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**A MODEL OF NON-BELIEF IN THE LAW OF LARGE
NUMBERS**

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A Model of Non-Belief in the Law of Large Numbers

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Abstract

People believe that, even in very large samples, proportions of binary signals might depart significantly from the population mean. We model this “non-belief in the Law of Large Numbers” by assuming that a person believes that proportions in any given sample might be determined by a rate different than the true rate. In prediction, a non-believer expects the distribution of signals will have fat tails, more so for larger samples. In inference, a non-believer remains uncertain and influenced by priors even after observing an arbitrarily large sample. We explore implications for beliefs and behavior in a variety of economic settings. (99 words)

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1 Introduction

Psychological research has identified systematic biases in people’s beliefs about the relationship between sample proportions and the population from which they are drawn. Following Tversky and Kahneman (1971), Rabin (2002) and Rabin and Vayanos (2010) model the notion that people believe in “the Law of Small Numbers (LSN),” exaggerating how likely it is that small samples will reflect the underlying population. Yet evidence indicates that people *also* do not believe in the Law of Large Numbers: they believe that even in very large random samples, proportions might depart significantly from the overall population rate. This paper develops a formal model of such “non-belief in the Law of Large Numbers,” which we abbreviate by NBLLN.¹ Our goal is not to explain the source of this bias, or to provide a unifying model of it and other biases, but rather to explore NBLLN’s implications and assess its potential importance in economic decision-making. Even though it is less well-known and has received far less attention than other biases, we argue that NBLLN explains extensive experimental evidence that people infer too little from large- and medium-sized samples. We show that NBLLN has a range of important economic consequences, including causing too little or too much risk-taking, a lack of demand for information, and a persistence of incorrect beliefs despite large amounts of data. In addition, we identify and explore conceptual challenges with modeling NBLLN that are also likely to arise in modeling other biases in statistical reasoning.

Kahneman and Tversky (1972) find that subjects seem to think sample proportions reflect a “universal sampling distribution,” virtually neglecting sample size. In doing so, subjects vastly exaggerate the probability of unbalanced ratios in large samples. For instance, independent of whether a fair coin is flipped 10, 100, or 1,000 times, the median subject thinks that there is about $\frac{1}{5}$ chance of getting between 45% and 55% heads, and about $\frac{1}{20}$ chance of between 75% and 85%. These beliefs are close to the right probabilities of $\frac{1}{4}$ and $\frac{1}{25}$ for the sample size of 10, but wildly miss the mark for the sample size of 1,000, where the sample is almost surely between 45% and 55% heads.

In Section 2, we develop our model of non-belief in the Law of Large Numbers in a simple setting, where a person is trying to predict the distribution of—or make an inference from—a fixed sample size. Throughout, we refer to our modeled non-believer in the Law of Large Numbers as *Barney*, and compare his beliefs and behavior to a purely Bayesian information processor, *Tommy*.² Formally, our model assumes that Tommy knows that the likelihood of different sample distributions

¹NBLLN is pronounced letter by letter, said with the same emphasis and rhythm as “Ahmadinejad.”

²“Tommy” is the conventional designation in the quasi-Bayesian literature to refer somebody who updates according to the dictums of the Reverend Thomas “Tommy” Bayes.

of an i.i.d. coin biased θ towards heads will be the “ θ -binomial distribution.” But Barney believes that large-sample proportions will be distributed according to a “ β -binomial distribution,” for some $\beta \in [0, 1]$ that itself is drawn from a distribution with mean θ . This model directly implies NBLLN: whereas Tommy knows that large samples will have proportions of heads very close to θ , Barney feels that the proportions in any given sample, no matter how large, might not be θ . Although the model largely reflects the “universal sample distribution” intuition from Kahneman & Tversky (1972), it also embeds *some* sensitivity to sample sizes, consistent with other evidence (e.g., Griffin and Tversky’s 1992 Study 1).³ Other models would share the basic features of NBLLN that we exploit in this paper; we discuss in Section 6 the merits and drawbacks of our particular formulation.

After defining the model, Section 2 describes some of its basic features for Barney’s predictions about the likelihood of different samples that might occur and his inferences from samples that have occurred. For instance, while Barney makes the same predictions as Tommy about sample sizes of 1, his beliefs about sample proportions are a mean-preserving spread of Tommy’s for samples of two or more signals. In situations of inference, we show that if Barney applies full Bayesian updating based on his wrong beliefs about the likelihood of different sample realizations, a realistic degree of NBLLN implies under-inference: Barney’s expected posterior ratio on different hypotheses is less extreme on average than Tommy’s. Importantly, for any proportion of signals—including the limit proportion generated by the true state—Barney fails to become fully confident even after infinite data. Consequently, Barney’s priors remain influential even after he has observed a large sample of evidence. In Appendix B, we review and meta-analyze the extensive experimental evidence on inference. Consistent with the general features of our model—and contrary to the widespread impression that overconfidence is the pervasive direction of mistakes in beliefs—this evidence clearly indicates that the typical finding is under-inference, and this under-inference is especially severe in large samples.⁴

³Even though we are not aware of any evidence on people’s beliefs regarding sample sizes larger than 1,000, our model imposes—consistent with Kahneman and Tversky’s (1972) interpretation—that Barney puts positive probability on sample proportions other than θ even in an infinite sample. We conjecture that people’s beliefs regarding much larger samples do indeed follow the same “universal sampling distribution” as for a sample size of 1,000. Nonetheless, we emphasize that even if the literal implications of our model for infinite sample sizes were not true, our large-sample limit results would still have substantial bite for the applications where we invoke them. This is because, as per the frequent reliance on large-sample limit results in econometrics, the Law of Large Numbers typically provides a good approximation for Tommy’s beliefs in the finite, moderately-sized samples that are realistic for those applications.

⁴Despite this evidence, NBLLN is not a phenomenon that has been widely embraced or emphasized by judgment researchers or behavioral economists. We surmise that this is largely because findings of under-inference have been associated with an interpretation called “conservatism” (e.g., Edwards, 1968)—namely, that people tend not to update their beliefs as strongly as Bayesian updating dictates—that does not mesh comfortably with base-rate neglect and other biases that often imply that people infer more strongly than Bayesian. In our view, summarizing people as overly conservative or not conservative enough is manifestly the wrong way to parse human judgment. By focusing on the concrete biases at play and highlighting the co-existence of NBLLN with other biases, we hope to make clear that there is no contradiction.

In the remainder of the paper, we draw out some of the consequences of NBLLN in a wide range of applications that cover many of the major areas in the economics of uncertainty, such as valuation of risky prospects, information acquisition, and optimal stopping. The applications highlight which features of the economic environment determine in which direction (e.g., more risk averse or less) Barney is biased relative to Tommy. Two themes run through these analyses and tie them together: Barney believes that anything can happen, and he under-infers from large samples.

Section 3 illustrates some of the economic implications of Barney’s mistaken beliefs about the likelihood of different samples. For example, Barney believes that the risk associated with a large number of independent gambles is greater than it actually is. This magnifies aversion to repeated risks, whether that risk aversion is due to diminishing marginal utility of wealth or (more relevantly) reference-dependent risk attitudes. Because he does not realize that the chance of aggregate losses becomes negligible, Barney may refuse to accept even infinite repetitions of a small, better-than-fair gamble. This refusal must come from a plausible model of risk preferences, such as loss aversion, generating the intrinsic aversion to small risks. But even such an improvement in assumptions about risk attitudes would not generate the observed behavior if a person believed in LLN. Benartzi and Thaler (1999), in fact, demonstrate clearly the role of both loss aversion and what we are calling NBLLN. However, in other contexts, where payoffs depend on extreme outcomes, Barney’s mistaken sampling beliefs could instead make him appear *less* risk averse than Tommy, such as playing a lottery in which whether he wins a prize depends on correctly guessing all of several numbers that will be randomly drawn.

Both Sections 2 and 3 analyze a model of NBLLN when there is a single, “given” sample that Barney will observe. Yet information does not always arrive in a single package of signals. A person may hear a series of individual reports from random strangers at cocktail parties about their car experiences, while also reading large-sample statistics of car performance. If Barney pools each of his interlocutors’ tales with the statistics from *Consumer Reports*, his inferences will be very different than if he separately updates his beliefs following each cocktail party anecdote, and then treats the *Consumer Reports* data as one big sample. Such cases confront us with a conceptual challenge intrinsic to the very nature of NBLLN: because Barney under-infers more for larger samples than smaller ones, he will infer differently if he lumps observations together versus separately. A model of NBLLN must involve a theory of how Barney groups information as a function of how it is presented to him and other features of his decision-making environment. With little empirical research to guide us, in Section 4 we discuss and formalize various combinations of assumptions on how Barney “retrospectively groups” signals—how he interprets evidence once he sees it—and “prospectively groups” signals—how he predicts ahead of time he will interpret evidence he might observe in the

future. Different combinations of assumptions may be warranted by different perceptual, framing, and decisionmaking environments. Of special interest is the possibility that Barney retrospectively groups signals differently than he prospectively anticipates he will. He may, for instance, plan to separately ask people at cocktail parties about their experiences, and prospectively focus on each conversation as if it is a separate signal; but then in retrospect, he may pool the conversations together as a large sample.

In Section 5, we explore Barney’s behavior in various environments involving learning and inference. Although for a given large number of signals Barney will be less certain than Tommy, if Barney and Tommy can both acquire information endogenously, Barney can actually end up being *more* certain about the state of the world. But because Barney thinks that his inference would be limited even from an infinite sample, he unambiguously has a lower willingness to pay for a *large* sample of data than Tommy. This lack of demand for statistical data is a central implication of NBLLN and contributes to explaining why people often rely instead on sources of information that provide only a small number of signals, such as anecdotes from strangers, stories from one’s immediate social network, and limited personal experience. Indeed, direct real-world evidence of under-inference might be difficult to find precisely because people rarely choose to incur the costs of obtaining a large sample.

A number of conclusions follow from the central fact that, even if Barney does end up observing extensive evidence, he fails to reach appropriately strong confidence. For example, NBLLN acts as an “enabling bias” for distinct psychological biases, such as “vividness bias” and optimism about one’s own abilities or preferences, that would otherwise be rendered irrelevant by the Law of Large Numbers. Absent NBLLN, after having processed from *Consumer Reports* a summary of the experiences of thousands of random strangers, hearing one random stranger recount a vivid story about her car experience—even if the story is overweighted a hundredfold—could not plausibly affect an agent’s beliefs. Similarly, absent NBLLN, optimistic priors would give way to more realistic self-assessments after a lifetime of experience.

When agents gather information, variants of NBLLN predict not only that people will never figure out the truth when they rationally should, but that their efforts to learn may be enormously costly. Like Tommy, Barney will plan to quit his costly information acquisition once he reaches some threshold of confidence. And, like Tommy, he may stop gathering signals very quickly if information is decisive. But because Barney tends to infer far less from signals than Tommy, when the initial signals are mixed, Barney may continue trying to learn even after many signals. Indeed, if Barney prospectively anticipates separately updating using each arriving signal but actually pools them retrospectively, Barney may become stuck in a “learning trap”: he persistently expects to

soon be confident enough to stop experimenting, or buying information, but, because he never achieves the confidence he anticipates, continues his costly efforts forever.

In Section 6, we discuss why we think our model is more compelling than alternative possible explanations and modeling approaches—both fully rational and not fully rational—that might seem to accommodate the psychology evidence. Our model, of course, ignores other important departures from Bayesian inference—such as base-rate neglect and belief in the Law of Small Numbers—that seem separable from NBLLN. But it also omits features—including the psychophysics of diminishing sensitivity, as well as unwillingness to hold or express extreme beliefs—that, as alternative sources of under-inference from large samples, are less separable. In Appendix A, in fact, we present a (complicated) formal model embedding some of these other errors along with NBLLN. Guided by this formal model, in Appendix B we attempt to give a fairly exhaustive review of the empirical work on sampling predictions and inference. We believe this review makes clear that our model of NBLLN is capturing a broad empirical reality. Although in many settings there are alternative reasons why a person may rationally disregard a large amount of evidence—e.g., because the evidence is less relevant given the person’s preferences than other, smaller-sample sources of information—our review documents NBLLN only in settings where a Bayesian would fully attend to the evidence.

2 The Single-Sample Model

Throughout the paper, we study a stylized setting where an agent observes a set of binary signals, each of which takes on a value of either a or b . Given a *rate* $\theta \in \Theta \equiv (0, 1)$, signals are generated by a binomial (i.i.d.) process where the probability of an a -signal is equal to θ . Signals arrive in *clumps* of size N . We denote the set of possible ordered sets of signals of size $N \in \{1, 2, \dots\}$ by $S_N \equiv \{a, b\}^N$, and we denote an arbitrary clump (of size N) by $s \in S_N$.⁵ Let A_s denote the total number of a ’s that occur in the clump $s \in S_N$, so that $\frac{A_s}{N}$ is the proportion of a ’s that occur in a clump of N signals. For a real number x , we will use the standard notations “ $\lceil x \rceil$ ” to signify the smallest integer that is weakly greater than or equal to x and “ $\lfloor x \rfloor$ ” to signify the largest integer that is weakly less than or equal to x . For any random variable y that takes as possible values the elements of set Y , let beliefs by Tommy (who believes in the Law of Large Numbers) be denoted by cumulative distribution function $F_Y(\cdot)$, implying probability density function $f_Y(\cdot)$, expectation $E_Y(\cdot)$, and variance $Var_Y(\cdot)$. Let corresponding beliefs by Barney (the non-believer in the LLN) be denoted by $F_Y^\psi(\cdot)$, $f_Y^\psi(\cdot)$, $E_Y^\psi(\cdot)$, and $Var_Y^\psi(\cdot)$, where ψ is a parameter for the

⁵Note that we forego the conventional strategy of providing notation for a generic signal, indexed by its number. It is less useful here because (within a clump) what matters to Barney is just the number of a signals, not their order. Conserving notation here, in Section 5.4 we use t to index the clumps of signals.

degree of NBLLN defined below.

In this section, we develop our model of Barney for the case where he is considering a single clump of N signals. This case corresponds to most of the experimental evidence about NBLLN, which has been collected in settings where subjects were presented with a single, fixed sample of signals or outcomes, in which subjects presumably process all the information together. This special case also allows us to lay bare the essential features of how our model captures NBLLN. When generalizing the model, complicating conceptual challenges arise. Some of our analysis in fact concerns precisely these complications, but we defer discussion of these issues and ways to handle them until Section 4.

According to the Law of Large Numbers, the mean of a random sample equals the rate with probability 1 in the limit as the sample size gets large: For any interval $(\alpha_1, \alpha_2) \subseteq [0, 1]$,

$$\lim_{N \rightarrow \infty} \sum_{x=\lceil \alpha_1 N \rceil}^{\lceil \alpha_2 N \rceil} f_{S_N | \Theta}(A_s = x | \theta) = \begin{cases} 1 & \text{if } \theta \in (\alpha_1, \alpha_2) \\ 0 & \text{otherwise} \end{cases}.$$

How might we capture the possibility that Barney believes (say, as per an example in Kahneman and Tversky, 1972) that it is reasonably likely that at least 600 of 1000 births at a hospital in a given year are boys, even though he knows that boys are born at a rate of 50%? The essence of our model is to assume that Barney believes samples are generated as if a rate of θ , here 50%, means that the rate is θ on average, but might be higher or lower for any given sample. For a given true rate θ , we model Barney as believing that for the sample he is considering: first, a “subjective rate” $\beta \in [0, 1]$ is drawn from a distribution centered at θ . Then the i.i.d. sample of 1000 babies is generated using rate β . The key implication is that if a given value of β were the actual rate, it would (by the Law of Large Numbers!) exactly determine the proportion of signals in the limit of a very large sample. Therefore, the probability density that Barney assigns to any proportion β of signals (say, 60% of babies are boys) in a large sample is equal to the probability density that Barney assigns to the possibility that β equals that value. Although this modeling approach is “as if” Barney is unsure that the rate is θ , true parameter uncertainty is not at all our interpretation. Instead, consistent with the underlying NBLLN psychology, we interpret it as Barney’s belief that even his certainty that the underlying rate is θ is not a guarantee that the proportion in very large samples will approximate θ .⁶

⁶In keeping with this interpretation, Barney does not believe the realized β is a real feature of the coin, and is certainly not an object he makes inferences about. Instead, it is a representation of Barney’s subjective uncertainty that the θ will manifest itself in a given sample. By comparison, Acemoglu, Chernozukov, and Yildiz (2009) analyze a model that is formally similar to ours but *is* a model of parameter uncertainty. Similar to our Proposition 2 below, they show that a Bayesian agent fails to learn the state with certainty even after observing an infinite number of signals if he is uncertain about the meaning of signals. In contrast,

Formally, we assume that when Barney knows the rate is θ , he believes that signals are generated by a binomial (i.i.d.) process where the probability of an a -signal is equal to β . This $\beta \in [0, 1]$ is called the *subjective rate*, and it is drawn from a density $f_{\beta|\Theta}^\psi(\beta|\theta)$. We refer to $f_{\beta|\Theta}^\psi$ as Barney’s *subjective rate distribution* and assume that it has the following properties:

A1. For all β and θ , $F_{\beta|\Theta}^\psi(\beta|\theta)$ is absolutely continuous in β , and $f_{\beta|\Theta}^\psi(\beta|\theta)$ has full support on $(0, 1)$ and is point-wise continuous in θ . We will sometimes make the more restrictive assumption:

A1’. The same as A1, except $f_{\beta|\Theta}^\psi(\beta|\theta)$ has full support on $[0, 1]$.

A2. For all β and θ , $F_{\beta|\Theta}^\psi(\beta|1 - \theta) = 1 - F_{\beta|\Theta}^\psi(1 - \beta|\theta)$.

A3. For all θ and $\theta' > \theta$, $\frac{f_{\beta|\Theta}^\psi(\beta|\theta')}{f_{\beta|\Theta}^\psi(\beta|\theta)}$ is increasing in β .

A4. For all β and θ , $E_{\beta|\Theta}^\psi(\beta|\theta) = \theta$.

Assumptions A1 and A2 are mild and consistent with the experimentally-elicited densities of people’s beliefs about the distribution of signals in large samples. A1 differs from A1’ in allowing the density functions to converge to zero at proportions 0 and 1. The available experimental evidence cannot distinguish between A1 and A1’ because people’s beliefs about the likelihood of the most extreme sample proportions have not been elicited. While A1’ generally allows us to draw sharper theoretical conclusions, A1 accommodates our parametric example of the beta distribution (discussed below). Any results below that assume A1 also hold with A1’.

Assumptions A3 and A4 are substantive assumptions that do not follow easily from the psychology. Assumption A3 is a monotone likelihood ratio property: fixing any two rates, Barney believes that the likelihood of drawing any particular subjective rate given the high rate relative to the low rate is increasing in the subjective rate. This is a substantive assumption, and it is easy to imagine specifications of $f_{\beta|\Theta}^\psi(\beta|\theta)$ —especially in the spirit of the type of diminishing-sensitivity evidence discussed in Appendix A—that would violate A3. But A3 is in accord with the most directly relevant evidence, namely Griffin and Tversky’s (1992) Study 3, which examines a range of parameters of the sort that seems most likely for violating it.⁷ It holds for our main example of

we assume that Barney has a concrete belief about the meaning of signals, but he thinks that their meaning can vary from one sample to the next. In the dynamic applications we develop below, this interpretation will be much more than an aspiration to get the psychology right but rather an integral part of the formal model. For instance, in situations where Barney must predict further signals after observing his first 100 signals, we assume his expected proportions are still θ , rather than being influenced by the first signals as the parameter-uncertainty interpretation would suggest.

⁷In particular, Griffin and Tversky asked subjects to infer the likelihood that a coin is biased $\theta_A = .6$ in favor of heads rather than $\theta_B = .25$ in favor of heads, depending on different possible outcomes from flipping the coin 12 times. According to the (statistically erroneous) diminishing-sensitivity intuition, extreme samples, such as 10 heads out of 12, seem so unexpected in the case of either rate that they do not provide strong evidence about which rate is generating the flips. Yet consistent with A3, Griffin and Tversky find that subjects’ posterior beliefs in favor of the .6-biased coin are monotonically increasing in the number of heads.

the beta distribution and more generally is useful for establishing some of our results. Especially because the range of samples for which it is potentially false are inherently very unlikely, we think it is probably not an important caveat to our results. Assumption A4 says that the mean of Barney’s subjective rate distribution is the known objective rate. Although we rely on A4 extensively in the analysis, it is in fact violated in existing data. Below we discuss how it is violated, and in Appendix A, we propose a more comprehensive model that captures the psychology that we believe underlies the violations.

When Barney knows the rate is θ , he believes the likelihood of observing a particular clump of N signals, $s \in S_N$, is

$$f_{S_N|\Theta}^\psi(s|\theta) = \int_{\beta \in [0,1]} f_{S_N|\beta}(s|\beta) f_{\beta|\Theta}^\psi(\beta|\theta) d\beta, \quad (1)$$

where $f_{S_N|\beta}(s|\beta)$ is the (correct) probability of observing s if the rate were β , and this is averaged over the density of subjective rates, $f_{\beta|\Theta}^\psi(\beta|\theta)$. Consequently, Barney’s belief that a large sample will have a proportion of a signals in some range $[\alpha_1, \alpha_2]$ is exactly equal to Barney’s belief that the subjective rate β is in that range.⁸

Lemma 1. *Assume A1-A4. Barney does not believe in LLN: for any $\theta \in \Theta$ and interval $[\alpha_1, \alpha_2] \subseteq [0, 1]$,*

$$\lim_{N \rightarrow \infty} \sum_{x=\lceil \alpha_1 N \rceil}^{\lceil \alpha_2 N \rceil} f_{S_N|\Theta}^\psi(A_s = x|\theta) = F_{\beta|\Theta}^\psi(\beta = \alpha_2|\theta) - F_{\beta|\Theta}^\psi(\beta = \alpha_1|\theta) > 0.$$

Because we assume that Barney’s beliefs about the distribution of β puts positive probability density on the entire interval $(0, 1)$, the subjective-rate model captures the essence of our interpretation of NBLLN: Barney believes that the proportion of heads from flipping a coin *known to be fair* may not be 50% in any given sample, no matter how large.

Since (by the Law of Large Numbers) Barney’s belief about the distribution of signals in large samples coincides with his subjective rate distribution, the most appropriate density of $\beta|\theta$ would correspond to the empirical beliefs in studies such as those illustrated in Figure 1, drawn from Kahneman and Tversky (1972). The black, gray, and white bars—which correspond to people’s reported beliefs regarding samples of size 10, 100, and 1000, respectively—lie on top of each other. This distribution that is common across sample sizes, which presumably corresponds to people’s large-sample beliefs about sample proportions, could be directly assumed to be the density of $\beta|\theta$.

Although Assumptions A1 and A2 are consistent with Figure 1, A4 is not. Beliefs for $\theta = .5$,

⁸All proofs are elevated to Appendix C.

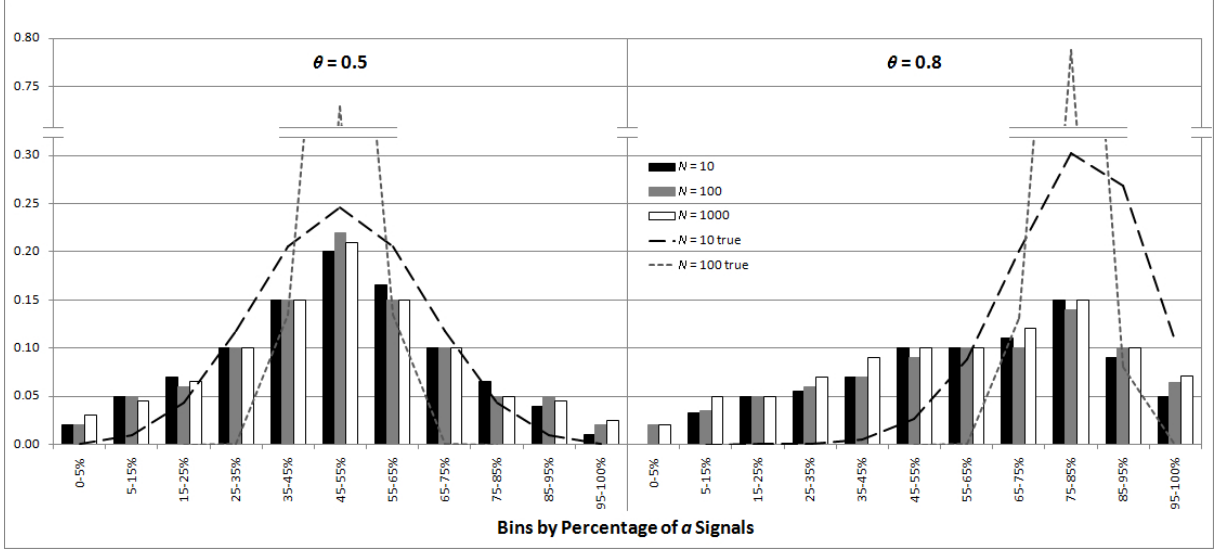


Figure 1: Evidence from Kahneman and Tversky (1972)

depicted in the left panel, naturally have mean approximately equal to .5. However, beliefs for $\theta = .8$, depicted in the right panel, have mean approximately equal to .6. The mean of the distribution of signals is displaced toward .5 apparently because the long tail of the distribution is fat. As we discuss in Appendices A and B, we believe that the fatness of the tail is in turn due to *flatness* of the tail. We attribute the flatness to a psychological bias—“sampling-distribution-tails diminishing sensitivity (SDTDS)” —in which people perceive very unlikely outcomes as similar to each other and hence similar in probability. Because flat tails are omitted from the model, it will not match some features of the empirical evidence, especially when the agent observes an extreme sample. We nonetheless justify A4 on two grounds: analytical convenience, and our contention that the violation of the assumption is best understood as a distinct psychological bias modeled in Appendix A.⁹

A *subjective sampling distribution* specifies an agent’s belief about the likelihood of each possible combination of signals when the rate θ is known. Whereas Lemma 1 shows that Barney’s subjective sampling distribution (for the number of a -signals) in the large-sample limit equals his “subjective rate distribution,” Proposition 1 shows that NBLLN also has implications for Barney’s finite-sample subjective sampling distributions.

Proposition 1. *Assume A1-A4. For any $\theta \in \Theta$ and $N \in \{1, 2, \dots\}$:*

⁹Analogously, we justify omitting the Law of Small Numbers (LSN) from the model for analytical convenience, and our contention that it is a distinct bias. However, our grounds for not incorporating LSN into the model are stronger because LSN will matter most when NBLLN will matter least—when the observed sample is small—whereas sampling-distribution-tails diminishing sensitivity is, like NBLLN, an especially extreme bias when the observed sample is large.

1. $E_{S_N|\Theta}^\psi \left(\frac{A_s}{N} | \theta \right) = E_{S_N|\Theta} \left(\frac{A_s}{N} | \theta \right) = \theta$.
2. $F_{S_N|\Theta} (A_s | \theta)$ second-order stochastically dominates (SOSD) $F_{S_N|\Theta}^\psi (A_s | \theta)$, and $Var_{S_N|\Theta}^\psi \left(\frac{A_s}{N} | \theta \right) \geq Var_{S_N|\Theta} \left(\frac{A_s}{N} | \theta \right)$ with strict inequality for $N > 1$.
3. $Var_{S_N|\Theta}^\psi \left(\frac{A_s}{N} | \theta \right)$ is strictly decreasing in N .
4. $F_{S_N|\Theta}^\psi (A_s | \theta')$ first-order stochastically dominates (FOSD) $F_{S_N|\Theta}^\psi (A_s | \theta)$ whenever $\theta' > \theta$.

Part 1 states that Barney, like Tommy, expects the average proportion of a 's in the sample to be θ . An immediate and important corollary of Part 1 is that Barney's beliefs coincide with Tommy's when $N = 1$. Part 2 states that Barney has a riskier subjective sampling distribution than Tommy. Combined with the fact that the mean of Barney's subjective sampling distribution is the same as Tommy's, this implies that Barney's subjective sampling distribution is a mean-preserving spread of Tommy's. This naturally implies that the variance of Barney's subjective sampling distribution is larger than Tommy's. Part 3 states that the variance of Barney's subjective sampling distribution (for the sample proportion) is strictly decreasing in N . Part 4 states that a higher true rate generates a rightward shift in Barney's entire subjective sampling distribution in the sense of first-order stochastic dominance.

We now turn to *inference* problems, where an agent with prior beliefs must infer from observed signals what the underlying rate is; e.g., determining the likelihood that a coin is head-biased rather than tail-biased, after observing a sample of coin flips. Let $\Theta \subseteq (0, 1)$ denote the set of rates that have positive prior probability. For simplicity, we assume Θ is a finite set. Without loss of generality, we consider the agent's beliefs about the relative likelihood of two of the rates $\theta_A > \theta_B$, given priors $0 < f_\Theta(\theta_A), f_\Theta(\theta_B) < 1$ with $f_\Theta(\theta_A) + f_\Theta(\theta_B) \leq 1$.

We maintain the standard assumption that an agent draws inferences by applying Bayes' Rule to his subjective sampling distributions. We do so both to highlight the role played per se by NBLLN, and because (as we discuss in Appendix B) our reading of the experimental evidence is that except for the well-established phenomenon of "base-rate neglect" (i.e., underweighting of priors), people's inferences are actually well-approximated by Bayes' Rule applied to their subjective sampling distributions.¹⁰ Consequently, Barney's beliefs after observing a particular clump $s \in S_N$ are $f_{\Theta|S_N}^\psi(\theta_A | s) = \frac{f_{S_N|\Theta}^\psi(s|\theta_A)f_\Theta(\theta_A)}{\sum_{\theta \in \Theta} f_{S_N|\Theta}^\psi(s|\theta)f_\Theta(\theta)}$ and $f_{\Theta|S_N}^\psi(\theta_B | s) = \frac{f_{S_N|\Theta}^\psi(s|\theta_B)f_\Theta(\theta_B)}{\sum_{\theta \in \Theta} f_{S_N|\Theta}^\psi(s|\theta)f_\Theta(\theta)}$. Because Barney updates Bayesianly given his model of the data-generating process, his subjective beliefs satisfy the

¹⁰Therefore, in applications where priors are equal—and hence base-rate neglect is neutralized as a factor—our assumption of Bayesian inference is fully appropriate. In applications where we do not assume equal priors, however, base-rate neglect could modify some of our results. Appendix A discusses and formalizes how to combine NBLLN with base-rate neglect.

“Law of Iterated Expectations”: Barney expects that for any sample size, the mean of his posterior beliefs will equal the mean of his prior beliefs: for any $N \geq 1$, $E_{\Theta|S_N}^\psi(\theta|s) = E_\Theta^\psi(\theta)$.

Due to the LLN, after observing a sufficiently large number of signals, Tommy will be arbitrarily close to knowing the true rate with certainty. In contrast, the central implication for inference of Barney’s NBLLN—which plays a large role in many of the applications later in this paper—is that Barney remains uncertain even after observing an *infinite* number of signals. To boot:

Proposition 2. *Assume A1-A4. Let $\theta \in \Theta$ be the true rate. Then for any $\theta_A, \theta_B \in \Theta$ and prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$ Barney draws limited inference even from an infinite sample: as $N \rightarrow \infty$, Barney’s posterior ratio converges almost surely (with respect to the true probability distribution over events) to a positive, finite number:*

$$\frac{f_{\Theta|S_N}^\psi(\theta_A|s)}{f_{\Theta|S_N}^\psi(\theta_B|s)} \xrightarrow{a.s.} \frac{f_{\mathbb{B}|\Theta}^\psi(\beta = \theta|\theta_A) f(\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta = \theta|\theta_B) f(\theta_B)}. \quad (2)$$

Because Barney’s asymptotic sampling distribution coincides with the subjective-rate distribution, his limit inference depends on the relative heights that the pdfs of the subjective-rate distributions for θ_A and θ_B assign to the proportion θ of a ’s. Since the subjective-rate distributions put positive density on every proportion in $(0, 1)$, Barney’s likelihood ratio will be finite. An immediate and important implication is that Barney’s priors—i.e., his ex ante theories about the world—influence his beliefs even in the limit of an infinite sample.

Tommy not only will learn the true rate for sure after observing a sufficiently large number of signals, he also correctly *anticipates* that a sufficiently large number of signals will make him certain of the true rate. In contrast, Barney mistakenly thinks that his posterior probability of rate θ_A after observing an infinite number of signals is a stochastic function of the true rate. The reason is that, even though Barney knows that his inferences in a large sample will be pinned down by the proportion of a ’s, he incorrectly thinks the proportion of a ’s is determined by the subjective rate, which could be any number between 0 and 1.

Proposition 3. *Assume A1-A4. Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and any prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$. Before having observed any data, Tommy believes: if the rate is θ_A , then his limit posterior probability that the rate is θ_A is 1. In contrast, before having observed any data, Barney believes: if the rate is θ_A , then his limit posterior probability that the rate is θ_A is a random variable that has positive density on a nondegenerate interval in $[0, 1]$. If we strengthen assumption A1 to A1’, then, in addition, the interval is closed and is a strict subset of $[0, 1]$.*

We next turn to inference in finite samples. Because Barney’s subjective sampling distribution

is correct when the sample size is 1, he will draw correct inferences in that case.

Proposition 4. *Assume A1-A4. Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$. For $N = 1$, Barney and Tommy infer the same. If $\theta_A = 1 - \theta_B$, then for any set of $N \in \{1, 2, \dots\}$ signal realizations $s \in S_N$, neither Tommy’s beliefs nor Barney’s beliefs change from the priors when $\frac{A_s}{N} = \frac{1}{2}$.*

In many of the inference experiments reviewed in Appendix B and in many of our applications involving inference, the two rates are “symmetric” in the sense that $\theta_A = 1 - \theta_B$, e.g., an urn might have either 60% red balls or 40% red balls. In that case, when exactly half the signals are a -signals, the sample is uninformative for both Barney and Tommy, and neither updates his beliefs about the rate.

For further analysis of the finite-sample case—as well as for some theoretical applications and empirical analysis—it is useful to have a parametric model of Barney’s subjective rate distribution. For some of our results, we impose the functional form of the beta distribution:

$$f_{B|\Theta}^\psi(\beta|\theta) = \beta^{\theta\psi-1} (1 - \beta)^{(1-\theta)\psi-1} \frac{\Gamma(\psi)}{\Gamma(\theta\psi) \Gamma((1-\theta)\psi)}, \quad (3)$$

where $0 < \psi < \infty$ is the exogenous parameter of the model, and $\Gamma(x) \equiv \int_{[0,\infty)} y^{x-1} e^{-y} dy$, defined on $x > 0$.¹¹ The properties A1-A4 are satisfied (see Appendix C, Lemma $\beta 5$), and this family of beta densities shares many qualitative features of people’s empirically-observed large-sample beliefs about the distribution of signals. A major advantage of this formulation is tractability: since the beta distribution is the conjugate prior for the binomial distribution, standard results from probability theory can be used to characterize Barney’s beliefs.¹² “Parameterized-Barney” is more biased for smaller ψ —with more dispersed subjective sampling distributions in the sense of SOSD—and Barney coincides with Tommy in the parameter limit $\psi \rightarrow \infty$.

Although we do not conduct a careful structural estimation, using evidence from studies that elicit subjects’ subjective sampling distribution as well as inference studies, we estimate that ψ falls within a range of 7-15. This parameterized model of Barney gives a sense of magnitudes for how Barney’s under-inference depends on the rates θ_A and θ_B . Suppose $\psi = 10$, Barney begins with

¹¹The more common way of writing this beta density is $f_{B|\Theta}^\psi(\beta|\theta) = \beta^{\theta\psi-1} (1 - \beta)^{(1-\theta)\psi-1} \frac{(\psi-1)!}{(\theta\psi-1)!((1-\theta)\psi-1)!}$. Our formulation is equivalent, except it allows for non-integer values of ψ . Recall that the Gamma Function, $\Gamma(x)$, is the standard generalization of the factorial function: it has the properties that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$, so that for any positive integer x , $\Gamma(x) = (x-1)!$.

