

Unidirectional liquid-metal magnetohydrodynamic flow in ducts and trenches



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Contents

1	Introduction	2
2	Literature review	7
2.1	Early work in magnetohydrodynamics	7
2.2	Two-dimensional analytical models	8
2.3	Quasi-3D modelling	9
2.4	Numerical methods	10
2.5	Incorporating thermal effects	11
2.6	Thermoelectric magnetohydrodynamics	12
3	Classical MHD Duct Flow	14
3.1	Modelling principles	14
3.1.1	Governing equations	14
3.1.2	Boundary conditions	16
3.2	Isothermal 2D MHD duct flow	18
3.2.1	Modelling assumptions	18
3.2.2	Governing equations	19
3.2.3	Boundary conditions	20
3.2.4	Nondimensionalisation	20
3.3	Asymptotic analysis	22
3.3.1	Distinguished limits	22
3.3.2	The case $r = \mathcal{O}(1)$	23
3.3.3	The case $r = \mathcal{O}(\epsilon^{2/3})$	33
3.4	Numerical solutions	36
3.5	Discussion	40

4	2D Trench Flow Problem	42
4.1	Modified constitutive laws	42
4.2	The Seebeck effect	42
4.3	The Peltier and Thomson effects	44
4.4	Problem derivation	45
4.4.1	Trench geometry	46
4.4.2	Modelling assumptions	46
4.4.3	Governing equations	47
4.4.4	Boundary conditions	49
4.4.5	Nondimensionalisation	56
4.4.6	Dimensionless problem	57
4.5	Steady problem	60
4.6	Numerical solutions	60
4.7	Thin wall approximation	62
4.8	Asymptotic analysis	65
4.8.1	Outer problem in the trench	67
4.8.2	Outer problem above the trench	67
4.8.3	Inner problem near the right wall	68
4.8.4	Inner problem near the base	71
4.8.5	Inner problem in bottom-right corner	78
4.8.6	Inner problem inside bottom of bottom-right corner	79
4.9	Parameter sweeps	80
4.9.1	Changing the horizontal wall dimensions, τ and ω .	81
4.9.2	Changing the vertical dimensions, ς and H	83
4.9.3	Geometrical parameter space	85
4.9.4	Changing the film thickness f and the trench angle θ	89
4.9.5	Varying the conductivity ratios Σ and K	89
4.9.6	Varying the applied field $\mathcal{B}^a$	90
4.9.7	Changing ψ , the applied field inclination	94
4.10	Comparison with existing data	98
4.11	Discussion	98
5	Conclusions and further work	102

List of Figures

1.1	Illustration of the Liquid Metal-Infused Trenches (LiMIT) concept .	6
3.1	Diagram of rectangular domain in Shercliff 1953 MHD duct flow problem, complete with regions and boundary layers	24
3.2	Plots of typical solutions to the Shercliff 1953 MHD duct flow problem	37
3.3	Horizontal and vertical cross-sections of typical solutions to the Shercliff 1953 MHD duct flow problem, as the Hartmann number increases	38
3.4	Asymptotic and numerical average velocities versus aspect ratio of duct in Shercliff 1953 MHD duct flow problem	39
4.1	A diagram illustrating the Seebeck effect, adapted from Shercliff's paper [78].	43
4.2	Generalised version of the geometry in a LiMIT-type setup	45
4.3	Geometrical trench parameters	52
4.4	Steady TEMHD problem for a single trench	59
4.5	Numerical solutions of temperature, velocity and magnetic field in a single rectangular trench, for typical parameters including those in a tokamak and in experiment	61
4.6	Right wall of trench including boundary conditions	63
4.7	Thin-wall problem for temperature inside the right wall of a trench	64
4.8	Thin-wall TEMHD problem in a LiMIT trench	65
4.9	Boundary layers and regions in thin-wall TEMHD trench problem .	66
4.10	Comparison of asymptotic and numerical solutions to thin-wall TEMHD trench problem near the base, in first limiting case	76

4.11	Comparison of asymptotic and numerical solutions to thin-wall TEMHD trench problem near the base, in second limiting case	77
4.12	Numerical solutions to rectangular trench problem, showing horizontal and vertical cross-sections of velocity and B-field as the sidewall thickness and trench width change	82
4.13	Numerical solutions to rectangular trench problem, showing horizontal and vertical cross-sections of velocity and B-field as the base thickness and trench height change	84
4.14	Average velocities and free-surface temperatures against geometrical trench parameters	86
4.15	Vertical velocity cross-sections against film thickness and trench slope angle	88
4.16	Horizontal and vertical cross-sections of velocity and B-field as electrical conductivity ratio changes	91
4.17	Horizontal and vertical cross-sections of velocity, B-field and temperature as the thermal conductivity ratio changes	92
4.18	Horizontal and vertical cross-sections of velocity and magnetic field as the reciprocal Hartmann number and applied B-field change . . .	93
4.19	Velocity as the applied B-field angle changes	96
4.20	Magnetic field as the applied B-field angle changes	97
4.21	Comparison with Xu's numerical analysis of LiMIT	101

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Chapter 1

Introduction

Nuclear fusion is the process by which two nuclei (typically isotopes of hydrogen) combine into one, releasing a lot of energy which can be harvested and converted into electrical energy. Nuclear fusion takes place inside the core of the Sun, which is made up of a superheated gas consisting of delocalised ions, known as a plasma, under extremely high pressures and temperatures, which allows the gas to remain confined and produce energy. One of the challenges with trying to replicate nuclear fusion commercially is that the gravitational forces that arise on Earth are nowhere near as strong as those inside the core of the Sun, which means that we need a different way to confine the plasma. This is achieved in practice by using strong electromagnets which keep the ions trapped inside the confinement region. Another challenge is that nuclear fusion requires temperatures comparable to those of the Sun to be able to fuse together nucleons. Researching how such extreme temperatures can be achieved has been of great interest since the 1950s, because nuclear fusion is a clean and powerful source of renewable energy, requiring raw materials which are in abundance, not likely to run out any time soon and can be extracted without long-term environmental damage. These include deuterium, which is readily found in sea water, and tritium, which can be bred during a fusion reaction.

The term *tokamak* is derived from the Russian phrase meaning “toroidal chamber with magnetic field”. A tokamak is a type of reactor consisting of a toroidal vacuum chamber in which plasma is trapped by magnetic fields which in turn are generated by large electrical currents passing through coils. These coils are orien-

tated in both poloidal and toroidal directions; following the small circles and large circles around a torus, respectively. Almost a dozen tokamaks were constructed at the Kuratchov Institute between 1955 and 1965, compared to 226 tokamaks around the world as of 2018 [5]. Numerous experiments were run in which the confinement times (during which the plasma temperature exceeds its critical ignition temperature) were significantly increased, and exceeded those seen in turbulent plasmas in other types of fusion reactors by at least threefold [3]. Global interest in tokamak fusion reactors was sparked following a collaboration between the Kuratchov Institute and the Culham Centre for Fusion Energy (CCFE), and the “Atoms for Peace Conference” in 1958 signalled the declassification of global nuclear fusion research. A more detailed summary of the historical development of tokamaks can be found in [83] whilst a highly condensed review of principles and work underpinning the *International Thermonuclear Experimental Reactor* (ITER) can be found in [4].

One might ask why tokamaks have not yet been perfected after being in development for over 70 years. This is at least partly down to the immense difficulty in choosing a material from which to comprise the *plasma-facing components* (PFC) inside a tokamak. The most important PFC is the first wall (so-called because it is the “first wall” with which the plasma comes into contact inside the reactor). Other PFCs inside a tokamak include the limiter, a small component augmented to the first wall to absorb much of the damage it would usually suffer, and the divertor, an axisymmetric panel found at the top and bottom of a tokamak on which waste heat and particle matter are deposited. For all PFCs, many different material properties need to be considered; for example: strength, ductility, thermal shock and fatigue resistance, thermal conductivity, stability and activation energy required for a solid-plasma reaction to occur [64]. Unfortunately, when candidate materials are exposed to plasma directly, many undesirable physical phenomena typically occur. Perhaps the most important of these is known as *recycling* [55], which arises from delocalised metal ions being released from the solid PFC material due to bombardment from the plasma. This can result in metal impurities contaminating the plasma, causing it to cool down and deteriorate in performance.

In the early development of tokamaks, PFCs were typically made or reinforced using solid metals. The choice of metals for this purpose can be broadly categorised

by their atomic number (referred to as Z henceforth). Low- Z elements (e.g. titanium and chromium) have relatively few electrons in their atoms and thus do not sacrifice much energy from the plasma in order to have their atoms stripped of electrons. However, they typically have low melting points and are thus highly susceptible to thermal damage, such as surface melting or even evaporation, during the operation of the tokamak. High- Z elements (e.g. molybdenum and tungsten) appear to have more desirable physical properties, such as higher melting points and higher activation energies. However, this trade-off is outweighed by the fact that high- Z elements are particularly prone to inducing *bremsstrahlung* radiation losses within the plasma. Bremsstrahlung can be thought of as “braking radiation” and this occurs when a charged particle is deflected by a nucleus of an atom through which it passes, decelerating in the process and releasing energy in the form of an electromagnetic wave. Applied to a tokamak, it is ions from the plasma which experience bremsstrahlung radiation losses as they interact with atoms inside nearby solid metal components. Much less high- Z impurity is required in the plasma than low- Z impurity for the same radiative heat losses; for instance, only about 0.1% of the plasma needs to comprise molybdenum impurities versus about 10% of it for boron impurities, in order for 10% of its overall energy to be lost to bremsstrahlung radiation [103]. The effect of recycling can be exacerbated if the ions released from the PFC surface are then neutralised, which results in a range of problems as they are no longer tied to magnetic fields [24].

Liquid PFCs have attracted great interest over the last few decades of tokamak research, usually comprising a liquid metal coating on top of a solid metal surface. Liquid metals are more flexible for dealing with the high heat fluxes that PFCs usually encounter, through advection, conduction, convection and evaporation. Surface tension allows a fluid to self-heal, thus limiting the time during which solid PFC surfaces come into contact with the plasma. Liquid metal on a PFC can also be replenished much more easily than one in solid form, keeping maintenance costs down. Liquid metals also go some way to addressing the issue arising from high- Z elements; low- Z liquid metals are particularly desirable because they do not significantly contribute to bremsstrahlung radiation losses and because they have relatively low melting points compared to high- Z metals. In fact, being low- Z is one of the main requirements of candidate liquid metals for PFCs, along with high

boiling point to minimise evaporation, high thermal conductivity, high specific heat capacity and low activation energy [57].

Especially over the last two decades, liquid lithium has been widely recognised as an attractive material to use for this purpose. Its performance-boosting characteristics were first observed upon its injection inside a tokamak in 2001 [53]. Since then, liquid lithium limiters have been shown to improve tokamak performance in virtually all aspects [52]. However, other suitable elements have been identified: gallium, aluminium, tungsten and tin [17]. Regardless of the choice of metal, the use of any liquid PFC has its own drawbacks, arising partly from the types of instabilities that they experience, such as Rayleigh-Taylor and Kelvin-Helmholtz instabilities. Under such high heat fluxes as 10 MW/m^2 , a phenomenon known as *dryout* can also occur when parts of the liquid lithium film evaporate. Although previous studies have pointed out that lithium vapour can help to reduce the incident heat flux onto a tokamak divertor [61], dryout is highly undesirable because it can result in considerable damage to surfaces of PFCs that are directly exposed to the plasma.

Several concepts for liquid metal PFCs have been developed; however, this present work will be concerned with a concept known as *Liquid Metal Infused Trenches (LiMIT)* developed at the University of Illinois at Urbana-Champaign (UIUC) [23], and shown in Figure 1.1. As the name suggests, LiMIT consists of an array of trenches whose walls are made of solid metal, resembling a heatsink as seen in computer appliances, as well as an electron beam designed to provide heat fluxes comparable to that seen in tokamak plasmas [39]. The whole LiMIT setup is wetted with liquid lithium, which is allowed to flow through the trenches thus resembling open-surface MHD channel flow. However, rather than being driven by a pressure gradient, as is traditionally the case in duct and channel flow, the flow in the LiMIT concept is due to a phenomenon known as the *Seebeck* (or *thermoelectric*) effect. In the presence of a mutually perpendicular temperature gradient and magnetic field, the Seebeck effect introduces a Lorenz driving force for the fluid [78].

The LiMIT concept has been reconstructed and observed in several setups with a range of heat fluxes (denoted Q henceforth), such as the HT-7 tokamak [67]. Computational experiments were carried out in 2014 [109] using COMSOL.

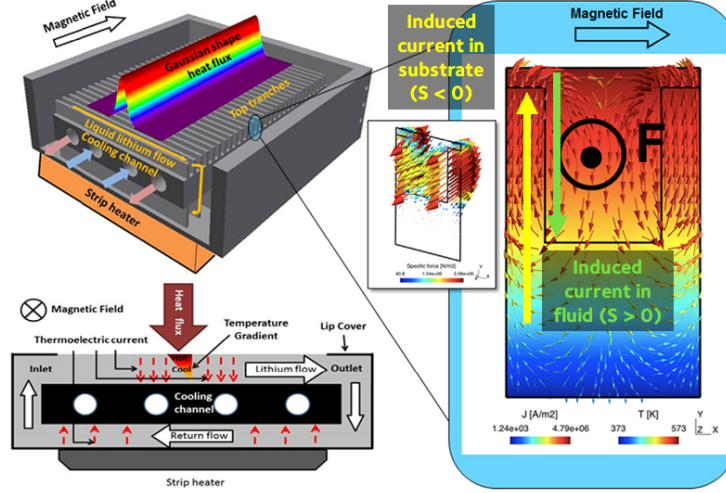


Figure 1.1: An illustration of the LiMIT concept, as shown in [23].

More details of liquid metal experiments inside fusion devices and a critical review can be found in [63] and the references therein.

This research project is in collaboration with Tokamak Energy, a nuclear fusion research company based in Didcot, Oxfordshire. They have built a prototype tokamak device known as the ST-40, which has successfully obtained temperatures of 100 million degrees Celsius. They are currently interested in analysing how liquid lithium can be used alongside a solid divertor plate in order to carry heat away from the plasma, thus allowing the device to run for longer durations.

In Chapter 2, we carry out a review of literature relevant to MHD duct flow and TEMHD duct flow. In Chapter 3, we carry out some analytical and numerical studies on classical 2D MHD duct flow. In Chapter 4, we derive a model of thermoelectrically driven MHD flow in a channel which includes all the essential physical properties, similar to the LiMIT concept from the University of Illinois at Urbana-Champaign (UIUC). We present an asymptotic analysis of the fluid flow, magnetic field and temperature in a rectangular channel where the walls are taken to be thin. We then compare the asymptotic predictions with solutions of the full model obtained through numerics. Finally, in Chapter 5, we will highlight some additional work that remains to be done.

Chapter 2

Literature review

2.1 Early work in magnetohydrodynamics

Although the field of fluid dynamics has existed for several centuries, interest in liquid metals only came about around World War II when liquid sodium was widely used; for example, as a valve coolant and for die-casting [51]. Liquid metals are notable because, although they are not intrinsically magnetic, they are electrically conducting liquids which carry charged particles, and thereby can impact upon magnetic fields. The term for the study of such fluids is *magnetohydrodynamics* (MHD). Mathematically speaking, MHD involves the coupling of the Navier-Stokes equations from fluid dynamics and Maxwell's equations from electromagnetism, both of which had been studied considerably in their own right but not reconciled properly until about a century ago. Early interest in MHD modelling emerged in 1937 with the work of Hartmann, whilst most of the seminal development of theoretical MHD duct flows, which dealt primarily with uniform external magnetic fields and ducts with constant cross-subsection, took place between 1956 and 1970 [36]. This theoretical modelling work, which often involved looking for asymptotic solutions for the flow rate, was frequently accompanied by experimental work involving flowmeters [74, 76]. Experimental work was the only way to validate theoretical modelling attempts, since computational methods were not yet widely implemented.

Shercliff [77] provided a brief critique of early work in electromagnetism and MHD up until 1965 (see references therein). Although it was shown to be possible

in principle to electromagnetically drive a liquid [62], MHD experiments were historically difficult to carry out due to limitations on the conducting liquids available at the time. Experimental progress did not improve notably until suitable liquid metals were made available through the emergence of metallurgic industries. Until then, the theoretical development of MHD was largely confined to astrophysical phenomena on such a large scale as to be impossible to verify experimentally.

MHD in a laboratory setting was demonstrated by Hartmann theoretically [27] and experimentally [28]. He showed that the flow of an electrically conducting liquid between two parallel plates can be affected by a constant applied magnetic field. He noted that electromagnetic effects flatten the velocity profile in the core, and that so-called *Hartmann layers* emerge at the edges, where the velocity rapidly drops in order to satisfy a no-slip condition at the walls. The *Hartmann number* is a dimensionless parameter given by $\text{Ha} = \mathcal{B}\mathcal{L}\sqrt{\sigma/\rho\nu}$, where $\mathcal{B}$ is the applied magnetic field, $\mathcal{L}$ is a characteristic length, σ is the electrical conductivity, ρ is the density and ν is the kinematic viscosity. This number is the square root of the ratio of electromagnetic terms to viscous terms in the Navier-Stokes equations. Much of the interest in MHD duct flow is ascertaining how the solutions to the MHD equations behave when Ha is large, and the velocity variations are effectively suppressed outside the Hartmann layers. In liquid metal applications relevant to nuclear fusion, the Hartmann number is typically in the range 10^2 to 10^4 .

2.2 Two-dimensional analytical models

Shercliff [85] later extended Hartmann's work into two dimensions by considering steady unidirectional flow with constant pressure gradient inside a rectangular duct whose walls are electrically insulating, in the presence of a constant applied magnetic field acting parallel to one pair of walls and perpendicular to the other. He demonstrated that Hartmann boundary layers of width $\mathcal{O}(\text{Ha}^{-1})$ exist near the walls perpendicular to the magnetic field, as well as side layers near the walls parallel to the field, of width $\mathcal{O}(\text{Ha}^{-1/2})$. This work prompted research into additional MHD aspects in traditionally established fluid mechanical phenomena [30], where it has also been shown that Hartmann layers naturally arise. Shercliff also considered other problems such as estimating the distance along a duct over which

the velocity profile develops after entry, for various cross-subsection shapes and for both conducting and non-conducting fluids [75]. There was much interest in obtaining asymptotic expansions for the average velocity inside a rectangular duct; for example, accurate to the first five terms (in increasing powers of Ha^{-1}) [105]. Magnetic fields have also been theoretically noted to have a stabilising influence on laminar MHD duct flow [49], and other types of layer, such as Ludford layers, have also been documented along the direction of the duct when the curvature of the wall changes significantly such that it no longer aligns with the magnetic field [34].

Electrical conductivity of the walls was since implemented in mathematical models of MHD duct flow; several limiting cases for a rectangular duct were considered by Hunt [33] and arbitrary conductivities were considered soon afterwards [15]. Chang and Lundgren [12] solved the governing equations inside both the duct and the conducting wall cross-subsection (which was assumed to be thin hitherto). Hunt [33] demonstrated that there can be large fluctuations in the velocity profile close to the conducting walls and that there can even be regions of flow reversal when Ha is large. It was later confirmed that the two limiting cases considered by Hunt are mathematically equivalent and that the thin wall restriction need not apply [10]. A comprehensive review of the work up to 1971 on MHD when the Hartmann number is large is found in [35] and [99], and MHD duct flow problems have also been tackled using variational approaches [7, 56, 93, 102]. Extremum principles were subsequently established, and used to derive asymptotic expansions for the optimal flow rate when the Hartmann number is large [85], to provide upper and lower bounds for the flow rate [86] and to deal with mixed boundary problems [87]. This series of works provides a rigorous foundation for the asymptotic expansions and boundary-layer arguments that were used by Shercliff up until then.

2.3 Quasi-3D modelling

Another dimensionless parameter that appears in the analysis of the MHD equations is called the *interaction parameter* $N = \text{Ha}^2/\text{Re}$, where Re is the Reynolds number. As well as the Hartmann number, N is also typically large in fusion

applications. If $N \gg 1$ then one can use the so-called *inertialess approximation* to make some progress with the asymptotics. This approach allows quasi-3D geometries to be considered, for example by Walker [99] who, considered MHD duct flow in ducts with parametrically-varying cross-subsections, for example, as the duct changes from a rectangle to a trapezoid [100]. Along with Hunt, he considered strong transverse magnetic fields acting on MHD flow in variable-area ducts. Walker demonstrated that there are “internal layers” in which the core solutions reconcile upstream and downstream of an expansion in the duct geometry. It has been shown experimentally and numerically that such expansions can alter the pressure drop even for moderately large values of N , even though the pressure drops should vanish as $N \rightarrow \infty$ [9, 22].

2.4 Numerical methods

Numerical methods for MHD duct flow problems began to emerge as early theoretical interest dwindled, and a range of geometries and physical phenomena have been successfully implemented; for example, in finite difference methods (FDMs) [16, 82, 84, 106], finite element methods (FEMs) [6, 32, 43, 71] including with penalty function formulations [98], boundary element methods (BEMs) [95] and more recently, exponential compact schemes [48]. Whilst numerical approaches have been helpful in lieu of experimental data, some setbacks have been highlighted. In particular, the classical MHD duct flow problem becomes singular as the Hartmann approaches infinity, which limits the range of Hartmann numbers that can be simulated and necessitates the use of a very fine mesh inside the boundary layers. Numerical convergence problems have also been demonstrated, particularly when the interaction parameter is large [65].

Some interesting phenomena have been uncovered using numerical methods. Classically documented behaviour such as Hartmann layers and velocity jets near conducting walls have been successfully resolved [89]. When the Reynolds number is not small, turbulent side layers can occur along with a laminar core [46], and triple-deck structures in the mean velocity profile arise [8]. Studies investigating the effects of a non-conducting channel insert have also been investigated [88].

However, simulating 2D MHD flow is of limited scope when designing intricate cooling systems inside fusion reactors [35, 66]. Numerical methods for MHD duct flow in three dimensions have been more prevalent in recent years [9,104] and have done an ample job of capturing detailed physics that would not be possible in two-dimensional setups. These include annular ducts [40], ducts with sudden contractions [44, 110], ducts with U-bends [29, 50, 101], ducts with channel inserts [88, 91], Other models that have been incorporated include power-law fluids [2] and forced turbulence [112]. MHD flows with obstacles inside ducts, including magnetic ones [19], have been studied, exposing phenomena such as cigar-shaped vortices [60]. Turbulence phenomena in low magnetic Reynolds number are reviewed in [45]. Commercial multiphysics packages such as COMSOL have been used successfully to solve certain MHD duct flow problems in 3D, including with heat effects [70].

2.5 Incorporating thermal effects

Due to its importance in the study of MHD generators, heat transfer in MHD duct flow has also been of high importance. Heat transfer is coupled with the MHD equations through viscous dissipation and Joule heating. Although these effects can be neglected in many problems, in MHD flow in the thermal entrance region of a flat duct, it was found that increasing the Hartmann number increases the temperature [13]. Consideration was also given to limiting values of the Nusselt number, highlighting discrepancies between electrically insulating ducts and MHD generators and accelerators [96]. It was also discovered that the heat transfer can be affected by other physical characteristics such as unequal wall conductances and wall heat fluxes [38]. The effects of MHD on heat transfer in a tokamak fusion reactor are discussed in [42].

Furthermore, properties which can comfortably be taken to be constant in isothermal problems, such as density and viscosity, are temperature-dependent in practice. In the present work, however, many physical constants relevant to the LiMIT-type setup are assumed to be constant, as they vary relatively weakly over the temperature ranges of interest. It is acknowledged that neglecting the variation of material properties over the temperature ranges of interest may result in significant deviations in heat transfer behaviour from what would actually be

observed; see [72] and references therein. The role of Hartmann and side layers in heat transfer has also been examined [111].

2.6 Thermoelectric magnetohydrodynamics

Thermoelectric magnetohydrodynamics (TEMHD) is the physical coupling of thermal effects with the magnetohydrodynamic problem, in which Ohm's law (relating the electric field to the current density) and Fourier's law (relating the heat flux to the electric current density) are both modified [78]. This means that it is possible for temperature gradients to create an electric field contribution, which in turn drives an electrical current. When combined with an applied magnetic field, this results in an additional Lorentz force contribution which can in turn provide a driving force which facilitates fluid flow. One manifestation of TEMHD effects is the so-called *Seebeck effect*, in which two electrical conductors sharing an interface can present an induced electrical current when a temperature gradient is applied along the interface. The LiMIT concept, as already discussed, serves as a good demonstration of this concept. The mechanism behind the Seebeck effect is a quantum phenomenon.

Although clearly there is a lot of literature on traditional MHD duct flow in different geometries, literature on TEMHD duct flow is much more sparse. Much of the work on TEMHD is within the context of dendritic growth and crystallisation (for example, see [41]) which is not relevant to the current work. Shercliff wrote a seminal paper demonstrating the key differences between MHD and TEMHD [78] and how the constitutive laws such as Ohm's law and Fourier's law are adjusted to take TEMHD effects into account. Shercliff also provided some studies on TEMHD within closed containers [79] and inside ducts with walls parallel to the magnetic field [80]. He also demonstrated that non-uniqueness can occur in TEMHD flows [81]. Since then, a significant amount of computational work involving the full physics has taken place, for example, in [92, 108, 109]. There have been several attempts by the engineers at UIUC to develop the work on free-surface TEMHD flow in channels, with view to mitigating dryout. For example, Szott [92] numerically investigated different trench setups as well as divertor geometries involving posts and established that the use of a bump at the bottom of the trench to displace

fluid away from the heat flux spike can go some way towards mitigating dryout. Divertor surface texturing has also been considered [69].

However, mathematically-focused work on TEMHD flow inside channels has been almost non-existent; in fact, Shercliff's early work is still referred to today by engineers [90], particularly those involved with LiMIT. Indeed, before the recent development of the LiMIT concept, nobody had attempted to use the impinging heat flux from the plasma to drive the flow in this way. To the best of our knowledge, there have been no attempts to systematically model free-surface TEMHD open-channel flow with a kinematic boundary condition at the free surface or find approximate solutions using asymptotic analysis, even in the two-dimensional case, even though some computations involving the full physics have taken place [108]. This is a gap in the TEMHD literature which the current work endeavours to address mathematically.

Chapter 3

Classical MHD Duct Flow

3.1 Modelling principles

3.1.1 Governing equations

We have *Maxwell's equations* in electromagnetism [26]:

$$\nabla \cdot (\varepsilon \mathbf{E}) = \varrho, \quad (3.1a) \quad \nabla \cdot \mathbf{B} = 0, \quad (3.1c)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3.1b) \quad \nabla \times \left(\frac{\mathbf{B}}{\mu} \right) = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \quad (3.1d)$$

where $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ is the electric field, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is the magnetic flux density, $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ is the current density, $\varrho = \varrho(\mathbf{x}, t)$ is the charge density, ε is the electric permittivity of the medium and μ is the magnetic permeability of the medium. The latter two quantities are material properties, and may be expressed equivalently as

$$\varepsilon = \varepsilon_0 (1 + \chi_e) \quad (3.2a) \quad \mu = \mu_0 (1 + \chi_m) \quad (3.2b)$$

where μ_0 and ε_0 are the values of μ and ε in a vacuum, χ_m is the magnetic susceptibility and χ_e is the electrical susceptibility, both of which are much less than unity in practice. We will assume ε and μ to be constant within each medium.

We also have *conservation of charge*,

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (3.3)$$

The (*incompressible*) *Navier-Stokes equations* in fluid mechanics, complete with the Lorenz force term, are given by [20]:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \rho \nu \nabla^2 \mathbf{u} + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}, \quad (3.4a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3.4b)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the fluid velocity, $p = p(\mathbf{x}, t)$ is the fluid pressure, $\mathbf{g}$ is the acceleration due to gravity, ρ is the fluid density and ν is the kinematic viscosity (both assumed constant).

The *heat equation*, complete with Joule heating and viscous dissipation terms, without any external heat sources, is given by

$$\rho c_v \left(\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T \right) = -\nabla \cdot \mathbf{q} + \mathbf{E} \cdot \mathbf{J} + \rho \nu (\nabla \otimes \mathbf{u}) : (\nabla \otimes \mathbf{u}), \quad (3.5)$$

where $T = T(\mathbf{x}, t)$ is the temperature, $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ is the conductive heat flux vector, while c_v and σ are the specific heat capacity and electrical conductivity of the medium. Although Joule heating is traditionally given by $|\mathbf{J}|^2/\sigma$ in many texts, this is only equivalent to $\mathbf{E} \cdot \mathbf{J}$ when $\mathbf{J} = \sigma \mathbf{E}$, which is the case for a conductor which is either stationary or not under the influence of a magnetic field absent. The form of the Joule heating term as shown in equation (3.5) is important when thermoelectric effects are considered, as in Chapter 4. Finally, the classical *Ohm's law* and *Fourier's law* are given by

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (3.6a) \quad \mathbf{q} = -k \nabla T, \quad (3.6b)$$

where σ is the electrical conductivity and k is the thermal conductivity.

An equivalent way of writing the momentum equation (3.4a) is

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \underline{\mathbf{T}} + \rho \mathbf{g} \quad \text{where} \quad \frac{D}{Dt} := \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \quad (3.7)$$

denotes the material derivative and $\underline{\mathbf{T}} = \underline{\boldsymbol{\tau}} + \underline{\mathbf{m}}$ is the total stress tensor, with the viscous stress tensor $\underline{\boldsymbol{\tau}}$ given by

$$\underline{\boldsymbol{\tau}} = -p \underline{\mathbf{I}} + \rho \nu \left(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^\top \right) \quad (3.8)$$

and the Maxwell stress tensor $\underline{\mathbf{m}}$ given by

$$\underline{\mathbf{m}} = \varepsilon \left(\mathbf{E} \otimes \mathbf{E} - \frac{1}{2} \mathbf{I} |\mathbf{E}|^2 \right) + \frac{1}{\mu} \left(\mathbf{B} \otimes \mathbf{B} - \frac{1}{2} \mathbf{I} |\mathbf{B}|^2 \right). \quad (3.9)$$

In typical liquid-metal MHD applications, velocities are sub-relativistic, that is,

$$\frac{\mathcal{L}}{c\mathcal{T}} \ll 1, \quad (3.10)$$

where $\mathcal{L}$ and $\mathcal{T}$ are characteristic length and time scales respectively, and $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the speed of light in a vacuum. As a result, several terms in the aforementioned governing equations can be safely neglected. In particular, the final term in equation (3.1d) is negligible, and the remaining relation $\mathbf{J} = \nabla \times \mathbf{B}/\mu$ can be used to eliminate $\mathbf{E}$ and $\mathbf{J}$ from the problem entirely. In doing so, we obtain a version of the *induction equation* which incorporates equations (3.1c) and (3.4b),

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B}, \quad (3.11)$$

where η is the magnetic diffusivity, defined by

$$\eta = \frac{1}{\mu\sigma}. \quad (3.12)$$

Similarly, the momentum equation (3.4a) may be written in the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \rho \nu \nabla^2 \mathbf{u} + \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g}. \quad (3.13)$$

3.1.2 Boundary conditions

The following boundary conditions are taken on the interface $\mathcal{S}$ between two media, each with labels $[+]$ and $[-]$. The unit normal vector $\hat{\mathbf{n}}$ is taken to point into the region $[+]$, and we use the notation $[F]_-^+ = F|_+ - F|_-$ to denote the jump in the quantity F across $\mathcal{S}$. The *electromagnetic jump conditions* are given by

$$\begin{aligned} [\varepsilon \mathbf{E} \cdot \hat{\mathbf{n}}]_-^+ &= \xi, & (3.14a) & \quad [\mathbf{B} \cdot \hat{\mathbf{n}}]_-^+ = 0, & (3.14c) \\ [\mathbf{E} \times \hat{\mathbf{n}}]_-^+ &= 0, & (3.14b) & \quad \left[\frac{1}{\mu} \mathbf{B} \times \hat{\mathbf{n}} \right]_-^+ = 0, & (3.14d) \end{aligned} \quad [\mathbf{J} \cdot \hat{\mathbf{n}}]_-^+ = 0, \quad (3.14e)$$

where $\xi = \xi(\mathbf{x}, t)$ is the surface charge density. The conditions (3.14a)–(3.14e)

apply at an interface between any two media, including when one of them is a vacuum.

At a fluid-fluid or fluid-solid interface, we impose the *no-slip boundary condition*,

$$[\mathbf{u}]_-^+ = 0. \quad (3.15)$$

At a moving interface, we also impose a *kinematic boundary condition*. Suppose that the interface between two fluids (or between a fluid and a vacuum) is given by the equation $F(\mathbf{x}, t) = 0$. Since the interface is moving with the fluid in time, the material derivative of F must also be zero, i.e.

$$\frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla) F = 0. \quad (3.16)$$

The *dynamic boundary condition* (i.e. stress balance) at a fluid-fluid or fluid-vacuum interface reads

$$[\underline{\mathbf{T}} \cdot \hat{\mathbf{n}}]_-^+ = \gamma \varkappa \hat{\mathbf{n}} - \nabla_S \gamma, \quad (3.17)$$

where $\varkappa$ is the surface curvature given by

$$\varkappa = \nabla \cdot \hat{\mathbf{n}}, \quad (3.18)$$

∇_S is the surface gradient operator, defined by

$$\nabla_S := \nabla - \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \nabla), \quad (3.19)$$

and γ is the surface tension of the interface.

