QUIVERS AND THE MODULAR REPRESENTATION THEORY OF FINITE GROUPS

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ABSTRACT

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Quivers and the modular representation theory of finite groups

The purpose of this thesis is to discuss the rôle of certain types of quiver which appear in the modular representation theory of finite groups. It is our concern to study two different types of quiver. First of all we construct the ordinary quiver of certain blocks of defect 2 of the symmetric group, and then apply our results to the alternating group and to the theory of partitions. Secondly, we consider connected components of the stable Auslander-Reiten quiver of certain groups G with normal subgroup N. The main interest lies in comparing the tree class of components of N-modules, with the tree class of components of these modules induced up to G.

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(a.k.a. the "without whom department")

"If anyone doubts my veracity, I can only say that I pity his lack of faith."

Baron Munchausen 1734-1794

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DEDICATION

Sweet love of youth, forgive, if I forget thee,
While the world's tide is bearing me along;
Other desires and other hopes beset me,
Hopes which obscure, but cannot do thee wrong!

"Remembrance"
Emily Jane Brontë 1818-1848

This work is dedicated,
with much love,
to the memory of Dawn McCarron.

INTRODUCTION

Let G be a finite group and k a field of characteristic p. One of the most important problems in the modular representation theory of G can be stated along the lines of the following: "given a p-block $\mathfrak{B}$ of G of defect d, what can be deduced about the structure of $\mathfrak{B}$?" This usually means that we are looking for the structure of the indecomposable projective modules (their composition factors, multiplicities and so on). In the case where $\mathfrak{B}$ has defect 0 or 1 (or more generally, when the defect group is cyclic), the beautiful results of Brauer and Dade (see [1]) give complete information in terms of the so-called Brauer tree. After some preliminary remarks in the first two chapters, we construct Brauer trees for cyclic blocks of some symmetric groups.

In fact there seems to have been very little actual block theory done with regard to the symmetric group, Σ_n. In 1940, T. Nakayama stated his characterisation of when an ordinary irreducible representation S^λ belongs to a p-block, in terms of the p-core [22, 6.1.21]. Once G. James gave an easy construction for all irreducible representations of Σ_n over any field [17] the way was open for a more thorough investigation of non-cyclic blocks. The main problem centres around finding the modular characters and the dimensions of these irreducible representations, or, equivalently, calculating the decomposition matrix. Using some new results, due to K. Schaper, we address ourselves to this problem of finding the decomposition numbers and then set about investigating elementary abelian p-blocks $\mathfrak{B}$ of rank 2 of Σ_{2p}. The main outcome of this is to compute the dimensions of all Ext^1 spaces of simple modules. As described in Chapter 4 this may be conveniently represented in the form of a 2-dimensional graph called the (ordinary) quiver $Q(\mathfrak{B})$ of $\mathfrak{B}$. $Q(\mathfrak{B})$ has a very pleasing symmetry: it is effectively a square which has been folded over along a diagonal. By inducing the vertices up to Σ_{2p+1} and beyond, it is possible to construct quivers for rank 2 blocks of symmetric groups of degree up to 3p-1. These results are to be found in Chapter 5, together with the corresponding story for alternating groups.

* * *

In 1975, M. Auslander and I. Reiten introduced the notion of the almost split sequence [4] in connection with the representation theory of artin algebras. Since then, investigation of these almost split, or AR-sequences has been intense, especially for modules over group algebras. As well as being intrinsically interesting, they are also very useful: for example they can be used to prove the non-singularity of the Benson-Parker inner product [5].

In fact we are not just interested in these AR-sequences *per se*; we are also keen to see how they "fit together" in a certain quiver called the Auslander-Reiten quiver $\Gamma(kG)$. This quiver is constructed from the indecomposable kG-modules together with the irreducible maps between them. In Chapter

6, we explain the connection between this quiver and the AR-sequences mentioned above; we also mention the main structure theorems of C. Riedtmann and P. Webb. The tree class associated to a connected component of the stable quiver, which Riedtmann's theorem produces will be our main concern, and this is explained in Chapter 7.

In the final two chapters we are motivated by a question raised by Solberg [36] concerning the relationship between possible tree classes of $\Gamma(kN)$ and $\Gamma(kG)$ where N is normal in G. More explicitly, suppose that M is a module lying in a component of $\Gamma(kN)$ where M is simple or periodic. Then M will lie in an A_∞ component. For simple M we investigate components of summands of $M^{\uparrow G}$, using little more than easy Clifford theory. For periodic M, which of course lie in "r-tubes", we would like to know when summands of $M^{\uparrow G}$ lie in s-tubes, where $s \gneq r$. These tubes we have christened "exceptional" and to construct some examples, we are obliged to use some pretty stiff cohomological methods.

CHAPTER 1

Preliminaries

This introductory chapter should be regarded as a preface to *both* parts of this thesis. It contains notation, remarks and standard theorems connected with representation theory. The first section deals with some absolutely basic material on finite group algebras, and further details may be found in any standard textbook like [9], [10] or [24], if required. The second section contains material which could be considered slightly more specialised: for example Loewy series and the Green Correspondence, together with the related ideas of vertices, blocks and defect groups. This material is discussed fully in [11] and [24].

1.1 Finite group algebras

To avoid unnecessary length, we shall use freely the language of group theory without comment: any gaps can be filled in by reference to [15], [16], which in fact also gives an account of ordinary and modular representation theory. It should be noted that, for typographical reasons, the only piece of non-standard notation we use is the symbol ":" to denote a semidirect product. We shall also assume familiarity with basic homological algebra (exact sequences and splittings, projectivity, projective covers, commutative diagrams and the Ext functor). For positively all aspects of this, see [10, Introduction]. Reference can also be made to [5, 1.4] or to more specialised works such as [8]. Any ring theoretic concepts which we use will be standard (see for example [2], [10]).

Let R be a complete d.v.r. with maximal ideal $\wp=\langle\pi\rangle$. Let k = R/$\wp$ be the residue class field, and let S denote the quotient field of R. Then the triple (k,R,S) is called a *p-modular system*. We say that a field F is *sufficiently large* if it contains the m'th roots of unity where m is the exponent of G. Then F is a splitting field for G and all its subgroups [10, Theorem 17.1]. By [10, §17B], for the p-modular system (k,R,S) with charS=0 and S sufficiently large, then k is sufficiently large. We shall be almost exclusively concerned with finitely generated left modules over the group algebra kG, for some finite group G and for k as described above. These conventions will be used throughout, and in particular one should note that k is a field of characteristic p, where p is prime and p divides the order of G.

If M is a kG-module, we assume familiarity with the notions of *indecomposability*, *simplicity*, (or *irreducibility*) and of *semisimplicity* of M. If M is projective-free, we call M a *core*.

By the Jordan-Hölder theorem as it applies to the kG-module M, M has a composition series whose factors A, B,...,Z say are unique up to appearance in the series. In this case, we write

$$M \sim A+B+\ldots+Z.$$

The *radical* of M, denoted J(M) is defined to be the intersection of all maximal submodules of M, and the *head* of M, denoted Hd(M) is the maximal semisimple factor module of M , so that Hd(M) = M/J(M). Denote by soc(M), the *socle* of M, defined as the maximal semisimple submodule of M. The projective cover of M will be denoted by P(M). It is well-known [11, §78] that if M is a core we can define ΩM to be the kernel of the surjective homomorphism $P(M) \to M$. The isomorphism class of ΩM is uniquely determined by the isomorphism class of M, and Ω is called the *Heller loop space operator*. So we have a short exact sequence (s.e.s.)

$$0 \to \Omega M \to P(M) \to M \to 0$$

Now define inductively $\Omega^n M := \Omega(\Omega^{n-1}M)$ for all integers n>1. M is said to be *periodic* if there exists an integer n>0 such that $M \cong \Omega^n M$. If n is the smallest such integer, then n is called the *period* of M.

If M and N are kG-modules, denote the space of k-maps by $\mathrm{Hom}_k(M,N)$; this space becomes a kG-module under the extension of the action

$$(gf)(m) = gf(g^{-1}m)$$

for $g \in G$, $f \in \mathrm{Hom}_k(M,N)$, $m \in M$. In particular, if k_G is the trivial kG-module (that is $g\lambda=\lambda$, $g \in G$, $\lambda \in k$), then $M^* := \mathrm{Hom}_k(M,k_G)$ is the *dual* of M; this is a G-space under the extension of the G-action

$$(gf)(m) = f(g^{-1}m)$$

where $g \in G$, $f \in M^*$. M is *self-dual* if $M \cong M^*$. Also the set of

G-invariant homomorphisms from M to N will be denoted by $\mathrm{Hom}_{kG}(M,N)$, which is a kG-module under the action given above. For the so-called 1-projective G-homomorphisms, see Section 9.2 in Part II.

The endomorphism ring $\mathrm{End}_{kG}(M)$ is a local ring if and only if the kG-module M is indecomposable, and this fact in turn implies

THEOREM 1.1.1 (Krull-Schmidt, [10]). *Suppose that M is as above. Then the indecomposable direct summands of M are uniquely determined up to isomorphism. In other words, if*

$$M \cong \sum_I \oplus M_i' \cong \sum_J \oplus M_j''$$

where the M_i' *and the* M_j'' *are indecomposable kG-modules, then there exists a bijection* $\phi: I \to J$ *such that* $M_i' \cong M''_{\phi(i)}$ *for all i.*∎

If M is isomorphic to a direct summand of N, we write $M|N$. If, further, M is indecomposable, we shall write $M\|N$.

We end this section with some further operations which one can perform on the kG-modules M and N. Denote by $M_{\downarrow H}$ the usual restriction of operators to a subgroup H of G. If χ is a character of M, then the restricted character will be denoted by χ_H. The tensor product [1], [2] of M and N, $M \otimes_k N$, is a kG-module in a natural way: if $g \in G$, $m \in M$, $n \in N$, then set

$$g(m \otimes n) = gm \otimes gn.$$

This gives a well-defined action of G on $M \otimes N$, because of the bilinearity of the definition, and it is clear that $M \otimes N$ is a module. The operation $\otimes$ is commutative, associative, distributive over addition, and so on.

We can use $\otimes$ to define the induced module. Let H be a subgroup of G and let A be a kH-module. We form the tensor product

$$A^{\uparrow G} := kG \otimes_{kH} A$$

which is a kG-module under the linear extension of the G-action

$$g(x\otimes a) = gx\otimes a$$

for $a\in A$, $g,x\in G$. If χ is a character of A then the induced character will be denoted by χ^G.

Finally we collect together some results relating induction, restriction and tensoring in a portmanteau theorem.

THEOREM 1.1.2 (see [10] or [24]). *Let M and N be kG- and kH- modules respectively. Then*

(i) $N^{\uparrow G} \cong \sum_{H\backslash G} g_i \otimes_{kH} N$ *where $H\backslash G$ denotes an arbitrary left transversal of H in G;*

(ii) (Nakayama Relations)

(a) $Hom_{kH}(M_{\downarrow H}, N) \cong Hom_{kG}(M, N^{\uparrow G})$

(b) $Hom_{kH}(N, M_{\downarrow H}) \cong Hom_{kG}(N^{\uparrow G}, M)$;

(iii) If M is projective then $M^{\uparrow G}$ is projective;

(iv) $(N\oplus V)^{\uparrow G} \cong N^{\uparrow G} \oplus V^{\uparrow G}$, *where V is a kH-module;*

(v) $(N^*)^{\uparrow G} \cong (N^{\uparrow G})^*$;

(vi) (Frobenius Reciprocity) $M\otimes_k N^{\uparrow G} \cong (M_{\downarrow H}\otimes_k N)^{\uparrow G}$;

(vii) (Transitivity of induction) Let $H\leq K\leq G$. Then $(N^{\uparrow K})^{\uparrow G} \cong N^{\uparrow G}$;

(viii) Short exact sequences have their exactness preserved by induction and restriction. ∎

We end this section with the statement of Mackey's Theorem. First some notation: if $L \leq G$ and M is a kL-module, we shall write $g \otimes M$ (or ${}^{g}M$ or gM) for the L^{g}-module with action

$$(glg^{-1})(g\otimes m) = g\otimes lm.$$

THEOREM 1.1.3 (Mackey Decomposition, [15, Satz 16.9] or [9, §44]). *Let $H,K \leq G$. Let N be a kK-module. Then*

$$N^{\uparrow G}{}_{\downarrow H} \cong \sum_{K\backslash G/H} \oplus((g_i \otimes N)_{\downarrow K^{g} \cap H})^{\uparrow H}$$

as kH-modules, where $K\backslash G/H$ denotes an arbitrary transversal of double (H,K)-cosets. ∎

Remark. We could have given an identical formulation of the above results for RG- and SG- modules, but since we deal primarily with kG-modules, it seems more natural just to look at this case.

1.2 Aspects of representation theory

This section deals with slightly more specialised topics, which nevertheless ought to be standard currency amongst the *cognoscenti*.

Let M be a kG-module. Then J(M) is a submodule, so it too has a radical and we denote $J(J(M))$ by $J^2(M)$. Repeating this argument, define $J^i(M)$; observe that there is an n such that $J^n(M)=0$ but $J^{n-1}(M)\neq 0$. This n is called the *Loewy length* of M, and the sequence of modules

$$M \supseteq J(M) \supseteq J^2(M) \supseteq \dots \supseteq J^{n-1}(M) \supseteq 0$$

together with their semisimple quotients is called the *Loewy series* of M. Dually, M/soc(M) is a module, so has a socle: we let $\mathrm{soc}_2(M)$ be the submodule of M containing soc(M) such that $\mathrm{soc}_2(M)/\mathrm{soc}(M)=\mathrm{soc}(M/\mathrm{soc}(M))$. Again we can inductively define $\mathrm{soc}_j(M)$. Now there is an m such that $\mathrm{soc}_m(M)=M$, and it is elementary to show that m=n, and the *socle series* is given by

$$0 \subseteq \mathrm{soc}(M) \subseteq \mathrm{soc}_2(M) \subseteq \dots \subseteq M$$

together with its quotients. We usually denote the Loewy length of M by $\ell(M)$. If the successive quotients of the Loewy (equivalently the socle) series are simple, then M is called *uniserial* (because in this case [1, I] it has a unique composition series), and we write $M=\mathcal{U}(S_1,\dots,S_n)$ where the S_i are simple such that $J^{i-1}(M)/J^i(M) \cong S_i$.

We say that M is *H-projective* for $H\leq G$ if there is a kH-module N such that $M|N^{\uparrow G}$. The following (by no means complete) list of properties of H-projective modules will prove useful. Such results are attributed to D.G. Higman.

THEOREM 1.2.1 ([10, §19], [5, pp 30-32]). *Let M be a kG-module and N a kH-module where $H\leq G$. Then*

(i) M is H-projective if and only if M is H^g-

projective;

(ii) *M is 1-projective if and only if M is projective;*

(iii) *If P is a Sylow p-subgroup of G then M is P-projective;*

(iv) *N is K-projective, for $K \leq H \Rightarrow N^{\uparrow G}$ is K-projective;*

(v) *For $S \leq_G H$, if M is S-projective then M is H-projective.* ∎

By (v) we may consider those subgroups V of G such that V is minimal with respect to the property that M is V-projective. A member of the set of such subgroups is called a *vertex* of M. By (iii) each vertex is a p-group. Also

THEOREM 1.2.2 ([1, p.66 Theorem 4]). *The vertices of an indecomposable kG-module form a conjugacy class of p-subgroups.* ∎

Let Q be a p-subgroup of G and set $N=N_G(Q)$. The following result is to be found in [24] and is known as the *Green Correspondence.*

THEOREM 1.2.3. *Let M be an indecomposable kG-module with vertex Q, and suppose that $H \geq N$. Then*

$$M_{\downarrow H} \cong fM \oplus \left(\bigoplus_i M_i \right)$$

where fM is indecomposable with vertex Q, and M_i is indecomposable with vertex contained in $Q^x \cap H$, for some $x \notin H$.

Let W be an indecomposable kH-module with vertex Q. Then

$$W^{\uparrow G} \cong gW \oplus (\bigoplus_i W_i)$$

where gW is indecomposable with vertex Q, and W_i *is indecomposable with vertex contained in* $Q^x \cap Q$ *for some* $x \notin H$.

Also $fgW \cong W$, *and* $gfM \cong M$. ■

fM and gW are called the *Green Correspondents* of M and W respectively.

We bring this introductory chapter to an end with a brief discussion of block theory. First we look at indecomposable summands of kG:

THEOREM 1.2.4 ([1, IV.13 Theorem 1]). *kG has a unique decomposition into a direct sum of subalgebras each of which is indecomposable as an algebra.* ■

The indecomposable subalgebras in this theorem are called the *blocks* of kG. If M is an indecomposable kG-module, then $\mathfrak{B}M=M$ for a certain block $\mathfrak{B}$, and $\mathfrak{B}'M=0$ for all other blocks $\mathfrak{B}'$. In this situation we say that M *lies in* (or *belongs to*) *the block* $\mathfrak{B}$. Notice that submodules, ~~quo~~quotient modules and direct sums of modules lying in a block $\mathfrak{B}$ also lie in $\mathfrak{B}$. The following is also relevant when we come to look at the ordinary quiver of a block in Part I:

PROPOSITION 1.2.5 ([24, I.10.8]). *Let* S_1 *and* S_2 *be* *simple kG-modules. Then S_1 and S_2 lie in the same block of kG if and only if there exists a sequence of simple modules*

$$S_1=T_1,\ T_2,\ \dots,\ T_n=S_2$$

such that $\mathrm{Ext}^1_{kG}(T_i,T_{i+1}) \neq 0$ for all i $(1\leq i\leq n-1)$. ∎

View kG as a $k(G\times G)$-module under the action

$$(g_1,g_2)a = g_1 a g_2^{-1}$$

for $a\in kG$, $g_1,g_2\in G$. Let δ be the diagonal map $G\to G\times G$. We have

THEOREM 1.2.6 ([1, IV.13 Theorem 4]). *If $\mathfrak{B}$ is a block of kG, then $\mathfrak{B}$ has a vertex as a $k(G\times G)$-module, of the form δD, where D is a p-subgroup of G.* ∎

The subgroups D of G referred to in Theorem 1.2.6 are a conjugacy class of p-subgroups of G, and are called the *defect groups* of $\mathfrak{B}$. If $|D|=p^d$, $\mathfrak{B}$ is said to be *of defect d*. Properties of blocks, including a discussion of Brauer Correspondence is to be found in [1], [24], [5]. Comments and results on blocks of defect 0, 1 and 2 are presented later in this work.

PART I

ORDINARY QUIVERS AND SYMMETRIC GROUPS

CHAPTER 2

The representation theory of the symmetric group

This chapter, read in conjunction with Chapter 1 is designed to provide the reader with all necessary background from the representation theory of the symmetric group. Such material will be used constantly throughout Chapters 3 to 5.

In the sections that follow, G will denote the symmetric group Σ_n of degree n. All the usual properties of G will be used without comment, for example results about order, conjugacy and characters. The starting point is G.D. James' construction [17] of the irreducible FG-modules over a field F of *arbitrary* characteristic. This depends heavily on the Specht module, the definition of which is given in Section 2.1. In subsequent chapters, the notion of the p-hook plays a central rôle, so we give a brief summary in Section 2.2. Later on we shall also need to calculate decomposition numbers for non-cyclic blocks of G, and so we take the opportunity, in Section 2.3 to place this important problem in a more general setting.

For further information on these and other topics, see the Lecture Notes of James [20], the definitive work of James & Kerber [22], and also §75 of [11].

2.1 Specht modules and simple kG-modules

DEFINITION 2.1.1. $\lambda=(\lambda_1,\lambda_2,\ldots)$ is a *partition* of n, written $\lambda\vdash n$, if λ_1, $\lambda_2,\ldots$ are non-negative integers, with $\lambda_1\geq\lambda_2\geq\ldots$ and $\sum_1^\infty \lambda_i = n$.

For example $(4,2^2,1)\vdash 9$. Clearly the number of partitions of n equals the number of inequivalent ordinary irreducible representations of G. If $\lambda\vdash n$, the *Young diagram* $[\lambda]$ is $\{(i,j): i,j\in\mathbb{Z},\ i\geq 1,\ 1\leq j\leq\lambda_i\}$. For example,

$$[(4,2^2,1)] = \begin{array}{llll} \times & \times & \times & \times \\ \times & \times & & \\ \times & \times & & \\ \times & & & \end{array}$$

If $(i,j)\in[\lambda]$, (i,j) is a *node* of $[\lambda]$. There are obvious definitions of rows and columns of $[\lambda]$. The *conjugate diagram* $[\lambda']$ is obtained by interchanging the rows and columns in $[\lambda]$. Of course this also produces a *conjugate partition* λ'.

If $\lambda,\mu\vdash n$ we say that λ *dominates* μ and write $\lambda\gg\mu$ if for all j

$$\sum_1^j \lambda_i \geq \sum_1^j \mu_i .$$

For example $(3,2,1)\gg(2^3)$. Note however that $(3,1^3)$ and (2^3) are incomparable under $\gg$. Note also that this notation is non-standard (for typographical reasons). The *dictionary* or *lexicographic* ordering, written $>$ is defined thus: $\lambda>\mu$ if and only if the least j such that $\lambda_j\neq\mu_j$ satisfies $\lambda_j>\mu_j$. Hence for example $(3,1^3)>(2^3)$. We say that λ,μ are *neighbours* with respect to $\gg$, written $\lambda\gg\!\bullet\mu$ if and only if $\lambda\gg\mu$, $\lambda\neq\mu$ and there is no $\alpha\vdash n$ such that $\lambda\underset{\neq}{\gg}\alpha\underset{\neq}{\gg}\mu$. For a pleasant characterisation involving diagrams, see [22, 1.4.10].

From now on, let $\lambda \vdash n$. Let $F=$ k or S.

DEFINITION 2.1.2. Let M^λ be the permutation module of G on cosets of Young subgroups

$$\Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \ldots := \Sigma_{\{1,\ldots,\lambda_1\}} \times \Sigma_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots .$$

Then the *Specht module* S^λ is defined as

$$S^\lambda := FGc^\lambda M^\lambda$$

where $c^\lambda := \sum_{\Sigma_{\lambda_1'} \times \Sigma_{\lambda_2'} \times \ldots} (\mathrm{sgn}\,\pi)\pi$. S^λ has dimension independent of F.

THEOREM 2.1.3 ([20, §4]). *Let* $F=\mathbb{Q}$. *Then* $\{S^\lambda : \lambda \vdash n\}$ *is a complete set of ordinary irreducible representations of* G. *Each* S^λ *is self-dual and absolutely irreducible.* ∎

Now take F=k. The key result in James' construction is:

THEOREM 2.1.4 (The submodule theorem [17]). *Let* X *be a* kG*-submodule of* M^λ. *Then either* $c^\lambda X=0$, *in which case* $X \subseteq (S^\lambda)^\perp$ *or* $c^\lambda X \neq 0$ *in which case* $X \supseteq S^\lambda$. ∎

So let $R^\lambda = \Sigma\{X$ a kG-submodule of S^λ: $c^\lambda X=0\}$. It is immediate that $c^\lambda R^\lambda=0$.

THEOREM 2.1.5. *Either* $c^\lambda S^\lambda=0$, *so that* $R^\lambda=S^\lambda$ *or* $c^\lambda S^\lambda \neq 0$, *implying that* $R^\lambda \neq S^\lambda$, *in which case* R^λ *is the unique maximal submodule of* S^λ. ∎

Let $D^\lambda=S^\lambda/R^\lambda$ so that either $D^\lambda=0$ or D^λ is simple by Theorem 2.1.5.

DEFINITION 2.1.6. $\lambda\vdash n$ is *p-singular* if for some i, $\lambda_{i+1}=\lambda_{i+2}=\ldots=\lambda_{i+p}>0$. Otherwise λ is *p-regular*.

For example $\lambda=(4,1)$ is a 2-regular partition of 5, while $(2,1^3)$ is 2-singular. By careful trickery, we can prove that the number of p-regular classes of G equals the number of p-regular partitions of n [20, 10.2]. Our main message is

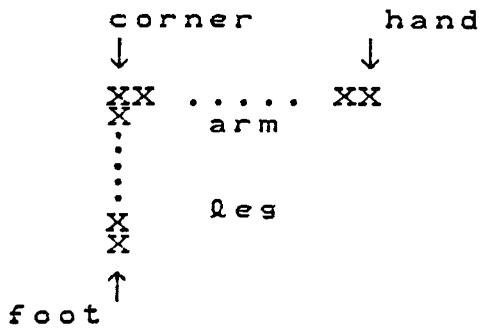
THEOREM 2.1.7 ([20, §11]). *Let* $\lambda\vdash n$. *Then*

(i) $D^\lambda=0$ *if and only if (char* $k=p$ *and)* λ *is p-singular.*

(ii) Let $K=k$. *Then* $\{D^\lambda:\ \lambda\vdash n,\ \lambda$ *p-regular*$\}$ *is a complete set of inequivalent irreducible kG-modules. Each* D^λ *is self-dual and absolutely irreducible.*■

2.2 *More notation*

Associated to the Young diagram is the concept of the hook. A *hook* is a partition of the form $(n-r,1^r)$, and has a Γ-shaped diagram



We are usually concerned with hooks lying as subsets of a given diagram. For example we give the hook

```
x x x x
x x-x-x
  |
x x x
```

with corner (2,2) inside $[(4^2,3)]$. The *length* of a hook is the number of nodes in it. The example just given has length 4. If we replace each node N in a diagram by the hook length of the hook with corner N then we obtain the *hook graph*. For the above this is

```
6 5 4 2
5 4 3 1
3 2 1
```

If (a,b) is a node of the diagram $[\lambda]$ we write h^{λ}_{ab} for the hook length.

A *skew* or *rim* hook is a connected part of the rim of $[\lambda]$ which one can remove to leave a proper diagram. We show below one of two possible skew 3-hooks

```
x x x x
x x x-x
    |
x x x
```

The *p-core* is obtained when we cannot remove any more skew p-hooks from the Young diagram. For example the 3-core of (7,5,4,3,2) is (4,2) as shown

```
x x x x x-x-x
x x x-x-x
x x-x-x
|
x x-x
| |
x x
```

The number of hooks removed is the *p-weight* (so here it is 5). By [22, 2.7.16] each diagram has a uniquely determined p-core.

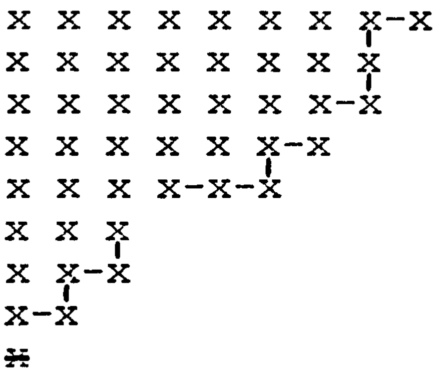
The importance of hooks arises in such theorems as the hook formula for dimensions:

$$\dim S^\lambda = \frac{n!}{\prod \{\text{hook lengths in } [\lambda]\}}$$

and the so-called Nakayama Conjecture

THEOREM 2.2.1 ([22, §6.1]). *S^λ and S^μ are in the same p-block of G if and only if they have the same p-core.*∎

We also need the slightly less well-known notion of the *p-edge*. This consists of p-segments all but at most one of which contains p points: the first p-segment comprises either the first p points of the rim of the diagram (starting with the longest row) or the entire rim if its length is less than p. The next p-segment is obtained similarly starting in the row next below that which contains the end of the last p-segment defined. Continue until the final row is reached. For example take p=5 and consider the partition $(9,8^2,7,6,3^2,2,1)\vdash 47$; take the 5-edge



to obtain $(7^2,6,5,3,2,1)$.

Remark. In most of these definitions 'p' need not be a prime, although in practice p=char k.

We shall make use of these definitions together with basic results like the Branching Theorem and Theorem 2.2.1 freely. In fact the Branching Theorem is used so often that we record it below for ease of reference.

DEFINITION 2.2.2. If $\lambda \vdash n$ we define

$$\lambda^{i-} = (\lambda_1,\ldots,\lambda_{i-1},\lambda_i-1,\lambda_{i+1},\ldots)$$
$$\lambda^{i+} = (\lambda_1,\ldots,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},\ldots).$$

THEOREM 2.2.3 (The Branching Theorem, [20, §9]). *If $\lambda \vdash n$ then we have for the restriction of S^λ to the stabiliser $H=\Sigma_{n-1}$ of the point n*

$$(S^\lambda)_{\downarrow H} = \sum_{\substack{i\\ \lambda_i>\lambda_{i+1}}} S^{\lambda^{i-}} .$$

If Σ_n denotes the stabiliser of the point $n+1$ in $G=\Sigma_{n+1}$ then we have for the induced representation

$$(S^\lambda)^{\uparrow G} = \sum_{\substack{i\\ \lambda_i<\lambda_{i-1}}} S^{\lambda^{i+}}$$

(*where* $\lambda_0:=\infty$).∎

2.3 The main problem: decomposition matrices

This section is nothing more than a brief survey of the problems involved in working out composition factors of certain modules for G. First suppose that given $\mu \vdash n$, we want to know about the composition factors of M^μ and S^μ. Over fields of characteristic zero, the situation is well-understood. The permutation module M^λ has a composition factor isomorphic to S^μ if and only if $\mu \gg \lambda$, and

$$M^\lambda \sim S^\lambda + \sum_{\mu\gg\lambda} b_\mu S^\mu$$

where the $b_\mu \in \mathbb{Z}_{\geq 0}$ are determined by Young's Rule. For this see [20, §14].

Now consider the same problem over $k(=\mathbb{Z}_p)$. We are mainly interested in the modular representations associated with the S^λ, and in particular their composition factors. Hence the actual modular representation we choose does not really matter: however it will simplify things greatly if we could start with a matrix representation of S^λ with entries in $\mathbb{Z}$, so that to produce the associated p-modular representation we need only reduce the coefficients modulo p. The reason this is possible is because of the following theorem.

THEOREM 2.3.1 ([22, 7.2.12]). *There is a basis L of S^λ with respect to which the entries in the matrix representing a permutation with respect to L, defined over $\mathbb{Q}$, are all integers. If the matrix entries are reduced modulo p, we obtain the matrix representing the same permutation with respect to L, defined over k.* ∎

An easy corollary of this is that S^λ defined over k is the p-modular representation $\overline{S^\lambda}$ of Σ_n obtained from S^λ defined over $\mathbb{Q}$. It is worth pointing out that this is certainly not the only way to construct a p-modular representation $\overline{S^\lambda}$ from S^λ since there are other matrix representations of Σ_n over $\mathbb{Z}$, $\mathbb{Q}$-equivalent to S^λ. The point is that while two such k-representations need not be k-equivalent, they will all have the same irreducible factors, which is after all what we are interested in. The matrix which records the multiplicities of the p-modular irreducible modules D^{λ} as factors of the reductions modulo p of the ordinary irreducible representations S^{μ}, is called the *decomposition matrix* for G. We denote this matrix by ${}^{n}D_{p}$. It can be shown ([22, 7.1.16, 7.2.13]) that ${}^{n}D_{p}$ has "wedge shape"

$$\begin{array}{cc} & D^{\mu}\ (\mu\ p\text{-regular}) \\ \begin{array}{c} S^{\mu} \\ (\mu\ p\text{-regular}) \end{array} & \left[\begin{array}{cccccc} 1 & & & & \bigcirc & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & * & & & \\ & & & & & \end{array}\right] \end{array}$$

provided that we place the p-regular partitions in dictionary ordering before all the p-singular partitions. In Chapters 3-5 we consider the problem of calculating ${}^{n}D_{p}$ where $p \leq n \leq 3p-1$ and p is odd for certain blocks of defect 2. However in general this calculation is still an open problem. For some "tricks of the trade" and some examples see [20, §24], [22, §6.3], [18], [19], [21] and [6]. See also Chapter 4, in which the method of Schaper is used, and Appendix B where a formal statement of his result (Theorem B.1.1) is given.

