associated to the heat equation

$$\partial_t u - \nu\Delta u = 0. \tag{2.1}$$

To be more precise, $u(x,t) = S(\nu t)u_0(x)$ denotes the unique solution of (2.1) in $\mathbb{R}^n \times (0,\infty)$ with initial data u_0. Moreover, if u_0 belongs to a suitable function space, the following formula is available.

$$S(\nu t)u_0(x) := \Gamma_\nu(\cdot,t) \star_x u_0 = \int_{\mathbb{R}^n} \Gamma_\nu(x-y,t)u_0(y)dy,$$

where

$$\Gamma_\nu(x,t) = \frac{1}{(4\pi\nu t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4\nu t}\right).$$

Finally, we use c or C to denote an absolute constant and we write $C(A,B,\ldots)$ when the constant depends on the parameters $A, B, \ldots$.

2.2 Lorentz spaces

In this work $L^{p,\infty}(\Omega)$ $(0 < p < \infty)$ stands for the *weak* $L_p(\Omega)$ space of functions f such that

$$\|f\|_{L^{p,\infty}(\Omega)} := \sup_{\gamma>0} \left\{\gamma|\{x \in \Omega : |f(x)| > \gamma\}|^{\frac{1}{p}}\right\} < \infty.$$

It is not difficult to show that $L_p(\Omega) \subset L^{p,\infty}(\Omega)$, and this holds for Ω with finite measure or not (see [22] for more properties of this function space). Those are a special case of the Lorentz spaces $L^{p,q}(\Omega)$ (with $0 < p, q \leq \infty$) which consist, when $p, q \neq \infty$, of functions f such that

$$\|f\|_{L^{p,q}(\Omega)} := p^{\frac{1}{q}} \left(\int_0^\infty s^{q-1}|\{x \in \Omega : |f(x)| > s\}|^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty,$$

with $L^{\infty,q} = \{0\}$ whenever $0 < q < \infty$ and $L^{p,p} = L_p$ for every $0 < p \leq \infty$.

The following inequalities will be very useful later.

Proposition 2.2.1. *(See [22]) Let (X,μ) be a measure space.*

1. *Let $f \in L^{p,\infty}(X,\mu)$ for some $0 < p < \infty$ and let E be a subset of X such that $\mu(E) < \infty$; then*

$$\int_E |f(x)|^q d\mu(x) \leq \frac{p}{p-q}\mu(E)^{1-\frac{q}{p}}\|f\|^q_{L^{p,\infty}(X,\mu)}.$$

2. *We have next the so-called O'Neil's inequality; this is a version of Hölder's inequality for Lorentz spaces and in particular weak Lebesgue spaces. We have*

$$\|fg\|_{L^{r,s}} \leq C_{p,q,s_1,s_2}\|f\|_{L^{p,s_1}}\|g\|_{L^{q,s_2}},$$

where $0 < p, q, r \leq \infty$, $0 < s_1, s_2 \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}$.

We recall now the following known decomposition property in weak Lebesgue spaces; for a proof, we refer to Lemma 3.1 in [51].

Lemma 2.2.1. *Take* $1 < t < r < s < \infty$*, and suppose that* $g \in L^{r,\infty}(\mathbb{R}^3)$*; for any* $N > 0$*, set* $\bar{g}^N := g\mathbb{1}_{|g|\le N}$ *and* $\hat{g}^N := g - \bar{g}^N$*. Then*

$$\|\bar{g}^N\|^s_{L_s(\mathbb{R}^3)} \le \frac{s}{s-r} N^{s-r}\|g\|^r_{L^{r,\infty}(\mathbb{R}^3)} - N^s \left|\{x \in \mathbb{R}^3 : |g(x)| > N\}\right|$$

and

$$\|\hat{g}^N\|^t_{L_t(\mathbb{R}^3)} \le \frac{r}{r-t} N^{t-r}\|g\|^r_{L^{r,\infty}(\mathbb{R}^3)}.$$

2.3 Solvability of the Lamé system and related estimates

Lemma 2.3.1. *Set* $Q_T := \mathbb{R}^3\times]0,T[$ *(*$T > 0$*); let* $f \in L_{s,l}(Q_T)$ *and* $F \in L_{s,l}(Q_T;\mathbb{R}^{3\times 3})$ *with* $1 < s, l < \infty$*, and* $\kappa \ge 0$*. There exist two functions* v *and* w *that uniquely solve the systems*

$$\begin{cases} \partial_t v - \Delta v - \kappa\nabla \operatorname{div} v = f \quad \text{in } Q_T \\ v|_{t=0} = 0 \quad \text{in } \mathbb{R}^3 \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w - \Delta w - \kappa\nabla \operatorname{div} w = \operatorname{div} F \quad \text{in } Q_T \\ w|_{t=0} = 0 \quad \text{in } \mathbb{R}^3 \end{cases}$$

such that

$$\|\partial_t v\|_{L_{s,l}(Q_T)} + \|\nabla^2 v\|_{L_{s,l}(Q_T)} \le c(\kappa, s, l)\|f\|_{L_{s,l}(Q_T)}, \quad \|\nabla w\|_{L_{s,l}(Q_T)} \le c(\kappa, s, l)\|F\|_{L_{s,l}(Q_T)};$$

$$\|\nabla v\|_{L_{l_1}(0,T;L_{s_1}(\mathbb{R}^3))} \le c(\kappa, s, s_1, l, l_1)T^{\frac{1}{2}\left(1-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_1}+\frac{3}{s_1}\right)}\|f\|_{L_{s,l}(Q_T)} \ \forall T \ge 0,$$
$$\text{with } s \le s_1 \le \infty,\ l \le l_1 \le \infty \text{ such that } 1 - \frac{2}{l} - \frac{3}{s} + \frac{2}{l_1} + \frac{3}{s_1} > 0;$$

$$\|v\|_{L_{l_2}(0,T;L_{s_2}(\mathbb{R}^3))} \le c(\kappa, s, s_2, l, l_2)T^{\frac{1}{2}\left(2-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_2}+\frac{3}{s_2}\right)}\|f\|_{L_{s,l}(Q_T)} \ \forall T \ge 0,$$
$$\text{with } s \le s_2 \le \infty,\ l \le l_2 \le \infty \text{ such that } 2 - \frac{2}{l} - \frac{3}{s} + \frac{2}{l_2} + \frac{3}{s_2} > 0;$$

$$\|w\|_{L_{l_3}(0,T;L_{s_3}(\mathbb{R}^3))} \le c(\kappa, s, s_3, l, l_3)T^{\frac{1}{2}\left(1-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_3}+\frac{3}{s_3}\right)}\|F\|_{L_{s,l}(Q_T)} \ \forall T \ge 0,$$
$$\text{with } s \le s_3 \le \infty,\ l \le l_3 \le \infty \text{ such that } 1 - \frac{2}{l} - \frac{3}{s} + \frac{2}{l_3} + \frac{3}{s_3} > 0.$$

Finally, we have Hölder continuity of v *as soon as* $\mu := 2 - \frac{2}{l} - \frac{3}{s} > 0$ *and for* w*, when* $\alpha := 1 - \frac{2}{l} - \frac{3}{s} > 0$*; to be more precise, we have the following estimates:*

$$|v(z_1) - v(z_2)| \le c(\kappa, s, l)\left(|x_1 - x_2| + \sqrt{|t_1 - t_2|}\right)^{\mu} \|f\|_{L_{s,l}(Q_T)}$$

and similarly

$$|w(z_1) - w(z_2)| \le c(\kappa, s, l)\left(|x_1 - x_2| + \sqrt{|t_1 - t_2|}\right)^{\alpha} \|F\|_{L_{s,l}(Q_T)},$$

for all $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in Q_T$*.*

Sketch of proof. Uniqueness is straightforward. For the existence part, we present only the proof for the function w since the function v's case follows the same ideas.
There exists a function q (using the Newtonian representation for solutions of the Poisson equation together with singular integral theory) such that

$$\Delta q = \operatorname{div}\operatorname{div}(F),$$

and

$$\|q\|_{L_{s,l}(Q_T)} \leq c\|F\|_{L_{s,l}(Q_T)}$$

Next, we introduce the function $F_0 := \operatorname{div}(F - qI_3)$ and let us notice that $\operatorname{div} F_0 = 0$ in $\mathcal{D}'(Q_T)$. From well-known solvability results for the heat equation (see e.g. [39, 41]), we have the existence of two functions w^1 and w^2 such that

$$\begin{cases} \partial_t w^1 - (1+\kappa)\Delta w^1 = \nabla q \quad \text{in } Q_T \\ \qquad w^1|_{t=0} = 0 \quad \text{in } \mathbb{R}^3 \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w^2 - \Delta w^2 = F_0 \quad \text{in } Q_T \\ \qquad w^2|_{t=0} = 0 \quad \text{in } \mathbb{R}^3 \end{cases} \tag{2.2}$$

Moreover, the following estimate is available: For all $T > 0$,

$$\|\nabla w^1\|_{L_{s,l}(Q_T)} + \|\nabla w^2\|_{L_{s,l}(Q_T)} \leq c(\kappa, s, l)\|F\|_{L_{s,l}(Q_T)}.$$
$$\|w^1\|_{L_{l_3}(0,T;L_{s_3}(\mathbb{R}^3))} + \|w^2\|_{L_{l_3}(0,T;L_{s_3}(\mathbb{R}^3))} \leq c(\kappa, s, s_3, l, l_3)T^{\frac{1}{2}\left(1-\frac{2}{l}-\frac{3}{s}+\frac{3}{s_1}\right)}\|F\|_{L_{s,l}(Q_T)},$$
$$\text{with } s \leq s_3 \leq \infty,\ l \leq l_3 \leq \infty \text{ such that } 1 - \frac{2}{l} - \frac{3}{s} + \frac{3}{s_1} > 0.$$

The latter estimate comes from well-known properties of the volume heat potential but the former is a bit more subtle; we refer to [38] (Theorem 1.1) for a proof of this statement. Next, using Campanato's characterisation of Hölder continuous functions (see the exact statement in the miscellaneous section), one gets without too much difficulty (and with the help of Poincaré's inequality on balls) that

$$|w^1(z_1) - w^1(z_2)| + |w^2(z_1) - w^2(z_2)| \leq c(\kappa, s, l)\left(|x_1 - x_2| + \sqrt{|t_1 - t_2|}\right)^\alpha \|F\|_{L_{s,l}(Q_T)},$$

for all $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in Q_T$ as long as $\alpha > 0$.
Finally, notice that $\operatorname{curl} w^1 = \operatorname{div} w^2 = 0$ in Q_T and $\Delta w^1 = \nabla \operatorname{div} w^1$ (at least in the sense of distributions); and we are done by setting $w := w^1 + w^2$. □

Remark 13. The estimates we obtained in the Lemma can be made explicit with respect to κ; see for instance Proposition 5.6.2 below where this is done.

We also have a solvability result for the Lamé system in the case of the Cauchy problem.

Lemma 2.3.2. *Let $u_0 \in L_{s_1}(\mathbb{R}^3)$ with $1 \leq s_1 < \infty$; there exists a unique function u such that*

$$\begin{cases} \partial_t u - \Delta u - \kappa \nabla \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \times]0, +\infty[\\ \qquad\qquad u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3. \end{cases} \qquad (\kappa \geq 0)$$

Moreover, we have

$$\|u(\cdot, t) - u_0\|_{L_{s_1}(\mathbb{R}^3)} \to 0 \text{ as } t \to 0^+$$

and the following estimates hold:

$$\|\partial_t^m \nabla^k u(\cdot, t)\|_{L_s(\mathbb{R}^3)} \leq c(\kappa, s, s_1) \frac{\|u_0\|_{s_1, \mathbb{R}^3}}{t^{m + \frac{k}{2} + \frac{3}{2}(\frac{1}{s_1} - \frac{1}{s})}}, \tag{2.3}$$

and

$$\|u\|_{L_{s,l}(\mathbb{R}^3 \times]0, \infty[)} \leq c(\kappa, s, s_1) \|u_0\|_{s_1, \mathbb{R}^3}, \tag{2.4}$$

with $\frac{1}{l} = \frac{3}{2}(\frac{1}{s_1} - \frac{1}{s})$.

Sketch of proof. Once again, uniqueness is straightforward. As in the proof of the previous lemma, we find a function q that solves the equation

$$\Delta q = \operatorname{div} u_0,$$

and such that

$$\|\nabla q\|_{s_1, \mathbb{R}^3} \leq c(s_1) \|u_0\|_{s_1, \mathbb{R}^3}.$$

Now, set $u_0^{(1)} := \nabla q$ and $u_0^{(2)} := u_0 - u_0^{(1)}$. Next, define $u^{(1)} = S((1 + \kappa)t) u_0^{(1)}$ and $u^{(2)} = S(\kappa t) u_0^{(2)}$. The solutions $u^{(1)}$ and $u^{(2)}$ satisfy the estimates (2.3) and (2.4). The first estimate comes from the convolution structure of these solutions, combined with a straightforward application of Young's inequality and scaling arguments; the second one is a bit tricky and was proved in [41] (Chapter 3, Theorem 9.1). And we are done by setting $u = u^{(1)} + u^{(2)}$, for the same reasons as in the previous lemma. $\square$

Next, we have the following local regularity result for the time-dependent Lamé system.

Lemma 2.3.3 (Local regularity)**.** *Let $u \in L_2(Q)$ (and $\kappa \geq 0$) such that*

$$\partial_t u - \Delta u - \kappa \nabla \operatorname{div} u = 0 \text{ in } \mathcal{D}'(Q).$$

Then, for any $k = 0, 1, 2, \ldots$ and any $0 < \varrho < 1$, there exists $C = C(\kappa, \varrho, k) > 0$ such that

$$\sup_{(x,t) \in Q(\varrho)} |\nabla^k u(x, t)| \leq C \left(\fint_Q |u|^2 dz \right)^{\frac{1}{2}}.$$

Proof. We see without too much difficulty that

$$\int_{Q((1+\varrho)/2)} |\nabla u|^2 dz \leq c(\kappa, \varrho) \int_Q |u|^2 dz, \tag{2.5}$$

(for an arbitrary $0 < \varrho < 1$) and

$$\partial_t \operatorname{curl} u - \Delta \operatorname{curl} u = 0 \quad \text{and} \quad \partial_t \operatorname{div} u - (1+\kappa)\Delta \operatorname{div} u = 0,$$

in the sense of distributions. Next, from well-known local regularity results for the heat equation (see e.g. [39, 41])

$$\sup_{(x,t)\in Q((1+3\varrho)/4)} |\nabla^k \operatorname{curl} u(x,t)| \leq C(\varrho, k) \left(\int_{Q((1+\varrho)/2)} |\operatorname{curl} u|^2 dz \right)^{\frac{1}{2}}$$

$$\sup_{(x,t)\in Q((1+3\varrho)/4)} |\nabla^k \operatorname{div} u(x,t)| \leq C(\kappa, \varrho, k) \left(\int_{Q((1+\varrho)/2)} |\operatorname{div} u|^2 dz \right)^{\frac{1}{2}}$$

for any $k = 0, 1, 2 \ldots$ Thus, from (2.5), we have

$$\sup_{(x,t)\in Q((1+3\varrho)/4)} (|\nabla^k \operatorname{curl} u(x,t)| + |\nabla^k \operatorname{div} u(x,t)|) \leq c(\kappa, \varrho, k) \left(\int_Q |u|^2 dz \right)^{\frac{1}{2}}, \tag{2.6}$$

for any $k = 0, 1, 2 \ldots$ Finally, using the identity

$$-\Delta u = \operatorname{curl}(\operatorname{curl} u) - \nabla \operatorname{div} u \text{ in } \mathcal{D}'(Q),$$

together with the stationary analogue of the first estimate in Theorem 2.4.9 of [39] (for instance) and taking into account (2.6), we have that the lemma is proved. □

2.4 Miscellaneous

We record in this section some auxiliary results we will need to close arguments in this work.

The following covering lemma is the parabolic analogue of the well-known Vitali's covering lemma for balls.

Lemma 2.4.1 (Vitali's lemma, See e.g. [6]). *Let $\mathcal{J}$ be a family of parabolic cylinders $Q^* = Q^*((x,t), r)(:= B(x,r)\times]t - \frac{7}{8}r^2, t + \frac{1}{8}r^2[)$ contained in a bounded subset of $\mathbb{R}^3 \times \mathbb{R}$. Then there exists a countable subfamily $\mathcal{J}' = \{Q_i^* = Q^*((x^i, t_i), r_i)\}$ such that*

$$Q_i^* \cap Q_j^* = \varnothing \quad \text{for} \quad i \neq j,$$

$$\forall Q^* \in \mathcal{J}, \exists Q_i^* \in \mathcal{J}' \text{ such that } Q^* \subset Q^*((x^i, t_i), 5r_i).$$

The following proposition gives us a parabolic analogue of Gehring reverse Hölder lemma.

Proposition 2.4.1 (See e.g. [18] Proposition 1.3)**.** *Let* $g \geq 0$ *in* Q *and satisfy with some constant* $q > 1$

$$\fint_{Q(z_0,R)} g^q dz \leq b \left(\fint_{Q(z_0,R)} g dz \right)^q + \theta \int_{Q(z_0,4R)} g^q dz$$

for every $z_0 \in Q$ *and* $Q(z_0, 4R) \subset Q$*. Then there exists a constant* $\theta_0 = \theta_0(q, n)$ *such that, if* $\theta < \theta_0$*, then* $g \in L_{p,loc}(Q)$ *for* $p \in [q, q + \epsilon[$ *and*

$$\left(\fint_{Q(z_0,R)} g^p dz \right)^{\frac{1}{p}} \leq c \left(\fint_{Q(z_0,R)} g^q dz \right)^{\frac{1}{q}} \quad \text{for all } Q(z_0, 4R) \subset Q;$$

the constant c *and* ϵ *depending on* b, q, θ *and* n *only.*

We have now the following iteration lemma.

Lemma 2.4.2 (See e.g. [14] Lemma 5.2)**.** *Let* $f : [r/2, r] \to [0, +\infty[$ *be bounded and satisfy*

$$f(t) \leq \theta f(s) + \frac{A}{(s-t)^2} + \frac{B}{(t-s)^q},$$

for some $\theta < 1$ *and all* $r/2 \leq t < s \leq r$*. Then, there exists a constant* $C = C(\theta, q) > 0$ *such that*

$$f\left(\frac{r}{2}\right) \leq C \left(\frac{A}{r^2} + \frac{B}{r^q} \right).$$

Next, we have the following parabolic analogue of Campanato characterisation of Hölder continuous functions. Our statement of this result is taken from Gregory Seregin's lecture notes "Parabolic PDEs"; an easy adaptation to the parabolic case of the proof of a similar result in [24] (Chapter 3, Theorem 3.1) yields also this proposition.

Proposition 2.4.2 (Campanato's Hölder continuity criteria)**.** *Let* $u \in L_1(Q_T)$ *with* $Q_T := \Omega \times]0, T[$*, and* Ω *a domain of* $\mathbb{R}^n$*. Assume that there exist numbers* $A > 0$ *and* $0 < \alpha < 1$ *such that*

$$\fint_{Q(z_0,r)} |u - (u)_{z_0,r}| dz \leq A r^\alpha$$

whenever $Q(z_0, r) \subset Q_T$*.*

For any $\Omega' \subset\subset \Omega$ *and any* $\delta > 0$*, the following statements are valid:*

1. *for* $z_0 \in \overline{\Omega'} \times [\delta, T]$*, there exists* $\lim_{r \to 0} (u)_{z_0,r}$ *and we let*

$$u(z_0) = \lim_{r \to 0} (u)_{z_0,r};$$

2.

$$|u(z_0) - (u)_{z_0,r}| \leq c(n,\alpha) A r^{\alpha};$$

3.

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega'}\times[\delta,T])} \leq c(\Omega,\Omega',T,\delta,n,\alpha)\left[A + \|u\|_{L_1(Q_T)}\right].$$

We conclude this chapter with a special case of the Schauder's inequality in bounded domains

Theorem 2.4.3 (See e.g. [37] Chapter 10, Theorem 10.2.2 & Theorem 10.3.3)**.** *For any* $f \in C^{\delta,\delta/2}(Q)$ *and* $g \in C^{2+\delta,1+\delta/2}(Q)$ *(here* $Q := B\times]-1,0[$*) there exists a unique function* $u \in C^{2+\delta,1+\delta/2}(Q)$ *satisfying the equation* $\partial_t u - \Delta u = f$ *in* Q *and equal to* g *in* $\partial' Q$*. Moreover, there is a constant* N *depending only on* δ *such that:*

$$|u|_{C^{2+\delta,1+\delta/2}(Q)} \leq N(|f|_{C^{\delta,\delta/2}(Q)} + |g|_{C^{2+\delta,1+\delta/2}(Q)}).$$

Remark 14. Here

$$|u|_{C^{2+\delta,1+\delta/2}(Q)} := |\partial_t u|_{C^{\delta,\delta/2}(Q)} + \sum_{i,j=1}^{n} |\partial_{ij} u|_{C^{\delta,\delta/2}(Q)}.$$

Chapter 3

Existence results

3.1 Introduction

In this chapter, we discuss in more detail the existence results, stated in the Introduction, for the systems $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$; we will focus more on the system $\mathcal{L}_\kappa = 0$ since things are the same for $\mathcal{DL}_\kappa = 0$. Let us also point out that we will mostly focus on establishing a priori estimates, which, according to, now, over a century of analysis of PDEs, constitute the key step in proving existence. We do not dive much into the details of the proof of existence, since the details of this analysis, for our models, do not bring that much originality, whatsoever, to this work.

3.2 Existence of local-energy solutions

The first step in proving this result is a short-time existence result, which relies on the following a priori estimate; this estimate was first proved in [43] for the incompressible Navier-Stokes system; here we follow the proof given in [30] (see Lemma 2.2) which is much simpler than the original one.

Lemma 3.2.1 (A priori estimate for local-energy solutions)**.** *Let* $u_0 \in L_{2,loc}(\mathbb{R}^3)$ *be such that for some* $R > 0$, $\alpha(R) := \sup_{x_0\in\mathbb{R}^3} \int_{B(x_0,R)} |u_0(x)|^2 dx < \infty$, *and let* u *be a local-energy solution to the system* $\mathcal{L}_\kappa = 0$ *or* $\mathcal{DL}_\kappa = 0$ *with initial data* u_0*. Then, there exists some small absolute number* $\mu > 0$ *such that for* $0 < \lambda < \mu \min\{1, \alpha(R)^{-2}R^2, (1+\kappa)^{-1}\}$, *we have*

$$\sup_{x_0\in\mathbb{R}^3} \operatorname*{ess\,sup}_{0<t<\lambda R^2} \int_{B(x_0,R)} \frac{|u(x,t)|^2}{2} dx \\ + \sup_{x_0\in\mathbb{R}^3} \int_0^{\lambda R^2} \int_{B(x_0,R)} \left(|\nabla u(x,t)|^2 + \kappa |\operatorname{div} u(x,t)|^2\right) dxdt \leq C_0 \alpha(R),$$

with C_0 an absolute large constant.

Proof. We focus here only on the system $\mathcal{L}_\kappa = 0$ since things are the same for the system $\mathcal{DL}_\kappa = 0$. Let $0 \le \phi \in C_0^\infty(B)$ such that $\phi \equiv 1$ in $B(1/2)$ and $\phi \equiv 0 \in B \setminus B(3/4)$; let $x_0 \in \mathbb{R}^3$, $R > 0$ and set $\phi_{x_0,2R}(x) = \phi((x - x_0)/2R)$. We have, from the local energy inequality satisfied by u, that:

$$\begin{aligned}
&\int_{B(x_0,2R)} \frac{|u(x,t)|^2}{2}\phi_{x_0,2R}(x)dx \\
&\quad + \int_0^t \int_{B(x_0,2R)} \left(|\nabla u|^2 + \kappa(\operatorname{div} u)^2\right)\phi_{x_0,2R}dxds \le \int_{B(x_0,2R)} \frac{|u_0(x)|^2}{2}\phi_{x_0,2R}(x)dx \\
&\quad + \int_0^t \int_{B(x_0,2R)} \frac{|u|^2}{2}\Delta\phi_{x_0,2R}dxds + \int_0^t \int_{B(x_0,2R)} (\frac{|u|^2}{2} - \kappa \operatorname{div} u)u \cdot \nabla\phi_{x_0,2R}dxds.
\end{aligned}$$

Now, set

$$\begin{aligned}
A_R(\lambda) = &\sup_{x_0 \in \mathbb{R}^3} \operatorname*{ess\,sup}_{0<t<\lambda R^2} \int_{B(x_0,R)} \frac{|u(x,t)|^2}{2}dx \\
&+ \sup_{x_0 \in \mathbb{R}^3} \int_0^{\lambda R^2} \int_{B(x_0,R)} \left(|\nabla u(x,t)|^2 + \kappa|\operatorname{div} u(x,t)|^2\right)dxdt \quad (\lambda > 0).
\end{aligned}$$

For a.e. $t \in]0, \lambda R^2[$, we have

$$\begin{aligned}
&\int_{B(x_0,R)} \frac{|u(x,t)|^2}{2}dx + \int_0^t \int_{B(x_0,R)} |\nabla u|^2 dxds \\
&+ \int_0^t \int_{B(x_0,2R)} \kappa|\operatorname{div} u|^2\phi_{x_0,2R}dxds \le C\Bigg(\alpha(R) + \lambda A_R(\lambda) + \frac{1}{R}\int_0^{\lambda R^2} \int_{B(x_0,2R)} |u|^3 dxds \\
&\qquad + \int_0^{\lambda R^2} \int_{B(x_0,2R)} \kappa|\operatorname{div} u|\phi_{x_0,2R}^{\frac{1}{2}}|u||\nabla\phi_{x_0,2R}^{\frac{1}{2}}|dxds\Bigg).
\end{aligned}$$

Next, by a known multiplicative inequality, we have

$$\int_0^{\lambda R^2} \int_{B(x_0,2R)} |u|^3 dxds \le C\lambda^{\frac{1}{4}}R^{\frac{1}{2}}A_R(\lambda)^{\frac{3}{2}} \quad (\text{if } \lambda \le 1),$$

and thanks to Young's inequality that

$$\begin{aligned}
\int_0^{\lambda R^2} \int_{B(x_0,2R)} \kappa|\operatorname{div} u|\phi_{x_0,2R}^{\frac{1}{2}}|u||\nabla\phi_{x_0,2R}^{\frac{1}{2}}|dxds \le \frac{1}{2}\int_0^t \int_{B(x_0,2R)} \kappa|\operatorname{div} u|^2\phi_{x_0,2R}dxds \\
+ C\kappa\lambda A_R(\lambda).
\end{aligned}$$

Consequently, we get that

$$\int_{B(x_0,R)} \frac{|u(x,t)|^2}{2}dx + \int_0^t \int_{B(x_0,R)} |\nabla u|^2 dxds$$
$$+\frac{1}{2}\int_0^t \int_{B(x_0,2R)} \kappa |\operatorname{div} u|^2 \phi_{x_0,2R} dxds \le C\left[\alpha(R) + (1+\kappa)\lambda A_R(\lambda) + \lambda^{\frac{1}{4}} R^{-\frac{1}{2}} A_R(\lambda)^{\frac{3}{2}}\right],$$

for a.e. $t \in]0, \lambda R^2[$ and all $x_0 \in R^3$. Therefore

$$A_R(\lambda) \le C\left[\alpha(R) + (1+\kappa)\lambda A_R(\lambda) + \lambda^{\frac{1}{4}} R^{-\frac{1}{2}} A_R(\lambda)^{\frac{3}{2}}\right];$$

By choosing $\lambda \le \min\{1, (2C(1+\kappa))^{-1}\}$, we find that

$$A_R(\lambda) \le 2C\left(\alpha(R) + \lambda^{\frac{1}{4}} R^{-\frac{1}{2}} A_R(\lambda)^{\frac{3}{2}}\right),$$

and from there the conclusion follows by standard continuation arguments. □

Remark 15. One way to use this result for our existence problem is by considering the following regularised system (if we are looking at the system $\mathcal{L}_\kappa = 0$):

$$\begin{cases} \partial_t v - \Delta v - \kappa \nabla \operatorname{div} v + (v)_\epsilon \cdot \nabla v + \dfrac{v}{2} \operatorname{div}(v)_\epsilon = 0 \quad \text{in } \mathbb{R}^3 \times]0, \infty[, \\ v|_{t=0} = u_{0,\epsilon} \quad \text{in } \mathbb{R}^3; \end{cases} \tag{3.1}$$

where $(v)_\epsilon(\cdot,t) := \varrho_\epsilon \star_x v(\cdot,t)$ and ϱ_ϵ is the usual mollification kernel; and $(u_{0,\epsilon})_{\epsilon>0}$ is a sequence of functions in $C_0^\infty(\mathbb{R}^3)$ such that $\|u_{0,\epsilon} - u_0\|_{L_{2,unif}} \to 0$. Taking the limit $\epsilon \to 0^+$ yields the short-time existence. From this point, it's a matter of following line-by-line the arguments in [43, 44] or Appendix B in [60] to conclude the global in time existence. Since this global in time aspect of the local-energy solutions will not be touched in this work, we skip the details here. The only thing we will need in the sequel (see e.g. proof of Theorem 5.3.1 below) is the above a priori estimate.

3.3 $L_{p,\infty}$ properties for energy solutions

We give now the proof of Theorem 1.2.3.

Proof. The energy identity is straightforward once we have the L_p−estimates; this is due to the extra-regularity we get from these new estimates. Thus, we will focus here on establishing the p−related estimates. We consider once again the regularised system (3.1), with this time, u_0 as initial data (instead of its smooth approximation). We set for simplicity

$$F^\epsilon := (v)_\epsilon \cdot \nabla v + \frac{v}{2} \operatorname{div}(v)_\epsilon;$$

next, we introduce the functions $\hat{v}, F^{(1)}, F^{(2)}, w, w^{(1)}$ and $w^{(2)}$ such that $\hat{v}$ uniquely solves (see Lemma 2.3.2)

$$\begin{cases} \partial_t \hat{v} - \Delta \hat{v} - \kappa \nabla \operatorname{div} \hat{v} = 0 \quad \text{in } \mathbb{R}^3 \times]0,\infty[\\ \hat{v}|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3; \end{cases}$$

$F^{(1)} = \nabla p$ such that

$$\Delta p(\cdot,t) = \operatorname{div} F^{\epsilon}(\cdot,t), \tag{3.2}$$

and $F^{(2)} = F^{\epsilon} - F^{(1)}$, and finally $w = v - \hat{v} = w^{(1)} + w^{(2)}$ where the functions $w^{(i)}$ $(i = 1,2)$ uniquely solve (see Lemma 2.3.1)

$$\begin{cases} \partial_t w^{(1)} - (1+\kappa)\Delta w^{(1)} = -F^{(1)} \text{ in } \mathbb{R}^3 \times \mathbb{R}_+ \\ w^{(1)}|_{t=0} = 0 \text{ in } \mathbb{R}^3 \end{cases}, \quad \begin{cases} \partial_t w^{(2)} - \Delta w^{(2)} = -F^{(2)} \text{ in } \mathbb{R}^3 \times \mathbb{R}_+ \\ w^{(2)}|_{t=0} = 0 \text{ in } \mathbb{R}^3. \end{cases}$$

We also introduce the following norm

$$\|\cdot\|_{\mathcal{Y}_T} := \|\cdot\|_{L_{\frac{5p}{3}}(\mathbb{R}^3 \times]0,T[)} + \|\sqrt{t}\nabla\cdot\|_{L_{\frac{5p}{3}}(\mathbb{R}^3 \times]0,T[)}. \tag{3.3}$$

We have (for $i = 1,2$) that

$$\begin{aligned} \|\sqrt{t}F^{(i)}\|_{L_{\frac{5p}{6}}(\mathbb{R}^3 \times]0,T[)} &\leq c\|\sqrt{t}F^{\epsilon}\|_{L_{\frac{5p}{6}}(\mathbb{R}^3 \times]0,T[)} \\ &\leq c\|v\|_{L_{\frac{5p}{3}}(\mathbb{R}^3 \times]0,T[)} \|\sqrt{t}\nabla v\|_{L_{\frac{5p}{3}}(\mathbb{R}^3 \times]0,T[)} \\ &\leq c\|v\|^2_{\mathcal{Y}_T}, \end{aligned} \tag{3.4}$$

and we claim that we get from the previous estimate, that

$$\|w^{(i)}\|_{\mathcal{Y}_T} \leq c(\kappa,p) T^{\frac{1}{2}-\frac{3}{2p}} \|v\|^2_{\mathcal{Y}_T} \quad (i = 1,2); \tag{3.5}$$

It is enough to prove the claim for $w^{(2)}$ only, since the $w^{(1)}$ case can be reduced to the previous one after rescaling. We have the following representation formula for $w^{(2)}$ (similar representation for $w^{(1)}$)

$$w^{(2)}(x,t) = \int_0^t \Gamma(\cdot, t-s) \star_x F^{(2)}(\cdot,s) ds \tag{3.6}$$

By Young's inequality, and by using scaling arguments for the heat kernel Γ, we get that

$$\begin{aligned} \|w^{(2)}(\cdot,t)\|_{L_{\frac{5p}{3}}(\mathbb{R}^3)} &\leq c(p) \int_0^t \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}(\mathbb{R}^3)}}{\sqrt{s}(t-s)^{\frac{3}{2}\times\frac{3}{5p}}} ds \\ &\leq c(p) \left(\int_0^{t/2} \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}}}{\sqrt{s}(t-s)^{\frac{3}{2}\times\frac{3}{5p}}} ds + \int_{t/2}^t \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}}}{\sqrt{s}(t-s)^{\frac{3}{2}\times\frac{3}{5p}}} ds \right) \\ &= c(p) \left[I_1(t) + I_2(t) \right]; \end{aligned}$$

we set for simplicity $Q_T = \mathbb{R}^3 \times]0,T[$; we have on the one hand that

$$\begin{aligned} I_1(t) &\le \frac{c(p)}{t^{\frac{3}{2}\times\frac{3}{5p}}} \left(\int_0^{t/2} \frac{ds}{s^{\frac{1}{2}\times\frac{5p}{5p-6}}} \right)^{\frac{5p-6}{5p}} \|\sqrt{t}F^{(2)}\|_{L_{\frac{5p}{6}}(Q_T)} \\ &=: c(p) t^{\frac{5p-12}{10p} - \frac{3}{2}\times\frac{3}{5p}} \|\sqrt{t}F^{(2)}\|_{L_{\frac{5p}{6}}(Q_T)}; \end{aligned}$$

from this, we get that

$$\left(\int_0^T I_1(t)^{\frac{5p}{3}} dt \right)^{\frac{3}{5p}} \le c(p) T^{\frac{1}{2}-\frac{3}{2p}} \|\sqrt{t}F^{(2)}\|_{L_{\frac{5p}{6}}(Q_T)}.$$

On the other hand, we have

$$I_2(t) \le c \int_{t/2}^t \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}}}{(t-s)^{\frac{1}{2}+\frac{3}{2}\times\frac{3}{5p}}} ds,$$

which also yields, by an application of Young's inequality, that

$$\left(\int_0^T I_2(t)^{\frac{5p}{3}} dt \right)^{\frac{3}{5p}} \le c(p) T^{\frac{1}{2}-\frac{3}{2p}} \|\sqrt{t}F^{(2)}\|_{L_{\frac{5p}{6}}(Q_T)}.$$

Consequently, we have that

$$\|w^{(2)}\|_{L_{\frac{5p}{3}}(Q_T)} \le c(p) T^{\frac{1}{2}-\frac{3}{2p}} \|\sqrt{t}F^{(2)}\|_{L_{\frac{5p}{6}}(Q_T)}. \tag{3.7}$$

Using once again, the representation formula we had for $w^{(2)}$, together with similar computations as above, we find that

$$\begin{aligned} \|\sqrt{t}\nabla w^{(2)}(\cdot,t)\|_{L_{\frac{5p}{3}}(\mathbb{R}^3)} &\le \frac{c(p)}{t^{\frac{3}{2}\times\frac{3}{5p}}} \int_0^{t/2} \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}(\mathbb{R}^3)}}{\sqrt{s}} ds \\ &\quad + c(p) \int_{t/2}^t \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}}}{(t-s)^{\frac{1}{2}+\frac{3}{2}\times\frac{3}{5p}}} ds \\ &=: c(p) \left[J_1(t) + J_2(t) \right]; \end{aligned}$$

$J_1(t)$ is treated in a similar way as $I_1(t)$, and $J_2(t)$ as $I_2(t)$; and the claim is proved.

Now, thanks to Lemma 2.3.2 together with estimate (3.5), we get that

$$\|v\|_{\mathcal{Y}_T} \le c(\kappa,p) T^{\frac{1}{2}-\frac{3}{2p}} \|v\|_{\mathcal{Y}_T}^2 + c_0(p)\|u_0\|_{L_p(\mathbb{R}^3)}; \tag{3.8}$$

Choosing

$$T_* = \left(\frac{3}{16 c(\kappa,p) c_0(p) \|u_0\|_{L_p(\mathbb{R}^3)}} \right)^{\frac{2p}{p-3}},$$

we get, by using continuation agreements that

$$\|v\|_{\mathcal{Y}_T} \leq \frac{4c_0(p)}{3}\|u_0\|_{L_p(\mathbb{R}^3)}, \tag{3.9}$$

for all $0 \leq t \leq T_*$.

Finally, in order to get the $L_{p,\infty}(Q_{T_*})$ estimate for v all we have to do is to get it for the functions $w = v - \hat{v} = w^{(1)} + w^{(2)}$. To achieve this, we consider one more time the integral representation of $w^{(1)}$ and $w^{(2)}$. For $w^{(2)}$ for instance, we have for all $t \in]0, T_*[$

$$\begin{aligned}
\|w^{(2)}(\cdot,t)\|_{L_p(\mathbb{R}^3)} &\leq c(p)\int_0^t \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}(\mathbb{R}^3)}}{\sqrt{s}(t-s)^{\frac{3}{10p}}}ds \\
&\leq \frac{c(p)}{t^{\frac{3}{10p}}}\int_0^{t/2} \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}(\mathbb{R}^3)}}{\sqrt{s}}ds \\
&+ c(p)\int_{t/2}^t \frac{\|\sqrt{s}F^{(2)}(\cdot,s)\|_{L_{\frac{5p}{6}}(\mathbb{R}^3)}}{(t-s)^{\frac{1}{2}+\frac{3}{10p}}}ds \\
&\leq c(p)t^{\frac{1}{2}-\frac{3}{2p}}\|v\|^2_{\mathcal{Y}_T} \\
&\leq c(\kappa,p)\|u_0\|_{L_p};
\end{aligned}$$

we have the same result for $w^{(1)}$; thus, in the end, we obtain

$$\|v\|_{L_\infty(0,T_*;L_p(\mathbb{R}^3))} \leq c(\kappa,p)\|u_0\|_{L_p}.$$

In order, to see that the initial condition is achieved in the desired sense, we use the fact that

$$\begin{aligned}
\|v(\cdot,t) - u_0\|_{L_p(\mathbb{R}^3)} &\leq \|\underbrace{v(\cdot,t) - \hat{v}(\cdot,t)}_{w^{(1)}+w^{(2)}}\|_{L_p(\mathbb{R}^3)} + \|\hat{v}(\cdot,t) - u_0\|_{L_p(\mathbb{R}^3)} \\
&\leq c(\kappa,p)t^{\frac{1}{2}-\frac{3}{2p}}\|v\|^2_{\mathcal{Y}_{T_*}} + \|\hat{v}(\cdot,t) - u_0\|_{L_p(\mathbb{R}^3)},
\end{aligned}$$

which guarantees $\|v(\cdot,t) - u_0\|_{L_p(\mathbb{R}^3)} \to 0$ as $t \to 0^+$ (thanks to Lemma 2.3.2). Taking the limit $\epsilon \to 0^+$, we have that the proof is completed. □

We move now to the proof of Proposition 1.2.1.

Proof of Proposition 1.2.1. The proof of this result is quite well known in the literature; see for instance [16], Theorem 7.1. We focus here on $\mathcal{L}_\kappa = 0$ since things are the same for the system $\mathcal{D}\mathcal{L}_\kappa = 0$; the key step in proving this result is establishing the following estimate:

$$\|w(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_0^t \left(\|\nabla w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)} + \kappa\|\operatorname{div} w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}\right) dt'$$
$$\leq 2\int_0^t \int_{\mathbb{R}^3} (w\cdot\nabla w)\cdot u_*(x,t')dxdt' + \int_0^t\int_{\mathbb{R}^3} u_*\cdot w \operatorname{div} w(x,t')dxdt', \quad (3.10)$$

for all $t \in [0,T_*]$ and with $w = u - u_*$. In order to obtain estimate (3.10), we start from the energy inequality for u, then use the fact that $u = w + u_*$, and then make all the necessary expansions; to be more precise, we obtain the following:

$$\|w(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + \|u_*(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_{\mathbb{R}^3} w\cdot u_*(x,t)dx$$
$$+2\int_0^t \left(\|\nabla w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)} + \kappa\|\operatorname{div} w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}\right) dt'$$
$$+2\int_0^t \left(\|\nabla u_*(\cdot,t')\|^2_{L_2(\mathbb{R}^3)} + \kappa\|\operatorname{div} u_*(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}\right) dt'$$
$$+4\int_0^t\int_{\mathbb{R}^3} (\nabla w : \nabla u_* + \kappa \operatorname{div} w \operatorname{div} u_*)dxdt' = \|u_0\|^2_{L_2(\mathbb{R}^3)};$$

next, we use the energy identity for u_*; and then, in order to deal with the remaining unpleasant terms, we multiply the equation $\mathcal{L}_\kappa u_* = 0$ by w, and finally, perform the appropriate integration by parts and algebraic computations. Let us point out that, the computations and integration by parts we mentioned previously are legitimate thanks to the fact that we have the extra-regularity $u_* \in L_{p,\infty}(\mathbb{R}^3\times]0,T_*[) \cap L_{\frac{5p}{3}}(\mathbb{R}^3\times]0,T_*[)$ and $\nabla u_* \in L_2(\mathbb{R}^3\times]0,\infty[)$.

Let us conclude now the proof of the proposition. We denote for simplicity the right-hand side of (3.10) by RHS; we have thanks to Hölder's inequality

$$RHS \leq c\left(\int_0^t \|\nabla w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt'\right)^{\frac{1}{2}} \times \left(\int_0^t \|w(\cdot,t')\|^2_{L_6(\mathbb{R}^3)}dt'\right)^{\frac{3}{2\times 5}}$$
$$\times\left(\int_0^t \|u_*(\cdot,t')\|^5_{L_5(\mathbb{R}^3)}\|w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt'\right)^{\frac{1}{5}}$$
$$\leq c\left(\int_0^t \|\nabla w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt'\right)^{1-\frac{1}{5}}\left(\int_0^t \|u_*(\cdot,t')\|^5_{L_5(\mathbb{R}^3)}\|w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt'\right)^{\frac{1}{5}}$$
$$\leq \int_0^t \|\nabla w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt' + c\int_0^t \|u_*(\cdot,t')\|^5_{L_5(\mathbb{R}^3)}\|w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt'.$$

Thus, we obtain that

$$\|w(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} \leq c\int_0^t \|u_*(\cdot,t')\|^5_{L_5(\mathbb{R}^3)}\|w(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt', \quad \text{for all } t\in[0,T_*];$$

notice that since $u_* \in L_{\frac{10}{3}}(\mathbb{R}^3\times]0,\infty[) \cap L_{\frac{5p}{3}}(\mathbb{R}^3\times]0,T_*[)$ (with $p \geq 3$), we are always guaranteed that $u_* \in L_5(\mathbb{R}^3\times]0,T_*[)$. By applying Gronwall's lemma, we obtain that $w \equiv 0$ in $\mathbb{R}^3 \times [0,T_*]$; and the proposition is proved. $\square$

Finally, we obviously have that Theorem 1.2.2 comes from the above two results.

3.4 Existence of strong solutions

We focus in this section on the bounded smooth domain case, but we make sure that the tools we use involve, as little as possible, any compactness property of our domain; this guarantees that our proofs can be extended to the unbounded domain case.

Let us begin the proof of Theorem 1.2.4; we focus here only on the system $\mathcal{L}_\kappa = 0$ since things works in the same way for the system $\mathcal{DL}_\kappa = 0$.

Proof. We consider the bounded domain analogue of the system (3.1)

$$\begin{cases} \partial_t v - \Delta v - \kappa\nabla \operatorname{div} v + (v)_\epsilon \cdot \nabla v + \dfrac{v}{2}\operatorname{div}(v)_\epsilon = 0 \quad \text{in } \Omega\times]0,\infty[, \\ v|_{t=0} = u_0 \quad \text{in } \Omega; \quad v(\cdot,t)|_{\partial'\Omega} = 0 \ \forall t > 0; \end{cases} \tag{3.11}$$

where

$$(v)_\epsilon(x,t) = \int_\Omega \varrho_\epsilon(x-y)v(y,t)dy,$$

with ϱ_ϵ being the usual mollification kernel. Existence of a "nice" v solution to (3.11) is fairly standard now; we also see without much trouble that the only challenging point in the proof of existence of a strong solution is the third point of the definition. To achieve this, let us set

$$y(t) := \int_\Omega |\nabla v(x,t)|^2 dx.$$

By multiplying the first equation in (3.11) by Δv, then integrating by part and finally using Hölder's inequality, we obtain

$$\frac{1}{2}y'(t) + \int_\Omega |\Delta v|^2 dx + \kappa \int_\Omega |\nabla \operatorname{div} v|^2 dx \leq c\|v\|_{6,\Omega}\|\nabla v\|_{3,\Omega}\|\Delta v\|_{2,\Omega};$$

this yields after using Young's inequality

$$y'(t) + \int_\Omega |\Delta v|^2 dx + \kappa \int_\Omega |\nabla \operatorname{div} v|^2 dx \leq c\|v\|_{6,\Omega}^2\|\nabla v\|_{3,\Omega}^2. \tag{3.12}$$

We evaluate now the right-hand side in the above. By the Gagliardo-Nirenberg inequality, we have

$$\|v\|_{6,\Omega}^2 \leq cy,$$

and by an interpolation inequality (together with Sobolev embedding) that

$$\|\nabla v(\cdot,t)\|_{3,\Omega}^2 \leq c(\Omega) y^{1/2}(t) \left(\int_\Omega |\nabla^2 v(\cdot,t)|^2 dx + y(t) \right)^{1/2}; \tag{3.13}$$

but by the Calderon-Zygmund inequality, we know that

$$\int_\Omega |\nabla^2 v(\cdot,t)|^2 dx \leq c(\Omega) \int_\Omega |\Delta v(\cdot,t)|^2 dx; \tag{3.14}$$

Furthermore, by the Poincaré-Wirtinger inequality, combined with the fact that $\int_\Omega \nabla v dx = 0$ (by Green's formula), we obtain

$$y(t) \leq c(\Omega) \int_\Omega |\nabla^2 v(\cdot,t)|^2 dx. \tag{3.15}$$

In the end we obtain:

$$y'(t) + \int_\Omega |\Delta v|^2 dx + \kappa \int_\Omega |\nabla \operatorname{div} v|^2 dx \leq c(\Omega) y^3(t); \tag{3.16}$$

notice that if we were in the unbounded domain case (i.e. $\mathbb{R}^3$), estimate (3.15) wouldn't have been needed, since, we could use the Gagliardo-Nirenberg inequality directly in the interpolation inequality (3.13).
Consequently, we have

$$\frac{y'(t)}{1+y^2(t)} \leq c(\Omega) y(t),$$

which yields after integration

$$\begin{aligned} \arctan(y(t)) &\leq \arctan(\|\nabla u_0\|_{2,\Omega}^2) + c(\Omega) \int_0^T y(s) ds \\ &\leq \arctan(\|\nabla u_0\|_{2,\Omega}^2) + c(\Omega) \|u_0\|_{2,\Omega}^2 \\ &< \frac{\pi}{2}, \end{aligned}$$

i.e. $y(t) \leq C$, for all $t > 0$. Passing to the limit $\epsilon \to 0^+$ yields the desired result; and the proof is complete. □

Now, let us prove Theorem 1.2.5.

Proof. Let us go back to estimate (3.16); We have that

$$\frac{y'}{y^3} \leq c(\Omega),$$

for all $t > 0$. The integration with respect to time give us:

$$\frac{1}{y^2(0)} - \frac{1}{y^2(t)} \leq 2c(\Omega)t$$

and thus

$$y^2(t)(1 - 2c(\Omega)ty^2(0)) \leq y^2(0).$$

We let $T'_* = 1/(4cy^2(0))$. Then $1 - 2cty(0)^2 \geq 1/2$ and $y(t) \leq \sqrt{2}y(0)$ for $0 < t \leq T'_*$ and the proof is completed after passing to the limit $\epsilon \to 0^+$. □

Chapter 4

Partial regularity results for our toy-models

4.1 Introduction

In this chapter we establish some partial regularity results for our toy-models. By this we mean that, assuming an extra control on a certain norm of the solution, we aim to derive further regularity properties. The class of quantities at the heart of this analysis are called *scale invariant quantities*, which are quantities $F(u, r)$ (with u a solution) such that:

$$F(u^\lambda, 1) = F(u, \lambda),$$

for all $\lambda > 0$ and $u^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$.

Analogous results to the ones we establish in this chapter are also available for the incompressible Navier-Stokes system (see e.g. [60] and references therein for a record of those results). However, by making use of the absence of the pressure in our models, we are able to establish some regularity results which are not available for the Navier-Stokes equations at this point in time; we record those in Chapter 6: Some New Regularity Results.

Unless mentioned otherwise, the notation and definitions used in this chapter are contained in Chapter 2: Preliminary Material and the results and text base of this chapter are based on [27, 28]. We focus here only on models in non-divergence form since our results hold also true for the divergence form counterparts with some simplicity due to the divergence structure of the nonlinearity.

4.2 A generalised ϵ-regularity result

We prove in this section an ϵ-regularity criterion similar to that of Caffarelli-Kohn-Nirenberg (see [6]) for a generalised counterpart of the system $\mathcal{L}_\kappa = 0$. Our setting is as follows:
Let $\mathcal{O}$ be an open subset of $\mathbb{R}^{3+1}$ and $a, b \in L_m(\mathcal{O})$ with $m > 5$. A function u is called *suitable weak solution* to

$$\partial_t u - \Delta u - \kappa \nabla \operatorname{div} u + u\cdot\nabla u + \frac{u}{2}\operatorname{div} u + \frac{a}{2}\cdot\nabla u + \operatorname{div}(u\otimes\frac{a}{2}) + \operatorname{div}(b\otimes u) - \frac{b}{2}\operatorname{div} u = 0, \quad (4.1)$$

in $\mathcal{O}$ if $u \in L_{2,\infty}(\mathcal{O}) \cap W^{1,0}_2(\mathcal{O})$, satisfies (4.1) in the sense of distribution in $\mathcal{O}$ and

$$\partial_t \frac{|u|^2}{2} - \Delta\frac{|u|^2}{2} - \kappa \operatorname{div}(u \operatorname{div} u) + |\nabla u|^2 + \kappa(\operatorname{div} u)^2 + \operatorname{div}\left((u+a)\frac{|u|^2}{2}\right) \\ + u\cdot\operatorname{div}(b\otimes u) - \frac{1}{2}u\cdot b\operatorname{div} u \leq 0 \quad (4.2)$$

in the sense of distributions in $\mathcal{O}$. Let us point out that the term $u\cdot\operatorname{div}(a\otimes u)$ is indeed a distribution and should be understood in the following way

$$\langle u\cdot\operatorname{div}(a\otimes u), \phi\rangle = -\int_{\mathcal{O}} a_i u_j u_{i,j}\phi dxdt - \int_{\mathcal{O}} a_i u_j u_i \phi_{,j} dxdt,$$

for all $\phi \in C^\infty_0(\mathcal{O})$. All the other terms in (4.2) obviously make sense because of the energy class of u and the integrability condition on a, b. The existence of a suitable weak solution to (4.1) can be achieved, by similar means as what we presented in the previous chapter.
The first main theorem of this section reads as follows.

