Higher-Dimensional Realizability for Intensional Type Theory

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Oriel College
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A thesis submitted for the degree of
Doctor of Philosophy

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Abstract

We develop realizability models of intensional type theory, based on groupoids, wherein realizers themselves carry higher-dimensional structure. In the spirit of realizability this is intended to formalise a higher-dimensional (topological, homotopical) BHK interpretation whereby evidence for an identification is a path.

The parameter over which we build realizability models is a "realizer category". These are equipped with an interval *qua* internal co-groupoid, which facilitates a notion of homotopy in the ambient category, as well as a fundamental groupoid construction on it. In groupoidal realizability, objects of a base groupoid are realized by points in the fundamental groupoid of some object from the realizer category, and the isomorphisms from the base groupoid are realized by paths in that fundamental groupoid.

We first explain why a naive formulation of groupoidal assemblies is not fit for modelling type theory; this motivates studying *partitioned* groupoidal assemblies.

The main result of the thesis is that, when the realizer category is finitely complete in a suitable sense, the ensuing category of partitioned groupoidal assemblies is a path category with weak dependent products, hence a model of a version of intensional (1-truncated) type theory with dependent products and without function extensionality. When the underlying realizer category is "untyped", there exists an impredicative universe of 1-types, given by the modest fibrations.

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To Mum and Dad, who instilled within me a love of learning,
and encouraged me to follow my dreams.

And Julia, for being with me throughout this process.

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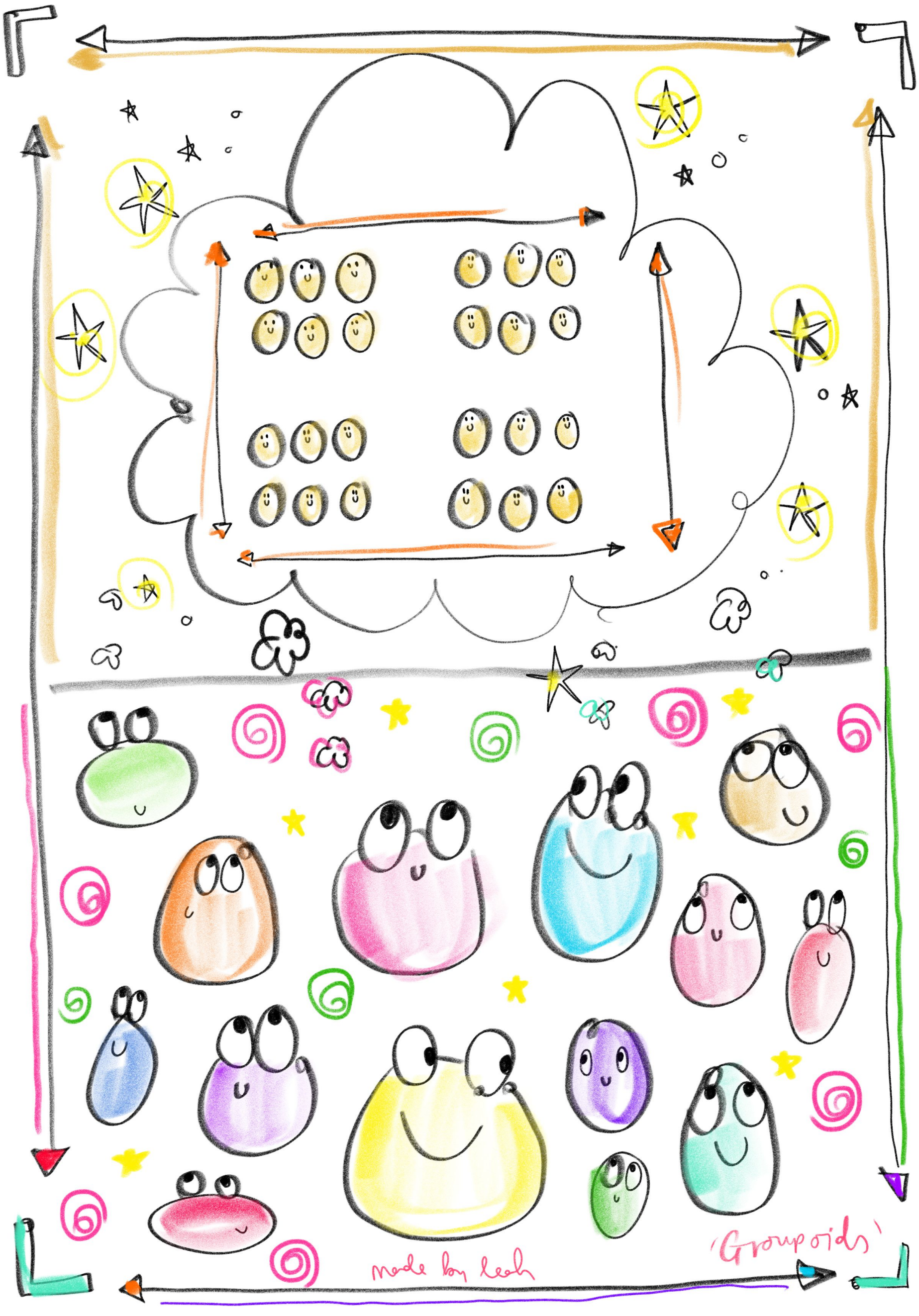
The parameter over which we build realizability models is a "realizer category". These are equipped with an interval *qua* internal co-groupoid, which facilitates a notion of homotopy in the ambient category, as well as a fundamental groupoid construction on it. In groupoidal realizability, objects of a base groupoid are realized by points in the fundamental groupoid of some object from the realizer category, and the isomorphisms from the base groupoid are realized by paths in that fundamental groupoid.

We first explain why a naive formulation of groupoidal assemblies is not fit for modelling type theory; this motivates studying *partitioned* groupoidal assemblies.

The main result of the thesis is that, when the realizer category is finitely complete in a suitable sense, the ensuing category of partitioned groupoidal assemblies is a path category with weak dependent products, hence a model of a version of intensional (1-truncated) type theory with dependent products and without function extensionality. When the underlying realizer category is "untyped", there exists an impredicative universe of 1-types, given by the modest fibrations.



An artist's interpretation of groupoids, based on keywords: symmetry, inverse, many, group. By *Made by Leah*.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Contents

1.1 Motivation

1.1.1 Realizability and BHK

The subject of realizability goes back to Stephen Cole Kleene's "number realizability" interpretation of Heyting arithmetic [Kle45]. The motivation was to make precise the computational content of constructive arithmetic; this allows for the extraction of programs from proofs in Heyting arithmetic.

A set of natural numbers (encoding computable functions), the realizers, is assigned to each (closed) formula of arithmetic. So we first fix an effective enumeration $\{-\}$ of all computable functions, so that $\{n\}(-)$ is the n^{th} computable function. Moreover, we help ourselves to a primitive recursive pairing function

$$\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

with primitive recursive projections

$$\pi_1, \pi_2 : \mathbb{N} \to \mathbb{N}$$

such that

$$\langle \pi_1(n), \pi_2(n) \rangle = n$$

The assignment of realizers to formulas is defined by recursion on the structure of formulas. We write $n \Vdash \varphi$ to mean that n realizes (belongs to the set of realizers of) φ.

- $n \Vdash t = t'$ iff $t = t'$.
- $n \Vdash \varphi \wedge \psi$ iff $\pi_1(n) \Vdash \varphi \wedge \pi_2(n) \Vdash \psi$.
- $n \Vdash \varphi \vee \psi$ iff one of the following hold:
  - $\pi_1(n) = 0$ and $\pi_2(n) \Vdash \varphi$.
  - $\pi_1(n) = 1$ and $\pi_2(n) \Vdash \psi$.
- $n \Vdash \varphi \to \psi$ iff for all $m \Vdash \varphi$: $\{n\}(m)\downarrow$ and $\{n\}(m) \Vdash \psi$.
- $n \Vdash \exists x.\varphi(x)$ iff $\pi_2(n) \Vdash \varphi\left(\pi_1(n)\right)$.
- $n \Vdash \forall x.\varphi(x)$ iff for all $m \in \mathbb{N}$: $\{n\}(m)\downarrow$ and $\{n\}(m) \Vdash \varphi(m)$.

The number realizability interpretation is sound for Heyting arithmetic: if a formula φ is provable in Heyting arithmetic then it is realized by some natural number. One can prove the disjunction and existence properties for Heyting arithmetic as corollaries:

- Disjunction property: if $\varphi \vee \psi$ is provable then either φ is provable or ψ is provable.
- Existence property: if $\exists x.\varphi(x)$ is provable then there is a natural number n such that $\varphi(n)$.

These are hallmarks of a constructive theory. On the other hand, there are formulas that are realized but not provable in Heyting arithmetic.

Realizability is often said to formalise the Brouwer–Heyting–Kolmogorov (BHK) interpretation [Hey30, Hey31, Kol32, Hey34, Hey56]. This is an informal, constructive explanation of logical operators in terms of what counts as evidence for a given proposition.[1]

- Evidence for a conjunction of propositions is a pair consisting of evidence for each of the conjuncts.
- Evidence for a disjunction of propositions is a pair consisting of an identifier for one of the disjuncts as well as evidence for the identified proposition.
- Evidence for an implication is a process/method/function that, when given evidence for the antecedent, produces evidence for the consequent.
- Evidence for an existential proposition is a pair consisting of an object (from the domain of discourse) and evidence that this object satisfies the body of the proposition.
- Evidence for a universal proposition is again a process/method/function that, when given any object (of the domain of discourse), produces evidence that this object satisfies the body of the proposition.

What counts as evidence for the atomic proposition $t = t'$ is not specified. One strategy is to take some dummy object to count as evidence for the proposition $t = t'$ when t is identified with t' in the metatheory; there is no evidence for the formula $t = t'$ when (in the metatheory) t is not identified with t'. Thus we are meant to be able to recognise when t is identified with t'

---

[1] Note though that there are realizability interpretations of classical theories (see [Kri01, Kri03, Kri09] and, for a categorical take, [Str13]).

in the metatheory, perhaps by performing some computation. But a realizer for $t = t'$ provides no additional information about the proposition apart from the fact that it holds.

In realizability, formally specified realizers play the role that evidence does in the BHK interpretation. Moreover, informal terms like "process", "method" and "function" are given a precise meaning. For instance, in number realizability, natural numbers play the role of evidence, and processes/methods/functions are computable functions (encoded as natural numbers).

Realizability is a fascinating area, with a history that meanders through mathematical, computational and philosophical terrain. The reader who wishes to understand this journey in greater detail is directed to van Oosten's essay [Oos01]. Presently, we fast-forward to the late 1970s, when the categorical treatment of realizability took off with Hyland's introduction of the *effective topos* **Eff** [Hyl82].[2]

Realizability categories may be loosely characterised as universes of computable mathematics. Each realizability category is constructed over some model of computation (understood in a broad sense). Often this is a partial combinatory algebra (PCA). These are abstract models of untyped computation. A PCA consists of a binary "application" operation over a carrier set. Certain distinguished elements (combinators) satisfying certain laws are required to exist, which allows the PCA to mimic λ-abstraction. The paradigmatic example is "Kleene's first algebra" $\mathcal{K}_1$, whose carrier set is $\mathbb{N}$. The application operation is:

$$(-) \cdot (-) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$n \cdot m := \{n\}(m)$$

---

[2] Perhaps, then, a better metaphor for the history of—and continued work on—realizability is not a single river, however meandering, but a river that branches into multiple streams, those streams occasionally remerging, and with tributaries joining along the way.

Eff is constructed over $\mathcal{K}_1$, and as such, its internal logic extends Kleene's number realizability.

There are other realizability categories besides realizability toposes: categories of assemblies and partitioned assemblies, and modest subcategories thereof (these will be defined in Section 2.4). These are all categories of "datatypes" (sets) whose "values" (elements) are "implemented" in the "programming language" of the underlying model of computation. Functions must be implemented in this language also, that is, they must be computable (as is the case in realizability toposes).

One way to cash out the claim that realizability categories are *universes* of computable mathematics is to say that they model rich and expressive type theories, indeed impredicative (polymorphic) and dependent type theories such as the *calculus of constructions* [CH88] (implemented in the theorem provers *Coq* and *Lean*). However, traditional realizability categories only model *extensional* type theory, that is, type theory in which any two identity proofs (terms of an identity type) are themselves equal.

The statement that any two identity proofs are themselves equal is known as UIP (for the *uniqueness* of *identity proofs*). For a little while, it was not known whether UIP is derivable in type theory. But eventually, Hofmann and Streicher [HS98] showed that UIP is independent of core Martin-Löf type theory [ML75], giving a model in groupoids, wherein identity proofs are interpreted by isomorphisms: there may, of course, be many distinct morphisms within a given hom set of a groupoid. The groupoid model was a catalyst for further investigation into *intensional* type theory (ITT): type theory without UIP (though it was already known to have the computational feature of decidable type checking [Hof95a]). Types were later shown to carry the structure of weak ∞-groupoids [vdBG10, Lum10] (see also [Bou16]). Ultimately, this lead to *homotopy type theory* (HoTT), a system which takes full advantage of being free to refute UIP.

1.1.2 Higher realizability and higher BHK

In the setting of intensional type theory, we view types as spaces, terms as points in a space, identity proofs as paths in a space and higher identity proofs as higher paths, ie. homotopies: this is the homotopy interpretation of type theory [Uni13].

With this in mind, we revisit the BHK interpretation. In particular, instead of a dummy object standing as evidence for the atomic proposition $t = t'$, we offer the clause:

- Evidence for an identity $t = t'$ is a path from t to t'.

Just as with "process" and "method" and "function" before, here "path" is understood informally. Whereas before we relied on being able to recognise when two objects are equal meta-theoretically, we now rely on being able to tell, meta-theoretically, when there is a path between two objects. So we are lead by this BHK clause to thinking of objects as living in spaces of some sort. We will call the BHK interpretation with this clause the "higher-dimensional BHK interpretation" (or the "topological" or "homotopical BHK interpretation").

> **The overarching aim of this thesis is to construct higher-dimensional categorical realizability models of ITT that formalize the higher-dimensional BHK interpretation.**

This means that realizers need to carry higher-dimensional structure, so that a realizer of an identity proof can be a path of some sort.

All this being said, it is consistent that types in plain ITT are all discrete (consider the set model). HoTT makes some additions to ITT. The two key additions are *higher inductive types* and Voevodsky's *univalence axiom*. A higher inductive type is like a regular inductive type (such as the type $\mathbb{N}$ of natural numbers, or the coproduct $A + B$ of types A and B), except that its constructors may be valued in its (iterated) identity types. An example is the circle $\mathbb{S}^1$, with constructors:

$$\begin{aligned}&\mathsf{base} : \mathbb{S}^1\\&\mathsf{loop} : \mathsf{Id}_{\mathbb{S}^1}(\mathsf{base}, \mathsf{base})\end{aligned}$$

Still, as it stands, it is consistent that $\mathbb{S}^1$ is discrete. The addition of the univalence axiom makes types behave homotopically as expected. Roughly speaking, the univalence axiom states that equivalence of types is equivalent to identity of types in the universe $\mathcal{U}$.

$$(A \simeq B) \simeq \mathsf{Id}_{\mathcal{U}}(A, B)$$

Univalence is an extensionality principle for the universe of types: it characterises the identity type of two types in the universe as the type of equivalences between them. It contradicts UIP in general. Philosophically, it oozes structuralism [Awo13], and practically, it sets on a formal footing the informal (in traditional set-theoretic foundations) practice of identifying equivalent structures. With the univalence axiom, Licata and Shulman proved that the fundamental group of $\mathbb{S}^1$ is identical to $\mathbb{Z}$ [LS13]. Indeed, HoTT facilitates a synthetic development of a great deal of homotopy theory.

The BHK interpretation is closely related to Martin-Löf's meaning explanations [ML82]. In the paradigm of meaning explanations, one starts with an untyped programming language together with a deterministic operational semantics, and constructs a type theory over this. Through a system of relations, one specifies which values of the original language are canonical types and which are canonical members of canonical types. In fact, for technical reasons, one specifies when two values are equal canonical types and when two values are equal members of canonical types. Thus we have a system of partial equivalence relations (PERs)—the relations are not reflexive because we do not want that all values are canonical types and members of canonical types.

The meaning of judgements A type and $M \in A$ is specified in terms of the computational behaviour of the programs A and M. The former means that A evaluates to a canonical type; the latter means that M evaluates to a canonical member of the canonical type corresponding to A. This meaning is then extended by "functionality" to open (hypothetical) judgements:

$$a_1 : A_1, ..., a_n : A_n \gg A = B \ \mathsf{type}$$
$$a_1 : A_1, ..., a_n : A_n \gg M = N \in A$$

The meaning of type formers is given compositionally. For example, a canonical member of the product type $A \times B$ is a pair $\langle M, N \rangle$ consisting of members $M \in A$ and $N \in B$. Likewise, a canonical member of the function type $A \to B$ is an abstraction $\lambda x.M$, where $x : A \gg M \in B$. A logico-computational reading of $M \in A$ is that the program M computes evidence for the truth of the proposition A. Thus, a canonical member of the product type $A \times B$ is a program containing a pair of programs: one computing evidence for A and the other, evidence for B. A canonical member of the function type $A \to B$ is a program that, when given a program that computes evidence for the truth of A, uses it to compute evidence for the truth of B. The connection to BHK is apparent.

A type theory constructed in this way is called a "computational type theory" in order to distinguish it from a type theory defined inductively by inference rules, sometimes called a "formal type theory" (this is our default understanding of "type theory"). As a PER construction over an (untyped) programming language, a computational type theory can be seen as realizability interpretations of a formal type theory for which it is sound. We will see in Section 2.4 that PERs are also part of the story of categorical realizability.

Angiuli, Harper and Wilson introduce a *cubical* generalisation of Martin-Löf's meaning explanations [AHW17] (see also [AH17]). The starting point is an untyped cubical programming language; the result is computational *higher* type theory (CHTT).

Cubical languages allow the direct manipulation of finite-dimensional cubes thanks to "dimension names", ie. variables thought of as ranging over the unit interval (not interval *qua* higher inductive type). In HoTT, univalence (like function extensionality) is added by postulating a term that inhabits a certain type. No computational behaviour of this term is specified. The addition of an axiom in this way breaks *canonicity*: it is no longer the case that terms of base type are judgementally equal (evaluate) to *canonical* terms (eg. numerals $\mathsf{succ}(...(\mathsf{succ}(0)))$ in the case of $\mathbb{N}$). In CTT [CCHM15], on the other hand, univalence (and function extensionality) is *derivable*—and canonicity holds [Hub19].

CHTT is sound for the rules of CTT (note that there are a variety of flavours of CTT); hence the former can be seen as a realizability interpretation of the latter. CHTT was later extended by Angiuli, Hou and Harper to incorporate univalent universes [AFH18], and by Cavallo and Harper to incorporate higher inductive types [CH19].

Of all existing approaches to realizability for ITT or HoTT, CHTT is closest in spirit to our approach in this thesis. This is based on the fact that, in both approaches, realizers themselves carry higher-dimensional structure. We use the term "higher-dimensional realizability" to indicate this. However, in this thesis, we aim to develop a *general, categorical* approach to realizability models of ITT. We are also interested in *impredicative universes*, so far not treated in CHTT.

In other work on realizability for intensional or homotopy type theory, realizers do not carry higher-dimensional structure. Hofstra and Warren [HW13] equip the syntax of 1-truncated ITT with a notion of realizability, allowing them to show that the syntactic groupoid associated to the type theory generated by a graph has the same homotopy type as the free groupoid on this graph.

Further work in this area was motivated by the search for *impredicative and univalent* universes of higher homotopy types. Uemura [Uem18] gave a model with such a universe in the category of cubical assemblies over $\mathcal{K}_1$, ie. presheaves on a category of cubes, valued in (set-based) assemblies over $\mathcal{K}_1$. Interestingly, Uemura [Uem18, p. 16] states that the cubical assemblies model:

> *does not seem to be what should be called a realizability ∞-topos, a higher dimensional analogue of a realizability topos. One problem is that, in the cubical assembly model, realizers seem to play no role in its internal cubical type theory.*

It seems that there is a gap between realizers (which come from $\mathbb{N}$, itself a 0-type) on the one hand, and the higher-dimensional aspect of the model on the other hand.

Swan and Uemura [SU21] show that Church's thesis (all functions on the naturals are computable, a hallmark of realizability models) fails in cubical assemblies, though it holds in a reflective subuniverse—and is thus consistent with univalence. Another point of interest is *propositional resizing*, which states that every proposition is equivalent to a "small" one (one living in the lowest universe). Uemura's model does not satisfy propositional resizing.

Van den Berg exhibits **Eff** as the homotopy category of a path category in which there is an impredicative and univalent universe of propositions which, in contrast to cubical assemblies, *does* satisfy propositional resizing [vdB18b]. A more complicated path category contains an impredicative and univalent universe of sets satisfying propositional resizing. Path categories are the main notion of model of type theory employed in this thesis, and will be introduced properly in Section 2.2.

1.2 Contributions

> **The overarching contribution of this thesis is the foundation of a general, categorical theory of higher-dimensional realizability.**

Here is a chapter-by-chapter summary:

- Chapter 2 is made up of background material on type theory, path categories—our primary notion of model of type theory—categories with families, and categorical realizability. With the latter, we start by reviewing the (best-known) theory of realizability over PCAs, but finish with realizability over categories. This is a nice platform from which to jump into higher-dimensional realizability, which utilises realizer categories.

- Chapter 3 is all about realizer categories, the gadgets over which we build realizability models. The novel idea is that, suitable for higher-dimensional realizability, realizer categories come equipped with an interval *qua* internal co-groupoid. The interval supplies a notion of homotopy internal to the ambient category, and also a fundamental groupoid construction on it. Realizer categories come in two forms: typed and untyped. Untyped realizer categories are required to contain a *universal object*, which in particular is an up-to-homotopy model of the untyped λ-calculus. In Section 3.3.1, we describe an example of an untyped realizer category coming from categorified domain theory: the category of Scott complete categories.

- Chapter 4 is a first pass at groupoidal assemblies. We present a naive formalisation of groupoidal assemblies, before explaining why they are not suitable for building models of ITT. This has to do with the inability to model context extension (Σ-types) and the identity type family. This motivates the next chapter.

- Chapter 5 is the meat of the thesis. In Section 5.1, we define a very natural category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ of *partitioned groupoidal assemblies* over the realizer category $(\mathbb{C}, \mathbb{I})$. In Section 5.2, we show that this category can be extended to a (2,1)-category whose 2-cells are homotopies with respect to an interval. If the realizer category $(\mathbb{C}, \mathbb{I})$ is finitely complete as a (2,1)-category, then so is $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$.

  In Section 5.3, we show that, when $(\mathbb{C}, \mathbb{I})$ is a finitely complete (2,1)-category, $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ forms a path category with weak homotopy dependent products. This means that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is a model of a version of 1-truncated ITT with dependent functions and without function extensionality. By a results of van den Berg and Moerdijk relating to homotopy exact completions of path categories, we also obtain models of type theory with function extensionality.

  Following this, in Section 5.3.3 we investigate impredicative universes. This involves switching to an untyped realizer category $(\mathbb{C}, \mathbb{I}, U)$, where U is a universal object in $\mathbb{C}$. The category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ of partitioned groupoidal assemblies over the untyped realizer category $(\mathbb{C}, \mathbb{I}, U)$ is also a path category with weak homotopy dependent products, again assuming that $(\mathbb{C}, \mathbb{I})$ is finitely complete. We define the class of *modest fibrations*, which gives rise to an *impredicative universe of 1-types* in the path category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$.

  In Section 5.4, we go on to consider another kind of object classifier, inspired by the notion of *generic proof* [Men00]. We show that there is a weak (in multiple senses) sort of classifier for fibrations in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ with small fibres.

  Finally, in Section 5.5, we consider the indexed perspective. We establish a Grothendieck correspondence between fibrations of partitioned groupoidal assemblies and *indexed partitioned groupoidal assemblies*. It is hard to imagine having come up with the latter notion without going through the former.

- Chapter 6 ruminates on an important feature of groupoidal realizability, before discussing future work.

Chapter 2

Background

Contents

2.1 Type theory

This section serves to orient us towards the type-theoretic matters that we are concerned with in this thesis.

Type theory has two kinds of equality: *judgemental* equality and *propositional* (or *typal*) equality. Showing that a propositional equality holds involves constructing a proof term, that is, evidence internal to type theory for the identification. Showing that a judgemental equality holds does not involve finding evidence as such. In this sense, propositional and judgemental equality are like, respectively, equality of *sense* and of *reference* à la Frege [Fre48].

The identity type family is given by the following rules. There is the formation rule:

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash a, b : A}{\Gamma \vdash \mathsf{Id}_A(a, b) \ \mathsf{type}} \ \text{Id-F}$$

Identity types have a single "reflexivity" constructor:

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}_a : \mathsf{Id}_A(a, a)} \ \text{Id-I}$$

The elimination rule is (at least in HoTT) sometimes referred to as "path induction":

$$\frac{\begin{array}{c}\Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash C \ \mathsf{type} \\ \Gamma, z : A \vdash c : C\,[z, z, \mathsf{refl}_z / x, y, p] \qquad \Gamma \vdash a, b : A \qquad \Gamma \vdash q : \mathsf{Id}_A(a, b)\end{array}}{\Gamma \vdash J(C, c, a, b, q) : \mathsf{Id}_A(a, b)} \ \text{Id-E}$$

Finally there is a β-rule:

$$\frac{\begin{array}{c}\Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash C \ \mathsf{type} \\ \Gamma, z : A \vdash c : C\,[z, z, \mathsf{refl}_z / x, y, p] \qquad \Gamma \vdash a : A\end{array}}{\Gamma \vdash J\,(C, c, a, a, \mathsf{refl}_a) \equiv c[a/z] : C\,[a, a, \mathsf{refl}_a / x, y, p]} \ \text{Id-}\beta$$

But computation rules can be relaxed so that, instead of prescribing judgemental equalities, they posit proof terms. The propositional β-rule for identity types looks like this:

$$\frac{\begin{array}{c}\Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash C \ \mathsf{type} \\ \Gamma, z : A \vdash c : C\,[z, z, \mathsf{refl}_z / x, y, p] \qquad \Gamma \vdash a : A\end{array}}{\Gamma \vdash \beta(C, c, a) : \mathsf{Id}_{C[a, a, \mathsf{refl}_a / x, y, p]}\left(J\,(C, c, a, a, \mathsf{refl}_a)\,, c[a/z]\right)} \ \text{Id-}\beta\text{-P}$$

As already discussed, Hofmann and Streicher's groupoid model [HS98] taught us that identity types may (or may not) have multiple inhabitants that are not themselves identified in an iterated identity type. That is, one cannot derive UIP:

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash a, b : A \qquad \Gamma \vdash p, q : \mathsf{Id}_A(a, b)}{\Gamma \vdash \mathsf{uip}(A, a, b, p, q) : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p, q)} \ \text{UIP}$$

The groupoid model does, however, validate the following rule. This rule says that any pair of 2-dimensional paths inhabiting the same iterated identity type are themselves identified in a twice-iterated identity type.

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash a, b : A \qquad \Gamma \vdash p, q : \mathsf{Id}_A(a, b) \qquad \Gamma \vdash \alpha, \beta : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p, q)}{\Gamma \vdash \mathsf{uip}(A, a, b, p, q) : \mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,q)}(\alpha, \beta)}$$

A type theory with this rule is said to be 1-truncated. The groupoidal realizability models in this thesis are all 1-truncated. More generally, we can consider n-truncated type theory.

Without any extra assumptions on identity types, an ∞-groupoid structure (which may or may not be truncated at some level) can be defined on each type [vdBG10, Lum10]. Composition, units and inverses can be defined using path induction; the groupoid laws hold up to higher, coherent identity proofs. Voevodsky gave a model in simplicial sets [KK21] that inspired the univalence axiom.

Types may in turn be considered terms of a universe type $\mathcal{U}$. To be more precise, one often takes (especially for models) inhabitants of $\mathcal{U}$ to be "codes" for types, that can be decoded into bona fide types:

$$\frac{\Gamma \vdash A : \mathcal{U}}{\Gamma \vdash \mathsf{El}(A) \ \mathsf{type}}$$

We refer to both A and $\mathsf{El}(A)$ as "small types".

We have to specify under which type formers the universe is closed. For example, $\mathcal{U}$ is closed under dependent products iff the following rule holds.

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma, x : A \vdash B : \mathcal{U}}{\Gamma \vdash \Pi(x : A).B : \mathcal{U}} \ \mathcal{U}\text{-}\Pi$$

This rule stipulates that for a family of small types indexed by a small type the dependent function is itself a small type. If we have a family of small types indexed by a possibly "large type" (one not living in $\mathcal{U}$), but nevertheless allow the dependent product to live in the universe then we have an *impredicative* type theory.

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, x : A \vdash B : \mathcal{U}}{\Gamma \vdash \Pi(x : A).B : \mathcal{U}} \ \mathcal{U}\text{-}\Pi\text{-IMPR}$$

The impredicative type theory that we have described here should be compared with *System F* [Gir72, Rey74]. System F is not a dependent type theory, but one can translate judgements of System F into impredicative dependent type theory. One application of impredicativity is *impredicative encodings* of inductive types (as an alternative to postulating them, thus helping minimise the formal system—what we need to trust in an implementation). System F encodings do not satisfy their full universal properties; equivalently, they do not validate η. In the dependent setting, this means that they lack a dependent eliminator. [AFS18] refines System F encodings of inductive types translated into impredicative ITT so that they satisfy their full universal properties. The method of refinement makes crucial use of identity types, not available in System F. The method is extended to treat some higher inductive types.