CHAPTER 3

Blocks of symmetric groups with cyclic defect groups

When we have a block of a group G of defect 0 then the structure is particularly easy to describe: the block is a simple algebra and the decomposition matrix is the 1×1 identity matrix, [1, IV]. In this chapter we concentrate on constructing the decomposition matrix of blocks of defect 1 of Σ_i where $p \leq i \leq 2p-1$ for all odd primes p. By the Brauer-Dade-Green-Peacock theory described in [1, V], this is equivalent to constructing the Brauer tree for each block. We do this in Section 3.1 by first constructing the tree for Σ_p and then using (mathematical) induction. In Section 3.2, using these trees, we are able to define and give examples of a certain type of directed graph called a quiver. For a cyclic block, the quiver has a particularly simple shape, in contrast to the more complicated beast constructed for other blocks in the next two chapters.

3.1 The Brauer tree

Recall (from [1],[14]) that if $\mathfrak{B}$ is a block of a finite group G, with cyclic defect group then $\mathfrak{B}$ is a *Brauer tree algebra*, in the sense that there is a Brauer tree which describes the structure of the indecomposable projective kG-modules. For the actual algorithm, see the cited references. In our case we shall take an arbitrary cyclic block (with defect group C_p, cyclic of order p) of $G=\Sigma_i$ where $p \le i \le 2p-1$ and construct the Brauer tree.

Recall that the "Nakayama Conjecture" (Theorem 2.2.1) gives a way of sorting characters into blocks. Hence we can state our main theorem:

THEOREM 3.1.1. *The Brauer tree for any block of defect 1 (that is with defect group C_p, p prime) for Σ_i, (where $p \le i \le 2p-1$) in characteristic p is obtained as follows:*

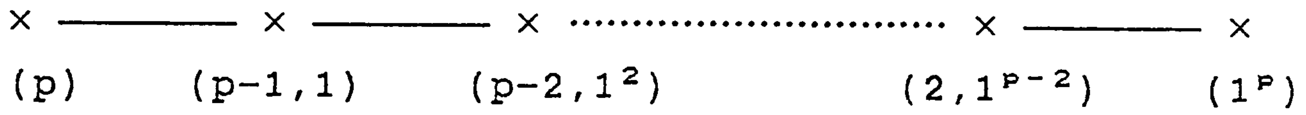
(i) Write down the p-core.

(ii) Write down all possible ways of adding a skew p-hook to the core in order to leave a proper diagram.

(iii)Each such diagram produces a character in the block (the one corresponding to the partition defined by the diagram). Place these in an open polygon by means of the dictionary ordering on partitions.

(iv) The polygon thus constructed is the Brauer tree of the block.

Remarks. (*i*) In particular, the Brauer tree of the principal block of Σ_p mod p is



.....(*)

where we write (λ) instead of the character χ^{λ}. This tree is certainly well-known [22 ,Ex.6.3]. One can think of the main theorem as being a generalisation to cyclic blocks with a non-empty p-core.

(*ii*) Of course this description of the Brauer tree will work for an arbitrary block of defect 1 with a modification of the given proof.

Proof of theorem. We use induction on i, the degree of Σ_i.

The base step is to prove that (*) is as claimed, (noting that it makes sense here only to consider the principal block). Let $H=\Sigma_{p-1}$ and $G=\Sigma_p$. Begin by inducing characters of H to G, using Theorem 2.2.3. The characters χ of H we consider are the ones corresponding to hook partitions. This is because we need a summand of χ^G lying in $\mathfrak{B}_0(G)$. Let 1_H be the trivial character of H. Then

$$(\chi^{(p-1)})^G = (1_H)^G = \chi^{(p)} + \chi^{(p-1,1)}$$

$$(\chi^{(p-2,1)})^G = \chi^{(p-1,1)} + \chi^{(p-2,2)} + \chi^{(p-2,1^2)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\chi^{(p-i,1^{i-1})})^G = \chi^{(p-i+1,1^{i-1})} + \chi^{(p-i,2,1^{i-2})} + \chi^{(p-i,1^i)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\chi^{(1^{p-1})})^G = \chi^{(1^p)} + \chi^{(2,1^{p-2})}$$

Observe now that the operation of reducing modulo p commutes with the operation of inducing. So reduce the characters listed above modulo p. However H is a p'-group so every kH-module M is projective, by Maschke's Theorem, and so $M^{\uparrow G}$ is projective, by Theorem 1.1.2(iii). So

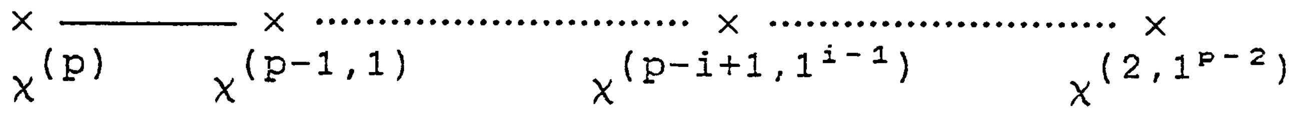
(1) all the characters listed above are projective characters of G (by Theorem 1.1.2(iii)).

Now $(1_H)^G = \chi^{(p)} + \chi^{(p-1,1)}$ lies in the principal block so must be a character of an indecomposable projective module. Hence $\chi^{(p)}$ and $\chi^{(p-1,1)}$ must be neighbours in the Brauer tree. Also, $\chi^{(p)}=1_G$ is of course irreducible, so lies at the end of the stem. We are now able to begin

Consider next, for $0<i<p$, the general character of the form

$$(\chi^{(p-i.1^{i-1})})^G = \chi^{(p-i+1,1^{i-1})} + \chi^{(p-i,2,1^{i-2})} + \chi^{(p-i,1^i)}$$

By (1), this is the character of a projective module. By Nakayama, the second term on the right is not in $\mathfrak{B}_0(G)$ hence it splits off. What remains is a character of an indecomposable projective module, so $\chi^{(p-i+1,1^{i-1})}$ and $\chi^{(p-i,1^i)}$ are neighbours. The tree is now



The final step is to consider $(\chi^{(1^{p-1})})^G = \chi^{(1^p)} + \chi^{(2,1^{p-2})}$. This lies in $\mathfrak{B}_0(G)$ so is the character of a projective indecomposable module. Once again $\chi^{(1^p)}$ and $\chi^{(2,1^{p-2})}$ are neighbours in the tree with the ordinary irreducible $\chi^{(1^p)}$ at the end.

The proof that (*) is the tree for Σ_p is now complete.

So assume the result is true for Σ_k where $k\geq p$. Consider a general p-core of a block of Σ_k. This has k-p nodes where $k-p<2p$, and we suppose it is of shape

$$m\left\{\begin{array}{l} x\ x\ .\ .\ .\ .\ .\ .\ .\ .\ x \\ x\ x\ \ .\ .\ .\ .\ x \\ \ \ \ \ .\ .\ .\ .\ .\ \qquad\quad . \\ x\ x.\ .\ x \end{array}\right.$$

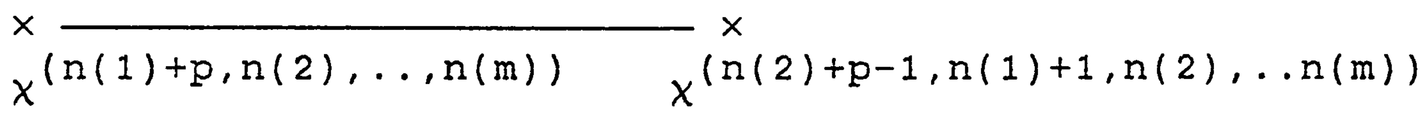
For each of the m rows suppose that row j has n(j) nodes. Clearly $\sum_{j=1}^{m} n(j) = k-p$. By induction we construct the Brauer tree of the block associated with this core. Our algorithm tells us that to construct the tree, we imagine the core being fixed on the page, and then we write down the possible ways a string of p beads can wrap itself around the rim of the core. Start with

$$\begin{array}{l} \leftarrow\ n(1) \rightarrow \leftarrow\ \ p\ \ \rightarrow \\ x\ x\ \ldots\ x\ x\ x\ \ldots\ x \\ \\ \quad\ \ .\ \ .\ \ .\ \ . \end{array}$$

giving the partition (n(1)+p,n(2),...,n(m)). Proceed next to

$$\begin{array}{l} \leftarrow\ n(1)\quad\ \rightarrow \\ x\ x\ .....\ x\ x\ x\ .....\ x \\ x\ x...x...x\ x \\ \\ \quad\ \ .\ \ .\ \ .\ \ . \end{array}$$

where the top row has n(1)+p-(n(1)-n(2)+1)=n(2)+p-1 nodes, and the second row has n(1)+1 nodes, giving the partition (n(2)+p-1,n(1)+1,n(2),...,n(m)). By induction the tree begins



We now need the diagram of the general partition λ, say. This is shown in Figure 1 below in which the added p-hook is shown using o o o notation.

```
x x   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .x
x x  .  .  .  .  .  .  .  .  .  .  .  .  .  .  x

       .  .  .  .  .  .  .  .  .  .  .  .  .

x x   .  .  .  .  .  .  .   x o.   .  .  .o         ←j'th
x x  .  .  .  .  .x o  .  . o o
                   o

.  .  .  .


          o
x x  . . . .  x o
x x...x o . . o o                                   ←k'th
    .  .  .  .

x...x                                               ←m'th
```

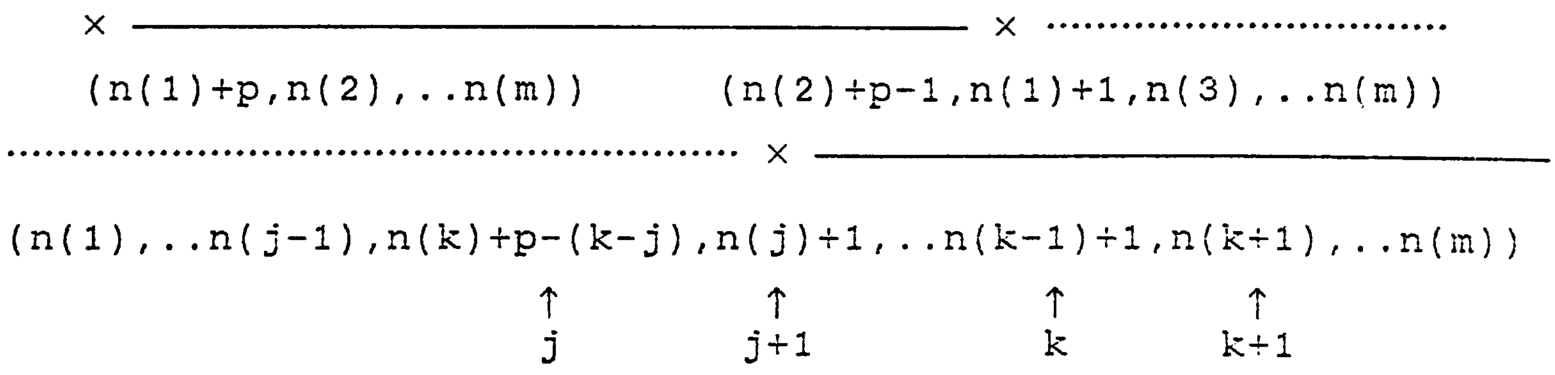
FIGURE 1: [λ]

Suppose that one adds ℓ nodes to the j'th row, where ℓ≤n(j-1)-n(j). Assume that the skew hook which we have added continues on to the k'th row. We find an expression for x=ℓ+n(j). We have

$$x=n(j)+p-(n(k-1)-n(k)+1+n(k-2)-n(k-1)+1+..+n(j)-n(j-1)+1)$$

$$=n(k)+p-(k-j)$$

Also, for the remaining rows, the r+1'th row will now contain n(r)+1 nodes (j≤r<k). We now draw the tree (with the χ signs omitted) in Figure 2.



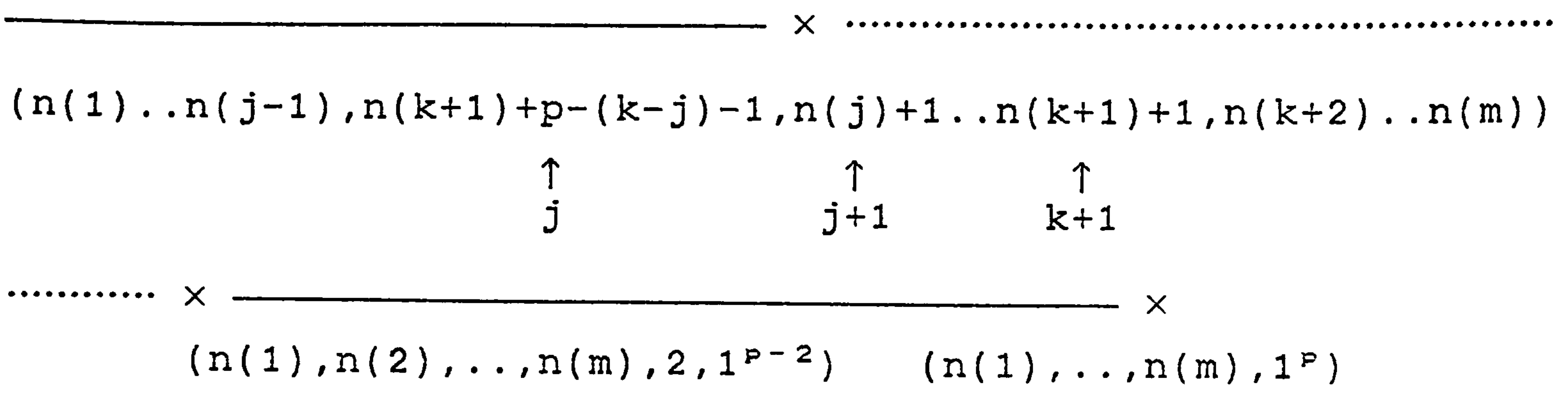


FIGURE 2: tree for Σ_k.

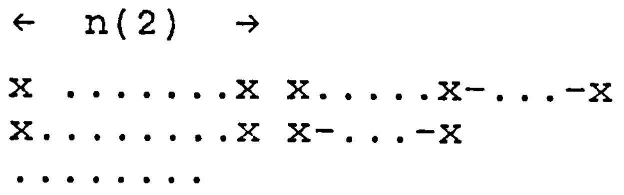
We now induce to Σ_{k+1} using the Branching Theorem. Denote the projective characters in the Brauer tree by $\chi_1,\ldots\chi_r$. Induce the general χ shown above: $\chi^G =$
$$\Big[\big(n(1)+1,..,n(m)\big)+..+\big(n(1),..,n(j-1)+1,..,n(m)\big)+\big(n(1),..,n(k)+p-(k-j)+1,..,n(m)\big)+\big(n(1),..,n(j)+2,..,m(m)\big)+..\big(n(1),..,n(m)+1\big)\Big]+\Big[\big(n(1)+1,..,n(m)\big)^*+..+\big(n(1),..,n(j-1)+1,..,n(m)\big)^*+\big(n(1),..,n(k+1)+p-(k-j)-1,..,n(m)\big)+\big(n(1),..,n(j)+2,..,n(m)\big)+..+\big(n(1),..,n(m)+1\big)^*\Big]$$
where the * indicates that the partition has different middle terms to the one appearing within the first set of squared brackets.

The next task is to separate these representations into blocks (which of course uses Theorem 2.2.1), and hence construct the tree. First we look at $\chi_1{}^G =$ $\chi^{(n(1)+p+1,n(2),..,n(m))+\cdots}$ $+\chi^{(n(2)+p,n(1)+1,..,n(m))+\cdots}$.

This first character has p-core

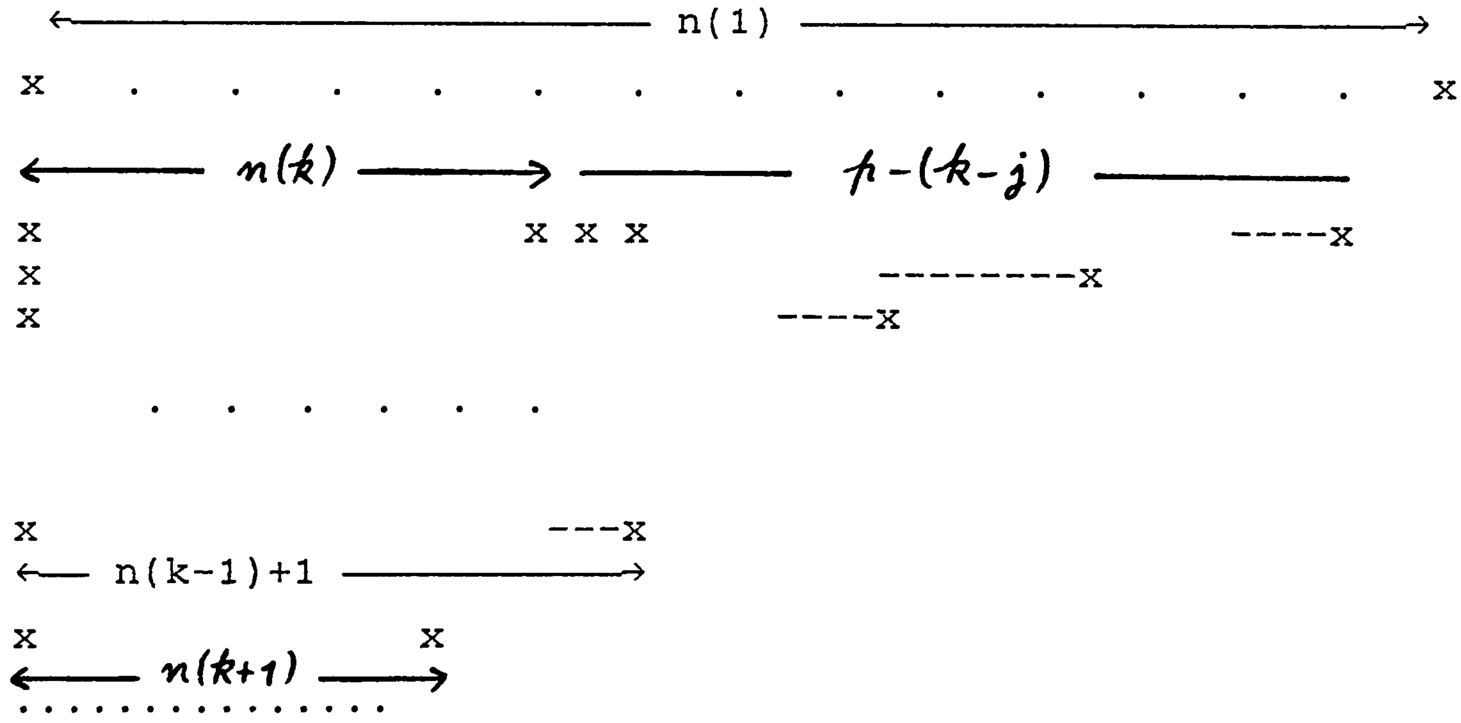
```
← n(1)+1  →
x ......x x x-x- ...-x
x ..x                         (**)
. . .
```

and we take this as the core defining the block we want. All other characters in this first batch have p-core not of the form (**). In the second batch of characters, the second character has core



which is precisely (**). All other characters are in a different block because they do not have p-core (**). So by standard arguments (see the base step) the tree begins

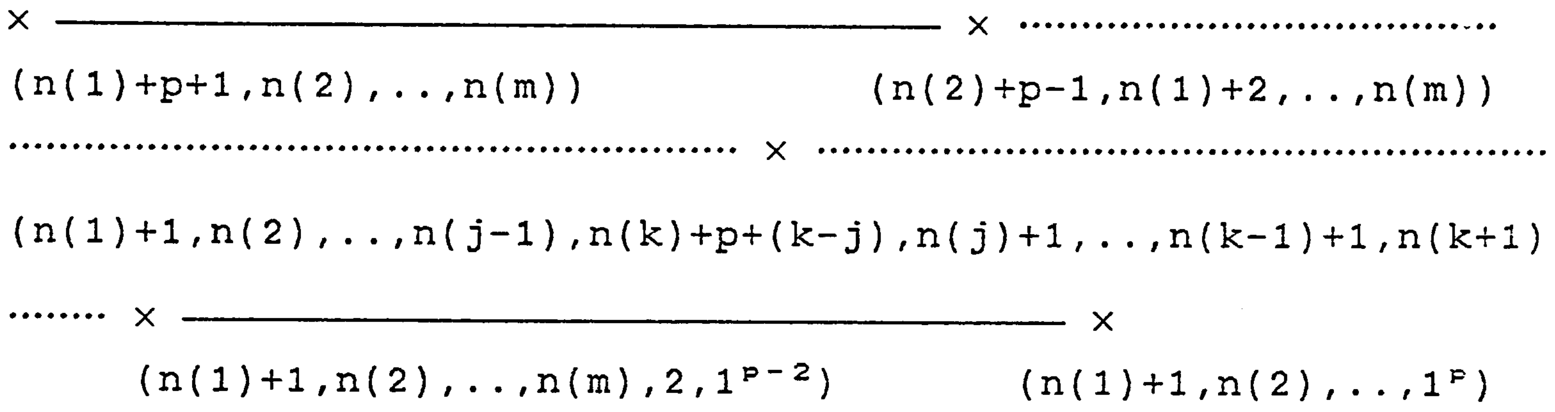
The next step is to deal with the general middle term which we have called χ^G. Recall that the core must have n(1)+1 nodes in the first row and hence in the first batch of characters the one with partition (n(1)+1,...) is the only candidate. Removing a p-hook, we obtain (**):



Observe that this is just Figure 1 except with an extra node in the first row. Removing the hook, we obtain (**).

For the second batch of characters, only the first one has the form [λ], this time with the rim hook pushed one line further down (and the extra node). Removal of the hook gives (**) again.

The two characters we obtained from this analysis will be neighbours in the Brauer tree for Σ_{k+1} which is now constructed as



Our theorem now follows by induction.∎

Now we have obtained the tree in Theorem 3.1.1. Suppose instead we try to obtain a description as to where an arbitrary character in the block lies without actually having to construct the whole tree.

COROLLARY 3.1.2. *Let* χ^{λ} *be an arbitrary character appearing in one of the trees for* Σ_i *as given in Theorem 3.1.1. Then the serial number of the character in the tree is equal to the leg length of the p-hook which we can remove from [λ].*

Proof. We define the leg length to be the number of nodes in the leg plus 1. Note that in this proof we freely use the notation of Theorem 3.1.1.

Once again we do an induction on i, the degree of Σ_i. When i=p, all partitions appearing in the tree are hooks of the form $(p-i,1^i)$ and by the base step of Theorem 3.1.1, this occurs in the i+1'th position. But the leg length of the hook is i+1 as required.

Assume the result for Σ_k, where $k \geq p$ and construct the tree for Σ_k. Take the arbitrary partition as in Figure 1 and recall that its core is obtained by omitting the circles. This skew hook corresponds to a hook Γ [20, 18.4] of leg length k-j+1. By induction, this is the serial number of the character χ.

Consider now Σ_{k+1}. The tree is shown above, as is the core of the arbitrary character (see (**) on page 30). By comparing with Σ_k, the leg length is k-j+2 which is equal to the serial number as required.∎

3.2 *The ordinary quiver and cyclic blocks*

This section is designed as a preliminary to Chapters 4 and 5 in which the so-called ordinary quiver of a block is discussed. Here we give the definition together with some examples of the quiver for the case of cyclic blocks of the symmetric group.

DEFINITION 3.2.1. Let k be algebraically closed. Let $\mathfrak{B}$ be a block of a group algebra kG. The *ordinary quiver* of $\mathfrak{B}$

is the (finite) directed graph whose vertices are the irreducible kG-modules (up to isomorphism) lying in $\mathfrak{B}$, and where the number of arrows $S \to T$ is $\dim_k \mathrm{Ext}^1_{kG}(S,T)$ where S and T are irreducible, (or, equivalently the number of times T occurs in $J(P(S))/J^2(P(S))$. Notice that by Proposition 1.2.5, the quiver is connected.

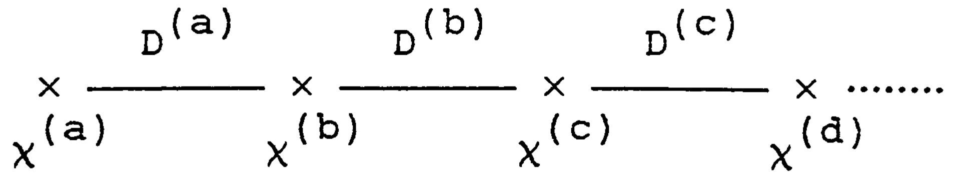
In some sense, one can think of the quiver, $Q(\mathfrak{B})$, of $\mathfrak{B}$ as measuring the complexity of $\mathfrak{B}$. For example $\mathfrak{B}$ is semisimple if and only if $\mathfrak{B}$ has no edges. We calculate some examples below; however for a much more detailed exposition see [30, p.96 *et seq.*].

Now recall from Section 2.1, the condition for two partitions λ, μ to be neighbours. Recall also the notation we introduced for this.

THEOREM 3.2.2. *Let $\mathfrak{B}$ be any cyclic block of $G=\Sigma_i$ where $p \le i \le 2p-1$. Let D^λ, D^μ be p-modular irreducibles lying in $\mathfrak{B}$. Then $\mathrm{Ext}^1_{kG}(D^\lambda, D^\mu) \neq 0$ if and only if $\mu \gg\!\bullet\, \lambda$ or $\lambda \gg\!\bullet\, \mu$.*

Remark. We comment on a slight abuse of notation here. By neighbours, we mean that there is no α such that $\lambda \gg \alpha \gg \mu$ and such that $D^{(\alpha)} \in \mathfrak{B}$. This convention excludes the obvious trivialities.

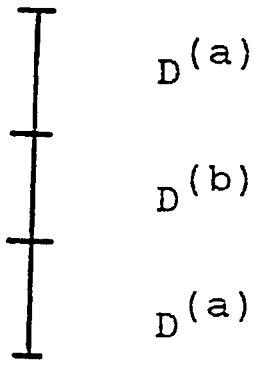
Proof of theorem. Construct the Brauer tree of $\mathfrak{B}$ as in Section 3.2. This may be represented as



Considering projective covers and using basic properties of

Brauer trees [1, §17] we obtain the Loewy series:

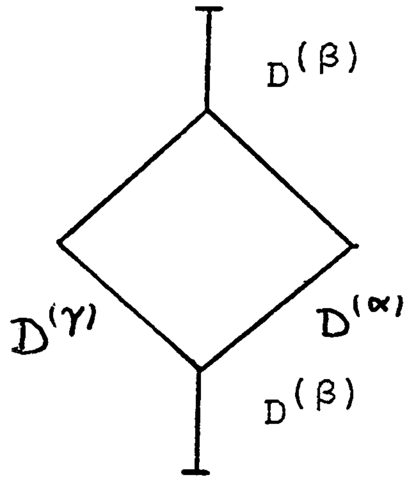
$P(D^{(a)}) =$



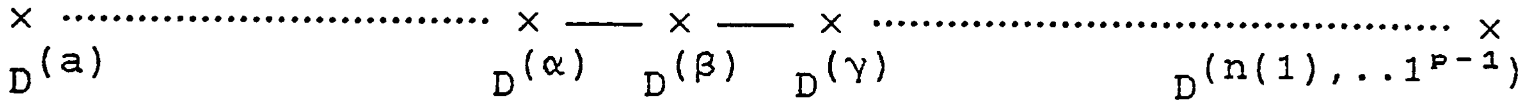
giving $\dim_k \mathrm{Ext}^1_{kG}(D^{(b)},D^{(a)})=1$.

Let $\chi=\chi^{\alpha}$ be as in Section 3.1 and let $\mu=\mu^{\beta}$ lie immediately to the right of it. Let $\nu=\nu^{\gamma}$ follow μ in the tree. We have a Loewy series

$P(D^{(\beta)}) =$



So $\dim_k(\mathrm{Ext}^1_{kG}(D^{(\beta)},D^{(\alpha)}))=1$ and $\dim_k(\mathrm{Ext}^1_{kG}(D^{(\gamma)},D^{(\beta)}))=1$. Hence obtain the ordinary quiver



Hence result.∎

Remark. Notice by Theorem 2.1.6(i) that the quiver has one less node than the tree.

CHAPTER 4

The ordinary quiver of the principal block of Σ_{2p}

4.1 Introduction

Let $\mathfrak{B}$ be a block of a finite group G. Then in Definition 3.2.1 we introduced the ordinary quiver of $\mathfrak{B}$. Theorem 3.2.2 then provided us with examples of such a quiver in the case where $G=\Sigma_i$ $p\le i\le 2p-1$, and $\mathfrak{B}$ was any cyclic p-block of G. In that case the quiver was just an open polygon, and was easily computed using the Brauer tree. Our aim in this chapter is to construct the quiver in the case where $G=\Sigma_{2p}$ and $\mathfrak{B}=\mathfrak{B}_0(G)$, the principal p-block of G. We also wish to compare our results with the corresponding cyclic theory.

The contents of this chapter are based on the work contained in [26] and we now summarise the main points. In Section 4.2 we state a few results about Brauer trees for Σ_{2p-1} which we need from Chapter 3. In Section 4.3 one can present the main theorem (Theorem 4.3.1) and exhibit the ordinary quiver $Q(\mathfrak{B})$ in Figure 3 (Appendix A). The proof is given in Sections 4.4 to 4.6. We take a modular irreducible D^λ for G (where $\lambda\vdash 2p$) and construct a certain filtration of the projective cover $P(D^\lambda)$ whose quotients are various compatible modules (see Definition 4.2.1). The structure of these compatibles, which are induced from irreducible kH-modules, is discussed in Section 4.4. This is done using decomposition numbers, calculated using a new result of K. Schaper [34], although of course one could have calculated these numbers *en passant* by Frobenius Reciprocity. For details of the method of Schaper, and for some examples of decomposition matrices of $\mathfrak{B}_0(G)$ for small primes, see Appendix B.

The compatibles can, in some sense, be fitted together, coming as they do from the Brauer trees for the blocks of defect 1 of H. This is described in detail in Section 4.5. Finally, in Section 4.6, we get a complete classification of those simple modules D^{μ} for which $\dim_k \mathrm{Ext}^1_{kG}(D^{\lambda},D^{\mu})=0$ or 1, and it is shown that there are no other possibilities simply by appealing to the "ordering" of compatibles in the filtrations.

The dimension of a given Ext^1 space is always 0 or 1 and the ordering of compatibles means that our quiver, which can be thought of as a pictorial representation of all the Ext^1 spaces, looks "locally" like a 2-dimensional lattice (away from the ends). We get some degeneration at the ends, but nonetheless the overall structure still produces for us a triangular-shaped lattice which is symmetric about the middle. Examples of this pleasing symmetry are given in Figures 4-6 (Appendix A).

4.2 Preliminary results

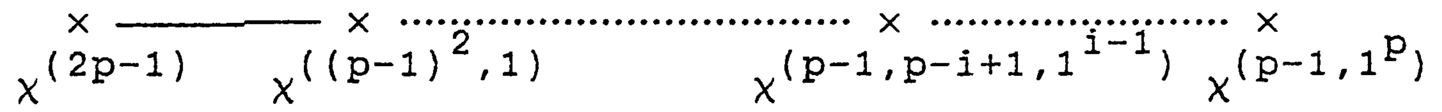
This short section is really designed to establish notation. Let k be a field of characteristic p, and let $H=\Sigma_{2p-1}$ and $G=\Sigma_{2p}$.