More useful forms of the boundary condition (3.17) can be derived by taking dot and cross products with the vector $\hat{\mathbf{n}}$, that is,

$$[(\underline{\mathbf{T}} \cdot \hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}]_-^+ = \gamma \varkappa, \quad (3.20a) \quad [(\underline{\mathbf{T}} \cdot \hat{\mathbf{n}}) \times \hat{\mathbf{n}}]_-^+ = -\nabla_S \gamma \times \hat{\mathbf{n}}. \quad (3.20b)$$

Note that the surface tension γ is a material property, which generally depends on temperature T which itself varies with position and time. The right-hand side of equation (3.20b) shows how the resulting surface tension gradient in general generates a shear stress at the interface. Recall that the total stress $\mathbf{T}$ consists of the viscous stress $\boldsymbol{\tau}$ and the magnetic stress $\mathbf{m}$. In the sub-relativistic limit,

the terms in (3.9) involving the electric field $\mathbf{E}$ are negligible, and using tensor algebra, one finds that

$$(\underline{\mathbf{m}} \cdot \hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = \frac{1}{2\mu} ((\mathbf{B} \cdot \hat{\mathbf{n}})^2 - |\mathbf{B} \times \hat{\mathbf{n}}|^2), \quad (3.21a)$$

$$(\underline{\mathbf{m}} \cdot \hat{\mathbf{n}}) \times \hat{\mathbf{n}} = \frac{1}{\mu} (\mathbf{B} \times \hat{\mathbf{n}}) (\mathbf{B} \cdot \hat{\mathbf{n}}). \quad (3.21b)$$

The boundary conditions (3.14c) and (3.14d) can then be used to show that

$$[(\underline{\mathbf{m}} \cdot \hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}]_{-}^{+} = \frac{1}{2} \left(\left[\frac{1}{\mu} \right]_{-}^{+} (\mathbf{B} \cdot \hat{\mathbf{n}})^2 - [\mu]_{-}^{+} \left| \frac{\mathbf{B} \times \hat{\mathbf{n}}}{\mu} \right|^2 \right), \quad (3.22a)$$

$$[(\underline{\mathbf{m}} \cdot \hat{\mathbf{n}}) \times \hat{\mathbf{n}}]_{-}^{+} = 0, \quad (3.22b)$$

thus demonstrating that the magnetic contribution to the stress balance at $\mathcal{S}$ arises purely in the normal direction as a result of the discontinuity in the magnetic permeability μ across $\mathcal{S}$. To summarise, the dynamic boundary condition thus produces

$$[(\underline{\boldsymbol{\tau}} + \underline{\mathbf{m}}) \cdot \hat{\mathbf{n}}]_{-}^{+} = \gamma \boldsymbol{\kappa}, \quad (3.23a)$$

$$[(\underline{\boldsymbol{\tau}} \cdot \hat{\mathbf{n}}) \times \hat{\mathbf{n}}]_{-}^{+} = \hat{\mathbf{n}} \times \boldsymbol{\nabla} T \frac{d\gamma}{dT}. \quad (3.23b)$$

We assume that the temperature T is continuous across any interface $\mathcal{S}$ and enforce continuity in the normal component of heat flux $\mathbf{q}$, i.e.

$$[T]_{-}^{+} = 0, \quad (3.24a)$$

$$[\mathbf{q} \cdot \hat{\mathbf{n}}]_{-}^{+} = 0. \quad (3.24b)$$

We also impose a far-field condition, that the overall magnetic field approaches a specified applied magnetic field $\mathbf{B}^a$:

$$\mathbf{B}(\mathbf{x}, t) \rightarrow \mathbf{B}^a(\mathbf{x}, t) \quad \text{as } |\mathbf{x}| \rightarrow \infty. \quad (3.25)$$

3.2 Isothermal 2D MHD duct flow

3.2.1 Modelling assumptions

We develop the relevant mathematical tools by analysing the canonical MHD duct flow problem of Shercliff [73]. This consists of isothermal unidirectional flow inside

a single rectangular duct with electrically insulating walls. Shercliff obtained the velocity field w (u in his paper) using separation of variables and a well-educated guess of an exponential solution inside boundary layers near the duct walls. He used this solution to obtain an approximate asymptotic expansion for the net flow rate $\bar{w}$ along a rectangular duct. Although this problem is somewhat idealised, it serves as a conceptual foundation from which to address MHD duct flow problems involving more physics.

We consider steady unidirectional flow along a uniform rectangular duct occupying the region $(x, y, z) \in [-\mathfrak{a}, \mathfrak{a}] \times [-\mathfrak{b}, \mathfrak{b}] \times (-\infty, \infty)$. The temperature is constant and the material outside is taken to be electrically insulating. We neglect gravity and suppose that the fluid flow is driven by a constant pressure gradient

$$\frac{dp}{dz} = -k\rho\nu, \quad (3.26)$$

where k is a constant. We suppose that the velocity and magnetic field, respectively, have the forms

$$\mathbf{u}(\mathbf{x}, t) = w(x, y)\mathbf{k}, \quad (3.27a) \quad \mathbf{B}(\mathbf{x}, t) = \mathcal{B}^a\mathbf{i} + b(x, y)\mathbf{k}, \quad (3.27b)$$

where $\mathcal{B}^a$ is the constant applied magnetic field, and b is the induced field. For now, the applied field is assumed to act perpendicular to the duct axis and parallel to one pair of walls ($y = \pm\mathfrak{b}$).

3.2.2 Governing equations

The *magnetic Reynolds number* is defined by $\text{Rm} = \mathcal{W}\mathcal{L}/\eta$, where $\mathcal{W}$ and $\mathcal{L}$ are typical velocity and length scales, respectively. In liquid metal applications, Rm is almost inevitably small, in which case it is well-known that the induced magnetic field is much smaller than the applied field [68]. When terms of order Rm are neglected, the induction equation (3.11) and the Navier-Stokes equation (3.13) become, respectively

$$\eta \left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} \right) + \mathcal{B}^a \frac{\partial w}{\partial x} = 0 \quad \text{in } |x| < \mathfrak{a}, |y| < \mathfrak{b}, \quad (3.28a)$$

$$\rho\nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\mathcal{B}^a}{\mu} \frac{\partial b}{\partial x} + k\rho\nu = 0 \quad \text{in } |x| < \mathfrak{a}, |y| < \mathfrak{b}. \quad (3.28b)$$

3.2.3 Boundary conditions

The boundary conditions are no-slip conditions at the walls ($w = 0$) and that the walls are electrically insulating ($b = 0$). The latter boundary condition comes from considering the induced field outside the duct. Since the external medium is assumed to be non-conducting, Maxwell's equations (3.1) imply that $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mathbf{0}$. When the magnetic field takes the assumed form (3.27b), it follows that b is constant outside the duct and therefore, because of the imposed far-field condition, must be equal to zero. We thus deduce the condition $b = 0$ on the boundary of the duct from the tangential magnetic field condition (3.14d) (neglecting any jump in the magnetic permeability μ).

3.2.4 Nondimensionalisation

As described in Chapter 2, the *Hartmann number* is a dimensionless parameter that characterises the ratio of electromagnetic to viscous effects in the Navier-Stokes equation. In this instance, it is defined in terms of the area of the duct, $\mathfrak{A} = 4\mathfrak{a}\mathfrak{b}$, by setting

$$\text{Ha} = \mathcal{B}^{\mathfrak{a}} \sqrt{\mathfrak{a}\mathfrak{b}} \sqrt{\frac{\sigma}{\rho\nu}}, \quad (3.29\text{a}) \quad \epsilon = \frac{1}{\text{Ha}}. \quad (3.29\text{b})$$

This choice enables us to vary the *aspect ratio*

$$r = \frac{\mathfrak{b}}{\mathfrak{a}}, \quad (3.30)$$

which in this case is defined by the height of the rectangular duct divided by its width. A small value of r corresponds to a wide but shallow duct, whilst a large value of r corresponds to a tall but very thin duct. Note that in this context, the cross-sectional area $\mathfrak{A}$ is kept constant. The reason for explicitly defining the reciprocal of the Hartmann number is because Ha is typically large in fusion applications, so we will be interested in the asymptotic regime where $\epsilon \ll 1$.

We map the duct cross-section onto a square by setting

$$x = \mathfrak{a}\hat{x}, \quad (3.31\text{a}) \quad y = \mathfrak{b}\hat{y}. \quad (3.31\text{b})$$

The velocity and induced magnetic field scalings are chosen as follows:

$$w = \mathcal{U}\hat{w}, \quad (3.31c) \quad b = \epsilon \text{Rm} \mathcal{B}^a \hat{b}, \quad (3.31d)$$

where the velocity scale $\mathcal{U}$ is to be determined and, as above, $\text{Rm} = \mathcal{U}\sqrt{a\mathfrak{b}}/\eta$ is the magnetic Reynolds number. When ϵRm is small, the induced field is much smaller than the applied field, as anticipated in section 3.2.2. An appropriate velocity scale can be found by balancing the driving pressure gradient with the drag exerted on the flow in the Hartmann layers (see below), which produces

$$\mathcal{U} = \epsilon \mathbf{k} a \mathfrak{b} = \frac{\mathbf{k}}{\mathcal{B}^a} \sqrt{\frac{\rho \nu a \mathfrak{b}}{\sigma}}, \quad (3.32)$$

and the corresponding scale for the induced field is then given by

$$\epsilon \text{Rm} \mathcal{B}^a = \frac{\rho \nu \mu \mathbf{k} \sqrt{a \mathfrak{b}}}{\mathcal{B}^a}. \quad (3.33)$$

To summarise, the dimensionless problem to be solved is

$$\epsilon \left(r \frac{\partial^2 \hat{b}}{\partial \hat{x}^2} + \frac{1}{r} \frac{\partial^2 \hat{b}}{\partial \hat{y}^2} \right) + \sqrt{r} \frac{\partial \hat{w}}{\partial \hat{x}} = 0 \quad \text{in } |\hat{x}| < 1, \quad |\hat{y}| < 1, \quad (3.34a)$$

$$\epsilon \left(r \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{1}{r} \frac{\partial^2 \hat{w}}{\partial \hat{y}^2} \right) + \sqrt{r} \frac{\partial \hat{b}}{\partial \hat{x}} + 1 = 0 \quad \text{in } |\hat{x}| < 1, \quad |\hat{y}| < 1, \quad (3.34b)$$

$$\hat{w} = 0 \quad \text{on } \hat{x} = \pm 1, \quad \hat{y} = \pm 1, \quad (3.34c)$$

$$\hat{b} = 0 \quad \text{on } \hat{x} = \pm 1, \quad \hat{y} = \pm 1, \quad (3.34d)$$

where we recall that r is the aspect ratio of the rectangular duct cross-section. With the simple boundary conditions (3.34c)-(3.34d), it is convenient to introduce the symmetry variables

$$\hat{V}(\hat{x}, \hat{y}) = \hat{w}(\hat{x}, \hat{y}) + \hat{b}(\hat{x}, \hat{y}), \quad (3.35a) \quad \hat{W}(\hat{x}, \hat{y}) = \hat{w}(\hat{x}, \hat{y}) - \hat{b}(\hat{x}, \hat{y}), \quad (3.35b)$$

to decouple the governing equations (3.34a) and (3.34b) to become

$$\epsilon \left(r \frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{1}{r} \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} \right) + \sqrt{r} \frac{\partial \hat{V}}{\partial \hat{x}} + 1 = 0 \quad \text{in } |\hat{x}| < 1, \quad |\hat{y}| < 1, \quad (3.36a)$$

$$\epsilon \left(r \frac{\partial^2 \hat{W}}{\partial \hat{x}^2} + \frac{1}{r} \frac{\partial^2 \hat{W}}{\partial \hat{y}^2} \right) - \sqrt{r} \frac{\partial \hat{W}}{\partial \hat{x}} + 1 = 0 \quad \text{in } |\hat{x}| < 1, \quad |\hat{y}| < 1, \quad (3.36b)$$

$$\hat{V} = 0 \quad \text{on } \hat{x} = \pm 1, \quad \hat{y} = \pm 1, \quad (3.36c)$$

$$\hat{W} = 0 \quad \text{on } \hat{x} = \pm 1, \quad \hat{y} = \pm 1. \quad (3.36d)$$

Moreover, there is a symmetrical relationship between $\hat{V}$ and $\hat{W}$, that is,

$$\hat{V}(\hat{x}, \hat{y}) = \hat{W}(-\hat{x}, \hat{y}), \quad (3.37)$$

which in effect means that only $\hat{V}$ needs to be determined, by solving equation (3.36a). The dimensionless velocity $\hat{w}$ can then be obtained via

$$\hat{w}(\hat{x}, \hat{y}) = \frac{1}{2}(\hat{V}(\hat{x}, \hat{y}) + \hat{V}(-\hat{x}, \hat{y})). \quad (3.38)$$

We note from equations (3.36a)-(3.36b) that $\hat{V}$ and $\hat{W}$ are both even functions of $\hat{y}$, that is,

$$\hat{V}(\hat{x}, \hat{y}) = \hat{V}(\hat{x}, -\hat{y}), \quad \hat{W}(\hat{x}, \hat{y}) = \hat{W}(\hat{x}, -\hat{y}) \quad (3.39)$$

which imply the symmetry conditions

$$\frac{\partial \hat{V}}{\partial \hat{y}}(\hat{x}, 0) = \frac{\partial \hat{W}}{\partial \hat{y}}(\hat{x}, 0) = 0. \quad (3.40)$$

In principle, this circumvents the need to solve the problem on the entire duct.

In the following sections we obtain solutions to (3.36a) using asymptotic analysis, in the limit as $\epsilon \rightarrow 0$, and numerical methods. We are particularly interested in determining how the dimensionless mean velocity

$$\bar{w} = \frac{1}{2} \int_0^1 \int_{-1}^1 \hat{V}(\hat{x}, \hat{y}) d\hat{x} d\hat{y} \quad (3.41)$$

depends on the parameters ϵ and r . Below we will address the specific problem of determining the aspect ratio r that maximises the flow rate $\bar{w}$ for a given ϵ .

3.3 Asymptotic analysis

3.3.1 Distinguished limits

Equation (3.36a) has two distinguished limits in r of interest, which are found by looking at possible balances of terms in pairs, namely $r = \mathcal{O}(1)$ and $r = \mathcal{O}(\epsilon^{2/3})$ as $\epsilon \rightarrow 0$. In each case, equation (3.36a) is to be solved by using matched asymptotic expansions. The first attempt to apply this technique to rectangular MHD duct flow was by Temperley and Todd in 1971 [94], although they included other effects such as wall conductivity and did not consider varying the duct geometry.

3.3.2 The case $r = \mathcal{O}(1)$

3.3.2.1 Asymptotic regions and matching conditions

A complete diagrammatical summary of the regions, boundary layers, boundary conditions and matching conditions is provided in Figure 3.1. We define the *left variable* $\hat{X}_L$ as satisfying $\hat{x} = -1 + \epsilon \hat{X}_L$, with $\hat{V}(\hat{x}, \hat{y}) = \hat{V}_L(\hat{X}_L, \hat{y})$. The *top variable* $\hat{Y}_T$ satisfies $\hat{y} = 1 - \epsilon^{1/2} \hat{Y}_T$, with $\hat{V}(\hat{x}, \hat{y}) = \hat{V}_T(\hat{x}, \hat{Y}_T)$, and the “*inner top*” variable $\hat{Y}_{T_I}$ satisfies $\hat{Y}_T = \epsilon^{1/2} \hat{Y}_{T_I}$, with $\hat{V}(x, y) = \hat{V}_{T_{IL}}(X_L, Y_{T_I})$. We also define the *right variable* $\hat{X}_R$ as satisfying $\hat{x} = 1 - \epsilon \hat{X}_R$, along with the *top-right variable* $\hat{y} = 1 - \epsilon \hat{Y}_{TR}$, with $\hat{V}(\hat{x}, \hat{y}) = \hat{V}_{TR}(\hat{X}_R, \hat{Y}_{TR})$. In Figure 3.1, curved boundaries are used to reflect the nature of the matching conditions (3.42f)–(3.42h), which each take limits of their inner variables towards infinity.

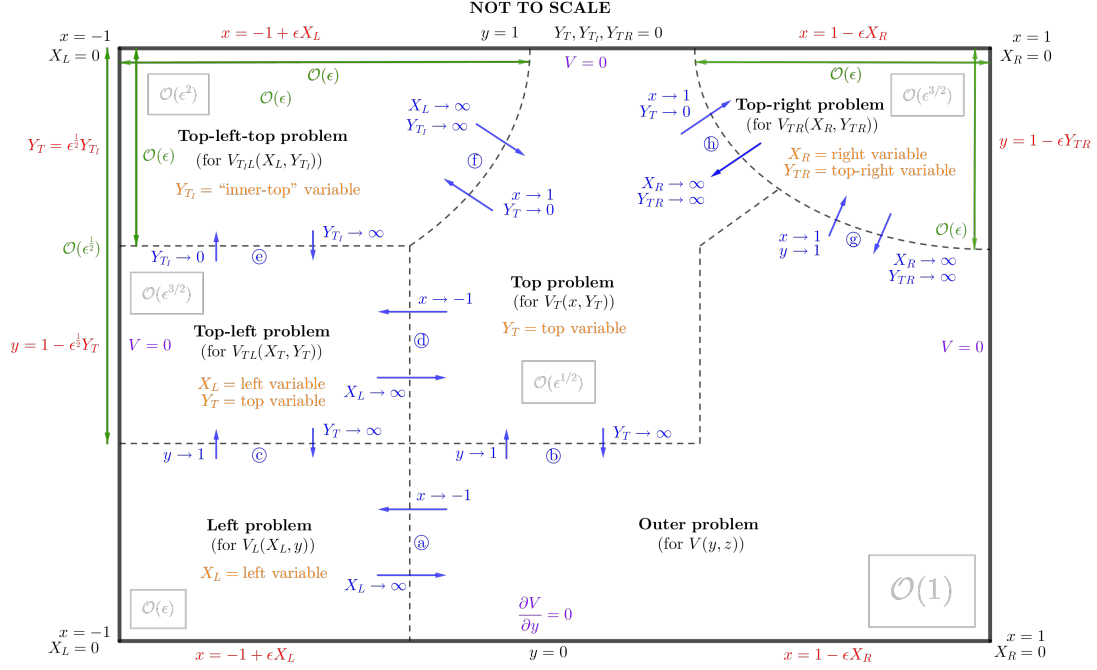


Figure 3.1: A diagrammatical representation of all regions and boundary layers for the Shercliff problem in the case $r = \mathcal{O}(1)$, including changes of variables and region names (black), inner variables (red), variable names (orange), boundary layer thicknesses (green), matchings (blue) and contributions that each region makes to the asymptotic expansion for the average velocity (3.43) (light grey, boxed).

The blue arrows crossing between the regions shown in Figure 3.1 are used as a basis for matching conditions, of which there are eight, namely (with the letters

corresponding to the circled labels in Figure 3.1).

$$\hat{V}(-1 + \epsilon \hat{X}_L, \hat{y}) \sim \hat{V}_L(\hat{X}_L, \hat{y}) \quad \text{as } \hat{X}_L \rightarrow \infty, \quad (3.42a)$$

$$\hat{V}(\hat{x}, 1 - \epsilon^{1/2} \hat{Y}_T) \sim \hat{V}_T(\hat{x}, \hat{Y}_T) \quad \text{as } \hat{Y}_T \rightarrow \infty, \quad (3.42b)$$

$$\hat{V}_L(\hat{X}_L, 1 - \epsilon^{1/2} \hat{Y}_T) \sim \hat{V}_{TL}(\hat{X}_L, \hat{Y}_T) \quad \text{as } \hat{Y}_T \rightarrow \infty, \quad (3.42c)$$

$$\hat{V}_T(-1 + \epsilon \hat{X}_L, \hat{Y}_T) \sim \hat{V}_{TL}(\hat{X}_L, \hat{Y}_T) \quad \text{as } \hat{X}_L \rightarrow \infty, \quad (3.42d)$$

$$\hat{V}_{TL}(\hat{X}_L, \epsilon \hat{Y}_{T_I}) \sim \hat{V}_{T_{IL}}(\hat{X}_L, \hat{Y}_{T_I}) \quad \text{as } \hat{Y}_{T_I} \rightarrow \infty, \quad (3.42e)$$

$$\hat{V}_T(-1 + \epsilon \hat{X}_L, \epsilon^{1/2} \hat{Y}_{T_I}) \sim \hat{V}_{T_{IL}}(\hat{X}_L, \hat{Y}_{T_I}) \quad \text{as } \hat{X}_L, \hat{Y}_{T_I} \rightarrow \infty, \quad (3.42f)$$

$$\hat{V}(1 - \epsilon \hat{X}_R, 1 - \epsilon \hat{Y}_{TR}) \sim \hat{V}_{TR}(\hat{X}_R, \hat{Y}_{TR}) \quad \text{as } \hat{X}_R, \hat{Y}_{TR} \rightarrow \infty, \quad (3.42g)$$

$$\hat{V}_T(1 - \epsilon \hat{X}_R, \epsilon \hat{Y}_{TR}) \sim \hat{V}_{TR}(\hat{X}_R, \hat{Y}_{TR}) \quad \text{as } \hat{X}_R, \hat{Y}_{TR} \rightarrow \infty. \quad (3.42h)$$

3.3.2.2 Flow-rate considerations

As stated above, we are particularly concerned with the asymptotic expansion of the (dimensionless) flow rate $\bar{w}$, given by (3.41). It is helpful to consider which regions in Figure 3.1 make the most important contributions to the flow rate, to avoid solving unnecessarily complex problems later on. By decomposing the integral (3.41) into different regions and splitting up the domain of integration, one can express the asymptotic expansion of $\bar{w}$ as

$$\bar{w} = \mathcal{I}_O + \epsilon^{1/2} \mathcal{I}_T + \epsilon \mathcal{I}_L + \epsilon^{3/2} \mathcal{I}_{TC} + \mathcal{O}(\epsilon^2) \quad (3.43)$$

where $\mathcal{I}_O$, $\mathcal{I}_L$ and $\mathcal{I}_{TC}$ are respectively the flow rate contributions from the core, the left boundary layer, and the corners (combined). To make sense of the boundary layers, we use the intermediate variables $\hat{\mathcal{X}}_L$, $\hat{\mathcal{X}}_R$ and $\hat{\mathcal{Y}}$, such that $\hat{\mathcal{X}}_L, \hat{\mathcal{X}}_R, \hat{\mathcal{Y}} \rightarrow 0$ and also $\hat{\mathcal{X}}_L/\epsilon, \hat{\mathcal{X}}_R/\epsilon, \hat{\mathcal{Y}}/\epsilon^{1/2} \rightarrow \infty$ as $\epsilon \rightarrow 0$. We have

$$\mathcal{I}_O = \frac{1}{2} \int_{-1+\hat{\mathcal{X}}_L}^1 \int_0^{1-\hat{\mathcal{Y}}} \hat{V}(\hat{x}, \hat{y}) d\hat{x} d\hat{y}, \quad (3.44)$$

$$\mathcal{I}_T = \frac{1}{2} \int_{-1+\hat{\mathcal{X}}_L}^{1-\hat{\mathcal{X}}_R} \int_{1-\hat{\mathcal{Y}}}^1 \hat{V}(\hat{x}, \hat{y}) d\hat{x} d\hat{y} = \frac{1}{2} \int_{-1+\hat{\mathcal{X}}_L}^{1-\hat{\mathcal{X}}_R} \int_0^{\hat{\mathcal{Y}}/\epsilon^{1/2}} \hat{V}_T(\hat{x}, \hat{Y}_T) d\hat{x} d\hat{Y}_T, \quad (3.45)$$

$$\mathcal{I}_L = \frac{1}{2} \int_{-1}^{-1+\hat{\mathcal{X}}_L} \int_0^{1-\hat{\mathcal{Y}}} \hat{V}(\hat{x}, \hat{y}) d\hat{x} d\hat{y} = \frac{1}{2} \int_0^{\hat{\mathcal{X}}_L/\epsilon} \int_0^{1-\hat{\mathcal{Y}}} \hat{V}_L(\hat{X}_L, \hat{y}) d\hat{X}_L d\hat{y}, \quad (3.46)$$

and

$$\mathcal{I}_{TC} = \frac{1}{2} \int_{1-\hat{y}}^1 \int_{-1}^{-1+\hat{x}_L} \hat{V}(\hat{x}, \hat{y}) d\hat{x} d\hat{y} + \frac{1}{2} \int_{1-\hat{y}}^1 \int_{1-\hat{x}_R}^1 \hat{V}(\hat{x}, \hat{y}) d\hat{x} d\hat{y}. \quad (3.47)$$

We will evaluate terms up to and including order ϵ when determining the asymptotic expansion (3.43), but no further. However, we will provide some insight into how solutions to the corner problems are obtained, should one like to compute terms of order $\epsilon^{3/2}$ and beyond.

3.3.2.3 Outer problem

The outer problem governs the dominant behaviour of the fluid inside the duct away from any boundary layers. The problem to be solved is equation (3.36a), and a regular expansion of the form

$$\hat{V}(\hat{x}, \hat{y}) \sim \hat{V}_0(\hat{x}, \hat{y}) + \epsilon \hat{V}_1(\hat{x}, \hat{y}) + \dots \quad (3.48)$$

can be used to solve the outer problem in its entirety. In doing so we note *a priori* that the solution for $\hat{V}$ cannot have a boundary layer at the wall $\hat{x} = 1$ (justification to follow shortly). It is hence possible to enforce the zero Dirichlet condition there directly, so $\hat{V}_0(1, \hat{y}) = 0$. Using this information leads to a leading-order outer solution of

$$\hat{V}_0(\hat{x}, \hat{y}) = \frac{1 - \hat{x}}{\sqrt{r}}. \quad (3.49)$$

Substituting this solution into equation (3.36a), along with the expansion (3.48), and comparing terms in order ϵ , one can deduce that $\hat{V}_1(\hat{x}, \hat{y}) = F_1(\hat{y})$ where $F_1(\hat{y})$ is an arbitrary function. The boundary condition $\hat{V}_1(1, \hat{y}) = 0$ implies that $F_1(\hat{y}) \equiv 0$ for all $\hat{y}$. This means that there is no first-order contribution to the outer solution of equation (3.36a) and one can also show that there are also no contributions at second order or higher, so that the leading-order solution (3.49) is exponentially accurate in the outer region.

3.3.2.4 Inner problem near left wall

To determine the inner solution at the left wall, we apply the change of variables $\hat{x} = -1 + \epsilon \hat{X}_L$. The ϵ scaling is a distinguished limit, and is consistent with the

well-known phenomenon of the Hartmann layer. After substituting in a regular expansion of the form

$$\hat{V}_L(\hat{X}_L, \hat{y}) \sim \hat{V}_{L,0}(\hat{X}_L, \hat{y}) + \epsilon \hat{V}_{L,1}(\hat{X}_L, \hat{y}) + \dots, \quad (3.50)$$

we obtain the leading-order equation

$$\sqrt{r} \frac{\partial^2 \hat{V}_{L,0}}{\partial \hat{X}_L^2} + \frac{\partial \hat{V}_{L,0}}{\partial \hat{X}_L} = 0. \quad (3.51)$$

Upon applying the transformed left-wall boundary condition and leading-order matching condition, namely

$$\hat{V}_{L,0}(0, \hat{y}) = 0, \quad (3.52)$$

$$\lim_{\hat{X}_L \rightarrow \infty} \hat{V}_{L,0}(\hat{X}_L, \hat{y}) = \lim_{\hat{x} \rightarrow -1} \hat{V}_0(\hat{x}, \hat{y}), \quad (3.53)$$

one obtains

$$\hat{V}_{L,0}(\hat{X}_L, \hat{y}) = \frac{2}{\sqrt{r}} \left(1 - e^{-\frac{\hat{X}_L}{\sqrt{r}}} \right) \quad (3.54)$$

as the solution. Using this expression, and using Van Dyke's matching principle, one can obtain an approximate composite expression for $\hat{V}$ at $\hat{y} = 0$:

$$\hat{V}(\hat{x}, 0) \approx \frac{1 - \hat{x}}{\sqrt{r}} - \frac{2}{\sqrt{r}} e^{-(1-\hat{x})/\epsilon}. \quad (3.55)$$

It is straightforward to confirm that there can be no boundary layer at the right wall $\hat{x} = 1$ by trying to repeat this exercise but by letting $\hat{x} = 1 - \delta \hat{X}_R$. It is impossible to satisfy the equivalent of the matching condition (3.53) in this case because the solution to the inner-right problem would contain a term which becomes exponentially large as $\hat{X}_R \rightarrow \infty$.

3.3.2.5 Inner problem near top wall

To examine the boundary layer at $\hat{y} = 1$, we transform equation (3.36a) by performing the change of variables $\hat{y} = 1 - \epsilon^{1/2} \hat{Y}_T$. The side-layer width $\epsilon^{1/2}$ is a distinguished limit and consistent with that seen in the literature [33]. By seeking a regular asymptotic expansion of the form

$$\hat{V}_T(\hat{x}, \hat{Y}_T) \sim \hat{V}_{T,0}(\hat{x}, \hat{Y}_T) + \epsilon \hat{V}_{T,1}(\hat{x}, \hat{Y}_T) + \dots, \quad (3.56)$$

and substituting into the transformed version of equation (3.36a), one obtains leading-order problem

$$\frac{1}{r} \frac{\partial^2 \hat{V}_{T,0}}{\partial \hat{Y}_T^2} + \sqrt{r} \frac{\partial \hat{V}_{T,0}}{\partial \hat{x}} + 1 = 0, \quad (3.57)$$

which is subject to Dirichlet and matching conditions

$$\hat{V}_{T,0}(\hat{x}, 0) = 0, \quad \lim_{\hat{Y}_T \rightarrow \infty} \hat{V}_{T,0}(\hat{x}, \hat{Y}_T) = \lim_{\hat{y} \rightarrow 1} \hat{V}_0(\hat{x}, \hat{y}). \quad (3.58)$$

This problem (3.57)–(3.58) can be solved using similarity solutions. Owing to the form of the outer solution (3.49), one seeks a solution to equation (3.57) of the form

$$\hat{V}_{T,0}(\hat{x}, \hat{Y}_T) = \left(\frac{1 - \hat{x}}{\sqrt{r}} \right) f(v) \quad \text{where} \quad v = \frac{r^{3/4} \hat{Y}_T}{\sqrt{1 - \hat{x}}}. \quad (3.59)$$

Thus one obtains a non-autonomous ODE boundary-value problem in $f(v)$:

$$f'' + \frac{v}{2} f' + 1 - f = 0, \quad f(0) = 0, \quad f(\infty) = 1, \quad (3.60)$$

which has solution

$$f(v) = 1 - \left(\frac{1}{2} v^2 + 1 \right) \operatorname{erfc} \left(\frac{v}{2} \right) + \frac{v}{\sqrt{\pi}} e^{-\frac{1}{4} v^2}. \quad (3.61)$$

The leading-order solution to the top problem is then given by (3.59). Similarly to before, one can obtain an approximate composite expansion of $\hat{V}$ along the line $\hat{x} = 0$:

$$\hat{V}(0, \hat{y}) \approx \frac{1}{\sqrt{r}} \left(f \left(\frac{r^{3/4}(1 - \hat{y})}{\sqrt{\epsilon}} \right) + f \left(\frac{r^{3/4}(1 + \hat{y})}{\sqrt{\epsilon}} \right) - 1 \right). \quad (3.62)$$

3.3.2.6 Inner problem near top-left corner

As noted by Temperley and Todd, corner regions arise where the Hartmann and side layers overlap. Corner solutions can be examined by rescaling both variables, so that $\hat{y} = 1 - \epsilon^{1/2} \hat{Y}_T$ and $\hat{x} = -1 + \epsilon \hat{X}_L$. We then let $\hat{V}(\hat{x}, \hat{y}) = \hat{V}_{TL}(\hat{X}_L, \hat{Y}_T)$ where TL stands for “top-left”. Applying this change of variables to equation (3.36a) leads to the overall problem

$$r^2 \frac{\partial^2 \hat{V}_{TL}}{\partial \hat{X}_L^2} + \epsilon \frac{\partial^2 \hat{V}_{TL}}{\partial \hat{Y}_T^2} + r^{3/2} \frac{\partial \hat{V}_{TL}}{\partial \hat{X}_L} + \epsilon r = 0, \quad (3.63)$$

which is subject to a Dirichlet boundary condition $\hat{V}_{\text{TL}}(0, \hat{Y}_{\text{T}}) = 0$, as well as matching conditions. By supposing a regular asymptotic expansion of the form

$$\hat{V}_{\text{TL}} \sim \hat{V}_{\text{TL},0} + \epsilon \hat{V}_{\text{TL},1} + \dots \quad (3.64)$$

as a solution to equation (3.63), one obtains the leading-order problem

$$\sqrt{r} \frac{\partial^2 \hat{V}_{\text{TL},0}}{\partial \hat{X}_{\text{L}}^2} + \frac{\partial \hat{V}_{\text{TL},0}}{\partial \hat{X}_{\text{L}}} = 0, \quad \hat{V}_{\text{TL},0}(0, \hat{Y}_{\text{T}}) = 0. \quad (3.65)$$

The general solution of equation (3.65) is

$$\hat{V}_{\text{TL},0}(\hat{X}_{\text{L}}, \hat{Y}_{\text{T}}) = F_{\text{TL},0}(\hat{Y}_{\text{T}}) \left(1 - e^{-\frac{\hat{X}_{\text{L}}}{\sqrt{r}}} \right), \quad (3.66)$$

where $F_{\text{TL},0}(Y_{\text{T}})$ is to be determined by the leading-order equivalent of matching condition (3.42d):

$$\lim_{\hat{X}_{\text{L}} \rightarrow \infty} \hat{V}_{\text{TL},0}(\hat{X}_{\text{L}}, \hat{Y}_{\text{T}}) = \lim_{\hat{x} \rightarrow -1} \hat{V}_{\text{T},0}(\hat{x}, \hat{Y}_{\text{T}}). \quad (3.67)$$

This procedure gives

$$\hat{V}_{\text{TL},0}(\hat{X}_{\text{L}}, \hat{Y}_{\text{T}}) = \frac{2}{\sqrt{r}} f \left(\frac{r^{3/4} \hat{Y}_{\text{T}}}{\sqrt{2}} \right) \left(1 - e^{-\frac{\hat{X}_{\text{L}}}{\sqrt{r}}} \right), \quad (3.68)$$

where f is again the function defined by equation (3.61).