2.2 Path categories

The main notion of model of type theory that we employ in this thesis is that given by *path categories*. Path categories, introduced by van den Berg and Moerdijk [vdBM18], are a slight variation on Brown's *categories of fibrant objects* [Bro73], with examples including the fibrant objects in any model category in which every object is cofibrant.

Path categories are a quite minimal setting for doing homotopy theory and modelling ITT. Moreover, van den Berg and Moerdijk formulated a notion of *homotopy exact completion* of path categories, which we use to obtain models of type theory with function extensionality, and also leads to a discussion of free completions of partitioned groupoidal assemblies and the wider programme of higher-dimensional realizability.

Definition 2.2.1 (Path category) A category $\mathbb{D}$ equipped with two classes of maps, fibrations and equivalences, is a path category iff the following axioms hold. We refer to a map that is both a fibration and an equivalence as an "acyclic fibration".

(PC1) Isomorphisms are fibrations and fibrations are closed under composition.

(PC2) The pullback of a fibration along any other map exists and is again a fibration.

(PC3) $\mathbb{D}$ has a terminal object 1 and every map $X \twoheadrightarrow 1$ is a fibration.

(PC4) Isomorphisms are equivalences.

(PC5) Equivalences satisfy the 2-out-of-6 property.

(PC6) Every object in $\mathbb{D}$ has a path object.

(PC7) Every acyclic fibration has a section.

(PC8) The pullback of an acyclic fibration along any other map exists and is again an acyclic fibration. ◊

(PC2) and (PC3) together imply that $\mathbb{D}$ has finite products. Regarding (PC6), a path object $\mathcal{P}X \in \mathbb{D}$ for an object $X \in \mathbb{D}$ is a factorisation of the diagonal by an equivalence r followed by a fibration $\langle s, t \rangle$.

$$\begin{tikzcd}
 & \mathcal{P}X \arrow[dr, two heads, "\langle s,t \rangle"] & \\
X \arrow[ur, "r"] \arrow[rr, "\Delta_X"'] & & X \times X
\end{tikzcd}$$

Observe that every finitely complete category $\mathbb{E}$ can be viewed as a "degenerate" path category in which every map is a fibration and isomorphisms are equivalences. In this case, the factorisation of the diagonal is $\Delta_X = \Delta_X \circ \mathsf{id}_X$, and so the interpretation of type theory in $\mathbb{E}$ validates UIP (in the sense that there is at most one section of Δ_X).

In general, path categories allow for a "non-split" interpretation of type theory in which computation rules hold propositionally; moreover, the syntactic category associated to a type theory with propositional computation rules carries the structure of a path category [vdB18a]. The basic setup of path categories models the structural rules of type theory, plus identity types; but further type formers may be added over and above this. We first describe a limited amount of the theory of path categories, mostly just enough to give an idea of the interpretation of ITT in path categories; but note that many standard results from homotopy theory hold in this relatively weak setting.

2.2.1 Some theory

First, we can define homotopies in a path category $\mathbb{D}$. Given two parallel maps $f, g : Y \to X$, a homotopy H from f to g is a map:

$$H : Y \to \mathcal{P}X$$

such that $f = sH$ and $g = tH$, where $\langle s, t\rangle : \mathcal{P}X \to X \times X$ is the fibration for some path object $\mathcal{P}X$ of X. The relation $\simeq$ of being homotopic gives rise to a congruence on $\mathbb{D}$, and one obtains the homotopy category $\mathbf{Ho}(\mathbb{D})$ of $\mathbb{D}$ by quotienting with respect to $\simeq$.

There is also a notion of fibrewise homotopy. But first we observe that any map $f : Y \to X$ in $\mathbb{D}$ factors as an equivalence w_f followed by a fibration p_f, ie. $f = p_f w_f$ (see [vdBM18, p. 3141] and [Bro73, p. 421]). We set

$w_f := [Y, rf]$, as in the following diagram.

$$\begin{tikzcd}
Y \arrow[r, dashed, "{[Y,rf]}"] \arrow[dr, equal] & Y \times_X \mathcal{P}X \arrow[r, "\pi_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}X \arrow[r, "t"] \arrow[d, "s"'] & X \\
& Y \arrow[r, "f"'] & X &
\end{tikzcd}$$

Then $p_f := t\pi_2$.

Given an object $X \in \mathbb{D}$, let $\mathbb{D}(X)$ be the full subcategory of the slice $\mathbb{D}/X$ spanned by the fibrations. $\mathbb{D}(X)$ is a path category, where a morphism in $\mathbb{D}(X)$ is a fibration or equivalence iff it is so in $\mathbb{D}$. The path object of a fibration $f \in \mathbb{D}(X)$ uses the above factorisation for the morphism $\Delta_f : Y \to Y \times_X Y$. The following is Proposition 2.6 in [vdBM18], and is proved on p. 428 of [Bro73].

Lemma 2.2.2 (Brown's lemma) *For any map $f : Y \to X$, the pullback functor*

$$f^* : \mathbb{D}(X) \to \mathbb{D}(Y)$$

preserves both fibrations and equivalences.

Let $f, g : Y \to X$ be two parallel arrows and $p : X \to I$ be a fibration such that $pf = pg$. A fibrewise homotopy H from f to g is a map:

$$H : Y \to P_I(X)$$

such that $sH = f$ and $tH = g$, where

$$\langle s, t\rangle : P_I(X) \to X \times_I X$$

is from the path object structure of $p \in \mathbb{D}(I)$. If $pf = pg$ is a fibration then H is a homotopy in $\mathbb{D}(I)$, but the definition makes sense even when this is not so. We write $f \simeq_I g$ if there is a fibrewise homotopy from f to g.

Definition 2.2.3 ((Weak) homotopy dependent products) A path category $\mathbb{D}$ has weak homotopy dependent products iff for any two fibrations $f : X \to J$ and $\alpha : J \to I$ there is an object

$$\Pi_\alpha(F) : \Pi_\alpha X \to I$$

in $\mathbb{D}(I)$ together with a an "evaluation map"

$$\mathsf{ev} : \alpha^*\Pi_\alpha X \to X$$

over J satisfying the following weak universal property: if there is a map $g : Y \to I$ and a map $m : \alpha^*Y \to X$ over J then there exists a map

$$n : Y \to \Pi_\alpha X$$

such that

$$m \simeq_J \alpha^* n$$

If n is unique with this property then $\mathbb{D}$ has (strong) homotopy dependent products. $\Diamond$

For degenerate path categories, having homotopy dependent products corresponds to being locally cartesian closed.

Carboni and Rosolini [CR00] proved that the exact completion $\mathbb{E}_{\mathsf{ex/lex}}$ of a category with finite limits is locally cartesian closed iff $\mathbb{E}$ is weakly locally cartesian closed (again, weakness corresponds to lack of uniqueness in the universal property).[1] Van den Berg and Moerdijk give a construction for the *homotopy exact completion* $\mathbf{Hex}(\mathbb{D})$ of a path category $\mathbb{D}$:

- Objects: homotopy equivalence relations. A fibration $p : R \to X \times X$ is a homotopy equivalence relation iff the following three conditions are met. We write p_1 for $\pi_1 p$ and p_2 for $\pi_2 p$.
  - There exists a map $\rho : X \to R$ such that $p\rho = \Delta_X$.
  - There exists a map $\sigma : R \to R$ such that $p_1\sigma = p_2$ and $p_2\sigma = p_1$.

---

[1] A regular category is one that admits a good notion of image factorisation. An exact category is a regular category that has quotients of equivalence relations.

  - There exists a map $\tau : R \times_X R \to R$ such that $p_1 q_1 = p_1 \tau$ and $p_2 q_2 = p_2 \tau$, where

$$\begin{tikzcd}
R \times_X R \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "p_1"] \\
R \arrow[r, "p_2"'] & X
\end{tikzcd}$$

- Morphisms: a morphism

$$f : (p : R \to X \times X) \to (q : S \to Y \times Y)$$

  is an equivalence class of morphisms $f : X \to Y$ for which there is a map $\phi : R \to S$ making the square

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d, "p"'] & S \arrow[d, "q"] \\
X \times X \arrow[r, "f \times f"'] & Y \times Y
\end{tikzcd}$$

  commute; two such morphisms $f, g : X \to Y$ are identified when there is a map $H : X \to S$ making the following triangle commute.

$$\begin{tikzcd}
 & S \arrow[d, "q"] \\
X \arrow[ur, "H"] \arrow[r, "\langle f,g \rangle"'] & Y \times Y
\end{tikzcd}$$

If $\mathbb{D}$ is a finitely complete category regarded as a degenerate path category, then $\mathbf{Hex}(\mathbb{D})$ coincides with $\mathbb{D}_{\mathsf{ex/lex}}$. It is also the case that $\mathbf{Hex}(\mathbb{D})$ is the ex/lex completion of the homotopy category of $\mathbb{D}$, which is obtained by quotienting hom-sets by homotopy.

In general, $\mathbf{Hex}(\mathbb{D})$ arises as the homotopy category of another path category, denoted $\mathbf{Ex}(\mathbb{D})$. The category $\mathbf{Ex}(\mathbb{D})$ has:

- Objects: same as $\mathbf{Hex}(\mathbb{D})$.
- Morphisms: a morphism

$$f : (p : R \to X \times X) \to (q : S \to Y \times Y)$$

> is a map $f : X \to Y$ for which there is a map $\phi : R \to S$ making the square
>
> $$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "p"'] & S \arrow[d, "q"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$
>
> commute.

We write $f \sim g$ if there exists a map $H : X \to S$ such that $\langle f, g \rangle = qH$; this constitutes a congruence on $\mathbf{Ex}(\mathbb{D})$. The fibrations and equivalences of $\mathbf{Ex}(\mathbb{D})$ are hand-picked so that $\sim$ becomes the homotopy relation in $\mathbf{Ex}(\mathbb{D})$.

A morphism $f : \rho \to q$ in $\mathbf{Ex}(\mathbb{D})$ is a fibration iff it is a fibration in $\mathbb{D}$ and there exists a map

$$\nabla : X \times_Y S \to R$$

in $\mathbb{D}$ such that $p_1 \nabla = \pi_1$ and $f p_2 \nabla = q_2 \pi_2$, where

$$\begin{tikzcd} X \times_Y S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Moreover, f is an equivalence in $\mathbf{Ex}(\mathbb{D})$ iff there is a map $g : q \to p$ in $\mathbf{Ex}(\mathbb{D})$ such that $fg \sim \mathsf{id}_Y$ and $gf \sim \mathsf{id}_X$.

The following theorem, which is [vdBM18, Thm. 5.10] combined with the discussion thereafter, is applied in Section 5.3.2 to obtain models of type theory with function extensionality.

Theorem 2.2.4 (van den Berg and Moerdijk) *If $\mathbb{D}$ has weak homotopy dependent products, then* $\mathbf{Hex}(\mathbb{D})$ *is locally cartesian closed and* $\mathbf{Ex}(\mathbb{D})$ *has (strong) homotopy dependent products.*

Finally, we come to the appropriate notion of universe in the setting of path categories. Note that a square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r] & W \end{tikzcd}$$

where f and g are fibrations, is a homotopy pullback iff the universal map $X \to Z \times_W Y$ is an equivalence.

Let $\mathcal{S}$ be a subclass of fibrations (the "small fibrations") in $\mathbb{D}$ that is pullback stable, closed under composition and contains all isomorphisms (note that [vdB18b], slightly more generally, allows stability under homotopy pullback). The following is the content of [vdB18b, Def. 2.2.5]

Definition 2.2.5 (Representation) A representation θ for the class $\mathcal{S}$ is a map $\theta : \Theta \to \Lambda$ in $\mathcal{S}$ such that any map in $\mathcal{S}$ arises as a homotopy pullback of θ along some map (a "characteristic map") in $\mathbb{D}$. ◊

A representation for $\mathcal{S}$ in a degenerate path category $\mathbb{D}$ is a map such that every map in $\mathcal{S}$ arises as the pullback of the representation along some map in $\mathbb{D}$. If $\mathcal{S}$ is closed under Π_α for *any* map $\alpha \in \mathbb{D}$ then $\mathcal{S}$ is *impredicative.*

2.2.2 Interpreting type theory

Let $\mathbb{D}$ be a path category with an impredicative subclass $\mathcal{S}$ of small fibrations and a representation θ for $\mathcal{S}$. We now describe at a high level the interpretation of ITT in $\mathbb{D}$.

- Contexts are interpreted by objects. A dependent type $\Gamma \vdash A \ \mathsf{type}$ is interpreted by a fibration

  $$[\![\Gamma \vdash A \ \mathsf{type}]\!] : [\![\Gamma, x : A]\!] \to [\![\Gamma]\!]$$

  thought of as the "projection substitution". In the context of extensional type theory, fibrations are often known as "display maps". Pullback stability of fibrations corresponds to types being closed under substitution. That substitution is modelled by pullback gives a "non-split" model of type theory. This is a well-documented coherence issue going back to Seely's interpretation of dependent type theory in locally cartesian closed categories [See84]: substitution in type theory is strictly associative, whereas pullback is associative up to isomorphism. Many non-split models can be "strictified" in some way (see [Cur93, Hof95b, CD11, CGH14, LW15]).

- A term $\Gamma \vdash a : A$ is modelled by a section

  $$\llbracket \Gamma \vdash a : A \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \Gamma, x : A \rrbracket$$

  of $\llbracket \Gamma \vdash A \ \mathsf{type} \rrbracket$.

- Identity types are modelled using path objects. This method goes back to the work of Awodey and Warren [AW09] in the setting of Quillen model categories. Path categories validate the rule Id-β-P from Section 2.1 but not the rule Id-β, ie. the computation rule for identity types holds propositionally in path categories.

- The representation is used to model a universe. In particular, the representation map corresponds to the judgement

  $$X : \mathcal{U} \vdash \mathsf{El}(X) \ \mathsf{type}$$

  Small types are modelled by small fibrations, which are also closed under substitution. That every small fibration can be obtained as a homotopy pullback of the representation means that small fibrations are equivalent to those that arise as $\Gamma \vdash \mathsf{El}(A)$ type for some $\Gamma \vdash A : \mathcal{U}$:

  $$\begin{tikzcd}
\llbracket \Gamma, x : \mathsf{El}(A) \rrbracket \arrow[r] \arrow[d, "\llbracket \Gamma \vdash \mathsf{El}(A)\ \mathsf{type} \rrbracket"'] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket X : \mathcal{U}, x : \mathsf{El}(X) \rrbracket \arrow[d, "\llbracket X : \mathcal{U} \vdash \mathsf{El}(X)\ \mathsf{type} \rrbracket"] \\
\llbracket \Gamma \rrbracket \arrow[r, "\llbracket \Gamma \vdash A : \mathcal{U} \rrbracket"'] & \llbracket X : \mathcal{U} \rrbracket
\end{tikzcd}$$

where we abuse notation slightly and write $\llbracket \Gamma \vdash A : \mathcal{U} \rrbracket$ for the composite $\pi_2 \circ \llbracket \Gamma \vdash A : \mathcal{U} \rrbracket$ in the diagram:

$$\begin{tikzcd}
\llbracket \Gamma, X : \mathcal{U} \rrbracket \arrow[r, "\pi_2"] \arrow[d, "\llbracket X:\mathcal{U} \vdash \mathcal{U}\ \text{type} \rrbracket"] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket X : \mathcal{U} \rrbracket \arrow[d, "!"] \\
\llbracket \Gamma \rrbracket \arrow[u, bend left, "\llbracket \Gamma \vdash A:\mathcal{U} \rrbracket"] \arrow[r, "!"'] & 1
\end{tikzcd}$$

(we are viewing $\llbracket \Gamma \vdash A : \mathcal{U} \rrbracket$ not as a term but as a morphism of contexts (substitution)).

- Dependent products are modelled by homotopy dependent products, but computation rules again hold propositionally. Weak dependent products corresponds to lack of η (and thus function extensionality). The rule $\mathcal{U}$-Π-IMPR from Section 2.1 is satisfied because $\mathcal{S}$ is impredicative.

2.3 Categories with families

We swiftly define *categories with families* (CwFs) [Dyb95] (see also [Hof97]), which are a split model of type theory. This is in order to provide context for the indexed perspective adopted in Section 4.2 and Section 5.5.

Let **Fam** be the category of families of sets:

- Objects: An object is a pair (X^0, X^1), where X^0 is a set and
$$\left(X^1_x\right)_{x \in X^0}$$
is a family of sets indexed by X^0.

- Morphisms: a morphism
$$\left(f^0, f^1\right) : \left(X^0, X^1\right) \to \left(Y^0, Y^1\right)$$
is a pair consisting of a function $f^0 : X^0 \to Y^0$ and a family of functions:
$$\left(f^1_x : X^1_x \to Y^1_{f_0(x)}\right)_{x \in X^0}$$

A CwF $\mathbb{E}$ is given by the following data:

- A category (of contexts) $\mathbb{E}$ with a terminal object (the empty conetxt).
- A functor:

  $$\mathcal{F} = (\mathsf{Ty}, \mathsf{Tm}) : \mathbb{D}^{\mathsf{op}} \to \mathbf{Fam}$$

  To explain this notation, we can decompose $\mathcal{F}$ as:

  $$\mathcal{F}(\Gamma) = \Big(\mathsf{Ty}(\Gamma), (\mathsf{Tm}(\Gamma, A))_{A \in \mathsf{Ty}(\Gamma)}\Big)$$

  $\mathsf{Ty}(\Gamma)$ is the set of types in context Γ and $\mathsf{Tm}(\Gamma, A)$ is the set of terms of type A in context Γ. The morphism part of $\mathcal{F}$ is substitution. CwFs are split models of type theory due to functoriality of $\mathcal{F}$.

- For each $\Gamma \in \mathbb{E}$ and each $A \in \mathsf{Ty}(\Gamma)$ a *comprehension*. That is, an object $\Gamma.A \in \mathbb{E}$ and two "projections":

  $$\begin{aligned} &\mathsf{p}(A) : \Gamma.A \to \Gamma \\ &\mathsf{v}_A \in \mathsf{Tm}\Big(\Gamma.A, \mathcal{F}(\mathsf{p}(A))^0(A)\Big) \end{aligned}$$

  such that for each $f : \Delta \to \Gamma$ and $M \in \mathsf{Tm}(\Delta, \mathcal{F}(f)^0(A))$ there exists a unique morphism

  $$\langle f, M \rangle_A : \Delta \to \Gamma.A$$

  such that:

  $$\begin{aligned} &\mathsf{p}(A) \circ \langle f, M \rangle_A = f \\ &\mathcal{F}\left(\langle f, M \rangle_A\right)^1_{\mathcal{F}(\mathsf{p}(A))^0(A)} (\mathsf{v}_A) = M \end{aligned}$$

  In particular, $\Gamma.A$ is the extended context $\Gamma, x : A$.

2.4 Categorical realizability models

The theory of realizability categories is often developed over partial combinatory algebras (PCAs). Throughout this section, we move from realizability over PCAs, through realizability over *typed* combinatory algebras, to realizability over categories. From there, it is a smooth transition to higher-dimensional realizability. [vO08, Str08, Bau22] are all good references on this material.

2.4.1 Over partial combinatory algebras

In this section we will show that the category $\mathbf{Asm}(\mathcal{A})$ of set-based assemblies over a PCA $\mathcal{A}$ models impredicative type theory. We will also show that the category $\mathbf{PAsm}(\mathcal{A})$ of partitioned (set-based) assemblies models a version of impredicative type theory without function extensionality (dependent products don't satisfy η). These models are presented as degenerate path categories, to make for an easier comparison with groupoidal realizability models developed later in the thesis.

PCAs are abstract models of untyped computation. Combinatory algebras were introduced by Schönfinkel [Sch24] and Curry [Cur30], whereas *partial* combinatory algebras were first studied by Feferman [Fef75]. A PCA

$$\mathcal{A} = (\mathcal{A}, (-) \cdot (-) : \mathcal{A} \times \mathcal{A} \to \mathcal{A})$$

consists of a partial binary operation $(-) \cdot (-)$ on a carrier set $\mathcal{A}$, such that there are distinguished "combinators" s and k satisfying:

$$\begin{aligned} &\mathsf{k} \cdot a \cdot b = a \\ &\mathsf{s} \cdot a \cdot b \downarrow \\ &\mathsf{s} \cdot a \cdot b \cdot c \simeq a \cdot c \cdot (b \cdot c) \end{aligned}$$

where $a \cdot b \downarrow$ means that the application $a \cdot b$ is defined, and $a \cdot b \simeq a' \cdot b'$ ("Kleene equality") means that both either both $a \cdot b$ and $a' \cdot b'$ are undefined or else they are both defined and equal elements of $\mathcal{A}$. Note that application associates to the left and we often write $a \cdot b$ simply as ab.

Example 2.4.1 We have already seen the paradigmatic example of a PCA, Kleene's first algebra $\mathcal{K}_1$. Another important example of a PCA is the untyped λ-calculus. Let Λ be the set of untyped λ-terms up to β-equivalence (and, of course, α-equivalence). Then we define:

$$\begin{aligned} &(-) \cdot (-) : \Lambda \times \Lambda \to \Lambda \\ &t \cdot u := tu \end{aligned}$$

In this case the combinators are:

$$\begin{aligned} &\mathsf{k} := \lambda xy.x \\ &\mathsf{s} := \lambda fgx.fx(gx) \end{aligned}$$

A *polynomial* over $\mathcal{A}$ is a formal expression built from the grammar:

$$t ::= x \mid a \mid t \cdot t$$

where x is taken from a countably infinite set $\mathcal{V}$ of variables and $a \in \mathcal{A}$. We can talk about free variables of a polynomial and the substitution $t[a_1, ..., a_n / x_1, ..., x_n]$ in t of elements $a_1, ..., a_n \in \mathcal{A}$ for variables $x_1, ..., x_n \in \mathcal{V}$. Moreover, closed polynomials have an obvious interpretation in $\mathcal{A}$. If t is a closed polynomial ($\mathsf{FV}(t) = \emptyset$) then we say that it is defined ($t \downarrow$) iff it is defined when interpreted in $\mathcal{A}$. $t \simeq t'$ holds for closed polynomials iff they are both undefined or else both defined and equal when interpreted in $\mathcal{A}$. By extension, if t is an open polynomial with $\mathsf{FV}(t) \subseteq \{x_1, ..., x_n\}$ then we say that t is defined iff for all $a_1, ..., a_n \in \mathcal{A}$ the substitution

$$t[a_1, ..., a_n / x_1, ..., x_n]$$

is defined. Finally, for open polynomials t, t' with $\mathsf{FV}(t), \mathsf{FV}(t') \subset \{x_1, ..., x_n\}$, we define $t \simeq t'$ iff for all $a_1, ..., a_n \in \mathcal{A}$:

$$t[a_1, ..., a_n / x_1, ..., x_n] \simeq t'[a_1, ..., a_n / x_1, ..., x_n]$$

Proposition 2.4.2 ([Sch24, Cur30, Fef75]) *PCAs are combinatorially complete.*

This means that PCAs can mimic λ-abstraction. In more detail: for every polynomial t over $\mathcal{A}$ and every variable x there exists a polynomial

$$\lambda x.t$$

with $\mathsf{FV}(\lambda x.t) \subseteq \mathsf{FV}(t) - \{x\}$ such that $(\lambda x.t)\downarrow$ and for all $a \in \mathcal{A}$:

$$(\lambda x.t)a \simeq t[a/x]$$

Sketch of proof: We define $\lambda x.t$ by induction on the structure of t:

$$\begin{aligned} &\lambda x.x := \mathsf{I} && \text{if } x \in \mathcal{V} \\ &\lambda x.t := \mathsf{K}t && \text{if } x \notin \mathsf{FV}(t) \\ &\lambda x.tt' := \mathsf{S}(\lambda x.t)(\lambda x.t') && \text{if } x \in \mathsf{FV}(t) \end{aligned}$$

□

We will make use of the fact that, in any PCA, we can pair elements and project elements from a pair. That is, in any PCA $\mathcal{A}$ there are elements $\mathsf{p}, \pi_1, \pi_2 \in \mathcal{A}$ such that for all $a, b \in \mathcal{A}$:

$$\begin{aligned} &\mathsf{p}ab\downarrow \\ &\pi_1(\mathsf{p}ab) = a \\ &\pi_2(\mathsf{p}ab) = b \end{aligned}$$

These are defined respectively:

$$\begin{aligned} &\mathsf{p} := \lambda xyz.zxy \\ &\pi_1 := \lambda z.z(\lambda xy.x) \\ &\pi_1 := \lambda z.z(\lambda xy.y) \end{aligned}$$

We now define the category $\mathbf{Asm}(\mathcal{A})$ of set-based assemblies over the PCA $\mathcal{A}$:

- Objects: an assembly X is a pair:

  $$(X, \Vdash_X)$$

  where $X \in \mathbf{Set}$ and

  $$\Vdash_X \subseteq \mathcal{A} \times X$$

  is the realizability relation, which is assumed to be total in that all elements of X have at least one realizer. We can think of X as a datatype whose "values" $x \in X$ are implemented by their realizers.

- Morphisms: a morphism

  $$f : (X, \Vdash_X) \to (Y, \Vdash_Y)$$

  of assemblies is a function $f : X \to Y$ such that there exists $e \in \mathcal{A}$ such that for all $x \in X$ and all $a \in \mathcal{A}$:

  $$a \Vdash_X x \quad \Rightarrow \quad ea \downarrow \ \wedge \ ea \Vdash_Y f(x)$$

  We say that the function f is realized (or tracked) by e, and write $e \Vdash f$.

An assembly X is partitioned when its realizability relation $\Vdash_X$ is actually a function:

$$\| - \|_X : X \to \mathcal{A}$$

The full subcategory of $\mathbf{Asm}(\mathcal{A})$ spanned by the partitioned assemblies is denoted $\mathbf{PAsm}(\mathcal{A})$.

An assembly X is modest when for any $x, x' \in X$ and any $a \in \mathcal{A}$

$$a \Vdash_X x \ \wedge \ a \Vdash_X x' \quad \Rightarrow \quad x = x'$$

The full subcategory of $\mathbf{Asm}(\mathcal{A})$ spanned by the modest assemblies is denoted $\mathbf{Mod}(\mathcal{A})$, but note that there is also the category of modest *and* partitioned assemblies.

Proposition 2.4.3 **Asm**$(\mathcal{A})$ *is finitely complete.*

Sketch of proof: The terminal assembly is $(1, \Vdash_1)$, where

$$a \Vdash_1 *$$

for all $a \in \mathcal{A}$.

The pullback of $g : (Y, \Vdash_Y) \to (Z, \Vdash_Z)$ along $f : (X, \Vdash_X) \to (Z, \Vdash_Z)$ is

$$\left(X \times_Z Y, \Vdash_{X \times_Z Y}\right)$$

where

$$a \Vdash_{X \times_Z Y} (x, y) \quad \Leftrightarrow \quad \pi_1 a \Vdash_X x \quad \wedge \quad \pi_2 a \Vdash_Y y \qquad \square$$

This allows us to view **Asm**$(\mathcal{A})$ as a degenerate path category. We will continue to show that **Asm**$(\mathcal{A})$ models impredicative type theory

Proposition 2.4.4 **Asm**$(\mathcal{A})$ *is cartesian closed. The exponential object Y^X is modest when Y is.*

Sketch of proof: The exponential of $(Y, \Vdash_Y)$ by $(X, \Vdash_X)$ is

$$(\mathbf{Asm}(\mathcal{A})(X, Y), \Vdash_{Y^X})$$

where

$$e \Vdash_{Y^X} f \quad \Leftrightarrow \quad e \Vdash f$$

The evaluation map, inherited from **Set**, is realized by $\lambda x.\pi_1 x(\pi_2 x)$.

Let Y be modest. Then $e \Vdash f$ and $e \Vdash g$ imply that for all $x \in X$ and all $a \Vdash_X x$ we have $ea \Vdash f(x)$ and $ea \Vdash g(x)$. By modesty of Y, this means that for all $x \in X$ we have $f(x) = g(x)$, ie. $f = g$. $\square$

A family of assemblies is simply a morphism in **Asm**$(\mathcal{A})$. A *modest* family $m : Y \to X$ of assemblies is a morphism in **Asm**$(\mathcal{A})$ such that for all

$x : 1 \to X$ the following pullback is a modest assembly.

$$\begin{tikzcd}
Y_x \arrow[r, "\pi_2"] \arrow[d, "x^* m"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "m"] \\
1 \arrow[r, "x"'] & X
\end{tikzcd}$$

That is, all fibres are modest. By the pullback lemma, modest families are closed under pullback and composition. Isomorphisms are trivially modest families. The modest families are our subclass of small fibrations (remember, every map is a fibration).