DEFINITION 4.2.1. Let M be a kG-module and suppose that M has a Loewy series such that $\ell(M)=3$. Suppose that M has simple head and simple socle isomorphic to A and semisimple heart $H(M)\cong B\oplus\cdots\cdots\oplus Z$. Then we shall say that M is *compatible* and we write

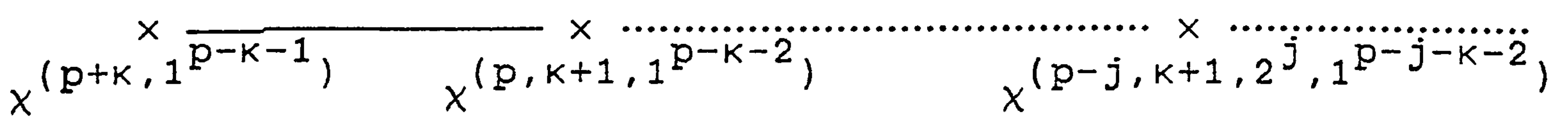
$$M = \mathcal{C}(A;\ B,\cdots\cdots,Z).$$

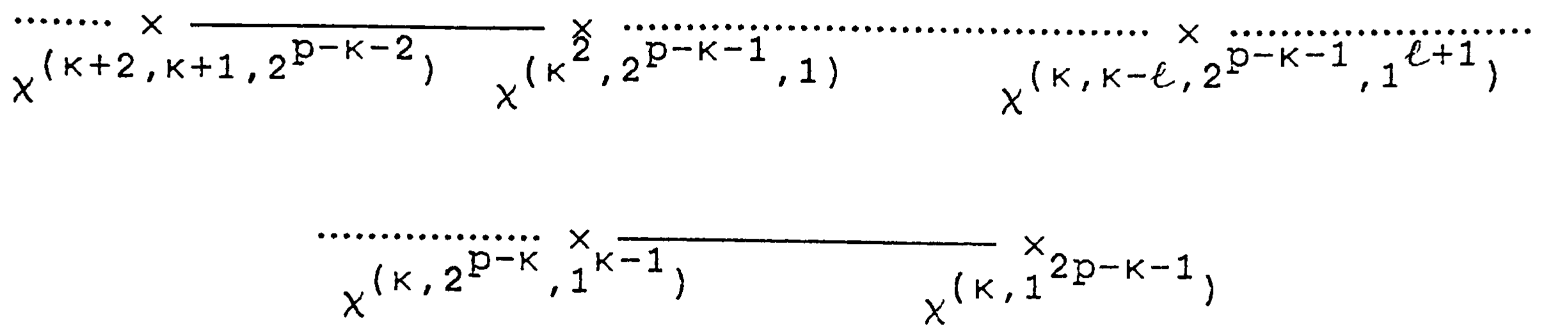
To prove the main result of this chapter, we induce simple kH-modules up to G and analyse their Loewy structure. Now $\mathfrak{B}_0(G)$ is the unique p-block of G of weight 2 and we want to ensure that the induced modules have a nonzero summand lying in this block. Clearly this happens if we consider simple kH-modules lying in p-blocks of H which have either p-core with partition $C_j=(j,1^{p-j-1})$ where $1\leq j\leq p-1$ or of shape $C_{2p-1}=(p,1^{p-1})$. It is possible, using Theorem 3.1.1 to give the Brauer trees of such blocks of H:

THEOREM 4.2.2. *Let $\mathfrak{B}_\kappa$ be the block of H with p-core C_κ, where $1\leq\kappa\leq p-1$. Then the Brauer tree of $\mathfrak{B}_{p-1}$ is*



and for $\mathfrak{B}_\kappa$ where $\kappa\leq p-2$ we have





for $0\le j\le p-\kappa-2$ *and* $0\le\ell\le\kappa-2$. ∎

Theorem 4.2.2 provides us with a knowledge of the decomposition numbers of blocks of H. At this point one could also have obtained the decomposition matrix of the principal block of G using a process called r-induction. However, we have preferred to calculate this using the method of Schaper (see [6] and [34]), and I am grateful to D. Benson for explaining this method to me. In general terms we obtain a matrix with $\frac{1}{2}(p-1)(p+1)$ columns (the number of p-modular irreducibles), and all but one column has four 1's; there is a unique column with three 1's. These facts seem to be part of the folklore.

4.3 Statement of the main theorem and some examples

Recall from Chapter 3 the definition of the (ordinary) quiver of a block. Let $\mathfrak{B}=\mathfrak{B}_0(G)$ be the principal p-block of G, and let $Q(\mathfrak{B})={}^{2p}Q_p$ be its quiver. Then we can now state our main result as

THEOREM 4.3.1. *If* $1 \leq \kappa \leq p-4$ *and* $0 \leq j \leq p-\kappa-4$ *we let*

$$u_\kappa = D^{(p+\kappa+1,1^{p-\kappa-1})},$$

$$v_{j,\kappa} = D^{(p-j,\kappa+2,2^j,1^{p-j-\kappa-2})},$$

$$w_\kappa = D^{((\kappa+3)^2,2^{p-\kappa-3})},$$

$$t_j = D^{(p-j,2^{j+1},1^{p-j-2})}.$$

Then in Figure 3 (Appendix A) we have

$$\dim_k \mathrm{Ext}^1_{kG}(v_{0,\kappa},u_\kappa) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(u_{\kappa+1},u_\kappa) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(v_{j,\kappa+1},v_{j,\kappa}) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(v_{j+1,\kappa},v_{j,\kappa}) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(w_\kappa,v_{p-\kappa-2,\kappa-1}) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(w_\kappa,v_{p-\kappa-3,\kappa}) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(w_\kappa,v_{p-\kappa-4,\kappa+1}) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(v_{j,1},t_j) = 1$$

$$\dim_k \mathrm{Ext}^1_{kG}(t_{j+1},t_j) = 1.$$

The Ext^1 *spaces obtained by interchanging the modules also have dimension 1. All other* Ext^1 *spaces are zero.*∎

Remarks. (*i*) In the figure, we write λ instead of D^λ.

(*ii*) Note the general shape of ${}^{2p}Q_p$: there are p-1 nodes down the left-hand and right-hand edges, and p-1 nodes along the top. The left-hand edge has the hook partitions lying on a line in their lexicographic ordering. However observe using Theorem 4.2.2 that the quiver for the principal block of Σ_i ($p \leq i \leq 2p-1$) consists of certain hooks lying on a line in this ordering. So in some sense Q($\mathfrak{B}$) is a "natural" generalisation of all these quivers of subgroups.

(*iii*) For analogues of the theorem for symmetric groups of higher degree, see Chapter 5 (in particular Theorem 5.4.3 and accompanying remarks).

EXAMPLES. In Appendix A we give three examples of ordinary quivers in Figures 4,5 and 6 for the primes 5,7 and 11.

4.4 Some Loewy series

We begin by constructing series for simple modules induced from p-blocks of H. However details of the proofs are given only for modules induced from the principal block of H. The method for other blocks is identical.

Concerning the reduction modulo p of a module, we refer the reader to Theorem 2.3.1. We often denote this using the "bar" notation, although in places where it is obvious what is intended the bar will be omitted. Recall also the definition and notation for uniserial and compatible modules.

LEMMA 4.4.1. *Let* $\mathfrak{B}_{p-1}$ *have p-core* C_{p-1}. *Let it have Brauer tree as shown in Theorem 4.2.2. Let the corresponding simple edges of the tree be* $S, T, \ldots, X, \ldots$. *Then the Loewy series of* $S^{\uparrow G}$, $T^{\uparrow G}, \ldots, X^{\uparrow G}, \ldots$ *are as follows:*

(i) $S^{\uparrow G} = \mathcal{U}(k_G, D^{(2p-1,1)}, k_G)$

(ii) $T^{\uparrow G} = \mathcal{C}(D^{(p,p-1,1)};\ D^{(2p-2,1^2)}, D^{(p^2)}, D^{((p-1)^2,2)})$

(iii) $X^{\uparrow G} = \mathcal{C}(D^{(p,p-i+1,1^{i-1})};\ D^{(2p-i,1^i)},$

$$D^{(p-1,p-i+1,2,1^{i-2})} \text{ where } i\geq 3.$$

Proof. Here we use decomposition numbers and the fact that the D^{λ} are self-dual (Theorem 2.1.6).

(i) By Theorem 2.2.3

$$(\chi^{(2p-1)})^{G} = \chi^{(2p)} + \chi^{(2p-1,1)}$$

and by Schaper the reduction of $(S^{(2p-1)})^{\uparrow G}$ has composition factors k, k and $D^{(2p-1,1)}$. By self-duality, the module must have the stated structure.

(ii) Now we seek the factors of $T^{\uparrow G}$. We know that as kH-modules, $P(S)=\mathfrak{U}(S,T,S)$ where $S=D^{(2p-1)}$ and $T=D^{((p-1)^2,1)}$ so that there is a s.e.s.

$$0 \to S \to \Omega S \to T \to 0.$$

Hence we obtain

$$0 \to S^{\uparrow G} \to (\Omega S)^{\uparrow G} \to T^{\uparrow G} \to 0 \qquad \ldots(1)$$

$S^{\uparrow G}$ is known from above. So we need to consider

$$\overline{(S^{(p-1)^2,1)})^{\uparrow G}}.$$

Inducing to the principal block

$$(\chi^{((p-1)^2,1)})^{G} = \chi^{(p,p-1,1)} + \chi^{((p-1)^2,2)}.$$

These have factors $D^{(2p)}$, $D^{(2p-2,1^2)}$, $D^{(p,p-1,1)}$, $D^{(p^2)}$ and $D^{(2p-1,1)}$, and $D^{(2p)}$, $D^{(p,p-1,1)}$ and $D^{((p-1)^2,2)}$. So using (1) we now have that

$$T^{\uparrow G} \sim D^{(2p-2,1^2)} + D^{(p^2)} + 2D^{(p,p-1,1)} + D^{((p-1)^2,2)}. \qquad \ldots(2)$$

(Notice that we list here only the factors in the principal block).

(iii) Now concentrate on factors for the general induced module. We claim that

$$X^{\uparrow G} \sim D^{(2p-i,1^i)} + 2D^{(p,p-i+1,1^{i-1})} + D^{(p-1,p-i+1,2,1^{i-2})} \quad \text{for } i\geq 3. \qquad \ldots(3)$$

For i=3 we know that $P(D^{((p-1)^2,1)}) = \mathcal{C}(T;\ S,U)$ where $U = D^{(p-1,p-2,1^2)}$ is the third irreducible module in the tree. As before we have an induced sequence

$$0 \to T^{\uparrow G} \to (\Omega T)^{\uparrow G} \to S^{\uparrow G} \oplus U^{\uparrow G} \to 0$$

and we already know the factors of $S^{\uparrow G}$ and $T^{\uparrow G}$ hence it is elementary to show that $U^{\uparrow G} \sim D^{(2p-3,1^3)} + 2D^{(p,p-2,1^2)} + D^{(p-1,p-2,2,1)}$.

Now suppose i>3. Then $P(D^{(p-1,p-i+2,1^{i-2})}) = \mathcal{C}(W;\ V,X)$ and so the induced sequence is

$$0 \to W^{\uparrow G} \to (\Omega W)^{\uparrow G} \to V^{\uparrow G} \oplus X^{\uparrow G} \to 0.$$

By induction $W^{\uparrow G} \sim D^{(2p-i+1,1^{i-1})} + 2D^{(p,p-i+2,1^{i-2})} + D^{(p-1,p-i+2,2,1^{i-3})}$. Using Schaper, the factors of $(\Omega W)^{\uparrow G}$ are obtained by looking at reductions of $S^{(p,p-i+1,1^{i-1})} + S^{(p-1,p-i+1,2,1^{i-2})}$. These are $D^{(2p-i,1^i)} + D^{(2p-i+1,1^{i-1})} + 2D^{(p,p-i+1,1^{i-1})} + 2D^{(p,p-i+2,1^{i-2})} + D^{(p-1,p-i+1,2,1^{i-2})} + D^{(p-1,p-i+2,2,1^{i-3})}$. (3) is now an elementary consequence.

Now we use (2) and (3) to complete the proof of the lemma. We do this in

LEMMA 4.4.2. *Let X be the general module appearing in Lemma 4.4.1, and let $M=X^{\uparrow G}$. Then*

(i) M is indecomposable.

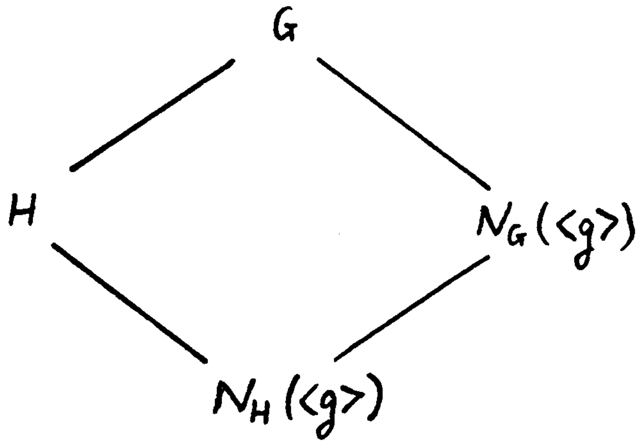
(ii) M has isomorphic (irreducible) head and socle, namely the unique composition factor, $\zeta(M)$, of multiplicity two.

(iii) M is compatible, of Loewy length 3, and $M=\mathcal{C}(\zeta(M);$ all remaining factors).

(iv) M is uniserial if and only if $X=S$, where S is as in Lemma 4.4.1.

Proof. When X=S, see Lemma 4.4.1.

Let $X\neq S$. Recall the Green Correspondence as detailed in Theorem 1.2.3. Let f_H denote the Green Correspondent inside H with respect to the vertex <g> of X, where g has order p.



Recall (from [9]) the definition of the outer tensor product, which we denote by #. Let $Q=\Sigma_p$, and let $N_0=N_Q(\langle g\rangle)$. Consider the subgroup $P=\Sigma_{p-1}$ of Q. The result we want to prove is tackled in several steps:

(4) Let $Z=f_H X$. Then $N=Z^{\uparrow N_0\times Q}\cong Z\ \#\ k_{\downarrow P}{}^{\uparrow Q}$ (identifying two different Σ_p's)

To see this note that $Z=Z_{\downarrow N_0}\ \#\ k$ since Z lies in

$\mathfrak{B}_0(N_0 \times Q)$. Hence by [9, 43.2] $N \cong (Z_{\downarrow N_0})^{\uparrow N_0} \# k_{\downarrow P}{}^{\uparrow Q}$. Also:

(5) $k_{\downarrow P}{}^{\uparrow Q}$ is indecomposable.

Note that because P is a p'-group, $k_{\downarrow P}$ is projective (with character $\chi^{(p-1)}$). Hence $k_{\downarrow P}{}^{\uparrow Q}$ is projective (with character $\chi^{(p)}+\chi^{(p-1,1)} = 1+\chi^{(p-1,1)}$), and so it is isomorphic to the projective cover of k_Q. As a result $k_{\downarrow P}{}^{\uparrow Q}$ is indecomposable.

(6) M is indecomposable.

Since Z is indecomposable and by (5), $k_{\downarrow P}{}^{\uparrow Q}$ is too, it follows by (4), that M is indecomposable. To obtain the factors, one can merely apply (2) and (3) directly.

Finally note that as M is self-dual, it has isomorphic head and socle which therefore has to be $\zeta(M)$. So the series either is uniserial, or has Loewy length 3. As all composition factors are self-dual (Theorem 2.1.6) and all but one have multiplicity one, M is compatible as required.∎

As before we let $\mathfrak{B}_\kappa$ be the p-block of H with p-core C_κ, where, by Lemma 4.4.1, we may assume that $\kappa \leq p-2$. Let i be the serial number of the module $M_i = S_i{}^{\uparrow G}$ appearing in the i'th position in the tree of Theorem 4.2.2.

THEOREM 4.4.3. *With the above notation, M_i has the following Loewy series:*

(i) $M_1 = \mathcal{C}(D^{(p+\kappa+1,1^{p-\kappa-1})};\ D^{(p+\kappa,1^{p-\kappa})},$
$D^{(p+\kappa+2,1^{p-\kappa-2})})$.

(*ii*) *For* $0 \le j \le p-\kappa-4$,

$$M_{j+2} = \mathcal{C}(D^{(p-j,\kappa+2,2^{j},1^{p-j-\kappa-2})}; D^{(p-j,\kappa+1,2^{j},1^{p-j-\kappa-1})}, D^{(p-j,\kappa+3,2^{j},1^{p-j-\kappa-3})}).$$

(*iii*) *For* $\kappa \le p-5$,

$$M_{p-\kappa-1} = \mathcal{C}(D^{(\kappa+3,\kappa+2,2^{p-\kappa-3},1)}; X, D^{(\kappa+3,\kappa+1,2^{p-\kappa-3},1^{2})}, D^{((\kappa+3)^{2},2^{p-\kappa-3})}), \text{ where}$$

$$X = \begin{cases} D^{((\kappa+4)^{2},2^{p-\kappa-4})}, & \kappa \le p-5 \\ D^{(p^{2})}, & \kappa = p-3 \\ k_G, & \kappa = p-2 \end{cases}.$$

(*iv*) $$M_{p-\kappa} = \mathcal{U}(D^{((\kappa+2)^{2},2^{p-\kappa-2})}, D^{(\kappa+2,\kappa+1,2^{p-\kappa-2},1)}, D^{((\kappa+2)^{2},2^{p-\kappa-2})}).$$

(*v*) *For* $\kappa > 2$,

$$M_{p-\kappa+1} = \mathcal{C}(D^{(\kappa+1,\kappa,2^{p-\kappa-1},1)}; X, D^{(\kappa+2,\kappa,2^{p-\kappa-2},1^{2})}, D^{((\kappa+1)^{2},2^{p-\kappa-1})}), \text{ where}$$

$$\begin{cases} D^{(\kappa^{2},2^{p-\kappa})}, & \kappa > 2 \\ 0, & \kappa = 1,2 \end{cases}.$$

(*vi*) *For* $1 < \ell < \kappa-2$, $\kappa > 2$,

$$M_{p-\kappa+\ell+1} = \mathcal{C}(D^{(\kappa+1,\kappa-\ell,2^{p-\kappa-1},1^{\ell+1})}; D^{(\kappa,\kappa-\ell,2^{p-\kappa},1^{\ell})}, D^{(\kappa+2,\kappa-\ell,2^{p-\kappa-2},1^{\ell+2})}).$$

Proof. As in Lemmas 4.4.1 and 4.4.2.∎

Finally, we also have

THEOREM 4.4.4. *There is a module*

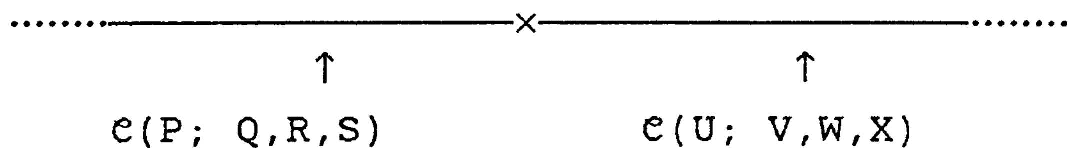
$$\mathcal{E}(D^{(p+1,1^{p-1})}; D^{(p+2,1^{p-2})}, D^{(p,2,1^{p-2})}).$$

Proof. Inducing the defect 0 representation $D^{(p,1^{p-1})}$ of H to G, we know that we must obtain a projective module having *a filtration with quotients* $S^{(p,1^p)}$, $S^{(p+1,1^{p-1})}$ and $S^{(p,2,1^{p-2})}$. The *modular composition factors* are easily computed using Schaper's result. *Using the extensions provided by Theorem 4.4.3, one can now pull out the required compatible module as the top quotient in this filtration.* ■

4.5 *A useful lemma*

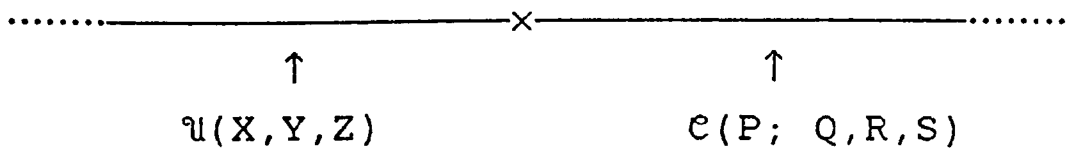
We shall see later that the results of Section 4.4 give us all the necessary information on the extension groups. The following result however makes the <u>construction</u> of the quiver much easier.

LEMMA 4.5.1. *(i) Consider a segment of the tree in Theorem 4.2.2 together with the series of the induced modules*



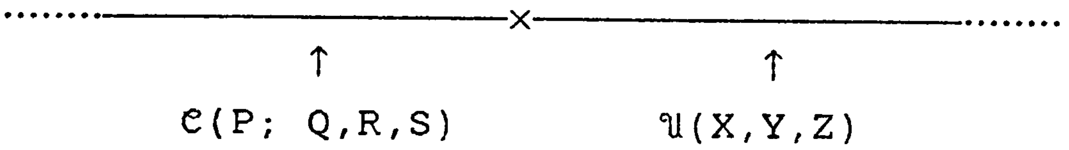
where at least one of the Q,V is zero. Then $\mathrm{Ext}^1_{kG}(U,P)\neq 0$.

(ii) Suppose that the induced modules have shape



where $Q=0$ *(say) if* $\kappa=2$. *Then* $\mathrm{Ext}^1_{kG}(P,X)\neq 0$.

(iii) Finally we can have, for $\kappa\leq p-2$



where $Q=0$ *(say) if* $\kappa=2$. *Then* $\mathrm{Ext}^1_{kG}(X,P)\neq 0$.

Remarks. (*i*) In part (ii) above we require $\kappa\geq 2$.

(*ii*) As yet we make no assertions concerning dimensions of the Ext^1 spaces. For this, see Section 4.6.

Proof of lemma. Clearly (iii) is dual to (ii), and in (i) we need only check for V=0 say. As in Section 4.4 we do the verification only for the principal block.

At this juncture, the reader is referred to Theorem 4.2.2 and Lemma 4.4.1. We write ϕ_λ for the Brauer character corresponding to λ.

(1) Begin by showing that $\mathrm{Ext}^1_{kG}(D^{(p,p-1,1)},k)\neq 0$.

To see this, note that the Schaper formula applied to $S^{(p,p-1,1)}$ gives us the character sum $2\phi_{(2p-1,1)} + \phi_{(2p)} + \phi_{(p^2)} + \phi_{(2p-2,1^2)}$. Now just observe that anything in the first Schaper layer extends the top layer. This simple argument is powerful and will be used again below.

(2) $\mathrm{Ext}^1_{kG}(D^{(p,p-2,1^2)},D^{(p,p-1,1)})\neq 0$.

The character sum is $\chi^{(2p-3,1^3)}-\chi^{(2p)}-\chi^{(p^2)}-\chi^{(p,p-1,1)} =$

$2\Phi_{(2p-2,1^2)} + \Phi_{(2p-3,1^3)} + \Phi_{(p,p-1,1)}$. So again $D^{(p,p-1,1)}$ is in the first layer.

(3) $\mathrm{Ext}^1_{kG}(D^{(p,p-i+1,1^{i-1})},D^{(p,p-i+2,1^{i-2})})\neq 0$ for $i\geq 4$.

The character sum is now $2\Phi_{(2p-i+1,1^{i-1})} + \Phi_{(2p-i,1^i)} + \Phi_{(p,p-i+2,1^{i-2})}$ and hence the result follows again.∎

4.6 *Conclusion of proof*

Our final task is to show that the dimension of each of the Ext^1 spaces studied above can only be 1, and that no other extensions exist.

PROPOSITION 4.6.1. *(i) Let L be any of the modules with Loewy series as given in Lemma 4.4.1 and Theorems 4.4.3 and 4.4.4. Then there is only one equivalence class of extension of the head of L by a simple module in the middle layer.*

(ii) Suppose we have two consecutive series, as in Lemma 4.5.1, with heads V and W respectively. Then

$$\dim_k \mathrm{Ext}^1_{kG}(W,V) = 1$$

(iii) There are no equivalence classes of extensions other than those calculated in (i) and (ii) above.

Proof. All three parts follow easily if we give the proof for modules induced from the principal block as the method for other modules is identical.

(1) $\dim_k \mathrm{Ext}^1_{kG}(D^{(p,p-1,1)},k) = 1$.

The two Loewy series (i) and (ii) of Lemma 4.4.1 define a filtration of P(k) with quotients

$$X = \mathfrak{U}(k,\ D^{(2p-1,1)},\ k),$$

$$Y = \mathcal{C}(D^{(p,p-1,1)};\ D^{(2p-2,1^2)},\ D^{(p^2)},\ D^{((p-1)^2,2)}),$$

$$X.$$

So the only possible extensions are those of type (i) or type (ii) above. Conversely these *do* exist by Lemmas 4.4.1 and 4.5.1.

(2) $\dim_k \mathrm{Ext}^1_{kG}((D^{(p,p-2,1^2)},\ D^{(p,p-1,1)}) = 1$.

The relevant filtration has quotients

$$Y,$$

$$\mathcal{C}(D^{(p,p-2,1^2)};\ D^{(2p-3,1^3)},\ D^{(p-1,p-2,2,1)}),$$

$$Y.$$

Hence the result follows as in (1).

(3) $\dim_k \mathrm{Ext}^1_{kG}(D^{(p,p-i,1^i)},\ D^{p,p-i+1,1^{i-1}})$ where $i \geq 4$.

It is clear that the method used above also works here.■

CHAPTER 5

Symmetric groups of higher degree, and alternating groups

5.1 Introduction

This section deals with the construction of the ordinary quiver of $\mathfrak{B}_0(\Sigma_i)$ where $2p \leq i \leq 3p-1$ modulo p. The point here is that we can use the results of Chapter 4 (and in particular Theorem 4.3.1) as a base step for an induction argument, to obtain analogous results for principal blocks of symmetric groups of higher degree. The main result is Theorem 5.2.1. By tensoring with the alternating character, we can obtain results about the conjugate block to the principal block: these results are contained in Theorem 5.2.2.

Having all these ordinary quivers at our disposal, we can, for the cases we have looked at, verify a conjecture of G. Mullineux [27] concerning a bijection of p-regular diagrams. For details, see Section 5.3. Admittedly this is hardly overwhelming evidence for the proof of the conjecture in general, but at least we have a proof which deals with an infinite number of cases.

In the last section of this chapter we investigate how far our results carry over to the alternating group, A_i $(2p \leq i \leq 3p-1)$. Use is made of results in this and the previous chapter to construct quivers for blocks of A_i, and we shall see that in all but two cases, the general shape is the same as that for Σ_i (see Theorem 5.4.4 and accompanying remarks).

5.2 The main theorem

Let $\pi=2p$. Then we are considering the group $G_\gamma=\Sigma_{\pi+\gamma}$ where $1\leq\gamma\leq p-1$. Let $\mathfrak{B}=\mathfrak{B}_0(G_\gamma)$ and let $Q=Q(\mathfrak{B})$ be its quiver. The figures referred to below are to be found in Appendix C.

THEOREM 5.2.1. *If* $1\leq\gamma\leq p-3$ *then* Q *is shown in Figure* 9. *If* $\gamma=p-2,p-1$ *then* Q *is shown in Figure* 10.

THEOREM 5.2.2. *Let* $\mathfrak{B}'$ *be the block of* G_γ *with p-core* $\left.\begin{matrix}\times\\ \times\\ \cdot\\ \dot{\times}\end{matrix}\right\}\gamma$. *Let* $Q'=Q(\mathfrak{B}')$. *Then if* $1\leq\gamma\leq p-3$, Q' *is shown in Figure* 11 *while if* $\gamma=p-2,p-1$ *then* Q' *is shown in Figure* 12.

Remarks. (*i*) As before in the figure we write λ instead of D^λ.

(*ii*) Compare with Theorem 4.3.1. We have the same general shape as in Figure 3 but notice that for $\Sigma_{\pi+\gamma}$ we obtain γ new extensions. For the principal block these appear as vertical bars up the left-hand side, and for the conjugate block they appear up the right-hand side.

(*iii*)The shape of the quiver in Theorem 5.2.2 may be easily seen if we "tensor" the quiver ${}^{\gamma}Q_p$ with the alternating character. To obtain the labelling however we must mimic the proof of Theorem 5.2.1, and the details are omitted. The conjecture of Mullineux is essentially a conjecture about the form of this labelling - see Section 5.3.

(*iv*) We give examples for p=5: these are shown in Figures 13 and 14.

Proof of Theorem 5.2.1. Proceed by induction on γ. Observe first that the case $\gamma=0$ is dealt with in Chapter 4. When $\gamma=p-2$ or $p-1$ a direct argument can be used, so we may assume that $0<\gamma<p-2$.

By induction the quiver may be represented as in Figure 9. The method will be to take the nodes of this quiver and induce to $\Sigma_{\pi+\gamma+1}$ and then obtain the Loewy series of the induced module. We do this in exactly the same way as the proof of Theorem 4.3.1 and we list the modules in the principal block in the table below.

D^{λ}	$(D^{\lambda})\uparrow^{\Sigma_{\pi+\gamma+1}}$
$k_{\Sigma_{\pi+\gamma}}$	$k_{\Sigma_{\pi+\gamma+1}}$
$(p+\kappa+1,\gamma+1,1^{p-\kappa-2})$	$(p+\kappa+1,\gamma+2,1^{p-\kappa-2})$ for $\kappa\geq\gamma+1$.

$(p+\gamma+1,\gamma+1,1^{p-\gamma-2})$

$$\begin{tikzcd}
& (p+\gamma+1,\gamma+2,1^{p-\gamma-2}) \arrow[dl, dash] \arrow[dr, dash] & \\
(p+\gamma+1,\gamma+3,1^{p-\gamma-3}) \arrow[dr, dash] & & {[p+\gamma+1,\gamma+1,1^{p-\gamma-1}]} \arrow[dl, dash] \\
& (p+\gamma+1,\gamma+2,1^{p-\gamma-2}) &
\end{tikzcd}$$

(Only when $\gamma=1$ is the module [.] missing; when $\gamma=p-3$, $k_{\Sigma_{\pi+\gamma+1}}$ is added to the second layer.)

$(p+\gamma,\kappa+2,1^{p-\kappa-2})$	$(p+\gamma+1,\kappa+2,1^{p-\kappa-2})$ for $0\le\kappa\le\gamma-1$
$(p+\gamma,p)$	$(p+\gamma+1,p)$
$(p+\gamma,\kappa+2,1^{p-\kappa-2})$	$(p+\gamma+1,\kappa+2,1^{p-\kappa-2})$ for $\kappa\ge\gamma+1$
$(p+\gamma,\gamma+2,1^{p-\gamma-2})$	$(p+\gamma,\gamma+2,1^{p-\gamma-1})$
$(p+\kappa,\gamma+1,1^{p-\kappa-1})$	$(p+\kappa,\gamma+2,1^{p-\kappa-1})$ for $\kappa\le\gamma-1$
$((p-j)^2,\gamma+2,2^{j-1})$	$((p-j)^2,\gamma+3,2^{j-1})$
$(p-j,\kappa+2,\gamma+2,2^{j-1},1^{p-j-\kappa-2})$	$(p-j,\kappa+2,\gamma+3,2^{j-1},1^{p-j-\kappa-2})$ for $\kappa>\gamma$

$(p-j,(\gamma+2)^2,2^{j-1},1^{p-j-\gamma-2})$	$(p-j,(\gamma+2)^2,2^{j-1},1^{p-j-\gamma-1})$
$(p-j,\gamma+1,\kappa+2,2^{j-1},1^{p-j-\kappa-1})$	$(p-j,\gamma+2,\kappa+2,2^{j-1},1^{p-j-\kappa-1})$
$(\gamma,(\kappa+3)^2,2^{p-\kappa-3})$	$(\gamma+1,(\kappa+3)^2,2^{p-\kappa-3})$
$(\gamma,\kappa+3,2^{p-\kappa-2},1^{\kappa+1})$	$(\gamma+1,\kappa+3,2^{p-\kappa-2},1^{\kappa+1})$

We complete the proof by considering the following result.

