

Amplification of Gravitationally Induced Entanglement

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Observation of gravitationally induced entanglement between two massive particles can be viewed as implying the existence of the nonclassical nature of gravity. However, weak interaction in the gravitational field is extremely small so that gravitationally induced entanglement is exceptionally challenging to test in practice. For addressing this key challenge, here we propose a criterion based on the logical contradictions of weak entanglement, which may boost the sensitivity of the signal due to the gravitationally induced entanglement. Specifically, we make use of the weak-value scenario and Einstein-Podolsky-Rosen (EPR) steering. We prove that it is impossible for a classical mediator to act on two local quantum objects to simulate amplified-weak-value phenomenon in two-setting EPR steering. Our approach can amplify the signal of gravitationally induced entanglement that were previously impossible to observe, by any desired factor that depends on the magnitude of the weak value. Our results not only open up the possibility of exploring nonclassical nature of gravity in the near future, but also pave the way for weak entanglement criterion of a more general nature.

I. INTRODUCTION

Quantum theory and general relativity, the two backbones of modern physics, have been verified with very high precision in their respective fields. Yet, it is hard to unify them into a unique corpus of laws. One possible route to that general theory is the quantisation of gravity, with the same spirit as other field theories. However, there is a long-standing debate whether gravity should be quantized [1–5]. Traditionally, it is believed that the effects of quantum gravity should occur at high energy scales or in the short length regime which are beyond the reach of current technology. Recently, there has been a revival of the idea of a tabletop probe, which highlights the interaction of the probe mass with the gravitational field generated by another mass [6–10]. Especially, two gravity-induced-entanglement tests, sometimes called the Bose-Marletto-Vedral (BMV) experiments [11, 12], have been proposed, which may be used to expose the quantum nature of gravity. BMV's protocol aims to provide a firm evidence on whether the gravitational field is mediated by the transfer of quantum information. Bose *et al.* suggest that if we admit the central principle of quantum information theory: entanglement between two systems cannot be created by local operations and classical communication (LOCC), then gravitationally induced entanglement indicates that gravity must be quantum [12]. On the other hand, Marletto and Vedral argue for this view through a more general information-theoretic argument [11, 13], which is based on constructor theory [14]. Specifically, one does not need to assume any specific dynamics law of mediators (in this case the gravitational field) to justify the conclusions of creating entanglement in the experimental proposal [11, 13]. In this article, we will focus on the quantum formalism. Till now, a variety of advanced theoretical and experimental proposals have been suggested to investigate the gravitationally induced entanglement and non-classicality [15–40].

Entanglement witnesses are a suitable method for measur-

ing the gravitationally induced entanglement [41, 42]. Unfortunately, due to the extremely weak strength of gravity, a “strong” and detectable entanglement signal might require a longer interaction time of massive particles in a superposition of two location (matter-wave-like interferometer), which poses a serious challenge to current experimental techniques. As we all know, a general rule of thumb is that the larger and heavier a particle is, the shorter its coherence time. In particular, the experiments must be implemented within the coherent time otherwise the loss of entanglement due to decoherence would prevent us from concluding anything about the quantum nature of gravity. Is it possible to detect weakly entangled signals with limited coupling time for a given mass of particles and a finite resolution or sensitivity of the measurement devices? Could we amplify the signals of these non-classical correlations? This is an issue that has not been mainly considered in previous studies [15–40] and is also the main motivation for our present paper.

There is a famous parametric amplification approach in quantum information field, called weak-value amplification [43–45]. Weak values have their root in quantum weak measurement, which describes a weak coupled measurements, proposed by Aharonov, Albert, and Vaidman [43]. Weak-value amplification exploits the fact that the post-selection of the weak measurement of a pointer can yield an amplified shift which is exceptionally sensitive to small changes in an interaction parameter. This has been successfully applied to the estimation of a range of small physical parameters [45], including beam deflection [46, 47], frequency shifts [48], phase shifts [49] and so on.

In this article, we propose a criterion for determining weak-gravitationally-induced entanglement, which makes use of a weak-value scenario and Einstein-Podolsky-Rosen (EPR) steering [50–52]. Specifically, we unify the weak measurements (weak value amplification scenarios) in the framework of EPR steering. Similar to the Bell test [53, 54], we consider two sets of measurement bases that can be randomly se-

lected, one of which is the normal measurement basis (e.g., the computational basis) and the other one corresponding to weak value amplification. We present a comparison of two predictions of the quantum and classical mediator, the measurement probability distribution and the measurement visibility. We show that in the case of weak entanglement, the classical mediator (in this case, the gravitational field) cannot simulate the results related to the measurement visibility of weak-value basis, thus ruling out the separable model. Concretely, our approach can amplify the signal of gravitationally induced entanglement by any desired factor that depends on the magnitude of the weak value. Compared to the previous protocols, our approach allows us to observe entangled signals that were previously impossible to observe. Besides, our criterion is not limited to the detection of weak entanglement in gravity. It is applicable to more general case of weak entanglement, including potentially macroscopic entanglement.

II. QUANTUM FORMALISM OF BMV EXPERIMENTS

Here we focus on the quantum formalism of BMV experiments. As shown in Fig. 1, BMV proposal is presented. Two quantum mass Q_A and Q_B are initially at distance from each other. Each mass individually undergoes Mach-Zehnder-type interference in parallel, and interacts with the other mass via the gravitational field which plays the role of the mediator \mathcal{M} . Under the assumption of locality, observation of gravitationally induced entanglement between $Q_A \oplus Q_B$ is the indirect evidence of nonclassicality (quantumness) of the mediator \mathcal{M} [11–13]. Specifically, The initial state of system Q_A and system Q_B is a separable state (by the first beam splitter), denoted as $\varrho_A \otimes \varrho_B = |+\rangle_A \langle +| \otimes |+\rangle_B \langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Since the masses on different paths interact via the gravitational field, the state of the composite system becomes, before they enter their respective final beam splitters, $\varrho_{AB} = \Lambda(\varrho_A \otimes \varrho_B)$, where $\Lambda(\cdot)$ is the map of channel (operation) acting on quantum systems Q_A and Q_B induced by the mediator \mathcal{M} . If the quantum state ϱ_{AB} can not be written as $\sum_i p_i \varrho_A^i \otimes \varrho_B^i$, then ϱ_{AB} is a entangled state, which indicates that the action $\Lambda(\cdot)$ is a entanglement operation. This results may be the evidence of quantumness for mediator \mathcal{M} [11, 13, 39]. On the contrary, a classical mediator can only produce unentangled quantum state for quantum systems Q_A and Q_B , that is $\varrho_{AB}^C = \Lambda^C(\varrho_A \otimes \varrho_B) = \sum_i p_i \varrho_A^i \otimes \varrho_B^i$, where $\Lambda^C(\cdot)$ denotes the effective channel induced by a classical mediator.

Weak-value scenario of BMV experiments. Under the time evolution of the joint state of the two masses is purely due to their mutual gravitational interaction [11, 13]. Thus, the channel $\Lambda(\cdot)$ of quantum mediator \mathcal{M} mentioned above (in Fig. 1) is an unitary operation. Specifically, the unitary is given as $U = \exp(-i \frac{H\tau}{\hbar}) = \cos(\frac{\Delta\phi\tau}{2\hbar})I \otimes I + i\sin(\frac{\Delta\phi\tau}{2\hbar})Z \otimes Z$, where $H = -\frac{\Delta\phi}{2}Z \otimes Z$ is the Hamiltonian of the two mass is [23, 24], in which $\Delta\phi = Gm_1m_2(\frac{1}{d} - \frac{1}{\sqrt{d^2+L^2}})$. After the unitary evolution, the joint quantum state of Q_A and

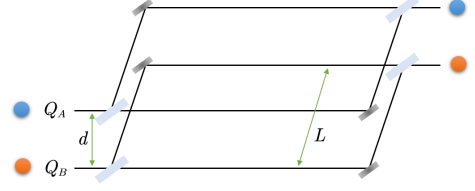


FIG. 1: Symmetric Bose-Marletto-Vedral experiment for testing gravitationally induced entanglement. There are two mass Q_A and Q_B . Each mass individually undergoes Mach-Zehnder-type interference in parallel, and interacts with the other mass via gravity.