¹²The functional form (3) has a few implications about asymmetric inference (i.e., inference problems where $\theta_A \neq 1 - \theta_B$) that do not have general intuitions related to NBLN. These are presented in Lemma $\beta 4$ in Appendix C. Since these properties would not generalize to other models that are equally consistent with existing evidence, we avoid stating implications that rely on these properties of the functional form.

equal priors on the two states, and the true rate is θ_A . If the difference between the rates is relatively large—with $\theta_A = 1 - \theta_B = .8$ —then the role of NBLLN is relatively small. In an infinite sample, Barney’s subjective posterior probability of rate θ_A will converge to .9998. However, if the two rates are closer together—with $\theta_A = 1 - \theta_B = .6$ —then in an infinite sample, Barney’s subjective posterior probability of rate θ_A will converge to only .69 (which is what Tommy’s posterior would be after only 6 heads and 4 tails!). As a reminder about the role of priors, this means if Barney initially had beliefs more extreme than 2.25:1 in favor of rate θ_B , he will, in an infinite sample, surely end up believing rate θ_B is more likely, even when θ_A is the true rate.

While most dramatic in large samples, NBLLN has implications for all sample sizes larger than 1. Since Barney’s subjective sampling distribution is too dispersed when $N > 1$, Barney will “generally” under-infer when the sample size is larger than 1. In order to make that claim precise, we measure Barney’s (and Tommy’s) “change in beliefs” by the absolute difference between his posterior probability that θ_A is the true rate and his prior probability: $|f_{\Theta|S_N}^\psi(\theta_A|s) - f_\Theta(\theta_A)|$. Unlike in large samples, in small samples it is no longer universally true that Barney under-infers relative to Tommy. For particular realizations Barney can over-infer or under-infer relative to Tommy—or even infer in the opposite direction, so that a sample that causes Tommy to think rate θ_A is more likely, causes Barney to think rate θ_B is more likely!¹³ Nonetheless, we believe that Barney under-infers *in expectation*, taken with respect to the true sampling distribution. Proposition 5 proves this statement for the case of ψ sufficiently small, but we conjecture that it holds for any $0 < \psi < \infty$.¹⁴

Proposition 5. *Assume Barney has the beta-distribution functional form given by equation (3). Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$, prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$ and a set of $N \in \{1, 2, \dots\}$ signal realizations $s \in S_N$. Regardless of whether the true rate is θ_A or θ_B , for ψ sufficiently small, the expected change in Barney’s beliefs is smaller than the expected change in Tommy’s beliefs. Furthermore, suppose $\theta_A = 1 - \theta_B$. Then for any sample of $N > 1$ signals such that $\frac{A_s}{N} \neq \frac{1}{2}$ and any ψ , Barney under-infers relative to Tommy. In addition, while Tommy’s inference depends solely on the difference in the number of a and b signals, Barney’s change in beliefs is smaller from larger samples with the same difference.*

Intuitively, on average Barney under-infers because he partially attributes the information in the

¹³For example, using the parameterized model, set $\psi = 10$, $\theta_A = .7$, and $\theta_B = .6$, and assume equal priors on the two states. Then if the realizations of 80 signals are 53 a -signals and 27 b -signals, then Tommy believes that state A is more likely, while Barney believes state B is more likely.

¹⁴We have simulated Barney’s and Tommy’s expected change in beliefs for a range of parameter values: for each of $\psi \in \{1, 2, \dots, 30\}$ and $N \in \{5, 10, 15, 20\}$, we examined each of $\theta_A, \theta_B \in \{.5, .6, .7, .8, .9\}$. We also ran a number of simulations for $\theta_A = .99$ and $.999$ and for $\psi = 100$. In every case we examined, Barney’s expected change in beliefs was smaller than Tommy’s.

realized sample to the subjective rate, rather than extracting all of the information about the true rate.

In symmetric inference problems (i.e., $\theta_A = 1 - \theta_B$), Proposition 5 shows that stronger comparisons can be made between Barney and Tommy: as long as the realized sample is informative, parameterized-Barney will under-infer, not just in expectation. Proposition 5 also notes a key feature of Barney’s updating that shows how it leads to a bias toward “proportional thinking” in inference along the lines suggested by researchers such as Griffin and Tversky (1992). Consider samples where the difference between the number of a -signals and the number of b -signals is the same, e.g., aa and $abababaa$. Tommy will draw the same inference from the two samples. But because his asymptotic sampling distributions depend on the proportion of a and b signals rather than their number, Barney infers less from the larger sample.

3 Prediction Applications

This section explores the implications of NBLLN for behavior that depends on an agent’s subjective sampling distribution. In these applications, a key feature of the model is Barney’s overestimation of the likelihood of extreme sample proportions. While in some settings (such as the investment context we discuss below), there are non-NBLLN reasons why people might perceive distributions to have “fat tails,” we emphasize that Barney believes that extreme outcomes are likely even for known and familiar i.i.d. random processes, such as coin flips. Moreover, NBLLN has the particular property that Barney’s beliefs become more distorted as the sample size increases, and this property also plays a central role in the applications.

3.1 Perceived Aggregate Risk

An economics professor at MIT once told his colleague Paul Samuelson that, whereas he would reject a bet for even odds to gain \$200 or lose \$100, he would accept 100 repetitions of that bet. Even though such behavior sounds reasonable to most of us, Samuelson (1963) reports the conversation so as to prove that it violates classical expected-utility theory. That is, a Tommy with classical expected-utility preferences (defined over final wealth) who does not exhibit unrealistically large wealth effects should be willing to take a single bet if and only if he is willing to take $N \geq 1$ independent plays of that bet. Intuitively, in the absence of wealth effects, preferring $K + 1$ bets to K bets is the same thing as preferring 1 bet on top of any realization of the K bets. By induction, preferring to take any positive number of the bets is the same as preferring to take one bet.

Yet, it is not just the “switching” that violates classical expected utility preferences, but the

aversion to the single bet to begin with. Rabin (2000a, 2000b) and Rabin and Thaler (2001) have followed others in noting that the degree of concavity required for classical expected-utility preferences, defined over wealth, to explain risk-averse behavior over the small stakes involved in a single play of the gamble is calibrationally implausible. Loss aversion—the tendency to feel a loss more intensely than an equal-sized gain—explains why the majority of people who turn down the one-shot gamble do so.¹⁵ A Tommy with a simple, piecewise-linear loss-averse utility function

$$u(w_0, z) = \begin{cases} w_0 + z & \text{if } z \geq 0 \\ w_0 + \lambda z & \text{if } z < 0 \end{cases}, \quad (4)$$

where w_0 is initial wealth and z is a monetary gain or loss, will refuse the one-shot bet as long as the coefficient of loss aversion, λ —often set equal to 2.25 (e.g., Tversky and Kahneman, 1991)—is greater than 2.¹⁶ However, a Tommy with typical loss-averse preferences would be extremely happy to accept 100 repetitions of the same gamble: while the expected gain is \$5,000, the chance of a net loss is only 1/700, and the chance of losing more than \$1,000 is only 1/26,000.

Despite being loss averse enough to turn down the one bet, nobody with fully rational beliefs would turn down 100 repetitions of this bet. Yet, unlike Samuelson’s colleague, many people would do so! In hypothetical questions from one study in Benartzi and Thaler (1999), for instance, 36% of participants said they would turn down a single scaled-down Samuelson type bet (win \$100 or lose \$50), but fully 25% also reject the 100-times repeated gamble.¹⁷

NBLLN helps explain why many people turn down these gambles.¹⁸ Barney exaggerates the

¹⁵Samuelson himself had speculated that it was the willingness to accept repeated plays of the bet that was the mistake, rather than the refusal to accept a single gamble. Samuelson’s conjecture that his colleague’s willingness to accept the repeated gamble was the result of a “fallacy of large numbers”—a mistaken belief that the riskiness of the gamble evaporates with a sufficiently large number of repetitions—is the *opposite* of NBLLN, and is contradicted by Benartzi and Thaler’s (1999) evidence, reported below, that people exaggerate the probability of a loss in the repeated bet.

¹⁶Although essentially correct for small gambles, assuming linear consumption utility can become problematic if bets are repeated so many times as to involve large amounts of wealth. However, if in our limit results below, we halve the stakes every time we double the number of repetitions, the linearity assumption is unobjectionable.

¹⁷In two other subject pools, they find 34% and 23% turn down a simple \$20/\$10 gamble, and *more* people—57% and 50%—turn down the repeated gamble. Keren (1991) finds similar results in incentivized single bets vs five-times-repeated bets; for related hypothetical evidence, see Keren and Wagenaar (1987) and Redelmeier and Tversky (1992). Klos, Weber and Weber (2005) replicate and extend Benartzi and Thaler’s findings. They present subjects with four lotteries, each of which may be played singly, repeated 5 times, or repeated 50 times. Subjects generally prefer the repeated gambles but vastly overestimate the probability of loss as well as the expected loss conditional on losing money. Klos, Weber and Weber also find that subjects incorrectly believe the probability of the monetary outcome ending up within a given interval around the expected value increases with the number of repetitions. This last finding is inconsistent with our model of NBLLN and may reflect a bias from focusing subjects’ attention on the expected value, or it may be consistent with “exact representativeness,” a bias we discuss in Appendix B.

¹⁸Our emphasis on how NBLLN helps explain why loss-averse people turn down the repeated bet is because of its calibration relevance, but it is worth noting that NBLLN also has implications for how expected-

probability that the repeated bet will turn out badly. Indeed, Benartzi and Thaler report evidence consistent with this explanation: when asked the probability of losing money after 150 repetitions of a 90%/10% bet to gain \$0.10/lose \$0.50, 81% of subjects overestimated the probability—and by an enormous margin. While the correct answer is .003, the average estimate was .24. To show that subjects’ mistaken beliefs were driving their choices, Benartzi and Thaler compared subjects’ willingness to accept the repeated bet with their willingness to accept a single-play bet that had the histogram of money outcomes implied by the repeated bet. While only 49% of the college-student subjects accepted 150 repetitions of the bet, 90% accepted the equivalent single-play bet, suggesting that the repeated bet would have been very attractive if subjects had correctly understood the distribution of outcomes.^{19,20}

Formally, while a loss-averse Tommy will always accept a better-than-fair bet if it is repeated enough times, a loss-averse Barney may—depending on how favorable the bet is and how loss-averse he is—turn down an infinitely-repeated bet.

Proposition 6. *Assume A1-A4. Suppose Barney and Tommy have simple, piecewise-linear loss-averse preferences as specified in (4). Fix any gamble (θ, h, t) , paying off $h > 0$ with probability θ and $-t$ with probability $1 - \theta$, that is better than fair: $\theta h > (1 - \theta)t$. For any $\lambda \geq 1$, there is some $N' \geq 1$ such that if $N > N'$, then Tommy will accept N repetitions of the gamble. In contrast, for Barney there is some threshold level of loss aversion $\hat{\lambda} > 1$ such that: if $\lambda < \hat{\lambda}$, then there is some N' sufficiently large such that Barney will accept N repetitions of the gamble for all $N > N'$; and if $\lambda \geq \hat{\lambda}$, then there is some N'' sufficiently large such that Barney will reject N repetitions of the gamble for all $N > N''$.*

Moreover, it is possible for Barney to exhibit the *opposite* pattern from Samuelson’s colleague, accepting the single bet but rejecting the 100-times repeated bet! Consistent with this possibility, the evidence cited above from Benartzi and Thaler (1999) found behavior in the opposite direction

utility-over-wealth agents respond to repetitions of bets. We can extend the “if” part of Samuelson’s theorem: if Barney rejects a bet at all initial wealth levels w_0 , then he would also reject any $N \geq 1$ independent plays of that bet. The “only if” direction does not extend, and a Barney who is just indifferent between accepting and rejecting a simple bet would, because he exaggerates the risk, strictly prefer to reject repeated versions of the gamble.

¹⁹Note also that something more than the type of “narrow bracketing” stressed by authors such as Tversky and Kahneman (1986), Kahneman and Lovallo (1993), Benartzi and Thaler (1995), Read, Loewenstein and Rabin (1999), Barberis, Huang and Thaler (2006), and Rabin and Weizsäcker (2009) seems to be playing a role. Those papers emphasize that people often react to a combination of risky bets as if they were deciding about each risky bet in isolation from all the others. While such neglect of the effects of aggregating risks may help explain why people reject the repeated gamble, it seems clear that even people who attend to the aggregate effects *misunderstand* these aggregate effects. Benartzi and Thaler (1999) make this especially clear by demonstrating directly that people asked the probability of aggregate loss of independent bets exaggerate along the lines predicted by NBLLN.

²⁰Benartzi and Thaler also elicited the effects of showing the histogram in the above hypothetical examples and showed that it reduces rejections from 25%, 57%, and 50% to, respectively, 14%, 10%, and 17%.

of Samuelson’s colleague in two out of their three studies.²¹

Using the one-parameter functional form for Barney, calibrations suggest that NBLLN goes much of the way, but *not* all of the way, in explaining why 25-57% of participants turned down the repeated bets in Benartzi and Thaler’s studies. For Tommy, the coefficient of loss aversion required to explain this data is absurdly high, in excess of 32,000. For Barney with $\psi = 10$, the required loss aversion is approximately 15—many orders of magnitude closer to reality but still much larger than reasonable estimates of λ , such as 2.25.²²

In addition to predicting that people perceive too much risk in repeated betting, NBLLN also predicts that people perceive too much risk in the closely-related context of long-term investing. Indeed, analogously to the evidence on repeated betting, Benartzi and Thaler (1999) found that university employees vastly overestimated the probability that equities would lose money over a thirty-year horizon. Moreover, the employees stated a far greater willingness to invest in equities when they were explicitly shown the thirty-year returns. While there are many reasons why individuals may invest less in equities for retirement than recommended by standard finance models, we suspect that NBLLN is an important contributing factor. As such, just as in other settings researchers may underestimate risk aversion by ignoring overconfidence when inferring risk preferences from investment behavior, researchers might therefore exaggerate the risk aversion of investors by ignoring NBLLN.

NBLLN also helps explain why people fail to fully recognize the benefits of diversification. Proposition 6 would go through essentially unchanged if the agent were mixing N independent gambles (rather than repeating a gamble N times). Tommy would always accept a portfolio of positive-expected-value gambles if N is sufficiently large. In contrast, if Barney is sufficiently loss averse, then regardless of how large N is, Barney may prefer not to hold this portfolio.

²¹Preferences that generate an aversion for multi-stage resolution of risk—such as the preferences proposed by Koszegi and Rabin (2009) or Dillenberger (2010)—could also predict rejections of repeated gambles. The psychology underlying this prediction, however, only seems plausible when the outcomes of each individual gamble are observed separately. In contrast, NBLLN predicts rejection precisely due to mistaken beliefs about the *combined* outcomes of the gambles. Furthermore, NBLLN predicts risk-seeking behavior in settings where these other models would not, e.g., in the lottery example in the next subsection.

²²Incorporating some of the biases missing from our model of NBLLN (such as the SDTDS bias) that would generate even fatter tails in subjects’ subjective sampling distribution helps reduce the required level of loss aversion even more—although still not to 2.25. Indeed, even if an individual exhibited the most extreme form of NBLLN *and* SDTDS (and possibly also probability weighting), putting equal weight on every possible outcome of the repeated gamble, the required level of loss aversion would be around 4 (because only 1/3 of the outcomes are losses). This exercise implies that some other bias is also implicated in turning down the repeated bet. Whatever it is might also explain why some subjects turn down the aggregate bet even when it is presented in histogram form.

3.2 Other Prediction Applications

In the example above, Barney's exaggerated belief about the likelihood of extreme sample proportions causes him to behave in a way that appears risk averse because his payoff depends on the total number of a -signals (i.e., winning outcomes) that occur. In situations where Barney's payoff instead depends on whether an extreme outcome occurs, the same exaggerated belief can lead to behavior that appears risk seeking. For example, suppose Barney is deciding whether to play a lottery game where he picks N numbers, he wins a prize if all his picks match the numbers chosen randomly by the lottery, and the probability that any given number he picks is chosen by the lottery is θ . If $N > 1$, then Barney believes his chance of matching all the number is higher than it is, and hence his willingness to pay to play this game is higher than Tommy's. The logic of NBLLN implies that the difference from Tommy is larger for larger N .

In contrast, in situations where Barney gets a payoff as long as an extreme outcome does *not* occur, Barney's behavior can appear to be especially risk averse. Suppose Barney attends a job fair at which he applies to $N > 1$ equally-valuable jobs, his chance of getting any particular job is θ , and he cannot accept more than one job. Since Barney believes his chance of getting at least one job offer is lower than it is, Barney's willingness to pay to attend the job fair is lower than Tommy's.

Another implication of Barney's distorted predictions is that he does not think that knowing the rate is as useful for predicting signals as it actually is. This can cause Barney's willingness to pay for information to differ from Tommy's (for reasons distinct from Barney's inferences, as explored in Section 5). For example, suppose an agent is making a one-shot decision whether or not to put his savings of \$100,000 in a risk-free bond that pays 0% interest or in an illiquid mutual fund for the next N years, but he does not know whether the fund is run by a talented manager (state A) or not (state B). If the manager is talented, then the probability that the fund will increase rather than decrease in value by 10% in any given year is θ_A , which is higher than the probability of the good return if the manager is not talented, $\theta_B < \theta_A$. Fix the agent's utility function over final wealth $u(w) = \sqrt{w}$, number of years $N = 2$, Barneyess parameter $\psi = 1$, prior $f_\Theta(\theta_A) = .5$, and $\theta_B = .4$.

If $\theta_A = .55$, then Barney's willingness to pay for a perfectly revealing signal about the state is higher than Tommy's, while if $\theta_A = .75$, then Barney's willingness to pay is lower. The key difference is what the agent would do in the absence of the signal. Due to the multiplicative returns of investing, Barney overvalues investing in the mutual fund relative to Tommy regardless of θ_A . If $\theta_A = .55$, then the likelihood of a good return is relatively low in both states, hence in the absence of information, either agent would put his savings in the bond. For Barney, knowing

the state has value because he would switch to investing in the fund if he knew the manager were talented, while Tommy would invest in the bond in either state. In contrast, if $\theta_A = .75$, then both agents would invest in the fund if ignorant but would switch to the bond if they knew the manager were not talented. Because Barney overvalues investing in the mutual fund if the manager is bad, finding out the manager’s type yields a smaller total gain for him than for Tommy.

4 NBLLN in Multiple-Sample Settings

For Tommy, it does not matter whether he treats 20 independent signals as one sample of 20, two samples of 10, or 20 samples of 1. In contrast, and intrinsic to the very meaning of NBLLN, Barney’s beliefs about the distribution of and inference from signals depend on how he divides them up into samples. In this section, we provide a framework for thinking about this issue, and—in order to facilitate studying the range of possible consequences of NBLLN in dynamic decision-making environments as well as to provide guidance for future experimental work—we formulate plausible alternative assumptions.²³ We review in Appendix B the scant and somewhat contradictory evidence we can find about which assumptions are appropriate; since there is so little evidence, our proposals are tentative. We keep our discussion in this section informal, relegating formal definitions to Appendix D.

Several distinctions will be useful. *Clumping* refers to how signals are objectively delivered to Barney by his environment. For example, when Barney asks a sequence of friends about their experience driving a Volvo, each friend’s report arrives as a separate clump, but when Barney reads a summary of 10,000 individuals’ experiences in *Consumer Reports*, the 10,000 signals arrive as a single clump. We refer to how Barney subjectively processes these clumps for the purposes of making forecasts and inferences as how he *groups* the signals. Barney forms beliefs regarding each group of signals as if a subjective rate β were drawn that applies only to that group, and hence the single-clump model from Section 2 can be applied to each group. Although economic models of decisionmaking do not traditionally specify how signals are clumped, our aspiration is have this be an exogenous assumption that is not *per se* related to NBLLN, and ideally is pinned down by observable characteristics of a situation. How clumps are grouped, on the other hand, must be a feature of any complete model of a departure from Bayesian information processing.

We distinguish two facets of how Barney groups data. The first is how he processes clumps into groups *retrospectively*—how he processes clumps he has already received. The second is how

²³Moreover, the implications of other non-Bayesian models of judgment biases—such as base-rate neglect—similarly exhibit sensitivity to how data are framed. As such, the range of approaches we outline here may prove useful for studying those other biases.

he processes clumps into groups *prospectively*. Prospective grouping determines his forecast about what data he will observe given his current beliefs *and* his forecast of what he will infer from that data after he observes it.

In each of the retrospective and prospective directions, we focus attention on two ways that Barney might process clumps of signals. If Barney groups the signals the same way he receives them from the environment, we call him *acceptive*. Acceptive Barney would process *Consumer Reports* as a single sample of 10,000, and then each of his friends’ reports as a separate sample. If Barney processes all of the clumps of signals he observes as a single, large group, we call him *pooling*. For example, pooling Barney would treat *Consumer Reports* data and his friends’ stories as a single, larger sample.²⁴

In principle, one could imagine Barney’s beliefs in a dynamic environment as being either retrospective-acceptive or retrospective-pooling, combined with being either prospective-acceptive or prospective-pooling. We argue that Barney cannot be prospective-pooling, however, in any environment where he expects to make a decision at a future date. More generally, we impose the following constraint on any model of NBLN: at any date where the agent makes a decision, he processes signals before and after that date as being in separate groups—and before that date, he knows he will do so. We consider this to be a modeling coherence constraint because it ensures that Barney’s NBLN from the single-clump model in Section 2 generalizes to every decision node in a multiple-clump setting.

To see this, suppose Barney knows that $\theta = .5$, reads a summary of 10,000 individuals’ experiences in *Consumer Reports*, and then must make a prediction about the next 1,000 signals he will observe. If, in violation of our modeling constraint, he were to process all 11,000 signals together as a single group after already observing the first 10,000 signals, then he would believe that the same subjective rate β applies to all 11,000 signals. Using the first 10,000 signals, he would update his belief about β from $f_{\beta|\Theta}^{\psi}(\beta|\theta = .5)$ to a density that puts almost all the probability mass on the observed proportion of a -signals, say 50%. Since the next clump is grouped with the earlier clump, his subjective sampling distribution for the next clump will put negligible weight on a proportion of a -signals outside a neighborhood of 50%. In his predictions about future signals, Barney would no longer exhibit NBLN. In contrast, our modeling constraint requires that Barney forms beliefs as if a new β is drawn from $f_{\beta|\Theta}^{\psi}(\beta|\theta)$ before the next 1,000 signals, so his subjective

²⁴While we conjecture that these two grouping processes cover a wide range of typical situations, we acknowledge that in certain situations, other grouping processes are psychologically plausible. For example, as we mention in Section 6, in the context of social learning it may be natural to group one’s own signal separately from everyone else’s, even if all the signals occur simultaneously. As another example, Barney may group signals according to the perceived similarity of the information source. After observing 10,000 datapoints from *Consumer Reports* followed by 10 friends’ reports obtained sequentially, for instance, Barney may retrospectively process the information as a group of 10,000 followed by a group of 10.

sampling distribution is exactly as in the single-clump model for $N = 1,000$. Precisely because it rules out learning about the subjective rate, this constraint distinguishes our model of NBLLN from the generalization of the model from Section 2 that one would employ if one interpreted it as a fully-rational model with uncertainty about the parameter β . (Barney’s predictions regarding a single clump and a single decision node, as in Sections 2 and 3, are of course covered by the prospective-acceptive case.)

Which processing assumptions are the most psychologically plausible? We hypothesize that in many situations, Barney will expect to process data more finely in the future than he actually will do retrospectively.²⁵ For example, if Barney plans to keep talking to friends one by one until he feels confident, he might think ahead with attention to each separate signal, focusing on how he will update from current beliefs after his next conversation. But then in retrospect, after he has talked to his next friend, he may quite naturally treat that friend’s information symmetrically with all the previous conversations and take stock of his current information by thinking together about all the advice he has received. Accordingly, in some of our applications, we explore the implications of Barney being retrospective-pooling and prospective-acceptive.

As an implication of all the other ways he is rational, Tommy always processes information the way that he expects to process information. We call this property *processing-consistency*. Despite his irrationality, Barney shares this property if his retrospective and prospective thought processes coincide. In particular, if Barney is retrospective-acceptive and prospective-acceptive, then he is processing-consistent and accurately forecasts what his own future beliefs will be after he observes a sequence of signals. In contrast, if Barney is retrospective-pooling and prospective-acceptive—as we have argued is often plausible—then he is not processing-consistent. As a result, he may behave in a time-inconsistent way, e.g., expecting to learn a lot from purchasing a large number of signals, but remaining uncertain after observing the signals and therefore preferring to purchase yet more signals. This time-inconsistency will play a role in some of the applications we study.

Besides lack of evidence, an additional and major reason the hypotheses in this section are tentative is that we have completely sidestepped the issue of when and how Barney might “think through” his beliefs more fully. Even if Barney is not processing-consistent, our *model* of Barney’s beliefs is internally consistent. Barney’s *beliefs* themselves, however, may not be internally consistent, and this raises additional conceptual and practical issues in applying a model of NBLLN.²⁶

²⁵We emphasize, however, that we are hypothesizing about what is typical, and we can imagine alternative possibilities in certain situations. For example, Barney might be retrospective-acceptive but pool signals he has not yet observed: before he talks to the 10 friends (say) he plans to talk to, he may not attend to the time separation of the information and not realize that he will update his beliefs story-by-story as he goes along. But retrospectively, he may be retrospective-acceptive, distinguishing colorful details of his friends’ stories.

²⁶The internal inconsistency we highlight here does not arise in “false-model Bayesian” models of biased

For example, a teacher could elicit Barney’s belief about the likelihood that a first signal will be a , the likelihood that a second signal will be a conditional on the first signal being a , and the likelihood that a sequence of two signals will be aa ; then the teacher could point out that the product of the first two does not equal the second. In fact, even our coherence constraint above could fail depending on the questions a teacher asked Barney. For example, suppose Barney expects to observe 10,000 signals from *Consumer Reports*, then make some payoff-relevant decision, and then observe another 1,000 signals. If a teacher asks Barney to forecast all 11,000 signals, then Barney would presumably do so according to the single-clump model with $N = 11,000$ —in violation of our modeling constraint. Even in the absence of a “teacher,” Barney might ask himself such questions.²⁷ While we flag these issues, and we think they are natural subjects of future research, we proceed in subsequent sections with the assumption that Barney does not think through the inconsistencies in his own beliefs.

5 Inference Applications

This section explores some implications of NBLLN for inference, applying variants of our model to information acquisition, learning, and experimentation. In all of these applications, we adopt a standard framework in which an agent is uncertain about which of two possible states of the world, $\omega \in \{A, B\}$, is true. State A has prior probability $0 < f_{\Theta}(\theta_A) < 1$, and state B has prior probability $f_{\Theta}(\theta_B) = 1 - f_{\Theta}(\theta_A)$. In state A , the probability of an a -signal is θ_A , and the probability of a b -signal is $1 - \theta_A$. In state B , the probability of an a -signal is $\theta_B < \theta_A$, and the probability of a b -signal is $1 - \theta_B$. Depending on the particular application, the agent can take actions, or observe outcomes and signals that can inform him about the state of the world.

While the applications in Section 3 focused only on the implications of Barney’s predictions, beliefs such as Barberis, Shleifer and Vishny (1998), Rabin (2002), and Rabin and Vayanos (2010). In these models, biases are formulated as agents holding the wrong theory as to the statistical structure of the world, but as being fully Bayesian in their interpretation of data within that structure. So long as all events that are possible in the true world are also possible in the agents’ imagined world, no internal inconsistency can arise. (Though even in these models, it may be very likely that the agent will observe a sequence of signals that he perceives to be very unlikely.) Processing-consistent variants of Barney likewise reduce to a “false-model Bayesian” theory. The processing-*inconsistent* variants of NBLLN, however, assume an intrinsically non-Bayesian thought process. The modeling challenges associated with internally-inconsistent beliefs are not specific to NBLLN and will arise in any model of belief formation that is fundamentally non-Bayesian.

²⁷Importantly, a decision Barney faces might itself naturally cause Barney to ask himself such questions. For example, imagine Barney is making a decision whose payoff depends on whether the state is A or B . He could purchase one signal, and then decide whether to purchase a second signal, or he could purchase two signals all at once at a discount. When deciding what to do, it seems natural that Barney would ask himself what he would conclude after observing each of the three possible outcomes: he observes 1 signal, 1 signal followed by 1 signal, and 2 signals together. Having explicitly asked himself about these possibilities, it seems odd that Barney would—as assumed if Barney is prospective-acceptive—expect to conclude less from the 2 signals together than the 2 signals individually.

Barney's behavior in the current setting may depend both on his predictions—what signals he expects to observe—and his inferences—what he expects to believe and actually believes after observing the signals. We therefore make use of the assumptions described in Section 4 about how Barney groups signals retrospectively and prospectively.

5.1 Lack of Demand for Large Samples

Suppose Barney is trying to decide what make of car to buy, a Volvo or a Lada.²⁸ The state is $\omega = A$ if the Volvo is superior and $\omega = B$ if the Lada is superior. Barney is choosing whether to acquire information by asking a friend, which will provide him with a single signal at cost $c_f > 0$, or by purchasing *Consumer Reports*, which will provide him with the aggregate information from a large number N of signals, at cost $c_r > c_f$. Barney is prospective-acceptive: he anticipates that if he purchases *Consumer Reports*, he will group all the signals together. After observing the information, Barney must take an action μ , either buying the Volvo ($\mu = \mu_A$) or the Lada ($\mu = \mu_B$). Barney's payoff is $u(\mu, \omega)$, which equals 1 if the action matches the state and 0 otherwise.

The comparison between Barney's and Tommy's valuations of an intermediate-sized sample is ambiguous: even though Barney might expect to infer less on average than Tommy would, Barney also overestimates the probability of an extreme outcome that would allow for stronger inferences. Therefore, perhaps counterintuitively, if Barney and Tommy can both choose the total number of signals to purchase, Barney can actually end up being *more* certain about the state of the world.

For a *large* sample, however, Barney unambiguously has a lower willingness to pay than Tommy does. Tommy correctly anticipates that—due to the Law of Large Numbers—reading *Consumer Reports* will make him virtually certain about the true state, but Barney expects to remain uncertain (see Proposition 3). Because Barney expects to learn less from *Consumer Reports* than Tommy does, Barney is more likely to ask the friend.

Proposition 7. *Assume A1-A4. Suppose Barney is prospective-acceptive. Fix payoffs $u(\mu, \omega)$, rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$, prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, and the cost of asking a friend $c_f > 0$. Suppose that knowing the state is valuable: $u(\mu_A, A) > u(\mu_B, A)$ and $u(\mu_A, B) < u(\mu_B, B)$. Furthermore, suppose that c_f is small enough that if *Consumer Reports* were not available, Tommy would ask the friend. If the number of signals N in *Consumer Reports* is sufficiently large, then there exist thresholds c'_r and c''_r with $c_f < c'_r < c''_r$ such that: if $c_r < c'_r$, then both Tommy and Barney buy *Consumer Reports*; if $c_r > c''_r$, then both Tommy and Barney ask the friend; and if $c_r \in (c'_r, c''_r)$, then Tommy buys *Consumer Reports* while Barney asks the friend.*

²⁸A Lada is a type of car. So is a Volvo.

Barney's lack of demand for statistical data is a central implication of NBLLN. We believe it is consistent with obvious facts: we live in a world in which people are not persuaded by statistical evidence that should be convincing, people do not demand such information, and such information is therefore rarely supplied by the market. Public health announcements are more effective if they feature vivid anecdotes rather than statistics, and the car-purchaser who actually consults *Consumer Reports* is the exception rather than the rule. In some cases, people may correctly expect large samples not to be very useful because preferences are heterogeneous, e.g., restaurant satisfaction ratings. When preference heterogeneity is less of an issue, however, e.g., the frequency of car battery failure, the lack of demand for statistical data is a major “dog that didn't bark” clue that implicates NBLLN.

The flip side of people's failure to demand large numbers of signals is their willingness to rely instead on sources of information that provide only a small number of signals. Indeed, given the amount of other information people may be able to obtain at relatively low cost, NBLLN helps explain why they nonetheless often instead rely on limited personal experience, stories from one's immediate social network, or anecdotes from strangers.

The lack of demand for large samples generated by NBLLN is especially severe when Barney is initially confident about the state or when each individual signal is relatively uninformative, i.e., θ_A is close to θ_B . Tommy understands that a sufficiently large number of such signals will nonetheless reveal the state. In contrast, when Barney has a confident prior or when signals are relatively uninformative, Barney may be unwilling to pay *any* positive cost for even an infinite number of signals!

Proposition 8. *Assume A1' and A2-A4. Suppose that the agent is deciding whether to buy Consumer Reports at cost c_r or not obtain any signals. Furthermore, fix payoffs $u(\mu, \omega)$ so that knowing the state is valuable: $u(\mu_A, A) > u(\mu_B, A)$ and $u(\mu_A, B) < u(\mu_B, B)$. For Tommy: for all rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and priors $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, there exists a threshold $c_r^* > 0$ such that if $c_r < c_r^*$, then as long as the number of signals N in Consumer Reports is sufficiently large, he buys Consumer Reports. In contrast, for prospective-acceptive Barney: (i) for all rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$, there exist priors $f_\Theta(\theta_A)$ such that for any $c_r > 0$ and any N , he does not buy Consumer Reports; and (ii) for all priors $f_\Theta(\theta_A)$ at which he is not indifferent between μ_A and μ_B , there exist rates $\theta_A, \theta_B \in \Theta$, where $\theta_A > \theta_B$, such that for any $c_r > 0$ and any N , he does not buy Consumer Reports.*

Unlike Proposition 7, Proposition 8 relies on A1' (not just A1): Barney thinks that he will draw a limited inference no matter how extreme the sample proportions turn out to be. If his priors are extreme enough or the rates are close enough together, then he thinks that the information

provided by an infinite number of signals will not affect whether he buys the Volvo or the Lada. Consequently, his willingness to pay for an infinite number of signals is zero.

On the flipside, even though Barney often under-infers when presented with information, he may nonetheless purchase information even when it will not have any objective value for him. This is because Barney believes that his posterior after observing a large sample is a random variable, and his willingness to pay is positive whenever he thinks he might draw an extreme enough inference to switch his action from what he would do given only his prior. Yet for a large sample, Barney’s posterior is in fact deterministic. Hence if Barney’s priors are extreme enough, and the cost of information small enough, he may incur the cost of purchasing Consumer Reports even though the information will almost surely not affect his action. In section 5.3 below, we illustrate how such a “learning trap” phenomenon can also occur in a dynamic setting through a different mechanism (namely, Barney’s incorrect beliefs about how he will process future signals).

5.2 Enabling “Vividness Bias”

Our example of Barney under-inferring from statistics in *Consumer Reports* is borrowed from Nisbett and Ross (1980), who used the contrast between pallid statistics and colorful anecdotes to illustrate “vividness bias.” According to psychological research on vividness bias, people overweight vivid evidence in reaching their judgments; a random stranger’s graphic description of the horrors that ensued when her car broke down while trying to pick up her child from school may weigh more heavily in our judgment of the brand of car she drives than summary statistics based on large samples of data.

However, NLLN is a confound for evidence that might otherwise be interpreted as vividness bias. Suppose the random stranger’s vividly-recounted car experience is grouped separately from dry statistics summarizing the experiences of a large random sample of similar strangers. In that case, the *comparative* over-use of vivid evidence may not be over-use of that evidence relative to its proper use, but an indication of under-use of the other evidence. Empirical work on vividness bias should attempt to disentangle it from NLLN.

Moreover, even if there is genuine over-use of evidence due to its vividness (and consequent emotional tug), NLLN is a theoretically necessary “enabling bias” for vividness bias to matter. Suppose that the story-teller’s car experience, due to its vividness, is counted as 30 data points instead of 1. Despite over-use of the anecdote, LLN implies that its impact will be swamped when a person also has access to a large data set. However, since NLLN causes the person to infer less from the large data set than she would infer from 30 data points, the vivid anecdote is enabled to have the predominant effect.

5.3 Sequential Information Acquisition

Imagine that Barney is (still) trying to decide whether to buy a Volvo or a Lada. But rather than polling a group of friends, he asks one at a time which car is better. Conditional on the responses that he receives, he can decide to ask more friends, or to stop and choose a car. Such sequential information acquisition raises some new issues and possibilities to which we now turn.

Formally, each period $t = 1, 2, \dots$, the agent can choose to purchase a single signal at cost $c > 0$ or take an action $\mu \in \{\mu_A, \mu_B\}$.²⁹ If the agent takes an action, he gets payoff $u(\mu, \omega)$, which equals 1 if the action matches the state ω and 0 otherwise, and the agent faces no further decisions. If the agent decides to purchase an additional signal, he sees the realization of the signal, and he proceeds to the next period. The agent lives forever and seeks to maximize the expected action payoff minus expected signal-purchase costs; if the agent purchases κ signals and then takes action μ , his utility is $u(\mu, \omega) - \kappa c$. We assume no discounting, so that the only reason an agent would stop acquiring information before being absolutely certain is the cost c of obtaining an additional signal.

