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Some rigidity results for the Hawking mass and a lower bound for the Bartnik capacity

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EP/N509796/1**Abstract**

We prove rigidity results involving the Hawking mass for surfaces immersed in a 3-dimensional, complete Riemannian manifold (M, g) with non-negative scalar curvature (respectively, with scalar curvature bounded below by -6). Roughly, the main result states that if an open subset $\Omega \subset M$ satisfies that every point has a neighbourhood $U \subset \Omega$ such that the supremum of the Hawking mass of surfaces contained in U is non-positive, then Ω is locally isometric to Euclidean \mathbb{R}^3 (respectively, locally isometric to the Hyperbolic 3-space \mathbb{H}^3). Under mild asymptotic conditions on the manifold (M, g) (which encompass as special cases the standard ‘asymptotically flat’ or, respectively, ‘asymptotically hyperbolic’ assumptions) the previous quasi-local rigidity statement implies a *global rigidity*: if every point in M has a neighbourhood U such that the supremum of the Hawking mass of surfaces contained in U is non-positive, then (M, g) is globally isometric to Euclidean \mathbb{R}^3 (respectively, globally isometric to the Hyperbolic 3-space \mathbb{H}^3). Also, if the space is not flat (respectively, not of constant sectional curvature -1), the methods give a small yet explicit and strictly positive lower bound on the Hawking mass of suitable spherical surfaces. We infer a

small yet explicit and strictly positive lower bound on the Bartnik mass of open sets (non-locally isometric to Euclidean \mathbb{R}^3) in terms of curvature tensors. Inspired by these results, in the appendix we propose a notion of ‘sup-Hawking mass’ which satisfies some natural properties of a quasi-local mass.

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1 | INTRODUCTION

The goal of this paper is to prove some rigidity results involving the Hawking mass for surfaces immersed in an asymptotically flat (respectively, asymptotically hyperbolic; and some suitable milder asymptotic assumptions), 3-dimensional, complete Riemannian manifold (M, g) with non-negative scalar curvature (respectively, with scalar curvature bounded below by -6). Let us start by recalling some motivations for studying such a geometric setting, coming from Mathematical General Relativity.

1.1 | Some background and motivation

In the framework of Mathematical General Relativity, a Riemannian manifold (M^3, g) with non-negative scalar curvature $\text{Sc}_g \geq 0$ represents a ‘time-symmetric space-like slice’ of a 4-dimensional space-time satisfying the so-called *Dominant Energy Condition* (DEC for short). The condition $\text{Sc}_g \geq 0$ is indeed a consequence of the DEC, coupled with the property of being a ‘time-symmetric space-like slice’ (that is, M has vanishing second fundamental form as a hypersurface in the ambient 4-dimensional space-time) satisfying the Einstein Constraint Equations.

The relationship between the notion of mass in General Relativity and the geometry of a manifold has been studied extensively in recent times. Let us recall some basic notation and fundamental results. First of all, let us mention that, unless otherwise specified, a manifold is allowed to have non-empty boundary.

Definition 1.1 (Asymptotically flat Riemannian manifold with horizon boundary). A complete Riemannian manifold (M^3, g) is said to be *asymptotically flat* (AF for short) if there exists a compact subset $\mathcal{K} \Subset M$ such that $M^3 \setminus \mathcal{K}$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_1(0)}$ via a map Ψ which induces a system of coordinates at infinity, and in such coordinates the metric is written as

$$g_{ij} = \delta_{ij} + \sigma_{ij} \quad \text{where} \quad |x|^{|\alpha|} |(\partial^\alpha \sigma_{ij})(x)| = \mathcal{O}(|x|^{-\tau}) \quad \text{as } |x| \rightarrow \infty, \quad (1)$$

for some $\tau > 1/2$ and all multi-indices α , with $|\alpha| = 0, 1, 2, 3$. We also require that the scalar curvature Sc of (M, g) is integrable. Moreover, if the boundary of M is non-empty, we assume that it is minimal and that there are no closed minimal surfaces intersecting the interior of M ; in this case we say that the manifold is *asymptotically flat with horizon boundary*.

Note that in Definition 1.1 and throughout, we only consider one-ended AF manifolds with horizon boundary. For an AF Riemannian manifold (M, g) with horizon boundary as above there is a natural notion of ‘total mass’ of (M, g) , well known under the name of ADM-mass (after Arnowitt–Deser–Misner [1]) and defined by

$$m_{\text{ADM}}(M, g) = \lim_{\rho \rightarrow \infty} \frac{1}{16\pi\rho} \int_{|x|=\rho} \sum_{i,j=1}^3 (\partial_i g_{ij} - \partial_j g_{ii}) x^j. \quad (2)$$

Bartnik [2] showed that $m_{\text{ADM}}(M, g)$ as defined in (2) is finite and independent of the chart at infinity.

The fundamental *Positive Mass Theorem*, proven first by Schoen–Yau [44] using minimal surfaces methods and then by Witten [52] via spinorial techniques, states that if (M^3, g) is an AF Riemannian manifold with non-negative scalar curvature, then $m_{\text{ADM}}(M, g) \geq 0$ and equality holds if and only if (M^3, g) is isometric to the Euclidean \mathbb{R}^3 .

Another landmark example of the interplay between mass and geometry is the *Riemannian Penrose Inequality*, proved independently by Huisken–Ilmanen [15] and Bray [4]. The Riemannian Penrose Inequality is a strengthening of the lower bound on the ADM mass in case of horizon boundary of an AF manifold with non-negative scalar curvature; namely, $m_{\text{ADM}}(M, g) \geq \sqrt{\frac{|\partial M|}{16\pi}}$.

In this paper, we further investigate the relationship between geometry and mass. More precisely between the geometric conditions of non-negative scalar curvature and asymptotic flatness (and some generalisations of it) on the one hand, and two notions of *quasi-local mass*: those of Hawking [11] and Bartnik [3]. Let us recall that, while in Newtonian gravity it is possible to define the mass contained in a region Ω simply by integration of a ‘mass density function’ over Ω , defining a corresponding concept in the setting of General Relativity is much more subtle. Indeed, due to the Equivalence Principle, there is no pointwise notion of ‘gravitational energy density’ (see, for instance, [48, Section 20.4] or [41]).

Nevertheless there are several proposals for a notion of ‘quasi-local mass’ contained in a region Ω (see, for instance, [22, Chapter 6] or [46]); we will focus on the Hawking [11] and Bartnik [3] quasi-local masses.

The Hawking mass

Definition 1.2 (Hawking mass). Let (M^3, g) be a Riemannian manifold with non-negative scalar curvature and let $\Sigma \hookrightarrow M$ be an immersed sphere. The Hawking mass of Σ is defined as

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{(16\pi)^3}} (16\pi - W(\Sigma)), \quad (3)$$

where $|\Sigma|$ is the area, $W(\Sigma) := \int_{\Sigma} H^2 dV_{\Sigma}$ is the Willmore functional and where we adopt the convention that the mean curvature H is the *sum* of the principal curvatures.

Evidently, if Σ is a minimal surface, then its Hawking mass is positive. It is also readily checked that the Hawking mass of a round sphere in \mathbb{R}^3 is equal to zero, as its Willmore functional equals 16π . A classical inequality due to Willmore [51] asserts that the round sphere is the unique

minimiser of the Willmore functional (up to scaling). Thus the Hawking mass of any surface in \mathbb{R}^3 is less or equal to 0, with equality if and only if the surface is a round sphere. This fact is already suggesting that it could be appropriate to consider the supremum of the Hawking mass, over a suitable class of surfaces (see the Appendix A.1 for an implementation of this idea for a notion of quasi-local mass).

Nevertheless, Christodoulou and Yau [8] showed that the Hawking mass is non-negative for stable mean curvature spheres in 3-manifolds with non-negative scalar curvature (see also the more recent work [28] by Miao–Wang–Xie). The popularity of the Hawking mass is arguably due to the very powerful monotone property (Eardley, Geroch, Jang–Wald [16]) along the Inverse Mean Curvature Flow, which was key in Huisken–Ilmanen’s [15] proof of the Riemannian Penrose inequality for a single black hole (see Bray [4] for the multiple black holes case and for a different proof).

The Bartnik mass

First of all let us mention that, after Bartnik [3] introduced the quasi-local mass named after him, several variants appeared in the literature. For convenience, here we adopt a version proposed in [4, 15]. We refer to the recent [17, 27] for reconciliation of some of the various versions and for a discussion of the topic.

Let (M^3, g) be an AF Riemannian manifold (possibly with horizon boundary) with non-negative scalar curvature and let $\Omega \subset M$ be a bounded open set with smooth topological boundary $\partial\Omega$. Recall that $\partial\Omega$ is said to be *outer-minimising* if $P(\Omega) \leq P(\Omega')$ for any set $\Omega' \subset M$ of finite perimeter (denoted with $P(\Omega')$) and finite volume such that $\Omega \subset \Omega'$. Define the *Bartnik mass* $m_B(\Omega)$ of Ω (also known as Bartnik capacity of Ω) as

$$m_B(\Omega) := \inf\{m_{\text{ADM}}(\tilde{M}, \tilde{g}) : (\tilde{M}, \tilde{g}) \in \mathcal{A}\}, \quad (4)$$

where \mathcal{A} is the set of AF manifolds (possibly with horizon boundary) with non-negative scalar curvature into which Ω isometrically embeds such that $\partial\Omega \subset \tilde{M}$ is outer-minimising.

Note that the Positive Mass Theorem (or the Riemannian Penrose inequality, in case all elements in \mathcal{A} have non-empty horizon boundary) immediately yields the non-negativity of the Bartnik mass. Another advantage of the Bartnik mass is that it satisfies a natural monotonicity property under inclusion

$$m_B(\Omega_1) \leq m_B(\Omega_2) \quad \forall \Omega_1 \Subset \Omega_2 \subset M, \text{ with } \partial\Omega_1 \text{ outer-minimising in } \Omega_2 \text{ and } \partial\Omega_2 \text{ outer minimising in } M. \quad (5)$$

Moreover, as a consequence of the proof of the Riemannian Penrose inequality via Inverse Mean Curvature Flow by Huisken–Ilmanen [15], it holds that

$$m_{\text{ADM}}(M, g) \geq m_H(\partial\Omega) \quad (6)$$

for every $\partial\Omega \subset M$ outer-minimising in the AF manifold (M, g) with non-negative scalar curvature.

Since every smooth extension $(\tilde{M}, \tilde{g}) \in \mathcal{A}$ induces the same mean curvature on $\partial\Omega \subset \tilde{M}$, the inequality (6) combined with the very definition (4) implies

$$m_B(\Omega) \geq m_H(\partial\Omega). \quad (7)$$

Computing the Bartnik mass of a subset Ω is in general a non-trivial task. For some recent works in this direction, see, for instance, Mantoulidis–Schoen [25], CabreroPacheco–Cederbaum–McCormick–Miao [6] and Miao–Xie [29]: while it is clear from the definition that it is conceivable to expect upper bounds by direct comparison with somewhat explicit competitors, the issue of finding explicit lower bounds is more subtle. The latter is one of the goals of the present paper.

1.2 | Statement of the main results

The first main result can be informally stated as follows: if the Hawking quasi-local mass is non-positive locally everywhere in an open set Ω having non-negative scalar curvature, then Ω is locally isometric to Euclidean \mathbb{R}^3 . Below is the precise statement:

Theorem 1.3 (Quasi-local rigidity Theorem 4.1). *Let (M^3, g) be a three-dimensional Riemannian manifold and let $\Omega \subset M$ be an open subset with non-negative scalar curvature. Assume that every $p \in \Omega \setminus \partial M$ admits a neighbourhood $U \subset M \setminus \partial M$ such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0. \quad (8)$$

Then $\Omega \setminus \partial M$ is locally isometric to Euclidean \mathbb{R}^3 .

In order to obtain a *global* rigidity result out of the *quasi-local* rigidity Theorem 1.3, it is useful to add a suitable asymptotic condition at infinity. A fairly mild asymptotic assumption is the next one (that to the best of our knowledge has not appeared in the literature before). In order to state it, recall that a sequence $\{p_n\} \subset M$ is said to be *diverging* if, for some (and thus for any) fixed $\bar{p} \in M$, we have $d(\bar{p}, p_n) \rightarrow \infty$ as $n \rightarrow \infty$, where d is the Riemannian distance function induced by g .

Definition 1.4. Let (M, g) be a complete Riemannian manifold. We say (M, g) is *asymptotically locally simply connected* (ALSC for short) if it is non-compact and for every $R > 0$, and any diverging sequence $\{p_n\} \subset M$, there exists $N = N(R) \geq 1$ such that the ball $B_R^g(p_n)$ is simply connected, for every $n > N(R)$.

Remark 1.5. Note that the ALSC condition is satisfied by AF manifolds since, in this case, the balls $B_R^g(p_n)$ are eventually diffeomorphic to Euclidean balls, which of course are simply connected. However, ALSC is a much milder condition than AF, as it merely requires a *local* (mild) topological control and no assumption on the metric tensor and on the *global* topology of M (outside of a compact set). Non-AF examples of ALSC manifolds include, for instance, asymptotically conical manifolds, the Bryant soliton [5] and, more generally, C^0 -asymptotically locally Euclidean manifolds [33].

Theorem 1.6 (Global rigidity Theorem 4.3). *Let (M^3, g) be a connected, complete Riemannian manifold without boundary and with non-negative scalar curvature. If every $p \in M$ admits a neighbourhood U satisfying the local non-positivity condition (8) for the Hawking mass, then (M^3, g) is isometric to a flat space form. Furthermore, if (M^3, g) is ALSC, then it is globally isometric to Euclidean \mathbb{R}^3 .*

We mention again that, inspired by the above rigidity results, in Appendix A.1 we propose a notion of ‘sup-Hawking mass’ which satisfies some natural properties of a quasi-local mass.

For the sake of the introduction, we confined the presentation to manifolds with non-negative scalar curvature. The reader interested in the extensions to the case of scalar curvature bounded below by a negative (or strictly positive constant) is referred to Section 6. Let us recall that such ambient spaces are particularly relevant when the cosmological constant in Einstein’s Equations is non-zero.

In order to prove the above rigidity results, we will compute accurate Taylor expansions of the Hawking mass on suitable spherical surfaces of small area (see Section 1.3 for some idea of the methods, and Proposition 3.2 for the precise statement). In Theorem 5.7, we will show that such spherical surfaces are outer-minimising, and thus provide a lower bound on the Bartnik mass (due to the monotonicity property (5) and the bound (7)). As a consequence, we obtain the next lower bound on the Bartnik mass in terms of curvature tensors.

Theorem 1.7 (Lower Bound on the Bartnik mass). *Let (M^3, g) be an AF Riemannian manifold with non-negative scalar curvature and with (possibly empty) horizon boundary ∂M . Let $\Omega \subset M$ be a bounded open set with smooth boundary $\partial\Omega$. Let $p \in \Omega \setminus \partial M$ and let $\bar{\rho} = \bar{\rho}(p) := \inf_{q \in \partial M \cup \partial\Omega} d(p, q)$. Then, for all $\rho \in (0, \bar{\rho}/2)$, the following lower bound holds:*

$$m_B(\Omega) \geq \frac{1}{12} \text{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \text{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \text{Sc}_p^2 \right) \rho^5 + \mathcal{O}_p(\rho^6), \quad (9)$$

where $S := \text{Ric} - \frac{1}{3} \text{Sc} g$ denotes the trace-free Ricci tensor and $\limsup_{\rho \rightarrow 0^+} \rho^{-6} |\mathcal{O}_p(\rho^6)| < \infty$.

In particular, if $m_B(\Omega) = 0$, then $\Omega \setminus \partial M$ is locally isometric to Euclidean \mathbb{R}^3 .

Remark 1.8. Note that (9) gives a strictly positive (yet small) lower bound on the Bartnik mass of Ω , provided that

- either: Ω contains a point p with $\text{Sc}_p > 0$;
- or: $\text{Sc} \equiv 0$ on Ω and there exists $p \in \Omega$ with $\|S_p\| \neq 0$.

As observed in the proof of Theorem 4.1, if Ω has non-negative scalar curvature and it does not have constant sectional curvature, then one of the two conditions above must be satisfied.

Remark 1.9. Note that the lower bound (9) is interesting only when Ω does not contain some connected component of ∂M . Indeed if Ω contains some connected component Σ of ∂M , then the Riemannian Penrose Inequality [4, 15] yields that $m_B(\Omega) \geq \sqrt{\frac{|\Sigma|}{16\pi}}$. This would give a *definite* lower bound on $m_B(\Omega)$ in contrast with the *infinitesimal* lower bound (9). However, if $\partial M \cap \Omega = \emptyset$, the lower bound (9) seems to be new and interesting.

Let us also mention the recent work [53] by Wiygul, where the first-order Taylor expansion of the Bartnik mass is computed for closed geodesic balls of small radius $\rho > 0$ and center $p \in M$, giving $\frac{1}{12} \text{Sc}_p \rho^3$. Under the additional condition that the Riemann curvature tensor vanishes at p , the first order Taylor expansion of the Bartnik mass for such geodesic balls is given by $\frac{1}{120} \Delta \text{Sc}(p) \rho^5$.

Note that these results are in accordance with the lower bound given in (9), which holds without the assumption that the Riemann curvature tensor vanishes at p .

Remark 1.10. The fact that $m_B(\Omega) = 0$ forces $\Omega \setminus \partial M$ to be locally isometric to Euclidean \mathbb{R}^3 was obtained with different methods by Huisken–Ilmanen (see [15, Positivity Property 9.2]). In the same paper, Huisken–Ilmanen write ‘careful examination of the proof (of the positivity property 9.2) should give a small, but explicit lower bound for $m_B(\Omega)$ in terms of the scalar curvature in a small region’. Our methods permit to implement this: indeed (9) gives a small, but quantitative lower bound for $m_B(\Omega)$ in terms of the scalar curvature (at first order, and in terms of other curvature tensors for higher order) in a small region (in our arguments, such a region will be a perturbed geodesic sphere of small radius).

1.3 | Some ideas of the proofs

In order to prove the quasi-rigidity Theorem 1.3, it is of course key to identify suitable competitors in order to test the condition (8). A first attempt would be to use geodesic spheres $S_{p,\rho}$ contained in the open set U . However, such surfaces are not ‘optimal enough’ for the Hawking mass. Indeed, using the expansions of the paper (which in turn build on top of [30, 31, 40]), one can check that

$$\begin{aligned} m_H(S_{p,\rho}) &= \sqrt{\frac{|S_{p,\rho}|_g}{(16\pi)^3}} \left(\frac{8\pi}{3} \text{Sc}_p \rho^2 - \left[\frac{4\pi}{27} \text{Sc}_p^2 - \frac{4\pi}{15} \Delta \text{Sc}(p) \right] \rho^4 + \mathcal{O}(\rho^5) \right) \\ &= \frac{1}{12} \text{Sc}_p \rho^3 - \left(\frac{1}{144} \text{Sc}_p^2 - \frac{1}{120} \Delta \text{Sc}(p) \right) \rho^5 + \mathcal{O}(\rho^6). \end{aligned}$$

Note that such expansions give no interesting information in case $\text{Sc} \equiv 0$. The idea is thus to ‘optimally perturb’ the geodesic spheres by suitable normal graphs, motivated by the fact (not strictly necessary for the arguments, but useful as a motivation) that the maximisers of the Hawking mass under small area constraint are indeed perturbed geodesic spheres. This principle has been already observed by Lamm–Metzger [20], who proved $W^{2,2}$ -closeness to a geodesic sphere under a small energy assumption, and by Laurain–Mondino [32], who proved smooth convergence to a geodesic sphere under a milder energy assumption. For the reader’s convenience, we will give a self-contained proof of this fact in the exact framework of the present paper in Proposition A.10 in the Appendix.

In order to compute the Hawking mass for such optimal competitors, in Lemma 3.1 we prove that the graph function $w_{p,\rho}$ for an optimally perturbed geodesic sphere must satisfy a precise expansion in terms of ρ and of curvature tensors at p ; namely it has the form

$$w_{p,\rho}(\Theta) = \left(-\frac{1}{6} \text{Ric}(\Theta, \Theta) + \frac{1}{18} \text{Sc}(p) \right) \rho^2 + \mathcal{O}_p(\rho^3), \quad (10)$$

where $\Theta \in \mathbb{S}^2$ is the parametrising coordinate, and $\limsup_{\rho \rightarrow 0} \rho^{-3} \|\mathcal{O}_p(\rho^3)\|_{C^{4,\alpha}(\mathbb{S}^2)} < \infty$.

Plugging such an expansion of the optimal normal graph $w_{p,\rho}$ into the definition of the Hawking mass gives, after some computational efforts, the expansion of m_H on such optimal competitors:

$$m_H(S_{p,\rho}(w_{p,\rho})) = \frac{1}{12} \text{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \text{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \text{Sc}_p^2 \right) \rho^5 + \mathcal{O}_p(\rho^6). \quad (11)$$

This is proved in Proposition 3.2, which represents the key technical result of the paper. Indeed, once (11) is proved, it is not hard to obtain the quasi-local rigidity Theorem 1.3 (see Theorem 4.1

in the body of the paper): basically the assumption $\text{Sc} \geq 0$ on Ω coupled with the condition that $m_H(S_{p,\rho}(w_{p,\rho})) \leq 0$ for every $p \in \Omega$ forces Ω to be Ricci flat and then flat, due to the expansion (11).

The global rigidity Theorem 1.6 (Theorem 4.3 in the body of the paper) follows from the quasi-local rigidity Theorem 1.3 and the classification of flat manifolds (see, for instance, [54]), plus a case by case analysis when applying the ALSC condition.

In order to obtain the lower bound on the Bartnik mass (Theorem 1.7) out of the expansion (11), we prove that the optimally perturbed geodesics spheres $S_{p,\rho}(w_{p,\rho})$ are outer-minimising. This is achieved in Theorem 5.7 via a blow-up argument. In order to gain strong enough compactness in such an argument, we employ the regularity theory for quasi-minimisers of the perimeter by Tamanini [47], refining previous celebrated results by De Giorgi [10].

Once it is established that $S_{p,\rho}(w_{p,\rho})$ are outer-minimising, Theorem 1.7 follows from the expansion (11) combined with the monotonicity property (5) and the bound (7).

2 | PRELIMINARIES AND NOTATION

- (1) We will use the convention that greek index letters (for example, μ, ν, η) varies from 1 to 3 while latin index letters (for example, i, j, k, l) vary from 1 to 2. We will adopt the Einstein convention for summation over repeated indices.
- (2) For a 3-dimensional Riemannian manifold (M^3, g) with Levi-Civita connection ∇ , the Riemann curvature endomorphism is given by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

for vector fields X, Y, Z on M . The associated Riemann curvature (0,4)-tensor is

$$\text{Rm}(X, Y, Z, W) = g(\mathcal{R}(Z, W)Y, X).$$

The Ricci curvature tensor is the trace over the first and third indices of Rm : that is, if $\{E_\mu\}_{\mu=1,2,3} \subset T_p M$ is an orthonormal basis, we have

$$\text{Ric}(X, Y) = \sum_{\mu=1}^3 \text{Rm}(E_\mu, X, E_\mu, Y), \quad \forall X, Y \in T_p M.$$

The scalar curvature, denoted by Sc , is the trace of the Ricci tensor:

$$\text{Sc}(p) = \sum_{\mu=1}^3 \text{Ric}(E_\mu, E_\mu).$$

We adopt the standard index notation $R_{\mu\nu} := \text{Ric}(E_\mu, E_\nu)$. A key tensor for this paper is the traceless Ricci tensor

$$S := \text{Ric} - \frac{1}{3} \text{Sc} \, g. \quad (12)$$

It is easily seen that $\|S\|^2 = \|\text{Ric}\|^2 - \frac{1}{3} \text{Sc}^2$.

