

EMBEDDING UNIVERSAL COVERS OF GRAPH MANIFOLDS IN PRODUCTS OF TREES

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ABSTRACT. We prove that the universal cover of any graph manifold quasi-isometrically embeds into a product of three trees. In particular we show that the Assouad-Nagata dimension of the universal cover of any closed graph manifold is 3, proving a conjecture of Smirnov.

A graph manifold is a compact connected 3-manifold (possibly with boundary) which admits a decomposition into Seifert fibred pieces, when cut along a collection of embedded tori and/or Klein bottles. In particular a graph manifold is a 3-manifold whose geometric decomposition admits no hyperbolic part. For this reason the class of graph manifold groups is rigid within the class of 3-manifold groups [KL95], moreover, such groups are classified up to quasi-isometry [BN08].

More details on graph manifolds and proofs of the above results can be found in [BDM09], [Ger94] and [KL98].

We show the following:

Theorem 1. *The universal cover of any graph manifold quasi-isometrically embeds in the product of three metric trees.*

One may wish to compare this theorem with the result by Buyalo and Schroeder [BS05] that \mathbb{H}^3 can be quasi-isometrically embedded in the product of three infinite valence simplicial trees. (This was refined to three infinite binary trees by [BDS07].)

As an application, we determine the Assouad-Nagata dimension (\dim_{AN}) - as defined by Assouad, [Ass82] - of the universal cover of closed graph manifolds. We denote the asymptotic Assouad-Nagata dimension by $asdim_{AN}$. Recall that the Assouad-Nagata dimension bounds from above the asymptotic dimension, first introduced by Gromov in [Gro93]. However, asymptotic dimension and asymptotic Assouad-Nagata dimension can differ radically, see for instance the examples in [BDL06]. The asymptotic Assouad-Nagata dimension of a group also bounds from above the dimension of its asymptotic cones [DH08] and if a group has finite Assouad-Nagata dimension then it has compression exponent 1 [Gal08].

The asymptotic dimension of universal covers of closed graph-manifolds is known to be 3, as mentioned in [Smi10], in view of results in [BD08]. Also, Smirnov [Smi10] showed that their Assouad-Nagata dimension is finite (at most 7) and conjectured that it actually equals 3. Theorem 1 implies his conjecture:

Corollary 2. *If \widetilde{M} is the universal cover of a closed graph-manifold then*

$$\dim_{AN}\widetilde{M} = asdim_{AN}\widetilde{M} = 3.$$

Proof. Asymptotic dimension never exceeds either of the aforementioned dimensions, so this provides the lower bound of 3 in both cases, also, by definition $\dim_{AN}\widetilde{M} \leq asdim_{AN}\widetilde{M}$. Results in [LS05] prove $asdim_{AN}X \leq n$ when X is an n -fold product of trees and $asdim_{AN}A \leq asdim_{AN}B$ whenever A admits a quasi-isometric embedding into B , so we get the upper bound using Theorem 1. \square

A graph manifold is said to be *non-geometric* if its decomposition into Seifert fibred pieces is non-trivial. Notice that if the decomposition is trivial then the universal cover is quasi-isometric to the product of a tree with \mathbb{R} .

Question 3. *Can fundamental groups of non-closed, non-geometric graph manifold quasi-isometrically embed into a product of 2 trees?*

Proof of Theorem 1. We only have to consider non-geometric *flip* graph manifolds. In fact - at the level of universal covers - any graph manifold is quasi-isometric to a flip graph manifold [KL98]. We do not need the definition of such manifolds, as we will recall the essential properties required. Let M be a flip graph manifold and let T be its Bass-Serre tree. The universal cover \widetilde{M} of M is constructed by suitably gluing certain metric spaces $X_v = F_v \times \mathbb{R}$, for v a vertex in T . Each F_v is the universal cover of a compact surface with non-empty boundary and so it admits a metric retraction $r_v : F_v \rightarrow T_v$, where $T_v \subseteq F_v$ is a tree, with the further properties that r_v is injective when restricted to any boundary component of F_v and there exists μ (not depending on v) such that for each $x \in F_v$ we have $d_{F_v}(x, r_v(x)) \leq \mu$. Finally, the gluings are performed as follows. Let v, v' be adjacent vertices. Then there exist parametrisations $\gamma_v : \mathbb{R} \rightarrow F_v, \gamma_{v'} : \mathbb{R} \rightarrow F_{v'}$ of boundary components of $F_v, F_{v'}$ so that $(\gamma_v(t), u) \in F_v \times \mathbb{R}$ is identified with $(\gamma_{v'}(u), t) \in F_{v'} \times \mathbb{R}$ for each $t, u \in \mathbb{R}$. This is explained, for example, in [BN08].

Step 1. The trees.

The first tree will just be the Bass-Serre tree $T_0 = T$. Let us define the other two trees T_1, T_2 as follows.

