Model Theory of Shimura Varieties

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Abstract

The purpose of this thesis is to provide model-theoretic structures to study Shimura varieties. Two main problems are studied.

Transcendence Properties of j. This part is aimed at proving a general transcendence property for the j function in the spirit of Schanuel's conjecture. Such a result was proven for the exponential function by M. Bays, J. Kirby and A. Wilkie, and it relies heavily on the Ax-Schanuel theorem. An analogue of Ax-Schanuel for the j function was proven by J. Pila and J. Tsimerman, and this allowed to adapt the strategy used for the exponential to j.

One concrete consequence of the main transcendence result is:

THEOREM. *Let $\tau \in \mathbb{R}$ be j-generic. Suppose $z_1, \ldots, z_n \in \mathbb{H}^+$ and $g \in \mathrm{GL}_2(\mathbb{Q}(\tau))^+$ are such that $z_1, \ldots, z_n, gz_1, \ldots, gz_n$ are in different $\mathrm{GL}_2(\mathbb{Q})^+$-orbits (pairwise). Then:*

$$\text{t.d.}\,(j(\bar{z}), j'(\bar{z}), j''(\bar{z}), j(g\bar{z}), j'(g\bar{z}), j''(g\bar{z})/\tau) \geq 3n.$$

The term *j-generic* is analogous to that of being *exponentially transcendental.* Using o-minimality, we prove that all but countably many values for τ can be used.

We also analyse a modular version of Schanuel's conjecture and prove that there are at most countably many essential counterexamples to this conjecture.

Categoricity of Shimura Varieties. Let $p : X^+ \to S(\mathbb{C})$ be a Shimura variety. We associate with it a countable two-sorted first-order language and analyse the complete first-order theory it determines. The question of interest in this part of the thesis of whether or not this theory is categorical. One immediate issue is that the theory cannot determine the size of the fibres of the map between the sorts, so we restrict our analysis to the models with the smallest possible fibres.

Categoricity for Shimura curves was established by C. Daw and A. Harris. From their result, it is immediate that some form of arithmetic open image condition is needed for the categoricity of general Shimura varieties. Our goal is to postulate the correct open image condition and prove that it is equivalent to categoricity. This open image condition is not known in general, however, using known cases of the Mumford-Tate conjecture, we obtain unconditional categoricity of two well-known Shimura varieties: $\mathcal{A}_2$ and $\mathcal{A}_3$, the moduli spaces of principally polarised abelian varieties of dimension 2 and 3, respectively.

Contents

Introduction

The purpose of this thesis is to provide model-theoretic structures to study Shimura varieties. Two types of problems are studied.

We first consider one-sorted models, in which both the variety and the uniformisation domain of the variety are seen as defined within the same field structure. Here we look at the problem of obtaining generic transcendence properties for the modular j function from its corresponding Ax-Schanuel theorem. Even though we only work with this particular function, at the end of §2.4.4 we explain why the methods we used are expected to generalise to Shimura curves.

The other problem we study is that of categoricity of Shimura varieties as two-sorted structures, that is, where the domain of the variety is treated a space separate from the field structure over which the variety is defined. This problem is tackled by extending strategies from geometric model theory explained in [**7**] and [**17**]. In this approach, we want to understand to what extent the universal cover of a Shimura variety is determined by its algebraic properties.

Beyond the main results we obtain regarding these two problems, the major motivation for this thesis has been to show that many results in the area of Diophantine geometry which have been proven for the exponential function or for abelian varieties can be replicated in the setting of Shimura varieties. In this thesis, we are particularly interested in finding ways to translate results concerning the so-called "special structure" of abelian varieties and of the multiplicative group, to that of Shimura varieties. Many important problems involving each of these varieties can be stated in terms of its corresponding special structures. In particular, in this thesis we will study variations of Schanuel's conjecture in Part 2, and a reformulation of the Mumford-Tate conjecture in Part 4. For more details on these and other important problems regarding the special structure, such as the Andreé-Oort and the Zilber-Pink conjectures, see [**44**].

This thesis is divided into four parts. Part 1 is devoted to some necessary background in model theory, the j function, and Shimura varieties. An overview of the other parts of the thesis is given below.

Transcendence Properties of j. The aim of Part 2 is to prove a generic transcendence property for the j function in the spirit of Schanuel's conjecture (see [**2**, Conjecture (S)]). Such a result is proven in [**9**] for the exponential function.