For Tommy, the characterization of optimal behavior is well-known (e.g., Wald, 1947). Each time Tommy purchases a signal, he updates his posterior beliefs. His optimal behavior is characterized by two probabilities, ν_l and ν_h , with $0 < \nu_l < \nu_h < 1$. If and only if the posterior probability of state A exceeds ν_h , he stops and takes action μ_A ; if and only if it goes below ν_l , he stops and takes action μ_B . Tommy continues to purchase signals as long as his posterior beliefs remains between ν_l and ν_h . But because his posterior ratio is a martingale process, Tommy will eventually feel strongly enough to take an action almost surely. Importantly, as $c \rightarrow 0$, $\nu_l \rightarrow 0$ and $\nu_h \rightarrow 1$, and so as information becomes cheaper, the agent requires more extreme beliefs to stop purchasing information.

Because we assume that the signals arrive one at a time and that Barney is prospective-acceptive, Barney expects to group the signals as samples of size 1. Since Barney *expects* to behave exactly like Tommy, his policy is the same as Tommy's, with the *same* thresholds ν_l and ν_h determining when he stops and takes an action. If Barney is retrospective-acceptive, his beliefs and behavior will be identical to Tommy's.

If he is retrospective-pooling, however, Barney's behavior can differ qualitatively from Tommy's. In this case, the impact of an additional signal on his posterior beliefs is smaller than for Tommy. Moreover, the marginal impact will approach zero as his sample of observed signals grows. This is

²⁹Here and later, we restrict Barney to purchasing information a single realization at a time. If we allowed Barney to choose how many signals he could purchase each period, Barney would have to think about what he would learn from purchasing two signals sequentially in order to compare it to purchasing two signals simultaneously. Modeling this thought process raises challenges—the same as those mentioned in footnote 27—that we sidestep in this paper.

because Barney's inference becomes more and more driven by the proportion of a -signals, which is less affected by an additional signal in a larger sample. However, Barney *believes* that an additional signal will matter, regardless of the sample size he has already observed. As a result, Barney can become stuck in a *learning trap*, in which he purchases signals forever, but they will never change his confidence in the state of the world enough for him to stop. The first part of Proposition 9 shows that such a learning trap can occur and becomes more likely the more signals he has already observed:

Proposition 9. *Assume Barney has the beta-distribution functional form given by equation (3). Fix payoffs $u(\mu, \omega)$, rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$. Suppose Barney is prospective-acceptive and retrospective-pooling.*

1. *For all $\bar{p} < 1$, there exists $\bar{c} > 0$ such that for all $c \leq \bar{c}$, Barney buys an infinite number of signals with probability $p > \bar{p}$. Furthermore, suppose that Barney, before buying any signals, has a positive probability of buying an infinite number of signals. Then for any $\varepsilon > 0$, there exists $N_\varepsilon > 2$ such that if Barney buys an additional signal after having already bought N_ε signals, the probability of Barney buying a finite number of signals from then on is less than ε .*
2. *Suppose $\theta_A = 1 - \theta_B$. Suppose Barney is willing to buy an additional signal when his posterior probability (of state A) is equal to q , and suppose Barney's posterior is q after observing N signals. If Barney's posterior probability of state A is q after observing $N' > N$ signals, then the probability that Barney will buy an infinite number of signals is weakly higher after he has observed the N' signals than it was after the N signals.*
3. *Again, suppose $\theta_A = 1 - \theta_B$ and the prior $f_\Theta(\theta_A) \geq .5$. For any $\varepsilon > 0$, there exists $N > 2$ such that if Barney chooses an action after buying at least N signals, then the likelihood ratio of Barney having taken the action that does not match the state to the action that matches the state is less than ε .*

If Barney ends up in a learning trap, then his welfare is unboundedly negative. Because a small c tempts him to wait longer, the probability of a learning trap becomes arbitrarily close to 1 as the signal cost c becomes arbitrarily small.³⁰

³⁰ Although we prove the results of Proposition 9, and of Proposition 10 in the next subsection, using our parameterized model of Barney, we believe the results in both propositions extend to any distribution satisfying A1-A4. Furthermore, if A1 is replaced with A1', then there exist situations in which Barney will purchase an infinite number of signals (in dynamic information acquisition) or take a sub-optimal action in every period (in experimentation) with probability 1—unlike in the current propositions, where these possibilities can occur but always with probability

The second part of the proposition states that, for a given c , Barney is more likely to get caught in a learning trap the more signals he has already observed, holding constant his posterior belief. For example, if the rates θ_A and θ_B are symmetric, then an equal number of a and b signals does not change Barney’s beliefs; hence Barney is more likely to end up purchasing an infinite number of signals after having observed $ababab$ than he was before he observed any signals.^{31,32}

This result points to a more general implication of NBLLN that emerges across a range of dynamic applications: it makes a bigger difference to his eventual beliefs for Barney than for Tommy whether he happens to observe strong evidence of the true state early or late in his learning process. The basic logic of NBLLN implies that Barney finds observing strong, early evidence, such as the group of signals aa , more persuasive that A is the true state than a group such as $abababaa$, even though these two groups are, objectively, similarly strong signals about the state (indeed, exactly equally strong when $1 - \theta_A = \theta_B$). As a result, if Barney observes strong evidence early on, he may stop trying to learn about the state after only a few signals, while if he observes ambiguous data early on, he may continue trying to learn even after many signals.

This insight also relates to the third part of Proposition 9 (which we prove for the case where Barney’s prior weakly favors the true state, although we believe it holds more generally). It states that if Barney *does* eventually stop purchasing signals, then the probability that he chooses the correct action converges to 1 as the number of signals increases. This is *not* true for Tommy. Tommy stops when the difference between the number of a -signals and b -signals exits some region, and the probability of observing any given difference is independent of the total number of signals observed. In contrast, Barney stops purchasing signals when the *proportion* of a -signals exits some region, and as the number of signals increases, this proportion becomes arbitrarily more likely to cross the threshold that favors the true state than to cross the threshold that favors the false state.

Perhaps surprisingly, an outsider observer who observes the agent purchasing a large number of

strictly less than 1. That is because, under A1’, Barney’s likelihood ratio is bounded away from zero and infinity. Hence Barney could be in a situation where no infinite sequence of signal realizations would affect his action (even though he perpetually believes there exist sequences that would).

³¹Furthermore, elaborating on this second part: there exists a \hat{N} such that for all $N' > \hat{N}$, if Barney’s posterior probability of state A is q after observing N' signals, then the probability that Barney will buy an infinite number of signals is *strictly* higher after he has observed the N' signals than after the N signals.

³²We have found several experiments that set up a dynamic information-purchase setting with a payoff structure similar to the model in the text and that compare subjects’ behavior with a Bayesian benchmark, which is calculated assuming expected-value maximization. Tversky and Edwards (1966), Pitz (1968), Wendt (1969), and Hershman and Levine (1970) found that subjects purchased too much information. In contrast, Fried and Peterson (1969) and Pitz and Barrett (1969) found that subjects purchased too little information. Moreover, Pitz and Barrett found that when the already-observed sample size was larger, holding constant the objective strength of evidence, subjects bought *fewer* additional signals. Also contrary to our model’s prediction, Sanders and Ter Linden (1967) Studies 1-3 found that, when the already-observed sample size was larger, subjects stopped acquiring information at a point where the objective evidence was weaker. In Sanders and Ter Linden’s experiments, however, the signals arrived at a rate of 2, 5, or 10 signals per second, which is so fast that the nature of the inference task is likely quite different than in other studies.

signals (but does not observe the realizations) should be more confident in betting that the agent took the correct action if the agent was Barney rather than Tommy. Tommy would only have purchased a large number of signals if the evidence were ambiguous up until the very end. Barney, in contrast, having observed a large number of signals, only takes an action when the cumulative evidence from many signals is overwhelming.

5.4 Experimentation and Learning About Oneself

Rather than purchasing signals about the quality of the car, Barney could instead take them for test drives, perhaps by renting them. Here, instead of an explicit cost, the cost of information acquisition is the cost of waiting to purchase the correct car. If the Volvo is the better car, then Barney is losing out every day he drives the Lada. For the same reasons that retrospective-pooling Barney could get caught in a learning trap in a dynamic information-acquisition setting, in such an experimentation setting he could end up remaining forever uncertain about the state. In our working paper (Benjamin, Rabin, and Raymond, 2012), we show that Barney could forever take an action that provides a suboptimal flow payoff in the mistaken expectation that the action will eventually provide useful information.

The failure to learn the truth from a great deal of feedback about the outcomes from one’s own actions has important ramifications for people’s beliefs about themselves. While there are reasons unrelated to NBLLN for why people have optimistic priors about their own abilities and preferences, we believe that NBLLN acts an “enabling bias” that explains how, despite a lifetime of experience with themselves, people remain uncertain about their own type, and optimistic priors do not give way to more realistic self-assessments. For example, NBLLN may explain why people persist in being overoptimistic about their ability on tasks that they regularly engage in. NBLLN may also explain why people remain uncertain regarding their own altruistic preferences—an otherwise-puzzling lack of knowledge that is a crucial ingredient for self-signaling to help explain altruistic behavior.

We formalize these ideas in a simple, two-action experimentation environment. The agent’s type is the state of the world, $\omega \in \{A, B\}$, about which the agent is uncertain. In each of an infinite number of periods $t = 1, 2, \dots$, the agent takes an action $\mu_t \in \{A, B\}$. After taking an action, the agent receives either a high payoff, $u^H(\mu_t)$, or a low payoff, $u^L(\mu_t) \leq u^H(\mu_t)$. The agent receives the payoff $u^H(\mu_t)$ with probability $\theta_\omega \in (.5, 1)$ and $u^L(\mu_t)$ with probability $1 - \theta_\omega$, and hence this outcome serves not only as a payoff but also as a signal about the state. If the agent is of type A , she earns a higher expected payoff from taking action A , while if she is of type B , she earns a higher expected payoff from taking action B ; formally, $\theta_A(u^H(A) - u^H(B)) \geq (1 - \theta_A)(u^L(B) - u^L(A))$ and

$\theta_B(u^H(B) - u^H(A)) \geq (1 - \theta_B)(u^L(A) - u^L(B))$. The agent discounts the future at rate $0 < \delta < 1$.

The first part of Proposition 10 examines the case where both actions are informative. As is well known, Tommy learns the true state and eventually takes his best action in every period. For Barney, if he is retrospective-pooling, then—following the logic of Proposition 2—his beliefs will converge to a limit posterior at which he will remain uncertain about the state. Moreover, the stronger his prior in favor of one of the states, the stronger his limit posterior in favor of that state. If Barney’s prior is strong enough, then even if it is incorrect, it may drive his actions in every period. This may describe situations where people begin with strong, but possibly incorrect, intuitions about their own abilities or preferences, and these intuitions are never fully corrected by experience.

Proposition 10. *Assume Barney has the beta-distribution functional form given by equation (3). Fix payoff functions $u^H(\mu_t)$ and $u^L(\mu_t)$, rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and discount factor δ . Suppose Barney is prospective-acceptive and retrospective-pooling.*

1. *Suppose $u^H(A) > u^L(A)$ and $u^H(B) > u^L(B)$. Without loss of generality, suppose the state is $\omega = B$. For all priors $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, Tommy’s belief that the state is B converges to 1 almost surely, and Tommy’s action converges to B almost surely. Barney’s belief that the state is B converges almost surely to a number in the interval $(0, 1)$, and this limit posterior is increasing in $f_\Theta(\theta_B)$. Moreover, if $f_\Theta(\theta_B)$ is sufficiently small, then there is positive probability that Barney takes action A in every period.*
2. *Suppose $u^H(A) > u^L(A)$ and $u^H(B) = u^L(B)$. For all priors $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, if the state is B , then almost surely at some finite T , Tommy will take action B for all periods $t \geq T$. For Barney, regardless of the state, there exists $0 < \underline{p} < 1$ such that for any prior $f_\Theta(\theta_A) \geq \underline{p}$, there is positive probability that Barney takes action A in every period. This probability is increasing in $f_\Theta(\theta_A)$ but is always strictly less than 1.*

The second part of the proposition considers a special case of the experimentation environment, a one-armed bandit problem that can be used to study learning about one’s own self-control. In state A , the agent can successfully exert self-control when faced with temptation, while in state B , the agent cannot resist temptation. In each period, the “risky” action, A , is to expose himself to temptation (e.g., buying potato chips at the supermarket to eat at home). The “safe” (or “commitment”) action, B , is to avoid the temptation (e.g., not buying the chips). The risky action has a higher payoff than the safe action in state A but a lower payoff in state B . A special feature of this setting is that if the agent chooses the safe action, thereby avoiding the tempting situation, he does not get any information about his own self-control.

This setting is a simplified version of the Planner-Doer model of learning self-control that Ali (2011) analyzes for the case of Tommy. Almost surely at some finite T , a Tommy without self-control will—regardless of his prior—take the safe action for all periods $t \geq T$. While it is possible that a Tommy who has self-control will always take the safe action and therefore never learn that he has self-control, the only Tommys who will take the risky action in the long run are Tommys with self-control. Most of us have the intuition that, despite repeatedly exposing themselves to temptation and succumbing to it, people remain perpetually optimistic about their own self-control—and yet this result for Tommy is the other way around!

In contrast, Proposition 10 shows that if Barney begins with a sufficiently strong belief in his own self-control ability, then it is possible that he will continue to take the risky action even in the long run, in the face of overwhelming evidence that he actually lacks self-control. Moreover, the likelihood that this occurs is increasing in his ex ante optimism.

6 Concluding Remarks

We have explored a range of applications where we think NBLLN matters. Yet for some other applications where NBLLN may matter, different or additional assumptions will be needed to close the model. For example, consider observational learning. Besides learning from his own experience or from gathered information (two situations we focused on in Section 5), an agent could also choose which car to buy by observing which car his neighbors have bought. In the theoretical literature, it is assumed that an infinite sequence of Tommys with common prior beliefs about the state each in turn observes a single signal along with the history of previous actions and then chooses his own action. Each receives a payoff equal to 1 if his action matches the state and 0 otherwise. Much of this work has emphasized that with probability 1, after some date each agent ignores his private signal, and a “herd” forms, with all agents thereafter choosing the same action (e.g., Banerjee, 1992; Bikhchandani, Hirshleifer and Welch, 1992). Although no agent directly observes more than one signal, NBLLN will influence observational learning if agents apply their non-belief to the signals they infer others are getting. In this setting, we think that the most psychologically plausible retrospective signal-processing assumption is that Barney groups all previous agents’ actions together and treats his own private signal, which is more vivid, as separate. In addition to such a grouping assumption, formally deriving the implications of NBLLN in this setting requires an extension of the framework introduced in Section 4 to address what Barney’s theory is as to how *other* agents draw inferences.

Our working paper (Benjamin, Rabin, and Raymond, 2012) contains a discussion of some

possible ways of handling this issue, along with a formal analysis of observational learning. The results parallel those for dynamic information acquisition from Section 5. Intuitively, if Barney groups together previous agents’ actions, then NBLLN implies that he will infer less from their actions than he should. Therefore, he will more often rely on his own signal, making it slower for a herd to *begin*, especially if early evidence is mixed. Moreover, since Barney needs to see more agents following their own signal before ignoring signals than does Tommy, when a herd *does* occur, it is more likely to be on the correct action. Finally, because Barney’s learning is limited even in an infinite sample, a qualitatively different kind of behavior is possible. If agents do not herd quickly enough, they will instead form what we call an “eddy”: following some period, every agent chooses the action corresponding to his own signal.

Another strategic setting in which NBLLN can matter a great deal is persuasion. Barney is again uncertain whether to buy a Volvo or a Lada. A Lada saleswoman has observed N signals about which car is better and can choose how to clump these signals when revealing them to Barney. If the saleswoman must reveal all the signals, and if Barney is retrospective-acceptive but unaware of his own NBLLN—and hence does not realize that the clumping of signals will affect his beliefs—then Barney will not draw any inferences about the state from the saleswoman’s behavior. In that case, she can maximally move Barney’s beliefs in favor of the Lada by clumping all the pro-Volvo signals together and separating out each pro-Lada signal. For any two distinct rates and any priors, if N is sufficiently large, then the saleswoman can make Barney arbitrarily confident that the Lada is superior.

There are conceptualizations of the tendency to under-infer from large samples that differ from that embedded in our model. One interpretation proposed for many cognitive biases is “ecological mismatch”: while a person’s thought process leads to biased beliefs for i.i.d. processes studied in the laboratory, the same thought process would generate appropriate beliefs for the typical, real-world random processes people encounter. For example, in the case of under-inference, Winkler and Murphy (1973) posit that people may treat independent signals as if they were positively correlated because their real-world experience is with positively correlated signals. Such positive correlation would generate excessively-dispersed subjective sampling distributions and under-inference but *not* NBLLN (because the Law of Large Numbers still “works” for positively correlated signals under mild regularity conditions; see Hu, Rosalsky, and Volodin, 2008). Moreover, while ecological-mismatch arguments often have merit, we think the argument is unappealing in this context because the bias we call NBLLN is evident in examples with which subjects have a great deal of real-world experience, such as coin-flipping.³³

³³We also note that in the case of the Law of Small Numbers, the opposite ecological-mismatch hypothesis

Many have proposed conceptualizing under-inference in large samples as one consequence of the “representativeness heuristic,” according to which people draw inferences based on the degree of similarity between features of a sample and features of a population from which the sample might have been drawn. Indeed, Kahneman and Tversky (1972) present evidence for what we call NBLLN in precisely this context. Although NBLLN certainly seems consistent with representativeness, it is not clear how the logic of representativeness predicts the prototypical case of under-inference: e.g., an agent who observes 600 heads and 400 tails continues to put non-trivial probability on the coin being fair. Representativeness could explain this kind of observation if it is interpreted as inferences based on proportions, combined with the additional assumptions of reasonably accurate inferences in small samples and insensitivity to sample size, but that combination of assumptions essentially amounts to our model.

A natural alternative modeling approach would be to build a theory of “sample-size neglect,” in which, loosely speaking, an agent forms beliefs about a sample of any size as if it were a “medium-sized” sample of, say, size 7. Such a model would imply under-inference for sample sizes larger than 7 and over-inference for sample sizes smaller than 7. This is a common conceptualization and one which we found compelling enough to consider as our first (and more parsimonious) approach. But we have come to the view that NBLLN and LSN are distinct phenomena. LSN is inherently linked to the gambler’s fallacy, the incorrect belief that in i.i.d. coin flips, a head becomes less likely than 50% following a streak of heads. Moreover, the gambler’s fallacy has as much force in large samples as in small samples. For example, Benjamin, Moore, and Rabin (2012) found that people think that the probability of a head following a streak of 9 heads from a fair coin is only 32%. If NBLLN were (like LSN) linked to beliefs about sequences of random events, then people would have to believe that a long streak of heads makes a subsequent head *more* likely.³⁴ Instead, evidence and intuition suggest that NBLLN is not due to *any* belief people have about the likelihood of particular sequences of random events. Rather, it is due to an erroneous understanding of how the likelihoods of sequences “aggregate up” into sampling distributions.

Indeed, we have come to the view that the most psychologically compelling alternative approach is often proposed: that people ordinarily deal with *negatively*-autocorrelated signals. Typical real-world processes would have to have a fairly complicated form involving short-run negative autocorrelation and long-run positive autocorrelation to rationalize both the Law of Small Numbers and NBLLN.

³⁴One could argue that the sample-size neglect theory, when linked to beliefs about sequences of random events, provides a parsimonious account of both the gambler’s fallacy and its apparent opposite, the “hot hand fallacy.” This is a false parsimony, however, because shoe-horning the gambler’s fallacy and the hot hand fallacy into the same psychological mechanism generates counterfactual predictions about when they occur. As noted, the gambler’s fallacy occurs even after a long streak of heads, and as far as we are aware, the hot hand fallacy has never been observed for coins. Instead, the hot hand fallacy is usually understood as occurring in situations where an agent believes that the random process alternates between “hot” and “cold” rates.

to modeling NBLLN would attempt to capture people’s failure to realize just how many combinations of a and b signals generate proportions close to the population mean. While we know of no attempt to formulate NBLLN along such lines, we believe that such a model would share the main features and predictions of our model.

While the logic of NBLLN unambiguously predicts that people will extract far too little information from large samples, there are strands of literature both within psychology and within economics on “over-confidence” in beliefs. Rather than viewing over-confidence and under-confidence as fundamental biases in themselves, we view both as outcomes to be explained as a function of the information a person is confronted with. Our model of NBLLN highlights a feature of the decision-making environment—namely, sample size—that affects the degree to which an agent will draw too weak an inference from evidence. In Appendix A we combine NBLLN with LSN, which generates a bias toward over-confidence in inferences, and the overall pattern we predict is: approximately correct inference for samples of size 1, over-inference in small samples larger than 1, and under-inference in large samples. LSN will exacerbate people’s tendency to rely on smaller samples. In dynamic settings, LSN will make it more likely for an agent to stop acquiring information after just a few signals—but, if the initial evidence does not cause the agent to stop, the agent will still draw inferences based on sample proportions and may get caught in a learning trap.

Our model of NBLLN is defined only when the signals are i.i.d. and binomial. There are some natural approaches to modeling NBLLN for non-i.i.d. signal sequences. Consider a binomial random process defined by a mapping from any initial rate, θ_0 , and any history of t observed signals, h_t , into a rate that the $(t + 1)^{\text{st}}$ signal will be an a -signal, $\theta(\theta_0, h_t)$. When Barney knows the initial rate is θ_0 , he forms his beliefs as if the initial rate were β , a random variable drawn from distribution $f_{\beta|\Theta}^{\psi}(\beta|\theta_0)$. For the first signal in a group, he believes that the probability of an a -signal is β , and for the $(t + 1)^{\text{th}}$ signal within that group, he believes that the probability of an a -signal is $\theta(\beta, h_t)$. This modeling approach can be applied not only when the signals truly are non-i.i.d., but also when an agent falsely believes they are non-i.i.d. due to another psychological bias (as in LSN; see Appendix A).

There are also natural extensions of our modeling approach to non-binomial cases. Suppose, for example, that the signals are normally distributed i.i.d. with known mean μ and variance σ^2 . We can imagine a cousin of Barney believes instead that signals are generated by a two-stage process, where a subjective mean ν is drawn from some distribution centered at μ , and then the signals are drawn from a normal distribution with mean ν and variance σ^2 . While Tommy believes that the mean of a large random sample of signals will converge to a point mass at μ , Barney’s cousin believes it will converge to the density of ν . We could assume that the density of ν corresponds to

the empirical large-sample beliefs, or for analytical tractability, we could assume that ν follows the conjugate prior distribution for the normal distribution, which is itself a normal distribution.

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Appendix to: “A Model of Non-Belief in the Law of Large Numbers”

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Appendix A: A Combined Model of Non-Bayesian Updating

In this appendix, we briefly outline models of other biases besides NBLLN that also appear to matter for inference, focusing on the same single-clump context of Section 2. These models help organize our review of the evidence in Appendix B. Although these models are far more cursory and preliminary than our model of NBLLN, we hope that formalizing these alternatives can both crisply differentiate them from NBLLN—clarifying in particular that none of them are the “opposite” of or inconsistent with NBLLN—and also be suggestive of how to develop these other biases along lines we have done with NBLLN.

At first glance, NBLLN appears to be directly at odds with another bias in beliefs, the Law of Small Numbers (LSN): Tversky and Kahneman (1971) formulated the term “Law of Small Numbers” to refer to the idea that people exaggerate the likelihood that small samples will reflect the underlying population. While NBLLN generally leads to under-inference, LSN generates over-inference. However, these two biases are neither logically nor psychologically inconsistent. Indeed, we believe it is the combination of the two which has led judgment researchers to posit a bias of “sample-size neglect” in which people overestimate the resemblance of small samples and underestimate the resemblance of large samples as if they simply do not see the relevance of sample size. We re-interpret what appears to be sample-size neglect as a combination of these two biases, and we show how the basic under-inference implications of NBLLN goes through after LSN is accounted for.

According to LSN, people exaggerate how much small samples resemble the population. In Rabin’s (2002) model, an agent forms beliefs about N i.i.d. draws that have known rate θ as if signals were drawn without replacement from an “urn” of size M that contains exactly θM a -signals

and $(1 - \theta)M$ b -signals.¹ To make sure that the agent continues to view the draws as random even after many signals have been realized, it can be assumed that $M > 2$, and the urn is “renewed” every odd number of draws. In other words, every odd-numbered draw, the signal is drawn from a refilled urn of size M , and every even-numbered draw, the signal is drawn from the urn of size $M - 1$ that is depleted by the previous signal’s draw. An agent who believes in LSN is called Freddy. The parameter M governs the strength of LSN, with Freddy becoming Tommy in the parameter limit $M \rightarrow \infty$. In Rabin’s model, for a given θ , the parameter M , in addition to being an integer larger than 2, must satisfy the constraint that θM is an integer. This constraint becomes problematic when combining the model of LSN with our model of NBLLN. In our model of NBLLN, the agent thinks the outcome of a random sample is determined by the subjective rate β , which is drawn from a distribution with full support on either $(0, 1)$ or $[0, 1]$ (depending on whether A1 or A1’ holds). Hence for any M , βM will be non-integer-valued with probability 1. For this reason, we propose a variant of Rabin’s model that, while still requiring that M is an integer larger than 2, remains well-defined even when θM is not an integer.

Our variant of Rabin’s model of LSN is identical to Rabin’s model except that instead of believing that the “urn” contains θM a -signals and $(1 - \theta)M$ b -signals, Freddy believes that it contains \tilde{A} a -signals and $(M - \tilde{A})$ b -signals, where \tilde{A} is an integer-valued random variable that equals $j \in \{0, 1, \dots, M\}$ with probability $\binom{M}{j} \theta^j (1 - \theta)^{M-j}$. In words, Freddy thinks that signals are drawn without replacement from an urn of size M , but he believes the composition of the urn is random, with the “average” urn containing θM a -signals and $(1 - \theta)M$ b -signals. When the rate is known to be θ , Freddy believes that the number of a -signals in the urn is a binomial random variable with parameters (θ, M) .

NBLLN is the belief that the mean of a random sample converges to a non-trivial distribution, rather than a precise estimate of the mean of the population, in the limit as the sample size gets large. We refer to an agent who believes in both LSN and NBLLN as Barney-Freddy, with beliefs denoted by $f^{\psi M}$. Like Barney, he predicts that the sample is drawn according to a subjective rate that may not equal the true rate. We assume that the subjective rate $\beta \in [0, 1]$ is drawn from a density $f_{\beta|\Theta}^{\psi}(\beta|\theta)$ determined by θ and Barneyess parameter ψ that satisfies Assumptions A1-A4 from Section 2. In accordance with LSN, Barney-Freddy thinks that signals are drawn without replacement from an “urn” of size M that contains \tilde{A} a -signals and $(M - \tilde{A})$ b -signals, where \tilde{A} is a binomial random variable with parameters (β, M) . Hence Barney-Freddy thinks that the “average” urn contains βM a -signals and $(1 - \beta)M$ b -signals. The integer $M > 2$ parameterizes

¹Rabin and Vayanos (2010) improves on Rabin’s (2002) model that we discuss here, generalizing LSN beyond the binomial case and without assuming the signal-generating process is *i.i.d.* We explore this more contrived simple model of LSN here for ease of combining it with NBLLN.

the degree of belief in LSN, and we assume that the urn is “renewed” and \tilde{A} is re-drawn every odd number of signals. As usual for the model of NBLN, the same subjective rate β applies to the entire clump.

Barney-Freddy both believes in the gambler’s fallacy—that is, he expects recent a -signals to be followed by b ’s and vice-versa—and believes that the sample mean of a large population will converge toward a full-support limit distribution. He does not observe the subjective rate β , but conditional on any given β , Barney-Freddy expects that if the even-numbered draw t is more likely to be an a signal if the $(t - 1)^{st}$ draw was b than if it was a . Hence even without knowing β , Barney-Freddy excessively expects that a and b signals will alternate between an odd draw and an even draw. Consequently, for reasonable calibrations of ψ and M , Barney-Freddy’s subjective sampling distribution in a finite sample will be too peaked, putting too little weight in the tails. On the other hand, Lemma A1 states that Barney-Freddy’s beliefs about a large sample converge to a full-support limit distribution. Like for Barney, Barney-Freddy’s limit density will be equal to $f_{\beta|\Theta}^{\psi}(\beta|\theta)$, regardless of the degree of belief in LSN. Intuitively, every odd-even pair of draws will have, on average, proportion β of a -signals. Hence by the Law of Large Numbers, the sample as a whole will tend toward having proportion β of a -signals almost surely.

Lemma A1. *Barney-Freddy does not believe in LLN: For any $\theta \in \Theta$ and interval $[\alpha_1, \alpha_2] \subseteq [0, 1]$,*

$$\lim_{N \rightarrow \infty} \sum_{x=\lfloor \alpha_1 N \rfloor}^{\lfloor \alpha_2 N \rfloor} f_{S_N|\Theta}^{\psi M}(A_s = x|\theta) = F_{\beta|\Theta}^{\psi}(\beta = \alpha_2|\theta) - F_{\beta|\Theta}^{\psi}(\beta = \alpha_1|\theta) > 0.$$

Not only are LSN and NBLN mutually consistent, but LSN actually *magnifies* NBLN. For smaller M , Barney-Freddy believes that odd-even signal pairs more frequently alternate, and hence his subjective sampling distribution converges to the limit distribution more quickly.

While it is true that Freddiness generates over-inference while Barneyess tends to generate under-inference, there is a clear pattern to when Barney-Freddy over-infers and when he under-infers. For reasonable calibrated values of ψ and M , Barney-Freddy will over-infer from small samples. For any values of ψ and M , it follows immediately from Lemma A1 that Barney-Freddy will under-infer when the sample size N is sufficiently large.

A well-known bias is base-rate neglect (Kahneman and Tversky, 1973), an underweighting of prior probabilities in drawing inferences. Instead of assuming that an agent updates according to Bayes’s Rule applied to his subjective sampling distributions, we can capture base-rate neglect by

assuming that the agent draws inferences according to:

$$f_{S_N|\Theta}^{\psi Mb}(\theta_A|s) = \frac{f_{S_N|\Theta}^{\psi M}(s|\theta_A) f_{\Theta}(\theta_A)^b}{f_{S_N|\Theta}^{\psi \phi \gamma}(s|\theta_A) f_{\Theta}(\theta_A)^b + f_{S_N|\Theta}^{\psi \phi \gamma}(s|\theta_B) f_{\Theta}(\theta_B)^b}$$

and

$$f_{S_N|\Theta}^{\psi Mb}(\theta_B|s) = \frac{f_{S_N|\Theta}^{\psi M}(s|\theta_B) f_{\Theta}(\theta_B)^b}{f_{S_N|\Theta}^{\psi M}(s|\theta_A) f_{\Theta}(\theta_A)^b + f_{S_N|\Theta}^{\psi M}(s|\theta_B) f_{\Theta}(\theta_B)^b},$$

where $f_{S_N|\Theta}^{\psi M}(s|\theta_A)$ and $f_{S_N|\Theta}^{\psi M}(s|\theta_B)$ are the subjective sampling distributions of an agent with Barneyess parameter $0 < \psi < \infty$ and Freddiness parameter $M > 2$, and $0 \leq b \leq 1$ parameterizes the degree of base-rate neglect. If $b = 1$, these formulae specialize to Bayes' Rule, where there is no base-rate neglect. If $b = 0$, the agent ignores base rates altogether, treating any prior probabilities as if they were 50-50. This formulation of base-rate neglect has been previously adopted in empirical work by (e.g., Grether, 1980) and concurrently in theoretical work by Bodoh-Creed (2010). Applying it theoretically in dynamic settings raises many of the same conceptual issues as NBLLN; the way in which the agent processes groups of signals will matter a great deal in how his beliefs evolve. For that reason, we believe the framework we have begun to develop in this paper for analyzing dynamic NBLLN may be of use for analyzing dynamic base-rate neglect as well.

A simple explanation for some of the evidence on people's beliefs is "extreme-belief aversion," an aversion to holding beliefs that are close to certainty. Consider a discrete probability density function, $f_X(\cdot)$, that puts positive probability on a set of possible outcomes x_1, x_2, \dots, x_J . We capture the idea of extreme-belief aversion by defining a mapping from the true probability density, $f_X(\cdot)$, to a subjective probability density that is less extreme,

$$f_X^{\xi}(x_i) = \frac{.5 + \xi \cdot (f_X(x_i) - .5)}{\sum_{j=1}^J (.5 + \xi \cdot (f_X(x_j) - .5))}.$$

The parameter $0 < \xi < 1$ describes the degree of extreme-belief aversion, with smaller values corresponding to greater bias. If $\xi = 1$, the subjective probabilities coincide with the true probabilities, while if $\xi = 0$, all outcomes x_1, x_2, \dots, x_J are treated as equally likely.

One interpretation of the transformed probability, $f_X^{\xi}(x)$, is that it represents the agent's truly-held beliefs. Another interpretation is that the agent actually holds beliefs $f_X(x)$ but *reports* beliefs that are transformed by the ξ function. While the latter is certainly plausible when Tommy's beliefs are very extreme—we can easily imagine a person saying she is 99% sure when her true belief is .9999—it does not address the evidence that beliefs inferred from betting behavior also exhibit

under-inference.

Qualitatively, extreme-belief aversion can explain both the excessively-dispersed subjective sampling distributions and the under-inference evidence that we discuss in Appendix B. An agent with extreme-belief aversion will have a more dispersed subjective sampling distribution than Tommy in large samples by virtue of compressing beliefs away from 0 and 1. An agent with extreme-belief aversion will also under-infer in situations where Tommy’s inference would be extreme, as would almost always occur when the sample is large. Extreme-belief aversion taken alone, however, implies that in any two inference problems where Tommy’s posterior is the same, people would hold (or at least report) the same belief. There is evidence contradicting this implication, indicating that extreme-belief aversion is not the only deviation from Bayesian belief formation that is going on.

For example, consider the experiment in Griffin and Tversky (1992), where $\theta_A = .6$. Tommy’s inference depends only on the difference between the number of a -signals and the number of b -signals. However, when the sample is 4 a ’s and 1 b , subjects’ median belief in favor of θ_A is .80, while when the sample is 10 a ’s and 7 b ’s, subjects’ median belief in favor of θ_A is .60. Tommy’s belief would be .77 in both cases, so people are under-inferring from the sample of size 17 and actually slightly over-inferring from the sample of size 5. Consistent with Proposition 5—but inconsistent with extreme-belief aversion being the only bias in beliefs—people infer less from the same difference in a and b signals when the sample is larger. Kahneman and Tversky (1972) and Kraemer and Weber (2004) also report evidence that beliefs are sensitive to sample size, holding constant the difference in the number of a and b signals.

While extreme-belief aversion may help to describe the evidence on people’s beliefs, and it may be a confound for other interpretations of biased beliefs, we do not review evidence for extreme-belief aversion, and we know of no evidence that our crude formulation is a close match for people’s thinking. Extreme-belief aversion may also lead to internal-inconsistency modeling challenges that we do not address. For example, it seems reasonable to assume that the transformation above could be applied to an agent’s sampling distribution or to the agent’s inferences, depending on which beliefs are being elicited. In that case, however, subjective sampling distributions and inferences will not in general be linked by Bayes’ Rule.

Combining the three biases above with NBLLN gives a complicated model that captures many features that could be applied to predict beliefs and behavior in economic settings. One insight that comes immediately out of the combination is that base-rate neglect—i.e., underweighting priors—is not the opposite of, or contradictory to, the way NBLLN leads people to underweight likelihood information. Indeed, as we have noted in some discussions above about vividness and other biases, NBLLN is, especially in understanding “multi-clump” information processing, likely a contributor

to the relevance of other biases. In many information-rich environments where full Bayesians would correctly become very confident independent of their priors, NBLLN is necessary for the question of whether people neglect base rates to be relevant. At the same time, we show above why NBLLN means that unless people completely neglect base rates, people’s initial beliefs matter even in the long run. In combination, in fact, NBLLN suggests that it is possible that the real and often important fact that people under-use base rates may be consistent with the possibility that base rates matter more for social and economic phenomena than fully rational models have supposed that they do.