Theorem 4.2.1 (ϵ-regularity criterion)**.** *Let u be a suitable weak solution to* (4.1) *in Q with $a, b \in L_m(Q)$ and $m > 5$. Then there exists an $\epsilon_0 = \epsilon_0(\kappa, m) > 0$ with the following properties: if*

$$\left(\fint_Q |u|^3 dz\right)^{\frac{1}{3}} + \left(\fint_Q |a|^m dz\right)^{\frac{1}{m}} + \left(\fint_Q |b|^m dz\right)^{\frac{1}{m}} \leq \epsilon_0, \quad (4.3)$$

then u is Hölder continuous in $\overline{Q(\frac{1}{2})}$ with exponent $\alpha = \alpha(m) \in]0,1[$ and

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q(\frac{1}{2})})} \leq C(\epsilon_0, \kappa, m). \quad (4.4)$$

Remark 16. The proof of this theorem we present here is inspired from the proof of a similar result in the case of the 3D incompressible Navier-Stokes equations given in [47, 60]. The particular case $a = b = 0$ was tackled in [55] (see Theorem 7.1). Finally, let us point out that the case $\kappa = 0$ and $a = b = 0$ was tackled in [28] using a more direct method which could also be extended to the general case (4.1) upon some minor changes.

The following auxiliary result gives us an estimate for a generalised time-dependent Lamé system and will be needed for the proof of Theorem 4.2.1.

Lemma 4.2.2. *Let $a, b \in L_m(Q;\mathbb{R}^3)$ and $F \in L_m(Q;\mathbb{R}^{3\times 3})$ such that*

$$\|a\|_{L_m(Q)} + \|b\|_{L_m(Q)} + \|F\|_{L_m(Q)} \leq M, \tag{4.5}$$

for some arbitrary $M > 0$ and $m > 5$. Let also $\lambda \in \mathbb{R}^3$ with $|\lambda| \leq M$ and $u \in L_{2,\infty}(Q) \cap W^{1,0}(Q)$ be such that

$$\operatorname*{ess\,sup}_{-1<t<0} \int_B |u(x,t)|^2 dx + \int_{-1}^{0}\int_B |\nabla u|^2 dxdt \leq M^2. \tag{4.6}$$

Assume u satisfies

$$\partial_t u - \Delta u - \kappa \nabla \operatorname{div} u + \lambda\cdot\nabla u + \frac{\lambda}{2}\operatorname{div} u + \frac{a}{2}\cdot\nabla u + \operatorname{div}(u\otimes\frac{a}{2}) + \operatorname{div}(b\otimes u) - \frac{b}{2}\operatorname{div} u = \operatorname{div} F,$$

in the sense of distributions. Then u is Hölder continuous in $\overline{Q(\frac{1}{2})}$ with exponent $\alpha = \alpha(m) \in]0,1[$ and

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q(\frac{1}{2})})} \leq C(\kappa, m, M).$$

Proof. Let $0 < \varrho < r \leq 1$ and $0 \leq \varphi_{\varrho,r} \in C_0^\infty(B)$ such that $\varphi_{\varrho,r} \equiv 1$ in $B((r+\varrho)/2)$ and $\varphi_{\varrho,r} \equiv 0$ in $B(r) \setminus B((3r+\varrho)/4)$. Next, let us set

$$F^{\varrho,r} := \varphi_{\varrho,r}\left(F - u\otimes\lambda - \frac{\lambda}{2}\otimes u - u\otimes\frac{a}{2} - b\otimes u\right), \; f^{\varrho,r} := \left(\frac{b}{2}\operatorname{div} u - \frac{a}{2}\cdot\nabla u\right)\varphi_{\varrho,r}.$$

If $u \in L_q(Q(r))$ and $\nabla u \in L_p(Q(r))$ $(p, q \geq 2)$ with

$$\frac{1}{m} + \frac{1}{q} < \frac{1}{p} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{5}\right) \quad \text{and} \quad \frac{1}{m} + \frac{1}{p} - \frac{2}{5} < \frac{1}{q} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{5}\right) \tag{4.7}$$

(e.g. (4.7) holds true for $q = 10/3$ and $p = 2$) then

$$F^{\varrho,r} \in L_{\frac{mq}{m+q}}(Q(r)) \text{ and } f^{\varrho,r} \in L_{\frac{mp}{m+p}}(Q(r)).$$

From Lemma 2.3.1, we have the existence of two functions $v^{\varrho,r}$ and $w^{\varrho,r}$ such that

$$\begin{cases} \partial_t v^{\varrho,r} - \Delta v^{\varrho,r} - \kappa\nabla \operatorname{div} v^{\varrho,r} = f^{\varrho,r} & \text{in } \mathbb{R}^3\times]-1,0[\\ v^{\varrho,r}|_{t=-1} = 0 \quad \text{in } \mathbb{R}^3 \end{cases}$$

and
$$\begin{cases} \partial_t w^{\varrho,r} - \Delta w^{\varrho,r} - \kappa \nabla \operatorname{div} w^{\varrho,r} = \operatorname{div} F^{\varrho,r} \quad \text{in } \mathbb{R}^3\times]-1,0[\\ w^{\varrho,r}|_{t=-1} = 0 \quad \text{in } \mathbb{R}^3. \end{cases}$$
Moreover the following estimates are available

$$\|w^{\varrho,r}\|_{L_{\hat{q}}(\mathbb{R}^3\times]-1,0[)} + \|\nabla w^{\varrho,r}\|_{L_{\frac{mq}{m+q}}(\mathbb{R}^3\times]-1,0[)} \leq c(m,p,q)\|F^{\varrho,r}\|_{L_{\frac{mq}{m+q}}(Q(r))},$$

as long as
$$\frac{1}{\hat{q}} > \frac{1}{q} + \frac{1}{m} - \frac{1}{5},$$

and
$$\|v^{\varrho,r}\|_{L_{\tilde{q}}(\mathbb{R}^3\times]-1,0[)} + \|\nabla v^{\varrho,r}\|_{L_{\tilde{p}}(\mathbb{R}^3\times]-1,0[)} \leq c(m,p,q)\|f^{\varrho,r}\|_{L_{\frac{mp}{m+p}}(Q(r))},$$

as long as
$$\frac{1}{\tilde{p}} > \frac{1}{p} + \frac{1}{m} - \frac{1}{5} \quad \text{and} \quad \frac{1}{\tilde{q}} > \frac{1}{p} + \frac{1}{m} - \frac{2}{5}.$$
Now, we choose $q_1 \geq 2$ and $p_1 \geq 2$ such that

$$\frac{1}{q_1} = \frac{1}{q} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{5}\right) \quad \text{and} \quad \frac{1}{p_1} = \frac{1}{p} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{5}\right); \tag{4.8}$$

because of (4.7), we have that

$$\begin{aligned} \|w^{\varrho,r}\|_{L_{q_1}(Q)} + \|\nabla w^{\varrho,r}\|_{L_{p_1}(Q)} &\leq c(\kappa,m,p,q)\|F^{\varrho,r}\|_{L_{\frac{mq}{m+q}}(Q(r))}, \\ \|v^{\varrho,r}\|_{L_{q_1}(Q)} + \|\nabla v^{\varrho,r}\|_{L_{p_1}(Q)} &\leq c(\kappa,m,p,q)\|f^{\varrho,r}\|_{L_{\frac{mp}{m+p}}(Q(r))}; \end{aligned} \tag{4.9}$$

moreover (4.7) is true for q and p replaced respectively by q_1 and p_1.
Finally, let us notice that

$$\partial_t(u - v^{\varrho,r} - w^{\varrho,r}) - \Delta(u - v^{\varrho,r} - w^{\varrho,r}) - \kappa\nabla \operatorname{div}(u - v^{\varrho,r} - w^{\varrho,r}) = 0 \tag{4.10}$$

in $Q((r+\varrho)/2)$; which together with Lemma 2.3.3 lead to $u \in L_{q_1}(Q(\varrho))$ and $\nabla u \in L_{p_1}(Q(\varrho))$ where $q_1 > q$ and $p_1 > p$. The goal now is to iterate this process. We set $q_0 = 10/3$, $p_0 = 2$ and we define a sequence (q_k) and (p_k) via the following recursive formula
$$\frac{1}{q_k} = \frac{1}{q_{k-1}} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{5}\right) \quad \text{and} \quad \frac{1}{p_k} = \frac{1}{p_{k-1}} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{5}\right)$$
and we find $u \in L_{q_k}(Q(2^{-1} + 2^{-(1+k)}))$ and $\nabla u \in L_{p_k}(Q(2^{-1} + 2^{-(1+k)}))$ for all $k \geq 0$. Now, for a large enough $k_0 = k_0(m)$, we have that

$$\frac{mq_{k_0}}{m + q_{k_0}} > 5 \text{ and } \frac{mp_{k_0}}{m + p_{k_0}} > \frac{5}{2};$$

thus, for $\varrho = 2^{-1} + 2^{-(2+k_0)}$ and $r = 2^{-1} + 2^{-(1+k_0)}$, we find that there exists an $\alpha = \alpha(m) \in]0,1[$ such that $v^{\varrho,r}, w^{\varrho,r} \in C^{\alpha,\frac{\alpha}{2}}(Q(2^{-1} + 2^{-(2+k_0)}))$. Using one more time (4.10) and Lemma 2.3.3, we get that $u \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q(1/2)})$. A careful tracking of the constants in the above process yields also the estimate claimed in the lemma; we omit the details. Thus the lemma is proved. □

We turn now to the proof of Theorem 4.2.1. First, we prove a so-called "oscillation lemma" which, roughly speaking asserts that if u has a *small oscillation* in Q, then the oscillation is even smaller in Q_θ for $0 < \theta < 1$ (see e.g. [60, 31] where an analogue of this result was proved for the incompressible Navier-Stokes equations).

In order to state our lemma and for what follows, some additional notion is needed; we introduce

$$Y(z_0, R; u) := \left(\fint_{Q(z_0,R)} |u - (u)_{z_0,R}|^3 dz \right)^{\frac{1}{3}}, \ Y_\theta(u) := Y(0, \theta; u).$$

Lemma 4.2.3. *Given any numbers $\theta \in]0, 1/3[$, $m > 5$ and $M > 0$, there are three constants $\alpha = \alpha(m) \in]0,1[$, $\epsilon = \epsilon(\theta, m, M) > 0$ and $C_1 = C_1(\kappa, m, M) > 0$ such that for any $\|a\|_{L_m(Q)} \leq M$, $\|b\|_{L_m(Q)} \leq c_0$, $c_0 > 0$ being a small absolute constant (to be specified later), and any suitable weak solution u to (4.1) in Q, satisfying the additional conditions*

$$|(u)_{,1}| \leq M, \quad Y_1(u) + |(u)_{,1}| \left(\fint_Q |a|^m dz \right)^{\frac{1}{m}} + |(u)_{,1}| \left(\fint_Q |b|^m dz \right)^{\frac{1}{m}} < \epsilon,$$

the following estimate is valid:

$$Y_\theta(u) \leq C_1 \theta^\alpha \left[Y_1(u) + |(u)_{,1}| \left(\fint_Q |a|^m dz \right)^{\frac{1}{m}} + |(u)_{,1}| \left(\fint_Q |b|^m dz \right)^{\frac{1}{m}} \right].$$

Proof. Assume that the statement is false. This means that there exist numbers $\theta \in]0, 1/3[$, $m > 5$ and $M > 0$ and sequences a^k, b^k and a sequence of suitable weak solutions u^k to (4.1) (with a and b replaced respectively by a^k and b^k) such that

$$|(u^k)_1| \leq M, \quad \|a^k\|_{L_m(Q)} \leq M, \ \|b^k\|_{L_m(Q)} \leq c_0,$$
$$Y_1(u^k) + |(u^k)_{,1}| \left(\fint_Q |a^k|^m dz \right)^{\frac{1}{m}} + |(u^k)_{,1}| \left(\fint_Q |b^k|^m dz \right)^{\frac{1}{m}} = \epsilon_k \to 0^+ \text{ as } k \to \infty,$$
$$Y_\theta(u^k) > C_1 \theta^\alpha \epsilon_k,$$

for all $k \in \mathbb{N}$. Next, we introduce the following functions

$$v^k := \frac{u^k - (u^k)_{,1}}{\epsilon_k}, \quad -F^k := \frac{(u^k)_{,1} \otimes a^k}{2\epsilon_k} + \frac{b^k \otimes (u^k)_{,1}}{\epsilon_k};$$

we have

$$\left(\fint_Q |v^k|^3 dz\right)^{\frac{1}{3}} + \left(\fint_Q |F^k|^m dz\right)^{\frac{1}{m}} \leq 4 \quad \text{and } Y_\theta(v^k) > C_1\theta^\alpha. \tag{4.11}$$

Moreover,

$$\begin{aligned}\partial_t v^k - \Delta v^k - \kappa\nabla \operatorname{div} v^k + \epsilon_k\left(v^k\cdot\nabla v^k + \frac{v^k}{2}\operatorname{div} v^k\right) + (u^k)_{,1}\cdot\nabla v^k + \frac{(u^k)_{,1}}{2}\operatorname{div} v^k \\ + \frac{a^k}{2}\cdot\nabla v^k + \operatorname{div}(v^k\otimes\frac{a^k}{2}) + \operatorname{div}(b^k\otimes v^k) - \frac{b^k}{2}\operatorname{div} v^k = \operatorname{div}(F^k),\end{aligned} \tag{4.12}$$

in the sense of distributions on Q and

$$\begin{aligned}\partial_t\frac{|v^k|^2}{2} - \Delta\frac{|v^k|^2}{2} - \kappa\operatorname{div}(v^k\operatorname{div} v^k) + |\nabla v^k|^2 + \kappa(\operatorname{div} v^k)^2 + \operatorname{div}\left((\epsilon_k v^k + a^k + (u^k)_{,1})\frac{|v^k|^2}{2}\right) \\ + v^k\cdot\operatorname{div}((u^k)_{,1}\otimes\frac{v^k}{2} + b^k\otimes v^k - F^k) - \frac{1}{2}v^k\cdot b^k\operatorname{div} v^k \leq 0,\end{aligned} \tag{4.13}$$

in the sense of distributions on Q. From the previous inequality, we get that

$$\begin{aligned}&\frac{1}{2}\int_B |v^k(x,t)|^2\phi(x,t)dx + \int_{-1}^t\int_B |\nabla v^k|^2\phi dz + \kappa\int_{-1}^t\int_B(\operatorname{div} v^k)^2\phi dz \\ &\quad\leq \int_{-1}\int_B\frac{|v^k|^2}{2}(\partial_t\phi + \Delta\phi)dz + \int_{-1}^t\int_B\frac{|v^k|^2}{2}(\epsilon_k v^k + a^k + (u^k)_{,1})\cdot\nabla\phi dz \\ &+\int_{-1}^t\int_B\phi\nabla v^k : ((u^k)_{,1}\otimes\frac{v^k}{2} + b^k\otimes v^k - F^k)dz + \int_{-1}^t\int_B v^k\otimes\nabla\phi : ((u^k)_{,1}\otimes\frac{v^k}{2} + b^k\otimes v^k - F^k)dz \\ &\quad - \kappa\int_{-1}^t\int_B v^k\cdot\nabla\phi\operatorname{div} v^k dz + \frac{1}{2}\int_{-1}\int_B\phi v^k\cdot b^k\operatorname{div} v^k dz,\end{aligned}$$

for any $0 \leq \phi \in C_0^\infty(B\times]-1,1[)$. Next, we define

$$E_k(r) := \frac{1}{2}\operatorname*{ess\,sup}_{-r^2<t<0}\int_{B(r)}|v^k(x,t)|^2dx + \int_{-r^2}\int_{B(r)}[|\nabla v^k(x,t)|^2 + \kappa(\operatorname{div} v^k(x,t))^2]dz,$$

for any $0 < r < 1$. Our goal now is to get a uniform estimate (with respect to k) for E_k. To this end we start by recalling the following well-known interpolation inequality

$$\|v^k\|^2_{L_{\frac{10}{3}}(Q(r))} \leq CE_k(r),$$

with $C > 0$ being an universal constant. Then, for any $1/2 < r_1 < r_2 \leq 1$, if we choose appropriately the test function ϕ in the above local energy inequality together with the help of Hölder's inequality and the estimates on $v^k,\ a^k,\ b^k$ and F^k, we have that:

$$
\begin{aligned}
E_k(r_1) \leq{} & \frac{C}{(r_2-r_1)^2} + \frac{C}{r_2-r_1}\Bigg\{\epsilon_k \int_Q |v^k|^3 dz + \left(\int_Q |v^k|^3 dz\right)^{\frac{2}{3}} \left(\int_Q |a^k|^3 dz\right)^{\frac{1}{3}} \\
& + |(v^k)_{,1}| \int_Q |v^k|^2 dz\Bigg\} + E_k(r_2)^{\frac{1}{2}} \Bigg\{ |(u^k)_{,1}| \left(\int_Q |v^k|^2 dz\right)^{\frac{1}{2}} \\
& + \left(\int_Q |b^k|^5 dz\right)^{\frac{1}{5}} \left(\int_{Q(r_2)} |v^k|^{\frac{10}{3}} dz\right)^{\frac{3}{10}} + \left(\int_Q |F^k|^2 dz\right)^{\frac{1}{2}}\Bigg\} \\
& + \frac{C}{r_2-r_1}\Bigg\{ |(u^k)_{,1}| \int_Q |v^k|^2 dz + \left(\int_Q |v^k|^3 dz\right)^{\frac{2}{3}} \left(\int_Q |a^k|^3 dz\right)^{\frac{1}{3}} \\
& \left(\int_Q |v^k|^2 dz\right)^{\frac{1}{2}} \left(\int_Q |F^k|^2 dz\right)^{\frac{1}{2}}\Bigg\} + E_k(r_2)^{\frac{1}{2}} \left\{\frac{C(\kappa)}{(r_2-r_1)^2} \int_Q |v^k|^2 dz\right\}^{\frac{1}{2}} \\
\leq{} & \frac{C(\kappa, M)}{(r_2-r_1)^2} + C(M) E_k(r_2)^{\frac{1}{2}} + (\frac{1}{4} + C\|b^k\|_{L_m(Q)}) E_k(r_2) \\
\leq{} & \frac{C(\kappa, M)}{(r_2-r_1)^2} + (\frac{1}{2} + C\|b^k\|_{L_m(Q)}) E_k(r_2).
\end{aligned}
$$

Note that $\|b\|_{L_m(Q)} \leq c_0$ with c_0 small. Therefore, if we choose c_0 such that $Cc_0 < 1/2$, we can iterate the above estimate (see Lemma 2.4.2) and conclude that

$$E_k(3/4) \leq C(\kappa, M),\ \forall k \in \mathbb{N}. \tag{4.14}$$

This together with the fact that v^k satisfies (4.12) yield

$$\|\partial_t v^k\|_{L_{4/3}(-(3/4)^2,0;H^{-1}(B(3/4)))} \leq C(\kappa, M),\ \forall k \in \mathbb{N}, \tag{4.15}$$

where $H^{-1}(B(3/4))$ stands here for the dual of the Sobolev space $\overset{\circ}{L}{}^1_2(B(3/4))$. Now, from Aubin-Lions and Banach-Alaoglu compactness results, we can choose subsequences of $v^k,\ a^k,\ b^k,\ (u^k)_{,1}$ and F^k (which we still denote $v^k,\ a^k,\ b^k,\ (u^k)_{,1}$ and F^k) such that for some $\lambda \in \mathbb{R},\ v \in L_3(Q(3/4)),\ a, b, F \in L_m(Q(3/4))$, we have

$$
\begin{gathered}
v^k \to v \text{ strongly in } L_3(Q(3/4)),\ (u^k)_{,1} \to \lambda \\
a^k \rightharpoonup a,\ b^k \rightharpoonup b,\ F^k \rightharpoonup F \quad \text{weakly in } L_m(Q(3/4)).
\end{gathered}
$$

Moreover, we have that

$$
\begin{gathered}
\left(\int_{Q(3/4)} (|a| + |b| + |F|)^m dz\right)^{\frac{1}{m}} \leq C(M), \quad |\lambda| \leq M, \\
\operatorname*{ess\,sup}_{-(3/4)^2<t<0} \int_{B(3/4)} |v(x,t)|^2 dx + \int_{Q(3/4)} |\nabla v(x,t)|^2 dz \leq C(\kappa, M).
\end{gathered}
\tag{4.16}
$$

Finally, from (4.12), we see that

$$\partial_t v-\Delta v-\kappa\nabla \operatorname{div} v+\lambda\cdot\nabla v+\frac{\lambda}{2}\operatorname{div} v+\frac{a}{2}\cdot\nabla v+\operatorname{div}(v\otimes\frac{a}{2})+\operatorname{div}(b\otimes v)-\frac{b}{2}\operatorname{div} v=\operatorname{div} F,$$

in the sense of distributions in $Q(3/4)$. Thus, from Lemma 4.2.2, we have that

$$\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B(1/2)})}\leq C(\kappa,m,M),$$

for some $\alpha=\alpha(m)\in]0,1[$. This implies that

$$Y_\theta(v)\leq C(\kappa,m,M)\theta^\alpha; \tag{4.17}$$

but because of (4.11) and the strong L_3-convergence of v^k, we also have that $Y_\theta(v)\geq C_1\theta^\alpha$. This together with (4.17) give us

$$C_1\leq C(\kappa,m,M).$$

If from the beginning, C_1 is chosen so that $C_1>2C(\kappa,m,M)$, we arrive at a contradiction and the lemma is proved. $\square$

Lemma 4.2.3 admits the following iterations.

Lemma 4.2.4. *Given numbers $M>0$ and $m>5$, we choose $\theta\in]0,1/3[$ so that*

$$C_1\theta^{\alpha-\beta}<1 \quad \textit{and} \quad \theta<c_1 \tag{4.18}$$

where $c_1=c_1(m)>0$ is a small number to be specified later, C_1,α are as in Lemma 4.2.3 and $\beta:=\alpha/2$. Then, there exists $\epsilon_*=\epsilon_*(\kappa,\theta,m,M)<\epsilon$ sufficiently small, such that for any $\|a\|_{L_m(Q)}\leq M$, $\|b\|_{L_m(Q)}\leq c_0$ ($\epsilon=\epsilon(\theta,m,M)>0$ and $c_0>0$ being also as in Lemma 4.2.3), and any suitable weak solution u to (4.1) in Q, satisfying the additional conditions

$$|(u)_{,1}|\leq M/2,\quad Y_1(u)+M\left(\fint_Q|a|^m dz\right)^{\frac{1}{m}}+M\left(\fint_Q|b|^m dz\right)^{\frac{1}{m}}<\epsilon_*,$$

the following holds: $\forall k=1,2,\ldots,$ we have

$$|(u)_{,\theta^{k-1}}|\leq M,$$

$$Y_{\theta^{k-1}}(u)+\theta^{k-1}|(u)_{,\theta^{k-1}}|\left[\left(\fint_{Q(\theta^{k-1})}|a|^m dz\right)^{\frac{1}{m}}+\left(\fint_{Q(\theta^{k-1})}|b|^m dz\right)^{\frac{1}{m}}\right]<\epsilon_*,$$

$$Y_{\theta^k}(u)\leq\theta^\beta\left\{Y_{\theta^{k-1}}(u)+\theta^{k-1}|(u)_{,\theta^{k-1}}|\left[\left(\fint_{Q(\theta^{k-1})}|a|^m dz\right)^{\frac{1}{m}}+\left(\fint_{Q(\theta^{k-1})}|b|^m dz\right)^{\frac{1}{m}}\right]\right\}.$$

Proof. We prove the lemma by induction; the case $k = 1$ is true thanks to Lemma 4.2.3. Now, suppose the conclusion is true for $k \leq k_0$ ($k_0 \geq 1$) and let us show that it remains the case for $k = k_0 + 1$.

For all $k \leq k_0$, we have

$$\begin{aligned}
Y_{\theta^k}(u) &\leq \theta^\beta \left\{ Y_{\theta^{k-1}}(u) + \theta^{k-1} |(u)_{,\theta^{k-1}}| \left[\left(\fint_{Q(\theta^{k-1})} |a|^m dz \right)^{\frac{1}{m}} + \left(\fint_{Q(\theta^{k-1})} |b|^m dz \right)^{\frac{1}{m}} \right] \right\} \\
&\leq \theta^\beta \left\{ Y_{\theta^{k-1}}(u) + \theta^{(k-1)(1-5/m)} M \left[\left(\fint_Q |a|^m dz \right)^{\frac{1}{m}} + \left(\fint_Q |b|^m dz \right)^{\frac{1}{m}} \right] \right\} \\
&\leq \theta^\beta Y_{\theta^{k-1}}(u) + \theta^{k\beta_1} \epsilon_*,
\end{aligned}$$

with $\beta_1 = \min\{\beta, 1 - 5/m\}$. By iterating the last inequality, we get

$$Y_{\theta^k}(u) \leq \theta^{k\beta} Y_1(u) + k\theta^{k\beta_1}\epsilon_*, \quad \forall k \leq k_0.$$

Thus,

$$\begin{aligned}
|(u)_{,\theta^{k_0}}| &\leq \sum_{k=1}^{k_0} |(u)_{,\theta^k} - (u)_{,\theta^{k-1}}| + |(u)_{,1}| \\
&\leq \theta^{-5/3} \sum_{k=1}^{k_0} Y_{\theta^{k-1}}(u) + |(u)_{,1}| \\
&\leq \theta^{-5/3} \sum_{k=1}^{k_0} \left(\theta^{(k-1)\beta} + (k-1)\theta^{(k-1)\beta_1} \right) \epsilon_* + M/2 \\
&\leq \underbrace{\theta^{-5/3} \left((1-\theta^\beta)^{-1} + \sum_{k=0}^{\infty} k\theta^{k\beta_1} \right)}_{C(\theta,m)} \epsilon_* + M/2.
\end{aligned}$$

By choosing $\epsilon_* \leq M(2C(\theta, m))^{-1}$, we find that

$$|(u)_{,\theta^{k_0}}| \leq M.$$

Moreover,

$$\begin{aligned}
Y_{\theta^{k_0}}(u) + \theta^{k_0} |(u)_{,\theta^{k_0}}| \left[\left(\fint_{Q(\theta^{k_0})} |a|^m dz \right)^{\frac{1}{m}} + \left(\fint_{Q(\theta^{k_0})} |b|^m dz \right)^{\frac{1}{m}} \right] &\leq \theta^\beta \epsilon_* + \theta^{(1-5/m)k_0} \epsilon_* \\
&\leq \theta^\beta \epsilon_* + \theta^{(1-5/m)} \epsilon_* \\
&< \epsilon_*,
\end{aligned}$$

if we choose $\theta < c_1(m)$ to be small enough. Now, set

$$
\begin{gathered}
u^0(y,s) = \theta^{k_0} u(\theta^{k_0} y, \theta^{2k_0} s), \\
a^0(y,s) = \theta^{k_0} a(\theta^{k_0} y, \theta^{2k_0} s), \quad b^0(y,s) = \theta^{k_0} b(\theta^{k_0} y, \theta^{2k_0} s),
\end{gathered}
$$

with $(y,s) \in Q$. We show without trouble that u^0 is also a suitable weak solution to (4.1) with a and b replaced respectively by a^0 and b^0; moreover the conditions stated in Lemma 4.2.3 are satisfied for these new functions. Consequently, we have (thanks to Lemma 4.2.3) that

$$
Y_\theta(u^0) \leq \theta^\beta \left[Y_1(u^0) + |(u^0)_{,1}| \left(\fint_Q |a^0|^m dz \right)^{\frac{1}{m}} + |(u^0)_{,1}| \left(\fint_Q |b^0|^m dz \right)^{\frac{1}{m}} \right],
$$

that is

$$
Y_{\theta^{k_0+1}}(u) \leq \theta^\beta \left\{ Y_{\theta^{k_0}}(u) + \theta^{k_0} |(u)_{,\theta^{k_0}}| \left[\left(\fint_{Q(\theta^{k_0})} |a|^m dz \right)^{\frac{1}{m}} + \left(\fint_{Q(\theta^{k_0})} |b|^m dz \right)^{\frac{1}{m}} \right] \right\}.
$$

This concludes the induction and the proof of the lemma. □

By translation and dilatation, we obtain the following corollary.

Corollary 4.2.4.1. *Let u be a suitable weak solution to* (4.1) *in $Q(z_0,R)$ with*

$$
R^{1-5/m} \|a\|_{L_m(Q(z_0,R))} \leq M \quad \text{and} \quad R^{1-5/m} \|b\|_{L_m(Q(z_0,R))} \leq c_0
$$

with c_0 as in Lemma 4.2.3. Take θ, β and ϵ_* as in the previous lemma. If we have in addition that

$$
R|(u)_{z_0,R}| \leq M/2 \text{ and } RY(z_0,R;u) + RM \left[\left(\fint_{Q(z_0,R)} |a|^m dz \right)^{\frac{1}{m}} + \left(\fint_{Q(z_0,R)} |b|^m dz \right)^{\frac{1}{m}} \right] < \epsilon_*,
$$

then for all $k \geq 1$

$$
R|(u)_{Q(z_0,\theta^{k-1}R)}| \leq M,
$$

and

$$
\begin{aligned}
Y(z_0,\theta^k R;u) \leq \theta^\beta \Bigg[& RY(z_0,\theta^{k-1}R;u) + R\theta^{k-1} |(u)_{z_0,\theta^{k-1}R}| \left(\fint_{Q(z_0,\theta^{k-1}R)} |a|^m dz \right)^{\frac{1}{m}} \\
& + R\theta^{k-1} |(u)_{z_0,\theta^{k-1}R}| \left(\fint_{Q(z_0,\theta^{k-1}R)} |b|^m dz \right)^{\frac{1}{m}} \Bigg].
\end{aligned}
$$

We are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $M = 2021$ and choose θ according to Lemma 4.2.4. Define

$$A := \left(\fint_Q |u|^3 dz\right)^{\frac{1}{3}} + \left(\fint_Q |a|^m dz\right)^{\frac{1}{m}} + \left(\fint_Q |b|^m dz\right)^{\frac{1}{m}}.$$

Observe that

$$Q(z_0, 1/4) \subset Q \text{ if } z_0 \in Q(1/2),$$

$$\frac{1}{4}|(u)_{z_0,1/4}| \leq cA, \quad \left(\frac{1}{4}\right)^{1-5/m} (\|a\|_{L_m(Q(z_0,1/4))} + \|b\|_{L_m(Q(z_0,1/4))}) \leq c(m)A,$$

and

$$\frac{1}{4}Y(z_0, 1/4; u) + \frac{1}{4} \times M\left[\left(\fint_{Q(z_0,1/4)} |a|^m dz\right)^{\frac{1}{m}} + \left(\fint_{Q(z_0,1/4)} |b|^m dz\right)^{\frac{1}{m}}\right] \leq c(m)A;$$

Now, we choose

$$\epsilon_0 < \min\{2020/c, 2020/c(m), c_0/c(m), \epsilon_*/c(m)\}$$

with c_0 and ϵ_* as in Lemma 4.2.4. Consequently, by applying Corollary 4.2.4.1, we are able to prove (and by also using a similar iteration process as what we did in the proof of Lemma 4.2.4) that

$$Y(z_0, \theta^k/4; u) \leq C(\kappa, \theta, m, M)\theta^{k\beta_2},$$

for all $z_0 \in Q(1/2)$, $k \geq 1$ and with $\beta_2 = 1/2(\beta + \beta_1)$. Hölder continuity of u in $Q(\overline{1/2})$ follows from Campanato's type condition (see Proposition 2.4.2 in Chapter 2). The theorem is proved. □

Theorem 4.2.1 can be strengthened in the following manner, by removing the smallness condition on a and b.

Theorem 4.2.5 (Improved ϵ-regularity criterion). *Let u be a suitable weak solution to (4.1) in Q with $\|a\|_{L_m(Q)} + \|b\|_{L_m(Q)} \leq M$ for some $M > 0$ and $m > 5$. Then there exists an $\epsilon_1 = \epsilon_1(\kappa, m, M) > 0$ with the following properties: if*

$$\left(\fint_Q |u|^3 dz\right)^{\frac{1}{3}} \leq \epsilon_1, \tag{4.19}$$

then u is Hölder continuous in $\overline{Q(\frac{1}{2})}$ with exponent $\alpha = \alpha(m) \in]0, 1[$ and

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q(\frac{1}{2})})} \leq C(\epsilon_1, \kappa, m). \tag{4.20}$$

We skip the proof of Theorem 4.2.5 since it uses just a quite straightforward scaling argument, and is essentially a repetition of the proof of an analogous result for the incompressible Navier-Stokes system in [31] (see Theorem 2.2 in there).

The presence of the term $\kappa\nabla \operatorname{div} u$ does not appear to help with improving the regularity of zeros of $\mathcal{L}_\kappa$ (or $\mathcal{DL}_\kappa$). With that in mind, we only prove the following, more refined results, for the case $\kappa = 0$. As a matter of fact, recovering these results for the general case ($\kappa \neq 0$) is possible, by following the exact same lines as in the proofs we present here, except for the critical case of the Ladyzhenskaya-Prodi-Serrin criterion; we will discuss that below.

4.3 A Caffarelli-Kohn-Nirenberg type theorem

We prove now a Caffarelli-Kohn-Nirenberg type theorem for $\mathcal{L}_0 = 0$; we borrow ideas from the proof of a similar result for the Navier-Stokes system (see e.g. [60, 47, 6]). Our result is as follows.

Theorem 4.3.1 (A Caffarelli-Kohn-Nirenberg type result). *There exists a positive constant ϵ such that if for any suitable weak solution u to $\mathcal{L}_0 = 0$ in Q, we have*

$$\sup_{0<r<1} \frac{1}{r}\int_{Q(r)} |\nabla u|^2 dz < \epsilon, \tag{4.21}$$

then the map $z \mapsto u(z)$ is Hölder continuous in $\overline{Q(r_0)}$, with $0 < r_0 < 1$. Moreover, there exist absolute positive constants $c_{k,l}$ ($k, l = 0, 1, 2, \ldots$) such that

$$\max_{z\in\overline{Q(r_0)}} |\partial_t^l \nabla^k u(z)| \leq \frac{c_{k,l}}{r_0^{1+2l+k}}.$$

Remark 17. A similar result is also available for the model $\mathcal{DL}_0$.

Before giving the proof of this result let us emphasise that, unlike in the case of the standard Navier-Stokes system, this Caffarelli-Kohn-Nirenberg type result gives us for our model smoothness in time; and this is mainly due to the absence of pressure. The following scaled energy quantities

$$\begin{gathered} A(r) := \sup_{-r^2\leq t\leq 0} \frac{1}{r}\int_{B(r)} |u(x,t)|^2 dx, \quad E(r) := \frac{1}{r}\int_{Q(r)} |\nabla u|^2 dz \\ C(r) := \frac{1}{r^2}\int_{Q(r)} |u|^3 dz \end{gathered} \tag{4.22}$$

and the following auxiliary lemmata will be needed for the proof of the theorem.

Lemma 4.3.2 (An interpolation type result). *For all* $0 < r \le \varrho < 1$*,*

$$C(r) \le c\left[\left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho) + \left(\frac{\varrho}{r}\right)^3 A^{3/4}(\varrho)E^{3/4}(\varrho)\right].$$

Proof. We have

$$\begin{aligned}\int_{B(r)} |u|^2 dx &= \int_{B(r)} \left(|u|^2 - [|u|^2]_{,\varrho}\right) dx + \left(\frac{r}{\varrho}\right)^3 \int_{B(\varrho)} |u|^2 dx \\ &\le \int_{B(\varrho)} \left||u|^2 - [|u|^2]_{,\varrho}\right| dx + \left(\frac{r}{\varrho}\right)^3 \int_{B(\varrho)} |u|^2 dx.\end{aligned}$$

By Poincaré's inequality on the ball, we have

$$\int_{B(\varrho)} \left||u|^2 - [|u|^2]_{,\varrho}\right| dx \le c\int_{B(\varrho)} |\nabla u||u| dx,$$

(where c, as usual, is an absolute positive constant). Therefore, we get

$$\begin{aligned}\int_{B(r)} |u|^2 dx &\le c\varrho\left(\int_{B(\varrho)} |\nabla u|^2 dx\right)^{1/2}\left(\int_{B(\varrho)} |u|^2\right)^{1/2} + \left(\frac{r}{\varrho}\right)^3 \int_{B(\varrho)} |u|^2 dx \\ &\le c\varrho^{3/2} A^{1/2}(\varrho)\left(\int_{B(\varrho)} |\nabla u|^2 dx\right)^{1/2} + \left(\frac{r}{\varrho}\right)^3 \varrho A(\varrho).\end{aligned} \tag{4.23}$$

By an interpolation inequality (and Sobolev embedding and Poincaré's inequality on the ball), we obtain that

$$\begin{aligned}\int_{B(r)} |u|^3 dx &\le c\left[\left(\int_{B(r)} |\nabla u|^2 dx\right)^{3/4}\left(\int_{B(r)} |u|^2\right)^{3/4} + \frac{1}{r^{3/2}}\left(\int_{B(r)} |u|^2 dx\right)^{3/2}\right] \\ &\le c\left\{\varrho^{3/4}A^{3/4}(\varrho)\left(\int_{B(r)} |\nabla u|^2 dx\right)^{3/4} + \frac{1}{r^{3/2}}\left[c\varrho^{3/2}A^{1/2}(\varrho)\left(\int_{B(\varrho)} |\nabla u|^2 dx\right)^{1/2}\right.\right. \\ &\qquad \left.\left. + \left(\frac{r}{\varrho}\right)^3 \varrho A(\varrho)\right]^{3/2}\right\} \\ &\le c\left\{\left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho) + \left(\int_{B(\varrho)} |\nabla u|^2 dx\right)^{3/4}\left[\varrho^{3/4} + \frac{\varrho^{9/4}}{r^{3/2}}\right] A^{3/4}(\varrho)\right\}.\end{aligned}$$

Integrating the latter inequality in t on $]t_0 - r^2, t_0[$, we get

$$\begin{aligned}\int_{Q(r)} |u|^3 dz &\le c\left\{r^2\left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho) + \left[\varrho^{3/4} + \frac{\varrho^{9/4}}{r^{3/2}}\right] A^{3/4}(\varrho) r^{1/2}\left(\int_{Q(\varrho)} |\nabla u|^2 dx\right)^{3/4}\right\} \\ &\le c\left\{r^2\left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho) + \left[\varrho^{3/4} + \frac{\varrho^{9/4}}{r^{3/2}}\right] A^{3/4}(\varrho) r^{1/2} E^{3/4}(\varrho)\varrho^{3/4}\right\}.\end{aligned}$$

Finally, by noticing that

$$\left[\varrho^{3/4} + \frac{\varrho^{9/4}}{r^{3/2}}\right] r^{1/2}\varrho^{3/4} = \left[\left(\frac{\varrho}{r}\right)^{3/2} + \left(\frac{\varrho}{r}\right)^{3}\right] r^2 \leq 2\left(\frac{\varrho}{r}\right)^{3} r^2,$$

we have that the proof is completed. $\square$

Lemma 4.3.3. *For any* $0 < R < 1$,

$$A(R/2) + E(R/2) \leq c\left[C^{2/3}(R) + A^{1/2}(R)C^{1/3}(R)E^{1/2}(R)\right].$$

Proof. By choosing a suitable cut-off function in the energy inequality satisfied by u, we get the following estimate

$$\begin{aligned} A(R/2) + E(R/2) \leq c\Bigg[\frac{1}{R^3}\int_{Q(R)} |u|^2 dz + \frac{1}{R^2}\int_{Q(R)} \big||u|^2 - [|u|^2]_{,R}\big|\,|u| dz \\ + \int_{-R^2}^{0} [|u|^2]_{,R} \int_{B(R)} \frac{1}{R}|\nabla u| dx dt\Bigg]. \end{aligned} \tag{4.24}$$

First, let us notice that

$$\frac{1}{R^3}\int_{Q(R)} |u|^2 dz \leq cC^{2/3}(R);$$

next,

$$\begin{aligned} \int_{-R^2}^{0} [|u|^2]_{,R} \int_{B(R)} \frac{1}{R}|\nabla u| dx dt &= c\int_{-R^2}^{0} \left(\frac{1}{R^3}\int_{B(R)} |u|^2 dx\right)^{1/2} \left(\frac{1}{R^3}\int_{B(R)} |u|^2 dx\right)^{1/2} \\ &\quad\times \left(\frac{1}{R}\int_{B(R)} |\nabla u| dx\right) dt \\ &\leq c\frac{A(R)^{\frac{1}{2}}}{R} \left(\frac{1}{R^3}\int_{Q(R)} |u|^2 dz\right)^{1/2} \\ &\quad\times \left(\int_{-R^2}^{0} \left(\frac{1}{R}\int_{B(R)} |\nabla u| dx\right)^2 dt\right)^{1/2} \\ &\leq c\frac{A(R)^{\frac{1}{2}}}{R} C^{1/3}(R) R E^{1/2}(R) \\ &\leq cA(R)^{1/2}C^{1/3}(R)E^{1/2}(R). \end{aligned}$$

We deal with the remaining term in the following way:

$$\begin{aligned}
\int_{Q(R)} \left||u|^2 - [|u|^2]_{,R}\right| |u| dz &\leq \int_{-R^2}^{0} \left(\int_{B(R)} \left||u|^2 - [|u|^2]_{,R}\right|^{3/2} \right)^{2/3} \left(\int_{B(R)} |u|^3 \right)^{1/3} \\
&\leq c \int_{-R^2}^{0} \left(\int_{B(R)} |\nabla u|^2 dx \right)^{1/2} \left(\int_{B(R)} |u|^2 dx \right)^{1/2} \\
&\quad \times \left(\int_{B(R)} |u|^3 dx \right)^{1/3} dt \\
&\leq c R^{1/2} A^{1/2}(R) \left(\int_{Q(R)} |u|^3 dz \right)^{1/3} \\
&\quad \times \left(\int_{-R^2}^{0} \left(\int_{B(R)} |\nabla u|^2 dx \right)^{3/4} dt \right)^{2/3} \\
&\leq c R^{1/2+2/3} A^{1/2}(R) C^{1/3}(R) R^{1/3} \left(\int_{Q(R)} |\nabla u|^2 dz \right)^{1/2} \\
&\leq c R^2 A^{1/2}(R) C^{1/3}(R) E^{1/2}(R),
\end{aligned}$$

which concludes the proof. □

Proof of Theorem 4.3.1. It follows from Lemma 4.3.2 and the assumptions of Theorem 4.3.1 that:

$$C(r) \leq c \left[\left(\frac{\varrho}{r} \right)^3 A^{3/4}(\varrho) \epsilon^{3/4} + \left(\frac{r}{\varrho} \right)^3 A^{3/2}(\varrho) \right]. \tag{4.25}$$

By introducing, the new quantity

$$\mathcal{E}(r) := A^{3/2}(r),$$

we derive from Lemma 4.3.3 that

$$\mathcal{E}(r) \leq \left[C(2r) + A^{3/4}(2r) C^{1/2}(2r) \epsilon^{3/4} \right]. \tag{4.26}$$

Now let us assume that $0 < r \leq \varrho/2 < \varrho \leq 1$. Replacing r with $2r$ in (4.25), we can rewrite (4.26) as follows:

$$\begin{aligned}\mathcal{E}(r) \leq c &\Bigg[\left(\frac{\varrho}{r}\right)^3 A^{3/4}(\varrho)\epsilon^{3/4} + \left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho) \\ &+A^{3/4}(2r)\left(\left(\frac{\varrho}{r}\right)^3 A^{3/4}(\varrho)\epsilon^{3/4} + \left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho)\right)^{1/2} \epsilon^{3/4}\Bigg] \\ &\leq c\Bigg[\left(\frac{\varrho}{r}\right)^3 A^{3/4}(\varrho)\epsilon^{3/4} + \left(\frac{r}{\varrho}\right)^3 A^{3/2}(\varrho) + \left(\frac{\varrho}{r}\right)^{3/2+3/4} A^{3/4+3/8}(\varrho)\epsilon^{3/4+3/8} \\ &+ \left(\frac{\varrho}{r}\right)^{3/4} A^{3/4}(\varrho)\left(\frac{r}{\varrho}\right)^{3/2} A^{3/4}(\varrho)\epsilon^{3/4}\Bigg].\end{aligned}$$

Here, the obvious inequality $A(2r) \leq c\varrho A(\varrho)/r$ has been used. Applying Young's inequality with an arbitrary positive constant δ, we show that

$$\mathcal{E}(r) \leq c\left(\frac{r}{\varrho}\right)^{3/4}(\epsilon^{3/4}+1)\mathcal{E}(\varrho) + c\delta\mathcal{E}(\varrho) + c(\delta)\left(\epsilon^{3/2}\left(\frac{\varrho}{r}\right)^6 + \left(\frac{\varrho}{r}\right)^9 \epsilon^{9/2}\right).$$

Therefore,

$$\mathcal{E}(r) \leq c\left[\left(\frac{r}{\varrho}\right)^{3/4}(\epsilon^{3/4}+1)+\delta\right]\mathcal{E}(\varrho) + c(\delta)\left(\frac{\varrho}{r}\right)^9(\epsilon^{3/2}+\epsilon^{9/2}). \tag{4.27}$$

Inequality (4.27) holds for $r \leq \varrho/2$ and can be rewritten as follows:

$$\mathcal{E}(\vartheta\varrho) \leq c\left[\vartheta^{3/4}(\epsilon^{3/4}+1)+\delta\right]\mathcal{E}(\varrho) + c(\delta)\vartheta^{-9}(\epsilon^{3/2}+\epsilon^{9/2}) \tag{4.28}$$

for any $0 < \vartheta \leq 1/2$ and any $0 < \varrho \leq 1$.
Now, assuming that $\epsilon \leq 1$, let us fix ϑ and δ such that

$$2c\vartheta^{1/4} < 1/2, \quad 0 < \vartheta \leq 1/2, \quad c\delta < \vartheta^{1/2}/2. \tag{4.29}$$

Obviously, ϑ and δ are independent of ϵ. We find that

$$\mathcal{E}(\vartheta\varrho) \leq \vartheta^{1/2}\mathcal{E}(\varrho) + G \tag{4.30}$$

for any $0 < \varrho \leq 1$, where $G = G(\epsilon) \to 0$ as $\epsilon \to 0^+$.
By iterating (4.30), we obtain that

$$\mathcal{E}(\vartheta^k\varrho) \leq \vartheta^{k/2}\mathcal{E}(\varrho) + cG, \tag{4.31}$$

for any natural number k and any $0 < \varrho \leq 1$. Letting $\varrho = 1$, we obtain that

$$\mathcal{E}(\vartheta^k) \leq \vartheta^{k/2}\mathcal{E}(1) + cG. \tag{4.32}$$

Hence, it can be easily deduced from (4.32), that

$$\mathcal{E}(r) \leq c\left(r^{1/2}\mathcal{E}(1) + G(\epsilon)\right), \tag{4.33}$$

for all $0 < r \leq 1/2$. Now, (4.25) implies, for $0 < r \leq 1/4$, that

$$\begin{aligned} C(r) &\leq c\left[\mathcal{E}^{1/2}(2r)\epsilon^{3/4} + \mathcal{E}(2r)\right] \\ &\leq c\left[r^{1/4}\mathcal{E}^{1/2}\epsilon^{3/4} + G(\epsilon)^{1/2}\epsilon^{3/4} + r^{1/2}\mathcal{E}(1) + G(\epsilon)\right]. \end{aligned}$$

Choosing ϵ sufficiently small and r_0 also sufficiently small, we obtain that

$$|Q|^{-\frac{1}{3}}C(r_0)^{\frac{1}{3}} < \epsilon_0,$$

where ϵ_0 is as in Theorem 4.2.1 (with $a = b = 0$). Since u is a suitable weak solution in $Q(r_0)$, Theorem 4.2.1 and the scaling symmetry of our system yield the required statement, and the estimate holds for the case $k = l = 0$; the other cases can be obtained by a straightforward bootstrap argument. Thus, Theorem 4.3.1 is proved. □

Remark 18. As observed in [6] for the case of the Navier-Stokes system, Theorem 4.3.1 give us an information on the *size* of the set of potential singularities of a suitable weak solution u to $\mathcal{L}_0$. Indeed, let us introduce the following parabolic analogue of Hausdorff measures.

Definition 4.3.1 (See [6]). For any $X \subset \mathbb{R}^{3+1}$ and $k \geq 0$; the $k-$dimensional parabolic Hausdorff measure of X, denoted by $\mathcal{H}^k(X)$, is defined as follows:

$$\mathcal{H}^k(X) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(X),$$

where

$$\mathcal{H}^k_\delta(X) = \inf\left\{\sum_{r=1}^{\infty} r_i^k : \; X \subset \bigcup_i Q^*_{r_i}(x,t), \; r_i < \delta\right\} \tag{4.34}$$

and $Q^*_{r_i}(x,t) := B(x,r) \times]t - \frac{7}{8}r^2, t + \frac{1}{8}r^2[$. In particular, we have that $\mathcal{H}^4 = \mathcal{L}^4$ the Lebesgue measure on $\mathbb{R}^{3+1}$.

For u a suitable weak solution to $\mathcal{L}_0 = 0$ or $\mathcal{D}\mathcal{L}_0 = 0$ in Q, if we denote by Σ the set of potential singular points of u in Q then $\mathcal{H}^1(\Sigma) = 0$. In fact, we are able to prove a slightly stronger result. We introduce a measure $\mathcal{P}^1$ on $\mathbb{R}^{3+1}$ defined in a similar manner as $\mathcal{H}^1$ with the difference being that the cylinders $Q^*_{r_i}(x,t)$ in (4.34) are replaced with parabolic cylinders $Q_{r_i}(x,t)$; clearly

$$\mathcal{H}^1 \leq c\mathcal{P}^1,$$

where $c > 0$ is a universal constant. Then, we are able to show that $\mathcal{P}^1(\Sigma) = 0$. Indeed, let us start by noticing that by Theorem 4.3.1

$$z_0 = (x_0, t_0) \in \Sigma \Longrightarrow \limsup_{r \to 0^+} r^{-1} \int_{Q^*(z_0,r)} |\nabla u|^2 dx > \epsilon.$$

Now, let V be a neighbourhood of Σ in Q, and let $\delta > 0$. For each $z_0 = (x_0, t_0) \in \Sigma$, we can choose $Q^*(z_0, r)$ with $r < \delta$ such that

$$r^{-1} \int_{Q^*(z_0,r)} |\nabla u|^2 dx > \epsilon \quad \text{and} \quad Q^*(z_0, r) \subset V.$$

Applying Lemma 2.4.1 to this covering, we obtain a disjoint subcovering $\{Q^*(z_0^i, r_i)\}$ such that

$$\Sigma \subset \bigcup_i Q^*(z_0^i, 5r_i),$$

and we have

$$\begin{aligned} \sum_i r_i &\le \epsilon^{-1} \sum_i \int_{Q^*(z_0^i,r_i)} |\nabla u|^2 dx \\ &\le \epsilon^{-1} \int_V |\nabla u|^2 dx. \end{aligned} \tag{4.35}$$

Since we choose $r < \delta$, then we have

$$\sum_i r_i^4 \le \delta^3 \epsilon^{-1} \int_V |\nabla u|^2 dx,$$

which leads to $\mathcal{H}^4(\Sigma) = 0$, since δ is arbitrary; and a fortiori $\mathcal{L}^4(\Sigma) = 0$. Next, we have, still from (4.35), that

$$\mathcal{P}^1(\Sigma) \le c\epsilon^{-1} \int_V |\nabla u|^2 dx,$$

and this holds for every neighbourhood V of Σ. Since we showed previously that $\mathcal{L}^4(\Sigma) = 0$, it follows that $\mathcal{P}^1(\Sigma) = 0$. This result means that the set of potential singularities of u is *discrete*.