- (3) Let $(\mathring{M}, \mathring{g}) \hookrightarrow (M, g)$ be an isometrically immersed, closed, 2-sided, 2-dimensional surface with inward pointing normal unit vector N . The (scalar) second fundamental form h is defined by

$$h(X, Y) = g(\nabla_X Y, N) = -g(\nabla_X N, Y), \quad (13)$$

for $X, Y \in T\Sigma$ vector fields tangent to Σ . The two eigenvalues k_1 and k_2 of h at $p \in \Sigma$ are called *the principal curvatures*. We set

$$H := k_1 + k_2 \quad \text{and} \quad D := k_1 k_2,$$

where H is called *the mean curvature*. Let \mathring{g} denote the restriction of g to $T\Sigma$ (with matrix \mathring{g}_{ij} with respect to a fixed set of coordinates) and let \mathring{g}^{ij} denote the matrix of \mathring{g}^{-1} . It holds

$$H = \text{Tr}_{\mathring{g}}(h) = \mathring{g}^{ij} h_{ij} \quad D = \det(\mathring{g}^{ik} h_{kj}) = \frac{\det h_{ij}}{\det \mathring{g}_{kl}}.$$

- (4) Large positive constants are always denoted by C . The value of C is allowed to vary from formula to formula and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we either add subscripts to C , for example, C_δ , or we mention the dependence with parenthesis, for example, $C(\delta)$. Also constants with subscripts (or with parenthesis) are allowed to vary.

2.1 | Perturbed geodesic spheres

2.1.1 | Notation about perturbed geodesic spheres

Denote with \mathbb{S}^2 the standard unit sphere in the Euclidean 3-dimensional space \mathbb{R}^3 , $\Theta \in \mathbb{S}^2$ the radial unit vector with components Θ^μ parametrised by the polar coordinates $0 < \theta^1 < \pi$ and $0 < \theta^2 < 2\pi$ chosen in order to satisfy

$$\begin{cases} \Theta^1 = \sin \theta^1 \cos \theta^2 \\ \Theta^2 = \sin \theta^1 \sin \theta^2 \\ \Theta^3 = \cos \theta^1. \end{cases}$$

Call with Θ_i the coordinate vector fields on \mathbb{S}^2 :

$$\Theta_1 := \frac{\partial \Theta}{\partial \theta^1}, \quad \Theta_2 := \frac{\partial \Theta}{\partial \theta^2}, \quad (14)$$

and $\bar{\Theta}_i$ or $\bar{\Theta}_i$ the corresponding normalised vectors:

$$\bar{\Theta}_1 = \bar{\Theta}_1 := \frac{\Theta_1}{\|\Theta_1\|}, \quad \bar{\Theta}_2 = \bar{\Theta}_2 := \frac{\Theta_2}{\|\Theta_2\|}.$$

We next define the perturbed geodesic spheres $S_{p,\rho}(w)$ in the 3-dimensional Riemannian manifold (M, g) .

Fix a point $p \in M \setminus \partial M$ and consider the exponential map Exp_p centred at p . For $\rho > 0$ small enough, the sphere $\rho \mathbb{S}^2 \subset T_p M$ is contained in the injectivity radius of the exponential map. We call $S_{p,\rho}$ the *geodesic sphere* of center p and radius ρ , parametrised by

$$\Theta \in \mathbb{S}^2 \subset T_p M \mapsto \text{Exp}_p[\rho \Theta].$$

The perturbed geodesic spheres are normal graphs on geodesic spheres, by a function w belonging to the following suitable function spaces (chosen for technical reasons in order to apply Schauder estimates in Lemma 3.1).

Denote with $C^{4,\alpha}(\mathbb{S}^2)$ (or simply $C^{4,\alpha}$) the set of the C^4 functions on \mathbb{S}^2 whose fourth derivatives, with respect to the tangent vector fields, are α -Hölder continuous ($0 < \alpha < 1$). The Laplace–Beltrami operator on \mathbb{S}^2 is denoted by $\Delta_{\mathbb{S}^2}$ or, if there is no ambiguity, by Δ . The fourth order elliptic operator $\Delta(\Delta + 2)$ induces the following orthogonal splitting of $L^2(\mathbb{S}^2)$:

$$L^2(\mathbb{S}^2) = \text{Ker}[\Delta(\Delta + 2)] \oplus \text{Ker}[\Delta(\Delta + 2)]^\perp;$$

note that the splitting makes sense since the kernel is finite (four) dimensional, thus a closed subspace.

If we consider $C^{4,\alpha}(\mathbb{S}^2)$ as a subspace of $L^2(\mathbb{S}^2)$, we can define

$$C^{4,\alpha}(\mathbb{S}^2)^\perp := C^{4,\alpha}(\mathbb{S}^2) \cap \text{Ker}[\Delta(\Delta + 2)]^\perp.$$

Of course $C^{4,\alpha}(\mathbb{S}^2)^\perp$ is a Banach space with respect to the $C^{4,\alpha}$ norm; it is the space from which we will draw the perturbation w . If there is no confusion $C^{4,\alpha}(\mathbb{S}^2)^\perp$ will be called simply $C^{4,\alpha\perp}$.

We can now define the *perturbed geodesic spheres* that we will use as ‘test’ surfaces for the Hawking mass. Fix $p \in M$, $\rho > 0$ and a small $C^{4,\alpha}(\mathbb{S}^2)$ function w ; the perturbed geodesic sphere $S_{p,\rho}(w)$ is the surface parametrised by

$$\Theta \in \mathbb{S}^2 \mapsto \text{Exp}_p[\rho(1 - w(\Theta))\Theta].$$

The tangent vector fields on $S_{p,\rho}(w)$ induced by the canonical polar coordinates on \mathbb{S}^2 are denoted by Z_i .

Following the notation of [40], given $a \in \mathbb{N}$, we denote with $\mathcal{L}_p^{(a)}(w)$ an arbitrary linear combination of the function w together with its derivatives with respect to the tangent vector fields Θ_i up to order a . The coefficients of $\mathcal{L}_p^{(a)}$ may depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C = C_p > 0$ independent on $\rho \in (0, 1)$ such that

$$\|\mathcal{L}_p^{(a)}(w)\|_{C^{k,\alpha}(\mathbb{S}^2)} \leq C \|w\|_{C^{k+a,\alpha}(\mathbb{S}^2)}.$$

Similarly, given $a, b \in \mathbb{N}$, we denote with $\mathcal{Q}_p^{(b)(a)}(w)$ an arbitrary non-linear combination, of order at least b , of the function w together with its derivatives with respect to the tangent vector fields Θ_i up to order a such that $\mathcal{Q}_p^{(b)(a)}(0) = 0$, for every $p \in M$. The coefficients of the Taylor expansion of $\mathcal{Q}_p^{(b)(a)}(w)$ in powers of w and its partial derivatives may depend on ρ and p but, for

all $k \in \mathbb{N}$, there exists a constant $C = C_p > 0$ independent on $\rho \in (0, 1)$ such that

$$\|\mathcal{Q}_p^{(b)(a)}(w_2) - \mathcal{Q}_p^{(b)(a)}(w_1)\|_{C^{k,\alpha}(\mathbb{S}^2)} \leq C(\|w_2\|_{C^{k+a,\alpha}(\mathbb{S}^2)} + \|w_1\|_{C^{k+a,\alpha}(\mathbb{S}^2)})^{b-1} \times \|w_2 - w_1\|_{C^{k+a,\alpha}(\mathbb{S}^2)}, \quad (15)$$

provided $\|w_l\|_{C^a(\mathbb{S}^2)} \leq 1$, $l = 1, 2$. We also agree that $\mathcal{O}_p(\rho^d)$ denotes an arbitrary smooth function on \mathbb{S}^2 that might depend on p but which is bounded by a constant (possibly dependent on p) times ρ^d in $C^k(B_1(p))$ topology, for all $k \in \mathbb{N}$.

2.1.2 | Expansions of geometric quantities

In this subsection, we recall the Taylor expansion of the geometric quantities associated to the a perturbed geodesic sphere $S_{p,\rho}(w)$, appearing in the Willmore functional and its first derivative (for the proofs see, [40] and [31, Section 3.1]). These will be used in later sections and hold for any Riemannian 3-manifold.

For the following expansions, we will fix the (polar) coordinate vector fields Θ_i on \mathbb{S}^2 defined in (14), and the corresponding coordinate vector fields Z_i on $S_{p,\rho}(w)$; that is, we will use the notation $\dot{g}_{ij} := \dot{g}(Z_i, Z_j)$ (respectively, $(g_{\mathbb{S}^2})_{ij} := g(\Theta_i, \Theta_j)$) and analogously $h_{ij} := h(Z_i, Z_j)$. The derivatives of the function $w : \mathbb{S}^2 \rightarrow \mathbb{R}$ with respect to Θ_i are denoted by w_i . All the curvature terms, all the covariant derivatives and all the scalar products are meant to be evaluated at p (since we fixed normal coordinates centred at p , at p the metric is Euclidean).

The following expansions hold:

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} + \frac{1}{3}g(\mathcal{R}(\Theta, E_\mu)\Theta, E_\nu)(1-w)^2\rho^2 + \frac{1}{6}g(\nabla_\Theta\mathcal{R}(\Theta, E_\mu)\Theta, E_\nu)(1-w)^3\rho^3 \\ &\quad + \frac{1}{20}g(\nabla_{\Theta\Theta}^2\mathcal{R}(\Theta, E_\mu)\Theta, E_\nu)(1-w)^4\rho^4 \\ &\quad + \frac{2}{45}g(\mathcal{R}(\Theta, E_\mu)\Theta, E_\tau)g(\mathcal{R}(\Theta, E_\nu)\Theta, E_\tau)(1-w)^4\rho^4 \\ &\quad + \mathcal{O}_p(\rho^5) + \rho^5\mathcal{L}_p^{(0)}(w) + \rho^5\mathcal{Q}_p^{(2)(0)}(w) \\ \dot{g}_{ij} &= g_{ij}^{\mathbb{S}^2}(1-w)^2\rho^2 + w_iw_j\rho^2 + \frac{1}{3}g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^4\rho^4 \\ &\quad + \frac{1}{6}g(\nabla_\Theta\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^5\rho^5 + \frac{1}{20}g(\nabla_{\Theta\Theta}^2\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^6\rho^6 \\ &\quad + \frac{2}{45}\delta^{\mu\nu}g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_\mu)g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_\nu)(1-w)^6\rho^6 \\ &\quad + \mathcal{O}(\rho^7) + \rho^7\mathcal{L}_p^{(0)}(w) + \rho^7\mathcal{Q}_p^{(2)(0)}(w) \\ \dot{g}^{ij} &= g_{\mathbb{S}^2}^{ij}(1-w)^{-2}\rho^{-2} - g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}w_lw_k(1-w)^{-4}\rho^{-2} - \frac{1}{3}g_{\mathbb{S}^2}^{il}g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kj} \\ &\quad - \frac{1}{6}g_{\mathbb{S}^2}^{il}g(\nabla_\Theta\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kj}(1-w)\rho - \frac{1}{20}g_{\mathbb{S}^2}^{il}g(\nabla_{\Theta\Theta}^2\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kj}(1-w)^2\rho^2 \end{aligned} \quad (16)$$

$$\begin{aligned}
& -\frac{2}{45}\delta^{\mu\nu}g_{\mathbb{S}^2}^{il}g(\mathcal{R}(\Theta, \Theta_l)\Theta, E_\mu)g(\mathcal{R}(\Theta, \Theta_k)\Theta, E_\nu)g_{\mathbb{S}^2}^{kj}(1-w)^2\rho^2 \\
& +\frac{1}{9}g_{\mathbb{S}^2}^{il}g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kn}g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m)g_{\mathbb{S}^2}^{mj}(1-w)^2\rho^2 \\
& +\mathcal{O}_p(\rho^3)+\rho^3\mathcal{L}_p^{(0)}(w)+\rho^2\mathcal{Q}_p^{(2)(0)}(w)+\rho^{-2}\mathcal{Q}_p^{(4)(1)}(w).
\end{aligned} \tag{17}$$

For the details of the derivation of the next expansion, the interested reader can see Appendix A.3.

$$\begin{aligned}
h_{ij} &= g_{ij}^{\mathbb{S}^2}(1-w)\rho + (\text{Hess}_{\mathbb{S}^2}(w))_{ij}\rho \\
& +\frac{1}{2}g_{ij}^{\mathbb{S}^2}g_{\mathbb{S}^2}^{kl}w_kw_l\rho + w_kg_{\mathbb{S}^2}^{kl}\left(g_{jl}^{\mathbb{S}^2}w_i + g_{il}^{\mathbb{S}^2}w_j - g_{ij}^{\mathbb{S}^2}w_l\right)\rho \\
& +\frac{2}{3}g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^3\rho^3 \\
& +\frac{1}{6}w_kg_{\mathbb{S}^2}^{kn}g_{\mathbb{S}^2}^{ml}g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m)\left(\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2}\right)\rho^3 \\
& -\frac{1}{6}w_kg_{\mathbb{S}^2}^{kl}\left(\partial_i g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)\right)\rho^3 \\
& +\frac{5}{12}g(\nabla_\Theta \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)\rho^4 + \frac{3}{20}g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)\rho^5 \\
& +\frac{2}{15}\delta^{\mu\nu}g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_\mu)g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_\nu)\rho^5 \\
& +\mathcal{O}_p(\rho^6)+\rho^4\mathcal{L}_p^{(1)}(w)+\rho\mathcal{Q}_p^{(3)(2)}(w)+\rho^3\mathcal{Q}_p^{(2)(1)}(w).
\end{aligned} \tag{18}$$

Recalling that $H = g^{ij}h_{ij}$, the combination of (17) and (18) gives

$$\begin{aligned}
H &= 2\rho^{-1} + (2 + \Delta_{\mathbb{S}^2})w\rho^{-1} + 2w(w + \Delta_{\mathbb{S}^2}w)\rho^{-1} \\
& +\frac{1}{6}w_kg_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{kn}g_{\mathbb{S}^2}^{ml}g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m)\left(\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2}\right)\rho \\
& -\frac{1}{6}w_kg_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{kl}\left(\partial_i g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)\right)\rho \\
& -\frac{1}{3}g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)(\text{Hess}_{\mathbb{S}^2}(w))_{ij}\rho - \frac{1}{3}\text{Ric}(\Theta, \Theta)(1-w)\rho \\
& +\frac{1}{4}g_{\mathbb{S}^2}^{ij}g(\nabla_\Theta \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^2\rho^2 \\
& +\left[\frac{1}{10}g_{\mathbb{S}^2}^{ij}g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) + \frac{4}{45}g_{\mathbb{S}^2}^{ij}\delta^{\mu\nu}g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_\mu)g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_\nu) \right. \\
& \quad \left. -\frac{1}{9}g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)\right](1-w)^3\rho^3 \\
& +\mathcal{O}_p(\rho^4)+\rho^2\mathcal{L}_p^{(1)}(w)+\rho^{-1}\mathcal{Q}_p^{(3)(2)}(w)+\rho\mathcal{Q}_p^{(2)(1)}(w),
\end{aligned} \tag{19}$$

and thus

$$\begin{aligned}
 H^2 = & \left[4 + 4(2 + \Delta_{\mathbb{S}^2})w + 8w(w + \Delta_{\mathbb{S}^2}w) + ((2 + \Delta_{\mathbb{S}^2})w)^2 \right] \rho^{-2} \\
 & + \left[\frac{2}{3} w_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m)) \left(\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2} \right) \right. \\
 & - \frac{2}{3} \text{Ric}(\Theta, \Theta)(2 + \Delta_{\mathbb{S}^2}w) - \frac{4}{3} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k) (\text{Hess}_{\mathbb{S}^2}(w))_{ij} \\
 & \left. - \frac{2}{3} w_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kl} (\partial_i g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)) \right] \\
 & + \left[g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) \right] \rho \\
 & + \left[\frac{2}{5} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta}^2 \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) + \frac{16}{45} \delta^{\mu\nu} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_{\nu}) \right. \\
 & \left. - \frac{4}{9} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k) + \frac{1}{9} (\text{Ric}(\Theta, \Theta))^2 \right] \rho^2 \\
 & + \mathcal{O}_p(\rho^3) + \rho \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(1)}(w) + \rho^{-2} \mathcal{Q}_p^{(3)(2)}(w).
 \end{aligned} \tag{20}$$

The determinant of the first fundamental form will be useful to compute integrals on $S_{p,\rho}(w)$ and can be expanded as

$$\begin{aligned}
 \det \dot{g} = & \sin^2 \theta^1 \rho^4 \left[(1-w)^4 + g_{\mathbb{S}^2}^{ij} w_i w_j - \frac{1}{3} \text{Ric}(\Theta, \Theta)(1-w)^6 \rho^2 \right. \\
 & + \frac{1}{6} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^7 \rho^3 + \frac{1}{20} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta}^2 \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^8 \rho^4 \\
 & + \frac{2}{45} \delta^{\mu\nu} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_{\nu})(1-w)^8 \rho^4 \\
 & \left. + \frac{1}{9} g(\mathcal{R}(\Theta, \Theta_1)\Theta, \Theta_1) g(\mathcal{R}(\Theta, \bar{\Theta}_2)\Theta, \bar{\Theta}_2)(1-w)^8 \rho^4 - \frac{1}{9} g(\mathcal{R}(\Theta, \Theta_1)\Theta, \bar{\Theta}_2)^2 (1-w)^8 \rho^4 \right] \\
 & + \mathcal{O}_p(\rho^9) + \rho^9 \mathcal{L}_p^{(0)}(w) + \rho^6 \mathcal{Q}_p^{(2)(1)}(w) + \rho^4 \mathcal{Q}_p^{(4)(1)}(w).
 \end{aligned} \tag{21}$$

3 | HAWKING MASS OF AN OPTIMALLY PERTURBED GEODESIC SPHERE

Motivated by Proposition A.10 in the Appendix, it is natural to choose the class of perturbed geodesic spheres $S_{p,\rho}(w)$ as surfaces to ‘test’ the positivity of the Hawking mass. More precisely, by using the Euler–Lagrange equation of the Willmore functional (under area constraint), we will find an expansion of the perturbation w which is necessary for the perturbed geodesic sphere $S_{p,\rho}(w)$ to be a critical point (under area constraint). Let us stress that in Lemma 3.1 we do not claim to construct area-constrained Willmore surfaces centred at any point (that would be false, as a necessary condition would be that such a point is a critical point for the scalar curvature; see [20,

32]). The goal of Lemma 3.1 is thus merely to suggest an *ansatz* for an ‘optimal’ perturbation (so the reader could take for granted that a convenient choice of perturbation w is given by the expansion (22)). Such ‘optimally’ perturbed geodesic spheres will be the key geometric objects to prove our main theorems. Throughout the section, (M, g) will be an arbitrary Riemannian 3-manifold.

3.1 | An optimal perturbation

In this subsection, we compute the expansion (as $\rho \rightarrow 0$) that a perturbation w has to satisfy if the perturbed geodesic sphere $S_{p,\rho}(w)$ is a critical point (under area constraint) of the Willmore functional or, equivalently, of the Hawking mass.

Lemma 3.1. *For a fixed a compact subset $\mathcal{K} \Subset M \setminus \partial M$, there exists $\rho_0 > 0$, $r > 0$ and a map $w_{(\cdot,\cdot)} : \mathcal{K} \times (0, \rho_0] \rightarrow C^{4,\alpha}(\mathbb{S}^2)^\perp$, $(p, \rho) \mapsto w_{p,\rho}$ such that if $S_{p,\rho}(w)$ is a critical point of the Willmore functional under area constraint (or equivalently, a critical point of the Hawking mass under area constraint) with $(p, \rho, w) \in \mathcal{K} \times (0, \rho_0] \times B_{C^{4,\alpha}(\mathbb{S}^2)^\perp}(0, r)$, then $w = w_{p,\rho}$. Moreover, for every $(p, \rho) \in \mathcal{K} \times (0, \rho_0]$ the following expansion holds*

$$w_{p,\rho}(\Theta) = \left(-\frac{1}{6} \text{Ric}(\Theta, \Theta) + \frac{1}{18} \text{Sc}(p) \right) \rho^2 + \mathcal{O}_p(\rho^3), \quad (22)$$

where $\Theta \in \mathbb{S}^2$ is the parametrising coordinate, and $\limsup_{\rho \rightarrow 0} \rho^{-3} \|\mathcal{O}_p(\rho^3)\|_{C^{4,\alpha}(\mathbb{S}^2)} < \infty$.

Proof. Step 1: We show that, for every fixed compact subset $\mathcal{K} \Subset M$, there exists $\rho_0 > 0$, $r > 0$ and a map $w_{(\cdot,\cdot)} : \mathcal{K} \times (0, \rho_0] \rightarrow C^{4,\alpha}(\mathbb{S}^2)^\perp$, $(p, \rho) \mapsto w_{p,\rho}$ such that if $S_{p,\rho}(w)$ is a critical point of the Willmore functional under area constraint (or equivalently, a critical point of the Hawking mass under area constraint) with $(p, \rho, w) \in \mathcal{K} \times (0, \rho_0] \times B(0, r)$, then $w = w_{p,\rho}$ and $\lim_{\rho \rightarrow 0} \|w_{p,\rho}\|_{C^{4,\alpha}(\mathbb{S}^2)} = 0$.

If $S_{p,\rho}(w)$ is a critical point of the Willmore functional under area constraint then it satisfies the area-constrained Euler–Lagrange equation for the Willmore functional (see, for instance, [21] for a derivation of the formula):

$$2\Delta H + H(H^2 - 4D + 2\text{Ric}(N, N)) = \lambda H, \quad (23)$$

where $\lambda \in \mathbb{R}$ plays the role of Lagrange multiplier and N is the inward pointing unit normal vector. As proved in [32, Lemma 2.2], the Lagrange multiplier λ in (23) remains bounded under the assumptions of the lemma. Using the geometric expansions of Section 2.1.2, one can check that (23) for a perturbed geodesic sphere $S_{p,\rho}(w)$ gives (see, for instance, [30, Proposition 3.2])

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)w + \mathcal{O}_p(\rho^2) + \rho^2 \mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w) = 0. \quad (24)$$

In particular, setting $P : L^2(\mathbb{S}^2) \rightarrow \text{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)]^\perp$ the orthogonal projection, *a fortiori* (24) yields

$$P \left[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)w + \mathcal{O}_p(\rho^2) + \rho^2 \mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w) \right] = 0. \quad (25)$$

Since the operator $\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)$ is invertible on the space orthogonal to its Kernel and $w \in C^{4,\alpha}(\mathbb{S}^2)^\perp = \text{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)]^\perp \cap C^{4,\alpha}(\mathbb{S}^2)$, setting

$$K := [\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)]^{-1} : \text{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)]^\perp \subseteq L^2(\mathbb{S}^2) \rightarrow \text{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)]^\perp,$$

equation (25) is equivalent to the fixed point problem

$$w = K \circ P \left[\mathcal{O}_p(\rho^2) + \rho^2 \mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w) \right] =: F_{p,\rho}(w). \quad (26)$$

Using Schauder estimates, one can check that, for every fixed compact set $\mathcal{K} \Subset M$, there exist $\rho_0 > 0$ and $r > 0$ such that for all $p \in \mathcal{K}$ and $\rho \in [0, \rho_0]$ the map

$$F_{p,\rho} : B(0, r) \subset C^{4,\alpha}(\mathbb{S}^2)^\perp \rightarrow C^{4,\alpha}(\mathbb{S}^2)^\perp$$

is a contraction (see [30, Lemma 4.4] for the details). Thus, for every $p \in \mathcal{K}$ and $\rho \in (0, \rho_0]$, there exists a unique $w_{p,\rho} \in B(0, r) \subset C^{4,\alpha}(\mathbb{S}^2)^\perp$ such that the surface $S_{p,\rho}(w_{p,\rho})$ is an area-constrained Willmore surface. By continuous dependence on parameters of fixed points in contractions, it also follows that $\lim_{\rho \rightarrow 0} \|w_{p,\rho}\|_{C^{4,\alpha}(\mathbb{S}^2)} = 0$ (again, see [30, Lemma 4.4] for the details).