Proposition 2.4.5 $\mathbf{Asm}(\mathcal{A})$ *is locally cartesian closed. That is, for every* $f : Y \to Z$ *in* $\mathbf{Asm}(\mathcal{A})$ *there is a right adjoint*

$$\Pi_f : \mathbf{Asm}(\mathcal{A})/Y \to \mathbf{Asm}(\mathcal{A})/Z$$

to the pullback functor f^**. Moreover, if* $g : X \to Y$ *is a modest family then so is* $\Pi_f(g) : \Pi_f X \to Z$.

Sketch of proof: Let $g : X \to Y \in \mathbf{Asm}(\mathcal{A})$. Let P be the set of pairs (z, h) such that $z \in Z$ and

$$\begin{tikzcd}
Y_z \arrow[rr, "h"] \arrow[dr, hook] & & X \arrow[dl, "g"] \\
& Y &
\end{tikzcd} \tag{2.4.1}$$

Put

$$e \Vdash_P (z, h) \quad \Leftrightarrow \quad \pi_1 e \Vdash_Z z \;\; \wedge \;\; \pi_2 e \Vdash h$$

The assembly $(\Pi_f X, \Vdash_{\Pi_f X})$ is defined to be:

$$\Pi_f X := \{p \in P \mid \exists e \in \mathcal{A}.\, e \Vdash_P p\}$$
$$e \Vdash_{\Pi_f X} p \quad \Leftrightarrow \quad e \Vdash_P p$$

The map $\Pi_f(g) : \Pi_f X \to Z$ is the first projection, realized by the first projection.

Moreover, assume g is modest and let $e \Vdash_{\Pi_f X} (z,h),(z,h')$ for

$$(z,h),(z,h') \in (\Pi_f X)_z$$

both in the fibre over z. Then we know that $e \Vdash h,h' : Y_z \to X$. For any $y \in Y_z$ and any $a \Vdash_{Y_z} y$ we have that $ea \Vdash_X h(y), h'(y)$. As $h(y)$ and $h'(y)$ are in the same fibre of g by (2.4.1), modesty of g allows us to conclude that $h = h'$. □

Define a fully faithful functor:

$$\begin{aligned} \nabla : \mathbf{Set} &\to \mathbf{Asm}(\mathcal{A}) \\ X &\mapsto (X, \Vdash_X := \mathcal{A} \times X) \\ f &\mapsto f \end{aligned}$$

We reiterate that *every* element $a \in \mathcal{A}$ realizes every element $x \in X$. In this way, $\nabla(f)$—indeed any map into an object in the image of ∇—is realized by $\lambda x.x$.

Proposition 2.4.6 $\mathbf{Asm}(\mathcal{A})$ *has a representation for modest families.*

It is helpful to observe that a modest set $X = (X, \Vdash_X)$ induces a PER $P(X)$ on $\mathcal{A}$. We declare $(a,b) \in P(X)$ iff there exists a (necessarily unique) $x \in X$ such that $a, b \Vdash_X x$. Conversely, given a PER R on $\mathcal{A}$, we obtain a modest set $M(R)$ with underlying set the quotient

$$\mathsf{Dom}_R(\mathcal{A})/R$$

where

$$\mathsf{Dom}_R(\mathcal{A}) := \{a \in \mathcal{A} \mid aRa\}$$

(R is an equivalence relation on $\mathsf{Dom}_R(\mathcal{A})$). The realizability relation $\Vdash_{M(R)}$ is given by

$$a \Vdash_{M(R)} [b] \quad \Leftrightarrow \quad a \in [b]$$

A round trip in one direction is the identity: $P(M(R)) = R$. In $\mathsf{Mod}(\mathcal{A})$ there is an isomorphism:

$$\begin{aligned} \phi_X : X &\to M(P(X)) \\ x &\mapsto \{a \in \mathcal{A} \mid a \Vdash_X x\} \end{aligned}$$

realized by $\lambda x.x$. The inverse is:

$$\begin{aligned} \phi_X^{-1} : M(P(X)) &\to X \\ [a] &\mapsto \widetilde{a} \end{aligned}$$

where $\widetilde{a}$ is the unique element of X such that $a \Vdash_X x$, also realized by $\lambda x.x$.

In fact, there is an equivalence of categories:

$$\mathbf{Mod}(\mathcal{A}) \simeq \mathbf{PER}(\mathcal{A})$$

where $\mathbf{PER}(\mathcal{A})$ is the category of PERS on $\mathcal{A}$. A morphism $f : R \to S$ of PERs is a function

$$f : \mathsf{Dom}_R(\mathcal{A})/R \to \mathsf{Dom}_S(\mathcal{A})/S$$

between the quotients that is tracked: there exists $e \in \mathcal{A}$ such that

$$f([a]) = [ea]$$

Proof of Proposition 2.4.6: Let

$$\mathsf{PER}(\mathcal{A}) := \mathsf{Ob}\left(\mathbf{PER}(\mathcal{A})\right)$$

Define

$$\Lambda := \nabla\left(\mathsf{PER}(\mathcal{A})\right)$$

and:

$$\begin{aligned} \Theta &:= \{(R, A) \mid R \in \mathsf{PER}(\mathcal{A}) \wedge A \in \mathsf{Dom}(\mathcal{A})/R\} \\ a \Vdash_\Theta (R, A) \quad &\Leftrightarrow \quad a \in A \end{aligned}$$

The first projection $\theta : (R, A) \mapsto R$ is a modest family, realized by $\lambda x.x$.

Let $m : Y \to X$ be a modest family. We define its characteristic map:

$$\nu_m : X \to \Lambda$$
$$\nu_m(x) := \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid \exists y \in Y_x.\ a, b \Vdash_Y y\}$$

There is an isomorphism:

$$m_* : Y \to X \times_\Lambda \Theta$$
$$m_*(y) := (m(y), \nu_m(m(y)), \{a \in \mathcal{A} \mid a \Vdash_Y y\})$$

realized by $\lambda x.x$. The inverse $m^* : X \times_\Lambda \Theta \to Y$ sends $(x, \nu_m(x), A)$ to the unique $\hat{x} \in Y_x$ realized by the members of A. This is also realized by $\lambda x.x$.□

We now turn our attention to the category $\mathbf{PAsm}(\mathcal{A})$ of *partitioned* assemblies.

Proposition 2.4.7 $\mathbf{PAsm}(\mathcal{A})$ *is finitely complete.*

Sketch of proof: The terminal partitioned assembly is $(1, \| * \|_1 := a_0)$, for some chosen $a_0 \in \mathcal{A}$.

The pullback of $g : (Y, \|-\|_Y) \to (Z, \|-\|_Z)$ along $f : (X, \|-\|_X) \to (Z, \|-\|_Z)$ is

$$\big(X \times_Z Y, \|-\|_{X \times_Z Y}\big)$$

where

$$\|(x,y)\|_{X \times_Z Y}(x,y) := \mathsf{p}\|x\|_X\|y\|_Y \qquad \square$$

As morphisms of partitioned assemblies may have many realizers, but elements of partitioned assemblies are each allowed only a single realizer, so $\mathbf{PAsm}(\mathcal{A})$ does not have (strong) function spaces, but only weak ones.

Proposition 2.4.8 $\mathbf{PAsm}(\mathcal{A})$ *is weakly cartesian closed. The exponential object* Y^X *is modest when* Y *is.*

Sketch of proof: The underlying set of the exponential of $(Y, \Vdash_Y)$ by $(X, \Vdash_X)$ is

$$\{(f, e) \in \mathbf{PAsm}(\mathcal{A})(X, Y) \times \mathcal{A} \mid e \Vdash f\}$$

The realizability function is defined by the second projection.

The transpose $\widetilde{f} : Z \to Y^X$ of a function $f : Z \times X \to Y$ is given by

$$\widetilde{f}(z) := (f(z, -), e)$$

where e is some chosen realizer $e \Vdash f(z, -)$—this this the source of weakness of exponentials. $\widetilde{f}$ is realized by $\lambda x.e$.

Let Y be modest. Then $(f, e), (g, e) \in Y^X$ means that for any $x \in X$ we have $\|f(x)\|_Y = e\|x\|_X = \|g(x)\|_Y$. As Y is modest this means that $f(x) = g(x)$ for all $x \in X$, ie. $f = g$. $\square$

Modest families of partitioned assemblies are defined analogously to modest families of assemblies. Again, these constitute the subclass of small fibrations.

Proposition 2.4.9 **PAsm**$(\mathcal{A})$ *is weakly locally cartesian closed. That is, for every* $f : Y \to Z$ *in* **Asm**$(\mathcal{A})$ *there is a weak right adjoint*

$$\Pi_f : \mathbf{Asm}(\mathcal{A})/Y \to \mathbf{Asm}(\mathcal{A})/Z$$

to the pullback functor f^*. *Moreover, if* $g : X \to Y$ *is a modest family then so is* $\Pi_f(g) : \Pi_f X \to Z$.

Sketch of proof: Let $g : X \to Y \in \mathbf{Asm}(\mathcal{A})$. The underlying set of $\Pi_f X$ is the set of triples (z, h, e) such that $z \in Z$,

$$\begin{tikzcd}
Y_z \arrow[rr, "h"] \arrow[dr, hook] & & X \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

and $e \Vdash h$. Put

$$\|(z, h, e)\|_{\Pi_f X} := \mathsf{p}\|z\|_Z e$$

The map $\Pi_f(g) : \Pi_f X \to Z$ is the first projection, realized by the first projection.

Moreover, assume g is a modest family and that

$$\|(z, h, e)\|_{\Pi_f X} = \|(z, h', e')\|_{\Pi_f X}$$

As

$$\pi_2\|(z, h, e)\|_{\Pi_f X} = \pi_2\|(z, h', e')\|_{\Pi_f X}$$

we know that $e = e'$. As $e \Vdash h, h'$ and g is modest, we can obtain $h = h'$ as in the proof of Proposition 2.4.5. $\square$

Like $\mathbf{Asm}(\mathcal{A})$, the partitioned assemblies also have an impredicative universe. The inclusion of **Set** in $\mathbf{PAsm}(\mathcal{A})$ is different from that in $\mathbf{Asm}(\mathcal{A})$. Choose an arbitrary $a_0 \in \mathcal{A}$ and define a functor:

$$\begin{aligned} \nabla : \mathbf{Set} &\to \mathbf{PAsm}(\mathcal{A}) \\ X &\mapsto (X, \Vdash_X : x \mapsto a_0) \\ f &\mapsto f \end{aligned}$$

where $\nabla(f)$—indeed any map into an object in the image of ∇—is realized by $\lambda x.a_0$. This is sometimes called the "chaotic" inclusion.

Proposition 2.4.10 $\mathbf{PAsm}(\mathcal{A})$ *has a representation for modest families.*

Proof: Observe that modest partitioned assemblies are in bijective correspondence with subsets of $\mathcal{A}$. So define

$$\Lambda := \nabla\left(\mathcal{P}(\mathcal{A})\right)$$

and:

$$\Theta := \{(U, a) \mid U \in \mathcal{P}(\mathcal{A}) \wedge a \in U\}$$
$$\|(U, a)\|_\Theta := a$$

The first projection $\theta : (U, a) \mapsto U$ is a modest family. Given a modest family $m : Y \to X$ we define its characteristic map $\nu_m : X \to \Lambda$ such that $\nu_m(x)$ is the subset of $\mathcal{A}$ determined by Y_x. Then m is the pullback of θ by ν_m. ☐

Realizability toposes arise as ex/lex completions of categories of partitioned assemblies [RR90], and categories of general assemblies as reg/lex completions of categories of partitioned assemblies. This is summarised in the following schematic.

$$\begin{tikzcd}
\mathbf{PAsm}(\mathcal{A}) \arrow[r, "\text{reg/lex completion}"'] \arrow[rr, bend left, "\text{ex/lex completion}"] & \mathbf{Asm}(\mathcal{A}) \arrow[r, "\text{ex/reg completion}"'] & \mathbf{RT}(\mathcal{A})
\end{tikzcd}$$

More generally, Menni [Men00, Men03] proved that the exact completion $\mathbb{E}_{\mathsf{ex/lex}}$ of a finitely complete category $\mathbb{E}$ is a topos if and only if $\mathbb{E}$ is weakly locally cartesian closed and has a generic proof. The generic proof, which gives rise to the subobject classifier in the ex/lex completion, classifies maps up to "bi-implication" (not isomorphism), and a characteristic map is guaranteed only to exist—it need not be unique. Formally, a generic proof θ in a category $\mathbb{E}$ is a map $\theta : \Theta \to \Lambda$ such that for any map $f : Y \to X \in \mathbb{E}$ there is a morphism $\nu_f : X \to \Lambda$ such that there are maps f_* and f^* making the following diagram commute.

$$\begin{tikzcd}
Y \arrow[r, bend left, "f_*"] \arrow[dr, bend right, "f"'] & P \arrow[l, bend left, "f^*"] \arrow[r] \arrow[d, "(\nu_f)^*\theta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Theta \arrow[d, "\theta"] \\
& X \arrow[r, "\nu_f"'] & \Lambda
\end{tikzcd}$$

Proposition 2.4.11 (Menni) **PAsm**$(\mathcal{A})$ *has a generic proof.*

Proof: The representation for modest families of partitioned groupoidal assemblies from Proposition 2.4.10 is a generic proof in $\mathbf{PAsm}(\mathcal{A})$! □

2.4.2 Over typed combinatory algebras

As an intermediate between PCAs and categories, we discuss *typed* combinatory algebras (TCAs) [Lon99]. Note that we drop partiality at this point, for we do not treat partiality in higher-dimensional realizability.

A TCA $\mathcal{A}$ is a family of non-empty sets

$$(\mathcal{A}_A)_{A\in\mathcal{T}}$$

indexed by a non-empty set $\mathcal{T}$ of "types". $\mathcal{T}$ is closed under operations $\times$ and $\to$. There is also a family of typed application operations, one for each pair $(A, B) \in \mathcal{T}$:

$$(-) \cdot_{A,B} (-) : \mathcal{A}_{A\to B} \times \mathcal{A}_A \to \mathcal{A}_B$$

though again we tend to suppress the $\cdot_{A,B}$ (or at least their subscripts). The following typed combinators are required to exist, satisfying the given equations.

$$\begin{aligned}
&\mathsf{k} \in \mathcal{A}_{A\to B\to A} && \mathsf{k}ab = a\\
&\mathsf{s} \in \mathcal{A}_{(A\to B\to C)\to(A\to B)\to A\to C} && \mathsf{s}abc = ac(bc)\\
&\mathsf{p} \in \mathcal{A}_{A\to B\to A\times B} && \\
&\pi_1 \in \mathcal{A}_{A\times B\to A} && \pi_1(\mathsf{p}ab) = a\\
&\pi_2 \in \mathcal{A}_{A\times B\to B} && \pi_2(\mathsf{p}ab) = b
\end{aligned}$$

TCAs are combinatorially complete in a typed sense; see [Bau22] for more details.

An assembly X in the category $\mathbf{Asm}(\mathcal{A})$, where $\mathcal{A}$ is a *typed* combinatory algebra, comes with a "realizer type" $A \in \mathcal{T}$; realizers for the elements of the underlying set of X are drawn from $\mathcal{A}_A$. So "values" of X are implemented by objects of the same type in the language of the TCA. In full, the category has:

- Objects: triples:

  $$(X, A, \Vdash_X)$$

  where $X \in \mathbf{Set}$, $A \in \mathcal{T}$ and

  $$\Vdash_X \subseteq \mathcal{A}_A \times X$$

  is a total relation in that all elements of X have at least one realizer.

- Morphisms: a morphism

  $$f : (X, A, \Vdash_X) \to (Y, B, \Vdash_Y)$$

  of assemblies is a function $f : X \to Y$ such that there exists

  $$e \in \mathcal{A}_{A \to B}$$

  such that for all $x \in X$ and all $a \in \mathcal{A}_A$:

  $$a \Vdash_X x \quad \Rightarrow \quad ea \Vdash_Y f(x)$$

The category $\mathbf{Asm}(\mathcal{A})$ of assemblies over a TCA is finitely complete and locally cartesian closed. One can consider partitioned and modest subcategories as before. We will not rehearse all the details, but point out that constructions in $\mathbf{Asm}(\mathcal{A})$ use the corresponding (possibly weak) structure in the TCA. For example, the product of $(X, A, \Vdash_X)$ and $(Y, B, \Vdash_Y)$ is

$$(X \times Y, A \times B, \Vdash_{X \times Y})$$

where

$$a \Vdash_{X \times Y} (x, y) \quad \Leftrightarrow \quad \pi_1 a \Vdash_X x \ \wedge \ \pi_2 a \Vdash_Y y$$

There is, however, no impredicative universe: impredicativity is linked to untypedness. Lietz and Streicher [LS02, Theorem 4.2] showed that the following—and more—are equivalent, where $\mathcal{A}$ is a typed partial combinatory algebra (TPCA):

- $\mathcal{A}$ has a *universal type*, so is essentially untyped (such a type can be endowed with the structure of a PCA, the ensuing realizability categories over which are equivalent to the corresponding realizability categories over the original TPCA);
- a particular realizability category, whose objects are formal partial equivalence relations with respect to a hyperdoctrine built from $\mathcal{A}$, is a topos;
- $\mathbf{Asm}(\mathcal{A})$ has a generic mono: every mono arises as a pullback of the generic mono.

Similar results in slightly different settings are found in [Bir00b, RR01]. To be sure, the above result does not mention impredicative universes of modest sets, but we suspect that this manifestation of impredicativity is also inextricably linked to untypedness.

To round off this section, we hint at how $\mathbf{Asm}(\mathcal{A})$ can be organised into a CwF. This gives more context to Section 5.5. A *uniform family* of assemblies over an assembly $(X, A, \Vdash_X) \in \mathbf{Asm}(\mathcal{A})$ is a function

$$\mathbb{X} : X \to \mathbf{Asm}(\mathcal{A})$$

such that there is a uniform realizer type (second component) across the $\mathbb{X}(x)$.[2] Uniform families over X comprise $\mathsf{Ty}(X)$ in the CwF with category of contexts $\mathbf{Asm}(\mathcal{A})$. Note that $\mathbf{Asm}(\mathcal{A})$ for $\mathcal{A}$ a PCA can be organised into

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[2] We believe that the terminology here goes back to [BBS04], but is also used in [Bir99, Bir00a, Jac99, Bau00]. Of course, the need for a uniformity condition does not arise in the untyped case.

a CwF in the same way; though, of course, there is no need for a uniformity condition.

The extended context $X.\mathbb{X} \in \mathbf{Asm}(\mathcal{A})$ is:

$$(\Sigma(x \in X).\mathbb{X}(x), A \times B, \Vdash_{X.\mathbb{X}})$$

where $\Sigma(x \in X).\mathbb{X}(x)$ is the dependent sum in **Set**, and B is the uniform realizer type of the $\mathbb{X}(x)$, and

$$a \Vdash_{X.\mathbb{X}} (x, y) \quad \leftrightarrow \quad \pi_1 a \Vdash_X x \quad \wedge \quad \pi_2 a \Vdash_{\mathbb{X}(x)} y$$

Hence, we see the importance of uniformity; for what would the realizer type of $X.\mathbb{X}$ be if the $\mathbb{X}(x)$ had possibly distinct realizer types?

2.4.3 Over categories

Finally, we come to realizability over categories. This is what we shall generalise. Analyses of set-based realizability using realizer categories were given by Birkedal [Bir99, Bir00a, Bir00b] and Robinson-Rosolini [RR01]. The former constructs models of type theory in assemblies and modest sets over weakly closed partial cartesian categories (WCPC-categories, ie. categories with a notion of partial map that are weakly cartesian closed in a suitable sense) via a tripos-theoretic approach. The latter studies necessary conditions for the construction of realizability models with certain type-theoretic properties. Both consider when a realizability construction gives rise to a topos. Before these works, Lambek [Lam95] offered a category of PERs built over an arbitrary category, and Abramsky [Abr95] delivered a talk describing categories of assemblies and modest sets over an arbitrary category. Both show that the ensuing realizability categories inherit or improve features of the realizer category.

A category, like a TCA, gives a *typed* notion of realizability: we think of objects of the category as types. We work with categories that have terminal objects. If $\mathbb{C}$ is one of those, then we have a functor

$$\Pi := \mathbb{C}(1, -) : \mathbb{C} \to \mathbf{Set}$$

The category $\mathbf{Asm}(\mathbb{C})$ of set-based assemblies over $\mathbb{C}$ is given by:

- Objects: an assembly X is a triple:

  $$(X, A, \Vdash_X)$$

  where X is the underlying set, $A \in \mathbb{C}$ is the realizer type and

  $$\Vdash_X \subseteq \Pi A \times X$$

  is the realizability relation (which again is assumed to be total in that all elements of X have at least one realizer).

- Morphisms: a morphism

  $$f : (X, A, \Vdash_X) \to (Y, B, \Vdash_Y)$$

  of assemblies is a function $f : X \to Y$ such that there exists $e : A \to B$ such that for all $x \in X$ and all $a \in \Pi A$:

  $$a \Vdash_X x \quad \Rightarrow \quad \Pi(e)(a) \Vdash_Y f(x)$$

Partitioned and modest assemblies are defined as usual.

This situation encompasses that of assemblies over TCAs, if we assume that a TCA has a type $1 \in \mathcal{T}$ such that $\mathcal{A}_1 = 1$. Given such a TCA $\mathcal{A}$, we build a category $\mathbb{C}(\mathcal{A})$ by setting:

- Objects: $\mathtt{Ob}(\mathbb{C}(\mathcal{A})) := \mathcal{T}$.

- Morphisms: $\mathbb{C}(\mathcal{A})(A, B)$ is the set of "computable functions" $\mathcal{A}_A \to \mathcal{A}_B$. A function $k : \mathcal{A}_A \to \mathcal{A}_B$ is computable iff there exists $e \in \mathcal{A}_{A \to B}$ representing k: for all $a \in \mathcal{A}_A$ we have $k(a) = e \cdot a$.

Note that computable functions $\mathcal{A}_1 \to \mathcal{A}_A$ are in bijective correspondence with elements of $\mathcal{A}_A$: a function $k : \mathcal{A}_1 \to \mathcal{A}_A$ is represented by $\lambda(x : 1).k(*)$.

Proposition 2.4.12 *The category* $\mathbf{Asm}(\mathcal{A})$ *of assemblies over the TCA* $\mathcal{A}$ *(with a type* $1 \in \mathcal{T}$ *such that* $\mathcal{A}_1 = 1$*) is isomorphic to the category* $\mathbf{Asm}(\mathbb{C}(\mathcal{A}))$ *of assemblies over the category* $\mathbb{C}(\mathcal{A})$ *constructed from the TCA* $\mathcal{A}$*.*

Proof: Up to the identification of computable functions $\mathcal{A}_1 \to \mathcal{A}_A$ with elements of $\mathcal{A}_A$, the functors going back and forth between these two categories are identities on both objects and arrows. A function f realized by e in $\mathbf{Asm}(\mathcal{A})$ is realized by the computable function $e \cdot (-)$ in $\mathbf{Asm}(\mathbb{C}(\mathcal{A}))$; and a function g realized by the computable function k in $\mathbf{Asm}(\mathbb{C}(\mathcal{A}))$ is realized by e_k in $\mathbf{Asm}(\mathcal{A})$, where e_k represents k. □

The presence of a "universal object" $U \in \mathbb{C}$ turns the *a priori* typed notion of realizability provided by the category $\mathbb{C}$ into an untyped one. Universal objects will be discussed in Section 3.3 of the next chapter.

Chapter 3

Realizer categories

Contents

The parameter over which we construct a groupoidal realizability category is a "realizer category". Categories provide a general framework for realizability, as we have seen, and also are amenable to higher generalisations. The first supposition we make of realizer categories is that they are cartesian closed. If $\mathbb{C}$ is cartesian closed, for any $A, B, C \in \mathbb{C}$, we denote by

$$\lambda_{A,B,C} : \mathbb{C}\left(A \times B, C\right) \to \mathbb{C}\left(A, C^B\right)$$

the currying map, and by

$$\mu_{A,B,C} : \mathbb{C}\left(A, C^B\right) \to \mathbb{C}\left(A \times B, C\right)$$

the uncurrying map (we often just write λ and μ). These maps constitute a natural isomorphism of hom sets.

Cartesian closure provides a setting for a well-behaved notion of interval *qua* internal co-groupoid (à la Warren [War08, War12]).[1] This is the key piece of structure for doing *higher-dimensional* realizability. Ultimately, it provides an answer to the question: what sort of thing realizes isomorphisms in a groupoid? In a tad more detail, it supplies a notion of homotopy, thereby endowing the ambient category with the structure of a strict ω-category. Moreover, it facilitates a fundamental groupoid construction for objects of the ambient category. In groupoidal realizability, objects of a groupoid are realized by points in the fundamental groupoid of some object from the realizer category, and isomorphisms in the groupoid are realized by paths in that fundamental groupoid.

3.1 Intervals

The definition of internal categories seems to go back to Grothendieck [Gro61]. As we have said, [War08, War12] are good places to read about internal co-categories and co-groupoids.

Definition 3.1.1 (Interval) An interval $\mathbb{I}$ in a CCC $\mathbb{C}$ is a diagram of the form

$$\begin{tikzcd}
\mathbb{I}_0 \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] & \mathbb{I}_1 \arrow[l, "*"'] \arrow[loop above, "\sigma"] \arrow[r, "2"] \arrow[r, bend left, "i_0"] \arrow[r, bend right, "i_1"'] & \mathbb{I}_2 \arrow[r, bend left, "j_0"] \arrow[r, bend right, "j_1"'] & \mathbb{I}_3
\end{tikzcd}$$

where $\mathbb{I}_0$ is terminal. The diagram

$$\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[d, "1"'] & \mathbb{I}_1 \arrow[d, "i_1"] \\
\mathbb{I}_1 \arrow[r, "i_0"'] & \mathbb{I}_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

---

[1] It is an issue left for future work to see how much of the theory of groupoidal realizability goes through using less structured categories. Indeed, one of the benefits of realizability constructions is that they turn weaker structures into stronger ones.

is required to be a pushout. Maps $\mathbb{I}_1 \to A$ (for any A) are thought of as paths in A, and so, the pushout allows us to concatenate two paths $\alpha, \beta : \mathbb{I}_1 \to A$ that match nose to tail: $\beta 0 = \alpha 1$. The result is $[\beta, \alpha] : \mathbb{I}_2 \to A$ (a path with twice the length of α and β, if you like). Likewise, the following is a pushout.

$$\begin{tikzcd}
\mathbb{I}_1 \arrow[r, "i_1"] \arrow[d, "i_0"'] & \mathbb{I}_2 \arrow[d, "j_0"] \\
\mathbb{I}_2 \arrow[r, "j_1"'] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbb{I}_3
\end{tikzcd}$$

To round off the definition, the following co-groupoid axioms are required to hold. The first set makes sure that the endpoint (or source and target) maps $0, 1$ play nicely with co-composition 2 and co-identity $*$.

$$\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{I}_1 \arrow[d, "2"] \\
\mathbb{I}_1 \arrow[r, "i_0"'] & \mathbb{I}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{I}_1 \arrow[d, "2"] \\
\mathbb{I}_1 \arrow[r, "i_1"'] & \mathbb{I}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[dr, equal] & \mathbb{I}_1 \arrow[d, "*"] & \mathbb{I}_0 \arrow[l, "1"'] \arrow[dl, equal] \\
& \mathbb{I}_0 &
\end{tikzcd}$$

The second set makes sure that the inverse operation σ behaves as expected.

$$\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[dr, "1"'] & \mathbb{I}_1 \arrow[d, "\sigma"] & \mathbb{I}_0 \arrow[l, "1"'] \arrow[dl, "0"] \\
& \mathbb{I}_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "\sigma"] \arrow[dr, equal] & \mathbb{I}_0 \arrow[d, "\sigma"] \\
& \mathbb{I}_0
\end{tikzcd}$$

The next two axioms are co-identity and co-associativity respectively.

$$\begin{tikzcd}
& \mathbb{I}_1 \arrow[dl, "\mathbb{I}_1"'] \arrow[d, "2"] \arrow[dr, "\mathbb{I}_1"] & \\
\mathbb{I}_1 & \mathbb{I}_2 \arrow[l, "{[\mathbb{I}_1, 0*]}"] \arrow[r, "{[1*, \mathbb{I}_1]}"'] & \mathbb{I}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_1 \arrow[r, "2"] \arrow[d, "2"'] & \mathbb{I}_2 \arrow[d, "{[j_1 i_1, j_0 2]}"] \\
\mathbb{I}_2 \arrow[r, "{[j_1 2, j_0 i_0]}"'] & \mathbb{I}_1
\end{tikzcd}$$

Lastly, we have co-inverse laws.

$$\begin{tikzcd}
\mathbb{I}_1 \arrow[r, "2"] \arrow[d, "*"'] & \mathbb{I}_2 \arrow[d, "{[\mathbb{I}_1, \sigma]}"] \\
\mathbb{I}_0 \arrow[r, "1"'] & \mathbb{I}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_1 \arrow[r, "2"] \arrow[d, "*"'] & \mathbb{I}_2 \arrow[d, "{[\sigma, \mathbb{I}_1]}"] \\
\mathbb{I}_0 \arrow[r, "0"'] & \mathbb{I}_3
\end{tikzcd}$$

$\Diamond$

Given the central importance of realizer categories in this thesis, we make an official definition:

Definition 3.1.2 (Typed realizer category) A (typed) realizer category is a cartesian closed category $\mathbb{C}$ together with an interval $\mathbb{I} \in \mathbb{C}$. $\Diamond$

Example 3.1.3 The category **Gpd** of groupoids and functors is a (typed) realizer category, with an interval $\mathbb{I}$ whose object of co-arrows $\mathbb{I}_1 := \mathbf{I}$ is the "walking isomorphism":

$$\begin{tikzcd} 0 \arrow[r, bend left, "i"] & 1 \arrow[l, bend left, "i^{-1}"] \end{tikzcd}$$

The maps 0 and 1 pick out the corresponding endpoints of $\mathbf{I}$. The map σ sends $i \mapsto i^{-1}$.