LEMMA 5.2.3. *In the situation above, let*

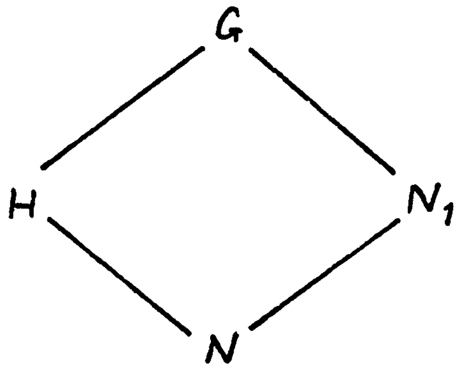
$$0 \to D^\lambda \to \overline{S^\mu} \to D^\mu \to 0$$

be a nonsplit extension of H-modules where $H=\Sigma_{\pi+\gamma}$. *Then in the s.e.s.*

$$0 \to (D^\lambda)\uparrow^G \to (\overline{S^\mu})\uparrow^G \to (D^\mu)\uparrow^G \to 0$$

where $G=\Sigma_{\pi+\gamma+1}$, *the middle term is indecomposable.*

Proof. Once again the method will be to use a form of Green Correspondence.



Let $D=C_p\times C_p$ and $\Omega=\Sigma_\pi$. Denote the outer tensor product of two modules by # as in the proof of Lemma 4.4.2. Let $N_0=N_\Omega(D)$,

$N=N_H(D)$ and $N_1=N_G(D)$ $(=N_0\times\Sigma_{\gamma+1})$. Let f_H denote the Green Correspondence inside H with respect to the vertex D of $M=\overline{S^{\mu}}$. Then by assumption M is indecomposable so has a Green Correspondent f_HM, lying in $\mathfrak{B}_0(N)$. Then $Z=f_HM=Z_{\downarrow N_0}\#k_{\Sigma_\gamma}$. Now

$$\begin{aligned} Z^{\uparrow N_1} &= Z^{\uparrow N_0\times\Sigma_{\gamma+1}} \\ &= (Z_{\downarrow N_0})^{\uparrow N_0} \# (k_{\Sigma_\gamma})^{\uparrow\Sigma_{\gamma+1}} \\ &= Z \# (k_{\Sigma_\gamma})^{\uparrow\Sigma_{\gamma+1}} \end{aligned}$$

and $(k_{\Sigma_\gamma})^{\uparrow\Sigma_{\gamma+1}}$ has only one summand in the principal block and so $Z^{\uparrow N_1}$ has only one summand in the principal block. The result about $M^{\uparrow G}$ is immediate because summands of $M^{\uparrow G}$ lying in $\mathfrak{B}_0(G)$ are in one-one correspondence with summands of $Z^{\uparrow N_1}$ lying in $\mathfrak{B}_0(N\times\Sigma_{\gamma+1})$.

This completes the proof of the lemma and also of the theorem, because by considering filtrations, all nonzero $\mathrm{Ext}^1_{k\Sigma_{\pi+n+1}}$ groups have dimension 1.∎

5.3 *Some comments on a conjecture of Mullineux*

Recall from Section 2.2 the definition of the p-edge of a Young diagram. Let $\lambda\vdash n$, with n arbitrary and, following [27], we make the following definition:

DEFINITION 5.3.1. *(i)* $e(\lambda)$:= number of points in the p-edge of λ.

(ii) $I(\lambda)$:= the partition obtained from λ by removing the p-edge from λ.

Observe that in the example on page 20, $e(\lambda)=16$ and $I(\lambda)$

$=(7^2,6,5,3,2,1)$.

Suppose we construct $\lambda=\lambda_\alpha$, $I(\lambda)=\lambda_{\alpha-1}$, $I(I(\lambda))=\lambda_{\alpha-2}$, $\ldots,\emptyset=\lambda_0$ in α steps. Let $e_i=e(\lambda_i)$, r_i=number of rows of λ_i. Clearly the sequences $\mathcal{E}=(e_1,\ldots e_\alpha)$, $\mathfrak{R}=(r_1,\ldots r_\alpha)$ together determine λ [27, 2.1].

DEFINITION 5.3.2. Let λ be p-regular. By [27, 2.2] each λ_i is p-regular. Define $\mathcal{S}=(s_1,\ldots s_\alpha)$ by

$$s_i = e_i-r_i+\epsilon_i \quad \text{where} \quad \epsilon_i=\begin{cases} 0 \ , & p|e_i \\ 1 \ , & \text{otherwise} \end{cases}.$$

Mullineux defines his bijection as follows:

THEOREM 5.3.3 [27, 4.1]). *The finite sequences* $\mathcal{E}$ *and* $\mathcal{S}$ *of Definition 5.3.2 determine a sequence of p-regular partitions* μ_i $(1\leq i\leq\alpha)$, *such that each* μ_i *has* s_i *rows and* $e(\mu_i)=$ $=e_i$. ∎

This map of p-regular partitions is clearly seen to be a bijection of order (at most) 2. Notice that the alternating representation of Σ_n induces a bijection (of order 2) upon the D^λ. Mullineux in [27] and [28] conjectured that this bijection agrees with the one constructed in Theorem 5.3.3 and gave a few examples for small primes, and when p>n. Our analysis gives the following result:

THEOREM 5.3.4. *The conjecture is true for all cyclic blocks of* Σ_i $(p\leq i\leq 2p-1)$ *and all elementary abelian blocks of* Σ_j $(2p\leq j\leq 3p-1)$.

Proof. In all cases notice that tensoring with $S^{(1^i)}$ merely sends the ordinary quiver of a block to the block characterised by the conjugate p-core. By Theorems 3.1.1, 4.3.1, 5.2.1 and 5.2.2, this gives the result.∎

Remark. Theorem 5.3.4 should be compared with Corollary 3.2 of [26]. We have now verified the conjecture up to degree 3p-1, but while this gives some evidence in an infinite number of cases, the conjecture is still far from being proved in general.

5.4 *Some remarks on alternating groups*

As soon as a result on the representation theory of Σ_n is proved it is usual to ask what happens vis-à-vis the alternating group A_n. Recall the following definition for an arbitrary kH-module M, where H is an arbitrary subgroup of an arbitrary group G.

DEFINITION 5.4.1. The *inertia group* T of M in G is

$$T=T(M)=\{g\in G:\ M\cong g\otimes M\}.$$

If T=G we say that M is *G-stable*.

Recall also the elementary result below:

LEMMA 5.4.2. *Assume T=H. Then $M^{\uparrow G}$ is indecomposable.*

Proof. If not, write $M^{\uparrow G}=M_1\oplus M_2$. Then By Theorem 1.1.3 $M_{i\downarrow H}\cong\sum_{\Gamma_i}\oplus(g_i\otimes M)$ where i=1,2 and $\Gamma_i\subseteq G\setminus H$ and $\Gamma_1\cap\Gamma_2=\emptyset$. Choose $g_i\in\Gamma_i$, i=1,2 and let $g_3=g_2g_1^{-1}$. Then $M_1\cong g_3M_1$ but $(g_3M_1)_{\downarrow H}\cong$

$\sum_{\Gamma_i} \oplus(g_3 g \otimes M)$ which is impossible because $g_3 g_1 \in \Gamma_2$.■

Remark. Note that $A_n \trianglelefteq \Sigma_n$ with a cyclic p'-quotient. This is a situation which will also be studied in Part II where much use will be made of Lemma 5.4.2.

There are two cases depending on whether a kA_n-module M is G-stable or not.

PROPOSITION 5.4.3. *Let $H=A_n$ $G=\Sigma_n$. Let ζ be the alternating character. Let M be a kH-module.*

(i) If M is G-stable then M extends to a kG-module M_1 and

$$M^{\uparrow G} \cong M_1 \oplus (\zeta \otimes M_1).$$

(ii)If $T(M)=H$ then $M^{\uparrow G}$ is indecomposable and $(M^{\uparrow G})_{\downarrow H} \cong M \oplus (g \otimes M)$ where $g \in G$ and $M \cong g \otimes M$.

Proof. (i) M extends to M_1 by a result to be proved in Part II, Section 7.3. So $M^{\uparrow G} = {M_1}_{\downarrow H}{}^{\uparrow G} = M_1 \otimes 1_H{}^{\uparrow G} = M_1 \oplus (\zeta \otimes M_1)$ and $\mathrm{Hom}(M^{\uparrow G}, M^{\uparrow G}) \cong \mathrm{Hom}(M, M^{\uparrow G}{}_{\downarrow H}) = \mathrm{Hom}(M, M \oplus M)$ so the two summands are different. Part (ii) is from Lemma 5.4.2.■

Remark. As we shall see later, the situation of (i) is called the *split* case and that of (ii) is called the *fuse* case.

Restriction of G-modules to H depends whether a module is equal to its core or not. See [22, 6.1.46]. We deal now with the consequences of this in case n=2p, 2p+1,...,3p-1.

There are two cases.

Case 1 $n=2p,2p+1$

Here there is a unique block of weight 2 namely the principal block. If λ and λ' are associated partitions and $D^{\lambda}\not\cong D^{\lambda'}$ then we write, inside $A_{\pi+\gamma}$ $(\gamma=0,1)$, the symbol ${}^{2}\lambda\downarrow$ as a vertex for the quiver, meaning that two kG-modules fuse to give a single kH-module. In case $\lambda=\lambda'$, we write, following [22, 2.5.7], $\lambda^{\pm}$ to indicate that the restricted module splits into two summands.

THEOREM 5.4.4. *With the above notation, the ordinary quiver for* $A_{\pi+\gamma}$, $\gamma=0,1$, *is given in Figure 15 in Appendix C.* ∎

Remarks. (*i*) One can obtain the correct value of λ at each vertex from this and the previous chapter.

(*ii*) The quiver for the alternating groups of Theorem 5.4.4 do not have the same "natural symmetry" as those constructed earlier. Observe however that it is essentially the quiver for the symmetric group which has "folded and frayed".

(*iii*)The dotted lines appearing at bottom right appear only for $\Sigma_{\pi+1}$.

(*iv*) Notice that we have obtained a unique 2-dimensional Ext^{1} group at the top right-hand side. This is shown by the double edge.

(*v*) In the example given in Appendix C (Figure 16), we give the quiver for A_{11} (mod 5).

Now we look at

Case 2 $2p+2 \leq n \leq 3p-1$

In this case no module in a block is associated to another module in the same block. So the quiver just fuses with one in another (namely the conjugate) block. The example shown in the Appendix (Figure 17) is A_{12} (mod 5).

PART II

AUSLANDER-REITEN QUIVERS
AND THEIR TREE CLASS

Chapter 6

Introduction to Auslander-Reiten theory

Let G be an arbitrary finite group and suppose that k is algebraically closed. In Chapters 7 to 9, we are concerned with Auslander-Reiten theory for modules over the finite group algebra Λ=kG. The underlying theory falls naturally into two separate but intimately linked areas, which are dealt with in Sections 6.1 and 6.2 below: namely almost split sequences and Auslander-Reiten quivers. The sequences are built up from so-called irreducible maps and were first studied in a series of papers by M. Auslander and I. Reiten for modules over artin algebras (see [4]) although here we deal only with modules over Λ. The AR-quiver is a 2-dimensional graph of these maps, as is explained in Section 6.1 where some well-known examples are also given; the exposition presented here is based on lectures of K. Erdmann. In Section 6.2 we give a summary of Webb's work on classifying the tree class of components of this quiver. The tree class is a directed tree which plays an important rôle in the Structure Theorem of Ch. Riedtmann [33]: we shall be much concerned in this thesis with components of tree class A_∞, A_∞^∞ and D_∞. Remarks about these and other types of components are to be found in Theorem 6.2.3.

6.1 Almost split sequences

Let M and N be nonzero indecomposable Λ-modules.

DEFINITION 6.1.1. The Λ-homomorphism $f:M\to N$ is said to be *irreducible* if and only if f is not an isomorphism and whenever $f=g\circ h$ (where g,h are Λ-homomorphisms) is a factorization of f, either g is split epi or h is split mono.

LEMMA 6.1.2. *If $f:M\to N$ is irreducible, then f is either an epimorphism or a monomorphism.*

Proof. By the homomorphism theorem, there are maps g,h such that

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\
& M/\ker(f) \arrow[ur, "h"'] &
\end{tikzcd}$$

with $\ker g=\ker(f)$, $\operatorname{im} h=\operatorname{im}(f)$. Then either h is a split epi, in which case h is an epi hence so is f, or else g is a split mono, which implies g is a mono, so f is too.∎

EXAMPLE. Let $N=P$, an indecomposable projective and let $M=J(N)$. Then the inclusion $\iota:M\hookrightarrow N$ is irreducible.

Proof. ι is nonsplit since M, N are indecomposable and $M\ncong N$. Let

$$\begin{tikzcd}
M \arrow[rr, hook, "\iota"] \arrow[dr, "\kappa"'] & & N \\
& A \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Now $\operatorname{im}\iota=M\subseteq\operatorname{im}\rho\subseteq N=P$. $P/J(P)$ is simple so $\operatorname{im}\rho=P$ or $\operatorname{im}\rho=J(P)$. If $\operatorname{im}\rho=P$ then ρ is onto and P is projective so ρ splits. If $\operatorname{im}\rho=J(P)=\operatorname{im}\iota$ then κ splits (with right inverse $\iota^{-1}\rho$).

DEFINITION 6.1.3. The *Auslander-Reiten quiver* (or AR-quiver for short) is a directed graph $\Gamma(\Lambda)$ whose vertices are the (isomorphism classes of) indecomposable Λ-modules, and such that there are d labelled edges

$$[M] \overset{d}{\substack{\longrightarrow \\ \cdots\cdots\cdots \\ \longrightarrow}} [N]$$

if and only if d is maximal with the property that there exist irreducible maps

$$\sum_{d} \oplus M \longrightarrow N$$

Remarks. (*i*) Usually we have $d \leq 1$.

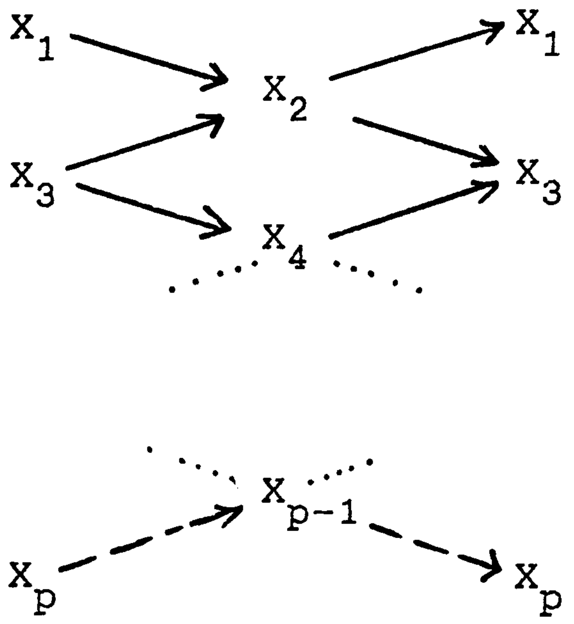
(*ii*) The definition is symmetric: it can be shown using Theorem 6.1.5 below, that d is the largest number such that there exist irreducible maps

$$M \longrightarrow \sum_{d} \oplus N.$$

Very often projective modules in the quiver provide unnecessary complications. To avoid this we introduce the notion of the stable quiver.

DEFINITION 6.1.4. The *stable* AR-quiver $\Gamma_s(\Lambda)$ is obtained from $\Gamma(\Lambda)$ by removing projective vertices and all arrows starting and ending at projectives.

EXAMPLE. Take $\Lambda = kP$ where $P = \langle x \rangle$ is cyclic of order p. Recalling the Jordan Canonical Form for x, let X_i be the i-dimensional indecomposable $(1 \leq i \leq p)$. Then the quiver looks like



where the stable quiver is obtained by removing the X_p and the dotted arrows.

Let

$$M^- = \{X: \exists \text{ arrow } [X] \to [M]\}$$

$$M^+ = \{X: \exists \text{ arrow } [M] \to [X]\}$$

and recall from Section 1.1 the definition of the Heller operator Ω. The following theorem was proved, by Auslander and Reiten for modules over an artin algebra, but we here just consider group algebras.

THEOREM 6.1.5 ([4]). *(i) If M is indecomposable and non-projective then there exists a (unique to isomorphism) indecomposable module Y such that $M^-=Y^+$ and there exists a nonsplit s.e.s.*

$$0 \longrightarrow Y \xrightarrow{\beta} \sum_{X_i \in M^-} \oplus\, d_i X_i \xrightarrow{\alpha} M \longrightarrow 0 \qquad \ldots\ldots(\xi)$$

where d_i is the number of arrows $[X_i]\to[M]$ (or the number of *arrows* $[Y]\to[X_i]$.

(ii) If Y *is indecomposable and non-injective (which is the same as being non-projective since* Λ *is symmetric) then there exists a unique indecomposable module* M *Such that* $Y^+=M^-$ *and a s.e.s.* (ξ).■

Remarks. (*i*) One writes $Y=\tau M$ where τ is the *Auslander-Reiten translate*. If Λ is a group algebra then $\tau\cong\Omega^2$. For details of the proof in the case of group algebras see [24, II.9.2].

(*ii*) (ξ) is called the *Auslander-Reiten sequence* or the *almost split sequence* (usually abbreviated to AR-sequence) *terminating in* M. We denote this by $\mathcal{A}(M)$.

(*iii*) Note finally that, as a result of this theorem, it is of interest not only to describe the indecomposable Λ-modules, but also the maps between modules. Hence we sometimes find it convenient to consider all the action taking place in the category modΛ of finitely-generated Λ-modules. The rationale behind this thinking is hinted at in [32].

EXAMPLE. It is an exercise to show that the only irreducible maps involving a projective indecomposable module $P=P(S)$ (S simple) are

$$\pi: P \overset{\text{can}}{\twoheadrightarrow} P/S = P/\mathrm{soc}(P)$$

and

$$j: J(P) \overset{\text{inc}}{\rightarrowtail} P$$

Thus if an AR-sequence involves a projective module, it is of this form. Let $\pi_1=\pi/J(P)$, $\bar{j}$=induced inclusion map. The following is an AR-sequence

$$0 \longrightarrow J(P) \xrightarrow{\begin{bmatrix}\pi_1\\ j\end{bmatrix}} J(P)/S \oplus P \xrightarrow{[\,\bar{j}\,,-\pi]} P/S \longrightarrow 0$$

where P=P(S), S simple.

Remarks. (*i*) In general, the quiver is locally finite (meaning that $|M^-|, |M^+| < \infty$, for all M). This is a consequence of Theorem 6.1.5.

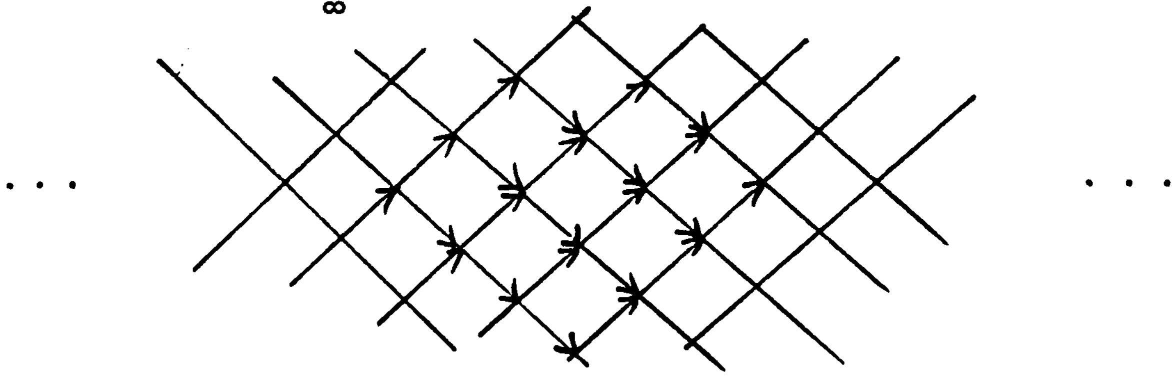
(*ii*) τ induces a translation on $\Gamma_s(\Lambda)$, and if $\tau=\Omega^2$ then Ω induces a graph automorphism on $\Gamma_s(\Lambda)$.

(*iii*) The components of α and β in (ξ) are the irreducible maps.

(*iv*) Since irreducible maps are not, by definition, isomorphisms there are no loops in the graph.

$\Gamma_s(kG)$ is an example of an abstract *stable representation quiver* of the type discussed by C. Riedtmann [33]. There, she proves a Structure Theorem which states (in our case) that if Θ is a connected component of $\Gamma_s(kG)$ then $\Theta \cong \mathbb{Z}B/\Pi$ where B is a directed tree and $\Pi \leq \mathrm{Aut}\,\mathbb{Z}B$ is an "admissible" group of automorphisms. $\mathbb{Z}B$ is defined as follows: the vertices are (n,x) $n\in\mathbb{Z}$, $x\in B$: for each arrow $x\to y$ there are two arrows $(n,x)\to(n,y)$ and $(n,y)\to(n-1,x)$. The translation is $\lambda(n,x) = (n+1,x)$. We call $\mathbb{Z}B$ the *universal cover* of Θ. B is called the *tree class* of Θ, and is denoted by $B(\Theta)$. For an excellent account of this see [5, §2.29].

EXAMPLE. $\Theta=\mathbb{Z}A_\infty^\infty$.



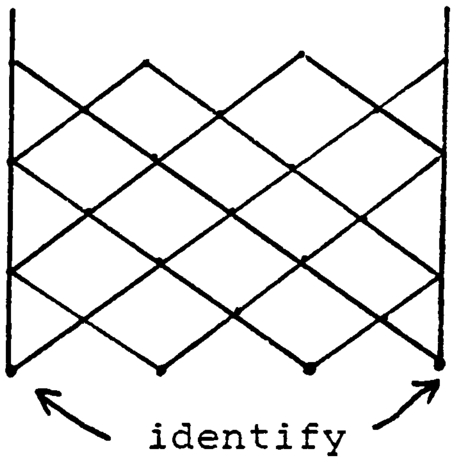
EXAMPLE. The components Θ of the stable AR-graph which will usually occur have the property that $\mathfrak{a}(M)$ has only two middle terms. The preceding example displays this feature. Other components of great interest are called *tubes*. They are of type $\mathbb{Z}A_\infty/\kappa$ for some integer κ. Modules in a stable component which is a tube may be labelled by elements of $\mathbb{Z}\times\mathbb{N}$ such that

(*a*) the irreducible maps go from $M(z,n)$ to $M(z,n+1)$ and from $M(z+1,n+1)$ to $M(z,n)$ and

(*b*) $M(z,n)\cong M(z+\kappa s,n)$ $\forall s\in\mathbb{Z}$, and the corresponding arrows are identified; also κ is the smallest such integer with this property.

The modules $M(z,1)$ form the *end* of the tube. The *r'th row* is given by the modules $M(z,r)$. Also κ is the *rank* of the tube. For an example of a 1-tube see the example following Definition 6.1.4; for a 3-tube, see below.

EXAMPLE. $\Theta=\mathbb{Z}A_\infty/3$.



3 vertices/row

In Chapter 9 we deal with periodic modules (see Section 9.1). We use there the fact that if M is periodic of period κ say, then it can be shown that the component of M is a

$\begin{cases} \kappa\text{-tube}, & \kappa \text{ odd} \\ \frac{\kappa}{2}\text{-tube}, & \kappa \text{ even} \end{cases}$. See Theorem 6.2.3 below.

6.2. *Auslander-Reiten quivers*

The paper of Webb [39] goes a long way to determining the tree class of connected components of $\Gamma_s(kG)$. There is a considerable simplification of this work by Okuyama [29], which uses the Benson-Parker inner product. We return to this in Chapter 7. Webb's main result is concerned with the existence of so-called additive functions from the vertices of the tree to $\mathbb{N}$. However since we shall be more concerned with the corollary to the theorem below, and not with the actual functions, we leave the interested reader to chase up the definition of subadditivity in [5, 2.30.2].

THEOREM 6.2.1 ([39, 2.2]). *The tree associated to a connected component of* $\Gamma_s(kG)$ *admits a subadditive function.* ∎

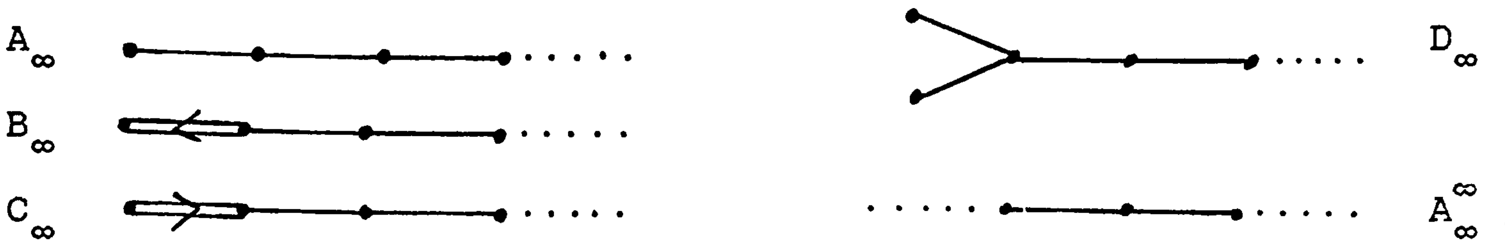
As a result, he proves

COROLLARY 6.2.2. *The tree class is among the following list:*

(a) finite Dynkin diagrams

(b) Euclidean diagrams

(c) infinite Dynkin diagrams:



■

For details of the proof and lots of pictures, see [5, 2.31] or [39, Theorem A].

We give below a summary of what is known to happen in each of the three cases given in Corollary 6.2.2.

THEOREM 6.2.3. *(i) If a component* Θ *of* $\Gamma_s(kG)$ *has a finite Dynkin diagram as its tree class then* Θ *consists of all the non-projective modules in a block of kG with cyclic defect group, [5, 2.31.8]. In this case the tree class is* A_n*, [13] or [31]. However the other finite Dynkin diagrams come up in algebras of finite representation type which are not blocks of finite group algebras.*

(ii) If Θ *has as tree class a Euclidean diagram, then there is a projective module attached to* Θ*, [5, 2.32.5], and* $p=2$*, [29, Theorem 2].*

(iii) If Θ *has infinitely many vertices and contains a periodic module, then the tree class is* A_∞*, [5, 2.31.11].*■

Remarks. *(i)* For further details about quiver components the reader is referred to sections 2.28 to 2.32 of Benson [5]. For more facts about almost split sequences defined over artin algebras, see the survey article of Reiten [32].

(ii) We shall not be concerned with the finite case of Theorem 6.2.3(i) since this is well-known.

In the next chapter we study (infinite) regular components. Θ is said to be *regular* if it contains no projective modules. It is then easy to prove that if, in addition, k is algebraically closed, then only A_∞, A_∞^∞ and D_∞ can occur as a tree class. (To see this, note that by the symmetry referred to early in Section 6.1 only diagrams of type A,D,E,$\tilde{A}$ (but not $\tilde{A}_{11}$), $\tilde{D}$ and $\tilde{E}$ occur. By regularity and Theorem 6.2.3(ii) only A,D and E occur. There can be no E diagrams by Remark (ii) above.

EXAMPLE. Let P be a normal subgroup of G and suppose that P is noncyclic, and not of order 4. Let Δ be the connected component of $\Gamma(kG)$ containing the trivial module k_G. Then Δ is regular. For a proof, see Linnell [25, Lemma 3.1].

CHAPTER 7

The tree class of components of abelian mod p groups

Suppose that we are given an AR-sequence as in Chapter 6. A natural question to ask is what sort of operations can be performed on the sequence which would preserve the property of being "almost split". It is a fact that the number of such operations is limited:

(a) *duality* ([5, 2.17.9]). If $0 \to X \to Y \to Z \to 0$ is an AR-sequence, then so is $0 \to Z^* \to Y^* \to X^* \to 0$

(b) *taking Homs* ([24, II.9.3]). Let Z be indecomposable and non-projective. Let $\mathcal{A}(Z)$ be $0 \to \tau Z \to E \to Z \to 0$. Assume that B is indecomposable, and not isomorphic to Z. Then the most we can say is that

$$0 \to \mathrm{Hom}_{kG}(B,\tau Z) \to \mathrm{Hom}_{kG}(B,E) \to \mathrm{Hom}_{kG}(B,\tau Z) \to 0$$

is exact (which is clear) and

$$0 \to \mathrm{Hom}_{kG}(Z,\tau Z) \to \mathrm{Hom}_{kG}(Z,E) \to \mathrm{Hom}_{kG}(Z,Z) \to \mathrm{soc}(\mathrm{Ext}^1_{kG}(Z,\tau Z)) \to 0$$

is exact (the truncation of the long exact sequence for Ext).

(c) *tensoring*. The usual problem here is that the tensor product of two indecomposable modules is usually decomposable, and there are very few methods to find the resulting summands. Under certain (often bizarre) conditions one can tensor an AR-sequence with some non-projective indecomposable module and obtain an AR-sequence after discarding projectives. For example, consider $\mathfrak{a}(I)$ where I is the trivial module. Let M be indecomposable such that $p \nmid \text{rank}_k M$. Then $\mathfrak{a}(M)=\mathfrak{a}(I)\otimes M$ is an AR-sequence up to projective summands as proved in [3]. There are generalisations of this, the most noteable to be found in [35], to which the reader is referred.

(*d*) *restricting* ([5, 2.17.10]). Recall from Section 1.2 the definition of the vertex of a module. Let H be a subgroup of G. Then an AR-sequence $\mathfrak{a}(Z)$ splits on restriction to H if and only if H does not contain a vertex of Z (or equivalently a vertex of τZ).

It is the aim of this chapter to consider the one obvious operation we have not mentioned: inducing from subgroups. We assume that the subgroup is normal so that we at least have the luxury of using a little Clifford theory to analyse the summands of an induced module. The scene is set in Section 7.1. In Sections 7.2 and 7.3 we prove a general theorem about inducing modules from a normal subgroup of an abelian mod p group. Other results along these lines (for more specialised groups) are proved in later chapters.

7.1 *Some definitions*

The motivation for studying the AR-theory of induced modules comes from the following result of Φ. Solberg [36], which falls out as a corollary of his results on strongly graded rings. Let G have a normal subgroup N and write

$\bar{G}=G/N$.

THEOREM 7.1.1 ([36, Corollary 7]). *Let* $F=kG\otimes_{kN}$ *:mod kN* $\longrightarrow$ *mod kG, and let H=restriction: mod kG* $\longrightarrow$ *mod kN. If* $|\bar{G}|$ *is invertible in k, then the functors F and H preserve AR-sequences (in the sense that if* $\mathfrak{a}(Z)$ *is given in mod kN respectively mod kG, then* $\mathfrak{a}(FZ)$ *respectively* $\mathfrak{a}(HZ)$ *is a direct sum of sequences in mod kG respectively mod kN.)*∎

Solberg claims that this could be used to compare the AR-quivers for kN and kG but gives no examples. In a recent paper [38], Uno considers the decomposition of an induced AR-sequence but is not really concerned with actual quivers *per se*. Throughout Part II we will be concerned with precisely the problem raised by Solberg for certain groups with a known quotient. More precisely,

DEFINITION 7.1.2. Let G have a normal subgroup N. Let $\mathfrak{X}$ be some group-theoretic property. We say that G is $\mathfrak{X}$ *mod p* if $\bar{G}$ has property $\mathfrak{X}$ and is a p'-group.

Remark. Examples will be given below and in Chapter 8. We comment that $\mathfrak{X}$ is usually taken to be "cyclic" or "abelian". If G is $\mathfrak{S}$ mod p, the underlying normal subgroup will always be denoted by N.

Now we fix some notation. Let G be abelian mod p. Let k be an algebraically closed field of characteristic p. M will always be an indecomposable kN-module.