Step 2: we show that the expansion (22) holds.

We set the ansatz

$$w_{p,\rho} = \rho^2 \bar{w}_p + \mathcal{O}_p(\rho^3), \quad (27)$$

where $\bar{w}_p \in C^{4,\alpha}(\mathbb{S}^2)^\perp$ depends on p but not on ρ and $\limsup_{\rho \rightarrow 0} \rho^{-3} \|\mathcal{O}_p(\rho^3)\|_{C^{4,\alpha}(\mathbb{S}^2)} < \infty$. In order to show that $w_{p,\rho}$ given in step 1 satisfies the ansatz (27) and the expansion (22), we need to improve the expansion of the Euler–Lagrange equation of the Willmore functional (23)–(24). To this aim, using the expansions of Section 2.1.2, one can check that (see the proof of [31, Proposition 3.9] for more details)

$$\begin{aligned} \Delta_{S_{p,\rho}(w)} H &= \frac{1}{\rho^3} \Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)w - \frac{1}{3\rho} \Delta_{\mathbb{S}^2} \text{Ric}(\Theta, \Theta) + \mathcal{O}_p(\rho^0) + \frac{1}{\rho^2} \mathcal{L}_p^{(4)}(w) + \frac{1}{\rho^3} \mathcal{Q}_p^{(2)(4)}(w), \\ H^2 - 4D &= \mathcal{O}_p(\rho^2) + \mathcal{L}_p^{(2)}(w) + \frac{1}{\rho^2} \mathcal{Q}_p^{(2)(2)}(w), \end{aligned}$$

which, plugged into (23), give

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)w = \left(\frac{1}{3} \Delta_{\mathbb{S}^2} \text{Ric}(\Theta, \Theta) - 2\text{Ric}(\Theta, \Theta) + \lambda \right) \rho^2 + \mathcal{O}_p(\rho^3) + \rho \mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w). \quad (28)$$

Inserting the ansatz (27), that we write as $w = \bar{w}\rho^2 + \mathcal{O}_p(\rho^3)$, in (28) yields

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)\bar{w} = \frac{1}{3} \Delta_{\mathbb{S}^2} \text{Ric}(\Theta, \Theta) - 2\text{Ric}(\Theta, \Theta) + \lambda. \quad (29)$$

We solve this PDE by Fourier methods, using the knowledge of the eigenfunctions of $\Delta_{\mathbb{S}^2}$. To this aim, writing the radial unit vector as $\Theta = x^\mu E_\mu \in T_p M$ where $\{E_\mu\}_{\mu=1,2,3}$ is an orthonormal basis

of $T_p M$, we have

$$\begin{aligned}\operatorname{Ric}(\Theta, \Theta) &= \operatorname{Ric}(x^\mu E_\mu, x^\nu E_\nu) = R_{\mu\nu} x^\mu x^\nu = \sum_{\mu \neq \nu} R_{\mu\nu} x^\mu x^\nu + \sum_{\mu} R_{\mu\mu} x^\mu x^\mu \\ &= \sum_{\mu \neq \nu} R_{\mu\nu} x^\mu x^\nu + \sum_{\mu} R_{\mu\mu} \left((x^\mu)^2 - \frac{1}{3} \right) + \frac{1}{3} \sum_{\mu} R_{\mu\mu},\end{aligned}$$

where $\sum_{\mu} R_{\mu\mu} = \operatorname{Sc}(p)$ is the scalar curvature at p and where we used that $\sum_{\mu} (x^\mu)^2 = 1$, since $\Theta \in \mathbb{S}^2$.

Recall that the eigenfunctions of $\Delta_{\mathbb{S}^2}$ relative to the second eigenvalue $\lambda_2 = -6$ are $x^\mu x^\nu$, $\mu \neq \nu$, and $(x^\mu)^2 - (x^\nu)^2$, $\mu \neq \nu$, hence

$$(x^1)^2 - \frac{1}{3} = \frac{1}{3} \left([(x^1)^2 - (x^2)^2] + [(x^1)^2 - (x^3)^2] \right)$$

is an element of the eigenspace relative to $\lambda_2 = -6$ (and analogously for the others $(x^\mu)^2$). Therefore,

$$\operatorname{Ric}(\Theta, \Theta) - \frac{1}{3} \operatorname{Sc}(p) = \sum_{\mu \neq \nu} R_{\mu\nu} x^\mu x^\nu + \sum_{\mu} R_{\mu\mu} \left((x^\mu)^2 - \frac{1}{3} \right)$$

is an eigenfunction of $\Delta_{\mathbb{S}^2}$ with eigenvalue -6 . We can then rewrite equation (29) as

$$\begin{aligned}\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2)\bar{w} &= \frac{1}{3} \Delta_{\mathbb{S}^2} \left(\operatorname{Ric}(\Theta, \Theta) - \frac{1}{3} \operatorname{Sc}(p) \right) - 2 \operatorname{Ric}(\Theta, \Theta) + \lambda \\ &= -4 \left(\operatorname{Ric}(\Theta, \Theta) - \frac{1}{3} \operatorname{Sc}(p) \right) - \frac{2}{3} \operatorname{Sc}(p) + \lambda.\end{aligned}$$

Setting the value of the Lagrange multiplier as $\lambda = \frac{2}{3} \operatorname{Sc}(p)$ and noting that

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2) \left[-\frac{1}{6} \left(\operatorname{Ric}(\Theta, \Theta) - \frac{1}{3} \operatorname{Sc}(p) \right) \right] = -4 \left(\operatorname{Ric}(\Theta, \Theta) - \frac{1}{3} \operatorname{Sc}(p) \right)$$

we conclude that

$$\bar{w} = -\frac{1}{6} \operatorname{Ric}(\Theta, \Theta) + \frac{1}{18} \operatorname{Sc}(p). \quad (30)$$

Summarising, we showed that $w_{p,\rho}$ as in the ansatz (27) with $\bar{w}_p = \bar{w}$ given in (30) solves (28) or, equivalently, (23). The proof is complete once we recall that, by step 1, the solution w of (28) is unique provided $\|w\|_{C^{4,\alpha}(\mathbb{S}^2)} < r$. \square

3.2 | Computation of the Hawking mass

Proposition 3.2. *Let $S_{p,\rho}(w_{p,\rho})$ be an optimally perturbed geodesic sphere, that is, $w_{p,\rho}$ is given in (22). Then the Hawking mass of $S_{p,\rho}(w_{p,\rho})$ has the following expansion:*

$$\begin{aligned}
m_H(S_{p,\rho}(w_{p,\rho})) &:= \sqrt{\frac{|S_{p,\rho}(w_{p,\rho})|}{(16\pi)^3}} (16\pi - W(S_{p,\rho}(w_{p,\rho}))) \\
&= \frac{1}{12} \text{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \text{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \text{Sc}_p^2 \right) \rho^5 + \mathcal{O}_p(\rho^6), \quad (31)
\end{aligned}$$

where Sc is the scalar curvature and $S := \text{Ric} - \frac{1}{3} \text{Sc} \, g$ is the traceless Ricci tensor.

Proof. In order to keep notation short, throughout the proof we will write w in place of $w_{p,\rho}$.

Step 1: the Willmore functional integrand.

To compute the Hawking mass, we find the expansion for the Willmore functional $W(S_{p,\rho}(w)) := \int_{S_{p,\rho}(w)} H^2 \, dV = \int_{\mathbb{S}^2} H^2 \sqrt{\det \dot{g}} \, d\theta^1 d\theta^2$. Using (21), the Taylor expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3)$ and the fact that $w = \mathcal{O}_p(\rho^2)$, we have:

$$\begin{aligned}
\sqrt{\det \dot{g}} &= \sin \theta^1 \rho^2 \left[(1-w)^2 + \frac{1}{2} g_{\mathbb{S}^2}^{ij} w_i w_j - \frac{1}{6} \text{Ric}(\Theta, \Theta) \rho^2 + \frac{2}{3} w \text{Ric}(\Theta, \Theta) \rho^2 \right. \\
&\quad + \frac{1}{12} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \rho^3 + \frac{1}{40} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \rho^4 \\
&\quad + \frac{1}{45} \delta^{\mu\nu} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta, \Theta_i) \Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j) \Theta, E_{\nu}) \rho^4 - \frac{1}{18} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \bar{\Theta}_2)^2 \rho^4 \\
&\quad \left. + \frac{1}{18} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) g(\mathcal{R}(\Theta, \bar{\Theta}_2) \Theta, \bar{\Theta}_2) \rho^4 - \frac{1}{72} \text{Ric}(\Theta, \Theta)^2 \rho^4 \right] + \mathcal{O}_p(\rho^7). \quad (32)
\end{aligned}$$

Multiplying (32) with (20), we obtain the integrand of the Willmore functional evaluated on a perturbed geodesic sphere:

$$\begin{aligned}
H^2 \sqrt{\det \dot{g}} &= \sin \theta^1 \left[\left[4 + 4\Delta_{\mathbb{S}^2} w + 4w\Delta_{\mathbb{S}^2} w + (\Delta_{\mathbb{S}^2} w)^2 + 2g_{\mathbb{S}^2}^{ij} w_i w_j \right] \right. \\
&\quad + \left[\frac{2}{3} w_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n) \Theta, \Theta_m) (\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2}) \right. \\
&\quad - \frac{2}{3} w_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kl} (\partial_i g(\mathcal{R}(\Theta, \Theta_j) \Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j)) \\
&\quad - \frac{4}{3} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_l) \Theta, \Theta_k) (\text{Hess}_{\mathbb{S}^2}(w))_{ij} - 2\text{Ric}(\Theta, \Theta) + 4w\text{Ric}(\Theta, \Theta) \\
&\quad - \frac{4}{3} \text{Ric}(\Theta, \Theta) \Delta_{\mathbb{S}^2} w \left. \right] \rho^2 + \left[\frac{4}{3} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \right] \rho^3 \\
&\quad + \left[\frac{1}{2} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) + \frac{4}{9} \delta^{\mu\nu} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta, \Theta_i) \Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j) \Theta, E_{\nu}) \right. \\
&\quad - \frac{4}{9} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) g(\mathcal{R}(\Theta, \Theta_l) \Theta, \Theta_k) \\
&\quad + \frac{2}{9} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) g(\mathcal{R}(\Theta, \bar{\Theta}_2) \Theta, \bar{\Theta}_2) \\
&\quad \left. - \frac{2}{9} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \bar{\Theta}_2)^2 + \frac{5}{18} \text{Ric}(\Theta, \Theta)^2 \right] \rho^4 + \mathcal{O}_p(\rho^5) \left. \right].
\end{aligned}$$

Inserting $w = \bar{w}\rho^2 + \mathcal{O}_p(\rho^3)$ yields

$$\begin{aligned}
 H^2 \sqrt{\det \bar{g}} = \sin \theta^1 & \left[4 + [4\Delta_{\mathbb{S}^2} \bar{w} - 2\text{Ric}(\Theta, \Theta)] \rho^2 + \left[\frac{4}{3} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \right] \rho^3 \right. \\
 & + \left[4\bar{w} \Delta_{\mathbb{S}^2} \bar{w} + (\Delta_{\mathbb{S}^2} \bar{w})^2 + 2g_{\mathbb{S}^2}^{ij} \bar{w}_i \bar{w}_j + 4\bar{w} \text{Ric}(\Theta, \Theta) - \frac{4}{3} \text{Ric}(\Theta, \Theta) \Delta_{\mathbb{S}^2} \bar{w} \right. \\
 & + \frac{2}{3} \bar{w}_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n) \Theta, \Theta_m) (\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2}) \\
 & - \frac{2}{3} \bar{w}_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kl} (\partial_i g(\mathcal{R}(\Theta, \Theta_j) \Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j)) \\
 & - \frac{4}{3} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_l) \Theta, \Theta_k) (\text{Hess}_{\mathbb{S}^2} \bar{w})_{ij} - \frac{2}{9} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \bar{\Theta}_2)^2 + \frac{5}{18} \text{Ric}(\Theta, \Theta)^2 \\
 & - \frac{4}{9} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) g(\mathcal{R}(\Theta, \Theta_l) \Theta, \Theta_k) \\
 & + \frac{2}{9} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) g(\mathcal{R}(\Theta, \bar{\Theta}_2) \Theta, \bar{\Theta}_2) + \frac{1}{2} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \\
 & \left. \left. + \frac{4}{9} \delta^{\mu\nu} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta, \Theta_i) \Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j) \Theta, E_{\nu}) \right] \rho^4 + \mathcal{O}_p(\rho^5) \right] \quad (33)
 \end{aligned}$$

Step 2: simplifying the Willmore functional integrand.

We will use the following computations for the derivatives of $\bar{w} = -\frac{1}{6} \text{Ric}(\Theta, \Theta) + \frac{1}{18} \text{Sc}(p)$:

$$\bar{w}_k := \partial_k \left(-\frac{1}{6} \text{Ric}(\Theta, \Theta) + \frac{1}{18} \text{Sc}(p) \right) = -\frac{1}{6} \partial_k (\text{Ric}(\Theta, \Theta)) = -\frac{1}{3} \text{Ric}(\Theta, \Theta_k) \quad (34)$$

$$\bar{w}_{kj} := \partial_j \left(-\frac{1}{3} \text{Ric}(\Theta, \Theta_k) \right) = -\frac{1}{3} (\text{Ric}(\Theta_j, \Theta_k) + \text{Ric}(\Theta, \Theta_{kj}))$$

which, combined with the fact that

$$\begin{aligned}
 \Theta_{11} &= -\Theta \\
 \Theta_{12} &= \Theta_{21} = \cot \theta^1 \Theta_2 \\
 \Theta_{22} &= -\sin \theta^1 \cos \theta^1 \Theta_1 - \sin^2 \theta^1 \Theta
 \end{aligned} \quad (35)$$

yields

$$\begin{aligned}
 \bar{w}_{11} &= -\frac{1}{3} (\text{Ric}(\Theta_1, \Theta_1) - \text{Ric}(\Theta, \Theta)) \\
 \bar{w}_{12} &= \bar{w}_{21} = -\frac{1}{3} (\text{Ric}(\Theta_1, \Theta_2) + \cot \theta^1 \text{Ric}(\Theta, \Theta_2)) \\
 \bar{w}_{22} &= -\frac{1}{3} (\text{Ric}(\Theta_2, \Theta_2) - \sin \theta^1 \cos \theta^1 \text{Ric}(\Theta, \Theta_1) - \sin^2 \theta^1 \text{Ric}(\Theta, \Theta)).
 \end{aligned} \quad (36)$$

Using (34) and recalling that $\Delta_{\mathbb{S}^2} \bar{w} = -6\bar{w}$, we can rewrite the first line of the terms multiplying ρ^4 in (33) as

$$\begin{aligned} & 4\bar{w}\Delta_{\mathbb{S}^2}\bar{w} + (\Delta_{\mathbb{S}^2}\bar{w})^2 + 2g_{\mathbb{S}^2}^{ij}\bar{w}_i\bar{w}_j + 4\bar{w}\text{Ric}(\Theta, \Theta) - \frac{4}{3}\text{Ric}(\Theta, \Theta)\Delta_{\mathbb{S}^2}\bar{w} \\ &= -\frac{5}{3}\text{Ric}(\Theta, \Theta)^2 + \frac{4}{9}\text{Sc}(p)\text{Ric}(\Theta, \Theta) + \frac{1}{27}\text{Sc}(p)^2 + \frac{2}{9}(\text{Ric}(\Theta, \Theta_1)^2 + \text{Ric}(\Theta, \bar{\Theta}_2)^2). \end{aligned} \quad (37)$$

The second line of the terms multiplying ρ^4 in (33) can be rewritten as

$$\begin{aligned} & \frac{2}{3}\bar{w}_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m) \left(\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2} \right) \\ &= \sum_k -\frac{2}{3}\bar{w}_k g_{\mathbb{S}^2}^{22} g_{\mathbb{S}^2}^{kk} g_{\mathbb{S}^2}^{11} g(\mathcal{R}(\Theta, \Theta_k)\Theta, \Theta_1) (2 \sin \theta^1 \cos \theta^1) \\ &= \frac{4 \cot \theta^1}{9} \text{Ric}(\Theta, \Theta_1) g(\mathcal{R}(\Theta, \Theta_1)\Theta, \Theta_1) + \frac{4 \cot \theta^1}{9} \text{Ric}(\Theta, \bar{\Theta}_2) g(\mathcal{R}(\Theta, \bar{\Theta}_2)\Theta, \Theta_1), \end{aligned} \quad (38)$$

where the first equality follows because the only non-zero terms occur when $i = j$, $k = n$ and $m = l$ (note that $\partial_i g_{jk}^{\mathbb{S}^2}$ is only non-zero when $j = k = 2$ and $i = 1$), and for the second equality we used (34). The third line of the terms multiplying ρ^4 in (33) can be rewritten as

$$\begin{aligned} & -\frac{2}{3}\bar{w}_k g_{\mathbb{S}^2}^{ij} g_{\mathbb{S}^2}^{kl} (\partial_i g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)) \\ &= \sum_{i,k} -\frac{2}{3}\bar{w}_k g_{\mathbb{S}^2}^{ii} g_{\mathbb{S}^2}^{kk} (2\partial_i g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_k) - \partial_k g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_i)) \\ &= \sum_{i,k} -\frac{2}{3}\bar{w}_k g_{\mathbb{S}^2}^{ii} g_{\mathbb{S}^2}^{kk} \left[2(g(\mathcal{R}(\Theta, \Theta_{ii})\Theta, \Theta_k) + g(\mathcal{R}(\Theta, \Theta_i)\Theta_i, \Theta_k) + g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_{ki})) \right. \\ &\quad \left. - g(\mathcal{R}(\Theta_k, \Theta_i)\Theta, \Theta_i) - g(\mathcal{R}(\Theta, \Theta_{ik})\Theta, \Theta_i) - g(\mathcal{R}(\Theta, \Theta_i)\Theta_k, \Theta_i) - g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_{ik}) \right] \\ &= -\frac{2}{3}\bar{w}_2 g_{\mathbb{S}^2}^{11} g_{\mathbb{S}^2}^{22} \left[2(g(\mathcal{R}(\Theta, \Theta_1)\Theta_1, \Theta_2) + g(\mathcal{R}(\Theta, \Theta_1)\Theta, \Theta_{12})) \right. \\ &\quad \left. - g(\mathcal{R}(\Theta_2, \Theta_1)\Theta, \Theta_1) - g(\mathcal{R}(\Theta, \Theta_{12})\Theta, \Theta_1) - g(\mathcal{R}(\Theta, \Theta_1)\Theta_2, \Theta_1) - g(\mathcal{R}(\Theta, \Theta_1)\Theta, \Theta_{12}) \right] \\ &\quad -\frac{2}{3}\bar{w}_1 g_{\mathbb{S}^2}^{22} g_{\mathbb{S}^2}^{11} \left[2(g(\mathcal{R}(\Theta, \Theta_{22})\Theta, \Theta_1) + g(\mathcal{R}(\Theta, \Theta_2)\Theta_2, \Theta_1) + g(\mathcal{R}(\Theta, \Theta_2)\Theta, \Theta_{21})) \right. \\ &\quad \left. - g(\mathcal{R}(\Theta_1, \Theta_2)\Theta, \Theta_2) - g(\mathcal{R}(\Theta, \Theta_{21})\Theta, \Theta_2) - g(\mathcal{R}(\Theta, \Theta_2)\Theta_1, \Theta_2) - g(\mathcal{R}(\Theta, \Theta_2)\Theta, \Theta_{21}) \right] \\ &\quad -\frac{2}{3}\bar{w}_2 g_{\mathbb{S}^2}^{22} g_{\mathbb{S}^2}^{22} \left[2(g(\mathcal{R}(\Theta, \Theta_{22})\Theta, \Theta_2) + g(\mathcal{R}(\Theta, \Theta_2)\Theta, \Theta_{22})) \right. \\ &\quad \left. - g(\mathcal{R}(\Theta, \Theta_{22})\Theta, \Theta_2) - g(\mathcal{R}(\Theta, \Theta_2)\Theta, \Theta_{22}) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{8}{9} \text{Ric}(\Theta, \bar{\Theta}_2) g(\mathcal{R}(\Theta, \Theta_1) \Theta_1, \bar{\Theta}_2) \\
&\quad + \frac{4}{9} \text{Ric}(\Theta, \Theta_1) (-\cot \theta^1 g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) + 2g(\mathcal{R}(\Theta, \bar{\Theta}_2) \bar{\Theta}_2, \Theta_1)) \\
&\quad - \frac{4 \cot \theta^1}{9} \text{Ric}(\Theta, \bar{\Theta}_2) g(\mathcal{R}(\Theta, \Theta_1) \Theta, \bar{\Theta}_2).
\end{aligned} \tag{39}$$

For the second equality above, we have used that $\Theta_{11} = -\Theta$ which means that all the $i = k = 1$ terms are zero due to the symmetries of the Riemann tensor. The last equality follows by applying (35) and (34). Combining (38) and (39) shows that the second and third lines of the fourth-order term in (33) become

$$\begin{aligned}
&\frac{8}{9} (\text{Ric}(\Theta, \bar{\Theta}_2) g(\mathcal{R}(\Theta, \Theta_1) \Theta_1, \bar{\Theta}_2) + \text{Ric}(\Theta, \Theta_1) g(\mathcal{R}(\Theta, \bar{\Theta}_2) \bar{\Theta}_2, \Theta_1)) \\
&= \frac{8}{9} (\text{Ric}(\Theta, \bar{\Theta}_2)^2 + \text{Ric}(\Theta, \Theta_1)^2),
\end{aligned} \tag{40}$$

where we have used the definition of the Ricci tensor and the symmetries of the Riemann tensor to rewrite $\text{Ric}(\Theta, \bar{\Theta}_2) = g(\mathcal{R}(\Theta, \Theta_1) \Theta_1, \bar{\Theta}_2)$ and $\text{Ric}(\Theta, \Theta_1) = g(\mathcal{R}(\Theta, \bar{\Theta}_2) \bar{\Theta}_2, \Theta_1)$.