$\mathbb{I}_2 := \mathbf{I}_2$ has three objects and, again, one arrow in each hom set.

$$\begin{tikzcd} 0 \arrow[r, bend left, "i_0"] & 1 \arrow[l, bend left, "i_0^{-1}"] \arrow[r, bend left, "i_1"] & 2 \arrow[l, bend left, "i_1^{-1}"] \end{tikzcd}$$

The maps i_0 and i_1 send $i \in \mathbf{I}$ to the synonymous (eponymous, even) morphisms in $\mathbb{I}_2$. The map 2 picks out the composite $i_1 i_0$.

Continuing the trend, $\mathbb{I}_3 := \mathbf{I}_3$ has four objects and one arrow in each hom set.

$$\begin{tikzcd} 0 \arrow[r, bend left, "i_0"] & 1 \arrow[l, bend left, "i_0^{-1}"] \arrow[r, bend left, "i_1"] & 2 \arrow[l, bend left, "i_1^{-1}"] \arrow[r, bend left, "i_2"] & 3 \arrow[l, bend left, "i_2^{-1}"] \end{tikzcd}$$

The map j_0 sends $i_0 \mapsto i_0$ and $i_1 \mapsto i_1$; the map j_1 sends $i_0 \mapsto i_1$ and $i_1 \mapsto i_2$.

Example 3.1.4 Let **hTop** be the category of topological spaces and homotopy classes of continuous maps. The full subcategory of **hTop** spanned by the CW complexes is cartesian closed. It has an interval $\mathbb{I}$ whose object of co-arrows is the usual interval $\mathbb{I}_1 = [0, 1]$. Note that the same diagram considered in **Top** is a non-example of an interval because the co-identity and co-associativity laws hold only up to homotopy.

3.1.1 Homotopies

As shown in [War12, War08], an interval $\mathbb{I} \in \mathbb{C}$ endows the ambient category with the structure of a strict ω-category (given that $\mathbb{C}$ is cartesian closed—which is more than enough). The higher cells are given by homotopies with respect to $\mathbb{I}$.

Definition 3.1.5 (Homotopy with respect to an interval) Let $(\mathbb{C}, \mathbb{I})$ be a realizer category. A homotopy

$$H : f \Rightarrow g : A \to B$$

with respect to the interval $\mathbb{I}$ is a map

$$H : A \times \mathbb{I}_1 \to B$$

making the following diagram in **C** commute.

$$\begin{tikzcd}
A \times \mathbb{I}_0 \arrow[r, "A\times 0"] \arrow[d, "\pi_1"'] & A \times \mathbb{I}_1 \arrow[d, "H"] & A \times \mathbb{I}_0 \arrow[l, "A\times 1"'] \arrow[d, "\pi_1"] \\
A \arrow[r, "f"'] & B & A \arrow[l, "g"]
\end{tikzcd}$$

◊

Given a homotopy $H : A \times \mathbb{I}_1 \to B$ we can find its domain and codomain respectively by:

$$\mathsf{Dom}(H) := H \circ \langle A, 0* \rangle$$

$$\mathsf{Cod}(H) := H \circ \langle A, 1* \rangle$$

(we use $*$ to denote the terminal map from any object to $\mathbb{I}_0 = 1$).

As the functor $(-) \times A$ possesses a right adjoint, the following square is a pushout.

$$\begin{tikzcd}
A \times \mathbb{I}_0 \arrow[r, "A\times 0"] \arrow[d, "A\times 1"'] \arrow[dr, phantom, "\ulcorner", very near end] & A \times \mathbb{I}_1 \arrow[d, "A\times i_1"] \\
A \times \mathbb{I}_1 \arrow[r, "A\times i_0"'] & A \times \mathbb{I}_2
\end{tikzcd}$$

If $H, H' : A \times \mathbb{I}_1 \to B$ such that $H'(A \times 0) = H(A \times 1)$, then the morphism $[H', H] : A \times \mathbb{I}_2 \to B$ is given by

$$\mu \left[\lambda(H' \circ \mathsf{swap}), \lambda(H \circ \mathsf{swap})\right] \circ \mathsf{swap} \tag{3.1.1}$$

If $\mathsf{Dom}(H') = \mathsf{Cod}(H)$ then their vertical composition is defined using this universal morphism:

$$H' \circ H := [H', H] \circ (A \times 2)$$

Note that "$\circ$" is being used in this equation for two different composition operations. This will happen a fair bit in this thesis, but we trust that the context allows the reader to discern what is going on.

The horizontal composition $H' * H$ of $H : A \times \mathbb{I}_1 \to B$ and $H' : B \times \mathbb{I}_1 \to C$ is given by the following composite in $\mathbb{C}$.

$$A \times \mathbb{I}_1 \xrightarrow{A \times \Delta} A \times (\mathbb{I}_1 \times \mathbb{I}_1) \cong (A \times \mathbb{I}_1) \times \mathbb{I}_1 \xrightarrow{H \times \mathbb{I}_1} B \times \mathbb{I}_1 \xrightarrow{H'} C$$

The identity homotopy at $f : A \to B$ is given by $f\pi_1 : A \times \mathbb{I}_1 \to B$, and the inverse of a homotopy $H : A \times \mathbb{I}_1 \to B$ is $H \circ (A \times \sigma)$.

3.1.2 Fundamental groupoids

By considering maps out of $\mathbb{I}$ into a fixed object $A \in \mathbb{C}$ we obtain the fundamental groupoid of A (we could go to higher dimensions, but this suffices for 1-groupoidal realizability). That is, we have a 2-functor:

$$\Pi = (-)^{\mathbb{I}} : \mathbb{C} \to \mathbf{Gpd}$$

A quick way to see this is because the contravariant hom functor takes colimits (used in the definition of an interval) to limits (used in the definition of a category).

The fundamental groupoid of A has as objects points in A, ie. maps $\mathbb{I}_0 \to A$. A morphism $\alpha : a \to b$ is a path α in A making the following diagram commute.

$$\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[dr, "a"'] & \mathbb{I}_1 \arrow[d, "\alpha"] & \mathbb{I}_0 \arrow[l, "1"'] \arrow[dl, "b"] \\
& A &
\end{tikzcd}$$

The composition of $\alpha : a \to b$ with $\beta : b \to c$ is defined:

$$\beta \circ \alpha := [\beta, \alpha] \circ 2$$

(2 re-parameterises the double-length path). The identity at a is $a*$ and the inverse of α is $\alpha\sigma$.

If $f : A \to B \in \mathbb{C}$, the action of

$$\Pi(f) : \Pi A \to \Pi B$$

is given by post-composition (in $\mathbb{C}$) with f. The composition law for functors holds because

$$f[\beta, \alpha] = [f\beta, f\alpha] \tag{3.1.2}$$

by the universal property of pushouts.

$$\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[d, "1"'] & \mathbb{I}_1 \arrow[d, "i_1"] \arrow[dd, bend left, "\beta"] & \\
\mathbb{I}_1 \arrow[r, "i_0"'] \arrow[dr, bend right, "\alpha"'] & \mathbb{I}_2 \arrow[d, "{[\beta,\alpha]}"'] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& A \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_0 \arrow[r, "0"] \arrow[d, "1"'] & \mathbb{I}_1 \arrow[d, "i_1"] \arrow[dr, bend left, "f\beta"] & \\
\mathbb{I}_1 \arrow[r, "i_0"'] \arrow[rr, bend right, "f\alpha"'] & \mathbb{I}_2 \arrow[r, "{[f\beta,f\alpha]}"'] \arrow[ul, phantom, "\ulcorner", very near start] & B
\end{tikzcd}$$

A natural isomorphism $\psi : F \Rightarrow G : \mathbb{D} \to \mathbb{E}$ between functors (between categories or groupoids) is equivalently given by a functor:

$$\psi : \mathbb{D} \times \mathbf{I} \to \mathbb{E}$$

such that $\psi(-, 0) = F$ and $\psi(-, 1) = G$. In this way, a homotopy $H : f \Rightarrow g : A \to B \in \mathbb{C}$ gives rise to a natural transformation

$$\Pi(H) : \Pi A \times \mathbf{I} \to \Pi B$$

by setting:

$$\begin{aligned}
\Pi(H)(a,0) = \Pi(H)(a) &:= H \circ \langle a, 0\rangle : \mathbb{I}_0 \to B \\
\Pi(H)(a,1) = \Pi(H)(a) &:= H \circ \langle a, 1\rangle : \mathbb{I}_0 \to B \\
\Pi(H)(\alpha,i) = \Pi(H)(\alpha) &:= H \circ \langle \alpha, \mathbb{I}_1\rangle : \mathbb{I}_1 \to B \\
\Pi(H)(\alpha,i^{-1}) = \Pi(H)(\alpha) &:= H \circ \langle \alpha, \sigma\rangle : \mathbb{I}_1 \to B
\end{aligned}$$

Given a homotopy $H : f \Rightarrow g : A \to B$ and a path $\alpha : a \to b \in \Pi(A)$, the path $\Pi(H)(\alpha, i)$ can be thought of as the diagonal path through a naturality square. The following lemma establishes that this diagonal is equal to each of the upper and lower boundaries.

Lemma 3.1.6 (Warren, personal communication) *With the above data, the following diagram commutes in* ΠB.

$$\begin{tikzcd}
\Pi(f)(a) \arrow[r, "H\circ\langle\alpha 0*{,}\mathbb{I}_1\rangle"] \arrow[d, "H\circ\langle\alpha{,}0*\rangle"'] \arrow[dr, "\Pi(H)(\alpha{,}i)"] & \Pi(g)(a) \arrow[d, "H\circ\langle\alpha{,}1*\rangle"] \\
\Pi(f)(b) \arrow[r, "H\circ\langle\alpha 1*{,}\mathbb{I}_1\rangle"'] & \Pi(g)(b)
\end{tikzcd}$$

Proof: We show that the diagonal is equal to the lower boundary; that these are equal to the upper boundary is a symmetric argument. Using the definition of composition in ΠB, we have to show:

$$H \circ \langle \alpha, \mathbb{I}_1\rangle = [H \circ \langle \alpha 1*, \mathbb{I}_1\rangle, H \circ \langle \alpha, 0*\rangle] \circ 2$$

which simplifies to

$$H \circ (\alpha \times \mathbb{I}_1) \circ [\langle 1*, \mathbb{I}_1\rangle, \langle \mathbb{I}_1, 0*\rangle] \circ 2$$

But

$$[\langle 1*, \mathbb{I}_1\rangle, \langle \mathbb{I}_1, 0*\rangle] \circ 2$$

is a decomposition of the diagonal $\Delta_{\mathbb{I}_1}$ [War12, p. 212]. □

3.2 Finitely complete realizer categories

In this section, we review a lemma due to Warren [War12, Sec. 2.3] concerning finitely complete (2,1)-categories arising from intervals. This lemma is used in Chapter 5 to prove certain results about $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ under the hypothesis that the realizer category $(\mathbb{C}, \mathbb{I})$, when viewed as a (2,1)-category, is finitely complete.

Let $(\mathbb{C}, \mathbb{I})$ be a realizer category. Define the (edge-symmetric) double category $\square\mathbb{C}(A, B)$ to have:

- Objects: maps $A \to B$ in $\mathbb{C}$.
- Both horizontal and vertical morphisms: maps $A \times \mathbb{I}_1 \to B$.
- 2-cells: commutative squares of maps $A \times \mathbb{I}_1 \to B$.

Denote by $\blacksquare\mathbb{C}(A, B)$ the (edge-symmetric) double category with the same objects and morphisms as $\square\mathbb{C}(A, B)$, but with:

- 2-cells: maps $A \times \mathbb{I}_1 \times \mathbb{I}_1 \to B$.

As a consequence of Lemma 3.1.6, every 2-cell $\phi \in \blacksquare\mathbb{C}(A, B)$ determines a 2-cell $\partial(\phi) \in \square\mathbb{C}(A, B)$:

$$\begin{tikzcd}
\phi_{00} \arrow[r, "\phi_{\mathbb{I}_1 0}"] \arrow[d, "\phi_{0\mathbb{I}_1}"'] \arrow[dr, phantom, "\Downarrow \partial(\phi)"] & \phi_{10} \arrow[d, "\phi_{1\mathbb{I}_1}"] \\
\phi_{01} \arrow[r, "\phi_{\mathbb{I}_1 1}"'] & \phi_{11}
\end{tikzcd}$$

where, eg. $\phi_{01} := \phi \circ \langle A, 0*, 1* \rangle : A \to B$ and $\phi_{\mathbb{I}_1 1} := \phi \circ \langle A\pi_1, \mathbb{I}_1\pi_2, 1 * \pi_2 \rangle : A \times \mathbb{I}_1 \to B$. This determines a double functor:

$$\partial : \blacksquare\mathbb{C}(A, B) \to \square\mathbb{C}(A, B) \tag{3.2.1}$$

Lemma 3.2.1 (Warren) $(\mathbb{C}, \mathbb{I})$ *is finitely complete as a (2,1)-category iff* ∂ *is an isomorphism of double categories.*

3.3 Untyped realizer categories

We know from [Bir00b, RR01, LS02] that "untypedness" is essential for impredicativity. Therefore, ahead of our investigation of impredicative universes in Section 5.3.3, we introduce *untyped* realizer categories. We turn the typed notion of realizability provided by a category into an untyped one by postulating a *universal object*. Traditionally, an object in a category is universal iff every object in the category is a retract of it. In the higher-dimensional setting, we allow this to hold up to homotopy.

Definition 3.3.1 (Universal object) An object U in a category $\mathbb{C}$ is universal iff every object $A \in \mathbb{C}$ is a retract of U up to homotopy, that is, for every object $A \in \mathbb{C}$ there is a section and retraction

$$A \xrightarrow{s_A} U \xrightarrow{r_A} A$$

together with a homotopy

$$\rho_A : r_A s_A \Rightarrow \mathsf{id}_A \qquad \Diamond$$

Definition 3.3.2 An untyped realizer category $(\mathbb{C}, \mathbb{I}, U)$ is a (typed) realizer category $(\mathbb{C}, \mathbb{I})$ together with a universal object $U \in \mathbb{C}$. ◊

3.3.1 Scott complete categories

In this section, we give an example of an untyped realizer category coming from categorified domain theory. Domain theory is a rich source of categories containing universal objects, going back to Scott's graph model [Sco76]. But we seek categories that also contain a non-trivial interval. In [Adá97], Adámek introduced Scott complete categories (SCCs), which are categorified Scott domains: posets become categories. Velebil later showed that the category of κ-small SCCs (for κ inaccessible) contains a universal object [Vel99].

Domain theory	Categorified domain theory
Poset: $x \leq y$	Category: $f : x \to y$
Bottom element and directed joins (pointed DCPO)	Directed colimits
Every element is directed join of finite/compact elements (algebraic)	Set of finitely presentable objects such that every element is directed colimit of finitely presentable objects (algebraic)
Every non-empty set with an upper bound has a join (consistently complete)	Every diagram with a cocone has a colimit (consistently cocomplete)

Table 3.1: Notions from domain theory and their categorical counterparts.

One way to understand domains is that $a \leq b$ means that there is a computational process that leads from a to b, where a and b are thought of as information states. In moving to categories, we *name* computational processes, so that $f : A \to B$ denotes a process f that takes us from A to B. Note that in the categorical setting it is possible to have distinct but isomorphic information states, whereas this is not possible in the posetal case. Many notions from domain theory involving posets have a natural analogue in category theory. Those needed for Scott domains and SCCs are collected in Table 3.1.

A Scott domain is an algebraic consistently complete pointed DCPO. An element c is finite (or compact) iff for every directed set D with a join $\vee D$, if $c \leq \vee D$ then there exists $d \in D$ such that $c \leq d$. This generalises to the notion of finitely presented object, ie. an object A such that $\mathsf{Hom}(A, -)$ preserves directed colimits. A category is said to be algebraic (in direct analogy with the posetal case) iff it has a set of finitely presentable objects such that every object in the category is a directed colimit of finitely presentable objects. A category is finitely accessible iff it has directed colimits and is algebraic. A category is said to be consistently cocomplete iff every diagram with a cocone has a colimit (for accessible categories, consistent cocompleteness is equivalent to consistent completeness). Thus, a Scott complete category is a finitely accessible algebraic consistently complete category.

There is a category **SCC** with:

- Objects: Scott complete categories.
- Morphisms: continuous—ie. directed colimit preserving—functors.

Moreover, if we consider **SCC** a 2-category with 2-cells natural transformations, then an *embedding-projection adjunction* is an adjunction in **SCC** such that the unit of the adjunction is the identity.

Theorem 3.3.3 ([Adá97, Theorem 3**])** **SCC** *is cartesian closed.*

Function spaces are categories of continuous functors and natural transformations between them. The proof that this is algebraic uses "step functors", by analogy with the proof of algebraicity for spaces of continuous functions between Scott domains, which uses "step functions".

Let κ be an inaccessible cardinal. For what follows, we re-calibrate the universe to κ. Sets are now sets of cardinality $< \kappa$ and classes are sets of cardinality κ; so "small" refers to having cardinality $< \kappa$. Categories are by default locally (κ-)small. Denote by **FCC** the category with:

- Objects: finitely consistently cocomplete (FCC) categories, ie. categories in which every finite diagram with a cocone has a colimit.
- Morphisms: FCC embeddings. An FCC embedding $F : \mathbb{D} \to \mathbb{E}$ between FCC categories is a full embedding of categories (a fully faithful and injective-on-objects functor) such that for every non-empty finite diagram $D : \mathbb{J} \to \mathbb{D}$, if FD has a cocone then D has a cocone and F preserves the colimit of D.

A full subcategory $\mathbb{D}'$ of an SCC category $\mathbb{D}$ such that every finitely presentable object in $\mathbb{D}$ is isomorphic to precisely one object in $\mathbb{D}'$ is FCC. We choose for each SCC $\mathbb{D}$ such a subcategory $\mathbb{D}_{\mathsf{fp}}$ (all such choices are equivalent).

Theorem 3.3.4 ([Vel99]) **SCC** *has a universal object.*

Remember, this is really the category of Scott complete locally κ-small categories (unfortunately the result does not hold for the cardinal $\kappa = \aleph_0$ [TV99]). Velebil's result is established by the following argument:

- Every SCC category $\mathbb{D}$ is, up to equivalence, the free completion by directed colimits of $\mathbb{D}_{\mathsf{fp}}$ [MP89].
- Every FCC embedding $\mathbb{D}_{\mathsf{fp}} \to \mathbb{E}_{\mathsf{fp}}$ induces an embedding-projection adjunction $\mathbb{D} \to \mathbb{E}$ between their free completions by directed colimits (see the discussion following Lemma 2.3 in [Vel99]).
- **FCC** has a weakly terminal object [Vel99, Corollary 5.2]. The proof of this result uses Trnková's embedding theorem [Trn66].

Note that the equivalence in the first bullet point is automatically an equivalence in **SCC** because equivalences are continuous. In addition to this, in order to conclude that **SCC** is an untyped realizer category, we must also check that invertible 2-cells in **SCC** are given by homotopies with respect to the interval from groupoids (see Example 3.1.3). We supply the following two lemmas that together verify this.

Lemma 3.3.5 *The interval* $\mathbb{I} \in$ **Gpd** *from Example 3.1.3 belongs to* **SCC**.

Lemma 3.3.6 *If functors* $F, G : \mathbb{D} \to \mathbb{E}$ *are continuous then a functor*

$$\psi : \mathbb{D} \times \mathbf{I} \to \mathbb{E}$$

such that $\psi(-,0) = F$ *and* $\psi(-,1) = G$ *is itself continuous.*

Proof of Lemma 3.3.5: $\mathbb{I}_0 = 1$, $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ are respectively the groupoids with one, two, three and four objects, where each hom set in each groupoid has exactly one member. All these groupoids have directed colimits—indeed they have colimits more generally (the terminal groupoid **1** does and all the groupoids are equivalent).

Every object in each of these groupoids is finitely presentable. The functor $\mathsf{Hom}(x, -) : I_i \to \mathbf{Set}$ for any $i = 0, 1, 2, 3$ and any $x \in I_i$ is the constantly 1 functor and so preserves directed colimits because the colimit of any constant diagram indexed by a connected category—and thus by a directed set—is the constant value. Thus the I_i are finitely accessible: each object $x \in I_i$ is the directed colimit of the constantly x diagram from **1**.

Further, any cocone in I_i has a colimit because I_i is contractible. Finally, the functors in the co-groupoid diagram are all equivalences and hence continuous. $\square$

Proof of Lemma 3.3.6: A functor $D : \mathbb{J} \to \mathbb{D} \times \mathbf{I}$ is naturally isomorphic to the functor $D' : \mathbb{J} \to \mathbb{D} \times \mathbf{I}$ given by

$$D'(-) := (\pi_1(D(-)), 0)$$

Then we have:

$$\begin{aligned}
\psi\left(\mathsf{colim}_{j\in\mathbb{J}}\, D'_j\right) &\cong \psi\left(\mathsf{colim}_{j\in\mathbb{J}}\left(\pi_1\left(D_j\right), 0_j\right)\right) \\
&\cong \psi\left(\mathsf{colim}_{j\in\mathbb{J}}\left(\pi_1\left(D_j\right)\right), \mathsf{colim}_{j\in\mathbb{J}}\, 0_j\right) \\
&\cong \psi\left(\mathsf{colim}_{j\in\mathbb{J}}\left(\pi_1\left(D_j\right)\right), 0\right) \\
&= F\left(\mathsf{colim}_{j\in\mathbb{J}}\left(\pi_1\left(D_j\right)\right)\right) \\
&\cong \mathsf{colim}_{j\in\mathbb{J}}\left(F\left(\pi_1\left(D_j\right)\right)\right) \\
&= \mathsf{colim}_{j\in\mathbb{J}}\left(\psi\left(\pi_1\left(D_j\right), 0_j\right)\right) \\
&\cong \mathsf{colim}_{j\in\mathbb{J}}\left(\psi\left(D'_j\right)\right)
\end{aligned}$$

in the category of cocones to $\psi D'$, where the second step uses the fact that colimits in the product of two categories are computed pointwise, the third step uses the fact that the colimit of a constant diagram indexed by a directed set is the constant value, and the fifth step uses the continuity of F.

To conclude, as a natural isomorphism of functors induces an isomorphism between the categories of cocones to each of these functors, we have the desired isomorphism:

$$\psi\left(\mathsf{colim}_{j\in\mathbb{J}} D_j\right) \cong \mathsf{colim}_{j\in\mathbb{J}}\left(\psi\left(D_j\right)\right) \qquad \square$$

To summarise the results of this section:

Corollary 3.3.7 **SCC** *is an untyped realizer category.*

3.3.2 Issues of size

This seems an appropriate place to comment on some issues of size. These crop up when dealing with untyped realizer categories, and so when investigating impredicative universes (Section 5.3.3). We denote by **Set** the category of small sets and functions, and by **Bij** $\subseteq$ **Set** the subcategory of sets and bijections. Both these categories are themselves locally small. We use **Gpd** for the category of small groupoids, whereas **GPD** is used for the category of locally small groupoids. We require that realizer categories $\mathbb{C}$ are locally small so that the fundamental groupoid functor

$$\Pi : \mathbb{C} \to \mathbf{Gpd}$$

indeed lands in small groupoids. Of course, when we do realizability over $\mathbb{SCC}$, groupoids in **Gpd** are κ-small and groupoids in **GPD** are locally κ-small, for κ inaccessible.

Chapter 4

Groupoidal assemblies: take one

Contents

In this chapter, we take a first pass at groupoidal assemblies. We define a category—indeed (2,1)-category— of (naive) groupoidal assemblies and show that it is cartesian closed. Unfortunately, this category is not suitable for modelling intensional type theory; we give two reasons why.

4.1 Naive groupoidal assemblies

The category $\mathbf{GAsm}(\mathbb{C}, \mathbb{I})$ of (naive) assemblies over the typed realizer category $(\mathbb{C}, \mathbb{I})$ has:

- Objects: an assembly X is a triple

  $$(X, A, \Vdash_X)$$

  where $X \in \mathbf{Gpd}$, $A \in \mathbb{C}$ is the realizer type, and

  $$\Vdash_X \subseteq \Pi A \times X$$

is a relation internal to **Gpd**, that is, a subgroupoid of the above product.

We write the relation $\Vdash_X$ inline:

$$\begin{aligned} a \Vdash_X x \;\; &\text{means} \;\; (a : \mathbb{I}_0 \to A, x \in X) \in \Vdash_X \\ \alpha \Vdash_X p \;\; &\text{means} \;\; (a : \mathbb{I}_1 \to A, p \in X) \in \Vdash_X \end{aligned}$$

Thus, objects of X are realized by points in ΠA, and morphisms of X are realized by paths in ΠA: this accomplishes our aim of formalising the higher-dimensional BHK interpretation. We require that all objects and arrows have at least one realizer.

- Morphisms: a morphism

  $$F : (X, A, \Vdash_X) \to (Y, B, \Vdash_Y)$$

  of assemblies is a functor $F : X \to Y$ such that there exists a map $e : A \to B$ in $\mathbb{C}$ satisfying:

  $$\alpha \Vdash_X p \implies \Pi(e)(\alpha) \Vdash_Y F(p)$$

  (this clause implies the expected one for objects—just consider identities). We write $e \Vdash F$ when e realizes F.

  Composition and identities are inherited from **Gpd**; realizers of composites are given by composites of realizers, identities are realized by identities, and inverses are realized by inverses.

In the traditional theory, we can understand assemblies as datatypes, whose values are implemented in the language provided by the PCA/TCA/ realizer category. Here we can use the same intuition, where datatypes may now have "higher" values (cf. higher inductive types [Uni13]).

We can, in fact, extend $\mathbf{GAsm}(\mathbb{C}, \mathbb{I})$ to a (2,1)-category. A 2-cell

$$\begin{tikzcd}
(X, A, \Vdash_{A,X}) \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & (Y, B, \Vdash_{B,Y}) \arrow[Rightarrow, from=U, to=D, "\phi"]
\end{tikzcd}$$

is a natural transformation

$$\phi : X \times \mathbf{I} \to Y$$

such that there exits a realizer $H : A \times \mathbb{I}_1 \to B$ satisfying:

$$\alpha \Vdash_X p \;\Rightarrow\; \Pi(H)(\alpha, i) \Vdash_Y \phi(p, i)$$

We write $H \Vdash \phi$ when H realizes ϕ.

If $H \Vdash \phi : F \Rightarrow F'$ and $H' \Vdash \psi : F' \Rightarrow F''$ where $F, F', F'' : X \to Y$ then the vertical composition $\psi \circ \phi$ is realized by the vertical composition $H' \circ H$. Let $p : x \to x'$ and $\alpha \Vdash_X p$. Then we use Lemma 3.1.6 to determine that:

$$\Pi\left(H' \circ H\right)(\alpha) = \left[\left(H' \circ H\right) \circ \langle \alpha 1*, \mathbb{I}_1 \rangle, \left(H' \circ H\right) \circ \langle \alpha, 0* \rangle\right] \circ 2$$

Looking at each component of the weakly universal map in the above expression, first, by interchange in $\mathbb{C}$ we have:

$$\left(H' \circ H\right) \circ \langle \alpha 1*, \mathbb{I}_1 \rangle = \left[H' \circ \langle \alpha 1*, \mathbb{I}_1 \rangle, H \circ \langle \alpha 1*, \mathbb{I}_1 \rangle\right] \circ 2$$

Secondly we have:

$$\left(H' \circ H\right) \circ \langle \alpha, 0* \rangle = H \circ \langle \alpha, 0* \rangle$$

That is, $\Pi(H' \circ H)(\alpha)$ is equal to the following composite in ΠB.

$$\bullet \xrightarrow{H \circ \langle \alpha, 0* \rangle} \bullet \xrightarrow{H \circ \langle \alpha 1*, \mathbb{I}_1 \rangle} \bullet \xrightarrow{H' \circ \langle \alpha 1*, \mathbb{I}_1 \rangle} \bullet$$

But:

$$\begin{aligned}
H \circ \langle \alpha, 0* \rangle &\Vdash_Y \phi(p, 0) = F(p) \\
H \circ \langle \alpha 1*, \mathbb{I}_1 \rangle &\Vdash_Y \phi(x', i) \\
H' \circ \langle \alpha 1*, \mathbb{I}_1 \rangle &\Vdash_Y \psi(x', i)
\end{aligned}$$

and so their composite

$$\Pi\left(H'\circ H\right)(\alpha)\Vdash_Y\left(\psi(x',\rightarrow)\circ\phi(x',\rightarrow)\circ\phi(p,0)\right)=(\psi\circ\phi)(p,i)$$

as required.