3.3.2.7 “Inner-top”-left corner problem

We note that the $\hat{Y}_{\text{T}}$ derivative is negligible at leading-order in equation (3.65), and it is therefore not possible to impose the Dirichlet boundary condition $\hat{V}_{\text{TL}}(\hat{X}_{\text{L}}) = 0$. Although the leading-order solution (3.68) satisfies this condition fortuitously, there may still be an “inner-inner” region where $\hat{Y}_{\text{T}}$ is small such that $\hat{Y}_{\text{T}}$ derivatives regain their importance. This corner region including higher-order corrections was analysed by Cook et. al. [18]. The rescaling

$$\hat{Y}_{\text{T}} = \epsilon^{1/2} \hat{Y}_{\text{T}_1}, \quad \hat{V}(\hat{x}, \hat{y}) = \epsilon^{1/2} \hat{V}_{\text{T}_1\text{L}}(\hat{X}_{\text{L}}, \hat{Y}_{\text{T}_1}), \quad (3.69)$$

results in the leading-order problem

$$r^2 \frac{\partial^2 \hat{V}_{\text{T}_1\text{L},0}}{\partial \hat{X}_{\text{L}}^2} + \frac{\partial^2 \hat{V}_{\text{T}_1\text{L},0}}{\partial \hat{Y}_{\text{T}_1}^2} + r^{3/2} \frac{\partial \hat{V}_{\text{T}_1\text{L},0}}{\partial \hat{X}_{\text{L}}} = 0, \quad \hat{V}_{\text{T}_1\text{L},0}(\hat{X}_{\text{L}}, 0) = \hat{V}_{\text{T}_1\text{L},0}(0, \hat{Y}_{\text{T}_1}) = 0, \quad (3.70)$$

along with the matching conditions

$$\hat{V}_{\text{TIL},0}(\hat{X}_L, \hat{Y}_{\text{TI}}) \sim 2\hat{Y}_{\text{TI}} \left(1 - e^{-\frac{\hat{X}_L}{\sqrt{r}}}\right) \sqrt{\frac{2}{\pi}} r^{1/4} \text{ as } \hat{Y}_{\text{TI}} \rightarrow \infty, \quad (3.71)$$

and

$$\hat{V}_{\text{TIL},0}(\hat{X}_L, \hat{Y}_{\text{TI}}) \sim 2\hat{Y}_{\text{TI}} r^{1/4} \sqrt{\frac{2}{\pi}} \text{ as } \hat{X}_L, \hat{Y}_{\text{TI}} \rightarrow \infty. \quad (3.72)$$

We notice that the whole problem in this region is satisfied by the far-field behaviour (3.71). Moreover, it may be shown that this solution is unique; the linearity of this problem permits the solution $\hat{V}_{\text{TIL},0}$ to be expressed as the sum of a complementary function $\hat{V}_{\text{TIL},0}^{[C]}$ (which satisfies a homogeneous far-field condition) and a particular integral $\hat{V}_{\text{TIL},0}^{[P]}$. The problem for $\hat{V}_{\text{TIL},0}^{[C]}$ can then be solved using transform methods, and one can show that it cannot satisfy a zero far-field condition unless it is identically zero. The leading-order solution for $\hat{V}_{\text{TIL}}$ is therefore simply given by

$$\hat{V}_{\text{TIL},0}(\hat{X}_L, \hat{Y}_{\text{TI}}) = \hat{W}_{\text{TIL},0}^{[P]}(\hat{X}_L, \hat{Y}_{\text{TI}}) = 2\hat{Y}_{\text{TI}} r^{1/4} \sqrt{\frac{2}{\pi}} \left(1 - e^{-\frac{\hat{X}_L}{\sqrt{r}}}\right). \quad (3.73)$$

3.3.2.8 Inner problem near top-right corner

The top-right corner problem is much more interesting than the inner top-left problem, and is included here for completeness. One can show that there is no analogue of the top-left “outer” solution on the right-hand side. By applying the change of variables $\hat{y} = 1 - \epsilon \hat{Y}_{\text{TR}}/\sqrt{r}$, $x = 1 - \epsilon \sqrt{r} \hat{X}_R$, and $\hat{V}(\hat{x}, \hat{y}) = \epsilon \hat{W}_{\text{TR}}(\hat{X}_R, \hat{Y}_{\text{TR}})$ to equation (3.36a), one obtains the full problem:

$$\frac{\partial^2 \hat{W}_{\text{TR}}}{\partial \hat{X}_R^2} + \frac{\partial^2 \hat{W}_{\text{TR}}}{\partial \hat{Y}_{\text{TR}}^2} - \frac{\partial \hat{W}_{\text{TR}}}{\partial \hat{X}_R} + 1 = 0, \quad \hat{W}_{\text{TR}}(0, \hat{Y}_{\text{TR}}) = 0, \quad \hat{W}_{\text{TR}}(\hat{X}_R, 0) = 0, \quad (3.74)$$

as well as matching conditions. By evaluating the leading-order top solution (3.57) in terms of the top-right variables, one obtains

$$\hat{V}_{\text{T},0} \left(1 - \epsilon \sqrt{r} \hat{X}_R, \epsilon^{1/2} \frac{\hat{Y}_{\text{TR}}}{\sqrt{r}}\right) = \epsilon \hat{X}_R f \left(\frac{\hat{Y}_{\text{TR}}}{\sqrt{\hat{X}_R}}\right), \quad (3.75)$$

with $f(v)$ defined as in (3.61). The prefactor of ϵ on the right-hand side of equation (3.75) is what necessitates the factor of ϵ in the above rescaling of $\hat{V}$. We deduce the leading-order matching condition

$$\hat{W}_{TR,0}(\hat{X}_R, \hat{Y}_{TR}) \sim \hat{X}_R f\left(\frac{\hat{Y}_{TR}}{\sqrt{\hat{X}_R}}\right) \text{ as } \hat{X}_R, \hat{Y}_{TR} \rightarrow \infty. \quad (3.76)$$

From equation (3.61), we deduce the asymptotic behaviour of $f(v)$ as $v \rightarrow 0$ and $v \rightarrow \infty$ respectively:

$$f(v) \sim \frac{2v}{\sqrt{\pi}} + \mathcal{O}(v^2) \text{ as } v \rightarrow 0, \quad f(v) \sim 1 \text{ as } v \rightarrow \infty, \quad (3.77)$$

The matching condition (3.76) therefore implies that

$$\hat{W}_{TR}(\hat{X}_R, \hat{Y}_{TR}) \sim \hat{X}_R \quad \text{as } \hat{Y}_{TR} \rightarrow \infty \text{ with } \hat{X}_R \text{ fixed}, \quad (3.78a)$$

$$\hat{W}_{TR}(\hat{X}_R, \hat{Y}_{TR}) \sim \frac{2}{\sqrt{\pi}} \hat{Y}_{TR} \sqrt{\hat{X}_R} \quad \text{as } \hat{X}_R \rightarrow \infty \text{ with } \hat{Y}_{TR} \text{ fixed}. \quad (3.78b)$$

In order to account for far-field behaviour (3.78), we make the substitution

$$\hat{W}_{TR}(\hat{X}_R, \hat{Y}_{TR}) = \hat{X}_R - \hat{\phi}(\hat{Y}_{TR}, \hat{X}_R) \quad (3.79)$$

and this turns the problem into

$$\frac{\partial^2 \hat{\phi}}{\partial \hat{X}_R^2} + \frac{\partial^2 \hat{\phi}}{\partial \hat{Y}_{TR}^2} - \frac{\partial \hat{\phi}}{\partial \hat{X}_R} = 0, \quad \hat{\phi}(0, \hat{Y}_{TR}) = 0, \quad \hat{\phi}(\hat{X}_R, 0) = \hat{X}_R, \quad (3.80a)$$

$$\hat{\phi}(\hat{X}_R, \hat{Y}_{TR}) \sim \hat{X}_R \left(1 - f\left(\frac{\hat{Y}_{TR}}{\sqrt{\hat{X}_R}}\right)\right) \text{ as } \hat{Y}_{TR}, \hat{X}_R \rightarrow \infty. \quad (3.80b)$$

The problem (3.80) can be solved via the *Fourier sine transform* (with respect to $\hat{Y}_{TR}$)

$$\tilde{\phi}(X_R, \omega) = \int_0^\infty \hat{\phi}(\hat{X}_R, \hat{Y}_{TR}) \sin(\omega \hat{Y}_{TR}) d\hat{Y}_{TR} \quad (3.81)$$

which transforms the problem into

$$\frac{\partial^2 \tilde{\phi}}{\partial \hat{X}_R^2} - \frac{\partial \tilde{\phi}}{\partial \hat{X}_R} - \omega^2 \tilde{\phi} = -\omega \hat{X}_R, \quad \tilde{\phi}(0, \omega) = 0, \quad \tilde{\phi}(\hat{X}_R, \omega) \sim \frac{\hat{X}_R}{\omega} \text{ as } \hat{X}_R \rightarrow \infty. \quad (3.82)$$

We thus obtain

$$\hat{\phi}(\hat{X}_R, \omega) = \frac{\hat{X}_R}{\omega} - \frac{1}{\omega^3} \left(1 - e^{-\frac{1}{2}\hat{X}_R(\sqrt{1+4\omega^2}-1)} \right) \quad (3.83)$$

and the solution to the original problem (3.74) for $\hat{W}_{\text{TR}}$ is given by

$$\hat{W}_{\text{TR}}(\hat{X}_R, \hat{Y}_{\text{TR}}) = \hat{X}_R - \frac{2}{\pi} \int_0^\infty \left(\frac{\hat{X}_R}{\omega} - \frac{1}{\omega^3} \left(1 - e^{-\frac{1}{2}\hat{X}_R(\sqrt{1+4\omega^2}-1)} \right) \right) \sin(\omega \hat{Y}_{\text{TR}}) d\omega. \quad (3.84)$$

We will not evaluate the integral in (3.84) in full; however, we can make the substitution $\omega = \psi/\sqrt{\hat{X}_R}$, and use the fact that

$$e^{-\frac{1}{2}\hat{X}_R(\sqrt{1+4\psi^2/\hat{X}_R}-1)} \sim e^{-\psi^2} + \frac{e^{-\psi^2}\psi^4}{\hat{X}_R} + \mathcal{O}\left(\frac{1}{\hat{X}_R^2}\right) \text{ as } \hat{X}_R \rightarrow \infty \quad (3.85)$$

to demonstrate that the solution (3.84) agrees with the far-field condition (3.78). Indeed, we have

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\psi^2 - 1 + e^{-\psi^2}}{\psi^3} \right) \sin(\psi v) d\psi = \frac{1}{2} (2 + v^2) \operatorname{erfc}(v) - \frac{v}{\sqrt{\pi}} e^{-\frac{1}{4}v^2}, \quad (3.86)$$

and the right-hand side (3.86) is precisely $1 - f(v)$. Therefore, by using results (3.84) and (3.86), we indeed have that

$$\hat{W}_{\text{TR}}(\hat{X}_R, \hat{Y}_{\text{TR}}) \sim \hat{X}_R f(v) \text{ as } \hat{X}_R, \hat{Y}_{\text{TR}} \rightarrow \infty. \quad (3.87)$$

3.3.2.9 Asymptotic flow rate

We now wish to evaluate the flow rate contributions in the regime $r = \mathcal{O}(1)$ by substituting the solutions we have obtained so far into integrals (3.44)–(3.46). Once this is done, the intermediate variables are dealt with by taking the limits $\mathcal{Y}, \mathcal{X}_R \rightarrow 0$ and also $\mathcal{Y}/\epsilon^{1/2}, \mathcal{X}/\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$. To evaluate integral (3.44) up to order ϵ , the leading-order solutions in the core, left wall and top wall are sufficient. Strictly speaking, we would also need the first-order contribution to the core solution, $V_1(x, y)$, but this is identically zero. By substituting the solutions in each region into the integrals, and taking the aforementioned limits, one obtains an asymptotic expansion for the average flow rate:

$$\bar{w} \sim \frac{1}{\sqrt{r}} - \frac{16}{15r^{5/4}} \sqrt{\frac{2}{\pi}} \epsilon^{1/2} + \mathcal{O}(\epsilon) \text{ as } \epsilon \rightarrow 0. \quad (3.88)$$

We note that the flow rate appears to diverge as the aspect ratio r tends to zero. However, the expression (3.88) stops being an asymptotic expansion when the first correction stops being asymptotically smaller than the leading term. This observation confirms our previous claim that $r = \mathcal{O}(\epsilon^{2/3})$ is a second distinguished limit for equation (3.36a). The asymptotic approximation (3.88) for the flow rate in each regime will later be compared with numerical solutions, after we have analysed the regime $r = \mathcal{O}(\epsilon^{2/3})$.

3.3.3 The case $r = \mathcal{O}(\epsilon^{2/3})$

3.3.3.1 Rescaled problem We now repeat the overall procedure from the previous section, but after making the change of variable $r = \varrho \epsilon^{2/3}$ where $\varrho = \mathcal{O}(1)$ as $\epsilon \rightarrow 0$. However, one finds that it also becomes necessary to scale $\hat{V} = \epsilon^{-1/3} \hat{U}$ so the governing equation (3.36a) becomes

$$\varrho \epsilon^{4/3} \frac{\partial^2 \hat{U}}{\partial x^2} + \frac{1}{\varrho} \frac{\partial^2 \hat{U}}{\partial y^2} + \sqrt{\varrho} \frac{\partial \hat{U}}{\partial x} + 1 = 0. \quad (3.89)$$

By attempting to scale with inner variables and looking for distinguished limits, one can establish that the only relevant boundary layer in this problem is the one at the left wall, with width $\mathcal{O}(\epsilon^{4/3})$. There is no boundary layer at the top wall this time; as r is gradually reduced from $\mathcal{O}(1)$ to $\mathcal{O}(\epsilon^{2/3})$ (that is, the duct is compressed vertically but stretched horizontally) the top boundary layer seen in Figure 3.1 expands and merges with the outer region. Along with Dirichlet boundary conditions $\hat{U}(\pm 1, \hat{y}) = 0$ and $\hat{U}(\hat{x}, \pm 1) = 0$, only one matching condition is needed to relate the inner and outer solutions, which is

$$\hat{U}(-1 + \epsilon^{4/3} \hat{X}_L, \hat{y}) \sim \hat{U}_L(\hat{X}_L, \hat{y}) \text{ as } \hat{X}_L \rightarrow \infty. \quad (3.90)$$

3.3.3.2 Flow-rate contributions The integral flow rate is expressible in terms of $\hat{U}(\hat{x}, \hat{y})$ as

$$\bar{w} = \frac{1}{2\epsilon^{1/3}} \int_{-1}^1 \int_0^1 \hat{U}(\hat{x}, \hat{y}) d\hat{x} d\hat{y}. \quad (3.91)$$

This integral can be expanded in the form

$$\bar{w} \sim \epsilon^{-1/3} \mathcal{I}_O + \epsilon \mathcal{I}_L + \mathcal{O}(\epsilon^{5/3}), \quad (3.92)$$

where $\mathcal{I}_O$ and $\mathcal{I}_L$ are the contributions to the flow rate from outer and inner solutions respectively, given by

$$\mathcal{I}_O = \frac{1}{2} \int_0^1 \int_{-1+\mathcal{X}}^1 \hat{U}(\hat{x}, \hat{y}) d\hat{x} d\hat{y}, \quad (3.93a)$$

$$\mathcal{I}_L = \frac{1}{2} \int_0^1 \int_0^{\mathcal{X}/\epsilon^{4/3}} \hat{U}_L(\hat{X}_L, \hat{y}) d\hat{X}_L d\hat{y}. \quad (3.93b)$$

We use the intermediate variable $\mathcal{X}$ to split up the domain of integration, such that $\hat{\mathcal{X}} \rightarrow 0$ but $\hat{\mathcal{X}}/\epsilon^{4/3} \rightarrow \infty$ as $\epsilon \rightarrow 0$.

3.3.3.3 Outer problem The form of equation (3.89) suggests a regular asymptotic expansion solution of the form $\hat{U} \sim \hat{U}_0 + \epsilon^{4/3} \hat{U}_1 + \dots$. The leading-order problem, complete with boundary conditions, takes the form

$$\frac{1}{\varrho} \frac{\partial^2 \hat{U}_0}{\partial \hat{y}^2} + \sqrt{\varrho} \frac{\partial \hat{U}_0}{\partial \hat{x}} + 1 = 0, \quad \hat{U}_0(\pm 1, \hat{y}) = 0, \quad \hat{U}_0(\hat{x}, 1) = 0. \quad (3.94)$$

The linearity of the PDE (3.94) allows the solution $\hat{U}_0$ to be expressed as the sum of a complementary function $\hat{U}_0^{[C]}$ and a particular integral $\hat{U}_0^{[P]}$, respectively satisfying

$$\frac{1}{\varrho} \frac{\partial^2 \hat{U}_0^{[C]}}{\partial \hat{y}^2} + \sqrt{\varrho} \frac{\partial \hat{U}_0^{[C]}}{\partial \hat{x}} = 0, \quad \frac{1}{\varrho} \frac{\partial^2 \hat{U}_0^{[P]}}{\partial \hat{y}^2} + \sqrt{\varrho} \frac{\partial \hat{U}_0^{[P]}}{\partial \hat{x}} = -1. \quad (3.95)$$

It is straightforward to verify that a particular integral solution to equation (3.95) is

$$\hat{U}_0^{[P]}(\hat{x}, \hat{y}) = \frac{\varrho}{2} (1 - \hat{y}^2), \quad (3.96)$$

and one can also show by using separation of variables that

$$\hat{U}_0^{[C]}(\hat{x}, \hat{y}) = -\frac{16\varrho}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} e^{-\frac{\pi^2}{\epsilon^{3/2}} (n+\frac{1}{2})^2 (1-\hat{x})} \cos\left(n + \frac{1}{2}\right) \pi \hat{y}. \quad (3.97)$$

Furthermore, by substituting the leading-order solution into the first-order version of equation (3.89),

$$\frac{1}{\varrho} \frac{\partial^2 \hat{U}_1}{\partial \hat{y}^2} + \sqrt{\varrho} \frac{\partial \hat{U}_1}{\partial \hat{x}} = -\varrho \frac{\partial^2 \hat{U}_0}{\partial \hat{x}^2}, \quad (3.98)$$

one can show that the first-order correction to the outer solution is given by

$$\hat{U}_1(\hat{x}, \hat{y}) = \frac{\pi}{\varrho^{3/2}}(\hat{x} - 1) \sum_{n=0}^{\infty} (-1)^n (2n+1) e^{-\frac{\pi^2}{\varrho^{3/2}}(n+\frac{1}{2})^2(1-\hat{x})} \cos\left(n + \frac{1}{2}\right) \pi \hat{y}. \quad (3.99)$$

3.3.3.4 Inner problem near left wall By rescaling equation (3.89) using $\hat{x} = -1 + \epsilon^{4/3} \hat{X}_L$, one transforms the governing equation (3.89) into

$$\varrho \frac{\partial^2 \hat{U}_L}{\partial \hat{X}_L^2} + \frac{\epsilon^{4/3}}{\varrho} \frac{\partial^2 \hat{U}_L}{\partial \hat{y}^2} + \epsilon^{4/3} + \sqrt{\varrho} \frac{\partial \hat{U}_L}{\partial \hat{X}_L} = 0, \quad (3.100)$$

which, upon using the asymptotic expansion $\hat{U}_L \sim \hat{U}_{L,0} + \epsilon^{4/3} \hat{U}_{L,1} + \dots$, becomes

$$\sqrt{\varrho} \frac{\partial^2 \hat{U}_{L,0}}{\partial \hat{X}_L^2} + \frac{\partial \hat{U}_{L,0}}{\partial \hat{X}_L} = 0 \quad (3.101)$$

at leading order. The solution to the leading-order problem (3.101) satisfying the boundary condition $\hat{U}_{L,0}(0, \hat{y}) = 0$ and the leading-order matching condition

$$\hat{U}_{L,0}(\hat{X}_L, \hat{y}) \rightarrow \hat{U}_0(-1, \hat{y}) \text{ as } \hat{X}_L \rightarrow \infty \quad (3.102)$$

is

$$\hat{U}_{L,0}(\hat{X}_L, \hat{y}) = \hat{U}_0(-1, \hat{y}) \left(1 - e^{-\frac{\hat{X}_L}{\sqrt{\varrho}}}\right). \quad (3.103)$$

3.3.3.5 Asymptotic flow rate By looking at the asymptotic expansion for the flow rate in equation (3.92), we can see that, if we want to evaluate only the contributions that are of order $\mathcal{O}(\epsilon^{-1/3})$ and $\mathcal{O}(\epsilon)$, then U_0 , U_1 and $U_{L,0}$ are all needed. By substituting the solutions (3.96), (3.97), and (3.99) into the asymptotic expansion (3.92), performing all the appropriate integrals and taking the limit $\mathcal{X} \rightarrow 0$ and $\mathcal{X}/\epsilon^{4/3} \rightarrow \infty$ as $\epsilon \rightarrow 0$, one can show that

$$\begin{aligned} \bar{w} \sim & \underbrace{\epsilon^{-1/3} \left(\frac{\varrho}{3} - \frac{\varrho^{5/2}}{15} + \frac{64\varrho^{5/2}}{\pi^6} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} e^{-\frac{(2n+1)^2 \pi^2}{2\varrho^{3/2}}} \right)}_{q_0(\varrho)} \\ & + \epsilon \left(-\frac{\varrho^{3/2}}{3} + \frac{8}{\pi^4} \sum_{n=0}^{\infty} \left(\frac{4\varrho^{3/2} + \pi^2(2n+1)^2}{(2n+1)^4} \right) e^{-\frac{(2n+1)^2 \pi^2}{2\varrho^{3/2}}} \right) + \dots \text{ as } \epsilon \rightarrow 0. \end{aligned} \quad (3.104)$$

3.3.3.6 Comparing both regimes Having obtained asymptotic expansions in both regimes $r = \mathcal{O}(1)$ and $r = \mathcal{O}(\epsilon^{2/3})$, it is instructive to validate them against each other by demonstrating that they match at leading-order. We note that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} e^{-\epsilon \frac{(2n+1)^2 \pi^2}{2r^{3/2}}} \sim \frac{\pi^6}{960} - \frac{\epsilon \pi^6}{192r^{3/2}} + \frac{\epsilon^2 \pi^6}{64r^3} + \mathcal{O}(\epsilon^{5/2}), \quad (3.105a)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} e^{-\epsilon \frac{(2n+1)^2 \pi^2}{2r^{3/2}}} \sim \frac{\pi^4}{96} - \frac{\epsilon \pi^4}{16r^{3/2}} + \mathcal{O}(\epsilon^{3/2}), \quad (3.105b)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} e^{-\epsilon \frac{(2n+1)^2 \pi^2}{2r^{3/2}}} \sim \frac{\pi^2}{8} + \mathcal{O}(\epsilon^{1/2}). \quad (3.105c)$$

The above series expansions are truncated in different places, because at some point in each one, there will be a term in the expansion whose power of $(2n+1)^2$ cancels with the negative power of $(2n+1)^2$, leading to divergent series. To make sense of the terms involving divergent series, one can use Euler-Maclaurin summation to approximate them, but this is not discussed here. After making the substitution $\varrho = r\epsilon^{-2/3}$ and using the aforementioned results (3.105a)–(3.105c), one can show that the expression (3.104) reduces to

$$\bar{w} \sim \frac{1}{\sqrt{r}} + \mathcal{O}(\epsilon), \quad (3.106)$$

which is exactly the same leading-order term as in the asymptotic expansion (3.88).

3.4 Numerical solutions

We use a finite element method (FEM) to solve equations (3.34a) and (3.34b) subject to Dirichlet boundary conditions. Numerical solutions are obtained using COMSOL Multiphysics, owing to its ease of use and flexibility in obtaining numerical solutions to PDE problems with a range of boundary conditions. Representative numerical solutions are shown in Figure 3.2. Rather than using the symmetry relationships to recover $\hat{w}$ and $\hat{b}$ from $\hat{V}$ and $\hat{W}$, the corresponding PDE problems for each variable have been solved. Fortunately, we can see from the plots that these relationships are satisfied anyway, namely that the plot of $\hat{V}$ is reflected horizontally to obtain that of $\hat{W}$, and that the latter pair of variables are superimposed to obtain plots for the former pair.

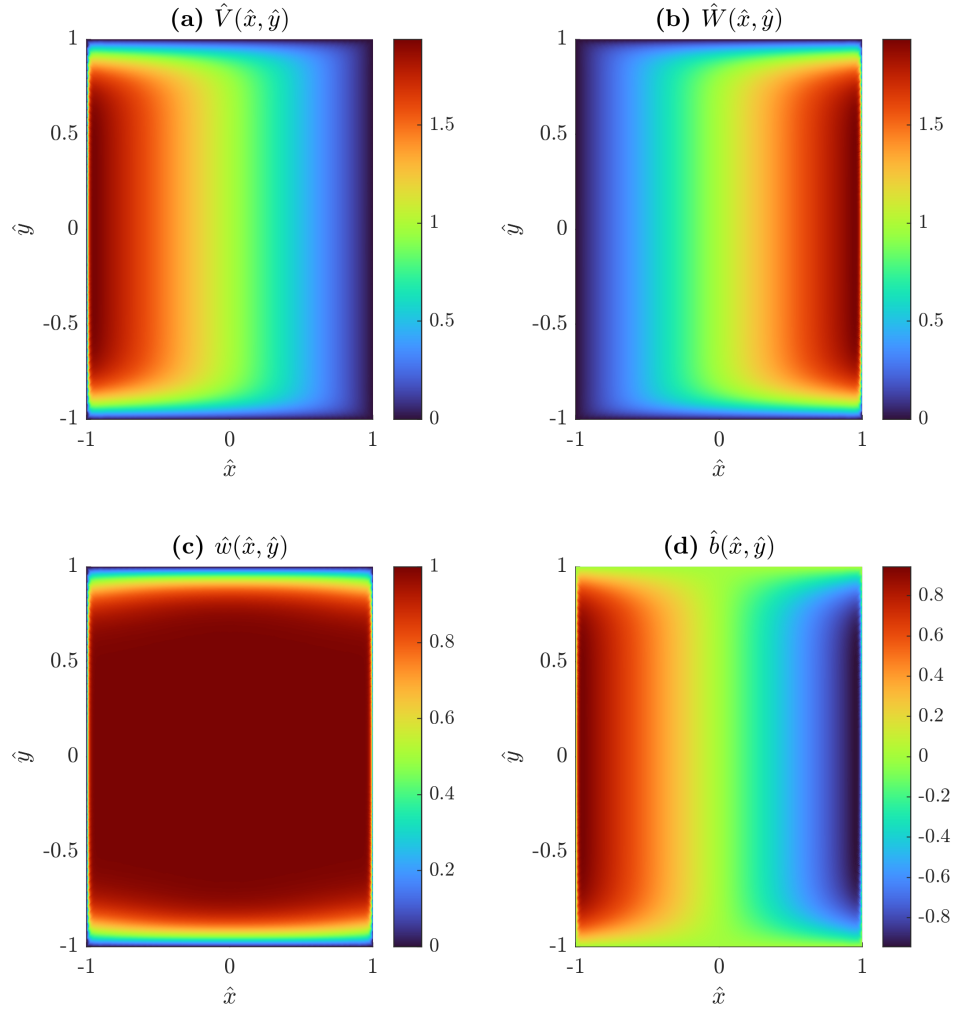


Figure 3.2: Plots of the various functions obtained as part of the 2D MHD duct flow problem, where $r = 1$ and $\epsilon = 0.01$.

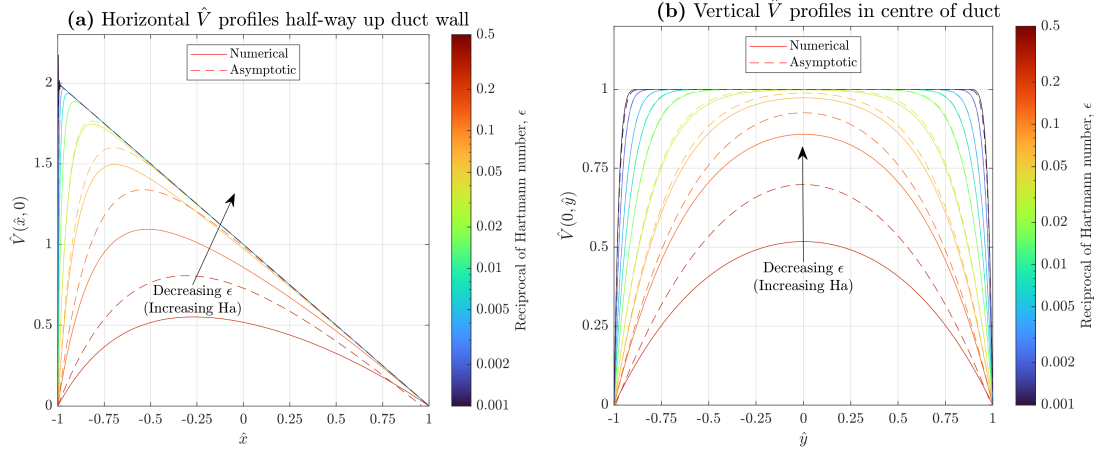


Figure 3.3: Results of using FEM to solve the problem for $\hat{V}$, along the central lines **(a)** $\hat{y} = 0$, and **(b)** $\hat{x} = 0$.

From Figure 3.2, we make several deductions. Plot **(a)** has a boundary layer near the left wall but not near the right, and the opposite is true for plot **(b)**. Although boundary layers of width $\mathcal{O}(\epsilon)$ were obtained in our solution to the problem for $\hat{V}$, these are not obvious from plot **(a)**. This is possibly due to the similarity solution structure, which adds a significant amount of curvature in the variation near that wall. However, the characteristic boundary-layer structure is made much more obvious in plot **(c)**, where there are very thin Hartmann layers near the side walls of $\mathcal{O}(\epsilon)$, whilst there are weaker “side” layers of width $\mathcal{O}(\epsilon^{1/2})$ near the top and bottom walls, which are parallel to the magnetic field. We also notice that plot **(d)** demonstrates a dipole structure in the induced magnetic field.

From Figure 3.3, we deduce that the approximate expansions (3.55) and (3.62) are in good agreement with the solutions for $\hat{V}$ obtained via numerical methods, and this agreement becomes better when ϵ is decreased. In particular, we deduce that $\hat{V}$ approaches a linear function as $\epsilon \rightarrow 0$ from plot **(a)**, which agrees with the asymptotics. Furthermore, by the symmetry relation (3.38), the velocity $\hat{w}$ is obtained by reflecting the solution for $\hat{V}$ in the $\hat{y}$ -axis and adding the two results. Applying this operation will understandably result in a function that is approximately constant apart from in regions close to the walls. This observation is made even more obvious by plot **(b)**.

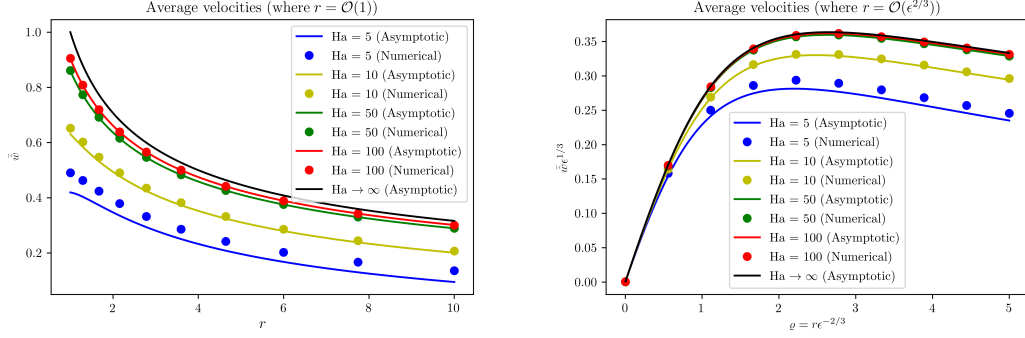


Figure 3.4: Plots of asymptotic average velocities versus those obtained from numerical solutions, in the cases $r = \mathcal{O}(1)$ (left) and $r = \mathcal{O}(\epsilon^{2/3})$ (right). **NB:** These plots were produced in Firedrake (www.firedrakeproject.org) rather than COMSOL.

The asymptotic expansions (3.88) and (3.104) for the average flow rate are compared with the corresponding numerical solutions in Figure 3.4. We can see that the asymptotic and numerical solutions are in good agreement with each other, even for moderate values of Ha , and that the agreement improves as Ha increases, which is to be expected. Importantly, we observe from the curves in Figure 3.4 that the average velocity increases and then decreases again, as r increases. This is consistent with what UIUC observed in their numerical analysis using COMSOL [109]. The maximum value of $\bar{w}\epsilon^{-1/3}$ in Figure 3.4 is seen to arise at $q \approx 2.6$ as $Ha \rightarrow \infty$. This is indeed confirmed by the maximum value of the leading-order term in the asymptotic expansion (3.106) as a function of q . One can use a numerical method to maximise this function (denoted $q_0(q)$) to find that $q_{\max} \approx 2.65144$ and $q_0(q_{\max}) \approx 0.363646$. This tells us that the maximum flow rate $\bar{w} \sim 0.363646\epsilon^{-1/3}$ occurs approximately at the optimal aspect ratio $r_{\max} \sim 2.65144\epsilon^{2/3}$ as $\epsilon \rightarrow 0$. These findings are interesting, because the optimal duct becomes thinner as the applied magnetic field $\mathcal{B}^a$ increases, and the maximum flow rate scales with $(a\mathbf{b}/\mathcal{B}^a)^{2/3}$ rather than $\sqrt{a\mathbf{b}}/\mathcal{B}^a$, as is traditionally the case in Hartmann flow. There is also no specific reason why the shape of the duct cross-section has to be limited to a rectangle; it is in principle possible to turn this work into a shape optimisation problem, where one attempts to maximise the flow rate (for a given value of Ha) over all duct shapes of unit area.

3.5 Discussion

In this chapter, we focused on isothermal 2D MHD duct flow as per the classical Shercliff 1953 problem. Despite this being an old problem that was studied over half a century ago, examining it as a canonical example of what could occur in MHD duct flow provided some useful insights which might be naturally-occurring in MHD problems involving more physics. Using asymptotic analysis, we confirmed the presence of Hartmann layers, of width $\mathcal{O}(\text{Ha}^{-1})$ near the walls perpendicular to the applied magnetic field, and so-called “side layers” of width $\mathcal{O}(\text{Ha}^{-1/2})$ near the walls parallel to the applied magnetic field. When the aspect ratio $r = \mathcal{O}(1)$, both of these types of boundary layer are present, as well as “corner” regions where these boundary layers overlap. In the case where $r = \mathcal{O}(\text{Ha}^{-2/3})$, the Hartmann layer changes width to $\mathcal{O}(\text{Ha}^{-4/3})$, whilst the “side” layers disappear entirely.

In Figure 3.2 we demonstrated how the physical variables $\hat{w}$ and $\hat{b}$ can be clearly inferred from the variables $\hat{V}$ and $\hat{W}$. The nature of the electrically insulating and no-slip boundary conditions on the walls is what allows the latter two variables to be used, effectively reducing the order of the problem so that second-order decoupled PDEs can be solved, as opposed to fourth-order coupled PDEs for the former two variables. In this figure, plot (a) confirms the existence of a boundary layer near the left wall, but not the right, as well as thicker boundary layers near the top and bottom walls. By combining the solutions seen in plots (a) and (b), one can obtain solutions for the velocity in (c) and the magnetic field in (d).

Figure 3.3 demonstrates that the asymptotic and numerical solutions for $\hat{V}$ agree with each other as $\epsilon \rightarrow 0$, whilst Figure 3.4 demonstrates excellent agreement between the asymptotic and numerical results for the average velocity. Furthermore, we showed that when $r = \mathcal{O}(\text{Ha}^{-2/3})$, there is a duct aspect ratio that optimises the flow rate.

There are numerous extensions to this work that could be carried out; for example, allowing the applied magnetic field to vary obliquely, so that it no longer aligns parallel or perpendicular to the walls, and allowing the electrical conductivities of the walls to vary. In the former instance, it is still possible to use symmetry variables $\hat{V}$ and $\hat{W}$ to study the problem, but this is not the case when the walls are electrically conducting, because the nature of the boundary conditions on $\hat{w}$

and $\hat{b}$ are no longer of the same type. These extensions are very straightforward to carry out numerically using finite-element packages, although when using matched asymptotic expansions, the problems can only really be tackled in certain distinguished limits involving the applied B-field angle and the wall conductivity.

In the next chapter we apply the same concepts to a less idealised geometry, namely TEMHD flow along a trench driven by an applied temperature gradient. In doing so, we no longer restrict ourselves by varying the aspect ratio of the duct or channel, but we allow scope to vary the side wall and base dimensions independently of one another, amongst other geometrical and physical parameters.

Chapter 4

2D Trench Flow Problem

4.1 Modified constitutive laws

In this section, we provide a more general version of the problem discussed in Chapter 3, which makes it more representative of a situation seen in Liquid Metal Infused Trenches (LiMIT). We consider 2D thermoelectric magnetohydrodynamic flow with a free surface, between electrically and thermally conducting walls.

The governing equations and boundary conditions are identical to those shown in Chapter 3, except for the modified Ohm and Fourier laws:

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B} - S \nabla T) \quad (4.1a) \qquad \mathbf{q} = -k \nabla T + ST \mathbf{J}, \quad (4.1b)$$

where S is the *Seebeck coefficient*. The additional terms arise from the Seebeck, Peltier and Thomson effects, and mathematically they originate from the Onsager reciprocal relations from thermodynamics [11]. A full mathematical derivation of these exact terms is beyond the scope of this thesis. However, we will discuss how the physical description of these effects is consistent with the forms of the additional terms in equations (4.1a) and (4.1b).