Q_B becomes

$$\begin{aligned} |\Psi\rangle &= U|+\rangle_A \otimes |+\rangle_B \\ &= \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_A |+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle_A |-\rangle_B, \end{aligned} \quad (1)$$

which is a two-qubit entangled state. Without loss generality, $|\Psi\rangle$ can be rewritten in other basis. For the qubit case, the identity can be expressed as $I = |\epsilon\rangle\langle\epsilon| + |\epsilon^\perp\rangle\langle\epsilon^\perp|$, satisfying $\langle\epsilon^\perp|\epsilon\rangle = 0$. Here we define $|\epsilon\rangle = \epsilon|0\rangle - \sqrt{1-\epsilon^2}|1\rangle$ and $|\epsilon^\perp\rangle = \sqrt{1-\epsilon^2}|0\rangle + \epsilon|1\rangle$, where ϵ is a real positive number. Now the composite state becomes

$$\begin{aligned} |\Psi\rangle &= (|\epsilon\rangle\langle\epsilon| + |\epsilon^\perp\rangle\langle\epsilon^\perp|) \otimes I |\Psi\rangle \\ &= \alpha|\epsilon\rangle_A \otimes |\tilde{\chi}_\epsilon\rangle_B + \beta|\epsilon^\perp\rangle_A \otimes |\tilde{\chi}_{\epsilon^\perp}\rangle_B \end{aligned} \quad (2)$$

where $\alpha = \langle\epsilon|+\rangle$, $\beta = \langle\epsilon^\perp|+\rangle$, $|\tilde{\chi}_\epsilon\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|-\rangle$, and $|\tilde{\chi}_{\epsilon^\perp}\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|-\rangle$. As can be seen, the weak values are embedded in the quantum states $|\tilde{\chi}_\epsilon\rangle$ and $|\tilde{\chi}_{\epsilon^\perp}\rangle$, which are given as $A_w^\epsilon = \frac{\langle\epsilon|Z|+\rangle}{\langle\epsilon|+\rangle} = \frac{\epsilon+\sqrt{1-\epsilon^2}}{\epsilon-\sqrt{1-\epsilon^2}}$ and $A_w^{\epsilon^\perp} = \frac{\langle\epsilon^\perp|Z|+\rangle}{\langle\epsilon^\perp|+\rangle} = \frac{-\epsilon+\sqrt{1-\epsilon^2}}{\epsilon+\sqrt{1-\epsilon^2}} = -\frac{1}{A_w^\epsilon}$ respectively. From the nature of $|\Psi\rangle$, it follows that if the quantum system Q_A is projected into $|\epsilon\rangle$ ($|\epsilon^\perp\rangle$), then the quantum state of system Q_B will collapse to (unnormalized) state $|\tilde{\chi}_\epsilon\rangle$ ($|\tilde{\chi}_{\epsilon^\perp}\rangle$), and vice versa. The amplified weak-value A_w^ϵ can be achieved if the result of the collapse of quantum system Q_A to $|\epsilon\rangle$ when $\alpha = \langle\epsilon|+\rangle$ is very small. From this perspective, the generation of weak value can be explained as it originated from EPR steering [50–52], which is determined by the measurements of one of the parties [55].

As we have shown above, the weak value A_w^ϵ ($A_w^{\epsilon^\perp}$) determine the form of the quantum state $|\tilde{\chi}_\epsilon\rangle$ ($|\tilde{\chi}_{\epsilon^\perp}\rangle$). The larger the weak value A_w^ϵ , the bigger (smaller) the component $|-\rangle$ of the quantum state $|\tilde{\chi}_\epsilon\rangle$ ($|\tilde{\chi}_{\epsilon^\perp}\rangle$). According to the theory of weak-value amplification, the quantum state $|\tilde{\chi}_\epsilon\rangle$ is more likely to be accurately measured with big weak value A_w^ϵ when the phase parameter $\frac{\Delta\phi\tau}{2\hbar}$ is extremely small. So one may use such amplification phenomenon to enhance sensitivity of signal of gravitationally induced entanglement. Unfortunately, the weak-value amplification approach is specific to parametric amplification, and it cannot be used directly to rule

out the possibility of classical models. Therefore, one need to find an entanglement criterion with weak-value amplification to exclude the model of classical mediator.

III. EPR STEERING AND WEAK-VALUE SCENARIO

Here we focus on how to construct an entanglement criterion with the weak-value amplification. As we have shown above, the weak-value scenario is a special case of EPR steering, which corresponds to one-measurement setting. It is known that the experimental results of one-measurement setting in EPR steering can be easily simulated by a local model. Therefore, from this point of view, the amplified weak value in BMV experiments may be simulated by a classical mediator. That is, one cannot determine the quantumness of gravity directly with a weak value amplification scenario. In general, EPR steering scenario needs at least two different measurement bases (two-measurement setting) to determine whether the joint quantum state is steerable (entangled) or not. Hence, one may consider exploiting the EPR-steering scenario to determine entanglement while keeping the measurement basis corresponding to the weak value amplification as one of the two measurement bases for EPR steering.

Nowadays, EPR steering has been heavily studied, including the detection of various linear and nonlinear inequalities (see review [56]). There are also some quantum steering paradox based on logical contradictions [57–59]. However, we will show that none of these can be directly used for the verification of weakly amplified versions of quantum steering. The reason is that all of these depend on the expectation value, which is related to the probability (the probability of weak amplification is very low). This could lead to experimental errors masking the true entangled signal. In the following, we consider not only the probability distribution of the steered quantum states, but also introduce a physical quantity, the *visibility* of the measurement (Π) of one of subsystem, i.e. \mathcal{Q}_B . We express this quantity in terms of $V = \text{Tr}(\varrho_B \Pi)$, where ϱ_B is the steered normalized density matrix of \mathcal{Q}_B . We will show that the genuine entanglement signal is hidden in the visibility. Satisfying all conditions of probability distribution and visibility allows us to exclude any separable state model.

IV. WEAK ENTANGLEMENT CRITERION

In general EPR steering scenario, there are two parities, one of which is trusted the other is untrusted. In that case, Local hidden state (LHS) model is considered [52] to simulate the predictions of genuine EPR steering. Fortunately, in the following, we do not need to make use of LHS model (separable model is considered) to analyze the steering since two parities are trusted (controlled by ourselves) and the system \mathcal{Q}_A and \mathcal{Q}_B are genuine quantum states. As mentioned above, the quantum states generated by quantum mediator is $|\Psi\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_A|+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle_A|-\rangle_B$ [60]. Here we set two measurement basis for \mathcal{Q}_A are $\{|0\rangle, |1\rangle\}$ and

$\{|\epsilon\rangle, |\epsilon^\perp\rangle\}$ respectively, we have four steered but not normalized quantum states

$$\begin{aligned}\tilde{\rho}_B^{(0|A)}(Q) &= \frac{1}{2}|\phi_+\rangle\langle\phi_+|_B, \\ \tilde{\rho}_B^{(1|A)}(Q) &= \frac{1}{2}|\phi_-\rangle\langle\phi_-|_B, \\ \tilde{\varrho}_B^{(\epsilon|A)}(Q) &= |\alpha|^2 \text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|)|\chi_\epsilon\rangle\langle\chi_\epsilon|_B, \\ \tilde{\varrho}_B^{(\epsilon^\perp|A)}(Q) &= |\beta|^2 \text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|_B,\end{aligned}\tag{3}$$

where $\alpha = \langle\epsilon|+\rangle$, $\beta = \langle\epsilon^\perp|+\rangle$, $|\phi_\pm\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle \pm i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle$, $|\chi_\epsilon\rangle\langle\chi_\epsilon| = \frac{|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|}{\text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|)}$, $|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}| = \frac{|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|}{\text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)}$ and $\tilde{\varrho}_B$ represent the unnormalized quantum states ((Q) represents the quantum mediator). Eqn. (3) indicates that when we project the quantum states of \mathcal{Q}_A to $|0\rangle, |1\rangle, |\epsilon\rangle, |\epsilon^\perp\rangle$, we get the quantum states of \mathcal{Q}_B are $|\phi_+\rangle, |\phi_-\rangle, |\chi_\epsilon\rangle, |\chi_{\epsilon^\perp}\rangle$ with probabilities

$$\begin{aligned}\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon,\chi_\epsilon)}, p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\} \\ = \{\frac{1}{2}, \frac{1}{2}, |\alpha|^2 \text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|), |\beta|^2 \text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)\},\end{aligned}\tag{4}$$