Turning now to the next term in (33), we have

$$\begin{aligned}
&-\frac{4}{3} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta, \Theta_l) \Theta, \Theta_k) (\text{Hess}_{\mathbb{S}^2} \bar{w})_{ij} \\
&= -\frac{4}{3} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) (\bar{w}_{11} - \Gamma_{11}^i \bar{w}_i) - \frac{8}{3 \sin^2 \theta^1} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_2) (\bar{w}_{12} - \Gamma_{12}^i \bar{w}_i) \\
&\quad - \frac{4}{3 \sin^4 \theta^1} g(\mathcal{R}(\Theta, \Theta_2) \Theta, \Theta_2) (\bar{w}_{22} - \Gamma_{22}^i \bar{w}_i) \\
&= \frac{4}{9} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) (\text{Ric}(\Theta_1, \Theta_1) - \text{Ric}(\Theta, \Theta)) + \frac{8}{9} g(\mathcal{R}(\Theta, \Theta_1) \Theta, \bar{\Theta}_2) \text{Ric}(\bar{\Theta}_2, \Theta_1) \\
&\quad + \frac{4}{9} g(\mathcal{R}(\Theta, \bar{\Theta}_2) \Theta, \bar{\Theta}_2) (\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - \text{Ric}(\Theta, \Theta)) \\
&= \frac{4}{9} (g(\mathcal{R}(\Theta, \Theta_1) \Theta, \Theta_1) \text{Ric}(\Theta_1, \Theta_1) + g(\mathcal{R}(\Theta, \bar{\Theta}_2) \Theta, \bar{\Theta}_2) \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2)) \\
&\quad + \frac{4}{9} \text{Ric}(\Theta, \Theta)^2 - \frac{8}{9} \text{Ric}(\bar{\Theta}_2, \Theta_1)^2 \\
&= -\frac{4}{9} [\text{Ric}(\Theta_1, \Theta_1) (\text{Ric}(\Theta_1, \Theta_1) + g(\mathcal{R}(\Theta_1, \bar{\Theta}_2) \Theta_1, \bar{\Theta}_2)) + \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) (\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) \\
&\quad + g(\mathcal{R}(\Theta_1, \bar{\Theta}_2) \Theta_1, \bar{\Theta}_2))] + \frac{4}{9} \text{Ric}(\Theta, \Theta)^2 - \frac{8}{9} \text{Ric}(\bar{\Theta}_2, \Theta_1)^2 \\
&= -\frac{4}{9} (\text{Ric}(\Theta_1, \Theta_1)^2 + \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2)^2) + \frac{2}{9} \text{Sc}(p) (\text{Ric}(\Theta_1, \Theta_1) + \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2)) \\
&\quad - \frac{4}{9} \text{Ric}(\Theta, \Theta) (\text{Ric}(\Theta_1, \Theta_1) + \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2)) + \frac{4}{9} \text{Ric}(\Theta, \Theta)^2 - \frac{8}{9} \text{Ric}(\bar{\Theta}_2, \Theta_1)^2 \\
&= -\frac{2}{9} \text{Sc}(p)^2 + \frac{2}{9} \text{Sc}(p) \text{Ric}(\Theta, \Theta) + \frac{8}{9} \text{Ric}(\Theta_1, \Theta_1) \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) + \frac{4}{9} \text{Ric}(\Theta, \Theta)^2 - \frac{8}{9} \text{Ric}(\bar{\Theta}_2, \Theta_1)^2,
\end{aligned} \tag{41}$$

where for the second equality we used (34), (36) and the fact that the only non-vanishing Christoffel symbols for \mathbb{S}^2 are $\Gamma_{12}^2 = \Gamma_{21}^2 = \cot \theta^1$ and $\Gamma_{22}^1 = -\sin \theta^1 \cos \theta^1$. For the third and fourth equalities, we used the following identities

$$\begin{aligned} g(\mathcal{R}(\Theta, \Theta_1)\Theta, \Theta_1) &= -\text{Ric}(\Theta_1, \Theta_1) - g(\mathcal{R}(\Theta_1, \bar{\Theta}_2)\Theta_1, \bar{\Theta}_2) \\ g(\mathcal{R}(\Theta, \bar{\Theta}_2)\Theta, \bar{\Theta}_2) &= -\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - g(\mathcal{R}(\Theta_1, \bar{\Theta}_2)\Theta_1, \bar{\Theta}_2) \\ g(\mathcal{R}(\Theta_1, \bar{\Theta}_2)\Theta_1, \bar{\Theta}_2) &= -\frac{1}{2}\text{Sc}(p) + \text{Ric}(\Theta, \Theta) \\ \text{Ric}(\Theta_1, \Theta_1) + \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) &= \text{Sc}(p) - \text{Ric}(\Theta, \Theta). \end{aligned} \quad (42)$$

Next, we note that the eleventh and fourteenth fourth-order terms in (33) cancel out as

$$\begin{aligned} &\frac{4}{9}\delta^{\mu\nu}g_{\mathbb{S}^2}^{ij}g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_\mu)g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_\nu) \\ &= \frac{4}{9}g_{\mathbb{S}^2}^{ij}[g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_1)g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_1) + g(\mathcal{R}(\Theta, \Theta_i)\Theta, \bar{\Theta}_2)g(\mathcal{R}(\Theta, \Theta_j)\Theta, \bar{\Theta}_2)] \\ &= \frac{4}{9}g_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{lk}g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l)g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_k), \end{aligned}$$

where we have used the orthonormal basis $\{E_1 = \Theta, E_2 = \Theta_1, E_3 = \bar{\Theta}_2\}$. The ninth term becomes

$$-\frac{2}{9}g(\mathcal{R}(\Theta, \Theta_1)\Theta, \bar{\Theta}_2)^2 = -\frac{2}{9}\text{Ric}(\Theta_1, \bar{\Theta}_2)^2 \quad (43)$$

and, finally, the twelfth

$$\begin{aligned} &\frac{2}{9}g(\mathcal{R}(\Theta, \Theta_1)\Theta, \Theta_1)g(\mathcal{R}(\Theta, \bar{\Theta}_2)\Theta, \bar{\Theta}_2) \\ &= \frac{2}{9}[-\text{Ric}(\Theta_1, \Theta_1) - g(\mathcal{R}(\Theta_1, \bar{\Theta}_2)\Theta_1, \bar{\Theta}_2)][-\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - g(\mathcal{R}(\Theta_1, \bar{\Theta}_2)\Theta_1, \bar{\Theta}_2)] \\ &= \frac{2}{9}\left[-\text{Ric}(\Theta_1, \Theta_1) + \frac{1}{2}\text{Sc}(p) - \text{Ric}(\Theta, \Theta)\right] \cdot \left[-\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) + \frac{1}{2}\text{Sc}(p) - \text{Ric}(\Theta, \Theta)\right] \\ &= \frac{2}{9}\text{Ric}(\Theta_1, \Theta_1)\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) + \frac{1}{18}\text{Sc}(p)^2 - \frac{2}{9}\text{Ric}(\Theta, \Theta)[\text{Sc}(p) - \text{Ric}(\Theta_1, \Theta_1) \\ &\quad - \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - \text{Ric}(\Theta, \Theta)] - \frac{1}{9}\text{Sc}(p)[\text{Ric}(\Theta_1, \Theta_1) + \text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2)] \\ &= \frac{2}{9}\text{Ric}(\Theta_1, \Theta_1)\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - \frac{1}{18}\text{Sc}(p)^2 + \frac{1}{9}\text{Sc}(p)\text{Ric}(\Theta, \Theta). \end{aligned} \quad (44)$$

again using (42). Substituting (37), (38), (39), (40), (41), (43) and (44) into (33) yields

$$\begin{aligned} H^2\sqrt{\det \bar{g}} &= \sin \theta^1 \left[4 + [4\Delta_{\mathbb{S}^2}\bar{w} - 2\text{Ric}(\Theta, \Theta)]\rho^2 + \left[\frac{4}{3}g_{\mathbb{S}^2}^{ij}g(\nabla_\Theta \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) \right]\rho^3 \right. \\ &\quad \left. + \left[-\frac{17}{18}\text{Ric}(\Theta, \Theta)^2 - \frac{13}{54}\text{Sc}(p)^2 + \frac{7}{9}\text{Sc}(p)\text{Ric}(\Theta, \Theta) \right] \right] \end{aligned} \quad (45)$$

$$+ \frac{10}{9} (\text{Ric}(\Theta, \Theta_1)^2 + \text{Ric}(\Theta, \bar{\Theta}_2)^2) + \frac{10}{9} (\text{Ric}(\Theta_1, \Theta_1)\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - \text{Ric}(\Theta_1, \bar{\Theta}_2)^2) \\ + \frac{1}{2} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta}^2 \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) \Big] \rho^4 + \mathcal{O}(\rho^5) \Big].$$

Step 3: computation of the integral of (45).

We now integrate each term, and finally we will combine the results. For the following, it will be useful to observe that

$$\int_{\mathbb{S}^2} (x^\mu)^2 dV_{\mathbb{S}^2} = \frac{4\pi}{3}, \quad \int_{\mathbb{S}^2} (x^\mu)^2 (x^\nu)^2 dV_{\mathbb{S}^2} = \frac{4\pi}{15} \text{ for } \mu \neq \nu, \quad \int_{\mathbb{S}^2} (x^\mu)^4 dV_{\mathbb{S}^2} = \frac{4\pi}{5}. \quad (46)$$

Recalling that $\Theta = (\sin \theta^1 \cos \theta^2, \sin \theta^1 \sin \theta^2, \cos \theta^1) = (x, y, z)$, we have

$$\Theta_1 = (\cos \theta^1 \cos \theta^2, \cos \theta^1 \sin \theta^2, -\sin \theta^1) = \left(\frac{xz}{\sqrt{x^2 + y^2}}, \frac{yz}{\sqrt{x^2 + y^2}}, -\sqrt{x^2 + y^2} \right) \\ \bar{\Theta}_2 = (-\sin \theta^2, \cos \theta^2, 0) = \left(-\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}, 0 \right). \quad (47)$$

• **Terms of order ρ^2 .** Note that

$$0 = \int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} \bar{w} dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} \bar{w} \sin \theta^1 d\theta^1 d\theta^2$$

and use (46) to compute

$$-2 \int_{\mathbb{S}^2} \text{Ric}(\Theta, \Theta) dV_{\mathbb{S}^2}(\Theta) = -\frac{8\pi}{3} \text{Sc}(p).$$

• **Terms of order ρ^3 .** Note that by anti-symmetry it holds

$$\int_{\mathbb{S}^2} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) dV_{\mathbb{S}^2}(\Theta) = 0.$$

• **Terms of order ρ^4 .**

Using (46), one can compute that (see, for instance, (4.17) in [40])

$$\int_{\mathbb{S}^2} (\text{Ric}(\Theta, \Theta))^2 dV_{\mathbb{S}^2}(\Theta) = \frac{4\pi}{15} (2\|\text{Ric}\|^2 + \text{Sc}(p)^2)$$

and that (see, for instance, [31, p. 780])

$$\int_{\mathbb{S}^2} (\text{Ric}(\Theta_1, \Theta_1)\text{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) - \text{Ric}(\Theta_1, \bar{\Theta}_2)^2) dV_{\mathbb{S}^2}(\Theta) = -\frac{2\pi}{3} (\|\text{Ric}\|^2 - \text{Sc}(p)^2).$$

We next compute the term $\int_{\mathbb{S}^2} (\text{Ric}(\Theta, \Theta_1)^2 + \text{Ric}(\Theta, \bar{\Theta}_2)^2) dV_{\mathbb{S}^2}$. Taking into account that $R_{ij} = R_{ji}$ and (47), we can write

$$\begin{aligned} \text{Ric}(\Theta, \Theta_1)^2 &= \left(R_{11} \frac{x^2 z}{\sqrt{x^2 + y^2}} + R_{12} \frac{2xyz}{\sqrt{x^2 + y^2}} + R_{13} \frac{z^2 x - x(x^2 + y^2)}{\sqrt{x^2 + y^2}} + R_{22} \frac{y^2 z}{\sqrt{x^2 + y^2}} \right. \\ &\quad \left. + R_{23} \frac{z^2 y - y(x^2 + y^2)}{\sqrt{x^2 + y^2}} - R_{33} z \sqrt{x^2 + y^2} \right)^2 \\ \text{Ric}(\Theta, \bar{\Theta}_2)^2 &= \left(-R_{11} \frac{xy}{\sqrt{x^2 + y^2}} + R_{12} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} + R_{22} \frac{xy}{\sqrt{x^2 + y^2}} - R_{31} \frac{yz}{\sqrt{x^2 + y^2}} \right. \\ &\quad \left. + R_{32} \frac{xz}{\sqrt{x^2 + y^2}} \right)^2. \end{aligned}$$

Observe that we can ignore any polynomial term in the integrand which has an odd power, since it will integrate to zero by anti-symmetry. Inspection shows that after the brackets are expanded, the only terms that will consist entirely of even powers of x , y and z are those containing the following coefficients of the Ricci tensor:

$$R_{11}^2, R_{11}R_{22}, R_{11}R_{33}, R_{12}^2, R_{13}^2, R_{22}^2, R_{22}R_{33}, R_{23}^2, R_{33}^2.$$

We compute these terms as follows, using that $x^2 + y^2 + z^2 = 1$:

$$\begin{aligned} &\left(R_{11} \frac{x^2 z}{\sqrt{x^2 + y^2}} \right)^2 + \left(-R_{11} \frac{xy}{\sqrt{x^2 + y^2}} \right)^2 = R_{11}^2 \left(\frac{x^4 z^2 + x^2 y^2}{x^2 + y^2} \right) \\ &= R_{11}^2 \left(\frac{x^2(x^2(1 - x^2 - y^2) + y^2)}{x^2 + y^2} \right) = R_{11}^2 \left(\frac{x^2(x^2 + y^2)(1 - x^2)}{x^2 + y^2} \right) = R_{11}^2 (x^2 - x^4), \\ &2 \left(R_{11} \frac{x^2 z}{\sqrt{x^2 + y^2}} \right) \left(R_{22} \frac{y^2 z}{\sqrt{x^2 + y^2}} \right) - 2 \left(R_{11} \frac{xy}{\sqrt{x^2 + y^2}} \right) \left(R_{22} \frac{xy}{\sqrt{x^2 + y^2}} \right) \\ &= 2R_{11}R_{22} \left(\frac{x^2 y^2 (z^2 - 1)}{x^2 + y^2} \right) = -2R_{11}R_{22} x^2 y^2, \\ &-2 \left(R_{11} \frac{x^2 z}{\sqrt{x^2 + y^2}} \right) \left(R_{33} z \sqrt{x^2 + y^2} \right) = -2R_{11}R_{33} x^2 z^2, \\ &\left(R_{12} \frac{2xyz}{\sqrt{x^2 + y^2}} \right)^2 + \left(R_{12} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \right)^2 = R_{12}^2 \left(\frac{4x^2 y^2 z^2 + x^4 + y^4 - 2x^2 y^2}{x^2 + y^2} \right) \\ &= R_{12}^2 \left(\frac{4x^2 y^2 (1 - x^2 - y^2) + x^4 + y^4 - 2x^2 y^2}{x^2 + y^2} \right) \end{aligned}$$

$$\begin{aligned}
&= R_{12}^2 \left(\frac{x^4 + y^4 + 2x^2y^2 - 4x^4y^2 - 4x^2y^4}{x^2 + y^2} \right) \\
&= R_{12}^2 \left(\frac{(x^2 + y^2)^2 - 4x^2y^2(x^2 + y^2)}{x^2 + y^2} \right) \\
&= R_{12}^2(x^2 + y^2 - 4x^2y^2) \\
&= R_{12}^2(1 - z^2 - 4x^2y^2),
\end{aligned}$$

$$\begin{aligned}
\left(R_{13} \frac{z^2x - x(x^2 + y^2)}{\sqrt{x^2 + y^2}} \right)^2 + \left(-R_{31} \frac{yz}{\sqrt{x^2 + y^2}} \right)^2 &= R_{13}^2 \left(\frac{(z^2x - x(1 - z^2))^2 + y^2z^2}{1 - z^2} \right) \\
&= R_{13}^2 \left(\frac{4z^4x^2 - 4z^2x^2 + x^2 + y^2z^2}{1 - z^2} \right) \\
&= R_{13}^2 \left(\frac{4z^4x^2 - 4z^2x^2 + 1 - y^2 - z^2 + y^2z^2}{1 - z^2} \right) \\
&= R_{13}^2 \left(\frac{4z^2x^2(z^2 - 1) + (1 - z^2)(1 - y^2)}{1 - z^2} \right) \\
&= R_{13}^2(1 - y^2 - 4z^2x^2),
\end{aligned}$$

$$-2 \left(R_{22} \frac{y^2z}{\sqrt{x^2 + y^2}} \right) \left(R_{33}z\sqrt{x^2 + y^2} \right) = -2R_{22}R_{33}y^2z^2,$$

$$\begin{aligned}
\left(R_{22} \frac{y^2z}{\sqrt{x^2 + y^2}} \right)^2 + \left(R_{22} \frac{xy}{\sqrt{x^2 + y^2}} \right)^2 &= R_{22}^2 \left(\frac{y^4z^2 + x^2y^2}{x^2 + y^2} \right) \\
&= R_{22}^2 \left(\frac{y^2(y^2(1 - x^2 - y^2) + x^2)}{x^2 + y^2} \right) \\
&= R_{22}^2 \left(\frac{y^2(x^2 + y^2)(1 - y^2)}{x^2 + y^2} \right) = R_{22}^2(y^2 - y^4),
\end{aligned}$$

$$\begin{aligned}
\left(R_{23} \frac{z^2y - y(x^2 + y^2)}{\sqrt{x^2 + y^2}} \right)^2 + \left(R_{32} \frac{xz}{\sqrt{x^2 + y^2}} \right)^2 &= R_{23}^2 \left(\frac{(z^2y - y(1 - z^2))^2 + x^2z^2}{1 - z^2} \right) \\
&= R_{23}^2 \left(\frac{4z^4y^2 - 4z^2y^2 + y^2 + x^2z^2}{1 - z^2} \right) \\
&= R_{23}^2 \left(\frac{4z^4y^2 - 4z^2y^2 + 1 - x^2 - z^2 + x^2z^2}{1 - z^2} \right)
\end{aligned}$$

$$\begin{aligned}
&= R_{23}^2 \left(\frac{4z^2 y^2 (z^2 - 1) + (1 - z^2)(1 - x^2)}{1 - z^2} \right) \\
&= R_{23}^2 (1 - x^2 - 4z^2 y^2),
\end{aligned}$$

$$\left(-R_{33} z \sqrt{x^2 + y^2} \right)^2 = R_{33}^2 z^2 (x^2 + y^2) = R_{33}^2 z^2 (1 - z^2) = R_{33}^2 (z^2 - z^4).$$

Thus, using (46), we get

$$\begin{aligned}
&\int_{\mathbb{S}^2} \text{Ric}(\Theta, \Theta_1)^2 + \text{Ric}(\Theta, \bar{\Theta}_2)^2 dV_{\mathbb{S}^2} \\
&= \int_{\mathbb{S}^2} R_{11}^2 (x^2 - x^4) - 2R_{11}R_{22}x^2y^2 - 2R_{11}R_{33}x^2z^2 + R_{12}^2 (1 - z^2 - 4x^2y^2) \\
&\quad + R_{13}^2 (1 - y^2 - 4z^2x^2) + R_{22}^2 (y^2 - y^4) - 2R_{22}R_{33}y^2z^2 \\
&\quad + R_{23}^2 (1 - x^2 - 4z^2y^2) + R_{33}^2 (z^2 - z^4) dV_{\mathbb{S}^2} \\
&= \frac{8\pi}{15} (R_{11}^2 + R_{22}^2 + R_{33}^2 + 3R_{12}^2 + 3R_{13}^2 + 3R_{23}^2) \\
&\quad - \frac{8\pi}{15} (R_{11}R_{22} + R_{11}R_{33} + R_{22}R_{33}) \\
&= \frac{4\pi}{5} (R_{11}^2 + R_{22}^2 + R_{33}^2 + 2R_{12}^2 + 2R_{13}^2 + 2R_{23}^2) \\
&\quad - \frac{4\pi}{15} (R_{11}^2 + R_{22}^2 + R_{33}^2 + 2R_{11}R_{22} + 2R_{11}R_{33} + 2R_{22}R_{33}) \\
&= \frac{4\pi}{5} \left[\|\text{Ric}\|^2 - \frac{1}{3} \text{Sc}(p)^2 \right] \\
&= \frac{4\pi}{5} \|S_p\|^2, \tag{48}
\end{aligned}$$

where $S := \text{Ric} - \frac{1}{3} \text{Sc } g$ is the traceless Ricci tensor.

To rewrite the final ρ^4 term, first note that $\nabla_{\Theta} \Theta_i = 0$ because Θ_i does not depend on the radial coordinate and the Christoffel symbols vanish at p since we chose normal coordinates. Thus, by compatibility of the metric:

$$\begin{aligned}
\frac{1}{2} g_{\mathbb{S}^2}^{ij} g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) &= \frac{1}{2} g_{\mathbb{S}^2}^{ij} \nabla_{\Theta\Theta}^2 g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) + \mathcal{O}_p(\rho) \\
&= \frac{1}{2} \nabla_{\Theta\Theta}^2 \left(g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \right) + \mathcal{O}_p(\rho) \\
&= -\frac{1}{2} \nabla_{\Theta\Theta}^2 \text{Ric}(\Theta, \Theta) + \mathcal{O}_p(\rho).
\end{aligned}$$

Now we can integrate, using index notation and (46), as follows:

$$\begin{aligned}
 & \int_{\mathbb{S}^2} -\frac{1}{2} \nabla_{\Theta\Theta}^2 \text{Ric}(\Theta, \Theta) dV_{\mathbb{S}^2} \\
 &= -\frac{1}{2} \int_{\mathbb{S}^2} \nabla_{\mu} \nabla_{\nu} R_{\sigma\tau} x^{\mu} x^{\nu} x^{\sigma} x^{\tau} dV_{\mathbb{S}^2} \\
 &= -\nabla_{\mu} \nabla_{\nu} R_{\sigma\tau} \frac{1}{2} \int_{\mathbb{S}^2} x^{\mu} x^{\nu} x^{\sigma} x^{\tau} dV_{\mathbb{S}^2} \\
 &= -\frac{4\pi}{30} \left(3 \sum_{\mu} \nabla_{\mu} \nabla_{\mu} R_{\mu\mu} + \sum_{\mu \neq \nu} \nabla_{\mu} \nabla_{\mu} R_{\nu\nu} + \sum_{\mu \neq \nu} \nabla_{\mu} \nabla_{\nu} R_{\mu\nu} + \sum_{\mu \neq \nu} \nabla_{\mu} \nabla_{\nu} R_{\nu\mu} \right) \\
 &= -\frac{2\pi}{15} \sum_{\mu\nu} (\nabla_{\mu} \nabla_{\mu} R_{\nu\nu} + 2 \nabla_{\mu} \nabla_{\nu} R_{\mu\nu}) \\
 &= -\frac{4\pi}{15} \Delta \text{Sc}(p), \tag{49}
 \end{aligned}$$

where in the last line we have used the contracted Bianchi identity. Thus, integrating integrating (45) we obtain

$$\begin{aligned}
 W(S_{p,\rho}(w)) &= \int_{\mathbb{S}^2} H^2 \sqrt{\det g} dV_{\mathbb{S}^2} \\
 &= 16\pi - \frac{8\pi}{3} \text{Sc}(p) \rho^2 + \left[-\frac{34\pi}{135} (2\|\text{Ric}\|^2 + \text{Sc}(p)^2) - \frac{26\pi}{27} \text{Sc}(p)^2 + \frac{28\pi}{27} \text{Sc}(p)^2 \right. \\
 &\quad \left. + \frac{8\pi}{9} \|S\|^2 - \frac{20\pi}{27} (\|\text{Ric}\|^2 - \text{Sc}(p)^2) - \frac{4\pi}{15} \Delta \text{Sc}(p) \right] \rho^4 + \mathcal{O}_p(\rho^5) \\
 &= 16\pi - \frac{8\pi}{3} \text{Sc}(p) \rho^2 + \left[\frac{4\pi}{27} \text{Sc}(p)^2 - \frac{16\pi}{45} \|S_p\|^2 - \frac{4\pi}{15} \Delta \text{Sc}(p) \right] \rho^4 + \mathcal{O}_p(\rho^5). \tag{50}
 \end{aligned}$$

Step 4: conclusion.