4.2 The Seebeck effect

The Seebeck effect is where a temperature difference between two electrical conductors produces a potential difference across the interface. To see how this manifests itself from equation (4.1a), consider an electrical current circulating around such an interface $\mathcal{C}$ between two electrical conductors (each with Seebeck coefficients S_1

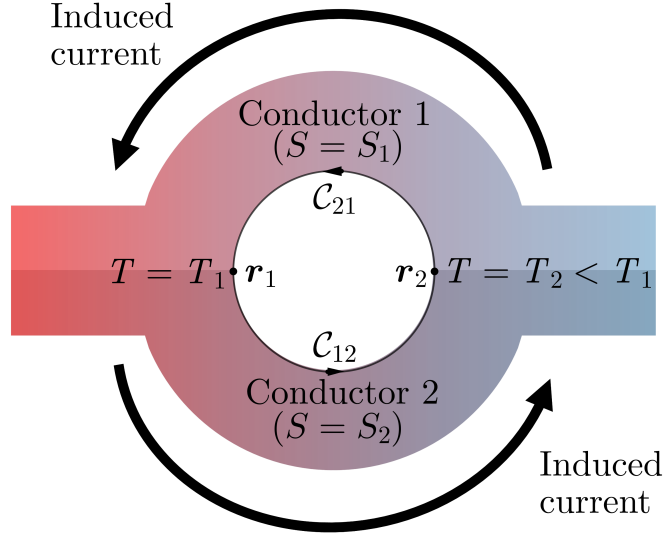


Figure 4.1: A diagram illustrating the Seebeck effect, adapted from Shercliff’s paper [78].

and S_2), surrounding a fictitious infinitesimal “hole”, as shown in Figure 4.1. Assume that the interface $\mathcal{C}$ between the two conductors is stationary, so that $\mathbf{u} = 0$ on $\mathcal{C}$. If one defines $\mathbf{E} = -\nabla V$, where V is the electric potential, then provided that there is no contact resistance or electrochemical effects along $\mathcal{C}$, taking the line integral of this gradient around $\mathcal{C}$ will result in zero; this is effectively Kirchhoff’s law around closed loops. Therefore, if equation (4.1a) is rearranged for the electric field $\mathbf{E}$ and integrated around $\mathcal{C}$, one obtains an expression for the *Seebeck e.m.f.* (electromotive force) $\mathcal{E}$, given by

$$\mathcal{E} = \oint_{\mathcal{C}} \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{r}. \quad (4.2)$$

Meanwhile, taking the line integral of $S\nabla T$ around $\mathcal{C}$, using the facts that $\mathcal{C}_{12} \rightarrow -\mathcal{C}_{21}$ as the hole is closed and that the Seebeck coefficient depends on temperature, one can use the chain rule to show that

$$\mathcal{E} = \int_{\mathcal{C}_{12}} (S_2 - S_1) \nabla T \cdot d\mathbf{r} = \int_{T_1}^{T_2} \Delta S dT \quad (4.3)$$

where $\Delta S = S_2 - S_1$ is the difference in Seebeck coefficient between conductors 1 and 2. We deduce that there is no Seebeck e.m.f. if either $T_1 = T_2$ or $S_1 = S_2$.

In practice, it is not possible to determine the individual Seebeck coefficients S_1 and S_2 ; rather, it is the difference in Seebeck coefficients (sometimes referred to as *thermoelectric power*) that is measured with respect to a specific material, for example, with respect to stainless steel [23]. We note that the Seebeck coefficient S is negative for most materials, but it happens to be positive for lithium.

It should be emphasised that the extra Seebeck term in (4.1a) does not result in any additional terms in the induction equation (3.11), even when the temperature-dependence of the Seebeck coefficient S is taken into account. This follows from a straightforward application of the chain and product rules:

$$\nabla \times (S \nabla T) = \frac{dS}{dT} \nabla T \times \nabla T = 0. \quad (4.4)$$

4.3 The Peltier and Thomson effects

The Peltier effect is in a sense the converse effect of the Seebeck effect; it is the presence of heating or cooling at an electrified interface between two electrical conductors. The Thomson effect is a continuous analogue of this effect, in which the temperature and thermophysical properties may change throughout the medium. By considering the contribution that these effects make to the heat equation (3.5), taking the divergence of the $ST\mathbf{J}$ part of the heat flux vector $\mathbf{q}$ gives, assuming steady-state (since $\nabla \cdot \mathbf{J} = 0$ by continuity of charge (3.3)),

$$\nabla \cdot (ST\mathbf{J}) = \left(T \frac{dS}{dT} + S \right) \mathbf{J} \cdot \nabla T \quad (4.5)$$

This term represents the generation of heating or cooling when an electrical current is present. However, by evaluating the $\mathbf{E} \cdot \mathbf{J}$ Joule heating term as seen in the heat equation (3.5), using the modified Fourier law (4.1b), we obtain

$$\mathbf{E} \cdot \mathbf{J} = \frac{|\mathbf{J}|^2}{\sigma} - (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{J} + S\mathbf{J} \cdot \nabla T. \quad (4.6)$$

When evaluating the right-hand side of the heat equation (3.5), we note that the $S\mathbf{J} \cdot \nabla T$ terms in each of equations (4.5) and (4.6) cancel, since it is the negative of equation (4.5) which is calculated as part of the $-\nabla \cdot \mathbf{q}$ term with the heat flux

vector $\mathbf{q}$ given by (4.1b). As a result, we have

$$\rho c_v \frac{DT}{Dt} = k \nabla^2 T + \frac{|\mathbf{J}|^2}{\sigma} + \rho \nu (\nabla \otimes \mathbf{u}) : (\nabla \otimes \mathbf{u}) - \frac{dS}{dT} \mathbf{J} \cdot \nabla T + (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{J} \quad (4.7)$$

In the case where the Seebeck coefficient is treated as a constant, as it is in our work, we note that the Seebeck effect plays no role in the heat equation *per se*. It only appears in the heat flux jump boundary condition (3.24b) at the free surface, and also the tangential electric field jump condition (3.14b) on the fluid-solid interface.

4.4 Problem derivation

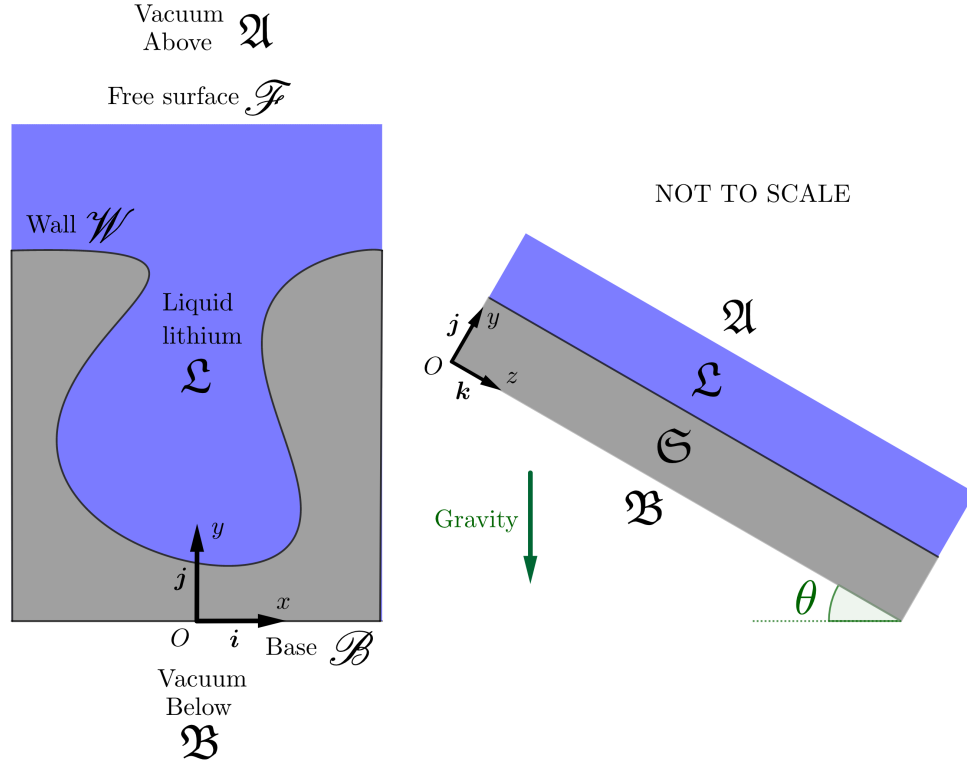


Figure 4.2: A diagram illustrating a generalised version of the geometry in a LiMIT-type setup.

4.4.1 Trench geometry

As illustrated in Figure 4.2, the problem contains four domains: the vacuum above ($\mathfrak{A}$), the liquid lithium ($\mathfrak{L}$), the solid substrate ($\mathfrak{S}$) and the vacuum below ($\mathfrak{B}$). We have three boundaries: the free surface ($\mathcal{F}$), the trench wall ($\mathcal{W}$) and the trench base ($\mathcal{B}$). The free surface $\mathcal{F}$ is assumed to be above the trench wall $\mathcal{W}$, so that the wall is always wetted by the liquid lithium and there are no regions of the wall that are left exposed to the region $\mathfrak{A}$. The trench wall $\mathcal{W}$ separates the regions $\mathfrak{L}$ and $\mathfrak{S}$, and the trench base $\mathcal{B}$ separates the regions $\mathfrak{S}$ and $\mathfrak{B}$. Both of the boundaries $\mathcal{W}$ and $\mathcal{B}$ remain fixed.

In the geometry of interest, x and y represent in-plane coordinates at a fixed point along the trench, whilst z represents distance along the trench in a downward direction. The geometry is (for now) assumed to be uniform in the z -direction, with the free surface given by $y = h(x, t)$.

Fixing $z \in [0, \mathcal{L}]$, the boundaries in the problem can be parametrised by

$$\mathbf{r}^{\mathcal{F}} = x\mathbf{i} + h(x, t)\mathbf{j} + z\mathbf{k}, \quad \text{for } x \in \left[-\frac{1}{2}\mathcal{W}, \frac{1}{2}\mathcal{W}\right], \quad (4.8a)$$

$$\mathbf{r}^{\mathcal{W}} = x^{\mathcal{W}}(s)\mathbf{i} + y^{\mathcal{W}}(s)\mathbf{j} + z\mathbf{k}, \quad \text{for } s \in [0, \ell^{\mathcal{W}}], \quad (4.8b)$$

$$\mathbf{r}^{\mathcal{B}} = x\mathbf{i} + z\mathbf{k}, \quad \text{for } x \in \left[-\frac{1}{2}\mathcal{W}, \frac{1}{2}\mathcal{W}\right], \quad (4.8c)$$

where $\mathcal{W}$ is the width of the trench, and s represents arc-length along the trench wall $\mathcal{W}$ with $\ell^{\mathcal{W}}$ being the total length of the interface, such that increasing s involves traversing the boundary of the liquid region $\mathfrak{L}$ anticlockwise. To enforce periodic geometry and boundary conditions, we restrict ourselves to wall shape curves that start at the point $(-\frac{1}{2}\mathcal{W}, \mathcal{H})$ and end at the point $(\frac{1}{2}\mathcal{W}, \mathcal{H})$, where $\mathcal{H}$ represents the height of the trench, i.e.

$$x^{\mathcal{W}}(0) = -\frac{1}{2}\mathcal{W}, \quad y^{\mathcal{W}}(0) = \mathcal{H}, \quad x^{\mathcal{W}}(\ell^{\mathcal{W}}) = \frac{1}{2}\mathcal{W}, \quad y^{\mathcal{W}}(\ell^{\mathcal{W}}) = \mathcal{H}. \quad (4.9)$$

4.4.2 Modelling assumptions

We do not explicitly model the plasma above the trench other than that it provides a heat source at the free surface, such that

$$\mathbf{q}^{\mathcal{L}} \cdot \hat{\mathbf{n}}^{\mathcal{F}} = Q_{\text{in}}(x, t) \text{ for } x \in \mathcal{F}. \quad (4.10)$$

We also forgo any mathematical modelling of the solid mechanics of the trench, for example thermal expansion, fracture or melting of the trench wall. We do not explicitly model the liquid flowing in the cooling pipes below the trench; for now the base of the trench is effectively treated as a perfect heatsink. We also assume that all material properties (such as density, viscosity and conductivity) are constants, so that any temperature dependencies may be neglected in the first instance.

We assume that the flow is unidirectional and that all dependent variables (apart from the pressure p) are independent of z , so that $\mathbf{u}(\mathbf{x}, t) = w(x, y, t)\mathbf{k}$, $T(\mathbf{x}, t) = T(x, y, t)$, and $p(\mathbf{x}, t) = p(x, y, z, t)$. We assume that the applied magnetic field $\mathbf{B}^a$ is independent of position and allow for the possibility that it is orientated at an oblique angle ψ to the horizontal direction, that is,

$$\mathbf{B}^a(\mathbf{x}, t) = B^a(t) (\mathbf{i} \cos \psi + \mathbf{j} \sin \psi). \quad (4.11)$$

Meanwhile, we assume that the induced magnetic field $\mathbf{b}$ is expressible in terms of an axial induced magnetic field b and an induced magnetic field potential A via

$$\mathbf{b}(\mathbf{x}, t) = b(x, y, t)\mathbf{k} + \nabla \times (A(x, y, t)\mathbf{k}) = b(x, y, t)\mathbf{k} + \frac{\partial A}{\partial y}\mathbf{i} - \frac{\partial A}{\partial x}\mathbf{j}. \quad (4.12)$$

This form of the induced magnetic field automatically satisfies the solenoidal property (3.1c).

In subsequent work, any dependent variables and material properties will be superscripted with labels corresponding to the domain in which they are defined; for example, $A^{\mathfrak{A}}$, $A^{\mathfrak{L}}$, $A^{\mathfrak{S}}$ and $A^{\mathfrak{B}}$ all denote the value of A in the upper vacuum, the liquid region, the solid region and the lower vacuum respectively. Furthermore, many physical properties reduce to zero in vacua; in particular, $\sigma^{\mathfrak{A}} = \sigma^{\mathfrak{B}} = k^{\mathfrak{A}} = k^{\mathfrak{B}} = S^{\mathfrak{A}} = S^{\mathfrak{B}} = \rho^{\mathfrak{A}} = \rho^{\mathfrak{B}} = c_p^{\mathfrak{A}} = c_p^{\mathfrak{B}} = \nu^{\mathfrak{A}} = \nu^{\mathfrak{B}} = 0$, and $\chi_e^{\mathfrak{A}} = \chi_e^{\mathfrak{B}} = \chi_m^{\mathfrak{A}} = \chi_m^{\mathfrak{B}} = 0$, so that $\mu^{\mathfrak{A}} = \mu^{\mathfrak{B}} = \mu_0$. We will forgo any superscripting of ν and γ since they are only defined in the liquid region $\mathfrak{L}$.

4.4.3 Governing equations

We proceed to simplify all the governing equations provided in Chapter 3.1 without working with the electric field $\mathbf{E}$ or the current density $\mathbf{J}$ directly. We note that

the assumption of unidirectional flow relies on the surface tension being constant at the free surface $\mathcal{F}$, so for now we neglect the possible dependence of γ on T . We also note that, with unidirectional flow, the kinematic boundary condition (3.16) reduces to $\partial h / \partial t = 0$, so the free-surface height must be steady.

4.4.3.1 In the liquid $\mathcal{L}$

In the liquid, the three components of the induction equation (3.11) become

$$\frac{\partial^2 A^\mathcal{L}}{\partial t \partial y} = -\frac{dB^a}{dt} \cos \psi + \eta^\mathcal{L} \frac{\partial}{\partial y} (\nabla^2 A^\mathcal{L}), \quad (4.13a)$$

$$-\frac{\partial^2 A^\mathcal{L}}{\partial t \partial x} = -\frac{dB^a}{dt} \sin \psi - \eta^\mathcal{L} \frac{\partial}{\partial x} (\nabla^2 A^\mathcal{L}), \quad (4.13b)$$

$$\frac{\partial b^\mathcal{L}}{\partial t} = \eta^\mathcal{L} \nabla^2 b^\mathcal{L} + B^a(t) \left(\cos \psi \frac{\partial w}{\partial x} + \sin \psi \frac{\partial w}{\partial y} \right) + \left(\frac{\partial A^\mathcal{L}}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial A^\mathcal{L}}{\partial x} \frac{\partial w}{\partial y} \right), \quad (4.13c)$$

whilst the three components of the Navier-Stokes equation (3.4a) become

$$0 = -\frac{\partial p}{\partial x} + \frac{1}{\mu^\mathcal{L}} B^a(t) (\nabla^2 A^\mathcal{L}) \sin \psi - \frac{1}{\mu^\mathcal{L}} b^\mathcal{L} \frac{\partial b^\mathcal{L}}{\partial x} - \frac{1}{\mu^\mathcal{L}} \frac{\partial A^\mathcal{L}}{\partial x} (\nabla^2 A^\mathcal{L}) \quad (4.14a)$$

$$0 = -\frac{\partial p}{\partial y} - \frac{1}{\mu^\mathcal{L}} B^a(t) (\nabla^2 A^\mathcal{L}) \cos \psi - \frac{1}{\mu^\mathcal{L}} b^\mathcal{L} \frac{\partial b^\mathcal{L}}{\partial y} - \frac{1}{\mu^\mathcal{L}} \frac{\partial A^\mathcal{L}}{\partial y} (\nabla^2 A^\mathcal{L}) - \rho g \cos \theta \quad (4.14b)$$

$$\begin{aligned} \rho^\mathcal{L} \frac{\partial w}{\partial t} = & -\frac{\partial p}{\partial z} + \rho^\mathcal{L} \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \rho g \sin \theta \\ & + \frac{1}{\mu} B^a(t) \left(\frac{\partial b^\mathcal{L}}{\partial y} \sin \psi + \frac{\partial b^\mathcal{L}}{\partial x} \cos \psi \right) + \frac{1}{\mu^\mathcal{L}} \left(\frac{\partial b^\mathcal{L}}{\partial x} \frac{\partial A^\mathcal{L}}{\partial y} - \frac{\partial b^\mathcal{L}}{\partial y} \frac{\partial A^\mathcal{L}}{\partial x} \right) \end{aligned} \quad (4.14c)$$

and the heat equation (3.5) becomes

$$\begin{aligned} \rho^\mathcal{L} c^\mathcal{L} \frac{\partial T^\mathcal{L}}{\partial t} = & k^\mathcal{L} \left(\frac{\partial^2 T^\mathcal{L}}{\partial x^2} + \frac{\partial^2 T^\mathcal{L}}{\partial y^2} \right) + \rho^\mathcal{L} \nu \left(\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right) \\ & + \frac{\eta^\mathcal{L}}{\mu^\mathcal{L}} \left(\left(\frac{\partial b^\mathcal{L}}{\partial y} \right)^2 + \left(\frac{\partial b^\mathcal{L}}{\partial x} \right)^2 + (\nabla^2 A^\mathcal{L})^2 \right) \\ & + \frac{1}{\mu^\mathcal{L}} w B^a(t) \left(\frac{\partial b^\mathcal{L}}{\partial y} \sin \psi + \frac{\partial b^\mathcal{L}}{\partial x} \cos \psi \right) + \frac{1}{\mu^\mathcal{L}} w \left(\frac{\partial A^\mathcal{L}}{\partial y} \frac{\partial b^\mathcal{L}}{\partial x} - \frac{\partial A^\mathcal{L}}{\partial x} \frac{\partial b^\mathcal{L}}{\partial y} \right). \end{aligned} \quad (4.15)$$

4.4.3.2 In the solid $\mathfrak{S}$

In the solid region $\mathfrak{S}$, there is no flow, so we have

$$\frac{\partial^2 A^\mathfrak{S}}{\partial t \partial y} = -\frac{dB^a}{dt} \cos \psi + \eta^\mathfrak{S} \frac{\partial}{\partial y} (\nabla^2 A^\mathfrak{S}), \quad (4.16a)$$

$$-\frac{\partial^2 A^\mathfrak{S}}{\partial t \partial x} = -\frac{dB^a}{dt} \sin \psi - \eta^\mathfrak{S} \frac{\partial}{\partial x} (\nabla^2 A^\mathfrak{S}), \quad (4.16b)$$

$$\frac{\partial b^\mathfrak{S}}{\partial t} = \eta^\mathfrak{S} \nabla^2 b^\mathfrak{S}, \quad (4.16c)$$

$$\rho^\mathfrak{S} c^\mathfrak{S} \frac{\partial T^\mathfrak{S}}{\partial t} = k^\mathfrak{S} \nabla^2 T^\mathfrak{S} + \frac{\eta^\mathfrak{S}}{\mu^\mathfrak{S}} \left(\left(\frac{\partial b^\mathfrak{S}}{\partial y} \right)^2 + \left(\frac{\partial b^\mathfrak{S}}{\partial x} \right)^2 + (\nabla^2 A^\mathfrak{S})^2 \right) \quad (4.16d)$$

4.4.3.3 In the vacua above ($\mathfrak{A}$) and below ($\mathfrak{B}$)

In the vacua, there is no flow and no current, so we simply have

$$\nabla^2 A^\mathfrak{A} = 0, \quad (4.17a) \quad \nabla^2 A^\mathfrak{B} = 0, \quad (4.17c)$$

$$b^\mathfrak{A} = 0, \quad (4.17b) \quad b^\mathfrak{B} = 0. \quad (4.17d)$$

4.4.4 Boundary conditions

4.4.4.1 At the free surface $\mathcal{F}$

Simplifying all of the boundary conditions outlined in Chapter 3.1.2, we obtain

$$A^\mathfrak{A} = A^\mathfrak{E}, \quad (4.18a)$$

$$(1 + \chi^\mathfrak{E}) \frac{\partial A^\mathfrak{A}}{\partial n} = \frac{\partial A^\mathfrak{E}}{\partial n}, \quad (4.18b)$$

$$b^\mathfrak{E} = 0, \quad (4.18c)$$

$$\frac{\partial w}{\partial y} - \frac{\partial w}{\partial x} \frac{\partial h}{\partial x} = 0, \quad (4.18d)$$

$$k^\mathfrak{E} \frac{\partial T^\mathfrak{E}}{\partial n} - Q_{\text{in}}(\mathbf{x}, t) \sqrt{1 + \left(\frac{\partial h}{\partial x} \right)^2} = 0, \quad (4.18e)$$

$$\begin{aligned}
p = & \gamma \varkappa - \frac{\chi^{\mathfrak{L}}}{2\mu_0 (1 + \chi^{\mathfrak{L}}) \left(1 + \left(\frac{\partial h}{\partial x}\right)^2\right)} \left\{ \left(1 + \left(\frac{\partial h}{\partial x}\right)^2\right) b^2 \right. \\
& + \left(B^{\mathfrak{a}}(t) \left(\cos \psi - \frac{\partial h}{\partial x} \sin \psi \right) + \frac{\partial A^{\mathfrak{L}}}{\partial y} + \frac{\partial h}{\partial x} \frac{\partial A^{\mathfrak{L}}}{\partial x} \right)^2 \\
& \left. + (1 + \chi^{\mathfrak{L}}) \left(B^{\mathfrak{a}}(t) \left(\sin \psi - \frac{\partial h}{\partial x} \cos \psi \right) - \frac{\partial A^{\mathfrak{L}}}{\partial x} - \frac{\partial h}{\partial x} \frac{\partial A^{\mathfrak{L}}}{\partial y} \right)^2 \right\} \quad (4.18f)
\end{aligned}$$

4.4.4.2 On the substrate wall $\mathscr{W}$

The boundary conditions on the substrate wall become

$$A^{\mathfrak{S}} = A^{\mathfrak{L}}, \quad (4.19a)$$

$$\begin{aligned}
& (1 + \chi^{\mathfrak{L}}) \frac{\partial A^{\mathfrak{S}}}{\partial n} - (1 + \chi^{\mathfrak{S}}) \frac{\partial A^{\mathfrak{L}}}{\partial n} \\
& + B^{\mathfrak{a}}(t) (\chi^{\mathfrak{S}} - \chi^{\mathfrak{L}}) \left(\frac{dx^{\mathscr{W}}}{ds} \cos \psi + \frac{dy^{\mathscr{W}}}{ds} \sin \psi \right) = 0, \quad (4.19b)
\end{aligned}$$

$$\nabla^2 A^{\mathfrak{L}} - \frac{1}{\Sigma_{\mathfrak{S}\mathfrak{L}}} \left(\frac{1 + \chi^{\mathfrak{L}}}{1 + \chi^{\mathfrak{S}}} \right) \nabla^2 A^{\mathfrak{S}} = 0, \quad (4.19c)$$

$$(1 + \chi^{\mathfrak{L}}) b^{\mathfrak{S}} - (1 + \chi^{\mathfrak{S}}) b^{\mathfrak{L}} = 0, \quad (4.19d)$$

$$\frac{1}{1 + \chi^{\mathfrak{L}}} \frac{\partial b^{\mathfrak{L}}}{\partial n} - \frac{1}{1 + \chi^{\mathfrak{S}}} \frac{1}{\Sigma_{\mathfrak{S}\mathfrak{L}}} \frac{\partial b^{\mathfrak{S}}}{\partial n} + \mu_0 (S^{\mathfrak{L}} - S^{\mathfrak{S}}) \frac{\partial T}{\partial s} = 0, \quad (4.19e)$$

$$\begin{aligned}
& -k^{\mathfrak{L}} \frac{\partial T^{\mathfrak{L}}}{\partial n} + k^{\mathfrak{S}} \frac{\partial T^{\mathfrak{S}}}{\partial n} - \frac{S^{\mathfrak{L}} T^{\mathfrak{L}}}{\mu^{\mathfrak{L}}} \frac{\partial b^{\mathfrak{L}}}{\partial s} + \frac{S^{\mathfrak{S}} T^{\mathfrak{S}}}{\mu^{\mathfrak{S}}} \frac{\partial b^{\mathfrak{S}}}{\partial s} = 0, \quad (4.19f)
\end{aligned}$$

$$T^{\mathfrak{L}} = T^{\mathfrak{S}} \text{ on } \mathscr{W}, \quad (4.19g)$$

where the *solid-to-fluid electrical conductivity ratio* is given by

$$\Sigma_{\mathfrak{S}\mathfrak{L}} = \frac{\sigma^{\mathfrak{S}}}{\sigma^{\mathfrak{L}}}. \quad (4.19h)$$

4.4.4.3 On the base $\mathcal{B}$

We have

$$A^{\mathfrak{S}} = A^{\mathfrak{B}}, \quad (4.20a)$$

$$\frac{\partial A^{\mathfrak{S}}}{\partial y} - (1 + \chi^{\mathfrak{S}}) \frac{\partial A^{\mathfrak{B}}}{\partial y} = \chi^{\mathfrak{S}} B^a(t) \cos \psi, \quad (4.20b)$$

$$b^{\mathfrak{S}} = 0, \quad (4.20c)$$

$$-k^{\mathfrak{S}} \frac{\partial T^{\mathfrak{S}}}{\partial y} - \frac{S^{\mathfrak{S}} T^{\mathfrak{S}}}{\mu^{\mathfrak{S}}} \frac{\partial b^{\mathfrak{S}}}{\partial x} = Q_{\text{out}}(x, t) \text{ on } \mathcal{B}, \quad (4.20d)$$

where $Q_{\text{out}}(x, t)$ represents the rate of heat removal at the base. In practice, the LiMIT trench setup is designed to keep the fluid at a consistent temperature. Therefore, the boundary condition (4.20d) is replaced by the simple Dirichlet boundary condition

$$T^{\mathfrak{S}} = \mathcal{T}_0 \text{ on } \mathcal{B}, \quad (4.21)$$

where $\mathcal{T}_0$ is the temperature maintained by the cooling system.

Name	Symbol	Typical values		
		Default	SLiDE	ST40
Trench length (mm)	$\mathcal{L}$	100	101.6	274
Trench width (mm)	$\mathcal{W}$	1	3	0.5
Trench wall height (mm)	$\mathcal{H}$	1	3	1
Film thickness (mm)	$\mathcal{F}$	0.1	0.25	0.1
Side wall thickness (mm)	t_S	0.15	0.5	0.1
Base thickness (mm)	t_B	0.15	1	0.1
Downward slope angle (rad)	θ	0.1	0	0.316
Applied toroidal B-field (T)	$\mathcal{B}^a$	1	0.07	3
Applied heat flux (MWm^{-2})	$\mathcal{Q}$	1	10	10

Table 4.1: Default dimensional parameter values that will be used for numerical simulations, as well as typical values as seen in setups such as SLiDE, and (eventually) the ST40, which have been ascertained through private communication with both UIUC and Tokamak Energy. The geometrical parameters for an assumed rectangular trench are defined in Figure 4.3.

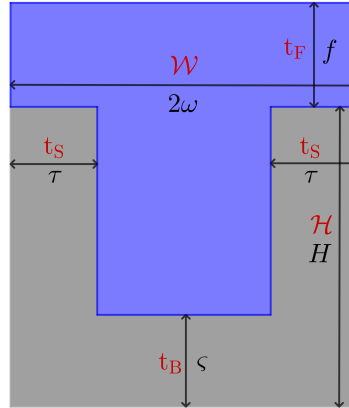


Figure 4.3: A diagram summarising the dimensionless geometrical parameters associated with a rectangular trench, including their dimensional analogues in red.

Name	Symbol	SI Unit	Lithium	SS 316	Tungsten
Electrical conductivity	σ	S m^{-1}	3.46×10^6	1.04×10^6	8.23×10^6
Thermal conductivity	k	$\text{W m}^{-1} \text{K}^{-1}$	4.80×10^1	1.79×10^1	1.40×10^2
Magnetic susceptibility	χ_m	None	$\approx 1.7 \times 10^{-6}$	$\approx 2 \times 10^{-2}$	$\approx 3.2 \times 10^{-7}$
Specific heat capacity	c_p	$\text{m}^2 \text{s}^{-1}$	5.67×10^3	5.35×10^2	5.03×10^1
Seebeck coefficient	S	V K^{-1}	2.4×10^{-5}	-2.69×10^{-6}	5.58×10^{-6}
Mass density	ρ	kg m^{-3}	5.05×10^2	7.83×10^3	1.92×10^4
Kinematic viscosity	ν	$\text{m}^2 \text{s}^{-1}$	1.06×10^{-6}	-	-
Surface tension	γ	N m^{-1}	3.81×10^{-1}	-	-

Table 4.2: Mean values of different thermophysical properties for different materials over the range $T \in [200^\circ\text{C}, 400^\circ\text{C}]$ aside from the Seebeck coefficient which has been averaged over $T \in [200^\circ\text{C}, 277^\circ\text{C}]$ due to lack of measurement data at higher temperatures. These have been obtained from [14, 21, 23, 31, 37, 47, 54, 97, 104].

Name	Symbol	Formula	Typical values		
			Default	SLiDE	ST40
Dimensionless trench half-width	ω	-	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Dimensionless side-wall thickness	τ	$\frac{t_S}{W}$	0.15	$\frac{1}{6}$	0.2
Dimensionless base thickness	ς	$\frac{t_B}{W}$	0.15	$\frac{1}{3}$	0.2
Dimensionless wall height	H	$\frac{\mathcal{H}}{W}$	1	1	2
Dimensionless film thickness	f	$\frac{\mathcal{F}}{W}$	0.1	$\frac{1}{12}$	0.2
Reciprocal of Hartmann number	ϵ	$\frac{1}{\mathcal{B}^a W} \sqrt{\frac{\rho^{\mathcal{E}} \nu}{\sigma^{\mathcal{E}}}}$	0.0124	0.0592	0.0249
Solid-to-fluid electrical conductivity ratio	Σ	$\frac{\sigma^{\mathcal{S}}}{\sigma^{\mathcal{F}}}$	2.3786	0.3006	2.3786
Solid-to-fluid thermal conductivity ratio	K	$\frac{k^{\mathcal{S}}}{k^{\mathcal{F}}}$	2.9167	0.3729	2.9167

Table 4.3: Default model dimensionless parameter values that will be used for numerical simulations, as well as typical values as seen in setups such as SLiDE (made of stainless steel 316), and the ST40 (made of tungsten). These values can be deduced from the thermophysical property values in Table 4.2.

Name	Symbol	Formula	Interpretation	Typical value
Reynolds number	Re	$\frac{\mathcal{U}\mathcal{W}}{\nu}$	$\frac{\text{Inertial effects}}{\text{Viscous effects}}$	10^3
Magnetic Reynolds number	Rm	$\frac{\mathcal{U}\mathcal{W}}{\eta^\mathcal{E}}$	$\frac{\text{Electromagnetic induction}}{\text{Electromagnetic diffusion}}$	10^{-3}
Hartmann number	Ha	$\mathcal{B}^a\mathcal{W}\sqrt{\frac{\sigma^\mathcal{E}}{\rho^\mathcal{E}\nu}}$	$\frac{\text{Electromagnetic effects}}{\text{Viscous effects}}$	10^2
Capillary number	Ca	$\frac{\rho^\mathcal{E}\nu\mathcal{U}}{\gamma}$	$\frac{\text{Viscous forces}}{\text{Surface tension}}$	10^{-3}
Froude number	Fr	$\frac{\mathcal{U}}{\sqrt{g\mathcal{W}}}$	$\frac{\text{Inertial forces}}{\text{Gravitational forces}}$	10^1
Prandtl number	Pr	$\frac{c^\mathcal{E}\rho^\mathcal{E}\nu}{k^\mathcal{E}}$	$\frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}}$	10^{-1}
Brinkman number	Br	$\frac{\rho^\mathcal{E}\nu\mathcal{U}^2}{k^\mathcal{E}\Delta\mathcal{T}}$	$\frac{\text{Viscous dissipation}}{\text{Thermal conduction}}$	10^{-7}
Marangoni number	Ma	$\frac{c^\mathcal{E}\mathcal{W}}{k^\mathcal{E}\nu}\frac{d\gamma}{dT}$	$\frac{\text{Marangoni effects}}{\text{Viscous effects}}$	10^1
Thermoelectric parameter	Te	$\frac{\sigma\Delta S\mathcal{B}\mathcal{L}^2\Delta\mathcal{T}}{\rho\nu^2}$	$\frac{\text{Thermoelectric effects}}{\text{Viscous effects}}$	10^7

Table 4.4: Summary of dimensionless parameters that arise in TEMHD, along with the “thermoelectric parameter” from [25], and order-of-magnitude estimates based on scalings chosen in our model as well as values from Tables 4.1 and 4.2, and $\mathcal{U} \approx 1 \text{ m s}^{-1}$.

4.4.5 Nondimensionalisation

Typical trench geometrical parameters are specified in Table 4.1, thermophysical material property values are shown in Table 4.2, model dimensionless parameters are shown in Table 4.3 and typical order-of-magnitude estimates of common dimensionless groups with respect to the scalings we choose in our model are shown in Table 4.4. We give three different sets of representative parameter values. One is relevant to the “SLiDE” setup which has been used to simulate the LiMIT concept experimentally [109], while another is based on the envisaged design of the ST40 tokamak. Finally, we also give “default” values which are used as a base case when we perform parameter sweeps in Section 4.9.