respectively. If there exists a separable model (classical mediator) can fake the results of Eqn. (27), then one does not convinced that \mathcal{Q}_A can steer \mathcal{Q}_B 's quantum state (namely, \mathcal{Q}_A and \mathcal{Q}_B are unentangled). Otherwise the separable model contradicts with the quantum predictions. However, since the precision of the measurement devices is limited [61], we may not be able to measure the signal of weak entanglement. One can verify that when the entanglement is extremely weak (i.e., $\frac{\Delta\phi\tau}{2\hbar}$ is very small), the probability $p^{(\epsilon,\chi_\epsilon)} = |\alpha|^2 \text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|) = |\alpha|^2 [\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]$ will be very small so that the measurement device with resolution $\gamma \geq p^{(\epsilon,\chi_\epsilon)} / (p^{(\epsilon,\chi_\epsilon)} + p^{(\epsilon^\perp,\chi_{\epsilon^\perp})})$ can not distinguish whether \mathcal{Q}_A and \mathcal{Q}_B are entangled or separable or not [62]. Therefore, we cannot directly determine weak entanglement in this way. Similarly, entanglement witness and other inequality methods to calculate the expectation value will also fail in this case.

Here we note that the Eqn. (3) and (27) may cover the predictions of visibility of the measurement of \mathcal{Q}_B (steered state) when the heralded probability is very small, i.e. $p^{(\epsilon,\chi_\epsilon)} \rightarrow 0$. Here we show that the visibility of measurement of system \mathcal{Q}_B is more robust and powerful to detect weak entanglement. Without loss generality, we define the visibility of projective measurement Π_i is

$$V_{\Pi_i}(\varrho_B) = \text{Tr}(\Pi_i \varrho_B),\tag{5}$$

One can see that, for a pure qubit, the maximal visibility is 1 while for a mixed state, it is impossible to obtain the visibility equals to 1. Let's set $\{\Pi_0, \Pi_1, \Pi_2, \Pi_3\}$ are $\{|\phi_+\rangle\langle\phi_+|, |\phi_-\rangle\langle\phi_-|, |\chi_\epsilon\rangle\langle\chi_\epsilon|, |\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\}$ respectively.

Now we have four sets of measurement visibility

$$\begin{aligned} V_{\Pi_0}(\varrho_B^{(0|A)}) &= \text{Tr}(|\phi_+\rangle\langle\phi_+|\varrho_B^{(0|A)}), \\ V_{\Pi_1}(\varrho_B^{(1|A)}) &= \text{Tr}(|\phi_-\rangle\langle\phi_-|\varrho_B^{(1|A)}), \\ V_{\Pi_2}(\varrho_B^{(\epsilon|A)}) &= \text{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^{(\epsilon|A)}), \\ V_{\Pi_3}(\varrho_B^{(\epsilon^\perp|A)}) &= \text{Tr}(|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\varrho_B^{(\epsilon^\perp|A)}), \end{aligned} \quad (6)$$

where $\varrho_B^{(0|A)}, \varrho_B^{(1|A)}, \varrho_B^{(\epsilon|A)}, \varrho_B^{(\epsilon^\perp|A)}$ are the steered and normalized quantum state of Q_B . Obviously, the steered states of Eqn. (3) satisfy that all above V are equal to 1. That is

$$V_{\Pi_0}^Q(\varrho_B^{(0|A)}) = V_{\Pi_1}^Q(\varrho_B^{(1|A)}) = V_{\Pi_2}^Q(\varrho_B^{(\epsilon|A)}) = V_{\Pi_3}^Q(\varrho_B^{(\epsilon^\perp|A)}) = 1, \quad (7)$$

$$V_{\Pi_2}^C(\varrho_B^{(\epsilon|A)}) = \text{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^{(\epsilon|A)}) = \frac{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} \langle\chi_\epsilon|\phi_+\rangle\langle\phi_+|\chi_\epsilon\rangle + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)} \langle\chi_\epsilon|\phi_-\rangle\langle\phi_-|\chi_\epsilon\rangle + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)} + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}. \quad (8)$$

When the entanglement is extremely weak, without loss generality, we set $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$, $\sin(\frac{\Delta\phi\tau}{2\hbar}) = \frac{\Delta\phi\tau}{2\hbar}$. The measurement basis $\{|\epsilon\rangle, |\epsilon^\perp\rangle\}$ is chosen to realize weak-value amplification (i.e. $A_w^\epsilon = k \frac{1}{\frac{\Delta\phi\tau}{2\hbar}}$ and $A_w^{\epsilon^\perp} = \frac{1}{A_w^\epsilon} \approx \frac{\Delta\phi\tau}{2k\hbar} \ll 1$, where k is a coefficient) when $\epsilon \rightarrow \frac{1}{\sqrt{2}}$ and we have $\alpha = \langle\epsilon|+\rangle \approx 0$, $\beta = \langle\epsilon^\perp|+\rangle \approx 1$ and $\alpha k \ll 1$. Upon substituting these approximations into Eqn. (8) (discard the second-order small quantity $|\alpha|^2$, $(\frac{\Delta\phi\tau}{2\hbar})^2$ and set $1 \pm (\frac{\Delta\phi\tau}{2\hbar})^2 \approx 1$), we obtain [62]

$$V_{\Pi_2}^C(\varrho_B^{(\epsilon|A)}) \approx \frac{1}{1+k^2}, \quad (9)$$

while $V_{\Pi_0}^C \approx V_{\Pi_1}^C \approx V_{\Pi_3}^C \approx 1$. One can see that this result is contradictory to the results in Eqn. (7). The measured visibility of weak entanglement is all equal to 1, however, the separable model has a $\frac{1}{1+k^2}$. If $k = 1$, we have $V_{\Pi_2}^C(\varrho_B^{(\epsilon|A)}) \approx \frac{1}{2}$. This is a logical contradiction of weak entanglement. It is clear that the distinguishability of measurement visibility is much greater than the probability distribution of measurement. Therefore, the signal of weak entanglement is amplified. Another implication of amplifying entanglement seems to be that we can reduce the experimental requirement in tests of gravitationally induced entanglement. Given the sensitivity of measurement device, our scheme can achieved $X = A_w^\epsilon$ saving for coupling strength of gravity. For example, if $A_w^\epsilon = 10^4$, we can reduce the mass of two systems by 10 times, and shorten the coupling time by 100 times [62]. Our methodology does not depend on a specific physical system. Hence, different physical systems may realize amplification of gravitationally induced entanglement by this approaches. Besides, our weak entanglement criterion remains valid at a certain degree of decoherence and the limited

in which label Q indicates quantum prediction. As we analyzed before, in the case of extremely weak entanglement, the results of Eqn. (27) (probability distribution) can be simulated by a separable state ϱ_{AB}^C [62]. Naturally, one may wonder whether separable states ϱ_{AB}^C can also emulate the measurement visibility. Our finding is that the visibility of measurement setting $V_{\Pi_2}(\varrho_B^{(\epsilon|A)})$ corresponding to the weak value amplification cannot be simulated. Specifically, the classical visibility of measurement Π_2 is given as [62]

precision of measurement device [62].

V. TESTS OF GRAVITATIONALLY INDUCED ENTANGLEMENT

As a result of technological advance in quantum manipulation of matter at larger mass scales [63–65] and in gravitational measurements at smaller mesoscopic mass scales [66], probing nonclassical nature of gravity becomes possible. In Ref [12, 40], the spin degrees of freedom of the particles are used to construct the Stern-Gerlach interferometry to test quantum gravity. Remarkably, there are many other physical realisations apart from spin degrees of freedom on probing gravitationally induced entanglement [15–38], such as neutrino-like oscillations [15], optomechanics [18] and atomic interferometers [27, 62].

Very recently, there is a promising experimental proposal that uses two-level systems coupled to a massive resonator (a harmonic oscillator) to probe gravitationally induced entanglement [68]. Unlike Ref. [27], it enhance the gravitational interaction of two two-level systems by a massive particle (as a mediator), where the effective gravity-induced coupling strength is increased by a factor of $\frac{g_b}{g_w}$ [68]. Our criterion can also be applied to this scenario to achieve additional amplification.