In order to conclude, we have to compute the expansion for $|S_{p,\rho}(w)|_g$. To this aim, inserting (22) into (32) yields the following expansion for the area of optimally perturbed spheres:

$$|S_{p,\rho}(w)|_g = |\mathbb{S}^2|_{\delta} \rho^2 \left[1 - \frac{1}{18} \text{Sc}_p \rho^2 + \mathcal{O}(\rho^4) \right]. \tag{51}$$

Substituting (50) and (51) into (3), we thus obtain

$$\begin{aligned}
 m_H(S_{p,\rho}(w_{p,\rho})) &= \frac{\sqrt{|\mathbb{S}^2|_{\delta} \rho}}{\sqrt{(16\pi)^3}} \left[1 - \frac{1}{36} \text{Sc}_p \rho^2 + \mathcal{O}(\rho^4) \right] \\
 &\quad \times \left[\frac{8\pi}{3} \text{Sc}_p \rho^2 + \left(\frac{4\pi}{15} \Delta \text{Sc}(p) + \frac{16\pi}{45} \|S_p\|^2 - \frac{4\pi}{27} \text{Sc}_p^2 \right) \rho^4 + \mathcal{O}(\rho^5) \right] \\
 &= \frac{1}{12} \text{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \text{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \text{Sc}_p^2 \right) \rho^5 + \mathcal{O}(\rho^6).
 \end{aligned}$$

□

4 | RIGIDITY RESULTS IN CASE OF NON-NEGATIVE SCALAR CURVATURE

Theorem 4.1. *Let (M^3, g) be a three-dimensional Riemannian manifold and let $\Omega \subset M$ be an open subset with non-negative scalar curvature. Assume that every $p \in \Omega \setminus \partial M$ admits a neighbourhood $U \subset M \setminus \partial M$ such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0, \quad (52)$$

or, more generally, that

$$\limsup_{\rho \downarrow 0} \rho^{-5} m_H(S_{p,\rho}(w_{p,\rho})) \leq 0, \quad \forall p \in \Omega \setminus \partial M, \quad (53)$$

where $S_{p,\rho}(w_{p,\rho})$ is the optimally perturbed geodesic sphere with $w_{p,\rho}$ given in Lemma 3.1. Then $\Omega \setminus \partial M$ is locally isometric to Euclidean \mathbb{R}^3 .

Proof. The combination of the assumption (53) with the expansion (31) yields

$$\limsup_{\rho \downarrow 0} \rho^{-5} \left[\frac{1}{12} \text{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \text{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \text{Sc}_p^2 \right) \rho^5 + \mathcal{O}(\rho^6) \right] \leq 0, \quad \forall p \in \Omega \setminus \partial M. \quad (54)$$

Since Sc is assumed to be non-negative, letting $\rho \downarrow 0$ and looking at the order ρ^{-2} in the expansion (54), we first infer that

$$\text{Sc} \equiv 0 \quad \text{on } \Omega \setminus \partial M. \quad (55)$$

Plugging (55) into (54) and looking now at the order ρ^0 in the expansion, we deduce that

$$S \equiv 0 \quad \text{on } \Omega \setminus \partial M. \quad (56)$$

Putting together (55) and (56) gives

$$\text{Ric} \equiv 0 \quad \text{on } \Omega \setminus \partial M. \quad (57)$$

Recalling that in dimension three the Riemann curvature tensor can be written as $\text{Rm} = \text{Ric} \otimes g - \frac{1}{4} \text{Sc} g \otimes g$ where \otimes is the Kulkarni–Nomizu product (see, for instance, [23, Corollary 7.26]) we conclude that $\text{Rm} \equiv 0$ on $\Omega \setminus \partial M$, as desired. \square

Recall the Definition 1.4 of asymptotically locally simply connected (ALSC for short). The global rigidity result (Theorem 4.3, below) will follow by the combination of Theorem 4.1 with the following proposition of independent interest.

Proposition 4.2. *Let (M^3, g) be a connected, complete, ALSC Riemannian manifold without boundary and with zero sectional curvature. Then (M^3, g) is globally isometric to Euclidean \mathbb{R}^3 .*

Proof. By the classical Killing–Hopf Theorem, we have that (M^3, g) is isometric to a space form \mathbb{R}^3/Γ , for some discrete sub-group Γ of the affine transformations $\mathbb{E}(3)$ of \mathbb{R}^3 acting freely (see, for instance, [54, Theorem 3.3.2]). Denote with $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^3/\Gamma \simeq M$ the covering map. Since $\mathbb{E}(3)$ is isomorphic to the semi-direct product $O(3) \ltimes \mathbb{R}^3$, we can write an arbitrary element $\gamma \in \Gamma$ as $\gamma = (r, a)$ where $r \in O(3)$ and $a \in \mathbb{R}^3$.

Denote with $n \in \{0, 1, 2, 3\}$ the rank of the maximal abelian subgroup of Γ . After choosing an appropriate orthonormal basis of \mathbb{R}^3 adapted to Γ , we can write $\mathbb{R}^3 = \mathbb{R}^n \times \mathbb{R}^{3-n}$ in such a way that each $\gamma \in \Gamma$ can be expressed uniquely as

$$\gamma = \delta \times \psi, \quad \text{where } \delta = (r|_{\mathbb{R}^n}, a), \quad a \in \mathbb{R}^n, \quad \text{and } \psi = (r|_{\mathbb{R}^{3-n}}, 0).$$

We next discuss separately the different cases for the value of n .

- $n = 0$. In this case, $\Gamma \subset O(3)$. Since Γ acts freely, it follows that $\Gamma = \{\text{Id}\}$ must be trivial and thus (M, g) is isometric to \mathbb{R}^3 .
- $n = 1$. We claim that for any $\gamma \in \Gamma$, the corresponding translation $a \in \mathbb{R}$ must vanish. Assume by contradiction this is not the case and fix such a $\gamma \in \Gamma$ with corresponding non-trivial translation $0 \neq a \in \mathbb{R}$. Fix an arbitrary point $(u, v) \in \mathbb{R} \times \mathbb{R}^2$ and consider the curve $\alpha(s) = (u + sa, v) \subset \mathbb{R}^3$ for $s \in [0, 1]$. It is easily seen that, for any radius $R > |a|$, $\pi \circ \alpha$ is a loop contained in $B_R^M(\pi(u, v)) \subset M$ and that $\pi \circ \alpha$ is not contractible in $B_R^M(\pi(u, v))$. Indeed, if $\pi \circ \alpha$ were homotopic to the constant loop $c(\cdot) \equiv \pi(u, v)$ then, by the uniqueness of path liftings in the covering space (see, for instance, [36, Lemma 54.1, Chapter 9]), the lifts of $c(\cdot)$ and $\pi \circ \alpha$ starting at (u, v) would end at the same point. But this is not true as the lift of $c(\cdot)$ is the constant loop in $\mathbb{R} \times \mathbb{R}^2$ taking value (u, v) , whereas the lift of $\pi \circ \alpha$ is α , which ends at $(u + a, v) \neq (u, v)$. Therefore, $B_R^M(\pi(u, v)) \subset M$ is not simply connected for every $R > a$ and every $(u, v) \in \mathbb{R} \times \mathbb{R}^2$, contradicting the ALSC assumption on M . This concludes the proof that for every $\gamma \in \Gamma$ the corresponding translation $a \in \mathbb{R}$ must vanish. It follows that $\Gamma \subset O(3)$ and thus, as in the case $n = 0$, we infer that $\Gamma = \{\text{Id}\}$ must be trivial and (M, g) is isometric to \mathbb{R}^3 .
- $n = 2$. We claim that for any $\gamma \in \Gamma$, the corresponding translation $a \in \mathbb{R}^2$ must vanish. Assume by contradiction this is not the case and fix such a $\gamma \in \Gamma$ with corresponding non-trivial translation $0 \neq a \in \mathbb{R}^2$. Fix an arbitrary point $(u, v) \in \mathbb{R}^2 \times \mathbb{R}$ and consider the curve $\alpha(s) = (u + sa, v) \subset \mathbb{R}^3$ for $s \in [0, 1]$. The analogous argument as in the case $n = 1$ shows that, for any radius $R > |a|$, $\pi \circ \alpha$ is a loop contained in $B_R^M(\pi(u, v)) \subset M$ and that $\pi \circ \alpha$ is not contractible in $B_R^M(\pi(u, v))$. This contradicts the ALSC assumption and thus, as in the $n = 1$ case, we conclude that $\Gamma = \{\text{Id}\}$ must be trivial and (M, g) is isometric to \mathbb{R}^3 .
- $n = 3$. In this case, the quotient $M = \mathbb{R}^3/\Gamma$ is compact, contradicting the ALSC assumption.

Therefore, the only possibility is $\Gamma = \{\text{Id}\}$ and (M, g) is isometric to \mathbb{R}^3 . □

Theorem 4.3. *Let (M^3, g) be a connected, complete, ALSC Riemannian manifold without boundary and with non-negative scalar curvature. Assume that every $p \in M$ admits a neighbourhood satisfying (52) or, more generally, that the infinitesimal non-positivity of m_H (53) holds for every $p \in M$.*

Then (M^3, g) is globally isometric to Euclidean \mathbb{R}^3 .

Proof. The result follows by the combination of Theorem 4.1 and Proposition 4.2. □

5 | A LOWER BOUND ON THE BARTNIK MASS

In this section, we are interested in asymptotically flat Riemannian manifolds as defined in the introduction, with non-negative scalar curvature.

In order to give a lower bound on the Bartnik mass, in the next subsection, we show that perturbed geodesic spheres $S_{p,\rho}(w)$ are outer-minimising. We will then combine the monotonicity of the Bartnik mass (5), with the fact that the Bartnik mass is bounded from below by the Hawking mass (7) and the geometric expansions of the Hawking mass on perturbed geodesic spheres obtained in the previous sections to get the result.

5.1 | Some auxiliary lemmas about sets of finite perimeter

The goal of this section is to establish some auxiliary lemmas that will be useful in the proof of the main result in the next section, that is Theorem 5.7.

For this section, we consider a general Riemannian manifold (M, g) of dimension three. Given a Borel subset $E \subset M$ and an open subset $U \subset M$, we denote by

$$P_g(E, U) := \sup \left\{ \int_E \operatorname{div}_g(X) dV_g : X \text{ is a } C^1 \text{ vector field with } \operatorname{supp} X \subset U \setminus \partial M, \|X\|_{\infty, g} \leq 1 \right\} \quad (58)$$

the perimeter of E relative to U . When $U = M$, we simply write $P_g(E) := P_g(E, M)$. Note that ∂M never contributes to the perimeter.

Given a point $p \in M \setminus \partial M$, let $\operatorname{Inj}_p > 0$ be the injectivity radius at p and denote with $\phi_p^p : B_{\operatorname{Inj}_p}^g(p) \rightarrow \mathbb{R}^3$ a normal coordinate chart centred at p with respect to the Riemannian metric g . It will be convenient to consider the rescaled metrics $g_\rho := \rho^{-2}g$, for $\rho \in (0, 1]$. It is a standard fact that, in g_ρ -normal coordinates centred at p , it holds

$$\|(g_\rho)_{\mu\nu} - \delta_{\mu\nu}\|_{C^k(B_{\rho^{-1}}^{g_\rho}(p))} \leq C \rho^2 r^2, \quad \forall \rho \in (0, 1], r \in (0, \operatorname{Inj}_p], \quad (59)$$

for some suitable $C = C(p, k) > 0$.

From the very definition of perimeter (58), it is easily seen that

$$P_{g_\rho}(E, U) = \rho^{-2} P_g(E, U), \quad \text{for every Borel subset } E \subset M \text{ and } U \subset M \text{ open.} \quad (60)$$

We next establish some lemmas linking the relative perimeter $P_{g_\rho}(E, B_r^{g_\rho}(q))$ of a finite perimeter set $E \subset M$, and the Euclidean relative perimeter $P_\delta(\phi_{g_\rho}(E), B_r^\delta(\phi_{g_\rho}(q)))$ of its image $\phi_{g_\rho}(E) \subset \mathbb{R}^3$. We remark that, in such relative perimeter calculations, we can ignore the fact that E may not be contained in the domain of ϕ_{g_ρ} because it is enough that $B_r^{g_\rho}(q)$ is contained there (which, for small ρ , it will be). To this aim, we will distinguish between the normal coordinate charts centred at either at a point p or q nearby, denoting them by $\phi_{g_\rho}^p$ and $\phi_{g_\rho}^q$, respectively.

Lemma 5.1. Fix a point $p \in M \setminus \partial M$. Then, there exist constants $C = C(p) > 0$, $\bar{\rho} = \bar{\rho}(p) > 0$ such that for every finite perimeter set $E \subset M$, every $\rho \in (0, \bar{\rho}]$ and every $q \in B_{10}^{g_\rho}(p)$ it holds

$$\left| P_\delta(\phi_{g_\rho}^p(E), B_r^\delta(\phi_{g_\rho}^p(q))) - P_{g_\rho}(E, B_r^{g_\rho}(q)) \right| \leq C r^2 P_{g_\rho}(E, B_r^{g_\rho}(q)) + C r^4, \quad \forall r \in (0, 1]. \quad (61)$$

Proof. Let $q \in B_{10}^{g_\rho}(p)$ and consider g_ρ -normal coordinates centred at q . Note that, by smooth dependence of initial data in the geodesic equation, the constant appearing in (59) (when applied to normal coordinates centred at $q \in B_{10}^{g_\rho}(p)$) may be chosen independently on q , so it still only depends on p .

For a C^1 -vector field X supported in $B_r^{g_\rho}(q)$ with $\|X\|_{\infty, \delta} \leq 1$, we thus have

$$\int_{E \cap B_r^{g_\rho}(q)} \operatorname{div}_{g_\rho}(X) dV_{g_\rho} = \int_{\phi_{g_\rho}^q(E \cap B_r^{g_\rho}(q))} [(1 + \mathcal{O}_p(r^2)) \operatorname{div}_\delta(X) + \mathcal{O}_p(r)] dV_\delta.$$

We infer that

$$\begin{aligned} & \left| \int_{E \cap B_r^{g_\rho}(q)} \operatorname{div}_{g_\rho}(X) dV_{g_\rho} - \int_{\phi_{g_\rho}^q(E \cap B_r^{g_\rho}(q))} \operatorname{div}_\delta(X) dV_\delta \right| \\ & \leq C r^2 \int_{\phi_{g_\rho}^q(E \cap B_r^{g_\rho}(q))} |\operatorname{div}_\delta(X)| dV_\delta + C r V_\delta(B_r^\delta(\phi_{g_\rho}^q(q))) \\ & \leq C r^2 \int_{\phi_{g_\rho}^q(E \cap B_r^{g_\rho}(q)) \cap \{\operatorname{div}_\delta(X) > 0\}} \operatorname{div}_\delta(X) dV_\delta + C r^2 \int_{\phi_{g_\rho}^q(E \cap B_r^{g_\rho}(q)) \cap \{\operatorname{div}_\delta(X) < 0\}} -\operatorname{div}_\delta(X) dV_\delta + C r^4 \\ & \leq C r^2 P_\delta(\phi_{g_\rho}^q(E), \phi_{g_\rho}^q(B_r^{g_\rho}(q))) + C r^4, \end{aligned}$$

where in the first inequality we used that $V_\delta(\phi_{g_\rho}^q(E) \cap B_r^\delta(\phi_{g_\rho}^q(q))) \leq V_\delta(B_r^\delta(\phi_{g_\rho}^q(q))) \leq C r^3$, and in the third inequality we have used that X is admissible in the definition of perimeter if and only if $-X$ is too.

Taking the supremum over all C^1 -vector fields X supported in $B_r^{g_\rho}(q)$ with $\|X\|_{\infty, \delta} \leq 1$ gives the claim. \square

Lemma 5.2. Fix a point $p \in M \setminus \partial M$. Then, there exist constants $C = C(p) > 0$, $\bar{\rho} = \bar{\rho}(p) > 0$ such that for every finite perimeter set $E \subset M$, every $\rho \in (0, \bar{\rho}]$ and every $q \in B_{10}^{g_\rho}(p)$, it holds

$$\left| P_\delta(\phi_{g_\rho}^p(E), B_r^\delta(\phi_{g_\rho}^p(q))) - P_\delta(\phi_{g_\rho}^q(E), B_r^\delta(\phi_{g_\rho}^q(q))) \right| \leq C P_{g_\rho}(E, B_{10}^{g_\rho}(p)) \rho^2, \quad \forall r \in (0, 1]. \quad (62)$$

Proof. From (59) and the smooth dependence on coefficients in the geodesic equation, it is standard to check that for any point $q \in B_{10}^{g_\rho}(p)$ the g_ρ -exponential map satisfies

$$\left\| \exp_q^{g_\rho} - \exp_{\phi_{g_\rho}^p(q)}^\delta \right\|_{C^1(B_{10}^{g_\rho}(p))} \leq C(p) \rho^2,$$

where we consider both maps (for $\exp_q^{g_\rho}$, via the pullback metric) as diffeomorphisms on a ball in \mathbb{R}^3 . Note that $\exp_{\phi_{g_\rho}^p(q)}^\delta = \mathcal{T}_{\phi_{g_\rho}^p(q)}$, where $\mathcal{T}_{\phi_{g_\rho}^p(q)}$ is the translation by the vector $\phi_{g_\rho}^p(q)$. Since the normal coordinate chart is the inverse of the exponential map, this means

$$\left\| \phi_{g_\rho}^q - \mathcal{T}_{-\phi_{g_\rho}^p(q)} \right\|_{C^1(B_{10}^{g_\rho}(p))} \leq C(p) \rho^2.$$

It follows that

$$\left\| \phi_{g_\rho}^p - \mathcal{T}_{\phi_{g_\rho}^p(q)} \circ \phi_{g_\rho}^q \right\|_{C^1(B_{10}^{g_\rho}(p))} \leq C(p) \rho^2.$$

Since the translation is an isometry of \mathbb{R}^3 , it is not hard to see that the last estimate implies the claim. \square

We next recall a well-known consequence of the monotonicity formula for a finite perimeter set which is stationary for the perimeter functional; actually this holds more generally for stationary rectifiable varifolds, see, for instance, [45, Chap. 4.3].

Lemma 5.3. *Let (M, g) be a Riemannian manifold, $U \subset M \setminus \partial M$ be a relatively compact open subset, and $E \subset M$ be a set of finite perimeter which is stationary for $P_g(\cdot, U)$, the perimeter functional relative to U (that is, zero first variation restricted to U). Then there exist $C = C(U, P_g(E, U)) > 0$, $r_0 = r_0(U, P_g(E, U)) > 0$ such that*

$$P_g(E, B_r^g(q)) \leq Cr^2, \quad \text{for all } B_r^g(q) \subset U \text{ and all } r \in (0, r_0]. \quad (63)$$

The following lemma can be proved along the same lines as [35, (6–9)]:

Lemma 5.4. *Let (M, g) be a Riemannian manifold, $U \subset M \setminus \partial M$ be a relatively compact open subset with $C^{1,1}$ boundary. Then there exist $r_0 = r_0(U) > 0$ and $C = C(U) > 0$ such that*

$$P_g(U) \leq P_g(E) + Cr^3, \quad \text{for all finite perimeter sets } E \text{ with } E \Delta U \Subset B_r^g(q), \text{ all } q \in \bar{U} \\ \text{and all } r \in (0, r_0].$$

We conclude this short section by recalling a regularity result of Tamanini [47], refining previous celebrated works by De Giorgi [10].

To this aim recall that, given a finite perimeter set $E \subset \mathbb{R}^3$ and an open bounded subset $V \subset \mathbb{R}^3$, the excess of E relative to V is defined as

$$\Psi(E, V) := P_\delta(E, V) - \inf\{P_\delta(F, V) \mid F \Delta E \Subset V\}.$$

Theorem 5.5 ([47, Theorem 1]). *Let $U \subset \mathbb{R}^3$ be an open subset and $E \subset \mathbb{R}^3$ be a set of finite perimeter. Assume there exist $\alpha = \alpha(E, U) \in (0, 1)$, $C = C(E, U) > 0$ and $R = R(E, U) > 0$ such that*

$$\Psi(E, B_r(q)) \leq Cr^{2+2\alpha}, \quad \text{for all } q \in U \text{ and } r \in (0, R). \quad (64)$$

Then, up to replacing E by the suitable almost every (a.e.) representative, it holds that the reduced boundary $\partial^* E$ coincides with the topological boundary ∂E , and ∂E is a $C^{1,\alpha}$ -hypersurface in U .

Moreover, assuming that (64) holds uniformly for a sequence $(E_k)_{k \in \mathbb{N}}$ convergent to E_∞ in L^1_{loc} -topology, then E_∞ satisfies (64) as well; moreover for any sequence of points $q_k \in \partial E_k$ converging to $q_\infty \in \partial E_\infty$, the outward-pointing unit normal to ∂E_k at q_k converges to the outward-pointing unit normal to ∂E_∞ at q_∞ .

5.2 | Perturbed geodesic spheres are outer-minimising

Let (M, g) be a Riemannian manifold. For a Borel subset $\Omega \subset M$, we denote with $|\Omega|_g$ its volume and, if in addition it is a set of finite perimeter, with $P_g(\Omega)$ its perimeter.

The isoperimetric profile function $I_{(M,g)} : [0, \infty) \rightarrow [0, \infty)$ of (M, g) is defined by

$$I_{(M,g)}(v) := \inf\{P_g(\Omega) : \Omega \subset M \text{ is a finite perimeter set with } |\Omega|_g = v\}. \quad (65)$$

If (M, g) is an AF, complete, three dimensional Riemannian manifold, the following holds

$$\lim_{v \downarrow 0} v^{-2/3} I_{(M,g)}(v) = \begin{cases} (36\pi)^{1/3} & \text{if } \partial M = \emptyset \\ (18\pi)^{1/3} & \text{if } \partial M \neq \emptyset. \end{cases} \quad (66)$$

This can be proved along the same lines as [38], noticing that the AF assumption guarantees that the pointed limit manifolds at infinity used in the proof of [38] coincide with the Euclidean space \mathbb{R}^3 (thus one can relax the assumption of $C^{4,\alpha}$ bounded geometry, used in [38] to guarantee that the limit manifolds at infinity are $C^{3,\alpha}$ with $C^{2,\alpha}$ Riemannian metric).

Lemma 5.6. *Let (M, g) be an AF, complete, 3-dimensional Riemannian manifold with (possibly empty) horizon boundary. Then, for every $V_0 > 0$ there exists $C = C(V_0) > 0$ such that*

$$P_g(\Omega) \geq C |\Omega|_g^{2/3}, \quad \text{for every } \Omega \subset M \text{ subset of finite perimeter, with } |\Omega|_g \in (0, V_0]. \quad (67)$$

Proof. From [37], we know that $I : (0, \infty) \rightarrow (0, \infty)$ is a continuous function. Using (66), we conclude that the function $[0, V_0] \ni v \mapsto v^{-2/3} I_{(M,g)}(v)$ is continuous and never vanishes, giving the claim (67). \square

The goal of the present section is to prove the next theorem, which will allow us to use the expansion for the Hawking mass of perturbed spheres in order to get a lower bound on the Bartnik mass in Theorem 1.7.