As a consequence of Theorem 4.3.1, we have the following result.

Proposition 4.3.1. *There exists a positive constant* ϵ_* *such that if for any suitable weak solution* u *to* $\mathcal{L}_0 = 0$ *in* Q*, we have*

$$\sup_{0<r<1} \sup_{-r^2 \le t \le 0} \frac{1}{r} \int_{B(r)} |u(x,t)|^2 dx < \epsilon_*, \tag{4.36}$$

then $z = 0$ *is a regular point of* u*, i.e. there exists* $0 < r_0 < 1$ *such that* u *is Hölder continuous in* $\overline{Q(r_0)}$*.*

Proof. Similarly to what we had in the proof of Theorem 4.3.1 (i.e. combining Lemma 4.3.2 and Lemma 4.3.3), we find for $0 < r \leq \varrho/2 < \varrho \leq 1$ that

$$E(r) \leq c \left[\left(\frac{r}{\varrho}\right)^2 \epsilon_* + \left(\frac{\varrho}{r}\right)^2 \epsilon_*^{1/2} E^{1/2}(\varrho) + \left(\frac{r}{\varrho}\right)^{1/2} \epsilon_* E^{1/2}(\varrho) + \left(\frac{\varrho}{r}\right)^{3/2} \epsilon_*^{3/4} E^{3/4}(\varrho) \right];$$

the latter can be rewritten in the following way

$$E(\vartheta\varrho) \leq c \left[\vartheta^2 \epsilon_* + \vartheta^{-2} \epsilon_*^{1/2} E^{1/2}(\varrho) + \vartheta^{1/2} \epsilon_* E^{1/2}(\varrho) + \vartheta^{-3/2} \epsilon_*^{3/4} E^{3/4}(\varrho) \right],$$

for any $0 < \vartheta \leq 1/2$ and any $0 < \varrho \leq 1$. Now, by applying Young's inequality, we get

$$E(\vartheta\varrho) \leq \vartheta^{1/2} E(\varrho) + c\vartheta^{-15/2}(\epsilon_* + \epsilon_*^2 + \epsilon_*^3), \tag{4.37}$$

which gives us after iteration

$$E(\vartheta^k \varrho) \leq \vartheta^{k/2} E(\varrho) + c\vartheta^{-15/2}(\epsilon_* + \epsilon_*^2 + \epsilon_*^3) \quad (\forall k \in \mathbb{N})$$

for all $0 < \vartheta \leq 1/2$ and all $0 < \varrho \leq 1$. From this, and by using similar arguments as in the proof of Theorem 4.3.1, we obtain that

$$E(r) \leq c \left[r^{1/2} E(1) + \epsilon_* + \epsilon_*^2 + \epsilon_*^3 \right], \ \forall r \in]0, 1/2]$$

thus

$$\sup_{0<r<r_0} E(r) \leq c \left[r_0^{1/2} E(1) + \epsilon_* + \epsilon_*^2 + \epsilon_*^3 \right] \quad (\text{with } r_0 \in]0, 1/2]). \tag{4.38}$$

Finally we choose r_0 and ϵ_* sufficiently small such that the right-hand side in (4.38) is less than ϵ (see Theorem 4.3.1); and the proof is completed by using the scaling symmetry of our system together with Theorem 4.3.1. □

4.4 The Ladyzhenskaya-Prodi-Serrin type theorems

A pair of exponents $(s, l) \in [3, \infty] \times [2, \infty]$ is said to be of Ladyzhenskaya-Prodi-Serrin type if it satisfies the following

$$\frac{3}{s} + \frac{2}{l} = 1. \tag{4.39}$$

When it comes to proving existence of strong solutions (and a fortiori Hölder continuity of those solutions) for a parabolic PDE, a necessary condition is that the lower order coefficients (advective and convective coefficients) of the equation belong to the space $L_{s,l}$ with (s, l) as in (4.39) and $s > 3$ (see e.g. [41] Chapter 3, Sections 6 to 10). The limit case $s = 3$ is more subtle; it was shown for instance in [3] that for a Fokker-Planck type equation with convective term b in $L_{3,\infty}$, there exists a 'smooth' solution that blows up at finite time. Therefore it is quite natural, from a regularity point of view, to wonder what happens when we assume that a solution to our system $\mathcal{L}_0 = 0$ (or $\mathcal{DL}_0 = 0$) belongs to the Ladyzhenskaya-Prodi-Serrin class. As we shall see, this class also provides us with scale invariant quantities that, unlike what we have seen so far, do not require any smallness condition to lead to regularity.

4.4.1 The case $s > 3$ of the Ladyzhenskaya-Prodi-Serrin condition

We use here techniques introduced in [58] for as similar question in the case of the incompressible 3D Navier-Stokes system; our first result is as follows (and is a crucial intermediate step in order to prove the first main result of this section, Theorem 4.4.3)

Theorem 4.4.1. *Let v be a weak solution to $\mathcal{L}_0 = 0$ in Q that belongs to the energy class:*

$$v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q).$$

There exists a universal positive constant $\epsilon_0 > 0$ such that if

$$\sup_{0<r<1} \|v\|^l_{s,l,Q(r)} < \epsilon_0, \tag{4.40}$$

provided (4.39) *and $s > 3$, then $z = 0$ is a regular point.*

Remark 19. Under condition (4.40) (finiteness is actually enough), by testing the system with $v\phi^2$, where $\phi \in C_0^\infty(Q)$ and integrating by part, we directly get that v is a suitable weak solution. A local energy identity is even available, by following ideas presented in Appendix A.

For the reader's convenience, we recall the following parabolic Sobolev type embeddings; we can see this thanks to Lemma 2.3.1.

Proposition 4.4.1. *Let Ω be a bounded domain with Lipschitz boundary. Let T be a positive parameter, and let $Q_T \equiv \Omega\times]0,T[$. Suppose $m, n, s, l, d, r \in [1,\infty]$ and put $\alpha = 2 - \frac{3}{m} - \frac{2}{n} + \frac{3}{s} + \frac{2}{l}$, $\beta = 1 - \frac{3}{m} - \frac{2}{n} + \frac{3}{d} + \frac{2}{r}$. The space $W^{2,1}_{m,n}(Q_T)$ is embedded continuously in $L_{s,l}(Q_T)$ if $m \leq s$, $n \leq l$, and $\alpha > 0$, or $1 \leq m \leq s < \infty$, $1 < n \leq l < \infty$, and $\alpha = 0$. Furthermore, $W^{2,1}_{m,n}(Q_T)$ is embedded continuously in $W^{1,0}_{d,r}(Q_T)$ if $m \leq d$, $n \leq r$, and $\beta > 0$, or $1 \leq m \leq d < \infty$, $1 < n \leq r < \infty$, and $\beta = 0$.*

With this proposition at hand, we are able to prove the following result:

Proposition 4.4.2. *Let Ω be a bounded domain with a sufficiently smooth boundary, and let T be a positive parameter. Suppose*

$$\begin{gathered} 1 < m < s, \quad 1 < n < l, \quad \frac{3}{s} + \frac{2}{l} = 1, \\ f \in L_{m,n}(Q_T), \quad u \in L_{s,l}(Q_T). \end{gathered} \tag{4.41}$$

There exists a positive number $\epsilon = \epsilon(m, n, s, l, Q_T)$ such that if

$$\|v\|_{s,l,Q_T} \leq \epsilon, \tag{4.42}$$

then the initial-boundary value problem

$$\partial_t u - \Delta u + v \cdot \nabla u + \frac{1}{2}(\operatorname{div} u)v = f \quad \text{in } Q_T, \quad u|_{\partial' Q_T} = 0 \tag{4.43}$$

has a unique solution, which satisfies the estimate

$$\|u\|_{W^{2,1}_{m,n}(Q_T)} \leq c(m, n, s, l, Q_T)\|f\|_{m,n,Q_T}. \tag{4.44}$$

Proof. Given $w \in W^{2,1}_{m,n}(Q_T)$, we have (by Hölder's inequality)

$$\|v \cdot \nabla w\|_{m,n,Q_T} + 1/2\|(\operatorname{div} w)v\|_{m,n,Q_T} \leq \|v\|_{s,l,Q_T}\|\nabla w\|_{d,r,Q_T} \leq \epsilon\|\nabla w\|_{d,r,Q_T}$$

with

$$d = \frac{sm}{s-m}, \quad \text{and } r = \frac{ln}{l-n} \quad \text{(well defined, see (4.41))}.$$

On the other hand, we have

$$\|\nabla w\|_{d,r,Q_T} \leq c(m, n, s, l, Q_T)\|w\|_{W^{2,1}_{m,n}(Q_T)},$$

which is valid because

$$1 - \frac{3}{m} - \frac{2}{n} + \frac{3}{d} + \frac{2}{r} = 0,$$

and thanks to Proposition 4.4.1. Hence, we get that

$$\|v \cdot \nabla w\|_{m,n,Q_T} + 1/2\|(\operatorname{div} w)v\|_{m,n,Q_T} \leq \epsilon c(m, n, s, l, Q_T)\|w\|_{W^{2,1}_{m,n}(Q_T)}. \tag{4.45}$$

Next, we introduce the linear operator $L : X \to X(:= \{u \in W^{2,1}_{m,n}(Q_T) : u|_{\partial' Q_T} = 0\})$ defined as follows: $Lw = u$, means that, for a given $w \in X$, u is the unique solution of the initial-boundary value problem

$$\partial_t u - \Delta u + v \cdot \nabla w + \frac{1}{2}(\operatorname{div} w)v = f \quad \text{in } Q_T, \quad u|_{\partial' Q_T} = 0.$$

We readily have from (4.45) that the operator L is a contraction if ϵ is sufficiently small and therefore we have existence of a unique solution $u \in W^{2,1}_{m,n}(Q_T)$ to (4.43). In order to prove (4.44), we use the fact that

$$\begin{aligned}\|u\|_{W^{2,1}_{m,n}(Q_T)} &\leq c(m,n,Q_T)\|f - v\cdot\nabla u - 1/2(\operatorname{div} u)v\|_{m,n,Q_T}\\ &\leq \epsilon c(m,n,s,l,Q_T)\|u\|_{W^{2,1}_{m,n}(Q_T)} + c(m,n,Q_T)\|f\|_{m,n,Q_T}.\end{aligned}$$

Choosing ϵ sufficiently small, we have that (4.44) is proved, which completes the proof. $\square$

Proof of Theorem 4.4.1. Let $\phi_{a,b}$ $(0 < a < b \leq 1)$ be a cut-off function of class $C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ such that

$$\begin{gathered}0 \leq \phi_{a,b} \leq 1, \quad \phi_{a,b} \equiv 1 \quad \text{in } B(a)\times]-a^2, a^2[,\\ \phi_{a,b} \equiv 0 \quad \text{off } B(b)\times]-b^2, b^2[.\end{gathered}$$

Let us consider the auxiliary initial-boundary value problem

$$\partial_t u - \Delta u + v\cdot\nabla u + \frac{1}{2}(\operatorname{div} u)v = F(v) \quad \text{in } Q, \quad u|_{\partial' Q} = 0, \tag{4.46}$$

where

$$F(v) := v.(v\otimes\nabla\phi_{a,b}) + 1/2(v.\nabla\phi_{a,b})v - 2\nabla\phi_{a,b}\cdot\nabla v + v(\partial_t\phi_{a,b} - \Delta\phi_{a,b}).$$

Let m, n such that $1 < m, n < 2$ and $3/m + 2/l \geq 5/2$. Let us notice that, since $s > 3$, we have that $2 \leq l < \infty$. Setting

$$d = \frac{sm}{s-m}, \quad \text{and } r = \frac{ln}{l-n},$$

it is not difficult to see that $d \leq 2s/(s-2) < 6$ and $3/d + 2/r \geq 3/2$. Hence, $\|v\|_{d,r,Q} < \infty$, since it belongs to the energy class $L_{2,\infty}(Q) \cap W^{1,0}_2(Q)$. We choose ϵ_0 (see (4.33)) such that $\epsilon_0^{1/l} < \epsilon$, with $\epsilon = \epsilon(m,n,s,l,Q)$ as in Proposition 4.4.2. Therefore, we have

$$\|v.(v\otimes\nabla\phi_{a,b}) + 1/2(v.\nabla\phi_{a,b})v\|_{m,n,Q} \leq c(a,b)\|v\|_{s,l,Q(b)}\|v\|_{d,r,Q(b)} \leq \epsilon c(a,b)\|v\|_{d,r,Q(b)}.$$

Hence

$$\|F(v)\|_{m,n} \leq c(a,b) < \infty. \tag{4.47}$$

By Proposition 4.4.2, we deduce that there exists a unique function u that is a solution to (4.46) such that

$$\|u\|_{W^{2,1}_{m,n}(Q)} \leq c(a,b). \tag{4.48}$$

By the uniqueness result in Proposition 4.4.2, we have that

$$u = \phi_{a,b} v \quad \text{in } Q.$$

Since a, b are arbitrary, we deduce that $v \in W^{2,1}_{m,n,loc}(Q)$. Now, let us set $m_1 = \min\{2m/(3-m), 3\}$. We see that

$$v \in W^{1,0}_{m_1,n}(Q(b)).$$

Next, we put $d_1 = m_1 s/(s - m_1)$ (well-defined since $m_1 \leq 3 < s$) and we calculate the number $\delta = 2 - \frac{3}{m} - \frac{2}{n} + \frac{3}{d_1} + \frac{2}{r}$; this number is equal to $1/m_1$ if $m_1 < 3$ and to $2 - 3/m \geq 1/3$ if $m_1 = 3$. Thus, by Proposition 4.4.1

$$v \in L_{d_1,r}(Q(b)).$$

So, as before, we obtain that $F(v) \in L_{m_1,n}(Q)$ and applying Proposition 4.4.2 (we reduce ϵ_0 if necessary) and using the same arguments, we deduce that $v \in W^{2,1}_{m_1,n,loc}(Q)$. Repeating this process (we reduce ϵ_0 if necessary at each step), we obtain a non-decreasing sequence m_k (given by the induction formula $m_k = \min\{2m_{k-1}/(3 - m_{k-1}), 3\}$) such that $v \in W^{2,1}_{m_k,n,loc}(Q)$. We show that after finitely many steps, we shall have $m_k = 3$. Indeed, otherwise,

$$1 < m_k < m_{k+1} = \frac{2m_k}{3 - m_k} < 3.$$

The sequence m_k in that case has a limit d. Clearly,

$$d = \frac{2d}{3-d} \geq m > 1,$$

but this is impossible.
From now on we assume that $v \in W^{2,1}_{3,n,loc}(Q)$ and we call ϵ_1 the result of the successive *finite* reductions of ϵ_0 (obviously $\epsilon_1 \leq \epsilon_0$). We put

$$l_1 = \frac{2n}{2-n}\delta,$$

and suppose that $\delta \in]0,1[$ is subject to the restriction $n > \frac{2}{1+\delta}$. It is easy to check that, with such l_1 and δ, we have

$$l_1 > 2, \quad s_1 = \frac{3n\delta}{n(1+\delta) - 2} > \frac{3n}{2(n-1)} > 3 \tag{4.49}$$

and

$$\frac{3}{s_1} + \frac{2}{l_1} = 1. \tag{4.50}$$

We fix m_1 so that $\frac{3n}{2(n-1)} < m_1 < s_1$, and calculate $d_1 = \frac{s_1 m}{s_1 - m}$, $r_1 = \frac{l_1 n}{l_1 - n}$. By a space-type embedding theorem,

$$v \in W^{1,0}_{m_1,n,loc}(Q).$$

Applying Proposition 4.4.1, we conclude that $v \in L_{s_1,l_1,loc}(Q)$, since

$$2 - \frac{3}{3} - \frac{2}{n} + \frac{3}{s_1} + \frac{2}{l_1} = 2 - \frac{2}{n} > 0,$$

and $v \in L_{d_1,r_1,loc}(Q)$, because

$$2 - \frac{3}{3} - \frac{2}{n} + \frac{3}{d_1} + \frac{2}{r_1} = \frac{3}{m_1} > 0.$$

Next, reducing the number b if necessary, we obtain

$$\|v\|_{s_1,l_1,Q(b)} < \epsilon(m_1, n, s_1, l_1, Q).$$

Therefore, as before, we deduce that

$$v \in W^{2,1}_{m_1,n}(Q(a)).$$

Since

$$\mu = 2 - \frac{3}{m_1} - \frac{2}{n} > 0,$$

we get that there exists an $r \in]0,1[$ such that v is Hölder continuous in the closure of $Q(r)$, which concludes the proof of the theorem (once again we can see this thanks to Lemma 2.3.1). □

It is worth mentioning that a similar ϵ-regularity result is also available for the limit case $s = 3$. In fact, we have an even stronger result.

Proposition 4.4.3. *Let v be a weak solution to $\mathcal{L}_0 = 0$ in Q that belongs to the energy class:*

$$v \in L_{2,\infty}(Q) \cap W^{1,0}_2(Q).$$

There exists a universal positive constant ϵ_0 such that if

$$\operatorname*{ess\,sup}_{-1<t<0} \|v(\cdot,t)\|_{L^{3,\infty}(Q)} < \epsilon_0, \tag{4.51}$$

then $z = 0$, is a regular point. Here $L^{3,\infty}$ is the weak L_3−space (see 'Chapter 2' for definition) which contains the standard Lebesgue space L_3.

One way to see this result is through the following lemma.

Lemma 4.4.2. *Let* $u \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ *be a weak solution to* $\mathcal{L}_0 = 0$ *in* Q *such that*

$$M := \operatorname*{ess\,sup}_{-1<t<0} \|u(\cdot,t)\|_{L^{3,\infty}(B)} < \infty.$$

Then, for any $Q(z_0,r) \subset Q$ *(with* $z_0 = (x_0,t_0)$*), we have*

$$\frac{1}{r} \operatorname*{ess\,sup}_{t_0-(\frac{r}{2})^2<t<t_0} \int_{B(x_0,\frac{r}{2})} |u(x,t)|^2 dx + \frac{1}{r} \int_{Q(z_0,\frac{r}{2})} |\nabla u|^2 dz \leq c\left(M^2 + M^4\right). \tag{4.52}$$

Proof. Once again, under the assumptions of the lemma, testing our equation with $u\phi^2$ (with $\phi \in C_0^\infty(B\times]-1,1[)$) yields that u is a suitable weak solution in Q. Next, we introduce the following cut-off function: $0 \leq \varphi_{\varrho,r} \in C_0^\infty(B(x_0,r)\times]-r^2,r^2[)$ (with $0 < \varrho < r < 1$) such that $\varphi_{\varrho,r} \equiv 1$ in $B(x_0,\varrho)\times]-\varrho^2,\varrho^2[$, $|\nabla^k \varphi_{\varrho,r}| \leq c/(r-\varrho)^k$ (with $k = 1,2$) and $|\partial_t \varphi_{\varrho,r}| \leq c/(r^2-\varrho^2)$. Now, take $\varphi_{\varrho,r}^2$ as a test function in the local energy inequality satisfied by u; we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{t_0-r^2<t<t_0} \int_{B(x_0,r)} \varphi_{\varrho,r}^2(x,t)|u(x,t)|^2 dx + \int_{Q(z_0,r)} \varphi_{\varrho,r}^2 |\nabla u|^2 dz &\leq \frac{c}{(r-\varrho)^2} \int_{Q(z_0,r)} |u|^2 dz \\
+ c\left[\frac{1}{(r-\varrho)^2}\int_{Q(z_0,r)} |u|^2 dz\right]^{\frac{1}{2}} &\left(\int_{Q(z_0,r)} \varphi_{\varrho,r}^2 |u|^4 dz\right)^{\frac{1}{2}},
\end{aligned} \tag{4.53}$$

where c is an absolute constant. We just have now to estimate the right-hand side of the previous inequality. For the first term, we use the fact that (see Proposition 2.2.1)

$$\int_{B(x_0,r)} |u(x,t)|^2 dx \leq 3|B(r)|^{1/3} \|u(\cdot,t)\|_{L_{3,\infty}(B)}^2 \leq crM^2, \tag{4.54}$$

for a.e. $t \in]t_0 - r^2, t_0[$. Thus,

$$\frac{1}{(r-\varrho)^2} \int_{Q(z_0,r)} |u|^2 dz \leq \frac{cr^3}{(r-\varrho)^2} M^2. \tag{4.55}$$

Next, we estimate the second factor of the last term in (4.53); we have thanks to Proposition 2.2.1 and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
\|\varphi_{\varrho,r}^{1/2} u(\cdot,t)\|_{4,B(r)} &= \|\varphi_{\varrho,r}^{1/2} |u|^{1/2} |u|^{1/2}\|_{L_{4,4}(B(x_0,r))} \\
&\leq c\|\varphi_{\varrho,r}^{1/2} |u|^{1/2}\|_{L_{12,4}(B(x_0,r))} \times \||u|^{1/2}\|_{L_{6,\infty}(B(x_0,r))} \\
&= c\|\varphi_{\varrho,r} u\|_{L_{6,2}(B(x_0,r))}^{1/2} \times \|u\|_{L_{3,\infty}(B(x_0,r))}^{1/2} \\
&\leq cM^{1/2} \|\nabla(\varphi_{\varrho,r} u)(\cdot,t)\|_{L_2(B(x_0,r))}^{\frac{1}{2}},
\end{aligned}$$

for a.e. $t \in]t_0 - r^2, t_0[$; this leads to

$$\left(\int_{Q(z_0,r)} \varphi_{\varrho,r}^2 |u|^4 dz\right)^{\frac{1}{2}} \leq cM \left[\left(\frac{1}{(r-\varrho)^2}\int_{Q(z_0,r)} |u|^2 dz\right)^{\frac{1}{2}} + \left(\int_{Q(z_0,r)} \varphi_{\varrho,r}^2 |\nabla u|^2\right)^{1/2}\right]. \quad (4.56)$$

Taking into account (4.53) and (4.56), we get

$$\operatorname*{ess\,sup}_{t_0-r^2<t<t_0} \int_{B(x_0,r)} \varphi_{\varrho,r}^2 |u|^2(x,t)dx + \int_{Q(z_0,r)} \varphi_{\varrho,r}^2 |\nabla u|^2 dz \leq \frac{cr^3}{(r-\varrho)^2}(M^2 + M^4). \quad (4.57)$$

Taking now $\varrho = r/2$ and dividing both side of the previous inequality by r, we have that the proof is completed. □

Proof of Proposition 4.4.3. From the previous lemma, we have that

$$\sup_{0<r<1} \frac{2}{r} \int_{Q(\frac{r}{2})} |\nabla u|^2 dz \leq c(\epsilon_0^2 + \epsilon_0^4),$$

and we choose ϵ_0 sufficiently small such that $c(\epsilon_0^2 + \epsilon_0^3) < \epsilon$, where ϵ is given by Theorem 4.3.1; the proof is completed thanks to Theorem 4.3.1. □

Remark 20. It is worth mentioning that the smallness assumption in Theorem 4.4.1 for the non-limit case (i.e. $s > 3$) can be easily removed; the reason being that $\|v\|_{s,l,Q(r)}^l \to 0$ as $r \to 0$. Therefore, this observation allows us to strengthen Theorem 4.4.1 in order to obtain our first main result of this section.

Theorem 4.4.3. *Let v be as in Theorem 4.4.1. Assume that*

$$v \in L_{s,l}(Q)$$

such that (4.39) *holds and $s > 3$. Then $z = 0$ is a regular point.*

For the limit case $s = 3$, the previous result is still true and this relies heavily on the structure of the nonlinearity; however things are much more complicated and we dedicate the next section to the proof of such statement.

4.4.2 The limit case $s = 3$ of the Ladyzhenskaya-Prodi-Serrin condition

We start by stating (without proof) the crucial results we shall use for the proof of our main theorem.

Unique continuation through spatial boundaries

We denote $Q(R,T) := B(R)\times]0,T[\subset \mathbb{R}^3 \times \mathbb{R}$ and consider a vector-valued function $u = (u_1, u_2, u_3)$, satisfying the three conditions:

$$u \in W^{2,1}_2(Q(R,T);\mathbb{R}^3); \tag{4.58}$$

$$|\partial_t u + \Delta u| \leq c_1(|u| + |\nabla u|) \quad \text{a.e. in } Q(R,T), \tag{4.59}$$

for some positive constant c_1;

$$|u(x,t)| \leq C_k(|x| + \sqrt{t})^k, \tag{4.60}$$

for all $k = 0, 1, \dots$, for all $(x,t) \in Q(R,T)$, and for some positive constants C_k. Here,

$$W^{2,1}_2(Q(R,T);\mathbb{R}^3) := \left\{ u : |u| + |\nabla u| + |\nabla^2 u| + |\partial_t u| \in L_2(Q(R,T)) \right\}.$$

Theorem 4.4.4. *Assume that a function u satisfies conditions* (4.58)-(4.60)*. Then, $u(x,0) = 0$ for all $x \in B(R)$.*

Backward uniqueness for the heat operator in the half-space

Let $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$ and $Q_+ := \mathbb{R}^n_+ \times]0,1[$. We consider a vector-valued function $u : Q_+ \to \mathbb{R}^n$, which is “sufficiently regular” (such that the following estimates make sense) and satisfies

$$|\partial_t u + \Delta u| \leq c_1(|u| + |\nabla u|) \quad \text{a.e in } Q_+, \tag{4.61}$$

for some positive constant c_1 and

$$u(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^n_+. \tag{4.62}$$

We also assume the following growth and integrability property for u

$$|u(x,t)| \leq e^{M|x|^2} \tag{4.63}$$

for all $(x,t) \in Q_+$ and for some $M > 0$.

$$u, \nabla u, \partial_t u, \nabla^2 u \in L_{2,loc}(Q_+). \tag{4.64}$$

Theorem 4.4.5. *Assume that u satisfies* (4.61)-(4.64)*. Then $u \equiv 0$ in Q_+.*

For a proof of the above two theorems, see e.g. [60], Appendix A. The second main result of this section is as follows.

Theorem 4.4.6. *For any $M > 0$, there exists a positive constant $\epsilon = \epsilon(M)$ with the following property: if $u \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ is a weak solution to $\mathcal{L}_0 = 0$ in Q that satisfies in addition*

$$\operatorname*{ess\,sup}_{-1<t<0} \|u(\cdot,t)\|_{L^{3,\infty}(B)} \leq M, \tag{4.65}$$

and

$$\|u(\cdot,0)\|_{L^{3,\infty}(B(1/2))} \leq \epsilon, \tag{4.66}$$

then $z = 0$ is a regular point of u.

Before giving the proof of this theorem let us make a couple of observations.

Remark 21. We justify why it makes sense to talk about values at all time for $\|u(\cdot,t)\|_{L^{3,\infty}(B(1/2))}$. We go back to Lemma 4.4.2 and its proof to see that the following holds:

$$\|u\|_{L_4(Q(5/6))} + \|\nabla u\|_{L_2(Q(5/6))} \leq C(M).$$

Thus, by Hölder's inequality, we obtain

$$\|u \cdot \nabla u + \frac{u}{2} \operatorname{div} u\|_{L_{4/3}(Q(5/6))} \leq C(M)$$

and a fortiori

$$\int_{Q(3/4)} \left(|\partial_t u|^{4/3} + |\nabla^2 u|^{4/3}\right) dz \leq C(M).$$

The latter implies that

$$u \in C([-(3/4)^2, 0]; L_{4/3}(B(3/4))). \tag{4.67}$$

Now, let us introduce

$$\Pi = \left\{t \in]-(3/4)^2, 0[: \|u(\cdot,t)\|_{L^{3,\infty}(Q(3/4))} \leq M\right\};$$

because the set $]-(3/4)^2, 0[\backslash\Pi$ is negligible, we have that for all $t_0 \in [-(3/4)^2, 0]$, there exists a sequence $(t_0^n) \subset \Pi$ such that $t_0^n \to t_0$. Now, let $\phi \in C_0^\infty(B(3/4))$; we have

$$\begin{aligned}\int_{B(3/4)} u(x,t_0)\phi(x)dx &= \int_{B(3/4)} (u(x,t_0) - u(x,t_0^n))\phi(x)dx + \int_{B(3/4)} u(x,t_0^n)\phi(x)dx \\ &\leq \|u(\cdot,t_0) - u(\cdot,t_0^n)\|_{L_{4/3}(B(3/4))}\|\phi\|_{L_4(B(3/4))} + M\|\phi\|_{L^{\frac{3}{2},1}(B(3/4))}.\end{aligned}$$

Now, taking the limit $n \to \infty$ in the above and taking into account (4.67), we find that

$$\int_{B(3/4)} u(x,t_0)\phi(x)dx \leq M\|\phi\|_{L^{\frac{3}{2},1}(B(3/4))},$$

for all $t_0 \in [-(3/4)^2, 0]$ and all $\phi \in C_0^\infty(B(3/4))$. This yields, since $L^{3,\infty}$ is the dual of the Lorentz space $L^{\frac{3}{2},1}$ (see for instance [22]), that

$$\sup_{-(3/4)^2 \le t \le 0} \|u(\cdot,t)\|_{L^{3,\infty}(B(3/4))} \le M; \tag{4.68}$$

this makes in particular (4.66) a justified assumption.

Remark 22. The second observation is that Theorem 4.4.6 implies that for $u \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ a weak solution to $\mathcal{L}_0 = 0$ in Q such that $\|u\|_{3,\infty,Q} < \infty$, we have that $z = 0$ is a regular point of u. To see this, we use the fact that

$$\|u\|_{L^{3,\infty}(\mathcal{X})} \le \|u\|_{L_3(\mathcal{X})},$$

($\mathcal{X}$ being any measure space) and we set $M := \|u\|_{3,\infty,Q}$. Using similar arguments as in the previous remark, we get that

$$\sup_{-(3/4)^2 \le t \le 0} \|u(\cdot,t)\|_{L_3(B(3/4))} \le M.$$

Consequently, we have that there exists $0 < r_M \le 3/4$ such that $\|u(\cdot,0)\|_{L_3(B(r_M))} \le \epsilon$ with ϵ as in the statement of Theorem 4.4.6. Finally if $r_M > 1/2$, we apply directly Theorem 4.4.6 to u and we are done, otherwise, we define $v^M(x,t) := 2r_M u(2r_M x, 4r_M^2 t)$ (with $(x,t) \in Q$) and we apply Theorem 4.4.6 to v^M instead.

We start now the proof of Theorem 4.4.6.

Proof of Theorem 4.4.6. We argue by contradiction. Assume that there exists a number $M > 0$ such that for any $\epsilon_k > 0$ (with $\epsilon_k \to 0^+$ as $k \to \infty$) one can find a weak solution $u^k \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ to $\mathcal{L}_0 = 0$ in $\mathcal{D}'(Q)$ satisfying

$$\operatorname*{ess\,sup}_{-1<t<0} \|u^k(\cdot,t)\|_{L^{3,\infty}(B)} \le M \quad \text{and} \quad \|u^k(\cdot,0)\|_{L^{3,\infty}(B(1/2))} \le \epsilon_k, \tag{4.69}$$

and for which $z = 0$ is a singular point. Thus, from Proposition 4.3.1, we can find a sequence (r_k) such that $r_k \to 0$ as $k \to \infty$ and

$$\frac{1}{r_k} \sup_{-r_k^2 \le t \le 0} \int_{B(r_k)} |u^k(x,t)|^2 > \epsilon_\star. \tag{4.70}$$

We extend now u^k to the whole space $\mathbb{R}^{3+1}$ by 0 and denote the extension by $\bar{u}^k$. Next, we introduce the function $U^k(y,s) = r_k \bar{u}^k(r_k y, r_k^2 s)$; we have obviously that

$$\|U^k\|_{L_\infty(\mathbb{R};L^{3,\infty}(\mathbb{R}^3))} = \|u^k\|_{L_\infty(-1,0;L^{3,\infty}(B))} \le M,$$

which implies that (U^k) converges to v weakly$-\star$ in $L_\infty(\mathbb{R}; L^{3,\infty}(\mathbb{R}^3))$ up to a subsequence still denoted by (U^k). Using similar arguments as in Remark 21, we get

$$\begin{gathered}\int_{\tilde{Q}} |\nabla U^k|^2 dz \leq c(|\tilde{Q}|, M),\\ \int_{\tilde{Q}} \left(|U^k|^4 + |\nabla^2 U^k|^{4/3} + |\partial_t U^k|^{4/3}\right) dz \leq c(|\tilde{Q}|, M),\end{gathered} \tag{4.71}$$

for all parabolic cylinders $\tilde{Q} \subset\subset \mathbb{R}^{3+1}$ and all k. This leads to

$$U^k \to v \quad \text{in } C([a,b]; L_2(\Omega)), \tag{4.72}$$

for any $-\infty < a < b < +\infty$ and $\Omega \subset\subset \mathbb{R}^3$. Let us summarise the key information we get for the limit v.

$$\begin{gathered}\|v\|_{L_\infty(\mathbb{R};L^{3,\infty}(\mathbb{R}^3))} \leq M;\\ \int_{\tilde{Q}} \left(|v|^4 + |\nabla v|^2 + |\nabla^2 v|^{4/3} + |\partial_t v|^{4/3}\right) dz \leq c(|\tilde{Q}|, M), \quad \forall \tilde{Q} \subset\subset \mathbb{R}^{3+1};\\ v \in C([a,b]; L_2(\Omega)), \forall -\infty < a < b < +\infty \text{ and } \Omega \subset\subset \mathbb{R}^3;\\ v \text{ satisfies equation } \mathcal{L}_0 = 0 \text{ a.e. in } \mathbb{R}^{3+1},\\ 2\int_{\mathbb{R}}\int_{\mathbb{R}^3} \phi |\nabla v|^2 dz = \int_{\mathbb{R}}\int_{\mathbb{R}^3} \left\{|v|^2(\partial_t \phi + \Delta\phi) + v\cdot\nabla\phi |v|^2\right\} dz, \ \forall \phi \in C_0^\infty(\mathbb{R}^{3+1}).\end{gathered} \tag{4.73}$$

From (4.70), we have

$$\sup_{-1\leq s\leq 0} \int_B |U^k(y,s)|^2 dy = \frac{1}{r_k} \sup_{-r_k^2 \leq t \leq 0} \int_{B(r_k)} |u^k(x,t)|^2 dx > \epsilon_\star;$$

thus, thanks to (4.72),

$$\sup_{-1\leq s\leq 0} \int_B |v(y,s)|^2 dy > \epsilon_\star. \tag{4.74}$$

Now, fix $R > 0$ and choose r_k such that $1/2r_k > R$ (this is legitimate since $r_k \to 0^+$). We have from the second estimate in (4.69) that

$$\|U^k(\cdot, 0)\|_{L^{3,\infty}(B(R))} \leq \epsilon_k,$$

for all k such that $1/2r_k > R$. This implies that $U^k(\cdot,0) \to 0$ strongly in $L^{3,\infty}(B(R))$ and, a fortiori, strongly in $L_2(B(R))$. Because of (4.72) and since R was chosen arbitrarily, we get that

$$v(\cdot, 0) \equiv 0 \quad \text{in } \mathbb{R}^3.$$

The next step is to apply, successively, our backward uniqueness and unique continuation results. In order to do so, we need some suitable uniform estimates. Let $T_2 > 0$, we have thanks to the Fubini theorem that

$$\begin{aligned}\left|\{(y,s)\in\mathbb{R}^3\times]-4T_2,0[: |v|>\gamma\}\right| &= \int_{-4T_2}^{0}\left|\{y\in\mathbb{R}^3 : |v(y,s)|>\gamma\}\right| ds\\ &\le \gamma^{-3}\int_{-4T_2}^{0}\|v(\cdot,s)\|_{L^{3,\infty}(\mathbb{R}^3)}ds\\ &\le 4T_2\gamma^{-3}M<\infty,\end{aligned}$$

for any fixed $\gamma > 0$. Thus, for any $\bar{\epsilon} > 0$, there exists $R_2 = R_2(\bar{\epsilon}, T_2, \gamma) > 4$ (with the parameters $\bar{\epsilon}$ and γ to be chosen later) such that

$$\left|\{(y,s)\in\mathbb{R}^3\setminus B(R_2/4)\times]-4T_2,0[: |v|>\gamma\}\right|\le\bar{\epsilon}. \tag{4.75}$$

Now, take $z_0 \in \mathbb{R}^3 \setminus B(R_2/2)\times]-2T_2, 0[$; we have

$$\begin{aligned}\int_{Q(z_0,1)}|v|^3dz &= \int_{Q(z_0,1)\cap\{|v|\le\gamma\}}|v|^3dz+\int_{Q(z_0,1)\cap\{|v|>\gamma\}}|v|^3dz\\ &\le\gamma^3|Q|+\left(\int_{Q(z_0,1)}|v|^4dz\right)^{\frac{3}{4}}\left|\{(y,s)\in\mathbb{R}^3\setminus B(R_2/4)\times]-4T_2,0[: |v|>\gamma\}\right|^{\frac{1}{4}}\\ &\le\gamma^3|Q|+C(M)\bar{\epsilon}^{\frac{1}{4}}\quad\text{(see (4.73))}.\end{aligned}$$

Finally, we choose γ and $\bar{\epsilon}$ such that $\gamma^3|Q| + C(M)\bar{\epsilon}^{\frac{1}{4}} < |Q|\epsilon_0^3$, with ϵ_0 as in Theorem 4.2.1 (with $a = b = 0$). This directly gives us by bootstrap arguments that

$$\max_{z\in Q(z_0,1/2)}|\nabla^k v(z)|\le c_{0k},$$

for any $k = 0, 1, \ldots$, and the latter holds for any $z_0 \in \mathbb{R}^3 \setminus B(R_2/2)\times]-2T_2, 0[$. Consequently, we have

$$|\partial_t v-\Delta v|\le K(|v|+|\nabla v|)\quad\text{in }\mathbb{R}^3\setminus\overline{B(R_2)}\times]-T_2,0], \tag{4.76}$$

for a suitable $K > 0$. So, from Theorem 4.4.5, we get that

$$v\equiv 0\quad\text{in }\mathbb{R}^3\setminus\overline{B(R_2)}\times]-T_2,0], \tag{4.77}$$

and this holds for any $T_2 > 0$. From (4.73), we have that $\|v(\cdot,t_*)\|_{\overset{\circ}{L}{}^1_2(B(2R_2))} < \infty$ for a.e. $-\infty < t_* < 0$. The short time well-posedness of strong solutions established in the previous chapter and estimate (4.77) together with Theorem 4.4.4 guarantee that

$$v(\cdot,t_*)\equiv 0\quad\text{in }\mathbb{R}^3,$$

and the latter holds for all $-\infty < t_* < 0$ and a fortiori

$$v\equiv 0\quad\text{in }\mathbb{R}^{3+1},$$

which contradicts (4.74); and the proof is complete. □

Concluding remarks: We do not know, at this moment of time, whether the backward uniqueness (Theorem 4.4.5) and unique continuation theorem (Theorem 4.4.4) hold also true for the time-dependent Lamé system; we suspect, however, that it is the case. Consequently, we can't prove Theorem 4.4.6 in the case of operators $\mathcal{L}_\kappa$ and $\mathcal{DL}_\kappa$ for $\kappa > 0$. This will be the object of future investigations.

Finally, one could wonder what happens if condition (4.66) is removed in Theorem 4.4.6; in other words whether we could get regularity only from condition (4.65) like in Remark 22 for the case of L_3. Unfortunately, like in the case of the Navier-Stokes system, we are unable to conclude to regularity or blow-up. This will also be the object of future investigations.

Chapter 5

Approximation of forward self-similar solutions to the 3D Navier-Stokes system

5.1 Introduction

In this chapter, we construct forward self-similar solutions to the 3D incompressible Navier-Stokes system as the singular limit, when $\kappa \to \infty$ of forward self-similar solutions to $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$. Unless mentioned otherwise, the notation and definitions used in this chapter are contained in Chapter 2: Preliminary Material and the results and text base of this chapter are based on [27, 29].

Let us give more context to the problem at hand in this chapter. In [31], H. Jia and V. Šverák proved the existence of the so-called forward self-similar solution $u \in C^\infty(\mathbb{R}^3 \times]0,\infty[)$ to the Cauchy problem for the homogeneous incompressible Navier-Stokes system (1.1)-(1.3), where the initial data u_0 is an arbitrary large (-1)-homogeneous divergence-free vector valued field; see also for instance [44, 4, 9] for a different construction and [19] for a proof of global well-posedness under some smallness condition on the initial data u_0. By definition, the velocity u is invariant with respect to the Navier-Stokes scaling, i.e., $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ for all $\lambda > 0$. Such a problem can be reduced to the existence of a solution to the following stationary system

$$-\Delta U + U \cdot \nabla U - \frac{x}{2} \cdot \nabla U - \frac{U}{2} + \nabla P = 0, \qquad \operatorname{div} U = 0 \tag{5.1}$$

in $\mathbb{R}^3$, where the profile U is subjected to the following boundary condition at infinity

$$|U(x) - u_0(x)| = o(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{5.2}$$

Then

$$u(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$$

is a local energy weak solution to (1.1)-(1.3) in the sense of Lemarie-Rieusset (see [43], [44] and also [34]). An important idea of Jia and Šverák is to use the possible instantaneous non-uniqueness of forward self-similar solutions in order to construct different weak Leray-Hopf solutions with the same L_2-initial data. In fact, they state a sufficient condition on the spectrum of the linearised problem ensuring non-uniqueness of forward self-similar solutions. So far, it is an open problem whether the above condition holds for a certain (-1)-homogeneous initial data. However, numerical experiments, see [23], demonstrate that there are initial data for which the above spectral condition is satisfied.

As aforementioned at the beginning of this section we are going to approximate this forward self-similar solution u to the incompressible Navier-Stokes system with forward solutions to $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$. There is a hope that this might help us to get a better understanding of the non-uniqueness phenomenon for the Navier-Stokes system, since we will have a priori two candidates in our hands. For our approximating systems, we have the following elliptic systems for the profile U^κ:

$$-\Delta U^\kappa - \kappa\nabla\operatorname{div} U^\kappa + U^\kappa\cdot\nabla U^\kappa + \frac{U^\kappa}{2}\operatorname{div} U^\kappa - \frac{x}{2}\cdot\nabla U^\kappa - \frac{U^\kappa}{2} = 0 \quad \text{in } \mathbb{R}^3, \tag{5.3}$$

if we are solving $\mathcal{L}_\kappa = 0$ or

$$-\Delta U^\kappa - \kappa\nabla\operatorname{div} U^\kappa + \operatorname{div}\left(U^\kappa\otimes U^\kappa + \frac{|U^\kappa|^2}{2}I_3\right) - \frac{x}{2}\cdot\nabla U^\kappa - \frac{U^\kappa}{2} = 0 \quad \text{in } \mathbb{R}^3, \tag{5.4}$$

if we are solving $\mathcal{DL}_\kappa = 0$ instead. We require the following asymptotic on U^κ:

$$|U^\kappa(x) - u_0(x)| = o(|x|^{-1}) \quad \text{as } |x|\to\infty. \tag{5.5}$$

Then the corresponding solution to the Cauchy problem has the form

$$u^\kappa(x,t) = \frac{1}{\sqrt{t}}U^\kappa\left(\frac{x}{\sqrt{t}}\right)$$

for $x\in\mathbb{R}^3$ and $t>0$.

Before we continue our discussion let us mention that we can find in the literature results about constructing solutions of the Cauchy or initial boundary value problem for the Navier-Stokes system (1.1) as the limit of solutions of the Cauchy or initial boundary value problem for the system $\mathcal{L}_\kappa = 0$; this was done, for instance, for initial data u_0 in the Lebesgue spaces L_2 (see e.g. [42, 55, 67]) and L_3 (see [55]). In all these papers, the key step was to find the right notion of solution in order to establish some uniform global (in space) energy estimates for the solutions of $\mathcal{L}_\kappa = 0$. If we consider

$(-1)-$homogeneous initial data, the weak Lebesgue space $L^{3,\infty}(\mathbb{R}^3)$ appears to be the natural choice to carry out our analysis. In the case of the Cauchy problem for the incompressible Navier-Stokes system with initial data in $L^{3,\infty}(\mathbb{R}^3)$, a notion of weak solution which enables the derivation of a global energy inequality was introduced in [2]; the authors called those solutions *global weak* $L^{3,\infty}-$*solutions*. Consequently, an analogous notion of their solution for our models $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$ appears to be the right setting for our work.

Let us state now our results. We will focus mostly on the operator $\mathcal{L}_\kappa$ since all the results and computations in this case hold also for the operator $\mathcal{DL}_\kappa$ with some simplicity due to the divergence structure of the nonlinearity. The first result we present here is about the existence of forward self-similar solutions to the Cauchy problem for the systems $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$ with the initial condition

$$u^\kappa|_{t=0} = u_0. \tag{5.6}$$

Indeed, given $\kappa \geq 0$, one can prove the existence of a forward self-similar solution to the Cauchy problem $\mathcal{L}_\kappa = 0$ and (5.6) or $\mathcal{DL}_\kappa = 0$ and (5.6), using the same method as in [31]. Our contribution here is that we use the notion of global weak $L^{3,\infty}$-solution which is slightly stronger than the notion of weak Lemarie-Rieusset solutions. In order to present the corresponding definitions, we need the semigroup $S_\kappa(t)$ associated to the Lamé system, i.e., $v^\kappa(x,t) = S_\kappa(t)u_0(x)$, where v^κ is a solution to the Cauchy problem:

$$\partial_t v^\kappa - \Delta v^\kappa - \kappa \nabla \operatorname{div} v^\kappa = 0 \quad \text{in } Q_+ \tag{5.7}$$

and

$$v^\kappa|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3. \tag{5.8}$$

For example, in the case of $\mathcal{L}_\kappa = 0$, the definition of a global weak $L^{3,\infty}-$solution is as follows.

Definition 5.1.1. We say that u^κ is a global weak $L^{3,\infty}-$solution to the Cauchy problem $\mathcal{L}_\kappa = 0$ and (5.6) in Q_+, with $u_0 \in L^{3,\infty}(\mathbb{R}^3)$, if the function

$$w^\kappa = u^\kappa - v^\kappa \tag{5.9}$$

has the following properties:

$$\sup_{0<t<T} \int\limits_{\mathbb{R}^3} |w^\kappa(x,t)|^2 dx + \int\limits_0^T \int\limits_{\mathbb{R}^3} |\nabla w^\kappa|^2 dx dt \leq C(T) < \infty \tag{5.10}$$

for all $T > 0$;

$$\partial_t w^\kappa - \Delta w^\kappa - \kappa \nabla \mathrm{div}\, w^\kappa + u^\kappa \cdot \nabla u^\kappa + \frac{u^\kappa}{2} \mathrm{div}\, u^\kappa = 0 \tag{5.11}$$

in the sense of distributions; the function

$$t \mapsto \int_{\mathbb{R}^3} w^\kappa(x,t) \cdot w(x) dx, \tag{5.12}$$

is continuous at each $t \geq 0$ for all $w \in L_2(\mathbb{R}^3)$;

$$\|w^\kappa(\cdot,t)\|_{L_2(\mathbb{R}^3)} \to 0 \quad \text{as } t \to 0^+; \tag{5.13}$$

for a.a. $t \in]0,T[$, the local energy inequality

$$\begin{aligned} \frac{1}{2}\int_0^t |u^\kappa(x,t)|^2 \phi(x,t) dx + \int_0^t \int_{\mathbb{R}^3} \left(|\nabla u^\kappa|^2 + \kappa |\mathrm{div}\, u^\kappa|^2 \right) \phi(x,t) dx dt \\ \leq \int_0^t \int_{\mathbb{R}^3} \frac{|u^\kappa|^2}{2} (\partial_t \phi + \Delta \phi) dx dt + \int_0^t \int_{\mathbb{R}^3} (\frac{|u^\kappa|^2}{2} - \kappa \mathrm{div}\, u^\kappa) u^\kappa \cdot \nabla \phi dx dt. \end{aligned} \tag{5.14}$$

is valid for each non-negative test function $\phi \in C_0^\infty(Q_+)$.

Remark 23. We have also an analogous definition for the Cauchy problem $\mathcal{DL}_\kappa = 0$ and (5.6).

Remark 24. It is easy to show that w^κ is a turbulent solution in the Leray sense (see [45]). In other words, for all $t \in [0,T]$,

$$\begin{aligned} \frac{1}{2}\int\limits_{\mathbb{R}^3} |w^\kappa(x,t)|^2 dx + \int\limits_s^t \int\limits_{\mathbb{R}^3} (|\nabla w^\kappa|^2 + \kappa |\mathrm{div}\, w^k|^2) dx dt' \\ \leq \frac{1}{2}\int\limits_{\mathbb{R}^3} |w^\kappa(x,s)|^2 dx + \int\limits_s^t \int\limits_{\mathbb{R}^3} (v^\kappa \otimes w^\kappa + v^\kappa \otimes v^\kappa) : \nabla w^k dx dt' \\ + \int\limits_s^t \int\limits_{\mathbb{R}^3} \frac{1}{2} v^\kappa \cdot w^\kappa \mathrm{div}\, w^\kappa dx dt' \end{aligned} \tag{5.15}$$

for a.a. $s \in [0,T]$, including $s = 0$.

We are able now to formulate our first result.

Theorem 5.1.1. *Assume that $u_0 \in L^{3,\infty}(\mathbb{R}^3)$. There exists at least one global weak $L^{3,\infty}-$solution u^κ to the Cauchy problem $\mathcal{L}_\kappa = 0$ and (5.6). Moreover, the global energy* *estimate*

$$\|w^\kappa(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + \int_0^t\int_{\mathbb{R}^3} |\nabla w^\kappa(x,s)|^2 dxds \\ + \kappa\int_0^t\int_{\mathbb{R}^3} |\operatorname{div} w^\kappa|^2 dxds \leq c_0 t^{\frac{1}{2}}\left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)}\right), \quad (5.16)$$

holds for all $\kappa, t > 0$ and for an absolute positive constant c_0.

Because we do not know if the solution we constructed in the above theorem is unique, there is no guarantee that a $(-1)-$homogeneous initial data produces a scale invariant solution. But we are able to prove the following result.