VI. DISCUSSION AND CONCLUSION

Historically, there is a one measurement steering protocol the same as the amplification by LOCC. In particular, Gisin's paper in 1995, who called it "Hidden quantum nonlocality revealed by local filters" [69]. However, this local filter is es-

essentially a Positive Operator Valued Measures (POVM) and needs to be performed with the help of an additional Hilbert space, such as an additional ancillary qubit. The POVM measurement will increase the experimental difficulty because it requires additional coupling to a new quantum system and measuring it. Certainly, if one do not consider the difficulty of measurement, then one may perform two types of entanglement amplification. The first type of amplification can be achieved by using the local filter method, and the second type of amplification is achieved by the way we propose in the paper.

From a fundamental perspective, our work combines weak value theory and quantum correlation theory for the first time. We show that weak-valued amplification in the two-setting protocol is impossible to be simulated classically. Our results also support the fact that weak values are quantum, whereas in the past it was controversial whether weak values were quantum or not [70–74].

Our scheme is applicable to any weakly entangled pure state, while allowing for the presence of partial decoherence and noise. It be expected to significantly reduce the requirements for experiments, allowing for the test of gravitationally induced entanglement in the near future. From a more general point of view, our results is a general weak entanglement criterion. We reveal how the hidden weakly entangled information is re-presented as it is. Compared to the previous protocols, our approach allows us to observe entangled signals that were previously impossible to observe. As an outlook, we expect that the criterion can be extended to the more general mixed states, which may make it more possible to detect the entanglement of macroscopic objects.

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Appendix A : separable state model for classical mediator

A. Separable state model

Here we analyze the model of separable states for \mathcal{Q}_A and \mathcal{Q}_B (induced by a classical mediator) that can simulate the

results of quantum entanglement. As we mentioned before, the quantum states generated by classical mediator is $\varrho_{AB} = \Lambda^C(\varrho_A \otimes \varrho_B) = \sum_i p_i \varrho_A^i \otimes \varrho_B^i$, we have

$$\begin{aligned}\tilde{\varrho}_B^{(0|A)}(C) &= \sum_i p_i \text{Tr}(|0\rangle\langle 0| \varrho_A^i) \varrho_B^i, \\ \tilde{\varrho}_B^{(1|A)}(C) &= \sum_i p_i \text{Tr}(|1\rangle\langle 1| \varrho_A^i) \varrho_B^i, \\ \tilde{\varrho}_B^{(\epsilon|A)}(C) &= \sum_i p_i \text{Tr}(|\epsilon\rangle\langle \epsilon| \varrho_A^i) \varrho_B^i, \\ \tilde{\varrho}_B^{(\epsilon^\perp|A)}(C) &= \sum_i p_i \text{Tr}(|\epsilon^\perp\rangle\langle \epsilon^\perp| \varrho_A^i) \varrho_B^i,\end{aligned}\tag{10}$$

where $\tilde{\varrho}_B(C)$ is the unnormalized quantum state with the classical mediator. If a classical mediator can simulate all the results of a quantum mediator, it must satisfy $\tilde{\varrho}_B^{(0|)}(Q) = \tilde{\varrho}_B^{(0|)}(C)$, $\tilde{\varrho}_B^{(1|)}(Q) = \tilde{\varrho}_B^{(1|)}(C)$, $\tilde{\varrho}_B^{(\epsilon|)}(Q) = \tilde{\varrho}_B^{(\epsilon|)}(C)$ and $\tilde{\varrho}_B^{(\epsilon^\perp|)}(Q) = \tilde{\varrho}_B^{(\epsilon^\perp|)}(C)$. So we have

$$\begin{aligned}p^{(0, \chi_{\phi+})} |0\rangle\langle 0|_A \otimes |\phi_+\rangle\langle \phi_+|_B \\ = \sum_i p_i \text{Tr}(|0\rangle\langle 0| \varrho_A^i) \text{Tr}(|\phi_+\rangle\langle \phi_+| \varrho_B^i) \varrho_A^i \otimes \varrho_B^i, \\ p^{(1, \chi_{\phi-})} |1\rangle\langle 1|_A \otimes |\phi_-\rangle\langle \phi_-|_B \\ = \sum_i p_i \text{Tr}(|1\rangle\langle 1| \varrho_A^i) \text{Tr}(|\phi_-\rangle\langle \phi_-| \varrho_B^i) \varrho_A^i \otimes \varrho_B^i, \\ p^{(\epsilon, \chi_\epsilon)} |\epsilon\rangle\langle \epsilon|_A \otimes |\chi_\epsilon\rangle\langle \chi_\epsilon|_B \\ = \sum_i p_i \text{Tr}(|\epsilon\rangle\langle \epsilon| \varrho_A^i) \text{Tr}(|\chi_\epsilon\rangle\langle \chi_\epsilon| \varrho_B^i) \varrho_A^i \otimes \varrho_B^i, \\ p^{(\epsilon^\perp, \chi_{\epsilon^\perp})} |\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes |\chi_{\epsilon^\perp}\rangle\langle \chi_{\epsilon^\perp}|_B \\ = \sum_i p_i \text{Tr}(|\epsilon^\perp\rangle\langle \epsilon^\perp| \varrho_A^i) \text{Tr}(|\chi_{\epsilon^\perp}\rangle\langle \chi_{\epsilon^\perp}| \varrho_B^i) \varrho_A^i \otimes \varrho_B^i,\end{aligned}\tag{11}$$

where

$$\begin{aligned}\{p^{(0, \phi_+)}, p^{(1, \phi_-)}, p^{(\epsilon, \chi_\epsilon)}, p^{(\epsilon^\perp, \chi_{\epsilon^\perp})}\} \\ = \{\frac{1}{2}, \frac{1}{2}, \alpha^2 \text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle \tilde{\chi}_\epsilon|), \beta^2 \text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle \tilde{\chi}_{\epsilon^\perp}|)\}.\end{aligned}\tag{12}$$

It is well-known that a pure state cannot be obtained by a convex sum of other different states, namely, a density matrix of pure state can only be expanded by itself. Let us look at Eq. (11), because the left-hand side is proportional to a pure state, without loss of generality, one has

$$\begin{aligned}
p^{(0,\phi_+)} &= p_j \text{Tr}(|0\rangle\langle 0| \varrho_A^j) \text{Tr}(|\phi_+\rangle\langle \phi_+| \varrho_B^j) & \text{and} & \quad \varrho_A^j \otimes \varrho_B^j = |0\rangle\langle 0|_A \otimes |\phi_+\rangle\langle \phi_+|_B, \\
p^{(1,\phi_-)} &= p_k \text{Tr}(|1\rangle\langle 1| \varrho_A^k) \text{Tr}(|\phi_-\rangle\langle \phi_-| \varrho_B^k) & \text{and} & \quad \varrho_A^k \otimes \varrho_B^k = |1\rangle\langle 1|_A \otimes |\phi_-\rangle\langle \phi_-|_B, \\
p^{(\epsilon, \chi_\epsilon)} &= p_m \text{Tr}(|\epsilon\rangle\langle \epsilon| \varrho_A^m) \text{Tr}(|\chi_\epsilon\rangle\langle \chi_\epsilon| \varrho_B^m) & \text{and} & \quad \varrho_A^m \otimes \varrho_B^m = |\epsilon\rangle\langle \epsilon|_A \otimes |\chi_\epsilon\rangle\langle \chi_\epsilon|_B, \\
p^{(\epsilon^\perp, \chi_{\epsilon^\perp})} &= p_n \text{Tr}(|\epsilon^\perp\rangle\langle \epsilon^\perp| \varrho_A^n) \text{Tr}(|\chi_{\epsilon^\perp}\rangle\langle \chi_{\epsilon^\perp}| \varrho_B^n) & \text{and} & \quad \varrho_A^n \otimes \varrho_B^n = |\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes |\chi_{\epsilon^\perp}\rangle\langle \chi_{\epsilon^\perp}|_B.
\end{aligned} \tag{13}$$

Naturally, one may consider mixing these four pure states with corresponding probabilities to construct a separable model to simulate the prediction of quantum mediator. Since two basis