Theorem 5.7. *Let (M, g) be an AF, complete, 3-dimensional Riemannian manifold with non-negative scalar curvature and with (possibly empty) horizon boundary ∂M . Fix $p \in M \setminus \partial M$. Then there exist $\rho_0 = \rho_0(p) > 0, r_0 = r_0(p) > 0$ such that the perturbed geodesic spheres $S_{p,\rho}(w)$ are outer-minimising for every $\rho \in (0, \rho_0]$ and every $w \in C^1(\mathbb{S}^2)$ with $\|w\|_{C^1(\mathbb{S}^2)} \leq r_0$.*

Throughout this section, we will denote by $B_{p,\rho}(w)$ the perturbed geodesic ball enclosed by $S_{p,\rho}(w)$.

Proof. Using the standard L^1_{loc} -compactness and lower-semicontinuity of the perimeter, observe that, for any $\rho > 0$ and $w \in C^1(\mathbb{S}^2)$, there exists a set of finite perimeter $\Omega_{p,\rho,w} \subset M$ minimising the perimeter among all sets of finite perimeter and finite volume containing $B_{p,\rho}(w)$. Such $\Omega_{p,\rho,w} \subset M$ is called *minimising hull* of $B_{p,\rho}(w)$. Clearly

$$P_g(\Omega_{p,\rho,w}) \leq P_g(B_{p,\rho}(w)), \quad 0 < |B_{p,\rho}(w)|_g \leq |\Omega_{p,\rho,w}|_g. \quad (68)$$

We will show that, up to choosing the suitable a.e. representative, $\Omega_{p,\rho,w} = B_{p,\rho}(w)$ for any $\rho, \|w\|_{C^1}$ small enough. Note that, under such a smallness condition, it holds that $B_{p,\rho}(w) \subseteq M \setminus \partial M$; thus, in the blow up arguments of steps 2–4, it is not restrictive to assume that $\partial M = \emptyset$.

Step 1. We claim that there exists $C = C_p \geq 1$ such that

$$0 < C^{-1} \leq \liminf_{\rho \downarrow 0} \frac{|\Omega_{p,\rho,w}|_g}{\rho^3} \leq \limsup_{\rho \downarrow 0} \frac{|\Omega_{p,\rho,w}|_g}{\rho^3} \leq C < \infty, \quad \forall \|w\|_{C^1(\mathbb{S}^2)} \leq 1. \quad (69)$$

The lower bound is a direct consequence of the second inequality in (68), thus we are left to show the upper bound.

From [7, Theorem C.2] (see also [18, Theorem 3] after [14]), we know that $\lim_{v \rightarrow \infty} \mathcal{I}_{(M,g)}(v) = +\infty$. Thus, (68) implies that there exists $V_0 > 0$ such that $|\Omega_{p,\rho,w}|_g \leq V_0$ for all $\rho \in (0, 1]$, $\|w\|_{C^1} \leq 1$.

Hence, the upper bound follows from the isoperimetric inequality (67) and the perimeter bound in (68):

$$\limsup_{\rho \downarrow 0} \frac{|\Omega_{p,\rho,w}|_g}{\rho^3} \stackrel{(67)}{\leq} C \limsup_{\rho \downarrow 0} \frac{P_g(\Omega_{p,\rho,w})^{3/2}}{\rho^3} \stackrel{(68)}{\leq} C \limsup_{\rho \downarrow 0} \frac{P_g(B_{p,\rho}(w))^{3/2}}{\rho^3} < \infty.$$

Step 2. Blow up and L^1_{loc} -convergence to a Euclidean ball.

In this step, we blow up the Riemannian manifold (M, g) at p with scaling rate ρ^{-1} as $\rho \downarrow 0$, and we show that the ‘rescaled $\Omega_{p,\rho,w}$ ’ converge as finite perimeter sets to the Euclidean ball of unit radius $B_1^\delta(0) \subset \mathbb{R}^3$.

To this aim, consider the rescaled Riemannian metric $g_\rho := \rho^{-2} g$ and observe that the rescaled pointed manifolds (M, g_ρ, p) converge to $(\mathbb{R}^3, \delta, 0)$ in the smooth pointed Cheeger-Gromov sense as $\rho \downarrow 0$; that is, calling $B_{\rho^{-1}}^\delta(0) \subset \mathbb{R}^3$ the Euclidean ball of radius ρ^{-1} centred at 0, for every $\rho \in (0, 1]$ there exists a smooth map $\psi_\rho : B_{\rho^{-1}}^\delta(0) \rightarrow M$ which is diffeomorphic onto its image such that $\psi_\rho(0) = p$ and $\psi_\rho^* g_\rho \rightarrow \delta$ smoothly locally on \mathbb{R}^3 as $\rho \downarrow 0$.

Combining the smooth pointed Cheeger-Gromov convergence with the compactness/lower semicontinuity of the perimeter, it follows that for every sequence $\rho_n \downarrow 0$ there exists a subsequence (not relabeled) and a set of finite perimeter $\bar{\Omega} \subset \mathbb{R}^3$ such that $\psi_{\rho_n}^{-1}(\Omega_{p,\rho_n,w_n}) \subset B_{\rho_n^{-1}}^\delta(0) \subset \mathbb{R}^3$ converges in $L^1_{loc}(\mathbb{R}^3)$ to $\bar{\Omega}$ and

$$|\bar{\Omega}|_\delta \leq \liminf_{n \rightarrow \infty} |\Omega_{p,\rho_n,w_n}|_{g_{\rho_n}} = \liminf_{n \rightarrow \infty} \rho_n^{-3} |\Omega_{p,\rho_n,w_n}|_g \stackrel{(69)}{<} \infty \quad \text{for all } \rho_n, \|w_n\|_{C^1(\mathbb{S}^2)} \rightarrow 0. \quad (70)$$

$$P_\delta(\bar{\Omega}) \leq \liminf_{n \rightarrow \infty} P_{g_{\rho_n}}(\Omega_{p,\rho_n,w_n}) = \liminf_{n \rightarrow \infty} \rho_n^{-2} P_g(\Omega_{p,\rho_n,w_n})$$

Since by construction $B_{p,\rho_n}(w_n) \subset \Omega_{p,\rho_n,w_n}$ and $\psi_{\rho_n}^{-1}(B_{p,\rho_n}(w_n)) \rightarrow B_1^\delta(0)$ smoothly as $n \rightarrow \infty$, it also holds

$$\begin{aligned} B_1^\delta(0) \subset \bar{\Omega}, \quad \lim_{n \rightarrow \infty} \rho_n^{-2} P_g(B_{p,\rho_n}(w_n)) &= \lim_{n \rightarrow \infty} P_{g_{\rho_n}}(B_{p,\rho_n}(w_n)) \\ &= P_\delta(B_1^\delta(0)), \quad \forall \rho_n, \|w_n\|_{C^1(\mathbb{S}^2)} \rightarrow 0. \end{aligned} \quad (71)$$

Recalling that $P_g(\Omega_{p,\rho,w}) \leq P_g(B_{p,\rho_n}(w_n))$, the combination of (70) and (71) yields:

$$B_1^\delta(0) \subset \bar{\Omega}, \quad |\bar{\Omega}|_\delta < \infty, \quad P_\delta(\bar{\Omega}) \leq P_\delta(B_1^\delta(0)).$$

The rigidity in the Euclidean isoperimetric inequality yields that $|\bar{\Omega} \Delta B_1^\delta(0)|_\delta = 0$. By the arbitrariness of the sequences (ρ_n) and (w_n) , we conclude that

$$\psi_\rho^{-1}(\Omega_{p,\rho,w}) \rightarrow B_1^\delta(0) \text{ in } L_{loc}^1(\mathbb{R}^3) \text{ as } \rho \rightarrow 0, \quad \|w\|_{C^1(\mathbb{S}^2)} \rightarrow 0. \quad (72)$$

Step 3. Improving the convergence via regularity theory.

Let us first fix some notation. Given a point $p \in M$, denote with $\phi_p^p : B_{\text{Inj}_p}^g(p) \rightarrow \mathbb{R}^3$ a normal coordinate chart centred at p with respect to the Riemannian metric g . Again, we consider the rescaled metrics $g_\rho := \rho^{-2}g$, for $\rho \in (0, 1]$. Notice that we can (and will) choose ψ_ρ from step 2 to be $\psi_\rho = (\phi_{g_\rho}^p)^{-1}$.

We first claim that there exist constants $C = C(p) > 0$ and $\rho_0 = \rho_0(p)$ such that

$$\begin{aligned} \left| P_\delta(\phi_{g_\rho}^p(\Omega_{p,\rho,w}), B_r^\delta(\phi_{g_\rho}^p(q))) - P_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) \right| &\leq Cr^4, \\ \text{for all } r \in (0, 1], \rho \in (0, \rho_0], q \in B_{10}^{g_\rho}(p), \|w\|_{C^1} &\leq 1. \end{aligned} \quad (73)$$

First, for any part of $\partial\Omega_{p,\rho,w}$ coinciding with the submanifold $S_{p,\rho}(w) = \partial B_{p,\rho}(w)$ we can use that

$$P_{g_\rho}(B_{p,\rho}(w), B_r^{g_\rho}(q)) \leq Cr^2, \quad \text{for all } r \in (0, 1], \rho \in (0, \rho_0(p)], q \in B_{10}^{g_\rho}(p), \|w\|_{C^1} \leq 1. \quad (74)$$

Second, away from the intersection points with $S_{p,\rho}(w)$, by construction $\Omega_{p,\rho,w}$ is locally perimeter minimising with respect to the metric g_ρ . Thus we can apply Lemma 5.3 with $E = \Omega_{p,\rho,w}$ to obtain

$$\begin{aligned} P_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) &\leq Cr^2, \quad \text{for all } r \in (0, 1], \rho \in (0, \rho_0], \|w\|_{C^1} \leq 1, q \in B_{10}^{g_\rho}(p) \\ \text{with } B_r^{g_\rho}(q) \cap S_{p,\rho}(w) &= \emptyset. \end{aligned} \quad (75)$$

The constant coming from Lemma 5.3 is independent of ρ because the $\Omega_{p,\rho,w}$ have uniformly bounded perimeter and g_ρ is smoothly converging to the Euclidean metric δ on $B_{11}^\delta(0)$.

Combining (74) and (75) with Lemma 5.1 gives the claim (73).

We next claim that the sequence $\phi_{g_\rho}^p(\Omega_{p,\rho,w})$ satisfies the conditions of Theorem 5.5.

Let $U = B_{10}^\delta(0)$. Note that for $\rho_0 = \rho_0(p) > 0$ small enough, we have that $\text{Inj}_p^{g_\rho} > 11$, so that $U = \phi_{g_\rho}^p(B_{10}^{g_\rho}(p))$ for all $\rho \in (0, \rho_0]$.

Let $F \subset M$ be such that $F \Delta \Omega_{p,\rho,w} \in B_r^{g_\rho}(q)$, and define $F' := F \cup B_{p,\rho}(w)$. Then, by the minimising assumption on $\Omega_{p,\rho,w}$, we have

$$P_{g_\rho}(\Omega_{p,\rho,w}) \leq P_{g_\rho}(F').$$

From standard properties of the perimeter, we have that

$$P_{g_\rho}(F \cup B_{p,\rho}(w)) + P_{g_\rho}(F \cap B_{p,\rho}(w)) \leq P_{g_\rho}(F) + P_{g_\rho}(B_{p,\rho}(w)).$$

Applying Lemma 5.4 with $U = B_{p,\rho}(w)$ gives that there exists $\bar{r} = \bar{r}(p) \in (0, 1]$, $C = C(p) > 0$ such that

$$P_{g_\rho}(B_{p,\rho}(w)) \leq P_{g_\rho}(G) + Cr^3, \quad \text{for all } G \Delta B_{p,\rho}(w) \in B_r^{g_\rho}(q), q \in B_{10}^{g_\rho}(p), r \in (0, \bar{r}].$$

Letting $G = F \cap B_{p,\rho}(w)$ and combining the three previous inequalities gives

$$P_{g_\rho}(\Omega_{p,\rho,w}) \leq P_{g_\rho}(F') \leq P_{g_\rho}(F) + Cr^3, \quad \text{for all } F \Delta \Omega_{p,\rho,w} \in B_r^{g_\rho}(q), q \in B_{10}^{g_\rho}(p), r \in (0, \bar{r}].$$

Since the sets $\Omega_{p,\rho,w}$ and F coincide outside of $B_r^{g_\rho}(q)$, the last estimate is equivalent to

$$P_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) \leq P_{g_\rho}(F, B_r^{g_\rho}(q)) + Cr^3, \quad \text{for all } F \Delta \Omega_{p,\rho,w} \in B_r^{g_\rho}(q), q \in B_{10}^{g_\rho}(p), r \in (0, \bar{r}]. \quad (76)$$

We can finally estimate the excess:

$$\begin{aligned} & \Psi(\phi_{g_\rho}^p(\Omega_{p,\rho,w}), B_r^\delta(\phi_{g_\rho}^p(q))) \\ &:= P_\delta(\phi_{g_\rho}^p(\Omega_{p,\rho,w}), B_r^\delta(\phi_{g_\rho}^p(q))) \\ & \quad - \inf \left\{ P_\delta(\phi_{g_\rho}^p(F), B_r^\delta(\phi_{g_\rho}^p(q))) | \phi_{g_\rho}^p(F) \Delta \phi_{g_\rho}^p(\Omega_{p,\rho,w}) \in B_r^\delta(\phi_{g_\rho}^p(q)) \right\} \\ & \stackrel{(73),(61)}{=} P_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(r^4) \\ & \quad - \inf \left\{ (1 + \mathcal{O}(r^2)) P_{g_\rho}(F, B_r^{g_\rho}(q)) + \mathcal{O}(r^4) | \phi_{g_\rho}^p(F) \Delta \phi_{g_\rho}^p(\Omega_{p,\rho,w}) \in B_r^\delta(\phi_{g_\rho}^p(q)) \right\} \\ & \stackrel{(76)}{\leq} P_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(r^4) - [(1 + \mathcal{O}(r^2))(P_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) - Cr^3) + \mathcal{O}(r^4)] \\ & \stackrel{(75)}{\leq} Cr^3, \quad \text{for all } r \in (0, \bar{r}], \|w\|_{C^1} \leq 1, q \in B_{10}^\delta(0), \end{aligned} \quad (77)$$

for some constant $C = C(p) > 0$. Thus, the family $\phi_{g_\rho}^p(\Omega_{p,\rho,w})$ satisfies the assumptions of Theorem 5.5 with $\alpha = \frac{1}{2}$. We infer that $\partial \Omega_{p,\rho,w}$ are $C^{1,1/2}$ surfaces, and the outward pointing unit normals of $\partial \Omega_{p,\rho,w}$ converge to the outward pointing unit normal of $\partial B_1^\delta(0)$ as $\rho, \|w\|_{C^1} \rightarrow 0$. This implies that there exists $r_0 = r_0(p) > 0$ and $\rho_0 = \rho_0(p) > 0$ such that, for all $\|w\|_{C^1} \leq r_0$ and

$\rho \in (0, \rho_0]$, the surfaces $\partial\Omega_{p,\rho,w}$ are $C^{1,1/2}$ graphs over $\partial B_1^\delta(0)$. Moreover, such graphs converge to 0 in the $C^{1,1/2}$ topology as $\rho \rightarrow 0$, $\|w\|_{C^1} \rightarrow 0$.

Step 4. Conclusion by a first variation argument.

First of all, note that for $\rho > 0$ small enough (depending only on p), the surface $\phi_{g_\rho}^p(S_{p,\rho}(w))$ is a graph over $\partial B_1^\delta(0) = \mathbb{S}^2$. Combining this fact with Step 3, we get that for $\rho > 0$ and $\|w\|_{C^1}$ small enough (depending only on p) the surface $\partial\phi_{g_\rho}^p(\Omega_{p,\rho,w})$ is a graph over $\phi_{g_\rho}^p(S_{p,\rho}(w))$. Thus $\partial\Omega_{p,\rho,w}$ is parameterized by

$$\partial\Omega_{p,\rho,w} = \{\exp_q^{g_\rho}(u_{\rho,w}(q)\hat{N}(q)) : q \in S_{p,\rho}(w)\}$$

for some function $u_{\rho,w} \in C^1(S_{p,\rho}(w))$. Note that $u_{\rho,w} \geq 0$, since by assumption $S_{p,\rho}(w) \subset \Omega_{p,\rho,w}$.

Using that both $\partial\phi_{g_\rho}^p(\Omega_{p,\rho,w})$ and $\phi_{g_\rho}^p(S_{p,\rho}(w))$ converge to $B_1^\delta(0)$ as $\rho \rightarrow 0$, $\|w\|_{C^1} \rightarrow 0$, we also have

$$\|u_{\rho,w}\|_{C^1} \rightarrow 0 \text{ as } \rho \rightarrow 0, \|w\|_{C^1} \rightarrow 0. \quad (78)$$

For fixed $\rho > 0$, consider the Banach space $C^1(S_{p,\rho}(w))$ of graph functions over $S_{p,\rho}(w)$. The area functional in g_ρ -metric, $A_{g_\rho} : C^1(S_{p,\rho}(w)) \rightarrow \mathbb{R}$, is Fréchet differentiable at 0, with derivative $d(A_{g_\rho})_0 \in \mathcal{L}(C^1(S_{p,\rho}(w)), \mathbb{R})$ such that

$$A_{g_\rho}(0 + h) = A_{g_\rho}(0) + d(A_{g_\rho})_0(h) + o(\|h\|_{C^1}), \quad \text{for all } h \in C^1(S_{p,\rho}(w)).$$

In particular, setting $h = u_{\rho,w}$ gives

$$A_{g_\rho}(0 + u_{\rho,w}) = A_{g_\rho}(0) + d(A_{g_\rho})_0(u_{\rho,w}) + o(\|u_{\rho,w}\|_{C^1}).$$

Comparing this to the first variation (Gateaux derivative) of A_{g_ρ} , we see that

$$d(A_{g_\rho})_0(u_{\rho,w}) = \int_{S_{p,\rho}(w)} H_{S_{p,\rho}(w)}^{g_\rho} u_{\rho,w} dV_{g_\rho}.$$

Therefore

$$A_{g_\rho}(0 + u_{\rho,w}) = A_{g_\rho}(0) + \int_{S_{p,\rho}(w)} H_{S_{p,\rho}(w)}^{g_\rho} u_{\rho,w} dV_{g_\rho} + o(\|u_{\rho,w}\|_{C^1}).$$

Note that the left-hand side coincides with $A_{g_\rho}(\partial\Omega_{p,\rho,w}) = P_{g_\rho}(\Omega_{p,\rho,w})$. Moreover, for small ρ , we have that

$$H_{S_{p,\rho}(w)}^{g_\rho} = 2 + \mathcal{O}(\rho^2) > 0,$$

indeed, by (19) we know $H_{S_{p,\rho}(w)}^g = 2\rho^{-1} + \mathcal{O}(\rho)$, which gets multiplied by a factor ρ due to the scaling of the metric. Thus, for small ρ , we get

$$A_{g_\rho}(\partial\Omega_{p,\rho,w}) \geq A_{g_\rho}(0) = A_{g_\rho}(S_{p,\rho}(w)),$$

with equality if and only if $u_{\rho,w} \equiv 0$.

Since by construction $\partial\Omega_{p,\rho,w}$ is the minimising hull of $B_{p,\rho}(w)$, we have $A_{g_\rho}(\partial\Omega_{p,\rho,w}) \leq A_{g_\rho}(S_{p,\rho}(w))$. Thus $u_{\rho,w} \equiv 0$, that is $\partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$. Hence $S_{p,\rho}(w)$ is outer-minimising for $\rho > 0$ and $\|w\|_{C^1}$ sufficiently small (smallness depending only on p). \square

5.3 | Proof of the Bartnik mass Theorem 1.7

Firstly, recall that (7) gives

$$m_B(\Omega) \geq m_H(\partial\Omega).$$

For every perturbed geodesic sphere $S_{p,\rho}(w) \subset \Omega$, with $\rho > 0$ and $\|w\|_{C^1}$ sufficiently small (only depending on p), we have

$$m_H(S_{p,\rho}(w)) \leq m_B(B_{p,\rho}(w)) \leq m_B(\Omega), \quad (79)$$

where the first inequality follows from (7) applied to $S_{p,\rho}(w)$ and the second follows by the monotonicity property (5), which applies due to the outer-minimising property of $S_{p,\rho}(w)$ proved in Theorem 5.7. For every $p \in \Omega \setminus \partial M$ and for $\rho > 0$ sufficiently small (depending only on p), let $w_{p,\rho} \in C^{4,\alpha}(\mathbb{S}^2)^\perp \subset C^1(\mathbb{S}^2)$ be given by Lemma 3.1, that is, $w_{p,\rho}$ is the optimal perturbation extremizing (in $C^{4,\alpha}(\mathbb{S}^2)^\perp$) the Hawking mass under area constraint. Combining the expansion (31) and the inequality (79), specialised to $w = w_{p,\rho}$, gives the claimed lower bound on the Bartnik mass:

$$m_B(\Omega) \geq \frac{1}{12} \text{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \text{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \text{Sc}_p^2 \right) \rho^5 + \mathcal{O}(\rho^6).$$

If $m_B(\Omega) = 0$, from (79) we infer that $m_H(S_{p,\rho}(w))$ is non-positive for every perturbed geodesic sphere of sufficiently small radius. We can then apply Theorem 4.1 to Ω to infer that $\Omega \setminus \partial M$ is locally isometric to Euclidean \mathbb{R}^3 .

6 | OTHER RIGIDITY RESULTS

In this section, we collect other rigidity results involving the Hawking mass in various different settings. As the reader will appreciate, the proofs will be quite straightforward due to the work done in the previous sections (in particular, we will make repeated use of the expansion of the Hawking mass obtained in Proposition 3.2).

6.1 | Rigidity for the generalised Hawking mass (that is, for non-zero cosmological constant)

The standard Hawking mass (3) is relevant when the ambient space is a Riemannian 3-manifold with non-negative scalar curvature. Such metrics are natural when the cosmological constant is zero. Instead, when the cosmological constant Λ is negative (respectively, positive), it is more natural to consider metrics with scalar curvature bounded below by a negative (respectively,

positive) constant. Indeed, the Dominant Energy Condition coupled with the Einstein Constraint Equations imply that the scalar curvature of a totally geodesic space-like hypersurface (that is, the so-called time-symmetric case) is bounded below by 2Λ .

When Λ is negative (respectively, positive) it is standard to choose the normalization $\Lambda = -3$ (respectively, $\Lambda = 3$) and it is natural to compare the geometry of a totally geodesic space-like hypersurface with a space-form of constant sectional curvature $K = -1$ (respectively, $K = 1$).

When $\Lambda \in \{-3, 0, 3\}$, it is also natural to modify the Hawking mass as follows (see, for instance, [39]).