Theorem 5.1.2. *Let $u_0 \in C^\infty(\mathbb{R}^3\setminus\{0\})$ be such that $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$. Then, given $\kappa \geq 0$, there exists a smooth solution U^κ to the boundary value problem (5.3) and (5.5) satisfying the decay estimates:*

$$|\partial^\alpha\left(U^\kappa(x) - V^\kappa(x)\right)| \leq \frac{C(\alpha,\kappa,u_0)}{(1+|x|)^{3+|\alpha|}}, \qquad V^\kappa(x) := v^\kappa(x,1),$$

for all $\alpha \in \mathbb{N}^3$ (with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$) and for all $x \in \mathbb{R}^3$. Moreover, $u^\kappa(x,t) = \frac{1}{\sqrt{t}}U^\kappa(\frac{x}{\sqrt{t}})$ is a global weak $L^{3,\infty}-$solution to the system $\mathcal{L}_\kappa = 0$ with initial data u_0 (see Definition 5.1.1).

Remark 25. The result in the above theorem holds also true for the boundary value problem (5.4) and (5.5); with some simplifications in the computations due to the divergence structure of the nonlinearity. We do not give the details of the computations for the sake of brevity.

Now, we state our main contributions. They concern behaviour of solutions to the boundary value problem (5.3) and (5.5) as $\kappa \to \infty$. To this end, let us make a simple remark: if $\operatorname{div} u_0 = 0$, then $v^\kappa(x,t) = v(x,t) = S(t)u_0(x)$, where $S(t)$ is a semigroup associated with the usual heat equation.

Theorem 5.1.3. *Let $u_0 \in C^\infty(\mathbb{R}^3\setminus\{0\})$ be such that $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$ and* $\operatorname{div} u_0 = 0$. *Let $W^\kappa(x) = U^\kappa(x) - V(x)$, where U^κ is a smooth solution to the boundary value problem (5.3) and (5.5) constructed in Theorem 5.1.2 and $V(x) = v(x,1)$. Then the following estimates are valid:*

$$\|W^\kappa\|^2_{W^1_2(\mathbb{R}^3)} + \kappa\|\operatorname{div} U^\kappa\|^2_{L_2(\mathbb{R}^3)} \leq c\left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)}\right), \quad (5.17)$$

and

$$\|\kappa\operatorname{div} U^\kappa\|^2_{L_2(\mathbb{R}^3)} \leq c\left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)}\right)^2, \quad (5.18)$$

where $c > 0$ is a universal constant. Moreover,

$$\int_{\mathbb{R}^3} |\nabla^2 W^\kappa|^2 dx + \kappa^2 \int_{\mathbb{R}^3} |\nabla \mathrm{div}\, W^\kappa|^2 dx \leq C(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}). \tag{5.19}$$

Theorem 5.1.4. *Under the assumptions of Theorem 5.1.3, there exists a subsequence, still indexed by κ, such that*

$$W^\kappa \to W, \qquad \nabla W^\kappa \to \nabla W, \qquad \nabla^2 W^\kappa \to \nabla^2 W \tag{5.20}$$

and

$$\kappa\, \mathrm{div}\, W^\kappa \rightharpoonup P \qquad \kappa \nabla \mathrm{div}\, W^\kappa \rightharpoonup \nabla P \tag{5.21}$$

in $L_2(\mathbb{R}^3)$ as $\kappa \to \infty$, where the limiting functions $U = V + W$ and P have the following properties:

(i)

$$|\partial^\alpha W(x)| \leq \frac{C(\alpha, u_0)}{(1 + |x|)^{3+|\alpha|}}; \tag{5.22}$$

for all $\alpha \in \mathbb{N}^3$ (with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$) and $x \in \mathbb{R}^3$;

(ii) the function $u(x,t) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$ (together with $p(x,t) = \frac{1}{t} P(\frac{x}{\sqrt{t}})$) is a global weak $L^{3,\infty}-$solution (in the sense of Barker-Seregin-Šverák [2]) to the incompressible Navier-Stokes system (1.1) *with initial data u_0.*

Remark 26. Once more, the above result holds also for the system (5.4).

Before we state the proofs of our theorems, let us give a couple of important properties of the semigroup $S_\kappa(t)$ that will be useful later; they are a slight refinement of estimates derived in Chapter 2 Lemma 2.3.2.

Proposition 5.1.1. *Let $S_\kappa(t)$ be the semigroup associated with the Lamé system, see the Cauchy problem (5.7) and (5.8). Then*

$$\|S_\kappa(t) u_0\|_{L^{3,\infty}(\mathbb{R}^3)} \leq c \|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}, \quad \forall t \geq 0. \tag{5.23}$$

For $1 \leq s_1 \leq s$, we have

$$\|S_\kappa(t) u_0\|_{L_s(\mathbb{R}^3)} \leq c(s, s_1) \left[1 + (1+\kappa)^{-\frac{1}{l}}\right] t^{-\frac{1}{l}} \|u_0\|_{L_{s_1}(\mathbb{R}^3)}, \tag{5.24}$$

where

$$\frac{1}{l} = \frac{3}{2}\left(\frac{1}{s_1} - \frac{1}{s}\right).$$

Proof. The classical Calderon-Zygmund estimate combined with real interpolation methods allow us to show the existence of a unique function q_0 up to a constant (which doesn't matter here since we are interested in the gradient of q_0) such that $\Delta q_0 = \operatorname{div} u_0$ and

$$\|\nabla q_0\|_{L^{3,\infty}(\mathbb{R}^3)} \leq c\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}. \tag{5.25}$$

Set $u_0^{(1)} := \nabla q_0$ and $u_0^{(0)} := u_0 - u_0^{(1)}$; notice that $\operatorname{div} u_0^{(0)} = 0$ and $\operatorname{curl} u_0^{(1)} = 0$ by definition. It is easy to check that $\Delta v^1 = \nabla \operatorname{div} v^1$ and $\operatorname{div} v^0 = 0$, where

$$\partial_t v^1 - (1+\kappa)\Delta v^1 = 0, \qquad v^1|_{t=0} = u_0^{(1)}$$

and

$$\partial_t v^0 - \Delta v^0 = 0, \qquad v^0|_{t=0} = u_0^{(0)}.$$

Moreover,

$$S_\kappa(t)u_0(\cdot) = v^0(\cdot,t) + v^1(\cdot,t) = S((1+\kappa)t)u_0^{(1)} + S(t)u_0^{(0)}. \tag{5.26}$$

Finally, from (5.25) and (5.26), from the convolution structure of the heat potential and from Young's inequality for weak type spaces (see [22] Theorem 1.2.13), we get (5.23).
For (5.24), we see have instead

$$\|u_0^{(0)}\|_{L_{s_1}(\mathbb{R}^3)} + \|u_0^{(1)}\|_{L_{s_1}(\mathbb{R}^3)} \leq c(s_1)\|u_0\|_{L_{s_1}(\mathbb{R}^3)}.$$

And once again, using the representation formula for solutions of the heat equation, Young's inequality and scaling arguments, we have (5.24). This concludes the proof. □

Let us begin now the proofs of the theorems.

5.2 Proof of Theorem 5.1.1

We will only justify the estimate (5.16) of the theorem because once we get this a priori estimate, the machinery used to prove existence is reminiscent to what we did in Chapter 3; we divide the proof into two steps:

Step I. Let $u^\kappa(x,t)=S_\kappa(t)u_0+w^\kappa$ be a global weak $L^{3,\infty}-$solution to $\mathcal{L}_\kappa=0$ with initial data $u_0\in L^{3,\infty}(\mathbb{R}^3)$. We apply Lemma 2.2.1 to get $u_0=\bar{u}_0^N+\hat{u}_0^N$ and we introduce the following functions

$$\bar{v}^N(x,t):=S_\kappa(t)\bar{u}_0^N(x), \tag{5.27}$$

$$\hat{v}^N(x,t):=S_\kappa(t)\hat{u}_0^N(x) \tag{5.28}$$

and

$$w^N(x,t):=w^\kappa(x,t)+\hat{v}^N(x,t)(=u^\kappa(x,t)-\bar{v}^N(x,t)), \tag{5.29}$$

for all $(x,t)\in Q_+$ (here we omit the dependence of $\bar{v}^N$, $\hat{v}^N$ and w^N with respect to κ for the sake of simplicity). By assumption and taking for instance $r=3,t=2$ in Lemma 2.2.1, we have that

$$\sup_{0<s<t}\|w^N(\cdot,s)\|^2_{L_2(\mathbb{R}^3)}+\int_0^t\int_{\mathbb{R}^3}|\nabla w^N(x,s)|^2dxds<\infty\quad\forall t>0,$$

$$\lim_{t\to 0^+}\|w^N(\cdot,t)-\hat{u}_0^N\|_{L_2(\mathbb{R}^3)}=0, \tag{5.30}$$

and

$$\partial_t w^N-\Delta w^N-\kappa\nabla\mathrm{div}\,w^N+u^\kappa\cdot\nabla u^\kappa+\frac{u^\kappa}{2}\mathrm{div}\,u^\kappa=0 \text{ in } Q_+. \tag{5.31}$$

From the construction of the semigroup $S_\kappa(t)$ in Proposition 5.1.1, it follows that if $a\in L_s(\mathbb{R}^3)$ $(1<s<\infty)$ then $\|S_\kappa(t)a-a\|_{L_s(\mathbb{R}^3)}\to 0$ as $t\to 0^+$; which gives us (5.30). Let $0\le\varphi\in C_0^\infty(B)$ be such that $\varphi\equiv 1$ in $B(1/2)$ and $\varphi\equiv 0$ in $B\setminus B(3/4)$; we define, for every $R>0$, $\varphi_R(x):=\varphi(x/R)$. Now, from equation (5.31) and the definition (5.29), we can get that, for all $t>0$,

$$\begin{aligned}
&\frac{1}{2}\int_{B(R)}|w^N(x,t)|^2\varphi_R(x)dx+\int_0^t\int_{B(R)}|\nabla w^N|^2\varphi_R dxds\\
&+\kappa\int_0^t\int_{B(R)}|\mathrm{div}\,w^N|^2\varphi_R dxds=\frac{1}{2}\int_{B(R)}|\hat{u}_0^N(x)|^2\varphi_R dx+\frac{1}{2}\int_0^t\int_{B(R)}|w^N|^2\Delta\varphi_R dxds\\
&\quad-\kappa\int_0^t\int_{B(R)}\mathrm{div}\,w^N w^N\cdot\nabla\varphi_R dxds+\frac{1}{2}\int_0^t\int_{B(R)}|w^N|^2w^N\cdot\nabla\varphi_R dxds\\
&\quad+\int_0^t\int_{B(R)}\Big((w^N\cdot\nabla w^N)\cdot\bar{v}^N-(\bar{v}^N\cdot\nabla w^N)\cdot w^N+\frac{1}{2}\bar{v}^N\cdot w^N\mathrm{div}\,w^N\Big)\varphi_R dxds\\
&\quad+\int_0^t\int_{B(R)}(\bar{v}^N\cdot\nabla w^N)\cdot\bar{v}^N\varphi_R dxds+\int_0^t\int_{B(R)}w^N\cdot\nabla\varphi_R\bar{v}^N\cdot w^N dxds\\
&\quad+\int_0^t\int_{B(R)}\bar{v}^N\cdot\nabla\varphi_R\bar{v}^N\cdot w^N dxds;
\end{aligned}\tag{5.32}$$

where, for simplicity, we write the right-hand side of the previous identity as follows:

$$\frac{1}{2}\int_{B(R)} |\hat{u}_0^N(x)|^2 \varphi_R dx + \sum_{k=1}^{7} I_k(R).$$

The aim now is to estimate the I_k's and take the limit $R \to \infty$. To this end, we need the following known estimate:

$$\|a\|_{s,l,Q_T} \leq c(s,l)|a|_{2,Q_T}, \tag{5.33}$$

for $2 \leq s \leq 6$ and l satisfying

$$\frac{3}{s} + \frac{2}{l} = \frac{3}{2};$$

here

$$|a|_{2,Q_T} = \left(\operatorname*{ess\,sup}_{0<t<T} \|a(\cdot,t)\|^2_{L_2(\Omega)} + \|\nabla a\|^2_{2,Q_T} \right)^{\frac{1}{2}}.$$

We have

$$\begin{aligned} I_1(R) + I_2(R) + I_3(R) \leq \frac{ct}{R^2} \sup_{0<s<t} \|w^N(\cdot,s)\|^2_{L_2(\mathbb{R}^3)} \\ + \frac{\kappa c t^{\frac{1}{2}}}{R} \sup_{0<s<t} \|w^N(\cdot,s)\|_{L_2(\mathbb{R}^3)} \left(\int_0^t \int_{\mathbb{R}^3} |\nabla w^N(x,s)|^2 dx ds \right)^{\frac{1}{2}} \\ + \frac{ct^{\frac{1}{2}}}{R} \left(\int_0^t \left(\int_{\mathbb{R}^3} |w^N(x,s)|^3 dx \right)^{\frac{4}{3}} ds \right)^{\frac{1}{2}} \to 0 \text{ as } R \to \infty. \end{aligned}$$

Next,

$$I_4(R) \leq \frac{5}{2} \int_0^t \left(\int_{B(R)} |\nabla w^N|^2 \varphi_R dx \right)^{\frac{1}{2}} \left(\int_{B(R)} |w^N \varphi_R^{\frac{1}{2}}|^3 dx \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx \right)^{\frac{1}{6}} ds,$$

but by interpolation, we find

$$\begin{aligned} \|w^N \varphi^{\frac{1}{2}}\|_{L_3(B(R))} &\leq \|w^N \varphi^{\frac{1}{2}}\|^{\frac{1}{2}}_{L_2(B(R))} \|w^N \varphi^{\frac{1}{2}}\|^{\frac{1}{2}}_{L_6(B(R))} \\ &\leq c \left(\int_{B(R)} |w^N|^2 \varphi_R dx \right)^{\frac{1}{4}} \left[\left(\int_{B(R)} |\nabla w^N|^2 \varphi_R dx \right)^{\frac{1}{4}} \right. \\ &\quad \left. + \left(\int_{B(R)} |\nabla \varphi_R^{\frac{1}{2}}|^2 |w^N|^2 dx \right)^{\frac{1}{4}} \right]; \end{aligned}$$

consequently

$$
\begin{aligned}
I_4(R) &\leq c\int_0^t \left(\int_{B(R)} |\nabla w^N|^2\varphi_R dx\right)^{\frac{3}{4}} \left(\int_{B(R)} |w^N|^2\varphi_R dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx\right)^{\frac{1}{6}} ds \\
&\quad + \frac{c}{R^{\frac{1}{2}}}\int_0^t \left(\int_{B(R)} |\nabla w^N|^2\varphi_R dx\right)^{\frac{1}{2}} \left(\int_{B(R)} |w^N|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx\right)^{\frac{1}{6}} ds \\
&\leq \epsilon_1 \int_0^t \int_{B(R)} |\nabla w^N|^2\varphi_R dx + c(\epsilon_1)\int_0^t \left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx\right)^{\frac{2}{3}} \left(\int_{B(R)} |w^N|^2\varphi_R dx\right) ds \\
&\quad + I_4^{(0)}(R),
\end{aligned}
$$

with $\epsilon_1 > 0$ and

$$
\begin{aligned}
I_4^{(0)}(R) &:= \frac{c}{R^{\frac{1}{2}}}\int_0^t \left(\int_{B(R)} |\nabla w^N|^2\varphi_R dx\right)^{\frac{1}{2}} \left(\int_{B(R)} |w^N|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx\right)^{\frac{1}{6}} ds \\
&\leq \frac{ct^{\frac{1}{2}}}{R^{\frac{1}{2}}}\|\bar{u}_0^N\|_{L_6(\mathbb{R}^3)} \sup_{0<s<t}\|w^N(\cdot,s)\|_{L_2(\mathbb{R}^3)} \left(\int_0^t\int_{\mathbb{R}^3} |\nabla w^N(x,s)|^2 dxds\right)^{\frac{1}{2}} \\
&\to 0 \text{ as } R\to\infty,
\end{aligned}
$$

where Proposition 5.1.1 was used (with $s = s_1 = 6$) in the last inequality.
For $I_5(R)$, we have

$$
\begin{aligned}
I_5(R) &\leq \epsilon_2 \int_0^t\int_{B(R)} |\nabla w^N|^2\varphi_R dx + c(\epsilon_2)\int_0^t\int_{\mathbb{R}^3} |\bar{v}^N|^4 dxds \\
&\leq \epsilon_2 \int_0^t\int_{B(R)} |\nabla w^N|^2\varphi_R dx + c(\epsilon_2)Nt\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)},
\end{aligned}
$$

where we used Proposition 5.1.1 (with $s = s_1 = 4$) and Lemma 2.2.1 (with $s = 4$ and $r = 3$) in the last inequality.
Finally,

$$
\begin{aligned}
I_6(R) + I_7(R) &\leq \frac{ct^{\frac{1}{4}}}{R^{\frac{1}{4}}}\|\bar{u}_0^N\|_{L_4(\mathbb{R}^3)} \left(\int_0^t \left(\int_{\mathbb{R}^3} |w^N|^4 dx\right)^{\frac{2}{3}} ds\right)^{\frac{3}{4}} \\
&\quad + \frac{ct}{R}\|\bar{u}_0^N\|^2_{L_4(\mathbb{R}^3)} \sup_{0<s<t}\|w^N\|_{L_2(\mathbb{R}^3)} \to 0 \text{ as } R\to\infty.
\end{aligned}
$$

Summarising our efforts, we get (from (5.32)) that

$$
\begin{aligned}
&\frac{1}{2}\int_{B(R)} |w^N(x,t)|^2\varphi_R(x)dx + \int_0^t\int_{B(R)} |\nabla w^N|^2\varphi_R dxds \\
&+\kappa\int_0^t\int_{B(R)} |\operatorname{div} w^\kappa|^2\varphi_R dxds \leq \frac{1}{2}\int_{B(R)} |\hat{u}_0^N(x)|^2\varphi_R dx + (\epsilon_1+\epsilon_2)\int_0^t\int_{B(R)} |\nabla w^N|^2\varphi_R dxds \\
&\quad + c(\epsilon_1)\int_0^t \left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx\right)^{\frac{2}{3}} \left(\int_{B(R)} |w^N|^2\varphi_R dx\right) ds + c(\epsilon_2)Nt\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)} + J(R),
\end{aligned}
$$

with $J(R) \to 0$ as $R \to \infty$. Choosing suitable ϵ_1 and ϵ_2, and using Proposition 5.1.1 and Lemma 2.2.1, in order to get

$$\left(\int_{\mathbb{R}^3} |\bar{v}^N|^6 dx\right)^{\frac{2}{3}} \leq cN^2 \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)},$$

we find that

$$\begin{aligned}\int_{B(R/2)} |w^N(x,t)|^2 dx + \int_0^t \int_{B(R/2)} |\nabla w^N|^2 dxds \\ + \kappa \int_0^t \int_{B(R/2)} |\operatorname{div} w^\kappa|^2 dxds \leq \int_{B(R)} |\hat{u}_0^N(x)|^2 dx + \\ c\left(N^2 \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \int_0^t \int_{B(R)} |w^N(x,s)|^2 dxds + Nt\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)}\right) + 2J(R).\end{aligned}$$

Finally, taking the limit $R \to \infty$ in the above inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^3} |w^N(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dxds \\ + \kappa \int_0^t \int_{\mathbb{R}^3} |\operatorname{div} w^\kappa|^2 dxds \leq \int_{\mathbb{R}^3} |\hat{u}_0^N(x)|^2 dx + \\ c\left(N^2 \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \int_0^t \int_{\mathbb{R}^3} |w^N(x,s)|^2 dxds + Nt\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)}\right), \quad (5.34)\end{aligned}$$

for all $t > 0$. By Applying Gronwall's lemma to (5.34), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |w^N(x,t)|^2 dx \leq \left(\int_{\mathbb{R}^3} |\hat{u}_0^N(x)|^2 dx + N^{-1}\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}\right) \\ \times \exp(cN^2 t\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)}), \quad (5.35)\end{aligned}$$

for all $t > 0$. Now, by substituting (5.35) in (5.34) and using the fact that

$$\int_{\mathbb{R}^3} |\hat{u}_0^N(x)|^2 dx \leq cN^{-1}\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)},$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |w^N(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dxds \\ + \kappa \int_0^t \int_{\mathbb{R}^3} |\operatorname{div} w^\kappa|^2 dxds \leq cN^{-1}\left(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)}\right) \exp(cN^2 t\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)}) \\ + cNt\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)}, \quad (5.36)\end{aligned}$$

for all $t > 0$ and $N > 0$.

Step II. Firstly, let us notice that

$$\|\hat{v}^N(\cdot,t)\|^2_{L_2(\mathbb{R}^3)}+2\int_0^t\int_{\mathbb{R}^3}\left[|\nabla\hat{v}^N|^2+\kappa(\operatorname{div}\hat{v}^N)^2\right]dxds=\|\hat{u}^N_0\|^2_{L_2(\mathbb{R}^3)}(\leq cN^{-1}\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)}),$$

for all $t>0$. Secondly, going back to the definition of w^N (see (5.29)) and using the above identity, we see that

$$\begin{aligned}\int_{\mathbb{R}^3}|w^\kappa(x,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla w^\kappa|^2dxds\\+\kappa\int_0^t\int_{\mathbb{R}^3}|\operatorname{div}w^\kappa|^2dxds\leq cN^{-1}\left(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}+\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)}\right)\exp(cN^2t\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)})\\+cNt\|u_0\|^3_{L^{3,\infty}(\mathbb{R}^3)},\end{aligned}\tag{5.37}$$

for all $t>0$ and $N>0$. Selecting now

$$N=\frac{1}{\sqrt{2ct\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)}}},$$

we finally find

$$\begin{aligned}\int_{\mathbb{R}^3}|w^\kappa(x,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla w^\kappa|^2dxds\\+\kappa\int_0^t\int_{\mathbb{R}^3}|\operatorname{div}w^\kappa|^2dxds\leq c_0t^{\frac{1}{2}}\left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)}+\|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)}\right),\end{aligned}\tag{5.38}$$

for all $t>0$ and a universal constant $c_0>0$. The same machinery works for the system $\mathcal{DL}_\kappa=0$. This concludes the proof.

5.3 Proof of Theorem 5.1.2

Our proof of existence relies on some suitable a priori decay estimates at infinity for the forward self-similar profile; we obtain these thanks to the so-called *local in space near initial time smoothness* applied to our systems $\mathcal{L}_\kappa=0$ and $\mathcal{DL}_\kappa=0$, which was first introduced in [31] in the context of the Navier-Stokes system.

5.3.1 Local in space near initial time smoothness of local-energy solutions

Our first result is as follows; it is a quite general result an can find applications in different settings from what we are presenting here.

Theorem 5.3.1. *Let* $u_0 \in L_{2,loc}(\mathbb{R}^3)$ *be such that* $\alpha := \sup_{x_0\in\mathbb{R}^3} \int_B |u_0(x)|^2 dx < \infty$. *Suppose in addition that* $M := \|u_0\|_{L_m(B)} < \infty$ *with* $m > 3$. *Let us decompose* $u_0 = u_0^1 + u_0^2$ *with* $u_0^1|_{B(4/3)} = u_0$, *supp* $u_0^1 \subset\subset B(2)$ *and* $\|u_0^1\|_{L_m(\mathbb{R}^3)} \leq M$. *Now, let* a *be the locally in time defined mild solution to the system* $\mathcal{L}_\kappa = 0$ *with initial data* u_0^1. *Then, there exists a time* $T = T(\alpha, \kappa, m, M) > 0$ *such that any local-energy solution* u *(see Definition 1.2.2) to* $\mathcal{L}_\kappa = 0$ *satisfies:*

$$\|u - a\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{B(1/2)\times[0,T]})} \leq C(\alpha, \kappa, m, M),$$

for some $\gamma = \gamma(m) \in]0, 1[$.

Proof. Let us start by discussing the decomposition of u_0 in the theorem; introduce the cut-off function $0 \leq \varphi \in C_0^\infty(B)$ such that $\varphi \equiv 1$ in $B(4/3)$ and $\varphi \equiv 0$ in $B(2) \setminus B(3/2)$. We have the required splitting if we set $u_0^1 := u_0\varphi$ and $u_0^2 := u_0(1 - \varphi)$.
Next, by assumption a solves the Cauchy problem for the system $\mathcal{L}_\kappa = 0$ with initial data u_0^1 in $\mathbb{R}^3 \times [0, T_1]$, where $T_1 = T_1(\kappa, m, M) > 0$ (see 'Chapter 3'); moreover, the following estimates hold:

$$\begin{aligned} \|a\|_{C([0,T_1];L_2(\mathbb{R}^3))} + \|\nabla a\|_{L_2(\mathbb{R}^3\times(0,T_1))} &\leq C(m, M), \\ \|a\|_{L_\infty(0,T_1;L_m(\mathbb{R}^3))} + \|a\|_{L_{\frac{5m}{3}}(\mathbb{R}^3\times(0,T_1))} &\leq C(\kappa, m, M). \end{aligned} \tag{5.39}$$

Now, we set $v := u - a$ and we observe that

$$\partial_t v - \Delta v - \kappa\nabla \operatorname{div} v + v \cdot \nabla v + \frac{v}{2} \operatorname{div} v + \frac{a}{2} \cdot \nabla v + \operatorname{div}(v \otimes \frac{a}{2}) + \operatorname{div}(a \otimes v) - \frac{a}{2} \operatorname{div} v = 0,$$

in the sense of distributions in $\mathbb{R}^3\times]0, T_1[$. Moreover, because u and a satisfy a local energy inequality, we see that

$$\partial_t \frac{|v|^2}{2} - \Delta\frac{|v|^2}{2} - \kappa \operatorname{div}(v \operatorname{div} v) + |\nabla v|^2 + \kappa(\operatorname{div} v)^2 + \operatorname{div}\left((v + a)\frac{|v|^2}{2}\right) \\ + v \cdot \operatorname{div}(a \otimes v) - \frac{1}{2} v \cdot a \operatorname{div} v \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3\times]0, T_1[). \tag{5.40}$$

In other words, v is a suitable weak solution to the generalised counterpart of $\mathcal{L}_\kappa = 0$ (see (4.1)) with $b = a$. Note also that $\lim_{t\to 0^+} \|v(\cdot, t) - u_0^2\|_{L_2(B(x_0,1))} \to 0$ for all $x_0 \in \mathbb{R}^3$, $u_0^2|_{B(4/3)=0}$ (thus we have as a byproduct $\lim_{t\to 0^+} \|v(\cdot, t)\|_{L_2(B(4/3))} = 0$) and from Lemma 3.2.1, there exists $0 < T_2 = T_2(\alpha, \kappa, m, M) < T_1$ such that

$$\operatorname*{ess\,sup}_{0<t<T_2} \frac{1}{2} \int_{B(2)} |v(x, t)|^2 dx + \int_0^{T_2} \int_{B(2)} |\nabla v(x, t)|^2 dxdt \leq C(\alpha, m, M). \tag{5.41}$$

From the local energy (5.40) for v and the fact that $\lim_{t\to 0^+}\|v(\cdot,t)\|_{L_2(B(4/3))}=0$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{B(4/3)}|v(x,t)|^2\phi(x)dx+\int_0^t\int_{B(4/3)}|\nabla v|^2\phi dz+\int_0^t\int_{B(4/3)}\kappa(\operatorname{div}v)^2\phi dz\\ \leq\int_0^t\int_{B(4/3)}\frac{|v|^2}{2}\Delta\phi dz+\int_0^t\int_{B(4/3)}\frac{|v|^2}{2}(v+a)\cdot\nabla\phi dz\\ +\int_0^t\int_{B(4/3)}a\otimes v:\nabla v\phi dz+\int_0^t\int_{B(4/3)}a\otimes v:v\otimes\nabla\phi dz\\ -\kappa\int_0^t\int_{B(4/3)}v\cdot\nabla\phi\operatorname{div}vdz+\frac{1}{2}\int_0^t\int_{B(4/3)}\phi v\cdot a\operatorname{div}vdz,\quad(5.42)\end{aligned}$$

for a.e. $t\in]0,T_2[$ and $0\leq\phi\in C_0^\infty(B(4/3))$ such $\phi\equiv 1$ in B. The previous estimate and a repeated use of Hölder's inequality (with estimate (5.41) at hand) yield:

$$\begin{aligned}\int_B\frac{|v(x,t)|^2}{2}dx+\int_0^t\int_B|\nabla v|^2dz+\int_0^t\int_B\kappa(\operatorname{div}v)^2dz\\ \leq C(\alpha,\kappa,m,M)(t+t^{\frac{1}{10}}+t^{\frac{2m-3}{5m}}+t^{\frac{m-3}{5m}}),\quad(5.43)\end{aligned}$$

for a.e. $t\in]0,T_2[$.
Now, fix $t_0\in]0,T_2[$ to be specified later. Then extend v to $B\times]-1+t_0,t_0[$ by setting $v\equiv 0$ in $B\times]-1+t_0,0[$. Extend also a to $B\times]-1+t_0,t_0[$ by setting $a\equiv 0$ for $t<0$. Clearly the extended function v is still a suitable weak solution to (4.1), with the extended a, in $B\times]-1+t_0,t_0[$. Indeed, the fact that $\lim_{t\to 0^+}\|v(\cdot,t)\|_{L_2(B)}=0$ ensures that $\partial_t v$ and $\partial_t\frac{|v|^2}{2}$ will not cause any problem across $\{t=0\}$. Finally, because of (5.43), if we choose $t_0=t_0(\alpha,\kappa,m,M)<T_2$ sufficiently small, we can apply Theorem 4.2.5 and conclude that v is Hölder continuous in $B(1/2)\times[0,t_0]$, for some $\gamma=\gamma(m)\in]0,1[$. This concludes the proof of the theorem. □

Theorem 5.3.1 allows us to prove the following result.

Theorem 5.3.2 (Local Hölder regularity of local-energy solutions)**.** *Let $u_0\in L_{2,loc}(\mathbb{R}^3)$ such that $\alpha:=\sup_{x_0\in\mathbb{R}^3}\int_B|u_0(x)|^2dx<\infty$. Suppose in addition that we have $M:=\|u_0\|_{C^{\gamma,\frac{\gamma}{2}}(B(2))}<\infty$. Then, there exists a $T=T(\alpha,\gamma,\kappa,M)>0$ such that any local-energy weak solution u to $\mathcal{L}_\kappa=0$ satisfies:*

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{B(1/4)}\times[0,T])}\leq C(\alpha,\gamma,\kappa,M).$$

Sketch of proof. With the same notation as in Theorem 5.3.1, we have that the support $\operatorname{supp}u_0^1\subset\subset B(2)$ and $\|u_0^1\|_{C^\gamma(\mathbb{R}^3)}\leq CM$. Consequently, $u-a$ is Hölder continuous with some exponent $\beta \in]0, \gamma[$ in $\overline{B(1/2)} \times [0, T_1]$ where $T_1 = T_1(\alpha, \kappa, M) > 0$. Since the initial data u_0^1 for a is in $C^\gamma(\mathbb{R}^3)$, it is not too difficult to show that $a \in C^{\gamma, \frac{\gamma}{2}}(\mathbb{R}^3 \times [0, T_1])$. Therefore, u is Hölder continuous with exponent β in $\overline{B(1/2)} \times [0, T_1]$. From this point, a bootstrap argument with repeated use of Lemma 2.3.1, Lemma 2.3.1 and Lemma 2.3.3 yields the required Hölder continuity of u; and a careful tracking of the constants gives us the estimate in the theorem. This concludes the proof. □

Remark 27. The above results also hold true for the model $\mathcal{DL}_\kappa = 0$, with some simplification introduced by the divergence structure of nonlinearity.

We can now establish the a priori estimates we need for our proof of existence.

5.3.2 A priori estimates for forward self-similar solutions to our models

We recall that we choose our initial data $u_0 = (u_0^1, u_0^2, u_0^3)$ which is a (-1)-homogeneous vector field so that $u_0|_{\partial B_1} \in C^\infty(\partial B_1)$. In this case, one can show that

$$|\partial^\alpha u_0(x)| \leq \frac{C(\alpha, u_0)}{|x|^{1+|\alpha|}}, \quad \forall \alpha \in \mathbb{N}^3.$$

Our first result is the following decay estimate.

Theorem 5.3.3 (A priori estimate for forward self-similar solutions). *Let u_0 be as above and let u be a scale invariant global weak $L^{3,\infty}$–solution to the system $\mathcal{L}_\kappa = 0$ or $\mathcal{DL}_\kappa = 0$. Then, the solution profile $U(\cdot)(:= u(\cdot, 1))$ belongs to $C^\infty(\mathbb{R}^3)$ and*

$$|\partial^\alpha (U - S_\kappa(1)u_0)(x)| \leq \frac{C(\alpha, \kappa, u_0)}{(1 + |x|)^{3+|\alpha|}},$$

for all $\alpha \in \mathbb{N}^3$ (with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$).

Proof. Let us start by pointing out that our proof of this theorem follows the same ideas as the proof of a similar result obtained in [31] for the incompressible Navier-Stokes system (the difference here being that the leading term in our system is the Lamé operator).

We know that $u(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$, $t > 0$, where $U(\cdot) = S_\kappa(1)u_0 + W(\cdot, 1)$, and the estimates for W are given by Theorem 5.1.1. Thus, we have that

$$\sqrt{t^*} \int_{B(1/\sqrt{t^*})} |U(y)|^2 dy + \sqrt{t^*} \int_{B(1/\sqrt{t^*})} |\nabla U(y)|^2 dy \leq C(\kappa, \|u_0\|_{C(\partial B)})(1 + \sqrt{t^*}), \quad (5.44)$$

for all $t^* > 0$.

On the other hand, for all $|x_0| = 8$, we have $u_0 \in C^\infty(B(x_0, 4))$. Therefore, by Theorem

5.3.2 and by some simple bootstrapping arguments (with estimate (5.16) at hand for the case of the system $\mathcal{L}_\kappa = 0$), we have that there exists a $T_2 = T_2(\kappa, u_0) > 0$ such that

$$\|\partial_t \partial^\alpha u\|_{L_\infty(\overline{B(x_0,1/8)}\times[0,T_2])} \leq C(\alpha, \kappa, u_0), \tag{5.45}$$

for all global weak $L^{3,\infty}-$solutions u to the system $\mathcal{L}_\kappa = 0$ or the system $\mathcal{DL}_\kappa = 0$ with initial data u_0. Since, for all $\lambda > 0$, the scaled function $u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is also a global weak $L^{3,\infty}-$solution to $\mathcal{L}_\kappa = 0$ or $\mathcal{DL}_\kappa = 0$ with initial data u_0, then (5.45) holds also for u^λ and we find that

$$|\lambda^{1+|\alpha|}\partial^\alpha u(\lambda x_0, \lambda^2 t) - \partial^\alpha u_0(x_0)| \leq C(\alpha, \kappa, u_0)t.$$

Setting $y = x_0/\sqrt{t}$, and by using the homogeneity of $\partial^\alpha u_0$, we have:

$$|\partial^\alpha(U - u_0)(y)| \leq \frac{C(\alpha, \kappa, u_0)}{|y|^{3+|\alpha|}}, \quad \forall |y| > \frac{8}{\sqrt{T_2}}. \tag{5.46}$$

Now, we choose $t^* = t^*(\kappa, u_0)$ in (5.44) sufficiently small so that

$$\int_{B(\frac{16}{\sqrt{T_2}})} \big(|U(y)|^2 + |\nabla U(y)|^2 dy\big) \leq C(\kappa, u_0). \tag{5.47}$$

Since the profile U satisfies either (5.3) or (5.4), elliptic theory estimates give us:

$$\|U\|_{C^k(\overline{B(9/\sqrt{T_2})})} \leq C(\kappa, k, u_0) \quad (k = 0, 1, 2 \ldots) \tag{5.48}$$

Going back to the definition of the semigroup $S_\kappa(t)$ in Proposition 5.1.1, we obtain that

$$\|\partial^\alpha S_\kappa(1)u_0\|_{L_\infty(\mathbb{R}^3)} \leq C(\alpha, \kappa, u_0)$$

and

$$|\partial^\alpha(S_\kappa(1)u_0 - u_0)(x)| \leq |\partial^\alpha(S(1)u_0^{(0)} - u_0^{(0)})(x)| \\ + |\partial^\alpha(S(1+\kappa)u_0^{(1)} - u_0^{(1)})(x)| \leq \frac{C(\alpha, \kappa, u_0)}{|x|^{3+|\alpha|}}$$

by the known properties of the heat equation. This concludes the proof of the theorem. □

Another important step in the proof of the main result of this section is as follows.

Proposition 5.3.1 (Decay for the linearly singularly forced Lamé system)**.** *Let $f \in C(\mathbb{R}^3)$ and suppose that $w \in L_\infty(0,T; L_\gamma(\mathbb{R}^3))$ for any $T > 0$ and for some $\gamma \geq 1$ and moreover*

$$\begin{cases} \partial_t w - \Delta w - \kappa \nabla \operatorname{div} w = t^{-\frac{3}{2}} f(\frac{x}{\sqrt{t}}) \quad in\ Q_+ \\ \lim\limits_{t \to 0^+} \|w(\cdot,t)\|_{L_\gamma(\mathbb{R}^3)} = 0. \end{cases} \tag{5.49}$$

Then:
(i) If $\tilde{w}$ satisfies also the above conditions, then $\tilde{w} = w$. Consequently $w(\lambda x, \lambda^2 t) = w(x,t)$ for all $\lambda > 0$.
(ii) If $M := \sup_{x\in\mathbb{R}^3}(1+|x|)^3|f(x)| < \infty$, then by setting $W(x) = w(x,1)$ we get: $\|W\|_{C^{1+\alpha}(B(R))} \le c(\alpha,\kappa,R)M$ for $\alpha \in (0,1)$ and

$$\sup_{x\in\mathbb{R}^3}\left[(1+|x|)^2|W(x)| + (1+|x|)^3|\nabla W(x)|\right] \le C(\kappa)M.$$

(iii) Similarly, if $M := \sup_{x\in\mathbb{R}^3}(1+|x|)^4|f(x)| < \infty$, then

$$\sup_{x\in\mathbb{R}^3}\left[(1+|x|)^3|W(x)| + (1+|x|)^4|\nabla W(x)|\right] \le C(\kappa)M.$$

Proof. (i) First, we observe that if a function $w \in L_\infty(0,T;L_{\gamma_1}(\mathbb{R}^3)+L_{\gamma_2}(\mathbb{R}^3))$ (for all $T>0$) is such that $\lim_{t\to 0^+}\|w(\cdot,t)\|_{L_{\gamma_1}(\mathbb{R}^3)+L_{\gamma_2}(\mathbb{R}^3)} = 0$ and

$$\partial_t w - \Delta w - \kappa\nabla \mathrm{div}\, w = 0 \quad \text{in } Q_+,$$

then $w \equiv 0$.
(ii) Second, we start with establishing a decay estimate for $\nabla\mathrm{div}\, w$. For this, notice that

$$\mathrm{div}\, w(x,t) = \int_0^t\int_{\mathbb{R}^3}\nabla\Gamma_\kappa(x-y,t-s)\cdot f(\frac{y}{\sqrt{s}})s^{-\frac{3}{2}}dyds, \tag{5.50}$$

with

$$\Gamma_\kappa(x,t) = \frac{1}{[4\pi(1+\kappa)t]^{\frac{3}{2}}}\exp\left(-\frac{|x|^2}{4(1+\kappa)t}\right).$$

A simple computation gives us

$$|\nabla\mathrm{div}\, W(x)| \le c(\kappa)M\int_0^1\int_{\mathbb{R}^3}\frac{1}{\left(|x-y|+\sqrt{1-s}\right)^4}\frac{1}{\left(|y|+\sqrt{s}\right)^3}dyds \le C(\kappa)M|x|^{-3}$$

for $|x|>8$. Since W satisfies the following system

$$-\Delta W - \kappa\nabla\mathrm{div}\, W - \frac{x}{2}\cdot\nabla W - \frac{W}{2} = f \quad \text{in } \mathbb{R}^3, \tag{5.51}$$

and (see for instance (5.50) combined with known estimates for the volume heat potential)

$$\|\mathrm{div}\, W\|_{L_2(\mathbb{R}^3)} + \|\nabla\mathrm{div}\, W\|_{L_{\frac{3}{2}}(\mathbb{R}^3)} \le C(\kappa)M,$$

therefore, elliptic estimates for the equation

$$-(1+\kappa)\Delta\mathrm{div}\, W - x\cdot\nabla\mathrm{div}\, W - \frac{3}{2}\mathrm{div}\, W = \mathrm{div}\, f$$

guarantee the estimate

$$\|\nabla \mathrm{div}\, W\|_{L_\infty(B(12))} \leq C(\kappa)M.$$

Consequently, we have

$$\sup_{x\in\mathbb{R}^3} \left[(1+|x|)^3|\nabla \mathrm{div}\, W(x)|\right] \leq C(\kappa)M.$$

Now, if we set $g(x) := f(x) + \nabla \mathrm{div}\, W(x)$, then

$$\partial_t w - \Delta w = t^{-\frac{3}{2}} g(\frac{x}{\sqrt{t}}) \quad \text{in } Q_+$$

and thus

$$w(\cdot,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t) g(\frac{y}{\sqrt{s}}) s^{-\frac{3}{2}} ds.$$

As previously, we can show that

$$\begin{aligned} |W(x)| &\leq c(\kappa)M \int_0^1 \int_{\mathbb{R}^3} \frac{1}{\left(|x-y|+\sqrt{1-s}\right)^3} \frac{1}{(|y|+\sqrt{s})^3} dyds \\ &\leq C(\kappa)M|x|^{-3}\log|x|, \end{aligned}$$

and

$$|\nabla W(x)| \leq c(\kappa)M \int_0^1 \int_{\mathbb{R}^3} \frac{1}{\left(|x-y|+\sqrt{1-s}\right)^4} \frac{1}{(|y|+\sqrt{s})^3} dyds \leq C(\kappa)M|x|^{-3},$$

for $|x| > 8$. The continuity estimates in $B(12)$ follow from standard elliptic estimates for (5.51).

(iii) This assertion is proved by using the same ideas as in part (ii); the difference here being that the source term has a faster decay at infinity (which make things easier in this case). □

Now, we are able to give a proof of the existence of a scale-invariant global weak $L^{3,\infty}$−weak solution to our models $\mathcal{L}_\kappa = 0$ and $\mathcal{DL}_\kappa = 0$. The proof is based on Schauder's fixed point theorem applied in a suitable function space in order to establish the existence of a solution to systems (5.3) or (5.4).

Proof of Theorem 5.1.2. We introduce the following function space

$$X = \left\{ V \in C^1(\mathbb{R}^3) : \sup_{x\in\mathbb{R}^3} \left[(1+|x|)^2|V(x)| + (1+|x|)^3|\nabla V(x)|\right] < \infty \right\} \tag{5.52}$$

endowed with the natural norm

$$\|V\|_X = \sup_{x\in\mathbb{R}^3} \left[(1+|x|)^2|V(x)| + (1+|x|)^3|\nabla V(x)|\right]; \tag{5.53}$$

the choice of this function space is motivated by Theorem 5.3.3 and a need for compactness as we shall see below.

We are going to use the same notation as in the proof of Proposition 5.1.1. Because of the scaling symmetry of u_0, we get that $u_0^{(0)}$ and $u_0^{(1)}$ are also $(-1)-$homogeneous. Moreover, elliptic estimates guarantee that $u_0^{(1)}, u_0^{(0)} \in C^\infty(\partial B)$ and we have

$$|\partial^\alpha u_0^{(1)}(x)| + |\partial^\alpha u_0^{(0)}(x)| \leq \frac{C(\alpha, u_0)}{|x|^{1+|\alpha|}}.$$

Consequently,

$$|\partial^\alpha S_\kappa(1)u_0(x)| \leq |\partial^\alpha v^0(x,1)| + |\partial^\alpha v^1(x,1)| \leq \frac{C(\alpha,\kappa,u_0)}{(1+|x|)^{1+|\alpha|}}$$

by the properties of the heat equation.

Next, introduce a parameter $\mu \in [0,1]$. Let us consider the following problem: find U such that

$$-\Delta U - \kappa\nabla \mathrm{div}\, U + U\cdot\nabla U + \frac{U}{2}\mathrm{div}\, U - \frac{x}{2}\cdot\nabla U - \frac{U}{2} = 0 \quad \text{in } \mathbb{R}^3, \tag{5.54}$$

and $|U(x) - V_\mu| = o(|x|^{-1})$ as $|x| \to \infty$, where $V_\mu(x) = S_\kappa(1)(\mu u_0)(x)$. We will seek U in the form

$$U = V_\mu + W, \quad \text{where } W \in X. \tag{5.55}$$

It is clear that $u(x,t) = \frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}})$ is a global weak $L^{3,\infty}-$solution to $\mathcal{L}_\kappa = 0$ with initial data μu_0 if and only if $U(x)$ satisfies the elliptic system (5.54) and $U(x) = V_\mu + W$ for some $W \in X$ (by Theorem 5.3.3). Thus, we have reduced the problem to finding $W \in X$ such that

$$\begin{aligned}-\Delta W - \kappa\nabla \mathrm{div}\, W - \frac{x}{2}\cdot\nabla W - \frac{W}{2} &= -W\cdot\nabla W - V_\mu\cdot\nabla W - W\cdot\nabla V_\mu - V_\mu\cdot\nabla V_\mu \\ &\quad - \frac{W}{2}\mathrm{div}\, W - \frac{V_\mu}{2}\mathrm{div}\, W - \frac{W}{2}\mathrm{div}\, V_\mu - \frac{V_\mu}{2}\mathrm{div}\, V_\mu \quad \text{in } \mathbb{R}^3.\end{aligned} \tag{5.56}$$

Notice that if we set $w(x,t) := \frac{1}{\sqrt{t}}W(\frac{x}{\sqrt{t}})$, we have that

$$\begin{cases}\partial_t w - \Delta w - \kappa\nabla \mathrm{div}\, w = t^{-\frac{3}{2}}F(\frac{x}{\sqrt{t}}) & \text{in } Q_+ \\ w|_{t=0} = 0 \quad \text{in } \mathbb{R}^3,\end{cases} \tag{5.57}$$

where

$$\begin{aligned}F = &-W\cdot\nabla W - V_\mu\cdot\nabla W - W\cdot\nabla V_\mu - V_\mu\cdot\nabla V_\mu \\ &- \frac{W}{2}\mathrm{div}\, W - \frac{V_\mu}{2}\mathrm{div}\, W - \frac{W}{2}\mathrm{div}\, V_\mu - \frac{V_\mu}{2}\mathrm{div}\, V_\mu\end{aligned}$$

has the decay properties in Proposition 5.3.1 provided $W \in X$. Conversely, for a function F with the decay estimates as in Proposition 5.3.1, system (5.57) is uniquely solvable and we denote the solution profile at time $t = 1$ by $\mathcal{G}(F) \in X$, i.e., $\mathcal{G}(F)(x) := w(x, 1)$. Obviously, $\mathcal{G}$ is a linear operator. The latter allows us to reformulate (5.56) as follows:

$$\begin{aligned}\text{Find } W \in X \text{ such that } W = \mathcal{G}(&W \cdot \nabla W - V_\mu \cdot \nabla W - W \cdot \nabla V_\mu - V_\mu \cdot \nabla V_\mu \\ &- \frac{W}{2}\operatorname{div} W - \frac{V_\mu}{2}\operatorname{div} W - \frac{W}{2}\operatorname{div} V_\mu - \frac{V_\mu}{2}\operatorname{div} V_\mu). \end{aligned} \tag{5.58}$$

Now, let us define an operator $K : X \times [0, 1] \to X$ as follows: $\forall V \in X$, $\mu \in [0, 1]$,

$$\begin{aligned} K(V, \mu) = \mathcal{G}(V_\mu \cdot \nabla V_\mu + \frac{V_\mu}{2}\operatorname{div} V_\mu) + \mathcal{G}(&V \cdot \nabla V + V_\mu \cdot \nabla V + V \cdot \nabla V_\mu \\ &+ \frac{V}{2}\operatorname{div} V + \frac{V_\mu}{2}\operatorname{div} V + \frac{V}{2}\operatorname{div} V_\mu). \end{aligned} \tag{5.59}$$

Notice that $\mathcal{G}(V_\mu \cdot \nabla V_\mu + \frac{V_\mu}{2}\operatorname{div} V_\mu) = \mu^2 \mathcal{G}(V \cdot \nabla V + \frac{V}{2}\operatorname{div} V)$ has a one-dimensional range (thus it is compact). In order to see that the second term is compact, let us consider a bounded sequence $V^{(j)}$ in X together with $\mu_j \in [0, 1]$ and set

$$\begin{aligned} G^{(j)} = \mathcal{G}(V^{(j)} \cdot \nabla V^{(j)} + V_{\mu_j} \cdot \nabla V^{(j)} + V^{(j)} \cdot \nabla V_{\mu_j} + \frac{V^{(j)}}{2}\operatorname{div} V^{(j)} + \frac{V_{\mu_j}}{2}\operatorname{div} V^{(j)} \\ + \frac{V^{(j)}}{2}\operatorname{div} V_{\mu_j}). \end{aligned}$$

The arguments of the operator $\mathcal{G}$ in the above formula have a decay $(1 + |x|)^{-4}$ or better, uniformly in j. From Proposition 5.3.1, we find that

$$\begin{aligned} \sup_j \|G^{(j)}\|_{C^{1+\alpha}(B(R))} < \infty, \quad \forall R > 0 \\ \sup_j \sup_{x \in \mathbb{R}^3} \left[(1 + |x|)^3 |G^{(j)}(x)| + (1 + |x|)^4 |\nabla G^{(j)}(x)|\right] < \infty, \end{aligned}$$

which implies the desired compactness in X by known arguments. Continuity follows the exact same arguments as for the compactness. Consequently, to solve the problem at hand, i.e.

$$\text{Find } W \in X \text{ such that } W + K(W, \mu) = 0, \qquad \mu \in [0, 1], \tag{5.60}$$

we can apply Leray-Schauder theory (see e.g. [50] or Theorem 11.6 in [20]). All the required a priori estimates are given by Theorem 5.3.3 and Proposition 5.3.1, thus the only thing to be verified is the solvability of the problem for small enough μ. But this can be easily done by an application of the implicit function theorem to our functional. The same reasoning applies to the model $\mathcal{DL}_\kappa = 0$. This concludes the proof. □

5.4 Proof of Theorem 5.1.3

Since estimate (5.17) follows from estimate (5.16), our main goal now is to prove the following bound

$$\int_0^T \int_{\mathbb{R}^3} |\kappa \operatorname{div} w^\kappa|^2 dxds \leq cT^{\frac{1}{2}} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^2 \tag{5.61}$$

for all $T > 0$ and for an absolute constant $c > 0$; see the notation in Section 1. Its proof is divided into three parts.

Part I [A priori estimates]. We focus here, only, on the system $\mathcal{L}_\kappa = 0$ since the reasoning is the same for the system $\mathcal{DL}_\kappa = 0$ with some simplifications due to the divergence structure of the non-linearity. We recall that $u^\kappa = w^\kappa + v$ and

$$U^\kappa(x) := u^\kappa(x,1) \quad \text{and} \quad u^\kappa(x,t) = \frac{1}{\sqrt{t}} U^\kappa\left(\frac{x}{\sqrt{t}}\right)$$
$$v(x,t) = S(t)u_0(x) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad W^\kappa := U^\kappa - V$$

(notice that $\operatorname{div} v = 0$ in $Q_+ := \mathbb{R}^3 \times]0,\infty[$ since $\operatorname{div} u_0 = 0$) and

$$\partial_t w^\kappa - \Delta w^\kappa - \kappa \nabla \operatorname{div} w^\kappa = -(w^\kappa \cdot \nabla w^\kappa + \frac{w^\kappa}{2} \operatorname{div} w^\kappa) - (v \cdot \nabla w^\kappa + \frac{v}{2} \operatorname{div} w^\kappa + w^\kappa \cdot \nabla v + v \cdot \nabla v) \tag{5.62}$$

in Q_+.