$a = \{0, 1\}$ and $b = \{\epsilon, \epsilon^\perp\}$ are randomly selected (with probability $\frac{1}{2}$). Therefore, the separable state induced by classical mediator can be written as

$$\begin{aligned}
\varrho_{AB}^C &= \Lambda^C(\varrho_A \otimes \varrho_B) = \frac{1}{2} [p^{(0,\phi_+)} |0\rangle\langle 0|_A \otimes |\phi_+\rangle\langle \phi_+|_B + p^{(1,\phi_-)} |1\rangle\langle 1|_A \otimes |\phi_-\rangle\langle \phi_-|_B + \\
&\quad \frac{1}{2} [p^{(\epsilon, \chi_\epsilon)} |\epsilon\rangle\langle \epsilon|_A \otimes |\chi_\epsilon\rangle\langle \chi_\epsilon|_B + p^{(\epsilon^\perp, \chi_{\epsilon^\perp})} |\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes |\chi_{\epsilon^\perp}\rangle\langle \chi_{\epsilon^\perp}|_B].
\end{aligned} \tag{14}$$

It should be noted that if our test only using a single basis $b = \{\epsilon, \epsilon^\perp\}$, it is easy to find a separable state to simulate the results of quantum mediator (one can verify it). Similar to quantum steering scenario, two or more than two basis are considered, in theory, there is no classical quantum mediator can simulate it (probability distribution). However, ideal projective measurements can not be implemented in experiments since they need infinite resource costs [61]. That is, the measurement device has limited resolution. One can verify that when the entanglement is extremely weak (i.e., $\frac{\Delta\phi\tau}{2\hbar}$ is very small), the probability $p^{(\epsilon, \chi_\epsilon)} = \alpha^2 \text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle \tilde{\chi}_\epsilon|) = \alpha^2 [\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar}) |A_w^\epsilon|^2]$ will be very small so that the measurement device with resolution $\eta \geq p^{(\epsilon, \chi_\epsilon)} / (p^{(\epsilon, \chi_\epsilon)} + p^{(\epsilon^\perp, \chi_{\epsilon^\perp})})$ can not distinguish whether \mathcal{Q}_A and \mathcal{Q}_B are entangled or separable or not.

Example. Suppose $\frac{\Delta\phi\tau}{2\hbar}$ is very small and A_w^ϵ is very large, we approximate $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$ and $\sin(\frac{\Delta\phi\tau}{2\hbar}) \approx \frac{\Delta\phi\tau}{2\hbar}$. So we have $|\tilde{\chi}_{\epsilon^\perp}\rangle \approx |+\rangle$ and $|\tilde{\chi}_\epsilon\rangle = |+\rangle + i\frac{\Delta\phi\tau}{2\hbar} A_w^\epsilon |-\rangle$. Here we set $\frac{\Delta\phi\tau}{2\hbar} A_w^\epsilon = 1$, the quantum state $|\tilde{\chi}_\epsilon\rangle$ becomes $|+\rangle + i|-\rangle$.

So we have

$$\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon, \chi_\epsilon)}, p^{(\epsilon^\perp, \chi_{\epsilon^\perp})}\} = \{\frac{1}{2}, \frac{1}{2}, 2\alpha^2, \beta^2\}. \tag{15}$$

Recall $\alpha = \langle \epsilon|+\rangle$, $\beta = \langle \epsilon^\perp|+\rangle$, $|\epsilon\rangle = \epsilon|0\rangle - \sqrt{1-\epsilon^2}|1\rangle$ and $|\epsilon^\perp\rangle = \sqrt{1-\epsilon^2}|0\rangle + \epsilon|1\rangle$. If we want to get a big weak value $A_w^\epsilon = \frac{\langle \epsilon|Z|+\rangle}{\langle \epsilon|+\rangle} = \frac{\epsilon + \sqrt{1-\epsilon^2}}{\epsilon - \sqrt{1-\epsilon^2}}$, ϵ should close to $\frac{1}{\sqrt{2}}$. So $\alpha \approx \frac{\sqrt{2}}{A_w^\epsilon} = \sqrt{2} \frac{\Delta\phi\tau}{2\hbar}$ and $\beta \approx 1$. Therefore, Eqn. (15) becomes

$$\{p^{(0,+)}, p^{(1,+)}, p^{(-, \chi_\epsilon)}, p^{(+, +)}\} = \{\frac{1}{2}, \frac{1}{2}, 4(\frac{\Delta\phi\tau}{2\hbar})^2, 1\}. \tag{16}$$

It is easy to verify that the above results can be simulated by a separable state $|+\rangle_A \otimes |+\rangle_B$ (a more accurate model should be in the form of ϱ_{AB}^C) if $4(\frac{\Delta\phi\tau}{2\hbar})^2$ is small which may be masked by the noise of measurement device.

B. Visibility of measurement for separable model

Before calculate the visibility of measurement, we need to find the reduced density matrix $\varrho_B^{\langle 0|A}$, $\varrho_B^{\langle 1|A}$, $\varrho_B^{\langle \epsilon|A}$ and $\varrho_B^{\langle \epsilon^\perp|A}$ for ϱ_{AB}^C , which are as follow

$$\begin{aligned}\varrho_B^{\langle 0|A} &= \frac{\text{Tr}_A(|0\rangle\langle 0|_A \otimes I_B \varrho_{AB}^C)}{\text{Tr}(|0\rangle\langle 0|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}p^{(0,\phi_+)}|\phi_+\rangle\langle\phi_+| + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)}|\chi_\epsilon\rangle\langle\chi_\epsilon| + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|}{\frac{1}{2}p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}; \\ \varrho_B^{\langle 1|A} &= \frac{\text{Tr}_A(|1\rangle\langle 1|_A \otimes I_B \varrho_{AB}^C)}{\text{Tr}(|1\rangle\langle 1|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}p^{(1,\phi_-)}|\phi_-\rangle\langle\phi_-| + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon,\chi_\epsilon)}|\chi_\epsilon\rangle\langle\chi_\epsilon| + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|}{\frac{1}{2}p^{(1,\phi_-)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}; \\ \varrho_B^{\langle \epsilon|A} &= \frac{\text{Tr}_A(|\epsilon\rangle\langle \epsilon|_A \otimes I_B \varrho_{AB}^C)}{\text{Tr}(|\epsilon\rangle\langle \epsilon|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)}|\phi_+\rangle\langle\phi_+| + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)}|\phi_-\rangle\langle\phi_-| + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}|\chi_\epsilon\rangle\langle\chi_\epsilon|}{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)} + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}; \\ \varrho_B^{\langle \epsilon^\perp|A} &= \frac{\text{Tr}_A(|\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes I_B \varrho_{AB}^C)}{\text{Tr}(|\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}(1-\epsilon^2)p^{(0,\phi_+)}|\phi_+\rangle\langle\phi_+| + \frac{1}{2}\epsilon^2 p^{(1,\phi_-)}|\phi_-\rangle\langle\phi_-| + \frac{1}{2}p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|}{\frac{1}{2}(1-\epsilon^2)p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}.\end{aligned}\tag{17}$$

Now substitute them into the Eqn. (28), we get the visibility of measurement of ϱ_{AB}^C :

$$\begin{aligned}V_{\Pi_0}^C(\varrho_B^{\langle 0|A}) &= \text{Tr}(|\phi_+\rangle\langle\phi_+| \varrho_B^{\langle 0|A}) = \frac{\frac{1}{2}p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)}\langle\phi_+|\chi_\epsilon\rangle\langle\chi_\epsilon|\phi_+\rangle + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\langle\phi_+|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\phi_+\rangle}{\frac{1}{2}p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}; \\ V_{\Pi_1}^C(\varrho_B^{\langle 1|A}) &= \text{Tr}(|\phi_-\rangle\langle\phi_-| \varrho_B^{\langle 1|A}) = \frac{\frac{1}{2}p^{(1,\phi_-)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon,\chi_\epsilon)}\langle\phi_-|\chi_\epsilon\rangle\langle\chi_\epsilon|\phi_-\rangle + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\langle\phi_-|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\phi_-\rangle}{\frac{1}{2}p^{(1,\phi_-)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}; \\ V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|A}) &= \text{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon| \varrho_B^{\langle \epsilon|A}) = \frac{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)}\langle\chi_\epsilon|\phi_+\rangle\langle\phi_+|\chi_\epsilon\rangle + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)}\langle\chi_\epsilon|\phi_-\rangle\langle\phi_-|\chi_\epsilon\rangle + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)} + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}; \\ V_{\Pi_3}^C(\varrho_B^{\langle \epsilon^\perp|A}) &= \text{Tr}(|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}| \varrho_B^{\langle \epsilon^\perp|A}) = \frac{\frac{1}{2}(1-\epsilon^2)p^{(0,\phi_+)}\langle\chi_{\epsilon^\perp}|\phi_+\rangle\langle\phi_+|\chi_{\epsilon^\perp}\rangle + \frac{1}{2}\epsilon^2 p^{(1,\phi_-)}\langle\chi_{\epsilon^\perp}|\phi_-\rangle\langle\phi_-|\chi_{\epsilon^\perp}\rangle + \frac{1}{2}p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}{\frac{1}{2}(1-\epsilon^2)p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}},\end{aligned}\tag{18}$$

where $|\phi_\pm\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle \pm i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle$, $|\chi_\epsilon\rangle = \frac{\cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|-\rangle}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]^{\frac{1}{2}}}$, and $|\chi_{\epsilon^\perp}\rangle = \frac{\cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|-\rangle}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]^{\frac{1}{2}}}$.