7.2 *Regular components and abelian mod p groups*

Let M lie in a regular component Δ of $\Gamma(kN)$, so that by our comments at the end of Section 6.2, $B(\Delta)\cong A_\infty$, A_∞^∞ or D_∞. We ask what are the components of summands of $M^{\uparrow G}$. By the structure theorem for finite abelian groups we write $G/N \cong C_{\ell_1}\times\cdots\cdots\times C_{\ell_r}$ with $C_{\ell_j}=\langle g_j\rangle$. Then $M^{\uparrow G}{}_{\downarrow N} = \sum_{i,j} \oplus\, g_j{}^{i}\otimes M$.

Recall from Definition 5.4.1 the definition of the inertia group T=T(M) of M. Notice that for an abelian mod p group, we are once again in the split-fuse situation, so we want to calculate T. Observe however that by the following result it is enough to consider only G-stable modules.

LEMMA 7.2.1. *With the above notation, write*

$$M^{\uparrow T} = \sum \oplus M_i$$

where the M_i *are indecomposable kT-modules. Then* $M_i{}^{\uparrow G}$ *is indecomposable.*

Proof (Ward & Willems, [16]). Let $\mathcal{T}$ be a transversal of T in G. Now $N\trianglelefteq G$ so $N\trianglelefteq T$ and $M^{\uparrow T}{}_{\downarrow N}$ is the direct sum of kN-modules of the form $x\otimes M$ $(x\in T)$, and by definition of T, $x\otimes M\cong M$. Hence by Theorem 1.1.1

$$M_{i\downarrow N} \cong M\oplus\cdots\cdots\oplus M \qquad \ldots\ldots(*)$$

where we have m_i summands say. So given $g\in G$

$$g\otimes M_i \cong \underbrace{(g\otimes M)\oplus\cdots\cdots\oplus(g\otimes M)}_{m_i}$$

which means

$$M_i\uparrow^G\downarrow_N \cong \left[\sum_{\mathcal{T}} \oplus\; t\otimes M_i\right]\downarrow_N \cong \sum_{\mathcal{T}} \oplus\; \left[(t\otimes M)\oplus\cdots\cdots(t\otimes M)\right]\downarrow_N$$

.....(**)

Write $M_i\uparrow^G = A\oplus B$. Now $M_i \| M_i\uparrow^G\downarrow_T$ so without loss, suppose $M_i | A\downarrow_T$. Write $A\downarrow_T = M_i'\oplus A'$ where $M_i'\cong M_i$. Then

$$A\downarrow_N = A_1\oplus\cdots\cdots\oplus A_{m_i}\oplus A'\downarrow_N$$

by (*) where $A_j\cong M$, that is

$$A\downarrow_N = tA\downarrow_N = tA_1\oplus\cdots\cdots\cdots\oplus tA_{m_i}\oplus tA'\downarrow_N$$

where $tA_j \cong t\otimes M$ as kN-modules.

It follows that $t\otimes M | A\downarrow_N$ with multiplicity at least m_i. Choosing $t\neq t'\in\mathcal{T}$, $t\otimes M\not\cong t'\otimes M$ as kN-modules. By (**) $M_i\uparrow^G\downarrow_N | A\downarrow_N$. But then since $\dim_k M_i\uparrow^G \geq \dim_k A$, we must have B=0, which shows that $M_i\uparrow^G$ is indecomposable.∎

Remark. There is no restriction on the quotient G/N in the proof of this lemma, although for what follows it is crucial that G/N has the properties stated above.

LEMMA 7.2.2 (Willems [16, Theorem VII.9.9]). *Let G be cyclic mod p, and let M be G-stable. Then there exists a kG-module W such that* $W\downarrow_N\cong M$.∎

This can be easily extended:

LEMMA 7.2.3. *Lemma 7.2.2 holds for abelian mod p groups.*

Proof. Write G/N as a product of cyclic groups as above. Then we do induction on r, the number of such factors. The case r=1 is Lemma 7.2.2. Now let r>1. Find a subgroup $H \trianglelefteq G$ such that H strictly contains N and $H/N \cong C_{\ell_1} \times \cdots\cdots C_{\ell_{r-1}}$, with $G/H \cong C_{\ell_r}$.

Since M is G-stable it is of course H-stable, hence by induction there is a kH-module Π such that $\Pi_{\downarrow N} \cong M$. By Lemma 7.2.1 we can assume Π is indecomposable and is G-stable. By Lemma 7.2.2 (which is the base step) there is a kG-module P such that $P_{\downarrow H} \cong \Pi$. Hence $P_{\downarrow N} = (P_{\downarrow H})_{\downarrow N} \cong M$.∎

In this case we have that $M^{\uparrow G}$ splits into ℓ non-isomorphic summands, where $\ell = \prod_i \ell_i$. With all these preliminaries out of the way, we can now proceed to the main result.

7.3 *The main theorem*

THEOREM 7.3.1. *Let p be odd. Let $\mathcal{C}$ be a regular component of $\Gamma(kH)$, where H and G are as above and k is algebraically closed. Let $M \in \mathcal{C}$ be indecomposable, and G-stable. Then $M^{\uparrow G}$ has summands lying in the following components* $\hat{\mathcal{C}}$ of G.

	$B(\mathcal{C})$	$B(\hat{\mathcal{C}})$	$\hat{\mathcal{C}}$
(i)	A_∞	A_∞	$\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\kappa$
(ii)	A_∞^∞	A_∞^∞ or D_∞	$\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}A_\infty^\infty/\sigma$ or $\mathbb{Z}D_\infty$
(iii)	D_∞	A_∞^∞ or D_∞	$\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}A_\infty^\infty/\sigma$ or $\mathbb{Z}D_\infty$

where $\kappa \in \mathbb{N}$ and σ is reflection in a τ-orbit.

Proof. (i) Consider Okuyama's construction of a subadditive function on a connected component of $\Gamma_s(kG)$ or $\Gamma_s(kN)$ as detailed in [29]. We use his notation. Observe that d_N and d_G can be defined using the same module Y inside N and G. Let $M^{\uparrow G}=\sum \oplus M_i$. Then

$$d_N(M)\le d_G(M_i) \quad (\forall i) \qquad \ldots\ldots(1)$$

To see this, choose s to be G-invariant. Notice that this does not affect subadditivity. We have

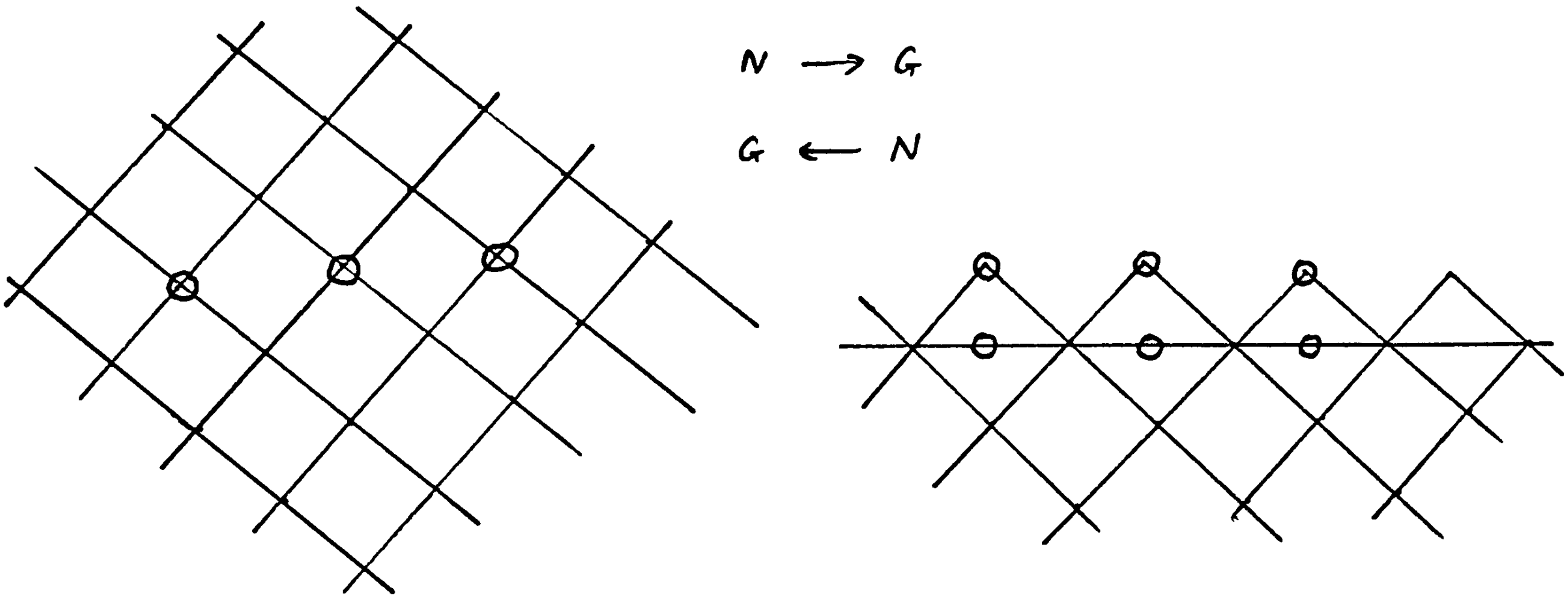
$$\begin{aligned} d_G(M_i) &= (s,M_i) = (s_0{}^{\uparrow G},M_i) \\ &= (s_0,M_{i\downarrow N}) = d_N(M_{i\downarrow N}) \\ &= r(s_0,M) = rd_N(M) \end{aligned}$$

where $s_0=Y^{\uparrow N}\oplus(\Omega^{-1}Y)^{\uparrow N}-I^{\uparrow N}$, $s=s_0{}^{\uparrow G}$ and r is the number of summands of $M_{i\downarrow N}$. Since $1\le r\le|G:N|$, (1) follows.

Now recall that if d is any subadditive function on a connected labelled graph and d is unbounded, then $T\cong A_\infty$, [5, 2.30.6(iv)]. So in our case, since $d_N(M_i)$ is unbounded, it must follow, by (1) that $d_G(M_i)$ is unbounded so we have A_∞ upstairs.

Finally observe that the component itself is either $\mathbb{Z}A_\infty$ or a tube [12]. This is because any non-trivial automorphism is a "rotation" of order $\kappa>0$. We investigate this in Chapter 8.

(ii),(iii) To see how A_∞^∞ and D_∞ can arise from each other, notice first that if the module M splits when induced then so do all its AR-translates. One can visualize this better by considering the following diagram.



The component Δ is isomorphic to $\mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}A_\infty^\infty/\sigma$, for $B(\Delta) \cong A_\infty^\infty$, but for $B(\Delta)\cong D_\infty$ there are no nontrivial automorphisms induced by conjugation.∎

Remarks. *(i)* For a thought-provoking discussion of graph automorphisms, see [12, §1].

(ii) Unfortunately, we are at present unable to provide any actual examples of (ii) and (iii) occurring.

CHAPTER 8

Simple modules in the Auslander-Reiten quiver

In this chapter, we assume all the basic terminology and results of Chapter 6. Here we shall be dealing with simple modules for certain classes of groups, and hence trying to answer Solberg's question (see Section 7.1) in this case. The results derived here fall naturally into two parts. In Section 8.1 we shall be looking at simple modules for abelian mod p groups and more generally for M_p mod p groups, in the case where k is algebraically closed of odd characteristic. The assumptions allow us immediately to write down what the simple G/N modules are. An example showing how the theory breaks down when p=2 is also given. In Section 8.2, we look at modules for an abelian mod p group over a non-closed field. The motivation for this comes from Benson's rendition of Galois descent, (as enshrined in §2.33 of [5]), and using the ideas there we can derive Theorem 8.2.4. This computes the tree class for simple FG-modules and can be viewed as the analogue of Theorem 8.1.3.

8.1 Simple modules I

Let k be algebraically closed of characteristic p, where p is odd. Let $P=O_p(G)$, the maximal normal p-subgroup of G, and suppose that P is non-cyclic. With respect to P suppose that G is abelian mod p and write $\bar{G}=G/O_p(G)$ and let $|\bar{G}|=m$. We have

PROPOSITION 8.1.1. *The trivial module* k_P *of* P *lies in a component of the stable part of the component of* $\Gamma(kP)$ *whose tree class is* A_∞.

Proof. This deep result is proved in [39, Theorem F] and [25, Theorem 1.1].∎

PROPOSITION 8.1.2. *(i)* $k\bar{G}$*-modules are the* kG*-modules on which* P *acts trivially.*

(ii) The simple kG*-modules are precisely the simple* $k\bar{G}$*-modules.*

Proof. (i) If X is a $k\bar{G}$-module then consider X as a kG-module via

$$(p\bar{g})x = \bar{g}x$$

($x\in X$, $p\in P$, $\bar{g}\in\bar{G}$) which works because $P \trianglelefteq G$.

(ii) If S is a simple $k\bar{G}$-module, then S is a kG-module by (i), so is still simple.

Let S be a simple kG-module. We have to show that P acts trivially on S. Let $S_0=\{m\in S\colon dm=m\ \forall d\in P\}$. It is enough to show that $S_0=S$. Clearly S_0 is a kG-submodule of S so either $S_0=0$ or $S_0=S$. View S as a G-space under the obvious action. Let $\Phi=\langle dm\colon d\in P\rangle$ so that Φ is a finite abelian p-group inside S.

Then $\Phi=\dot{\bigcup}\Phi_i$, a disjoint union of orbits Φ_i each of whose lengths divides $|P|$. Hence

$$
\begin{aligned}
n &= \text{the number of orbits of length } 1 \\
&\equiv 0 \pmod{p}
\end{aligned}
$$

But $n \neq 0$ because $\{0\}$ is an orbit and so $n \geq p$. So choose $0 \neq m$ in an orbit of size 1. Then $dm=m$ $\forall d \in P$ that is $m \in S_0$, whence $S_0 \neq \emptyset$ as required.∎

Recall that $\bar{G}$ is abelian, and let $S_1, \ldots, S_m$ be the simple $k\bar{G}$-modules. Then we have

THEOREM 8.1.3. *(i)* $k_P{\uparrow^G} = \sum_i \oplus S_i$;

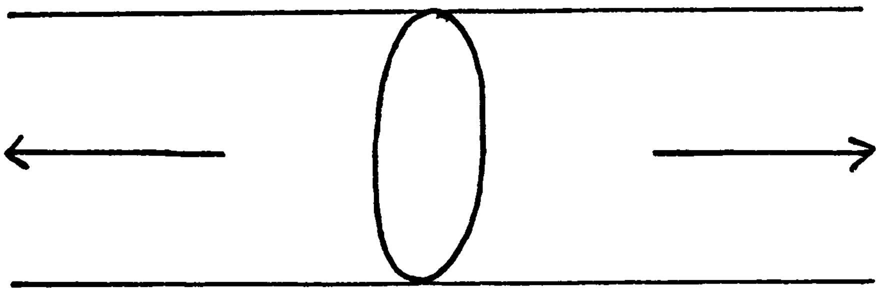
(ii) S_i *lies at the end of a component of* $\Gamma(kG)$ *whose tree class is* A_∞;

(iii) All the S_i *lie in different components.*

Remarks. *(i)* We require p to be odd. For let F be a field containing the primitive third roots of unity. Let $\Lambda = FA_4$ and observe that $A_4 \cong V_4 : C_3$. Let S_0, S_1 and S_2 be the three simple modules in characteristic 2. By the example on page 67, we have the following AR-sequence

$$0 \rightarrow J(P_0) \rightarrow P_0 \oplus S_1 \oplus S_2 \rightarrow P_0/S_0 \rightarrow 0$$

with $P_i = P(S_i)$, and hence we obtain the component containing all 3 simple modules, and looking like



As before, $k_{V_4}{\uparrow^{A_4}} = S_0 \oplus S_1 \oplus S_2$ and k_V lies in a $\tilde{A}_{12}$ component (see [5, Appendix]).

(ii) An example of a group satisfying the hypothesis of this theorem is the matrix group

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in GF(p^2),\ a \neq 0 \right\}$$

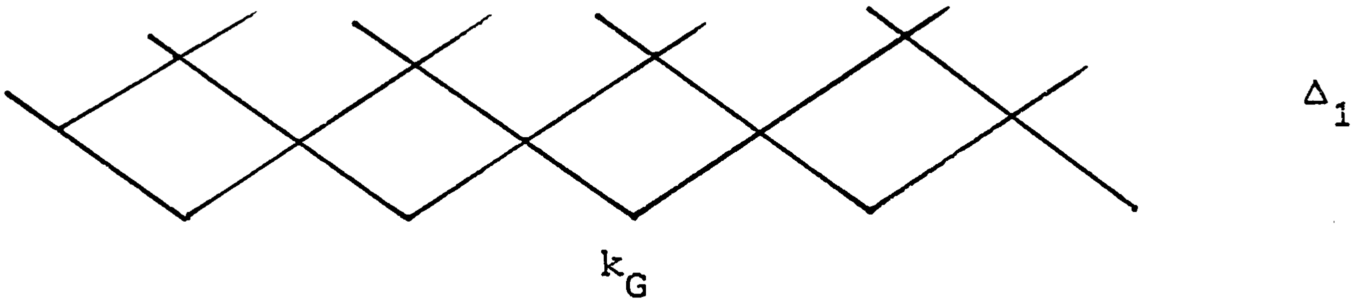
with

$$P = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in GF(p^2) \right\}$$

and of course $G \cong (C_p \times C_p) : C_{p^2-1}$.

Proof of theorem. (i) The simple kG-modules are 1-dimensional because $\bar{G}$ is abelian (using Proposition 8.1.2). Label these modules such that $S_1 = k_G$. Since $\dim S_i = 1$, $\dim S_{i\downarrow P} = 1$ hence $S_{i\downarrow P} = k_P$. Since $\bar{G}$ is a p'-group $S_i \| (S_{i\downarrow P})^{\uparrow G}$ that is $S_i \| k_P^{\uparrow G}$. Since $\dim k_P^{\uparrow G} = |G:P| \dim k_P = |\bar{G}|$, (i) follows.

(ii) It is clear that k_G lies at the end of its component Δ_1



Let $i>1$ and consider the component Δ_i of S_i. Applying the theorem of Schulz [35] to modules in Δ_1 after tensoring with S_i, we conclude that $\Delta_i = \Delta_1 \otimes S_i$ after removing projectives. We have to show that the tree class is again A_∞. Let H_G be the *heart* of $P(k_G)$, which by [39, Theorem E] is indecomposable. $\Delta_1 \otimes S_i$ looks like

$$\begin{tikzcd}
 & & \bullet \arrow[dr] & \vdots & \bullet \arrow[dr] & & \\
\cdots & \bullet \arrow[ur] \arrow[dr] & & \Omega H_G \otimes S_i \arrow[ur] \arrow[dr] & & \bullet & \cdots \\
 & & \Omega^{-1} S_i \arrow[ur] & & S_i \arrow[ur] & &
\end{tikzcd}$$

To show that this is a A_∞ component, we must show that $X=\Omega H_G \otimes S_i$ is indecomposable after removing projectives.

Suppose not, then $X=X_1 \oplus X_2$ say with $X_i \neq 0$. By a result of Benson and Carlson [11, Theorem 81.79], $k_G \| S_i \otimes S_j$ if and only if $S_i^* \cong S_j$. Then for such i and j,

$$\begin{aligned} \Omega H \otimes S_i = X_1 \oplus X_2 &\Rightarrow \Omega H \otimes S_i \otimes S_j = (X_1 \otimes S_j) \oplus (X_2 \otimes S_j) \\ &\Rightarrow \Omega H = (X_1 \otimes S_j) \oplus (X_2 \otimes S_j) \\ &\Rightarrow \Omega H \text{ is decomposable} \end{aligned}$$

and this is impossible by our observation above. The fact that S_i lies at the end of Δ_i is immediate.

(iii) Suppose that two simple modules lie in the same component. Then we have a graph automorphism of finite order. By (ii) we know that the simple modules have to lie at the end of the component, so the automorphism must be a translation. Since translations have infinite order we are done.∎

DEFINITION 8.1.4. A finite group is called an *M_p-group* if the simple modules in characteristic p are monomial (that

Proof of Theorem 8·1·5 (continued).

(ii) We must again show that $\Omega H \otimes S_i$ is indecomposable. Notice that ΩH is absolutely indecomposable and that S_i can be viewed as a simple kG-module. Then by [16, Theorem 9·12 a)], the tensor is an indecomposable kG-module.

(iii) follows from (ii) as above. ■

is induced from 1-dimensional representations of subgroups of G).

THEOREM 8.1.5. *Suppose that in the statement of Theorem 8.1.3, instead of assuming that G is abelian mod p, we assume that G is* M_p *mod p. Then*

$$k_P{}^{\uparrow G} = \sum \oplus a_iS_i \text{ for some } a_i\in\mathbb{N}$$

Also the conclusions of parts (ii) and (iii) of Theorem 8.1.3 hold here.

Proof.

(i) For the first part, let S be a simple $k\bar{G}$-module. Then $S=T\uparrow_H^G$ where T is a 1-dimensional kH-module of a subgroup H of G. By Theorem 1.1.3

$$S_{\downarrow P} = (T^{\uparrow G})_{\downarrow P} \cong \sum_{HgP} \oplus ({}^gT_{\downarrow H^g\cap P})^{\uparrow P}$$

Since T is 1-dimensional, ${}^gT$ is 1-dimensional, hence ${}^gT_{\downarrow H^g\cap P}$ is 1-dimensional. But $H^g\cap P\le P$ so we get ${}^gT_{\downarrow H^g\cap P}=k_{H^g\cap P}=k_D$ say. Since m is a p'-number we get $S\|S_{\downarrow P}{}^{\uparrow G}$ that is $S\|ak_D{}^{\uparrow P}$ where $a=|G:H|$. Then since dim $k_D{}^{\uparrow G}=|G:H^g\cap P|$ we are done.

8.2 Simple modules II

We assumed throughout the last section that k was algebraically closed. However here we do not demand that this be the case. So we let K be a field such that $K\supset k$. We are concerned with representations over both k and K.

PROPOSITION 8.2.1. K_G *lies at the end of an* A_∞

component.■

Of course this follows from [39, Theorem F] which holds irrespective of whether the field is closed or not. It is a natural question to ask if results similar to those in Section 8.1 hold for other simple modules S_i.

The set-up will be as follows: we take $k=GF(p)$ with p odd and we view k as the prime subfield of $K=GF(p^2)$. We assume G is $C_p \times C_p : C_{p^2-1}$ because it seems easier to prove results in the case when representations over K are all 1-dimensional. Let $F=k$ or K. Notice that there are simple kG-modules which are not 1-dimensional. As before we set $m=|\bar{G}|$.

LEMMA 8.2.2. Let $S_1,..,S_m$ be all the simple KG-modules. Then over k there are r 1-dimensional simple modules and $\frac{1}{2}(m-r)$ sets of conjugate 2-dimensional simple modules. Here r is the cardinality of the fixed field of the Frobenius automorphism, γ.

Proof. Observe that the Galois group $\mathcal{G}=Gal(K/k)=\langle\gamma\rangle$ is cyclic of order 2. There are $r=p-1$ fixed points, giving 1-dimensional simple modules $M_1,..,M_{p-1}$ which are absolutely simple. The remaining $v=(p^2-1-p+1)=p(p-1)$ simple modules $N_1,...N_v$ are 2-dimensional and have the property that

$$N_i \otimes K = S_i \oplus S_i^{\gamma}$$

where S_i^{γ} is the algebraic conjugate of S_i under γ.■

We can, by [5, 2.33.2] concentrate on finding in which components S_i and S_i^{γ} lie. Let S_i and S_i^{γ} lie at the end of the components Δ_i and $\bar{\Delta}_i$ respectively of $\Gamma(KG)$. Let δ_i be the component of $\Gamma(kG)$ in which N_i lies. In the theorem which follows we use the method of Galois descent mentioned above. Recall that the Galois group $\mathcal{G}$ acts on the set of indecomposable KG-modules in the obvious way.

DEFINITION 8.2.3. In the above situation we say that the Δ_i, and $\bar{\Delta}_i$ *lie above* δ_i. If $\mathcal{C}$ is a connected component of the stable quiver of KG-modules we define the *decomposition group* $\mathcal{G}_{\mathcal{C}}$ to be $\mathrm{Stab}_{\mathcal{G}}(\mathcal{C})$, and the *decomposition field* K^d to be the fixed field of $\mathcal{G}_{\mathcal{C}}$.

THEOREM 8.2.4. *Δ_i and $\bar{\Delta}_i$ are isomorphic as components of $\Gamma(KG)$. They have the same tree class which is A_∞. Moreover $\delta_i \cong \Delta_i$.*

Proof. Assume that $\Delta_i \ncong \bar{\Delta}_i$. In our case, since $\mathcal{G}=C_2$ we have $\mathcal{G}_{\Delta_i}=\mathcal{G}_{\bar{\Delta}_i}=1$ and hence by the Fundamental Theorem of Galois Theory $K^d=K$.

Let $\Delta_i^{(1)}$ and $\Delta_i^{(2)}$ be the images of Δ_i under $\mathcal{G}$. So we can assume that $\Delta_i^{(1)}=\Delta_i$, $\Delta_i^{(2)}=\bar{\Delta}_i$. Clearly $\Delta_i^{(j)}$ lie above δ_i (for j=1,2). Let $\mathfrak{Y}_i$ be the component of the stable quiver of KG-modules over which Δ_i lies, so that $\mathfrak{Y}_i \cong \Delta_i$. By [5, 2.33.4]

(1) Δ_i is the only component of the stable quiver of KG-modules lying over $\mathfrak{Y}_i$.

(2) There is a natural isomorphism $\mathfrak{Y}_i \cong \delta_i$.

(3) $\mathfrak{Y}$ is the quotient of Δ_i by the action of $\mathcal{G}_{\Delta_i}$.

(1) and (3) are redundant, but (2) gives us that $\mathfrak{Y}_i \cong \delta_i$. Notice also that the argument is symmetrical: let $\mathfrak{J}_i$ be the component of KG-modules over which $\bar{\Delta}_i = \Delta_i^{(2)}$ lies. Then as before $\Delta_i^{(2)} = \mathfrak{J}_i$ and $\mathfrak{J}_i \cong \delta_i$. Then

$$\bar{\Delta}_i = \Delta_i^{(2)} = \mathfrak{J}_i \cong \delta_i \cong \mathfrak{Y} \cong \Delta_i$$

which shows that $\Delta_i \cong \bar{\Delta}_i$. In fact we show below that

THEOREM 8.2.5. *Δ_i has tree class A_∞.*

which has an immediate

COROLLARY 8.2.6. *S_i and S_i^γ lie in isomorphic different components.*

Proof. If not then they clearly must lie at the end of an A_∞ component. The only possible graph automorphism of finite order has to be a translation (which of course has infinite order). Hence the component is a tube which is impossible.∎

Proof of Theorem. B_∞ and C_∞ cannot occur because if they did then the edge $\Omega S_i \to H$ is labelled (1,2) or (2,1). These are the dimensions of the space $\mathrm{Irr}(\Omega S_i, H)$ over the residue class fields of $\mathrm{End}_{KG}(\Omega S_i)$ and $\mathrm{End}_{KG}(H)$ respectively. Since both ΩS_i and H are absolutely indecomposable both endomorphism rings are <u>local</u>. Hence the dimensions must be equal.

A_∞^∞ cannot occur, again because the heart of $P(S_i)$ is indecomposable as a KG-module. We now show that D_∞ cannot occur.

Suppose the tree class is D_∞ and let H_i be the heart of S_i. Now part of the component looks like

$$\begin{tikzcd}
 & & Q_1 \arrow[drr, dash, dashed] & & \\
U \arrow[rr] \arrow[urr, dash, dashed] & & H_i \arrow[rr] \arrow[dr] & & V \\
 & \Omega S_i \arrow[ur] \arrow[dr] & & \Omega^{-1} S_i & \\
 & & P(S_i) \arrow[ur] & &
\end{tikzcd}$$

U and V are modules. We show that $\mathcal{A}(V)$ has no projective middle summand. For if it does then $\mathcal{A}(V)$ is

$$0 \to U \to H_i \oplus Q \to V \to 0$$

so that by an example in Section 6.1, Q is a projective indecomposable module with heart H_i. Let $Q=P(S_j)$. Now $U=\Omega S_j \ncong \Omega S_i$ so $Q \ncong P(S_i)$. But

$$\begin{aligned} |P|\dim S_j \le \dim Q &= \dim H_i + 2\dim S_j \\ &= |P|-2+2\dim S_j \end{aligned}$$

so that $|P|-2 \ge (|P|-2)\dim S_j$. So $\dim Q=|P|=\dim P(S_i)$ and $Q=P(S_i)\otimes S_j$, $H_i=H_i\otimes S_j$. There exists an AR-sequence

$$0 \to M \to H_i \oplus (Q\otimes S_j) \to N \to 0$$

with $Q\otimes S_j \ncong Q$. Since the tree class is D_∞ we need $Q\otimes S_j \cong P(S_i)$ and $S_j\otimes S_j \cong S_i$. Now S_j acts via $g\in\bar{G}$ as a scalar λ, and we have just shown $\lambda=-1$. So $\dim H_i$ is odd.

Let $G_0=P:\langle g\rangle$. $P(S_i)_{\downarrow G_0}$ and $Q_{\downarrow G_0}$ are indecomposable with isomorphic hearts. Under the algebraic closure $\bar{K}$, of K the irreducible representations of <g> are 1-dimensional. Tensoring with S_j now gives a fixed point free permutation of order 2 of these irreducibles hence

$$(*)\quad \left.\begin{array}{l} M \text{ semisimple } KG_0\text{-module} \\ M\otimes S_j \cong M \end{array}\right\} \Rightarrow 2 \mid \dim M$$

Observe that $P(S_i)$ is uniserial with its middle layer X of odd dimension. But X is also semisimple as a KG_0-module with $X\otimes S_j \cong X$ hence by (*) dim X is odd. It follows that $\mathcal{a}(V)$ is

$$0 \to U \to H_i \to V \to 0$$

with $U^* \cong V$ and in particular, dimU=dimV and dim H_i is odd, the final contradiction. Now using the fact that H_i is absolutely indecomposable we complete the proof.∎

Correction: We observe that the component $\Delta_i = \Delta \otimes S_i$ where Δ is the component containing K_G. It is then immediate that Δ_i has tree class A_∞.

CHAPTER 9

Periodic modules in the Auslander-Reiten quiver

In this final chapter we are concerned with periodic modules over certain group algebras. We know, from Theorem 6.2.3(iii) that such a periodic module, M say, lies in a κ-tube for some $\kappa>0$. By Theorems 1.1.2 and 6.2.3, $M^{\uparrow G}$ lies in a λ-tube, for some $\lambda\geq\kappa$. Hence, motivated by Example 9.1.1 below we are particularly interested in the case where $\lambda\gneq\kappa$; such tubes we call the *exceptional tubes*.

The first example is considered in Section 9.1. We take as our group a "mod p analogue" of the alternating group A_4 and construct a family of p-dimensional periodic modules which lie in 1-tubes. Theorem 9.1.4 gives a simple condition for these tubes to be exceptional. In Section 9.2 we aim to construct more periodic modules by considering pullback diagrams of the modules considered in Section 9.1. This uses some homological techniques, and the author is particularly grateful to Dave Benson for suggesting that consideration of the long exact sequence for Ext might bear fruit. Finally (Section 9.3) we very briefly sketch one possible way of constructing exceptional tubes for other group algebras.