We can subdivide the vertices of T into disjoint families V_1, V_2 such that if $v, v' \in V_i$ then $d_T(v, v')$ is even. Set $T'_i = \bigsqcup_{v \in V_i} T_v$. We wish now to define an equivalence relation \sim on T'_i , and we will set $T_i = T'_i / \sim$. Suppose that $v, v' \in V_i$, $v \neq v'$ and there exists w such that $d_T(v, w) = d_T(v', w) = 1$. We will set $x \sim_d x'$, for $x \in T_v, x' \in T_{v'}$, if there exist y, y' with $r_v(y) = x, r_{v'}(y') = x'$ such that the points in $F_w \times \mathbb{R}$ identified with $(y, 0) \in F_v \times \mathbb{R}, (y', 0) \in F_{v'} \times \mathbb{R}$ have the same \mathbb{R} -coordinate. To ensure an equivalence relation, we set \sim to be the transitive closure of \sim_d .

It is very easy to check that $T_i = T'_i / \sim$ is a metric tree with only countably many branching points. In fact, it can be described as the increasing union of metric spaces $\{X_k\}_{k \in \mathbb{N}}$ such that X_0 is a tree and X_{k+1} is obtained from X_k by identifying a line in X_k with a line in some tree.

Step 2. The components of the embedding.

Define $f_0 : \widetilde{M} \rightarrow T_0$ to be any map such that for all $x \in \widetilde{M}$, $x \in F_{f_0(x)} \times \mathbb{R}$ and define $f_i : \widetilde{M} \rightarrow T_i$ as follows. For each v , we let $\pi_v : F_v \times \mathbb{R} \rightarrow F_v$ be the projection on the first factor, and as usual denote the equivalence classes of \sim with square brackets.

If $x \in F_v \times \mathbb{R}$ for some $v \in V_i$, then set $f_i(x) = [r_v(\pi_v(x))]$. Otherwise we have $x \in F_w \times \mathbb{R}$ for $w \notin V_i$. Let $v \in V_i$ be any vertex such that $d_T(v, w) = 1$. Set $f_i(x) = [p]$ where $p \in T_v$ is such that $(p, 0)$ has, as a point in $F_w \times \mathbb{R}$, the same \mathbb{R} -coordinate as x . This does not depend on the choice of v , by the equivalence relation.

Step 3. The product map is a quasi-isometric embedding.

Define $f : \widetilde{M} \rightarrow \prod T_i$ to be $\prod f_i$. We wish to show that f is a quasi-isometric embedding. The easier inequality is $d(f(x), f(y)) \leq Kd(x, y) + C$: the maps π_v and r_v are non-expanding, so f_1 and f_2 are readily checked to be 1-Lipschitz, while f_0 satisfies $d_{T_0}(f_0(x), f_0(y)) \leq d_{\widetilde{M}}(x, y) / \rho + 1$ where

$$0 < \rho = \inf\{d_{\widetilde{M}}(x, x') : x \in X_v, x' \in X_{v'}, d_{T_0}(v, v') = 2\}.$$

For the other inequality we start with a geodesic δ in $\prod T_i$ connecting $f(x)$ to $f(y)$ and construct a path γ in \widetilde{M} connecting x to y such that $l(\gamma) \leq Kl(\delta) + C$. Let δ_1, δ_2 be the projections of δ on the factors. One may wish to compare the paths we obtain in this way with the “special paths” described in [Sis11].

Suppose that $x \in X_{v_0}$, $y \in X_{v_n}$ and let v_0, \dots, v_n be the vertices of T in the geodesic connecting v_0 to v_n . For $j = 0, \dots, n$ let $i(j) \in \{1, 2\}$ be such that $v_j \in V_{i(j)}$ and choose $\alpha_j \subseteq \delta_{i(j)}$ so that $\alpha_j \subseteq [r_{v_j}(F_{v_j})]$. We will also require that the final point of α_j is the starting point of α_{j+2} , that the starting point of α_0 is $f_{i(0)}(x)$ and that the final point of α_n is $f_{i(n)}(y)$. This can be easily arranged using the fact that each $[r_v(F_v)]$ is convex in the corresponding T_i .

For $j = 0, \dots, n-1$, let t_j be the \mathbb{R} -coordinate as a point in $F_{v_j} \times \mathbb{R}$ of $(p_j, 0) \in F_{v_{j+1}} \times \mathbb{R}$, where p_j is the starting point of α_{j+1} . Also, let t_n be the \mathbb{R} -coordinate of $y \in F_{v_n} \times \mathbb{R}$.

For $j = 0, \dots, n$ let γ_j be the path $\alpha_j \times t_j$ in X_{v_j} . Notice that the distance between the final point of γ_j and the starting point of γ_{j+1} is at most 2μ . So, we can concatenate in a suitable order the γ_j 's and n geodesics of length at most 2μ to obtain a path γ from x to y . Clearly $l(\gamma_j) = l(\alpha_j)$ so

$$l(\gamma) \leq \sum l(\gamma_j) + 2n\mu = l(\delta_1) + l(\delta_2) + 2n\mu = d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) + 2n\mu.$$

As $d(f_0(x), f_0(y)) \geq n-2$ we have

$$l(\gamma) \leq d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) + 2\mu d(f_0(x), f_0(y)) + 4\mu,$$

and we are done. \square

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