THEOREM 0.0.1 (see [**9**, Theorem 1.2]). *Let F be any exponential field, let $\lambda \in F$ be exponentially transcendental, and let $\overline{x} \in F^n$ be such that* $\exp(\overline{x})$ *is multiplicatively independent. Then:*

$$\text{t.d.}(\exp(\overline{x}), \exp(\lambda\overline{x})/\lambda) \geq n.$$

The notation we are using is as follows: for A and B subsets of a given field of characteristic zero, we write t.d.(A/B) to denote the transcendence degree of the field extension $\mathbb{Q}(A, B)/\mathbb{Q}(B)$. Given some elements $z_1, \ldots, z_n$ in the field F, we write $\overline{z}$ to denote the tuple $(z_1, \ldots, z_n)$, and if f is a function, then $f(\overline{z})$ denotes the tuple $(f(z_1), \ldots, f(z_n))$.

Theorem 0.0.1 relies heavily on the so-called Ax-Schanuel theorem (see [**2**, Theorem 3]). In [**45**], the authors prove an analogue of the Ax-Schanuel theorem for the j function (see Theorem 2.4.1), so it is natural to wonder if a strategy similar to the one used in [**9**] would provide a transcendence result for j.

To do this we will define "j-fields" in a similar way to how exponential fields are defined in [**28**]. Traditionally, the j function is understood as a modular function defined on the upper-half plane, but we will extend it to be defined on the upper and lower half planes so that $j : \mathbb{H}^+ \cup \mathbb{H}^- \to \mathbb{C}$, where given $z \in \mathbb{H}^-$ we define $j(z) := \overline{j(\overline{z})}$; here $\overline{z}$ is the complex conjugate of z (even though the symbol is the same, it should be clear from context when $\overline{z}$ denotes complex conjugation or a tuple of elements). We have extended j to the lower half-place because the condition for an element $g \in \mathrm{GL}_2(\mathbb{Q})$ to preserve $\mathbb{H}^+ \cup \mathbb{H}^-$ setwise can be decribed only using field structure: namely $\det(g) \neq 0$. If we only consider the upper-half plane, we would have to ask that $\det(g) > 0$, which would require us to include an order relation in our j-field (the importance of this will become apparent in the proof of Lemma 2.2.3). Given the nature of the transcendence properties of j that we are interested in, it does not matter if we extend the domain of j in this way.

The full statement of our main transcendence result for j, Theorem 2.4.9, is rather technical, but one immediate consequence to a concrete setting is the theorem below. The term *j-generic* is analogous to that of *exponential transcendence*; we give a precise definition in §2.3.

THEOREM 0.0.2 (see Corollary 2.4.4). *Let $\tau \in \mathbb{R}$ be j-generic. Suppose $z_1, \ldots, z_n \in \mathbb{H}^+ \cup \mathbb{H}^-$ and $g \in \mathrm{GL}_2(\mathbb{Q}(\tau))$ are such that $z_1, \ldots, z_n, gz_1, \ldots, gz_n$ are in different $\mathrm{GL}_2(\mathbb{Q})$-orbits (pairwise). Then:*

$$\text{t.d.}\,(j(\overline{z}), j'(\overline{z}), j''(\overline{z}), j(g\overline{z}), j'(g\overline{z}), j''(g\overline{z})/\tau) \geq 3n.$$

Using o-minimality, we prove that all but countably many values for τ can be used (see Remark 2.3.22), although, just as we do not have any explicit examples of exponentially transcendental numbers, no explicit j-generic numbers are known.

The key to obtaining Theorem 2.4.9 is in showing that j-fields prove to be an adequate setting in which to interpret the Ax-Schanuel theorem for j (see Proposition 2.4.2), that is to say, we can define certain pregeometries (here named gcl and $j$cl) on j-fields that encode the contents of this result.

Part 2 is organized as follows. After defining j-fields axiomatically, we study the family of pregeometries gcl_F obtained from the action of a subgroup of $\mathrm{GL}_2(F)$ on a field K, where F is a subfield of K. We also introduce here the notion of "geodesic disjointness".

Next comes a section on j-derivations, which are derivations which respect the j function and its derivatives. Using standard techniques of o-minimality, we show that $\mathbb{C}$ has non-trivial j-derivations. We also define the operator $j$cl.

The last section of Part 2 contains the transcendence results, the most important being Theorem 2.4.9 which generalises Theorem 0.0.2. We also analyse a modular version of Schanuel's conjecture and prove that, like in the case of the original Schanuel's conjecture, in $\mathbb{C}$ there are at most countably many essential counterexamples of this conjecture.

The results from Part 2 were published in **[22]**.