One last bias sits less comfortably with the others, and is harder to integrate. Many experiments make clear that peoples’ subjective sampling distributions have flatter tails than our model of NBLLN by itself can explain. We attribute the flatness to “sampling-distribution-tails diminishing sensitivity (SDTDS),” a bias in which people perceive very unlikely outcomes as similar to each other and hence similar in probability. Consider 10 flips of a coin that is biased .8 in favor of heads. We know of no direct evidence but conjecture that most people would judge the likelihood of observing 1 head as very close to the likelihood of observing 0 heads, even though observing 1 head is actually 40 times more likely.

SDTDS can be formalized by assuming that an agent forms beliefs as if the likelihood of sample realizations far from the average are more similar to each other (and to the average) than they truly are. For a sample of size N from a θ -biased coin, let σ denote the standard deviation of the sample proportion $\frac{A_N}{N}$ (which equals $\sqrt{\frac{\theta(1-\theta)}{N}}$ for Tommy but not for Barney-Freddy). Let the perceived distance between the realized $\frac{A_N}{N}$ and θ be $\sigma\gamma\left(\frac{\frac{A_N}{N}-\theta}{\sigma}\right)$, where the twice-differentiable “sample-perception function” $\gamma : (-1, 1) \rightarrow (-1, 1)$ has the properties (1) $\gamma(0) = 0$, (2) $0 < \gamma' < 1$, (3) $\gamma''(x) < 0$ for all $x > 0$, and $\gamma''(x) > 0$ for all $x < 0$, and (4) $\gamma(x) = -\gamma(-x)$ for all x .

Property (1) says that the agent perceives a sample proportion of θ accurately. Property (2) ensures that the agent perceives any other sample proportion as more similar to θ than it actually is. The key concavity and convexity assumption (3) means that neighboring samples are perceived as more similar to each other the further they are from θ . For this reason, the agent makes little distinction between outcomes that fall far in the tails of his subjective sampling distribution. Property (4) specifies that γ is symmetric around 0 so that the misperception is symmetric around θ .

Roughly speaking, a person who exhibits SDTDS with sample-perception function γ judges the probability of a sample $s \in S_N$ with proportion $\frac{A_N}{N}$ of a -signals as if it were the sample $\Lambda(s)$, which has proportion $\theta + \sigma\gamma\left(\frac{\frac{A_N}{N}-\theta}{\sigma}\right)$ a -signals. Formally, $f_{S_N|\Theta}^{\psi\phi\gamma}(s|\theta) = \frac{f_{S_N|\Theta}^{\psi\phi}(\Lambda(s)|\theta)}{\sum_{s' \in S_N} f_{S_N|\Theta}^{\psi\phi}(\Lambda(s')|\theta)}$, where $f_{S_N|\Theta}^{\psi\phi}$ is the subjective sampling distribution for $\psi\phi$ -Barney-Freddy, and the denominator is a

normalization that ensures that the subjective sampling distribution adds up to 1.² While NBLLN by itself leads to subjective sampling distributions that have flat tails, SDTDS is an additional force for flat tails. When the rate θ is not .5, the mean of the agent’s subjective sampling distribution no longer equals the mean of the objective sampling distribution because the “long tail” of the distribution is overweighted. Both of these features—flatter tails than can be accommodated with a reasonably-calibrated model of NBLLN and a mean shifted toward the long tail—are present in the subjective sampling distributions measured by Kahneman and Tversky (1972).

Unlike both our model of NBLLN and the Rabin (2002) model of LSN (and consequently also unlike the model of Barney-Freddy presented in this Appendix), SDTDS predicts under-inference from samples of size 1. In fact, the evidence we review in Appendix B suggests that people *do* under-infer in a sample of size 1. We note, however, that no experiment has disentangled the SDTDS explanation of under-inference from an alternative explanation of extreme-belief aversion. Nonetheless, in inference problems with normally-distributed signals, we suspect that SDTDS does help explain why subjects under-infer in a sample of size 1 when the realized signal is extreme. In contrast, when signals are binomial as in our analysis in the main text, we believe that the psychology of SDTDS — extreme events in the same tail of a distribution being judged as similar in probability — is unlikely to apply because there are no two extreme events in the same tail. Moreover, if SDTDS predicts under-inference from a sample of size 1, then it necessarily also implies incorrect predictions about the likelihood of a single signal realization, an implication that we view as implausible when signals are binomial.

Unfortunately, our formulation of SDTDS cannot be so easily integrated with NBLLN—or with a model combining NBLLN with LSN and base-rate neglect—because it does not arm somebody with a theory of the sequence of signals, only the frequency of signals within a sample. While surely a form of it could be specified that embeds a concrete theory of permutations within the sample (allowing for instance a person to believe all different sequences are equally likely), we do not know that the existing evidence provides a guide, nor do we believe the psychology underlying it translates easily into situations where an agent is cognizant of the ordering of signals. Moreover, an improved model or a better formulation than we have found to interpret existing evidence may be more compatible with the models of other biases than we have supposed.

²In general, $\Lambda(s)$ could be a sample outside the support of the objective sampling distribution. For example, it may be a sample with 3.2 a ’s. The expression above is nonetheless well-defined as long as the density $f_{S_N|\Theta}^{\psi\phi}(\cdot|\theta)$ can be evaluated at that sample, as it can for a binomial objective sampling distribution.

Appendix B: Experimental Evidence

In this appendix, we report on all papers we could identify with experimental results related to prediction and inference for binomial signals. Table B1 lists the papers, whether their evidence relates to prediction or inference, their experimental subject population, and their incentive structure. Most of the studies we review did not incentivize subjects' responses; we will also discuss how the evidence from the few incentivized experiments relates to the unincentivized studies.

Table B1. Experimental evidence on NBLN for binomial signals.

Author(s)	Year	Prediction or Inference?	Subjects	Incentives?
Beach, Wise, and Barclay	1970	inference	169 male undergrads	no
Camerer	1987	inference	74 undergrads	financial market
Chinnis and Peterson	1968	inference	40 male undergrads	no
Dave and Wolfe	2003	inference	40 undergrads	BDM for probability
DeSwart	1972	inference	21 male undergrads	no
DeSwart	1972	inference	18 male undergrads	no
Donnell and DuCharme	1975	inference	24 male undergrads	no
Gettys and Manley: Study 1	1968	inference	20 undergrads	no
Gettys and Manley: Study 2	1968	inference	28 undergrads	no
Green, Halbert, and Robinson	1965	inference	32 grad students	paid for guess about state
Grether	1980	inference	341 undergrads	paid for guess about state
Grether: Studies 1 and 2	1992	inference	97 undergrads	paid for guess about state
Grether: Study 3	1992	inference	55 summer students	BDM for probability
Griffin and Tversky: Study 1	1992	inference	35 undergrads	paid for accurate posterior
Griffin and Tversky: Study 2	1992	inference	40 undergrads	paid for accurate posterior
Griffin and Tversky: Study 3	1992	inference	50 undergrads	no
Kahneman and Tversky: prediction	1972	prediction	unclear	no
Kahneman and Tversky: inference	1972	inference	560 high school students	no
Kraemer and Weber	2004	inference	51 students (most grad)	paid for accurate posterior
Marks and Clarkson	1972	inference	68 undergrads	no
Nelson, Bloomfield, Hales, and Libby: Study 1	2001	inference	27 MBA students	financial market
Peterson and Miller	1965	inference	42 undergrads	no
Peterson and Swensson: Study 1	1968	inference	15 male undergrads	no
Peterson and Swensson: Study 2	1968	inference	18 male undergrads	no
Peterson, DuCharme, and Edwards: Study 1	1968	prediction	41 male undergrads	no
Peterson, DuCharme, and Edwards: Study 2	1968	both	24 male undergrads	no
Peterson, Schneider, and Miller	1965	inference	44 undergrads	no

Author(s)	Year	Prediction or Inference?	Subjects	Incentives?
Phillips and Edwards: Study 1	1966	inference	5 male undergrads	no
Phillips and Edwards: Study 2	1966	inference	48 male undergrads	paid for accurate posterior
Phillips and Edwards: Study 3	1966	inference	48 male undergrads	no
Pitz	1967	inference	28 undergrads	no
Sanders	1968	inference	32 undergrads	bets on the state
Sasaki and Kawagoe	2007	inference	1033 employees	no
Strub	1969	inference	12 male undergrads	paid for guess about state
Teigen: Study 1	1974	prediction	22 undergrads	no
Teigen: Study 2	1974	prediction	73 undergrads	no
Wheeler and Beach	1968	both	17 male undergrads	paid for beliefs (prediction), and bets (inference)

0.1 Evidence on Subjective Sampling Distributions

We begin by assessing subjective sampling distributions. We have found only 6 experiments from 4 papers in which researchers explicitly elicited experimental participants’ beliefs about the likelihood of each possible sample.³ None of these elicitations were incentivized. For 5 of these studies—Kahneman and Tversky (1972), Peterson, DuCharme and Edwards’s (1968) Study 1, Wheeler and Beach (1968), and both of Teigen’s (1974) studies—the data are displayed in the paper, and we have reproduced the graphs in Figures 1, B4, B5, and B1, respectively (we display both of Teigen’s studies together), shown below in the order we discuss them.

Among the papers, Kahneman and Tversky (1972) elicited sample-proportion beliefs for the largest sample sizes. As discussed in the Introduction, they find that subjective sampling distributions are “constant in proportions” for $N = 10, 100$, and 1000 . There is no noticeable tightening of the distribution even for $N = 1000$; while in fact there is less than a .01 chance of the proportion of heads falling outside the range 45% to 55%, subjects’ distributions assign probability .79 to a proportion outside that range.⁴

A straightforward implication of our model of NBLLN is that subjective sampling distributions will be flatter than the objective sampling distributions. In all 7 experiments, with the exception of the $N = 3$ conditions of one experiment, the researchers indeed concluded that subjective sampling distributions are excessively close to uniform.⁵

³Cohen and Hansel (1955) also elicited subjective sampling distributions, but we cannot compare their data with our model because they did not tell their subjects the rate that was generating the signals.

⁴At the time of this draft, two of the authors (Benjamin and Rabin, in joint work with Don Moore) have also collected data on people’s subjective sampling distributions for $N = 10$ and 1000 . We designed the experiment to measure subjective sampling distributions in several different ways to deal with potential confounds such as extreme-belief aversion. When we elicit distributions in the same manner as Kahneman and Tversky, we replicate their results almost exactly. While our preliminary findings support NBLLN for $N = 1000$, we also find that the evidence for NBLLN for $N = 10$ —and hence presumably also the evidence about smaller sample sizes reviewed below—is confounded by other explanations.

⁵Unlike the other 6 studies, Teigen (1974) asked subjects about the probability of each possible outcome

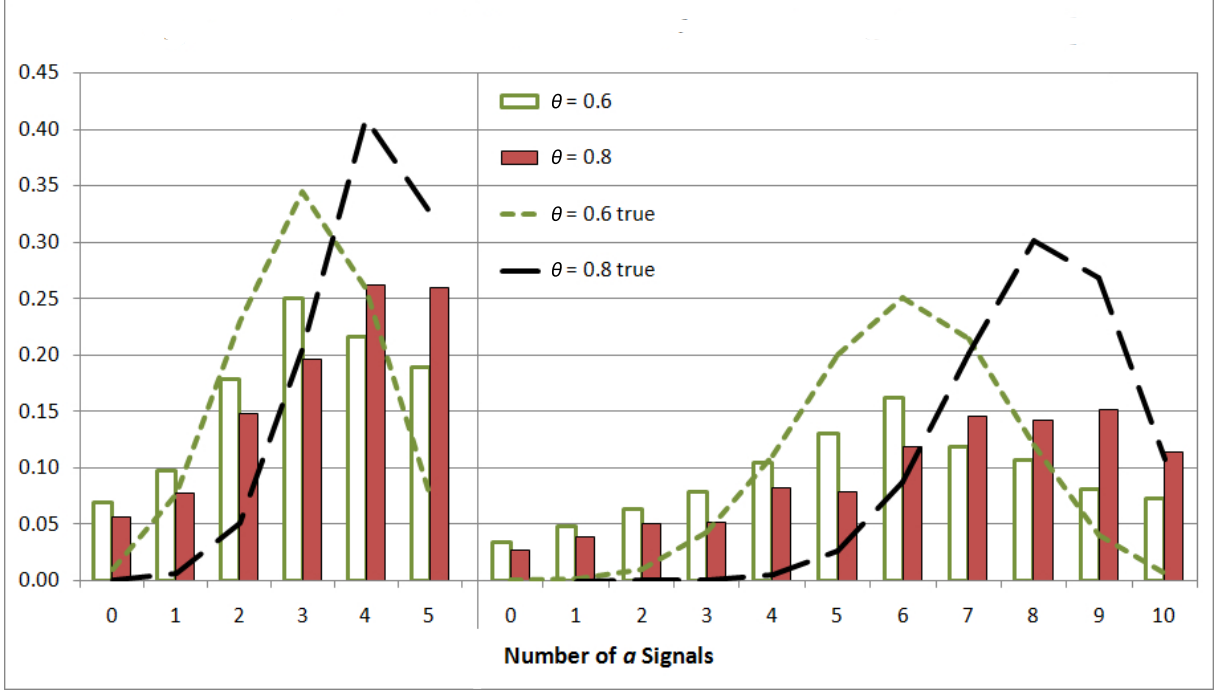


Figure B1: Median probability estimates, $N=5$ and $N=10$ (Teigen, 1974)

A feature of our model of NBLN is that people have correct beliefs about samples of size 1. We know of no evidence on this point, but our own introspection suggests it is virtually a tautology that if a person knows the rate of a -signals is θ , the person thinks the probability is θ that a single draw will be a .⁶ However, our model of NBLN implies that for any sample size larger than 1, no matter how small, the subjective sampling distribution will be too close to uniform. Peterson, DuCharme and Edwards's (1968) Study 1 $N = 3$ conditions are the only cases where researchers found subjective sampling distributions that are *not* too flat. For all three rates they studied ($\theta = .6, .7, .8$), the $N = 3$ subjective sampling distribution nearly coincides with the objective sampling distribution. We interpret this evidence as consistent with the combined effects of NBLN and LSN, rather than either of the biases considered alone. The combined model we discuss in Appendix A predicts that subjective sampling distributions will be correct for $N = 1$ and too flat for N large. For plausible parameter values, there will be a non-monotonicity for intermediate N : due to LSN, the subjective sampling distribution may be too peaked when N is larger than 1 but small. The relative strength of NBLN grows with N , so the subjective sampling distribution may be nearly

separately, without requiring that the probabilities sum to 1. He found that the probabilities summed to greater than 1, and subjects often assigned probabilities that were too high to every outcome. When the subjective sampling distributions were normalized to sum to 1, they were too flat.

⁶Of course there is a good reason no one has done this experiment: the correct answer about the probability of an a -signal is obvious if the experimenter has just said that it is θ . Nonetheless, we discuss below evidence from experiments on inference from a single signal. The correct answer to an inference problem is not obvious, and indeed subjects often do not get the correct answer, tending somewhat to under-infer on average.

correct for some intermediate N .

Assumption A4 of our model is that the subjective sampling distribution has the same mean as the objective sampling distribution, regardless of sample size. In contrast, the evidence indicates that when $\theta \neq .5$, the mean of the subjective sampling distribution is generally between the objective mean, θN , and $.5$. Kahneman and Tversky (1972) explicitly comment that “the mean is displaced towards the long tail” (p.440), and this pattern is visually evident in all of the figures except the $N = 3$ cases discussed above. From a modeling perspective, we believe it is appropriate for our model to have the counterfactual feature of an accurate mean for the subjective sampling distribution because it allows us to draw out the implications of believing that the limit distribution has full support, without mixing in the implications of an inaccurate mean. Moreover, we speculate that the “displaced mean” is the result of a psychologically distinct bias, “sampling-distribution-tails diminishing sensitivity (SDTSD),” sketched in Appendix A, and our calibrated model of NBLLN generates subjective sampling distributions that come closer to matching the data in Figure 1 when we additionally incorporate SDTSD into the model (calculations not shown).

There is mixed evidence about whether training and feedback affect subjective sampling distributions. Wheeler and Beach (1968) elicited subjective sampling distributions for a sample of size $N = 8$ for rates $\theta = .6$ and $.8$; these are shown in Figure B5. Next, their subjects were faced with 100 asymmetric binomial inference problems ($\theta_A = .8$ and $\theta_B = .4$). After each problem, the subject was told the true rate for that problem. Subjective sampling distributions were elicited again, the subjects responded to 100 more symmetric binomial inference problems with feedback, and the subjective sampling distributions were elicited a final time.⁷ Comparing the initial subjective sampling distributions with the final ones, the final subjective sampling distributions are less flat. For $\theta = .8$, the final subjective sampling distribution is quite close to the objective sampling distribution. For $\theta = .6$, the final subjective sampling distribution is actually more peaked than the objective sampling distribution. On the other hand, Peterson, DuCharme, and Edwards (1968, Study 2) conducted an experiment with four stages: (1) subjects were faced with symmetric binomial inference problems (with no feedback); (2) subjective sampling distributions were elicited for each combination of $N = 3, 5, 8$, and $\theta = .6, .7, .8$; (3) subjects were shown the objective sampling distributions; and (4) subjects were faced with another series of symmetric binomial inference problems. Subjects’ responses in the inference problems were similar in stage 4 as in stage 1, suggesting that showing subjects the objective sampling distributions had little effect on beliefs.

⁷These subsequent elicitations are not shown in Figure B5.

0.2 Evidence on Inference

We have found 33 studies from 26 papers measuring inferences from samples about which of two equally-likely rates, θ_A and θ_B , generated the samples. Most of these binomial inference problems are symmetric in the sense that $\theta_A = 1 - \theta_B$. Unless otherwise noted, all the studies we mention are symmetric. We focus first on the studies where the prior probabilities, $f_{\Theta}(\theta_A)$ and $f_{\Theta}(\theta_B)$, are equal. Equal priors neutralizes the role of base-rate neglect. We study below how inferences are affected by unequal priors.

To compare the degree of under- or over-inference across studies, note that Bayes' Rule can be written as $\frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} = \frac{f_{S_N|\Theta}(s|\theta_A)f_{\Theta}(\theta_A)}{f_{S_N|\Theta}(s|\theta_B)f_{\Theta}(\theta_B)}$. Since the signals are binomial, the rates are symmetric, and the priors are equal, Bayes' Rule can be expressed as $\frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} = \left(\frac{\theta_A}{1-\theta_A}\right)^{\frac{2A_s-N}{N} \times N}$. Taking the natural log twice and rearranging,

$$\ln \ln \frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} - \ln \left(\frac{2A_s - N}{N} \right) - \ln \ln \left(\frac{\theta_A}{1 - \theta_A} \right) = \ln N. \quad (1)$$

It is possible in 9 of the papers to identify the value of θ_A for the inference problem, the actual sample observed by subjects, and subjects' mean or median reported posterior. Using the experimental data, Figure B2 plots the left-hand side of equation (1) against $\ln N$.⁸ If the subjects' beliefs were Bayesian, the points should cluster along the identity line (the dashed line in Figure B2). Instead, most points fall below the identity line, indicating that subjects generally infer less in favor of rate θ_A than a Bayesian would. Moreover, the best-fitting regression line (the solid line in the figure) has a slope smaller than 1, suggesting that in the metric defined by the left-hand side of equation (1), the under-inference is greater for larger N .

⁸The left-hand side is well-defined only for inference problems such that $\frac{\theta_A}{1-\theta_A} > 1$ (that is, $\theta_A > .5$) and $\frac{A_s}{N} > \frac{1}{2}$ (that is, over half the realized signals are a 's). Hence, as written, equation (1) only applies to such cases. Although this holds for only 63 of the 99 regression observations in Figure B2 and Table B2, we can include additional regression observations by relabeling the rates and sample proportions that we plug into formula (1). In inference problems such that $\frac{\theta_A}{1-\theta_A} > 1$ and $\frac{A_s}{N} < \frac{1}{2}$, we express Bayes' Rule as $\frac{f_{\Theta|S_N}(\theta_B|s)}{f_{\Theta|S_N}(\theta_A|s)} = \left(\frac{\theta_B}{1-\theta_B}\right)^{\frac{2A_s-N}{N} \times N} = \left(\frac{\theta_A}{1-\theta_A}\right)^{\frac{N-2A_s}{N} \times N}$, so equation (1) becomes $\ln \ln \frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} - \ln \left(\frac{N-2A_s}{N} \right) - \ln \ln \left(\frac{\theta_A}{1 - \theta_A} \right) = \ln N$. This allows us to use an additional 32 regression observations. Finally, we can use a further 4 regression observations for which $\theta_A < .5$ and $\frac{A_s}{N} < \frac{1}{2}$ by expressing Bayes' Rule as $\frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} = \left(\frac{\theta_A}{1-\theta_A}\right)^{\frac{2A_s-N}{N} \times N} = \left(\frac{1-\theta_A}{\theta_A}\right)^{\frac{N-2A_s}{N} \times N}$, and we can take the log-log of this equation. In the case of 3 inference problems, $\frac{A_s}{N} = \frac{1}{2}$, so the Bayesian posterior ratio is equal to 1, and it is impossible to define what constitutes "over-inference" or "under-inference." Those 3 datapoints are dropped from the Figure B2 and Table B2.

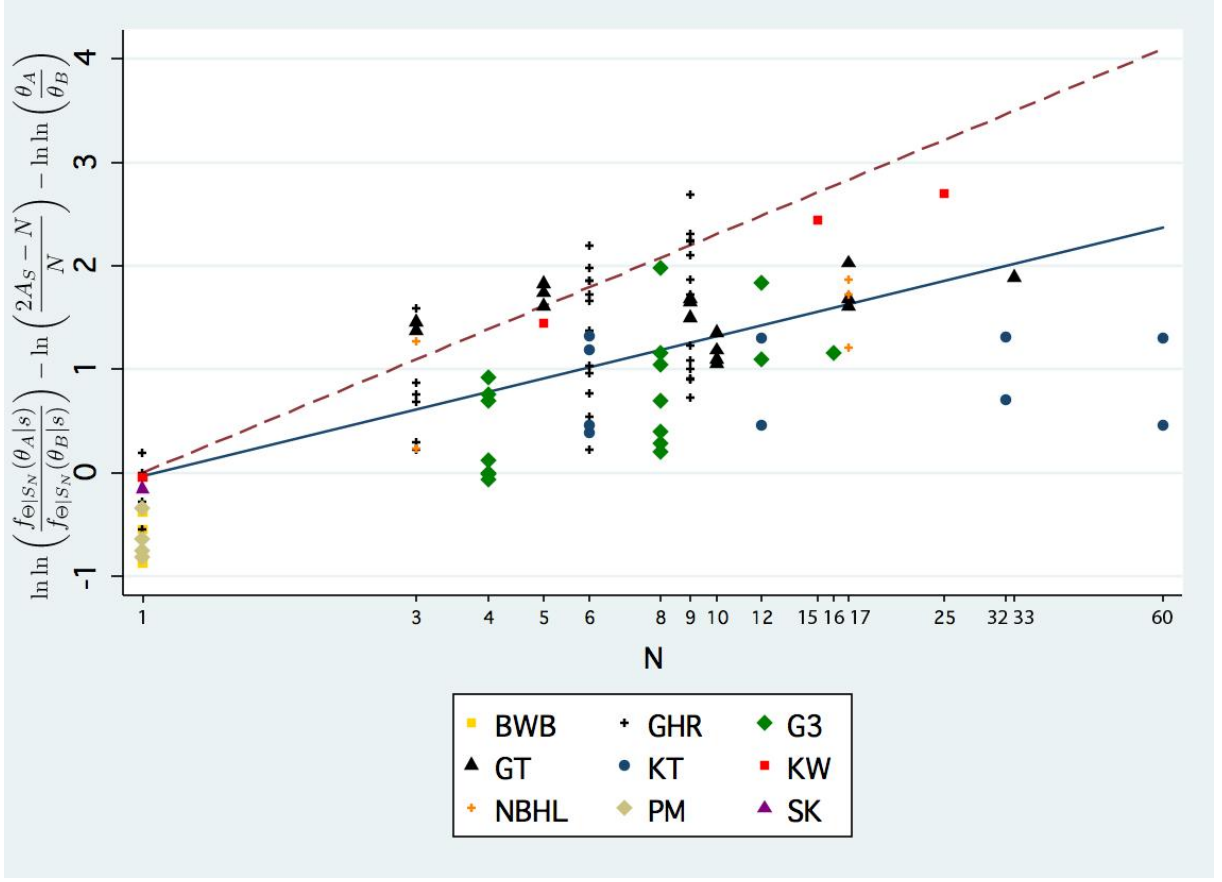


Figure B2: Inference with symmetric rates and equal priors

Notes: The x-axis is depicted on the natural log scale. For all datapoints in the figure, subjects knew that prior probabilities of the two rates were equal. The dotted line represents the null hypothesis of Bayesian updating, and the solid line is the best-fitting regression line from column 2 of Table B2. The includes studies are: BWB = Beach, Wise, and Barclay (1970); GHR = Green, Halbert, and Robinson (1965); G3 = Grether's (1992) Study 3; GT = Griffin and Tversky's (1992) Study 1; KT = Kahneman and Tversky (1972); KW = Kraemer and Weber (2004); NBHL = Nelson, Bloomfield, Hales, and Libby's (2001) Study 1; PM = Peterson and Miller (1965); SK = Sasaki and Kawagoe (2007).

To estimate the degree of under-inference and to probe its robustness, we rewrite (1) as a regression equation:

$$\ln \ln \frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} = \gamma_0 + \gamma_1 \ln N + \gamma_2 \ln \left(\frac{2A_s - N}{N} \right) + \gamma_3 \ln \ln \left(\frac{\theta_A}{1 - \theta_A} \right) + \varepsilon. \quad (2)$$

The null hypothesis of Bayesian updating is $\gamma_0 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$. In Table B2, we estimate versions of equation (2) with several different restrictions on the coefficients and data.

Table B2. Inference with symmetric rates and equal priors.

	(1)	(2)	(3)	(4)	(5)
	Restriction:	Restriction:	Restriction:	Coeffs	Coeffs unrestricted
	$\gamma_1 = \gamma_2 = \gamma_3 = 1$	$\gamma_2 = \gamma_3 = 1$	$\gamma_3 = 1$	unrestricted	Incentivized only
$\ln N$		0.587 (0.083)	0.470 (0.109)	0.485 (0.085)	0.675 (0.086)
$\ln \left(\frac{2A_s - N}{N} \right)$			0.776 (0.116)	0.914 (0.109)	0.957 (0.121)
$\ln \ln \left(\frac{\theta_A}{1 - \theta_A} \right)$				0.333 (0.101)	0.421 (0.104)
Constant	-0.753 (0.076)	-0.031 (0.143)	-0.002 (0.142)	-0.145 (0.108)	-0.168 (0.130)
R^2	0.000	0.426	0.360	0.603	0.763
#obs	99	99	99	99	47
#papers	9	9	9	9	3

Notes: Results are from OLS regressions, with standard errors in parentheses. The dependent variable is as described in the text. Coefficients for blank entries are restricted to equal 1. The fifth column restricts the data to incentivized experiments.

Column 1 estimates just γ_0 , under the restriction that $\gamma_1 = \gamma_2 = \gamma_3 = 1$. The estimate, $\hat{\gamma}_0 = -.753$, is significantly smaller than zero, indicating that the average pattern is under-inference.

Column 2 estimates both γ_0 and γ_1 , while restricting $\gamma_2 = \gamma_3 = 1$. The predicted regression line is plotted as the solid line in Figure B2. The regression confirms that the degree of under-inference is related to sample size; $\hat{\gamma}_1 = .587$ is significantly smaller than 1 (but greater than 0). Moreover, once the degree of under-inference is allowed to depend on sample size, there is no residual under-inference to be picked up by the constant term: $\hat{\gamma}_0 = -.031$, which is not statistically distinguishable from zero. Breaking down the data by study (not shown in the table), every study that manipulates sample size, while holding constant other features of the experiment—Peterson, Schneider, and Miller (1965); Pitz (1967); Peterson, DuCharme, and Edwards’s (1968) Study 2; Kahneman and Tversky (1972); Griffin and Tversky’s (1992) Study 1; Nelson, Bloomfield, Hales, and Libby’s (2001) Study 1; Kraemer and Weber (2004)—concludes that there is greater under-

inference in larger samples.⁹

Column 3 of Table B2 relaxes the regression model still further, estimating γ_0 , γ_1 , and γ_2 , with the only remaining restriction being $\gamma_3 = 1$. The coefficient on the proportion of a signals, $\hat{\gamma}_2 = .776$, is just statistically distinguishable from 1, and the estimates $\hat{\gamma}_0$ and $\hat{\gamma}_1$ are not substantially different in column 3 compared with column 2.

Column 4 estimates all four coefficients. Qualitatively, the main effect of relaxing the $\gamma_3 = 1$ restriction on the conclusions from column 3 is that the coefficient on the proportion of a -signals, $\hat{\gamma}_2 = .914$, is no longer statistically distinguishable from 1. Hence we cannot reject the hypotheses that $\gamma_0 = 0$ and $\gamma_2 = 1$, but $\hat{\gamma}_1$ is significantly smaller than 1. However, column 4 makes clear that the extremeness of the rates—the extent to which $\theta_A = 1 - \theta_B$ departs from .5—also matters for the degree of biased inference. The coefficient $\hat{\gamma}_3 = .333$ is much smaller than 1. This means that subjects under-infer by more the further is θ_A from .5. Breaking down the data by study (not shown in the table), every study that manipulates θ_A , while holding constant other features of the experiment either concludes that there is greater under-inference for θ_A further from .5 (Green, Halbert, and Robinson, 1965; Phillips and Edwards’s, 1966 Study 1 and 3; Peterson and Miller, 1965; Peterson and Swensson’s, 1968 Studies 1 and 2; Peterson, DuCharme, and Edwards’s, 1968 Study 2; Sanders, 1968; Donnell and DuCharme, 1975; Kahneman and Tversky, 1972) or finds it without explicitly stating it (Chinnis and Peterson, 1968; Beach, Wise, and Barclay, 1970; Shu and Wu, 2003).¹⁰

Griffin and Tversky (1992) use the term “discriminability” to describe the phenomenon of under-inference becoming more severe when θ_A and θ_B are further apart. In Griffin and Tversky’s (1992; Study 3) particularly clear evidence from asymmetric inference problems, subjects were asked to infer the likelihood of rate θ_A where the rates have equal priors, the sample has size $N = 12$, and number of a -signals is $A_s = 7, 8, 9$, or 10. When the rates are close together, $(\theta_A, \theta_B) = (.6, .5)$, the subjects exhibit slight over-inference: a Bayesian’s posteriors for these four inference problems would be .54, .64, .72, and .80, respectively, while subjects’ median posteriors were .55, .66, .75, and .85. When the rates are further apart, $(\theta_A, \theta_B) = (.6, .25)$, subjects exhibited massive under-inference: whereas a Bayesian’s posteriors in these problems would be .95, .98, .998, and .999, respectively, subjects’ posteriors were .60, .70, .80, and .90.

⁹For Green, Halbert, and Robinson (1965), we can also reach this conclusion by estimating the regression equation (2) on the data reported just in that paper. There are a number of other studies that manipulate sample size but do not analyze or display the data in a way that makes it clear how sample size affects the degree of bias in inference: Sanders (1968); Peterson and Swensson’s (1968) Study 2; Beach, Wise, and Barclay (1970); Marks and Clarkson (1972); and DeSwart (1972a, 1972b).

¹⁰DeSwart (1972a, 1972b) manipulates how far θ_A is from .5 but does not analyze or display the data in a way that makes it clear how it affects the degree of under-inference.

Griffin and Tversky’s evidence cannot be fully explained by extreme-belief aversion (that simply maps an objective posterior into a less extreme subjective one) because, for example, subjects’ posterior of .80 is identical whether $\theta_B = .5$ and 10 a -signals were observed or $\theta_B = .25$ and 9 a -signals were observed, but the objective posteriors are quite different in those two cases.

Our one-parameter model of NBLLN does predict $\gamma_3 < 1$, i.e., “more severe” under-inference (to be precise, Barneys likelihood ratio being closer to 1 than Tommys likelihood ratio, conditional on any realized sample) when the symmetric rates are further apart. Nonetheless, we suspect that SDTSD also contributes to explaining the flat tails of subjective sampling distributions and the discriminability phenomenon in inference.

Column 5 estimates the same regression as column 4, but with the data restricted to incentivized experiments. There are only three such studies—Green, Halbert, and Robinson (1965), Nelson, Bloomfield, Hales, and Libby (2001), and Kraemer and Weber (2004)—but the results in column 5 are largely similar to column 4, except that both $\hat{\gamma}_1 = .675$ and $\hat{\gamma}_3 = .421$ are larger, suggesting greater sensitivity to sample size and to rates when accurate inferences are rewarded. Nonetheless, both coefficients remain far less than 1, indicating substantial biases relative to Bayesian inference.

Training in inference appears to reduce but not eliminate under-inference. When subjects were told after each inference which state actually occurred, they became less biased over time but still under-inferred by the end of the experiment (Phillips and Edwards’s, 1966, Study 2; Camerer, 1987). Strub (1969) found under-inference among subjects who had received 114 hours of lecture sessions, demonstrations, problem-solving sessions, and other training in dealing with probabilities, including prior participation in inference experiments. When subjects were told after each inference what the normatively correct inference is, they very quickly learned to report more extreme beliefs, but they do not seem to have learned to draw better inferences. While reporting more extreme beliefs led to more accurate beliefs in problems where their pre-training beliefs were not extreme enough, it led to less accurate beliefs in problems where their pre-training beliefs were accurate (Donnell and DuCharme, 1975).

A few studies have found over-inference. In all cases, N is relatively small, and θ_A is relatively close to .5.¹¹ Griffin and Tversky’s Study 1 (1992; $\theta_A = .6$) , which compared inference from samples of size 3, 5, 9, 17, and 33, found over-inference for $N = 3$ and 5 and under-inference for the others. Nelson, Bloomfield, Hales, and Libby’s Study 1 (2001; $\theta_A = .6$) conducted an experimental asset market where payoffs depended on correct inferences from samples of size 3 and 17. Subjects under-inferred for $N = 17$ and over-inferred for $N = 3$. There is some evidence, though it is weak,

¹¹Peterson and Swennson’s (1968) Study 1 finds over-inference for $N = 1$ and $\theta_A = .6, .67, .75, .9$ in the first half of their data. In the same inference problems in the second half of their data from Study 1, and in both halves from Study 2, however, they find under-inference.

that over-inference in favor of a particular state occurs when the realized sample exactly matches the expected sample in that state, a phenomenon that has been called “exact representativeness.” In an experimental asset market, Camerer (1987; $N = 3$, $\theta_A = .67$) found that the price of a state-contingent asset that pays off if state A is true was too high—indicating over-inference in favor of state A —when the observed sample contained exactly 2 a -signals and 1 b -signal, and symmetrically over-inference in favor of state B when the sample proportions were reversed. In an experimental asset market with asymmetric rates of $\theta_A = .67$ and $\theta_B = .5$, Grether (1980; $N = 6$) similarly found evidence indicating over-inference in favor of state A when the realized sample was 4 a ’s and 2 b ’s and over-inference in favor of state B when the realized sample was 3 a ’s and 3 b ’s. In a similar experiment, Grether (1992) found less support for “exact representativeness.” Neither NBLN alone, nor NBLN combined with SDTSD, can explain over-inference.

Such over-inference can, however, be explained by the Law of Small Numbers (LSN). As shown in Appendix A, when NBLN and LSN are combined in a single model, small N is a necessary condition for over-inference. Moreover, we have argued that SDTSD will tend to generate under-inference when θ_A and θ_B are far apart.