The problem is nondimensionalised in the following way. We scale all length-scales with the width of the trench $\mathcal{W}$, that is:

$$x = \mathcal{W}\hat{x} \quad \text{and} \quad y = \mathcal{W}\hat{y}, \quad (4.22)$$

and similar nondimensionalisations hold for local coordinates (s, n) tangential and normal to a boundary respectively. Although the dimensionless trench width may be set to unity without loss of generality, it is convenient to allow it to vary, and we denote it by ω . We choose the timescale

$$\mathcal{T} = \frac{\epsilon \mathcal{W}^2}{\nu}, \quad (4.23)$$

which is $\mathcal{O}(10^{-2}\text{s})$ in practice. According to Tokamak Energy, the applied magnetic field in ST40 will likely increase from 0T to 3T within half a second; while plasma instabilities such as edge-localised modes (ELMs) can last between 0.1 and 1ms. The timescale (4.23) should be able to capture the dynamic response in both cases. For the electromagnetic dependent variables, we choose

$$A = \chi^{\mathfrak{S}} \mathcal{W} \mathcal{B}^{\mathfrak{a}} \hat{A} \quad (4.24\text{a}) \quad b = \epsilon \text{Rm} \mathcal{B}^{\mathfrak{a}} \hat{b}, \quad (4.24\text{b})$$

for the fluid dynamical dependent variables, we choose

$$w = \frac{\mathcal{Q} \Delta S}{k^{\mathfrak{S}} \mathcal{B}^{\mathfrak{a}}} \hat{w}, \quad (4.24\text{c}) \quad p = \frac{\gamma}{\mathcal{W}} \hat{p}, \quad (4.24\text{d})$$

and for the temperature, we choose

$$T = T_0 + \left(\frac{\mathcal{Q}\mathcal{W}}{k^{\mathfrak{L}}} \right) \hat{T}. \quad (4.24e)$$

Along with the dimensionless parameters shown in Table 4.4, we also have

$$\epsilon = \frac{1}{\text{Ha}}, \quad (4.25a) \quad \Sigma_{\mathfrak{SL}} = \frac{\sigma_{\mathfrak{S}}}{\sigma_{\mathfrak{L}}}, \quad (4.25c)$$

$$\Gamma = \epsilon \frac{\text{Re}}{\text{Fr}^2} \sin \theta, \quad (4.25b) \quad K_{\mathfrak{SL}} = \frac{k_{\mathfrak{S}}}{k_{\mathfrak{L}}}, \quad (4.25d)$$

where $\epsilon \ll 1$ in practice. $\Sigma_{\mathfrak{SL}}$ is the solid-to-liquid electrical conductivity ratio, so $\Sigma_{\mathfrak{SL}} \ll 1$ represents a trench wall which is perfectly electrically insulating or a fluid that is perfectly electrically conducting, whilst $\Sigma_{\mathfrak{SL}} \gg 1$ represents the opposite. A similar analogy holds for $K_{\mathfrak{SL}}$. Both of these constants are $\mathcal{O}(1)$ in practice. The dimensionless constant Γ comes from the gravity term in the z -component of the Navier-Stokes equations. It is generally small but not completely negligible, so we include this term for the time being.

The temperature T is scaled with the impinging heat flux at the free surface, and the velocity scale is chosen such that the impinging heat flux combined with the difference in Seebeck coefficient $\Delta S = S^{\mathfrak{L}} - S^{\mathfrak{S}}$ drives the flow. The velocity scale in (4.24c) represents a compromise between the impinging heat flux $\mathcal{Q}$ and the applied magnetic field $\mathcal{B}^a$. Everything else being equal, increasing the plasma temperature or laser intensity will linearly increase the flow speed, while increasing the applied magnetic field $\mathcal{B}^a$ will decrease the flow speed. This prediction shows qualitative agreement with the computational work carried out by UIUC [109].

4.4.6 Dimensionless problem

4.4.6.1 Induction equations

Neglecting terms of order $\chi^{\mathfrak{L}}$ and Rm/Re , we have, to leading order,

$$\frac{\epsilon \chi^{\mathfrak{L}}}{\text{Rm}} \hat{\nabla}^2 \hat{A} = \frac{d\hat{B}^a}{d\hat{t}} (\hat{y} \cos \psi - \hat{x} \sin \psi), \quad (4.26a)$$

$$\epsilon \hat{\nabla}^2 \hat{b} = -\hat{B}^a(\hat{t}) \left(\cos \psi \frac{\partial \hat{w}}{\partial \hat{x}} + \sin \psi \frac{\partial \hat{w}}{\partial \hat{y}} \right). \quad (4.26b)$$

If we assume that the flow is steady, then $\hat{A}$ satisfies Laplace's equation to leading order. In any case, we note that $\hat{A}$ decouples from equation (4.26b) for the induced magnetic field $\hat{b}$ along the trench.

4.4.6.2 Fluid equations

Neglecting the pressure gradient term in equation (4.14c), which is of $\mathcal{O}(\epsilon\delta/\text{Ca})$, equations (4.14) become

$$0 = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\text{Ca}}{\epsilon^3 \text{Re}} \frac{d\hat{B}^a}{dt} \hat{B}^a(\hat{t}) (\hat{y} \cos \psi - \hat{x} \sin \psi) \sin \psi, \quad (4.27a)$$

$$0 = -\frac{\partial \hat{p}}{\partial \hat{y}} - \frac{\text{Ca}}{\epsilon^3 \text{Re}} \frac{d\hat{B}^a}{dt} \hat{B}^a(\hat{t}) (\hat{y} \cos \psi - \hat{x} \sin \psi) \cos \psi - \frac{\text{CaRe}}{\text{Fr}^2} \cos \theta, \quad (4.27b)$$

$$\frac{\partial \hat{w}}{\partial \hat{t}} = -\frac{\epsilon\delta}{\text{Ca}} \frac{\partial \hat{p}}{\partial \hat{z}} + \epsilon \left(\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{y}^2} \right) + \hat{B}^a(\hat{t}) \left(\frac{\partial \hat{b}}{\partial \hat{y}} \sin \psi + \frac{\partial \hat{b}}{\partial \hat{x}} \cos \psi \right) + \Gamma. \quad (4.27c)$$

If steady flow is assumed, then equations (4.27) imply that the pressure $\hat{p}$ is of the form

$$\hat{p} = \mathbf{p}_0 - \mathbf{p}_1 \hat{z} - \frac{\text{CaRe}}{\text{Fr}^2} \cos \theta \hat{y}, \quad (4.28)$$

where $\mathbf{p}_0$ and $\mathbf{p}_1$ are constants. Meanwhile, when terms of order $\chi^2 \text{Ca}/\epsilon^2 \text{Rm} \approx 0.04$ are neglected, the normal stress boundary condition (3.23a) reduces to $\hat{p} = \hat{\kappa}$ at $\hat{y} = \hat{h}(x)$, where $\hat{\kappa}$ is the dimensionless curvature. Given periodic boundary conditions at $\hat{x} = \pm\omega$, it follows that the free surface must be flat, i.e. that $\hat{\kappa} = 0$ to lowest order. It also follows that $\mathbf{p}_1 = 0$, so the pressure gradient term in equation (4.27c) can be dropped. In steady state (where $\hat{B}^a = 1$), it therefore just remains to solve

$$\epsilon \nabla^2 \hat{b}^\mathcal{L} + \cos \psi \frac{\partial \hat{w}}{\partial \hat{x}} + \sin \psi \frac{\partial \hat{w}}{\partial \hat{y}} = 0, \quad (4.29a)$$

$$\epsilon \nabla^2 \hat{w} + \cos \psi \frac{\partial \hat{b}^\mathcal{L}}{\partial \hat{x}} + \sin \psi \frac{\partial \hat{b}^\mathcal{L}}{\partial \hat{y}} = -\Gamma. \quad (4.29b)$$

4.4.6.3 Heat equation

Most of the terms in the heat equation (4.15) are negligible to leading order, leaving

$$\text{Pr} \frac{\partial \hat{T}}{\partial \hat{t}} = \epsilon \left(\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{y}^2} \right). \quad (4.30)$$

The temperature $\hat{T}$ therefore decouples from all the other dependent variables to leading order, even in the unsteady case. We note that, Pr is moderately small compared to ϵ but not completely negligible.

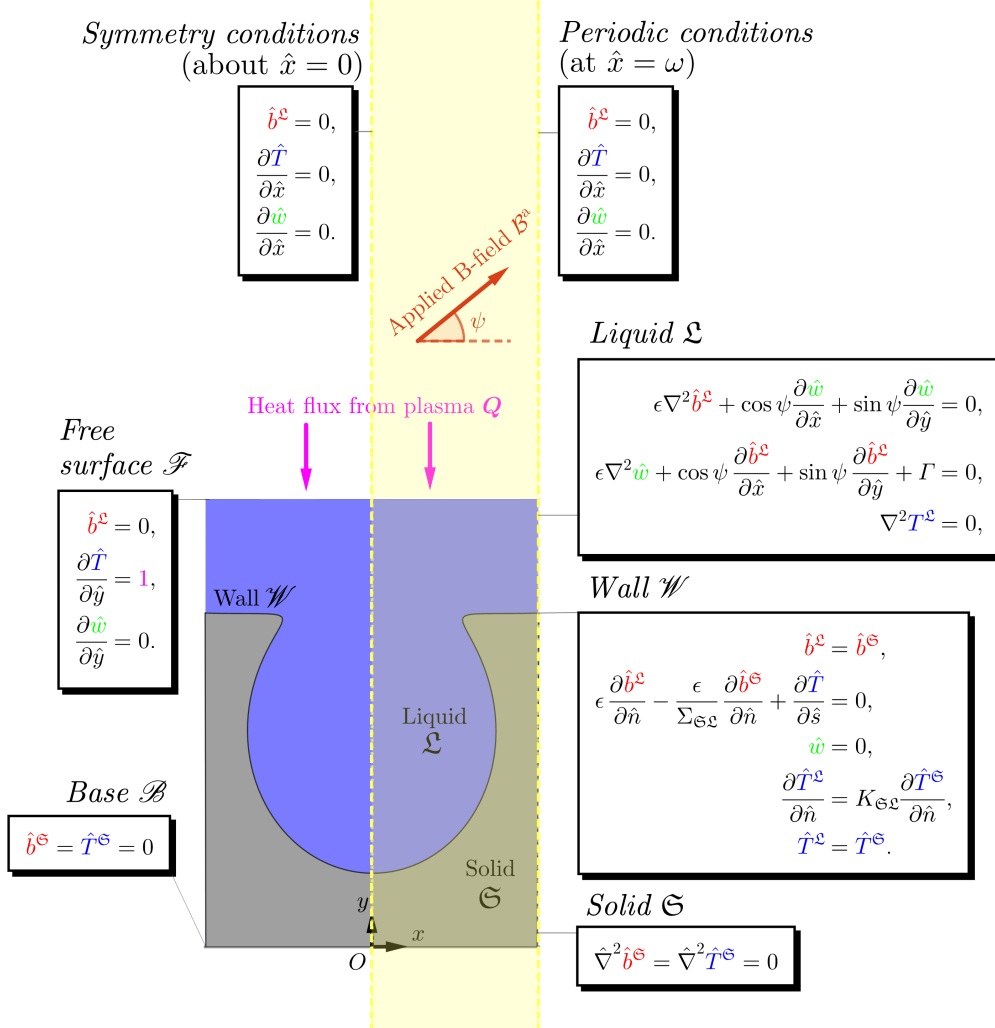


Figure 4.4: A summary of the steady TEMHD trench problem to leading order, assuming symmetry about $\hat{x} = 0$.

4.5 Steady problem

In subsequent subsections, we neglect time dependence of the applied magnetic field and the impinging heat flux at the free surface. To this end, we set $\hat{B}^a(t) = \hat{Q}^a(x, t) = 1$ and assume the dependent variables are likewise time-independent. With this simplification, the dimensionless problem to leading order is shown in Figure 4.4. Note that this figure does not show the equations for the pressure $\hat{p}$ or the induced magnetic field potential $\hat{A}$, because they decouple from the main problem at leading order, and are thus not of primary interest. For simplicity, we restrict attention to trenches with reflectional symmetry about $\hat{x} = 0$.

In the absence of gravity, the only inhomogeneous term in the problem is the right-hand side of the thermal free-surface boundary condition, representing the imposed heat flux, which drives the whole system. The resulting temperature gradient induces a magnetic field through the Seebeck boundary condition at the solid-liquid interface, and the induced field in turn generates the flow along the trench. Otherwise, the flow is driven by a combination of the normalised heat flux at the free surface, and the $+F$ term in the momentum equation, which incorporates gravitational forcing, but is also indicative of what might happen if the pressure gradient is constant, as it was in Chapter 3.

4.6 Numerical solutions

We solve the problem shown in Figure 4.4 numerically using finite element methods employed in COMSOL Multiphysics. Two-dimensional surface plots of the temperature, velocity and magnetic field in each of the three parameter cases shown in Tables 4.1 and 4.3 are shown in Figure 4.5.

In all three cases, the temperature appears to be approximately a linear function of $\hat{y}$, reaching its peak at the free surface where the external heat flux is being applied. The temperature profiles in plots **(a)** and **(g)** show some variation in the $\hat{x}$ -direction near the walls due to the change in thermal conductivity between the liquid and the trench walls, and that shown in plot **(d)** demonstrates a subtle amount of variation near the top-left and top-right corners.

Meanwhile, the velocity plots **(b,e,h)** show an obvious boundary layer structure which arises because the MHD problem becomes singular as $\epsilon \rightarrow 0$. There are

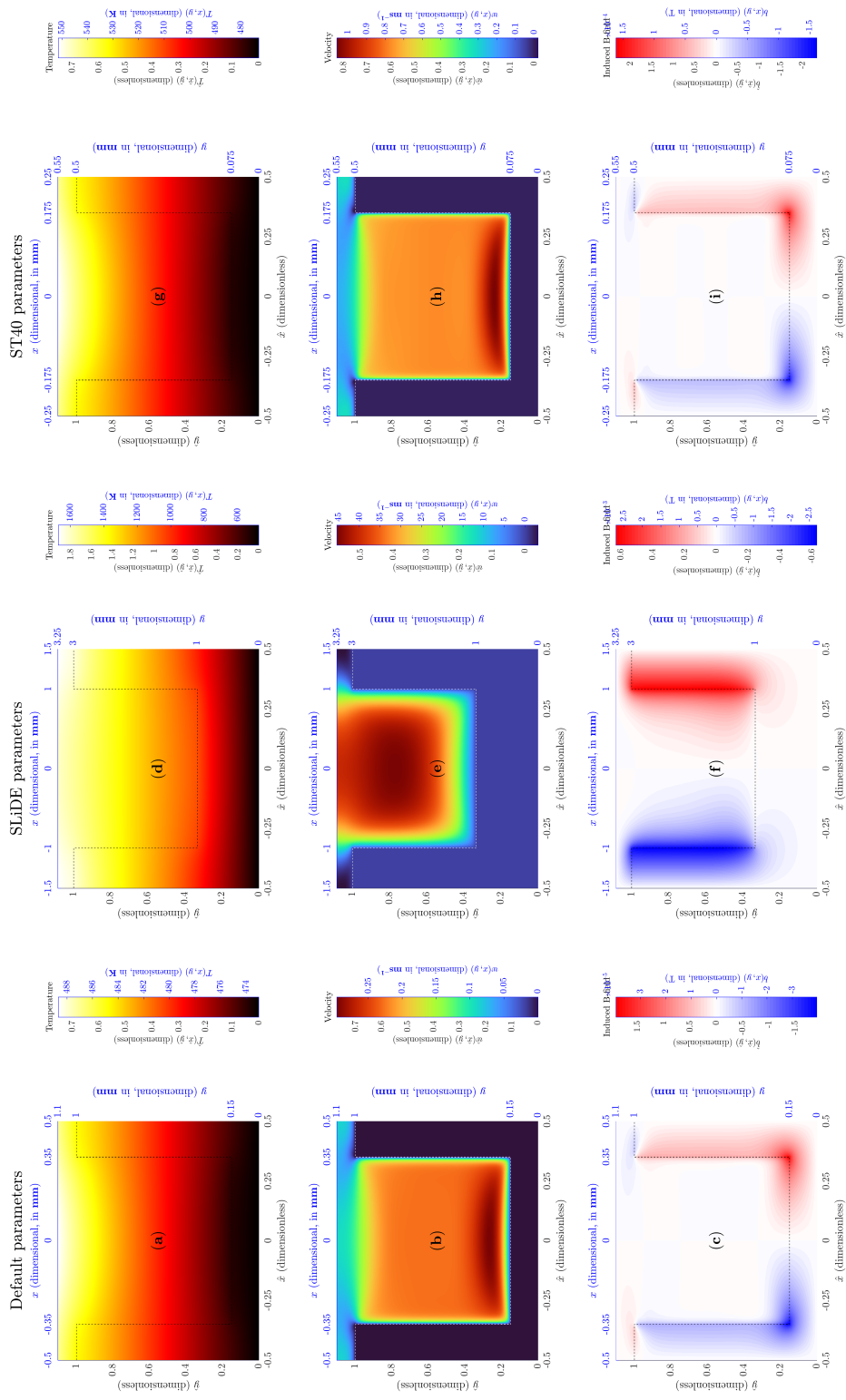


Figure 4.5: Plots of the temperatures, velocities and induced magnetic fields for each of the parameter cases shown in Tables 4.1 and 4.3.

boundary layers of width $\mathcal{O}(\epsilon)$ near the vertical walls, as well as weaker “side” layer of width $\mathcal{O}(\epsilon^{1/2})$ near the base and near the tops of the walls. This behaviour is consistent with the predictions made below using asymptotics in Section 4.8. These layers are regions where MHD drag suddenly becomes important. Outside these layers, it appears that the velocity inside the trench approaches a constant as $\epsilon \rightarrow 0$. The boundary layer structure can also be seen in the induced magnetic field plots **(c,f,i)**, and their behaviour shows a magnetic field dipole.

Another important feature revealed in particular by plots **(b)** and **(h)** is that there is a conducting-base velocity jet near the bottom of the trench. This jet arises due to fact that the electrically conducting base is in direct contact with a vacuum, which limits the amount of space through which electrical current loops can close. As a result, they have to be closed partially in the fluid. The velocity jet appears near the wall parallel to the magnetic field, as observed by Hunt [33].

Regarding the SLiDE setup, we note that the maximum temperature shown in plot **(d)** is close to the boiling point of lithium (1615 K), and that the velocities shown in plot **(e)** are much faster than what would be observed in practice. This could be because the magnetic field in this case is relatively weak (0.07T) whilst the impinging heat flux is fusion-relevant (10MW/m²), both of which increase the velocity scaling in equation (4.24c). Furthermore, liquid lithium becomes corrosive at temperatures above 680K, and this has not been considered in our modelling work.

4.7 Thin wall approximation

In general, the leading-order steady problem shown in Figure 4.4 can only be tackled analytically when the thicknesses of the side wall and base are small. For example, the side wall of the trench is shown in Figure 4.6, assuming that the wall thickness τ is constant.

Given the symmetry condition $\hat{b}^{\mathcal{L}} = 0$ at $\hat{x} = \omega$, we can approximate $\partial \hat{b}^{\mathcal{L}} / \partial \hat{x} \sim -\hat{b}^{\mathcal{L}} / \tau + \mathcal{O}(\tau)$, at $\hat{x} = \omega - \tau$, so the induced field in the liquid approximately satisfies the Robin boundary condition

$$\epsilon \frac{\partial \hat{b}^{\mathcal{L}}}{\partial \hat{x}} + \frac{\epsilon}{\Sigma_{\mathcal{L}}^{\mathcal{S}} \tau} \hat{b}^{\mathcal{L}} = \frac{\partial \hat{T}^{\mathcal{L}}}{\partial \hat{y}} \text{ at } \hat{x} = \omega. \quad (4.31)$$

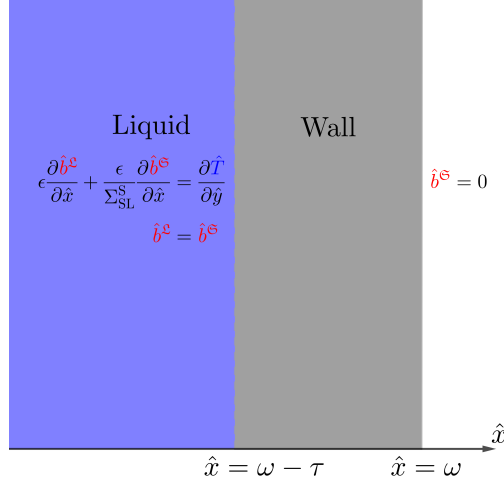


Figure 4.6: A diagram showing the right-hand wall of the trench and the corresponding boundary conditions on the induced magnetic field.

Similarly, at the base (assumed to have constant thickness $\varsigma \ll 1$), we obtain the effective boundary condition

$$\epsilon \frac{\partial \hat{b}^{\mathcal{L}}}{\partial \hat{y}} - \frac{\epsilon}{\Sigma_{\text{SL}}^{\text{B}}} \hat{b}^{\mathcal{L}} = -\frac{\partial \hat{T}^{\mathcal{L}}}{\partial \hat{x}} \text{ at } \hat{y} = 0. \quad (4.32)$$

Here and below, the superscripts S and B refer to the values at the side-walls and the base, respectively. Next we turn our attention to the thermal boundary conditions. At the base, an analogous approximation to the above results in the boundary condition

$$\frac{\partial \hat{T}^{\mathcal{L}}}{\partial \hat{y}} - \frac{K_{\text{SL}}^{\text{B}}}{\varsigma} \hat{T}^{\mathcal{L}} = 0 \text{ at } \hat{y} = 0. \quad (4.33)$$

Following the same approach at the side wall, we just get $\partial \hat{T}^{\mathcal{L}} / \partial \hat{x} = 0$ at leading order, and more effort is required to capture the effect of wall thermal conductivity. As shown in Figure 4.7, we make the local coordinate transformation $\hat{x} = (\omega - \tau) + \tau \hat{X}$ and $\hat{T}^{\text{S}}(\hat{x}, \hat{y}) = \Phi(\hat{X}, \hat{y})$.

By integrating across the solid region, we find the exact relation

$$\frac{\text{d}^2}{\text{d}\hat{y}^2} \int_0^1 \Phi(\hat{X}, \hat{y}) \text{d}\hat{X} = \frac{1}{K_{\text{SL}}^{\text{S}}} \frac{\partial \hat{T}}{\partial \hat{x}}(\omega - \tau, \hat{y}) \quad (4.34)$$

which represents heat conservation in the side wall. But also we deduce from Figure 4.7 that $\Phi(\hat{X}, \hat{y}) \sim \hat{T}^{\mathcal{L}}(\omega, \hat{y})$ to lowest order in τ , and hence deduce the

$$\begin{aligned}
& \hat{x} = (\omega - \tau) + \tau \hat{X} \\
& \hat{T}^{\mathfrak{S}}(\hat{x}, \hat{y}) = \Phi(\hat{X}, \hat{y}) \\
& \frac{\partial^2 \Phi}{\partial \hat{X}^2} + \tau^2 \frac{\partial^2 \Phi}{\partial \hat{y}^2} = 0 \\
& \frac{\partial \hat{T}^{\mathfrak{E}}}{\partial \hat{x}} - K_{\text{SL}}^{\mathfrak{S}} \frac{\partial \Phi}{\partial \hat{X}} = 0 \\
& \Phi = \hat{T}^{\mathfrak{E}} \\
& \hat{X} = 0 \qquad \hat{X} = 1 \\
& \frac{\partial \Phi}{\partial \hat{X}} = 0
\end{aligned}$$

Figure 4.7: The thin-wall problem for the temperature inside the side wall.

effective boundary condition

$$\frac{\partial^2 \hat{T}^{\mathfrak{E}}}{\partial \hat{y}^2} = \frac{1}{K_{\text{SL}}^{\mathfrak{S}} \tau} \frac{\partial \hat{T}^{\mathfrak{E}}}{\partial \hat{x}} \quad \text{at } \hat{x} = \omega. \quad (4.35)$$

The full thin-wall problem is illustrated in Figure 4.8 where we introduce several further dimensionless parameters

$$c_{\text{S}} = \frac{\epsilon}{\Sigma_{\mathfrak{SE}}^{\text{S}} \tau}, \quad (4.36\text{a}) \qquad \kappa_{\text{S}} = \frac{1}{K_{\mathfrak{SE}}^{\text{S}} \tau}, \quad (4.36\text{c})$$

$$c_{\text{B}} = \frac{\epsilon^{1/2}}{\Sigma_{\mathfrak{SE}}^{\text{B}} \varsigma}, \quad (4.36\text{b}) \qquad \kappa_{\text{B}} = \frac{K_{\mathfrak{SE}}^{\text{B}}}{\varsigma}. \quad (4.36\text{d})$$

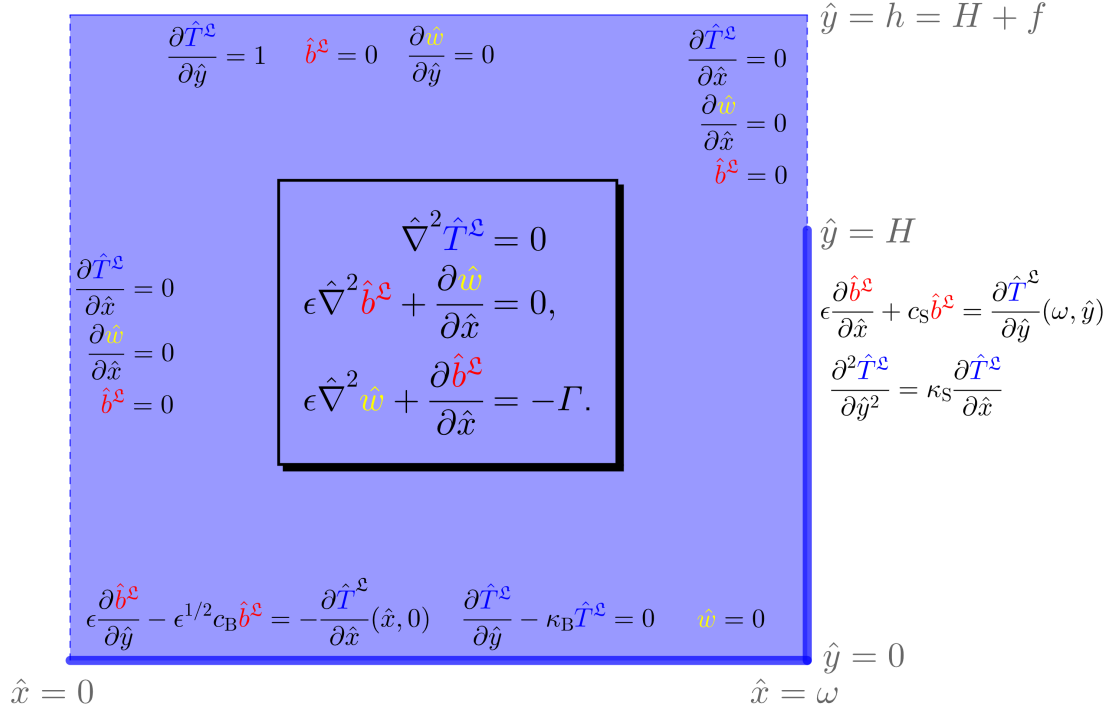


Figure 4.8: Summary of the thin-wall problem for a rectangular trench.

In practice, the side walls and the base are made of the same material; however, the most interesting behaviour occurs when the above constants are all $\mathcal{O}(1)$ while $\epsilon \rightarrow 0$, where ϵ is the reciprocal of the Hartmann number. We note that the temperature in the thin-wall problem is still decoupled from the magnetohydrodynamic problem, and is to be determined numerically in general. We proceed to solve the remaining magnetohydrodynamic problem, treating the liquid temperature as an in principle known function in the subsequent analysis.

4.8 Asymptotic analysis

We solve the 2D TEMHD trench problem in the limit of small ϵ using matched asymptotic expansions and solving problems in a range of regions around the domain. To simplify matters, we take $\psi = 0$, so that the applied magnetic field is strictly in the toroidal direction in a tokamak configuration.

As in Chapter 3, one can show that there is a boundary layer of thickness $\mathcal{O}(\epsilon)$ near $\hat{x} = \omega$, below the top of the trench wall, and a “side” layer near the

base $\hat{y} = 0$ of thickness $\mathcal{O}(\epsilon^{1/2})$. The flows above and below the trench walls are coupled through a “cross-over” region near $\hat{y} = H$, which is also of width $\mathcal{O}(\epsilon^{1/2})$. To obtain complete estimates of the velocity and induced magnetic field, one needs to perform local rescalings of the problem close to each of these boundary layers. In some cases, more than one rescaling is required, in so-called “corner” regions, as first noted in [94]. A diagram illustrating all the regions in which rescalings need to be performed, as well as their nomenclature, is shown in Figure 4.9. The problem in the “cross-over” region is beyond the scope of this thesis, and will not be solved.

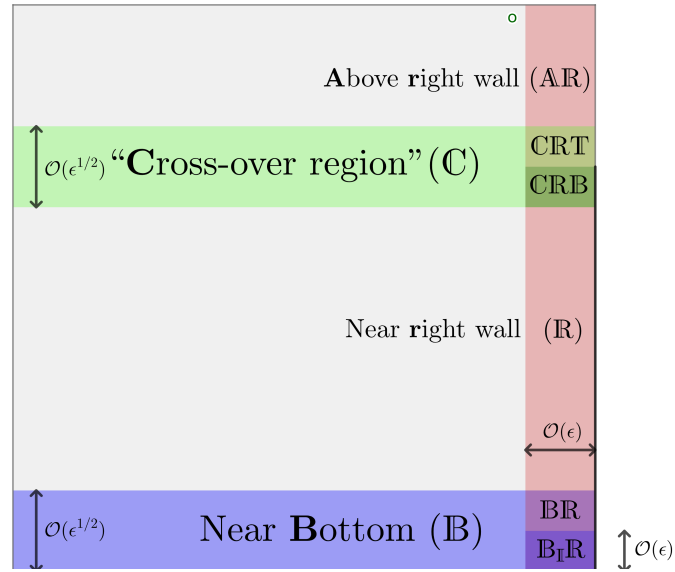


Figure 4.9: A diagram illustrating all the boundary layers and regions in which the dependent variables need to be rescaled.

4.8.1 Outer problem in the trench

The outer problem in the bulk of the trench remains as shown inside the shadowed box in Figure 4.8. If we seek solutions of the forms

$$\hat{b}^{\mathcal{E}} \sim \hat{b}_0^{\mathcal{E}} + \epsilon \hat{b}_1^{\mathcal{E}} + \dots \quad (4.37a) \quad \hat{w} \sim \hat{w}_0 + \epsilon \hat{w}_1 + \dots \quad (4.37b)$$

then one can find a solution that satisfies all the boundary conditions, apart from those at $\hat{x} = 0$. At leading order, such solutions are given by

$$\hat{w}_0(\hat{x}, \hat{y}) = \hat{w}_0^-(\hat{y}) \quad \text{in } y < H, \quad (4.38a) \quad \hat{b}_0^{\mathcal{E}}(\hat{x}, \hat{y}) = -\Gamma \hat{x} \quad \text{in } y < H. \quad (4.38b)$$

The outer problem on its own does not completely determine the form of $\hat{w}_0^-(\hat{y})$, which must therefore be found by matching with solutions inside the boundary layers. Furthermore, we anticipate that this function can be different above and below the top of the wall.

4.8.2 Outer problem above the trench

When $\hat{y} > H$, integrating the Navier-Stokes equation in Figure 4.8 from $\hat{x} = 0$ to $\hat{x} = \omega$, we get

$$0 = \int_0^\omega \left(\epsilon \hat{\nabla}^2 \hat{w} + \frac{\partial \hat{b}^{\mathcal{E}}}{\partial \hat{x}} + \Gamma \right) d\hat{x}, \quad (4.39)$$

$$= \epsilon \frac{d^2}{d\hat{y}^2} \int_0^\omega \hat{w}(\hat{x}, \hat{y}) d\hat{x} + \Gamma \omega, \quad (4.40)$$

which implies that

$$\frac{d}{d\hat{y}} \int_0^\omega \hat{w}(\hat{x}, \hat{y}) d\hat{x} = \frac{\Gamma \omega}{\epsilon} (H + f - \hat{y}). \quad (4.41)$$

We deduce that $\hat{w}$ must be order $1/\epsilon$ above the trench, and therefore let

$$\hat{w}(\hat{x}, \hat{y}) = \frac{1}{\epsilon} \hat{W}(\hat{x}, \hat{y}) \quad \text{in } \hat{y} > H. \quad (4.42)$$

Then, we get that $\hat{W}(\hat{x}, \hat{y})$ satisfies the exact relation

$$\int_0^\omega \hat{W}(\hat{x}, \hat{y}) d\hat{x} = \frac{\Gamma\omega}{2}(\hat{y} - \hat{H})(H + 2f - \hat{y}) + \epsilon c_1 \quad (4.43)$$

where c_1 is an $\mathcal{O}(1)$ constant. Now when we seek the solutions in the form

$$\hat{b}^\mathcal{E} \sim \hat{b}_0^\mathcal{E} + \epsilon^2 \hat{b}_1^\mathcal{E} + \dots, \quad (4.44)$$

$$\hat{W} \sim \hat{W}_0 + \epsilon^2 \hat{W}_1 + \dots, \quad (4.45)$$

we find that $\hat{W}_0$ is independent of $\hat{x}$ and thus

$$\hat{W}_0(\hat{x}, \hat{y}) = \frac{\Gamma}{2}(\hat{y} - H)(H + 2f - \hat{y}), \quad (4.46)$$

and it then follows that $\hat{b}^\mathcal{E}(\hat{x}, \hat{y}) = 0$. Proceeding to higher order, one finds that $\hat{W}_1, \hat{W}_2, \dots$ are all constant, while $\hat{b}$ is zero to all algebraic orders in ϵ when $\hat{y} > H$.