If $H\Vdash\phi:F\Rightarrow G:X\to Y$ and $H'\Vdash\psi:F'\Rightarrow G':Y\to Z$ then the horizontal composite $\psi*\phi$:

$$X\times\mathbf{I}\xrightarrow{X\times\Delta_{\mathbf{I}}}X\times(\mathbf{I}\times\mathbf{I})\cong(X\times\mathbf{I})\times\mathbf{I}\xrightarrow{\phi\times\mathbf{I}}Y\times\mathbf{I}\xrightarrow{\psi}Z$$

is realized by the horizontal composite $H*H'$, for if $\alpha\Vdash_X p$ then

$$\left(H*H'\right)\circ\langle\alpha,\mathbb{I}_1\rangle=H\circ\left\langle H'\circ\langle\alpha,\mathbb{I}_1\rangle,\mathbb{I}_1\right\rangle$$

It is straightforward to check that identity homotopies realize identity natural transformations and inverse homotopies realize inverse natural transformations.

Proposition 4.1.1 **GAsm**$(\mathbb{C},\mathbb{I})$ *is cartesian closed.*

Proof: The terminal object in **GAsm**$(\mathbb{C},\mathbb{I})$ is:

$$\mathbf{1}:=(\mathbf{1},\mathbb{I}_0,\Vdash_{\mathbf{1}})$$

where $\mathsf{id}_{\mathbb{I}_0}$ is the sole realizer of the sole object of **1** and $*:\mathbb{I}_1\to\mathbb{I}_0$ is the sole realizer of the sole morphism of **1**.

The binary product of assemblies $(X,A,\Vdash_X)$ and $(Y,B,\Vdash_Y)$ is:

$$(X\times Y,A\times B,\Vdash_{X\times Y})$$

where

$$a:\mathbb{I}_0\to A\times B\Vdash_{X\times Y}(x,y)$$

holds iff both:

$$\pi_1 a:\mathbb{I}_0\to A\Vdash_X x\qquad\pi_2 a:\mathbb{I}_0\to B\Vdash_Y y$$

and where

$$\alpha : \mathbb{I}_1 \to A \times B \Vdash_{X \times Y} (p, q)$$

holds iff both:

$$\pi_1 \alpha : \mathbb{I}_1 \to A \Vdash_X p \qquad \pi_2 \alpha : \mathbb{I}_1 \to B \Vdash_Y q$$

This defines a subgroupoid of $\Pi(A \times B) \times (X \times Y)$. For composition, consider:

$$\alpha : a \to b \Vdash_{X_1 \times X_2} (p_1, p_2) : (x_1, x_2) \to (y_1, y_2)$$
$$\beta : b \to c \Vdash_{X_1 \times X_2} (q_1, q_2) : (y_1, y_2) \to (z_1, z_2)$$

Then we know:

$$\pi_i \alpha : \pi_i a \to \pi_i b \Vdash_{X_i} p_i : x_i \to y_i$$
$$\pi_i \beta : \pi_i b \to \pi_i c \Vdash_{Y_i} q_i : y_i \to z_i$$

So

$$\pi_i \beta \circ \pi_i \alpha : \pi_i a \to \pi_i b \Vdash_{X_i} q_i \circ p_i : x_i \to z_i$$

By Equation (3.1.2), we have

$$\pi_i (\beta \circ \alpha) : \pi_i a \to \pi_i b \Vdash_{X_i} q_i \circ p_i : x_i \to z_i$$

and so we can deduce

$$\beta\alpha : a \to z \Vdash_{X_1 \times X_2} (q_1 p_1, q_2 p_2) : (x_1, x_2) \to (z_1, z_2)$$

as required. Similar "componentwise" arguments work for identities and inverses.

The exponential of $(Y, B, \Vdash_Y)$ by $(X, A, \Vdash_X)$ is

$$\left(\mathbf{GAsm}(\mathbb{C}, \mathbb{I})(X, Y), B^A, \Vdash_{Y^X}\right)$$

where:

$$e : \mathbb{I}_0 \to B^A \Vdash_{Y^X} F \;\Leftrightarrow\; \mu(e) \circ \langle *, A \rangle : A \to B \Vdash F$$
$$H : \mathbb{I}_1 \to B^A \Vdash_{Y^X} \phi \;\Leftrightarrow\; \mu(H) \circ \mathsf{swap} : A \times \mathbb{I}_1 \to B \Vdash \phi$$

This defines a subgroupoid of

$$\Pi(B^A) \times \mathbf{GAsm}(\mathbb{C}, \mathbb{I})(X, Y)$$

For composition, consider:

$$H : e \to f \Vdash_{Y^X} \phi : F \to F'$$
$$K : f \to g \Vdash_{Y^X} \psi : F' \to F''$$

We want to show that

$$K \circ H : e \to g \Vdash_{Y^X} \psi\phi : F \to F''$$

We know that

$$\mu(H) \circ \mathsf{swap} : \mu(e) \circ \langle *, A \rangle \Rightarrow \mu(f) \circ \langle *, A \rangle$$
$$\mu(K) \circ \mathsf{swap} : \mu(f) \circ \langle *, A \rangle \Rightarrow \mu(g) \circ \langle *, A \rangle$$

are homotopies because:

$$\begin{aligned}
\mu(H) \circ \mathsf{swap} \circ (A \times 0) &= \mu(H) \circ (A \times 0) \circ \mathsf{swap} \\
&= \mu(H \circ 0) \circ \mathsf{swap} \\
&= \mu(e) \circ \mathsf{swap} \\
&= \mu(e) \circ \langle *, A \rangle
\end{aligned}$$

where the second step is by naturality of μ, and similarly:

$$\mu(H) \circ \mathsf{swap} \circ (A \times 1) = \mu(f) \circ \langle *, A \rangle$$
$$\mu(K) \circ \mathsf{swap} \circ (A \times 0) = \mu(f) \circ \langle *, A \rangle$$
$$\mu(K) \circ \mathsf{swap} \circ (A \times 1) = \mu(g) \circ \langle *, A \rangle$$

Thus, if we show that

$$\mu(K \circ H)\mathsf{swap}$$

is equal to the composition

$$(\mu(K)\mathsf{swap}) \circ (\mu(H)\mathsf{swap})$$

of homotopies, then we are done. We calculate:

$$\begin{aligned}
&(\mu(K)\mathsf{swap}) \circ (\mu(H)\mathsf{swap}) \\
=&\, [\mu(K) \circ \mathsf{swap}, \mu(H) \circ \mathsf{swap}] \circ (A \times 2) \\
=&\, [\lambda\,(\mu(K) \circ \mathsf{swap} \circ \mathsf{swap})\,, \lambda\,(\mu(H) \circ \mathsf{swap} \circ \mathsf{swap})] \circ \mathsf{swap} \circ (A \times 2) \\
=&\, [\lambda\,(\mu(K))\,, \lambda\,(\mu(H))] \circ \mathsf{swap} \circ (A \times 2) \\
=&\, [\mu(K), \mu(H)] \circ \mathsf{swap} \circ (A \times 2) \\
=&\, [\mu(K), \mu(H)] \circ (A \times 2) \circ \mathsf{swap} \\
=&\mu\,([K, H] \circ 2) \circ \mathsf{swap} \\
=&\mu\,([K, H] \circ 2) \circ \mathsf{swap} \\
=&\mu(K \circ H)\mathsf{swap}
\end{aligned}$$

where the second step applies Equation (3.1.1). Similar arguments work for identities and inverses. □

4.2 Two problems

The aim is to model ITT. We will use CwFs to frame the following discussion (we believe that it is clearer that way). That is, we will attempt to define a CwF with category of contexts $\mathbf{GAsm}(\mathbb{C}, \mathbb{I})$. At the level of underlying groupoids, this should be the Hofmann-Streicher model [HS98]. First we define our dependent types as uniform families; these are analogous to the set-based notion (see Section 2.4.2). However, after defining uniform families, we expose two problems with them (or with $\mathbf{GAsm}(\mathbb{C}, \mathbb{I})$): one related to modelling context extension and the other to modelling identity types.

Definition 4.2.1 A uniform family $\mathbb{X}$ of groupoidal assemblies indexed by a groupoidal assembly $(X, A, \Vdash_X)$ is a functor

$$\mathbb{X} : X \to \mathbf{GAsm}(\mathbb{C}, \mathbb{I})$$

where there is a uniform realizer type across the $\mathbb{X}(x)$. $\Diamond$

The set of uniform families over X is $\mathsf{Ty}(X)$.

4.2.1 Context extension

In order to model context extension, we must in particular construct an assembly $X.\mathbb{X}$ out of a uniform family $\mathbb{X} \in \mathsf{Ty}(X)$. The underlying groupoid of this assembly, following the Hofmann-Streicher model, should be the Grothendieck construction of the underlying indexed groupoid of $\mathbb{X}$ (ie. $\mathbb{X}$ post-composed with the underlying groupoid functor).

Explicitly, the Grothendieck construction $X.\mathbb{X}$ has:

- Objects: pairs

$$(x \in X, u \in \mathbb{X}(x))$$

- Morphisms: A morphism

$$(p, r) : (x, u) \to (x', u')$$

  consists of a morphism $p : x \to x'$ in X and a morphism

$$r : \mathbb{X}(p)(u) \to u' \in \mathbb{X}(x')$$

  Composition is defined:

$$(q, s) \circ (p, r) := (qp, s \circ \mathbb{X}(q)(r))$$

  Identities are pairs of identities and the inverse of (p, r) is

$$\left(p^{-1}, \mathbb{X}\left(p^{-1}\right)\left(q^{-1}\right)\right)$$

Let B be the uniform realizer type of the $\mathbb{X}(x)$. The groupoid $X.\mathbb{X}$ is a dependent sum, whose objects and morphisms are pairs of such, where the type of the second component depends on the first component. Therefore, following BHK, the realizer type should be $A \times B$.

We expect the realizability relation for objects to be given by:

$$c \Vdash_{X.\mathbb{X}} (x,u) \;\Leftrightarrow\; \pi_1 c \Vdash_X x \;\wedge\; \pi_2 c \Vdash_{\mathbb{X}(y)} u$$

and for morphisms by:

$$\gamma \Vdash_{X.\mathbb{X}} (p,r) \;\Leftrightarrow\; \pi_1 \gamma \Vdash_X p \;\wedge\; \pi_2 \gamma \Vdash_{\mathbb{X}(y)} r$$

But this does not give a well-defined subgroupoid of

$$\Pi\,(A \times B) \times (X.\mathbb{X})$$

Assume that $\gamma : c \to c' \Vdash_{X.\mathbb{X}} (p,r) : (x,u) \to (x',u')$. Then we need, among other things, that $c \Vdash_{X.\mathbb{X}} (x,u)$, and so in particular that $\pi_2 c \Vdash_{X.\mathbb{X}} u$. However, from $\gamma \Vdash_{X.\mathbb{X}} (p,r)$ we can deduce $\pi_2 c \Vdash_{X.\mathbb{X}} \mathsf{Dom}(r) = \mathbb{X}(p)(u)$, but not necessarily $\pi_2 c \Vdash_{X.\mathbb{X}} u$ because $\mathbb{X}(p)^{-1}$ isn't necessarily realized by id_B.

4.2.2 Identity types

Let $X = (X, A, \Vdash_{A,X})$ be an assembly. We now attempt to define a uniform family of assemblies

$$\mathsf{Id}_X : X \times X \to \mathbf{GAsm}(\mathbb{C}, \mathbb{I})$$

corresponding to the identity type family over X.

For the object part of the functor:

$$(x_1, x_2) \mapsto X(x_1, x_2) = \left(X(x_1, x_2), A^{\mathbb{I}_1}, \Vdash_{X(x_1,x_2)}\right)$$

where the hom-set $X(x_1, x_2)$ is viewed as a discrete groupoid, and we have:

$$\alpha : \mathbb{I}_0 \to A^{\mathbb{I}_1} \Vdash_{X(x_1,x_2)} p \;\Leftrightarrow\; \mu(\alpha) \circ \langle *, \mathbb{I}_1 \rangle : \mathbb{I}_1 \to A \Vdash_{A,X} p$$

$$\mathsf{id}_\alpha : \mathbb{I}_1 \to A^{\mathbb{I}_1} \Vdash_{X(x_1,x_2)} \mathsf{id}_p \;\Leftrightarrow\; \alpha : \mathbb{I}_0 \to A^{\mathbb{I}_1} \Vdash_{X(x_1,x_2)} p$$

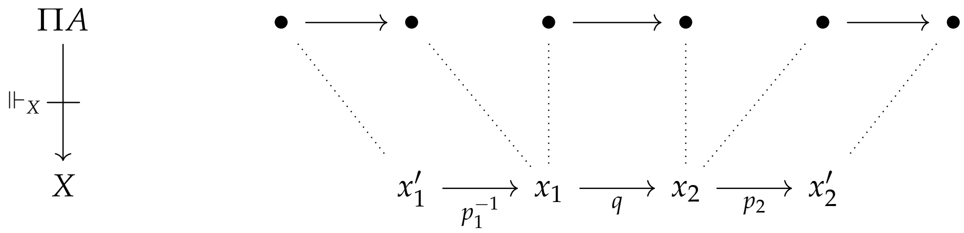

Figure 4.1: In naive assemblies, realizers of composable morphisms need not themselves be composable.

That is, a point α in the "path space" $A^{\mathbb{I}_1}$ realizes an object p of $X(x_1, x_2)$ iff when uncurried to be a path $\widetilde{\alpha}$ in A it realizes p regarded as an isomorphism in X. Again, this accords with the topological BHK interpretation.

Given $p_1 : x_1 \to x_1'$ and $p_2 : x_2 \to x_2'$, we define the morphism part of the functor to be:

$$\mathsf{Id}_X\,(p_1, p_2) := K_{p_1,p_2} : X(x_1, x_2) \to X(x_1', x_2')$$

where

$$K_{p_1,p_2}(q) := p_2 q p_1^{-1} \tag{4.2.1}$$

But the map (4.2.1) needs a realizer. One idea is that a realizer for this map should compose realizers of p_1^{-1}, q, p_2. The problem here is that realizers of composable morphisms need not themselves be composable, as visualised in Figure 4.1.

Notice that for *partitioned* assemblies, ie. those for which the realizability relation is in fact a *functor*

$$\Vdash_X : X \to \Pi A$$

this problem does not arise. With this, we begin to segue into the next chapter on *partitioned* groupoidal assemblies. The issue with context extension is resolved by letting morphisms of partitioned groupoidal assemblies be realized up to natural isomorphism... as shall be explained.

Chapter 5

Partitioned groupoidal assemblies

Contents

A better approach to higher-dimensional realizability is afforded by partitioned groupoidal assemblies. The justification for this is that we are able to define a very natural category of partitioned groupoidal assemblies, yielding a model—*qua* path category—of a version of 1-truncated ITT (without function extensionality, which is to be expected). When the realizer category is untyped, there exists an impredicative universe of 1-types, closed under identity types, given by the modest fibrations. This mirrors the traditional story. The relative complexity of partitioned assemblies versus (general, not-necessarily-partitioned) assemblies and realizability toposes seems to be amplified in the groupoidal context.

5.1 The category of partitioned groupoidal assemblies

We now define the constituents of the category $\mathbf{PGAsm}(\mathbb{C},\mathbb{I})$ of partitioned groupoidal assemblies over a typed realizer category $(\mathbb{C},\mathbb{I})$ (see Definition 3.3.2). This should be compared with the $\mathcal{F}$-construction of Robinson-Rosolini [RR01]. Later, when we come to universes, we will consider partitioned groupoidal assemblies over an untyped realizer category. Partitioned groupoidal assemblies are allowed to have locally small underlying groupoids.

Definition 5.1.1 (Partitioned groupoidal assembly) A partitioned groupoidal assembly X is a triple

$$(X, A, \|-\|_X)$$

where $X \in \mathbf{GPD}$ is a groupoid, $A \in \mathbb{C}$ is an object from the realizer category and

$$\|-\|_X : X \to \Pi A$$

is a functor. ◊

Thus, isomorphisms in the underlying groupoid of a partitioned groupoidal assembly are still realized by paths in some fundamental groupoid, as per the higher-dimensional BHK interpretation.

Definition 5.1.2 (Morphism of partitioned groupoidal assemblies) A morphism

$$F : (X, A, \|-\|_X) \to (Y, B, \|-\|_Y)$$

of partitioned groupoidal assemblies is a functor $F : X \to Y$ such that there exists $e : A \to B$ and a natural isomorphism ϵ:

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "\|{-}\|_X"'] & Y \arrow[d, "\|{-}\|_Y"] \\
\Pi A \arrow[r, "\Pi(e)"'] \arrow[ur, Rightarrow, shorten=12pt, "\epsilon"'] & \Pi B
\end{tikzcd}$$

◇

One way to understand this is that morphisms of partitioned groupoidal assemblies are required to be realized only up to natural isomorphism. The idea is that Fx, which is implemented by $\|Fx\|_Y$, could just as well be implemented by $\Pi(e)\|x\|_X$, as witnessed by ϵ_x. Another way to view this is that the *pair* (e, ϵ) realizes (or tracks) the functor F. We write $(e, \epsilon) \Vdash F$ to mean that (e, ϵ) realizes F. If $(e, \epsilon) \Vdash F : X \to Y$ and $(e', \epsilon') \Vdash G : Y \to Z$, then the composite $GF : X \to Z$ is realized by

$$\big(e'e, (\epsilon' * F) \circ (\Pi(e') * \epsilon)\big)$$

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "\|{-}\|_X"'] & Y \arrow[r, "G"] \arrow[d, "\|{-}\|_Y"] & Z \arrow[d, "\|{-}\|_Z"] \\
\Pi A \arrow[r, "\Pi(e)"'] \arrow[ur, Rightarrow, shorten=12pt, "\epsilon"'] & \Pi B \arrow[r, "\Pi(e')"'] \arrow[ur, Rightarrow, shorten=12pt, "\epsilon'"'] & \Pi C
\end{tikzcd}$$

Recall from Section 2.4 that, in the case of set-based realizability over a PCA $\mathcal{A}$, a modest partitioned assembly X can be characterised as an *injective* function $X \to \mathcal{A}$. Fully faithful functors are a sensible notion of monomorphism in the 2-category of groupoids.

Definition 5.1.3 (Modest partitioned groupoidal assembly) A groupoidal assembly $(X, A, \|-\|_X)$ is modest iff $\|-\|_X : X \to \Pi A$ is fully faithful. ◇

This generalises the traditional case, for if $\mathbb{I}$ is a discrete interval and X a discrete groupoid (set) then, given fullness:

$$\|x\|_X = \|x'\|_X \text{ implies } X(x, x') \neq \emptyset$$

which yields $x = x'$.

Proposition 5.1.4 $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ *is weakly cartesian closed. Moreover, the weak exponential object* Y^X *is modest if* Y *is.*

Proof: The terminal object is $(\mathbf{1}, \mathbb{I}_0, \|-\|_{\mathbf{1}} : * \mapsto \mathbb{I}_0)$. Let

$$(X, A, \|-\|_X), (Y, B, \|-\|_Y) \in \mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$$

Their product is

$$(X \times Y, A \times B, \|-\|_{X \times Y})$$

where

$$\|-\|_{X \times Y} := \langle \|-\|_X, \|-\|_Y \rangle$$

The weak exponential is

$$Y^X := \left(\mathsf{Real}\left(Y^X\right), B^A, \|-\|_{Y^X}\right)$$

where $\mathsf{Real}(Y^X)$ is the groupoid whose objects are triples

$$\left(F : X \to Y, e \in \Pi(B^A), \epsilon\right)$$

such that

$$(\mu(e) \circ \langle *, A \rangle, \epsilon) \Vdash F$$

and whose morphisms

$$(F, e, \epsilon) \to (G, e', \epsilon')$$

are pairs

$$(\psi : F \Rightarrow G, f : e \to e')$$

such that there exists a natural isomorphism ζ satisfying

$$\zeta(-, 0, -) = \epsilon \qquad \zeta(-, 1, -) = \epsilon' \tag{5.1.1}$$

as well as

$$(\mu(f) \circ \mathsf{swap}, \zeta) \Vdash \psi$$

In actual fact, these conditions uniquely determine ζ. A quick way to see this is using Lemma 3.2.1: The conditions (5.1.1), together with the fact that ζ is a natural isomorphism of the required shape, determine the boundary of ζ. As **GPD** is a finitely complete (2,1)-category, we invoke Lemma 3.2.1 to conclude that ζ is the unique natural isomorphism with the given specification. The realizability functor is the second projection.

The evaluation morphism is defined:

$$\begin{aligned} &\mathsf{ev} : Y^X \times X \to Y \\ &\mathsf{ev}(F, e, \epsilon, x) := Fx \\ &\mathsf{ev}(\psi, f, p) := \psi(p, i) \end{aligned}$$

and realized by (ev, ϵ'), where we overload notation and use ev for the evaluation map in $\mathbb{C}$ and where:

$$\epsilon'_{(F,e,\epsilon,x)} := \epsilon_x$$

See Figure 5.1.

For the universal property, suppose we have a morphism

$$K : (Z, C, \|-\|_Z) \times X \to Y$$

Pick a realizer $(e, \epsilon) \Vdash K$. With this we define the transpose of K.

$$\widetilde{K} : Z \to Y^X$$

On objects we have

$$\widetilde{K}(z) := (K(z, -), e^z, \epsilon^z)$$

$$\begin{tikzcd}
& \begin{array}{c}\mathsf{ev}\circ\langle e,\|x\|_X\rangle\\ =\Pi(\mu(e))\|x\|_X\end{array} \arrow[dl, "\begin{array}{c}\mathsf{ev}\circ\langle e,\|p\|_X\rangle\\ =\Pi(\mu(e))\|p\|_X\end{array}"'] \arrow[dd, "\begin{array}{c}\mathsf{ev}\circ\langle f,\|x\|_X\rangle\\ =\Pi(\mu(f))\|x\|_X\end{array}"] \arrow[rr, "\epsilon_x"] & & \|Fx\|_Y \arrow[dl, "\|F(p)\|_Y"] \arrow[dd, "\|\psi_x\|_Y"] \\
\begin{array}{c}\mathsf{ev}\circ\langle e,\|x'\|_X\rangle\\ =\Pi(\mu(e))\|x'\|_X\end{array} \arrow[rr, "\epsilon_{x'}"' near start] \arrow[dd, "\begin{array}{c}\mathsf{ev}\circ\langle f,\|x'\|_X\rangle\\ =\Pi(\mu(f))\|x'\|_X\end{array}"'] & & \|Fx'\|_Y \arrow[dd, "\|\psi_{x'}\|_Y" near start] & \\
& \begin{array}{c}\mathsf{ev}\circ\langle e',\|x\|_X\rangle\\ =\Pi(\mu(e'))\|x\|_X\end{array} \arrow[dl, "\begin{array}{c}\mathsf{ev}\circ\langle e',\|p\|_X\rangle\\ =\Pi(\mu(e'))\|p\|_X\end{array}"] \arrow[rr, "\epsilon'_x" near start] & & \|Gx\|_Y \arrow[dl, "\|G(p)\|_Y"] \\
\begin{array}{c}\mathsf{ev}\circ\langle e',\|x'\|_X\rangle\\ =\Pi(\mu(e'))\|x'\|_X\end{array} \arrow[rr, "\epsilon'_{x'}"'] & & \|Gx'\|_Y &
\end{tikzcd}$$

Figure 5.1: Cube exhibiting the naturality of ϵ'.

where:

$$e^z := \lambda\,(e \circ (\|z\|_Z \times A))$$

$$\epsilon^z_x := \epsilon_{(z,x)}$$

This works because

$$\begin{aligned}
&\Pi\left(\mu\left(\lambda\left(e \circ \left(\|z\|_Z \times A\right)\right)\right) \circ \langle *, A\rangle\right) \|x\|_X \\
=&\mu\left(\lambda\left(e \circ \left(\|z\|_Z \times A\right)\right)\right) \circ \langle *, A\rangle \circ \|x\|_X \\
=&e \circ \left(\|z\|_Z \times A\right) \langle *, A\rangle \circ \|x\|_X \\
=&e \circ \left(\|z\|_Z \times A\right) \circ \langle \mathbb{I}_0, \|x\|_X\rangle \\
=&e \circ \langle \|z\|_Z, \|x\|_X\rangle \\
=&\Pi(e)\|(z,x)\|_{Z\times X}
\end{aligned}$$

and

$$\epsilon_{(z,x)} : \Pi(e)\|(z,x)\|_{Z\times X} \to \|K(z,x)\|_Y$$

On morphisms we define

$$\widetilde{K}(r : z \to z') := (\psi^r, f^r)$$

where:

$$\begin{aligned}
\psi^r(p,i) &:= K(r,p) \\
f^r &:= \lambda\left(e \circ \left(\|r\|_Z \times A\right)\right)
\end{aligned}$$

Let Y be modest. To show that Y^X is modest, given $(F,e,\epsilon),(G,e',\epsilon') \in \mathsf{Real}(Y^X)$ we conclude that any $f : e \to e'$ uniquely determines a morphism $(\psi, f) : (F,e,\epsilon) \to (G,e',\epsilon')$ by fully faithfulness of $\|-\|_Y$ because there is a unique morphism (the image under $\|-\|_Y$ of ψ_x) making the following diagram commute.

$$\begin{tikzcd}
\|Fx\|_Y \arrow[r, dashed, "\|\psi_x\|_Y"] & \|Gx\|_Y \\
\begin{array}{c}\Pi(f)\langle 0,\|x\|_X\rangle \\ =\Pi(e)\|x\|_X\end{array} \arrow[u, "\epsilon_x"] \arrow[r, "\Pi(f)\langle \mathbb{I}_1,\|x\|_X\rangle"'] & \begin{array}{c}\Pi(f)\langle 1,\|x\|_X\rangle \\ =\Pi(e')\|x\|_X\end{array} \arrow[u, "\epsilon'_x"']
\end{tikzcd}$$

□

5.2 As a (2,1)-category

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ possesses an interval of its own—which is in fact modest. The object of co-arrows is:

$$\mathbf{I} := (\mathbf{I}, \mathbb{I}_1, \|-\|_{\mathbf{I}})$$
$$\|0\|_{\mathbf{I}} := 0 \qquad \|1\|_{\mathbf{I}} := 1 \qquad \|i\|_{\mathbf{I}} := \mathbb{I}_1$$

The other parts of the co-groupoid diagram are obtained by similarly matching up the corresponding parts of the co-groupoids in **GPD** and $\mathbb{C}$.

$$\mathbf{I}_2 := (\mathbf{I}_2, \mathbb{I}_2, \|-\|_{\mathbf{I}_2})$$
$$\|0\|_{\mathbf{I}_2} := i_0 0 \qquad \|1\|_{\mathbf{I}_2} := i_0 1 = i_1 0 \qquad \|2\|_{\mathbf{I}_2} := i_1 1$$
$$\|i_0\|_{\mathbf{I}_2} := i_0 \qquad \|i_1\|_{\mathbf{I}_2} := i_1$$
$$\mathbf{I}_3 := (\mathbf{I}_3, \mathbb{I}_3, \|-\|_{\mathbf{I}_3})$$
$$\|0\|_{\mathbf{I}_2} := j_0 i_0 0 \qquad \|1\|_{\mathbf{I}_2} := j_0 i_0 1 = j_0 i_1 0 = j_1 i_0 0$$
$$\|2\|_{\mathbf{I}_2} := j_0 i_1 1 = j_1 i_0 1 = j_1 i_1 0 \qquad \|3\|_{\mathbf{I}_2} := j_1 i_1 1$$
$$\|i_0\|_{\mathbf{I}_2} := j_0 i_0 \qquad \|i_1\|_{\mathbf{I}_2} := j_0 i_1 = j_1 i_0 \qquad \|i_2\|_{\mathbf{I}_2} := j_1 i_1$$

The underlying functors of the morphisms $i_0, i_1, 2, j_0, j_1 \in \mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ are as in Example 3.1.3 and are realized by the maps $i_0, i_1, 2, j_0, j_1 \in \mathbb{C}$ respectively.

We will show that $\mathbf{I}_2$ is the pushout of $0, 1$; a similar argument works for $\mathbf{I}_3$ with i_0, i_1. Suppose that we are in the following situation:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "0"] \arrow[d, "1"'] & \mathbf{I}_1 \arrow[d, "i_1"] \arrow[ddr, bend left, "G"] & \\
\mathbf{I}_1 \arrow[r, "i_0"'] \arrow[drr, bend right, "F"'] & \mathbf{I}_2 & \\
& & (X, A, \|-\|_X)
\end{tikzcd}$$

where $(e, \epsilon) \Vdash F$. The functor $[G, F] : \mathbf{I}_2 \to X$ is the universal morphism in **GPD**. Define

$$d := e0*$$

and note that

$$\Pi(d)\|0\|_{\mathbf{I}_2} = \Pi(d)\|1\|_{\mathbf{I}_2} = \Pi(d)\|2\|_{\mathbf{I}_2} = e0$$

and

$$\Pi(d)\|i_0\|_{\mathbf{I}_2} = \Pi(d)\|i_1\|_{\mathbf{I}_2} = e0* = \mathsf{id}_{e0}$$

We can track $[G, F]$ with (d, δ), where:

$$\begin{aligned}
\delta_0 &:= \epsilon_0 : e0 \to \|F0\|_X \\
\delta_1 &:= \epsilon_1 e : e0 \to \|F1\|_X \\
\delta_2 &:= \|G(i)\|_X \epsilon_1 e : e0 \to \|G1\|_X
\end{aligned}$$

which is natural using naturality of ϵ.