9.1 Periodic modules of dimension p

Let $G=(C_p\times C_p):C_{p^2-1}$ as before and let k be a field of characteristic p. Instead of considering irreducible kG-modules, we concentrate on periodic modules (recall from Section 1.1 that if Ω is the Heller loop space operator then a module M is *periodic* if $\Omega^n M \cong M$ for some n; and that the least such n is called the *period* of M). The motivation for studying these modules comes from the following example:

EXAMPLE 9.1.1 ([7], [5, Appendix]). Let F be an algebraically closed field of characteristic 2 and let P be the Klein 4-group, so that $P=\langle x\rangle \times \langle y\rangle$. In the notation of [5], let $M= V_{1,\lambda}$ (where $\lambda \in \mathbb{P}^1(F)$), of dimension 2 and period 1. So $V_{1,\lambda}$ lies at the end of a 1-tube. M is the representation

$$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad y \mapsto \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \qquad (\lambda \neq \infty)$$

$$x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad y \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad (\lambda = \infty)$$

and it lies in a component of tree class A_∞:

$$V_{1,\lambda} \rightleftarrows V_{2,\lambda} \rightleftarrows V_{3,\lambda} \rightleftarrows \cdots\cdots$$

Now let $A=A_4=\langle x,y,z:\ x^2=y^2=z^2=1,\ xy=yx,\ x^z=y,\ y^z=xy\rangle \cong P{:}C_3$. Let T be the inertia group of M in A. Let ω, $\bar{\omega}$ be the primitive cube roots of unity in F. Then

PROPOSITION 9.1.2. *(i) If* $\lambda \in [\mathbb{P}^1(F)/\langle z\rangle] \setminus \{\omega,\bar{\omega},1\}$ *then* $T=P$, *and so by Lemma 5.4.1* $W_{1,\lambda}=M^{\uparrow A}$ *is indecomposable and lies in a 1-tube.*

(ii) If $\lambda \in \{1,\omega,\bar{\omega}\}$ *then M is A-stable and we have*

$$W_{1,\omega}(\lambda) = M^{\uparrow A} = \begin{matrix} 1 \\ \omega \end{matrix} \oplus \begin{matrix} \bar{\omega} \\ 1 \end{matrix} \oplus \begin{matrix} \omega \\ \bar{\omega} \end{matrix},$$

a direct sum of three uniserial modules of Loewy length 2, each of which lies in a 3-tube (and similarly with ω replaced by $\bar{\omega}$).

(iii)The components of $W_{1,\lambda}$ and $W_{1,\omega}(\lambda)$ are shown below.

$$W_{1,\lambda} \rightleftarrows W_{2,\lambda} \rightleftarrows W_{3,\lambda} \rightleftarrows \cdots\cdots$$

$$\begin{tikzcd}
W_{1,\omega}(1) \arrow[dr] & & W_{1,\omega}(\bar{\omega}) \arrow[dr] & & W_{1,\omega}(\omega) \arrow[dr] & & W_{1,\omega}(1) \\
& W_{2,\omega}(\omega) \arrow[ur] \arrow[dr] & & W_{2,\omega}(1) \arrow[ur] \arrow[dr] & & W_{2,\omega}(\bar{\omega}) \arrow[ur] \arrow[dr] & \\
W_{3,\omega}(1) \arrow[ur] & & W_{3,\omega}(\bar{\omega}) \arrow[ur] & & W_{3,\omega}(\omega) \arrow[ur] & & W_{3,\omega}(1) \\
& & & \vdots & & &
\end{tikzcd}$$

(and similarly for ω replaced by $\bar{\omega}$). ∎

Now consider the general situation in odd characteristic, mentioned at the beginning of this section. Assume that k contains all the p^2-1'th roots of unity. Then we want to provide "mod p" analogues of the results of Proposition 9.1.2. So we are interested in finding periodic kP-modules X lying in κ-tubes such that $X^{\uparrow G}$ has summands lying in λ-tubes where $\lambda \gneq \kappa$. Our arguments employ the language and methods of cohomology.

Notice first of all that we can view G as the group of upper triangular matrices in $SL_2(p^2)$ with $P=C_p \times C_p$ (and note that we generalise this in Section 9.3). The following result provides us with plenty of 1-tubes of $\Gamma_s(kP)$.

LEMMA 9.1.3. *Periodic kP-modules all have period at most 2, so all lie in 1-tubes.*

Proof. P is an abelian p-group, so just apply [8, Corollary 8.8].■

Let $P=\langle x\rangle\times\langle y\rangle$ and let

$$x \mapsto X = \begin{bmatrix} 1 & 1 & \cdot & & 0 \\ & 1 & \cdot & \cdot & \\ & & \cdot & \cdot & 1 \\ 0 & & & \cdot & 1 \end{bmatrix} = X_0+I_p$$

where $X_0 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \cdot & \cdot \\ & & \cdot & 1 \\ & & & 0 \end{bmatrix}$. Let $y \mapsto Y$. Since we require, amongst other things, a representation of P, Y must satisfy $XY=YX$ that is $Y\in C(X)$ so that from linear algebra

$$Y = I+\lambda_1 X_0+\ldots\ldots+\lambda_{p-1}X_0^{p-1}$$

is a polynomial in X_0. Observe also that $X_0=X-I$ and $Y_0=Y-I$ are strictly upper triangular and so $X^p=Y^p=I$. Hence

LEMMA 9.1.4. *Let $M=M(1,\lambda)$ be the module given by*

$$x \mapsto \begin{bmatrix} 1 & 1 & \cdot & & 0 \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & 1 \\ 0 & & & \cdot & 1 \end{bmatrix} \qquad y \mapsto \begin{bmatrix} 1 & \lambda_1 & \cdot & & \lambda_{p-1} \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \lambda_1 \\ 0 & & & \cdot & 1 \end{bmatrix}$$

Then M is a kP-module of dimension p.■

LEMMA 9.1.5. *If $M=M(1,\lambda)$, then M is periodic of period 2.*

Proof. Let $V_r(M)$ be the (rank) variety of M (as defined in [8, §5]). We show that $V_r(M)$ is a line. Using Carlson's notation [8, §5] we get

$$s = \left(\frac{p-1}{p}\right) \dim M = p-1,$$

and if $\alpha=(\alpha_1,\alpha_2)$,

$$A_\alpha = I+\alpha_1X_0+\alpha_2(\lambda_1X_0+\lambda_2X_0^2+\ldots\lambda_{p-1}X_0^{p-1})$$

This is free as a $k<u_\alpha>$-module if and only if $\mathrm{rank}(A_\alpha-I) = p-1$. So $\alpha\in V_r(M)$ if and only if every $(p-1)\times(p-1)$ submatrix of

$$A_\alpha-I = \begin{bmatrix} 0 & \alpha_1+\lambda_1\alpha_2 & \lambda_2\alpha_2 & \cdots & \lambda_{p-1}\alpha_2 \\ & 0 & \alpha_1+\lambda_1\alpha_2 & \ddots & \vdots \\ & & \ddots & \ddots & \lambda_2\alpha_2 \\ & & & \ddots & \alpha_1+\lambda_1\alpha_2 \\ & & & & 0 \end{bmatrix}$$

has determinant equal to 0. Now this occurs

$$\Leftrightarrow (\alpha_1+\lambda_1\alpha_2)^{p-1} = 0$$

$$\Leftrightarrow \alpha_1 = -\lambda_1\alpha_2$$

Hence there are two cases given as follows.

case 1: $\lambda_1=0$ Then $V_r(M) = \{0\}\cup\{(0,\alpha_2): \alpha_2\in k\}$, which is the affine line spanned by $(0,1)$.

case 2: $\lambda_1\neq0$ Then $V_r(M) = \{0\}\cup\{(-\lambda_1\alpha_2,\alpha_2): \alpha_2\in k\}$, and this time this is the affine line spanned by $(-\lambda_1,1)$.

This implies that $V_r(M)$ is the affine line spanned by $(-\lambda_1,1)$, which in turn implies that M is periodic. By Lemma 9.1.3 the period is 1 or 2. Now if $\Omega M\cong M$, then taking the projective cover

$$0 \to M \to kP \to M \to 0$$

and counting dimensions, $p^2=2p$ that is $p=2$ which is impossible.∎

Now we seek a classification of all periodic kP-modules of dimension p. Once again the rank variety plays a major rôle. Note also that for the modules given above we have:

LEMMA 9.1.6. *M is cyclic as a kP-module.*

Proof. The generator is $\zeta e_p+\mathfrak{x}$ where $\zeta\in k$, $\mathfrak{x}\in sp\{e_1,\ldots, e_{p-1}\}$ and $e_1,\ldots,e_p$ are the standard column vectors.∎

Henceforth the generator of M will be denoted by "v". Let M be an arbitrary periodic kP-module of dimension p.

LEMMA 9.1.7. *Either $M_{\downarrow\langle x\rangle}$ or $M_{\downarrow\langle y\rangle}$ is free.*

Proof. Suppose not. In Carlson's notation, write $x=1+1(x-1)+0(y-1)=u_\alpha$ where $\alpha=(1,0)\in V_r(M)$. Also $y=1+0(x-1)+1(y-1)=u_\beta$ where $\beta=(0,1)\in V_r(M)$. This contradicts the fact that $V_r(M)$ is a <u>single</u> line through the origin.∎

LEMMA 9.1.8. *Suppose that $M_{\downarrow\langle x\rangle}$ is free. Then a basis of M may be found, such that*

$$x\mapsto\begin{bmatrix}1 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & 1\end{bmatrix}\qquad y\mapsto\begin{bmatrix}1 & \lambda_1 & \cdots & \lambda_{p-1}\\ & \ddots & \ddots & \vdots\\ & & \ddots & \lambda_1\\ & & & 1\end{bmatrix}$$

where $\lambda_i\in k$. In other words, $M\cong M(1,\lambda)$.

Proof. The fact that $x \mapsto X_0+I_p$ is clear; since $Y \in C(X)$, Y is a polynomial in X.∎

THEOREM 9.1.9. *The periodic kP-modules of dimension p are precisely*

(i) $M(1,\lambda)$

(ii) $M(\mu,1)$

for any λ *and any* μ *with* $\mu_1=0$.

Proof. The modules are periodic, by Lemmas 9.1.4 and 9.1.5, and the two possibilities occur according as $M_{\downarrow <x>}$ is or is not free. The conditions on the parameters occur through comparing varieties.

Finally we find out when two polynomials define the same module. Suppose we have the representation (i) above together with the module $M(1,\mu)$

Make a change of basis using the nonsingular matrix T. As $T \in C(X)$, T is upper unitriangular. By comparing entries in the matrix equation

$$TY_\lambda = Y_\mu T$$

we get that $\lambda_i=\mu_i$ $(1 \le i \le p-1)$.∎

It is now our concern to show that the M's defined above provide us with some "exceptional tubes" (see the introduction to the chapter).

Recall that $G=P:Q$ where $P=C_p \times C_p=\langle x\rangle \times \langle y\rangle$ and $Q=C_{p^2-1}=\langle g\rangle$. Let ω be a primitive element of Q. Then we may write $g = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$, and we can take $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}$.

Hence

$$x^g = \begin{bmatrix} 1 & \omega^2 \\ 0 & 1 \end{bmatrix} \qquad y^g = \begin{bmatrix} 1 & \omega^3 \\ 0 & 1 \end{bmatrix} \qquad \ldots(*)$$

Now since $P \trianglelefteq G$ we also have that $x^g = x^\alpha y^\beta$, $y^g = x^\gamma y^\delta$. This gives

$$x^g = \begin{bmatrix} 1 & \alpha+\beta\omega \\ 0 & 1 \end{bmatrix} \qquad y^g = \begin{bmatrix} 1 & \gamma+\delta\omega \\ 0 & 1 \end{bmatrix} \qquad \ldots(**)$$

Comparing (*) with (**) we require

$$\left.\begin{aligned} \alpha+\beta\omega &= \omega^2 \\ \gamma+\delta\omega &= \omega^3 \end{aligned}\right\} \qquad (\dagger)$$

LEMMA 9.1.10. *Such $\alpha, \beta, \gamma, \delta$ always exist.*

Proof. If $\omega^2 = \alpha+\beta\omega$ then

$$\begin{aligned} \omega^3 &= \omega(\alpha+\beta\omega) \\ &= \alpha\omega+\beta(\alpha+\beta\omega) \\ &= \alpha\beta+(\alpha+\beta^2)\omega \end{aligned}$$

so we may take $\gamma=\alpha\beta$, $\delta=\alpha+\beta^2$. That $(\dagger)$ holds is because there always exists a polynomial $x^2+ax+b \in \mathbb{Z}_p[x]$ which is irreducible over $\mathbb{Z}_p$. ∎

Consider $M=M(1,\lambda)$ as above. If $\alpha, \beta, \gamma, \delta$ are as in Lemma 9.1.10 then

$$x^\alpha \mapsto \begin{bmatrix} 1 & \alpha & \binom{\alpha}{2} & \cdot & & & 1 \\ & 1 & \cdot & \cdot & & & \vdots \\ & & \cdot & \cdot & \cdot & & \vdots \\ & & & \cdot & \cdot & \cdot & \vdots \\ & & & & \cdot & \cdot & \alpha \\ & & & & & \cdot & 1 \end{bmatrix}$$

$$y^{\beta} \mapsto \begin{bmatrix} 1 & \beta\lambda_1 & \binom{\beta}{2}\lambda_1^2+\beta\lambda_2 & & \\ & 1 & \beta\lambda_1 & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & \end{bmatrix} = \begin{bmatrix} 1 & f_1(\beta,\lambda_1) & f_2(\beta,\lambda_1,\lambda_2) & \\ & 1 & f_1(\beta,\lambda_1) & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

where $f_r(\beta,\lambda_1,\ldots\lambda_r)$ has degree $\left\{\begin{array}{ll} r-s+1 & \text{in } \lambda_s \\ r & \text{in } \beta \end{array}\right.$. So we have, for $g\otimes M$,

$$x \mapsto x^{\alpha}y^{\beta} = \begin{bmatrix} 1 & \alpha+\beta\lambda_1 & \binom{\beta}{2}\lambda_1^2+\alpha\beta\lambda_1+\beta\lambda_2+\binom{\alpha}{2} & \\ & 1 & \alpha+\beta\lambda_1 & \\ & & 1 & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & g_1(\alpha,\beta,\lambda_1) & g_2(\alpha,\beta,\lambda_1,\lambda_2) & \\ & 1 & g_1(\alpha,\beta,\lambda_1) & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

where $g_r(\alpha,\beta,\lambda_1,\ldots\lambda_r)$ has degree $\left\{\begin{array}{l} r \text{ in } \alpha,\beta \\ r-s+1 \text{ in } \lambda_s \end{array}\right.$. The matrix for $y \mapsto x^{\gamma}y^{\delta}=\hat{y}$ is found by replacing α,β by γ,δ. Recall also that $\gamma=\alpha\beta$ and $\delta=\alpha+\beta^2$.

Under conjugation by the diagonal matrix

$$\mathrm{diag}((\alpha+\beta\lambda_1)^{p-1},\ldots,1),$$

$\hat{X}$ will have 1's on the diagonal and the superdiagonal. One can always conjugate inside the stabiliser of this flag, so we can conjugate to X again. This is best seen from an example:

EXAMPLE 9.1.11. Take p=3. We have

$$\begin{bmatrix} 1 & \alpha+\beta\lambda_1 & k=\binom{\beta}{2}\lambda_1^2+\beta\lambda_2+\alpha\beta\lambda_1+\binom{\alpha}{2} \\ 0 & 1 & \alpha+\beta\lambda_1 \\ 0 & 0 & 1 \end{bmatrix}^{diag((\alpha+\beta\lambda_1)^2,\alpha+\beta\lambda_1,1)} = \begin{bmatrix} 1 & 1 & (\alpha+\beta\lambda_1)^{-2}k \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = S;$$

and

$${}_{S}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1-(\alpha+\beta\lambda_1)^{-2}k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we have to see what happens to $\hat{Y}$ under these two conjugations. It is easy to see that this produces the matrix

$$\begin{bmatrix} 1 & (\alpha+\beta\lambda_1)^{-1}(\gamma+\delta\lambda_1) & (\alpha+\beta\lambda_1)^{-2}\{\binom{\delta}{2}\lambda_1^2+\gamma\delta\lambda_1+\delta\lambda_2+\binom{\gamma}{2}\} - \\ & & (\gamma+\delta\lambda_1)(\alpha+\beta\lambda_1)^{-3}\{\binom{\beta}{2}\lambda_1^2+\alpha\beta\lambda_1+\beta\lambda_2+\binom{\alpha}{2}\} \\ & 1 & (\alpha+\beta\lambda_1)^{-1}(\gamma+\delta\lambda_1) \\ & & 1 \\ & & & \ddots \end{bmatrix}$$

$$\begin{bmatrix} 1 & g_1^{-1}h_1(\gamma,\delta,\lambda_1) & g_1^{-2}h_2(\gamma,\delta,\lambda_1,\lambda_2) \\ & 1 & g_1^{-1}h_1(\gamma,\delta,\lambda_1) \\ & & 1 \\ & & & \ddots \end{bmatrix}$$

where the h_i are polynomials in $\gamma,\delta,\lambda_1,\ldots\lambda_{p-1}$ and $g_1^{-1}=(\alpha+\beta\lambda_1)^{-1}$.

EXAMPLE 9.1.12. Take $p=3$. Let $\alpha+\beta\lambda_1=t$. Then

$$\begin{bmatrix} 1 & \gamma+\delta\lambda_1 & \psi = \binom{\delta}{2}\lambda_1^2+\delta\lambda_2+\gamma\delta\lambda_1+\binom{\gamma}{2} \\ 0 & 1 & \gamma+\delta\lambda_1 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{diag}((\alpha+\beta\lambda_1)^2,\alpha+\beta\lambda_1,1)} = \begin{bmatrix} 1 & t^{-1}(\gamma+\delta\lambda_1) & t^{-2}\psi \\ 0 & 1 & t^{-1}(\gamma+\delta\lambda_1) \\ 0 & 0 & 1 \end{bmatrix} = T$$

and

$$T^{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1-t^{-2}\kappa \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & t^{-1}(\gamma+\delta\lambda_1) & t^{-2}\psi - t^{-3}\kappa(\gamma+\delta\lambda_1) \\ 0 & 1 & t^{-1}(\gamma+\delta\lambda_1) \\ 0 & 0 & 1 \end{bmatrix}$$

So we have an upper triangular algebraic action on the space of λ's (given by the h's), which constitutes a rational representation of $\mathbb{Z}/(p^2-1)$. We require to know what the fixed space is, and our analysis produces

THEOREM 9.1.14. *There are finitely many G-stable kP-modules of the above form provided that λ_1 is a p^2-1'th root of unity, and the coefficients of λ_i in h_i $(i>1)$ are nontrivial.*

Remark. The question of finding other "exceptional tubes" is considered in Section 9.2.

Proof. The fixed space is given by solving the equations $\{h_r(\lambda_1,\ldots\lambda_r)=\lambda_r\}$, which of course is always possible by algebraic closure.

Begin by solving the equation

$$h_1(\lambda_1)=\lambda_1$$

that is

$$\lambda_1\delta+\gamma = \lambda_1(\lambda_1\beta+\alpha)$$

This produces the equation $\beta(\lambda_1{}^2-\beta\lambda_1-\alpha)=0$ upon substituting for γ and δ. Since $\beta\neq 0$ (else $g\in C(P)$) $\lambda_1=\omega$ or $\bar{\omega}$, which is a p^2-1'th root of unity, by equation (†) above.

Now continue to solve the equations $h_r(\lambda_1,\ldots\lambda_r)=\lambda_r$ to obtain but a finite number of G-stable modules.∎

EXAMPLE 9.1.15. Take $p=3$. The first equation is $\lambda_1{}^2=\alpha+\beta\lambda_1$. The second equation, linear in λ_2 is

$$\lambda_2 = \frac{\binom{\delta}{2}\lambda_1{}^2+\gamma\delta\lambda_1+\binom{\gamma}{2}-\lambda_1\{\binom{\beta}{2}\lambda_1{}^2+\alpha\beta\lambda_1+\binom{\alpha}{2}\}}{\lambda_1{}^4+\beta\lambda_1-\delta}$$

This gives a unique value of λ_2 for each 8'th root of unity λ_1.

Any G-stable M will be such that $M^{\uparrow G}$ lies in a $\frac{1}{2}(p^2-1)$-tube:

LEMMA 9.1.16. *Let M be a module given in Theorem 9.1.9, and let it be G-stable as in Theorem 9.1.14. Let $S(\zeta)$ be the simple kG-module corresponding to the p^2-1'th root of unity ζ. Then $M^{\uparrow G}$ is a direct sum of p uniserial modules of Loewy length p:*

$$M^{\uparrow G} = \begin{array}{c} S(1) \\ S(\lambda_1) \\ \vdots \\ S(\lambda_1{}^{p-1}) \end{array} \oplus \ldots\ldots \oplus \begin{array}{c} S(\lambda_1^{p^2-p}) \\ S(\lambda_1^{p^2-p+1}) \\ \vdots \\ S(\lambda_1^{p^2-1}) \end{array}$$

Proof. Clear from the above discussion.∎

9.2 *Pullbacks of periodic modules*

Suppose that M_1, M_2 are two p-dimensional periodic kP-modules which lie in exceptional tubes (in the sense of the previous section). Let $\phi: M_1 \to M_2$ be a kP-homomorphism and let $\psi: P(M_1) \to M_1$ be the projective cover of M_1. Let E be the submodule of $P(M_1)\oplus M_2$ consisting of those elements (x,y) where x and y have the same image in M_1 (the *pullback* of ϕ and ψ). In this section we aim to show that, under certain conditions, E also lies in an exceptional tube.

First we need a little more notation. In general, if X and Y are kG-modules, set

$$H = (X,Y) = \mathrm{Hom}_k(X,Y)$$

Write $F= (X,Y)^G = \mathrm{Hom}_{kG}(X,Y)$ for the set of fixed points of G on H. Recall from [24, II.2.7] that $\phi\in F$ is *1-projective* if it can be factored through P(X). We denote the ideal (in F) of 1-projective maps by $(X,Y)^G_1$. Denote the quotient by $(X,Y)^{1,G}$.

In what follows we need to understand the space $(M_1,M_2)^P$ where $M_1=M(1,\lambda)$ and $M_2=M(1,\mu)$. This is covered in the next couple of results. Recall first that

LEMMA 9.2.1. *If* $M_i=\langle v_i\rangle$ *then a k-basis of* M_i *is* $\{v_i, (x-1)v_i,\ldots,(x-1)^{p-1}v_i\}$ *where* $i=1,2$.∎

Let $\phi\in(M_1,M_2)$. As $M_i=\langle v_i\rangle$, ϕ is completely determined by $m=\phi(v_1)$. Now

$$m = \sum_{i=0}^{p-1} \xi_i (x-1)^i v_2 \tag{1}$$

for some $\xi_i \epsilon k$ which depend on ϕ of course. Notice that, ϕ extends linearly to a unique k<x>-homomorphism. We seek conditions under which it is a k<y>-homomorphism also (and hence $\phi \epsilon\ (M_1, M_2)^P$). We require

$$\phi((y-1)v_1) = (y-1)\phi(v_1). \tag{2}$$

Now, we expand both sides

$$\text{L.H.S.} = \phi\Big(\sum_{j=1}^{p-1} \lambda_j (x-1)^j v_1 \Big)$$

$$= \sum_j \lambda_j (x-1)^j m$$

$$= \sum_j \sum_{i=0}^{p-1} \lambda_j \xi_i (x-1)^{i+j} v_2 \tag{3}$$

Also

$$\text{R.H.S.} = (y-1)m$$

$$= (y-1)\Big(\sum_i \xi_i (x-1)^i v_2 \Big)$$

$$= \Big(\sum_{\kappa=1}^{p-1} \sum_i \mu_\kappa \xi_i (x-1)^{i+\kappa}\Big) v_2$$

By equating R.H.S. and L.H.S. we require

LEMMA 9.2.2. *With the above notation, in order that* $\phi\in(M_1,M_2)^P$, *we require*:

$$\sum_{\nu=0}^{p-1}\Big(\sum_{i=0}^{p-1}\xi_i\lambda_{\nu-i}\Big).(x-1)^{\nu}v_2 = \sum_{\nu=0}^{p-1}\Big(\sum_{i=0}^{p-1}\xi_i\mu_{\nu-i}\Big).(x-1)^{\nu}v_2$$

where λ_0, $\mu_0=0$.■

The system of equations in ξ,λ,μ which this produces will be used in

LEMMA 9.2.3. *Let* M_i *be as above. Then*

$$\dim_k(M_1,M_2)^P = r+1$$

where r *is maximal such that* $\lambda_i=\mu_i$ $(1\le i\le r)$ *and* $\lambda_{r+1}\neq\mu_{r+1}$.

Proof. For $0\le j\le r$ we define maps

$$\begin{aligned}\phi_j\colon M_1 &\to M_2\\ v_1 &\mapsto (x-1)^{p-r+j-1}\,v_2\end{aligned}$$

which clearly extends to a k-homomorphism. Notice that, in the notation of (1) above,

$$\xi_i(\phi_j) = \begin{cases}0, & i\neq p-i+j-1\\ 1, & i=p-i+j-1\end{cases}.$$

Both sides of the identity in Lemma 9.2.2 reduce to

$$\sum_{\alpha=1}^{r}\lambda_\alpha(x-1)^{p+\alpha-r-1+j}v_2$$

for each j, and so the ϕ_j are kP-homomorphisms. Clearly they form a linearly independent set.

Conversely, let $\phi\in(M_1,M_2)^P$ be arbitrary. Solving the equations arising from Lemma 9.2.2 it is easy to show that

$\xi_i(\phi)=0$ if $i\neq p-r-1+j$ and hence $\phi\in sp\{\phi_0,\ldots\phi_r\}$. So the ϕ_j form a basis as required.■

Suppose now that $\phi\colon M_2 \to M_1$ where the M_i are of the above form. Let $\psi\colon P(M_1) \to M_1$ be the projective cover of M_1. Let $E=E_\phi$ be the pullback of (ϕ,ψ) with projections (π_1,π_2):

$$\begin{tikzcd}
E_\phi \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\phi"] \\
P(M_1) \arrow[r, "\psi"'] & M_1
\end{tikzcd}$$

...(*)

In order that the upper sequence be nonsplit we require that ϕ is not 1-projective. This follows from

PROPOSITION 9.2.4. *Let M_i be as above. Assume that $\lambda_1=\mu_1$. Then ϕ is not 1-projective.*

Proof. By [24, 6.11], $d=\dim_k(M_1,M_2)_1^P$ equals the number of summands of $P(k_P)$ as a direct summand of $M_1^*\otimes M_2$. Now $\dim_k(M_1^*\otimes M_2)=p^2$ so that $d\leq 1$. Hence we have to investigate when $d=1$. In this case ,

$$\begin{aligned}(M_1,M_2)^P &= (M_1^*\otimes M_2)^P \\ &= soc(M_1^*\otimes M_2)\end{aligned}$$

This is 1-dimensional because $M_1^*\otimes M_2\cong kP$. Hence $\dim_k(M_1,M_2)=1$, and by Lemma 9.2.3 $\lambda_1\neq\mu_1$.■

We now have to show that E_ϕ is indecomposable.

THEOREM 9.2.5. *In the above notation, if $\lambda_1=\mu_1$ then E_ϕ*

is indecomposable.

Proof. From (*) we have a s.e.s.

$$0 \to E_\phi \to P(M_1)\otimes M_2 \to M_1 \to 0$$

Write $M_1=M(1,\lambda)$, and $M_2=M(1,\mu)$. Notice that by Lemma 9.2.3 $\dim_k(M(1,\mu),\ M(1,\lambda)^P = 1+(\#\text{ initial } \lambda_i=\#\text{ initial } \mu_i)$ and denote this number by r so that $1<r<p$. Using the fact that the M_i are periodic of period 2, the l.e.s. for Ext can be written as

$$\begin{tikzcd}
& \mathrm{Ext}^1(M_1,M_2) \arrow[r, "\phi_1^*"] & \mathrm{Ext}^1(M_2,M_1) \arrow[dr, "\pi_1^*"] & \\
\mathrm{Ext}^2(E,M_1) \arrow[ur, "\partial"] & & & \mathrm{Ext}^1(E,M_1) \arrow[dl, "\partial"] \\
& \mathrm{Ext}^2(M_2,M_1) \arrow[ul, "\pi_2^*"] & \mathrm{Ext}^2(M_1,M_1) \arrow[l, "\phi_2^*"'] &
\end{tikzcd}$$

Here the ϕ_j^* and the π_j^* $(j=1,2)$ are the induced maps (in dimension j), and the ∂'s are the connecting homomorphisms.

Let $s=\dim(\mathrm{im}\phi)$ so that $1\le s<r$. The claim is that $\mathrm{Ext}^i(N,M_1)\neq 0$ for $N\|E$ and all i. This follows from the more general lemma proved below.

LEMMA 9.2.6. *Let* X *and* Y *be modules for a p-group* P, *and suppose that* $X^*\otimes Y$ *is not projective. Suppose that* $V(X)\cap V(Y)\neq\{0\}$. *Then* $\mathrm{Ext}^i_{kP}(X,Y)\neq 0$ *for all* i.

Proof (due to D. Benson). Observe that $\mathrm{Ext}^i(X,Y)\cong \mathrm{Ext}^i(X^*\otimes Y,k)$. Construct a <u>minimal</u> projective resolution of

$X^{*}\otimes Y$ as

$$P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \rightarrow X^{*}\otimes Y \rightarrow 0$$

As $X^{*}\otimes Y$ is non-projective, all the $P_i \neq 0$, and by minimality all the ∂'s map into the radical of P_{i-1}. Hence any map

$$\begin{tikzcd} P_i \arrow[r, "\partial"] \arrow[dr] & P_{i-1} \arrow[d] \\ & k \end{tikzcd}$$

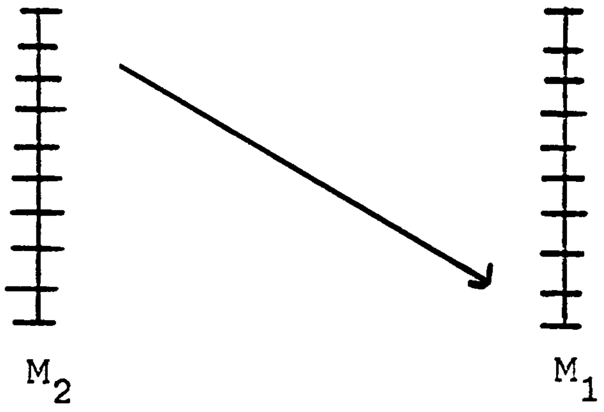
which is nonzero composed with the boundary map is nonzero. However $\mathrm{Ext}^{i}(X,Y)=\mathrm{Hom}(P_i,k)\neq 0$.∎

Now we can finish the proof:

Begin by constructing the s.e.s.

$$0 \rightarrow \mathrm{coker}\phi^{*} \rightarrow \mathrm{Ext}^{i}(E,M_1) \rightarrow \mathrm{ker}\phi^{*} \rightarrow 0 \qquad \ldots(**)$$

We show that $\phi_2{}^{*}$ is surjective. Recall that $\mathrm{Ext}^2_{kP}(M_i,M_j) \cong (\Omega^2 M_i,M_j)^{1,P}$ so that the map $\phi_2{}^{*}$ is a map End $M_1 \rightarrow (M_2,M_1)$ given by composition. The domain and codomain are uniserial:



and hence $\phi_2{}^{*}$ must be onto.

It now follows from (**) that $\mathrm{Ext}^2(E, M_1) = \mathrm{ker}\phi^{*}$. But this latter space is indecomposable as a module over End M_1, which is just a p-dimensional local ring. It follows that E is indecomposable.∎

We are now left with the problem of determining when two maps $M_2 \to M_1$ give isomorphic E's.