4.8.3 Inner problem near the right wall

Letting $\hat{x} = \omega - \epsilon \hat{X}_R$, so that $\hat{w}(\hat{x}, \hat{y}) = \hat{w}_R(\hat{X}_R, \hat{y})$ and $\hat{b}^\mathcal{E}(\hat{x}, \hat{y}) = \hat{b}_R^\mathcal{E}(\hat{X}_R, \hat{y})$, the problem in the boundary layer near the right wall reads

$$\frac{\partial^2 \hat{b}_R^\mathcal{E}}{\partial \hat{X}_R^2} + \epsilon^2 \frac{\partial^2 \hat{b}_R^\mathcal{E}}{\partial \hat{y}^2} - \frac{\partial \hat{w}_R}{\partial \hat{X}_R} = 0 \quad \text{in } \hat{X}_R > 0, \quad 0 < \hat{y} < H, \quad (4.47a)$$

$$\frac{\partial^2 \hat{w}_R}{\partial \hat{X}_R^2} + \epsilon^2 \frac{\partial^2 \hat{w}_R}{\partial \hat{y}^2} - \frac{\partial \hat{b}_R^\mathcal{E}}{\partial \hat{X}_R} + \epsilon \Gamma = 0 \quad \text{in } \hat{X}_R > 0, \quad 0 < \hat{y} < H, \quad (4.47b)$$

$$\hat{w}_R = 0 \quad \text{on } \hat{X}_R = 0, \quad 0 < \hat{y} < H, \quad (4.47c)$$

$$-\frac{\partial \hat{b}_R^\mathcal{E}}{\partial \hat{X}_R} + c_S \hat{b}_R^\mathcal{E} - \frac{\partial \hat{T}^\mathcal{E}}{\partial \hat{y}}(\omega, \hat{y}) = 0 \quad \text{on } \hat{X}_R = 0, \quad 0 < \hat{y} < H, \quad (4.47d)$$

as well as matching conditions with the outer solution (4.38b). Upon seeking asymptotic solutions of the forms

$$\hat{b}_R^\mathcal{E} \sim \hat{b}_{R,0}^\mathcal{E} + \epsilon \hat{b}_{R,1}^\mathcal{E} + \dots, \quad (4.48a) \quad \hat{w}_R \sim \hat{w}_{R,0} + \epsilon \hat{w}_{R,1} + \dots, \quad (4.48b)$$

one obtains the leading-order problem

$$\frac{\partial^2 \hat{b}_{R,0}^{\mathcal{E}}}{\partial \hat{X}_R^2} - \frac{\partial \hat{w}_{R,0}}{\partial \hat{X}_R} = 0 \quad \text{in } \hat{X}_R > 0, \quad 0 < \hat{y} < H, \quad (4.49a)$$

$$\frac{\partial^2 \hat{w}_{R,0}}{\partial \hat{X}_R^2} - \frac{\partial \hat{b}_{R,0}^{\mathcal{E}}}{\partial \hat{X}_R} = 0 \quad \text{in } \hat{X}_R > 0, \quad 0 < \hat{y} < H, \quad (4.49b)$$

$$\hat{w}_{R,0} = 0 \quad \text{on } \hat{X}_R = 0, \quad 0 < \hat{y} < H, \quad (4.49c)$$

$$-\frac{\partial \hat{b}_{R,0}^{\mathcal{E}}}{\partial \hat{X}_R} + c_S \hat{b}_{R,0}^{\mathcal{E}} - \frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega, \hat{y}) = 0 \quad \text{on } \hat{X}_R = 0, \quad 0 < \hat{y} < H, \quad (4.49d)$$

along with leading-order matching conditions

$$\hat{w}_{R,0}(\hat{X}_R, \hat{y}) \rightarrow \hat{w}_0^-(\hat{y}) \text{ as } \hat{X}_R \rightarrow \infty, \quad (4.49e)$$

$$\hat{b}_{R,0}^{\mathcal{E}}(\hat{X}_R, \hat{y}) \rightarrow -\Gamma\omega \text{ as } \hat{X}_R \rightarrow \infty. \quad (4.49f)$$

The problem (4.49) has solution

$$\hat{b}_{R,0}^{\mathcal{E}}(\hat{X}_R, \hat{y}) = \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega, \hat{y}) + \omega \Gamma c_S \right) e^{-\hat{X}_R} - \omega \Gamma, \quad (4.50a)$$

$$\hat{w}_{R,0}(\hat{X}_R, \hat{y}) = \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega, \hat{y}) + \omega \Gamma c_S \right) \left(1 - e^{-\hat{X}_R} \right), \quad (4.50b)$$

which also determines the leading-order outer velocity in the trench, namely

$$\hat{w}_0^-(\hat{y}) = \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega, \hat{y}) + \omega \Gamma c_S \right). \quad (4.50c)$$

This simple formula for the bulk velocity demonstrates the importance of temperature gradients in driving TEMHD flow. As will be demonstrated below, the expression (4.50c) shows good agreement with numerical results for the bulk velocity, even when the side-wall thickness τ is not especially small. The formula (4.50c) also resembles that which was obtained by Xu [108] in the limit of large Ha.

The solutions we have obtained thus far allow us to create a composite expression based on Van Dyke's matching principle. Provided that we are not close to

any horizontal boundary layers near the top or bottom of the trench, for a wall of thickness τ , we have

$$\hat{w}(\hat{x}, \hat{y}) \approx \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega - \tau, \hat{y}) + (\omega - \tau) \Gamma c_S \right) \left(1 - 2e^{-(\omega - \tau)/\epsilon} \cosh \left(\frac{\hat{x}}{\epsilon} \right) \right), \quad (4.51a)$$

$$\hat{b}^{\mathcal{E}}(\hat{x}, \hat{y}) \approx \frac{2}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega - \tau, \hat{y}) + (\omega - \tau) \Gamma c_S \right) e^{-(\omega - \tau)/\epsilon} \sinh \left(\frac{\hat{x}}{\epsilon} \right) - \Gamma \hat{x}. \quad (4.51b)$$

From equation (4.51b), one can construct an approximate expression for the overall induced magnetic field including that inside the solid, assuming that it is approximately linear inside the trench walls $|\hat{x}| \geq \omega - \tau$,

$$\hat{b}(\hat{x}, \hat{y}) \approx \begin{cases} \hat{b}^{\mathcal{E}}(\hat{x}, \hat{y}) & \text{if } |\hat{x}| \leq \omega - \tau, \\ \hat{b}^{\mathcal{E}}(\omega - \tau, \hat{y}) \left(\frac{\omega - |\hat{x}|}{\tau} \right) & \text{if } |\hat{x}| > \omega - \tau. \end{cases} \quad (4.52)$$

We will see that the above expressions show excellent agreement with the numerical results in Section 4.9, even when the side-wall thickness τ is not particularly small.

4.8.4 Inner problem near the base

Upon rescaling $\hat{y} = \epsilon^{1/2} \hat{Y}_B$, with $\hat{w}(\hat{x}, \hat{y}) = \hat{w}_B(\hat{x}, \hat{Y}_B)$ and $\hat{b}^\mathcal{E}(\hat{x}, \hat{y}) = \hat{b}_B^\mathcal{E}(\hat{x}, \hat{Y}_B)$, the problem near the boundary layer at the bottom becomes

$$\epsilon \frac{\partial^2 \hat{b}_B^\mathcal{E}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{b}_B^\mathcal{E}}{\partial \hat{Y}_B^2} + \frac{\partial \hat{w}_B}{\partial \hat{x}} = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.53a)$$

$$\epsilon \frac{\partial^2 \hat{w}_B}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}_B}{\partial \hat{Y}_B^2} + \frac{\partial \hat{b}_B^\mathcal{E}}{\partial \hat{x}} + \Gamma = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.53b)$$

$$\hat{w}_B = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.53c)$$

$$-\epsilon^{1/2} \frac{\partial \hat{b}_B^\mathcal{E}}{\partial \hat{Y}_B} + \epsilon^{1/2} c_B \hat{b}_B^\mathcal{E} + \frac{\partial \hat{T}^\mathcal{E}}{\partial \hat{x}}(\hat{x}, 0) = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.53d)$$

$$\frac{\partial \hat{w}_B}{\partial \hat{x}} = 0 \quad \text{on } \hat{x} = 0, \quad (4.53e)$$

$$\hat{b}_B^\mathcal{E} = 0 \quad \text{on } \hat{x} = 0, \quad (4.53f)$$

along with matching conditions as $\hat{Y}_B \rightarrow \infty$ and as $\hat{x} \rightarrow \omega$. For boundary condition (4.53d) to make sense, we appear to need either $c_B = \mathcal{O}(\epsilon^{-1/2})$ or the induced field $\hat{b}^\mathcal{E}$ to be larger than assumed up until now. However, we note that the flow in the LiMIT setup is believed to be principally driven by the imposed vertical temperature gradient. Indeed, we note that the problem for $\hat{T}^\mathcal{E}$ illustrated in Figure 4.8 is satisfied identically by $\hat{T}^\mathcal{E}(\hat{x}, \hat{y}) = \hat{y} + 1/\kappa_B$ in the physically relevant regime where $\kappa_B \gg 1$. We therefore assume henceforth that

$$\frac{\partial \hat{T}^\mathcal{E}}{\partial \hat{x}}(\hat{x}, 0) = 0. \quad (4.53g)$$

Upon seeking asymptotic expansion solutions of the forms

$$\hat{b}_B^\mathcal{E} \sim \hat{b}_{B,0}^\mathcal{E} + \epsilon \hat{b}_{B,1}^\mathcal{E} + \dots \quad (4.54a) \quad \hat{w}_B \sim \hat{w}_{B,0} + \epsilon \hat{w}_{B,1} + \dots \quad (4.54b)$$

we obtain leading-order problem

$$\frac{\partial^2 \hat{b}_{B,0}^{\mathcal{E}}}{\partial \hat{Y}_B^2} + \frac{\partial \hat{w}_{B,0}}{\partial \hat{x}} = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.55a)$$

$$\frac{\partial^2 \hat{w}_{B,0}}{\partial \hat{Y}_B^2} + \frac{\partial \hat{b}_{B,0}^{\mathcal{E}}}{\partial \hat{x}} + \Gamma = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.55b)$$

$$\hat{w}_{B,0} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.55c)$$

$$\frac{\partial \hat{b}_{B,0}^{\mathcal{E}}}{\partial \hat{Y}_B} - c_B \hat{b}_{B,0}^{\mathcal{E}} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.55d)$$

$$\frac{\partial \hat{w}_{B,0}}{\partial \hat{x}} = 0 \quad \text{on } \hat{x} = 0, \quad (4.55e)$$

$$\hat{b}_{B,0}^{\mathcal{E}} = 0 \quad \text{on } \hat{x} = 0, \quad (4.55f)$$

$$\hat{w}_{B,0} \rightarrow \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}} (\omega, 0) + \omega \Gamma c_S \right) \text{ as } \hat{Y}_B \rightarrow \infty, \quad (4.55g)$$

$$\hat{b}_{B,0}^{\mathcal{E}} \rightarrow -\Gamma \hat{x} \text{ as } \hat{Y}_B \rightarrow \infty, \quad (4.55h)$$

as well as the matching condition

$$(1 + c_S) \hat{w}_{B,0} + c_S \hat{b}_{B,0}^{\mathcal{E}} = \frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}} (\omega, 0) \quad \text{at } \hat{x} = \omega. \quad (4.55i)$$

The matching condition (4.55i) arises by rescaling both $\hat{x}$ and $\hat{y}$ as shown below, in the “inner-bottom-right corner” problem. This problem is entirely analogous to the leading-order version of the problem represented by equations (4.47), in that the solutions share the same exponential structure and satisfy the boundary conditions at $\hat{x} = \omega$, although they cannot satisfy the boundary conditions at $\hat{y} = 0$, and this would necessitate a further rescaling giving a “bottom-right-bottom” problem.

It is possible to decompose the problem given by equations (4.55) into two independent problems, each of which has only one inhomogeneous boundary condition. By subtracting off the far-field behaviour as $\hat{Y}_B \rightarrow \infty$,

$$\hat{w}_{B,0}(\hat{x}, \hat{Y}_B) = \tilde{\hat{w}}_{B,0}(\hat{x}, \hat{Y}_B) + \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}} (\omega, 0) + \omega \Gamma c_S \right), \quad (4.56a)$$

$$\hat{b}_{B,0}^{\mathcal{E}}(\hat{x}, \hat{Y}_B) = \tilde{\hat{b}}_{B,0}^{\mathcal{E}}(\hat{x}, \hat{Y}_B) - \Gamma \hat{x}, \quad (4.56b)$$

and decomposing the solution into two parts, such that

$$\tilde{w}_{B,0}(\hat{x}, \hat{Y}_B) = \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathfrak{E}}}{\partial \hat{y}}(\omega, 0) + \omega \Gamma c_S \right) \tilde{w}_{B,0}^{[1]}(\hat{x}, \hat{Y}_B) + \Gamma \tilde{w}_{B,0}^{[2]}(\hat{x}, \hat{Y}_B), \quad (4.57a)$$

$$\tilde{b}_{B,0}^{\mathfrak{E}}(\hat{x}, \hat{Y}_B) = \frac{1}{1 + c_S} \left(\frac{\partial \hat{T}^{\mathfrak{E}}}{\partial \hat{y}}(\omega, 0) + \omega \Gamma c_S \right) \tilde{b}_{B,0}^{\mathfrak{E}[1]}(\hat{x}, \hat{Y}_B) + \Gamma \tilde{b}_{B,0}^{\mathfrak{E}[2]}(\hat{x}, \hat{Y}_B), \quad (4.57b)$$

we find that the two problems to be solved are given by

$$\frac{\partial^2 \tilde{b}_{B,0}^{\mathfrak{E}[1]}}{\partial \hat{Y}_B^2} + \frac{\partial \tilde{w}_{B,0}^{[1]}}{\partial \hat{x}} = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.58a)$$

$$\frac{\partial^2 \tilde{w}_{B,0}^{[1]}}{\partial \hat{Y}_B^2} + \frac{\partial \tilde{b}_{B,0}^{\mathfrak{E}[1]}}{\partial \hat{x}} = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.58b)$$

$$\tilde{w}_{B,0}^{[1]} = -1 \quad \text{on } \hat{Y}_B = 0, \quad (4.58c)$$

$$\frac{\partial \tilde{b}_{B,0}^{\mathfrak{E}[1]}}{\partial \hat{Y}_B} - c_B \tilde{b}_{B,0}^{\mathfrak{E}[1]} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.58d)$$

$$\frac{\partial \tilde{w}_{B,0}^{[1]}}{\partial \hat{x}} = 0 \quad \text{on } \hat{x} = 0, \quad (4.58e)$$

$$\tilde{b}_{B,0}^{\mathfrak{E}[1]} = 0 \quad \text{on } \hat{x} = 0, \quad (4.58f)$$

$$\tilde{w}_{B,0}^{[1]} \rightarrow 0 \quad \text{as } \hat{Y}_B \rightarrow \infty, \quad (4.58g)$$

$$\tilde{b}_{B,0}^{\mathfrak{E}[1]} \rightarrow 0 \quad \text{as } \hat{Y}_B \rightarrow \infty, \quad (4.58h)$$

$$(1 + c_S) \tilde{w}_{B,0}^{[1]} + c_S \tilde{b}_{B,0}^{\mathfrak{E}[1]} = 0 \quad \text{on } \hat{x} = \omega, \quad (4.58i)$$

and

$$\frac{\partial^2 \tilde{b}_{B,0}^{[2]}}{\partial \hat{Y}_B^2} + \frac{\partial \tilde{w}_{B,0}^{[2]}}{\partial \hat{x}} = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.59a)$$

$$\frac{\partial^2 \tilde{w}_{B,0}^{[2]}}{\partial \hat{Y}_B^2} + \frac{\partial \tilde{b}_{B,0}^{[2]}}{\partial \hat{x}} = 0 \quad \text{in } \hat{Y}_B > 0, \quad 0 < \hat{x} < \omega, \quad (4.59b)$$

$$\tilde{w}_{B,0}^{[2]} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.59c)$$

$$\frac{\partial \tilde{b}_{B,0}^{[2]}}{\partial \hat{Y}_B} - c_B \tilde{b}_{B,0}^{[2]} = -c_B \hat{x} \quad \text{on } \hat{Y}_B = 0, \quad (4.59d)$$

$$\frac{\partial \tilde{w}_{B,0}^{[2]}}{\partial \hat{x}} = 0 \quad \text{on } \hat{x} = 0, \quad (4.59e)$$

$$\tilde{b}_{B,0}^{[2]} = 0 \quad \text{on } \hat{x} = 0, \quad (4.59f)$$

$$\tilde{w}_{B,0}^{[2]} \rightarrow 0 \quad \text{as } \hat{Y}_B \rightarrow \infty, \quad (4.59g)$$

$$\tilde{b}_{B,0}^{[2]} \rightarrow 0 \quad \text{as } \hat{Y}_B \rightarrow \infty, \quad (4.59h)$$

$$(1 + c_S) \tilde{w}_{B,0}^{[2]}(\omega, \hat{Y}_B) + c_S \tilde{b}_{B,0}^{[2]}(\omega, \hat{Y}_B) = 0 \quad \text{on } \hat{x} = \omega. \quad (4.59i)$$

In either of the two cases, the solution to the problem may be expressed in the form (with $j = 1, 2$)

$$\tilde{w}_{B,0}^{[j]}(\hat{x}, \hat{Y}_B) = \Re \left(\sum_{n=0}^{\infty} \mathcal{U}_n^{[j]} e^{-\mu_n \hat{Y}_B} \cosh \mu_n^2 \hat{x} \right) \quad (4.60a)$$

$$\tilde{b}_{B,0}^{[j]}(\hat{x}, \hat{Y}_B) = -\Re \left(\sum_{n=0}^{\infty} \mathcal{U}_n^{[j]} e^{-\mu_n \hat{Y}_B} \sinh \mu_n^2 \hat{x} \right) \quad (4.60b)$$

where

$$\mu_n = \frac{1}{\sqrt{\omega}} \sqrt{\phi + \left(n + \frac{1}{2}\right) i\pi} \quad \text{with} \quad \phi = \operatorname{arctanh} \left(\frac{c_S}{1 + c_S} \right), \quad (4.61)$$

The complex coefficients $\mathcal{U}_n$ are determined in principle from the remaining

inhomogeneous boundary conditions; either (4.58c) or (4.59d), i.e.

$$\Re \left(\sum_{n=0}^{\infty} \mathcal{U}_n^{[j]} \cosh \mu_n^2 \hat{x} \right) = \begin{cases} -1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \end{cases} \quad (4.62a)$$

$$\Re \left(\sum_{n=0}^{\infty} (\mu_n + c_B) \mathcal{U}_n^{[j]} \sinh \mu_n^2 \hat{x} \right) = \begin{cases} 0 & \text{if } j = 1, \\ -c_B \hat{x} & \text{if } j = 2, \end{cases} \quad (4.62b)$$

It is not obvious how to evaluate the coefficients $\mathcal{U}_n$ in general, although there are two limiting cases in which they can be found using Fourier series. These limiting cases are **(i)** $c_S \rightarrow 0$, and **(ii)** $c_B \rightarrow \infty$.

4.8.4.1 Case 1: $c_S \rightarrow 0$

In this first limiting case, corresponding to $\phi \rightarrow 0$, the coefficients are given by

$$\mathcal{U}_n^{[1]} = \frac{2(-1)^{n+1}}{(n + \frac{1}{2}) \pi} \left(1 - \frac{i \sqrt{(n + \frac{1}{2}) \pi}}{\sqrt{(n + \frac{1}{2}) \pi + c_B \sqrt{2\omega}}} \right), \quad (4.63a)$$

$$\mathcal{U}_n^{[2]} = \frac{i(-1)^n (2\omega)^{3/2} c_B}{\left(\sqrt{(n + \frac{1}{2}) \pi + c_B \sqrt{2\omega}} \right) (n + \frac{1}{2})^2 \pi^2}. \quad (4.63b)$$

4.8.4.2 Case 2: $c_B \rightarrow \infty$

In the second limiting case, the conditions (4.58c) and (4.59d) may be summarised as

$$\begin{aligned} & \cosh \left(\frac{\phi \hat{x}}{\omega} \right) \sum_{n=0}^{\infty} \Re (\mathcal{U}_n^{[j]}) \cos \left(\left(n + \frac{1}{2} \right) \frac{\pi \hat{x}}{\omega} \right) \\ & - \sinh \left(\frac{\phi \hat{x}}{\omega} \right) \sum_{n=0}^{\infty} \Im (\mathcal{U}_n^{[j]}) \sin \left(\left(n + \frac{1}{2} \right) \frac{\pi \hat{x}}{\omega} \right) = \begin{cases} -1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \end{cases} \end{aligned} \quad (4.64a)$$

$$\begin{aligned} & \sinh \left(\frac{\phi \hat{x}}{\omega} \right) \sum_{n=0}^{\infty} \Re (\mathcal{U}_n^{[j]}) \cos \left(\left(n + \frac{1}{2} \right) \frac{\pi \hat{x}}{\omega} \right) \\ & - \cosh \left(\frac{\phi \hat{x}}{\omega} \right) \sum_{n=0}^{\infty} \Im (\mathcal{U}_n^{[j]}) \sin \left(\left(n + \frac{1}{2} \right) \frac{\pi \hat{x}}{\omega} \right) = \begin{cases} 0 & \text{if } j = 1, \\ -\hat{x} & \text{if } j = 2. \end{cases} \end{aligned} \quad (4.64b)$$

It follows that

$$\mathcal{U}_n^{[1]} = \frac{2(-1)^{n+1} i \cosh \phi}{\mu_n^2 \omega}, \quad \mathcal{U}_n^{[2]} = 2(-1)^{n+1} i \left(\frac{\cosh \phi}{\mu_n^4 \omega} - \frac{\sinh \phi}{\mu_n^2} \right). \quad (4.65)$$

Figures 4.10 and 4.11 show that the bottom boundary layer solutions just found show excellent agreement with the numerical solutions to the original trench problem when the walls are made sufficiently thin. The only discrepancies are due to the boundary layer on the right-hand wall, which are not picked up by the inner solution near the base.

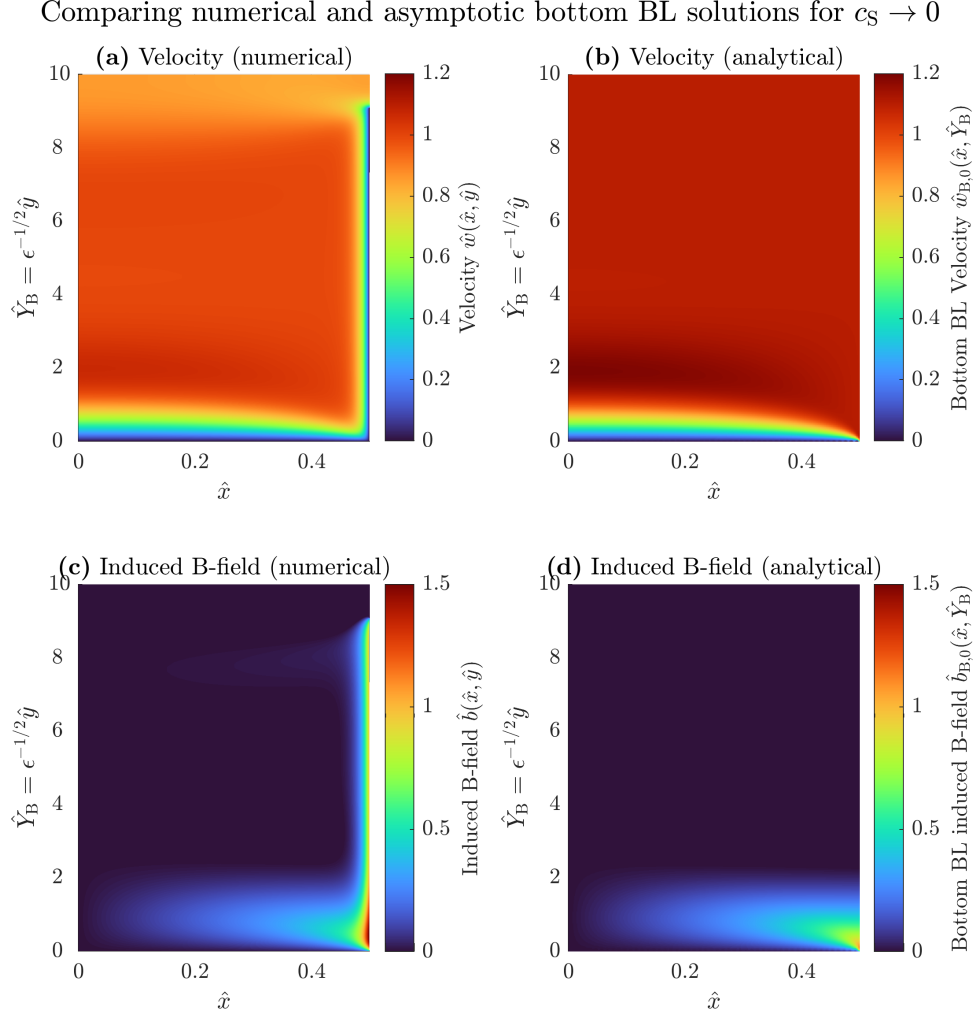


Figure 4.10: Plots of (a,c) numerical solutions to the original trench problem where $\epsilon = 0.0124$, $\tau = \varsigma = 10^{-3}$, $c_S = 10^{-6}$, $c_B = 46.874$, $\omega = 0.5$, $\Gamma = 0.0299$, $\frac{\partial \hat{T}}{\partial y}(\omega - \tau, \varsigma) = 1.1005$, and the analytical solutions (b,d) to the bottom boundary-layer problem (4.55) with the same parameters.

Comparing numerical and asymptotic bottom BL solutions for $c_B \rightarrow \infty$

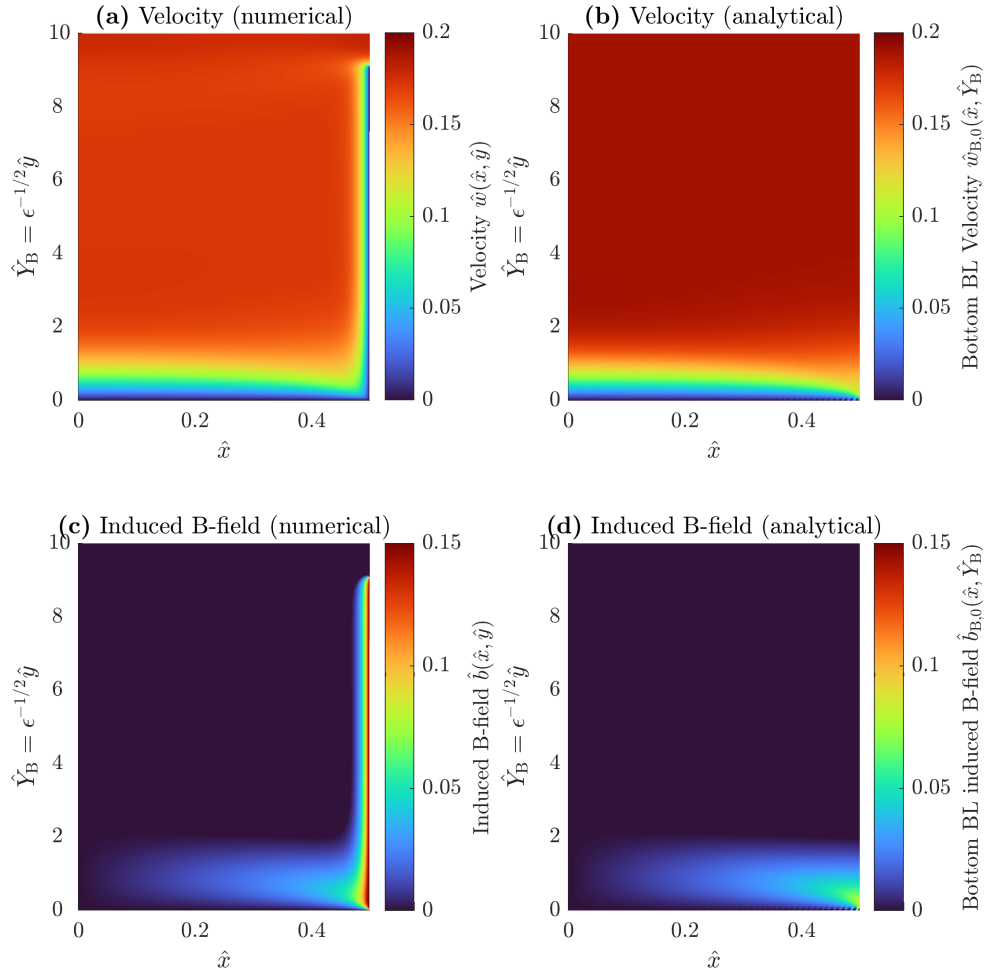


Figure 4.11: Plots of **(a,c)** numerical solutions to the original trench problem where $\epsilon = 0.0124$, $\tau = \varsigma = 10^{-3}$, $c_B = 10^6$, $c_S = 5.2292$, $\omega = 0.5$, $\Gamma = 0.0299$, $\frac{\partial \hat{T}}{\partial y}(\omega - \tau, \varsigma) = 1.1005$, and the analytical solutions **(b,d)** to the bottom boundary-layer problem (4.55) with the same parameters.

4.8.5 Inner problem in bottom-right corner

After rescaling both $\hat{x} = \omega - \epsilon \hat{X}_R$ and $\hat{y} = \epsilon^{1/2} \hat{Y}_B$, with $\hat{w}(\hat{x}, \hat{y}) = \hat{w}_{BR}(\hat{X}_R, \hat{Y}_B)$ and $\hat{b}^\mathcal{E}(\hat{x}, \hat{y}) = \hat{b}_{BR}^\mathcal{E}(\hat{X}_R, \hat{Y}_B)$, we get

$$\frac{\partial^2 \hat{b}_{BR}^\mathcal{E}}{\partial \hat{X}_R^2} + \epsilon \frac{\partial^2 \hat{b}_{BR}^\mathcal{E}}{\partial \hat{Y}_B^2} - \frac{\partial \hat{w}_{BR}}{\partial \hat{X}_R} = 0 \quad \text{in } \hat{X}_R > 0, \quad \hat{Y}_B > 0, \quad (4.66a)$$

$$\frac{\partial^2 \hat{w}_{BR}}{\partial \hat{X}_R^2} + \epsilon \frac{\partial^2 \hat{w}_{BR}}{\partial \hat{Y}_B^2} - \frac{\partial \hat{b}_{BR}^\mathcal{E}}{\partial \hat{X}_R} + \epsilon \Gamma = 0 \quad \text{in } \hat{X}_R > 0, \quad \hat{Y}_B > 0, \quad (4.66b)$$

$$\hat{w}_{BR} = 0 \quad \text{on } \hat{X}_R = 0, \quad (4.66c)$$

$$-\frac{\partial \hat{b}_{BR}^\mathcal{E}}{\partial \hat{X}_R} + c_S \hat{b}_{BR}^\mathcal{E} - \frac{\partial \hat{T}^\mathcal{E}}{\partial \hat{y}} \left(\omega, \epsilon^{1/2} \hat{Y}_B \right) = 0 \quad \text{on } \hat{X}_R = 0, \quad (4.66d)$$

$$\hat{w}_{BR} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.66e)$$

$$\frac{\partial \hat{b}_{BR}^\mathcal{E}}{\partial \hat{Y}_B} + c_B \hat{b}_{BR}^\mathcal{E} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.66f)$$

along with matching conditions with the base inner problem analysed above. Upon seeking asymptotic expansion solutions of the forms

$$\hat{w}_{BR} \sim \hat{w}_{BR,0} + \epsilon \hat{w}_{BR,1} + \dots, \quad (4.67a) \quad \hat{b}_{BR}^\mathcal{E} \sim \hat{b}_{BR,0}^\mathcal{E} + \epsilon \hat{b}_{BR,1}^\mathcal{E} + \dots, \quad (4.67b)$$

the leading order problem is

$$\frac{\partial^2 \hat{b}_{BR,0}^\mathcal{E}}{\partial \hat{X}_R^2} - \frac{\partial \hat{w}_{BR,0}}{\partial \hat{X}_R} = 0 \quad \text{in } \hat{X}_R > 0, \quad \hat{Y}_B > 0, \quad (4.68a)$$

$$\frac{\partial^2 \hat{w}_{BR,0}}{\partial \hat{X}_R^2} - \frac{\partial \hat{b}_{BR,0}^\mathcal{E}}{\partial \hat{X}_R} = 0 \quad \text{in } \hat{X}_R > 0, \quad \hat{Y}_B > 0, \quad (4.68b)$$

$$\hat{w}_{BR,0} = 0 \quad \text{on } \hat{X}_R = 0, \quad (4.68c)$$

$$-\frac{\partial \hat{b}_{BR,0}^\mathcal{E}}{\partial \hat{X}_R} + c_S \hat{b}_{BR,0}^\mathcal{E} - \frac{\partial \hat{T}^\mathcal{E}}{\partial \hat{y}} (\omega, 0) = 0 \quad \text{on } \hat{X}_R = 0, \quad (4.68d)$$

$$\hat{w}_{BR,0} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.68e)$$

$$\frac{\partial \hat{b}_{BR,0}^\mathcal{E}}{\partial \hat{Y}_B} + c_B \hat{b}_{BR,0}^\mathcal{E} = 0 \quad \text{on } \hat{Y}_B = 0, \quad (4.68f)$$

along with matching conditions

$$\hat{w}_{\text{BR},0}(\hat{X}_{\text{R}}, \hat{Y}_{\text{B}}) \rightarrow \hat{w}_{\text{B},0}(\omega, \hat{Y}_{\text{B}}) \quad \text{as } \hat{X}_{\text{R}} \rightarrow \infty, \quad (4.69\text{a})$$

$$\hat{b}_{\text{BR},0}^{\mathcal{E}}(\hat{X}_{\text{R}}, \hat{Y}_{\text{B}}) \rightarrow \hat{b}_{\text{BR},0}^{\mathcal{E}}(\omega, \hat{Y}_{\text{B}}) \quad \text{as } \hat{X}_{\text{R}} \rightarrow \infty. \quad (4.69\text{b})$$

Due to equations (4.68a)-(4.68d), the solution only parametrically depends on $\hat{Y}_{\text{B}}$, and therefore cannot satisfy the final boundary conditions (4.68f) and (4.68e). The fact that some of the boundary conditions are still not satisfied necessitates a further rescaling inside the corner region as shown below.

The solution to problem (4.68) is given by

$$\hat{w}_{\text{BR},0}(\hat{X}_{\text{R}}, \hat{Y}_{\text{B}}) = \hat{w}_{\text{B},0}(\omega, \hat{Y}_{\text{B}}) \left(1 - e^{-\hat{X}_{\text{R}}}\right), \quad (4.70\text{a})$$

$$\hat{b}_{\text{BR},0}^{\mathcal{E}}(\hat{X}_{\text{R}}, \hat{Y}_{\text{B}}) = \hat{b}_{\text{BR},0}^{\mathcal{E}}(\omega, \hat{Y}_{\text{B}}) + \hat{w}_{\text{B},0}(\omega, \hat{Y}_{\text{B}})e^{-\hat{X}_{\text{R}}}. \quad (4.70\text{b})$$

where the boundary condition (4.68d) is satisfied provided

$$(1 + c_{\text{S}}) \hat{w}_{\text{B},0}(\omega, \hat{Y}_{\text{B}}) + c_{\text{S}} \hat{b}_{\text{B},0}^{\mathcal{E}}(\omega, \hat{Y}_{\text{B}}) = \frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega, 0). \quad (4.71)$$

Thus, as provided, we obtain the matching condition (4.55i) required to solve the inner problem near the base.