Now, we introduce the functions $w^{\kappa,1}$, $\hat{w}^{\kappa,i}$ and p_i^κ ($i = 1,2,3$) as solutions to the following Cauchy problems:

$$\begin{cases} \partial_t w^{\kappa,1} - \Delta w^{\kappa,1} - \kappa \nabla \operatorname{div} w^{\kappa,1} = -(w^\kappa \cdot \nabla w^\kappa + \frac{w^\kappa}{2} \operatorname{div} w^\kappa) & \text{in } Q_+ \\ w^{\kappa,1}|_{t=0} = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.63}$$

$$\begin{cases} \partial_t \hat{w}^{\kappa,1} - \Delta \hat{w}^{\kappa,1} + \nabla p_1^\kappa = -(v \cdot \nabla w^\kappa + \frac{1}{2} \operatorname{div}(v \otimes w^\kappa)) & \text{in } Q_+ \\ \operatorname{div} \hat{w}^{\kappa,1} = 0 & \text{in } Q_+ \\ \hat{w}^{\kappa,1}|_{t=0} = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.64}$$

$$\begin{cases} \partial_t \hat{w}^{\kappa,2} - \Delta \hat{w}^{\kappa,2} + \nabla p_2^\kappa = -\frac{1}{2} w^\kappa \cdot \nabla v & \text{in } Q_+ \\ \operatorname{div} \hat{w}^{\kappa,2} = 0 & \text{in } Q_+ \\ \hat{w}^{\kappa,2}|_{t=0} = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.65}$$

and

$$\begin{cases} \partial_t \hat{w}^{\kappa,3} - \Delta \hat{w}^{\kappa,3} + \nabla p_3^\kappa = -v\cdot\nabla v \quad \text{in } Q_+ \\ \operatorname{div} \hat{w}^{\kappa,3} = 0 \quad \text{in } Q_+ \\ \hat{w}^{\kappa,3}|_{t=0} = 0 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{5.66}$$

The proof of the unique solvability of the above Cauchy problems (5.63)-(5.66) in the energy class

$$\sup_{0<t<T} \Big(\|w^{\kappa,1}(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + \|\hat{w}^{\kappa,i}(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} \Big) + \int_0^T \int_{\mathbb{R}^3} \big(|\nabla w^{\kappa,1}(x,t)|^2 + |\nabla \hat{w}^{\kappa,i}(x,t)|^2 \big)\, dxdt < +\infty, \ \forall T>0 \tag{5.67}$$

and

$$\int_0^T \int_{\mathbb{R}^3} |p_i^\kappa(x,t)|^2 dxdt < \infty, \ \forall T>0 \ (i=1,2,3) \tag{5.68}$$

is more or less standard; see the estimates below and in Part III.
We now derive uniform (in κ) L_2-estimates for the pressure functions p_i^κ in $Q_T := \mathbb{R}^3\times]0,T[$. From (5.64), we have

$$-\Delta p_1^\kappa = \operatorname{div}\operatorname{div}(w^\kappa \otimes v + \frac{1}{2} v\otimes w^\kappa),$$

thus

$$\begin{aligned} \|p_1^\kappa(\cdot,t)\|_{L_2(\mathbb{R}^3)} &\le c\| |w^\kappa(\cdot,t)|\cdot|v(\cdot,t)| \|_{L_2(\mathbb{R}^3)} \\ &\le c\|w^\kappa(\cdot,t)\|_{L^{6,2}(\mathbb{R}^3)} \|v(\cdot,t)\|_{L^{3,\infty}(\mathbb{R}^3)} \\ &\le c\|u_0\|_{L^{3,\infty}} \|\nabla w^\kappa(\cdot,t)\|_{L_2(\mathbb{R}^3)}, \end{aligned}$$

for a.e. $t\in]0,T[$. Consequently, taking into account Theorem 5.1.1, we find

$$\int_0^T \int_{\mathbb{R}^3} |p_1^\kappa(x,t)|^2 dxdt \le cT^{\frac{1}{2}} \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \Big(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \Big), \tag{5.69}$$

for all $T>0$.
Next, we have

$$-\Delta p_2^\kappa = \operatorname{div}(\frac{1}{2} w^\kappa \cdot \nabla v),$$

and thus

$$\begin{aligned} \|\nabla p_2^\kappa(\cdot,t)\|_{L^{\frac{6}{5},2}(\mathbb{R}^3)} &\le c\|w^\kappa(\cdot,t)\|_{L_2(\mathbb{R}^3)} \|\nabla v(\cdot,t)\|_{L^{3,\infty}(\mathbb{R}^3)} \\ &\le \frac{c}{t^{\frac{1}{2}}} \|w^\kappa(\cdot,t)\|_{L_2(\mathbb{R}^3)} \|u_0\|_{L^{3,\infty}(\mathbb{R}^3)} \quad \text{for a.e. } t\in]0,T[, \end{aligned}$$

where the last inequality is a consequence of Young's inequality for weak type spaces (see [22], Theorem 1.2.13) and an $L_1(\mathbb{R}^3)$-estimate for the gradient of the heat kernel. Consequently, by applying Sobolev's embedding theorem and Theorem 5.1.1, we find

$$\int_0^T \int_{\mathbb{R}^3} |p_2^\kappa(x,t)|^2 dxdt \leq cT^{\frac{1}{2}} \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right), \tag{5.70}$$

for all $T > 0$. Finally, we have

$$-\Delta p_3^\kappa = \operatorname{div}\operatorname{div}(v \otimes v),$$

and thus

$$\begin{aligned} \|p_3^\kappa(\cdot,t)\|_{L_2(\mathbb{R}^3)} &\leq c\|v(\cdot,t)\|^2_{L_4(\mathbb{R}^3)} \\ &\leq \frac{c}{t^{\frac{1}{4}}} \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \quad \text{for a.e. } t \in]0,T[, \end{aligned}$$

where again the last estimate follows from the convolution structure of the heat potential and the corresponding inequalities. Consequently,

$$\int_0^T \int_{\mathbb{R}^3} |p_3^\kappa(x,t)|^2 dxdt \leq cT^{\frac{1}{2}} \|u_0\|^4_{L^{3,\infty}}, \tag{5.71}$$

for all $T > 0$. In conclusion, we have obtained that

$$\sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3} |p_i^\kappa(x,t)|^2 dxdt \leq cT^{\frac{1}{2}} \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right), \tag{5.72}$$

for all $T > 0$.

Going back once more to Theorem 5.1.1, we see that

$$\|W^\kappa\|^2_{L_2(\mathbb{R}^3)} + \|\nabla W^\kappa\|^2_{L_2(\mathbb{R}^3)} \leq c \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right);$$

therefore, by Sobolev embedding, we have

$$\|W^\kappa\|_{L_3(\mathbb{R}^3)} \leq c \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^{\frac{1}{2}},$$

where $c > 0$ is an absolute constant independent of κ. Consequently, we have

$$\sup_{0<t<\infty} \|w^\kappa(\cdot,t)\|_{L_3(\mathbb{R}^3)} \leq c \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^{\frac{1}{2}}. \tag{5.73}$$

From the latter estimate, we have (thanks again to Theorem 5.1.1) that

$$\|w^\kappa \cdot \nabla w^\kappa + \frac{w^\kappa}{2} \operatorname{div} w^\kappa\|^2_{L_{\frac{6}{5},2}(\mathbb{R}^3 \times]0,T[)} \leq cT^{\frac{1}{2}} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^2. \tag{5.74}$$

Now, let us introduce the functions (the justification of whose existence is similar to the one of $\hat{w}^{\kappa,2}$ and p_2^κ)

$$z^\kappa \in L_{2,\infty}(Q_T) \cap W_2^{1,0}(Q_T) \quad \forall T > 0,$$

and

$$q^\kappa \in L_2(Q_T) \quad \forall T > 0,$$

such that

$$\begin{cases} \partial_t z^\kappa - \Delta z^\kappa + \nabla q^\kappa = -(w^\kappa \cdot \nabla w^\kappa + \frac{w^\kappa}{2} \operatorname{div} w^\kappa) \quad \text{in } Q_+ \\ \operatorname{div} z^\kappa = 0 \quad \text{in } Q_+ \\ z^\kappa|_{t=0} = 0 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{5.75}$$

From the equation

$$-\Delta q^\kappa = \operatorname{div}(w^\kappa \cdot \nabla w^\kappa + \frac{w^\kappa}{2} \operatorname{div} w^\kappa),$$

estimate (5.74) and thanks to Sobolev embedding, we see that the pressure q^κ can be chosen so that, for all $T > 0$, the following inequality is valid:

$$\int_0^T \int_{\mathbb{R}^3} |q^\kappa(x,t)|^2 dxdt \leq cT^{\frac{1}{2}} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^2. \tag{5.76}$$

Now, we set $\bar{w}^{\kappa,1} := w^{\kappa,1} - z^\kappa$ and we see that

$$\begin{cases} \partial_t \bar{w}^{\kappa,1} - \Delta \bar{w}^{\kappa,1} - \kappa \nabla \operatorname{div} \bar{w}^{\kappa,1} = \nabla q^\kappa \quad \text{in } Q_+ \\ \bar{w}^{\kappa,1}|_{t=0} = 0 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{5.77}$$

Similarly to what has been done in the proof of Theorem 5.1.1, we get for all $t > 0$ the energy estimate:

$$\frac{1}{2} \int_{\mathbb{R}^3} |\bar{w}^{\kappa,1}(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla \bar{w}^{\kappa,1}(x,s)|^2 dxds$$
$$+ \kappa \int_0^t \int_{\mathbb{R}^3} |\operatorname{div} \bar{w}^{\kappa,1}(x,s)|^2 dxds \leq \left(\int_0^t \int_{\mathbb{R}^3} |q^\kappa|^2 dxds \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^3} |\operatorname{div} \bar{w}^{\kappa,1}|^2 dxds \right)^{\frac{1}{2}}.$$

We multiply both sides of the above inequality by κ and use Young's inequality to obtain

$$\kappa \int_{\mathbb{R}^3} |\bar{w}^{\kappa,1}(x,t)|^2 dx + \kappa \int_0^t \int_{\mathbb{R}^3} |\nabla \bar{w}^{\kappa,1}(x,s)|^2 dxds$$
$$+ \int_0^t \int_{\mathbb{R}^3} |\kappa \operatorname{div} \bar{w}^{\kappa,1}(x,s)|^2 dxds \leq \int_0^t \int_{\mathbb{R}^3} |q^\kappa|^2 dxds;$$

thus, thanks to (5.76), we find that

$$\kappa \int_{\mathbb{R}^3} |\bar{w}^{\kappa,1}(x,T)|^2 dx + \kappa \int_0^T \int_{\mathbb{R}^3} |\nabla \bar{w}^{\kappa,1}(x,s)|^2 dxds \\ + \int_0^T \int_{\mathbb{R}^3} |\kappa \mathrm{div}\, \bar{w}^{\kappa,1}(x,s)|^2 dxds \leq cT^{\frac{1}{2}} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^2,$$

for all $T > 0$. Since $\mathrm{div}\, z^\kappa = 0$, we finally obtain that

$$\int_0^T \int_{\mathbb{R}^3} |\kappa \mathrm{div}\, w^{\kappa,1}(x,s)|^2 dxds \leq cT^{\frac{1}{2}} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right)^2, \tag{5.78}$$

for all $T > 0$. Let us mention that the trick to get the space-time uniform L_2-estimate for the term $\kappa \mathrm{div}\, w^{\kappa,1}$ is inspired by a similar estimate established in [42] for the stationary Lamé system and will be used once more in the next step.

Part II. We introduce the following functions

$$\hat{w}^\kappa := \sum_{i=1}^{3} \hat{w}^{\kappa,i}, \quad w^{\kappa,2} := w^\kappa - w^{\kappa,1} - \hat{w}^\kappa \quad \text{and } p^\kappa := \sum_{i=1}^{3} p_i^\kappa;$$

we get that

$$\begin{cases} \partial_t w^{\kappa,2} - \Delta w^{\kappa,2} - \kappa \nabla \mathrm{div}\, w^{\kappa,2} = \nabla p^\kappa \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\ w^{\kappa,2}|_{t=0} = 0 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{5.79}$$

and again, by means reminiscent to what has been done in the proof of Theorem 5.1.1, we find for all $t > 0$ another energy estimate:

$$\frac{1}{2} \int_{\mathbb{R}^3} |w^{\kappa,2}(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla w^{\kappa,2}(x,s)|^2 dxds \\ + \kappa \int_0^t \int_{\mathbb{R}^3} |\mathrm{div}\, w^{\kappa,2}(x,s)|^2 dxds \leq \left(\int_0^t \int_{\mathbb{R}^3} |p^\kappa|^2 dxds \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^3} |\mathrm{div}\, w^{\kappa,2}|^2 dxds \right)^{\frac{1}{2}}.$$

Multiplying both sides of the previous inequality by κ and using Young's inequality, we get

$$\kappa \int_{\mathbb{R}^3} |w^{\kappa,2}(x,t)|^2 dx + \kappa \int_0^t \int_{\mathbb{R}^3} |\nabla w^{\kappa,2}(x,s)|^2 dxds \\ + \int_0^t \int_{\mathbb{R}^3} |\kappa \mathrm{div}\, w^{\kappa,2}(x,s)|^2 dxds \leq \int_0^t \int_{\mathbb{R}^3} |p^\kappa(x,s)|^2 dxds,$$

and thanks to (5.72), we finally find that

$$\kappa \int_{\mathbb{R}^3} |w^{\kappa,2}(x,T)|^2 dx + \kappa \int_0^T \int_{\mathbb{R}^3} |\nabla w^{\kappa,2}(x,t)|^2 dxdt \\ + \int_0^T \int_{\mathbb{R}^3} |\kappa \mathrm{div}\, w^{\kappa,2}|^2 dxds \leq cT^{\frac{1}{2}} \|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} \left(\|u_0\|^2_{L^{3,\infty}(\mathbb{R}^3)} + \|u_0\|^4_{L^{3,\infty}(\mathbb{R}^3)} \right), \tag{5.80}$$

for all $T > 0$. By noticing that $\operatorname{div} \hat{w}^\kappa = 0$ and taking into account (5.78), we arrive at estimate (5.61).

Part III. All that is left in order to finish the proof of (5.61) is to justify the construction of the functions $w^{\kappa,1}$, $\hat{w}^{\kappa,i}$ and p^κ_i $(i = 1, 2, 3)$ such that (5.67) and (5.68) hold. For the functions $\hat{w}^{\kappa,i}$ and p^κ_i, the existence comes from results for the inhomogeneous heat equation in the whole space together with estimates established earlier for the terms:

$$v \cdot \nabla w^\kappa + \frac{1}{2}\operatorname{div}(v \otimes w^\kappa), \quad \frac{1}{2} w^\kappa \cdot \nabla v \quad \text{and} \quad v \cdot \nabla v.$$

Therefore, we skip the details for the sake of brevity.
Now for the construction of $w^{\kappa,1}$, we go back to the function q^κ such that

$$-\Delta q^\kappa = \operatorname{div}(w^\kappa \cdot \nabla w^\kappa + \frac{w^\kappa}{2}\operatorname{div} w^\kappa),$$

and introduce the function $Q^\kappa := \nabla q^\kappa + w^\kappa \cdot \nabla w^\kappa + \frac{1}{2} w^\kappa \operatorname{div} w^\kappa$ such that $\operatorname{div} Q^\kappa = 0$ and the following estimates holds (see (5.74))

$$\|Q^\kappa\|_{L_{\frac{6}{5},2}(Q_T)} + \|\nabla q^\kappa\|_{L_{\frac{6}{5},2}(Q_T)} < \infty,$$

for all $T > 0$.
Now, let us look for the solution to the Cauchy problem (5.63) in the form $w^{\kappa,1} = w^{\kappa,11} + w^{\kappa,12}$, where

$$\sup_{0<t<T} \|w^{\kappa,1i}(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + \int_0^T \int_{\mathbb{R}^3} |w^{\kappa,1i}(x,t)|^2 dxdt < \infty, \quad (i = 1, 2)$$

for all $T > 0$ and

$$\begin{cases} \partial_t w^{\kappa,11} - \Delta w^{\kappa,11} = Q^\kappa \text{ in } Q_+ \\ w^{\kappa,11}|_{t=0} = 0 \quad \text{in } \mathbb{R}^3, \end{cases} \text{and} \begin{cases} \partial_t w^{\kappa,12} - (1+\kappa)\Delta w^{\kappa,12} = \nabla q^\kappa \text{ in } Q_+ \\ w^{\kappa,12}|_{t=0} = 0 \quad \text{in } \mathbb{R}^3. \end{cases}$$

Now, notice that $\operatorname{div} w^{\kappa,11} = 0$ and $\operatorname{curl} w^{\kappa,12} = 0$ in Q_+ and $\Delta w^{\kappa,12} = \nabla \operatorname{div} w^{\kappa,12}$ and we are done by setting $w^{\kappa,1} := w^{\kappa,11} + w^{\kappa,12}$. This concludes the proof of estimate (5.61).

Now, our goal is to establish a uniform H^2-estimate (5.19) (with respect to κ) for the forward self-similar profile U^κ. We recall once again that:

$$U^\kappa = V + W^\kappa, \ \operatorname{div} u_0 = 0,$$
$$|\partial^\alpha W^\kappa(x)| \leq \frac{c(\kappa, u_0)}{(1+|x|)^{3+|\alpha|}}, \quad |\partial^\alpha V(x)| \leq \frac{c(u_0)}{(1+|x|)^{1+|\alpha|}}.$$

The above decay estimates make legitimate the upcoming integration by parts. Let us also recall the following known formulae:

$$\begin{aligned} v\cdot\Delta v = -|\nabla v|^2 + \Delta\frac{|v|^2}{2}, \quad \Delta v = \nabla\mathrm{div}\, v - \mathrm{curl}\,(\mathrm{curl}\, v) \\ \mathrm{div}\,(x\cdot\nabla v) = x\cdot\nabla\mathrm{div}\, v + \mathrm{div}\, v, \quad \mathrm{curl}\,(x\cdot\nabla v) = x\cdot\nabla\mathrm{curl}\, v + \mathrm{curl}\, v. \end{aligned} \tag{5.81}$$

We find from (5.3) the following equation for W^κ:

$$\begin{aligned} -\Delta W^\kappa - \kappa\nabla\mathrm{div}\, W^\kappa - \frac{x}{2}\cdot\nabla W^\kappa - \frac{W^\kappa}{2} + U^\kappa\cdot\nabla W^\kappa + \frac{U^\kappa}{2}\mathrm{div}\, W^\kappa \\ + U^\kappa\cdot\nabla V = 0 \quad \text{in } \mathbb{R}^3. \end{aligned} \tag{5.82}$$

Now, let us multiply system (5.82) by ΔW^κ and integrate over the whole space to obtain (thanks also to the formulas (5.81)):

$$\begin{aligned} \int_{\mathbb{R}^3}|\Delta W^\kappa|^2dx + \kappa\int_{\mathbb{R}^3}|\nabla\mathrm{div}\, W^\kappa|^2dx \leq \kappa\int_{\mathbb{R}^3}\nabla\mathrm{div}\, W^\kappa\cdot\mathrm{curl}\,(\mathrm{curl}\, W^\kappa)dx \\ -\frac{1}{2}\int_{\mathbb{R}^3}(x\cdot\nabla W^\kappa)\cdot\nabla\mathrm{div}\, W^\kappa dx + \frac{1}{2}\int_{\mathbb{R}^3}(x\cdot\nabla W^\kappa)\cdot\mathrm{curl}\,(\mathrm{curl}\, W^\kappa)dx \\ +\frac{1}{2}\int_{\mathbb{R}^3}|\nabla W^\kappa|^2dx - \frac{1}{4}\int_{\mathbb{R}^3}\Delta|W^\kappa|^2dx + c\||U^\kappa||\nabla W^\kappa|\|_{L_2(\mathbb{R}^3)}\|\nabla^2W^\kappa\|_{L_2(\mathbb{R}^3)} \\ + \|U^\kappa\|_{L_4(\mathbb{R}^3)}\|\nabla V\|_{L_4(\mathbb{R}^3)}\|\nabla^2W^k\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{5.83}$$

Next, we have

$$\begin{aligned} \||U^\kappa||\nabla W^\kappa|\|_{L_2(\mathbb{R}^3)} \leq c\|U^\kappa\|_{L_6(\mathbb{R}^3)}\|\nabla W^\kappa\|_{L_3(\mathbb{R}^3)} \\ \leq c(\|V\|_{L_6(\mathbb{R}^3)} + \|\nabla W^k\|_{L^2(\mathbb{R}^3)})\|\nabla W^k\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}\|\nabla^2W^k\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \\ \leq c(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)})\|\nabla^2W^k\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}, \end{aligned}$$

$$\|\nabla V\|_{L_4(\mathbb{R}^3)} \leq c\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)},$$

and

$$\|\nabla^2W^\kappa\|_{L_2(\mathbb{R}^3)} \leq c\|\Delta W^\kappa\|_{L_2(\mathbb{R}^3)}.$$

Using inequality (5.83), successive integration by parts, the above estimates and the energy estimates for W^κ, we find

$$\int_{\mathbb{R}^3}|\nabla^2W^\kappa|^2dx + \kappa\int_{\mathbb{R}^3}|\nabla\mathrm{div}\, W^\kappa|^2dx \leq c(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}). \tag{5.84}$$

Finally, let us establish the uniform L_2-estimate for $\kappa\nabla\mathrm{div}\, W^\kappa$. To achieve this, we set

$$F^\kappa := -\Delta W^\kappa - \frac{W^\kappa}{2} + U^\kappa\cdot\nabla W^\kappa + \frac{U^\kappa}{2}\mathrm{div}\, W^\kappa + U^\kappa\cdot\nabla V,$$

and see from the above computations that

$$\|F^\kappa\|_{L_2(\mathbb{R}^3)} \leq c(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}).$$

Now, writing

$$\kappa\nabla\mathrm{div}\, W^\kappa = -\frac{x}{2}\cdot\nabla W^\kappa + F^\kappa,$$

we get

$$\begin{aligned}\kappa^2\int_{\mathbb{R}^3}|\nabla\mathrm{div}\, W^\kappa|^2dx &\leq -\frac{1}{2}\int_{\mathbb{R}^3}(x\cdot\nabla W^\kappa)\cdot\kappa\nabla\mathrm{div}\, W^\kappa dx\\ &\quad+\|F^\kappa\|_{L_2(\mathbb{R}^3)}\|\kappa\nabla\mathrm{div}\, W^\kappa\|_{L_2(\mathbb{R}^3)}\\ &\leq -\frac{\kappa}{4}\int_{\mathbb{R}^3}(\mathrm{div}\, W^\kappa)^2dx + \|F^\kappa\|_{L_2(\mathbb{R}^3)}\|\kappa\nabla\mathrm{div}\, W^\kappa\|_{L_2(\mathbb{R}^3)}.\end{aligned}$$

Using Cauchy's inequality, we obtain that

$$\kappa^2\int_{\mathbb{R}^3}|\nabla\mathrm{div}\, W^\kappa|^2dx \leq c(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}). \tag{5.85}$$

5.5 Proof of Theorem 5.1.4

From Theorem 5.1.3, it follows that there exists a subsequence (still denoted in the same way as W^κ) such that

$$W^\kappa \rightharpoonup W, \qquad \nabla W^\kappa \rightharpoonup \nabla W, \qquad \nabla^2 W^\kappa \rightharpoonup \nabla^2 W \tag{5.86}$$

and

$$\kappa\mathrm{div}\, w^k \rightharpoonup P \qquad \kappa\nabla\mathrm{div}\, w^k \rightharpoonup \nabla P \tag{5.87}$$

in $L_2(\mathbb{R}^3)$. We can also state that

$$W^k \to W \tag{5.88}$$

in $L_{4,loc}(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$, and

$$\sup_\kappa\sup_{x\in\mathbb{R}^3}|W^k(x)| < \infty. \tag{5.89}$$

Having the above convergence, it is easy to show that the limit functions U and P satisfy the profile equations (5.1). Now, let us justify strong convergence. Let $\bar{U}^\kappa := U^\kappa - U = W^\kappa - W$. Next, we get from $\mathcal{L}_\kappa = 0$ and (5.1) that

$$\begin{aligned}&-\Delta\bar{U}^\kappa - \kappa\nabla\mathrm{div}\,\bar{U}^\kappa + \nabla P + U^\kappa\cdot\nabla\bar{U}^\kappa + \frac{\bar{U}^\kappa}{2}\mathrm{div}\, U^\kappa + \frac{U}{2}\mathrm{div}\, U^\kappa\\ &\qquad + \bar{U}^\kappa\cdot\nabla U - \frac{x}{2}\cdot\nabla\bar{U}^\kappa - \frac{1}{2}\bar{U}^\kappa = 0 \quad \text{in } \mathbb{R}^3.\end{aligned} \tag{5.90}$$

Multiplying the previous equation by $\bar{U}^\kappa$ and integrating on the whole of $\mathbb{R}^3$, we obtain (the latter can be verified by suitable cut-off and passing to the limit with the help of (5.86) and (5.87)) that

$$\int_{\mathbb{R}^3}|\nabla\bar{U}^\kappa|^2dx+\kappa\int_{\mathbb{R}^3}|\operatorname{div}\bar{U}^\kappa|^2dx+\frac{1}{4}\int_{\mathbb{R}^3}|\bar{U}^\kappa|^2dx\leq\frac{\|P\|_{L_2(\mathbb{R}^3)}}{\kappa}\|\kappa\operatorname{div}U^\kappa\|_{L_2(\mathbb{R}^3)}$$
$$+\frac{1}{2\kappa}\|U\|_{L_4(\mathbb{R}^3)}\|\bar{U}^\kappa\|_{L_4(\mathbb{R}^3)}\|\kappa\operatorname{div}U^\kappa\|_{L_2(\mathbb{R}^3)}+\int_{\mathbb{R}^3}|\bar{U}^\kappa|^2|\nabla U|dx.$$

Since the L_4-norms of U and $\bar{U}^\kappa$ are uniformly bounded, it is enough to show that the last term on the right-hand side of the above inequality tends to zero as $\kappa\to\infty$. Indeed,

$$\int_{\mathbb{R}^3}|\bar{U}^\kappa|^2|\nabla U|dx\leq\Big(\int_{\mathbb{R}^3}|\bar{U}^\kappa|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\bar{U}^\kappa|^2|\nabla U|^2\Big)^{\frac{1}{2}}.$$

The first factor on the right-hand side of the latter inequality is bounded while the second one tends to zero by Lebesgue's theorem, see (5.88) and (5.89).
Now, let us notice the following identity

$$\sup_{0<\tau\leq t}\int_{\mathbb{R}^3}|u^\kappa(x,\tau)-u(x,\tau)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla(u^\kappa-u)|^2dxd\tau$$
$$=\sqrt{t}\int_{\mathbb{R}^3}(|\bar{U}^\kappa|^2+2|\nabla\bar{U}^\kappa|^2)dx,$$

where $u(x,t)=\frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}})$. It is not so difficult to deduce from the above identity that in fact u is a global weak $L^{3,\infty}$-solution to the Navier-Stokes system. To this end, one needs to take into account semigroup estimates and the above strong convergence.
Now, since $u(x,t)=\frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}})$ is a global weak $L^{3,\infty}$-solution to the Navier-Stokes system, the estimates (5.22) are true as well. This is shown in [31].
To show the strong convergence of $\nabla^2\bar{U}^\kappa$ in $L_2(\mathbb{R}^3)$ to zero, it is sufficient to multiply equation (5.90) by $\Delta\bar{U}^\kappa$ and use the strong convergence of $\bar{U}^\kappa$ and $\nabla\bar{U}^\kappa$, and the weak convergence of $\nabla^2\bar{U}^\kappa$ in $L_2(\mathbb{R}^3)$ to zero. To be more precise, we obtain from (5.90) (similarly to what we did for (5.83)) that:

$$\int_{\mathbb{R}^3}|\Delta\bar{U}^\kappa|^2dx+\kappa\int_{\mathbb{R}^3}|\nabla\operatorname{div}\bar{U}^\kappa|^2dx\leq\frac{\|\nabla P\|_{L_2(\mathbb{R}^3)}}{\kappa}\|\kappa\nabla\operatorname{div}\bar{U}^\kappa\|_{L_2(\mathbb{R}^3)}$$
$$+\|U^\kappa\|_{L_4(\mathbb{R}^3)}\|\nabla\bar{U}^\kappa\|_{L_4(\mathbb{R}^3)}\|\Delta\bar{U}^\kappa\|_{L_2(\mathbb{R}^3)}+\frac{1}{2\kappa}\|\bar{U}^\kappa\|_{L_4(\mathbb{R}^3)}\|\kappa\operatorname{div}U^\kappa\|_{L_4(\mathbb{R}^3)}\|\Delta\bar{U}^\kappa\|_{L_2(\mathbb{R}^3)}$$
$$+\frac{1}{2\kappa}\|U\|_{L_4(\mathbb{R}^3)}\|\kappa\operatorname{div}U^\kappa\|_{L_4(\mathbb{R}^3)}\|\Delta\bar{U}^\kappa\|_{L_2(\mathbb{R}^3)}$$
$$+\|\bar{U}^\kappa\|_{L_4(\mathbb{R}^3)}\|\nabla U\|_{L_4(\mathbb{R}^3)}\|\Delta\bar{U}^\kappa\|_{L_2(\mathbb{R}^3)}+\int_{\mathbb{R}^3}|\nabla\bar{U}^\kappa|^2dx;\quad(5.91)$$

Using the uniform estimates we obtained in Theorem 5.1.3 and the above strong convergences, we have that the proof is complete.

When we look at the decay estimate for $W^\kappa := U^\kappa - V^\kappa$ in Theorem 5.1.2 and the estimate (5.22) for the limit W, a natural question is whether we can make this estimate for W^κ uniform in κ; in other words we wish to show that

$$|\partial^\alpha W^\kappa(x)| \leq \frac{C(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}} \quad (\text{with } \alpha \in \mathbb{N}^3)$$

for all $\kappa \geq 0$. We tackle this question in the next section.

5.6 Uniform decay estimates

From now on we use the notation $(U^{\kappa_n})_{n\geq 0}$ for the sequence of profiles that yields the limit U. The final main theorem of this chapter is as follows:

Theorem 5.6.1. *For all $n \geq 0$, we have $W^{\kappa_n} = W^{+,\kappa_n} + W^{-,\kappa_n}$ where W^{+,κ_n} is a divergence-free vector field and W^{-,κ_n} is a curl-free vector field such that:*

$$|\partial^\alpha W^{+,\kappa_n}(y)| \leq \frac{c(\alpha, u_0)}{(1+|y|)^{3+|\alpha|}} \quad \text{and} \quad |\partial^\alpha W^{-,\kappa_n}(y)| \leq \frac{c(\alpha, u_0)}{(1+|y|)^{2+|\alpha|}},$$

for all $y \in \mathbb{R}^3$ and $\alpha \in \mathbb{N}^3$. Moreover, we get for all $\alpha \in \mathbb{N}^3$ that:

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} (1+|y|)^{|\alpha|+3^-} |\partial^\alpha (W^{+,\kappa_n} - W)(y)| = 0$$

and

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} (1+|y|)^{|\alpha|+2^-} |\partial^\alpha W^{-,\kappa_n}(y)| = 0.$$

The proof relies on the following observation.

Lemma 5.6.2. *For all $\epsilon > 0$, there exists a $T_\epsilon > 0$ such that, for all $|x_0| > 3$, we have*

$$\sup_{0<t<T_\epsilon} \|w^{\kappa_n}(\cdot, t)\|_{L_3(B(x_0))} \leq \epsilon, \quad \forall n \geq 0.$$

Proof. We have

$$\begin{aligned}
\|w^{\kappa_n}(\cdot,t)\|_{L_3(B(x_0))} &= \left(\int_{B(\frac{x_0}{\sqrt{t}}, \frac{1}{\sqrt{t}})} |W^{\kappa_n}|^3 dy \right)^{\frac{1}{3}} \\
&\leq \left(\int_{B(\frac{x_0}{\sqrt{t}}, \frac{1}{\sqrt{t}})} |W^{\kappa_n} - W|^3 dy \right)^{\frac{1}{3}} + \left(\int_{B(\frac{x_0}{\sqrt{t}}, \frac{1}{\sqrt{t}})} |W|^3 dy \right)^{\frac{1}{3}}.
\end{aligned}$$

Since $\|W^{\kappa_n} - W\|_{L_3(\mathbb{R}^3)} \to 0$, we have that there exists an integer $N(\epsilon) > 0$ such that $\|W^{\kappa_n} - W\|_{L_3(\mathbb{R}^3)} < \epsilon/2$ for all $n \geq N(\epsilon)$ and $R_\epsilon > 0$ such that $\|W\|_{L_3(|x|>R_\epsilon)} < \epsilon/2$. Consequently, we have that

$$\sup_{0<t\leq \frac{4}{R_\epsilon^2}} \|w^{\kappa_n}(\cdot,t)\|_{L_3(B(x_0))} \leq \epsilon, \quad \forall n \geq N(\epsilon).$$

This readily leads to the desired result. $\square$

Next, we have the following uniform strong solution estimate.

Proposition 5.6.1. *There exists a* $T_0 > 0$ *such that, for all* $|x_0| > 3$, *we have*

$$\sup_{0<t\leq T_0} \int_{B(x_0,1/2)} |\nabla w^{\kappa_n}(x,t)|^2 dx + \int_0^{T_0} \int_{B(x_0,1/2)} |\nabla^2 w^{\kappa_n}(x,t)|^2 dxdt$$
$$+ \kappa_n \int_0^{T_0} \int_{B(x_0,1/2)} |\nabla \operatorname{div} w^{\kappa_n}(x,t)|^2 dxdt \leq c(u_0, T_0), \quad \forall n \geq 0.$$

Proof. Let us start by recalling that our divergence-free $(-1)-$homogeneous initial data u_0 belongs to $C^\infty(\mathbb{R}^3 \setminus \{0\})$. Therefore, we have the following bounds:

$$\|\partial_t^k \nabla^l v\|_{L_\infty(B(x_0)\times[0,\infty[)} \leq c(k,l,u_0), \quad \text{for } k,l = 0,1,2,\dots \tag{5.92}$$

where $v(x,t) := S(t)u_0(x)$.
We recall that the w^{κ_n}'s satisfy the following equation:

$$\partial_t w^{\kappa_n} - \Delta w^{\kappa_n} - \kappa_n \nabla \operatorname{div} w^{\kappa_n} + w^{\kappa_n} \cdot \nabla w^{\kappa_n}$$
$$+ \frac{w^{\kappa_n}}{2} \operatorname{div} w^{\kappa_n} = -(v \cdot \nabla w^{\kappa_n} + \frac{v}{2} \operatorname{div} w^{\kappa_n}) - w^{\kappa_n} \cdot \nabla v - v \cdot \nabla v \quad \text{in } \mathbb{R}^3, \tag{5.93}$$

with $w^{\kappa_n}(\cdot,0) \equiv 0$ in $\mathbb{R}^3$. Introduce now the cut-off function $0 \leq \varphi \in C_0^\infty(B)$ such that $\varphi \equiv 1$ in $B(1/2)$ and $\varphi \equiv 0$ in $B \setminus B(3/4)$; set $\varphi_{x_0}(x) = \varphi(x - x_0)$. By testing (5.93) with $\varphi_{x_0}\Delta(w^{\kappa_n}\varphi_{x_0})$, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\int_{B(x_0)}|\nabla(w^{\kappa_n}\varphi_{x_0})|^2dx+\int_{B(x_0)}|\Delta(w^{\kappa_n}\varphi_{x_0})|^2dx+\kappa_n\int_{B(x_0)}|\nabla(\operatorname{div}w^{\kappa_n}\varphi_{x_0})|^2dx\\
&=\int_{B(x_0)}(2\nabla\varphi_{x_0}\cdot\nabla w^{\kappa_n}+\Delta\varphi_{x_0}w^{\kappa_n})\cdot\Delta(w^{\kappa_n}\varphi_{x_0})+\int_{B(x_0)}\kappa_n\operatorname{div}w^{\kappa_n}\left[2\nabla\varphi_{x_0}\cdot\Delta(w^{\kappa_n}\varphi_{x_0})\right.\\
&\left.-2(\nabla\varphi_{x_0}\cdot\nabla w^{\kappa_n})\cdot\nabla\varphi_{x_0}-\Delta\varphi_{x_0}\nabla\varphi_{x_0}\cdot w^{\kappa}+\varphi_{x_0}\nabla^2\varphi_{x_0}:\nabla w^{\kappa_n}+\varphi_{x_0}w^{\kappa_n}\cdot\nabla\varphi_{x_0}\right]dx\\
&\quad+\int_{B(x_0)}w^{\kappa_n}\cdot\nabla(w^{\kappa_n}\varphi_{x_0})\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx-\int_{B(x_0)}w^{\kappa_n}\cdot\nabla\varphi_{x_0}w^{\kappa_n}\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx\\
&\quad+\frac{1}{2}\int_{B(x_0)}\operatorname{div}(w^{\kappa_n}\varphi_{x_0})w^{\kappa_n}\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx-\frac{1}{2}\int_{B(x_0)}w^{\kappa_n}\cdot\Delta(w^{\kappa_n}\varphi_{x_0})w^{\kappa_n}\cdot\nabla\varphi_{x_0}dx\\
&\qquad+\int_{B(x_0)}v\cdot\nabla(w^{\kappa_n}\varphi_{x_0})\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx-\int_{B(x_0)}v\cdot\varphi_{x_0}w^{\kappa_n}\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx\\
&\qquad+\frac{1}{2}\int_{B(x_0)}\operatorname{div}(w^{\kappa_n}\varphi_{x_0})v\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx-\frac{1}{2}\int_{B(x_0)}v\cdot\Delta(w^{\kappa_n}\varphi_{x_0})w^{\kappa_n}\cdot\nabla\varphi_{x_0}dx\\
&\qquad\qquad-\int_{B(x_0)}\varphi_{x_0}(w^{\kappa_n}\cdot\nabla v+v\cdot\nabla v)\cdot\Delta(w^{\kappa_n}\varphi_{x_0})dx.
\end{aligned}$$

Consequently, we find for all $0<t<T_\epsilon$, with T_ϵ as in Lemma 5.6.2,

$$\begin{aligned}
&\frac{d}{dt}\int_{B(x_0)}|\nabla(w^{\kappa_n}\varphi_{x_0})|^2dx+\int_{B(x_0)}|\nabla^2(w^{\kappa_n}\varphi_{x_0})|^2dx\\
&\quad\le\|v\|^2_{L_\infty(B(x_0)\times[0,\infty[)}\int_{B(x_0)}|\nabla(w^{\kappa_n}\varphi_{x_0})|^2dx+c\epsilon\int_{B(x_0)}|\nabla^2(w^{\kappa_n}\varphi_{x_0})|^2dx\\
&\qquad+c\int_{B(x_0)}|\kappa\operatorname{div}w^{\kappa_n}|^2dx+\int_{B(x_0)}|w^{\kappa_n}|^4dx+\|v\cdot\nabla v\|^2_{L_\infty(B(x_0)\times[0,\infty[)}\\
&\quad+c(1+\|v\|^2_{L_\infty(B(x_0)\times[0,\infty[)}+\|\nabla v\|^2_{L_\infty(B(x_0)\times[0,\infty[)})\int_{B(x_0)}(|w^{\kappa_n}|^2+|\nabla w^{\kappa_n}|^2)dx,
\end{aligned}$$

with $c>0$ an absolute constant. Choosing $\epsilon=1/2c$ and denoting for simplicity $T_0=T_{1/2c}$, we get thanks to our uniform estimates and Gronwall's inequality that the proposition is proved. □

The goal now is to bootstrap the previous regularity result to get some uniform $L_\infty-$estimates for $\partial_t w^{\kappa_n}$ and its spatial derivatives. To this end, we need the following auxiliary results.

Proposition 5.6.2. *Set $Q_T:=\mathbb{R}^3\times]0,T[$ $(T>0)$ and let $f\in L_{s,l}(Q_T)$ with $1<s,l<\infty$. There exists a function v that uniquely solves the system*

$$\begin{cases}\partial_t v-\Delta v-\kappa\nabla\operatorname{div}v=f & \text{in } Q_T\\ v|_{t=0}=0 & \text{in } \mathbb{R}^3\end{cases}$$

such that

$$\|\partial_t v\|_{L_{s,l}(Q_T)} + \|\nabla^2 v\|_{L_{s,l}(Q_T)} + \kappa\|\nabla \operatorname{div} v\|_{L_{s,l}(Q_T)} \le c(s,l)\|f\|_{L_{s,l}(Q_T)};$$

$$\|\nabla v\|_{L_{s_1,l_1}(Q_T)} \le c(s,s_1,l,l_1)[1+(1+\kappa)^{\frac{1}{2}(-1-\frac{3}{s}+\frac{3}{s})}]T^{\frac{1}{2}\left(1-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_1}+\frac{3}{s_1}\right)}\|f\|_{L_{s,l}(Q_T)} \ \forall T \ge 0,$$

$$\textit{with } s \le s_1 \le \infty,\ l \le l_1 \le \infty \textit{ such that } 1-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_1}+\frac{3}{s_1} > 0;$$

$$\|v\|_{L_{s_2,l_2}(Q_T)} \le c(s,s_2,l,l_2)[1+(1+\kappa)^{\frac{1}{2}(-\frac{3}{s}+\frac{3}{s_2})}]T^{\frac{1}{2}\left(2-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_2}+\frac{3}{s_2}\right)}\|f\|_{L_{s,l}(Q_T)} \ \forall T \ge 0,$$

$$\textit{with } s \le s_2 \le \infty,\ l \le l_2 \le \infty \textit{ such that } 2-\frac{2}{l}-\frac{3}{s}+\frac{2}{l_2}+\frac{3}{s_2} > 0.$$

Proof. The proof of this result follows the same ideas as the ones used in the proof of Lemma 2.3.1. The difference here being that the dependence in κ is made explicit.

There exists a function q such that: $\Delta q = \operatorname{div} f$ and $\|\nabla q\|_{L_{s,l}(Q_T)} \le c\|f\|_{L_{s,l}(Q_T)}$. Next introduce $f^0 := -\nabla q + f$ and notice that $\operatorname{div} f^0 = 0$. Now, from known-solvability results for the heat equation, we have the existence of two functions v^1 and v^2 such that

$$\begin{cases} \partial_t v^1 - (1+\kappa)\Delta v^1 = \nabla q & \text{in } Q_T \\ w^1|_{t=0} = 0 & \text{in } \mathbb{R}^3 \end{cases} \quad \text{and} \quad \begin{cases} \partial_t v^2 - \Delta v^2 = f^0 & \text{in } Q_T \\ w^2|_{t=0} = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{5.94}$$

Notice that $\operatorname{curl} v^1 \equiv 0$ and $\operatorname{div} v^2 \equiv 0$ in Q_T and $\Delta v^1 = \nabla \operatorname{div} v^1$. Moreover, the following estimates are available.

$$\|\partial_t v^2\|_{L_{s,l}(Q_T)} + \|\nabla^2 v^2\|_{L_{s,l}(Q_T)} \le c(s,l)\|f\|_{L_{s,l}(Q_T)}.$$

$$\|\nabla v^2\|_{L_{l_1}(0,T;L_{s_1}(\mathbb{R}^3))} \le c(s,s_1,l,l_1)T^{\frac{1}{2}\left(1-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_3}+\frac{3}{s_3}\right)}\|f\|_{L_{s,l}(Q_T)},$$

$$\text{with } s \le s_1 \le \infty,\ l \le l_1 \le \infty \text{ such that } 1-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_1}+\frac{3}{s_1} > 0;$$

$$\|v^2\|_{L_{l_2}(0,T;L_{s_2}(\mathbb{R}^3))} \le c(s,s_2,l,l_2)T^{\frac{1}{2}\left(2-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_2}+\frac{3}{s_2}\right)}\|f\|_{L_{s,l}(Q_T)},$$

$$\text{with } s \le s_2 \le \infty,\ l \le l_2 \le \infty \text{ such that } 1-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_2}+\frac{3}{s_2} > 0.$$

In the case of v^1, using scaling arguments, we get:

$$\|\partial_t v^1\|_{L_{s,l}(Q_T)} + \kappa\|\nabla^2 v^1\|_{L_{s,l}(Q_T)} \le c(s,l)\|f\|_{L_{s,l}(Q_T)}.$$

$$\|\nabla v^1\|_{L_{l_1}(0,T;L_{s_1}(\mathbb{R}^3))} \le c(s,s_1,l,l_1)(1+\kappa)^{\frac{1}{2}(-1-\frac{3}{s}+\frac{3}{s_1})}T^{\frac{1}{2}\left(1-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_3}+\frac{3}{s_3}\right)}\|f\|_{L_{s,l}(Q_T)},$$

$$\text{with } s \le s_1 \le \infty,\ l \le l_1 \le \infty \text{ such that } 1-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_1}+\frac{3}{s_1} > 0;$$

$$\|v^1\|_{L_{l_2}(0,T;L_{s_2}(\mathbb{R}^3))} \le c(s,s_2,l,l_2)(1+\kappa)^{\frac{1}{2}(-\frac{3}{s}+\frac{3}{s_2})}T^{\frac{1}{2}\left(2-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_2}+\frac{3}{s_2}\right)}\|f\|_{L_{s,l}(Q_T)},$$

$$\text{with } s \le s_2 \le \infty,\ l \le l_2 \le \infty \text{ such that } 1-\frac{2}{l}-\frac{3}{s}+\frac{3}{l_2}+\frac{3}{s_2} > 0.$$

We complete the proof by setting $v := v^1 + v^2$. □

Our second auxiliary result is as follows:

Proposition 5.6.3. *Let* $x_0 \in \mathbb{R}^3$, $T_0 > 0$ *and suppose that a function* F *is such that*

$$\|\partial^\alpha F\|_{L_\infty(B(x_0)\times]0,T_0[)} \leq M(\alpha), \quad \forall \alpha \in \mathbb{N}^3.$$

Let $w \in L_2(B(x_0)\times]0,T_0[)$ *be such that* $\kappa \operatorname{div} w \in L_2(B(x_0)\times]0,T_0[)$ *and*

$$\begin{cases} \partial_t w - \Delta w - \kappa \nabla \operatorname{div} w = F \quad \text{in } B(x_0)\times]0,T_0[\\ \qquad\qquad w|_{t=0} = 0 \quad \text{in } B(x_0). \end{cases}$$

Then we have

$$\|\partial^\alpha \partial_t w\|_{L_2(B(x_0,2/3)\times]0,T_0[)} + \|\partial^{\alpha+2} w\|_{L_2(B(x_0,2/3)\times]0,T_0[)} + \kappa \|\partial^{\alpha+1} \operatorname{div} w\|_{L_2(B(x_0,2/3)\times]0,T_0[)} \leq C,$$

for all $\alpha \in \mathbb{N}^3$ *and with* $C = C(\alpha, M(\alpha), T_0, \|w\|_{L_2(B(x_0)\times]0,T_0[)}, \kappa \|\operatorname{div} w\|_{L_2(B(x_0)\times]0,T_0[)})$. *From this, we see that*

$$\|\partial_t \partial^\alpha w\|_{L_2(]0,T_0[;C^{0,\frac{1}{2}}(\overline{B(x_0,2/3)}))} + \kappa \|\partial^\alpha \operatorname{div} w\|_{L_2(]0,T_0[;C^{0,\frac{1}{2}}(\overline{B(x_0,2/3)}))} \leq C,$$

$$\|\partial^\alpha w\|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{B(x_0,1/2)}\times[0,T_0])} \leq C,$$

with α *and* C *as above.*

Proof. We only sketch the proof since the arguments we need here are quite standard; let us introduce the cut-off function $0 \leq \varphi \in C_0^\infty(B)$ such that $\varphi \equiv 1$ in $B(1)$ and $\varphi \equiv 0$ in $B(2) \setminus B(3/2)$; set $\varphi_{x_0,k}(x) = \varphi\left((x - x_0)/(1 - \sum_{l=1}^k 1/4^l)\right)$. We see that

$$\partial_t(\varphi_{x_0,k}\partial^\alpha \operatorname{div} w) - (1+\kappa)\Delta(\varphi_{x_0,k}\partial^\alpha \operatorname{div} w) = -(1+\kappa)\partial^\alpha \operatorname{div} w \Delta\varphi_{x_0,k} \\ - 2(1+\kappa)\operatorname{div}(\partial^\alpha \operatorname{div} w \nabla \varphi_{x_0,k}) + \varphi_{x_0,k}\partial^\alpha \operatorname{div} F, \quad (5.95)$$

in $B(x_0)\times]0,T_0[$. Set for simplicity $w^{\alpha,\kappa,x_0,k} = (1+\kappa)\varphi_{x_0,k}\partial^\alpha \operatorname{div} w$. In the case $\alpha = 0$ and $k = 1$, we get by testing (5.95) with $w^{0,\kappa,x_0,1}$ followed by integration by parts that

$$\int_0^{T_0}\int_{B(x_0)} |\nabla w^{0,\kappa,x_0,1}|^2 dxdt \leq c(T_0)\left(\|(1+\kappa)\operatorname{div} w\|^2_{L_2(B(x_0)\times]0,T_0[)} \right. \\ \left. + \|\operatorname{div} F\|^2_{L_\infty(B(x_0)\times]0,T_0[)}\right);$$

thus, the right-hand side of (5.95) belongs to $L_2(\mathbb{R}^3\times]0,T_0[)$ when $\alpha = 0$ and $k = 2$. Consequently, we get, by known estimates for the heat equation together with scaling arguments, that

$$\|\nabla^2 w^{0,\kappa,x_0,2}\|_{L_2(\mathbb{R}^3\times]0,T_0[)} \leq c(T_0)\left(\|(1+\kappa)\operatorname{div} w\|_{L_2(B(x_0)\times]0,T_0[)} \right. \\ \left. + \|\operatorname{div} F\|_{L_\infty(B(x_0)\times]0,T_0[)}\right).$$

Repeating this process for $\alpha \neq 0$, we get that the desired inequality is proved for the term $\kappa \operatorname{div} w$.
Finally, we write

$$\partial_t w - \Delta w = -\kappa \nabla \operatorname{div} w + F \quad \text{in } B(x_0) \times]0, T_0[;$$

By making use of the previous conclusion, and using similar arguments as above, we have that the proposition is proved. □

We are able to prove now Theorem 5.6.1.