Recall that $\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon,\chi_\epsilon)}, p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\} = \{\frac{1}{2}, \frac{1}{2}, \alpha^2 \text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|), \beta^2 \text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)\}$, $\text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|) = \cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2$, $\text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|) = \cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2$, $|\langle\phi_+|\chi_\epsilon\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}$, $|\langle\phi_+|\chi_{\epsilon^\perp}\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}$, $|\langle\phi_-|\chi_\epsilon\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}$, $|\langle\phi_-|\chi_{\epsilon^\perp}\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}$, one has

$$\begin{aligned}V_{\Pi_0}^C(\varrho_B^{\langle 0|A}) &= \frac{\frac{1}{4} + \frac{1}{2}\epsilon^2\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2] + \frac{1}{2}(1-\epsilon^2)\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}{\frac{1}{4} + \frac{1}{2}\epsilon^2\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2] + \frac{1}{2}(1-\epsilon^2)\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}; \\ V_{\Pi_1}^C(\varrho_B^{\langle 1|A}) &= \frac{\frac{1}{4} + \frac{1}{2}(1-\epsilon^2)\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2] + \frac{1}{2}\epsilon^2\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}{\frac{1}{4} + \frac{1}{2}(1-\epsilon^2)\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2] + \frac{1}{2}\epsilon^2\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}; \\ V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|A}) &= \frac{\frac{1}{4}\epsilon^2 \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]} + \frac{1}{4}(1-\epsilon^2) \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]} + \frac{1}{2}\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}{\frac{1}{4}\epsilon^2 + \frac{1}{4}(1-\epsilon^2) + \frac{1}{2}\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}; \\ V_{\Pi_3}^C(\varrho_B^{\langle \epsilon^\perp|A}) &= \frac{\frac{1}{4}(1-\epsilon^2) \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]} + \frac{1}{4}\epsilon^2 \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]} + \frac{1}{2}\beta^2\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2}{\frac{1}{4}(1-\epsilon^2) + \frac{1}{4}\epsilon^2 + \frac{1}{2}\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}.\end{aligned}\tag{19}$$

C. Examples

can realize weak-value amplification (i.e. $A_w^\epsilon \approx \frac{1}{\frac{\Delta\phi\tau}{2\hbar}}$ and

When the entanglement is extremely weak, $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$, $\sin(\frac{\Delta\phi\tau}{2\hbar}) = \frac{\Delta\phi\tau}{2\hbar} \sim 0$. The measurement basis $\{|\epsilon\rangle, |\epsilon^\perp\rangle\}$

$A_w^{\epsilon\perp} = \frac{1}{A_w^\epsilon} \approx \frac{\Delta\phi\tau}{2\hbar} \ll 1$ when $\epsilon \rightarrow \frac{1}{\sqrt{2}}$ and we have $\alpha = \langle \epsilon | + \rangle \approx 0$, $\beta = \langle \epsilon^\perp | + \rangle \approx 1$. Upon substituting these approximations into Eqn. (19) (discard the second-order small quantity $|\alpha|^2$, $(\frac{\Delta\phi\tau}{2\hbar})^2$ and set $1 \pm (\frac{\Delta\phi\tau}{2\hbar})^2 \approx 1$), we obtain

$$\begin{aligned} V_{\Pi_0}^C(\varrho_B^{(0|A)}) &\approx 1; \\ V_{\Pi_1}^C(\varrho_B^{(1|A)}) &\approx 1; \\ V_{\Pi_2}^C(\varrho_B^{(\epsilon|A)}) &\approx \frac{1}{2}; \\ V_{\Pi_3}^C(\varrho_B^{(\epsilon^\perp|A)}) &\approx 1. \end{aligned} \quad (20)$$

One can see that this result is contradictory to the results analyzed in our main text. The measured visibility of weak entanglement is all equal to 1, however, the separable model has a $\frac{1}{2}$. This is a logical contradiction.

The more general case is that we set $A_w^\epsilon = k \frac{1}{\frac{\Delta\phi\tau}{2\hbar}}$, where is the coefficient. We also assure $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$, $\sin(\frac{\Delta\phi\tau}{2\hbar}) = \frac{\Delta\phi\tau}{2\hbar}$, $A_w^{\epsilon\perp} = \frac{1}{A_w^\epsilon} \approx \frac{\Delta\phi\tau}{2k\hbar} \ll 1$ and $\alpha k \ll 1$. One has

$$\begin{aligned} V_{\Pi_0}^C(\varrho_B^{(0|A)}) &\approx 1; \\ V_{\Pi_1}^C(\varrho_B^{(1|A)}) &\approx 1; \\ V_{\Pi_2}^C(\varrho_B^{(\epsilon|A)}) &\approx \frac{1}{1+k^2}; \\ V_{\Pi_3}^C(\varrho_B^{(\epsilon^\perp|A)}) &\approx 1. \end{aligned} \quad (21)$$

Appendix B: Experimental consideration

We have shown that steering scenarios with weak value amplification are not possible to be simulated classically. In the following, we focus on the analysis of the steered state for weak-value amplification, so as to consider the feasibility of the experiment for detecting weak-gravitationally-induced entanglement. Let set $m_1 = m_2 = m$ and suppose that the coupling strength of gravity is weak. After the weak gravity interaction and projective measurement ($\langle \epsilon |$) on \mathcal{Q}_A , the system \mathcal{Q}_B becomes $\varrho_B^{(\epsilon|A)}(Q) = |\chi_\epsilon\rangle\langle\chi_\epsilon|_B$ with probability $|\alpha|^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]$.

One can see that there is a weak value, which is $A_w^\epsilon = \frac{\langle \epsilon | Z | + \rangle}{\langle \epsilon | + \rangle} = \frac{\epsilon + \sqrt{1-\epsilon^2}}{\epsilon - \sqrt{1-\epsilon^2}}$. If the coupling strength of interaction of gravity is 0, that is $\Delta\phi = 0$. In this case, the final state of system \mathcal{Q}_B is $|+\rangle$, which do not carry any weak-value information. That is there is no entanglement between these two quantum system. On the contrary, when the coupling strength is not 0 but small, that is, there is a gravitationally induced phase, the final state of system \mathcal{Q}_B becomes $|\chi_\epsilon\rangle \approx \frac{1}{[1 + |\frac{\Delta\phi}{2\hbar} A_w^\epsilon|^2]^{\frac{1}{2}}}(|+\rangle + i\frac{\Delta\phi}{2\hbar} A_w^\epsilon |-\rangle)$. One can see that the quantum state of \mathcal{Q}_B is depend on the weak value A_w^ϵ . Suppose that we measure an observable $\hat{\Pi}_2 = |\chi_\epsilon\rangle\langle\chi_\epsilon|$ on the system \mathcal{Q}_B . The observable of the displacement of Π_2 is the expectation value of the final state minus the expectation

value of the initial state ($|+\rangle$) of \mathcal{Q}_B , and we get

$$\begin{aligned} \langle \Delta \hat{\Pi}_2 \rangle_Q &= \text{Tr}(\Pi_2 \varrho_B^{(\epsilon|A)}) - \text{Tr}(\Pi_2 |+\rangle\langle +|) \\ &= V_{\Pi_2}^Q(\varrho_B^{(\epsilon|A)}) - \frac{1}{[1 + |\frac{\Delta\phi}{2\hbar} A_w^\epsilon|^2]} \\ &= 1 - \frac{1}{[1 + |\frac{\Delta\phi}{2\hbar} A_w^\epsilon|^2]}. \end{aligned} \quad (22)$$

One can see that if $\frac{\Delta\phi}{2\hbar} A_w^\epsilon = 1$, $\langle \Delta \hat{\Pi}_2 \rangle_Q = \frac{1}{2}$. Even though the heralded probability $2|\alpha|^2 = 1 - 2\epsilon\sqrt{1-\epsilon^2}$ is small, after many runs of experiment, we can still observe a clear shift of the quantum state, which is a signal of entanglement between two system introduced by gravity (average shift will be 0 without entanglement generation).