A distinctive feature of our theory—a feature that differentiates it from alternative theories of under-inference from large samples discussed in Section 7 and Appendix A—is the prediction that inferences from a sample of size 1 will be correct. Sample-size neglect predicts over-inference for samples of size 1, while extreme-belief aversion predicts under-inference for samples of any size, including 1. There are 11 experiments that measure inference when $N = 1$. Peterson, Schneider, and Miller (1965; $\theta_A = .6$), Dave and Wolfe (2003; $\theta_A = .7$), Peterson and Swenson’s Study 2 (1968; $\theta_A = .6, .67, .75, .9$), and Gettys and Manley’s (1968) Studies 1 and 2 (which used a variety of asymmetric inference problems) found substantial under-inference. Chinnis and Peterson (1968; $\theta_A = .67, .8$), Kraemer and Weber (2004; $\theta_A = .6$), and Sasaki and Kawagoe (2007; $\theta_A = .67$) found slight under-inference, very close to Bayesian, and Peterson and Swenson’s (1968; $\theta_A = .6, .67, .75, .9$) Study 1 found over-inference in the first half of their data and under-inference in the second half. In a mix of symmetric and asymmetric problems, Peterson and Miller (1965) found under-inference for $(\theta_A, \theta_B) = (.83, .17)$, $(\theta_A, \theta_B) = (.71, .2)$, and $(\theta_A, \theta_B) = (.67, .33)$, and over-inference for $(\theta_A, \theta_B) = (.6, .43)$. Green, Halbert, and Robinson (1965; $\theta_A = .6, .8$) found inferences very close to Bayesian when $N = 1$. The evidence is mixed but with more of the studies leaning toward under-inference.¹²

¹²Presumably, the many papers on base-rate neglect also contain evidence on inferences from samples of size 1. Virtually none of them have 50-50 priors, however, so it is difficult to disentangle biased inference from base-rate neglect. We do not review this literature systematically, but to give a flavor of what it may indicate, we examined Bar-Hillel’s (1980) seminal paper. Our impression is that the evidence in Bar-Hillel’s paper roughly mirrors the evidence from experiments on single-signal inference reviewed above. The

As an aside, we can use the same balls-and-urns experiments to study how priors affect inference and thereby measure the prevalence and extent of base-rate neglect in these studies. Taking the log of Bayes' Rule and rearranging:

$$\ln \frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} - \ln \frac{f_{S_N|\Theta}(s|\theta_A)}{f_{S_N|\Theta}(s|\theta_B)} = \ln \frac{f_{\Theta}(\theta_A)}{f_{\Theta}(\theta_B)}. \quad (3)$$

As above, from the published experiments, we obtain for each inference problem the values θ_A , θ_B , $f_{\Theta}(\theta_A)$, the actual sample observed by subjects, and subjects' mean or median reported posterior. The right-hand side of equation (3) can be readily calculated from $f_{\Theta}(\theta_A)$, as can the first term on the left-hand side, $\ln \frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)}$, from subjects' posterior. In order not to confound base-rate neglect with other biases (that affect inference even when priors are equal), we calculate the second term on the left-hand side, $\ln \frac{f_{S_N|\Theta}(s|\theta_A)}{f_{S_N|\Theta}(s|\theta_B)}$, as the predicted value, $\ln \frac{\widehat{f_{S_N|\Theta}(s|\theta_A)}}{\widehat{f_{S_N|\Theta}(s|\theta_B)}}$, from the previously-estimated regression equation (2). We use only the symmetric-rate data. Figure B3 plots the left-hand side of equation (3) against $\ln \frac{f_{\Theta}(\theta_A)}{f_{\Theta}(\theta_B)}$. Regardless of whatever other biases may affect inference, if the subjects correctly incorporate base rates into their inferences, the data should lie along the identity line (the dashed line in Figure B3). However, the best-fitting line (the solid line in the figure) has a slope less than 1, indicating that subjects' inferences are too insensitive to the prior probabilities.

full distribution of subjects' reported posteriors can be eyeballed from histograms reported for Bar-Hillel's Studies 1, 2, 3, 7 and 8, each of which presents an inference problem where a single signal is indicative of the less likely of two states that subjects are given base rates for. We divide the 222 subjects' responses into four categories. Because the signal strength always was in the opposite direction of the base rate, the 33% of subjects whose posteriors equaled the base rate or weaker must have been either under-inferring from the signal or (as is presumably unlikely) "over-using" the base rate. By contrast, 9% of subjects reported posteriors stronger than the signal, almost surely indicating over-use of the signal (since otherwise they must be reversing the base rate). 31% of subjects reported posteriors of exactly the signal strength. Although not logically necessary, we share the presumption of Bar-Hillel and most researchers in this area that these subjects were almost surely simply using the signal strength and ignoring the base rate altogether. The remaining 27% of subjects reported posteriors strictly between the correct Bayesian posteriors and the posteriors that would completely ignore the base rate. It is unclear how many of these subjects were over-using or under-using the signal because anyone under-using the base rate could have been either over-inferring or under-inferring from the signal. From these data taken together, it seems likely that between 9% and 36% of the subjects were over-inferring from the signal, and at least 33% of the subjects were under-using the signal.

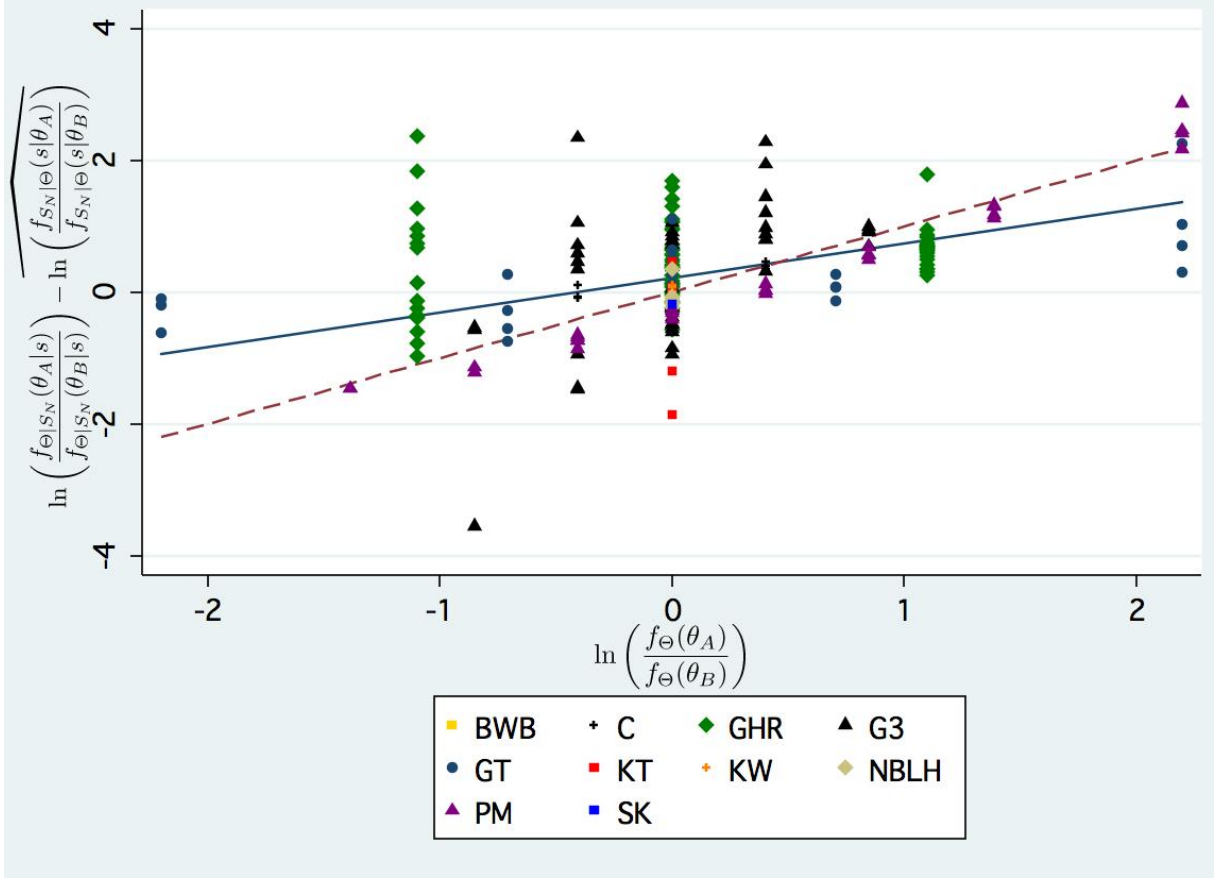


Figure B3: Base-rate neglect in symmetric-rate inference problems.

Notes: The dotted line represents the null hypothesis of Bayesian updating, and the solid line is the best-fitting regression line from column 1 of Table B5. The includes studies are: BWB = Beach, Wise, and Barclay (1970); C = Camerer (1987); GHR = Green, Halbert, and Robinson (1965); G3 = Grether's (1992) Study 3; GT = Griffin and Tversky's (1992) Study 1; KT = Kahneman and Tversky (1972); KW = Kraemer and Weber (2004); NBHL = Nelson, Bloomfield, Hales, and Libby's (2001) Study 1; PM = Peterson and Miller (1965); SK = Sasaki and Kawagoe (2007).

To formally investigate the degree of base-rate neglect, we rewrite (3) as a regression equation:

$$\ln \frac{f_{\Theta|S_N}(\theta_A|s)}{f_{\Theta|S_N}(\theta_B|s)} - \ln \frac{f_{S_N|\Theta}(s|\theta_A)}{f_{S_N|\Theta}(s|\theta_B)} = \varphi_0 + \varphi_1 \ln \frac{f_{\Theta}(\theta_A)}{f_{\Theta}(\theta_B)} + \varepsilon. \quad (4)$$

The null hypothesis of Bayesian updating corresponds to $\varphi_0 = 0$ and $\varphi_1 = 1$.

The first column of Table B3 estimates equation (4) on the full sample. While we can reject the null hypothesis that $\varphi_0 = 0$, it is clear from Figure B5 that the deviation is small. On the other hand, the fitted value $\hat{\varphi}_1 = .601$ is significantly less than 1 (and larger than 0). The second column estimates the equation using only the data with unequal prior probabilities, but the results

are similar. Finally, the third column restricts the data to incentivized experiments. While $\hat{\varphi}_0$ is now essentially zero, $\hat{\varphi}_1 = .405$ indicates even stronger base-rate neglect in these data.

Table B3. Base-rate neglect.

	(1)	(2)	(5)
	All data	Only unequal priors	Only incentivized
$\ln \frac{f_{\Theta}(\theta_A)}{f_{\Theta}(\theta_B)}$	0.524 (0.078)	0.510 (0.081)	0.360 (0.115)
Constant	0.216 (0.053)	0.299 (0.087)	-0.130 (0.084)
R^2	0.242	0.303	0.176
#obs	209	110	48
#papers	10	5	3

Notes: Results are from OLS regressions, with standard errors in parentheses. The dependent variable is as described in the text.

0.3 Consistency Between Subjective Sampling Distributions and Inference

A core feature of the fully rational model that our model retains is that people draw inferences that are consistent with Bayes’s Rule applied to their subjective sampling distributions. Although surely not a perfect fit, we believe (like other researchers before us) that this feature is approximately right except insofar as people neglect base rates. There is indirect supportive evidence from the qualitative correspondence between the evidence on subjective sampling distributions (reviewed in section B.1) and the evidence on inferences (reviewed in section B.2). There is direct evidence from two studies that measured subjective sampling distributions and inferences for the same subject.

Peterson, DuCharme, and Edwards (1968, Study 2) conducted symmetric inference experiments with every combination of $N = 3, 5, 8$, and $\theta_A = 1 - \theta_B = .6, .7, .8$. Then subjects drew subjective sampling distributions for the nine binomial distributions (shown in Figure B4). Peterson, DuCharme, and Edwards plotted subjects’ inferences against what their inferences would be if they applied Bayes’s Rule to their subjective sampling distributions. Peterson, DuCharme, and Edwards found that “most points cluster extremely close to the identity line.”

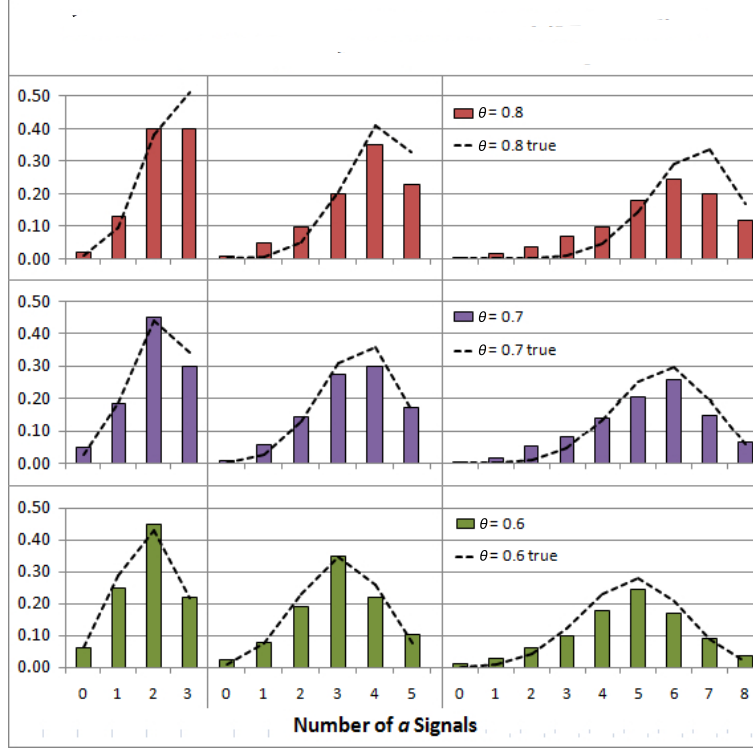


Figure B4: Median probability estimates, $N=3$, $N=5$, and $N=8$ (Peterson DuCharme and Edwards 1968)

Wheeler and Beach (1968) elicited subjects' subjective sampling distributions for a sample of size $N = 8$ for rates $\theta = .6$ and $.8$ (see Figure B5) and then asked subjects to make bets in an inference task. Wheeler and Beach inferred subjective posteriors from the bets, under the assumption that subjects sought to maximize expected winnings. The correlation between the median subjective log-posterior (calculated from the first 20 inference problems) and the median subjective log-likelihood (calculated from applying Bayes's Rule to the subjective sampling distribution elicited at the very beginning of the experiment) was .90. For individual subjects' data, the median correlation was .85.

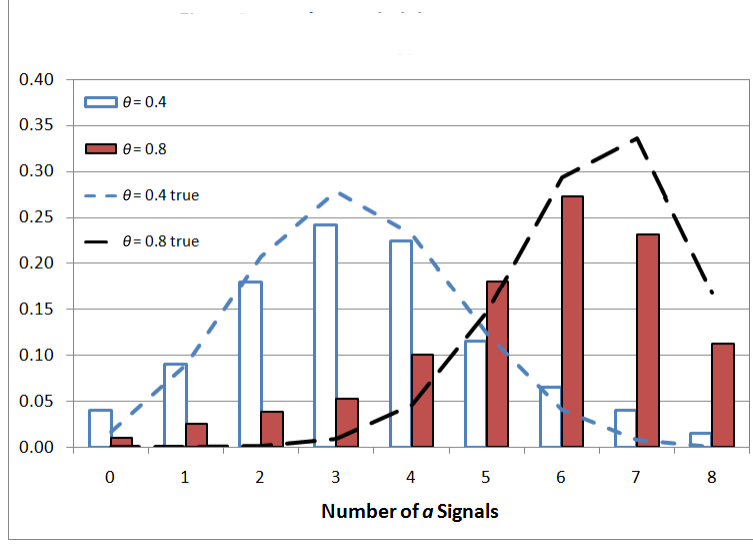


Figure B5: Median probability estimates, N=8 (Wheeler and Beach 1968)

0.4 Inference For Sequential Clumps

In Section 4, we lay out various possible dynamic extensions of our model of NBLLN. Unfortunately, there are no experiments that elicit people’s beliefs about what they will infer conditional on observing samples in the future. Here we review the few experiments that compare inferences from a sample presented simultaneously with a sample drawn sequentially.

In symmetric inference with $\theta = .6$ and $N = 48$, Peterson, Schneider, and Miller (1965) presented the sample clumped as a single sample of size 48, four samples of size 12, twelve samples of size 4, or forty-eight samples of size 1. Subjects reported their updated beliefs after each clump was shown. The results are somewhat difficult to interpret because the data are averaged across clumps and realizations. Nonetheless, Peterson et al.s finding that under-inference is more severe when clumps contain more signals is consistent with either retrospective-acceptive or retrospective-pooling.

Grether (1992, Study 3) confronted subjects with (incentivized) symmetric, binomial inference problems, with rates $\theta_A = 1 - \theta_B = .2, .3, .4, .6, .7, .8$, and priors $f_{\Theta}(\theta_A) = .3, .4, .5, .6, .7$. The sample size always began as $N = 4$. In some cases, however, after subjects made their inference, they were asked to make an updated inference after an additional 4 signals were drawn, up to a maximum of 12 signals in total. Although only aggregate statistics are reported in the paper, David Grether sent us the subject-level data. In a few cases, we can learn about how subjects process the signals by comparing their inferences before and after they receive a clump. For example, in one situation, the rates were $\theta_A = 1 - \theta_B = .2$, the prior probabilities of the rates were equal, and the first four signals were all b ’s. The next four signals were 2 a ’s and 2 b ’s. The objective posterior

probability of rate θ_B is the same after all eight as after the first four: .9961. However, the subjects' subjective posterior (that is, the median across subjects) is .95 after the first four signals and .70 after all eight. This pattern of additional uninformative signals causing the subjective posterior to move toward .5 is consistent with retrospective-pooling Barney, not retrospective-acceptive Barney. The same pattern holds in all three other test cases in Grether's data.¹³

In contrast, Kraemer and Weber (2004) present evidence that supports retrospective-acceptive Barney. For an incentivized, symmetric binomial inference problem with $\theta_A = 1 - \theta_B = .6$ and $N = 5$, subjects presented with a sample of 3 a 's and 2 b 's gave mean posterior probability for rate θ_A of .585. Other subjects who were instead shown the same signals as two separate samples, one with 3 a 's and 0 b 's and one with 0 a 's and 2 b 's, gave mean posterior probability for state A of .56, which is marginally statistically different. Similarly, with $\theta_A = 1 - \theta_B = .6$ and $N = 25$, subjects' mean posterior probability for rate θ_A was .56 when the sample was 13 a 's and 12 b 's, but .53 (strongly statistically distinguishable from .56) when two samples of 13 a 's and 0 b 's, and then 0 a 's and 12 b 's were presented sequentially. The fact that subjects make different inferences in these two cases is inconsistent with being retrospective-pooling, but not inconsistent with being retrospective-acceptive.¹⁴

Shu and Wu's (2003) Study 3 appears to be inconsistent with any of the dynamic extensions of our model that we consider. They conduct a symmetric binomial inference problem with samples of size $N = 10$ and three different levels of the rates, $\theta_A = 1 - \theta_B = .6, .75$, or $.9$. In one condition, subjects observed the 10 signals one at a time before stating a posterior belief. In the other conditions, subjects observed the 10 signals in clumps of 2 signals each or 5 signals each. While for some realizations of the 10-signal sets subjects draw less extreme inferences when the signals arrive in larger clumps—as predicted for retrospective-acceptive Barney—the results on average

¹³In the 2nd test case, $f_{\Theta}(\theta_A) = .4$, $\theta_A = 1 - \theta_B = .3$, and the first eight signals were 1 a and 7 b 's. The next four signals were 2 a 's and 2 b 's. While the objective posterior probability of rate θ_B is identically .9908, subjects' median subjective posterior fell from .9 to .5. In the 3rd test case, $f_{\Theta}(\theta_A) = .7$, $\theta_A = 1 - \theta_B = .4$, and the first four signals were 2 a 's and 2 b 's, and the next four signals were 2 a 's and 2 b 's. The objective posterior probability of rate θ_A remains .7, but subjects' median subjective posterior fell from .775 to .7. In the 4th test case, $f_{\Theta}(\theta_A) = .3$, $\theta_A = 1 - \theta_B = .7$, and the first four and next four signals were 2 a 's and 2 b 's. The objective posterior probability of rate θ_A remains .3, but subjects' median subjective posterior increased from .42 to .51. If we examine subjects' mean posterior probabilities, the pattern is robust in the first three test cases (.87 to .66, .72 to .59, and .73 to .68) and hard to interpret in the fourth (.54 to .58).

¹⁴Unlike the experimental subjects, however, retrospective-acceptive Barney will under-infer weakly *more* from two clumps than from one. This is because an additional clump provides information about an additional draw of β from the distribution, which is weakly more informative about the true rate than an additional signal about a fixed β . That being said, a combination of retrospective-acceptive Barney with SDTSD could explain why the two extreme clumps lead to a weaker overall inference than the single clump.

Kraemer and Weber (2004) also have a third experimental treatment that generates ambiguous evidence. In the $N = 5$ case, when subjects are presented with a sample of 1 a and 1 b , followed by a sample of 2 a 's and 1 b , their mean posterior is .575, in between the other two treatments and not statistically distinguishable from either. Similarly, in the $N = 25$ case, the mean posterior is .555 when subjects are presented with a sample of 6 a 's and 6 b 's, followed by a sample of 7 a 's and 6 b 's.

go in the opposite direction.¹⁵ While it may be possible to reconcile Shu and Wu’s results with a combination of NBLN and the dynamics of base-rate neglect, that combined model should be worked out to see if it systematically reverses some of the conclusions we reach in Section 5.

0.5 Evidence For Non-Binomial Distributions

While the vast majority of simple inference experiments have been conducted with binomial signals, there are a few studies with other distributions. The results overall are consistent with NBLN and SDTSD applying beyond binomial subjective sampling distributions.

There are a handful of studies where signals are multinomial. For example, in Beach’s (1968) experiment, there were two decks of cards, a Red Deck and a Green Deck. Each card had a letter from A to F written on it. The Red and Green Decks had equal priors, but each deck had different proportions of the lettered cards. The subjects were shown $N = 3$ cards, one card at a time, and reported their subjective probability that the cards were being drawn from the Red Deck, as opposed to the Green Deck, after each draw. The likelihood ratios for the cards ranged from 1:2.5 to 3:1. For example, the likelihood ratio for card F was 1:2. A second group of subjects faced the same inference task with the same likelihood ratios for each card, but with the absolute probabilities scaled down for some cards and scaled up for others. For example, for the first group of subjects, the probability of an F card was .03 for the Red Deck and .06 for the Green Deck; for the second group of subjects, the probability of an F card was .16 for the Red Deck and .32 for the Green Deck. The first main finding is that subjects under-inferred on average. The other finding was that, for a given objective likelihood ratio, Group 1 under-inferred more for cards where Group 1’s probabilities for that card were scaled down relative to Group 2’s. Our interpretation is that when Group 1’s probabilities are scaled down, the observed sample lies further in the tails on the subjective sampling distribution for both decks. SDTSD predicts more extreme under-inference in such cases.

Under-inference is also the general finding in the other multinomial experiments we could find that compared subjects’ posteriors with Bayesian posteriors (Phillips, Hays, and Edwards’s Study 1, 1966; Dale, 1968; Martin, 1969; Martin and Gettys, 1969; Chapman, 1973). However, there are two exceptions: (1) Phillips, Hays, and Edwards (1966) varied the sample size of signals observed by subjects and, while finding under-inference for $N = 3, 5$, and 9, found essentially Bayesian inference for $N = 1$, and (2) Dale (1968) reported that in one particular trial where the data happened to

¹⁵Sanders (1968) and Beach, Wise, and Barclay (1970) also compare inferences from simultaneously-presented samples with inferences from sequentially-presented samples, but it is difficult to interpret their findings because the results for the simultaneously-presented samples are averaged across different sample sizes.

exactly match one of the multinomial rates, about 1/8 of the subject over-inferred—the most over-inference he observed on any trial. Martin (1969), Martin and Gettys (1969), and Chapman (1973) also reported that subjects’ under-inferred by more when an observed sample warranted a more extreme conclusion. That finding could be due to NBLN, SDTSD, or both.

DuCharme (1970) conducted a normal-signal inference experiment where subjects observed samples of size 1. He found under-inference when the sample was relatively far in the tails of both distributions, consistent with SDTSD, although he interpreted his results as meaning that people are reluctant to report extreme probabilities. Gustafson, Shukla, Delbecq, and Walster (1973) told subjects the average heights and weights of Midwestern college-age men and women. Subjects were then asked to draw inferences from samples of size 1; specifically, they were asked a series of questions such as, “The observed height of a person is 68 inches. Is the person more likely to be a male or female? How much more likely?” Gustafson et al. found that subjects over-inferred when the objective likelihood ratio was relatively small and under-inferred when the objective likelihood ratio was relatively large. Assuming that subjects believed that the sampling distributions for height and weight were normal distributions, this result means that subjects over-inferred when the sample was relatively close to the men’s or women’s mean height or weight and under-inferred when the sample was relatively far in the tails of both distributions. In two studies, DuCharme and Peterson (1968) familiarized subjects with normal distributions for male and female heights and then elicited subjects’ beliefs that a sample was being drawn from the population of men or of women. Subjects’ posteriors were nearly Bayesian when $N = 1$, but subjects under-inferred for samples of size $N = 4$.

Peterson and Phillips (1966) conducted an experiment where the rate generating binary signals was drawn from a uniform distribution on $[0, 1]$. Subjects observed 48 binary signals and after each signal had to specify a 33% confidence interval for the rate. Subjects’ confidence intervals were almost always too wide, indicating that subjects under-inferred from the data about the rate.

Appendix C: Proofs

0.6 Preliminary and Appendix A Results

According to Whitt's (1979) Theorem 4, A3 implies that for any N and any $\theta_A, \theta_B \in \Theta$, Barney's likelihood ratio $\frac{f_{S_N|\Theta}^\psi(s|\theta_A)}{f_{S_N|\Theta}^\psi(s|\theta_B)}$ is strictly increasing in the number of a -signals in the sample. We will repeatedly use this fact in the proofs.

The preliminary results all apply to the beta-distribution functional form, given by equation (3) in the main text. Let $p_0 \equiv \frac{f_\Theta(\theta_A)}{f_\Theta(\theta_B)}$ denote the agent's prior ratio, let $\Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B) \equiv \frac{f_{S_N|\Theta}^\psi(s|\theta_A)}{f_{S_N|\Theta}^\psi(s|\theta_B)}$ denote Barney's likelihood ratio after observing a clump of N signals $s \in S_N$, and let $\Pi_{\Theta \times \Theta|S_N}^\psi(\theta_A, \theta_B|s) \equiv \frac{f_{\Theta|S_N}^\psi(\theta_A|s)}{f_{\Theta|S_N}^\psi(\theta_B|s)} = p_0 \Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B)$ denote his posterior ratio.

Lemma $\beta 1$. *Assume Barney has the beta-distribution functional form given by equation (3). Suppose Barney is prospective-acceptive, knows the rate is $\theta \in \Theta$, and will observe a random clump of N signals $s \in S_N$. Then he believes the probability of observing A_s a -signals in the sequence s is:*

$$f_{S_N|\Theta}^\psi(s|\theta_A) = \frac{\Gamma(\psi)}{\Gamma(\psi+N)} \frac{\Gamma(\theta\psi + A_s)}{\Gamma(\theta\psi)} \frac{\Gamma((1-\theta)\psi + N - A_s)}{\Gamma((1-\theta)\psi)} \frac{\Gamma(N+1)}{\Gamma(A_s+1)\Gamma(N-A_s+1)}.$$

Proof:

$$\begin{aligned} & f_{S_N|\Theta}^\psi(s|\theta_A) \\ &= \int_0^1 f_{S_N|\beta}^\psi(s|\beta) f_{\beta|\Theta}^\psi(\beta|\theta_A) d\beta \\ &= \int_0^1 \frac{\Gamma(N+1)}{\Gamma(A_s+1)\Gamma(N-A_s+1)} \beta^{A_s} (1-\beta)^{N-A_s} \frac{\Gamma(\psi)}{\Gamma(\theta\psi)\Gamma((1-\theta)\psi)} \beta^{\theta\psi-1} (1-\beta)^{(1-\theta)\psi-1} d\beta \\ &= \frac{\Gamma(\psi)}{\Gamma(\psi+N)} \frac{\Gamma(\theta\psi + A_s)}{\Gamma(\theta\psi)} \frac{\Gamma((1-\theta)\psi + N - A_s)}{\Gamma((1-\theta)\psi)} \frac{\Gamma(N+1)}{\Gamma(A_s+1)\Gamma(N-A_s+1)} \times \\ & \quad \int_0^1 \frac{\Gamma(\psi+N)}{\Gamma(\theta\psi + A_s)\Gamma((1-\theta)\psi + N - A_s)} \beta^{\theta\psi+A_s-1} (1-\beta)^{(1-\theta)\psi+N-A_s-1} d\beta \\ &= \frac{\Gamma(\psi)}{\Gamma(\psi+N)} \frac{\Gamma(\theta\psi + A_s)}{\Gamma(\theta\psi)} \frac{\Gamma((1-\theta)\psi + N - A_s)}{\Gamma((1-\theta)\psi)} \frac{\Gamma(N+1)}{\Gamma(A_s+1)\Gamma(N-A_s+1)}, \end{aligned}$$

where the fourth equality follows because the term being integrated is the pdf of a beta distribution, and the integral of a pdf is equal to 1. □

Lemma $\beta 2$. Assume Barney has the beta-distribution functional form given by equation (3) and is retrospective pooling. Consider two states of the world, A and B , associated with known rates $\theta_A, \theta_B \in \Theta$ and a prior ratio $p_0 = \frac{f(\theta_A)}{f(\theta_B)} \in (0, \infty)$. Barney's likelihood ratio after observing a clump of N signals $s \in S_N$ which has A_s a -signals is:

$$\Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B) = \frac{\Gamma(\theta_A \psi + A_s) \Gamma((1 - \theta_A) \psi + N - A_s)}{\Gamma(\theta_B \psi + A_s) \Gamma((1 - \theta_B) \psi + N - A_s)} \frac{\Gamma(\theta_B \psi) \Gamma((1 - \theta_B) \psi)}{\Gamma(\theta_A \psi) \Gamma((1 - \theta_A) \psi)}.$$

Moreover, in the “symmetric inference” case where $\theta_A = 1 - \theta_B$,

$$\Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B) = \frac{\Gamma(\theta_A \psi + A_s) \Gamma((1 - \theta_A) \psi + N - A_s)}{\Gamma(\theta_B \psi + A_s) \Gamma((1 - \theta_B) \psi + N - A_s)}$$

Proof: This is an immediate implication of the previous lemma. □

Lemma $\beta 3$. Assume Barney has the beta-distribution functional form given by equation (3). Consider two states of the world, A and B , associated with known rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A \geq \theta_B$. Suppose Barney has observed N signals, $s \in S_N$, and he observes his $N + 1^{st}$ signal. Then Barney's likelihood ratio is closer to $\Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B)$ if he pools the $N + 1^{st}$ signal together with the others than if he groups it separately: that is, if the $N + 1^{st}$ signal is an a -signal, then

$$\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B) \Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B) \geq \Pi_{S_{N+1}|\Theta \times \Theta}^\psi(s \cup a|\theta_A, \theta_B),$$

and if $N + 1^{st}$ signal is a b -signal, then

$$\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B) \Pi_{S_N|\Theta \times \Theta}^\psi(s|\theta_A, \theta_B) \leq \Pi_{S_{N+1}|\Theta \times \Theta}^\psi(s \cup a|\theta_A, \theta_B).$$

Proof: We will prove the claim in the lemma that pertains if the final signal is an a -signal; the proof for the other case is analogous. Denote by A_N the number of a -signals in s_N . If Barney pools

the a -signal with the others, then his likelihood ratio is

$$\begin{aligned}
& \Pi_{S_{N+1}|\Theta \times \Theta}^\psi (s \cup a | \theta_A, \theta_B) \\
&= \frac{\Gamma(\theta_A \psi + A_N + 1) \Gamma((1 - \theta_A) \psi + N + 1 - A_N - 1) \Gamma(\theta_B \psi) \Gamma((1 - \theta_B) \psi)}{\Gamma(\theta_B \psi + A_N + 1) \Gamma((1 - \theta_B) \psi + N + 1 - A_N - 1) \Gamma(\theta_A \psi) \Gamma((1 - \theta_A) \psi)} \\
&= \frac{(\theta_A \psi + A_N) \Gamma(\theta_A \psi + A_N) \Gamma((1 - \theta_A) \psi + N - A_N) \Gamma(\theta_B \psi) \Gamma((1 - \theta_B) \psi)}{(\theta_B \psi + A_N) \Gamma(\theta_B \psi + A_N) \Gamma((1 - \theta_B) \psi + N - A_N) \Gamma(\theta_A \psi) \Gamma((1 - \theta_A) \psi)} \\
&= \frac{\theta_A \psi + A_N}{\theta_B \psi + A_N} \Pi_{S_N|\Theta \times \Theta}^\psi (s | \theta_A, \theta_B),
\end{aligned}$$

where the second equality uses Lemma $\beta 2$, and the third equality uses the fact that $\Gamma(n+1) = n\Gamma(n)$.

If Barney instead separately groups the final signal, then his likelihood ratio is

$$\begin{aligned}
\Pi_{S_1|\Theta \times \Theta}^\psi (a | \theta_A, \theta_B) \Pi_{S_N|\Theta \times \Theta}^\psi (s | \theta_A, \theta_B) &= \frac{\theta_A \psi}{\theta_B \psi} \Pi_{S_N|\Theta \times \Theta}^\psi (s | \theta_A, \theta_B) \\
&= \frac{\theta_A}{\theta_B} \Pi_{S_N|\Theta \times \Theta}^\psi (s | \theta_A, \theta_B).
\end{aligned}$$

By assumption, $\frac{\theta_A}{\theta_B} \geq 1$. Therefore, for all $k > 0$,

$$\frac{\theta_A}{\theta_B} \geq \frac{\theta_A \psi + k}{\theta_B \psi + k} \geq 1.$$

The result follows. □

Lemma $\beta 4$. *Assume Barney has the beta-distribution functional form given by equation (3) and is retrospective pooling. Consider two states of the world, A and B , associated with known rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and a prior ratio $p_0 = \frac{f(\theta_A)}{f(\theta_B)} \in (0, \infty)$.*

- *Barney draws the same inferences as Tommy when the observed sequence is ab or ba .*
- *If the sample is all a 's or all b 's, Barney under-infers relative to Tommy.*
- *For any true state $\theta \in (0, 1)$, Barney believes that: as $N \rightarrow \infty$, his posterior ratio $\frac{f_{\Theta|S_N}^\psi(\theta_A|s)}{f_{\Theta|S_N}^\psi(\theta_B|s)}$ will converge in distribution to a random variable that has full support on $(0, \infty)$.*

Proof:

- Barney's posterior after one a and one b signal is

$$\begin{aligned}
& p_0 \frac{\Gamma(\theta_A \psi + 1) \Gamma((1 - \theta_A) \psi + 1) \Gamma(\theta_B \psi) \Gamma((1 - \theta_B) \psi)}{\Gamma(\theta_B \psi + 1) \Gamma((1 - \theta_B) \psi + 1) \Gamma(\theta_A \psi) \Gamma((1 - \theta_A) \psi)} \\
&= p_0 \frac{\Gamma(\theta_A \psi) \theta_A \psi \Gamma((1 - \theta_A) \psi) (1 - \theta_A) \psi \Gamma(\theta_B \psi) \Gamma((1 - \theta_B) \psi)}{\Gamma(\theta_B \psi) \theta_B \psi \Gamma((1 - \theta_B) \psi) (1 - \theta_B) \psi \Gamma(\theta_A \psi) \Gamma((1 - \theta_A) \psi)} \\
&= p_0 \frac{\theta_A (1 - \theta_A)}{\theta_B (1 - \theta_B)},
\end{aligned}$$

which equals Tommy's posterior.

- Barney's posterior after N a -signals and 0 b -signals is:

$$\begin{aligned}
& p_0 \frac{\Gamma(\theta_A \psi + N) \Gamma((1 - \theta_A) \psi) \Gamma(\theta_B \psi) \Gamma((1 - \theta_B) \psi)}{\Gamma(\theta_B \psi + N) \Gamma((1 - \theta_B) \psi) \Gamma(\theta_A \psi) \Gamma((1 - \theta_A) \psi)} \\
&= p_0 \frac{\Gamma(\theta_A \psi + N) \Gamma(\theta_B \psi)}{\Gamma(\theta_B \psi + N) \Gamma(\theta_A \psi)} \\
&= p_0 \frac{(\theta_A \psi + N - 1)(\theta_A \psi + N - 2) \dots (\theta_A \psi) \Gamma(\theta_A \psi) \Gamma(\theta_B \psi)}{(\theta_B \psi + N - 1)(\theta_B \psi + N - 2) \dots (\theta_B \psi) \Gamma(\theta_B \psi) \Gamma(\theta_A \psi)} \\
&= p_0 \frac{(\theta_A \psi + N - 1)(\theta_A \psi + N - 2) \dots (\theta_A \psi)}{(\theta_B \psi + N - 1)(\theta_B \psi + N - 2) \dots (\theta_B \psi)}.
\end{aligned}$$

Note that the numerator and denominator each has N terms in it. Tommy's posterior ratio is

$$p_0 \frac{(\theta_A)^N}{(\theta_B)^N},$$

so also for Tommy, the numerator and denominator each has N terms. Furthermore, since $\frac{\theta_A}{\theta_B} > 1$, for all $k > 0$, $\frac{\theta_A + k}{\theta_B + k} < \frac{\theta_A}{\theta_B}$. Therefore,

$$p_0 \frac{(\theta_A \psi + N - 1)(\theta_A \psi + N - 2) \dots (\theta_A \psi)}{(\theta_B \psi + N - 1)(\theta_B \psi + N - 2) \dots (\theta_B \psi)} < p_0 \frac{(\theta_A)^N}{(\theta_B)^N}.$$

Moreover, the likelihood ratio clearly favors state A for both Tommy and Barney. Hence, Barney under-infers relative to Tommy. The case of N b -signals and 0 a -signals proceeds analogously.