Proof of Theorem 5.6.1. We take again a cut-off function $0 \leq \varphi \in C_0^\infty(B)$ such that $\varphi \equiv 1$ in $B(1)$ and $\varphi \equiv 0$ in $B(2) \setminus B(3/2)$; set $\varphi_{x_0}(x) = \varphi(4(x - x_0))$ and solve (with the help of Proposition 5.6.2)

$$\begin{cases} \partial_t \hat{w} - \Delta \hat{w} - \kappa \nabla \hat{w} = -(u^{\kappa_n} \cdot \nabla u^{\kappa_n} + \dfrac{u^{\kappa_n}}{2} \operatorname{div} u^{\kappa_n}) \varphi_{x_0} \quad \text{in } \mathbb{R}^3 \times]0, T_0[\\ \hat{w}|_{t=0} = 0 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{5.96}$$

Notice that because of Proposition 5.6.1, estimate (5.92) and by multiplicative inequality, we have that the right-hand side of (5.96_1) belongs to $L_{s,l}(\mathbb{R}^3 \times]0, T_0[)$ with $3/s + 2/l = 2$ ($s \leq 6$) where the bound on that norm depends only on u_0 and T_0. Next, by setting $\bar{w} := w^{\kappa_n} - \hat{w}$ and applying Proposition 5.6.3 to $\bar{w}$ (with $F \equiv 0$), we get that

$$\|w^{\kappa_n}\|_{L_{s_0,l_0}(B(x_0,1/4) \times]0,T_0[)} + \|\nabla w^{\kappa_n}\|_{L_{s_1,l_1}(B(x_0,1/4) \times]0,T_0[)} \leq c(u_0, T_0)$$

where $3/s_0 + 2/l_0 > 0$ and $3/s_1 + 2/l_1 > 1$; and the same integrability holds for u^{κ_n} because of (5.92). By repeating this process (after applying ∂^α to term $u^{\kappa_n} \cdot \nabla u^{\kappa_n} + \frac{1}{2} u^{\kappa_n} \operatorname{div} u^{\kappa_n}$ and suitably changing the cut-off function at each step), we end up with

$$\|\partial^\alpha u^{\kappa_n}\|_{L_\infty(B(x_0,3/16) \times]0,T_0[)} \leq c(\alpha, u_0, T_0), \quad \forall \alpha \in \mathbb{N}^3. \tag{5.97}$$

Since, we have

$$\partial_t w^{\kappa_n} - \Delta w^{\kappa_n} - \kappa \nabla \operatorname{div} w^{\kappa_n} = -(u^{\kappa_n} \cdot \nabla u^{\kappa_n} + \frac{u^{\kappa_n}}{2} \operatorname{div} u^{\kappa_n}) \quad \text{in } B(x_0, 3/16) \times]0, T_0[\tag{5.98}$$

with $w^{\kappa_n}(\cdot, 0) \equiv 0$, we get thanks to (5.97) and Proposition 5.6.3 that

$$\|\partial_t \partial^\alpha w^{\kappa_n}\|_{L_2(]0,T_0[;C^{0,1/2}(\overline{B(x_0,1/8)}))} \leq c(\alpha, u_0, T_0); \tag{5.99}$$

and this also holds if we replace u^{κ_n} by $u^{\lambda,\kappa_n}(x,t) := \lambda u^{\kappa_n}(\lambda x, \lambda^2 t)$ for all $\lambda > 0$ and $x \in \mathbb{R}^3, t > 0$. Consequently, we have that

$$\lambda^{|\alpha|+1} |\partial^\alpha w^{\kappa_n}(\lambda x_0, \lambda^2 t)| \leq c(\alpha, u_0, T_0) t^{1/2},$$

for all $|x_0| > 3$, $t \in]0, T_0[$ and all $\lambda > 0$. Now, take $\lambda = 1/\sqrt{t}$ and $y = x_0/\sqrt{t}$; we get:

$$|\partial^\alpha W^{\kappa_n}(y)| \leq \frac{c(\alpha, u_0, T_0)}{|y|^{2+|\alpha|}}, \quad \forall |y| > \frac{3}{\sqrt{T_0}}. \tag{5.100}$$

Now recall that

$$-\Delta W^{\kappa_n} - \kappa_n \nabla \operatorname{div} W^{\kappa_n} - \frac{y}{2} \cdot \nabla W^{\kappa_n} - \frac{W^{\kappa_n}}{2} = -(U^{\kappa_n} \cdot \nabla U^{\kappa_n} + \frac{U^{\kappa_n}}{2} \operatorname{div} U^{\kappa_n}) \quad \text{in } \mathbb{R}^3$$

with $U^{\kappa_n} = W^{\kappa_n} + e^{\Delta} u_0$ and we have shown that (see Theorem 5.1.3)

$$\|W^{\kappa_n}\|_{H^2(\mathbb{R}^3)} + \kappa_n \|\operatorname{div} W^{\kappa_n}\|_{H^1(\mathbb{R}^3)} \leq c(u_0), \quad \forall n \geq 0.$$

From this, using similar energy arguments as what we did in the proof of Proposition 5.6.3, we get that

$$\|\partial^\alpha W^{\kappa_n}\|_{L_\infty(B(5/\sqrt{T_0}))} \leq c(\alpha, u_0, T_0). \tag{5.101}$$

We have proved that

$$|\partial^\alpha W^{\kappa_n}(y)| \leq \frac{c(\alpha, u_0)}{(1 + |y|)^{2+|\alpha|}}, \quad \forall y \in \mathbb{R}^3, \ \forall n \geq 0. \tag{5.102}$$

Now, we introduce the vector-fields $W^{-,\kappa_n} := \nabla Q^{\kappa_n}$ and $W^{+,\kappa_n} := W^{\kappa_n} - W^{-,\kappa_n}$ where $\Delta Q^{\kappa_n} = \operatorname{div} W^{\kappa_n}$ and $\|W^{+,\kappa_n}\|_{H^2(\mathbb{R}^3)} + \|W^{-,\kappa_n}\|_{H^2(\mathbb{R}^3)} \leq c(u_0)$. Obviously, we have $\operatorname{div} W^{+,\kappa_n} \equiv 0 \equiv \operatorname{curl} W^{-,\kappa_n}$. Next, we set

$$w^{+,\kappa_n}(x,t) = \frac{1}{\sqrt{t}} W^{+,\kappa_n}(\frac{x}{\sqrt{t}});$$

taking the "curl" in (5.98), we obtain

$$\partial_t \operatorname{curl} w^{+,\kappa_n} - \Delta \operatorname{curl} w^{+,\kappa_n} = -\operatorname{curl}(u^{\kappa_n} \cdot \nabla u^{\kappa_n} + \frac{u^{\kappa_n}}{2} \operatorname{div} u^{\kappa_n}),$$

from which we get, thanks to (5.97), that

$$\|\partial_t \partial^\alpha \operatorname{curl} w^{+,\kappa_n}\|_{L_\infty(B(x_0,1/8)\times]0,T_0[)} + \|\partial^\alpha \operatorname{curl} w^{+,\kappa_n}\|_{L_\infty(B(x_0,1/8)\times]0,T_0[)} \leq c(\alpha, u_0, T_0).$$

From this, we get in a similar manner to what we did above that

$$|\partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y)| \leq \frac{c(\alpha, u_0)}{(1 + |y|)^{4+|\alpha|}}, \quad \forall y \in \mathbb{R}^3, \ \forall n \geq 0. \tag{5.103}$$

We recover now the decay for W^{+,κ_n}. Let us start by noticing that $\int_{\mathbb{R}^3} \partial^\alpha \operatorname{curl} W^{+,\kappa_n} dx = 0$ for all $\alpha \in \mathbb{N}^3$. The case $\alpha \neq 0$ follows easily from (5.103); however, for $\alpha = 0$, we proceed as follows. Since $\operatorname{curl} W^{+,\kappa_n} \in L_1(\mathbb{R}^3)$, we have that the Fourier transform G of $\operatorname{curl} W^{+,\kappa_n}$ is continuous on $\mathbb{R}^3$. Moreover since $\operatorname{div}(\operatorname{curl} W^{+,\kappa_n}) = 0$, we have that

$x \cdot G(x) = 0$ for all $x \in \mathbb{R}^3$; in particular $\xi \cdot G(r\xi) = 0$, for all $r > 0$ and all $\xi \in \mathbb{R}^3$. Thus, taking $r \to 0^+$, we get $0 = G(0) = \int_{\mathbb{R}^3} \operatorname{curl} W^{+,\kappa_n} dx$. Next, notice that because $\operatorname{div} \partial^\alpha W^{+,\kappa_n} = 0$, we have that $-\Delta \partial^\alpha W^{+,\kappa_n} = \operatorname{curl}(\operatorname{curl} \partial^\alpha W^{+,\kappa_n})$. Consequently, we have the following representation formula:

$$\begin{aligned}
\partial^\alpha W^{+,\kappa_n}(x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\operatorname{curl}(\partial^\alpha \operatorname{curl} W^{+,\kappa_n})(y)}{|x-y|} dy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y) dy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\frac{x-y}{|x-y|^3} - \frac{x}{|x|^3} \right) \wedge \partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y) dy \\
&= \frac{1}{4\pi} \int_{\{|y| > \frac{1}{2}|x|\} \cap \{|x-y| > 2|x|\}} \frac{x-y}{|x-y|^3} \wedge \partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y) dy \\
&\quad - \frac{1}{4\pi} \frac{x}{|x|^3} \wedge \int_{|y| > \frac{1}{2}|x|} \partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y) dy \\
&\quad + \frac{1}{4\pi} \int_{\{|y| > \frac{1}{2}|x|\} \cap \{|x-y| \leq 2|x|\}} \frac{x-y}{|x-y|^3} \wedge \partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y) dy \\
&\quad + \frac{1}{4\pi} \int_{|y| \leq \frac{1}{2}|x|} \left(\int_0^1 \nabla \left(\frac{x}{|x|^3} \right) (x - sy) ds \; y \right) \wedge \partial^\alpha \operatorname{curl} W^{+,\kappa_n}(y) dy \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The goal now is to estimate the I_i's. We have $|\nabla(x/|x|^3)| \leq c|x|^{-3}$ and for $|y| \leq \frac{1}{2}|x|$, we have $|x - sy| \geq |x|/2$ (with $0 < s < 1$); thus we have

$$|I_4| \leq \frac{c(\alpha, u_0)}{|x|^3} \int_{|y| \leq \frac{1}{2}|x|} \frac{|y|}{(1+|y|)^{4+|\alpha|}} dy \leq \frac{c(\alpha, u_0)}{(1+|y|)^{3+|\alpha|}}. \tag{5.104}$$

Next,

$$|I_3| \leq \frac{c(\alpha, u_0)}{(1+|x|)^{4+|\alpha|}} \int_{|x-y| \leq 2|x|} \frac{dy}{|x-y|^2} dy \leq \frac{c(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}}. \tag{5.105}$$

Finally,

$$|I_1 + I_2| \leq \frac{c(\alpha, u_0)}{|x|^2} \int_{|y| > \frac{1}{2}|x|} \frac{dy}{(1+|y|)^{4+|\alpha|}} dy \leq \frac{c(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}}. \tag{5.106}$$

Consequently, we have proved that

$$|\partial^\alpha W^{+,\kappa_n}(y)| \leq \frac{c(\alpha, u_0)}{(1+|y|)^{3+|\alpha|}}, \quad \forall y \in \mathbb{R}^3, \forall n \geq 0, \tag{5.107}$$

and thus we have proved all the required decay estimates.

For the convergence part, we have shown in the paper [27] that $\kappa_n \| \operatorname{div} W^{-,\kappa_n} \|_{L_2(\mathbb{R}^3)} \leq$

$c(u_0)$; this estimate together with the pre-compactness we have from estimates (5.103) and (5.107) implies that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} (1+|y|)^{|\alpha|+2^-} |\partial^\alpha W^{-,\kappa_n}(y)| = 0 \quad \forall \alpha \in \mathbb{N}^3.$$

Similar arguments lead to

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} (1+|y|)^{|\alpha|+2^-} |\partial^\alpha (W^{+,\kappa_n} - W)(y)| = 0 \quad \forall \alpha \in \mathbb{N}^3.$$

And this concludes the proof of the theorem. □

Chapter 6

Some new regularity results

6.1 Introduction

In the same spirit as what we started in 'Chapter 4', we present in this chapter some additional partial regularity results which, compared to what is known for the regularity theory of the 3D incompressible Navier-Stokes system, are relatively new. These results rely heavily on the absence of the non-locality introduced by the pressure. Those results are part of my paper [28].

6.2 A higher integrability result

The first main result of this chapter states as follows.

Theorem 6.2.1 (Higher space-time integrability of the gradient)**.** *Let u be a suitable weak solution to $\mathcal{L}_0 = 0$ (or $\mathcal{DL}_0 = 0$) in Q such that*

$$\operatorname*{ess\,sup}_{-1<t<0} \|u(\cdot,t)\|_{BMO^{-1}(B)} < \infty.$$

Then we have that

$$\int_{Q(1/2)} |\nabla u|^{2+\delta} dz < \infty,$$

with $\delta > 0$. Here, $f \in BMO^{-1}(B;\mathbb{R}^3)$ shall be understood as there being an $F \in BMO(B;\mathbb{R}^{3\times 3})$ such that $f = \operatorname{div} F$; we do not need anything more than this property in our proof but the interested reader may find more details about the space BMO^{-1} in [36].

The assumption that $u \in L_\infty(-1,0;BMO^{-1}(B))$ arises naturally when one studies regularity of energy solutions for parabolic equations. We refer to [61] for instance where it was shown, in the case of parabolic equations with singular divergence free-drift in $L_\infty(-1,0;BMO^{-1}(B))$, that what the authors call "suitable weak solutions" for their equation are, in fact, Hölder continuous; let us mention that, regarding those authors' so-called suitable weak solutions, we were able to prove in [26] (see 'Appendix A') that standard energy solutions fulfil their suitable weak solution criterion. Moreover, they also showed in [61] that this function space is optimal for getting regularity (when we have the divergence-free condition).

It is also worth pointing out that there is no such higher integrability result for the 3D incompressible Navier-Stokes system at this time, and as mentioned at the beginning of this chapter, this is mainly due to the presence of the pressure (see e.g. [59] where this problem was considered).

Proof. The proof is divided into two steps.

Step I. We begin by establishing a Caccioppoli's type inequality. To formulate it, we need to introduce additional notation. Let us fix nonnegative cut-off functions $\varphi \in C_0^\infty(B(2))$ such that $\varphi \equiv 1$ in B and χ with the following properties:

$$\chi(t) = \begin{cases} 0 \text{ for } t \leq -4, \\ (t+4)/3 \text{ for } -4 < t \leq -1, \\ 1 \text{ for } t > -1. \end{cases}$$

Now, for an arbitrary point $z_0 = (x_0, t_0)$ such that $Q(z_0, 2R) \subset Q$, we set

$$\chi_{t_0,2R}(t) = \chi((t-t_0)/R^2), \quad \varphi_{x_0,2R}(x) = \varphi((x-x_0)/R),$$

and we introduce the special mean value

$$u_{x_0,2R}(t) = \int_{B(x_0,2R)} u(x,t)\varphi^2_{x_0,2R}(x)dx \left(\int_{B(x_0,2R)} \varphi^2_{x_0,2R}(x)dx\right)^{-1}.$$

We set $\hat{u}_{x_0,2R}(x,t) := u(x,t) - u_{x_0,2R}(t)$ and introduce the matrix-valued function $F = (F_{ij}) \in L_\infty(-1,0;BMO(B;\mathbb{R}^{3\times 3}))$ which is such that $u = \operatorname{div} F$.

Our Caccioppoli's inequality reads as follows

$$\begin{aligned}
&\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}(x,t_0')|^2 \varphi^2_{x_0,2R}(x)\chi^2_{t_0,2R}(t_0')dx + 2\int_{-1}^{t_0'}\int_{B(x_0,2R)} \chi^2_{t_0,2R}\varphi^2_{x_0,2R}|\nabla \hat{u}_{x_0,2R}|^2 dz \\
&\qquad \leq \int_{-1}^{t_0'}\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^2(\varphi^2_{x_0,2R}\partial_t\chi^2_{t_0,2R} + \chi^2_{t_0,2R}\Delta\varphi^2_{x_0,2R})dz \\
&\qquad - \int_{-1}^{t_0'}\int_{B(x_0,2R)} (F_{ik} - [F_{ik}]_{x_0,2R})(\varphi^2_{x_0,2R})_{,ik}|\hat{u}_{x_0,2R}|^2\chi^2_{t_0,2R}dz \\
&\qquad - 2\int_{-1}^{t_0'}\int_{B(x_0,2R)} (F_{ik} - [F_{ik}]_{x_0,2R})\chi^2_{t_0,2R}(\varphi^2_{x_0,2R})_{,i}\hat{u}^j_{x_0,2R}(\hat{u}^j_{x_0,2R})_k dz \\
&\qquad - \int_{-1}^{t_0'} \chi^2_{t_0,2R} u_{x_0,2R}\cdot\left(\int_{B(x_0,2R)} \hat{u}_{x_0,2R}\varphi^2_{x_0,2R}\operatorname{div}\hat{u}_{x_0,2R}dx\right)dt, \qquad (6.1)
\end{aligned}$$

for almost every $t_0' \in]-t_0-(2R)^2, t_0[$, for all $z_0 = (x_0,t_0) \in Q$ and all $R > 0$ satisfying the additional condition $Q(z_0,2R) \subset Q$.

We will need some information on the regularity of $u_{x_0,2R}$ in the proof of (6.1). What we can show is that

$$\dot{u}_{x_0,2R}(:= \frac{d}{dt}u_{x_0,2R}) \in L_{\frac{3}{2}}(]-1,0[), \qquad (6.2)$$

and that is all we actually need to make our computations rigorous. To see this, we take as a test function in $\mathcal{L}_0 = 0$,

$$w_i^j(x,t) = \delta_{ij}\varphi^2_{x_0,2R}(x)\eta(t),$$

where δ_{ij} is the Kronecker symbol and η is an arbitrary function in $C_0^\infty(]-1,0[)$. As a result, we get that

$$\begin{aligned}
\dot{u}^i_{x_0,2R}(t) = &-\left(\int_{B(x_0,2R)} \nabla u_i\cdot\nabla\varphi^2_{x_0,2R}(x)dx + \int_{B(x_0,2R)} u\cdot\nabla u_i\varphi^2_{x_0,2R}(x)dx\right. \\
&\left. + \frac{1}{2}\int_{B(x_0,2R)} u_i\varphi^2_{x_0,2R}(x)\operatorname{div} u dx\right)\left(\int_{B(x_0,2R)} \varphi^2_{x_0,2R}(x)dx\right)^{-1}, \qquad (6.3)
\end{aligned}$$

which readily gives (6.2).

Next, we replace $u(x,t)$, in his local energy inequality, by $\hat{u}_{x_0,2R}(x,t) + u_{x_0,2R}(t)$ and take as a test function $\phi = \chi^2_{t_0,2R}\varphi^2_{x_0,2R}$. The terms that do not contain any spatial derivatives can be transformed as follows:

$$\begin{aligned}
&\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}(x,t_0') + u_{x_0,2R}(t_0')|^2\chi^2_{t_0,2R}(t_0')\varphi^2_{x_0,2R}(x)dx \\
&= \int_{B(x_0,2R)} |\hat{u}_{x_0,2R}(x,t_0')|^2\chi^2_{t_0,2R}(t_0')\varphi^2_{x_0,2R}(x)dx + \chi^2_{t_0,2R}(t_0')|u_{x_0,2R}(t_0')|^2\int_{B(x_0,2R)} \varphi^2_{x_0,2R}(x)dx,
\end{aligned}$$

and

$$\int_{-1}^{t_0'}\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}+u_{x_0,2R}|^2\partial_t\chi_{t_0,2R}^2\varphi_{x_0,2R}^2dz=\int_{-1}^{t_0'}\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^2\partial_t\chi_{t_0,2R}^2\varphi_{x_0,2R}^2dz$$
$$+\int_{B(x_0,2R)}\varphi_{x_0,2R}^2(x)dx\left(\chi_{t_0,2R}^2(t_0')|u_{x_0,2R}(t_0')|^2-2\int_{-1}^{t_0'}u_{x_0,2R}(t)\cdot\dot{u}_{x_0,2R}(t)dt\right).$$

Notice that

$$\int_{-1}^{t_0'}\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}+u_{x_0,2R}|^2\chi_{t_0,2R}^2\Delta\varphi_{x_0,2R}^2dz=\int_{-1}^{t_0'}\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^2\chi_{t_0,2R}^2\Delta\varphi_{x_0,2R}^2dz$$
$$-2\int_{-1}\chi_{t_0,2R}^2u_{x_0,2R}\cdot\left(\int_{B(x_0,R)}\nabla\hat{u}_{x_0,2R}\nabla\varphi_{x_0,2R}^2dx\right)dt.$$

Taking into account the previous expansions and (6.3), the local energy inequality becomes

$$\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}(x,t_0')|^2\varphi_{x_0,2R}^2(x)\chi_{t_0,2R}^2(t_0')dx+\int_{-1}^{t_0'}\int_{B(x_0,2R)}\chi_{t_0,2R}^2\varphi_{x_0,2R}^2|\nabla\hat{u}_{x_0,2R}|^2dz$$
$$\leq\int_{-1}^{t_0'}\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^2(\varphi_{x_0,2R}^2\partial_t\chi_{t_0,2R}^2+\chi_{t_0,2R}^2\Delta\varphi_{x_0,2R}^2)dz$$
$$+\int_{-1}^{t_0'}\int_{B(x_0,2R)}u\cdot\nabla\varphi_{x_0,2R}^2|u|^2\chi_{t_0,2R}^2dz$$
$$+2\int_{-1}^{t_0'}\chi_{t_0,2R}^2u_{x_0,2R}\cdot\left(u\cdot\nabla u\varphi_{x_0,2R}^2+\frac{1}{2}u\varphi_{x_0,2R}^2\operatorname{div}u dx\right)dt.\quad(6.4)$$

All that is left is to transform the last two terms in (6.4) and notice that some cancellations occur when putting them together. We have

$$\int_{-1}^{t_0'}\int_{B(x_0,2R)}u\cdot\nabla\varphi_{x_0,2R}^2|u|^2\chi_{t_0,2R}^2dz=A_0+\int_{-1}^{t_0'}\int_{B(x_0,2R)}u\cdot\nabla\varphi_{x_0,2R}^2|\hat{u}_{x_0,2R}|^2\chi_{t_0,2R}^2dx,$$

with

$$A_0=\int_{-1}^{t_0'}\chi_{t_0,2R}^2|u_{x_0,2R}|^2\int_{B(x_0,2R)}u\cdot\nabla\varphi_{x_0,2R}^2dxdt+2\int_{-1}\chi_{t_0,2R}^2u_{x_0,2R}\cdot\left(u\cdot\nabla\varphi_{x_0,2R}^2\hat{u}_{x_0,2R}\right)dt.$$

Using the fact that $u=\operatorname{div}F$, we get by integrating by parts that

$$\int_{-1}^{t_0'}\int_{B(x_0,2R)}u\cdot\nabla\varphi_{x_0,2R}^2|\hat{u}_{x_0,2R}|^2\chi_{t_0,2R}^2dx$$
$$=-\int_{-1}^{t_0'}\int_{B(x_0,2R)}(F_{ik}-[F_{ik}]_{x_0,2R})(\varphi_{x_0,2R}^2)_{,ik}|\hat{u}_{x_0,2R}|^2\chi_{t_0,2R}^2dz$$
$$-2\int_{-1}^{t_0'}\int_{B(x_0,2R)}(F_{ik}-[F_{ik}]_{x_0,2R})\chi_{t_0,2R}^2(\varphi_{x_0,2R}^2)_{,i}\hat{u}_{x_0,2R}^j(\hat{u}_{x_0,2R}^j)_kdz.$$

Now, by performing one more integration by part in the last term of (6.4), we obtain that

$$\begin{aligned}2\int_{-1}^{t_0'} \chi^2_{t_0,2R}u_{x_0,2R}\cdot\left(u\cdot\nabla u\varphi^2_{x_0,2R}+\frac{1}{2}u\varphi^2_{x_0,2R}\operatorname{div} u dx\right)dt &= -A_0\\ -\int_{-1}^{t_0'}\chi^2_{t_0,2R}u_{x_0,2R}\cdot\left(\int_{B(x_0,2R)}\hat{u}_{x_0,2R}\varphi^2_{x_0,2R}\operatorname{div}\hat{u}_{x_0,2R}dx\right)dt,&\end{aligned}$$

which concludes the proof of (6.1) after putting all the terms together.

Step II. We derive now a reverse Hölder inequality using the Caccioppoli's inequality established previously. Using simple known arguments, we get from (6.1) the following estimate

$$\begin{aligned}I &:= \int_B |\hat{u}_{x_0,2R}(x,t_0)|^2\varphi^2_{x_0,2R}(x)dx + 2\int_{-1}^{t_0}\int_B \chi^2_{x_0,2R}\varphi^2_{x_0,2R}|\nabla\hat{u}_{x_0,2R}|^2 dz\\ &\leq c\left(\frac{1}{R^2}\int_{Q(z_0,2R)}|\hat{u}_{x_0,2R}|^2dz+\frac{1}{R^2}\int_{Q(z_0,2R)}|F-[F]_{x_0,2R}||\hat{u}_{x_0,2R}|^2dz\right.\\ &\quad+\frac{1}{R}\int_{Q(z_0,2R)}|F-[F]_{x_0,2R}|(|\nabla\hat{u}_{x_0,2R}|\varphi_{x_0,2R}\chi_{t_0,2R})|\hat{u}_{x_0,2R}|dz\\ &\quad\left.+\int_{-1}^{t_0}|u_{x_0,2R}(t)|\int_B(|\nabla\hat{u}_{x_0,2R}|\varphi_{x_0,2R}\chi_{t_0,2R})|\hat{u}_{x_0,2R}|dz\right)\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

Next, we estimate the I_i's. For this we introduce $s\in]1,2[$ and obtain the following

$$\begin{aligned}I_2 &\leq \int_{t_0-(2R)^2}^{t_0}\left(\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{s}}\left(\int_{B(x_0,2R)}|F-[F]_{x_0,2R}|^{\frac{s}{2s-2}}dx\right)^{\frac{2s-2}{s}}dt\\ &\leq CR^{2(\frac{3}{s'}-1)}\int_{t_0-(2R)^2}^{t_0}\left(\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{s}}dt,\end{aligned}$$

with $C=C(s,\Gamma)>0$ (where we set for simplicity $\Gamma := \operatorname{ess\,sup}_{-1<t<0}\|F(\cdot,t)\|_{BMO(B)}$) and as usual $s'=s/(s-1)$. Similarly, we have

$$\begin{aligned}I_3 &\leq \frac{C(s)}{R}R^{\frac{3}{s'}}\operatorname*{ess\,sup}_{t_0-(2R)^2<t<t_0}\ \sup_{B(x_0,2R)\subset B}\left(\frac{1}{|B(2R)|}\int_{B(x_0,2R)}|F-[F]_{x_0,2R}|^{s'}dx\right)^{\frac{1}{s'}}\\ &\quad\times\int_{t_0-(2R)^2}^{t_0}\left(\int_{B(x_0,2R)}|\nabla\hat{u}_{x_0,2R}|^2\varphi^2_{x_0,2R}\chi^2_{t_0,2R}dx\right)^{\frac{1}{2}}\left(\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{2s}}dt\\ &\leq CR^{\frac{3}{s'}-1}\left(\int_{Q(z_0,2R)}|\nabla\hat{u}_{x_0,2R}|^2\varphi^2_{x_0,2R}\chi^2_{t_0,2R}dz\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{t_0-(2R)^2}^{t_0}\left(\int_{B(x_0,2R)}|\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{s}}dt\right)^{\frac{1}{2}},\end{aligned}$$

with $C = C(s, \Gamma) > 0$. Next, notice that $u_{x_0,2R}(t)$ can be rewritten as follows

$$u_{x_0,2R}(t) = -\frac{1}{R^3 \int_{B(2)} \varphi^2(x)dx} \int_{B(x_0,2R)} (F_{ik} - [F_{ik}]_{x_0,2R})(\varphi^2_{x_0,2R})_{,k} dx;$$

thus,

$$|u_{x_0,2R}(t)| \le \frac{C}{R},$$

for all $t \in [-1, 0]$ and $C = C(s, \varphi, \Gamma) > 0$. We get, as before,

$$I_4 \le C R^{\frac{3}{s'}-1} \left(\int_{Q(z_0,2R)} |\nabla \hat{u}_{x_0,2R}|^2 \varphi^2_{x_0,2R} \chi^2_{t_0,2R} dz \right)^{\frac{1}{2}} \times \left(\int_{t_0-(2R)^2}^{t_0} \left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}},$$

with $C = C(s, \varphi, \Gamma) > 0$.
Summarising our efforts, we have

$$\int_B |\hat{u}_{x_0,2R}(x, t_0)|^2 \varphi^2_{x_0,2R}(x) dx + \int_{-1}^{t_0} \int_B \chi^2_{x_0,2R} \varphi^2_{x_0,2R} |\nabla \hat{u}_{x_0,2R}|^2 dz \le C(s, \varphi, \Gamma) R^{2(\frac{3}{s'}-1)} \int_{t_0-(2R)^2}^{t_0} \left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt. \quad (6.5)$$

It is worth mentioning that a careful analysis of the constant in the previous inequality shows the following dependency in Γ:

$$C = c(s, \varphi)(1 + \Gamma^2).$$

From this point the rest of the proof follows line by line the proof of a similar result in [61]. We, nonetheless, present the proof here for the sake of completeness. Upon assuming $s \in]1, 3/2[$, one can find without difficulty numbers $0 < \lambda < 1$, $0 < \mu < 1$ and $1 < r < 2$ such that

$$\frac{2s}{2-s} = 2\lambda + \frac{3r}{3-r}\mu,$$
$$\lambda + \mu = 1,$$
$$\frac{3r}{3-r}\mu \frac{2-s}{s} = 1.$$

Using these numbers, we derive from (6.5) that

$$\begin{aligned}\int_{Q(z_0,R)} |\nabla \hat{u}_{x_0,2R}|^2 dz &\leq CR^{2(\frac{3}{s'}-1)} \int_{t_0-(2R)^2}^{t_0} \left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^{2\lambda+\frac{3r}{3-r}\mu} dx \right)^{\frac{2-s}{s}} dt \\ &\leq CR^{2(\frac{3}{s'}-1)} \int_{t_0-(2R)^2}^{t_0} \left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^2 dx \right)^{\frac{2-s}{s}\lambda} \\ &\quad \times \left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^{\frac{3r}{3-r}} dx \right)^{\frac{2-s}{s}\mu} dt.\end{aligned}$$

The last term can be estimated with the help of Sobolev's inequality

$$\begin{aligned}\int_{Q(z_0,R)} |\nabla \hat{u}_{x_0,2R}|^2 dz \leq CR^{2(\frac{3}{s'}-1)} &\operatorname*{ess\,sup}_{t_0-(2R)^2<t<t_0} \left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}(x,t)|^2 dx \right)^{\frac{1}{2}} \\ &\times R^{2\frac{r-1}{r}} \left(\int_{Q(z_0,2R)} |\nabla \hat{u}_{x_0,2R}|^r \right)^{\frac{1}{r}}, \quad (6.6)\end{aligned}$$

with $C = C(s, \varphi, \Gamma) > 0$. To estimate the first multiplier on the right-hand side of the last inequality, we proceed as follows. By the Poincaré-Sobolev inequality, we have that

$$\left(\int_{B(x_0,2R)} |\hat{u}_{x_0,2R}|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} \leq c(s) R^{3\frac{2-s}{s}-1} \int_{B(x_0,2R)} |\nabla \hat{u}_{x_0,2R}|^2 dx. \quad (6.7)$$

Combining this with (6.5), we get that

$$\int_B |\hat{u}_{x_0,2R}(x,t_0)|^2 \varphi^2_{x_0,2R}(x) dx \leq C \int_{Q(z_0,2R)} |\nabla \hat{u}_{x_0,2R}|^2 dx.$$

Assuming now that $Q(z_0, 3R) \subset Q$, we have the following estimate:

$$\operatorname*{ess\,sup}_{t_0-(2R)^2<t<t_0} \int_{B(x_0,2R)} |\hat{u}_{x_0,2R}(x,t)|^2 dx \leq C \int_{Q(z_0,3R)} |\nabla \hat{u}_{x_0,2R}|^2 dz, \quad (6.8)$$

where $C = C(s, \varphi, \Gamma) > 0$. We are now ready to estimate the fist multiplier on the right-hand side of (6.6). We apply (6.8) in the following way

$$\begin{aligned}\int_{B(x_0,2R)} |u(x,t) - u_{x_0,2R}(t)|^2 dx &\leq c \int_{B(x_0,2R)} |u(x,t) - u_{x_0,4R}(t)|^2 dx \\ &\leq C \int_{Q(z_0,3R)} |\nabla \hat{u}_{x_0,2R}|^2 dz,\end{aligned}$$

for almost every $t \in]t_0 - (2R)^2, t_0[$. Finally, (6.6) becomes

$$\begin{aligned}\frac{1}{|Q(R)|} \int_{Q(z_0,R)} |\nabla u|^2 dz \leq C &\left(\frac{1}{|Q(6R)|} \int_{Q(z_0,6R)} |\nabla u|^2 dz \right)^{\frac{1}{2}} \\ &\times \left(\frac{1}{|Q(2R)|} \int_{Q(z_0,2R)} |\nabla u|^r dz \right)^{\frac{1}{r}},\end{aligned}$$

which holds for some $r \in]1,2[$ and any $Q(z_0,6R) \subset Q$. As before, $C = C(s,\varphi,\Gamma) > 0$. This leads to (see Theorem 2.4.1) the existence of $p > 2$ such that $\nabla u \in L_p(Q(R))$, for any $R \in]0,1[$. Moreover, the following estimate is valid

$$\left(\frac{1}{|Q(R)|}\int_{Q(z_0,R)} |\nabla u|^p dz\right)^{\frac{1}{p}} \leq C\left(\frac{1}{|Q(6R)|}\int_{Q(z_0,6R)} |\nabla u|^2 dz\right)^{\frac{1}{2}}, \qquad (6.9)$$

for all $Q(z_0,6R) \subset Q$ with $6R <\mathrm{dist}(x_0,\partial B)$ and $t_0 - (6R)^2 > -1$. Moreover, the constant $C > 0$ depends only on Γ. This concludes the proof of Theorem 6.2.1. $\square$

We conclude this section by mentioning a consequence of Theorem 6.2.1. The latter gives us a refinement of the set of singular points of the solution u (this is systematically true as soon as one has a higher integrability of the gradient of u). Indeed, we have, thanks to Hölder's inequality, that

$$\frac{1}{r}\int_{Q(r)} |\nabla u|^2 dz \leq c(\delta)\left(\frac{1}{r^{1-2\delta}}\int_{Q(r)} |\nabla u|^{2+\delta} dz\right)^{\frac{2}{2+\delta}}; \qquad (6.10)$$

consequently, Theorem 4.3.1 guarantees that there exists a constant $\epsilon_1 = \epsilon_1(\delta) > 0$ such that if

$$\sup_{0<r<1} \frac{1}{r^{1-2\delta}}\int_{Q(r)} |\nabla u|^{2+\delta} dz < \epsilon_1,$$

then $z = 0$ is a regular point. From this, we derive in a similar way as what we did in Remark 18 in Chapter 4, that the $(1-2\delta)$-dimensional parabolic Hausdorff measure of the set of singular points of a suitable weak solutions $u \in L_\infty(-1,0;BMO^{-1}(B))$ in Q is null. If $\delta \geq 1/2$ then it is easy to see from (6.10) that $z = 0$ is a regular point for u. Unfortunately, we are not able to prove or disprove this statement for $\delta < 1/2$ at the moment.

6.3 A partial boundary regularity result

In order to state the next main result of this chapter, we need the following boundary analogue of suitable weak solutions.

Definition 6.3.1. We say that the function u is a *boundary suitable weak solution* to $\mathcal{L}_0 = 0$ in Q_+ if:

1. $u \in L_{2,\infty}(Q_+) \cap W_2^{1,0}(Q_+)$, and satisfies $\mathcal{L}_0 = 0$ in the sense of distributions in Q_+, with the following no-slip boundary condition

   $$u|_{x_3=0} = 0$$

   in the sense of the trace;

2. The following local energy inequality holds true

$$\int_{B_+} \phi |u(x,t)|^2 dx + 2\int_{-1}^{t}\int_{B_+} \phi |\nabla u|^2 dxds \leq \int_{-1}^{t}\int_{B_+} |u|^2(\partial_t \phi + \Delta\phi) dxds + \int_{-1}^{t}\int_{B_+} u\cdot\nabla\phi |u|^2 dxds,$$

for a.e $t \in]-1,0[$ with $\phi \in C_0^\infty(B\times]-1,1[)$.

The second main result of this chapter is as follows.

Theorem 6.3.1. *Let u be a boundary suitable weak solution to $\mathcal{L}_0 = 0$ in Q_+ such that $u \in L_\infty(Q_+)$. Then, we have*

$$u \in C^\infty(\overline{Q_+(a)}),$$

with $0 < a < 1$.

Before giving the proof of this result, let us point out that the higher regularity we obtained does not necessarily occur in the case of the 3D incompressible Navier-Stokes system for which a counter-example was constructed (see [62][1] where a bounded boundary suitable weak solution with an unbounded gradient was constructed).

Proof. Let $\chi \in C_0^\infty(]-1,1[)$ and $\phi \in C_0^\infty(B)$ be two cut-off functions such that $0 \leq \chi, \phi \leq 1$, $\chi \equiv 1$ in $]-(1/2)^2,(1/2)^2[$ and $\chi \equiv 0$ in $]-1,1[\setminus]-(3/4)^2,(3/4)^2[$. Similarly $\phi \equiv 1$ in $B(1/2)$ and $\phi \equiv 0$ in $B \setminus B(3/4)$. Now, let us set for simplicity $\psi(x,t) := \chi(t)\phi(x)$ and introduce the functions $v_i := u_i\psi$ and $F_i := -(u\cdot\nabla u_i + u_i \operatorname{div} u)\psi - (2\nabla u_i\cdot\nabla\psi + u_i\Delta\psi) + u_i\partial_t\psi$ (with $i = 1,2,3$). We have, at least in the sense of distributions in Q_+, that

$$\partial_t v_i - \Delta v_i = F_i,$$

with $F_i \in L_2(Q_+)$. We define

$$\bar{F}_i(x,t) := \begin{cases} F_i(x_1,x_2,x_3,t) \text{ in } \{x_3 > 0\} \\ -F_i(x_1,x_2,-x_3,t) \text{ in } \{x_3 \leq 0\} \end{cases}$$

the odd extension of F_i and consider the initial boundary value problem

$$\begin{cases} \partial_t \bar{v}_i - \Delta \bar{v}_i = \bar{F}_i \text{ in } Q \\ \bar{v}_i|_{\partial' Q} = 0. \end{cases}$$

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[1] Their work was motivated by K. Kang "Unbounded normal derivative for the Stokes system near boundary", Math. Ann., 331, 87–109 (2005).

Standard parabolic theory guarantees the existence of a unique solution $\bar{v}_i$ that satisfies the following estimate

$$\|\bar{v}_i\|_{W_2^{2,1}(Q)} \leq c\|\bar{F}_i\|_{L_2(Q)},$$

where c is an absolute positive constant. This uniqueness of $\bar{v}_i$ together with the parity of $\bar{F}_i$ ensure us that $\bar{v}_i$ is also odd with respect to the x_3 variable. From this, we deduce that

$$v_i = \bar{v}_i|_{\{x_3>0\}}, \tag{6.11}$$

which gives us $v_i \in W_2^{2,1}(Q_+)$ and by embedding $v_i \in W_{p_1}^{1,0}(Q_+)$ where

$$p_1 = \frac{10}{3} > 2;$$

thus we have that $F_i \in L_{p_1}(Q_+(1/2))$. Starting again the above machinery, but this time with $F_i \in L_{p_1}(Q_+(1/2))$ (the cut-off functions have to be changed in a suitable manner), and iterating, we end up with

$$F_i \in W_p^{1,0}(Q_+(a_p)),$$

with $0 < a_p < 1$ and for all $p \in [1, \infty[$. Going back, one more time, to the equation of $\bar{v}_i$, we have that

$$\partial_t \bar{v}_{i,j} - \Delta \bar{v}_{i,j} = \bar{F}_{i,j}$$

where $i, j = 1, 2, 3$. But we have, if we fix $p > 5/2$, that $\bar{F}_{i,j} \in L_p(Q(a))$, and by standard parabolic theory we obtain that $\bar{v}_{i,j} \in C^{\alpha,\alpha/2}(\overline{Q(a/2)})$ with $\alpha = 2 - 5/p > 0$. Next, using (6.11), we get that

$$v_{i,j} \in C^{\alpha,\alpha/2}(\overline{Q_+(a/2)}),$$

for $i, j = 1, 2, 3$ and $0 < a < 1$. Next, let us notice that by choosing appropriately the cut-off functions, we have now that $F_i \in C^{\alpha,\alpha/2}(\overline{Q_+(a/4)})$; and therefore, we get that (even if it means to change the domain on which we solve the equation of $\bar{v}_i$ into $Q(a/4)$):

$$\bar{v}_i \in C^{\alpha+2,\alpha/2+1}(\overline{Q(a/4)}),$$

thanks to Schauder estimates (see Theorem 2.4.3 in Chapter 2). Using once more (6.11), we obtain that

$$v_i \in C^{\alpha+2,\alpha/2+1}(\overline{Q_+(a/4)}) \quad \text{and } F_{i,j} \in C^{\alpha,\alpha/2}(\overline{Q_+(a/8)});$$

Repeating this process, we have that the theorem is proved. □

6.4 The radially symmetric case

The final main result of this chapter is concerned with the case where our suitable weak solution is radially symmetric. Let us point our that the divergence-free condition prevents such situation to occur in the case of the incompressible Navier-Stokes system. We are able to prove regularity of such solutions in that case for our models. This gives us, in contrast with the above regularity results, a geometric condition for obtaining regularity. It is also worth mentioning the work by Plecháč and Šverák in [54] where they were able to construct, in the whole space, a smooth radially symmetric solution to $\mathcal{L}_\kappa = 0$ (and $\mathcal{D}\mathcal{L}_\kappa = 0$) with a given, suitable decay at infinity (see the Introduction for a discussion concerning this result). Here, we consider the radial symmetry property only locally and still were able to conclude regularity; to be more precise, we have the following result.

Theorem 6.4.1. *Let u be a suitable weak solution to $\mathcal{L}_0 = 0$, which is moreover radially symmetric i.e.*

$$u(x,t) = -v(|x|,t)x.$$

Then, we have that

$$u \in C^\infty(\overline{Q(1/2)}).$$

Before continuing our development, let us point out that because of Theorem 4.3.1 i.e. our version of the Caffarelli-Kohn-Nirenberg theorem, we have that if u admits singular points, and those points should necessary belong to the set $\{0\}\times]-1,0[$; this is due to the fact that the 1-dimensional parabolic Hausdorff measure of the set of singular points of u in Q is equal to zero.

Assume now that $z_{t_0} = (0,t_0)$ (with $-1 < t_0 < 0$) is a singular point of u. Making use again of Theorem 4.3.1, we may construct, upon use of a space-time shift and using the natural scaling of $\mathcal{L}_0 = 0$ (i.e. $u^\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$ with $\lambda > 0$ is also a solution to $\mathcal{L}_0 = 0$ if u is) a function $\tilde{u}$ with the following properties:

1. $\tilde{u} \in L_{2,\infty}(Q) \cap W^{1,0}_2(Q)$ and obey $\mathcal{L}_0 = 0$ in the sense of distributions in Q;
2. $\tilde{u} \in L_\infty(B\times]-1,-a^2[)$ for all $a \in]0,1[$;
3. for all $r_1 \in]0,1[$ such that $\tilde{u} \in L_\infty(\{r_1 < |x| < 1\}\times]-1,0[)$;
4. $\tilde{u}(x,t) = -\tilde{v}(|x|,t)x$, and the origin $z = 0$ is a singular point of $\tilde{u}$.

To see the previous assertion, we proceed in the following manner. Because of the observations made in the previous paragraph, we have that there exists an $r > 0$ such that the first three points hold true in $Q(z_{t_0}, r)$. Defining now $\tilde{u}(x,t) = ru(rx, t_0 + r^2 t)$ (with $(x,t) \in Q$), we readily get that the new function satisfies all the above points.

Next, we have the following proposition which is the key step in proving Theorem 6.4.1.

Proposition 6.4.1. *The solution $\tilde{u}$ constructed above has the following property:*

$$\sup_{z \in \overline{Q(1/2)}} |x|^{2/3} |\tilde{u}(x,t)| < \infty.$$

A straightforward consequence of this proposition is that the origin is actually a regular point for $\tilde{u}$. Indeed, one can readily show that $\tilde{u} \in L_{3,\infty}(Q(1/2))$ which is a necessary condition for regularity (see Remark 22 in Chapter 4). The point $z = 0$ being a regular point of $\tilde{u}$ is therefore a contradiction. Consequently, we have that $z = (0, t_0)$ is a regular point and this concludes the proof of Theorem 6.4.1. The only thing left is to prove the proposition.

Proof. For simplicity, we drop in the sequel, the tilde symbol for u and v. We readily have the following equation for v:

$$v_t = v_{rr} + \frac{4}{r} v_r + \frac{3}{2} r v v_r + \frac{5}{2} v^2 \tag{6.12}$$

for $(r,t) \in]0,1[\times]-1,0[$. Let us introduce the function

$$v^{(1)}(r,t) = r^{1+2/3} v(r,t).$$

We have the following equation for $v^{(1)}$:

$$v^{(1)}_t - v^{(1)}_{rr} - \left(\frac{4}{3r} + \frac{3}{2r^{2/3}} v^{(1)}\right) v^{(1)}_r + \frac{20}{9r^2} v^{(1)} = 0, \tag{6.13}$$

for $(r,t) \in]0,1[\times]-1,0[$. Let $a \in]0,1/2[$ and $\varepsilon \in]0,1/2[$. Our goal now is to apply a weak maximum principle to (6.13) in $]\varepsilon, 1/2[\times]-1/4, -a^2[$; indeed, notice that v is smooth in the closure of $]\varepsilon, 1/2[\times]-1/4, -a^2[$. We have that

$$\max_{(r,t) \in [\varepsilon,1/2] \times [-1/4,-a^2]} |v^{(1)}(r,t)| = \max \left\{ \max_{\varepsilon \le r \le 1/2} |v^{(1)}(r, -\frac{1}{4})|, \max_{-1/4 \le t \le -a^2} |v^{(1)}(\frac{1}{2}, t)|, \max_{-1/4 \le t \le -a^2} |v^{(1)}(\varepsilon, t)| \right\}.$$

But from the second and third point of the properties we enumerated for $\tilde{u}$ above, and by noticing that $v^{(1)}(0,t) = 0$ for all $t \in]-1/4, -a^2[$, we deduce that there exists a finite positive constant $C = C(u)$ independent of a such that

$$\max_{(r,t)\in[0,1/2]\times[-1/4,-a^2]} |v^{(1)}(r,t)| \leq C,$$

for all $a \in]0, 1/2[$. Consequently, we have that

$$\max_{(r,t)\in[0,1/2]\times[-1/4,0]} |v^{(1)}(r,t)| \leq C, \tag{6.14}$$

which concludes the proof. □

Chapter 7

Liouville type theorems

7.1 Introduction

So far, we have been trying to understand the regular or singular behaviour of the system $\mathcal{L}_0 = 0$, essentially, by means of an energy inequality. We present in this chapter a different approach which, roughly speaking, requires us to take a much *closer* look at the (potential) singular points of our suitable weak solutions. We recall that a point z_0 is called a regular point of u (with u being a solution of our model) if there exists a parabolic cylinder, say $Q(z_0, r)$ such that u is Hölder continuous in $\overline{Q(z_0, r)}$; and z_0 is said to be a singular or blow-up point if it is not a regular point. We classify singular points into two classes, but before giving those, let us explain our motivation. We recall the following scale-invariant quantities:

$$\begin{gathered} A(v, z_0; r) := \operatorname*{ess\,sup}_{t_0 - r^2 < t < t_0} \frac{1}{r} \int_{B(x_0, r)} |v(x,t)|^2 dx, \\ C(v, z_0; r) := \frac{1}{r^2} \int_{Q(z_0,r)} |v(x,t)|^3 dz, \; E(v, z_0; r) := \frac{1}{r} \int_{Q(z_0,r)} |\nabla v(x,t)|^2 dz, \end{gathered} \tag{7.1}$$

with $z_0 = (x_0, t_0)$. We showed that upon assuming some smallness condition on the quantities A, C and E, one gets regularity for our system. Assume now that the smallness assumption is replaced by finiteness only; the question is: **"can we still say something about the regularity of our solution?"**. We showed in Chapter 4 that the answer is still *yes* for the Ladyzhenskaya-Prodi-Serrin quantities but we cannot, at this point in time, say the same or the opposite for the quantities (7.1). Therefore we cannot exclude the possibility that one can have a singularity at a point z_0 with one of the above quantities (7.1) being finite (upon taking a supremum). In that case, we call z_0 a *type I singularity or blow-up*. Any other singularity is said to be of *type II*.

7.2 Some properties of Type-I blow-ups

A key observation about type I singularities is recorded in the following Proposition.

Proposition 7.2.1. *Let v be a suitable weak solution to $\mathcal{L}_0 = 0$ in Q. If*

$$\min\{\sup_{0<r<1} A(r), \sup_{0<r<1} C(r), \sup_{0<r<1} E(r)\} < \infty,$$

then

$$\sup_{0<r<1} \{A(r) + C(r) + E(r)\} < \infty;$$

Here we use the abbreviations $A(r) = A(v, 0; r)$ etc.

We divide the proof into the following three lemmata.

Lemma 7.2.1. *Assume that we are given a suitable weak solution v in Q to our system $\mathcal{L}_0 = 0$. Let, in addition,*

$$\sup_{0<r\leq 1} E(r) =: E_0 < \infty. \tag{7.2}$$

Then, there exists a positive constant d depending only on E_0 such that

$$A^{3/2}(r) + C(r) \leq d\left(r^{1/2}A^{3/2}(1) + 1\right), \tag{7.3}$$

for all $0 < r \leq 1/4$.

Lemma 7.2.2. *Assume that we are given a suitable weak solution v in Q to our system $\mathcal{L}_0 = 0$. Let, in addition,*

$$\sup_{0<r\leq 1} C(r) =: C_0 < \infty. \tag{7.4}$$

Then, there exists a universal positive constant c such that

$$A(r) + E(r) \leq c\left(C_0 + C_0^{2/3}\right), \tag{7.5}$$

for all $0 < r \leq 1/2$.

Lemma 7.2.3. *Assume that we are given a suitable weak solution v in Q to our system $\mathcal{L}_0 = 0$. Let, in addition,*

$$\sup_{0<r\leq 1} A(r) =: A_0 < \infty. \tag{7.6}$$

Then, there exists a positive constant e depending only on A_0 such that

$$C^{3/4}(r) + E(r) \leq e\left(r^2E(1) + 1\right), \tag{7.7}$$

for all $0 < r \leq 1/2$.