4.8.6 Inner problem inside bottom of bottom-right corner

After rescaling both $\hat{x} = \omega - \epsilon \hat{X}_{\text{R}}$ and $\hat{Y}_{\text{B}} = \epsilon^{1/2} \hat{Y}_{\text{B}_\text{I}}$, so that $\hat{w}(\hat{x}, \hat{y}) = \hat{w}_{\text{B}_\text{IR}}(\hat{X}_{\text{R}}, \hat{Y}_{\text{B}_\text{I}})$ and $\hat{b}^{\mathcal{E}}(\hat{x}, \hat{y}) = \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}}(\hat{X}_{\text{R}}, \hat{Y}_{\text{B}_\text{I}})$, we get, to leading-order,

$$\frac{\partial^2 \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}}}{\partial \hat{X}_{\text{R}}^2} + \frac{\partial^2 \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}}}{\partial \hat{Y}_{\text{B}_\text{I}}^2} - \frac{\partial \hat{w}_{\text{B}_\text{IR}}}{\partial \hat{X}_{\text{R}}} = 0 \quad \text{in } \hat{X}_{\text{R}} > 0, \quad \hat{Y}_{\text{B}_\text{I}} > 0, \quad (4.72\text{a})$$

$$\frac{\partial^2 \hat{w}_{\text{B}_\text{IR}}}{\partial \hat{X}_{\text{R}}^2} + \frac{\partial^2 \hat{w}_{\text{B}_\text{IR}}}{\partial \hat{Y}_{\text{B}_\text{I}}^2} - \frac{\partial \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}}}{\partial \hat{X}_{\text{R}}} = 0 \quad \text{in } \hat{X}_{\text{R}} > 0, \quad \hat{Y}_{\text{B}_\text{I}} > 0, \quad (4.72\text{b})$$

$$\hat{w}_{\text{B}_\text{IR}} = 0 \quad \text{on } \hat{X}_{\text{R}} = 0, \quad (4.72\text{c})$$

$$-\frac{\partial \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}}}{\partial \hat{X}_{\text{R}}} + c_{\text{S}} \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}} - \frac{\partial \hat{T}^{\mathcal{E}}}{\partial \hat{y}}(\omega, 0) = 0 \quad \text{on } \hat{X}_{\text{R}} = 0, \quad (4.72\text{d})$$

$$\hat{w}_{\text{B}_\text{IR}} = 0 \quad \text{on } \hat{Y}_{\text{B}_\text{I}} = 0, \quad (4.72\text{e})$$

$$\frac{\partial \hat{b}_{\text{B}_\text{IR}}^{\mathcal{E}}}{\partial \hat{Y}_{\text{B}_\text{I}}} = 0 \quad \text{on } \hat{Y}_{\text{B}_\text{I}} = 0, \quad (4.72\text{f})$$

and matching conditions

$$\hat{w}_{\text{BIR}}(\hat{X}_{\text{R}}, \hat{Y}_{\text{BI}}) \sim \begin{cases} \hat{w}_{B,0}(\omega, 0) & \text{as } \hat{X}_{\text{R}} \rightarrow \infty, \\ \hat{w}_{\text{BR},0}(\hat{X}_{\text{R}}, 0) & \text{as } \hat{Y}_{\text{BI}} \rightarrow \infty, \end{cases} \quad (4.72\text{g})$$

$$\hat{b}_{\text{BIR}}^{\mathcal{E}}(\hat{X}_{\text{R}}, \hat{Y}_{\text{BI}}) \sim \begin{cases} \hat{b}_{\text{BR},0}^{\mathcal{E}}(\omega, 0) & \text{as } \hat{X}_{\text{R}} \rightarrow \infty, \\ \hat{b}_{\text{BR},0}^{\mathcal{E}}(\hat{X}_{\text{R}}, 0) & \text{as } \hat{Y}_{\text{BI}} \rightarrow \infty. \end{cases} \quad (4.72\text{h})$$

We refrain from attempting to solve this problem as it appears to be intractable to do so analytically. The best way to address this problem is to solve it numerically; however, a numerical method has been used to solve the full problem anyway (in the previous subsection).

4.9 Parameter sweeps

We now carry out some parameter sweeps on the steady trench problem, keeping all parameter values fixed at the default values seen in Tables 4.1 and 4.3, unless specified otherwise. The numerical solutions shown (solid) include comparison with predictions provided by the asymptotics (dashed). To facilitate varying the trench geometry while avoiding unphysical parameter regimes (for example, if the wall height is smaller than the base thickness), we introduce various physically relevant geometrical ratios which are summarised in Table 4.5.

Name	Symbol	Formula	Range
Aspect ratio	r_{A}	$\frac{H}{\omega}$	$(-\infty, \infty)$
Overfilling ratio	r_{F}	$\frac{\omega f}{(\omega - \tau)(H - \varsigma)}$	$(0, \infty)$
Side-wall fraction	r_{S}	$\frac{\tau}{\omega}$	$(0, 1)$
Base fraction	r_{B}	$\frac{\varsigma}{H}$	$(0, 1)$

Table 4.5: A summary of dimensionless ratios used to characterise the geometry of a LiMIT-style trench.

In performing parameter sweeps, we show the effect of changing each parameter on the horizontal and vertical velocity profiles, as well as the horizontal magnetic

field profiles. The vertical magnetic field profiles are omitted because these are equal to zero in practice, and highly susceptible to numerical noise. The temperature is plotted only in cases where it is affected by the parameter being swept over. Also, by horizontal and vertical profiles, we mean cross-sections of the dependent variables at $\hat{y} = \frac{1}{2}(H + \varsigma)$ and at $\hat{x} = 0$ respectively.

As a cautionary note, we emphasise that the vertical velocity profiles are not described perfectly accurately by the asymptotic approximations. This is because the asymptotic approximation only captures the behaviour in the bulk and near the Hartmann layers. This does not take into account the boundary-layer behaviour near the bottom, and hence the no-slip condition appears not to be satisfied, and also demonstrates a singularity at $\hat{y} = H$ due to the presence of the wall corner. The asymptotic vertical velocity profiles have therefore been truncated near the base, and near the wall corner $\hat{y} = H$.

4.9.1 Changing the horizontal wall dimensions, τ and ω

The effects of changing the side wall thickness τ while keeping the trench width ω fixed are shown in Figures 4.12(a-c), whilst Figures 4.12(d-f) show the effects of changing ω whilst keeping τ fixed.

The horizontal velocity profiles (a) demonstrate that increasing the wall thickness decreases the maximum velocity of the fluid enclosed between the walls. The $\cap$ -shape of the velocity profile is retained apart from when the wall is at its thickest, at which point the Hartmann layers appear to merge and create an approximately parabolic velocity profile when $\tau \approx 0.4$. The dashed curves show that the numerical results are in excellent agreement with the asymptotic prediction (4.51a). The horizontal magnetic field profiles in plot (c) show approximate piecewise linear behaviour, which becomes sharper and more pronounced when the walls are thin. This is because making the walls thinner results in a stronger electrical current density passing through them, since their electrical conductivity remains constant. Outside the Hartmann layers, the induced field is very small as expected, given the small value of I here, namely 0.0299. Plot (b) shows that the boundary layer at the bottom of the trench becomes thinner and less curved as the wall thickness increases, suppressing the velocity jet. The asymptotic prediction (4.51a) works well inside the trench and away from the bottom boundary layer. Above the trench,

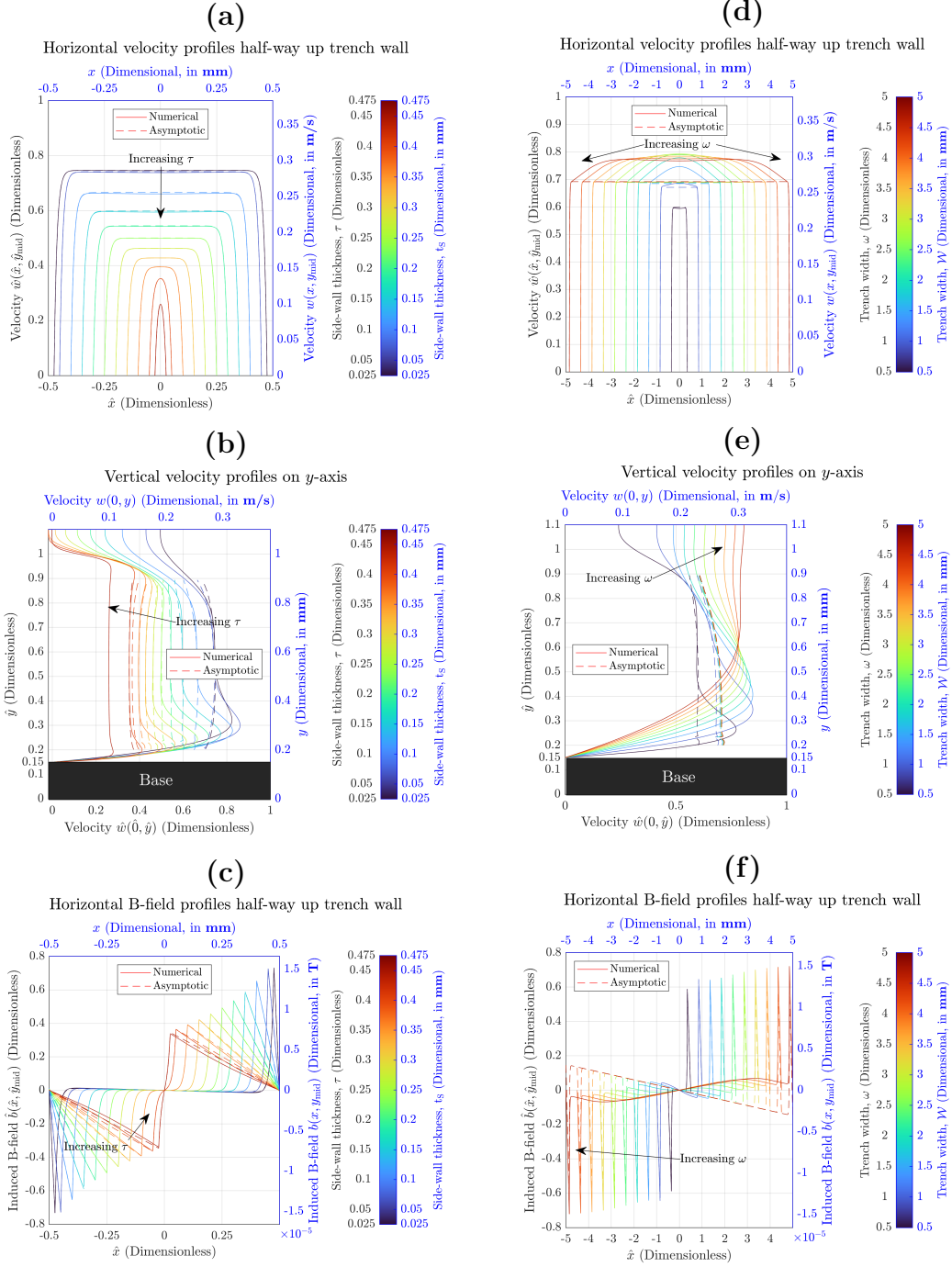


Figure 4.12: Numerical plots of (a,d) horizontal velocity profiles half-way up the trench wall, (b,e) vertical velocity profiles along the $\hat{y}$ -axis, and (c,f) horizontal magnetic field profiles half-way up the trench wall, for dimensionless side-wall thickness $\tau \in \{0.005, 0.05, 0.1, \dots, 0.45, 0.495\}$ (in (a,b,c)) and dimensionless trench width $\omega \in \{0.4, 0.6, \dots, 1.8, 2\}$ (in (d,e,f)).

the velocity seems to approach a constant value somewhat lower than that inside the trench. Making the walls thicker causes the speed of the flow above the walls to decrease because of the increased drag due to the tops of the trench walls. Also, the “cross-over region” near the tops of the walls becomes narrower and less curved as the side walls are brought closer together.

Figure 4.12(**d-f**) shows the effects of varying the dimensionless trench width ω whilst keeping the side wall thickness τ fixed. We recall that the problem has been non-dimensionalised with a typical trench with $\mathcal{W}$. It is therefore possible to set $\omega = \frac{1}{2}$ without loss of generality, in which case varying the trench width varies the values of the dimensionless parameters ϵ and Γ . Instead, we find it more convenient to keep ϵ and Γ fixed but allow ω to vary. We note that the colours in plots (**a-c**) are oppositely correlated to those in (**d-f**), i.e. that making the walls thinner has a similar effect to making the trench wider.

Figure 4.12(**d**) shows that the velocity inside the trench increases with increasing ω , but only up to a point. At larger values of ω there is “bulge” in the velocity profile, raising it slightly above the asymptotic prediction (shown using dashed curves). Plot (**e**) shows the cause of this discrepancy - as the trench becomes wider, the conducting-wall jet near the base becomes elongated, thus being captured by the central horizontal line $\hat{y} = \frac{1}{2}(H + \varsigma)$. The intensity of the magnetic field dipole shown in plot (**f**) does not vary anywhere near as significantly as in plot (**c**), because the effective conductivity of the side walls does not change as they move apart. However, the magnetic field gradient in the bulk changes sign as ω increases, again because of infiltration by the bottom boundary layer.

4.9.2 Changing the vertical dimensions, ς and H

The effects of changing the base thickness ς and wall height H are demonstrated in Figure 4.13, assuming that the free surface thickness f is kept constant. The horizontal velocity plots in plots (**a**) and (**d**) show that the bulk velocity approaches a limiting value as $\varsigma \rightarrow 0$ or $H \rightarrow \infty$, in which cases the asymptotic and numerical predictions are in generally good agreement. However, as either $\varsigma \rightarrow H$ or $H \rightarrow \varsigma$, an M-shaped velocity is observed because the conducting-base jet intersects the central horizontal line $\hat{y} = \frac{1}{2}(H + \varsigma)$, thus revealing a velocity profile characteristic of conducting-wall MHD duct flow [34]. The asymptotic

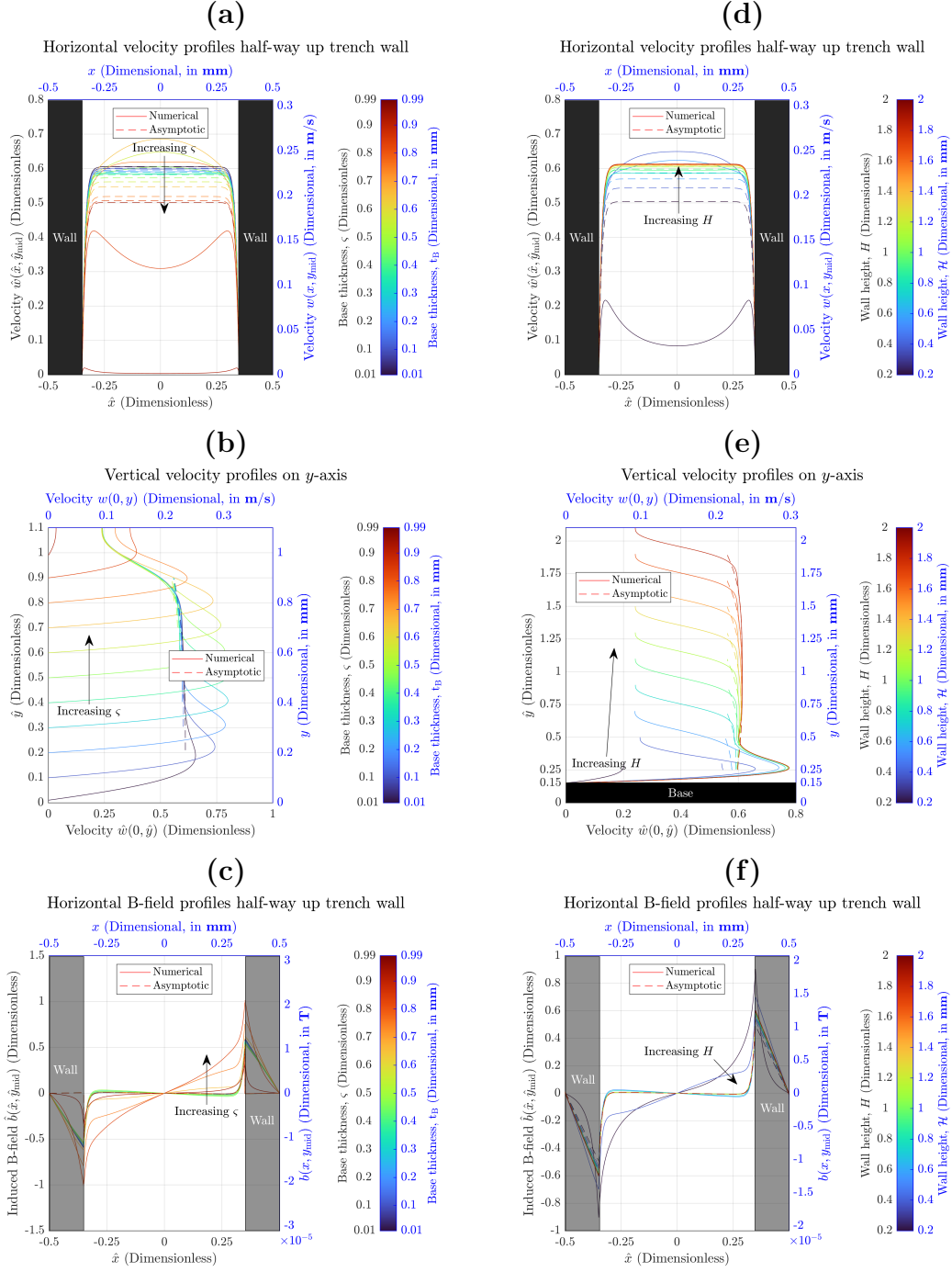


Figure 4.13: Numerical plots of (a,d) horizontal velocity profiles half-way up the trench wall, (b,e) vertical velocity profiles along the $\hat{y}$ -axis, and (c,f) horizontal magnetic field profiles half-way up the trench wall, for dimensionless base thickness $\varsigma \in \{0.01, 0.1, 0.2, \dots, 0.9, 0.99\}$ (in (a,b,c)) and dimensionless wall height $H \in \{0.2, 0.4, \dots, 1.8, 2\}$ (in (d,e,f)).

approximation (4.51a) for the horizontal velocity profile breaks down in both of these limits where the base becomes close to the top of the walls. The velocity profiles ultimately collapse to zero due to the lack of a TEMHD effect arising from the vanishing side walls. Moreover, the boundary layers near the bottom and top of the trench as well as that near the top of the trench become superimposed and annihilate one another. In plot **(b)**, we see that making the base thicker shifts the velocity profiles upwards, and although this strengthens the conducting-base jet, it does so only up to a point (around 0.8 m/s). The core shape of the velocity profile is largely unaffected as ς changes.

Increasing H has a similar shifting effect on the velocity profile near the top as shown in plot **(e)**, although several features become prominent in this case; for example, the conducting-base jet, whose behaviour does not change past $H = 0.6$. In both cases, the bulk velocity profile demonstrates a uniform shape away from the jet and the top of the wall, and in this region the asymptotics and numerics show good agreement. In plots **(c,f)**, we notice again that the asymptotic approximation for the magnetic field holds well, except when the base and top of the walls are brought very close together.

4.9.3 Geometrical parameter space

We now examine the effects on the global flow properties of varying the trench geometry. We consider three averaged quantities that describe the global flow characteristics, namely the average flow rates above and below the trench walls, defined by

$$\hat{w}_A = \frac{1}{2\omega f} \int_H^{H+f} \int_{-\omega}^{\omega} \hat{w}(\hat{x}, \hat{y}) d\hat{x} d\hat{y}, \quad (4.73a)$$

$$\hat{w}_B = \frac{1}{2(\omega - \tau)(H - \varsigma)} \int_{\varsigma}^H \int_{-\omega+\tau}^{\omega-\tau} \hat{w}(\hat{x}, \hat{y}) d\hat{x} d\hat{y}, \quad (4.73b)$$

respectively, and the average surface temperature, defined by

$$\hat{T}_F = \int_{-\omega}^{\omega} \hat{T}^{\mathcal{L}}(\hat{x}, H + f) d\hat{x}. \quad (4.73c)$$

For the LiMIT concept to be successful, reasonably high flow rates both above and within the trench should be maintained, whilst the surface temperature should not get too high.

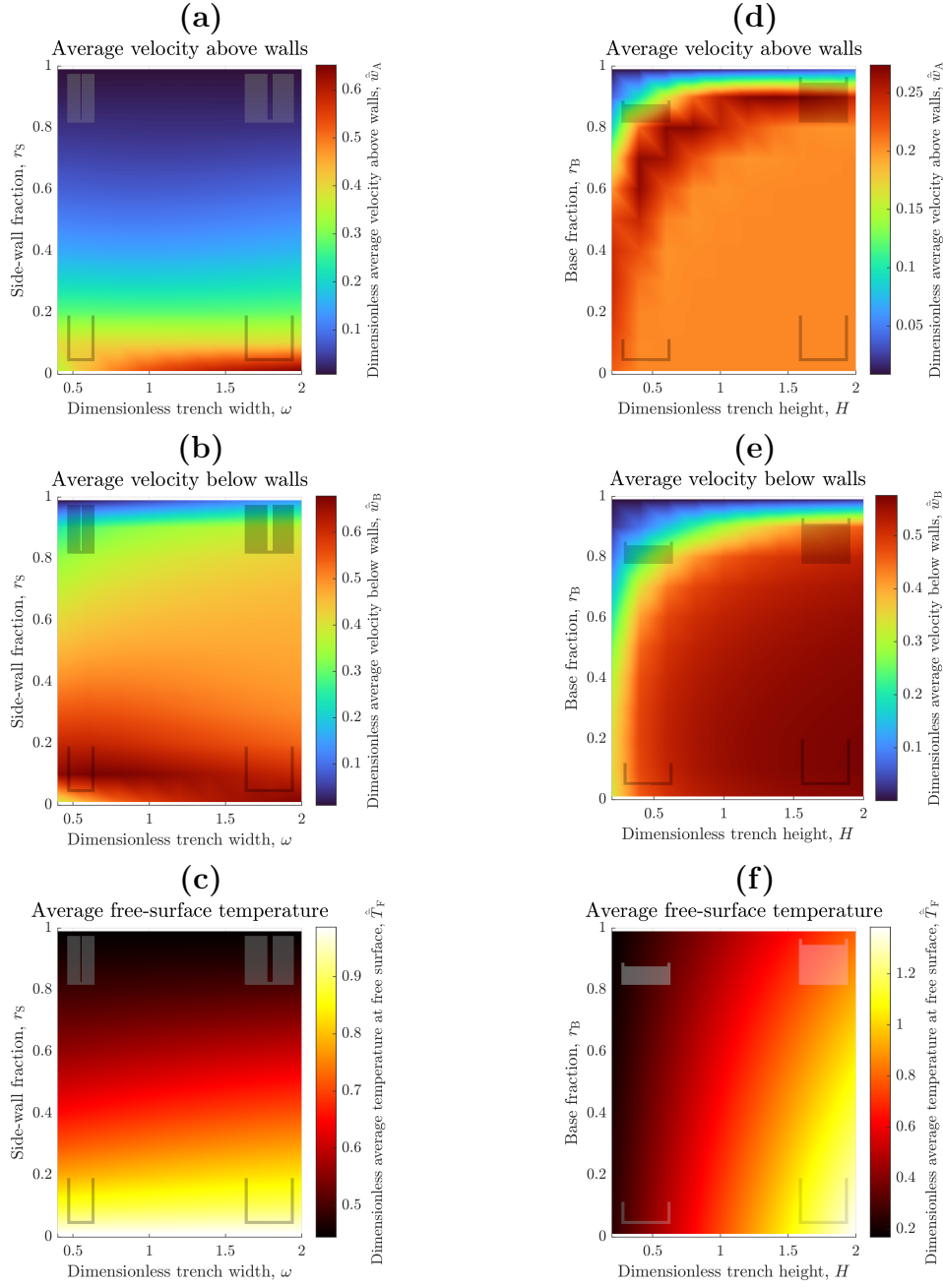


Figure 4.14: The dependence of the average velocity above and inside the trench ((a,d) and (b,c) respectively) and the average free-surface temperature (c,f) on the trench width ω and side-wall fraction r_S (a-c) and on the trench height H and base fraction r_B (d-f), for $\omega \in \{0.5, 1, \dots, 5\}$, $H \in \{0.2, 0.4, \dots, 2\}$, and $r_S, r_B \in \{0.01, 0.1, \dots, 0.9, 0.99\}$.

We start by varying the horizontal wall dimensions ω and r_s in Figure 4.14, **(a,b,c)**. Plot **(a)** demonstrates that the average velocity above the walls appears largely unaffected by making the trench wider, but making the trench walls thicker results in a significant drop. Unsurprisingly, when the side walls are up to approximately 5% of the width of the trench, the average velocities above and below the trench walls are similar values, and both appear to be optimised when the trench is not particularly wide, as shown in plot **(b)**. Otherwise, the two calculated average velocities can vary markedly, so that the flow speed inside the trench cannot readily be inferred by observation of the free surface. As plot **(c)** shows, making the trench wider does not greatly affect the average surface temperature. However, making the walls thinner does result in a hotter free surface, because there is a smaller volume of solid wall through which heat can conduct, recalling that the solid is roughly three times as conducting as the liquid. On the other hand, making the trench walls too thick results in very little flow either above or below the walls.

Next we turn our attention to plots **(d,e,f)** in Figure 4.14, where the effects of varying the wall height H and the base fraction r_B are examined. The the velocities above and below the walls appear to show similar distributions. However, plot **(d)** reveals a curved band of optimal values for the flow above the trench, where there is only relatively moderate flow inside the trench. On the left of this curved region, we have a tall trench with a very high base, so the trench walls resemble short “stumps”. On the other hand, an optimal flow below the walls in plot **(e)** is achieved when the trench is tall, but with a relatively thin base.

This latter regime is also where the free surface is hottest, as plot **(f)** shows, due to there being less conducting wall for heat flux to pass through. In practice, divertor designers want to ensure a good flow rate of lithium both above and inside the trench, as well as limiting the lithium temperature. With this in mind, we propose that trench dimensions along the band in Figure 4.14**(d)** represent a good trade-off between increasing the average velocities above and below the walls, and reducing the temperature of the free surface. This observation suggests that a “trenchlet”-style configuration could be a good option where there is a thick divertor plate and only slightly higher trench walls.

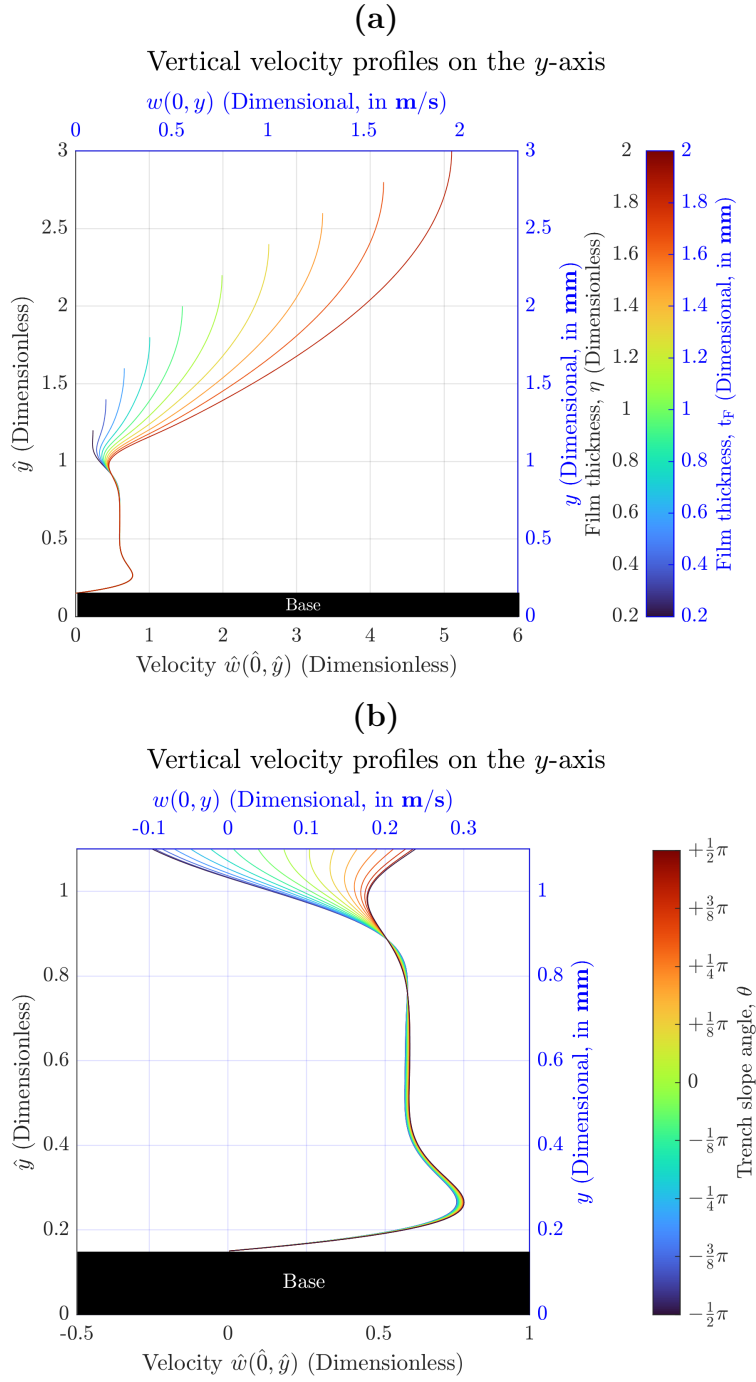


Figure 4.15: Plots showing the effect of varying **(a)** the film thickness f , **(b)** the trench slope angle θ , on the vertical velocity distributions.

4.9.4 Changing the film thickness f and the trench angle θ

The effects of varying the film thickness f and the trench slope angle θ are shown in Figure 4.15. From plot (a), we can see that overfilling the trench has a marked impact on the nature of the flow above the walls without significantly affecting the flow inside the trench. In particular, we can see that for $\hat{y} > H$, the velocity profile appears to be a quadratic function of $\hat{y}$ that attains a maximum at the free surface. This observation is in qualitative agreement with the predictions provided by the asymptotics, namely the result (4.46). For $f > 1$, the maximum velocity value at the free surface is a global maximum, and indeed making the free surface too thick can result in flows above the walls that vastly exceed those inside the trench. Recalling that $\theta = 0.1$ in this case, we infer that even moderate trench slope angles can result in large flow rates above the walls. We note, though, that in practice the intention is to use the thermoelectric effects as the principal driver of the flow rather than relying on gravity, at least because similar divertor plates may be used at the top of ST40, where gravity acts in the opposite direction (relatively speaking).

As Figure 4.15 (b) shows, the velocity profile between the walls is mostly unaffected by the slope angle θ , because TEMHD is the dominant driving mechanism there. On the other hand, increasing θ increases the flow speed above the trench walls, and in the case where $f = 0.1$ as plotted, making the trench point vertically downwards leads to flow speeds above the walls which compare to those below. Making $\theta < 0$ slows down the flow above the trench and can lead to backflows. The fluid above the walls experience both the drag from the TEMHD-driven flow inside the trench and the gravitational forcing, and when $\theta < 0$ these effects act in opposite directions.

4.9.5 Varying the conductivity ratios Σ and K

The effect of varying the solid-to-fluid electrical and thermal conductivity ratios Σ and K are shown in Figures 4.16 and 4.17, assuming that each is the same for the side walls and the base. In Figures 4.16(a,b), we observe that as $\Sigma \rightarrow 0$ (i.e. the wall becomes a perfect insulator) the velocity profiles collapse to zero. Mathematically, this is because the boundary condition (4.31) becomes $\hat{b}^{\mathcal{E}} = 0$,

and the temperature gradient thus becomes irrelevant despite being the key driving mechanism behind the flow in this problem. This effect demonstrates again the importance of the return current in the trench walls in driving the flow. On the other hand, increasing Σ increases the flow rate, but only up to a point, and as $\Sigma \rightarrow \infty$ (i.e. the wall becomes a perfect conductor) the velocity approaches a limiting profile. In plot **(a)** this value is slightly greater than 0.6, which coincides with the asymptotic approximation (4.46) in the limit where $c_s \rightarrow 0$. Indeed, in all cases the asymptotic approximation (4.46) for the velocity works well away from the bottom boundary layer. The velocity jet near the base which is evident in plot **(b)** becomes stronger as Σ increases, but disappears when $\Sigma < 1$. Plot **(c)** shows that the Hartmann layers in the induced magnetic field are suppressed as Σ increases.

Meanwhile, in Figure 4.17, we observe that, as $K \rightarrow 0$ (i.e. the wall becomes a perfect thermal insulator), the liquid lithium becomes driven faster, as shown in plot **(a)** and there is clear electrical current in the Hartmann layers and side walls as shown in plot **(b)**. However, the temperature gradients in plot **(d)** become unphysically large, because the heat flux imposed at the free surface is unable to escape through the trench base. Meanwhile, as $K \rightarrow \infty$ (so that the wall becomes a perfect thermal conductor), the wall carries away all the heat from the fluid, suppressing any temperature gradients and therefore not generating the thermoelectric currents required to drive the flow. Looking at the temperature plots in **(d)**, we see that the piecewise linear approximation

$$T(0, y) \approx \begin{cases} \frac{\hat{y}}{K} & \text{if } 0 < \hat{y} \leq \varsigma, \\ \frac{\varsigma}{K} - \varsigma + \hat{y} & \text{if } \varsigma < \hat{y} \leq H + f, \end{cases} \quad (4.74)$$

provides a reasonable approximation when K is small and is exact when $K = 1$. However, at larger values of K , the temperature profile ceases to be approximately piecewise linear.

4.9.6 Varying the applied field $\mathcal{B}^a$

As illustrated in Figure 4.18, we proceed to discuss the effect of varying the reciprocal Hartmann number ϵ . Recall that ϵ is proportional to $1/\mathcal{B}^a$, while $\Gamma = (k^{\mathcal{E}} g \mathcal{W} / \mathcal{Q}^a \Delta \mathcal{S}) \sqrt{\rho^{\mathcal{E}} / \sigma^{\mathcal{E}} \nu}$ is independent of $\mathcal{B}^a$. Provided the material and

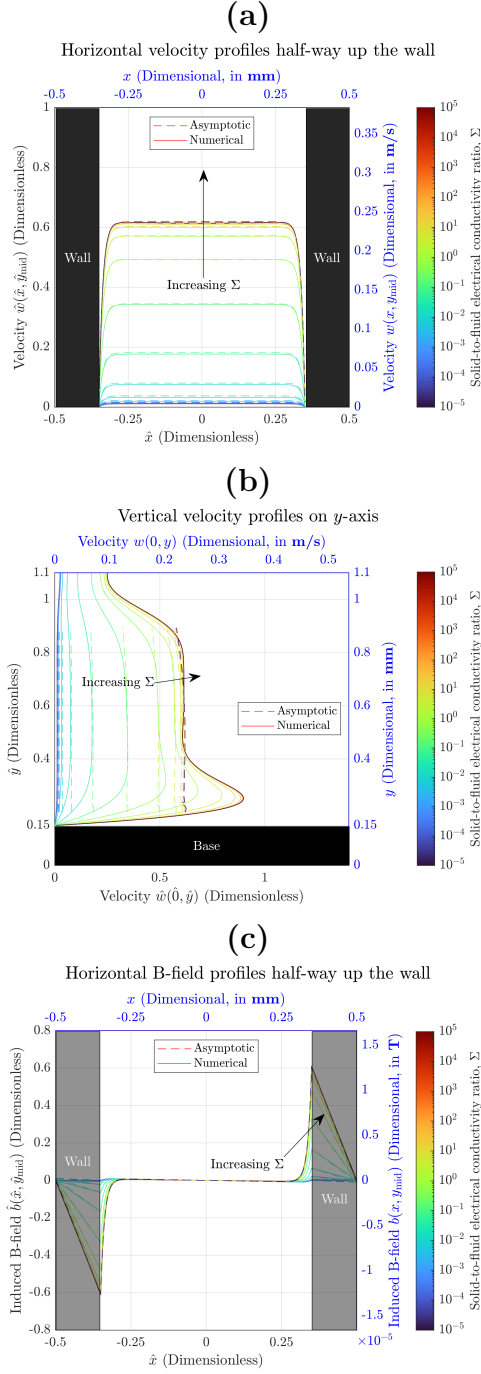


Figure 4.16: Plots illustrating the effect of changing Σ on (a) the horizontal velocity profiles half-way up the trench wall, (b) the vertical velocity profiles on the $\hat{y}$ -axis, and (c) the horizontal induced magnetic field profiles half-way up the wall, all with $\Sigma = 10^n$ where $n \in \{-5, -4.5, \dots, 4.5, 5\}$.