We cannot automatically deduce from [War12] that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$, with homotopies as 2-cells, is a strict ω-category, because it is only *weakly* cartesian closed. Therefore, we will show by hand that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is a (2,1)-category.

Let $\phi : F \Rightarrow G : X \to Y$ and $\psi : G \Rightarrow H : X \to Y$ be homotopies in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ (ie. realized natural transformations, considered as special functors). Their vertical composition $\psi \circ \phi = \psi\phi : F \Rightarrow H : X \to Y$ is defined as in $\mathbf{Gpd}$:

$$\begin{aligned}
&\psi\phi : X \times \mathbf{I} \to Y \\
&\psi\phi(f : x \to y, i) := \psi_y \circ \phi_y \circ F(f) = H(y) \circ \psi_x \circ \phi_x
\end{aligned}$$

Let $(e^{\phi}, \epsilon^{\phi}) \Vdash \phi$ and $(e^{\psi}, \epsilon^{\psi}) \Vdash \psi$. We obtain a realizer $(e^{\psi\phi}, \epsilon^{\psi\phi}) \Vdash \psi\phi$ as follows. First let $e^{\psi\phi} := e^{\phi}$. Then define:

$$\begin{aligned}
\epsilon^{\psi\phi}_{(x,0)} &:= \epsilon^{\phi}_{(x,0)} : e^{\phi}\ \|(x,0)\|_{X\times\mathbf{I}} \to \|Fx\|_Y \\
\epsilon^{\psi\phi}_{(x,1)} &:= \|\psi_x\|_Y \circ \epsilon^{\phi}_{(x,1)} : e^{\phi}\ \|(x,0)\|_{X\times\mathbf{I}} \to \|Gx\|_Y \to \|Fx\|_Y
\end{aligned}$$

That this is natural is captured in the following diagram.

$$\begin{tikzcd}
e^{\phi} \left\|(x,0)\right\|_{X\times\mathbf{I}} \arrow[r, "\epsilon^{\psi}\phi_{(x,0)}=\epsilon^{\phi}_{(x,0)}"] \arrow[d, "e^{\phi}\|(f,0)\|_{X\times\mathbf{I}}"] \arrow[dd, bend right=60, swap, "e^{\phi}\|(f,i)\|_{X\times\mathbf{I}}"] & \left\|Fx\right\|_{Y} \arrow[d, swap, "\substack{\|\psi\phi(f,0)\|_{Y}\\=\|F(f)\|_{Y}}"] \arrow[ddd, bend left=40, "\|\psi\phi(f,i)\|_{Y}"] \\
e^{\phi} \left\|(y,0)\right\|_{X\times\mathbf{I}} \arrow[r, "\epsilon^{\phi}_{(y,0)}"] \arrow[d, "e^{\phi}\|(y,i)\|_{X\times\mathbf{I}}"] & \left\|Fy\right\|_{Y} \arrow[d, swap, "\substack{\|\psi\phi(f,0)\|_{Y}\\=\|F(f)\|_{Y}}"] \\
e^{\phi} \left\|(y,1)\right\|_{X\times\mathbf{I}} \arrow[r, swap, "\epsilon^{\phi}_{(y,1)}"] \arrow[dr, swap, "\epsilon^{\psi\phi}_{(y,1)}"] & \left\|Gy\right\|_{Y} \arrow[d, swap, "\|\psi_{y}\|_{Y}"] \\
 & \left\|Hy\right\|_{Y}
\end{tikzcd}$$

Now let $\phi : F \Rightarrow G : X \to Y$ and $\psi : H \Rightarrow K : Y \to Z$. Their horizontal composition $\psi * \phi : HF \Rightarrow KG : X \to Z$ is defined as in **Gpd**:

$$\psi\phi : X \times \mathbf{I} \to Z$$

$$\psi * \phi(x,0) := HFx$$

$$\psi * \phi(x,1) := KGx$$

$$\psi * \phi(f : x \to y, i) := \psi_{Gy} \circ H\left(\phi_{y}\right) \circ H(F(f)) = K(G(f)) \circ \psi_{Gx} \circ H\left(\phi_{x}\right)$$

Let $(e^{\phi}, \epsilon^{\phi}) \Vdash \phi$, $(e^{\psi}, \epsilon^{\psi}) \Vdash \psi$ and $(e^{H}, \epsilon^{H}) \Vdash H$. We obtain a realizer $(e^{\psi*\phi}, \epsilon^{\psi*\phi}) \Vdash \psi\phi$ as follows.

$$\begin{aligned}
e^{\psi*\phi} &:= e^{H}e^{\phi} \\
\epsilon^{\psi*\phi}_{(x,0)} &:= \left(\left(\epsilon^{H} * \phi\right) \circ \left(\Pi\left(e^{H}\right) * \epsilon^{\phi}\right)\right)_{(x,0)} \\
\epsilon^{\psi*\phi}_{(x,1)} &:= \left\|\psi_{Gx}\right\|_{Z} \circ \left(\left(\epsilon^{H} * \phi\right) \circ \left(\Pi\left(e^{H}\right) * \epsilon^{\phi}\right)\right)_{(x,1)}
\end{aligned}$$

That this is natural is captured in Figure 5.2.

$$\begin{tikzcd}
\Pi\left(e^H\right)\left(e^\phi\|(x,0)\|_{X\times\mathbf{I}}\right) \arrow[r, "\Pi\left(e^H\right)\left(\epsilon^\phi_{(x,0)}\right)"] \arrow[d, "\Pi\left(e^H\right)\left(e^\phi\|(f,0)\|_{X\times\mathbf{I}}\right)"] \arrow[rr, bend left, "\epsilon^{\psi*\phi}_{(x,0)}"] \arrow[dd, bend right=60, "\Pi\left(e^H e^\phi\right)\|(f,i)\|_{X\times\mathbf{I}}"'] & \Pi\left(e^H\right)\|Fx\|_y \arrow[r, "\epsilon^H_{Fx}"] \arrow[d, "\Pi\left(e^H\right)\|F(f)\|_Y"] & \|HFx\|_Z \arrow[d, "\|H(F(f))\|_Z"'] \arrow[ddd, bend left=60, "\|(\psi*\phi)(f,i)\|_Z"] \\
\Pi\left(e^H\right)\left(e^\phi\|(y,0)\|_{X\times\mathbf{I}}\right) \arrow[r, "\Pi\left(e^H\right)\left(\epsilon^\phi_{(y,0)}\right)"] \arrow[d, "\Pi\left(e^H\right)\left(e^\phi\|(y,i)\|_{X\times\mathbf{I}}\right)"] & \Pi\left(e^H\right)\|Fy\|_y \arrow[r, "\epsilon^H_{Fy}"] \arrow[d, "\Pi\left(e^H\right)\left\|\phi_y\right\|_Y"] & \|HFy\|_Z \arrow[d, "\|H(F(f))\|_Z"'] \\
\Pi\left(e^H\right)\left(e^\phi\|(y,1)\|_{X\times\mathbf{I}}\right) \arrow[r, "\Pi\left(e^H\right)\left(\epsilon^\phi_{(y,1)}\right)"] \arrow[rrd, bend right, "\epsilon^{\phi*\psi}_{(y,1)}"'] & \Pi\left(e^H\right)\|Gy\|_y \arrow[r, "\epsilon^H_{Gy}"] & \|HGy\|_Z \arrow[d, "\left\|\psi_{Gy}\right\|_Z"'] \\
 & & \|KGy\|_Z
\end{tikzcd}$$

Figure 5.2: Naturality of $\epsilon^{\psi*\phi}$.

The identity 2-cell on $F : X \to Y$ is given by the identity natural transformation, and realized by

$$\left(e^F\pi_1, \left(\epsilon^F * \pi_1\right) \circ \left(\Pi\left(e^F\right) * \|-\|_X\pi_1\right)\right)$$

where $(e^F, \epsilon^F) \Vdash F$. The inverse of $\phi : F \Rightarrow G : X \to Y$ is given by the inverse natural transformation $\phi^{-1} : G \Rightarrow F : X \to Y$. Assuming (e^ϕ, ϵ^ϕ) is a realizer for ϕ, then ϕ^{-1} is realized by $(e^\phi, \widetilde{\epsilon})$, where:

$$\begin{aligned}
&\widetilde{\epsilon} : \Pi(e^\phi) \circ \|-\|_{X\times\mathbf{I}} \Rightarrow \|-\|_Y \circ \phi^{-1} \\
&\widetilde{\epsilon}_{(x,0)} := \epsilon^\phi_{(x,1)} \\
&\widetilde{\epsilon}_{(x,1)} := \epsilon^\phi_{(x,0)}
\end{aligned}$$

The axioms for (2,1)-categories hold in virtue of them holding for **Gpd**.

Observe that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is a quotient of the comma 2-category $\mathbf{GPD} \downarrow \Pi$ (a quotient because realizers for (higher) morphisms are only required to exist, not carried around as data).[1]

5.2.1 Finite completeness

In the theme of realizability categories inheriting (or improving) structure from the realizer category, we will now show that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is finitely complete as a (2,1)-category when the realizer category is. Here we also mention the work of Shulman [Shu21] on regular and exact completions of 2-categories. These constructions apply to any finitely complete 2-category, and so, by what follows, to $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ in particular.

In order to prove this, we will use a Lemma 3.2.1. The upshot of this lemma for us is that, given $(\mathbb{C}, \mathbb{I})$ is finitely complete as a (2,1)-category, we may obtain a realizer $\mathbb{I}_1 \times \mathbb{I}_1 \to A$ by providing a commutative square of paths $\mathbb{I}_1 \to A$.

Proposition 5.2.1 *If $(\mathbb{C}, \mathbb{I})$ is finitely complete as a (2,1)-category then so is $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$.*

---

[1] Cf. Mac Lane's "super comma category" [Mac71].

Proof: A (2,1)-category is finitely complete iff it has a terminal object, pullbacks and pseudo pullbacks (comma objects). A terminal object was exhibited in Proposition 5.1.4.

The (strict) pullback of $G : (Y, B, \|-\|_Y) \to (Z, C, \|-\|_Z)$ along $F : (X, A, \|-\|_X) \to (Z, C, \|-\|_Z)$ is given by

$$F^*Y := (F^*Y, A \times B, \|-\|_{F^*Y})$$

where F^*Y is the pullback of groupoids and the realizability functor is given by

$$\|-\|_{F^*Y} := \langle \|-\|_X, \|-\|_Y \rangle$$

The projection functors are realized by projection morphisms from $\mathbb{C}$. If $S : (W, D, \|-\|_W) \to X$ and $T : W \to Y$ are such that $FS = GT$ then we obtain a universal morphism:

$$\begin{aligned} &[S, T] : W \to F^*Y \\ &[S, T](-) := (S(-), T(-)) \end{aligned}$$

that is realized by

$$\big(\langle e, e' \rangle : D \to A \times B, \langle \epsilon, \epsilon' \rangle\big)$$

where $(e, \epsilon) \Vdash S$ and $(e', \epsilon') \Vdash T$.

The pseudo pullback of $G : Y \to Z$ along $F : X \to Z$ is given by

$$F \downarrow G := \left(F \downarrow G, A \times B \times C^{\mathbb{I}_1}, \|-\|_{F\downarrow G}\right)$$

where $F \downarrow G$ is the pseudo pullback of groupoids (comma groupoid), and the realizability functor is defined:

$$\begin{aligned} \|(x, y, r : Fx \to Gy)\|_{F\downarrow G} &:= \langle \|x\|_X, \|y\|_Y, \lambda \|r\|_Z \rangle \\ \big\|(p : x \to x', q : y \to y')\big\|_{F\downarrow G} &:= \left\langle \|p\|_X, \|q\|_Y, \lambda \left(\partial^{-1} \|(p, q, r, r')\|_Z\right) \right\rangle \end{aligned}$$

where $\|(p,q,r,r')\|_Z$ denotes the commutative square:

$$\begin{tikzcd}
\|Fx\|_Z \arrow[r, "\|F(p)\|_Z"] \arrow[d, "\|r\|_Z"'] & \|Fx'\|_Z \arrow[d, "\|r'\|_Z"] \\
\|Gy\|_Z \arrow[r, "\|G(q)\|_Z"'] & \|Gy'\|_Z
\end{tikzcd}$$

Note that this is where we use finite-completeness of $(\mathbb{C},\mathbb{I})$, or rather its consequence that ∂ is an isomorphism. The projection functors are realized by projection morphisms from $\mathbb{C}$.

If we have morphisms $S : W \to X$ and $T : W \to Y$ and a 2-cell $\psi : FS \Rightarrow GT$ then we obtain a universal morphism:

$$\begin{aligned}
&[S,T,\psi] : W \to F \downarrow G \\
&[S,T,\psi](w) := (Sw, Tw, \psi(w,i)) \\
&[S,T,\psi](v \to w') := (S(v), T(v))
\end{aligned}$$

We construct a realizer $(e,\epsilon) \Vdash [S,T,\psi]$ from realizers $(e^S,\epsilon^S) \Vdash S$, $(e^T,\epsilon^T) \Vdash T$ and $(e^\psi,\epsilon^\psi) \Vdash \psi$ as follows.

$$\begin{aligned}
&e : D \to A \times B \times C^{\mathbb{I}_1} \\
&e := \left\langle e^S, e^T, \lambda\left(e^\psi\right)\right\rangle
\end{aligned}$$

Furthermore:

$$\epsilon_w := \left\langle \epsilon^S_w, \epsilon^T_w, \lambda\left(\partial^{-1}\left(\epsilon^\psi_{(w,i)}\right)\right)\right\rangle$$

where $\epsilon^\psi_{(w,i)}$ denotes the following (naturality) square of paths in ΠC.

$$\begin{tikzcd}
e^\psi\|(w,0)\|_{W\times\mathbf{I}} \arrow[r, "\epsilon^\psi_{(w,0)}"] \arrow[d, "e^\psi\|(w,i)\|_{W\times\mathbf{I}}"'] & \|\psi(w,0)\|_Z = \|FSw\|_Z \arrow[d, "\|\psi(w,i)\|_Z"] \\
e^\psi\|(w,1)\|_{W\times\mathbf{I}} \arrow[r, "\epsilon^\psi_{(w,1)}"'] & \|\psi(w,1)\|_Z = \|GTw\|_Z
\end{tikzcd}$$

ϵ is natural due (for the third component) to the double functoriality of ∂^{-1}. □

5.3 As a path category

Throughout this section, we assume that the realizer category $(\mathbb{C}, \mathbb{I})$ is finitely complete as a (2,1)-category. We show that the category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ of partitioned groupoidal assemblies over $(\mathbb{C}, \mathbb{I})$ models a version of 1-truncated ITT without function extensionality. When the realizer category is untyped, there is an impredicative universe of 1-types, closed under identity types.

The notion of model that we employ is that given by path categories. Recall from Section 2.2 that a path category comes equipped with two classes of maps: fibrations and equivalences. Dependent types are modelled by fibrations.

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is a path category when the fibrations are taken to be those morphisms whose underlying functor is a Grothendieck fibration (equivalently, isofibration) and the equivalences are taken to be equivalences internal to the (2,1)-category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$, ie. morphisms with an inverse up to realized natural isomorphism.

(PC1)-(PC3) hold due to standard results about Grothendieck fibrations. (PC4) and (PC5) hold for equivalences in any 2-category. (PC6) gets its own section following this one. This just leaves (PC7) and (PC8), which we return to shortly.

Lemma 5.3.1 *Let* $X = (X, A, \|-\|_X) \in \mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ *and suppose that we have an equivalence:*

$$F : X \to Y \qquad G : Y \to X \qquad \phi : \mathsf{id}_X \Rightarrow GF \qquad \psi : \mathsf{id}_Y \Rightarrow FG$$

in **GPD**. *Then* Y *can be equipped with the structure of a partitioned groupoidal assembly such that the above equivalence is elevated to one in* $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$. *Moreover, if* X *is modest then so is* Y.

Proof: Y is given the realizer type A and the realizability functor

$$\|-\|_Y := \|-\|_X \circ G : Y \to \Pi A$$

The functor G is clearly realized by $(\mathsf{id}, \mathsf{id})$. The functor F is realized by $(\mathsf{id}, \|-\|_X * \phi)$. The natural transformation ϕ is realized by (π_1, ϵ^{ϕ}), where:

$$\epsilon^{\phi}_{(x,0)} := \mathsf{id}_{\|x\|_X} \qquad\qquad \epsilon^{\phi}_{(x,1)} := \|\phi_x\|_X$$

The case for ψ is completely symmetric.

The functor G, being an equivalence of groupoids, is fully faithful and essentially surjective. So if X is modest then $\|-\|_Y$ is the composition of two fully faithful functors, and hence is fully faithful itself. ☐

Assuming AC, every fibration has a cleavage. Thus, if the underlying functor of $F : (X, A, \|-\|_X) \to (Y, B, \|-\|_Y)$ is a (cloven) fibration, we denote the chosen lift of $q : y = Fx \to y'$ at x by $\overline{q}(x) : x \to q^*(x)$. A morphism $q : y \to y'$ induces a morphism $q^* : X_y \to X_{y'}$ of partitioned groupoidal assemblies between the fibre assemblies. The fibre assembly X_y has the fibre groupoid X_y as its underlying groupoid. The realizer type is A, and the realizability functor $\|-\|_{X_y}$ is given by

$$X_y \hookrightarrow X \xrightarrow{\|-\|_X} \Pi A$$

The following diagram encapsulates the underlying functor of q^*, which is standard.

$$\begin{tikzcd}
X \arrow[dd, "F"'] & x \arrow[r, "\overline{q}(x)"] \arrow[d, "p"'] & q^*(y) \arrow[d, dashed, "q^*(p)"] \\
 & x' \arrow[r, "\overline{q}(x')"'] & q^*(x') \\
Z & z \arrow[r, "q"'] & x'
\end{tikzcd}$$

q^* is realized by $(\mathsf{id}_A, \epsilon)$, where

$$\epsilon_x := \|\overline{q}(x)\|_X$$

That is, the induced functor q^* can be elevated to a morphism of partitioned groupoidal (fibre) assemblies.

Now back to the path category axioms. For (P7), given an acyclic fibration $F : (X, A, \|-\|_X) \to (Y, B, \|-\|_Y)$, a weak inverse $G : Y \to X$ of F with realizer (e, ϵ), and 2-cell $\psi : GF \Rightarrow \mathsf{id}_X$, we define a section $S : Y \to X$ of F by:

$$\begin{aligned} S(y) &:= \left(\psi_y\right)^* (Gy) \\ S(p : y \to y') &:= \overline{\left(\psi_{y'}\right)}(Gy') \circ G(p) \circ \left(\overline{(\psi_y)}(Gy)\right)^{-1} \end{aligned}$$

This is realized by (e, ϵ'), where

$$\epsilon'_y := \overline{(\psi_y)}(Gy) \circ \epsilon_y$$

For (P8), let $F : (X, A, \|-\|_X) \to (Z, C, \|-\|_Z)$ be arbitrary and let $G : (Y, B, \|-\|_Y) \to (Z, C, \|-\|_Z)$ be an acyclic fibration with weak inverse $H : Z \to Y$ witnessed by natural isomorphisms $\psi : GH \Rightarrow \mathsf{id}_Z$ and $\phi : \mathsf{id}_Y \Rightarrow HG$. We construct a weak inverse $S : X \to F^*Y$ to the fibration $F^*(G) : F^*Y \to X$. Define $S := [X, T]$, where:

$$\begin{aligned} &T : X \to Y \\ &Tx := \psi^*_{Fx}(HFx) \\ &T(f) := \overline{\phi_{Fx'}}(HFx') \circ H(F(f)) \circ \left(\overline{\phi_{Fx}}(HFx)\right) \end{aligned}$$

Given realizers $(e^F, \epsilon^F) \Vdash F$ and $(e^H, \epsilon^H) \Vdash H$, a realizer for T is $(e^F e^H, \epsilon^T)$, where

$$\epsilon^T_x := \overline{\psi_{Fx}}(HFx) \circ \epsilon^H_{Fx} \circ \Pi\left(e^H\right)\left(\epsilon^F_x\right)$$

Clearly we have $F^*G \circ S = \mathsf{id}_X$. We now construct a natural isomorphism $\sigma : \mathsf{id}_{F^*Y} \Rightarrow S \circ F^*(G)$. We have

$$S\left(F^*(G)(x, y)\right) = S(x) = \left(x, \psi^*_{Fx}(HFx)\right)$$

So we define

$$\sigma_{(x,y)} := \left(\mathsf{id}_x, \sigma_y\right)$$

where σ_y is defined to be the following composite.

$$y \xrightarrow{\phi_y} HGy = HFx \xrightarrow{\overline{\psi_{Fx}}(HFx)} \psi^*_{Fx}(HFx)$$

This is realized by $(\pi_1 : (A \times B) \times \mathbb{I}_1 \to A \times B, \epsilon^\sigma)$, where:

$$\begin{aligned} \epsilon^\sigma_{(x,y,0)} &:= \langle \|x\|_X, \|y\|_Y \rangle \\ \epsilon^\sigma_{(x,y,1)} &:= \langle \|x\|_X, \|\sigma_y\|_Y \rangle \end{aligned}$$

There is a subclass of fibrations given by the modest fibrations.

Definition 5.3.2 (Modest fibration) A modest fibration in $\mathbf{GPAsm}(\mathbb{C}, \mathbb{I})$ is a fibration $M : Y \to X$ in such that for all $x : \mathbf{1} \to X$ the pullback

$$\begin{tikzcd} Y_x \arrow[r, "\pi_2"] \arrow[d, "x^*M"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "M"] \\ \mathbf{1} \arrow[r, "x"'] & X \end{tikzcd}$$

is modest, that is, for all fibre groupoids Y_x:

$$\|-\|_{Y_x} := \|-\|_Y \circ \pi_2 : Y_x \to \Pi U$$

is fully faithful. $\diamondsuit$

By the pullback lemma, modest fibrations are closed under pullback along arbitrary morphisms and closed under composition. Trivially, isomorphisms are modest fibrations.

Remark 5.3.3 By AC, any Grothendieck fibration F is fibrewise equivalent to a split one. Suppose F is a fibration in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$. By AC, its underlying functor is fibrewise equivalent to a split one. Lemma 5.3.1 allows us to upgrade this equivalence to one in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$.

Now suppose that $M : Y \to X$ is a modest fibration. Is the splitting

$$\begin{tikzcd} \widetilde{Y} \arrow[rr, "G", "\sim"'] \arrow[dr, "\widetilde{M}"'] & & Y \arrow[dl, "M"] \\ & X & \end{tikzcd}$$

of M again a modest fibration? By Brown's lemma (2.2.2) it is indeed: Take any $x : \mathbf{1} \to X$. Then, by Brown's lemma,

$$x^*G : \widetilde{Y}_x \to Y_y$$

is an equivalence between fibre assemblies. The realizability functor of $\widetilde{Y}_x$ is given by:

$$\begin{aligned} \|-\|_{\widetilde{Y}_x} &:= \|-\|_Y \circ G \circ j \\ &= \|-\|_Y \circ i \circ x^*G \\ &= \|-\|_{Y_x} \circ x^*G \end{aligned}$$

(where i, j are the relevant inclusions of fibres to total assemblies) which is, as the composition of two fully faithful functors, fully faithful.

Let us also mention that later (in Section 5.4) we will have reason to consider the subclass of fibrations given by the morphisms whose underlying functor is a Grothendieck fibration with small fibres.

5.3.1 Path objects

The remaining axiom (PC6) for path categories asserts that every object X has a path object $\mathcal{P}X$, ie. a factorisation of the diagonal by an equivalence followed by a fibration:

$$\begin{tikzcd} & \mathcal{P}X \arrow[dr, "(\mathsf{s},\mathsf{t})_X"] & \\ X \arrow[ur, "\mathsf{r}_X"] \arrow[rr, "\Delta_X"'] & & X \times X \end{tikzcd}$$

Theorem 5.3.4 *Every object in* $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ *has a path object. Moreover, the path object* $\mathcal{P}X$ *is modest when* X *is.*

Proof: Given $X = (X, A, \|-\|_X)$, the weak exponential object $X^{\mathbf{I}}$ (see proposition 5.1.4) is a path object $\mathcal{P}X$.

The equivalence $\mathsf{r}_X : X \to X^{\mathbf{I}}$ is given by:

$$\begin{aligned} \mathsf{r}_X(x) &:= (i \mapsto \mathsf{id}_x, \lambda\, \|\mathsf{id}_x\|_X, \mathsf{id}) \\ \mathsf{r}_X(p) &:= \left((i,i) \mapsto p, \lambda\left(\partial^{-1}\|p\|_X\right)\right) \end{aligned}$$

where the argument $\|p\|_X$ of ∂^{-1} denotes the commutative square whose horizontal edges are $\|p\|_X$ and whose vertical edges are id_x. r_X is realized by $(\lambda(\pi_1) : A \to A^{\mathbb{I}_1}, \mathsf{id})$.

The fibration $(\mathsf{s},\mathsf{t})_X : X^{\mathbf{I}} \to X \times X$ is given by:

$$\begin{aligned} (F, e, \epsilon) &\mapsto (F0, F1) \\ (\psi, f) &\mapsto (\psi(0,i), \psi(1,i)) \end{aligned}$$

realized by $(\langle \mathsf{eval} \circ \langle \mathsf{id}, 0\rangle, \mathsf{eval} \circ \langle \mathsf{id}, 1\rangle\rangle, \epsilon)$.

Suppose (F, e, ϵ) is in the fibre over (x_1, x_2). Then $F(i) : x_1 \to x_2$. We define the chosen lift $(\psi, f) : (F, e, \epsilon) \to (G, e, \epsilon')$ of $p = (p_1, p_2) : (x_1, x_2) \to (x_1', x_2')$ at (F, e, ϵ) by:

$$G(i) := p_2 \circ F(i) \circ p_1^{-1} \qquad \epsilon_0' := p_1 \circ \epsilon_0 \qquad \epsilon_1' := p_2 \circ \epsilon_1$$

and

$$\psi(i,i) := p_2 \circ F(i) = G(i) \circ p_1$$

We obtain the realizer

$$f := \lambda\left(\partial^{-1}(e)\right) : \mathbb{I}_1 \to A^{\mathbb{I}_1}$$

where e above is used to denote the following commutative square of paths in ΠA.

$$\begin{tikzcd} \Pi(\mu(e))(0) \arrow[r, equal, "\epsilon_0'^{-1} p_1 \epsilon_0"] \arrow[d, "\Pi(\mu(e))"'] & \Pi(\mu(e))(0) \arrow[d, "\Pi(\mu(e))"] \\ \Pi(\mu(e))(1) \arrow[r, equal, "\epsilon_1'^{-1} p_2 \epsilon_1"'] & \Pi(\mu(e))(1) \end{tikzcd}$$

□

5.3.2 Dependent products

The notion of dependent product suitable for path categories is that of *homotopy dependent product*, provided in Definition 2.2.3.

Theorem 5.3.5 $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ *has weak homotopy dependent products. Moreover, if* $F : Y \to Z$ *is a fibration and* $M : X \to Y$ *is a modest fibration, then the dependent product*

$$\Pi_F(M) : \Pi_F X \to Z$$

is a modest fibration.

With this theorem we have that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ models a version of ITT with dependent products but not function extensionality.

Proof: Given fibrations $G : (X, A, \| - \|_X) \to (Y, B, \| - \|_Y)$ and $F : Y \to (Z, C, \| - \|_Z)$, the dependent product $\Pi_F X$ has an underlying groupoid whose objects are tuples (z, H, e, ϵ), where $z \in Z$, $H : F \downarrow z \to X$ in the slice over Y

$$\begin{tikzcd} F \downarrow z \arrow[rr, "H"] \arrow[dr, "\pi_1"'] & & X \arrow[dl, "G"] \\ & Y & \end{tikzcd} \tag{5.3.1}$$

(where $F \downarrow z$ is the "homotopy fibre", constructed as a pseudo pullback), $e \in \Pi(A^{B \times C^{\mathbb{I}_1}})$ and ϵ is a natural isomorphism such that $(\mu(e), \epsilon) \Vdash H$.

Observe that any $r : z \to z'$ induces a morphism $F \downarrow r : F \downarrow z \to F \downarrow z'$ of partitioned assemblies defined by $(y, u : Fy \to z) \mapsto (y, ru)$ and identity on morphisms; the morphism is realized by (e^r, ϵ^r), where:

$$\begin{aligned} e^r &:= \mathsf{id}_{B \times C^{\mathbb{I}_1}} \\ \epsilon^r_{(y,u)} &:= \left\langle B, \lambda \left(\partial^{-1} \| (r, u) \|_Z \right) \right\rangle \end{aligned}$$

and where $\|(r,u)\|_Z$ denotes the following commutative square.

$$\begin{tikzcd}
\|Fy\|_Z \arrow[r, equal] \arrow[d, "\|u\|_Z"'] & \|Fy\|_Z \arrow[d, "\|ru\|_Z"] \\
\|z\|_Z \arrow[r, "\|r\|_Z"'] & \|z'\|_Z
\end{tikzcd}$$

Naturality follows from the double-functoriality of ∂^{-1}.