Definition 6.1. Let $K \in \{-1, 0, 1\}$ and let (M^3, g) be a 3-dimensional Riemannian manifold with $\text{Sc} \geq 6K$. The generalized Hawking mass of an immersed sphere Σ in M is

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{(16\pi)^3}} \left(16\pi - \int_{\Sigma} (H^2 + 4K) dV_{\Sigma} \right). \quad (80)$$

Arguing along the lines as the proof Theorem 4.3, we obtain the following rigidity result involving the generalised Hawking mass.

Theorem 6.2. Let (M^3, g) be a connected, complete Riemannian manifold without boundary, with scalar curvature $\text{Sc} \geq 6K$ where $K \in \{-1, 0, 1\}$. If for every $p \in M$ there is a neighbourhood U of p such that the generalised Hawking mass of every embedded sphere contained in U is non-positive, then (M^3, g) is isometric to a space form of constant sectional curvature K .

Proof. Let us compute the generalized Hawking mass of the perturbed geodesic spheres as before. Note that the only difference with the standard Hawking mass is the extra term $4K \int_{\mathbb{S}^2} \sqrt{\det \dot{g}}$. Recalling (51), this is easily evaluated up to fourth order as

$$4K \int_{\mathbb{S}^2} \sqrt{\det \dot{g}} = 16K\pi\rho^2 - \frac{8K\pi}{9}\text{Sc}_p\rho^4 + \mathcal{O}_p(\rho^5).$$

Recalling (50), we obtain the following expansion for the generalized Hawking mass of $S_{p,\rho}(w_{p,\rho})$:

$$\begin{aligned} m_H(S_{p,\rho}(w_{p,\rho})) &= \sqrt{\frac{|S_{p,\rho}(w_{p,\rho})|}{(16\pi)^3}} \left[\left(\frac{8\pi}{3}\text{Sc}_p - 16K\pi \right) \rho^2 \right. \\ &\quad \left. + \left(\frac{4\pi}{15}\Delta\text{Sc}_p + \frac{16\pi}{45}\|S_p\|^2 - \frac{4\pi}{27}\text{Sc}_p^2 + \frac{8K\pi}{9}\text{Sc}_p \right) \rho^4 + \mathcal{O}_p(\rho^5) \right]. \end{aligned} \quad (81)$$

Assuming that $m_H(S_{p,\rho}(w_{p,\rho})) \leq 0$ for $\rho > 0$ sufficiently small yields $\text{Sc}_p \leq 6K$. Since we assumed that $\text{Sc}_p \geq 6K$ for all $p \in M$, we have

$$\text{Sc} \equiv 6K. \quad (82)$$

Inserting (82) into (81) and evaluating at sufficiently small $\rho > 0$ gives $\frac{16\pi}{45} \|S_p\|^2 \leq 0$ for every $p \in M$. Therefore, the trace-free Ricci tensor vanishes:

$$S \equiv 0. \quad (83)$$

Putting together (82) and (83) gives

$$\text{Ric} \equiv 2Kg.$$

Recalling that in dimension three the Riemann curvature tensor can be written as $\text{Rm} = \text{Ric} \otimes g - \frac{1}{4}\text{Sc} g \otimes g$ where \otimes is the Kulkarni–Nomizu product (see, for instance, [23, Corollary 7.26]), we conclude that g has constant sectional curvature K . \square

Remark 6.3. Note that (81) actually gives a strictly positive (yet small) lower bound on the generalised Hawking mass of the optimally perturbed geodesic sphere $S_{p,\rho}(w_{p,\rho})$, if $\text{Sc}_p > 6K$ or $\text{Sc} \equiv 6K$ & $\|S_p\| \neq 0$ (and, as observed in the proof, such a point $p \in M$ always exists if (M, g) does not have constant sectional curvature and $\text{Sc} \geq 6K$).

Even though the proof of Theorem 5.7 made use of the AF assumption, we expect that for $\rho > 0$ sufficiently small the surface $S_{p,\rho}(w_{p,\rho})$ is outward minimising also in a locally asymptotically hyperbolic framework. We did not push in that direction since it does not seem to be useful in order to obtain a lower bound on the hyperbolic analogue of the Bartnik mass, in the spirit of Theorem 1.7. Indeed, if for $\rho > 0$ sufficiently small the surface $S_{p,\rho}(w_{p,\rho})$ is outward-minimising, one can start the (weak) Inverse Mean Curvature Flow in the sense of Huisken–Ilmanen [15] and the generalised Hawking mass is monotone non-decreasing also in this setting (see, for instance, [39] for more details). However, as proved by Neves [39], it may happen that the asymptotic limit of the Hawking mass along the IMC flow exceeds the hyperbolic-ADM mass of the manifold, thus preventing us to repeat the proof of Theorem 1.7 in the $K = -1$ case.

6.2 | \mathbb{R}^3 and \mathbb{H}^3 rigidity in the homogeneous setting

In this section, we replace the ALSC assumption in the rigidity Theorem 4.3 with the homogeneity condition. Recall that a Riemannian manifold (M, g) is said to be *homogeneous* if its isometry group $\text{Isom}(M, g)$ acts transitively on M . In other words, if for every $p, q \in M$, there exists $\gamma \in \text{Isom}(M, g)$ such that $\gamma(p) = q$.

Whilst of course our spatial universe is not *exactly* homogenous, at cosmological scales the homogeneity provides a useful idealisation. Indeed, spatial homogeneity is a standard assumption in Cosmology. For instance, it leads to an exact solution of Einstein's field equations, known as the Robertson–Walker metric for space-time [12, 50].

Theorem 6.4. *Let $K \in \{-1, 0, 1\}$ and let (M^3, g) be a connected, homogeneous Riemannian manifold with scalar curvature $\text{Sc} \geq 6K$. If for every $p \in M$ there is a neighbourhood U of p such that the generalised Hawking mass (80) of every embedded sphere contained in U is non-positive, then (M^3, g) is isometric to:*

- \mathbb{H}^3 (if $K = -1$); or
- $\mathbb{R}^m \times \mathbb{T}^{3-m}$, for some $0 \leq m \leq 3$, where \mathbb{T}^{3-m} is a flat torus of dimension $3 - m$ (if $K = 0$); or

- \mathbb{S}^3/Γ for some finite subgroup of isometries $\Gamma < \text{Iso}(\mathbb{S}^3)$ acting freely on \mathbb{S}^3 (if $K = 1$).

Proof. Since homogeneity implies completeness and that $\partial M = \emptyset$, Theorem 6.2 yields that M has constant sectional curvature K . The conclusion now follows from the classical classification of homogenous spaces of constant sectional curvature (see for instance [54, Theorem 2.7.1]). Note that in the case $K = 1$, one can be more precise about the type of quotients appearing, at the price of a more technical statement. \square

6.3 | \mathbb{R}^3 and \mathbb{H}^3 rigidity under global asymptotic volume growth assumptions

In this section, we replace the ALSC assumption in the rigidity Theorem 4.3 with a global volume growth assumption (satisfied, for instance, by asymptotically flat spaces and asymptotically hyperbolic spaces). Such an assumption is obtained by comparing the volume growth of metric balls in the space under consideration with metric balls in an appropriate model space, as the radius goes to infinity.

Definition 6.5. Let $K \in \{-1, 0\}$ and let (M^3, g) be a complete Riemannian manifold without boundary and with $\text{Sc} \geq 6K$. We say that (M^3, g) satisfies the K -Global Asymptotic Volume property (K -GAVP) if

$$\limsup_{r \rightarrow \infty} \frac{\text{Vol}_g(B_r^g(p))}{\text{Vol}_K(r)} \geq 1, \quad (84)$$

where $\text{Vol}_K(r)$ denotes the volume of a metric ball of radius r in the 3-dimensional simply connected space of constant sectional curvature K .

There are various ways to define an asymptotically hyperbolic manifold. For the conformal compactification approach, see, for instance, Wang [49]. In closer analogy to Definition 1.1, we take the asymptotic chart approach (see, for instance, [9, 13, 43] for discussions on the physical relevance of such metrics):

Definition 6.6. A 3-dimensional Riemannian manifold (M, g) is said to be *asymptotically hyperbolic (AH)* if there is a compact subset $C \subset M$ and a diffeomorphism $\phi : M \setminus C \rightarrow \mathbb{R}^3 \setminus \overline{B_1(0)}$ such that the metric satisfies

$$|g_{\mu\nu} - (g_{\mathbb{H}^3})_{\mu\nu}| = \mathcal{O}(r^{-s})$$

for some $s > 0$ in the chart ϕ . Here $\mathbb{H}^3 = (\mathbb{R}^3, g_{\mathbb{H}^3})$ denotes the standard hyperbolic space with metric $g_{\mathbb{H}^3} = \frac{1}{1+r^2}dr^2 + r^2g_{\mathbb{S}^2}$ (in polar coordinates).

Remark 6.7. AF (respectively, AH) manifolds satisfy the 0-Global Asymptotic Volume property (respectively, -1 -GAVP). For the AF case, consider a straight line segment γ in \mathbb{R}^3 , parameterised on the interval $(0, \sqrt{r})$ by $\gamma(t) = \sqrt{rt}\hat{a}$, for some unit vector $\hat{a} \in \mathbb{R}^3$. It is easy to see that if (M, g) is AF with metric $g|_{M \setminus C} = \bar{g} + h$, with $h = \mathcal{O}(r^{-s})$ for some $s > 0$, then there exists a constant $A > 0$

such that

$$\text{Length}_g(\gamma) \leq A + (1 + Ar^{-s}) \text{Length}_\delta(\gamma), \quad \text{for all } r \geq 1,$$

giving that $B_r^\delta(p) \subset B_{Ar(1+Ar^{-s})}^g(p)$. Therefore $\text{Vol}_\delta(B_r^\delta(p)) \leq \text{Vol}_g(B_{A+r(1+Ar^{-s})}^g(p))$. Sending $r \rightarrow \infty$ yields the 0-GAVP:

$$\limsup_{r \rightarrow \infty} \frac{\text{Vol}_g(B_r^g(p))}{\text{Vol}_\delta(B_r^\delta(p))} \geq 1.$$

The hyperbolic case is analogous, replacing straight lines by minimising geodesics in \mathbb{H}^3 and applying Definition 6.6.

Theorem 6.8. *Let $K \in \{-1, 0\}$. Let (M^3, g) be a connected, complete Riemannian manifold without boundary, with scalar curvature $\text{Sc} \geq 6K$, and satisfying the K -Global Asymptotic Volume property. If for every $p \in M$ there is a neighbourhood U of p such that the generalised Hawking mass of every embedded sphere contained in U is non-positive, then (M, g) is isometric to \mathbb{H}^3 (if $K = -1$) or \mathbb{R}^3 (if $K = 0$).*

Proof. By recalling Theorem 6.2, the assumptions on the scalar curvature and Hawking mass imply that (M^3, g) has constant sectional curvature K . Therefore, the Ricci curvature of (M^3, g) is identically equal to $2Kg$. The Bishop–Gromov Theorem (see, for instance, [23, 42]) gives that the ratio

$$r \mapsto \frac{\text{Vol}_g(B_r^g(p))}{\text{Vol}_K(r)} \text{ is non-increasing and is bounded above by 1.}$$

Moreover, if equality holds for some $r > 0$, then the metric ball $B_r^g(p)$ in (M, g) is isometric to a metric ball of radius r in the simply connected 3-dimensional space of constant sectional curvature K . It follows that

$$\limsup_{r \rightarrow \infty} \frac{\text{Vol}_g(B_r^g(p))}{\text{Vol}_K(r)} \leq 1, \quad (85)$$

with equality if and only if for every $r > 0$ the metric ball $B_r^g(p)$ in (M, g) is isometric to a metric ball of radius r in the simply connected 3-dimensional space of constant sectional curvature K or, equivalently, if (M, g) is globally isometric to the simply connected 3-dimensional space of constant sectional curvature K .

Since the equality in (85) is forced by the K -GAVP, the result follows. \square

APPENDIX

A.1 | A sup-Hawking mass for AF manifolds with non-negative scalar curvature

Inspired by the results of the present paper, it is natural to propose a slight variant of the Hawking mass. Indeed, the standard Hawking mass, while being very useful in applications (for instance, in the proof of the Riemannian Penrose inequality via Inverse Mean Curvature Flow by Huisken–Ilmanen [15]), has some inconvenient features. For instance, it can be negative and it has no clear

monotone property under inclusion. Even though the list of properties that are desirable for a quasi-local mass is open for debate, let us mention some natural ones.

Let (M^3, g) be an asymptotically flat Riemannian manifold with non-negative scalar curvature and with (possibly empty) horizon boundary ∂M .

According to Bartnik [2], a ‘good’ notion of quasi-local mass $m(\Omega)$ for subsets $\Omega \subset M$ should satisfy:

- (i) $m(\Omega)$ should be uniquely defined for every domain Ω ;
- (ii) positivity: $m(\Omega) > 0$ unless $\Omega \subset \mathbb{R}^3$, in which case $m(\Omega) = 0$;
- (iii) monotonicity: if $\Omega_1 \subset \Omega_2 \subset M$, then $m(\Omega_1) \leq m(\Omega_2)$;
- (iv) asymptotic to ADM mass: If $\{\Omega_k\}_{k=1}^\infty$ is an exhaustion of M , then $m(\Omega_k) \rightarrow m_{\text{ADM}}(M, g)$ as $k \rightarrow \infty$.

Even if not explicitly requested by Bartnik, it is also natural to require:

- (v) Compatibility with Schwarzschild: Let $m_{\text{Sch}} > 0$ and consider $\mathbb{R}^3 \setminus B_{2m_{\text{Sch}}}^\delta(0) \simeq \mathbb{S}^2 \times [2m_{\text{Sch}}, \infty)$ endowed with the Schwarzschild metric $g^{m_{\text{Sch}}}$ of mass $m_{\text{Sch}} > 0$:

$$g^{m_{\text{Sch}}} := \left(1 - \frac{2m_{\text{Sch}}}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}. \quad (\text{A.1})$$

Then the quasi-local mass of every subset (satisfying suitable geometric conditions) containing the horizon $\{r = 2m_{\text{Sch}}\}$ is equal to m_{Sch} .

We propose the next definition.

Definition A.1 (sup-Hawking mass). Let (M^3, g) be an asymptotically flat Riemannian manifold in the sense of Definition 1.1 with non-negative scalar curvature and with (possibly empty) horizon boundary ∂M . For every open subset $\Omega \subset M$, denote with $\bar{\Omega}$ its topological closure and define the *sup-Hawking mass* as

$$m_{\text{SH}}(\Omega) := \sup\{m_H(\partial\Omega') \mid \Omega' \subset \bar{\Omega} \text{ such that } \partial\Omega' \simeq \mathbb{S}^2 \text{ is smooth and outer-minimising in } \bar{\Omega}\}. \quad (\text{A.2})$$

It is clear that, if $\partial\Omega \simeq \mathbb{S}^2$ is smooth, then

$$m_H(\partial\Omega) \leq m_{\text{SH}}(\Omega). \quad (\text{A.3})$$

A benefit of the proposed sup-Hawking mass (A.2) is that it satisfies (a suitable version) of all the requirements (i)–(v). Property (i) is clearly satisfied, so let us discuss the others. The proof of (ii) is a nice application of the present paper as it involves basically all of the main results.

Proposition A.2 (Validity of (ii)). Let (M, g) and $\Omega \subset M$ be as in Definition A.1.

Then $m_{\text{SH}}(\Omega) \geq 0$ with equality if and only if $\Omega \setminus \partial M$ is locally isometric to Euclidean \mathbb{R}^3 .

Proof. For every $p \in \Omega \setminus \partial M$, let $S_{p,\rho}(w_{p,\rho})$ be the optimally perturbed geodesic sphere constructed in Lemma 3.1. Of course, for $\rho > 0$ small enough, it holds that $S_{p,\rho}(w_{p,\rho}) \subset \Omega \setminus \partial M$.

From Theorem 5.7, $S_{p,\rho}(w_{p,\rho})$ is outer-minimising in M (and thus in $\bar{\Omega}$) for $\rho > 0$ small enough. Thus, from the very definition of the sup-Hawking mass (A.2) it holds that

$$m_H(S_{p,\rho}(w_{p,\rho})) \leq m_{SH}(\Omega), \quad \forall p \in \Omega \setminus \partial M, \quad \forall \rho \in (0, \bar{\rho}_p), \quad (\text{A.4})$$

for some suitable $\bar{\rho}_p > 0$ depending on $p \in \Omega \setminus \partial M$. The combination of Proposition 3.2 and Theorem 4.1 then yields that $m_{SH}(\Omega) \geq 0$ with equality only if $\Omega \setminus \partial M$ is locally isometric to Euclidean \mathbb{R}^3 . Conversely, if $\Omega \setminus \partial M$ is locally isometric to \mathbb{R}^3 then one can choose sufficiently small round spheres as competitors in (A.2) and obtain that $m_{SH}(\Omega) = 0$. \square

Proposition A.3 (Validity of (iii)). *Let (M, g) and $\Omega_1 \subset \Omega_2 \subset M$ be as in Definition A.1 with $\partial\Omega_1$ outer-minimising in $\bar{\Omega}_2$. Then $m_{SH}(\Omega_1) \leq m_{SH}(\Omega_2)$.*

Proof. If $\Omega' \subset \bar{\Omega}_1$ satisfies that $\partial\Omega' \simeq \mathbb{S}^2$ is smooth and outer-minimising in $\bar{\Omega}_1$ and $\partial\Omega_1$ is outer-minimising in $\bar{\Omega}_2$, then $\partial\Omega' \simeq \mathbb{S}^2$ is smooth and outer-minimising in $\bar{\Omega}_2$ as well. The monotonicity is then a direct consequence of the definition (A.2). \square

In order to prove (a suitable version of) property (iv), we first establish the next two results of independent interest.

Proposition A.4 (m_{ADM} is an upper bound for m_{SH}). *Let (M, g) and $\Omega \subset M$ be as in Definition A.1. Assume in addition that $\partial\Omega$ is outer-minimising in M . Then $m_{SH}(\Omega) \leq m_{ADM}(M, g)$.*

Proof. Since $\partial\Omega$ is outer-minimising in M , every $\Omega' \subset \bar{\Omega}$ with $\partial\Omega' \simeq \mathbb{S}^2$ smooth and outer-minimising in $\bar{\Omega}$ is also outer-minimising in M . Thus, one can run the (weak version of) Inverse Mean Curvature Flow by Huisken–Ilmanen [15] starting from Ω' and obtain that $m_H(\partial\Omega') \leq m_{ADM}(M, g)$. The claim follows now from the very definition (A.2) of $m_{SH}(\Omega)$. \square

Remark A.5 ($m_H \leq m_{SH} \leq m_B$). Since the upper bound in Proposition A.4 holds for any AF extension of Ω , it follows that

$$m_{SH}(\Omega) \leq m_B(\Omega) \leq m_{ADM}(M, g) \quad \text{for all } \Omega \subset M \text{ with } \partial\Omega \text{ outer-minimising,}$$

where $m_B(\Omega)$ denotes the Bartnik mass of Ω . Thus, recalling (A.3), we obtain:

$$m_H(\partial\Omega) \leq m_{SH}(\Omega) \leq m_B(\Omega) \leq m_{ADM}(M, g),$$

$$\forall \Omega \subset M \text{ with } \partial\Omega \simeq \mathbb{S}^2 \text{ smooth and outer-minimising.}$$

Lemma A.6 (Existence of an exhaustion asymptotic to the ADM mass). *Let (M, g) be as in Definition A.1. Let Σ_ρ be the coordinate sphere of radius $\rho \gg 1$ in an asymptotically flat coordinate chart and denote with B_ρ the bounded region enclosed by Σ_ρ . Then $\lim_{\rho \rightarrow \infty} m_{SH}(B_\rho) = m_{ADM}(M, g)$.*

Proof. It is well known that for $\rho \gg 1$ sufficiently large, the coordinate sphere Σ_ρ satisfies:

- Σ_ρ is outer-minimising in M (a careful reader has probably noticed that this fact can also be proven by a blow-down argument analogous to the proof by blow-up of Theorem 5.7);
- $\lim_{\rho \rightarrow \infty} m_H(\Sigma_\rho) = m_{ADM}(M, g)$, see, for instance, [22, Exercise 4.25].

The combination of the last property and (A.3) gives on the one hand that

$$m_{\text{ADM}}(M, g) = \lim_{\rho \rightarrow \infty} m_H(\Sigma_\rho) \leq \liminf_{\rho \rightarrow \infty} m_{SH}(B_\rho).$$

On the other hand, using Proposition A.4 and that Σ_ρ is outer-minimising, we infer that $m_{SH}(B_\rho) \leq m_{\text{ADM}}(M, g)$ for every $\rho \gg 1$. The conclusion follows. \square

We can now prove the following (suitable version of) property (iv).

Proposition A.7 (Validity of (iv)). *Let (M, g) be as in Definition A.1. If $\{\Omega_k\}_{k=1}^\infty$ is an exhaustion of M such that each $\partial\Omega_k$ is outer-minimising in M , then $m_{SH}(\Omega_k) \rightarrow m_{\text{ADM}}(M, g)$ as $k \rightarrow \infty$.*

Proof. First, using Proposition A.4 and that $\partial\Omega_k$ is outer-minimising, we infer that

$$m_{SH}(\Omega_k) \leq m_{\text{ADM}}(M, g), \quad \forall k \in \mathbb{N}.$$

Also, using that $\{\Omega_k\}_{k=1}^\infty$ is an exhaustion of M , we have that for every $\rho \gg 1$ there exists $k_0 > 0$ such that

$$B_\rho \subset \Omega_k, \quad \forall k \geq k_0,$$

where B_ρ is as in Lemma A.6. The monotonicity property of m_{SH} (see Proposition A.3) and the fact that $\partial B_\rho = \Sigma_\rho$ is outer-minimising in M (see the proof of Lemma A.6) yield that for every $\rho \gg 1$ sufficiently large there exists $k_0 > 0$ such that

$$m_{SH}(B_\rho) \leq m_{SH}(\Omega_k), \quad \forall k \geq k_0.$$

Recalling Lemma A.6, we thus obtain

$$m_{\text{ADM}}(M, g) = \lim_{\rho \rightarrow \infty} m_{SH}(B_\rho) \leq \liminf_{k \rightarrow \infty} m_{SH}(\Omega_k).$$

The conclusion follows by the combination of the first and last displayed formulas in the proof. \square

We next establish property (v).

Proposition A.8 (Validity of (v)). *Let $m_{Sch} > 0$ and consider $\mathbb{R}^3 \setminus B_{2m_{Sch}}^\delta(0) \simeq \mathbb{S}^2 \times [2m_{Sch}, \infty)$ endowed with the Schwarzschild metric $g^{m_{Sch}}$ of mass $m_{Sch} > 0$ as in (A.1). Let $\Omega \subset \mathbb{R}^3 \setminus B_{2m_{Sch}}^\delta(0)$ be an open subset such that $\partial\Omega$ is outer-minimising and Ω contains the horizon, that is, $\{r = 2m_{Sch}\} \subset \Omega$. Then $m_{SH}(\Omega) = m_{Sch}$.*

Proof. Since $m_{\text{ADM}}(g^{m_{Sch}}) = m_{Sch}$, the upper bound given in Proposition A.4 yields

$$m_{SH}(\Omega) \leq m_{Sch}.$$

To obtain the reversed inequality, observe that the horizon $\Sigma_{2m_{Sch}} := \{r = 2m_{Sch}\}$ is outer-minimising and satisfies $m_H(\Sigma_{2m_{Sch}}) = m_{Sch}$. The definition (A.2) of m_{SH} yields

$$m_{Sch} = m_H(\Sigma_{2m_{Sch}}) \leq m_{SH}(\Omega).$$

The conclusion follows by combining the two inequalities displayed in the proof. \square

We wish to conclude this appendix by suggesting a possible research direction, drawing a more precise connection between the sup-Hawking mass and the Bartnik mass.