Proof of Lemma 7.2.1. The proof of Lemma 7.2.1 uses, up to a slight modification, similar arguments to the ones we used in the proof of Theorem 4.3.1; we only sketch the proof here. The following analogue of (4.26) is available:

$$\mathcal{E}(\vartheta\varrho) \leq c\left[\vartheta^{3/4}(E_0^{3/4}+1)+\delta\right]\mathcal{E}(\varrho)+c(\delta)\vartheta^{-9}(E_0^{3/2}+E_0^{9/2}) \tag{7.8}$$

for any $0 < \vartheta \leq 1/2$ and any $0 < \varrho \leq 1$ (with $\mathcal{E} = A^{3/2}$). Fix now ϑ and δ such that

$$c\vartheta^{1/4}(E_0^{3/4}+1) < 1/2 \quad 0 < \vartheta \leq 1/2 \quad \text{and} \quad c\delta < \vartheta^{1/2}/2.$$

Now, we have

$$\mathcal{E}(\vartheta\varrho) \leq \vartheta^{1/2}\mathcal{E}(\varrho) + G(E_0).$$

Using standard iteration arguments, we obtain

$$\mathcal{E}(r) \leq r^{1/2}\mathcal{E}(1) + G(E_0),$$

for all $0 < r \leq 1/2$. Together with the latter, (4.23) gives us

$$C(r) \leq c[G(E_0) + r^{1/2}\mathcal{E}(1)];$$

and this concludes the proof. □

Proof of Lemma 7.2.2. By choosing a suitable cut-off function in the local energy inequality for v, we obtain

$$A(r/2) + E(r/2) \leq c\left[\frac{1}{r^3}\int_{Q(r)}|v|^2dz + \frac{1}{r^2}\int_{Q(r)}|v|^3dz\right]. \tag{7.9}$$

Next, notice that

$$\frac{1}{r^3}\int_{Q(r)}|v|^2dz \leq cC^{2/3}(r) \leq cC_0^{2/3},$$

which directly leads to (7.5). □

Proof of Lemma 7.2.3. Thanks to Lemma 4.3.2, we have that

$$C(r) \leq c\left[A^{3/4}(r)E^{3/4}(r) + A^{3/2}(r)\right] \leq c\left[A_0^{3/4}E^{3/4}(r) + A_0^{3/2}\right]. \tag{7.10}$$

Next, similarly to (7.9), we have that

$$\begin{aligned} E(r) &\leq c\left[C^{2/3}(2r) + C(2r)\right] \\ &\leq \left[\left(\frac{\varrho}{r}\right)^2 C(\varrho) + \left(\frac{\varrho}{r}\right)^{4/3} C^{2/3}(\varrho)\right], \end{aligned} \tag{7.11}$$

for any $0 < r \le \varrho/2 < \varrho \le 1$. To prove (7.11), the inequality $C(r) \le (\varrho/r)^2 C(\varrho)$, $0 < r \le \varrho$ has been used. From inequalities (7.10) and (7.11), we have that

$$E(r) \le c\left[\left(\frac{\varrho}{r}\right)^2\left(A_0^{3/4}E^{3/4}(\varrho) + A_0^{3/2}\right) + \left(\frac{\varrho}{r}\right)^{4/3}\left(A_0^{1/2}E^{1/2}(\varrho) + A_0\right)\right].$$

Next, we would like to use the fact that the power of $E(\varrho)$, in the inequality above, is less than one; Young's inequality is used with an arbitrary positive constant δ and we find

$$E(r) \le \delta E(\varrho) + C(\delta)\left(\frac{\varrho}{r}\right)^8\left(A_0^3 + A_0^{3/2} + A_0\right),$$

for any $0 < r \le \varrho/2 < \varrho \le 1$. By standard iteration arguments, we obtain

$$E(r) \le cr^2 E(1) + c\left(A_0^3 + A_0^{3/2} + A_0\right),$$

for any $0 < r \le 1/2$. Next, we can derive from (7.10) that

$$C^{4/3}(r) \le c\left(A_0 E(r) + A_0^2\right) \le e_1(A_0)(r^2 E(1) + 1),$$

which concludes the proof. $\square$

Proposition 7.2.1 is proved as well, thanks to the previous lemmata.

Another important class of scale-invariant quantities is

$$\begin{aligned} G_1(v, z_0; r) &= \sup_{z=(x,t)\in Q(z_0,r)} |x - x_0||v(z)|, \\ G_2(v, z_0; r) &= \sup_{z=(x,t)\in Q(z_0,r)} \sqrt{t_0 - t}|v(z)|. \end{aligned} \tag{7.12}$$

As usual the abbreviations $G_1(r) = G_1(v, 0; r)$ and $G_2(r) = G_2(v, 0; r)$ are used. Similarly to what we had for the quantities (7.1), we have the following result.

Proposition 7.2.2. *Let v be a suitable weak solution to $\mathcal{L}_0 = 0$ in Q.*

1. *(ε-regularity) There exists a positive constant ϵ such that if*

   $$\min\{G_1(1), G_2(1)\} < \epsilon,$$

   then $z = 0$ is a regular point.

2. *If $z = 0$ is a singular point such that $\min\{G_1(1), G_2(1)\} < \infty$ holds, then it is a type I singularity; in other words*

   $$\min\{G_1(1), G_2(1)\} < \infty \Longrightarrow \sup_{0<r<1}\{A(r) + C(r) + E(r)\} < \infty.$$

Proof. We divide the proof into two steps; the first one addresses the first point of the proposition while the second step addresses the second point.

Step 1. Since

$$\operatorname*{ess\,sup}_{-1<t<0} \|v(\cdot,t)\|_{L^{3,\infty}(B)} \leq c \sup_{(x,t)\in Q} |x||v(z)|,$$

we have, thanks to Proposition 4.4.3, that all we need to prove is

$$\sup_{(x,t)\in Q} \sqrt{-t}|v(x,t)| < \epsilon \Longrightarrow z=0 \text{ is a regular point;}$$

To achieve this, we are going to prove the following estimate:

$$\frac{1}{r} \operatorname*{ess\,sup}_{-(\frac{r^2}{2})<t<0} \int_{B(\frac{r}{2})} |v(x,t)|^2 dx + \frac{1}{r} \int_{Q(\frac{r}{2})} |\nabla v(x,t)|^2 dz \leq c\left(G_2(1)^2 + G_2(1)^6\right), \quad (7.13)$$

for $c>0$ an absolute constant and for all $0<r<1$.
We introduce the following cut-off functions: $0 \leq \chi_{\varrho,r} \in C_0^\infty(]-r^2,r^2[)$ and $0 \leq \varphi_{\varrho,r} \in C_0^\infty(B(x_0,r))$ (with $0<\varrho<r<1$) such that $\chi_{\varrho,r}, \varphi_{\varrho,r} \leq 1$, $\chi_{\varrho,r} \equiv 1$ in $]-\varrho^2,\varrho^2[$, $\varphi_{\varrho,r} \equiv 1$ in $B(x_0,\varrho)$, $|\nabla^k \varphi_{\varrho,r}| \leq c/(r-\varrho)^k$ (with $k=1,2$) and $|\chi_{\varrho,r}^{(l)}| \leq c/(r^2-\varrho^2)^l$ (with $l=0,1$). Now, we take $\phi_{\varrho,r}^2$ (with $\phi_{\varrho,r} := \chi_{\varrho,r}\varphi_{\varrho,r}$) as a test function in the local energy inequality satisfied by v to obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{-r^2<t<0} \int_{B(r)} \phi_{\varrho,r}^2(x,t)|v(x,t)|^2 dx + \int_{Q(r)} \phi_{\varrho,r}^2 |\nabla v|^2 dz \leq \frac{1}{2} \int_Q |v|^2 (\partial_t \phi_{\varrho,r}^2 + \Delta \phi_{\varrho,r}) dz \\ + \frac{1}{2} \int_Q |v|^2 v \cdot \nabla \phi_{\varrho,r}^2. \quad (7.14)\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{2}\int_Q |v|^2 \partial_t \phi_{\varrho,r}^2 dz &\leq c\frac{G_2(1)}{r^2-\varrho^2} \int_{-r^2}^{-\varrho^2} \frac{1}{\sqrt{-t}} \left(\int_B |v| \phi_{\varrho,r} dx \right) dt \\ &\leq c\frac{G_2(1)}{r-\varrho} r^{3/2} \left(\sup_{-r^2<t<0} \int_{B(r)} \phi_{\varrho,r}^2(x,t)|v(x,t)|^2 dx \right)^{1/2} \\ &\leq \frac{1}{8} \sup_{-r^2<t<0} \int_{B(r)} \phi_{\varrho,r}^2(x,t)|v(x,t)|^2 dx + c\frac{r^3}{(r-\varrho)^2} G_2(1)^2.\end{aligned}$$

Next,

$$\begin{aligned}\frac{1}{2}\int_Q |v|^2 \Delta \phi_{\varrho,r}^2 dz &= \int_Q |v|^2 \phi_{\varrho,r} \left(\Delta \phi_{\varrho,r} + |\nabla (\phi_{\varrho,r}^{1/2})|^2 \right) dz \\ &\leq c\frac{G_2(1)}{(r-\varrho)^2} \int_{-r^2}^0 \frac{1}{\sqrt{-t}} \left(\int_B |v| \phi_{\varrho,r} dx \right) dt \\ &\leq \frac{1}{8} \sup_{-r^2<t<0} \int_{B(r)} \phi_{\varrho,r}^2(x,t)|v(x,t)|^2 dx + c\frac{r^5}{(r-\varrho)^4} G_2(1)^2.\end{aligned}$$

Finally, we have

$$\begin{aligned}\frac{1}{2}\int_Q |v|^2 v\cdot\nabla\phi_{\varrho,r}^2 &= 2\int_Q |\phi_{\varrho,r}v|^{3/2}(v|v|^{1/2})\cdot\nabla(\phi_{\varrho,r}^{1/2})\\ &\le c\frac{G_2(1)^{3/2}}{r-\varrho}\int_{-r^2}^0 \frac{1}{(-t)^{3/4}}\left(\int_B |\phi_{\varrho,r}v|^{3/2}dx\right)dt\\ &\le \frac{1}{4}\sup_{-r^2<t<0}\int_{B(r)}\phi_{\varrho,r}^2(x,t)|v(x,t)|^2dx + c\frac{r^5}{(r-\varrho)^4}G_2(1)^6.\end{aligned}$$

Consequently, we get from (7.14) that

$$\begin{aligned}\operatorname*{ess\,sup}_{-\varrho^2<t<0}\int_{B(\varrho)}|v(x,t)|^2dx + \int_{Q(\varrho)}|\nabla v(x,t)|^2dz &\le c\frac{r^3}{(r-\varrho)^2}G_2(1)^2\\ &\quad + c\frac{r^5}{(r-\varrho)^4}(G_2(1)^2+G_2(1)^6);\end{aligned}$$

choosing $\varrho = r/2$, we have that (7.13) is proved. Thanks to Theorem 4.3.1, we have that the first point is proved.

Step 2. Once again, thanks to Proposition 4.4.3, estimate (7.13) above and Proposition 7.2.1, we have that the second point is proved. □

Remark 28. It is not clear at this point of time whether or not the reverse sense of the implication in the second point of the previous proposition is true, even in the case of the Navier-Stokes system.

7.3 The blow-up technique and Liouville type theorems

A powerful tool used in the study of (potential) singular points of the solution of a PDE is the *blow-up technique*. Roughly speaking, it consists of "blowing-up" the solution in the vicinity of the singular point and observing what happens in the limit. We give a simple, nonetheless rigorous, illustration of the previous statement, for our model. Let u be a suitable weak solution to $\mathcal{L}_0 = 0$ in $Q_+ = \mathbb{R}^3\times]0,+\infty[$ and we denote by T the first time of appearance of singularities in u.
We set

$$M(t) := \sup_{0<\tau\le t}\|u(\cdot,\tau)\|_{\infty,\mathbb{R}^3},$$

for $t\in]0,T[$. By definition of T, we have that

$$M(t)\to+\infty,$$

whenever $t \to T^-$.

Next, we can construct a sequence t_k such that $t_k \in]0, T[$, $t_k < t_{k+1}$, $t_k \to T$, and

$$(+\infty >) M_k := M(t_k) = |u(x_k, t_k)| \to +\infty,$$

as $k \to \infty$, with $x_k \in \mathbb{R}^3$. We define

$$u^k(y, s) = \lambda_k u(x, t), \tag{7.15}$$

where $x = x^k + \lambda_k y$, $t = t_k + \lambda_k^2 s$, and $\lambda_k = 1/M_k$. The scaled function u^k is also a suitable weak solution of our model (in $\mathbb{R}^3 \times] - \frac{t_k}{\lambda_k^2}, \frac{T - t_k}{\lambda_k^2}[$) and we have

$$|u^k| \leq 1 \quad \text{in } \mathbb{R}^3 \times] - \frac{t_k}{\lambda_k^2}, 0[, \tag{7.16}$$

$$|u^k(0, 0)| = 1. \tag{7.17}$$

The main theorem of this section reads as follows.

Theorem 7.3.1. *For all $a > 0$, the sequence (u^k) is uniformly Hölder continuous in the closure of $Q(a) = B(a) \times] - a^2, 0[$, and there exists a subsequence (u^{k_j}) of (u^k), which converges uniformly on each compact subsets of $\mathbb{R}^3 \times] - \infty, 0]$ to a vector field v with the following properties: v is a distributional solution of $\mathcal{L}_0 = 0$ in $Q_-(:= \mathbb{R}^3 \times] - \infty, 0[)$, $|v| \leq 1$ in Q_-, $\nabla v \in L_{2,loc}(Q_-)$ and $|v(0, 0)| = 1$.*

Proof. Let $a > 0$; there exists a positive number k_a such that $Q(a) \subset Q(2a) \subset \mathbb{R}^3 \times] - \frac{t_k}{\lambda_k^2}, 0[$, for all $k > k_a$. We have by standard arguments (we make use of the local energy inequality for the u_k's, (7.16), and a bootstrap argument) that

$$\|u^k\|_{2,\infty,Q(a)} + \|\nabla u^k\|_{2,Q(a)} \leq c(a), \tag{7.18}$$

$$\|u^k\|_{C^{0,\alpha}(\overline{Q(a)})} \leq c(a), \tag{7.19}$$

with $0 < \alpha \leq 1/2$. Hence, using the Ascoli-Arzela theorem together with the Cantor diagonal process, we obtain a subsequence, still denoted by u^k and a vector field v such that for any $a > 0$

$$\begin{aligned} u^k &\to v \quad \text{in } C^{0,\alpha}(\overline{Q(a)}), \\ \nabla u^k &\rightharpoonup \nabla v \quad \text{in } L_{2,loc}(Q_-). \end{aligned} \tag{7.20}$$

Since the u_k are suitable weak solutions to $\mathcal{L}_0 = 0$, we have thanks to the above limits that v is also a weak solution to our toy-model in the sense that: $v \in L_{2,\infty,loc}(Q_-) \cap W^{0,1}_{2,loc}(Q_-)$ and

$$\int_{Q_-} \left(v.\partial_t \phi + v \otimes v : \nabla \phi - \nabla v : \nabla \phi + \frac{1}{2} (\operatorname{div} v) v.\phi \right) dx dt = 0,$$

for all $\phi \in C_0^\infty(Q_-)$. Moreover, in view of (7.16) and (7.17), we have that $|v(z)| \leq 1$ for all $z \in Q_-$ and $|v(0,0)| = 1$. Therefore the proof is complete. $\square$

Remark 29. Let us notice that the vector field v constructed in the previous theorem is smooth and the following estimate is valid:

$$\sup_{(x,t)\in Q_-} |\partial_t^k \nabla^l v(x,t)| \leq c(k,l) < \infty$$

for any $k, l = 0, 1, \dots$

This is a consequence of the fact that v is also a suitable weak solution to $\mathcal{L}_0 = 0$, which in turn allows us to apply the ϵ-regularity theory developed in Chapter 4. Indeed, we have

$$\left(R^3 \fint_{Q(z_0,R)} |v|^3 dz \right)^{\frac{1}{3}} \leq R < \epsilon_1,$$

for $R > 0$ chosen so that $R = \epsilon_1/2$, where ϵ_1 is given in Theorem 4.2.5. Then, there are positive constants c_k with $k = 1, 2, \dots$ such that

$$|\nabla^k v(z_0)| \leq \frac{c_k}{R^{k+1}}$$

for any $z_0 \in Q_-$, by bootstrap arguments.

We introduce for simplicity the following terminology.

Definition 7.3.1. We say that u is a bounded ancient solution of $\mathcal{L}_0 = 0$ in Q_- if

$$\nabla u \in L_{2,loc}(Q_-) \quad \text{and} \quad u \in L_\infty(Q_-),$$

and it is a weak solution of $\mathcal{L}_0 = 0$ in Q_-; the previous remark shows that such a solution is actually smooth.

We can always construct a non-trivial bounded ancient solution of $\mathcal{L}_0 = 0$ in Q_-. Indeed, take

$$u(x,t) = \frac{4}{3} \times \frac{\mathrm{e}^{-x+t}}{1 + \mathrm{e}^{-x+t}},$$

which is a solution to

$$\partial_t u - u_{xx} + \frac{3}{2} u u_x = 0 \quad \text{in } \mathbb{R}\times]-\infty, 0[.$$

Now, define

$$U(x,t) = u(x_1, t) e_1$$

where $e_1 = (1, 0 \dots, 0) \in \mathbb{R}^n$; here $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t \in]-\infty, 0[$. This gives us the required solution. This suggests adding some additional information to the bounded ancient solutions in order to get a better insight into the regular or singular behaviour of our model.

We will focus here on type I blow-ups only. First, notice that if we assumed that the solution u introduced at the beginning of this section admits only type I singularities, then we will have that the limit solution v has the following additional property

$$\sup_{Q(z_0,r)\subset Q_-} \{A(v,z_0;r)+C(v,z_0;r)+E(v,z_0;r)\}<\infty. \tag{7.21}$$

The following natural question arise: **If v is a bounded ancient solution to $\mathcal{L}_0=0$ in $\mathbf{Q}_-$ such that** (7.21) **holds, do we have that v is constant?**
If one were to give a positive answer to this question, then in particular the limit v obtained in Theorem 7.3.1 would be constant. However, (7.21) would imply that this constant is zero; in other words $v\equiv 0$ in Q_-. But this contradicts the fact that $|v(0,0)|=1$. Consequently, this excludes any occurrence of type I singularities in (suitable) weak solutions of system $\mathcal{L}_0=0$.
In case one gives a negative answer to this question i.e. one finds a non-trivial bounded ancient solution v for which (7.21) holds, it is still possible to say something. Indeed, we have the following proposition, which was initially proved in [1] for the incompressible Navier-Stokes system.

Proposition 7.3.1. *Assume that there exists a non-trivial bounded ancient solution v for which* (7.21) *holds. Then, we can construct a suitable weak solution u in Q for which the point $z=0$ is a type I singularity. (Theorem 7.3.1 shows that those two statements are actually equivalent).*

Proof. Let v be a non-trivial bounded ancient solution of $\mathcal{L}_0=0$ satisfying (7.21). We have

$$\|v\|_{L_\infty(Q)}=:N>0; \tag{7.22}$$

We introduce the sequence $(v^{(k)})_{k\in\mathbb{N}}$ of suitable weak solutions

$$v^{(k)}=kv(kx,k^2t),\quad (x,t)\in Q(2). \tag{7.23}$$

Notice that, because of (7.21), we have that

$$\sup_{k\in\mathbb{N}}\sup_{Q(z_0,r)\subset Q(2)}\left\{A(v^{(k)},z_0;r)+C(v^{(k)},z_0;r)+E(v^{(k)},z_0;r)\right\}<\infty. \tag{7.24}$$

This allows us, by standard arguments, to find a suitable weak solution u in Q to our model, such that up to a subsequence still denoted $(v^{(k)})$

$$\nabla v^{(k)}\rightharpoonup\nabla u \text{ in } L_2(Q),\quad v^{(k)}\to u \text{ in } L_3(Q).$$

Moreover, from (7.22) and (7.23), we have

$$\|v^{(k)}\|_{L_\infty(Q(1/k))} = kN \to \infty. \tag{7.25}$$

The former guarantees that

$$\sup_{Q(z_0,r)\subset Q} \{A(u,z_0;r) + C(u,z_0;r) + E(u,z_0;r)\} < \infty.$$

We claim that the latter implies that $z = 0$ is a singular point of u. To see this, assume that it is not the case, i.e. $z = 0$ is a regular point. Thus there exists $0 < R < 1$ such that $u \in L_\infty(Q(R))$. Next, we have that

$$\begin{aligned}\left(r^3 \fint_{Q(r)} |u|^3 dz\right)^{\frac{1}{3}} &\le r\|u\|_{L_\infty(Q(R))} \\ &\le \epsilon_1/2, \quad \text{for } 0 < r < \min(R, \epsilon_1 2^{-1}\|u\|^{-1}_{L_\infty(Q(R))}).\end{aligned}$$

From the above strong convergence in $L_3(Q)$, we get that

$$\left(r^3 \fint_{Q(r)} |v^{(k)}|^3 dz\right) < \epsilon_1,$$

for k large enough and $0 < r < \min(R, \epsilon_1 2^{-1}\|u\|^{-1}_{L_\infty(Q(R))})$. This implies, thanks to Theorem 4.2.5, that there exists $0 < \varrho < r$ such that

$$\|v^{(k)}\|_{L_\infty(Q(\varrho))} \le c(\varrho),$$

for k large enough. This contradicts (7.25) and the claim is proved and so is the proposition. □

We conclude this section with a Liouville type result for system $\mathcal{L}_0 = 0$. Before stating it, we need to introduce the following cylindrical coordinates r, θ, x_3 which are such that $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $x_3 = x_3$. The velocity components in this new system of coordinates are going to be u_r, u_θ and u_3. We aim to study system $\mathcal{L}_0 = 0$ in the axially symmetric setting which means that:

$$u_{r,\theta} := \partial u_r/\partial\theta = 0, \quad u_{\theta,\theta} = 0, \quad u_{3,\theta} = 0.$$

Thus, our toy-model in the axially symmetric case is as follows:

$$\begin{cases} \partial_t u_r - (\Delta_{ax} u_r - \frac{u_r}{r^2}) + (u_r u_{r,r} + u_3 u_{r,3} - \frac{u_\theta^2}{r}) + \frac{1}{2}(\frac{u_r}{r} + u_{r,r} + u_{3,3})u_r = 0 \\ \partial_t u_\theta - (\Delta_{ax} u_\theta - \frac{u_\theta}{r^2}) + (u_r u_{\theta,r} + u_3 u_{\theta,3} - \frac{u_r u_\theta}{r}) + \frac{1}{2}(\frac{u_r}{r} + u_{r,r} + u_{3,3})u_\theta = 0 \\ \partial_t u_3 - \Delta_{ax} u_3 + (u_r u_{3,r} + u_3 u_{3,3}) + \frac{1}{2}(\frac{u_r}{r} + u_{r,r} + u_{3,3})u_3 = 0 \end{cases}$$

with

$$\Delta_{ax} f := \frac{1}{r}(r f_r)_{,r} + f_{,33}.$$

We are now able to state the last theorem of this section.

Theorem 7.3.2. *Let u be an arbitrary axially symmetric bounded ancient solution to $\mathcal{L}_0 = 0$ with zero swirl i.e $u_\theta = 0$, with null u_3 component and no dependence with respect to the x_3 variable; this boils down to having a single equation for the u_r component:*

$$\partial_t u_r - \frac{1}{r}(r u_{r,r})_{,r} + \frac{u_r}{r^2} + \frac{3}{2} u_r u_{r,r} + \frac{1}{2}\frac{u_r^2}{r} = 0 \quad in\ ]0,+\infty[\times]-\infty,0]. \tag{7.26}$$

We add the following additional assumption (which is still in the spirit of the type I blow-ups analysis)

$$\sup_{(r,t)\in\mathbb{R}_+\times]-\infty,0[} (r+\sqrt{-t})|u_r(r,t)| < \infty. \tag{7.27}$$

Then $u \equiv 0$.

Before giving the proof of this theorem, let us mention that condition (7.27) can be relaxed thanks to the following proposition, which we state here without proof; the proof is essentially a repetition, with no original differences with arguments used in [63] for a similar result in the case the incompressible Navier-Stokes equations.

Proposition 7.3.2. *Let v be an axially symmetric suitable weak solution in Q such that*

$$\sup_{(x,t)\in Q} \sqrt{-t}|v(x,t)| < \infty.$$

Then

$$\sup_{(x,t)\in Q(1/8)} |x'||v(x,t)| < \infty;$$

here $x = (x', x_3)$ and $r = |x'|$.

Proof of Theorem 7.3.2. The idea of the proof is the same as in the proof of Theorem 6.4.1. Set $v(r,t) := r^{1/3}u_r(r,t)$; from (7.26), we get the following equation for v:

$$\partial_t v - v_{,rr} + \left(\frac{1}{3r} + \frac{3}{2}\frac{v}{r^{1/3}}\right)v_{,r} + \frac{2}{3}\frac{v}{r^2} = 0.$$

By applying a weak maximum principle to the previous equation in $[\epsilon,\epsilon^{-1}]\times[-R^2,0]$, we get that

$$\max_{(r,t)\in[\epsilon,\epsilon^{-1}]\times[-R^2,0]} |v(r,t)| \le \max\left\{ \max_{-R^2\le t\le 0} |v(\epsilon,t)|, \max_{-R^2\le t\le 0} |v(\epsilon^{-1},t)|, \right. \\ \left. \max_{r\in[\epsilon,\epsilon^{-1}]} |v(r,-R^2)| \right\}. \tag{7.28}$$

The goal now is to pass to the limit $\epsilon \to 0^+$ and $R \to \infty$ successively in (7.28). We have, thanks to (7.27) that

$$|v(r,t)| \leq \frac{M}{r^{2/3}}, \quad \forall (r,t) \in \mathbb{R}_+ \times]-\infty, 0[,$$

with $M := \sup_{(r,t)\in\mathbb{R}_+\times]-\infty,0[}(r+\sqrt{-t})|u_r(r,t)|$; thus

$$\max_{-R^2\leq t\leq 0} |v(\epsilon^{-1},t)| \leq c\epsilon^{2/3} \to 0, \quad \text{as } \epsilon \to 0^+. \tag{7.29}$$

Next, we have that

$$\max_{-R^2\leq t\leq 0} |v(\epsilon,t)| \leq \epsilon^{1/3} \max_{-R^2\leq t\leq 0} |u_r(\epsilon,t)| \leq \epsilon^{1/3}\|u_r\|_\infty \to 0, \quad \text{as } \epsilon \to 0^+. \tag{7.30}$$

From (7.28)-(7.30), we have that

$$\sup_{r\geq 0, -R^2\leq t\leq 0} |v(r,t)| \leq \sup_{r\geq 0} |v(r,-R^2)|. \tag{7.31}$$

It is quite straightforward to prove, thanks to (7.27), that

$$\sup_{r\geq 0} |v(r,t)| \leq \frac{2^{2/3}}{3}\frac{M}{(-t)^{1/3}}; \tag{7.32}$$

this together with (7.31) gives us

$$\sup_{r\geq 0, -R^2\leq t\leq 0} |v(r,t)| \leq \frac{2^{2/3}}{3}\frac{M}{R^{2/3}} \to 0, \quad \text{as } R \to \infty.$$

Therefore $v \equiv 0$ and consequently $u_r \equiv 0$; and the theorem is proved. □

7.4 Self-similar blow-up

We call (Leray) backward self-similar solution to the toy-model $\mathcal{L}_0 = 0$ in $\mathbb{R}^3\times]0,T[$, solutions u of the form

$$u(x,t) = \lambda(t)U(\lambda(t)x), \text{ with } \lambda(t) := \frac{1}{\sqrt{2a(T-t)}}, \tag{7.33}$$

where $a > 0$ and the profile $U = (U_1, U_2, U_3)$ is defined in $\mathbb{R}^3$. If u satisfies $\mathcal{L}_0 = 0$, then we find the following equation for the profile U

$$-\Delta U + U\cdot\nabla U + \frac{1}{2}U \operatorname{div} U + ay\cdot\nabla U + aU = 0 \quad \text{in } \mathbb{R}^3. \tag{7.34}$$

Like in the case of the incompressible Navier-Stokes system, a non-zero solution to (7.34) should produce a solution, to the system $\mathcal{L}_0 = 0$, that blows up at finite time;

moreover, this blow-up is of type I. Consequently, this would also give a negative answer to the question of global well-posedness for system $\mathcal{L}_0 = 0$; but of course, we have to specify the appropriate solution class for u for this statement to make complete sense.

Our first result in this direction shows that the only self-similar solution in $\mathbb{R}^3\times]0,T[$ that satisfies a global energy inequality is zero. Such a result was first obtained in [53] for the 3D incompressible Navier-Stokes system, by a crucial use of the maximum principle applied to an analogue of the Bernoulli pressure. Here, it is not clear whether or not we can have such a maximum principle for a certain conserved quantity.

Theorem 7.4.1. *Let u be a self-similar solution to $\mathcal{L}_0 = 0$ in $\mathbb{R}^3\times]0,T[$ such that*

$$\operatorname*{ess\,sup}_{t\in]0,T[}\|u(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + \int_0^T\int_{\mathbb{R}^3}\|\nabla u\|^2_{L_2(\mathbb{R}^3)} < \infty. \tag{7.35}$$

Then, $u \equiv 0$.

Before giving the proof of Theorem 7.4.1, let us record some crucial properties of the profile U. We restrict ourselves, without loss of generality, to a local backward self-similar solution u in $Q := B\times]-1,0[$ and we take $a = 1/2$. More precisely, we have that u is a weak solution to $\mathcal{L}_0 = 0$ in Q, such that $u(x,t) = \lambda(t)U(\lambda(t)x)$ (for all $(x,t) \in Q$) with

$$\lambda(t) := \frac{1}{\sqrt{-t}}.$$

Proposition 7.4.1. *Assume u as before and that it is in addition a suitable weak solution in Q. Then, $U \in C^\infty(\mathbb{R}^3)$ and we have the following decay $U = O(|y|^{-1})$ as $|y| \to \infty$.*

Proof. Let us start by noticing that for u belonging to the energy class $L_{2,\infty}(Q) \cap L_2(-1,0;L_2(B))$, we have that $U \in W^1_{2,loc}(\mathbb{R}^3)$. We only show that $\nabla U \in L_{2,loc}(\mathbb{R}^3)$ since from that we can conclude that $U \in L_{2,loc}(\mathbb{R}^3)$

$$\begin{aligned}\int_{-1}^0 \|\nabla u(\cdot,t)\|^2_{2,B}dt &= \int_{-1}^0 \lambda(t)\|\nabla U\|^2_{2,B(\lambda(t))}dt\\ &= \int_{-1}^{-1/R^2} \lambda(t)\|\nabla U\|^2_{2,B(\lambda(t))}dt + \int_{-1/R^2}^0 \lambda(t)\|\nabla U\|^2_{2,B(\lambda(t))}dt\\ &\geq \|\nabla U\|^2_{2,B(R)}\int_{-1/R^2}^0 \lambda(t)dt.\end{aligned}$$

Thus $\|\nabla U\|_{2,B(R)} < \infty$ for all $R > 0$. Next, since u is a weak solution to $\mathcal{L}_0 = 0$, we have that $U \in W^1_{2,loc}(\mathbb{R}^3)$ is a weak solution to

$$- \Delta U + U \cdot \nabla U + \frac{1}{2} U \operatorname{div} U + \frac{1}{2} y \cdot \nabla U + \frac{1}{2} U = 0 \quad \text{in } \mathbb{R}^3. \tag{7.36}$$

From this, smoothness is readily obtained by using, what are by now, quite well-understood elliptic regularity (bootstrap) arguments; see for instance [60] Chapter 3, for the case of the stationary Navier-Stokes system, for an analogous result.

For the decay, we use ideas introduced by Tsai in [68] for a similar result in the case of the incompressible Navier-Stokes system. Assume that there exists a sequence $(y_k) \subset \mathbb{R}^3$ such that $|y_k| \to \infty$ and

$$|U(y_k)||y_k| > k.$$

Let us denote $y/|y|$ by $\hat{y}$. Up to a subsequence, still denoted by $(\hat{y}_k)$, we have that $\hat{y}_k \to x_*$ with $|x_*| = 1$. We claim that u is not bounded in any $Q_r(\sigma x_*, 0) := B(\sigma x_*, r) \times]-r^2, 0[$ for $\sigma \in]0, 1/2[$ and $r \in]0, 1/2[$. To see this, let us fix σ and r, and set $\lambda_0 := \lambda(-r^2) = 1/r$. For k large enough, we have $|y_k| > \sigma\lambda_0$ and $|\sigma\hat{y}_k - \sigma x_*| < r$. Let t_k be the time such that $\lambda(t_k)\sigma = |y_k|$. Hence, it easily follows that $\lambda(t_k) = \sigma^{-1}|y_k| > \lambda_0$ and hence $t_k \in] - r^2, 0[$. Therefore the point $(\sigma\hat{y}_k, t_k)$ is contained in $Q_r(\sigma x_*, 0) \subset Q$. On the other hand,

$$|u(\sigma\hat{y}_k, t_k)| = \lambda(t_k)|U(\lambda(t_k)\sigma\hat{y}_k)| = \sigma^{-1}|y_k||U(y_k)| > \sigma^{-1}k.$$

This proves the claim; in other words, all points of the segment $\{(\sigma x_*, 0) : \sigma \in]0, 1[\}$ are singular points for u. This is a contradiction with the fact that the set of singular points of u in the parabolic cylinder Q has a one-dimensional parabolic Hausdorff measure null (see Remark 18). This concludes the proof. $\square$

The following proposition allows us to identify the defect in U; by defect we mean the term that prevents U to have a faster decay.

Proposition 7.4.2. *Let $U \in C^\infty(\mathbb{R}^3)$ such that (7.34) holds and $U = O(|y|^{-1})$ as $|y| \to \infty$. Then we have*

$$U = \zeta + W,$$

where $\zeta, W \in L^{3,\infty}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3 \backslash \{0\})$. Moreover, we have that ζ is a (-1)-homogeneous vector field i.e. $\lambda\zeta(\lambda x) = \zeta(x)$, for all $\lambda > 0$. Next, we have that

$$\partial^\alpha W = O(|y|^{-3-|\alpha|}) \text{ as } |y| \to \infty,$$

for all $\alpha \in \mathbb{N}^3$.

Proof. We set $u(x,t) = \lambda(t)U(\lambda(t)x)$ with now $(x,t) \in \mathbb{R}^3\times]-1,0[$. It is not difficult to see that u is a suitable weak solution in any bounded parabolic cylinder contained in $\mathbb{R}^3\times]-1,0[$ (see for instance Lemma 4.4.2 for a way to get the required estimates). Let $x_0 \in \mathbb{R}^3 \setminus \{0\}$. For all $t \in]-r^2, 0[$ (with $0 < r < 1$), we have that

$$\begin{aligned}\int_{B(x_0,r)} |u(x,t)|^3 dx &= \int_{B(\lambda(t)x_0,\lambda(t)r)} |U|^3 dy \\ &\le c \left(\int_{B(\lambda(t)x_0,\lambda(t)r)} |U|^4 dy \right)^{3/4} (r\lambda(t))^{3/4},\end{aligned}$$

with c an universal positive constant. From the decay hypothesis we have on U, we know that $U \in L_4(\mathbb{R}^3)$, and thus for any $\epsilon > 0$, there exists $\rho > 0$ such that

$$\int_{|y|>\rho} |U|^4 dy < \epsilon.$$

We have on the other hand that $B(\lambda(t)x_0, \lambda(t)r) \subset \{|y| > \rho\}$ as soon as

$$0 < r < \frac{|x_0|}{1+\rho}.$$

For such r, we have that

$$\int_{B(x_0,r)} |u(x,t)|^3 dx < c\epsilon^{3/4} r^{3/4} \lambda(t)^{3/4}.$$

Thus,

$$\begin{aligned}\left(r^3 \fint_{Q((x_0,0),r)} |u(x,t)|^3 dxdt \right)^{\frac{1}{3}} &\le c \left(\frac{1}{r^2} \epsilon^{3/4} r^{3/4} \int_{-r^2}^{0} \frac{dt}{(-t)^{3/8}} \right)^{\frac{1}{3}} \\ &= c\epsilon^{1/4} \\ &< \epsilon_1,\end{aligned}$$

as soon as $\epsilon < \epsilon_1^4/c^4$, with ϵ_1 given by Theorem 4.2.5. Therefore, we have that

$$\sup_{z \in Q_{r/2}(x_0,0)} |\partial_t^l \nabla^k u(z)| < \frac{c_k}{r^{2l+k+1}}, \tag{7.37}$$

with $l, k = 0, 1, \ldots$, $c_{k,l}$ universal positive constants and r depending only on $|x_0|$ and U.

Once again, from the decay hypothesis we have on U, we have that U belongs to the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$; we also observe that

$$\|u(\cdot,t)\|_{L^{3,\infty}(\mathbb{R}^3)} = \|U\|_{L^{3,\infty}(\mathbb{R}^3)}, \quad \text{for all } t \in]-1,0[.$$

This observation together with (7.37) guarantees the existence of $\zeta \in L^{3,\infty}(\mathbb{R}^3)$ such that $u(\cdot,t)$ converges to ζ in the weak$-*$ topology of $L^{3,\infty}(\mathbb{R}^3)$ as $t \to 0^-$; moreover, one also has that $\nabla^l u(\cdot,t)$ converges to $\nabla^l \zeta$ uniformly on any compact set included in $\mathbb{R}^3 \setminus \{0\}$.

Let $\phi \in C_0^\infty(\mathbb{R}^3)$, we have on one hand that

$$\int_{\mathbb{R}^3} \lambda u(\lambda x, \lambda^2 t)\phi(x)dx = \int_{\mathbb{R}^3} u(x,t)\phi(x)dx \to \int_{\mathbb{R}^3} \zeta(x)\phi(x)dx,$$

as $t \to 0^-$ and for all $\lambda > 0$. On the other hand, we have

$$\int_{\mathbb{R}^3} \lambda u(\lambda x, \lambda^2 t)\phi(x)dx = \int_{\mathbb{R}^3} u(y, \lambda^2 t)\lambda^{-2}\phi(y/\lambda)dy \to \int_{\mathbb{R}^3} \zeta(y)\lambda^{-2}\phi(y/\lambda)dy,$$

as $t \to 0^-$. Thus by uniqueness, we have

$$\int_{\mathbb{R}^3} \zeta(x)\phi(x)dx = \int_{\mathbb{R}^3} \zeta(y)\lambda^{-2}\phi(y/\lambda)dy,$$

i.e.

$$\int_{\mathbb{R}^3} \zeta(x)\phi(x)dx = \int_{\mathbb{R}^3} \lambda\zeta(\lambda x)\phi(x)dx,$$

for all $\lambda > 0$; which yields $\zeta(x) = \lambda\zeta(\lambda x)$, for all $\lambda > 0$.

Let us set now $W := U - \zeta \in L^{3,\infty}(\mathbb{R}^3)$. We have that

$$\partial^\alpha u(x,t) = \partial^\alpha \zeta(x) - \int_t^0 \partial_t \partial^\alpha u(x,s)ds,$$

for all $t \in]t_1, 0[$ and $R_1 < |x| < R_2$ (where $-1 < t_1 < 0$ and $0 < R_1 < R_2 < \infty$). Using the self-similarity of u and the scaling invariance of ζ, we obtain that

$$\lambda(t)^{1+|\alpha|}\partial^\alpha W(\lambda(t)x) = -\int_t^0 \partial_t \partial^\alpha u(x,s)ds,$$

and therefore

$$|\lambda(t)^{1+|\alpha|}\partial^\alpha W(\lambda(t)x)| \leq M|t|,$$

for all $t \in]t_1, 0[$ and $R_1 < |x| < R_2$ where $M > 0$ depends only on t_1, R_1, R_2 and U (we get this thanks to (7.37)). From here, it is a matter of direct computation to show that $\partial^\alpha W = O(|y|^{-3-|\alpha|})$ as $|y| \to \infty$. This concludes the proof. □

Let us prove now Theorem 7.4.1.

Proof of Theorem 7.4.1. From (7.35), we have that $U \in L_2(\mathbb{R}^3)$ and $\nabla U \in L_2(\mathbb{R}^3)$; therefore, by Sobolev's inequality, we get that $U \in L_3(\mathbb{R}^3)$. Consequently, we have that $u \in L_{3,\infty}(\mathbb{R}^3 \times]0,T[)$, and a fortiori that u is a suitable weak solution in $\mathbb{R}^3 \times]0,T[$. Next, for $|x_0| = 1/2$, we have that

$$\int_{B(x_0,1/8)} |u(x,t)|^3 dx = \int_{B(\frac{x_0}{\sqrt{2a(T-t)}}, \frac{1}{8\sqrt{2a(T-t)}})} |U(y)|^3 dy \to 0 \text{ as } t \to T^-;$$

therefore, looking at the construction of ζ in Proposition 7.4.2, we have that $\zeta \equiv 0$. Thus, from Proposition 7.4.1 and Proposition 7.4.2, we have that $|U(y)| \leq c(1+|y|)^{-3}$ for all $y \in \mathbb{R}^3$, and consequently (thanks also to Remark 22)

$$\sup_{(x,t)\in\mathbb{R}^3\times[0,T]} \frac{1}{\sqrt{2a(T-t)}} \left| U\left(\frac{x}{\sqrt{2a(T-t)}}\right)\right| < \infty. \tag{7.38}$$

Using the decay of U (especially the fact that $|U|$ achieves its maximum at a point in $\mathbb{R}^3$), we have that (7.38) implies that

$$\|U\|_{L_\infty(\mathbb{R}^3)} \sup_{t\in[0,T]} \frac{1}{\sqrt{2a(T-t)}} < \infty,$$

which is true only if $U \equiv 0$; and the proof is complete. □

Even if Theorem 7.4.1 excludes the possibility of having a solution of $\mathcal{L}_0 = 0$ which is self-similar and has a global finite energy, it is nonetheless worth looking at the local problem. To be more precise, we would like to know whether or not we can have a non-trivial self-similar solution which satisfies the following local energy inequality

$$\operatorname*{ess\,sup}_{-1<t<0} \int_B |u(x,t)|^2 dx + \int_Q |\nabla u(x,t)|^2 dxdt < \infty, \tag{7.39}$$

and is moreover a suitable weak solution in Q. A positive answer will provide a solution to Cauchy problem for $\mathcal{L}_0 = 0$ with initial data in $L^{3,\infty} \cap L_\infty(\mathbb{R}^3)$ which blows-up at finite time. Although in the case of the incompressible Navier-Stokes equations the answer is negative (see for instance [68, 10]), we do not know at this stage whether or not it is also the case for our toy model $\mathcal{L}_0 = 0$. We can nonetheless exclude some particular solutions.

Proposition 7.4.3. *Let u be a suitable weak solution to $\mathcal{L}_0 = 0$ in Q, such that $u(x,t) = \lambda(t)U(\lambda(t)x)$ (for all $(x,t) \in Q$) with $\lambda(t) := \frac{1}{\sqrt{-t}}$. Assume, moreover, that U is radially symmetric i.e, there exists a scalar function ϕ such that*

$$U(y) = y\phi(|y|). \tag{7.40}$$

Then u is identically zero.

The proof of this Proposition readily comes from from Theorem 6.4.1 in the previous chapter.

Remark 30. From the proof of Theorem 7.4.1, one can see that for a solution $U \in C^\infty(\mathbb{R}^3)$ to system (7.36), such that $U \in L_3(\mathbb{R}^3)$ or $U = O(|y|^{-1-\epsilon})$ (as $|y| \to \infty$ and $\epsilon > 0$), we have that $U \equiv 0$.

Remark 31. All of the results above hold also true for system $\mathcal{DL}_0 = 0$.

Next, we provide additional qualitative information on the profile U by means of Liouville type results.

Proposition 7.4.4. *Let u be a suitable weak solution to $\mathcal{L}_0 = 0$ in Q, such that $u(x,t) = \lambda(t)U(\lambda(t)x)$ (for all $(x,t) \in Q$) with $\lambda(t) := \frac{1}{\sqrt{-t}}$. Then, none of the components of U has a constant sign on $\mathbb{R}^3$; in other words, they must vanish at some points in $\mathbb{R}^3$.*

Proof. We argue by contradiction. To fix our ideas, let us assume, without loss of generality, that $U_3 > 0$ in $\mathbb{R}^3$. A strong minimum principle applied to an obvious modification of equation $\mathcal{L}_0 u_3 = 0$ guarantees that

$$\min_{(x,t)\in\partial' Q(1/2)} u_3 > 0. \tag{7.41}$$

Next, by noticing that u_i/u_3 ($i = 1,2$) satisfy the following equation

$$\partial_t(\frac{u_i}{u_3}) - \Delta(\frac{u_i}{u_3}) + (u - 2\frac{\nabla u_3}{u_3}) \cdot \nabla(\frac{u_i}{u_3}) = 0 \quad \text{in Q},$$

we find that

$$\max_{(x,t)\in\overline{Q(1/2)}} |u_i/u_3| \leq \frac{\max_{(x,t)\in\partial' Q(1/2)} u_i}{\min_{(x,t)\in\partial' Q(1/2)} u_3};$$

As a byproduct, we have that

$$\frac{U_1}{U_3}, \frac{U_2}{U_3} \in L_\infty(\mathbb{R}^3). \tag{7.42}$$

Now, introduce the function $F(y) = \varphi(|y|)$ such that

$$-\Delta F + \frac{y}{4} \cdot \nabla F = 0 \quad \text{i.e.} \quad \varphi''(r) + (\frac{r}{4} - \frac{2}{r})\varphi' = 0$$

and $\varphi, \varphi' > 0$ with φ which grows exponentially as $r \to \infty$. Next, we have

$$\begin{aligned}-\Delta F + U \cdot \nabla F + \frac{F}{2}\operatorname{div} U + \frac{y}{2}F + \frac{F}{2} &= \left(U \cdot \frac{y}{|y|} + \frac{|y|}{4}\right)\varphi' + \frac{1}{2}(\operatorname{div} U + 1)\varphi \\ &\geq 0 \quad \text{for } |y| \geq R_0,\end{aligned}$$

with a suitable choice of R_0; this can be done thanks to Proposition 7.4.1 and Proposition 7.4.2.

Let us notice that the following two equations are available

$$\begin{aligned} -\Delta(\frac{U_i}{U_3}) + \left(U + \frac{y}{2} - 2\frac{\nabla U_3}{U_3}\right) \cdot \nabla(\frac{U_i}{U_3}) = 0 \quad \text{in } \mathbb{R}^3 \\ -\Delta(\frac{F}{U_3}) + \left(U + \frac{y}{2} - 2\frac{\nabla U_3}{U_3}\right) \cdot \nabla(\frac{F}{U_3}) \geq 0 \quad \text{in } \{|y| \geq R_0\}, \end{aligned} \tag{7.43}$$

(with $i = 1, 2$); set $M_i := \max_{|y|=R_0}(\frac{U_i}{U_3})$ and introduce the function $F_i^\epsilon := M_i + \epsilon\frac{F}{U_3}$ for all $\epsilon > 0$. Next, we have that

$$-\Delta(\frac{U_i}{U_3} - F_i^\epsilon) + \left(U + \frac{y}{2} - 2\frac{\nabla U_3}{U_3}\right) \cdot \nabla(\frac{U_i}{U_3} - F_i^\epsilon) \leq 0 \quad \text{in } \{|y| \geq R_0\};$$

by applying a maximum principle to the previous inequality, we find that

$$\max_{|y| \geq R_0} \left(\frac{U_i}{U_3} - F_i^\epsilon\right) \leq \max_{|y|=R_0} \left(\frac{U_i}{U_3} - F_i^\epsilon\right)^+ ;$$

however, on $\{|y| = R_0\}$, we have $U_i/U_3 - F_i^\epsilon \leq 0$. Therefore, we have

$$\frac{U_i}{U_3} \leq M_i + \epsilon\frac{F}{U_3},$$

for all $|y| \geq R_0$ and all $\epsilon > 0$. Taking, the limit $\epsilon \to 0^+$, we find that U_i/U_3 achieves its maximum in $\mathbb{R}^3$. By applying a strong maximum principle to the first equation in (7.43), we obtain that U_i/U_3 is constant; we denote this constant c_i.

Let us introduce the vector $b = (c_1, c_2, 1)$; the equation for U_3 becomes

$$-\Delta U_3 + (U + \frac{y + b}{2}) \cdot \nabla U_3 + \frac{U_3}{2} = 0 \quad \text{in } \mathbb{R}^3;$$

once again, by applying a maximum principle to the previous equation while taking into account the decay at infinity of U_3, we obtain that $U_3 \equiv 0$, which is a contradiction. □

Our hope is that this proposition will provide us with information that might allow us to conclude that the defect ζ is null even in the case one has only the local energy inequality (7.39).

Chapter 8

Conclusions

As mentioned in our introductory chapter, the work undertaken in this thesis is motivated by the question of knowing **whether our current incapacity to answer the regularity question for the Navier-Stokes system in 3D is mainly due to its non-locality**. The conclusion we have reached at this point of time is that, this has all to do with the supercriticality and little to do with the non-locality.

On the other hand, because of its supercriticality, the regularity question for the model $\mathcal{L}_0 = 0$, in itself, is a rather fascinating problem. Although, we haven't been able to solve this question in this work, we made, while trying, a couple of interesting observations; we record now two of these, which in our opinion, are the most striking and promising for solving this question:

- Upon augmenting the space variable, $\mathcal{L}_0 = 0$ can be transformed into a quasi-linear diffusion equation. Indeed, let u be a solution to $\mathcal{L}_0 = 0$, and set $\bar{x} = (x_0, x) \in \mathbb{R}^{1+3}$ and $\mathcal{U}(\bar{x}, t) = e^{x_0 + Nt} u(x, t)$ with N an arbitrary constant; now, introduce the matrix:

  $$\mathcal{A}^N = \begin{pmatrix} N & -\frac{u_1}{2} & -\frac{u_2}{2} & -\frac{u_3}{2} \\ -\frac{u_1}{2} & 1 & 0 & 0 \\ -\frac{u_2}{2} & 0 & 1 & 0 \\ -\frac{u_3}{2} & 0 & 0 & 1 \end{pmatrix};$$

  the spectrum of the matrix $\mathcal{A}_N$ is

  $$\operatorname{Spec}(\mathcal{A}^N) = \left\{ 1, 1, \frac{1}{2}(N + 1 - \sqrt{(N-1)^2 + |u|^2}), \frac{1}{2}(N + 1 + \sqrt{(N-1)^2 + |u|^2}) \right\};$$

  we find that

  $$\partial_t \mathcal{U} - \operatorname{div}_{\bar{x}} \left(\mathcal{A}^N \nabla_{\bar{x}} \mathcal{U} \right) = 0 \tag{8.1}$$

- Since the presence of the "$\operatorname{div} u$" term in $\mathcal{L}_0 = 0$ is one of the main reasons why we are unable to answer this regularity question for our model, having more information on the relationship between components of u might help dealing with this term. One way to see this relationship is through the following equation:

  $$\partial_t(\frac{u_i}{u_j}) - \Delta(\frac{u_i}{u_j}) + (u - 2\frac{\nabla u_j}{u_j}) \cdot \nabla(\frac{u_i}{u_j}) = 0. \tag{8.2}$$

  In order to really make use of this equation, a good knowledge of the term $\nabla u_j/u_j$ is necessary, which unfortunately we don't have at this point. Let us also point our that at the time we were completing this work, it was brought to our attention by Tao that the use of additional monotonicity or sign preservation properties have also been used by Chae in [7] to solve the regularity question for a non-local toy-model for the Navier-Stokes system; all the more reason to consider this problem in that direction.

Appendix A

A local energy identity for parabolic equations with divergence-free drift

A.1 Introduction

We are considering in this chapter parabolic equations of the type

$$\partial_t u - \operatorname{div}(a\nabla u) + b.\nabla u = 0,$$

where a is a bounded, symmetric and uniformly elliptic matrix and b a divergence-free vector field belonging to $L_\infty(BMO^{-1})$. Like in Chapter 6 above, we say here that a divergence-free vector field b belongs to the space BMO^{-1} if there exits a skew symmetric matrix d belonging to BMO such that $b = -\operatorname{div}(d)$. Therefore, the above equation can be rewritten as follows:

$$\partial_t u - \operatorname{div}(A\nabla u) = 0, \tag{A.1}$$

where $A = a + d$, with a as before and $d \in L_\infty(BMO)$ a skew symmetric matrix.