A morphism $(r,\psi,f) : (z,H,e,\epsilon) \to (z',H',e',\epsilon')$ in $\Pi_F X$ consists of a morphism $r : z \to z'$, a natural isomorphism $\psi : H \Rightarrow H' \circ (F \downarrow r)$ over Y and a path $f : e \to e' \circ e^r = e'$ such that there exists a natural isomorphism ζ satisfying:

$$\zeta(-,0,-) = \epsilon$$

$$\zeta(-,1,-) = \big(\epsilon' * (F \downarrow r)\big) \circ \big(\Pi(e') * \epsilon^r\big)$$

$$\begin{tikzcd}
F \downarrow z \arrow[r, "F\downarrow r"] \arrow[d, "\|-\|_{F\downarrow z}"'] & F \downarrow z' \arrow[r, "H'"] \arrow[d, "\|-\|_{F\downarrow z'}"'] & X \arrow[d, "\|-\|_X"] \\
\Pi\big(B \times C^{\mathbb{I}_1}\big) \arrow[r, equal] \arrow[ur, Rightarrow, "\epsilon^r"'] & \Pi(B \times C^{\mathbb{I}_1}) \arrow[r, "\Pi(e')"'] \arrow[ur, Rightarrow, "\epsilon'"'] & \Pi A
\end{tikzcd}$$

as well as

$$(\mu(f) \circ \mathsf{swap}, \zeta) \Vdash \psi$$

As in the proof of Proposition 5.1.4, these conditions uniquely determine ζ (its boundary is determined, so apply Lemma 3.2.1 to the finitely complete (2,1)-category **GPD**). The realizer type is $C \times A^{B \times C^{\mathbb{I}_1}}$ and the realizability functor is given by

$$\|-\|_{\Pi_F X} \coloneqq \langle \|-\|_Z \circ \pi_1, \pi_3 \rangle$$

The fibration $\Pi_F(G) : \Pi_F X \to Z$ is given by the first projection and realized by (π_1, id). The chosen lift of r at (z,H,e,ϵ) is

$$(r,\psi,f) : (z,H,e,\epsilon) \to (z',H',e,\epsilon')$$

where

$$\begin{aligned} H' &:= H \circ \left(F \downarrow r^{-1}\right) \\ \epsilon' &:= \left(\epsilon * F \downarrow r^{-1}\right) \circ (e * \epsilon^{r}) \\ \psi &:= \mathsf{id}_H \\ f &:= \mathsf{id}_e \end{aligned}$$

The third definition makes sense because

$$(F \downarrow r^{-1}) \circ (F \downarrow r) = \mathsf{id}_{F \downarrow z}$$

If $M : X \to Y$ is a modest fibration, then we can establish that $\Pi_F(M)$ is a modest fibration by a similar argument to that at the end of the proof of Proposition 5.1.4. Given $(z, H, e, \epsilon), (z, H', e', \epsilon') \in \Pi_F(M)_z$ in the fibre over z and $(y, u) \in F \downarrow z$, we know that $H(y, u)$ and $H'(y, u)$ are in the same fibre X_y of G by the commutativity of (5.3.1). Then any $f : e \to e'$ uniquely determines a morphism $(\mathsf{id}_z, \psi, f) \in \Pi_F(M)_z$ in the fibre over z by fully faithfulness of $\|-\|_{X_y}$, because for each $y \in Y_z$ there is a unique morphism (the image under $\|-\|_{X_y}$ of ψ_y) making the following diagram commute.

$$\begin{tikzcd}
\|Hx\|_{X_y} \arrow[r, dashed, "\|\psi_y\|_{X_y}"] & \|H'x\|_{X_y} \\
\begin{array}{c} \Pi(f)\langle 0, \|y\|_{X_y} \rangle \\ = \Pi(e)\|y\|_{X_y} \end{array} \arrow[u, "\epsilon_y"] \arrow[r, "\Pi(f)\langle \mathbb{I}_1, \|y\|_{X_y} \rangle"'] & \begin{array}{c} \Pi(f)\langle 1, \|y\|_{X_y} \rangle \\ = \Pi(e')\|y\|_{X_y} \end{array} \arrow[u, "\epsilon'_y"']
\end{tikzcd}$$

We now define the "evaluation map"

$$\mathsf{ev} : F^{*}\Pi_F X \to X$$

over Y, where F^{*} is pullback along F. First let us compute $F^{*}\Pi_F X$. Objects of the underlying groupoid of $F^{*}\Pi_F X$ are tuples (y, z, H, e, ϵ), where $y \in Y$, $z = Fy$, and H, e, ϵ are as above. A morphism

$$(q, r, \psi, f) : (y, z, H, e, \epsilon) \to (y', z', H', e', \epsilon')$$

consists of morphisms $q : y \to y'$ and $r = F(q) : z \to z'$, as well as ψ, f as described above. The realizer type of $F^*\Pi_F X$ is $B \times C \times A^B$ and the realizability functor is given by

$$\|-\|_{F^*\Pi_F X} \coloneqq \langle \|-\|_Y \circ \pi_1, \|-\|_Z \circ \pi_2, \pi_4 \rangle$$

Define the map ev by:

$$\begin{aligned} \mathsf{ev}(y, z, H, e, \epsilon) &\coloneqq H\left(y, \mathsf{id}_z\right) \\ \mathsf{ev}(q, r, \psi, f) &\coloneqq H'(q) \circ \psi_{(y, \mathsf{id}_z)} \end{aligned}$$

The argument q of H' is being considered as a morphism $(y, r) \to (y', \mathsf{id}_{z'})$ in $F \downarrow z'$. The map ev lives in the slice over Y thanks to (5.3.1). It is realized by

$$\left(\mathsf{ev} \circ \langle \pi_3, \pi_1 \rangle, \epsilon'\right)$$

where we overload notation and use ev for the evaluation map from $\mathbb{C}$, and

$$\epsilon'_{(y,z,H,e,\epsilon)} \coloneqq \epsilon_y$$

This is natural as we know that there exists a natural isomorphism ζ making the following diagram commute.

$$\begin{tikzcd}
\begin{matrix}\Pi(\mu(f))\ \langle 0, \|y\|_Y \rangle \\ = \mu(e)\|y\|_Y\end{matrix} \arrow[r, "\zeta(y,0,i)=\epsilon_y"] \arrow[d, "\Pi(\mu(f))\langle \mathbb{I}_1, \|y\|_Y \rangle"] \arrow[dd, bend right=60, "\Pi(\mu(f))\langle \mathbb{I}_1, q \rangle"'] & \|H(y, \mathsf{id}_z)\|_Y \arrow[d, "\|\psi_y\|_Y"] \\
\begin{matrix}\Pi(\mu(f))\langle 1, \|y\|_Y \rangle \\ = \mu(e')\|y\|_Y\end{matrix} \arrow[r, "\zeta(y,1,i)=\epsilon'_y"] \arrow[d, "\begin{matrix}\Pi(\mu(f))\langle \mathbb{I}_1, \|q\|_Y \rangle \\ = \mu(e')\|q\|_Y\end{matrix}"] & \begin{matrix}\|H'((F\downarrow r)(y, \mathsf{id}_z))\|_Y \\ = \|H'(y, r)\|_Y\end{matrix} \arrow[d, "\|H'(q)\|_Y"] \\
\begin{matrix}\Pi(\mu(f))\langle 1, \|y'\|_Y \rangle \\ = \mu(e')\|y'\|_Y\end{matrix} \arrow[r, "\epsilon'_{y'}"'] & \|H'\left(y', \mathsf{id}_{z'}\right)\|_Y
\end{tikzcd}$$

For the universal property, assume we have a map $R : (W, D, \|-\|_W) \to Z$ and a map $S : F^*W \to X$ over Y.

$$\begin{tikzcd}
F^*W \arrow[rr, "S"] \arrow[dr, "F^*R"'] & & X \arrow[dl, "G"] \\
& Y &
\end{tikzcd}$$

We construct a map T as in the following diagram.

$$\begin{tikzcd}
W \arrow[rr, "T"] \arrow[dr, "R"'] & & \Pi_F(X) \arrow[dl, "\Pi_F(G)"] \\
& Z &
\end{tikzcd}$$

Define:

$$Tw := (Rw, H, e, \epsilon)$$
$$T(v : w \to w') := (R(v), \psi, f)$$

where the components H, e, ϵ, ψ and f are defined below.

First:

$$H : F \downarrow Rw \to X$$
$$H := S \circ [\sigma_1, \sigma_2]$$

Here,

$$\sigma_1 : F \downarrow Rw \to Y$$
$$\sigma_1(y, r) := r^*(y)$$
$$\sigma_1(q : (y, r) \to (y', r')) := \overline{r'}(y') \circ q \circ \overline{r}(y)^{-1}$$

is realized by (π_1, δ), where

$$\delta_y := \|\overline{r}(y)\|_Y$$

and $\sigma_2 : F \downarrow Rw \to W$ is the constantly w functor, realized by $(\|w\|_{W}*, \delta')$, where $\delta_w := \mathsf{id}_{\|w\|_W}$. We can form $[\sigma_1, \sigma_2]$ because $F(r^*(y)) = Rw$ and

$$F\left(\overline{r'}(y') \circ q \circ \overline{r}(y)^{-1}\right) = r'qr^{-1} = r'qq^{-1}r'^{-1} = \mathsf{id}_{Rw}$$

Next, pick a realizer $(e^S, \epsilon^S) \Vdash S$ and define $e \in \Pi(A^{B\times C^{\mathbb{I}_1}})$ to be the exponential transpose of

$$1 \times \left(B \times C^{\mathbb{I}_1}\right) \xrightarrow{\langle \pi_1\pi_2, \pi_1\rangle} B \times 1 \xrightarrow{B\times\|w\|_W} B \times D \xrightarrow{e^S} A$$

The natural isomorphism ϵ is defined by the following pasting diagram.

$$\begin{tikzcd}
F \downarrow Rw \arrow[r, "{[\sigma_1,\sigma_2]}"] \arrow[d, "\|{-}\|_{F\downarrow Rw}"'] & F^*W \arrow[r, "S"] \arrow[d, "\|{-}\|_{F^*W}"'] & X \arrow[d, "\|{-}\|_X"] \\
\Pi\left(B \times \mathsf{C}^{\mathbb{I}_1}\right) \arrow[r, "\Pi(B\times\|w\|_{W^*})"'] \arrow[ur, Rightarrow, "{\langle\delta,\delta'\rangle}"'] & \Pi(B \times D) \arrow[r, "\Pi\left(e^S\right)"'] \arrow[ur, Rightarrow, "\epsilon^S"'] & \Pi A
\end{tikzcd}$$

The dependent products being constructed are weak because we have had to make a choice of realizer (e^S, ϵ^S) for each S.

As for $\psi : H \Rightarrow H' \circ (F \downarrow R(v))$, we must have:

$$\psi(y,r,0) = H(y,r) = S\left(r^*(y), w\right)$$
$$\psi(y',r',1) = H'(y', R(v)r') = S\left((R(v)r')^*(y'), w'\right)$$

So we define

$$\psi(q,i) := S\left(\overline{R(v)}\left((r')^*(y')\right), v\right) \circ H(q) = H'(q) \circ S\left(\overline{R(v)}\left(r^*(y)\right), v\right)$$

The argument q of H is regarded as a morphism $(y,r) \to (y',r')$ in $F \downarrow Rw$, whereas q *qua* argument of H' is regarded as a morphism $(y, R(v)r) \to (y', R(v)r')$ in $F \downarrow Rw'$. Finally, we define $f : e \to e'$ (recall that $(e,\epsilon) \Vdash H$) to be the exponential transpose of

$$\mathbb{I}_1 \times \left(B \times \mathsf{C}^{\mathbb{I}_1}\right) \xrightarrow{\langle \pi_1\pi_2, \pi_1\rangle} B \times \mathbb{I}_1 \xrightarrow{B\times\|v\|_W} B \times D \xrightarrow{e^S} A$$

To complete the proof we show that

$$S = \mathsf{ev} \circ F^*T$$

On objects:

$$\begin{aligned}
\mathsf{ev}\left(F^*T(y,w)\right) &= \mathsf{ev}(y, Tw) \\
&= \mathsf{ev}(y, Rw, H, e, \epsilon) \\
&= H\left(y, \mathsf{id}_{Rw}\right) \\
&= S(y,w)
\end{aligned}$$

and on morphisms:

$$\begin{aligned}
\mathsf{ev}\left(F^*T(q,v)\right) &= \mathsf{ev}(q, T(v)) \\
&= \mathsf{ev}(q, R(v), \psi, f) \\
&= H'\left(q : (y, R(v)) \to (y', \mathsf{id}_{Rw'})\right) \circ \psi_{(y, \mathsf{id}_{Rw})} \\
&= S\left(q \circ \left(\overline{R(v)}(y)\right)^{-1}, w'\right) \circ S\left(\overline{R(v)}(y), v\right) \\
&= S(q,v)
\end{aligned}$$

The fact that that $S = \mathsf{ev} \circ F^*T$ holds on the nose is because fibrations of groupoids are exponentiable. □

Corollary 5.3.6 *The path category* $\mathbf{Ex}(\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}))$ *has (strong) homotopy dependent products, and the category* $\mathbf{Hex}(\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}))$ *is locally cartesian closed.*

Proof: Apply Theorem 2.2.4. □

These categories thus provide models of type theory with function extensionality.

5.3.3 Impredicative universes

A significant amount of the interest in realizability models of type theory is due to their provision of impredicative universes. It is this aspect of groupoidal realizability to which we now turn.

We know, eg. from [Bir00b, RR01, LS02], that "untypedness" is essential for impredicativity. Hence the first step in our investigation of impredicative universes is to switch from typed realizer categories to untyped realizer categories; the latter were defined in Section 3.3.

Denote by $\mathbf{GPAsm}(\mathbb{C}, \mathbb{I}, U)$ the full subcategory of $\mathbf{GPAsm}(\mathbb{C}, \mathbb{I})$ spanned by those objects whose realizer type is U. As well as the path category structure, this subcategory inherits all categorical properties of the ambient category due to the following.

Proposition 5.3.7 *The inclusion*

$$\mathbf{PGAsm}(\mathbb{C},\mathbb{I},U) \hookrightarrow \mathbf{PGAsm}(\mathbb{C},\mathbb{I})$$

is an equivalence of (2,1)-categories.

Proof: It suffices to show essential surjectivity. For any $X = (X, A, \|-\|_X)$ we define an isomorphism

$$F : X \to X' = (X, U, \|-\|_{X'})$$

where

$$\|-\|_{X'} \coloneqq \Pi(s_A)\|-\|_X$$

The underlying functor of F is id_X, which is realized by (s_A, id). The underlying functor of F^{-1} is also id_X, this time realized by $(r_A, \Pi(\rho_A))$. $\square$

The following is the notion of representation given in Definition 2.2.5 when specialised to $\mathbf{PGAsm}(\mathbb{C},\mathbb{I},U)$ with $\mathcal{S}$ the modest fibrations (Definition 5.3.2).

Definition 5.3.8 (Representation for modest fibrations) A representation θ for modest fibrations is a modest fibration $\theta : \Theta \to \Lambda$ such that for every modest fibration $M : Y \to X$ there is a map $M_* : Y \to P$, where $P \coloneqq X \times_\Lambda \Theta$, such that $\nu_M \circ M = \theta \circ M_*$ and the induced map

$$[M_*, M] : Y \to P$$

is an equivalence $M \simeq (\nu_M)^*\theta$ in $\mathbf{PGAsm}(\mathbb{C},\mathbb{I},U)(X)$.

$$\begin{tikzcd}
Y \arrow[r, dashed, "{[M_*,M]}", "\sim"'] \arrow[rr, bend left, "M_*"] \arrow[dr, bend right, "M"'] & P \arrow[r] \arrow[d, "(\nu_M)^*\theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d, "\theta"] \\
& X \arrow[r, "\nu_M"'] & \Lambda
\end{tikzcd}$$

$\Diamond$

Theorem 5.3.9 **PGAsm**$(\mathbb{C}, \mathbb{I}, U)$ *has a representation for modest fibrations.*

Before giving the proof, we first define the "chaotic" inclusion

$$\nabla : \mathbf{GPD} \to \mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$$

of groupoids in partitioned groupoidal assemblies. Choose an arbitrary $a_0 \in \Pi U$ and set:

$$\begin{aligned} &\nabla X := (X, \|-\|_{\nabla X} : p \mapsto \mathsf{id}_{a_0}) \\ &\nabla(F) := F \end{aligned}$$

where $\nabla(F)$—indeed any map into an object in the image of ∇—is realized by $\lambda x.a_0$.

Proof of Theorem 5.3.9 Define

$$\Lambda := \nabla\left(\widehat{\Pi U}\right)$$

where $\widehat{\Pi U}$ is the groupoid of covariant **Bij**-valued presheaves on ΠU.

The underlying groupoid of Θ has as objects pairs (F, a), where $F \in \widehat{\Pi U}$ and $a \in \Pi U$ is such that $Fa \neq \emptyset$. A morphism $(\psi, \alpha) : (F, a) \to (G, b)$ consists of a natural isomorphism $\psi : F \Rightarrow G$ and a path $\alpha : a \to b$. The realizability functor $\|-\|_\Theta$ is given by the second projection. The modest fibration $\theta := \pi_1 : \Theta \to \Lambda$ is given by the first projection. Fix (F, a) and (F, b) in the fibre over $F \in \Lambda$. Every $\alpha : a \to b$ in ΠU uniquely determines a morphism (id_F, α) in the fibre over F. The chosen cartesian lift of $\psi : F \to G$ at (F, a) is $(\psi, \mathsf{id}_a) : (F, a) \to (G, a)$.

Let $M : Y \to X$ be a modest fibration. By Remark 5.3.3, we can (using AC) take the underlying Grothendieck fibration of M to be, up to equivalence in **PGAsm**$(\mathbb{C}, \mathbb{I}, U)$, split. Define the object part of its characteristic map $\nu_M : X \to \Lambda$ of M as follows. First,

$$\nu_M(x)(a) := \left\{\alpha : a \to a' \mid \exists y \in Y.\ My = x \wedge \|y\|_Y = a'\right\}_{\diagup\sim}$$

$$\begin{tikzcd}
\Pi U & & b \arrow[r, "\beta^{-1}"] & a \arrow[r, "\alpha"] & c \arrow[r, "\|\overline{p}(y_x^c)\|_Y"] & \|p^*(y_x^c)\|_Y \\
Y \arrow[u, "\|-\|_Y"] \arrow[d, "M"'] & & & & y_x^c \arrow[r, "\overline{p}(y_x^c)"] & p^*(y_x^c) \\
X & & & & x \arrow[r, "p"] & x'
\end{tikzcd}$$

Figure 5.3: Defining the characteristic map ν_M.

where the relation $\sim$ is isomorphism in the co-slice above a. Then

$$\nu_M(x)(\beta : a \to b)[\alpha] := [\alpha \circ \beta^{-1}]$$

To define $\nu_M(p : x \to x')(\beta, i)[\alpha]$, we pick a representative $\alpha : a \to c$ from $[\alpha]$. Further, we pick a $y_x^c \in Y$ such that $M(y_x^c) = x$ and $\|y_x^c\|_Y = c$. Using the fact that M is a fibration, we define

$$\nu_M(p : x \to x')(\beta, i)[\alpha] := \left[\|\overline{p}\,(y_x^c)\|_Y \circ \alpha \circ \beta^{-1}\right]$$

Figure 5.3 helps to visualise the definition.

This definition is independent of the choices made: Let $\alpha' : a \to d$ be a element of $[\alpha]$ isomorphic to α in the co-slice above a, and let $y_x^d \in Y$ be a choice of element such that $M(y_x^d) = x$ and $\|y_x^d\|_Y = d$. Then by modesty of M we have an isomorphism $y_x^c \to y_x^d$. Transporting this isomorphism along p and hitting it with $\|-\|_Y$ yields an isomorphism

$$\|\overline{p}\left(y_x^c\right)\|_Y \circ \alpha \circ \beta^{-1} \to \left\|\overline{p}\left(y_x^d\right)\right\|_Y \circ \alpha \circ \beta^{-1}$$

in the co-slice above b.

To show that $\nu_M(x)$ is functorial, first note that $\nu_M(\mathsf{id}_x)(\beta, i)[\alpha] = [\alpha \circ \beta^{-1}]$. For composition, we exhibit an isomorphism between

$$\|\overline{qp}\left(y_x^c\right)\|_Y \circ \alpha \circ \beta^{-1}$$

and

$$\left\|\overline{p}\left(y_{x'}^{\|p^*(y_x^c)\|_Y}\right)\right\|_Y \circ \|\overline{p}\left(y_x^c\right)\|_Y \circ \alpha \circ \beta^{-1}$$

in the slice over b, where $y_{x'}^{\|p^*(y_x^c)\|_Y}$ is a chosen element such that:

$$M\left(y_{x'}^{\|p^*(y_x^c)\|_Y}\right) = x'$$
$$\left\|y_{x'}^{\|p^*(y_x^c)\|_Y}\right\|_Y = \|p^*\left(y_x^c\right)\|_Y$$

Given that $p^*(y_{x'}^c)$ and $y_{x'}^{\|p^*(y_x^c)\|_Y}$ are both in the fibre over x' and the image of each under $\|-\|_Y$ is $\|p^*(y_x^c)\|_Y$, by fullness of $\|-\|_Y$ we obtain an isomorphism

$$\|p^*\left(y_x^c\right)\|_Y^{-1} : p^*\left(y_{x'}^c\right) \to y_{x'}^{\|p^*(y_x^c)\|_Y}$$

Transporting this isomorphism along q and then hitting with $\|-\|_Y$ gives the desired isomorphism.

The pullback $P := X \times_\Lambda \Theta$ has objects of the form (x, a) (we omit the component $\nu_M(x)$ that is determined by x) and morphisms of the form (p, α). To show that M is equivalent to $(\nu_M)^*\theta$ we first construct $M_* : Y \to \Theta$.

$$M_*(y) := (\nu_M(My), \|y\|_Y)$$
$$M_*(q) := (\nu_M(M(q)), \|q\|_Y)$$

This is realized by $(\mathsf{id}, \mathsf{id})$ and indeed satisfies $\nu_M \circ M = \theta \circ M_*$. So we get a universal map $[M_*, M]$.

Now we define a weak inverse $M^* : P \to Y$ to $[M_*, M]$. Given $(x, a) \in P$, we choose an element $[\gamma_{x,a}] \in \nu_M(x)(a)$ and a representative $\gamma_{x,a} : a \to c_{x,a}$ from $[\gamma_{x,a}]$ (we know $\nu_M(x)(a)$ is non-empty). Moreover, $v(\gamma_{x,a}) \in Y$ is a chosen element such that $M(v(\gamma_{x,a})) = x$ and $\|v(\gamma_{x,a})\|_Y = c_{x,a}$.

With this, on objects we define

$$M^* \left(x, a\right) := v \left(\gamma_{x,a}\right)$$

Now take a morphism

$$\left(p, \alpha\right) : \left(x, a\right) \to \left(x', b\right)$$

in P. We would like to define its image under M^* to be

$$\overline{p} \left(v \left(\gamma_{x,a}\right)\right) : v \left(\gamma_{x,a}\right) \to p^* \left(v \left(\gamma_{x,a}\right)\right) \tag{5.3.2}$$

but it is not necessarily the case that

$$p^* \left(v \left(\gamma_{x,a}\right)\right) = v \left(\gamma_{x',b}\right) \tag{5.3.3}$$

ie. the codomain may not align. Here we utilise modesty. Consider the following commutative diagram in ΠU.

$$\begin{tikzcd}
a \arrow[rr, "\alpha"] \arrow[d, "\gamma_{x,a}"'] & & b \arrow[d, "\gamma_{x',b}"] \\
c_{x,a} \arrow[dr, "\|\overline{p}(v(\gamma_{x,a}))\|_Y"'] & & c_{x',b} \\
& \|p^* \left(v \left(\gamma_{x,a}\right)\right)\|_Y \arrow[ur, dashed, "\delta_{p,\alpha}"'] &
\end{tikzcd}$$

We know that

$$p^*(v(\gamma_{x,a})), v(\gamma_{x',b}) \in Y_{x'}$$

and we have the morphism

$$\delta_{p,\alpha} : \|p^* \left(v \left(\gamma_{x,a}\right)\right)\|_Y \to c_{x',b} = \|v \left(\gamma_{x',b}\right)\|_Y$$

Thus, using fullness, we obtain a morphism

$$\left\|\delta_{p,\alpha}\right\|_{Y}^{-1} : p^{*}\left(v\left(\gamma_{x,a}\right)\right) \to v\left(\gamma_{x',b}\right)$$

in the fibre $Y_{x'}$. Therefore we can define

$$M^{*}\left(p,\alpha\right) := \left\|\delta_{p,\alpha}\right\|_{Y}^{-1} \circ \overline{p}\left(v\left(\gamma_{x,a}\right)\right)$$

which reduces to (5.3.2) in case (5.3.3) holds. Faithfulness ensures this is functorial. M^{*} is realized by $(\pi_2 r_{U\times U}, \epsilon)$, where

$$\epsilon_{(x,a)} := \gamma_{x,a} \circ \rho_a$$

There is a natural isomorphism $\sigma : \mathsf{id}_Y \Rightarrow M^{*}[M_{*}, M]$, defined

$$\sigma(q,i) := \|\gamma_{My',\|y'\|_Y}\|_{Y}^{-1} \circ q$$

and realized by $(\pi_1 r_{U\times U}, \epsilon)$, where:

$$\begin{aligned}
\epsilon_{(y,0)} &:= \left(\langle\|-\|_{Y}, \|-\|_{\mathbf{I}}\rangle * \Pi\left(\rho_{U\times U}\right) * \Pi(\pi_1)\right)_{(y,0)} \\
\epsilon_{(y,1)} &:= \gamma_{My,\|y\|_Y} \circ \epsilon_{(y,0)}
\end{aligned}$$

Conversely, there is a natural isomorphism $\tau : \mathsf{id}_P \Rightarrow [M_{*}, M]M^{*}$, defined

$$\tau\left(\left(p,\alpha\right),i\right) := \left(p, \gamma_{x',b} \circ \alpha\right)$$

and realized by $(\pi_1 r_{U\times U}, \epsilon')$, where:

$$\begin{aligned}
\epsilon'_{((x,a),0)} &:= \left(\langle\|{-}\|_{P}, \|-\|_{\mathbf{I}}\rangle * \Pi\left(\rho_{U\times U}\right) * \Pi(\pi_1)\right)_{((x,a),0)} \\
\epsilon'_{((x,a),1)} &:= \left(\mathsf{id}_x, \mathsf{id}_{\nu_M(x)}, \gamma_{x',b}\right) \circ \epsilon'_{((x,a),0)}
\end{aligned} \qquad \square$$

Therefore, $\mathbf{GPAsm}(\mathbb{C}, \mathbb{I}, U)$ is a model of 1-truncated ITT with an impredicative universe of 1-types, though without function extensionality.

5.4 Generic fibrations

Recall from Section 2.4.1 that the exact completion of a finitely complete category promotes a generic proof to a subobject classifier. We make two observations on this process:

- The exact completion turns a weaker structure into a stronger one. A generic proof classifies maps up to "bi-implication", and a characteristic map is guaranteed only to exist—it need not be unique. A subobject classifier, on the other hand, classifies monomorphisms up to isomorphism, and the characteristic map is unique.
- The exact completion reduces by one the truncation of the maps classified. A generic proof classifies *all* maps, whereas a subobject classifier classifies all *monomorphisms*.

A monomorphism $m : Y \hookrightarrow Z$ in a category $\mathbb{D}$ is (-1)-truncated because for any $f : X \to Z$ the hom set $(\mathbb{D}/Z)(f, m)$ is subterminal.

One of the things that a 2-topos is supposed to have is a classifying discrete opfibration (see [Web07]). The paradigmatic example is the classifier $\mathbf{Set}_{\bullet} \hookrightarrow \mathbf{Set}$ in **CAT** of discrete opfibrations with small fibres. For the duration of this section, we shall refer to fibrations with small fibres as "small fibrations" Interestingly, in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ there is a sort of generic proof for small fibrations: a generic small fibration (no modesty condition here). We exhibit this structure in case a higher exact completion follows a similar pattern to that involving generic proofs and subobject classifiers above. With that in mind, it may be that this would give rise in the exact completion to a classifying discrete (op)fibration. Without modesty, in order to define something like the equivalence M^* from the proof of Theorem 5.3.9 we need the small fibres restriction; and without modesty, the equivalence $Y \simeq P$ is demoted to a mere bi-implication.

Theorem 5.4.1 **PGAsm**$(\mathbb{C}, \mathbb{I}, U)$ *has a generic small fibration. This means that there is a small fibration* $\theta : \Theta \to \Lambda$ *such that for every small fibration* $F : Y \to X$ *there is a morphism* $\nu_F : X \to \Lambda$ *such that* F *and* $(\nu_F)^*\theta$ *are related by bi-implication, that is, there are functors* F_* *and* F^* *making the following diagram commute.*

$$\begin{tikzcd}
Y \arrow[r, bend left, "F_*"] \arrow[dr, bend right, "F"'] & P \arrow[l, bend left, "F^*"] \arrow[r] \arrow[d, "(\nu_F)^*\theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d, "\theta"] \\
 & X \arrow[r, "\nu_F"'] & \Lambda
\end{tikzcd}$$

Proof: We take

$$\Lambda := \nabla\left(\widehat{\Pi U}\right)$$

where now we use the notation $\widehat{\Pi U}$ to denote the category of covariant **Gpd**-valued presheaves on ΠU (ie. functors from ΠU to **Gpd**).