- $F_{\beta|\Theta}^\psi(\beta|\theta)$ for our parameterized model is absolutely continuous (since it is simply the cdf of the beta distribution). Therefore, the results of Lemma 1 hold. Lemma 1 implies that Barney's subjective sampling distribution for an infinite sample has pdf $\lim_{N \rightarrow \infty} f_{S_N|\Theta}^\psi(\frac{A_s}{N} = \alpha|\theta) = f_{\beta|\Theta}^\psi(\beta = \alpha|\theta)$ a.s. Therefore, after observing proportion α of a -signals in a large sample,

Barney anticipates having likelihood ratio $\frac{f_{\mathbb{B}|\Theta}^\psi(\beta=\alpha|\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta=\alpha|\theta_B)}$ a.s. Since the beta distribution satisfies the monotone likelihood ratio ordering (for a fixed ψ) as θ changes, by Whitt (1979), $\frac{f_{\mathbb{B}|\Theta}^\psi(\beta=\alpha|\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta=\alpha|\theta_B)}$ is strictly increasing in α .

Furthermore,

$$\begin{aligned} \frac{f_{\mathbb{B}|\Theta}^\psi(\beta=0|\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta=0|\theta_B)} &= \frac{\frac{\Gamma(\psi)}{\Gamma(\theta_A\psi)\Gamma((1-\theta_A)\psi)} 0^{\theta_A\psi-1} (1-0)^{(1-\theta_A)\psi-1}}{\frac{\Gamma(\psi)}{\Gamma(\theta_B\psi)\Gamma((1-\theta_B)\psi)} 0^{\theta_B\psi-1} (1-0)^{(1-\theta_B)\psi-1}} \\ &= \frac{\Gamma(\theta_B\psi)\Gamma((1-\theta_B)\psi)}{\Gamma(\theta_A\psi)\Gamma((1-\theta_A)\psi)} \frac{0^{(\theta_A-\theta_B)\psi}}{1^{(\theta_A-\theta_B)\psi}} = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{f_{\mathbb{B}|\Theta}^\psi(\beta=1|\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta=1|\theta_B)} &= \frac{\frac{\Gamma(\psi)}{\Gamma(\theta_A\psi)\Gamma((1-\theta_A)\psi)} 1^{\theta_A\psi-1} 0^{(1-\theta_A)\psi-1}}{\frac{\Gamma(\psi)}{\Gamma(\theta_B\psi)\Gamma((1-\theta_B)\psi)} 1^{\theta_B\psi-1} 0^{(1-\theta_B)\psi-1}} \\ &= \frac{\Gamma(\theta_B\psi)\Gamma((1-\theta_B)\psi)}{\Gamma(\theta_A\psi)\Gamma((1-\theta_A)\psi)} \frac{1^{(\theta_A-\theta_B)\psi}}{0^{(\theta_A-\theta_B)\psi}} = \infty. \end{aligned}$$

Because we are using the beta density for the subjective rate distribution, Barney's beliefs put full support on $\alpha \in (0, 1)$. Hence Barney thinks his large-sample likelihood ratio will converge in distribution to a random variable whose support is $\left(\frac{f_{\mathbb{B}|\Theta}^\psi(\beta=0|\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta=0|\theta_B)}, \frac{f_{\mathbb{B}|\Theta}^\psi(\beta=1|\theta_A)}{f_{\mathbb{B}|\Theta}^\psi(\beta=1|\theta_B)} \right) = (0, \infty)$. Since Barney's posterior ratio is his likelihood ratio times his prior ratio, the result follows. \square

Lemma $\beta 5$. *If Barney has the beta-distribution functional form given by equation (3), then his subjective-rate distribution satisfies A1-A4.*

Proof: We will prove each property in turn.

- Clearly the full-support property holds since $f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) = \frac{\Gamma(\psi)}{\Gamma(\theta\psi)\Gamma((1-\theta)\psi)} \beta^{\theta\psi-1} (1-\beta)^{(1-\theta)\psi-1} > 0$ for all $\beta \in (0, 1)$ and $\theta \in (0, 1)$. This function is also clearly point-wise continuous in θ since all the components are point-wise continuous. Note that $f_{\mathbb{B}|\Theta}^\psi(\beta|\theta)$ is defined for all $\beta \in [0, 1]$, is Lebesgue-integrable with respect to β and $F_{\mathbb{B}|\Theta}^\psi(\beta|\theta) = 0 + \int_0^\beta f_{\mathbb{B}|\Theta}^\psi(x|\theta) dx$ for all β in $[0, 1]$. Therefore, $F_{\mathbb{B}|\Theta}^\psi(\beta|\theta)$ is absolutely continuous with respect to β .
- Note that

$$f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) = \frac{\Gamma(\psi)}{\Gamma(\theta\psi)\Gamma((1-\theta)\psi)} \beta^{\theta\psi-1} (1-\beta)^{(1-\theta)\psi-1} = f_{\mathbb{B}|\Theta}^\psi(1-\beta|1-\theta).$$

Therefore,

$$\int_0^{1-x} f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta = \int_0^{1-x} f_{\mathbb{B}|\Theta}^\psi(1-\beta|1-\theta) d(1-\beta).$$

The left-hand side is $F_{\mathbb{B}|\Theta}^\psi(1-x|\theta)$. Using the change-of-variables $\tilde{\beta} = 1-\beta$, the right-hand side becomes

$$\begin{aligned} - \int_1^x f_{\mathbb{B}|\Theta}^\psi(\tilde{\beta}|1-\theta) d\tilde{\beta} &= \int_x^1 f_{\mathbb{B}|\Theta}^\psi(\tilde{\beta}|1-\theta) d\tilde{\beta} \\ &= 1 - \int_0^x f_{\mathbb{B}|\Theta}^\psi(\tilde{\beta}|1-\theta) d\tilde{\beta}. \end{aligned}$$

Hence

$$F_{\mathbb{B}|\Theta}^\psi(1-x|\theta) = 1 - F_{\mathbb{B}|\Theta}^\psi(x|1-\theta).$$

- Note that

$$\begin{aligned} \frac{f_{\mathbb{B}|\Theta}^\psi(\beta|\theta')}{f_{\mathbb{B}|\Theta}^\psi(\beta|\theta)} &= \frac{\frac{\Gamma(\psi)}{\Gamma(\theta'\psi)\Gamma((1-\theta')\psi)} \beta^{\theta'\psi-1} (1-\beta)^{(1-\theta')\psi-1}}{\frac{\Gamma(\psi)}{\Gamma(\theta\psi)\Gamma((1-\theta)\psi)} \beta^{\theta\psi-1} (1-\beta)^{(1-\theta)\psi-1}} \\ &= \frac{\Gamma(\theta\psi)\Gamma((1-\theta)\psi)}{\Gamma(\theta'\psi)\Gamma((1-\theta')\psi)} \left(\frac{\beta}{1-\beta}\right)^{(\theta'-\theta)\psi}. \end{aligned}$$

If $\theta' > \theta$, then this expression is increasing in β .

- The mean of the beta distribution with parameters $\theta\psi$ and $(1-\theta)\psi$ is $\frac{\theta\psi}{\theta\psi+(1-\theta)\psi} = \theta$.

□

Lemma A. Assume Barney has the beta-distribution functional form given by equation (3). Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A \geq \theta_B$, prior $p_0 = \frac{f(\theta_A)}{f(\theta_B)} \in (0, \infty)$ and some $0 < \underline{\lambda} < \bar{\lambda} < \infty$. If Barney's posterior ratio after one signal is in $(\underline{\lambda}, \bar{\lambda})$,

$$[p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)] \subseteq (\underline{\lambda}, \bar{\lambda}),$$

then for any natural number N , we can construct a sequence of N signals, $s \in S_N$, such that for every truncation of s —i.e., for every sequence comprising only the first $n \leq N$ signals of s —the posterior ratio after that truncation, $\Pi_{\Theta \times \Theta|S_n}^\psi(\theta_A, \theta_B|s_n)$, is in $(\underline{\lambda}, \bar{\lambda})$.

Proof: We prove this by induction. By hypothesis, the result is true for the truncation $n = 1$. Assume that the likelihood ratio after N signals is in $(\underline{\lambda}, \bar{\lambda})$, and we will show that we can add a signal, and it will still be in $(\underline{\lambda}, \bar{\lambda})$. Suppose that there are A_N a -signals and $N - A_N$ b -signals.

Since

$$[p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)] \subseteq (\underline{\lambda}, \bar{\lambda})$$

and $\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B) \leq 1 \leq \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)$, it follows that $p_0 \in (\underline{\lambda}, \bar{\lambda})$.

There are two cases. The first is that Barney's posterior ratio after the sequence of N signals is (weakly) less than p_0 , and so $\Pi_{S_N|\Theta \times \Theta}^\psi(s_N|\theta_A, \theta_B) \leq 1$. In this case add an a -signal. The new posterior ratio is $p_0 \Pi_{S_{N+1}|\Theta \times \Theta}^\psi(s_N \cup a|\theta_A, \theta_B)$. Now,

$$\begin{aligned} \bar{\lambda} \geq p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B) &\geq p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B) \Pi_{S_N|\Theta \times \Theta}^\psi(s_N|\theta_A, \theta_B) \\ &\geq p_0 \Pi_{S_{N+1}|\Theta \times \Theta}^\psi(s_N \cup a|\theta_A, \theta_B) \\ &\geq p_0 \Pi_{S_N|\Theta \times \Theta}^\psi(s_N|\theta_A, \theta_B) \geq \underline{\lambda} \end{aligned}$$

The first inequality is by hypothesis, the second is due to the fact that $\Pi_{S_N|\Theta \times \Theta}^\psi(s_N|\theta_A, \theta_B) \leq 1$, the third inequality is due to Lemma $\beta 3$, the fourth inequality holds since the posterior ratio increases when an additional signal is an a -signal, and the last inequality is again by hypothesis.

The second case is that the likelihood ratio after N signals is greater than p_0 , in which case add a b -signal, and the argument is analogous.

□

Lemma B. Assume Barney has the beta-distribution functional form given by equation (3). Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A \geq \theta_B$ and prior $p_0 = \frac{f(\theta_A)}{f(\theta_B)} \in (0, \infty)$. If $p_0 \in [\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)}]$, then for any N , there exists a sequence of signals $s \in S_N$ such that for all truncations of s to its first i signals, denoted s_i for $i = 0, 1, \dots, N$, $p_0 \Pi_{S_i|\Theta \times \Theta}^\psi(s_i|\theta_A, \theta_B) \in [\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)}]$.

Proof: The lemma is a direct implication of the following claim: If

$$p_0 \in \left[\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)} \right],$$

then for any n , there exists a sequence $\{s_i\}_{i=0}^n$, where for all $i = 0 \dots n$: (i) $s_i \in S_i$, (ii) s_i is a subsequence of s_{i+1} , and (iii)

$$p_0 \Pi_{S_{i+1}|\Theta \times \Theta}^\psi(s_i \cup a|\theta_A, \theta_B) \geq 1$$

and

$$p_0 \Pi_{S_{i+1}|\Theta \times \Theta}^\psi(s_i \cup b|\theta_A, \theta_B) \leq 1.$$

In words, we can construct a sequence of signals such that Barney's posterior ratio flips around 1

with each signal. We will prove this claim by induction and then show that it implies the lemma.

Clearly for $i = 0$ the claim is true, since the prior ratio is in $\left[\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)} \right]$. Now we will assume the statement is true up to i and prove it is true for $i + 1$.

There are two cases. In the first case, assume that after i signals, it is the a -signal that flips the posterior ratio around 1: $p_0 \Pi_{S_i|\Theta \times \Theta}^\psi(s_i|\theta_A, \theta_B) \leq 1$ and $p_0 \Pi_{S_{i+1}|\Theta \times \Theta}^\psi(s_i \cup a|\theta_A, \theta_B) \geq 1$ (since adding an a -signal must increase the posterior ratio). Assume that the claim is not true for $i + 1$: that is, there is no set of $i + 1$ signals for which $p_0 \Pi_{S_{i+2}|\Theta \times \Theta}^\psi(s_{i+1} \cup a|\theta_A, \theta_B) \geq 1$ and $p_0 \Pi_{S_{i+2}|\Theta \times \Theta}^\psi(s_{i+1} \cup b|\theta_A, \theta_B) \leq 1$. In particular, taking the set of n signals plus an a -signal,

$$p_0 \Pi_{S_{i+2}|\Theta \times \Theta}^\psi(s_i \cup a \cup a|\theta_A, \theta_B) \geq 1$$

and

$$p_0 \Pi_{S_{i+2}|\Theta \times \Theta}^\psi(s_i \cup a \cup b|\theta_A, \theta_B) \geq 1.$$

Because the claim is true up to n signals, however, we know that an additional b -signal must flip the posterior ratio below 1,

$$p_0 \Pi_{S_{i+1}|\Theta \times \Theta}^\psi(s_i \cup b|\theta_A, \theta_B) \leq 1.$$

But then we have identified a set of $i + 1$ signals for which the statement is true, namely $s_i \cup b$: we know that $s_i \cup b \cup b$ must also generate a posterior ratio below 1 (since adding another b -signal must decrease the likelihood ratio); and $s_i \cup b \cup a$ generates the same posterior ratio as $s_i \cup a \cup b$, which we know is above 1. So we have a contradiction. Therefore, either $s_i \cup a$ or $s_i \cup b$ must satisfy the statement. The proof for a b -signal proceeds analogously.

To see that the statement implies the lemma, assume WLOG that $p_0 \Pi_{S_i|\Theta \times \Theta}^\psi(s_i|\theta_A, \theta_B) \leq 1$, and that $p_0 \Pi_{S_{i+1}|\Theta \times \Theta}^\psi(s_i \cup a|\theta_A, \theta_B) \geq 1$. Since

$$p_0 \Pi_{S_{i+1}|\Theta \times \Theta}^\psi(s_i \cup a|\theta_A, \theta_B) \leq p_0 \Pi_{S_i|\Theta \times \Theta}^\psi(s_i|\theta_A, \theta_B) \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)$$

(due to Lemma $\beta 3$), it follows that

$$1 \leq p_0 \Pi_{S_i|\Theta \times \Theta}^\psi(s_i|\theta_A, \theta_B) \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B).$$

Hence

$$\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)} \leq p_0 \Pi_{S_i|\Theta \times \Theta}^\psi(s_i|\theta_A, \theta_B) \leq 1 \leq \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)}.$$

The proof for the case that a b -signal flips the posterior ratio below 1 proceeds analogously.

□

Lemma C. Assume Barney has the beta-distribution functional form given by equation (3). Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A \geq \theta_B$, prior ratio $p_0 = \frac{f_\Theta(\theta_A)}{f_\Theta(\theta_B)}$, and the true state $i \in \{A, B\}$ (with corresponding true rate θ_i). Denote Barney's limit likelihood ratio conditional on the true state as $\Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, \Omega = i)$. Suppose Barney is retrospective-pooling. For any $0 < \underline{\lambda} < \bar{\lambda} < \infty$ satisfying the following two statements:

1. $p_0 \Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, \Omega = i) \in (\underline{\lambda}, \bar{\lambda})$, and
2. $[p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)] \subseteq (\underline{\lambda}, \bar{\lambda})$ or $p_0 \in \left[\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)} \right] \subseteq (\underline{\lambda}, \bar{\lambda})$,

then there exists a set of infinite sequences \hat{S}_∞ that are realized with strictly positive probability, such that for any truncation s_i , $i \in \{1, 2, \dots\}$, of any $s_\infty \in \hat{S}_\infty$,

$$\Pi_{\Theta \times \Theta|S_i}^\psi(\theta_A, \theta_B|s_i) \in (\underline{\lambda}, \bar{\lambda}).$$

Proof: The proof proceeds in three steps:

1. Show that the prior ratio is in $(\underline{\lambda}, \bar{\lambda})$. This is true under either of the assumptions in condition 2 of the lemma.

- Since $\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B) \leq 1$ and $\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B) \geq 1$, the assumption

$$[p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)] \subseteq (\underline{\lambda}, \bar{\lambda})$$

implies $p_0 \in (\underline{\lambda}, \bar{\lambda})$.

- Alternatively, we directly assume that $p_0 \in \left[\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)} \right] \subseteq (\underline{\lambda}, \bar{\lambda})$.
- Note: If $\theta_A = \theta_B$, then Barney's posterior equals his prior for any sequence of signals. Since $p_0 \in (\underline{\lambda}, \bar{\lambda})$, the conclusion of the lemma follows trivially. Hence, for the remainder of the proof, assume $\theta_A > \theta_B$.

2. Show that for any finite N , we can construct a sequence of signals such that the posterior ratio is always in $(\underline{\lambda}, \bar{\lambda})$. There are two ways we can guarantee this:

- If $[p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)] \subseteq (\underline{\lambda}, \bar{\lambda})$, then it is true by Lemma A.

- Alternatively, if we assume $p_0 \in \left[\frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)}, \frac{1}{\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)} \right] \subseteq (\underline{\lambda}, \bar{\lambda})$, then it is true by Lemma B.

3. Show that there exists a large enough number of signals \hat{N} such that if the posterior ratio after \hat{N} signals is in $(\underline{\lambda}, \bar{\lambda})$, then with positive probability the posterior ratio will always be inside $(\underline{\lambda}, \bar{\lambda})$. The rest of the proof demonstrates this. The conclusion of the lemma then follows from combining steps 1-3, taking the sequence in step 2 to have \hat{N} signals, which (because it is a finite sequence) has positive probability.

Let $g(r) \equiv \Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, r)$ denote Barney's likelihood ratio in an infinite sample where the proportion of a -signals is r . This function will allow us to map from an observed proportion r to the posterior ratio that it would imply for an infinite sample, $p_0g(r)$. Even in dealing with finite samples, it will be useful to work with the $g(\cdot)$ function in describing the mapping from the proportion of a -signals to Barney's posterior ratio. In a finite sample, however, $p_0g(r)$ will only approximate Barney's posterior ratio, but we can bound the approximation error in a large enough sample; formally, using the Law of Large Numbers, for any $r, \epsilon > 0$, and $\nu > 0$, there exists an $N_{r,\epsilon,\nu}$ such that if $N \geq N_{r,\epsilon,\nu}$, then with probability at least $1 - \nu$, the posterior ratio after observing a proportion r a -signals out of N is within ϵ of $p_0g(r)$. Since all we will need for step 3 is that the posterior ratio remains within $(\underline{\lambda}, \bar{\lambda})$ with positive probability, the ν we pick does not matter. So fix ν .

Since $p_0g(r)$ is only an approximation of Barney's posterior ratio for a finite sample, we need to impose tighter bounds than $(\underline{\lambda}, \bar{\lambda})$ on where $p_0g(r)$ can wander in a finite sample in order to ensure that Barney's posterior ratio always remains within $(\underline{\lambda}, \bar{\lambda})$. To that end, fixing ϵ , we will now pick two other numbers, $\underline{\tau} < \bar{\tau}$, such that:

- $\underline{\tau} > \underline{\lambda} + \epsilon$, and $\bar{\tau} < \bar{\lambda} - \epsilon$;
- the limit posterior ratio conditional on the true state, $p_0\Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, \Omega = i)$, is in $(\underline{\tau}, \bar{\tau})$; and
- $g^{-1}(\underline{\tau})$ and $g^{-1}(\bar{\tau})$ are both rational.

Note that $\underline{\tau}$ and $\bar{\tau}$ satisfying the above properties exist because g is continuous and monotonic in r .

We now map these bounds on the posterior ratio onto the implied bounds on the proportion of a -signals. Let \underline{r} denote the proportion of a -signals such that $p_0g(\underline{r}) = \underline{\tau}$; and \bar{r} , the proportion of

a -signals such that $p_0g(\bar{r}) = \bar{\tau}$. By assumption both \underline{r} and \bar{r} are rational. Denote the proportion of realized a -signals after N signals by r_N .

We claim that for any particular \underline{r} and \bar{r} such that the true rate $\theta_i \in (\underline{r}, \bar{r})$, there is a large enough number of signals \tilde{N} such that if there have already been \tilde{N} observed with a proportion a -signals $r_{\tilde{N}} \in (\underline{r}, \bar{r})$, then with positive probability, $r_n \in (\underline{r}, \bar{r})$ for all numbers of signals $n \geq \tilde{N}$. This is because by Chebyshev's inequality, for any N and any $\delta > 0$, $P(|r_N - \theta_i| < \delta) \geq 1 - \frac{\theta_i(1-\theta_i)}{N^2\delta^2}$. Thus, the probability that the proportion of a -signals always stays within the bounds \underline{r}, \bar{r} is

$$P\left(|r_n - \theta_i| < \delta \forall n \geq \tilde{N} \text{ s.t. } |r_{\tilde{N}} - \theta_i| < \delta\right) \geq \prod_{n=\tilde{N}}^{\infty} \left(1 - \frac{\theta_i(1-\theta_i)}{n^2\delta^2}\right).$$

Note that for any $x > 0$ and $y < 1$, it is true that $y(1-x) > y-x$, and so we can bound the right-hand side of the previous equation by iteratively applying this inequality:

$$\prod_{n=\tilde{N}}^{\infty} \left(1 - \frac{\theta_i(1-\theta_i)}{n^2\delta^2}\right) > 1 - \sum_{n=\tilde{N}}^{\infty} \frac{\theta_i(1-\theta_i)}{n^2\delta^2}.$$

Hence there is positive probability that the proportion of a -signals always stays within (\underline{r}, \bar{r}) . While not relevant for this proof, we point out for reference in proofs below that this infinite series is a known convergent sum; so as $\tilde{N} \rightarrow \infty$, the right-hand side converges to 1.

Therefore, for any ϵ , we can choose a large enough number of signals, $\hat{N} = \max\{\tilde{N}, N_{\underline{r}, \epsilon, \nu}, N_{\bar{r}, \epsilon, \nu}\}$, so that if the proportion of a -signals after \hat{N} signals is within (\underline{r}, \bar{r}) , then with positive probability the proportion always remains within (\underline{r}, \bar{r}) . Therefore, with positive probability $p_0g(r)$ always remains within $(\underline{\tau}, \bar{\tau})$. Since $\hat{N} \geq N_{\underline{r}, \epsilon, \nu}$ and $\hat{N} \geq N_{\bar{r}, \epsilon, \nu}$, whenever $p_0g(r)$ is within $(\underline{\tau}, \bar{\tau})$, Barney's posterior ratio is within $(\underline{\lambda}, \bar{\lambda})$. □

Lemma A1. *Barney-Freddy does not believe in LLN: For any $\theta \in \Theta$ and interval $[\alpha_1, \alpha_2] \subseteq [0, 1]$,*

$$\lim_{N \rightarrow \infty} \sum_{x=\lfloor \alpha_1 N \rfloor}^{\lfloor \alpha_2 N \rfloor} f_{S_N|\Theta}^{\psi M}(A_s = x|\theta) = F_{\beta|\Theta}^{\psi}(\beta = \alpha_2|\theta) - F_{\beta|\Theta}^{\psi}(\beta = \alpha_1|\theta) > 0.$$

Proof: First, we claim that conditional on a particular subjective-rate β , the agent believes that the distribution of percentage of a -signals converges to a point mass on β . To see this, fix β , and suppose Barney-Freddy has observed N signals. We begin with the case where N is an even number (i.e., the agent has observed $\frac{N}{2}$ pairs of signals, where recall that the “urn” is renewed after each pair of signals). Considering each pair of signals as a single signal that has 4 possible values and a

mean value of 2β , LLN implies that Barney-Freddy believes that the distribution of percentage of a -signals converges to a point mass on β . This proves the claim for the case where N is an even number. Now, notice that for any $[\alpha_1, \alpha_2] \subseteq [0, 1]$ and $\varepsilon > 0$, we can find a large enough \hat{N} such that for all $N \geq \hat{N}$, if $\frac{A_{sN}}{N} \in (\alpha_1, \alpha_2)$, then $\frac{A_{sN}}{N+1} \in (\alpha_1 - \varepsilon, \alpha_2 + \varepsilon)$ and $\frac{A_{sN}+1}{N+1} \in (\alpha_1 - \varepsilon, \alpha_2 + \varepsilon)$. This observation, combined with the claim for even number N , proves the claim for odd number N .

The remainder of the proof is identical to the proof of Lemma 1 below. □

0.7 Main Text Results

Lemma 1. *Assume A1-A4. Barney does not believe in LLN: for any $\theta \in \Theta$ and interval $[\alpha_1, \alpha_2] \subseteq [0, 1]$,*

$$\lim_{N \rightarrow \infty} \sum_{x=\lfloor \alpha_1 N \rfloor}^{\lceil \alpha_2 N \rceil} f_{S_N|\Theta}^\psi(A_s = x|\theta) = F_{\mathbb{B}|\Theta}^\psi(\beta = \alpha_2|\theta) - F_{\mathbb{B}|\Theta}^\psi(\beta = \alpha_1|\theta) > 0.$$

Proof: We begin by proving a key fact. Consider some rational $\alpha \in [0, 1]$ (which will represent the percentage of a -signals) and an increasing sequence of integers m_1, m_2, \dots (which will represent sample sizes) such that αm_j (which will represent numbers of a -signals) is an integer for all $j = 1, 2, \dots$. Using the definition of Barneyiness,

$$\begin{aligned} \lim_{j \rightarrow \infty} F_{S_{m_j}|\Theta}^\psi(\alpha m_j|\theta) &= \lim_{j \rightarrow \infty} \int_0^1 F_{S_{m_j}|\mathbb{B}}(\alpha m_j|\beta) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta \\ &= \int_0^1 \lim_{j \rightarrow \infty} F_{S_{m_j}|\mathbb{B}}(\alpha m_j|\beta) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta \\ &= \int_0^\alpha \lim_{j \rightarrow \infty} F_{S_{m_j}|\mathbb{B}}(\alpha m_j|\beta) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta + \int_\alpha^1 \lim_{j \rightarrow \infty} F_{S_{m_j}|\mathbb{B}}(\alpha m_j|\beta) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta \\ &= \int_0^\alpha (1) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta + \int_\alpha^1 (0) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta \\ &= F_{\mathbb{B}|\Theta}^\psi(\beta = \alpha|\theta), \end{aligned}$$

where the second equality follows from the absolute continuity of $F_{\mathbb{B}|\Theta}^\psi(\beta|\theta)$, and the fourth follows from the Law of Large Numbers.

Now, by the above fact and the definition of the cdf, for any $[a_1, a_2] \subseteq [0, 1]$ with a_1, a_2 rational,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \sum_{x=\lfloor a_1 N \rfloor}^{\lceil a_2 N \rceil} f_{S_N|\Theta}^\psi(A_s = x|\theta) &= \lim_{N \rightarrow \infty} (F_{S_N|\Theta}^\psi(\lceil a_2 N \rceil|\theta) - F_{S_N|\Theta}^\psi(\lfloor a_1 N \rfloor|\theta)) \\
&= F_{\mathfrak{B}|\Theta}^\psi(\beta = a_2|\theta) - F_{\mathfrak{B}|\Theta}^\psi(\beta = a_1|\theta).
\end{aligned}$$

Consider a sequence of pairs of rational numbers, $(a_1^{(1)}, a_2^{(1)}), (a_1^{(2)}, a_2^{(2)}), \dots$ that converges to the pair of real numbers (α_1, α_2) . Taking the limit of the above equality along this sequence gives

$$\lim_{N \rightarrow \infty} \sum_{x=\lfloor \alpha_1 N \rfloor}^{\lceil \alpha_2 N \rceil} f_{S_N|\Theta}^\psi(A_s = x|\theta) = F_{\mathfrak{B}|\Theta}^\psi(\beta = \alpha_2|\theta) - F_{\mathfrak{B}|\Theta}^\psi(\beta = \alpha_1|\theta).$$

This is greater than 0 by the full-support assumption in A1. □

Proposition 1. *Assume A1-A4. For any $\theta \in \Theta$ and $N \in \{1, 2, \dots\}$:*

1. $E_{S_N|\Theta}^\psi\left(\frac{A_s}{N}|\theta\right) = E_{S_N|\Theta}\left(\frac{A_s}{N}|\theta\right) = \theta$.
2. $F_{S_N|\Theta}(A_s|\theta)$ second-order stochastically dominates (SOSD) $F_{S_N|\Theta}^\psi(A_s|\theta)$, and $\text{Var}_{S_N|\Theta}^\psi\left(\frac{A_s}{N}|\theta\right) \geq \text{Var}_{S_N|\Theta}\left(\frac{A_s}{N}|\theta\right)$ with strict inequality for $N > 1$.
3. $\text{Var}_{S_N|\Theta}^\psi\left(\frac{A_s}{N}|\theta\right)$ is strictly decreasing in N .
4. $F_{S_N|\Theta}^\psi(A_s|\theta')$ first-order stochastically dominates (FOSD) $F_{S_N|\Theta}^\psi(A_s|\theta)$ whenever $\theta' > \theta$.

Proof:

1. Since A_s is a binomial random variable with rate θ , $E_{S_N|\Theta}\left(\frac{A_s}{N}|\theta\right) = \theta$. Using the Law of Iterated Expectations, $E_{S_N|\Theta}^\psi\left(\frac{A_s}{N}|\theta\right) = E_{\mathfrak{B}|\Theta}^\psi[E_{S_N|\mathfrak{B}}\left(\frac{A_s}{N}|\beta\right)|\theta] = E_{\mathfrak{B}|\Theta}^\psi[\beta|\theta] = \theta$, where the last equality follows from A4.
2. To keep notation compact, let T denote a random variable whose distribution is binomial distribution with rate θ . Let Y denote the random variable whose distribution is $F_{S_N|\Theta}^\psi(A_s|\theta)$, i.e., the random variable induced by taking a binomial draw using rate β and then integrating over all possible β 's using $f_{\mathfrak{B}|\Theta}^\psi(\beta|\theta)$. We will show that Y is a mean preserving spread of the binomial distribution with rate θ , which implies second-order stochastic dominance.

For Y to be a mean preserving spread of T , it must be the case the $Y = T + Z$, where Z is a random variable with conditional mean $E[Z|T] = 0$. We will construct Z .

Recall that for any random variable V , its moment generating function $M_V(t) \equiv E[e^{tV}]$ (when it exists) completely characterizes the distribution of V and has the following useful properties: $E[V] = \frac{d}{dt} M_V(t)|_{t=0}$; for random variables V, V' and V'' , $V = V' + V''$ if and only if $M_V(t) = M_{V'}(t)M_{V''}(t)$; and for random variables V and V' , $M_V(t) = E[M_{V|V'}(t)|V']$.

Since the random variable $Y|\beta$ has a binomial distribution with parameter β , its moment generating function is $M_{Y|\beta}(t) = 1 - \beta + \beta e^t$. Therefore,

$$M_Y(t) = E[M_{Y|\beta}(t)|\beta] = \int_0^1 (1 - \beta + \beta e^t) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta,$$

which clearly exists since $M_{Y|\beta}(t)$ exists for each $\beta \in [0, 1]$. If there is some Z , then we know it must have moment generating function

$$M_Z(t) = \frac{M_Y(t)}{M_T(t)} = \frac{\int_0^1 (1 - \beta + \beta e^t) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta}{1 - \theta + \theta e^t}.$$

(With this moment generating function, the random variable $T + Z$ will have the same distribution as Y .) Hence, we will have proved that $F_{S_N|\Theta}^\psi(A_s|\theta)$ SOSD $F_{S_N|\Theta}(A_s|\theta)$ once we verify that $E[Z|T] = 0$. By construction, the random variable Z is independent of T (since Z 's moment generating function, and hence its distribution, does not depend on the realization of T). Therefore, $E[Z|T] = E[Z]$. Finally,

$$\begin{aligned} E[Z] &= \left. \frac{d}{dt} M_Z(t) \right|_{t=0} \\ &= \left. \frac{(1 - \theta + \theta e^t) \int_0^1 (\beta e^t) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta - \theta e^t \int_0^1 (1 - \beta + \beta e^t) f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta}{(1 - \theta + \theta e^t)^2} \right|_{t=0} \\ &= \frac{\int_0^1 \beta f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta - \theta \int_0^1 f_{\mathbb{B}|\Theta}^\psi(\beta|\theta) d\beta}{(1 - \theta + \theta)^2} = \theta - \theta = 0. \end{aligned}$$

This proves the first claim.

Using the Law of Total Variance,

$$\begin{aligned}
Var_{S_N|\Theta}^{\psi} \left(\frac{A_s}{N} | \theta \right) &= E_{\mathbb{B}|\Theta}^{\psi} \left[Var_{S_N|\mathbb{B}} \left(\frac{A_s}{N} | \beta \right) | \theta \right] + Var_{\mathbb{B}|\Theta}^{\psi} \left[E_{S_N|\mathbb{B}} \left(\frac{A_s}{N} | \beta \right) | \theta \right] \\
&= E_{\mathbb{B}|\Theta}^{\psi} \left[\frac{\beta(1-\beta)}{N} | \theta \right] + Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta] \\
&= \frac{\theta}{N} - \frac{Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta] + \theta^2}{N} + Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta] \\
&= \frac{\theta(1-\theta)}{N} + \frac{N-1}{N} Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta],
\end{aligned}$$

where the third equality uses $E_{\mathbb{B}|\Theta}^{\psi} (\beta^2 | \theta) = Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta] + [E_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta)]^2$. Since $Var_{S_N|\Theta}^{\psi} \left(\frac{A_s}{N} | \theta \right) = \frac{\theta(1-\theta)}{N}$, the result immediately follows.

3. From part 2,

$$Var_{S_N|\Theta}^{\psi} \left(\frac{A_s}{N} | \theta \right) = \frac{\theta(1-\theta)}{N} + \frac{N-1}{N} Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta].$$

Note that the highest variance of any distribution that has support on $[0, 1]$ and a mean of θ can have is $\theta(1-\theta)$; this is the variance of a distribution with mass $(1-\theta)$ on 0 and mass θ on 1. Any shifting of weight that preserves the mean must move positive mass closer to the mean. Since $f_{\mathbb{B}|\Theta}^{\psi}$ has full support (by A1) and mean θ (by A4), $Var_{\mathbb{B}|\Theta}^{\psi} [\beta | \theta] < \theta(1-\theta)$. The result immediately follows.