For example, in the Ref [11], the shift of quantity about entanglement is $p_1 = \sin^2(\frac{\Delta\phi}{2\hbar}) \approx \frac{\Delta\phi^2}{4\hbar^2}$ in the case of weak coupling, where p_1 is the the probabilities for the mass to emerge on path 1 (R). For showing the clear enhancement of our scheme, we give a simple example. If we set $A_w^\epsilon = 10^4$ and $\frac{\Delta\phi^2}{4\hbar^2} = 10^{-4}$, we have $\langle \Delta \hat{\Pi}_2 \rangle_Q = \frac{1}{2}$, while $p_1 \approx 10^{-4}$. That is we can enhance the sensitivity and resolution for detect the quantum gravity 0.5×10^4 times by using weak-value amplification scheme. However, in above example, the steered probability becomes $p = 2 \times 10^{-8}$ in weak-value scheme. Fortunately, for existing quantum technologies, the frequency of experiments can reach MHz and beyond (suppose the time of gravitational interaction within microseconds). That is we may have 10^6 runs in one second. The total run of experiments is about $p \times 10^6 \times 3600 \times 24 = 864$ each day. This is enough for us to achieve an accurate experimental estimation. Given a resolution of measurement, we achieved $X = 10^4$ saving for coupling strength of gravity. In other words, we can reduce the mass of two systems by 10 times, and shorten the coupling time by 100 times. This is a very experiment-friendly scheme, which increases the feasibility of testing gravitationally induced entanglement by using existing technology.

Appendix C: Observing quantum gravity using Limited resolution of measurement device

In a von Neumann type measurement, the pointer is shifted proportional to the eigenvalues of the measured observable

$$|\psi\rangle \otimes |\phi(q)\rangle \rightarrow \sum_a \langle a | \psi \rangle \cdot |a\rangle \otimes |\phi(q - g_0 a)\rangle, \quad (23)$$

where Ψ and $\phi(q)$ are the initial states of system and probe, respectively, the index a refers to the eigenbasis of the observable, q is the position of the probe, and g_0 is a coupling constant. The outcome of the measurement is then provided by reading the position of the probe.

In a ideal projective measurement the probe's initial state is narrower than the distance between the eigenvalues, i.e., $\langle \phi(q-a) | \phi(q-a') \rangle = \delta_{aa'}$, hence, reading the probe's position provides full information of the measured physical quantity and collapses the system into the corresponding eigenstate

of the observable. However, it has been show that the ideal projective measurements can not be implemented in experiments since they need infinite resource costs [61]. Therefore, the resolution of measurement devices are always limited, that is $|\langle\phi(q-a)|\phi(q-a')\rangle|^2 = \gamma \neq 0$. This is the noise come from the measurement process.

In the previous section, we consider two quantum system in initial state $|+\rangle|+\rangle$ interact each other by gravity, and one of them poselected to a almost completely orthogonal state $|\epsilon\rangle = \epsilon|0\rangle - \sqrt{1-\epsilon^2}|1\rangle$, where ϵ is close to $\frac{1}{\sqrt{2}}$. This postselected operation is exactly limited by the resolution of measurement γ . So the minimal overlap $\langle\epsilon|+\rangle = \sqrt{\gamma}$, which determines the upper limit of the weak value, $Max(A_w^\epsilon) \approx \frac{\sqrt{2}}{\sqrt{\gamma}}$.

Now let consider the case that the square of effective coupling strength $\frac{\Delta\phi^2}{4\hbar^2} = \gamma$. That is one can not measure the entanglement using traditional entanglement witness methods [11, 12], since the signal of gravitationally induced entanglement is covered by the noise of measurement device. In this case, weak-value based scheme is still work. One can obtain obvious signal of gravitationally induced entanglement $\langle\Delta\hat{\Pi}_2\rangle_Q = \frac{2}{3} \gg \gamma$.

Appendix D: Observing gravitationally induced entanglement with decoherence

In fact, the decoherence is exist in experiment. The longer time in single run, the more decoherence. Besides, the evolution of desired initial quantum state is not ideal because the system is inevitably coupled with the environment. Let us consider an environment-induced decoherence model (other decoherence models are out of our analysis) for the system \mathcal{Q}_A and \mathcal{Q}_B . Without loss generality, we consider the action of environment as a partially depolarizing channel, which is given as

$$N_E(\varrho) = (1-q)\varrho + q\frac{I}{d}, \quad (24)$$

where q corresponds to the degree of a system that has been decohered. Therefore, the final state before measurement has a mathematical form

$$\varrho_{AB} = (1-q)|\Psi\rangle\langle\Psi| + q\frac{I_A \otimes I_B}{4}, \quad (25)$$

where $|\Psi\rangle = \alpha|\epsilon\rangle_A \otimes |\tilde{\chi}_\epsilon\rangle_B + \beta|\epsilon^\perp\rangle_A \otimes |\tilde{\chi}_{\epsilon^\perp}\rangle_B$. Therefore, we can get four steered states

$$\begin{aligned} \tilde{\varrho}_B^{(0|A)}(Q) &= \frac{(1-q)}{2}|\phi_+\rangle\langle\phi_+|_B + \frac{q}{2}I_B, \\ \tilde{\varrho}_B^{(1|A)}(Q) &= \frac{(1-q)}{2}|\phi_-\rangle\langle\phi_-|_B + \frac{q}{2}I_B, \\ \tilde{\varrho}_B^{(\epsilon|A)}(Q) &= (1-q)|\alpha|^2\text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|)|\chi_\epsilon\rangle\langle\chi_\epsilon|_B + \frac{q}{2}I_B, \\ \tilde{\varrho}_B^{(\epsilon^\perp|A)}(Q) &= (1-q)|\beta|^2\text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|_B + \frac{q}{2}I_B. \end{aligned} \quad (26)$$

Further, we can project quantum states of \mathcal{Q}_B to $|\phi_+\rangle, |\phi_-\rangle, |\chi_\epsilon\rangle, |\chi_{\epsilon^\perp}\rangle$ with probabilities

$$\begin{aligned} &\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon,\chi_\epsilon)}, p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\} \\ &= \left\{\frac{1}{2} - \frac{q}{4}, \frac{1}{2} - \frac{q}{4}, (1-q)|\alpha|^2\text{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|) + \frac{q}{4}, (1-q)|\beta|^2\text{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|) + \frac{q}{4}\right\} \end{aligned} \quad (27)$$

Further, the visibility of measurement for quantum media-are

$$\begin{aligned} V_{\Pi_0}^{noise}(\varrho_B^{(0|A)}) &= \frac{1}{1+q}; \\ V_{\Pi_1}^{noise}(\varrho_B^{(1|A)}) &= \frac{1}{1+q}; \\ V_{\Pi_2}^{noise}(\varrho_B^{(\epsilon|A)}) &= 1 - \frac{q}{2\text{Tr}(\tilde{\varrho}_B^{(\epsilon|A)})}; \\ V_{\Pi_3}^{noise}(\varrho_B^{(\epsilon^\perp|A)}) &= 1 - \frac{q}{2\text{Tr}(\tilde{\varrho}_B^{(\epsilon^\perp|A)})} \end{aligned} \quad (28)$$

where $i = 0, 1, 2, 3$. Similarly, we can construct a separable state (14) model for comparison. If q is small, i.e. so that $V_{\Pi_2}^{noise}(Q) > V_{\Pi_2}^{noise}(C)$, we can still obtain the amplified signal of entanglement. Therefore, our weak entanglement criterion is applicable to some kinds of mixed states.