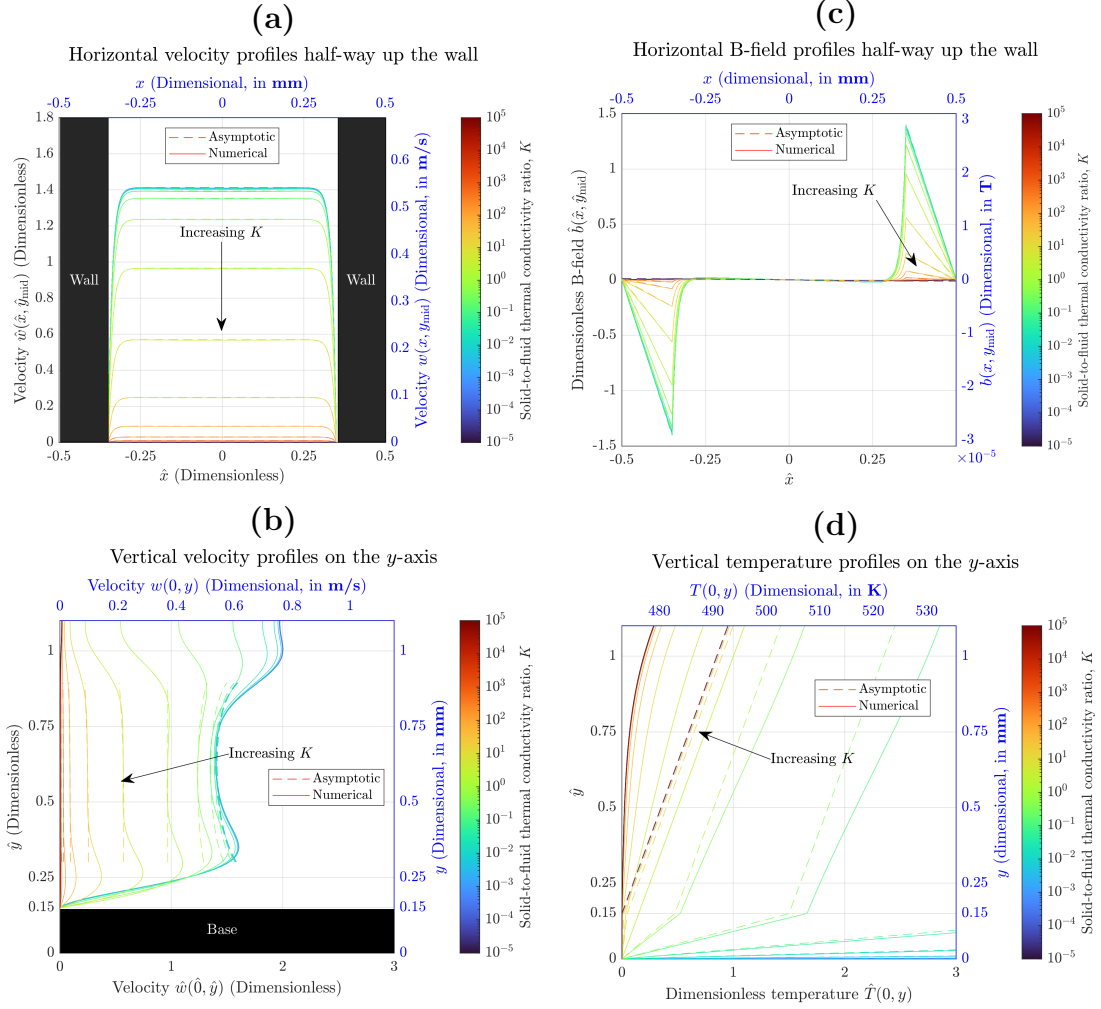


Figure 4.17: Plots illustrating the effect of changing K on (a) the horizontal velocity profiles half-way up the trench wall, (b) the vertical velocity profiles on the $\hat{y}$ -axis, (c) the horizontal induced magnetic field profiles half-way up the wall, and (d) the vertical temperature profiles on the $\hat{y}$ -axis, all with $K = 10^n$ where $n \in \{-5, -4.5, \dots, 4.5, 5\}$.

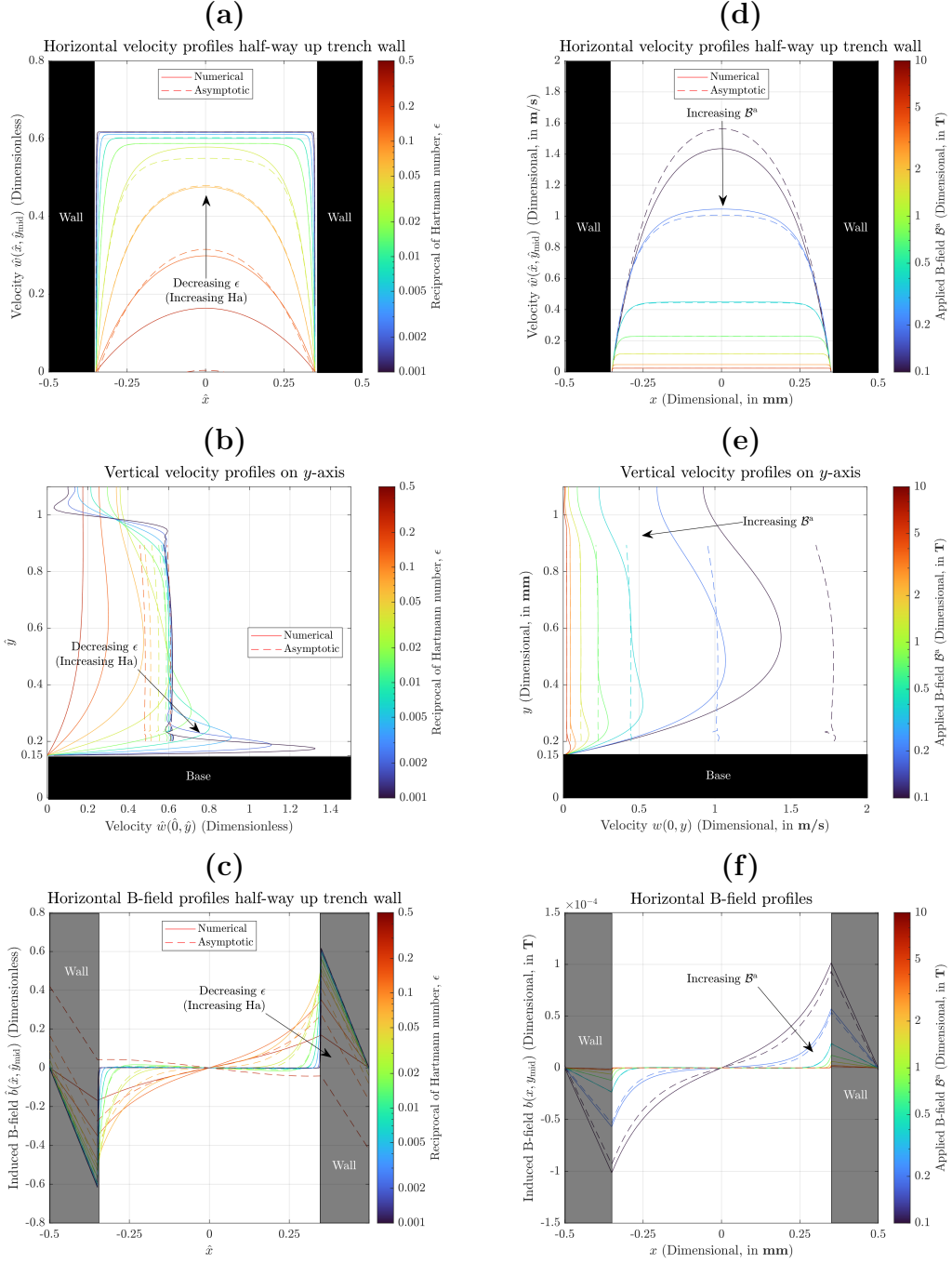


Figure 4.18: Plots illustrating the behaviour of *dimensionless* solutions in (a,b,c) for $\epsilon \in \{0.5, 0.2, 0.1, 0.05, 0.02, 0.01, 0.005, 0.002, 0.001\}$, and the *dimensional* solutions in (d,e,f) for $B^a \in \{0.1, 0.2, 0.5, 1, 2, 5, 10\}$ T.

geometrical properties are kept constant, varying ϵ can be considered to be mathematically equivalent to varying $\mathcal{B}^a$. However, when considering the impact on the dimensional solution, we must recall that the velocity scaling (4.24c) contains a factor of $1/\mathcal{B}^a$, which approaches zero as $\mathcal{B}^a \rightarrow \infty$. To make any practical judgements about what happens when the magnetic field strength is increased, one needs to refer to the dimensional solutions, or consider $\epsilon\hat{w}$.

In plot **(a)**, we clearly observe the singular behaviour of the horizontal velocity profiles as $\epsilon \rightarrow 0$, where the velocity approaches a constant equal to the temperature gradient on the right-hand wall outside Hartmann layers whose width is of $\mathcal{O}(\epsilon)$. In plot **(b)**, we observe that the conducting-wall jet at the base becomes much stronger as ϵ decreases. Meanwhile, the “cross-over region” joining the flows inside and above the trench becomes sharper as $\epsilon \rightarrow 0$, and the velocity above the trench walls consequently decreases. We also see that the induced magnetic field profile becomes an approximately piecewise linear function of $\hat{x}$, approaching a negligible quantity inside the fluid but with a finite gradient inside the walls.

However, when considering dimensional quantities, increasing the magnetic field strength $\mathcal{B}^a$ has the opposite effect to those seen in plots **(a-c)**. For example, in plot **(d)**, we see that increasing the magnetic field strength does create boundary layers near the walls, but simultaneously decreases the maximum velocity. This effect also captured in plot **(e)**, where, even though the velocity jets in plot **(b)** also appear, they are much less obvious. All of these effects occur because increasing $\mathcal{B}^a$ has the effect of retarding the flow everywhere. Finally, the same suppressing influence is also observed in the induced magnetic field in plot **(f)**, perhaps counterintuitively. However, this behaviour is in fact consistent with our chosen scaling for the induced magnetic field (4.24b), namely $\epsilon\text{Rm}\mathcal{B}^a = \mu^{\mathcal{L}}Q^a\Delta S\sqrt{\rho^{\mathcal{L}}\sigma^{\mathcal{L}}\nu}/k^{\mathcal{L}}\mathcal{B}^a$. This demonstrates that increasing the applied magnetic field strength decreases the strength of the induced field.

4.9.7 Changing ψ , the applied field inclination

Although the applied magnetic field angle ψ is unlikely to vary in practice, due to the toroidal symmetry of the divertor geometry and magnetic field in a tokamak, we nevertheless illustrate the effect that this has on the velocity and the induced magnetic field in Figures 4.19 and 4.20 respectively. The most obvious observation

from plot **(f)** in both figures is that the flow and magnetic field are reversed, as expected when $\psi = \pi$ and the applied magnetic field is pointing from right to left, rather than left to right. In Figure 4.19**(a)** with $\psi = \pi/6$, the typical Hunt velocity jet shifts over to the bottom-left corner, and the thickness of the boundary layer near the base decreases. We notice that this conducting jet appears to be stretched out in the direction of the magnetic field, as well as the small stagnant region in the top-right corner. It appears that the flow profiles are being distorted along dividing subcharacteristics, since the leading-order TEMHD trench problem becomes hyperbolic in the limit $\epsilon \rightarrow 0$. This phenomenon was also described analytically in [58], where the velocity gradient appears to become discontinuous across dividing characteristics passing through the corners. The appearance of distinct regions enclosed by such dividing characteristics was first predicted in [1]. In plot **(b)**, the velocity is close to zero near the subcharacteristic passing through the top-right corner in a neighbourhood of which the flow begins to reverse. In plot **(c)**, there is a perfectly symmetrical velocity dipole with the liquid in the right half of the trench flowing at an equal and opposite speed to that in the left, and the average velocity is equal to zero. The same behaviour as in plots **(a)** and **(b)** is demonstrated in plots **(d)** and **(e)**, albeit with reversed direction. In any case, it appears that the net velocity in the trench is maximised when the applied magnetic field is orthogonal to the temperature gradient, as expected.

Turning our attention to Figure 4.20, we notice that the induced field is close to zero outside a triangular region near the bottom-right corner in plots **(a,b)**. The dividing line is aligned with a sub-characteristic line inclined at an angle ψ to the horizontal, which encloses a region of high induced field strength, counteracting the dipole formed by the rest of the magnetic field profile. In in plot **(c)**, the applied field is parallel to the trench walls, where the induced magnetic field exhibits “side” layers of thickness of $\mathcal{O}(\epsilon^{1/2})$. These lead to the formation of a stronger electrical current passing through the walls, and hence a stronger Lorenz force which drives the fluid at a higher speed in both the left and right hand sides of the trench, although in opposite directions.

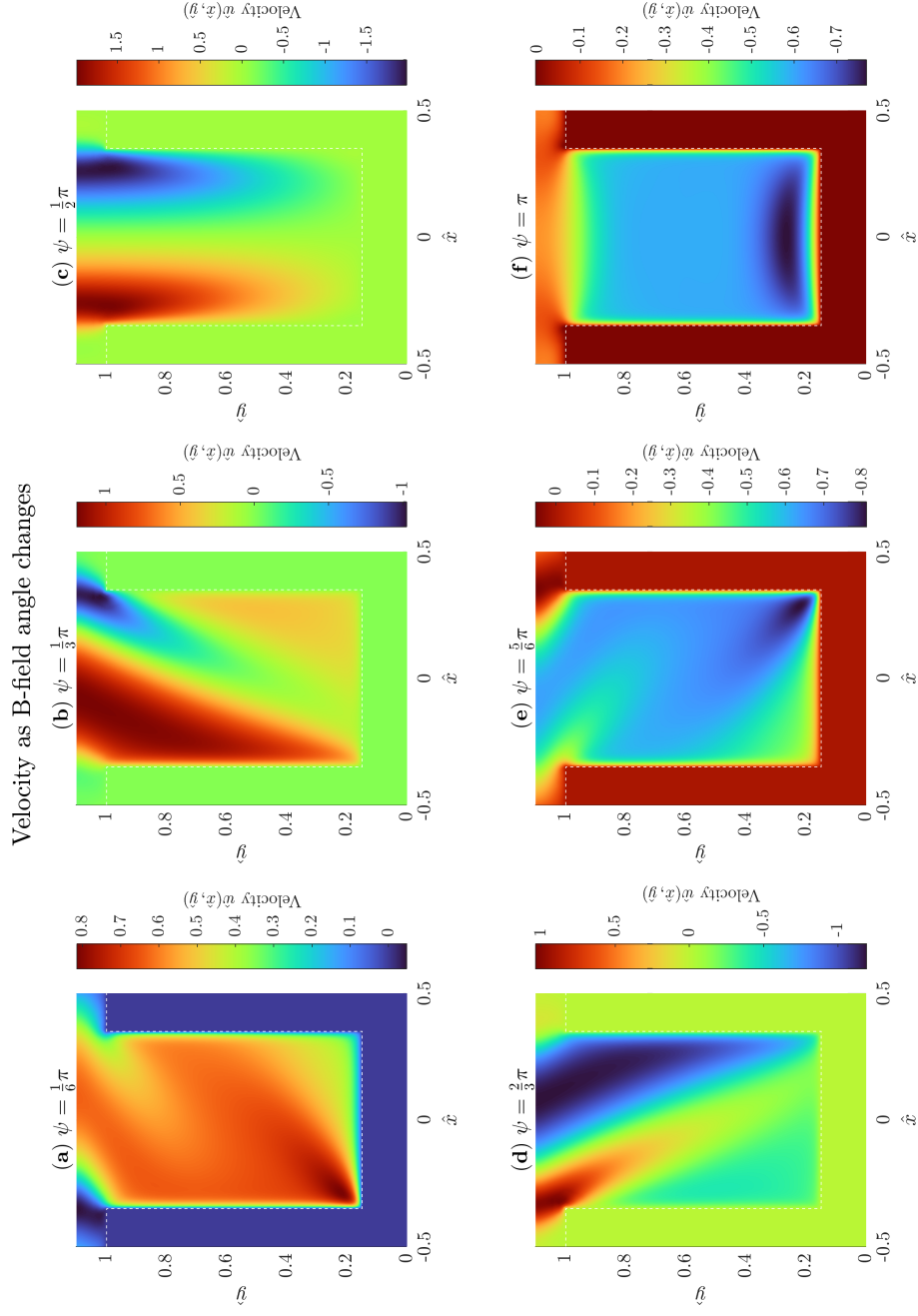


Figure 4.19: Two-dimensional surface plots illustrating the effect of changing the applied magnetic field angle ψ on the velocity field.

Induced B-field as applied B-field angle changes

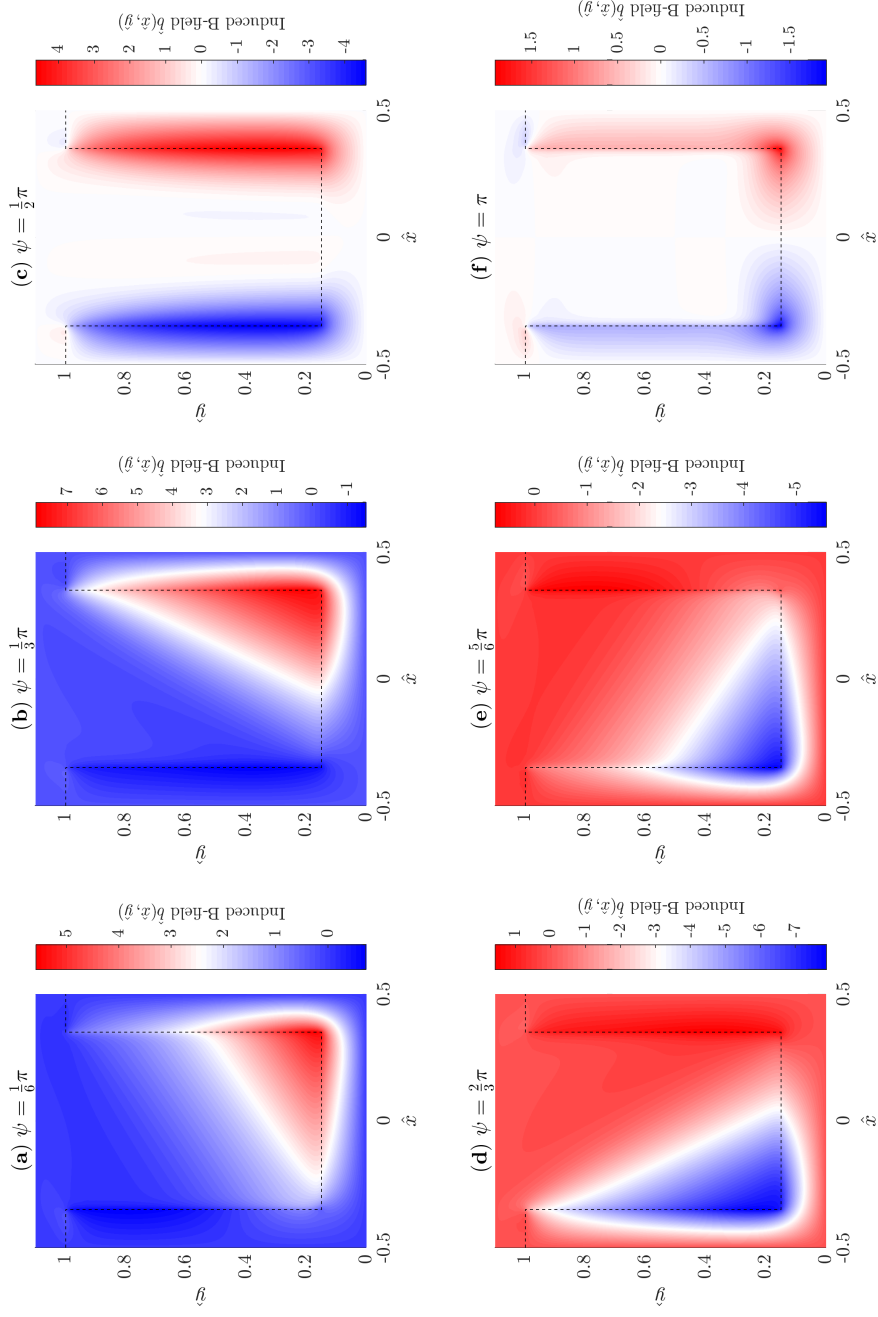


Figure 4.20: Two-dimensional surface plots illustrating the effect of changing the applied magnetic field angle ψ on the induced magnetic field.

4.10 Comparison with existing data

Due to the relative youth of the LiMIT concept and the use of TEMHD for this purpose, experimental and computational data are relatively scarce. We therefore draw parallels with Xu’s dissertation [108], which contains approximate results arising from a 1D analytical model, and a 3D numerical computational model. To start with, we compare with predictions of the temperature dependence on other parameters, by recreating Figures 4.29 and 4.31 from [108] in Figure 4.21. We notice that the results of our 2D numerical simulations show reasonable agreement with both of Xu’s 1D and 3D results, particularly for lower applied heat fluxes (namely 1 and 3MW/m²). However, we note that both our and Xu’s results predict temperatures that are unfeasible, because liquid lithium evaporates at approximately 1600K whilst becoming corrosive at approximately 700K. Xu’s 3D model and (in an ad hoc way) 1D model both incorporate heat transfer *along* the trench, which is neglected in our purely 2D calculations. In Xu’s results, the induced field and temperature are fully therefore coupled, while in ours they are not.

4.11 Discussion

In this Chapter, we derived a model of TEMHD flow inside a single trench, like that seen in the Liquid Metal-Infused Trenches concept. We provided an unsteady version of the problem, but carried out all the analysis in a steady form of the problem, where both the applied magnetic field and impinging heat flux at the free surface are constant. We used COMSOL Multiphysics to obtain numerical solutions to the rectangular trench problem. Solving the reduced version of the TEMHD problem after throwing away negligible terms, and programming the equations manually into COMSOL, resulted in numerical solutions that could be carried out at a small fraction of the computational cost of the problem involving the full physics. Doing so allowed us to carry out parameter sweeps easily and efficiently, resulting in insights that would otherwise be computationally demanding to obtain.

By solving the TEMHD trench problem when the walls are thin, we obtained approximate solutions in the limit of large Hartmann number which demonstrate

good agreement with the numerical solutions, even when the walls are not particularly thin. We also obtained a couple of limiting cases in which an analytical solution is possible to obtain, and found that this shows good qualitative agreement with the numerical solutions in those parameter regimes. From a physical point of view, we demonstrated that the velocity profiles are typically shaped like an upside-down “U”, with mostly constant behaviour near the centre of the trench apart from Hartmann layers and “side” layers near the side walls and base respectively. We also demonstrated that the horizontal magnetic field profiles are typically sawtooth-shaped. We demonstrated both asymptotically and numerically that there exist Hartmann and “side” layers near the side walls and base respectively, and numerically demonstrated that there are also velocity jets near the electrically conducting base.

Figure 4.12 shows that, as the side walls are made thicker, the Hartmann layers begin to merge, and the bulk velocity decreases and any velocity jets are flattened. The sharp sawtooth behaviour of the horizontal magnetic field profiles is ameliorated by making the side walls thicker, i.e. made less sharp. Meanwhile, making the trench wider has the effect of increasing the bulk velocity up to a point, whilst the horizontal magnetic field profiles are made sharper and taller but only up to a point. Meanwhile, Figure 4.13 demonstrates that making the base thicker has the effect of retarding the flow and making the magnetic field dipoles sharper, whilst making the trench wall higher serves to speed up the flow but up to a point, introducing a distinct velocity jet when the wall is high enough. For example, the shape studies involving changing the trench height and base thickness in Figure 4.14 demonstrate that a “trenchlet” configuration with a thick base and short side-walls is optimal in some sense; in fact, this type of trench has been used in the EAST tokamak.

Figure 4.15 demonstrates that making the free surface too high can result in massively faster speeds in the fluid region above the walls compared to that between the walls, and whilst changing the trench slope angle does not appreciably affect the TEMHD flow between the trench walls, it can significantly alter the flow above the walls, even resulting in a flow reversal in some cases. Figure 4.16 demonstrates that making the walls and base highly electrically conducting results in a fastest bulk flow in which a conducting-wall jet appears near the case, whilst

making them electrically insulating results in no flow. Meanwhile, Figure 4.17 shows that making the walls very thermally conducting results in no flow, whilst making them thermally insulating results in a fastest flow, although this observation should be treated with caution because the temperatures become unphysically large in this case. Decreasing ϵ (or equivalently increasing the Hartmann number) as in Figure 4.18 has the effect of flattening the velocity profiles in a similar way to plug flow, as well as inducing a sharp velocity jet near the base of the trench. However, in looking at the dimensional variables, somewhat counterintuitively, increasing the applied magnetic field strength $\mathcal{B}^a$ actually makes the flow slower, and decreases the strength of the induced magnetic field dipole.

Figures 4.19 and 4.20 demonstrate the presence of triangular-shaped regions in the solutions, which is not dissimilar to that seen in previous literature [59], and that if the direction of the magnetic field is reversed, then so is the distribution of velocity; this is surprisingly not the case for the induced magnetic field. Finally, Figure 4.21 shows that our model agrees reasonably well with those of Xu at lower heat fluxes, but the results we obtained diverge from those at higher heat fluxes.

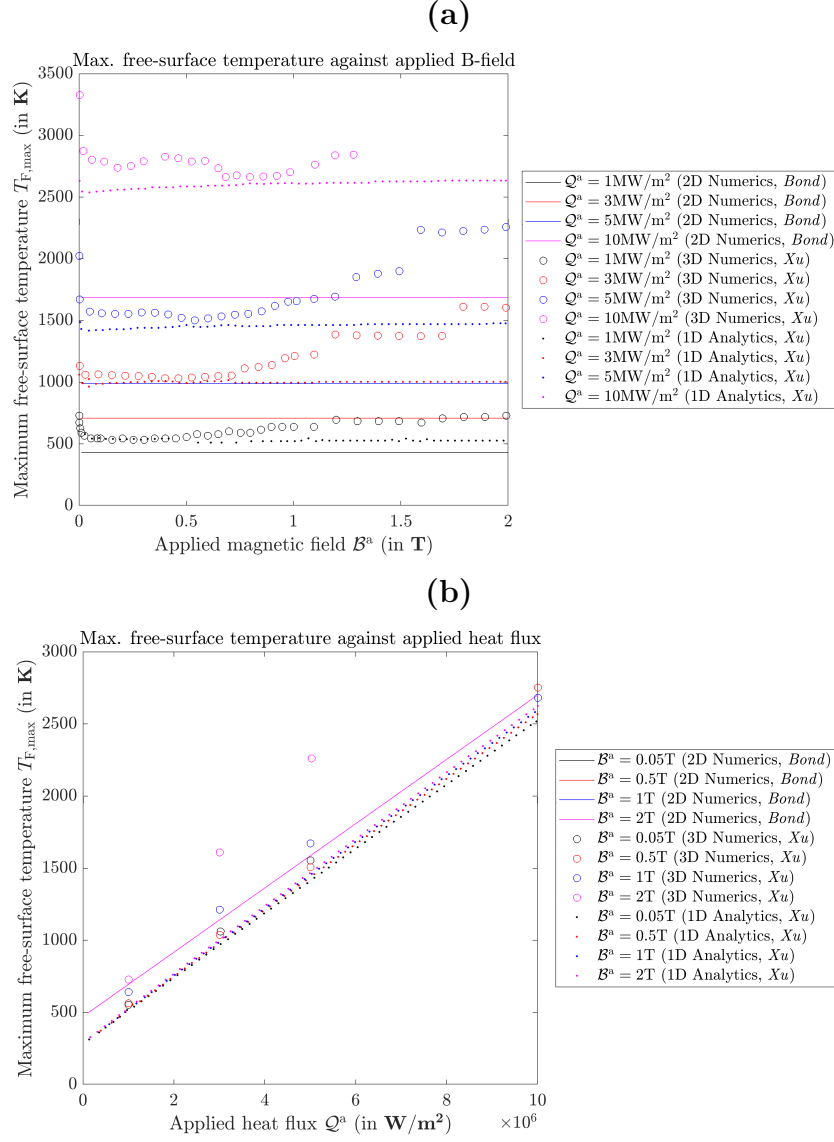


Figure 4.21: Plots of maximum free-surface temperature against (a) applied magnetic field, and (b) applied heat flux, and comparison with Xu's data incorporating stainless steel (SS316) trenches of 4mm total width, 5mm height and 1mm side-wall thickness. The electrical conductivity in this case is taken to be $S^{\ominus} = 1.6 \times 10^6 \text{S/m}$ and the thermal conductivity $k^{\ominus} = 16 \text{W/mK}$.

Chapter 5

Conclusions and further work

The work in this thesis is motivated by the need to understand the flow of liquid lithium in a trench geometry in the presence of large imposed magnetic field and heat flux.

To gain insight into the possible flow structure and the use of asymptotic and numerical solution techniques, in Chapter 3 we focus on the classical 1953 Shercliff problem in MHD duct flow, with isothermal flow and electrically insulating walls. We allow the aspect ratio r to vary and consider two different distinguished limits, solving the problem using matched asymptotic expansions in each case. In studying this simple canonical problem, both analytically and numerically, we observe some phenomena which are classically noticed in MHD duct flow problems; namely Hartmann layers and “side layers”. Although this problem has been studied for decades, the existence of very recent literature on adjustment of the aspect ratio demonstrates it remains relevant [107]. One interesting new contribution of Chapter 3 is the identification of a critical aspect ratio that optimised the flow along the duct for a given (large) Hartmann number.

In Chapter 4, we use the insights gained in Chapter 3 to perform an extensive hybrid numerical/asymptotic analysis of steady unidirectional TEMHD flow inside a trench akin to those seen in LiMIT. Treating the temperature as a known function to be determined numerically, we obtain asymptotic solutions for the velocity and magnetic field in the limit of large Hartmann number, demonstrating how temperature gradients drive fluid flow in a LiMIT-type setup. We use COMSOL Multiphysics to obtain numerical solutions to the problem in different param-

ter cases corresponding to both laboratory and fusion-relevant applications, and demonstrate good agreement with the asymptotic predictions.

In the case relevant to laboratory settings (such as SLiDE), we find that the predicted velocities exceed those that would be seen in practice. The definitive reason for this discrepancy is not fully understood. It could be an artifact resulting from the heat transfer into the cooling pipes beneath the trench not being explicitly modelled. Alternatively, it could be due to the way the average velocity in [108] is calculated or how the distributed heat flux is imposed (unlike in our model which assumes that it is uniform). The discrepancy could also be because the applied magnetic field in a laboratory setting is rather low (0.07T), which is a different regime from that considered in our model, where the Hartmann number is high.

Upon performing extensive parameter sweeps, we come to the following general conclusions. Firstly, making the side walls thicker can have a retarding effect on the fluid flow above the trench, whilst making the trenches wider can increase the flow speed up to a point. Increasing the base thickness can also increase the flow speed, provided that it is less than approximately 90% of the wall height due to the superposition of the crossover region and the conducting-base velocity jet. Increasing the wall height is found to reinforce the velocity jet near the conducting base, thus increasing the speed of the flow inside the trench, but only up to a point. The flow speed above the trench can be optimised by using either a trench that is very wide with short walls, or one whose base is very thick compared to the wall height. The latter scenario represents a good trade-off between optimising the flow and making sure that the free-surface temperature is relatively low.

If the trench is overfilled too much, then the liquid above the trench can flow at a speed several times that inside the trench, which can in turn pull fluid away from the trench. From the numerical simulations and contour plots, one can in principle determine how much the trench should be filled given a slope angle, so that velocities above and below the trench remain approximately coincident.

Although investigations involving changing the electrical and thermal conductivities have also taken place, demonstrating their impact on the flow properties, these quantities are in practice heavily constrained by the available choices of material used for constructing LiMIT. Nevertheless, they do confirm physical intuition; for example, if the side-walls are electrically insulating (i.e. Σ is small) then there

is no scope for thermoelectric currents to circulate around the side walls and drive the flow, so the flow velocity approaches zero. Meanwhile, if the wall is perfectly electrically conducting, then one obtains a velocity profile which shows approximate piecewise constant behaviour, and the electrical current through the wall reaches its peak. Both findings are consistent with the asymptotic predictions. Meanwhile, if the walls are thermally insulating (i.e. K is small), the liquid will become unfeasibly hot, leading to corrosion and evaporation, but if they are perfectly thermally conducting (i.e. K is large) then there will be no flow, and most of the temperature variation will be seen inside the base.

We confirm our asymptotic predictions by taking $\epsilon \ll 1$ in the numerics; in particular, the profile of the velocity in the bulk matches that of the temperature gradient on the right wall, aside from the side-wall boundary layers and the conducting-base jet and boundary layer. However, we emphasise that intuition is lost when considering only what happens to the dimensionless solutions, because the velocity scaling is inversely proportional to the applied magnetic field. When this is taken into account, our results agree with the findings from UIUC that increasing the magnetic field strength can have a deleterious effect on the flow velocity. Furthermore (perhaps counterintuitively) we find that increasing the applied magnetic field strength also has an attenuating effect on the induced magnetic field.

Changing the inclination of the applied magnetic field reveals some interesting behaviour which lines up with classically observed phenomena (e.g. those by Morley [58]), such as dividing sub-characteristics in the solutions. We confirm that reversing the orientation of the magnetic field reverses the direction of the flow, as expected, and that making the applied magnetic field vertical introduces regions of flow reversal, resulting in zero average velocity. Changing the applied field angle also results in triangular regions of concentrated induced magnetic field, and, perhaps surprisingly, the induced magnetic field shows somewhat similar behaviour when the applied field is vertical to when it is horizontal.

There are several interesting avenues that have arisen as a result of the modelling work we have presented. Firstly, one could attempt to model the cooling tubes beneath the trench in more detail, e.g. by replacing the Dirichlet boundary condition on the temperature on the base with a Robin boundary condition.

Secondly, despite in principle deriving an unsteady version of the trench problem, we have yet to fully consider unsteady effects and pose a time-dependent problem to solve using asymptotics and numerics. Tokamak Energy would be interested in investigating a “start-up” problem, in which the magnetic field is ramped up from zero to a field of several Teslas over a number of minutes, whilst the ST40 is in operation. Also of interest would be the effect of plasma instabilities such as edge-localised modes, which can occur over millisecond-length pulses. These could be emulated via large spikes in heat flux and/or applied field, and the effect of these spikes as the flow in the trench remain to be investigated.

Thirdly, we have focused on rectangular trenches in this modelling work, because because they are the most straightforward to design and manufacture. However, UIUC have access to 3D printing capabilities, which could in principle produce trenches of “novel” shapes including cross-sections that vary along the trench. It would be interesting to pose a shape optimisation problem, once it has been decided, in discussion with Tokamak Energy, what precisely they wish to optimise. In practice, the essential properties of the liquid lithium layer are its ability to withstand the flux of heat and exhaust produced from the plasma. For example, one could aim to maximise the flow speed above the trench, which we hypothesise to be a reasonable proxy for these properties.

We have assumed that the physically imposed properties, including the impinging heat flux and applied magnetic field, are all uniform along the direction of the trench. This is not true in practice; for example, the impinging heat flux is more likely to be concentrated around a strip near the centre of the trench (described by a Gaussian in UIUC’s modelling work) and the applied magnetic field also likely varies with distance along the trench. The next step is to take into account variations along the direction of the trench, so that the assumption of unidirectional flow is relaxed. To make computations more efficient, the trench can be assumed to be much longer than it is wide, so that a “slowly-varying” version of the model can be derived, in which distance along the z along the trench only varies the problem parametrically. In principle the problem could then be solved by considering a sequence of two-dimensional “slices” at different points along the trench.

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