4.

$$\begin{aligned}
F_{S_N|\Theta}^{\psi} (A_s | \theta) &= \sum_{i=0}^{A_s} \int_0^1 \frac{N!}{(N-i)!i!} \beta^i (1-\beta)^{N-i} f_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta) d\beta \\
&= \int_0^1 \left[\sum_{i=0}^{A_s} \frac{N!}{(N-i)!i!} \beta^i (1-\beta)^{N-i} \right] f_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta) d\beta \\
&= \int_0^1 F_{S_N|\mathbb{B}} (A_s | \beta) f_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta) d\beta,
\end{aligned}$$

where $F_{S_N|\mathbb{B}} (A_s | \beta)$ is the cdf of the binomial distribution conditional on a rate β . By A3, $F_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta')$ FOSD $F_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta)$ whenever $\theta' > \theta$, and therefore $\int_0^1 g(\beta) f_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta') d\beta > \int_0^1 g(\beta) f_{\mathbb{B}|\Theta}^{\psi} (\beta | \theta) d\beta$ for any function $g(\beta)$ that is decreasing in β . Since $F_{S_N|\mathbb{B}} (A_s | \beta)$ is decreasing in β , the result follows. □

Proposition 2. *Assume A1-A4. Let $\theta \in \Theta$ be the true rate. Then for any $\theta_A, \theta_B \in \Theta$ and prior $f_{\Theta}(\theta_A) = 1 - f_{\Theta}(\theta_B) \in (0, 1)$ Barney draws limited inference even from an infinite sample:*

as $N \rightarrow \infty$, Barney's posterior ratio converges almost surely (with respect to the true probability distribution over events) to a positive, finite number:

$$\frac{f_{\Theta|S_N}^{\psi}(\theta_A|s)}{f_{\Theta|S_N}^{\psi}(\theta_B|s)} \xrightarrow{a.s.} \frac{f_{B|\Theta}^{\psi}(\beta = \theta|\theta_A) f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta = \theta|\theta_B) f(\theta_B)}. \quad (5)$$

Proof: Using the Law of Large Numbers, and the true probability distribution over realizations of infinite samples (i.e., events):

$$\lim_{N \rightarrow \infty} \frac{f_{\Theta|S_N}^{\psi}(\theta_A|s)}{f_{\Theta|S_N}^{\psi}(\theta_B|s)} = \left(\lim_{N \rightarrow \infty} \frac{f_{S_N|\Theta}^{\psi}(s|\theta_A)}{f_{S_N|\Theta}^{\psi}(s|\theta_B)} \right) \frac{f(\theta_A)}{f(\theta_B)} = \frac{f_{B|\Theta}^{\psi}(\beta = \theta|\theta_A) f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta = \theta|\theta_B) f(\theta_B)} \text{ a.s.}$$

□

Proposition 3. Assume A1-A4. Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and any prior $f_{\Theta}(\theta_A) = 1 - f_{\Theta}(\theta_B) \in (0, 1)$. Before having observed any data, Tommy believes: if the rate is θ_A , then his limit posterior probability that the rate is θ_A is 1. In contrast, before having observed any data, Barney believes: if the rate is θ_A , then his limit posterior probability that the rate is θ_A is a random variable that has positive density on a nondegenerate interval in $[0, 1]$. If we strengthen assumption A1 to A1', then, in addition, the interval is closed and is a strict subset of $[0, 1]$.

Proof: The proof for Tommy is entirely standard and follows directly from the Law of Large Numbers. For Barney, recall that Lemma 1 implies that Barney's subjective sampling distribution for an infinite sample is $\lim_{N \rightarrow \infty} f_{S_N|\Theta}^{\psi}(\frac{A_s}{N} = \alpha|\theta) = f_{B|\Theta}^{\psi}(\beta = \alpha|\theta)$ a.s. Therefore, after observing proportion α of a -signals in a large sample, Barney anticipates having posterior ratio arbitrarily close to $\lim_{N \rightarrow \infty} \frac{f_{\Theta|S_N}^{\psi}(\theta_A|\frac{A_s}{N}=\alpha)}{f_{\Theta|S_N}^{\psi}(\theta_B|\frac{A_s}{N}=\alpha)} = \frac{f_{B|\Theta}^{\psi}(\beta=\alpha|\theta_A)f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta=\alpha|\theta_B)f(\theta_B)}$. Barney believes that this large-sample posterior ratio is a random variable because, according to A1, Barney's beliefs put full support on $\alpha \in (0, 1)$.

By A3 and Whitt (1979), $\frac{f_{B|\Theta}^{\psi}(\beta=\alpha|\theta_A)f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta=\alpha|\theta_B)f(\theta_B)}$ is strictly increasing in α . It follows that

$$\inf_{\alpha \in (0,1)} \frac{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_A) f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_B) f(\theta_B)} = \lim_{\alpha \rightarrow 0} \frac{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_A) f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_B) f(\theta_B)} < 1$$

and

$$\sup_{\alpha \in (0,1)} \frac{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_A) f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_B) f(\theta_B)} = \lim_{\alpha \rightarrow 1} \frac{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_A) f(\theta_A)}{f_{B|\Theta}^{\psi}(\beta = \alpha|\theta_B) f(\theta_B)} > 1.$$

Furthermore, by the Intermediate Value Theorem, the posterior ratio must take on each value in

$$\left(\inf_{\alpha \in (0,1)} \frac{f_{\mathbb{B}|\Theta}^{\psi}(\beta = \alpha|\theta_A) f(\theta_A)}{f_{\mathbb{B}|\Theta}^{\psi}(\beta = \alpha|\theta_B) f(\theta_B)}, \sup_{\alpha \in (0,1)} \frac{f_{\mathbb{B}|\Theta}^{\psi}(\beta = \alpha|\theta_A) f(\theta_A)}{f_{\mathbb{B}|\Theta}^{\psi}(\beta = \alpha|\theta_B) f(\theta_B)} \right)$$

for some $\alpha \in (0, 1)$. It follows that, as α ranges from 0 from 1, the limit posterior probability that the rate is θ_A takes values in a nondegenerate interval in $[0, 1]$. By A1, Barney assigns positive density to each possible realization of $\alpha \in (0, 1)$, and so all possible posteriors in that interval have a positive density of being realized. Under A1',

$$\lim_{N \rightarrow \infty} \frac{f_{\Theta|S_N}^{\psi}(\theta_A | \frac{A_s}{N} = 0)}{f_{\Theta|S_N}^{\psi}(\theta_B | \frac{A_s}{N} = 0)} = \frac{f_{\mathbb{B}|\Theta}^{\psi}(\beta = 0|\theta_A) f(\theta_A)}{f_{\mathbb{B}|\Theta}^{\psi}(\beta = 0|\theta_B) f(\theta_B)}$$

and

$$\lim_{N \rightarrow \infty} \frac{f_{\Theta|S_N}^{\psi}(\theta_A | \frac{A_s}{N} = 1)}{f_{\Theta|S_N}^{\psi}(\theta_B | \frac{A_s}{N} = 1)} = \frac{f_{\mathbb{B}|\Theta}^{\psi}(\beta = 1|\theta_A) f(\theta_A)}{f_{\mathbb{B}|\Theta}^{\psi}(\beta = 1|\theta_B) f(\theta_B)}$$

are simply real numbers strictly greater than 0 and less than 1. The result follows. \square

Proposition 4. *Assume A1-A4. Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and prior $f_{\Theta}(\theta_A) = 1 - f_{\Theta}(\theta_B) \in (0, 1)$. For $N = 1$, Barney and Tommy infer the same. If $\theta_A = 1 - \theta_B$, then for any set of $N \in \{1, 2, \dots\}$ signal realizations $s \in S_N$, neither Tommy's beliefs nor Barney's beliefs change from the priors when $\frac{A_s}{N} = \frac{1}{2}$.*

Proof: The first claim can be seen from the fact that for a single sample, the subjective sampling distribution of Tommy matches that of Barney.

To see that the second claim is true, suppose that Barney observes exactly half a and b signals, k of each signal (so that $N - k = k$). Barney's likelihood ratio is

$$\begin{aligned} \frac{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|\theta_A) d\beta}{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|\theta_B) d\beta} &= \frac{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|\theta_A) d\beta}{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|1-\theta_A) d\beta} \\ &= \frac{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|\theta_A) d\beta}{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(1-\beta|\theta_A) d\beta} \\ &= \frac{\int_0^1 \frac{N!}{k!k!} \beta^k (1-\beta)^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|\theta_A) d\beta}{\int_0^1 \frac{N!}{k!k!} (1-\beta)^k \beta^k f_{\mathbb{B}|\Theta}^{\psi}(\beta|\theta_A) d\beta} \\ &= 1, \end{aligned}$$

where the first equality comes from substituting $\theta_B = 1 - \theta_A$, and the second equality follows from

A2. This directly implies that Barney's likelihood ratio is equal to 1 if and only if Tommy's is, since the likelihood ratio is strictly increasing in the number of a -signals for both Barney and Tommy if $\theta_A > \frac{1}{2}$, and equal to 1 for any sample for both Barney and Tommy if $\theta_A = \frac{1}{2}$. \square

Proposition 5. *Assume Barney has the beta-distribution functional form given by equation (3). Fix rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$, prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$ and a set of $N \in \{1, 2, \dots\}$ signal realizations $s \in S_N$. Regardless of whether the true rate is θ_A or θ_B , for ψ sufficiently small, the expected change in Barney's beliefs is smaller than the expected change in Tommy's beliefs. Furthermore, suppose $\theta_A = 1 - \theta_B$. Then for any sample of $N > 1$ signals such that $\frac{A_s}{N} \neq \frac{1}{2}$ and any ψ , Barney under-infers relative to Tommy. In addition, while Tommy's inference depends solely on the difference in the number of a and b signals, Barney's change in beliefs is smaller from larger samples with the same difference.*

Proof: We will prove the first claim in two steps. First, we shall show that a sufficient condition is that Tommy's subjective sampling distribution is Blackwell-sufficient for Barney's. Second, we will show that this sufficient condition in fact holds for ψ sufficiently small.

First, suppose $F_{S_N|\Theta}(A_s|\theta_i)$ is Blackwell-sufficient for $F_{S_N|\Theta}^\psi(A_s|\theta_i)$ for $i \in \{A, B\}$. Using Blackwell (1951, 1953), note that $F_{S_N|\Theta}(A_s|\theta_i)$ is Blackwell-sufficient for $F_{S_N|\Theta}^\psi(A_s|\theta_i)$ if and only if for any continuous, convex function g ,

$$\begin{aligned} & \sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_i) g\left(\frac{f_{S_N|\Theta}(A_s = k|\theta_i)}{f_{S_N|\Theta}(A_s = k|\theta_{-i})}\right) \\ & \geq \sum_{k=0}^N f_{S_N|\Theta}^\psi(A_s = k|\theta_i) g\left(\frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_i)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_{-i})}\right) \end{aligned}$$

for $i \in \{A, B\}$ and $-i$ being the other state. Taking $g(x) = |p_0 - p_0 x|$, this implies

$$\begin{aligned} & \sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_i) \left| p_0 - p_0 \frac{f_{S_N|\Theta}(A_s = k|\theta_i)}{f_{S_N|\Theta}(A_s = k|\theta_{-i})} \right| \\ & \geq \sum_{k=0}^N f_{S_N|\Theta}^\psi(A_s = k|\theta_i) \left| p_0 - p_0 \frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_i)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_{-i})} \right| \end{aligned}$$

for $i \in \{A, B\}$.

From Proposition 1, we know that $f_{S_N|\Theta}^\psi(A_s|\theta_i)$ is a mean-preserving spread of $f_{S_N|\Theta}(A_s|\theta_i)$.

Since $g(x) = |p_0 - p_0 x|$ is a continuous convex function, it follows that

$$\begin{aligned} & \sum_{k=0}^N f_{S_N|\Theta}^\psi(A_s = k|\theta_i) \left| p_0 - p_0 \frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_i)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_{-i})} \right| \\ & \geq \sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_i) \left| p_0 - p_0 \frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_i)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_{-i})} \right| \end{aligned}$$

for $i \in \{A, B\}$. Chaining the two inequalities,

$$\begin{aligned} & \sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_i) \left| p_0 - p_0 \frac{f_{S_N|\Theta}(A_s = k|\theta_i)}{f_{S_N|\Theta}(A_s = k|\theta_{-i})} \right| \\ & \geq \sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_i) \left| p_0 - p_0 \frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_i)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_{-i})} \right|. \end{aligned}$$

Aggregating over states weighted by the priors,

$$\begin{aligned} & f(\theta_A) \left(\sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_A) \left| p_0 - p_0 \frac{f_{S_N|\Theta}(A_s = k|\theta_A)}{f_{S_N|\Theta}(A_s = k|\theta_B)} \right| \right) \\ & + f(\theta_B) \left(\sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_B) \left| p_0 - p_0 \frac{f_{S_N|\Theta}(A_s = k|\theta_B)}{f_{S_N|\Theta}(A_s = k|\theta_A)} \right| \right) \\ & \geq f(\theta_A) \left(\sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_A) \left| p_0 - p_0 \frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_A)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_B)} \right| \right) \\ & + f(\theta_B) \left(\sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_B) \left| p_0 - p_0 \frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_B)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_A)} \right| \right). \end{aligned}$$

This is what we were seeking to prove.

Now we will show that for $i \in \{A, B\}$ and $N > 1$, $F_{S_N|\Theta}(A_s|\theta_i)$ is Blackwell-sufficient for $F_{S_N|\Theta}^\psi(A_s|\theta_i)$ if ψ is sufficiently small. Since Barney's subjective sampling distribution is the same as Tommy's for $N = 1$, $F_{S_1|\Theta}(A_s|\theta_i)$ is Blackwell-equivalent to $F_{S_1|\Theta}^\psi(A_s|\theta_i)$ (regardless of the value of ψ). Note that as $\psi \rightarrow 0$, Barney's subjective sampling distribution converges to point masses at 0 and 1. Hence in the limit $\psi \rightarrow 0$, for any N , Barney expects to observe all a 's or all b 's; therefore, in the limit $\psi \rightarrow 0$, $F_{S_N|\Theta}^\psi(A_s|\theta_i)$ is Blackwell-equivalent to $F_{S_1|\Theta}^\psi(A_s|\theta_i)$. In contrast, for Tommy, $F_{S_N|\Theta}(A_s|\theta_i)$ for $N > 1$ is strictly Blackwell-sufficient for $F_{S_1|\Theta}(A_s|\theta_i)$ (e.g., Shaked and Tong, 1990). Therefore, for any $N > 1$, in the limit $\psi \rightarrow 0$, $F_{S_N|\Theta}(A_s|\theta_i)$ is Blackwell-sufficient for $F_{S_N|\Theta}^\psi(A_s|\theta_i)$. Using the necessary and sufficient inequality for Blackwell-sufficiency: for any

$N > 1$ and any continuous, convex function g , in the limit $\psi \rightarrow 0$,

$$\begin{aligned} & \sum_{k=0}^N f_{S_N|\Theta}(A_s = k|\theta_i) g\left(\frac{f_{S_N|\Theta}(A_s = k|\theta_i)}{f_{S_N|\Theta}(A_s = k|\theta_{-i})}\right) \\ & > \sum_{k=0}^N f_{S_N|\Theta}^\psi(A_s = k|\theta_i) g\left(\frac{f_{S_N|\Theta}^\psi(A_s = k|\theta_i)}{f_{S_N|\Theta}^\psi(A_s = k|\theta_{-i})}\right). \end{aligned}$$

Note that the right-hand side of this inequality is continuous in ψ . Therefore, the strict inequality holds for any sufficiently small $\psi > 0$, and so $F_{S_N|\Theta}(A_s|\theta_i)$ is Blackwell-sufficient for $F_{S_N|\Theta}^\psi(A_s|\theta_i)$ for any sufficiently small $\psi > 0$.

To see that the second claim in the proposition is true, consider some sample of size N , with A_s realizations of heads, and assume that $\theta_A > \frac{1}{2}$ and $A_s > N - A_s$. (The other cases have analogous proofs.) Tommy's likelihood ratio is

$$\frac{\frac{N!}{\theta_A!(N-\theta_A)!} \theta_A^{A_s} (1-\theta_A)^{N-A_s}}{\frac{N!}{\theta_A!(N-\theta_A)!} \theta_A^{N-A_s} (1-\theta_A)^{A_s}} = \left(\frac{\theta_A}{1-\theta_A}\right)^{2A_s-N},$$

which has $2A_s - N$ terms multiplied together. Barney's likelihood ratio is

$$\frac{\Gamma(\theta_A\psi + A_s)}{\Gamma((1-\theta_A)\psi + A_s)} \frac{\Gamma((1-\theta_A)\psi + N - A_s)}{\Gamma(\theta_A\psi + N - A_s)} = \frac{(\theta_A\psi + A_s - 1) \dots (\theta_A\psi + N - A_s)}{((1-\theta_A)\psi + A_s - 1) \dots ((1-\theta_A)\psi + N - A_s)}.$$

This also has $2A_s - N$ terms. Since by assumption $\frac{\theta_A\psi}{(1-\theta_A)\psi} = \frac{\theta_A}{1-\theta_A} > 1$, then for any $k > 0$,

$$\frac{\theta_A}{1-\theta_A} = \frac{\theta_A\psi}{(1-\theta_A)\psi} > \frac{\theta_A\psi + k}{(1-\theta_A)\psi + k} > 1,$$

which implies that

$$\left(\frac{\theta_A}{1-\theta_A}\right)^{2A_s-N} > \frac{(\theta_A\psi + A_s - 1) \dots (\theta_A\psi + N - A_s)}{((1-\theta_A)\psi + A_s - 1) \dots ((1-\theta_A)\psi + N - A_s)} > 1.$$

Hence Barney under-infers relative to Tommy.

Finally, consider two samples, one of size N with A_s a -signals, and the other of size $N' > N$ with A'_s a -signals. Suppose both samples have the same difference, d , between the number of a and b signals: $d = A_s - (N - A_s) = 2A_s - N$ and $d = 2A'_s - N'$. Since Tommy's likelihood ratio is $\left(\frac{\theta_A}{1-\theta_A}\right)^d$, it is the same for both samples. To consider Barney's inference, assume that $\theta_A > \frac{1}{2}$ and $d > 0$. (The other cases have analogous proofs.) Barney's likelihood ratio, stated above, has d

terms for both samples. Since by assumption $\frac{\theta_A \psi}{(1-\theta_A)\psi} = \frac{\theta_A}{1-\theta_A} > 1$, then for any $k' > k > 0$,

$$\frac{\theta_A \psi + k}{(1-\theta_A)\psi + k} > \frac{\theta_A \psi + k'}{(1-\theta_A)\psi + k'} > 1.$$

Note that $A'_s > A_s$. It follows that each term in Barney's likelihood ratio for the sample of size N is larger than the corresponding term in Barney's likelihood ratio for the sample of size N' (e.g., for the first term, $\frac{\theta_A \psi + A_s - 1}{(1-\theta_A)\psi + A_s - 1} > \frac{\theta_A \psi + A'_s - 1}{(1-\theta_A)\psi + A'_s - 1}$). Hence while Barney infers in favor of state A in both cases (like Tommy does), Barney's change in beliefs is smaller from the larger sample. \square

Proposition 6. *Assume A1-A4. Suppose Barney and Tommy have simple, piecewise-linear loss-averse preferences as specified in (4). Fix any gamble (θ, h, t) , paying off $h > 0$ with probability θ and $-t$ with probability $1 - \theta$, that is better than fair: $\theta h > (1 - \theta)t$. For any $\lambda \geq 1$, there is some $N' \geq 1$ such that if $N > N'$, then Tommy will accept N repetitions of the gamble. In contrast, for Barney there is some threshold level of loss aversion $\hat{\lambda} > 1$ such that: if $\lambda < \hat{\lambda}$, then there is some N' sufficiently large such that Barney will accept N repetitions of the gamble for all $N > N'$; and if $\lambda \geq \hat{\lambda}$, then there is some N'' sufficiently large such that Barney will reject N repetitions of the gamble for all $N > N''$.*

Proof: WLOG let the reference point be w_0 , and fix $\lambda \geq 1$. Denoting $G(z)$ as the distribution of monetary outcomes z , the expected utility of any lottery is:

$$\begin{aligned} & \int_0^\infty (w_0 + z) dG(z) + \int_{-\infty}^0 (w_0 + \lambda z) dG(z) \\ = & w_0 + \int_0^\infty z dG(z) + \lambda \int_{-\infty}^0 z dG(z). \end{aligned}$$

Clearly, this is better than the option of refusing the lottery if and only if

$$\int_0^\infty z dG(z) > -\lambda \int_{-\infty}^0 z dG(z) \Leftrightarrow \frac{\int_0^\infty z dG(z)}{-\int_{-\infty}^0 z dG(z)} < \lambda.$$

The monetary outcome of an N -times repeated gamble is $z = A_s h - (N - A_s)t = -Nt + A_s(h + t) = N(-t + \frac{A_s}{N}(h + t))$. Hence the agent earns a positive payoff from the gamble if and only if

$A_s \geq \left\lceil \frac{t}{h+t} N \right\rceil$. Substituting in, we find that for Tommy,

$$\frac{\int_0^\infty z dG(z)}{-\int_{-\infty}^0 z dG(z)} = \frac{\sum_{A_s=\lceil \frac{t}{h+t} N \rceil}^N (-t + \frac{A_s}{N} (h+t)) f_{S_N|\Theta}(A_s|\theta)}{-\sum_{A_s=0}^{\lceil \frac{t}{h+t} N \rceil-1} (-t + \frac{A_s}{N} (h+t)) f_{S_N|\Theta}(A_s|\theta)},$$

where θ is the known rate of the good outcome (and the N 's in the numerator and denominator cancel out).

Since the bet is better-than-fair, the probability of losing money goes to 0 as $N \rightarrow \infty$. Therefore, as $N \rightarrow \infty$, the denominator goes to 0. As $N \rightarrow \infty$, $\frac{A_s}{N}$ converges almost surely to θ , so the numerator goes to $-t + \theta(h+t) \geq 0$. Hence the ratio of the numerator to the denominator goes to infinity. This implies that Tommy always accepts the N -times repeated gamble for N sufficiently large.

Alternatively, for Barney,

$$\begin{aligned} \frac{\int_0^\infty z dG(z)}{-\int_{-\infty}^0 z dG(z)} &= \frac{\sum_{A_s=\lceil \frac{t}{h+t} N \rceil}^N (-t + \frac{A_s}{N} (h+t)) f_{S_N|\Theta}^\psi(A_s|\theta)}{-\sum_{A_s=0}^{\lceil \frac{t}{h+t} N \rceil-1} (-t + \frac{A_s}{N} (h+t)) f_{S_N|\Theta}^\psi(A_s|\theta)} \\ &\rightarrow \frac{\int_{\frac{t}{h+t}}^1 (-t + \beta(h+t)) f_{B|\Theta}^\psi(\beta|\theta) d\beta}{-\int_0^{\frac{t}{h+t}} (-t + \beta(h+t)) f_{B|\Theta}^\psi(\beta|\theta) d\beta} \end{aligned}$$

almost surely as $N \rightarrow \infty$. This is a finite, positive number. □

Proposition 7. *Assume A1-A4. Suppose Barney is prospective-acceptive. Fix payoffs $u(\mu, \omega)$, rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$, prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, and the cost of asking a friend $c_f > 0$. Suppose that knowing the state is valuable: $u(\mu_A, A) > u(\mu_B, A)$ and $u(\mu_A, B) < u(\mu_B, B)$. Furthermore, suppose that c_f is small enough that if Consumer Reports were not available, Tommy would ask the friend. If the number of signals N in Consumer Reports is sufficiently large, then there exist thresholds c'_r and c''_r with $c_f < c'_r < c''_r$ such that: if $c_r < c'_r$, then both Tommy and Barney buy Consumer Reports; if $c_r > c''_r$, then both Tommy and Barney ask the friend; and if $c_r \in (c'_r, c''_r)$, then Tommy buys Consumer Reports while Barney asks the friend.*

Proof: First, the fact that in the absence of *Consumer Reports*, Tommy will ask a friend imme-

diately implies the same about Barney, since they infer the same from a single signal.

We will prove the proposition in two steps. First we will show that both Tommy and Barney follow a threshold rule, purchasing *Consumer Reports* if and only if c_r is less than some threshold. Then we will show that Barney's threshold c'_r is less than Tommy's threshold c''_r .

Let q denote an arbitrary posterior ratio. Denote the set of possible q 's that could occur after asking a friend as Q_1 (the same for Tommy and Barney), Tommy's set of possible q 's after reading *Consumer Reports* as Q_N , and Barney's set of possible q 's after reading *Consumer Reports* as Q_N^ψ . Denote the expected value of taking the decision after asking a friend—which is the same for Tommy and Barney—as $V(1) = E_s[\max_\mu E_q[u(\mu, \omega)|q \in Q_1]|s \in S_1]$. (Note that this does not include the cost of asking a friend.) Denote the *subjective* expected value of taking the decision, after reading *Consumer Reports*, as $V(N) = E_s[\max_\mu E_q[u(\mu, \omega)|q \in Q_N]|s \in S_N]$ for Tommy and as $V^\psi(N) = E_s^\psi[\max_\mu E_q[u(\mu, \omega)|q \in Q_N^\psi]|s \in S_N]$ for Barney. Denote the value of knowing the state for sure, i.e., having full information, as $V(fi)$. Since the agents would like to condition their actions, $V(1) < V(fi)$.

For all $N' > N \geq 1$, a sample size of N' is strictly Blackwell-more-informative than a sample size of N . Therefore, for each of Tommy and Barney, the subjectively-anticipated distribution of posteriors after N' signals is a strict mean-preserving spread of the subjectively-anticipated distribution of posteriors after N signals. Because each of the two actions is better in one of the two states, $V(N') > V(N)$ and $V^\psi(N') > V^\psi(N)$; that is, $V(N)$ and $V^\psi(N)$ are both strictly increasing in N .

Since for any $N > 1$, $V(N) > V(1)$ and $V^\psi(N) > V(1)$, both Tommy and Barney follow a threshold rule.

Denote the agents' value of an infinite number of signals as $V(\infty) = \lim_{N \rightarrow \infty} V(N)$ and $V^\psi(\infty) = \lim_{N \rightarrow \infty} V^\psi(N)$. Since Tommy expects to learn the state almost surely after an infinite number of signals, $V(\infty) = V(fi)$. Since Barney believes his posterior will place positive weight on both states even in an infinite sample, and since Barney would like to condition his action on the state, $V^\psi(\infty) < V(fi)$. Therefore, there exists N' sufficiently large that for all $N > N'$, $V(N) > V^\psi(N)$. In that case, Barney's threshold c'_r is less than Tommy's threshold c''_r .

□

Proposition 8. *Assume A1' and A2-A4. Suppose that the agent is deciding whether to buy Consumer Reports at cost c_r or not obtain any signals. Furthermore, fix payoffs $u(\mu, \omega)$ so that knowing the state is valuable: $u(\mu_A, A) > u(\mu_B, A)$ and $u(\mu_A, B) < u(\mu_B, B)$. For Tommy: for all rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and priors $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, there exists a threshold $c_r^* > 0$ such that if $c_r < c_r^*$, then as long as the number of signals N in Consumer Reports*

is sufficiently large, he buys *Consumer Reports*. In contrast, for prospective-acceptive Barney: (i) for all rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$, there exist priors $f_\Theta(\theta_A)$ such that for any $c_r > 0$ and any N , he does not buy *Consumer Reports*; and (ii) for all priors $f_\Theta(\theta_A)$ at which he is not indifferent between μ_A and μ_B , there exist rates $\theta_A, \theta_B \in \Theta$, where $\theta_A > \theta_B$, such that for any $c_r > 0$ and any N , he does not buy *Consumer Reports*.

Proof: As in the previous proof, denote the *subjective* expected value of taking the decision, after observing N signals as $V(N)$ for Tommy and as $V^\psi(N)$ for Barney. Denote the subjective expected value of taking the decision without observing any signals as $V(\emptyset)$. Denote the value of knowing the state for sure, i.e., having full information, as $V(fi)$.

By construction $V(fi) > V(\emptyset)$. Denote $V(fi) - V(\emptyset) = c_r^*$. Tommy believes that as the number of signals goes to ∞ , his posterior converges almost surely to placing a weight of 1 on the true state. Hence $V(N)$ converges to $V(fi)$. Therefore, the proposition holds for Tommy.

Now, note that for both Barney and Tommy, there exists a threshold posterior ratio,

$$\tau = \frac{\frac{u(\mu_A, B) - u(\mu_B, B)}{u(\mu_B, A) + u(\mu_A, B) - u(\mu_B, B) - u(\mu_A, A)}}{1 - \frac{u(\mu_A, B) - u(\mu_B, B)}{u(\mu_B, A) + u(\mu_A, B) - u(\mu_B, B) - u(\mu_A, A)}},$$

such that if the posterior ratio is less than τ , the agent takes action μ_B ; if the posterior ratio is greater than τ the agent takes action μ_A ; and if the posterior ratio is equal to τ , the agent is indifferent between either action. Let $p_0 = \frac{f_\Theta(\theta_A)}{1 - f_\Theta(\theta_A)}$ be the prior ratio. Also note that (due to A1') Barney believes that there exist $0 < \underline{L} < \bar{L} < \infty$ such that as N goes to ∞ , his likelihood ratio will converge to a random variable L_∞ that has support over a bounded interval that is a subset of $[\underline{L}, \bar{L}]$. This means that his subjective posterior will converge to $p_0 L_\infty$ that has support on $[p_0 \underline{L}, p_0 \bar{L}]$.

To prove (i), notice that we can always find a value f^* large enough such that for all $f_\Theta(\theta_A) > f^*$, $p_0 \underline{L} > \tau$. This means that regardless of how many signals Barney observes, the value of the information contained in those signals is 0 (because Barney will take action μ_A no matter what). In this case, Barney will not purchase *Consumer Reports* for any positive price, regardless of the number of signals.

Next we prove (ii). Without loss of generality assume $p_0 > \tau$. Because the subjective-rate distribution is point-wise continuous in θ , for any $0 < \epsilon < 1$, we can find $\theta_A(\epsilon)$ and $\theta_B(\epsilon)$ close enough together such that $\epsilon < \frac{f_{B|\Theta}^\psi(\beta=0|\theta_A)}{f_{B|\Theta}^\psi(\beta=0|\theta_B)} < 1$. Recall that $\frac{f_{B|\Theta}^\psi(\beta=0|\theta_A)}{f_{B|\Theta}^\psi(\beta=0|\theta_B)} = \underline{L}$, the minimum possible limit likelihood ratio for Barney. Therefore, $\epsilon < \underline{L} < 1$. Furthermore, there exists an ϵ^* close enough to 1 such that for all $\epsilon^* < \epsilon < 1$, $p_0 \epsilon > \tau$. Therefore, for all $\epsilon^* < \epsilon < 1$, for rates $\theta_A(\epsilon)$ and $\theta_B(\epsilon)$, $p_0 \underline{L} > \tau$. As above, this implies that Barney will not purchase *Consumer Reports* for any

positive price, regardless of the number of signals.

□

Proposition 9. *Assume Barney has the beta-distribution functional form given by equation (3). Fix payoffs $u(\mu, \omega)$, rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and prior $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$. Suppose Barney is prospective-acceptive and retrospective-pooling.*

1. *For all $\bar{p} < 1$, there exists $\bar{c} > 0$ such that for all $c \leq \bar{c}$, Barney buys an infinite number of signals with probability $p > \bar{p}$. Furthermore, suppose that Barney, before buying any signals, has a positive probability of buying an infinite number of signals. Then for any $\varepsilon > 0$, there exists $N_\varepsilon > 2$ such that if Barney buys an additional signal after having already bought N_ε signals, the probability of Barney buying a finite number of signals from then on is less than ε .*
2. *Suppose $\theta_A = 1 - \theta_B$. Suppose Barney is willing to buy an additional signal when his posterior probability (of state A) is equal to q , and suppose Barney's posterior is q after observing N signals. If Barney's posterior probability of state A is q after observing $N' > N$ signals, then the probability that Barney will buy an infinite number of signals is weakly higher after he has observed the N' signals than it was after the N signals.*
3. *Again, suppose $\theta_A = 1 - \theta_B$ and the prior $f_\Theta(\theta_A) \geq .5$. For any $\varepsilon > 0$, there exists $N > 2$ such that if Barney chooses an action after buying at least N signals, then the likelihood ratio of Barney having taken the action that does not match the state to the action that matches the state is less than ε .*

Proof: Since Barney is prospective-acceptive and gets one signal at a time, he conceives of his problem as in the classic sequential information acquisition setting of Wald (1947). Therefore, we can characterize his optimal policy using two thresholds in the posterior ratio. Denote the lower and upper thresholds by $\underline{\lambda}$ and $\bar{\lambda}$, respectively. These thresholds are functions of the cost of signals and the payoffs for each action in each state (but not of the current posterior ratio). So long as Barney's current posterior ratio is in $(\underline{\lambda}, \bar{\lambda})$, he will continue to acquire information. If the posterior ratio ever is in the region $(0, \underline{\lambda}] \cup [\bar{\lambda}, 1)$, Barney will stop and take an action.

To prove the first part, assume WLOG that state A is the true state. Since Barney is retrospective-pooling, for any two values m and n such that $0 < m < n < \infty$, we can find a c' small enough so that if $c \leq c'$, then $\underline{\lambda} < m < n < \bar{\lambda}$. This is because, as in Wald (1947), the upper (lower) threshold is strictly increasing (decreasing) in c without bound.

Since $0 < p_0 < \infty$ and Barney's limit likelihood ratio $\Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, \Omega = i)$ is finite, his limit posterior ratio is bounded away from 0 and ∞ :

$$0 < p_0 \Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, \Omega = i) < \infty.$$

Moreover, since the posterior ratios after just a single a -signal or a single b -signal are bounded away from 0 and ∞ , we can find a small enough $\hat{c} > 0$ so that both the following statements are true for all $c < \hat{c}$:

1. $[p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), p_0 \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)] \subseteq (\underline{\lambda}, \bar{\lambda})$
2. $p_0 \Pi_{S_\infty|\Theta \times \Theta}^\psi(s_\infty|\theta_A, \theta_B, \Omega = i) \in (\underline{\lambda}, \bar{\lambda})$.

Therefore, by the proof of Lemma C, if $c < c'' \equiv \min\{c', \hat{c}\}$, then the probability that Barney purchases an infinite number of signals, conditional on still purchasing signals after N signals have already been purchased, is increasing in N and converges to 1 as N goes to infinity.

Fix $\bar{p} < 1$. By the above argument, for any $c < c''$, we find an \bar{N} such that the probability that Barney purchases an infinite number of signals, conditional on Barney still purchasing signals after \bar{N} signals have already been purchased, is larger than \bar{p} . Now, there exist $0 < m < n < \infty$ sufficiently far apart such that the probability equals 1 that Barney's posterior ratio lies the region (m, n) after he observes \bar{N} signals. We can find c''' sufficiently small that for all $c \leq c'''$, $\underline{\lambda} < m < n < \bar{\lambda}$. It follows that if $c < \bar{c} \equiv \min\{c'', c'''\}$, then Barney purchases an infinite number of signals with probability larger than \bar{p} .

Turning to the "Furthermore" claim, for Barney to stop purchasing signals, his posterior ratio must leave

$[\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)]$. Since Barney has a positive probability of purchasing an infinite number of signals, from the proof of Lemma C, we know that if he still wants to purchase an additional signal after having already purchased \hat{N} signals, the probability of the posterior ratio leaving $[\Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B), \Pi_{S_1|\Theta \times \Theta}^\psi(a|\theta_A, \theta_B)]$ is less than $1 - \sum_{i=\hat{N}}^\infty \frac{p(1-p)}{i^2 \delta^2}$, where $p = \Pi_{S_1|\Theta \times \Theta}^\psi(b|\theta_A, \theta_B)$ and δ is the distance between the limit proportion of a -signals and the closest proportion that induces Barney to stop purchasing signals. This expression is an upper bound on the probability of Barney purchasing a finite number of signals, if he still wants to purchase an additional signal after seeing \hat{N} signals. As noted in the proof of Lemma C, this upper bound converges to 0 as $\hat{N} \rightarrow \infty$; hence, for any $\epsilon > 0$, we can find a N_ϵ such that if Barney still wants to purchase an additional signal after seeing N_ϵ signals, then the probability of him purchasing a finite number of signals is less than ϵ .

Turning to the second part of the proposition, denote the situations where there is a current posterior ratio of q , which is a belief where the agent still strictly wants to experiment, after N and $N' > N$ signals as q_N and $q_{N'}$, respectively. Assume that given $q_{N'}$, a sequence of m signals, denoted s_m , causes the agent to stop purchasing signals and (WLOG) take action a , and that no truncation of s_m causes the agent to stop purchasing signals. We will show that given initial situation q_N , the sequence s_m also causes the agent to stop purchasing signals. This proves that the probability of the agent stopping is weakly higher given q_N than $q_{N'}$.

If any truncation of s_m causes the agent to stop given q_N , then we are done, so assume not.

Now we shall show that a sufficient condition for part (a) is that the posterior induced by k a -signals in a row is less in favor of state A when starting with $q_{N'}$ than when starting with q_N . To see this, reorder s_m so that it begins with a and b signals paired off in alternating orders. Note that this reordering does not change the posterior after s_m , only the path of posterior ratios. Since by assumption the agent ends up favoring action a , there must be more a -signals than b -signals, so the reordered s_m must end with some set of a -signals in a row. Denote this number as k . Note that after every pair of signals, a, b , the likelihood ratio must still be q , since the signal structure is symmetric. So therefore, starting with either q_N or $q_{N'}$, after $m - k$ signals (the pairs of a and b signals) of the reordered s_m , the likelihood ratio is still q . Hence we are now simply comparing the effect of k a -signals in a row, given $N + m - k$ preceding signals versus $N' + m - k$ preceding signals.