The underlying groupoid of Θ has as objects triples (F, a, m), where $F \in \widehat{\Pi U}$, $a \in \Pi U$ and $m \in Fa$. A morphism $(\psi, \alpha, \mu) : (F, a, m) \to (G, b, n)$ consists of a natural isomorphism $\psi : F \Rightarrow G$, a path $\alpha : a \to b$ and an isomorphism

$$\mu : \psi(\alpha, i)(m) \to n$$

The realizability functor is given by the second projection.

The fibration $\theta : \Theta \to \Lambda$ is given by the first projection. It is small because ΠU is small and presheaves in $\widehat{\Pi U}$ are valued in small groupoids. The chosen cartesian lift of $\psi : F \to G$ at (F, a, m) is

$$\left(\psi, \mathsf{id}_a, \mathsf{id}_{\psi_a(m)}\right) : (F, a, m) \to (G, a, \psi_a(m))$$

Let F be a small fibration. Define $\nu_F : X \to \Lambda$ as follows. The set of objects of the groupoid $\nu_F(x)(a)$ is

$$\{(y, \alpha) \mid Fy = x \wedge \alpha : a \to \|y\|_Y\}$$

A morphism $r : (y, \alpha) \to (y', \alpha')$ in $\nu_F(x)(a)$ is a morphism $r : y \to y'$ such that $F(r) = \mathsf{id}_x$ and the following triangle commutes.

$$\begin{tikzcd}
\|y\|_Y \arrow[rr, "\|r\|_Y"] & & \|y'\|_Y \\
& a \arrow[ul, "\alpha"] \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

The functor $\nu_F(x)(\beta : a \to b)$ is defined:

$$\nu_F(x)(\beta : a \to b)(y, \alpha) := \left(y, \alpha \circ \beta^{-1}\right)$$
$$\nu_F(x)(\beta : a \to b)(r : y \to y') := r$$

Finally, we define:

$$\nu_F(p : x \to x')(\beta, i)(y, \alpha) := \left(p^*(y), \|\overline{p}(y)\|_Y \circ \alpha \circ \beta^{-1}\right)$$
$$\nu_F(p : x \to x')(\beta, i)(r) := p^*(r)$$

This is well defined because $F(p^*(r)) = \mathsf{id}_{x'}$ and we have the commutative triangle

$$\begin{tikzcd}
\|y\|_Y \arrow[rr, "\|p^*(r)\|_Y"] & & \|y'\|_Y \\
& b \arrow[ul, "\|\overline{p}(y)\|_Y \circ \alpha \circ \beta^{-1}"] \arrow[ur, "\|\overline{p}(y')\|_Y \circ \alpha' \circ \beta^{-1}"'] &
\end{tikzcd}$$

as $\|p^*(r)\|_Y = \|\overline{p}(y') \circ r \circ (\overline{p}(y))^{-1}\|_Y$ by definition and r is a morphism in $\nu_F(x)(a)$.

The pullback $P := X \times_\Lambda \Theta$ has objects of the form $(x, a, (m_1, m_2))$ and morphisms of the form (p, α, r). To define $F_* : Y \to P$ we set:

$$F_*(y) := \left(Fy, \|y\|_Y, \left(y, \mathsf{id}_{\|y\|_Y}\right)\right)$$
$$F_*(q : y \to y') := \left(F(q), \|q\|_Y, q \circ \left(\overline{F(q)}(y)\right)^{-1}\right)$$

It is indeed the case that

$$\left(\nu_F(F(q)), \|q\|_Y, q \circ \left(\overline{F(q)}(y)\right)^{-1}\right) : \left(\nu_F(Fy), \|y\|_Y, \left(y, \mathsf{id}_{\|y\|_Y}\right)\right)$$
$$\to \left(\nu_F(Fy'), \|y'\|_Y, \left(y', \mathsf{id}_{\|y'\|_Y}\right)\right)$$

is a morphism in Θ, because

$$\nu_F\left(F(q)\right)\left(\|q\|_Y, i\right)\left(y, \mathsf{id}_{\|y\|_Y}\right) = \left(F(q)^*(y), \left\|\overline{F(q)}(y)\right\|_Y \circ \|q\|_Y^{-1}\right)$$

and:

$$\begin{aligned}
&q \circ \left(\overline{F(q)}(y)\right)^{-1} : F(q)^*(y) \to y' \\
&F\left(q \circ \left(\overline{F(q)}(y)\right)^{-1}\right) = F\left(q \circ \left(\overline{F\left(q^{-1}\right)}(y)\right)\right) = F\left(q \circ q^{-1}\right) = \mathsf{id}_{y'}
\end{aligned}$$

as well as

$$\begin{tikzcd}
\|y\|_Y \arrow[rr, "\left\|q \circ \left(\overline{F(q)}(y)\right)^{-1}\right\|_Y"] & & \|y'\|_Y \\
& b \arrow[ul, "\left\|\overline{F(q)}(y)\right\|_Y \circ \|q\|_Y^{-1}"] \arrow[ur, "\mathsf{id}_{\|y'\|_Y}"'] &
\end{tikzcd}$$

Clearly, $(\nu_F)^*\theta \circ F_* = F$. F_* is realized by

$$\left(s_{U\times U} \circ \langle e, \mathsf{id}\rangle, s_{U\times U} \circ \langle \epsilon, \mathsf{id}\rangle\right)$$

given a realizer $(e, \epsilon) \Vdash F$.

Finally we define $F^* : P \to Y$:

$$\begin{aligned}
F^*\left(x, a, (m_1, m_2)\right) &:= m_1 \\
F^*\left(p, \alpha, r\right) &:= r \circ \overline{p}(m_1) : m_1 \to p^*(m_1) \to m_1'
\end{aligned}$$

We have $F \circ F^* = (\nu_F)^*\theta$ because

$$F(r \circ \overline{p}(m_1)) = F(r) \circ F(\overline{p}(m_1) = \mathsf{id}_{p^*(m_1)} \circ p = p$$

F^* is realized by $(\pi_2 r_{U\times U}, \epsilon)$, where

$$\epsilon_{(x,a,(m_1,m_2))} := m_2 \qquad \square$$

5.5 A Grothendieck correspondence

We observed in Section 5.3 that, given a fibration $F : (X, A, \|-\|_X) \to (Y, B, \|-\|_Y)$ and a morphism $q : y \to y'$ in the base, the induced functor

$$q^* : X_y \to x_{y'}$$

between fibres is realized by $(\mathsf{id}_A, \epsilon)$, where

$$\epsilon_x := \|\overline{q}(x)\|_X$$

This is captured in the following diagram.

$$\begin{tikzcd}
X_y \arrow[rr, "q^*"] \arrow[dr, "\|-\|_{X_y}"'] & & X_{y'} \arrow[dl, "\|-\|_{X_{y'}}"] \\
& \Pi A \arrow[ur, Rightarrow, shorten=12mm, "\|\overline{q}(-)\|_X"] &
\end{tikzcd}$$

We now exhibit a Grothendieck correspondence between split fibrations in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ (fibrations in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ whose underlying functor is a split Grothendieck fibration) and *indexed* partitioned groupoidal assemblies.

Given a realizer category $(\mathbb{C}, \mathbb{I}, U)$, the category $\mathbf{GPD} \downarrow \Pi U$ has:

- Objects: those of of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$.
- Morphisms: a morphism

  $$(F, \epsilon) : (X, \|-\|_X) \to (Y, \|-\|_Y)$$

  consists of a functor $F : X \to Y$ and a natural isomorphism (the realizer):

  $$\begin{tikzcd}
  X \arrow[rr, "F"] \arrow[dr, "\|-\|_X"'] & & Y \arrow[dl, "\|-\|_Y"] \\
  & \Pi U \arrow[ur, Rightarrow, shorten=8mm, "\epsilon"] &
  \end{tikzcd}$$

The composition of $(F, \epsilon) : (X, \|-\|_X) \to (Y, \|-\|_Y)$ and $(G, \epsilon') : (Y, \|-\|_Y) \to (Z, \|-\|_Z)$ is given by:

$$\left(GF, \epsilon'_F \circ \epsilon\right)$$

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[rr, phantom, "\overset{\epsilon}{\Longrightarrow}"', shift right=6] \arrow[drr, "\|{-}\|_{X_y}"'] & & Y \arrow[rr, "G"] \arrow[rr, phantom, "\overset{\epsilon'}{\Longrightarrow}"', shift right=6] \arrow[d, "\|{-}\|_{X_y}"'] & & Z \arrow[dll, "\|{-}\|_{X_{y'}}"] \\
& & \Pi U & &
\end{tikzcd}$$

Identities are pairs of identities.

We emphasise that in this category, in contrast to $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$, functors carry around realizers as data (they are not just required to exist). The notation $\mathbf{GPD} \downarrow \Pi U$ is suggestive of the fact that this category is the underlying 1-category of the slice 2-category (more generally, comma 2-category) of the same name.

Definition 5.5.1 (Indexed partitioned groupoidal assembly) A partitioned groupoidal assembly indexed over $(X, \|-\|_X) \in \mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$, is a functor

$$\mathbb{X} : X \to \mathbf{GPD} \downarrow \Pi U \qquad \lozenge$$

We can now describe the aforementioned Grothendieck correspondence. Let $(X, \|-\|_X) \in \mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$. Explicitly, we will construct an indexed partitioned groupoidal assembly $\mathbb{Y} : X \to \mathbf{GPD} \downarrow \Pi U$ from a fibration in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ over X; and construct a fibration in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ over X from an indexed partitioned groupoidal assembly $\mathbb{Y} : X \to \mathbf{GPD} \downarrow \Pi U$.

Given a fibration $F : (Y, \|-\|_Y) \to X$ we define an indexed partitioned groupoidal assembly $\mathbb{Y} : X \to \mathbf{GPD} \downarrow \Pi U$ of partitioned groupoidal assemblies by:

$$\mathbb{Y}(x) := (Y_x, \|-\|_{Y_x})$$
$$\mathbb{Y}(p : x \to x') := (p^* : Y_x \to Y_{x'}, \|\overline{p}(-)\|_X)$$

where $\|-\|_{Y_x}$ is the following composite

$$Y_x \hookrightarrow Y \xrightarrow{\|-\|_Y} \Pi U$$

(see Section 5.3). This is functorial because

$$\overline{qp}(-) = \overline{q}\,(p^*(-)) \circ \overline{p}(-)$$

given that F is split.

Now suppose we are given an indexed partitioned groupoidal assembly

$$\mathbb{X} : X \to \mathbf{GPD} \downarrow \Pi U$$

We construct a fibration in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ over X as follows. The domain (or "total assembly") $X.\mathbb{X}$ of the fibration will have as its underlying groupoid the Grothendieck construction $X.\mathbb{X}$ of the indexed groupoid obtained by post-composing $\mathbb{X}$ with the underlying groupoid functor; that is:

- Objects: pairs $(x \in X, u \in \mathbb{X}(x))$.
- Morphisms: a morphism $(p, r) : (x, u) \to (x', u')$ is a pair consisting of a morphism $p : x \to x'$ in X and a morphism

  $$r : \mathbb{X}(p)(u) \to u'$$

  in $\mathbb{X}(x')$. Composition in this groupoid is given by

  $$(q, s) \circ (p, r) := (qp, s \circ \mathbb{X}(q)(r))$$

  Identities are pairs of identities.

The realizability functor is defined:

$$
\begin{aligned}
&\|-\|_{X.\mathbb{X}} : X.\mathbb{X} \to \Pi U \\
&\|(x,u)\|_{X.\mathbb{X}} \coloneqq s_{U\times U} \circ \left\langle \|x\|_X, \|u\|_{\mathbb{X}(x)} \right\rangle \\
&\|(p,r)\|_{X.\mathbb{X}} \coloneqq s_{U\times U} \circ \left\langle \|p\|_X, \|r\|_{\mathbb{X}(x')} \circ \epsilon_u^{\mathbb{X}(p)} \right\rangle
\end{aligned}
$$

where

$$\epsilon_u^{\mathbb{X}(p)} : \|u\|_{\mathbb{X}(x)} \to \|\mathbb{X}(p)(u)\|_{\mathbb{X}(x')}$$

is the component at u of the realizer (second component, natural isomorphism) of the morphism $\mathbb{X}(p)$ in **GPD** $\downarrow \Pi B$. To show that $\|-\|_{X.\mathbb{X}}$ is functorial, we first observe that the following diagram in ΠU commutes.

$$\begin{tikzcd}
 & & \|u''\|_{\mathbb{X}(x'')} \\
 & \|u'\|_{\mathbb{X}(x')} \arrow[ur, "\|s\|_{\mathbb{X}(x'')}\circ\epsilon_{u'}^{\mathbb{X}(q)}"] \arrow[r, "\epsilon_{u'}^{\mathbb{X}(q)}"] & \|\mathbb{X}(q)(u')\|_{\mathbb{X}(x'')} \arrow[u, "\|s\|_{\mathbb{X}(x'')}"'] \\
\|u\|_{\mathbb{X}(x)} \arrow[ur, "\|r\|_{\mathbb{X}(x')}\circ\epsilon_u^{\mathbb{X}(p)}"] \arrow[r, "\epsilon_u^{\mathbb{X}(p)}"'] \arrow[rr, bend right, "\epsilon_u^{\mathbb{X}(qp)}"'] & \|\mathbb{X}(p)(u)\|_{\mathbb{X}(x')} \arrow[u, "\|r\|_{\mathbb{X}(x')}"'] \arrow[r, "\epsilon_{\mathbb{X}(p)(u)}^{\mathbb{X}(q)}"'] & \begin{array}{c}\|\mathbb{X}(qp)(u)\|_{\mathbb{X}(x'')} \\ =\|\mathbb{X}(q)(\mathbb{X}(p)(u))\|_{\mathbb{X}(x'')}\end{array} \arrow[u, "\|\mathbb{X}(q)(r)\|_{\mathbb{X}(x'')}"']
\end{tikzcd}$$

Then, we calculate:

$$\begin{aligned}
&\|(q,s)\circ(p,r)\|_{X.\mathbb{X}}\\
=&\|(qp, s\circ\mathbb{X}(q)(r))\|_{X.\mathbb{X}}\\
=&s_{U\times U}\circ\left\langle \|qp\|_X, \|s\circ\mathbb{X}(q)(r)\|_{\mathbb{X}(x'')}\circ\epsilon_u^{\mathbb{X}(qp)}\right\rangle\\
=&s_{U\times U}\circ\left\langle [\|q\|_X,\|p\|_X]\circ 2, \|s\circ\mathbb{X}(q)(r)\|_{\mathbb{X}(x'')}\circ\epsilon_u^{\mathbb{X}(qp)}\right\rangle\\
=&s_{U\times U}\circ\left\langle [\|q\|_X,\|p\|_X]\circ 2, \left[\|s\|_{\mathbb{X}(x'')}\circ\epsilon_{u'}^{\mathbb{X}(q)}, \|r\|_{\mathbb{X}(x')}\circ\epsilon_u^{\mathbb{X}(p)}\right]\circ 2\right\rangle\\
=&s_{U\times U}\circ\left\langle [\|q\|_X,\|p\|_X], \left[\|s\|_{\mathbb{X}(x'')}\circ\epsilon_{u'}^{\mathbb{X}(q)}, \|r\|_{\mathbb{X}(x')}\circ\epsilon_u^{\mathbb{X}(p)}\right]\right\rangle\circ 2\\
=&s_{U\times U}\circ\left[\left\langle \|q\|_X, \|s\|_{\mathbb{X}(x'')}\circ\epsilon_{u'}^{\mathbb{X}(q)}\right\rangle, \left\langle \|p\|_X, \|r\|_{\mathbb{X}(x')}\circ\epsilon_u^{\mathbb{X}(p)}\right\rangle\right]\circ 2\\
=&\left[s_{U\times U}\circ\left\langle \|q\|_X, \|s\|_{\mathbb{X}(x'')}\circ\epsilon_{u'}^{\mathbb{X}(q)}\right\rangle, s_{U\times U}\circ\left\langle \|p\|_X, \|r\|_{\mathbb{X}(x')}\circ\epsilon_u^{\mathbb{X}(p)}\right\rangle\right]\circ 2\\
=&[\|(q,s)\|_{X.\mathbb{X}}, \|(p,r)\|_{X.\mathbb{X}}]\circ 2\\
=&\|(q,s)\|_{X.\mathbb{X}}\circ\|(p,r)\|_{X.\mathbb{X}}
\end{aligned}$$

where the fourth equality uses the commutativity of the preceeding diagram.

The fibration $X.\mathbb{X}\to X$, as we expect, is given by the first projection, which is realized by the first projection.

We could also work with a typed realizer category $(\mathbb{C},\mathbb{I})$, where we would then talk not about indexed partitioned groupoidal assemblies but *uniform families* of partitioned groupoidal assemblies. Uniform families are defined using the category $\mathbf{GPD}\downarrow\Pi$ that is given by:

- Objects: those of $\mathbf{PGAsm}(\mathbb{C},\mathbb{I})$.
- Morphisms: a morphism $(F,e,\epsilon):(X,A,\|-\|_X)\to(Y,B,\|-\|_Y)$ consists of a functor $F:X\to Y$, a morphism $e:A\to B$ and a natural isomorphism:

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "\|-\|_X"'] & Y \arrow[d, "\|-\|_Y"] \\
\Pi A \arrow[r, "\Pi(e)"'] \arrow[ur, Rightarrow, shorten=12pt, "\epsilon"'] & \Pi B
\end{tikzcd}$$

This category is like $\mathbf{PGAsm}(\mathbb{C},\mathbb{I})$ except that functors carry around realizers as extra data. Denote by $\mathbf{PGAsm}(\mathbb{C},\mathbb{I},1)$ the wide subcategory of $\mathbf{GPD}\downarrow\Pi$

with morphisms of the form $(F, \mathsf{id}, \epsilon)$. This implies that there can only be a morphism between assemblies with the same realizer type. A uniform family of partitioned groupoidal assemblies indexed by a partitioned groupoidal assembly $(X, A, \|-\|_X)$ is a functor

$$\mathbb{X} : X \to \mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, 1)$$

such that there is a uniform realizer type across all the $\mathbb{X}(x)$.

This analysis suggests that a split model of type theory—say a CwF—could be obtained using indexed partitioned groupoidal assemblies or uniform families of partitioned groupoidal assemblies. It seems that it would have been harder to come up with the notions of indexed partitioned groupoidal assembly and uniform family of partitioned groupoidal assemblies *directly*, rather than via fibrations of partitioned groupoidal assemblies. This is due to morphisms in the categories $\mathbf{GPD} \downarrow \Pi U$ and $\mathbf{GPD} \downarrow \Pi$ carrying realizers as extra data, which is not the case in categories of assemblies.

Chapter 6

Outlook

Contents

We hope that the development in this thesis is the beginning of an exciting programme of research on higher-dimensional categorical realizability, paralleling the extremely fruitful set-based programme, and provoking new questions in classical areas of theoretical computer science, such as domain theory and (models of) the λ-calculus.

6.1 A lesson

Before we discuss future work, we would like to draw out one part of the theory of groupoidal realizability that we think really makes the whole thing tick: that is that a realizer for a map of partitioned groupoidal assemblies consists of both an endomap e of U (in the untyped case) *and* a natural isomorphism ϵ.

Of course, the definition of morphism of partitioned groupoidal assemblies (5.1.2) is from a categorical perspective very natural: these morphisms are quotients of 1-cells from the comma 2-category $\mathbf{GPD} \downarrow \Pi$, analogous to the set-based case. This generalises Robinson-Rosolini's $\mathcal{F}$-construction [RR01].

But the lesson is also philosophically justified. We could think of e an implementing the underlying functor F up to isomorphism. Then Fx, which is implemented by $\|Fx\|_Y$, could just as well be implemented by $\Pi(e)\|x\|_X$, as witnessed by ϵ_x. But *together*, the pair (e, ϵ) tracks F accurately, ϵ being a uniform method (think BHK) to mediate between the actions of F and e on their respective levels. With this view, one might think of e and ϵ *together* as providing a realizer. Or one sticks to taking e to be the true implementation of F, with ϵ just a kind of "metatheoretic" assurance that the tracking holds up to "weak equality".

This higher-categorical notion of tracking is essential for a number of important constructions in this thesis. To pick out a few: the construction of the interval partitioned groupoidal assembly and the ensuing (2,1)-category of partitioned groupoidal assemblies (in Section 5.2), the construction of transport functors induced by morphisms in the base of a fibration (Section 5.3), and the construction of the impredicative universe (Theorem 5.3.9). If one were to require that functors are tracked on the nose, then it seems that one would end up with an impoverished category in terms of its structure (even taking an alternative definition of fibration, eg. where transport is tracked not by an identity in the first component).

6.2 Regular and exact completions of partitioned groupoidal assemblies

Perhaps the most pressing direction for future work is to investigate the regular and exact completions of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$. We have seen van den Berg and Moerdijk's **Hex** and **Ex** constructions, and we have also mentioned Shulman's regular and exact completion of 2-categories. We are not aware of any results pertaining to the preservation or improvement of features of the input category in the latter setting.

The following are questions of particular interest:

- Does the regular completion of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ contain an impredicative universe of 1-types?
- If so, does it contain an impredicative *and univalent* subuniverse of 0-types?
- What about propositional resizing?
- Is the exact completion of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ a realizability elementary (2,1)-topos?
- Does Church's thesis hold in the regular and exact completions?

The questions are of intrinsic interest; there has been a desire within the HoTT community to find realizability higher toposes (realizability models are a hallmark of constructive theories). One reason that we would like an affirmative answer to the first question is that the refined impredicative encodings of (higher) inductive types [AFS18] mentioned in Section 2.1 require function extensionality.

To be sure, we certainly want to consider higher-dimensional versions of regular and exact completions: a result due to Lumsdaine [Lum11] states that in any coherent (1-)category, any co-category is a co-equivalence relation, ie. it is a co-groupoid whose endpoint maps 0 and 1 are jointly epimorphic (from the point of view of fundamental groupoids, this means that any two parallel paths are equal). We escape this limitation in the higher setting: for example, the (2,1)-category **Gpd** is a (2,1)-topos but contains a co-groupoid that is not a co-equivalence relation.

6.3 Groupoidal realizability over the cubical untyped λ-calculus

The untyped λ-calculus is an important example of a PCA (see Example 2.4.1). Cubical type theory provides a computational interpretation of HoTT. Therefore, we believe that groupoidal realizability over the "cubical (untyped) λ-calculus" is an important case. The details are to be fully worked out, but the idea is that the cubical λ-calculus is what one obtains by removing types from a minimal cubical type theory. So there are two forms of judgement:

$$\Gamma \vdash t \; [\Psi]$$
$$\Gamma \vdash t = u \; [\Psi]$$

where Γ is a context of variables and Ψ is a context of dimension names. In Section 1.1.2, we discussed the cubical programming language of [AHW17]. While this comes with an operational semantics, the cubical λ-calculus comes with an equational theory. So, for instance, in the cubical λ-calculus, we may only infer a composition if the endpoints of the paths match in the usual way; however, one may write down "wild" compositions in the cubical programming language.

Scott [Sco80] showed how to construct a CCC with universal object from the untyped λ-calculus. One takes the idempotent completion (Karoubi envelope) of the monoid of λ-terms t satisfying

$$\lambda x.tx = t$$

(ie. η) where composition is given by

$$t \circ u := \lambda x.t(ux)$$

and the unit is $\lambda x.x$. A similar construction turns any PCA into a category; see [Bir99, Bir00b].

> Question: can one construct an untyped realizer category from the cubical untyped λ-calculus by some generalisation of Scott's method?

If the answer to the above question is "no", then is there some other structure (generalising untyped realizer categories) into which the cubical λ-calculus can be organised? [GJF19] presents a higher categorical generalisation of the Karoubi envelope, which may be helpful.

In lieu of an answer to the above questions, we have begun to investigate partitioned groupoidal assemblies formulated *directly* over the cubical λ-calculus (that is, instead of first organising it into a realizer category or some such structure). The idea is to replace ΠU, where $(\mathbb{C}, \mathbb{I}, U)$ is an untyped realizer category, with the following groupoid constructed from the syntax of the cubical λ-calculus (the "fundamental groupoid of the cubical untyped λ-calculus"):

- Objects: terms $\Gamma \vdash t\ [\cdot]$ in an empty dimension context.
- Morphisms: a morphism

  $$\alpha : (\Gamma \vdash t\ [\cdot]) \to (\Gamma \vdash u\ [\cdot])$$

  is an equivalence class of terms

  $$\Gamma \vdash \alpha\ [a]$$

  in singleton dimension context, such that:

  $$\Gamma \vdash \alpha[0/a] = t\ [\cdot]$$
  $$\Gamma \vdash \alpha[1/a] = u\ [\cdot]$$

  Two such morphisms $\Gamma \vdash \alpha\ [a]$ and $\Gamma \vdash \beta\ [a]$ are considered equivalent when:

  $$\Gamma \vdash \alpha[0/a] = \beta[0/a]\ [\cdot]$$
  $$\Gamma \vdash \alpha[1/a] = \beta[1/a]\ [\cdot]$$

(they are parallel), and there exists a term (square)

$$\Gamma \vdash H\ [a, b]$$

such that:

$$\begin{aligned}
&\Gamma \vdash H[0/b] = \alpha\ [a] \\
&\Gamma \vdash H[1/b] = \beta\ [a] \\
&\Gamma \vdash H[0/a] = \alpha[0/a] = \beta[0/a]\ [b] \\
&\Gamma \vdash H[1/a] = \alpha[1/a] = \beta[1/a]\ [b]
\end{aligned}$$

ie. $\Gamma \vdash H\ [a, b]$ is a homotopy rel. endpoints.

6.4 Other future work

- In the set-based case, does an impredicative universe of modest sets require untypedness (see Section 2.4.2)?
- Prove that $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is weakly locally pseudo cartesian closed (what would be a 2-categorical analogue of Theorem 5.3.5).
- Can the Grothendieck correspondence of Section 5.5 be elevated to an equivalence of categories?
- As another computational example of a realizer category, this time typed, we have begun to think about *groupoidal game semantics*. The idea is to have not just a set of moves, but a groupoid of moves and isomorphisms of moves. Strategies are then required to be functorial. This is a quite different approach to Yamada's game semantics for HoTT [Yam20]. We may hope to obtain an interval in a category of groupoidal games by taking the flat groupoidal games on the interval groupoid. However, basic categories of games often have only weak colimits plus commutative conversions (for example, eq. (3.1.2); see [AJ94]).

This raises the issue of doing groupoidal realizability over realizer categories with weak intervals, in the sense that the (strict) pushouts are replaced by weak pushouts plus commutative conversions.

It is—somewhat notoriously—tricky to construct games models of dependent type theory (see [Vák17, VJA18]), let alone *intensional* dependent type theory—this is despite the success of game semantics in the simply-typed setting (eg. as a solution to the full abstraction problem [AMJ94, HO00]). Realizability turns simply-typed structures into dependently-types ones; thus, doing realizability over games sidesteps the aforementioned difficulty.

- Find more examples of *untyped* realizer categories. A good place to start looking might be in the vicinity of the bicategory of *generalised species of structures* [FGHW08], which contains a higher-dimensional model of the untyped λ-calculus.
- See how much of the theory of groupoidal realizability can be developed using weakly cartesian closed realizer categories, bringing it more in line with [Bir00b].
- Are there would-be examples of realizer categories (with non-trivial intervals) with a notion of partiality?
- Is $\mathbf{PGAsm}(-)$ a functorial construction for a suitable notion of category of realizer categories (cf. [Lon95])?
- Is there a tripos-theoretic approach to higher-dimensional realizability?
- Applications (consistency, independence, conservativity, extraction of programs from proofs,...).
- Investigate weak ∞-groupoidal realizability. This should use a weaker notion of interval compared with that used in this thesis, in that the

co-groupoid axioms should be allowed to hold up to homotopy. An example should be $[0,1] \in \mathbf{Top}$—without any quotienting by homotopy.

- Can a realizer category $(\mathbb{C}, \mathbb{I})$ be equipped with the structure of a path category? Possibly, fibrations could be defined by the homotopy lifting property with respect to the interval $\mathbb{I}$, and equivalences as homotopy equivalences with respect to $\mathbb{I}$.

  If the above works, we may further ask if the fundamental groupoid functor is exact (preserves the terminal object, fibrations, equivalences and pullbacks along fibrations). If the answer to this is "yes", then the gluing construction of de Boer [dB18] (inspired by the gluing construction of Shulman [Shu15]) applies. This constructions bears some similarity with the construction of partitioned groupoidal assemblies. In the language of partitioned groupoidal assemblies, the gluing construction amounts to a situation where functors are tracked on the nose, realizers of functors are carried around as extra data, and realizability functors $\|-\|$ are fibrations. Path categories (or even less structured categories) may be good gadgets for doing realizability over.

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