Appendix E : atomic interferometers with a harmonic oscillator

Here we analyze a possible experimental proposal based on atomic interferometers with a harmonic oscillator [27]. We analyze how to apply our weak entanglement criterion to this scenario. Consider a harmonic oscillator \mathcal{Q}_A (a mechanical resonator) coupled to a two-state system \mathcal{Q}_B (an atom trapped in a double-well potential). Since \mathcal{Q}_B is a qubit state, one can set the position operator of the atom \mathcal{Q}_B to the Pauli matrix σ_z with the eigenstates $|L\rangle$ and $|R\rangle$, which represent the location of atom occupying, respectively. The gravitationally driven Hamiltonian of these two system is given as ($\hbar = 1$) [27]

$$H = wa^\dagger a + g(a + a^\dagger)\sigma_z, \quad (29)$$

where w , a^\dagger and a denote the frequency, creation and annihilation operators of the harmonic oscillator, respectively. The coupling coefficient g correspond to the gravitational interaction between atom and oscillator, satisfying $g \ll w$. Up to a global phase, the time-evolution operator can be rewritten as

$$U(t) = D^\dagger(\sigma_z\lambda)e^{-iwa^\dagger at}D(\sigma_z\lambda), \quad (30)$$

where $D(\zeta) \equiv \exp\{\zeta a^\dagger - \zeta^* a\}$ is the usual displacement operator and $\lambda = \frac{g}{w}$. Consider the oscillator is initialized in its ground state $|0\rangle_A$ and the atom is in the superposition of $|L\rangle_B$ and $|R\rangle_B$. After the time evolution, the composite quantum state becomes $|\Psi(t)\rangle = U(t)|0\rangle_A \otimes \frac{1}{\sqrt{2}}(|L\rangle_B + |R\rangle_B) = \frac{1}{\sqrt{2}}(|\eta\rangle_A \otimes |L\rangle_B + |-\eta\rangle_A \otimes |R\rangle_B)$, where the evolved states of the oscillator are coherent states $|\pm\eta\rangle_A = D(\pm\lambda(e^{-iwt} - 1))$.

If we implement the Hadamard gate to the two-level system \mathcal{Q}_B , we have

$$|\Psi(t)\rangle = \frac{1}{2c_+}|\text{cat}_{\eta_+}\rangle_A \otimes |L\rangle_B + \frac{1}{2c_-}|\text{cat}_{\eta_-}\rangle_A \otimes |R\rangle_B, \quad (31)$$

where $|\text{cat}_{\eta_+}\rangle_A = c_+(|\eta\rangle_A + |-\eta\rangle_A)$ and $|\text{cat}_{\eta_-}\rangle_A = c_-(|\eta\rangle_A - |-\eta\rangle_A)$ are the Schrödinger's cat state with $c_+ = \frac{1}{\sqrt{2(1+e^{-2|\eta|^2})}}$ and $c_- = \frac{1}{\sqrt{2(1-e^{-2|\eta|^2})}}$. Since $\langle \text{cat}_{\eta_-} | \text{cat}_{\eta_+} \rangle = 0$, one can address Eqn. (31) as two-qubit entangled state. However, the entanglement $|\Psi(t)\rangle$ is very weak ($\frac{1}{2c_-}$ is small) due to the fact that $|\eta|$ is very small (determined by gravitational interaction). Similar to Eqn. (2), one may choose a suitable basis for \mathcal{Q}_A (\mathcal{Q}_B) to a new form of $|\Psi(t)\rangle$, which has the amplified signal. For example, let's expand \mathcal{Q}_A to the basis $\{|v\rangle = \sin(\theta)|\text{cat}_{\eta_+}\rangle + \cos(\theta)|\text{cat}_{\eta_-}\rangle, |v^\perp\rangle = \cos(\theta)|\text{cat}_{\eta_+}\rangle - \sin(\theta)|\text{cat}_{\eta_-}\rangle\}$ with $\theta \ll 1$, and one has

$$|\Psi(t)\rangle = |v\rangle_A \otimes \left(\frac{\langle v | \text{cat}_{\eta_+} \rangle}{2c_+} |L\rangle_B + \frac{\langle v | \text{cat}_{\eta_-} \rangle}{2c_-} |R\rangle_B \right) + |v^\perp\rangle_A \otimes \left(\frac{\langle v^\perp | \text{cat}_{\eta_+} \rangle}{2c_+} |L\rangle_B + \frac{\langle v^\perp | \text{cat}_{\eta_-} \rangle}{2c_-} |R\rangle_B \right). \quad (32)$$

One can see that compared to the components of $|L\rangle_B$, the component of $|R\rangle_B$ is enlarged when \mathcal{Q}_A is projected to $|v\rangle_A$. If θ is small enough, one may have $\frac{\langle v | L \rangle}{2c_+} \sim \frac{\langle v | R \rangle}{2c_-}$ such that the component of $|R\rangle_B$ in the steered quantum state of \mathcal{Q}_B is boosted. According to weak entanglement criteria we proposed, this basis is the most significant ingredient to amplify the entangled signal. Similarly, another measurement basis needs to be selected, which may be $\{|+\rangle = \frac{1}{\sqrt{2}}|\text{cat}_{\eta_+}\rangle + \frac{1}{\sqrt{2}}|\text{cat}_{\eta_-}\rangle, |-\rangle = \frac{1}{\sqrt{2}}|\text{cat}_{\eta_+}\rangle - \frac{1}{\sqrt{2}}|\text{cat}_{\eta_-}\rangle\}$. A random selection of these two measurement bases yields the probability distribution and measurement visibility of system

\mathcal{Q}_B , which allows us to witness the gravitationally induced entanglement. In real scenarios, the resonator \mathcal{Q}_B may be a thermal state close to the ground state. We analyze this case in Appendix E.

In above analysis, we assume that the oscillator is initialized in its ground state $|0\rangle$. In a realistic implementation, due to the finite temperature (may be nK), the oscillator instead starts in a mixed state, such as a thermal state, denoted as $\varrho_{th} = \int d^2\zeta \frac{1}{\pi\bar{n}} e^{-|\zeta|^2/\bar{n}} |\zeta\rangle\langle\zeta|$. In this case, the evolving state of two system becomes

$$\varrho_{AB} = \int d^2\zeta \frac{1}{\pi\bar{n}} e^{-|\zeta|^2/\bar{n}} |\Psi_\zeta\rangle\langle\Psi_\zeta|, \quad (33)$$

where $|\Psi_\zeta\rangle = \frac{1}{2}(|\zeta + \eta\rangle_A + |\zeta - \eta\rangle_A) \otimes |L\rangle_B + \frac{1}{2}(|\zeta + \eta\rangle_A - |\zeta - \eta\rangle_A) \otimes |R\rangle_B$. Obviously, if $\zeta = 0$, $|\Psi_\zeta\rangle$ reduce to Eqn. (31). Here we will be concerned only with the projective measurements $\langle v |_A$ (corresponds to amplified entanglement), since the measurements of the other bases are trivial. The conditional state of the atom becomes

$$\varrho_B^{\langle v |_A} = \frac{\text{Tr}_A(|v\rangle\langle v|_A \otimes I_B \varrho_{AB})}{\text{Tr}(|v\rangle\langle v|_A \otimes I_B \varrho_{AB})}, \quad (34)$$

leading the measurement visibility

$$V_{\Pi_\mu}(\varrho_B^{\langle v |_A}) = \text{Tr}(\Pi_\mu \varrho_B^{\langle v |_A}) \quad (35)$$

where $\Pi_\mu = |\mu\rangle\langle\mu|$ and $|\mu\rangle = \left(\frac{\langle v | \text{cat}_{\eta_+} \rangle}{2c_+} |L\rangle_B + \frac{\langle v | \text{cat}_{\eta_-} \rangle}{2c_-} |R\rangle_B \right) / \sqrt{\left| \frac{\langle v | \text{cat}_{\eta_+} \rangle}{2c_+} \right|^2 + \left| \frac{\langle v | \text{cat}_{\eta_-} \rangle}{2c_-} \right|^2}$. When the thermal state is very close to the ground state, one may have $\frac{V_{\Pi_\mu}(\varrho_B^{\langle v |_A}) - V_{\Pi_\mu}^C(\varrho_B^{\langle v |_A})}{\gamma} = k > 1$, where γ is measurement sensitivity. Therefore we may still achieve k -fold magnification compare to the usual ones.

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