Note that given $q_{N'}$, there must have been more preceding a -signals than given q_N . Assume there were r' and r a -signals, respectively. Hence in the sequence $q_{N'} \cup s_m$, there is a total of $N' + m$ signals, and the number of a -signals is r' (the number of a -signals in the first N') plus $\frac{m-k}{2}$ (the number of a -signals in the alternating set) plus k (the number of a -signals in a row at the end). Therefore, using Lemma $\beta 2$, the agent's posterior ratio after $q_{N'} \cup s_m$ is

$$\begin{aligned}
& \Pi_{\Theta \times \Theta | S_{N'+m}}^{\psi}(\theta_A, \theta_B | q_{N'} \cup s_m) \\
&= p_0 \frac{\Gamma(\theta_A \psi + r' + \frac{m-k}{2} + k) \Gamma((1 - \theta_A) \psi + N' - r' + \frac{m-k}{2})}{\Gamma(\theta_B \psi + r' + \frac{m-k}{2} + k) \Gamma((1 - \theta_B) \psi + N' - r' + \frac{m-k}{2})} \\
&= p_0 \frac{\Gamma(\theta_A \psi + r' + \frac{m-k}{2}) \Gamma((1 - \theta_A) \psi + N' - r' + \frac{m-k}{2})}{\Gamma(\theta_B \psi + r' + \frac{m-k}{2}) \Gamma((1 - \theta_B) \psi + N' - r' + \frac{m-k}{2})} \prod_{i=1}^k \frac{\theta_A \psi + r' + \frac{m-k}{2} + i}{(1 - \theta_A) \psi + r' + \frac{m-k}{2} + i} \\
&= q \prod_{i=1}^k \frac{\theta_A \psi + r' + \frac{m-k}{2} + i}{(1 - \theta_A) \psi + r' + \frac{m-k}{2} + i}.
\end{aligned}$$

In the sequence $q_N \cup s_m$, there is a total of $N + m$ signals, and the number of a -signals is r (the number of a -signals in the first N) plus $\frac{m-k}{2}$ (the number of a -signals in the alternating set) plus

k (the number of a -signals in a row at the end). Therefore, analogously, the agent's posterior ratio after $q_N \cup s_m$ is

$$\Pi_{\Theta \times \Theta | S_{N'+m}}^\psi(\theta_A, \theta_B | q_N \cup s_m) = q \Pi_{i=1}^k \frac{\theta_A \psi + r + \frac{m-k}{2} + i}{(1 - \theta_A) \psi + r + \frac{m-k}{2} + i}.$$

Notice that for all i ,

$$1 < \frac{\theta_A \psi + r' + \frac{m-k}{2} + i}{(1 - \theta_A) \psi + r' + \frac{m-k}{2} + i} < \frac{\theta_A \psi + r + \frac{m-k}{2} + i}{(1 - \theta_A) \psi + r + \frac{m-k}{2} + i}$$

since $r' > r$ and both fractions are larger than 1. Therefore, since the agent stops purchasing signals given his posterior in situation $q_{N'}$, he must stop given his posterior in situation q_N , which favors state A even more strongly.

Now we turn to the third part of the proposition. Denote the prior ratio on state A by $p = \frac{f_\Theta(\theta_A)}{f_\Theta(\theta_B)}$. Denote the probability of Barney taking the correct action—WLOG, action μ_A —after purchasing N signals (given parameters $\theta_A, \theta_B, \psi, p$) as $V(A, \theta_A, \theta_B, \psi, N, p)$. Denote the probability of Barney taking the incorrect action μ_B as $V(B, \theta_A, \theta_B, \psi, N, p)$. For brevity, we will refer to these as $V(A, N, p)$ and $V(B, N, p)$, respectively. We will show that, for any $\nu > 0$, there exists an $N_\nu > 2$ such that if Barney still wants to purchase a signal after having already observed N_ν signals, then for all $N \geq N_\nu$, $\frac{V(B, N, p)}{V(A, N, p)} < \nu$. We will do this in several steps:

1. First, we will construct $V(i, N, p)$ in a way that will make it manageable to work with.
2. Second, we will fix equal priors $p = 1$, and we will show that, for any $\nu > 0$, there exists $N_\nu > 2$ such that for all $N \geq N_\nu$, $\frac{V(B, N, 1)}{V(A, N, 1)} < \nu$.
3. Third, we will show that for any priors $p > 1$ that strictly favor the true state and for all $N \geq N_\nu$, $\frac{V(B, N, p)}{V(A, N, p)} < \frac{V(B, N, 1)}{V(A, N, 1)}$. This is because relative to the equal priors case, there is a uniform shift of probability from likelihoods that favor Barney stopping and taking an incorrect action to a stopping and taking the correct action.

Now we shall do each step in order:

For the first step of the argument, because the signals are symmetric, we can define

$$\lambda \equiv \frac{1}{\Pi_{S_1 | \Theta \times \Theta}^\psi(a | \theta_A, \theta_B)} = \Pi_{S_1 | \Theta \times \Theta}^\psi(b | \theta_A, \theta_B).$$

The set of posterior ratios such that Barney still wants to purchase an additional signal, after having seen N signals, is denoted $Q(N) \subset [\lambda, \frac{1}{\lambda}]$, a set with a finite number of elements. Denote a

typical element by q .

We will construct V from a more elementary function: $\phi(A, q, N, M)$, the probability of Barney stopping purchasing signals and taking the μ_A action in the next M signals, when the current posterior ratio is q , and N signals have already been observed. Given p and N , there is a one-to-one mapping between the posterior ratio q and the “number of a signals” previously observed, which may not be a natural number for a particular q . Therefore we will consider the number of a and b signals as fixed at the levels implied by q .

After N signals have been observed, Barney stops purchasing signals and takes action μ_A only if the number of a -signals exceeds the number of b -signals by at least some number. Abusing notation slightly, denote by s_M^j an exact sequence of a and b signals from a set of M signals, with ordering j indexed from 0 to $2^M - 1$ (so s_M^j could contain any number of signals from 0 up to M signals). Denote the number of a -signals in s_M^j as $A(s_M^j)$. Define $a(s_M^j, N, q) \in \{0, 1\}$ as an indicator of whether Barney would have stopped purchasing signals and taken action μ_A with the exact ordering of signals in s_M^j , given that the posterior ratio was q after the first N signals. Now we can construct the probability of Barney stopping purchasing signals and taking action μ_A within the next M signals, given that the posterior ratio was q after the first N signals:

$$\phi(A, q, N, M) = \sum_{j=1}^{2^M-1} a(s_M^j, N, q) \theta_A^{A(s_M^j)} (1 - \theta_A)^{M-A(s_M^j)},$$

a polynomial in θ_A and $1 - \theta_A$.

To gain some intuition for ϕ , consider $q = 1$. Because the rates are symmetric, if we consider a set of j signals that causes Barney to stop purchasing signals and take the μ_A action, and we replace all the θ_A 's with $(1 - \theta_A)$'s, and vice-versa, then that set of signals will cause Barney to stop purchasing signals and take the μ_B action. Similarly, if we have a set of j signals that has not caused Barney to stop purchasing signals, then the same set of signals, but replacing θ_A 's with $(1 - \theta_A)$'s and vice-versa, will also not cause Barney to stop purchasing signals. Therefore, $\phi(B, 1, N, M)$ is the same function as $\phi(A, 1, N, M)$ but with all the θ_A 's replaced with $(1 - \theta_A)$'s and vice-versa. Now suppose either $q > 1$ or $q < 1$. Now $\phi(B, q, N, M)$ is not simply $\phi(A, q, N, M)$ with the rates switched. Because rates are symmetric, however, if we define $q' = \frac{1}{q}$, then $\phi(B, q', N, M)$ is the same function as $\phi(A, q, N, M)$ but with θ_A and $1 - \theta_A$ switched.

Define $\rho(N)$ as the minimum number of a -signals in a row that would need to occur after an initial set of N signals to put the posterior ratio above $\frac{1}{\lambda}$ (the upper bound of the interval $Q(N)$, as defined above), given that the posterior ratio was λ (the lower bound of the interval $Q(N)$) after the first N signals. $\rho(N)$ will be a useful quantity because, regardless of the posterior ratio after N

observations, it is the number of additional signals after which Barney stopping purchasing signals and taking either action could always possibly occur. Note that $\rho(N)$ is increasing in N .

Define $x(q, N, p) < N$ as the number of a -signals out of N such that $x(q, N, p)$ a -signals and $N - x(q, N, p)$ b -signals induces posterior ratio of q when the prior is p .

For each $x(q, N, p)$, we can construct the set of possible sequences that generates x a -signals and $N - x$ b -signals without causing Barney to stop purchasing signals at any point in the sequence. Denote the set of these sequences as $X(q, N, p)$. Each one of these sequences must have the same final number of a and b signals, but it must be the case that the difference between the number of a -signals and b -signals cannot have been too large at any point in the sequence, or else Barney would have stopped purchasing signals. Let $g(q, N, p) = \gamma(q, N, p) \theta_A^{x(q, N, p)} (1 - \theta_A)^{N - x(q, N, p)}$ denote the probability of the set of sequences that leads to x a -signals and $N - x$ b -signals without Barney ever having stopped purchasing signals, where γ is the number of sequences in the set.

Define $r_N(i)$ for $i = 0, 1, \dots$ recursively: $r_N(0) = N$ and $r_N(i) = r_N(i - 1) + \rho(r_N(i - 1))$ for $i > 0$. We now construct $V(A, N, p)$ as follows:

$$V(A, N, p) = \sum_{i=1}^{\infty} \sum_{q \in Q(r_N(i-1))} \phi(A, q, r_N(i-1), \rho(r_N(i-1))) g(q, r_N(i-1), p).$$

We define $V(B, N, p)$ analogously.

To understand this definition of V , each term is the probability of a Barney stopping purchasing signals and taking the correct action within the next $r_N(i)$ signals, conditional on having observed $r_N(i - 1)$ signals without stopping purchasing signals, starting with a prior ratio of p .

For the second step of the argument, now we will fix the prior ratio at 1 and show that for any $\nu > 0$, there exists $N_\nu > 2$ such that for all $N \geq N_\nu$, $\frac{V(B, N, 1)}{V(A, N, 1)} < \nu$.

We will begin by proving some key claims:

Claim 10.A *Let $a_i, b_i > 0$ for $i = 1, 2, \dots$, such that $\frac{a_i}{b_i} \leq \frac{a_{i+1}}{b_{i+1}}$. Then $\frac{a_1}{b_1} \leq \frac{\sum_{i=1}^N a_i}{\sum_{i=1}^N b_i} \leq \frac{a_N}{b_N}$.*

To see this is true, note that for positive a, b, c, d , if $0 < \frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Induction then shows that $\frac{\sum_{i=1}^N a_i}{\sum_{i=1}^N b_i}$ is bounded by the maximum and minimum $\frac{a_i}{b_i}$.

Claim 10.B *For any $\nu > 0$, there exists $N_\nu > 2$ such that for all $N \geq N_\nu$,*

$$\frac{\sum_{q \in Q(N)} \phi(B, q, N, \rho(N)) g(q, N, p)}{\sum_{q \in Q(N)} \phi(A, q, N, \rho(N)) g(q, N, p)} < \nu.$$

Together, Claims 10.A and 10.B imply the result we are seeking. Claim 10.A, combined with the fact that the probability of Barney stopping and taking an action is shrinking as the number of

signals gets large, implies that

$$\begin{aligned} \frac{V(B, N, p)}{V(A, N, p)} &= \frac{\sum_{i=1}^{\infty} \sum_{q \in Q(N)} \phi(B, q, r_N(i-1), r_N(i)) g(q, r_N(i-1), p)}{\sum_{i=1}^{\infty} \sum_{q \in Q(N)} \phi(A, q, r_N(i-1), r_N(i)) g(q, r_N(i-1), p)} \\ &\leq \frac{\sum_{q \in Q(N)} \phi(B, q, N, \rho(N)) g(q, N, p)}{\sum_{q \in Q(N)} \phi(A, q, N, \rho(N)) g(q, N, p)}. \end{aligned}$$

Claim 10.B then says that given any $\nu > 0$, for large enough N , this ratio is $< \nu$.

A sufficient condition for Claim 10.B is:

Claim 10.C *For any $\nu > 0$, there exists $N_\nu > 2$ such that for all $N \geq N_\nu$ and $q \in Q(N)$,*

$$\frac{\phi(B, q, N, \rho(N)) g(q, N, 1) + \phi(B, q', N, \rho(N)) g(q', N, 1)}{\phi(A, q, N, \rho(N)) g(q, N, 1) + \phi(A, q', N, \rho(N)) g(q', N, 1)} < \nu.$$

To see that Claim 10.C implies Claim 10.B, notice that $\frac{\sum_{q \in Q(N)} \phi(B, q, N, \rho(N)) g(q, N, p)}{\sum_{q \in Q(N)} \phi(A, q, N, \rho(N)) g(q, N, p)}$ is

$$\begin{aligned} &\leq \max_{q, q' \in Q(N) \text{ s.t. } q' = \frac{1}{q}} \frac{\phi(B, q, N, \rho(N)) g(q, N, 1) + \phi(B, q', N, \rho(N)) g(q', N, 1)}{\phi(A, q, N, \rho(N)) g(q, N, 1) + \phi(A, q', N, \rho(N)) g(q', N, 1)} \\ &< \nu \end{aligned}$$

where the first inequality follows from Claim 10.A and the second by Claim 10.C.

Therefore, we will have completed the second step of the argument once we prove Claim 10.C. We substitute:

$$\begin{aligned} &\phi(A, q, N, \rho(N)) g(q, N, 1) \\ &= g(q, N, 1) \sum_{j=1}^{2^{\rho(N)}-1} a(s_{\rho(N)}^j, N, q) \theta_A^{A(s_{\rho(N)}^j)} (1 - \theta_A)^{\rho(N) - A(s_{\rho(N)}^j)} \\ &= \gamma(q, N, 1) \theta_A^{x(q, N, 1)} (1 - \theta_A)^{N - x(q, N, 1)} \sum_{j=1}^{2^{\rho(N)}-1} a(s_{\rho(N)}^j, N, q) \theta_A^{A(s_{\rho(N)}^j)} (1 - \theta_A)^{\rho(N) - A(s_{\rho(N)}^j)}. \end{aligned}$$

Note that some of the sequences of signals starting with a posterior of q will cause Barney to stop purchasing signals before $\rho(N)$ signals. Imagine two sequences begin with the same initial z signals but differ in the signals that occur afterward. If the z^{th} signal causes Barney to stop purchasing signals, then these subsequent signals are irrelevant, and we can aggregate the two sequences together. Let $\zeta(q, \rho(N), N, z, i)$ denote the number of a -signals needed out of an additional $\rho(N)$ signals, starting with a posterior of q after N signals, for Barney to stop purchasing signals and take action i after exactly z signals, and Barney did not stop purchasing signals earlier. Let the

lowercase letter ς count the number of permutations of such sequences. Continuing to substitute:

$$\begin{aligned} & \phi(A, q, N, \rho(N))g(q, N, 1) \\ = & \gamma(q, N, 1)\theta_A^{x(q, N, 1)}(1 - \theta_A)^{N-x(q, N, 1)} \sum_{z=1}^{\rho(N)} \varsigma(q, \rho(N), N, z, A)\theta_A^{\zeta(q, z, N, A)}(1 - \theta_A)^{z-\zeta(q, z, N, A)}. \end{aligned}$$

Now, define

$$\begin{aligned} b_z(q', N) & \equiv \gamma(q', N, 1)\theta_A^{x(q', N, 1)}(1 - \theta_A)^{N-x(q', N, 1)}\varsigma(q', \rho(N), N, z, B)(1 - \theta_A)^{z-\zeta(q', z, N, B)}\theta_A^{\zeta(q', z, N, B)} \\ a_z(q', N) & \equiv \gamma(q, N, 1)\theta_A^{x(q, N, 1)}(1 - \theta_A)^{N-x(q, N, 1)}\varsigma(q, \rho(N), N, z, A)(1 - \theta_A)^{z-\zeta(q, z, N, A)}\theta_A^{\zeta(q, z, N, A)}. \end{aligned}$$

We can write

$$\frac{\phi(B, q, N, \rho(N))g(q, N, 1) + \phi(B, q', N, \rho(N))g(q', N, 1)}{\phi(A, q, N, \rho(N))g(q, N, 1) + \phi(A, q', N, \rho(N))g(q', N, 1)} = \frac{\sum_{z=1}^{\rho(N)} b_z(q, N) + \sum_{z=1}^{\rho(N)} b_z(q', N)}{\sum_{z=1}^{\rho(N)} a_z(q, N) + \sum_{z=1}^{\rho(N)} a_z(q', N)}$$

Starting from a prior ratio of 1, because the signals are symmetric, if x a -signals and $N - x$ b -signals leaves an agent with a posterior of q , then x b -signals and $N - x$ a -signals leaves an agent with a posterior of q' . Therefore $x(q', N, 1) = N - x(q, N, 1)$. Similarly, $\gamma(q', N, 1) = \gamma(q, N, 1)$. Also, starting with a posterior of q , if x additional a -signals and $N - x$ additional b -signals leaves Barney wanting to take stop purchasing signals and take action μ_A , then starting with a posterior of q' , x additional b -signals and $N - x$ additional a -signals leaves Barney wanting to stop purchasing signals and take action μ_B . Therefore $\zeta(q', z, N, B) = z - \zeta(q, z, N, A)$ and $\varsigma(q', \rho(N), N, z, B) = \varsigma(q, \rho(N), N, z, A)$.

Using these relationships,

$$\begin{aligned} \frac{b_z(q', N)}{a_z(q', N)} & = \frac{\gamma(q', N, 1)\theta_A^{x(q', N, 1)}(1 - \theta_A)^{N-x(q', N, 1)}\varsigma(q', \rho(N), N, z, B)(1 - \theta_A)^{z-\zeta(q', z, N, B)}\theta_A^{\zeta(q', z, N, B)}}{\gamma(q, N, 1)\theta_A^{x(q, N, 1)}(1 - \theta_A)^{N-x(q, N, 1)}\varsigma(q, \rho(N), N, z, A)(1 - \theta_A)^{z-\zeta(q, z, N, A)}\theta_A^{\zeta(q, z, N, A)}} \\ & = \frac{\gamma(q, N, 1)\theta_A^{N-x(q, N, 1)}(1 - \theta_A)^{x(q, N, 1)}\varsigma(q, \rho(N), N, z, A)(1 - \theta_A)^{\zeta(q, z, N, A)}\theta_A^{z-\zeta(q, z, N, A)}}{\gamma(q, N, 1)\theta_A^{x(q, N, 1)}(1 - \theta_A)^{N-x(q, N, 1)}\varsigma(q, \rho(N), N, z, A)(1 - \theta_A)^{z-\zeta(q, z, N, A)}\theta_A^{\zeta(q, z, N, A)}}, \end{aligned}$$

which is well-defined as long as $\gamma(q, N, 1)\varsigma(q, \rho(N), N, z, A) \neq 0$. (If $\gamma(q, N, 1)\varsigma(q, \rho(N), N, z, A) = 0$, then b_z drops out from $V(B, N, p)$, and a_z drops out from $V(A, N, p)$. Hence these terms can be

ignored.) Canceling and collecting terms:

$$\frac{b_z(q', N)}{a_z(q', N)} = \frac{(1 - \theta_A)^{x(q, N, 1) + \zeta(q, z, N, A)} (\theta_A)^{N + z - x(q, N, 1) - \zeta(q, z, N, A)}}{(\theta_A)^{x(q, N, 1) + \zeta(q, z, N, A)} (1 - \theta_A)^{N + z - x(q, N, 1) - \zeta(q, z, N, A)}}$$

Using Claim 10.A ,

$$\frac{\sum_z b_z(q', N)}{\sum_z a_z(q', N)} < \max_{z \text{ s.t. } \gamma(q, N, 1) \zeta(q', \rho(N), N, z, B) \neq 0} \frac{(1 - \theta_A)^{x(q, N, 1) + \zeta(q, z, N, A)} (\theta_A)^{N + z - x(q, N, 1) - \zeta(q, z, N, A)}}{(\theta_A)^{x(q, N, 1) + \zeta(q, z, N, A)} (1 - \theta_A)^{N + z - x(q, N, 1) - \zeta(q, z, N, A)}}.$$

The numerator is the probability that, starting with a prior ratio of 1, there will be a sequence of a and b signals, $N + z$ in total, that lands the agent just inside the zone where the agent stops purchasing signals and takes action μ_B . The denominator is the probability that, starting with a prior ratio of 1, there will be a sequence of a and b signals, $N + z$ in total, that lands the agent just inside the zone where the agent stops purchasing signals and takes action μ_A . This expression can be rearranged to yield

$$\left(\frac{1 - \theta_A}{\theta_A} \right)^{2x(q, N, 1) + 2\zeta(q, z, N, A) - N - z},$$

where the exponent, $2x(q, N, 1) + 2\zeta(q, z, N, A) - N - z \equiv \delta(N + z)$, is the minimum difference between the number of a and b signals out of $N + z$ signals in total at which Barney will stop purchasing signals; if there are $\delta(N + z)$ more a than b signals, then Barney will take action μ_A ; and if there are $\delta(N + z)$ more b than a signals, then Barney will take action μ_B . Notice that $\delta(N + z)$ is not a function of q . Because Barney's likelihood ratio asymptotically depends on the proportion of a signals, not the difference between the number of a and b signals, $\lim_{N+z \rightarrow \infty} \delta(N + z) = \infty$. Furthermore, since $\rho(N)$ is growing in N , then it follows that for any $\epsilon > 0$ we can find an N' large enough such that for all $q \in \left(\Pi_{S_1 | \Theta \times \Theta}^\psi(b | \theta_A, \theta_B), \Pi_{S_1 | \Theta \times \Theta}^\psi(a | \theta_A, \theta_B) \right)$ (and so for any $q \in Q$ for any N) and all $N > N'$,

$$\left(\frac{1 - \theta_A}{\theta_A} \right)^{2x(q, N, 1) + 2\zeta(q, z, N, A) - N - z},$$

is less than ϵ . It then follows directly from the previous claims that the upper bound on $\frac{V(B, N, p)}{V(A, N, p)}$, $\left(\frac{1 - \theta_A}{\theta_A} \right)^{\delta(N + z)}$, goes to 0 as $N \rightarrow \infty$.

Therefore, although Q varies with N , it must be the case that for any q in Q , the limit result holds. That is, the convergence is uniform over $q \in Q$.

For the third step of the argument, now assume that the prior rate favors A : $p > 1$. Note that nothing in the construction of the ϕ function changes. However, because p is now closer to $q > 1$ than it is to $q' < 1$, $\frac{g(q, N, p)}{g(q', N, p)} < \frac{g(q, N, 1)}{g(q', N, 1)}$. Using this fact, and the result proven above for the $p = 1$

case: for any $\nu > 0$, there exists $N_\nu > 2$ such that for all $N \geq N_\nu$, for any $q \in Q(N)$,

$$\frac{\phi(B, q, N, \rho(N))g(q, N, p)}{\phi(A, q', N, \rho(N))g(q', N, p)} < \frac{\phi(B, q, N, \rho(N))g(q, N, 1)}{\phi(A, q', N, \rho(N))g(q', N, 1)} < \nu,$$

and

$$\frac{\phi(B, q', N, \rho(N))g(q', N, p)}{\phi(A, q, N, \rho(N))g(q, N, p)} < \frac{\phi(B, q', N, \rho(N))g(q', N, 1)}{\phi(A, q, N, \rho(N))g(q, N, 1)} < \nu.$$

We argued above that Claim 10.A implies

$$\frac{V(B, N, p)}{V(A, N, p)} \leq \frac{\sum_{q \in Q(N)} \phi(B, q, N, \rho(N))g(q, N, p)}{\sum_{q \in Q(N)} \phi(A, q, N, \rho(N))g(q, N, p)}.$$

Using Claim 10.A again, along with the above inequalities, gives the result:

$$\frac{V(B, N, p)}{V(A, N, p)} < \nu.$$

□

Proposition 10. *Assume Barney has the beta-distribution functional form given by equation (3). Fix payoff functions $u^H(\mu_t)$ and $u^L(\mu_t)$, rates $\theta_A, \theta_B \in \Theta$ such that $\theta_A > \theta_B$ and discount factor δ . Suppose Barney is prospective-acceptive and retrospective-pooling.*

1. *Suppose $u^H(A) > u^L(A)$ and $u^H(B) > u^L(B)$. Without loss of generality, suppose the state is $\omega = B$. For all priors $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, Tommy's belief that the state is B converges to 1 almost surely, and Tommy's action converges to B almost surely. Barney's belief that the state is B converges almost surely to a number in the interval $(0, 1)$, and this limit posterior is increasing in $f_\Theta(\theta_B)$. Moreover, if $f_\Theta(\theta_B)$ is sufficiently small, then there is positive probability that Barney takes action A in every period.*
2. *Suppose $u^H(A) > u^L(A)$ and $u^H(B) = u^L(B)$. For all priors $f_\Theta(\theta_A) = 1 - f_\Theta(\theta_B) \in (0, 1)$, if the state is B , then almost surely at some finite T , Tommy will take action B for all periods $t \geq T$. For Barney, regardless of the state, there exists $0 < \underline{p} < 1$ such that for any prior $f_\Theta(\theta_A) \geq \underline{p}$, there is positive probability that Barney takes action A in every period. This probability is increasing in $f_\Theta(\theta_A)$ but is always strictly less than 1.*

Proof: We begin with the first part of the proposition. Since the high and low payoffs differ for both actions, both actions are informative. If the agent gets the high payoff, consider it an a signal; the low payoff, a b signal. In this case, standard results imply that as the number of signals goes

to infinity Tommy almost surely learns the true state. Furthermore, when Tommy is sufficiently confident about the state, he takes the action with the high payoff.

We know that as the number of signals Barney receives goes to infinity, his posterior ratio converges to the finite number

$$\frac{f_{B|\Theta}^\psi(\beta = \theta|\theta_A) f(\theta_A)}{f_{B|\Theta}^\psi(\beta = \theta|\theta_B) f(\theta_B)}$$

almost surely (where θ is the rate in the true state). This is increasing in the prior probability of state A .

Furthermore, because Barney is prospective-acceptive, we know that for the same current belief as Tommy, his current action will be the same. Hence Barney will only take action B if his current posterior ratio of state A to state B is below some threshold $\bar{\lambda}$. Let \bar{p} denote the largest prior ratio $\frac{f(\theta_A)}{f(\theta_B)}$ such that

$$1 > \frac{f_{B|\Theta}^\psi(\beta = \theta|\theta_A) f(\theta_A)}{f_{B|\Theta}^\psi(\beta = \theta|\theta_B) f(\theta_B)} \geq \bar{\lambda}$$

and

$$1 > \frac{f(\theta_A)}{f(\theta_B)} \geq \bar{\lambda}.$$

For all prior ratios p such that $1 > p > \bar{p}$,

$$1 > \frac{f_{B|\Theta}^\psi(\beta = \theta|\theta_A)}{f_{B|\Theta}^\psi(\beta = \theta|\theta_B)} p > \bar{\lambda}$$

and

$$1 > p > \bar{\lambda}.$$

By Lemma C, given prior p , with positive probability Barney's posterior ratio always stays within $(\bar{\lambda}, 1)$, and so he always takes action A .

Now we prove the second part of the proposition. Once again, because Barney is prospective-acceptive, given the same current belief as Tommy, he will choose the same action. Standard results imply that either agent takes action B if his current posterior ratio is below some threshold $\bar{\lambda}$; otherwise he takes action A (and is indifferent at $\bar{\lambda}$). In the present setting, whenever an agent takes action A , he receives a signal, but whenever he takes action B , he receives no signal. In this environment, the result concerning Tommy is entirely standard. For Barney, define \bar{p} as above, and by the same argument as above, for all prior ratios p such that $1 > p > \bar{p}$, with positive probability Barney's posterior ratio always stays within $(\bar{\lambda}, 1)$, and so he always takes action A .

The probability of always taking action A is increasing in the prior. To see this, consider priors

p', p'' such that $p'' \geq p' > \underline{p}$. Let $\hat{S}_\infty(p')$ denote the set of sequences that lead to Barney taking the A action every period given prior p' . Consider any particular sequence $s \in \hat{S}_\infty(p')$. Let s_N be the truncation of s after N signals. Note that $p' \frac{f_{\Theta|s_N}^\psi(\theta_A|s_N)}{f_{\Theta|s_N}^\psi(\theta_B|s_N)} \in [\bar{\lambda}, 1)$ for all $N = 1, 2, \dots$ by construction. Since $p'' \geq p'$,

$$p'' \frac{f_{\Theta|s_N}^\psi(\theta_A|s_N)}{f_{\Theta|s_N}^\psi(\theta_B|s_N)} \geq p' \frac{f_{\Theta|s_N}^\psi(\theta_A|s_N)}{f_{\Theta|s_N}^\psi(\theta_B|s_N)}$$

for all N , and so $p'' \frac{f_{\Theta|s_N}^\psi(\theta_A|s_N)}{f_{\Theta|s_N}^\psi(\theta_B|s_N)} \in [\bar{\lambda}, 1)$ for all N . Therefore given prior p'' , Barney will also always take action A for all sequences in $\hat{S}_\infty(p')$. Therefore the probability of always taking action A given prior p'' must be weakly higher than given prior p' .

The probability of always taking action A is always less than 1 because the limit likelihood ratio for Barney of state A to state B if he observes 0 a -signals is 0. Therefore, regardless of the prior, if Barney receives enough b signals in a row initially, his posterior ratio will fall below $\bar{\lambda}$.

□

Appendix D: Formal Description of the Multiple-Sample Model

To formalize the information-processing assumptions from Section 4, consider a decision problem in which up to $1 \leq T \leq \infty$ binary signals will be realized in total. To greatly simplify notation and clarify presentation, we assume that how the environment or his own thinking leads him to group signals is independent of the realizations of those signals and of earlier decisions.¹⁶ We define $T + 3$ partitions, each represented by a set of time periods, of the set of signals that fully characterize our predictions. We conceive of the first three of these partitions as embodying physical, informational, and perceptual assumptions rather than assumptions about Barney’s statistical reasoning. The first partition we define by the set of dates at which the agent knows he must make decisions, $D \subseteq \{0, 1, \dots, T\}$. If $\tau \in D$, the agent knows that he made or will make a decision after observing τ signals but before observing $\tau + 1$ signals. The agent’s payoff may depend on any or all of the decisions he makes, the signals that are realized, and the underlying state with which the signals are correlated. This opportunity-for-decisions partition, D , is of course specified in every economic model of decisionmaking.

The second partition is characterized by a set $C \subseteq \{0, 1, \dots, T - 1\}$ of dates where a new clump begins, such that if and only if $\tau \in C$, then signal τ is in a different clump than signal $\tau - 1$. We assume $0 \in C$. For example, if 11,000 signals arrives as a clump of 10,000 signals followed by a clump of 1,000 signals, then $T = 11,000$ and $C = \{0, 10,001\}$. There is one obvious basic restriction on clumping that we must make:

Clumping Assumption 0. $D \subseteq C$.

This assumption states that at any history where the agent makes a decision, subsequent signals arrive as a separate clump than previous signals. This restriction is inherent in the notion of clumps because the signals cannot have “arrived” together (at least in the relevant sense of the agent’s knowledge of their realizations) if the agent knows the realizations of only some of the signals. We treat this as a coherence assumption and always impose it.

Another coherence assumption on C is that Barney will not treat signals differently when he does not see the signals distinctly at all. If it is the case, unbeknownst to Barney, that 38 of the people in *Consumer Reports* statistics of 10,000 car owners have the last name Smith, we assume

¹⁶This is a substantive restriction, and we know of examples where it seems unrealistic, but we do not know how relaxing it will improve insights. Importantly, we are not assuming that *whether* Barney gathers more information is independent of what he has learned; our examples in Section 5 revolve around exactly such a decision by Barney. In those examples, Barney faces a decision after each signal whether to pay to observe another signal. To accommodate those examples in the framework of this section, it can be assumed that whenever Barney chooses not to observe further signals, decisions that “occur” at those future signals do not affect his payoff.

that Barney cannot (even if thusly motivated) treat Smiths as one sample and non-Smiths as another.¹⁷ Let $I \subseteq \{0, 1, \dots, T-1\}$ be a set of dates that defines the third partition, a partition of signals into equivalence classes of indistinguishable signals (so that if $\tau \in I$, then the signal at time τ is distinguishable from the signal at time $\tau-1$). Clearly it must be the case that $D \subseteq I$, but we also impose a second restriction on clumping:

Clumping Assumption 1. $C \subseteq I$.

Although economic models of decisionmaking do not traditionally specify a clumping partition, our aspiration is have the clumping partition be an exogenous assumption that is not *per se* related to NBLN, and ideally is pinned down by observable characteristics of a situation. In the interest of minimizing the number of assumptions that have to be specified anew for each new economic model (and hence limit degrees of freedom that might reduce the usefulness of the model), it would be especially attractive to pin down some rule for “clumping” that ties it to D or I . The two obvious candidates are:

Clumping Assumption 2(a). $C = D$.

Clumping Assumption 2(b). $C = I$.

While the three partitions characterized by D , C , and I reflect the physical and perceptual environment facing Barney, the remaining T partitions embed Barney’s NBLN psychology of how he separates out data. To capture the important possibility that Barney’s grouping might differ from different time perspectives, we assume that the agent may process the signals differently at different dates. At each $t = 0, 1, \dots, T$, there is a set $P_t \subseteq \{0, 1, \dots, T-1\}$ of dates where a new group begins. By letting each P_t contain elements both less than t and greater than t , each partition specifies both retrospective and prospective grouping of signals at that point in time. For all $\tau \leq t$, if $\tau \in P_t$, then after having observed t signals, the agent processes signal τ as being in a different group than signal $\tau-1$; while if $\tau \notin P_t$, then after having observed t signals, the agent processes signal τ as being in the same group as signal $\tau-1$. And for all $\tau > t$, $\tau \in P_t$ means Barney *anticipates* separating out signal τ from signal $\tau-1$. Whether he actually does so after observing signal τ is determined by $\{P_\tau, P_{\tau+1}, \dots\}$. We assume that $0 \in P_t$ for all t .

Our main modeling constraint, which rules out prospective-pooling whenever Barney will face a subsequent decision node, is formalized as:

¹⁷Conceivably, Barney could decide to label an indistinguishable group any way he wants, such as ordering them from 1 to 10,000, and then either perceptually or psychologically distinguish the signals based on this labeling. It seems a safe assumption that he will not do so.

Processing Assumption 0. For any $t = 0, 1, \dots, T$, if $\tau \in D$ and $\tau + 1 \geq t$, then $\tau + 1 \in P_t$.

This assumption states that at any date where the agent makes a decision, he processes signals before and after that date as being in separate groups—and that before that date, he knows he will do so. We consider this to be a modeling coherence assumption because it ensures that Barney’s NBLN from the single-clump model in Section 2 generalizes to every decision node in the multiple-clump model.

We additionally impose the coherence assumption that Barney cannot process in separate groups signals that he cannot distinguish from each other.

Processing Assumption 1. For any $t = 0, 1, \dots, T$, $I \subseteq P_t$.

In words, at every date, Barney’s processing partition must be a coarsening of his indistinguishability partition.

We now formalize some ways that Barney might form beliefs retrospectively and prospectively. Because D is the set of nodes where Barney’s beliefs are payoff-relevant, these definitions focus the assumptions on the P_t ’s where $t \in D$:

- **Retrospective-Pooling:** If $t \in D$ and $0 < \tau \leq t$, then $\tau \notin P_t$.
- **Retrospective-Acceptive:** If $t \in D$ and $0 < \tau \leq t$, then $\tau \in P_t \Leftrightarrow \tau \in C$.
- **Prospective-Acceptive:** If $t \in D$ and $\tau > t$, then $\tau \in P_t \Leftrightarrow \tau \in C$.

We omit defining “prospective-pooling” Barney because it is ruled out by Processing Assumption 0 in any decision problem with more than one decision node; at an earlier decision node, Barney cannot expect to pool together future signals that come before and after a future decision node. An agent is **processing-consistent** if he always processes information the way that he expects to process information: $P_t = P_{t'}$ for all $t, t' \in D$.

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