THEOREM 9.2.6. *Let* $\phi_1, \phi_2 : M_1 \to M_2$ *be such that they are not 1-projective. Then, in the following list, (i)⇒(ii)⇒(iii).*

(i) $im\phi_1 = im\phi_2$ *(which holds if and only if the images have the same dimension).*

(ii) There exists an $\alpha \in Aut\ M_2$ *such that* $\phi_2\alpha = \phi_1$

$$\begin{tikzcd}
M_2 \arrow[r, "\alpha"] \arrow[d, "\phi_1"'] & M_2 \arrow[dl, "\phi_2"] \\
M_1 &
\end{tikzcd}$$

(iii)If E_{ϕ_i} *is the extension determined by* ϕ_i, *then*

$$E_{\phi_1} \cong E_{\phi_2}$$

Proof. (ii)⇒(iii). This is clear because the map

$$\begin{aligned} \Phi : E_{\phi_1} &\longrightarrow E_{\phi_2} \\ (x,y) &\longmapsto (x,\alpha y) \end{aligned}$$

provides the isomorphism. (Note that if there were a commutative square

$$\begin{tikzcd}
M_2 \arrow[r, "\alpha"] \arrow[d, "\phi_1"'] & M_1 \arrow[d, "\phi_2"] \\
M_1 \arrow[r, "\beta"'] & M_1
\end{tikzcd}$$

then the same conclusion would hold, the isomorphism being given by $(x,y) \mapsto (x, \beta^{-1}\phi_2\alpha y)$).

$(i) \Rightarrow (ii)$. If the image spaces have dimension 1 then the ϕ are scalar multiples hence invertible. So $\alpha = \phi_2^{-1}\phi_1$ will do. Assume the image space has dimension >1.

Let $M_i = \langle e_i \rangle$, where $e_1 = \phi_1(e_2)$. Let $m \in M_2$ so that $m = ce_2 + \mathfrak{x}$ where $\mathfrak{x} \in e_2J$ is nilpotent. As $e_1 \in im\phi_1$, then by assumption

$e_1=\Phi_2(m)$ for some $m\in M_2$ of the above form. Define a map

$$\Psi:\ e_2 \mapsto m$$

If $z\in kP$ then e_2z is the general element of M_2 so we define α by

$$\alpha(e_2z)=mz.$$

For the desired result, we have to show that it is well-defined.

LEMMA 9.2.7. *(i) Let x be nilpotent. Then 1+x is a unit and so the sum of a nilpotent element and a unit is a unit.*

(ii) α is well-defined, in the sense that

$$e_2z=0 \Rightarrow mz=0$$

Proof. (i) This is trivial.

(ii) Suppose that $e_2z=0$. Now $m=ce_2+\mathfrak{X}$. As $\mathfrak{X}\in e_2J$ (the radical) we can write $m=ce_2+re_2=(c+r)e_2$ where $r\in J$ is nilpotent. By (i) $c+r$ is a unit u say. Hence $m=e_2u$.

In this case,

$$\begin{aligned} mz &= (ce_2+\mathfrak{X})z \\ &= e_2uz \\ &= e_2zu \\ &= 0. \end{aligned}$$

This proves the result.∎

COROLLARY 9.2.8. *In the notation of the above theorem, we obtain $\dim(\operatorname{im}\Phi_i)$ non-isomorphic extensions.*∎

9.3 *Periodic modules of other group algebras*

In Sections 9.1 and 9.2 we considered the group G which had a quotient $G/P \cong C_{p^2-1}$. Notice that C_{p^2-1} occurs as a subgroup of $SL_2(p)$. So one can naturally ask for analogues of the results of the previous sections in the case when G/P is some arbitrary subgroup of $SL_2(p)$. Note that all such subgroups have been classified in [37, 3.§6].

Now $Z(SL_2(p)) \cong C_2 = \langle\tau\rangle$ say. Then τ is an involution and $\tau=-I_2$. If $P=\langle x\rangle\times\langle y\rangle$ then τ inverts x and y, so that the action of C_2 on P is well-understood. Hence

$$x^\tau = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad y^\tau = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}$$
$$=x^{-1} \qquad\qquad = y^{-1}$$

so that effectively this is just a special case of Lemma 9.1.10 (with $\alpha=\delta=-1$ and $\beta=\gamma=0$).

Let M be one of the periodic kP-modules of dimension p appearing in Theorem 9.1.9. Let $M\otimes\tau$ be given by matrices with entries expressed in terms of the h_i, g_i as before. Then

THEOREM 9.3.1. *Provided that the coefficients of* λ_i *in* h_i *(*$i>1$*) are nontrivial there are infinitely many G-stable kP-modules M.*

This has the immediate

COROLLARY 9.3.2. *These G-stable M lie in exceptional tubes.*

Proof. The method is identical to that in Section 9.1. In the case given here, there is no condition on λ_1 to be satisfied. (In fact one should notice that this occurs precisely when λ_1 satisfies

$$\beta\lambda_1^2+(\alpha-\delta)\lambda_1-\gamma=0).\blacksquare$$

EXAMPLE. Take p=3. Then in Example 9.1.15, for any λ_1 we can put $\alpha=\delta=-1$, $\beta=\gamma=0$ to solve for λ_2. We get $\lambda_2=\lambda_1-\lambda_1^2$ and hence obtain an infinite number of exceptional tubes.

APPENDICES

APPENDIX A

This appendix contains the necessary figures alluded to in Chapter 4, Section 4.3.

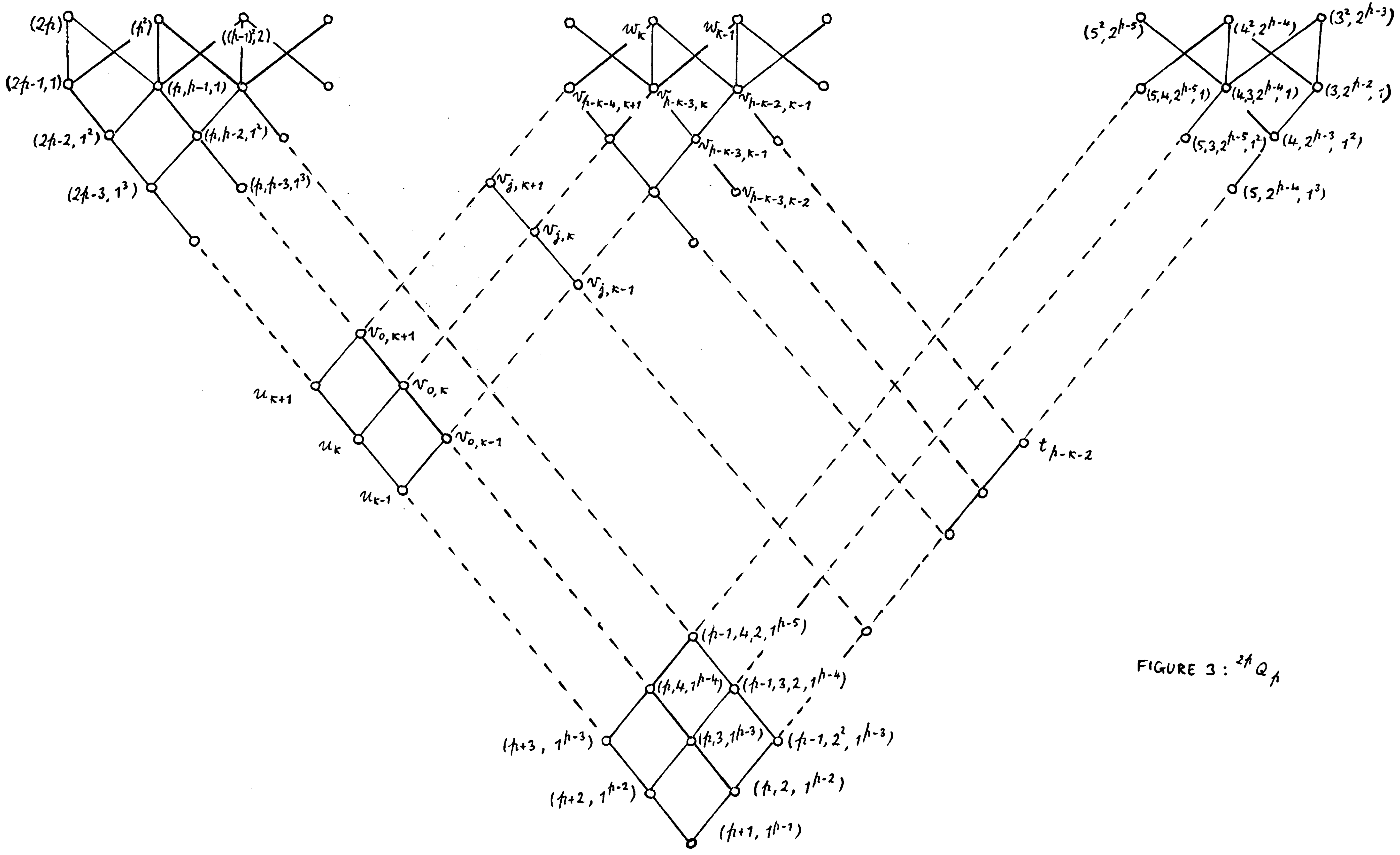


FIGURE 3 : ${}^{2h}Q_h$

FIGURE 4 : ${}^{10}Q_5$

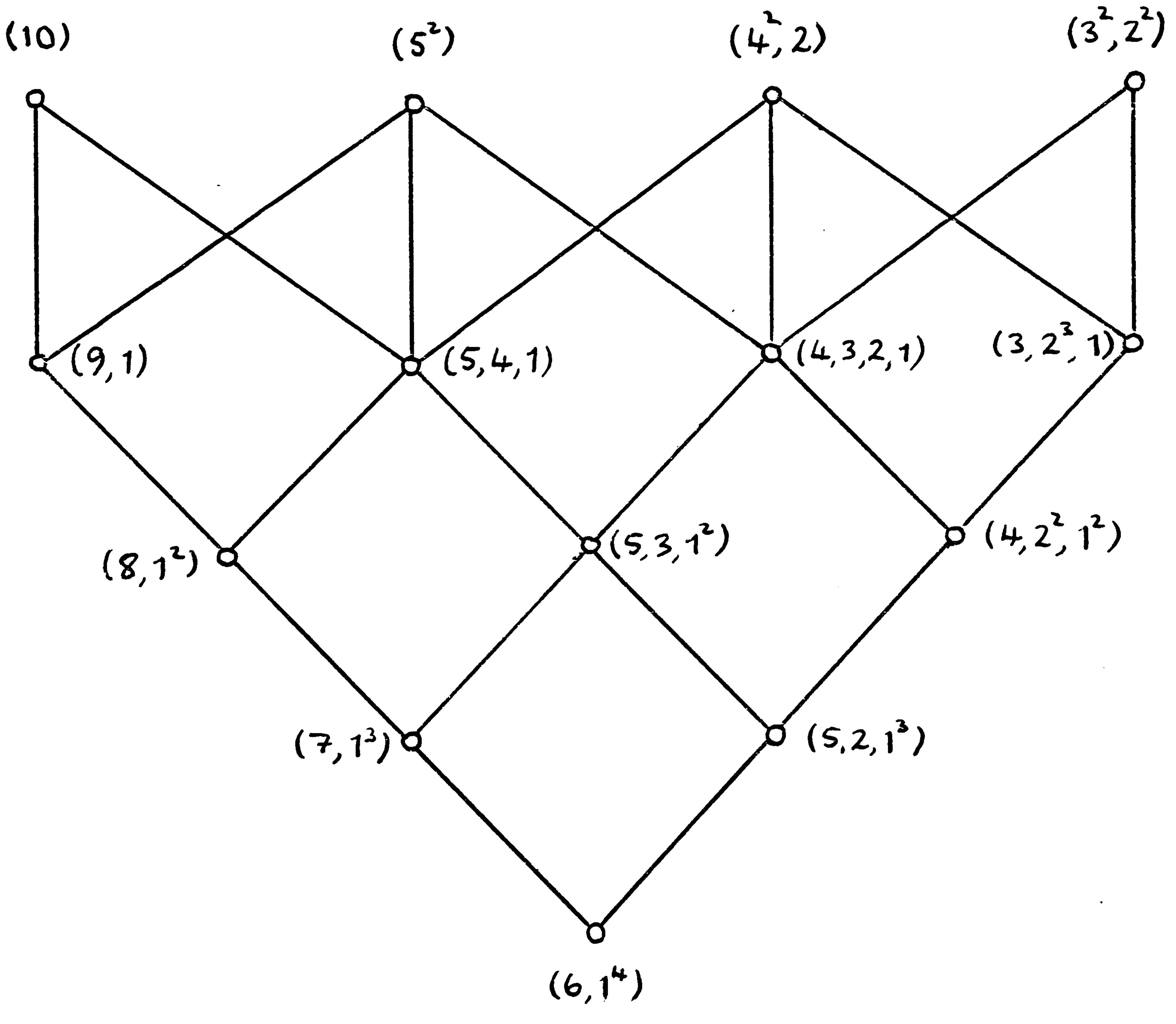
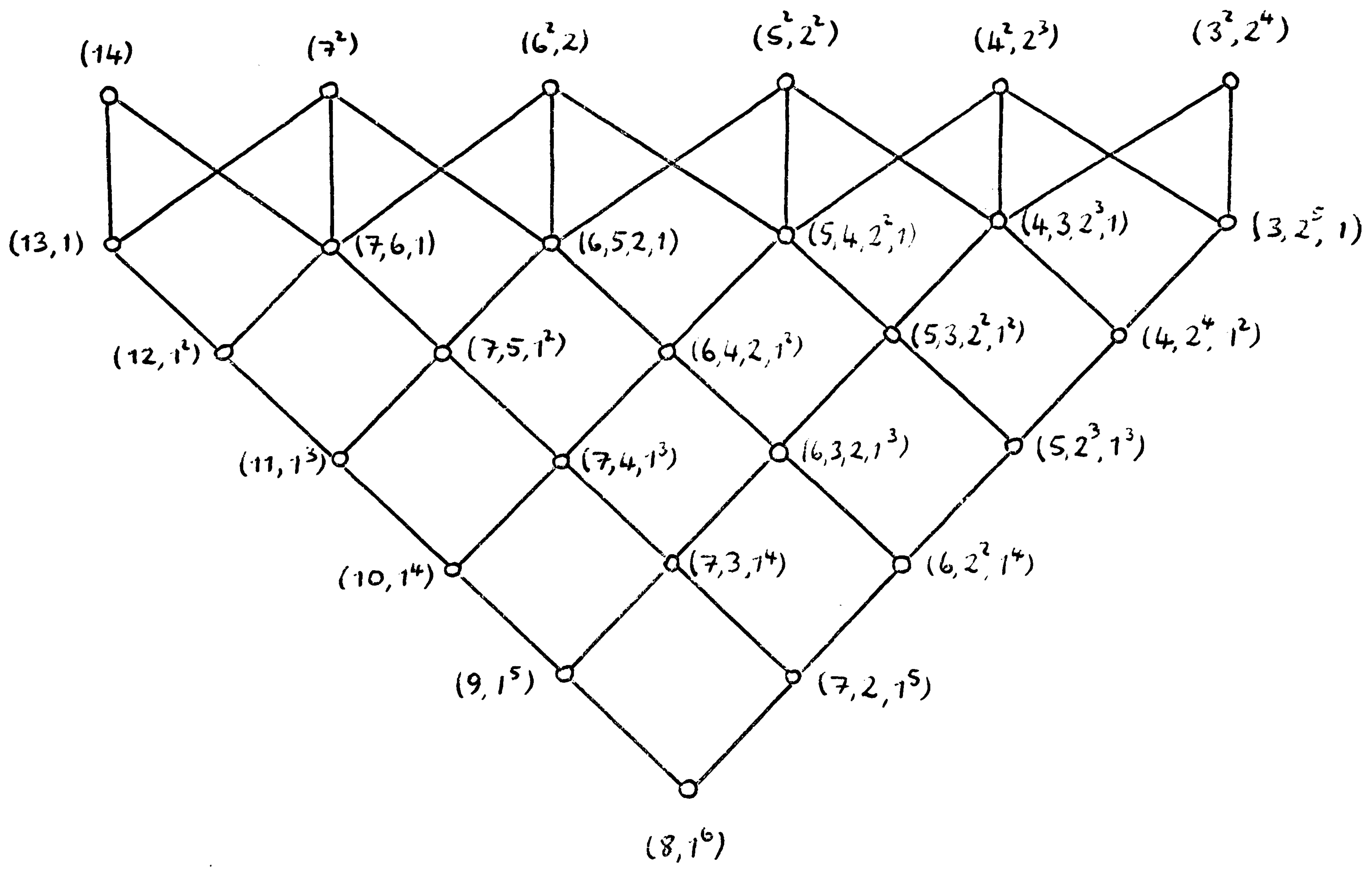


FIGURE 5 : ${}^{14}Q_7$



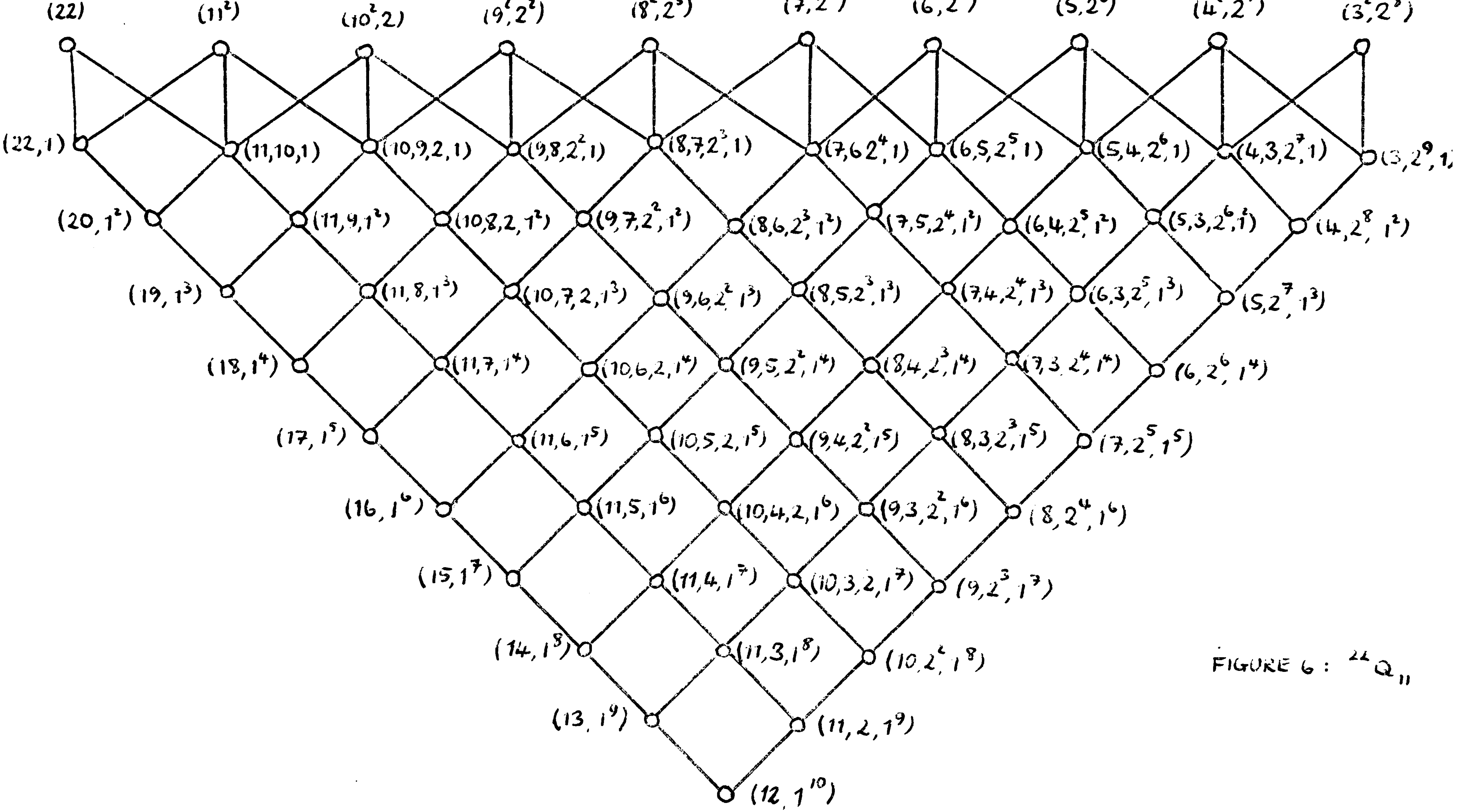


FIGURE 6: ${}^{22}Q_{11}$

APPENDIX B

The required material in [34] is still not readily available. This appendix is designed to give a brief indication as to how to calculate decomposition numbers using this material and to give a few examples. There is a useful summary of the work in [6].

B.1 General Theory

We use the notation and conventions of Part I. Recall that there is a natural symmetric bilinear form on M^λ (length 1, inner product 0) and this is inherited by S^λ (here $\lambda \vdash n$ has s parts and n is arbitrary). Let

$$S^\lambda(\alpha) = \{x \in S^\lambda : \forall y \in S^\lambda,\ p^\alpha \mid \langle x,y\rangle \}$$

which is clearly a submodule of S^λ. Filter $\overline{S^\lambda}=\overline{S^\lambda(0)}$ as

$$\overline{S^\lambda} \supseteq \overline{S^\lambda(1)} \supseteq \overline{S^\lambda(2)} \supseteq \ldots \supseteq 0$$

where $\overline{S^\lambda(\alpha)}$ is the image of $S^\lambda(\alpha)$ in $\overline{S^\lambda}$ under reduction modulo p. We comment that this is a departure from the previous use of the $\bar{\ }$ notation in Part I. Then Schaper's formula gives the values of the Brauer characters

$$\chi(\overline{S^\lambda(1)}) + \chi(\overline{S^\lambda(2)}) + \ldots$$

$$=\chi(\overline{S^\lambda(1)}/\overline{S^\lambda(2)}) + 2\chi(\overline{S^\lambda(2)}/\overline{S^\lambda(3)}) + 3\ldots$$

in terms of ordinary irreducibles.

Let ν_p be the p-adic valuation on integers (so that $\nu_p(p^\alpha m)=\alpha$ if $p \nmid m$). Recall also that if (x,y) is a node in the hook graph of λ, we denoted the hook length by $h^{\lambda}_{x,y}$. We have

THEOREM B.1.1 (Schaper, [34]). *The Brauer character*

$$\sum_{r>0} \chi(\overline{S^{\lambda}(r)})$$

agrees on p-regular conjugacy classes with the ordinary character

$$\sum_{1\le i<j\le s} \sum_{\ell=1}^{\lambda_j} \left(\nu_p(h^{\lambda}_{i\ell}) - \nu_p(h^{\lambda}_{j\ell})\right) \times$$
$$\chi(h^{\lambda}_{11}, h^{\lambda}_{21}, \ldots, h^{\lambda}_{i-1,1}, h^{\lambda}_{i1}+h^{\lambda}_{j\ell}, h^{\lambda}_{i+1,1}, \ldots, h^{\lambda}_{j1}-h^{\lambda}_{j\ell}, \ldots h^{\lambda}_{s1}). \blacksquare$$

See also [23] which gives an interesting forerunner of this result.

To apply Theorem B.1.1, suppose we have calculated the decomposition numbers for partitions dominated by λ. Then the formula tells us whether the decomposition numbers for λ are zero and gives an upper bound for those which are nonzero.

B.2 Examples

EXAMPLE B.2.1 ([23, §3]). Take $(5,4,3,2)\vdash 14$ and $p=5$.

$$H^{\lambda} = \begin{array}{lllll} 8 & 7 & 5 & 3 & 1 \\ 6 & 5 & 3 & 1 & \\ 4 & 3 & 1 & & \\ 2 & 1 & & & \end{array}$$

is the hook graph. The contribution from the second column can be found as from the Young diagram:

```
x x x x x
x x x-x
x x-x
x x
```

The skew (2,2)-hook is marked. Unwrapping this, then wrapping it on higher in the diagram gives

```
x x x x x x-x-x-x-x        x x x x x x        x x x x x
x x                        x x x-x-x-x        x x x-x-x
x                          x                  x x-x
x                          x                  x
```

Hence our character sum is

$$-\chi^{(10,2,1^2)}+\chi^{(6^2,1^2)}+\chi^{(5^2,3,1)}$$

where we have a positive sign if and only if the sum of the leg lengths of the skew hook and the new skew hook is odd.

EXAMPLE B.2.2. Consider the matrix $^{10}D_5$ (Figure 7), and suppose we have calculated, by some method, the first six rows of the matrix. We require factors of $\overline{S^{(5,4,1)}}$. Now $H^\lambda=$

7 5 4 3 1 which gives a character sum
5 3 2 1
1

$$\chi^{(8,1^2)}+\chi^{(10)}+\chi^{(5^2)}$$

which is

$$2\phi_{(9,1)}+\phi_{(8,1^2)}+\phi_{(10)}+\phi_{(5^2)}$$

upon reduction mod 5. The only dubiety lies with $D^{(9,1)}$. But $(\chi^{(9,1)})_{\Sigma_9}=\chi^{(8,1)}+\chi^{(9)}$ which is $\phi_{(9)}+\phi_{(8,1)}$ and hence the multiplicity is 1 not 2. This fills in the factors of the reduction of $S^{(5,4,1)}$. One can repeat this argument to obtain the complete matrix.

Remarks. (*i*) As mentioned in Chapter 2, while there are stacks of *ad hoc* methods for calculating ^nD_p, (for example see the references cited at the end of Section 2.3), there is no published algorithm which will work for all values of n. However the method outlined in Section B.1 seems to work exceptionally well for the cases we're interested in.

(*ii*) Two tables are given below (Figures 7&8) determined as in Section B.2. We give merely the p-regular part since the p-singular part is easily found from this using the character table.

FIGURE 7 : ${}^{10}D_5$

		1	8	28	56	70	34	217	266	56	34	217	28	1	8
		(10)	(9,1)	$(8,1^2)$	$(7,1^3)$	$(6,1^4)$	(5^2)	(5,4,1)	$(5,3,1^2)$	$(5,2,1^3)$	$(4^2,2)$	(4,3,2,1)	$(4,2^2,1^2)$	$(3^2,2^2)$	$(3,2^3,1)$
1	(10)	1													
9	(9,1)	1	1												
36	$(8,1^2)$		1	1											
84	$(7,1^3)$			1	1										
126	$(6,1^4)$				1	1									
42	(5^2)		1				1								
288	(5,4,1)	1	1	1			1	1							
567	$(5,3,1^2)$			1	1			1	1						
448	$(5,2,1^3)$				1	1			1	1					
252	$(4^2,2)$	1						1			1				
768	(4,3,2,1)						1	1	1		1	1			
567	$(4,2^2,1^2)$								1	1		1	1		
252	$(3^2,2^2)$						1					1		1	
288	$(3,2^3,1)$										1	1	1	1	1

FIGURE 8 : $^{14}D_7$

		1	12	66	220	495	792	924	417	4080	11078	13167	6996	792	2354	18679	26574	13167	495	2354	18679	11078	220	417	4080	66	1	12
		(14)	$(13,1)$	$(12,1^2)$	$(11,1^3)$	$(10,1^4)$	$(9,1^5)$	$(8,1^6)$	(7^2)	$(7,6,1)$	$(7,5,1^2)$	$(7,4,1^3)$	$(7,3,1^4)$	$(7,2,1^5)$	$(6^2,2)$	$(6,5,2,1)$	$(6,4,2,1^2)$	$(6,3,2,1^3)$	$(6,2^2,1^4)$	$(5^2,2^2)$	$(5,4,2^2,1)$	$(5,3,2^2,1^2)$	$(5,2^3,1^3)$	$(4^2,2^3)$	$(4,3,2^3,1)$	$(4,2^4,1^2)$	$(3^2,2^4)$	$(3,2^5,1)$
1	(14)	1																										
13	$(13,1)$	1	1																									
78	$(12,1^2)$		1	1																								
286	$(11,1^3)$			1	1																							
715	$(10,1^4)$				1	1																						
1287	$(9,1^5)$					1	1																					
1716	$(8,1^6)$						1	1																				
429	(7^2)		1						1																			
4576	$(7,6,1)$	1	1	1					1	1																		
15444	$(7,5,1^2)$			1	1					1	1																	
24960	$(7,4,1^3)$				1	1					1	1																
21450	$(7,3,1^4)$					1	1					1	1															
9504	$(7,2,1^5)$						1	1					1	1														
6435	$(6^2,2)$	1								1					1													
36608	$(6,5,2,1)$								1	1	1				1	1												
69498	$(6,4,2,1^2)$										1	1				1	1											
59904	$(6,3,2,1^3)$											1	1				1	1										
21450	$(6,2^2,1^4)$												1	1				1	1									
21450	$(5^2,2^2)$								1							1				1								
68640	$(5,4,2^2,1)$														1	1	1			1	1							
69498	$(5,3,2^2,1^2)$																1	1			1	1						
24960	$(5,2^3,1^3)$																	1	1			1	1					
21450	$(4^2,2^3)$														1						1			1				
36608	$(4,3,2^3,1)$																			1	1	1		1	1			
15444	$(4,2^4,1^2)$																					1	1		1	1		
6435	$(3^2,2^4)$																			1					1		1	
4576	$(3,2^5,1)$																							1	1	1	1	1

APPENDIX C

This appendix contains the necessary figures alluded to in Chapter 5.