G. Seregin and co-authors introduced, in their paper [61], the notion of *suitable weak solutions* to equation (A.1), which are distributional solutions that belong to the energy class $L_{2,\infty} \cap W_2^{1,0}$ and which satisfy a certain local energy inequality. In this chapter, we establish a local energy identity for distributional solutions of (A.1) which belong to the energy class $L_{2,\infty} \cap W_2^{1,0}$, and consequently, as a byproduct, we showed that the local energy inequality required in the definition of *suitable weak solutions* introduced in [61] is a direct consequence of being a distributional solution in the above energy class. Our work in this chapter serves also as a toolbox to show similar results for other (singular) PDEs.

A.2 Preliminaries

The space $BMO(\Omega; \mathbb{R}^{n\times n})$ of bounded mean oscillation functions is endowed with the following norm

$$\|d\|_{BMO(\Omega;\mathbb{R}^{n\times n})} := \sup\left\{ \frac{1}{|B(0,r)|} \int_{B(x_0,r)} |d - [d]_{x_0,r}|dx : B(x_0,r) \subset\subset \Omega \right\},$$

where, as usual, $[d]_{x_0,r}$ stands for the average of d over the ball $B(x_0,r)$; and the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is endowed with the following one

$$\|u\|_{\mathcal{H}^1(\mathbb{R}^n)} := \|\mathcal{M}u\|_{L_1(\mathbb{R}^n)},$$

where $\mathcal{M}u(x) := \sup_{\phi\in\mathcal{F}} \sup_{t>0} |(\phi_t \star u)(x)|$, with

$$\mathcal{F} := \{\phi \in C_0^\infty(B) : |\nabla\phi| \leq 1\}$$

and $\phi_t(x) := t^{-n}\phi(x/t)$. See for instance [64, 65] for more properties related to the above two function spaces.

We have the following well-known lemma regarding Hardy spaces.

Lemma A.2.1. *Let $u \in W_p^1(\mathbb{R}^n)$ and $v \in W_q^1(\mathbb{R}^n)$, with $1 < p < \infty$ and $1/p + 1/q = 1$; then $\partial_j u \partial_i v - \partial_j v \partial_i u \in \mathcal{H}^1(\mathbb{R}^n)$ for all $i, j = 1, \ldots, n$ and we have*

$$\|\partial_j u\partial_i v - \partial_j v\partial_i u\|_{\mathcal{H}^1} \leq C\|\nabla u\|_{L_p}\|\nabla v\|_{L_q}, \quad \forall i, j = 1, \ldots, n.$$

Proof. Set, as usual, for simplicity $\partial_i f = f_{,i}$. There holds:

$$u_{,j}\, v_{,i} - u_{,i}\, v_{,j} = ((u - [u]_{x_0,r})v_{,i}\,)_{,j} - ((u - [u]_{x_0,r})v_{,j}\,)_{,i}$$

for $x_0 \in \mathbb{R}^n$ and $r > 0$; therefore, we have for $\phi \in \mathcal{F}$ (defined as above)

$$\begin{aligned}
|\phi_t \star (u_{,j}\, v_{,i} - u_{,i}\, v_{,j}\,)(x)| &= \left| \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} \phi_{,j}\left(\frac{x-y}{t}\right)(u - [u]_{x_0,r})v_{,i}\, dy \right. \\
&\quad \left. - \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} \phi_{,i}\left(\frac{x-y}{t}\right)(u - [u]_{x_0,r})v_{,j}\, dy \right| \\
&\leq \frac{1}{t^{n+1}} \int_{B(x,t)} |u - [u]_{x_0,r}||\nabla v| dy \\
&\leq \frac{1}{t^{n+1}} \|u - [u]_{x,t}\|_{L_p(B(x,t))} \|\nabla v\|_{L_q(B(x,t))}, \\
&\leq \frac{C}{t^n} \|\nabla u\|_{L_p(B(x,t))} \|\nabla v\|_{L_q(B(x,t))}.
\end{aligned}$$

Therefore, we have

$$\mathcal{M}\left(u_{,j}\, v_{,i} - u_{,i}\, v_{,j}\right)(x) \leq C(\mathcal{M}^\sharp|\nabla u|^p(x))^{1/p}(\mathcal{M}^\sharp|\nabla v|^q(x))^{1/q},$$

where $\mathcal{M}^\sharp f$ is the Hardy-Littlewood maximal function defined as follows:

$$\mathcal{M}^\sharp f(x) := \sup\left\{\frac{1}{|B(r)|}\int_{B(x,r)}|f(y)|dy : 0 < r < \infty\right\}.$$

By the maximal theorem (see [64]) together with Hölder's inequality, we have that:

$$\mathcal{M}\left(u_{,j}\, v_{,i} - u_{,i}\, v_{,j}\right) \in L_1(\mathbb{R}^n) \text{ and } \|\mathcal{M}\left(u_{,j}\, v_{,i} - u_{,i}\, v_{,j}\right)\|_{L_1} \leq C(n,p)\|\nabla u\|_{L_p}\|\nabla v\|_{L_q}$$

which completes the proof of Lemma A.2.1. □

We recall now, for the sake of readability, some basic facts related to the spectral decomposition of the Laplace operator on a bounded domain Ω of $\mathbb{R}^n$, with smooth boundary. The Laplacian viewed as an unbounded operator from $L_2(\Omega)$ into itself has a discrete spectrum; we denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ (with $\lambda_n \to \infty$) its eigenvalues and $\{\phi_k\}_{k=1}^\infty$ the corresponding eigenvectors which form a Hilbert basis of $L_2(\Omega)$. Setting $\mathring{L}_2^1(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the Dirichlet semi-norm $\|u\|^2_{\mathring{L}_2^1(\Omega)} := \displaystyle\int_\Omega |\nabla u|^2$, and $H^{-1}(\Omega)$ to be the dual of $\mathring{L}_2^1(\Omega)$, we have the following lemma, which gives us a Hilbert basis of $\mathring{L}_2^1(\Omega)$ and a representation of the norm of H^{-1} by means of the eigenvectors and eigenvalues of the Laplace operator.

Lemma A.2.2. $(\phi_k/\sqrt{\lambda_k})_{k=1}^\infty$ *is a Hilbert basis of* $\mathring{L}_2^1(\Omega)$ *and as a direct consequence, we have that: if* $f \in H^{-1}(\Omega)$*, then*

$$\|f\|^2_{H^{-1}(\Omega)} = \sum_{k=1}^\infty f_k^2/\lambda_k,$$

where $f_k = \langle f, \phi_k\rangle$*.*

Proof. We first recall that the eigenvectors $(\phi_k)_{k=1}^\infty$ satisfy the following equation

$$\begin{cases} -\Delta\phi_k &= \lambda_k\phi_k \quad \text{in } \Omega \\ \quad u|_{\partial\Omega} &= 0; \end{cases}$$

hence $\phi_k \in L_2(\Omega)\cap\mathring{L}_2^1(\Omega)$. Let $u \in \mathring{L}_2^1(\Omega)$ such that $(u,\phi_k)_{\mathring{L}_2^1(\Omega)} = 0$ for all $k \geq 1$ (here $(u,v)_{\mathring{L}_2^1(\Omega)} := \int_\Omega \nabla u\cdot\nabla v dx$); this implies (we even have equivalence) that $(u,\phi_k)_{L_2(\Omega)} = 0$ for all $k = 1, 2, \ldots$ Since $(\phi_k)_{k=1}^\infty$ is a Hilbert basis of $L_2(\Omega)$, we deduce that $u = 0$. Therefore, $(\phi_k)_{k=1}^\infty$ is a complete basis of $\mathring{L}_2^1(\Omega)$. We also, readily, have that

$$(\phi_k, \phi_l)_{\mathring{L}_2^1(\Omega)} = \lambda_k\delta_{l,k},$$

where $\delta_{l,k}$ is the Kronecker symbol. Therefore, $(\phi_k/\sqrt{\lambda_k})_{k=1}^{\infty}$ is a Hilbert basis of $\overset{\circ}{L}{}_2^1(\Omega)$. Let $f \in H^{-1}(\Omega)$. From Riesz's representation theorem, we have that there exists a unique $u_f \in \overset{\circ}{L}{}_2^1(\Omega)$, which solves the equation $-\Delta u_f = f$ in Ω with homogeneous Dirichlet boundary condition and satisfies

$$\|f\|_{H^{-1}(\Omega)} = \|u_f\|_{\overset{\circ}{L}{}_2^1(\Omega)}.$$

Since $(\phi_k)_{k=1}^{\infty}$ is a Hilbert basis of $\overset{\circ}{L}{}_2^1(\Omega)$, we have thanks to Parseval's identity that

$$\begin{aligned}\|u_f\|^2_{\overset{\circ}{L}{}_2^1(\Omega)} &= \sum_{k=1}^{\infty} |(u_f, \phi_k/\sqrt{\lambda_k})_{\overset{\circ}{L}{}_2^1(\Omega)}|^2 \\ &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} |\langle f, \phi_k\rangle|^2,\end{aligned}$$

which concludes the proof. □

A.3 Main theorem

We now state the main result of this chapter.

Theorem A.3.1. *Let u which belongs to the energy class*

$$L_{2,\infty}(Q) \cap W_2^{1,0}(Q),$$

be such that

$$\int_Q u\partial_t\phi dz = \int_Q (A\nabla u).\nabla\phi dz \quad \forall \phi \in C_0^\infty(Q_1), \tag{A.2}$$

where $A = a + d$, with $a \in L_\infty(Q; \mathbb{R}^{n\times n})$ a symmetric matrix satisfying

$$\nu \mathbb{I}_n \leq a \leq \nu^{-1}\mathbb{I}_n$$

and $d \in L_\infty(-1, 0; BMO(B; \mathbb{R}^{n\times n}))$ a skew symmetric matrix. Then the following energy identity holds for a.e. $t_0 \in]-1, 0[$ and for all test functions $\phi \in C_0^\infty(B\times]-1, 1[)$:

$$\begin{aligned}\frac{1}{2}\int_B \phi(x, t_0)|u(x, t_0)|^2 dx + \int_{-1}^{t_0}\int_B \phi\nabla u.a\nabla u dz = \frac{1}{2}\int_{-1}^{t_0}\int_B |u|^2\partial_t\phi dz \\ - \int_{-1}^{t_0}\int_B (A\nabla u).\nabla\phi u dz.\end{aligned}$$

A.4 Proof of Theorem A.3.1

We start by proving a simple regularity result for the time derivative of u defined as in Theorem A.3.1.

Proposition A.4.1. *Let u be defined as in Theorem A.3.1. Then,*

$$\partial_t u \in L_2(-1,0;H^{-1}(B)).$$

Proof. We divide the proof into two steps

Step 1. Let us set

$$g(x,t) = A(x,t)\nabla u(x,t),$$

and consider the problem

$$\begin{cases} -\Delta U(\cdot,t) = \operatorname{div} g(\cdot,t) & \text{for a.e } t \in]-1,0[\\ U(\cdot,t)|_{\partial B} = 0. \end{cases} \tag{A.3}$$

Let $v \in C_0^\infty(B)$, we have:

$$\begin{aligned} \int_B \operatorname{div} g(\cdot,t) v dx &= -\int_B g(\cdot,t).\nabla v dx \\ &= -\int_B (a\nabla u).\nabla v dx - \int_B (d\nabla u).\nabla v dx \\ &=: A_1 + A_2. \end{aligned}$$

We have by a straightforward computation that

$$|A_1| \leq \|a\|_{L_\infty(Q)} \|\nabla u(\cdot,t)\|_{L_2(B)} \|\nabla v\|_{L_2(B)} \quad \text{for a.e } t \in]-1,0[.$$

On the other hand, we have thanks to the skew symmetry of d, that A_2 can be rewritten as follows

$$-A_2 = \frac{1}{2}\sum_{i,j=1}^{n} \int_B d_{ij}(u_{,j}\, v_{,i} - v_{,j}\, u_{,i})dx.$$

Denote by $\bar{u}$ the extension of u from B to $\mathbb{R}^n$ such that

$$\|\bar{u}(\cdot,t)\|_{W_2^1(\mathbb{R}^n)} \leq c\|u(\cdot,t)\|_{W_2^1(B)} \quad \text{for a.e } t \in]-1,0[,$$

where c depends only on n. Similarly, let us denote by $\bar{d}$ the extension of d from B to $\mathbb{R}^n$ such that

$$\|\bar{d}(\cdot,t)\|_{BMO(\mathbb{R}^n;\mathbb{R}^{n\times n})} \leq c\|d(\cdot,t)\|_{BMO(B;\mathbb{R}^{n\times n})} \quad \text{for a.e } t \in]-1,0[,$$

where, again, c depends only on n. In the latter case, to construct such an extension, one can use a reflection on the boundary (See, e.g., [33], where this is done for more general domains $\Omega \subset \mathbb{R}^n$). Therefore, because v is compactly supported in B, we have that

$$-A_2 = \frac{1}{2}\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \bar{d}_{ij}(\bar{u}_{,j}\, v_{,i} - v_{,j}\, \bar{u}_{,i})dx.$$

We have from Lemma A.2.1 that $\bar{u}_{,j}\, v_{,i} - v_{,j}\, \bar{u}_{,i} \in \mathcal{H}^1(\mathbb{R}^n)$ with

$$\|\bar{u}_{,j}\, v_{,i} - v_{,j}\, \bar{u}_{,i}\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|\nabla \bar{u}\|_{L_2(\mathbb{R}^n)}\|\nabla v\|_{L_2(\mathbb{R}^n)},$$

and since $BMO(\mathbb{R}^n)$ is the dual of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, we derive that

$$|A_2| \leq C\|\bar{d}\|_{L_\infty(-1,0;BMO(\mathbb{R}^n;\mathbb{R}^{n\times n}))}\|\nabla \bar{u}\|_{L_2(\mathbb{R}^n)}\|\nabla v\|_{L_2(\mathbb{R}^n)},$$

and a fortiori

$$|A_2| \leq C\|d\|_{L_\infty(-1,0;BMO(B;\mathbb{R}^{n\times n}))}\|\nabla u\|_{L_2(B)}\|\nabla v\|_{L_2(B)},$$

(with $C > 0$ depending only on n). Hence, we have that $\operatorname{div} g(\cdot,t) \in H^{-1}(B)$, with

$$\|\operatorname{div} g(\cdot,t)\|_{H^{-1}(B)} \leq C(n,a,d)\|\nabla u(\cdot,t)\|_{L_2(B)} \quad \text{for a.e } t \in]-1,0[.$$

Therefore, there exists a unique $U(\cdot,t) \in \overset{\circ}{L}{}^1_2(B)$ which solves (A.3) and such that

$$\|\nabla U(\cdot,t)\|_{L_2(B)} \leq C(n,a,d)\|\nabla u(\cdot,t)\|_{L_2(B)} \quad \text{for a.e } t \in]-1,0[.$$

We also deduce that $\nabla U \in L_2(Q)$ and

$$\|\nabla U\|_{L_2(Q)} \leq C(n,a,d)\|\nabla u\|_{L_2(Q)}.$$

Step 2. Now, we can rewrite (A.2) as follows

$$\int_Q u\partial_t\phi dz = \int_Q \nabla U.\nabla\phi dz \quad \forall \phi \in C_0^\infty(Q). \tag{A.4}$$

By a density arguments, we can test (A.4) with functions $\phi(x,t) = \chi(t)\phi_k(x)$, where $\chi \in C_0^\infty(]-1,0[)$ and ϕ_k is an eigenfunction (introduced in the second part of the preliminaries section; here we choose $\Omega = B$). Since $(\phi_k)_{k=1}^\infty$ is a Hilbert basis of $L_2(B)$, we can write u as follows

$$u(x,t) = \sum_{k=1}^{\infty} d_k(t)\phi_k(x),$$

where $d_k(t) = \int_B u(\cdot,t)\phi_k dx$; we also have

$$U(x,t) = \sum_{k=1}^{\infty} b_k(t)\phi_k(x),$$

where $b_k(t) = \int_B U(\cdot,t)\phi_k dx$. So we have, thanks to Lemma A.2.2, that

$$\|\nabla U\|^2_{L_2(Q)} = \int_{-1}^{0} \sum_{k=1}^{\infty} b_k^2(t)\lambda_k dt \leq C(n,a,d)\|\nabla u\|^2_{L_2(Q)} < \infty.$$

We have now

$$\begin{aligned}\int_{-1}^{0} d_k(t)\chi'(t)dt &= \int_{-1}^{0} \chi(t)\int_B \nabla U.\nabla\phi_k dxdt \\ &= \int_{-1}^{0} \chi(t)b_k(t)\lambda_k dxdt.\end{aligned}$$

So, $d_k'(t) = -b_k(t)\lambda_k$ and we deduce that

$$\partial_t u(x,t) = \sum_{k=1}^{\infty} d_k'(t)\phi_k(x),$$

where the convergence of this sum occurs in the space of distributions; thus we have, for every $w \in C_0^\infty(B)$ and $\chi \in C_0^\infty(]-1,0[)$, that

$$\begin{aligned}\int_{-1}^{0} \langle \partial u(\cdot,t), w\rangle\chi(t)dt &= -\lim_{N\to\infty}\int_{-1}^{0}\sum_{k=1}^{N} b_k(t)\lambda_k \int_B \phi_k(x)w(x)dx\chi(t)dt \\ &= \lim_{N\to\infty}\int_{-1}^{0}\sum_{k=1}^{N} b_k(t)\int_B \nabla\phi_k(x).\nabla w(x)dx\chi(t)dt \\ &\leq \|\nabla w\|_{L_2(B)}\int_{-1}^{0}\left(\sum_{k=1}^{\infty} b_k^2(t)\lambda_k\right)^{1/2}|\chi(t)|dt \quad \text{by Lemma A.2.2} \\ &\leq c(n,a,d)\|\nabla u\|_{L_2(Q)}\|\chi\|_{L_2(-1,0)}\|\nabla w\|_{L_2(B)},\end{aligned}$$

and the statement follows. □

Remark 32. Let $\varphi \in C_0^\infty(B)$. Then, we readily deduce from the above proposition that

$$\partial_t(u\varphi) \in L_2(-1,0;H^{-1}(B)).$$

Since obviously $u\varphi \in L_2(-1,0;\overset{\circ}{L}{}_2^1(B))$, we conclude that

$$u\varphi \in C([-1,0];L_2(B)).$$

Proof of Theorem A.3.1. We divide the proof into three steps

Step 1. Consider the following auxiliary equation

$$\begin{cases} \partial_t w - \operatorname{div}(A\nabla w) = F - \operatorname{div} G \quad \text{in } B\times]-1,0[, \\ \qquad\qquad w|_{\partial' Q} = 0, \end{cases} \tag{A.5}$$

where $F = u\partial_t\varphi - (A\nabla u).\nabla\varphi$ and $G = u(A\nabla\varphi)$. We have that the distribution $F - \operatorname{div} G$ belongs to $L_2(-1,0;H^{-1}(B))$. This is a direct consequence of the fact that $u\varphi$ is a distributional solution of (A.5) together with Remark 32 and Step 1 of the proof of Proposition A.4.1. But, for our purpose we are interested, more precisely, in the bounds of the terms which appear in the definition of $F - \operatorname{div} G$ and which belong to $L_2(-1,0;H^{-1}(B))$. Therefore let us consider a function $w \in C_0^\infty(B)$; then for a.e $t \in]-1,0[$

$$\begin{aligned} \int_B Fwdx + \int_B G.\nabla wdx &= \int_B u\partial_t\varphi wdx - \int_B (a\nabla u).\nabla\varphi wdx - \int_B (d\nabla u).\nabla\varphi wdx \\ &\quad + \int_B (a\nabla\varphi).\nabla wudx + \int_B (d\nabla\varphi).\nabla wudx \\ &= J_1 + J_2 + J_3, \end{aligned}$$

where

$$\begin{aligned} J_1 &:= \int_B u\partial_t\varphi w, \\ J_2 &:= -\int_B (a\nabla u).\nabla\varphi wdx + \int_B (a\nabla\varphi).\nabla wudx, \\ J_3 &:= -\int_B (d\nabla u).\nabla\varphi wdx + \int_B (d\nabla\varphi).\nabla wudx. \end{aligned}$$

We rewrite the term J_3 as follows

$$\begin{aligned} J_3 &= -\int_B d\nabla(uw).\nabla\varphi dx + \int_B (d\nabla w).\nabla\varphi udx + \int_B (d\nabla\varphi).\nabla wudx \\ &= -\int_B d\nabla(uw).\nabla\varphi dx \quad \text{(by the skew symmetry of } d\text{)}. \end{aligned}$$

But again, thanks to the skew symmetry of d, we have that

$$J_3 = -\frac{1}{2}\sum_{i,j=1}^{n} \int_B b_{i,j}\,[(uw)_{,j}\,\phi_{,i} - \phi_{,j}\,(uw)_{,i}\,]\,dx$$

and by performing the same computations as for A_2 in Step 1 of the proof of Proposition A.4.1, with the only difference being that we keep p arbitrary (instead of choosing $p = 2$ as in the proof of Proposition A.4.1), we obtain

$$|J_3| \leq C(n,a,d)\|\nabla(uw)\|_{L_p(B)}\|\varphi\|_{L_{p/(p-1)}(B)},$$

(for $1 < p < \infty$ to be suitably chosen in function of n) which implies that

$$|J_3| \leq C(n,a,d,\varphi)\left(\|w\nabla u\|_{L_p(B)} + \|u\nabla w\|_{L_p(B)}\right).$$

If $\mathbf{n \geq 3}$, we readily have for

$$1 < p < \min(2, \frac{n}{n-1}),$$

that (here $2^* = \frac{2n}{n-2}$)

$$\begin{aligned}\|w\nabla u\|_{L_p(B)} + \|u\nabla w\|_{L_p(B)} &\leq c(n)\left[\|w\|_{L_{2^*}(B)}\|\nabla u\|_{L_2(B)} + \|u\|_{L_{2^*}(B)}\|\nabla w\|_{L_2(B)}\right] \\ &\leq c(n)\left(\|\nabla u\|_{L_2(B)} + \|u\|_{L_2(B)}\right)\|\nabla w\|_{L_2(B)},\end{aligned}$$

where Sobolev embedding and Poincaré's inequality are used in the last estimate.

The case $\mathbf{n = 2}$ is a straightforward adaptation of the previous case (since $H^1(B)$ embeds continuously in every $L_s(B)$, $1 \leq s < \infty$), whereas for the case $\mathbf{n = 1}$, we take $p = 2$, use the fact that $H^1(B)$ is continuously embedded in $L_\infty(B)$ and Poincaré's inequality for the term in w.

Next, we have the following simple bound for the terms J_1 and J_2:

$$|J_1 + J_2| \leq C(n,\varphi)\left(\|\nabla u\|_{L_2(B)} + \|u\|_{L_2(B)}\right)\|\nabla w\|_{L_2(B)}.$$

So in conclusion, we have, for a.e $t \in]-1,0[$ that

$$\|F(\cdot,t) - \operatorname{div} G(\cdot,t)\|_{H^{-1}(B)} \leq C(n,a,d,\varphi)\left(\|\nabla u(\cdot,t)\|_{L_2(B)} + \|u(\cdot,t)\|_{L_2(B)}\right),$$

and we get a fortiori

$$\|F - \operatorname{div} G\|_{L_2(-1,0;H^{-1}(B))} \leq C(n,a,d,\varphi)\left(\|\nabla u\|_{L_2(Q)} + \|u\|_{L_{2,\infty}(Q)}\right) \tag{A.6}$$

Step 2. Let us now tackle the question of well-posedness of (A.5). Consider the time-indexed family of bilinear forms

$$\delta_t(w,v) := \int_B (A\nabla w).\nabla v dx.$$

Let us first notice that the map $t \in]-1,0[\mapsto \delta_t(w,v)$ is measurable for every $w, v \in \overset{\circ}{L}{}^1_2(B)$. Furthermore, we have by similar computations as those made in Step 1 in the proof of Proposition A.4.1, that there exists a constant $C = C(n,a,d) > 0$ independent of t such that

$$|\delta_t(w,v)| \leq C\|\nabla w\|_{L_2(B)}\|\nabla v\|_{L_2(B)},$$

for all $w, v \in \overset{\circ}{L}{}_2^1(B)$ i.e δ_t is a bounded bilinear operator on $\overset{\circ}{L}{}_2^1(B)$. We have, additionally, the following coercivity estimate

$$\begin{aligned}\delta_t(w,w) &= \int_B (A\nabla w).\nabla w dx\\ &= \int_B (a\nabla w).\nabla w dx \quad \text{(by the skew symmetry of } d\text{)},\\ &\geq \nu \int_B |\nabla w|^2 dx.\end{aligned}$$

In view of these previous estimates and the regularity proved for the right-hand side of (A.5) and considering the evolution triple $\overset{\circ}{L}{}_2^1(B) \subset L_2(B) \subset H^{-1}$, we have by applying Jacques-Louis Lions abstract theorem for well-posedness of evolution equations (see e.g. [48] or Theorem 10.9 in [5]) that there exists a unique solution

$$w \in C([-1,0]; L_2(B)) \cap L_2(-1,0; \overset{\circ}{L}{}_2^1(B)),$$

with

$$\partial_t w \in L_2(-1,0; H^{-1})$$

such that

$$\int_Q \partial_t w v dz + \int_Q (A\nabla w).\nabla v dz = \int_Q (F - \operatorname{div} G) v dz \tag{A.7}$$

for any $v \in C_0^\infty(Q)$. Let us notice that, from Remark 32, the fact that $u\varphi$ is a distributional solution of (A.5) and by the above uniqueness result:

$$w = u\varphi.$$

On another hand, by the regularity obtained for w, we can extend identity (A.7) to functions v in $L_2(-1,0; \overset{\circ}{L}{}_2^1(B))$ and therefore, test (A.7) with w itself. Thus, we get

$$\int_Q (A\nabla w).\nabla w dz = \int_Q (F - \operatorname{div} G) w dz. \tag{A.8}$$

Denote by L the left-hand side of the above identity. By the skew symmetry of d, we obtain that

$$L = \int_Q (a\nabla w).\nabla w dz;$$

therefore coming back to u, we get

$$\begin{aligned}L &= \int_Q a(\varphi\nabla u + u\nabla\varphi).(\varphi\nabla u + u\nabla\varphi) dz\\ &= \int_Q \varphi^2 (a\nabla u).\nabla u dz + 2\int_Q u\varphi(a\nabla u).\nabla\varphi dz + \int_Q u^2 (a\nabla\varphi).\nabla\varphi.\end{aligned}$$

Now, denote by R the right-hand side of (A.8); we easily obtain that

$$\begin{aligned}R &= \int_Q u\partial_t\varphi w dz - \int_Q (a\nabla u).\nabla\varphi w dz + \int_Q (a\nabla\varphi).\nabla w u - \int_Q d\nabla(uw).\nabla\varphi dz\\ &= \int_Q u^2\varphi\partial_t\varphi dz - \int_Q u\varphi(a\nabla u).\nabla\varphi dz + \int_Q \varphi u(a\nabla u).\nabla\varphi dz + \int_Q u^2(a\nabla\varphi).\nabla\varphi dz\\ &\quad -2\int_Q u\varphi(d\nabla u).\nabla\varphi dz.\end{aligned}$$

Therefore, (A.8) implies that

$$\begin{aligned}\int_Q \varphi^2(a\nabla u).\nabla u dz + 2\int_Q u\varphi(a\nabla u).\nabla\varphi dz = \int_Q u^2\varphi\partial_t\varphi dz\\ - 2\int_Q u\varphi(d\nabla u).\nabla\varphi dz, \quad \text{(A.9)}\end{aligned}$$

for all $\varphi \in C_0^\infty(Q)$. Let us notice that all the integrals in the above identity are finite, especially the last one on the right-hand side of (A.9). To see this we rewrite

$$\int_Q u\varphi(d\nabla u).\nabla\varphi dz = \int_Q d\nabla(u^2/2).\nabla(\varphi^2/2)dz$$

and use the same method as in the estimation of J_3 in the previous step.

Step 3. Now, we choose $\varphi(x,t) = \chi_\epsilon(t)\phi(x,t)$, where $\phi \in C_0^\infty(B\times]-1,1[)$ and $\chi_\epsilon(t) = 1$ if $t \le t_0 - \varepsilon$, $\chi_\epsilon(t) = (t_0 + \epsilon - t)/(2\epsilon)$, if $t_0 - \epsilon < t < t_0 + \epsilon$, and $\chi_\epsilon(t) = 0$ when $t \ge t_0 + \epsilon$, with $t_0 \in]-1, 0[$. Therefore passing to the limit $\epsilon \to 0$ in (A.9), we have that Theorem A.3.1 is proved. $\square$

Appendix B

Liouville type theorems for equations of fluid dynamics: the MHD system

We discussed in Chapter 7 the importance of having Liouville type results. We continue our investigations, in this chapter, with another important equation of fluid dynamics: the 2D stationary incompressible magneto-hydrodynamic (MHD) system

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = b \cdot \nabla b, \quad \operatorname{div} u = 0 \\ \qquad -\Delta b + u \cdot \nabla b - b \cdot \nabla u = 0 \end{cases} \text{in } \mathbb{R}^2 \tag{B.1}$$

In system (B.1), $u, b : \mathbb{R}^2 \to \mathbb{R}^2$ stand for the velocity and the magnetic field respectively, and p is the pressure term resulting from the incompressibility condition on u; this system and its time-dependent analogue are used to model electrically conductive fluids such as plasma, liquid metals, electrolytes etc. For additional physical background and mathematical theory, we refer to Schnack [56] and references therein.

The setting of our work is as follows:

$$\begin{gathered} u, b \in C^\infty(\mathbb{R}^2) \\ \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |\nabla b|^2 \right) dx < \infty. \end{gathered} \tag{B.2}$$

When $b = 0$ in system (B.1) (which becomes now the incompressible Navier-Stokes system), the question of the triviality of u under the above condition was first solved in [21], and was later, also established in [35] provided the velocity u is just bounded. Those two results rely heavily on the fact that one has a nice equation for the vorticity $w := u_{2,1} - u_{1,2}$

$$\partial_t w - \Delta w + u \cdot \nabla w = 0.$$

However, when the magnetic field b is not null, this nice equation for the vorticity is no more available, which makes this problem quite hard. In recent years many mathematicians tried to bring a complete understanding to the Liouville problem for this system but unfortunately without success; we cite the works of [8, 11, 57, 70] (and references therein) for interesting results, and more recently the work of Wang & Wang in [69] where they proved under some smallness condition on the L_1-norm of b that u and b were constants. Our work is more in the direction of [69]; our contribution is that we were able to uncover a nice drift-diffusion equation for the stream function associated to the magnetic field, and for which a maximum principle is available; this allows us to bring new insights to this problem and to improve the existing results in the literature. The results and text base of this chapter are based on ongoing work[1] with Simon Schulz (University of Cambridge) and Nicola De Nitti (Friedrich-Alexander-Universität Erlangen-Nürnberg).

Before stating our main results, let us point out that it is very common to see in the literature an added incompressibility condition $\operatorname{div} b = 0$ on the magnetic field; however, by doing so, the system becomes over-determined. We are able to prove that in our setting (B.2), the incompressibility of the magnetic field b can be derived from the system; this relies on the following auxiliary results.

Lemma B.0.1 (See [49] Appendix B, estimates B.9, B.10 & B.12)**.** *Let $f \in L_{2,loc}(\mathbb{R}^2)$ with finite Dirichlet energy*

$$\int_{\mathbb{R}^2} |\nabla f|^2 dx < \infty.$$

Then

$$\int_{B(R)} |f|^m dx \leq cR^2 (\log 2R)^{\frac{m}{2}} \left(\int_{\mathbb{R}^2} |\nabla f|^2 dx + \int_{B(1)} |f|^2 dx \right)^{\frac{m}{2}},$$

for all $m, R \geq 1$ and $c > 0$ an absolute constant.

Next, we have the following simple Liouville result.

Proposition B.0.1. *Let $\rho \in L_\infty(\mathbb{R}^2)$ and let a vector field $u \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^2)$ be such that* $\operatorname{div}(\rho u) = 0$. *Let $f \in L_2(\mathbb{R}^2)$ be such that*

$$-\Delta f + \rho u \cdot \nabla f = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^2). \tag{B.3}$$

Then $f \equiv 0$.

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[1] This work was submitted online some time after the submission of this thesis; see `https://arxiv.org/abs/2103.00551`

Proof. Let us start by noticing that classical regularity theory ensures that $f \in C^1(\mathbb{R}^2)$. Now, we introduce the following functions; let $h \in C^1(\mathbb{R})$ be piece-wise C^2 and such that

$$h(\omega) = \begin{cases} \omega^2, & \text{for } |\omega| \le \omega_0, \\ \omega_0(2|\omega| - \omega_0), & \text{for } |\omega| \ge \omega_0, \end{cases}$$

with $\omega_0 > 0$ fixed and shall be taken to infinity later on; notice that $h(\omega) \le \omega^2$ and $h(\omega) \le 2\omega_0|\omega|$ for all values of $\omega \in \mathbb{R}$. Next, let $0 \le \varphi \in C^\infty(B)$ such that $\varphi \equiv 1$ in $B(1/2)$ and $\varphi \equiv 0$ in $B \setminus B(3/4)$; set $\varphi_R(x) = \varphi(x/R)$ with $R \ge 1$.

We get, from multiplying equation (B.3) by $h'(f)\varphi_R$, that

$$\int_{\mathbb{R}^2} \varphi_R h''(f)|\nabla f|^2 dx = \int_{\mathbb{R}^2} h(f)\left(\Delta\varphi_R + \rho u \cdot \nabla\varphi_R\right) dx.$$

From this, we get that

$$\begin{aligned} \int_{\{|f| \le \omega_0\} \cap B(R/2)} |\nabla f|^2 dx &\le \frac{c}{R^2} \int_{R/2<|x|<3R/4} |f|^2 dx + \frac{c|[u]_R|\|\rho\|_{L_\infty}}{R} \int_{R/2<|x|<3R/4} |f|^2 dx \\ &+ \frac{c\omega_0\|\rho\|_{L_\infty}}{R} \int_{R/2<|x|<3R/4} |u - [u]_R||f| dx \\ &\le \frac{c}{R^2} \int_{R/2<|x|<3R/4} |f|^2 dx + c\|\rho\|_{L_\infty} \frac{(\log 2R)^{\frac{1}{2}}}{R} \times \\ &\times \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{B(1)} |u|^2 dx \right)^{\frac{1}{2}} \int_{R/2<|x|<3R/4} |f|^2 dx \\ &+ c\omega_0 \|\rho\|_{L_\infty} \|\nabla u\|_{L_2} \left(\int_{R/2<|x|<3R/4} |f|^2 dx \right)^{\frac{1}{2}} \\ &\to 0 \quad \text{as } R \to \infty, \end{aligned}$$

where we used Lemma B.0.1 to control the term $|[u]_R|$. Thus, we have proved that $|\nabla f|\mathbb{1}_{\{|f| \le \omega_0\}} \equiv 0$, for $\omega_0 > 0$ arbitrary; taking $\omega_0 \to \infty$, we have that the proposition is proved. □

Consequently, we have the following result.

Proposition B.0.2. *Let u, b be solutions to system* (B.1) *such that conditions* (B.2) *hold. Then,* $\operatorname{div} b \equiv 0$ *and there exists a stream function ψ (in $\mathring{L}^2_2(\mathbb{R}^2)$ the closure of $C_0^\infty(\mathbb{R}^2)$ with the semi-norm $\|\nabla^2 \cdot\|_{L_2(\mathbb{R}^2)}$) such that $b = (\psi_{,2}\ , -\psi_{,1})$.*

Proof. Taking the divergence in the second equation in (B.1), we get:

$$-\Delta(\operatorname{div} b) + u \cdot \nabla(\operatorname{div} b) = 0 \quad \text{in } \mathbb{R}^2.$$

Applying Proposition B.0.1 with $f = \operatorname{div} b$, $\rho = 1$, we get that $\operatorname{div} b = 0$, and the existence of the stream function ψ comes from the solvability of the equation $-\Delta\psi = \operatorname{curl} b \in L_2(\mathbb{R}^2)$ (with $\operatorname{curl} b = b_{2,1} - b_{1,2}$). $\square$

Remark 33. An analogue of Proposition B.0.2 is also available in higher dimensions, and is based on an adaptation of Proposition B.0.1 to higher dimensions.

B.1 Main results

Our first main result is stated as follows.

Theorem B.1.1. *Let* u, b *be solutions to system* (B.1) *such that conditions* (B.2) *hold. Assume that one of the following conditions holds:*

1. $b \in L_2(\mathbb{R}^2)$*;*
2. $u \in L_p(\mathbb{R}^2)$ *and* $b \in L_q(\mathbb{R}^2)$ *with* $p \in [2, \infty[$ *and* $q \in]2, \infty[$ *such that*

$$\frac{2}{\max(p,q)} + \frac{1}{p} \geq \frac{1}{2}. \tag{B.4}$$

Then, u *and* b *are constants.*

Remark 34. Under conditions (B.2), $b \in BMO(\mathbb{R}^2)$ (by embedding) and thanks to the interpolation between L_p and BMO spaces (see e.g. [12]), we have that the first point of the previous result is also true if we take $b \in L_p(\mathbb{R}^2)$, with $p \in [1, 2[$.

Our second result addresses the case where we allow the velocity u to grow; to be more precise, we have the following theorem.

Proposition B.1.1. *Let* u, b *be solutions to system* (B.1) *such that conditions* (B.2) *hold. Assume in addition that the following condition holds:*

$$u \in BMO^{-1}(\mathbb{R}^2).$$

Then,

1. *there exists an* $\alpha = \alpha(\|u\|_{BMO^{-1}}) > 0$ *such that if* $b \in L_p(\mathbb{R}^2)$ *with* $p \in]2, 2+\alpha[$*, then we have* $u, b \equiv 0$*;*
2. *or alternatively if* b *belongs also to* $BMO^{-1}(\mathbb{R}^2)$ *then* $u, b \equiv 0$.

Remark 35. The second point in the previous proposition was also proved in [11] in the 3D case; our proof of this result here is simpler and more straightforward.

B.2 Proof of Theorem B.1.1

We start by making the following observation; let ψ be as in Proposition B.0.2; then the second equation in (B.1) becomes:

$$\nabla^{\perp}\left(-\Delta\psi + u\cdot\nabla\psi\right) = 0;$$

here for a vector $v = (v_1, v_2) \in \mathbb{R}^2$, $v^{\perp} := (v_2, -v_1)$. Consequently, we get that

$$-\Delta\psi + u\cdot\nabla\psi = c_0, \quad (c_0 \in \mathbb{R}). \tag{B.5}$$

The next proposition provides more information on the potential values of c_0.

Proposition B.2.1. *Let u, b be solutions to system* (B.1) *such that conditions* (B.2) *hold. If we assume in addition that u or b belongs to $BMO^{-1}(\mathbb{R}^2)$, then $c_0 = 0$ (where c_0 as in* (B.5)*).*

Remark 36. Notice that, the conclusion of this proposition, obviously, still holds if we assume for instance $u \in L_p(\mathbb{R}^2)$ and $b \in L_q(\mathbb{R}^2)$ with $p, q \in [1, \infty[$; of course, this condition can be weakened even further.

Proof. We will present only the case $b \in BMO^{-1}(\mathbb{R}^2)$ since the case $u \in BMO^{-1}(\mathbb{R}^2)$ can be dealt with in a similar manner.

Because the Riesz transform (the Riesz transform of a function f is defined as $Rf := \nabla(-\Delta)^{\frac{1}{2}}f$) is a bounded operator from $BMO(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ (see for instance [65]), we have that the stream function ψ associated to b belongs to $BMO(\mathbb{R}^2)$.

Now, introduce the cut-off function $0 \le \varphi \in C_0^{\infty}(\mathbb{R}^2)$ such that $\varphi \equiv 1$ in $B(1/2)$ and $\varphi \equiv 0$ in $B \setminus B(3/4)$. Let $R > 1$ and set $\varphi_R(x) := \varphi(x/R)$. We get from (B.5) that

$$c_0 \int_{B(R)} \varphi_R dx = -\int_{B(R)} \Delta\psi\varphi_R dx + \int_{B(R)} u\cdot\nabla(\psi - [\psi]_{,R})\varphi_R.$$

This implies,

$$\begin{aligned} |c_0|R^2 \int_B \varphi dx &\le R\|\nabla b\|_{L_2}\|\phi\|_{L_2} + \left|\int_{B(R)} (\psi - [\psi]_R) u\cdot\nabla\varphi_R\right| \\ &\le c(\varphi)\left(R\|\nabla b\|_{L_2(\mathbb{R}^2)} + \|\psi\|_{BMO(\mathbb{R}^2)}\|u\|_{L_2(B(R))}\right). \end{aligned}$$

From Lemma B.0.1, we get that

$$|c_0| \le c(\varphi, b, u)\left(\frac{1}{R} + \frac{\sqrt{\log(2R)}}{R}\right) \to 0 \text{ as } R \to \infty;$$

and the proof is complete. □

Before, we give the proof of Theorem B.1.1, let us recall the following auxiliary results.

Lemma B.2.1 (Oscillation lemma, See [61] Theorem 4.2)**.** *Let* $v \in C(B) \cap H^1(B)$ *such that for any* $r \in]0,1[$ *the following maximum principle holds in* $B(r)$*:*

$$\max_{B(r)} v = \max_{\partial B(r)} v, \quad \min_{B(r)} v = \min_{\partial B(r)} v.$$

Then,

$$\sup_{x\in B(r)} |v(x) - v(0)| \leq \frac{c}{\sqrt{-\log r}} \|\nabla v\|_{L_2(B\setminus B(r))},$$

for all $r \in]0,1[$ *and with* $c > 0$ *an absolute constant.*

We are ready to give now the proof of Theorem B.1.1.

Proof of Theorem B.1.1. We divide the proof into 2 steps; the first one addresses the first point of the theorem and the second addresses the second point.

Step 1. Since $b \in L_2(\mathbb{R}^2)$ implies $b \in BMO^{-1}(\mathbb{R}^2)$, we have thanks to Proposition B.2.1 that

$$-\Delta\psi + u \cdot \nabla\psi = 0 \quad \text{in } \mathbb{R}^2.$$

By setting $\psi^R(x) := \psi(Rx)$ $(R > 0)$, we see that ψ^R satisfies the conditions in Lemma B.2.1; consequently, by taking $r = 1/2$ in the lemma, we obtain:

$$\sup_{x\in B(1/2)} |\psi^R(x) - \psi(0)| \leq c\|\nabla\psi^R\|_{L_2(B\setminus B(1/2))},$$

thus

$$\sup_{x\in B(R/2)} |\psi(x) - \psi(0)| \leq c\|b\|_{L_2(B(R)\setminus B(R/2))} \to 0 \text{ as } R \to \infty.$$

This implies that ψ is constant and therefore $b \equiv 0$; consequently we have that

$$-\Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^2,$$

with $\nabla u \in L_2(\mathbb{R}^2)$. This implies, thanks to [21], that u is constant. And the first point of the theorem is proved. Or alternatively, notice that the vorticity $w = \operatorname{curl} u$ satisfies the equation $-\Delta w + u \cdot \nabla w = 0$, and then apply Proposition B.0.1.

Step 2. By applying the divergence operator to the first equation in (B.1), we find that

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes u - b \otimes b) \quad \text{in } \mathbb{R}^2;$$

this guarantees, if we set $r = \max(p, q)$, that

$$\|p\|_{L_{\frac{r}{2}}(\mathbb{R}^2)} \leq c\left(\|u\|^2_{L_r(\mathbb{R}^2)} + \|b\|^2_{L_r(\mathbb{R}^2)}\right); \tag{B.6}$$

to obtain estimate (B.6), we used the hypothesis, the embedding of u, b in $BMO(\mathbb{R}^2)$ (due to (B.2)) and the interpolation between L_s and BMO spaces (see [12]).

Next, we rewrite the first equation in (B.1), in the following way:

$$-\Delta u + \nabla(\frac{|u|^2}{2} + p - \frac{|b|^2}{2}) - u^\perp \operatorname{curl} u = -b^\perp \operatorname{curl} b;$$

from this, we get that

$$-\Delta\frac{|u|^2}{2} + |\nabla u|^2 + u \cdot \nabla(\frac{|u|^2}{2} + p - \frac{|b|^2}{2}) = -u \cdot b^\perp \operatorname{curl} b.$$

Now notice that $\nabla\psi = -b^\perp$ and thanks to (B.5) (together with Remark 36), we have that $\operatorname{curl} b = u \cdot b^\perp$. Consequently, we have that

$$|\nabla u|^2 + |\operatorname{curl} b|^2 = \Delta\frac{|u|^2}{2} - u \cdot \nabla(\frac{|u|^2}{2} + p - \frac{|b|^2}{2}) \quad \text{in } \mathbb{R}^2. \tag{B.7}$$

We set for simplicity $Q := |u|^2/2 + p - |b|^2/2 \in L_{\frac{r}{2}}(\mathbb{R}^2)$; by multiplying (B.7) by our usual rescaled cut-off function φ_R and then integrating, we obtain that

$$\begin{aligned}\int_{B(R/2)} (|\nabla u|^2 + |\operatorname{curl} b|^2)dx &\leq \int_{B(R)\setminus B(R/2)} \frac{|u|^2}{2}\Delta\varphi_R dx + \int_{B(R)\setminus B(R/2)} u \cdot \nabla\varphi_R Q dx \\ &\leq \frac{c(\varphi, p)}{R^{\frac{4}{p}}}\left(\int_{B(R)\setminus B(R/2)} |u|^p dx\right)^{\frac{2}{p}} \\ &\quad + \frac{c(\varphi, p, q)}{R^{1-\frac{2}{s}}}\left(\int_{B(R)\setminus B(R/2)} |u|^p dx\right)^{\frac{1}{p}}\left(\int_{B(R)\setminus B(R/2)} |Q|^{\frac{r}{2}} dx\right)^{\frac{2}{r}},\end{aligned}$$

with $1/s = 1 - 1/p - 2/r$. Thus, the right-hand side in last inequality above, goes to zero as soon as $1 - 2/s \geq 0$ i.e. $1/p + 2/r \geq 1/2$, and the theorem is proved. □

Let us give now, the proof of Proposition B.1.1.

Proof of Proposition B.1.1. We divide the proof into 2 steps; the first step addresses the proof of the second point of the proposition whereas the second step addresses the proof of the first point

Step 1. Because $u, b \in BMO^{-1}(\mathbb{R}^2)$, we have that $u = \nabla^\perp \phi$ and $b = \nabla^\perp \psi$ with $\phi, \psi \in BMO(\mathbb{R}^2)$. Thanks, to Proposition B.2.1, we know that

$$-\Delta\psi + u \cdot \nabla\psi = 0;$$

by multiplying the above equation by $(\psi - [\psi]_{,R})\varphi_R^2$ (where φ_R is our usual rescaled cut-off function), and then integrating by part, we get that

$$\int_{B(R)} |\nabla\psi|^2 \varphi_R^2 dx = \frac{1}{2}\int_{B(R)} |\psi - [\psi]_{,R}|^2 \Delta\varphi_R^2 dx \\ + 2\int_{B(R)} \varphi_R \nabla\psi \cdot \nabla^\perp\varphi_R(\psi - [\psi]_{,R})(\phi - [\phi]_{,R})dx;$$

this implies, thanks to Young's inequality that

$$\int_{B(R/2)} |\nabla\psi|^2 dx \leq c(\varphi) \fint_{B(R)} |\psi - [\psi]_{,R}|^2 dx + \frac{c(\varphi)}{R^2}\int_{B(R)} |\psi - [\psi]_{,R}|^2 |\phi - [\phi]_{,R}|^2 dx \\ \leq c(\varphi)(\|\psi\|^2_{BMO(\mathbb{R}^2)} + \|\psi\|^2_{BMO(\mathbb{R}^2)}\|\phi\|^2_{BMO(\mathbb{R}^2)});$$

thus, by taking the limit $R \to \infty$, we get that $\nabla\psi \in L_2(\mathbb{R}^2)$ i.e. $b \in L_2(\mathbb{R}^2)$. By applying the first point of Theorem B.1.1, we have that the second point of the theorem is proved.

Step 2. Let $x_0 \in \mathbb{R}^2$ and $r > 0$; we set $\varphi_{x_0,r} := \varphi((x - x_0)/r)$. By repeating the computations in the previous step, but this time, with $\varphi_{x_0,r}$ instead of φ_R, we get that

$$\int_{B(x_0,r/2)} |\nabla\psi|^2 dx \leq c(\varphi)(1 + \|\phi\|^2_{BMO(\mathbb{R}^2)}) \left(\fint_{B(x_0,r)} |\psi - [\psi]_{x_0,r}|^4 dx\right)^{\frac{1}{2}} \\ \leq c(\varphi)(1 + \|\phi\|^2_{BMO(\mathbb{R}^2)}) r^2 \left(\fint_{B(x_0,r)} |\nabla\psi|^{\frac{4}{3}} dx\right)^{\frac{3}{2}}.$$

Thus, we have that

$$\left(\fint_{B(x_0,r)} |\nabla\psi|^2 dx\right)^{\frac{1}{2}} \leq c(\varphi)(1 + \|\phi\|_{BMO(\mathbb{R}^2)}) \left(\fint_{B(x_0,r)} |\nabla\psi|^{\frac{4}{3}} dx\right)^{\frac{3}{4}}, \tag{B.8}$$

for all $x_0 \in \mathbb{R}^2$ and $r > 0$. By applying the Gehring's reverse Hölder result (see e.g. Proposition 2.4.1) to (B.8), we obtain that there exists an $\alpha = \alpha(\|u\|_{BMO^{-1}(\mathbb{R}^2)}) > 0$ such that

$$\left(\fint_{B(1/6)} |\nabla\psi|^{2+\alpha} dx\right)^{\frac{1}{2+\alpha}} \leq c(1 + \|\phi\|_{BMO(\mathbb{R}^2)}) \left(\fint_{B(1)} |\nabla\psi|^2 dx\right)^{\frac{1}{2}}. \tag{B.9}$$

Now, we set $\psi^R(x) = \psi(Rx)$ and $\phi^R(x) = \phi(Rx)$; we find that

$$-\Delta\psi^R + \nabla^\perp\phi^R \cdot \nabla\psi^R = 0 \quad \text{in } \mathbb{R}^2;$$

thus (B.9) holds also for ψ replaced by ψ^R and ϕ replaced by ϕ^R; moreover, since $\|\phi^R\|_{BMO(\mathbb{R}^2)} = \|\phi\|_{BMO(\mathbb{R}^2)}$, we get that

$$\begin{aligned}\left(\int_{B(R/6)} |\nabla\psi|^{2+\alpha} dx\right)^{\frac{1}{2+\alpha}} &\leq \frac{c(1 + \|\phi\|_{BMO(\mathbb{R}^2)})}{R^{\frac{\alpha}{2+\alpha}-1+\frac{2}{p}}}\|\nabla\psi\|_{L_p(\mathbb{R}^2)} \\ &\to 0 \quad \text{as } R \to \infty,\end{aligned}$$

as soon as $p < 2 + \alpha$; and the proposition is proved. □

Concluding remarks: At this point, it is unclear whether the conclusion of Theorem B.1.1 still holds if the constraint (B.4) on the exponents is removed. This will be the object of future investigations.

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