Notice indeed that the sup-Hawking mass corresponds to ‘defining a quasi-local mass by taking the supremum from inside’ while the Bartnik mass corresponds to ‘defining a quasi-local mass by taking the infimum from outside’. In Remark A.5, we observed that for every $\Omega \subset M$ with $\partial\Omega$ outer-minimising it holds that $m_{SH}(\Omega) \leq m_B(\Omega)$, however, it is natural to expect that the two objects coincide (under appropriate conditions on Ω and/or M):

Open Problem: find appropriate (necessary and/or sufficient) conditions on Ω and/or M so that

$$m_{SH}(\Omega) = m_B(\Omega).$$

Remark A.9 (Extension to the case of non-null cosmological constant). By using the generalized Hawking mass (80), one can extend the definition (A.2) of sup-Hawking mass to the case when (M^3, g) has scalar curvature bounded below by $6K$, with $K \in \{-1, 0, 1\}$. Some of the good properties above would be retained:

- (i) is of course satisfied;
- (ii) is satisfied in an analogous form as in Proposition A.2: Let $\Omega \subset M$ be a bounded open set, then $m_{SH}(\Omega) \geq 0$ with equality if and only if $\Omega \setminus \partial\Omega$ is locally isometric to the space form of constant sectional curvature K .

This follows by localizing the proof of Theorem 6.2 (analogous to the proof of the quasi-local rigidity Theorem 4.1) and by extending the outer-minimizing property of $S_{p,\rho}(w)$ in Ω (see the proof of Theorem 5.7) to the case $Sc \geq -6K$ and bounded Ω ;

- (iii) is satisfied in the same form as in Proposition A.3.

However, property (iv), the upper bound analogous to Proposition A.4, and the analogue of Proposition A.8 have all little chances to hold for $K \neq 0$ (see Remark 6.3 for more details). This is why the presentation here is focused on the AF case with non-negative scalar curvature.

A.2 | Local area-constrained maximisers of m_H are perturbed geodesic spheres

In this paper, we often estimated the sup of the Hawking mass using perturbed geodesic spheres $S_{p,\rho}(w)$. Even if not strictly necessary for such arguments, in the next proposition, we prove that such a choice of competitors is very natural. Indeed we show that optimal competitors for the supremum of the Hawking mass among surfaces contained in a small ball are given by perturbed geodesic spheres $S_{p,\rho}(w)$ with $w \in C^{4,\alpha}(\mathbb{S}^2)^\perp$, $\|w\|_{C^{4,\alpha}} \leq C\rho^2$. Thus, such optimal competitors satisfy the expansions given in Lemma 3.1 and Proposition 3.2. For related results in this direction, see Lamm–Metzger [20], who proved $W^{2,2}$ -closeness to a geodesic sphere under a small energy assumption, and Laurain–Mondino [32], who proved smooth convergence to a geodesic sphere under a milder energy assumption.

Recall that, by definition

$$C^{4,\alpha}(\mathbb{S}^2)^\perp := C^{4,\alpha}(\mathbb{S}^2) \cap [\text{Ker}(\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2))]^\perp,$$

where $[\text{Ker}(\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2))]^\perp \subset L^2(\mathbb{S}^2)$ denotes the L^2 -orthogonal space to the finite (actually four) dimensional space $\text{Ker}(\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2} + 2))$.

Proposition A.10. *Let (M, g) be a three-dimensional Riemannian manifold and let $\Sigma_j \subset M$ be a sequence of maximisers of m_H under area constraint and Hausdorff converging to a point $\bar{p} \in M$. Then $\nabla \text{Sc}(\bar{p}) = 0$ and there exist $p_j \rightarrow \bar{p}, \rho_j \downarrow 0, w_j \in C^{4,\alpha}(\mathbb{S}^2)^\perp$ with $\limsup_{j \rightarrow \infty} \rho_j^{-2} \|w_j\|_{C^{4,\alpha}(\mathbb{S}^2)} < \infty$ such that, up to a subsequence, $\Sigma_j = S_{p_j, \rho_j}(w_j)$.*

Proof. First of all, recall that non-orientable closed two-dimensional surfaces cannot be embedded in \mathbb{R}^3 , but only immersed (that is, with self-intersections). Thus the Li–Yau inequality [24] implies

$$\inf\{W(\Sigma) : \Sigma \subset \mathbb{R}^3 \text{ closed non-orientable surface}\} \geq 32\pi > 16\pi. \quad (\text{A.5})$$

Moreover, from the proof of the Willmore conjecture by Marques–Neves [26], we know that

$$\inf\{W(\Sigma) : \Sigma \subset \mathbb{R}^3 \text{ closed surface with genus}(\Sigma) \geq 1\} \geq 8\pi^2 > 16\pi. \quad (\text{A.6})$$

Using normal coordinates centred at \bar{p} and estimating the difference between the Riemannian and Euclidean geometric quantities (see, for instance, [34]), it is not hard to check that (A.5) and (A.6) yield the existence of $C > 0$ such that for any sequence $\Sigma_j \subset M$ Hausdorff converging to a point p , with Σ_j either non-orientable or of genus at least one, it holds

$$\limsup_{j \rightarrow \infty} \frac{m_H(\Sigma_j)}{\sqrt{|\Sigma_j|}} \leq -C. \quad (\text{A.7})$$

On the other hand, it is easy to see that for geodesic spheres $S_{p, \rho}$, it holds

$$\limsup_{\rho \rightarrow 0} \frac{m_H(S_{p, \rho})}{\sqrt{|S_{p, \rho}|}} = 0. \quad (\text{A.8})$$

Thus, the assumptions that $\Sigma_j \subset M$ are a sequence of maximisers of m_H under area constraint and Hausdorff converging to a point $\bar{p} \in M$ yield that Σ_j must be a topological sphere, for j large enough, with $W(\Sigma_j) < 8\pi$. Now we can apply [32, Corollary 1.3] (see also [19]) to infer that $\nabla \text{Sc}(\bar{p}) = 0$ and that, if we rescale (M, g) around \bar{p} in such a way that the rescaled surfaces $\tilde{\Sigma}_j$ have fixed area 1, then $\tilde{\Sigma}_j$ converge smoothly (up to a subsequence) to a round sphere in the three-dimensional Euclidean space. In particular, Σ_j is a normal graph over a geodesic sphere, of small radius and graph function (in any C^k norm). Now, by a contraction mapping argument, one can find

$$p_j \rightarrow \bar{p}, \rho_j \downarrow 0, w_j \in C^{4,\alpha}(\mathbb{S}^2)^\perp \text{ with } \limsup_{j \rightarrow \infty} \rho_j^{-2} \|w_j\|_{C^{4,\alpha}(\mathbb{S}^2)} < \infty$$

such that $\Sigma_j = S_{p_j, \rho_j}(w_j)$. The proof of this last claim can be performed along the lines as [30, Lemma 5.3]: although the statement of [30, Lemma 5.3] is for critical points of W , the same proof holds verbatim more generally for area-constrained critical points using that the Lagrange multipliers are bounded, due to [32, Lemma 2.2]. \square

A.3 | Second fundamental form on a perturbed geodesic sphere

In this section, we give a self-contained proof of the Taylor expansion (18) for the second fundamental form on a perturbed geodesic sphere. We follow the approach in [40], however, we compute more terms, since they are needed for the main results in this paper.

First of all, let $Z_i, i = 1, 2$, be the coordinate vector fields on $S_{p, \rho}(w)$:

$$Z_i = \exp_* (\partial_{\Theta^i} \rho(1-w)\Theta) = \rho((1-w)\Theta_i - w_i \Theta).$$

In order to find an expression for the inward pointing unit normal vector to $S_{p, \rho}(w)$, consider

$$\tilde{N} := -\Theta + a^j Z_j,$$

where a^j are such that \tilde{N} is orthogonal to both Z_1 and Z_2 . Computing, we get

$$\begin{aligned} g(\tilde{N}, Z_i) &= g(-\Theta + a^j Z_j, Z_i) = -g(\Theta, Z_i) + a^j g(Z_j, Z_i) = -g(\Theta, \rho((1-w)\Theta_i - w_i \Theta)) + a^j \dot{g}_{ij} \\ &= \rho w_i + a^j \dot{g}_{ij}, \end{aligned}$$

where we used that $g(\Theta, \Theta) = 1$ and $g(\Theta, \Theta_i) = 0$. Therefore, to satisfy orthogonality, we need to choose a^j such that $a^j \dot{g}_{ij} = -w_i \rho$, or $a^j = -\dot{g}^{ij} w_i \rho$. In order to find the normalization constant, compute

$$\begin{aligned} g(\tilde{N}, \tilde{N}) &= g(-\Theta + a^j Z_j, -\Theta + a^i Z_i) \\ &= g(\Theta, \Theta) - g(\Theta, a^i Z_i) - g(\Theta, a^j Z_j) + g(a^j Z_j, a^i Z_i) \\ &= 1 - a^i g(\Theta, \rho((1-w)\Theta_i - w_i \Theta)) - a^j g(\Theta, \rho((1-w)\Theta_j - w_j \Theta)) + a^i a^j \dot{g}_{ij} \\ &= 1 + a^i w_i \rho + a^j w_j \rho + a^i a^j \dot{g}_{ij} \\ &= 1 - \dot{g}^{ij} w_i w_j \rho^2. \end{aligned}$$

Combining the last two relations, we get the (inward) unit normal vector:

$$N = (1 - \dot{g}^{kl} w_k w_l \rho^2)^{-\frac{1}{2}} (-\Theta - \dot{g}^{ij} w_i ((1-w)\Theta_j - w_j \Theta) \rho^2).$$

Using the Taylor expansion around 0 of $(1-x)^{-\frac{1}{2}}$ with $x = \dot{g}^{ij} w_i w_j \rho^2$, we get

$$g(\tilde{N}, \tilde{N})^{-\frac{1}{2}} = 1 + \frac{1}{2} \dot{g}^{ij} w_i w_j \rho^2 + \mathcal{Q}_p^{(4)(1)}(w). \quad (\text{A.9})$$

Combining with (17), we obtain

$$|N + \Theta + g_{\mathbb{S}^2}^{ij} w_i \Theta_j| = \rho^2 \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(1)}(w).$$

We next compute $\tilde{h}_{ij} := -g(\nabla_{Z_i} \tilde{N}, Z_j)$ as a first step for obtaining h_{ij} . We have

$$\begin{aligned}
 \tilde{h}_{ij} &= -g(\nabla_{Z_i}(-\Theta + a^k Z_k), Z_j) \\
 &= g(\nabla_{Z_i} \Theta, Z_j) - g(\nabla_{Z_i} a^k Z_k, Z_j) \\
 &= \frac{w_i}{1-w} g(\Theta, Z_j) - \frac{w_i}{1-w} g(\Theta, Z_j) + g(\nabla_{Z_i} \Theta, Z_j) - g(\nabla_{Z_i} a^k Z_k, Z_j) \\
 &= \frac{w_i}{1-w} g(\Theta, Z_j) + \frac{1}{1-w} [(1-w)g(\nabla_{Z_i} \Theta, Z_j) - w_i g(\Theta, Z_j)] - g(\nabla_{Z_i} a^k Z_k, Z_j) \\
 &= \frac{w_i}{1-w} g(\Theta, Z_j) + \frac{1}{1-w} g((1-w)\nabla_{Z_i} \Theta - w_i \Theta, Z_j) - g(\nabla_{Z_i} a^k Z_k, Z_j) \\
 &= \frac{w_i}{1-w} g(\Theta, Z_j) + \frac{1}{1-w} g(\nabla_{Z_i} ((1-w)\Theta), Z_j) - g(\nabla_{Z_i} a^k Z_k, Z_j). \tag{A.10}
 \end{aligned}$$

We compute the three terms in (A.10) separately. For the first one, we use the definition of Z_i and the fact that $g(\Theta, \Theta) = 1$ and $g(\Theta, \Theta_i) = 0$ to obtain

$$\frac{w_i}{1-w} g(\Theta, Z_j) = \frac{w_i}{1-w} g(\Theta, \rho((1-w)\Theta_j - w_j \Theta)) = -\frac{w_i w_j \rho}{1-w}. \tag{A.11}$$

Now consider ρ as a variable, giving

$$Z_0 := \exp_*(\partial_\rho(\rho(1-w)\Theta)) = (1-w)\Theta.$$

Since

$$\begin{aligned}
 g(\nabla_{Z_i}((1-w)\Theta), Z_j) &= Z_i(g((1-w)\Theta, Z_j)) - g((1-w)\Theta, \nabla_{Z_i} Z_j) \\
 &= Z_i(g((1-w)\Theta, \rho((1-w)\Theta_j - w_j \Theta))) - g((1-w)\Theta, \nabla_{Z_i} Z_j) \\
 &= Z_i(\rho(w-1)w_j) - g((1-w)\Theta, \nabla_{Z_i} Z_j) \\
 &= \rho(w_i w_j + w w_{ji} - w_{ji}) - g((1-w)\Theta, \nabla_{Z_i} Z_j)
 \end{aligned}$$

is symmetric in i and j , we have

$$g(\nabla_{Z_i}((1-w)\Theta), Z_j) = g(\nabla_{Z_j}((1-w)\Theta), Z_i).$$

Thus we compute

$$\begin{aligned}
 g(\nabla_{Z_i}((1-w)\Theta), Z_j) &= \frac{1}{2}(g(\nabla_{Z_i}((1-w)\Theta), Z_j) + g(\nabla_{Z_j}((1-w)\Theta), Z_i)) \\
 &= \frac{1}{2}(g(\nabla_{Z_i} Z_0, Z_j) + g(\nabla_{Z_j} Z_0, Z_i)) = \frac{1}{2}(g(\nabla_{Z_0} Z_i, Z_j) + g(\nabla_{Z_0} Z_j, Z_i)) \\
 &= \frac{1}{2} Z_0(g(Z_i, Z_j)) = \frac{1}{2} \partial_\rho g_{ij}, \tag{A.12}
 \end{aligned}$$

which sorts out the second term in (A.10). The final term becomes

$$\begin{aligned}
 g(\nabla_{Z_i} a^k Z_k, Z_j) &= Z_i(a^k g(Z_k, Z_j)) - a^k g(Z_k, \nabla_{Z_i} Z_j) \\
 &= Z_i(-\dot{g}^{lk} w_l \rho \dot{g}_{kj}) + \dot{g}^{lk} w_l \rho \dot{\Gamma}_{ij}^m \dot{g}_{km} \\
 &= -w_{ji} \rho + w_l \dot{\Gamma}_{ij}^l \rho \\
 &= -(\text{Hess}_{\dot{g}} w)_{ij} \rho.
 \end{aligned} \tag{A.13}$$

Substituting (A.11), (A.12) and (A.13) into (A.10) gives

$$\tilde{h}_{ij} = -\frac{w_i w_j \rho}{1-w} + \frac{1}{2(1-w)} \partial_\rho \dot{g}_{ij} + (\text{Hess}_{\dot{g}} w)_{ij} \rho. \tag{A.14}$$

To further expand \tilde{h}_{ij} , we combine the first two terms of (A.14). Differentiating (16) with respect to ρ gives

$$\begin{aligned}
 \partial_\rho \dot{g}_{ij} &= 2g_{ij}^{\mathbb{S}^2} (1-w)^2 \rho + 2w_i w_j \rho + \frac{4}{3} g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) (1-w)^4 \rho^3 \\
 &\quad + \frac{5}{6} g(\nabla_\Theta \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) (1-w)^5 \rho^4 + \frac{3}{10} g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) (1-w)^6 \rho^5 \\
 &\quad + \frac{4}{15} \delta^{\mu\nu} g(\mathcal{R}(\Theta, \Theta_i) \Theta, E_\mu) g(\mathcal{R}(\Theta, \Theta_j) \Theta, E_\nu) (1-w)^6 \rho^5 + \mathcal{O}_p(\rho^6) + \rho^6 \mathcal{L}_p^{(0)}(w) \\
 &\quad + \rho^6 \mathcal{Q}_p^{(2)(0)}(w).
 \end{aligned}$$

Hence, the first term in (A.14) cancels the second term of $\partial_\rho \dot{g}_{ij}$ after it is multiplied by $\frac{1}{2(1-w)}$. This leaves

$$\begin{aligned}
 \tilde{h}_{ij} &= g_{ij}^{\mathbb{S}^2} (1-w) \rho + (\text{Hess}_{\dot{g}} w)_{ij} \rho + \frac{2}{3} g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) (1-w)^3 \rho^3 \\
 &\quad + \frac{5}{12} g(\nabla_\Theta \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) (1-w)^4 \rho^4 + \frac{3}{20} g(\nabla_{\Theta\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) (1-w)^5 \rho^5 \\
 &\quad + \frac{2}{15} \delta^{\mu\nu} g(\mathcal{R}(\Theta, \Theta_i) \Theta, E_\mu) g(\mathcal{R}(\Theta, \Theta_j) \Theta, E_\nu) (1-w)^5 \rho^5 + \mathcal{O}_p(\rho^6) + \rho^6 \mathcal{L}_p^{(0)}(w) + \rho^6 \mathcal{Q}_p^{(2)(0)}(w).
 \end{aligned} \tag{A.15}$$

We finally compute $(\text{Hess}_{\dot{g}} w)_{ij}$. By definition, we have

$$(\text{Hess}_{\dot{g}} w)_{ij} = w_{ij} - \dot{\Gamma}_{ij}^k w_k = w_{ij} - \left[\frac{1}{2} \dot{g}^{kl} (\partial_i \dot{g}_{jl} + \partial_j \dot{g}_{il} - \partial_l \dot{g}_{ij}) \right] w_k. \tag{A.16}$$

Differentiating (16) term by term shows

$$\begin{aligned}
 \partial_i \dot{g}_{jl} &= \partial_i (g_{jl}^{\mathbb{S}^2}) (1-2w) \rho^2 - 2g_{jl}^{\mathbb{S}^2} w_i \rho^2 + \frac{1}{3} \partial_i (g(\mathcal{R}(\Theta, \Theta_j) \Theta, \Theta_l)) \rho^4 \\
 &\quad + \mathcal{O}_p(\rho^5) + \rho^4 \mathcal{L}_p^{(1)}(w) + \rho^2 \mathcal{Q}_p^{(2)(2)}(w).
 \end{aligned} \tag{A.17}$$

Combining (17) and (A.17), we get

$$\begin{aligned}\tilde{\Gamma}_{ij}^k &= \Gamma_{ij}^k + g_{\mathbb{S}^2}^{kl}(g_{ij}^{\mathbb{S}^2} w_l - g_{jl}^{\mathbb{S}^2} w_i - g_{il}^{\mathbb{S}^2} w_j) \\ &\quad + \frac{1}{6} g_{\mathbb{S}^2}^{kl} (\partial_l (g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_l)) + \partial_j (g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l)) - \partial_l (g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j))) \rho^2 \quad (\text{A.18}) \\ &\quad - \frac{1}{6} g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m) (\partial_i (g_{jl}^{\mathbb{S}^2}) + \partial_j (g_{il}^{\mathbb{S}^2}) - \partial_l (g_{ij}^{\mathbb{S}^2})) \rho^2 \\ &\quad + \mathcal{O}_p(\rho^3) + \rho^2 \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(2)}(w),\end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols of $g_{\mathbb{S}^2}$. Substituting (A.16) and (A.18) into (A.15) gives

$$\begin{aligned}\tilde{h}_{ij} &= g_{ij}^{\mathbb{S}^2} (1 - w) \rho + (\text{Hess}_{\mathbb{S}^2}(w))_{ij} \rho \\ &\quad + w_k g_{\mathbb{S}^2}^{kl} (g_{jl}^{\mathbb{S}^2} w_i + g_{il}^{\mathbb{S}^2} w_j - g_{ij}^{\mathbb{S}^2} w_l) \rho \\ &\quad + \frac{2}{3} g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) (1 - w)^3 \rho^3 \\ &\quad + \frac{1}{6} w_k g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m) (\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2}) \rho^3 \\ &\quad - \frac{1}{6} w_k g_{\mathbb{S}^2}^{kl} (\partial_i g(\mathcal{R}(\Theta, \Theta_j)\Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)) \rho^3 \\ &\quad + \frac{5}{12} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) \rho^4 + \frac{3}{20} g(\nabla_{\Theta}^2 \mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) \rho^5 \\ &\quad + \frac{2}{15} \delta^{\mu\nu} g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_{\nu}) \rho^5 \\ &\quad + \mathcal{O}_p(\rho^6) + \rho^4 \mathcal{L}_p^{(1)}(w) + \rho \mathcal{Q}_p^{(3)(2)}(w) + \rho^3 \mathcal{Q}_p^{(2)(1)}(w).\end{aligned}$$

Finally, to complete the proof, note that

$$\begin{aligned}h_{ij} &:= -g(\nabla_{Z_i} N, Z_j) = -g(\nabla_{Z_i} g(\tilde{N}, \tilde{N})^{-\frac{1}{2}} \tilde{N}, Z_j) \\ &= -Z_i(g(\tilde{N}, \tilde{N})^{-\frac{1}{2}}) g(\tilde{N}, Z_j) - g(\tilde{N}, \tilde{N})^{-\frac{1}{2}} g(\nabla_{Z_i} \tilde{N}, Z_j) \\ &= g(\tilde{N}, \tilde{N})^{-\frac{1}{2}} \tilde{h}_{ij},\end{aligned}$$

where we used that \tilde{N} is orthogonal to Z_j . Using (A.9), we obtain

$$\begin{aligned}h_{ij} &= g_{ij}^{\mathbb{S}^2} (1 - w) \rho + (\text{Hess}_{\mathbb{S}^2}(w))_{ij} \rho \\ &\quad + \frac{1}{2} g_{ij}^{\mathbb{S}^2} g_{\mathbb{S}^2}^{kl} w_k w_l \rho + w_k g_{\mathbb{S}^2}^{kl} (g_{jl}^{\mathbb{S}^2} w_i + g_{il}^{\mathbb{S}^2} w_j - g_{ij}^{\mathbb{S}^2} w_l) \rho \\ &\quad + \frac{2}{3} g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j) (1 - w)^3 \rho^3\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6} w_k g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta, \Theta_n) \Theta, \Theta_m) \left(\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2} \right) \rho^3 \\
& - \frac{1}{6} w_k g_{\mathbb{S}^2}^{kl} \left(\partial_i g(\mathcal{R}(\Theta, \Theta_j) \Theta, \Theta_l) + \partial_j g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_l) - \partial_l g(\mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \right) \rho^3 \\
& + \frac{5}{12} g(\nabla_{\Theta} \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \rho^4 + \frac{3}{20} g(\nabla_{\Theta}^2 \mathcal{R}(\Theta, \Theta_i) \Theta, \Theta_j) \rho^5 \\
& + \frac{2}{15} \delta^{\mu\nu} g(\mathcal{R}(\Theta, \Theta_i) \Theta, E_{\mu}) g(\mathcal{R}(\Theta, \Theta_j) \Theta, E_{\nu}) \rho^5 \\
& + \mathcal{O}_p(\rho^6) + \rho^4 \mathcal{L}_p^{(1)}(w) + \rho \mathcal{Q}_p^{(3)(2)}(w) + \rho^3 \mathcal{Q}_p^{(2)(1)}(w).
\end{aligned}$$

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