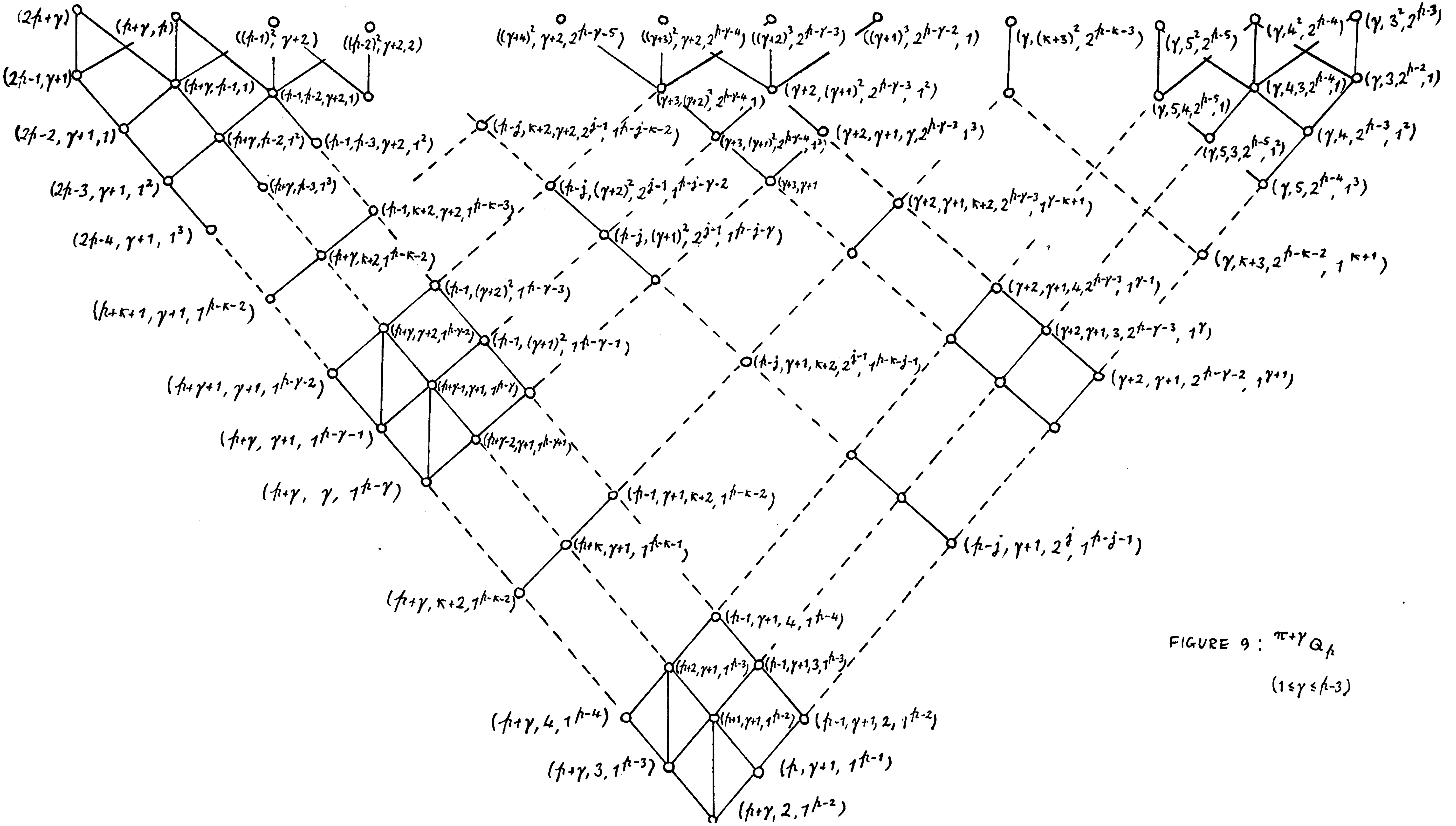


FIGURE 9 : $^{\pi+\gamma}Q_h$

$(1 \le \gamma \le h-3)$

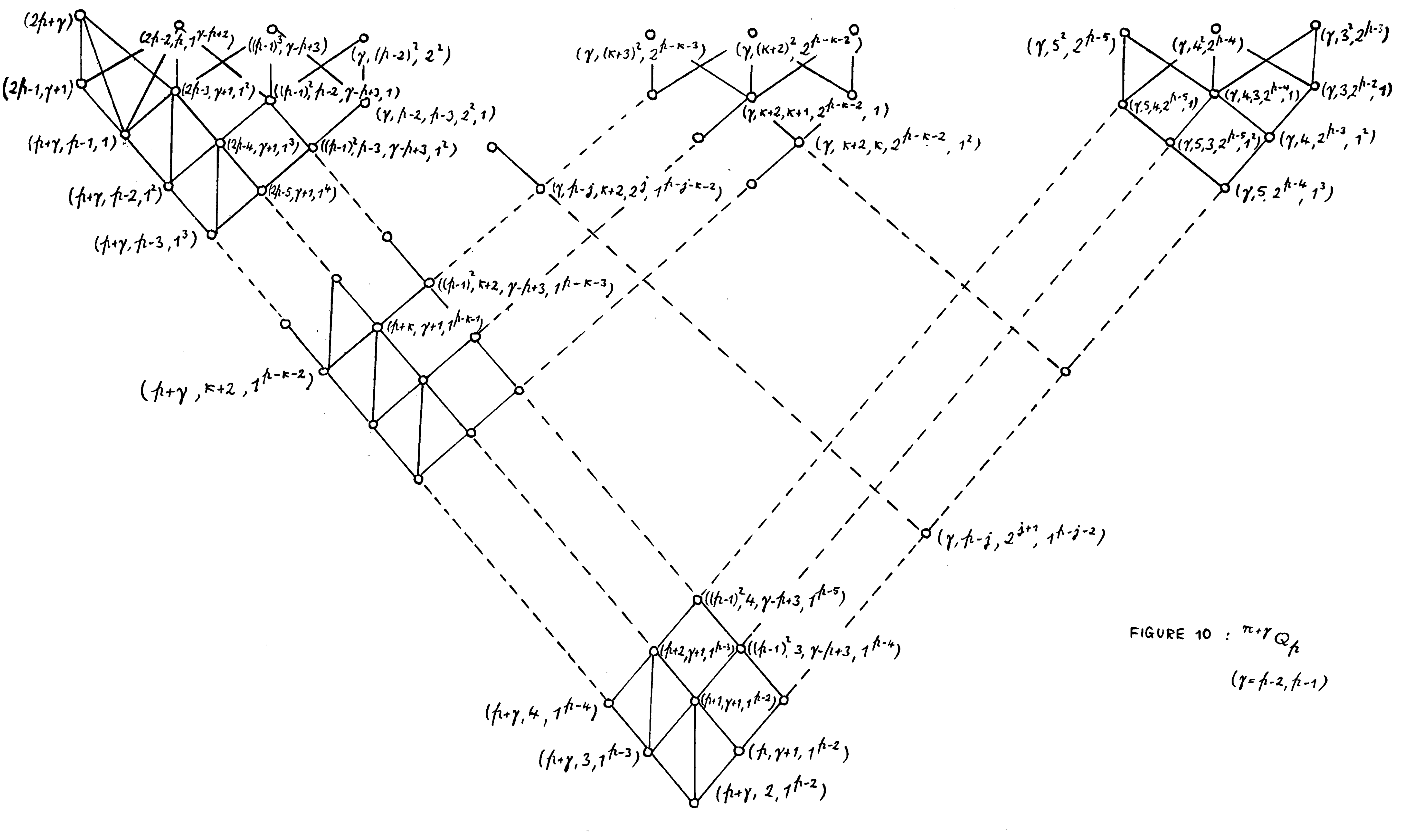


FIGURE 10 : $^{\pi+\gamma}Q_h$

$(\gamma=h-2,h-1)$

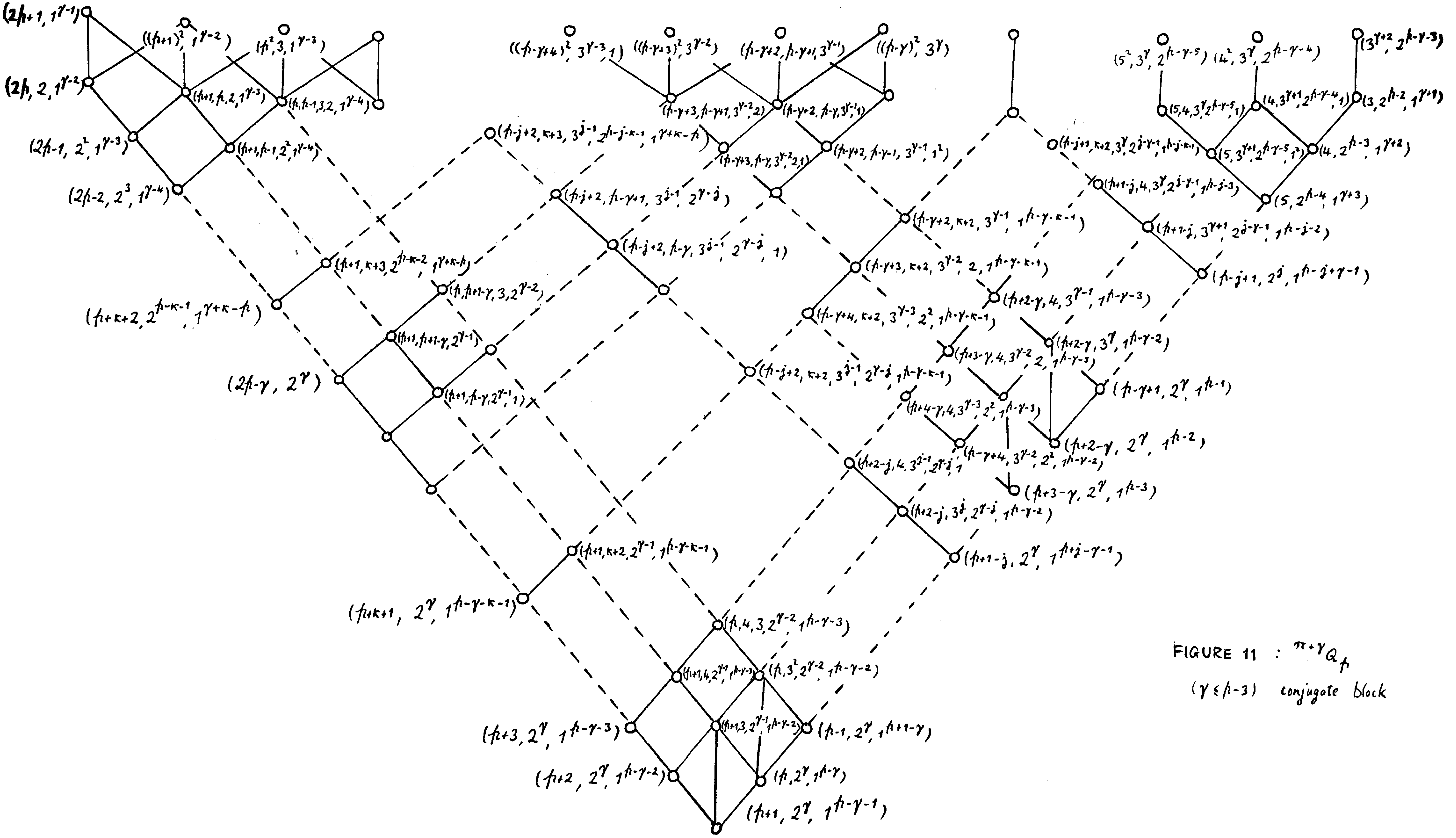


FIGURE 11 : $^{\pi+\gamma}Q_h$

$(\gamma \leq h-3)$ conjugate block

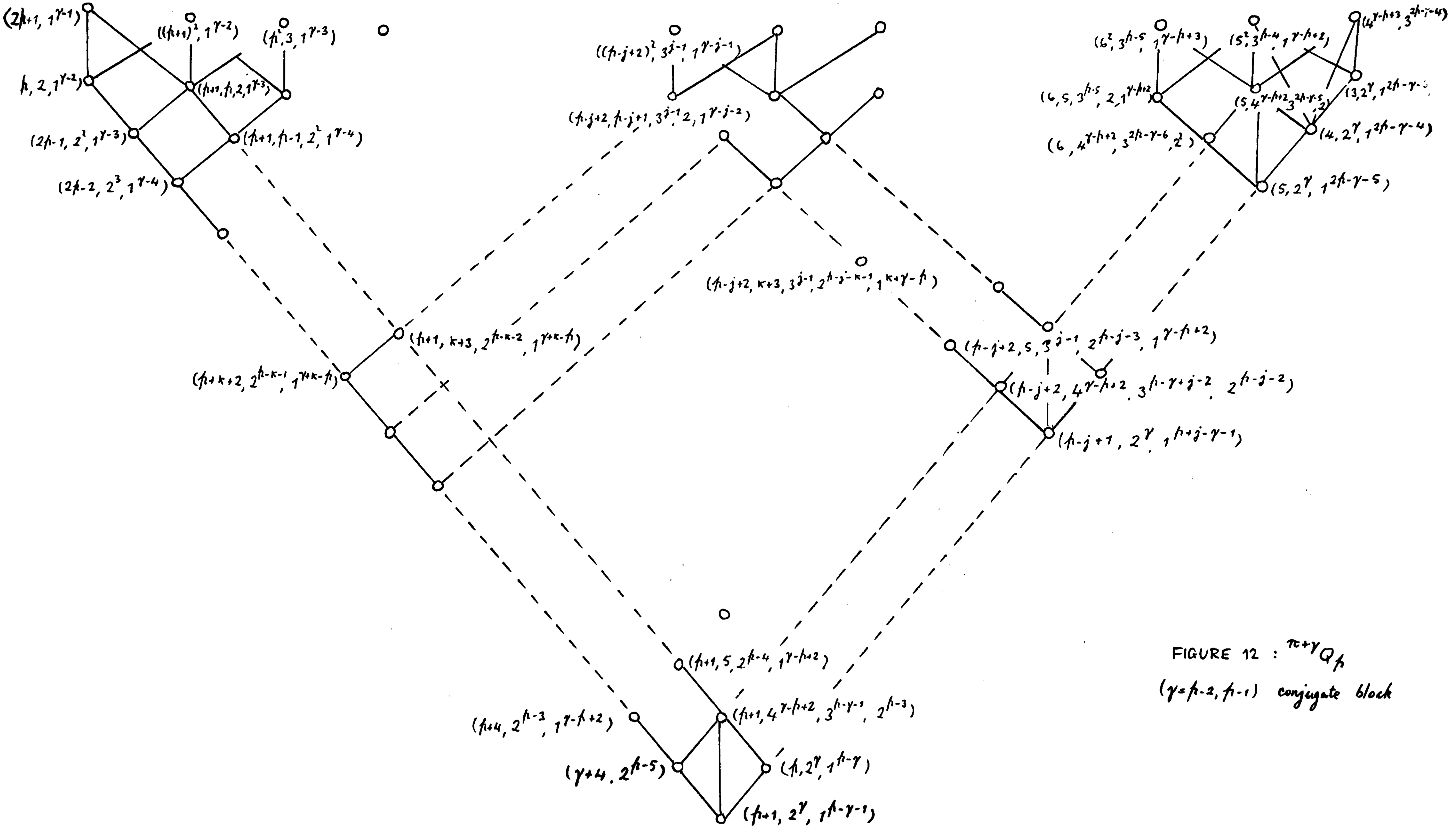


FIGURE 12 : ${}^{\pi+\gamma}Q_p$

$(\gamma = p-2, p-1)$ conjugate block

FIGURE 13 : ${}^{11}Q_5$

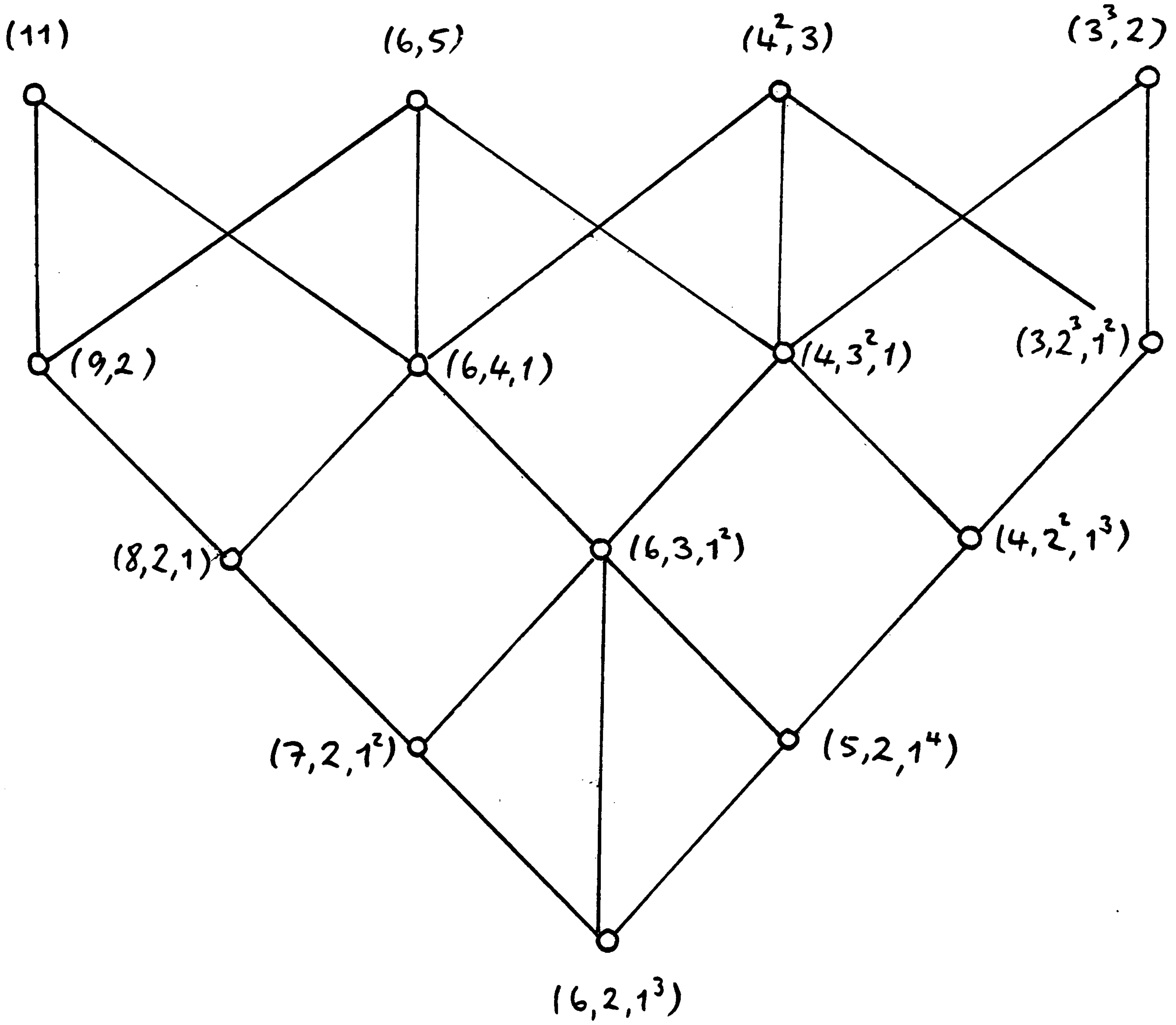
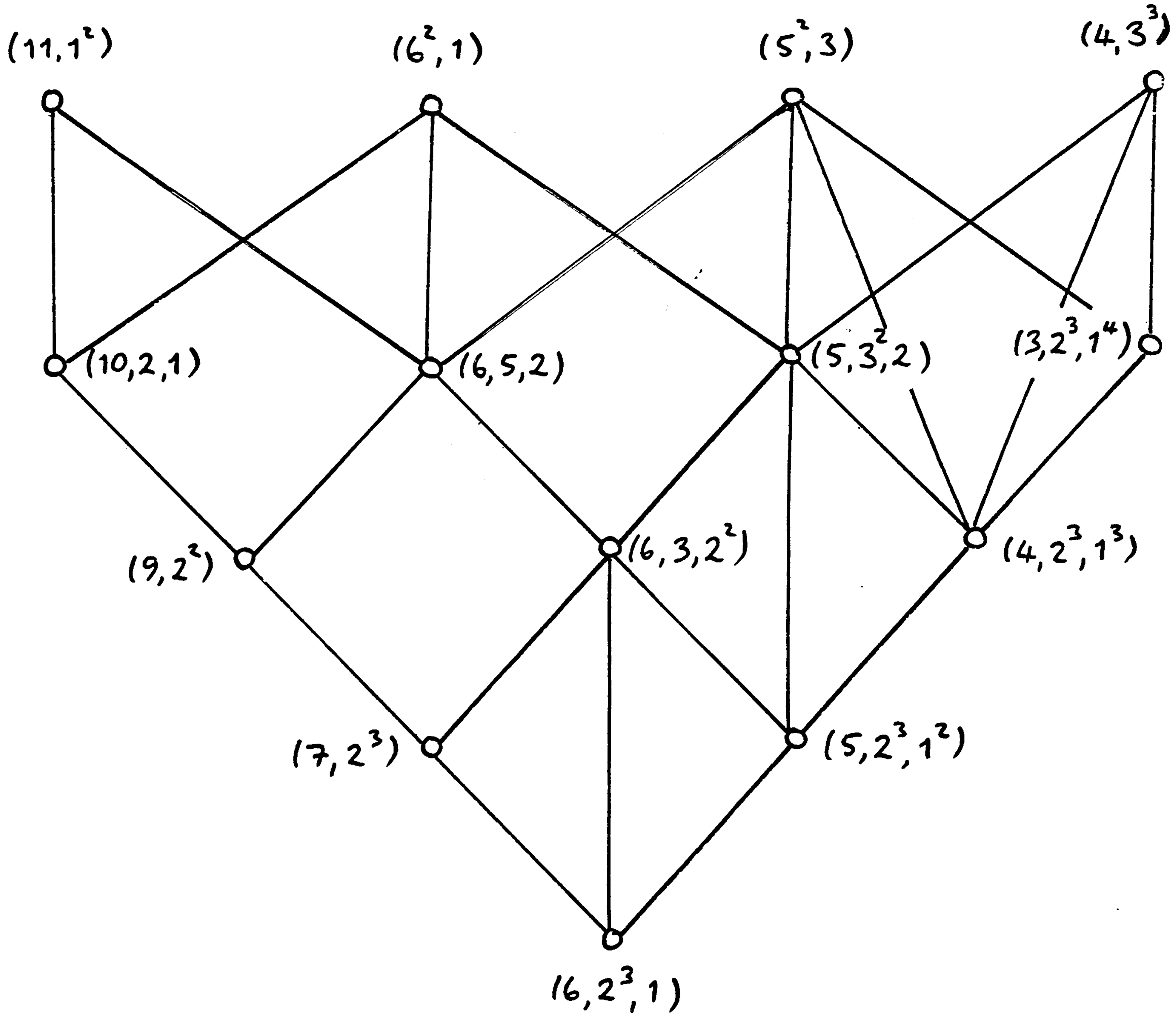


FIGURE 14 : ${}^{13}Q_5$



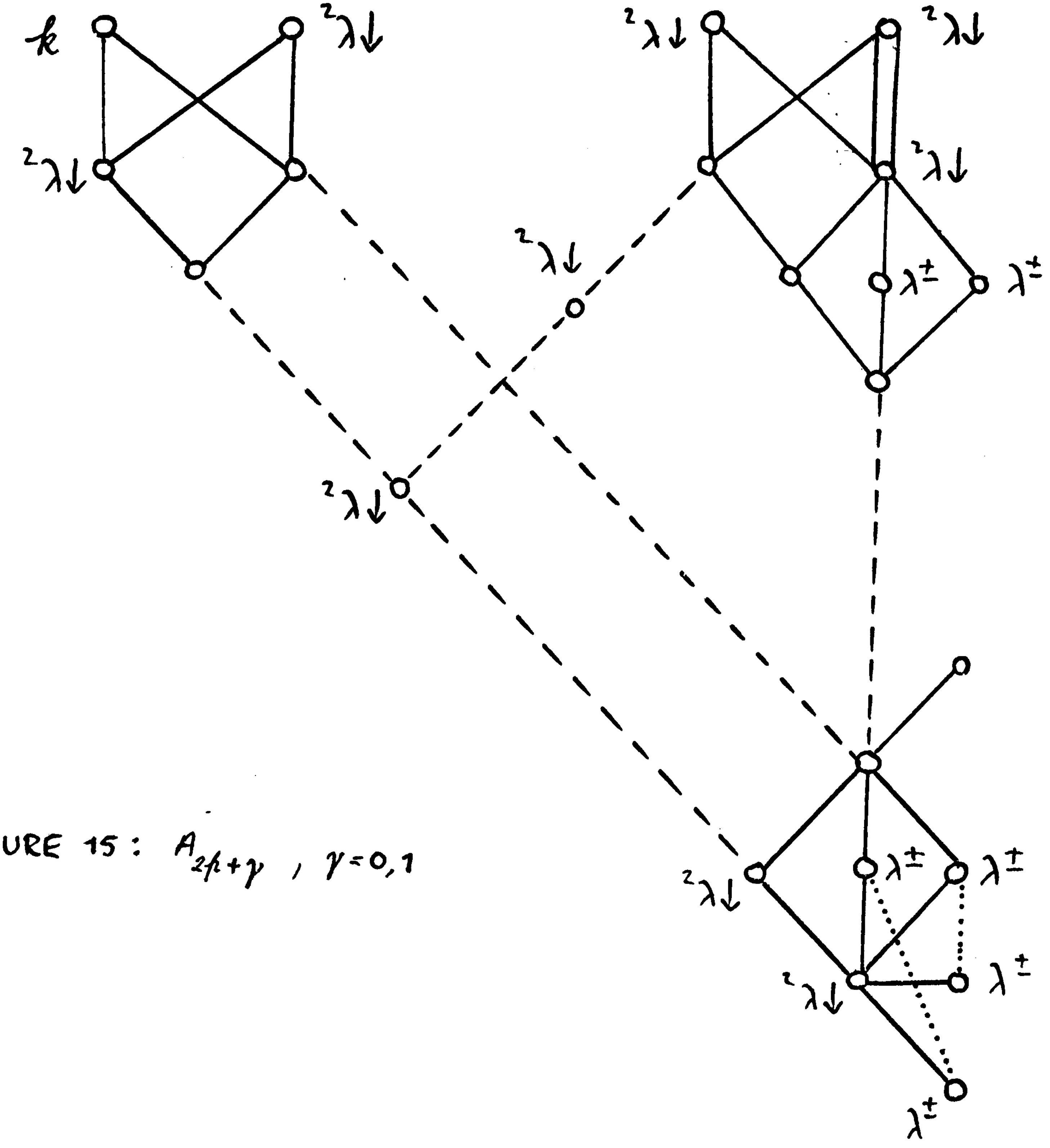


FIGURE 15: $A_{2\ell+\gamma}$, $\gamma=0,1$

FIGURE 16 : A_{11} (mod 5)

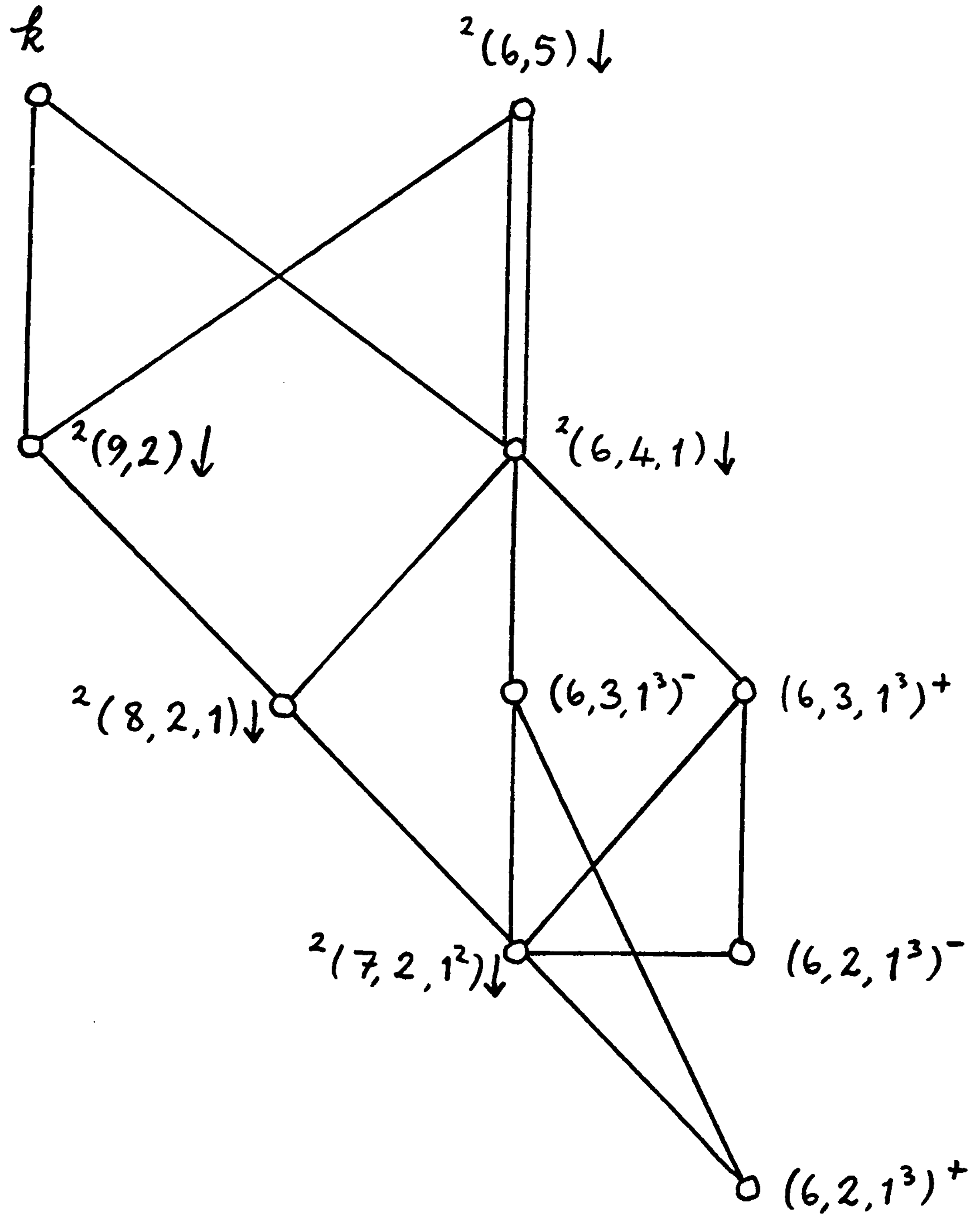
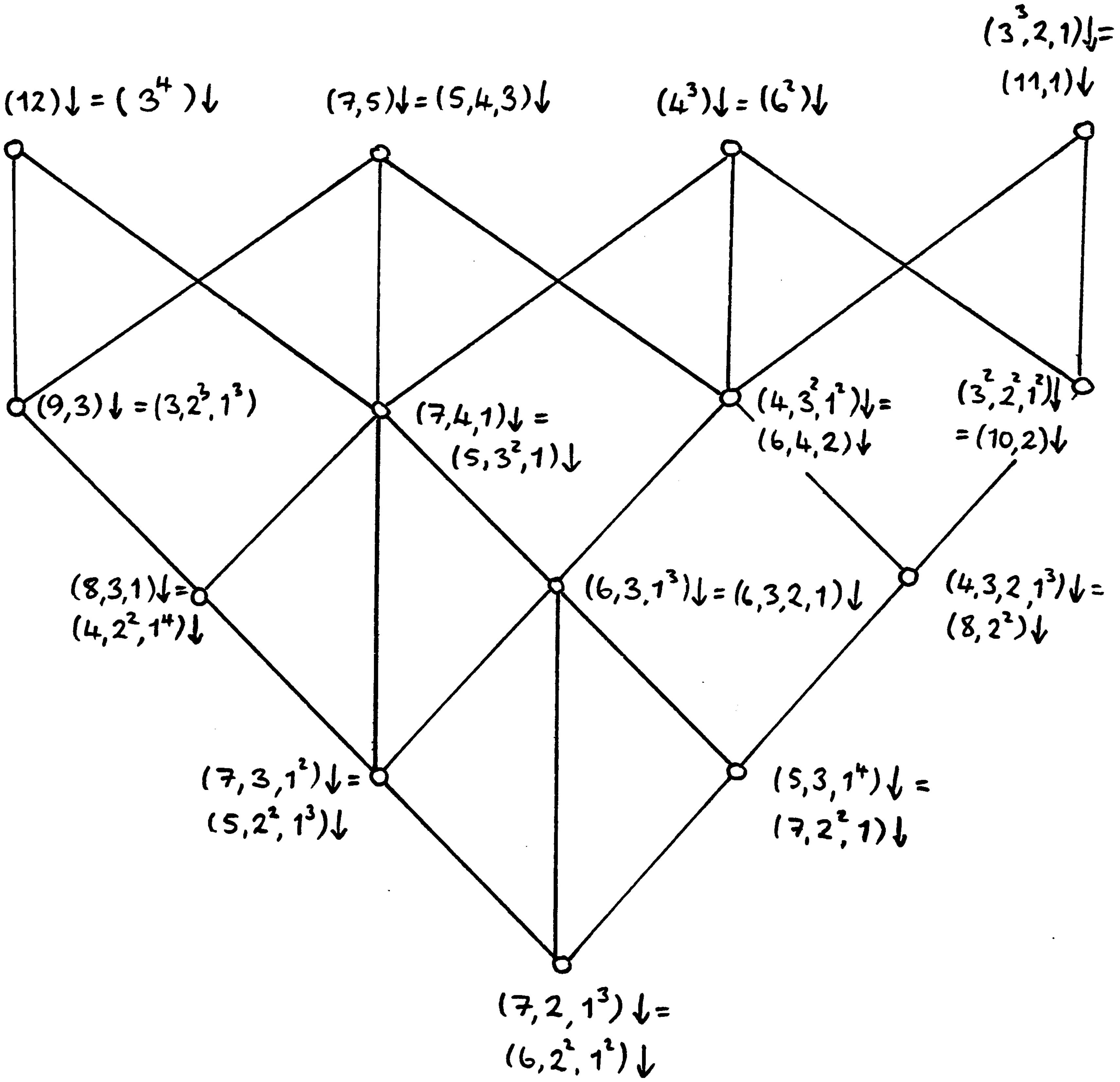


FIGURE 17 : A_{12} (mod 5)



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