Categoricity of Shimura Varieties. Parts 3 and 4 of the thesis are devoted to the categoricity problem for Shimura varieties. Suppose that $p : X^+ \to S(\mathbb{C})$ is a (connected) Shimura variety given by the Shimura datum (G, X), that $E = E(G, X)$ is the corresponding reflex field, and let Σ be the set of coordinates of special points in $S(\mathbb{C})$ (these terms will be appropriately defined in §1.3). One way of thinking about the categoricity problem for S is: if $q : X^+ \to S(\mathbb{C})$ is any other map (of sets) which defines a model of the first-order theory of p, is there a commutative diagram:

$$\begin{array}{ccc} X^+ & \xrightarrow{\xi} & X^+ \\ \downarrow p & & \downarrow q \\ S(\mathbb{C}) & \xrightarrow{\sigma} & S(\mathbb{C}), \end{array} \tag{0.0.1}$$

where $\xi : X^+ \to X^+$ is a $G^{\mathrm{ad}}(\mathbb{Q})^+$-equivariant bijection which respects special domains (see §1.3.4), and σ is a map induced by a field automorphism of $\mathbb{C}$ fixing $E(\Sigma)$? In other words: to what extent is p determined by the algebraic properties of special subvarieties of S? Therefore, having categoricity implies that the first-order theory of $p : X^+ \to S(\mathbb{C})$ (or, as we will see, a specific class of submodels) is an invariant of the Shimura variety.

Categoricity for Shimura curves was established in [**17**], and the authors of this work acknowledge in their paper that the language they used may not be expressive enough to treat the higher dimensional cases. Our first objective then is to provide an enriched language which keeps the nice properties of the language used in the one-dimensional case: quantifier elimination, a useful description of types, and superstability. The second objective is to give necessary and sufficient conditions for the theory of a Shimura variety of any dimension to be categorical. It is immediate from the work done in [**17**] (and from other similar categoricity results which will be addressed later) that some form of arithmetic open image condition is needed.

The main results of Part 4 (Theorems 4.2.10 and 4.3.14) show that the open image condition (which we split it into two conditions: FIC1 and FIC2) is equivalent to categoricity. This open image condition is known to hold in dimension 1 thanks to [**17**, §5], but not known in general. In fact, a version of this condition had already been independently conjectured by Pink in [**48**], see Conjecture 1, where he shows that Conjecture 1 is a generalisation of the Mumford-Tate conjecture. So our categoricity result is effectively a reinterpretation of these important conjectures from a model-theoretic point of view, that is, stating the problems only in terms of the properties of the special structure.

The proof of our main results will be approached adapting techniques from ℓ-atomicity and independent systems developed in [**7**] (based on work of Shelah ([**52**])) for studying commutative groups of finite Morley rank. We will also show an approach using quasiminimal classes giving the same result. We show both approaches as they emphasize different aspects of the problem, the methods of [**7**] focusing more on how to build big models from small ones, while the quasiminimal approach focuses on the existence of certain curves within the Shimura variety. As with other proofs of categoricity of arithmetic varieties, the methods used here show how abstract notions from model-theory and infinitary logic can have explicit interesting geometric consequences in arithmetic geometry.

Part 3 is organised as follows. In §3.1, §3.1.5 and §3.2 we present our model-theoretic structures for Shimura varieties, we define the special subvarieties condition (SS), give an axiomatisation for the complete first-order theory of $p : X^+ \to S(\mathbb{C})$, we prove quantifier elimination of this theory, and show some of its stability properties. We also define the standard fibres condition (SF).

Part 4 contains the categoricity results. In §4.2 and §4.3 we present our main theorems, first using quasiminimal classes, and then through independent systems. Finally, in §4.4 we show that in the case of $\mathcal{A}_2$ and $\mathcal{A}_3$ (these are the moduli spaces of principally polarised abelian varieties of dimensions 2 and 3, respectively) the categoricity conditions FIC1 and FIC2 can be obtained from the existing literature. As far as we know, this result is new.

This work follows an important series of results regarding categoricity of certain arithmetic varieties: [**17**] for categoricity of Shimura curves, [**7**] for categoricity of abelian and split semi-abelian varieties, [**8**] for categoricity conditions of pseudo-exponential maps, [**64**] for the general quasiminimal strategy for semi-abelian varieties, [**10**] and [**65**] for categoricity of multiplicative groups of algebraically closed fields, [**25**] for categoricity of abelian varieties under the continuum hypothesis, [**5**] for categoricity of elliptic curves without complex multiplication; among others.

Part 1

General Background

1.1. Model Theory

In this section we give some standard definitions and results from model theory needed for our work. We will always assume that the language $\mathcal{L}$ we are using is countable, and that any $\mathcal{L}$-theory we are considering has infinite models. For us the word *type* means *complete type*. Standard definitions and results from model theory can be found, for example, in [**57**].

DEFINITION. Let κ be an infinite cardinal. An $\mathcal{L}$-theory T is *κ-categorical* if all models of T of cardinality κ are isomorphic. We will say that T is *categorical* if it is κ-categorical for all infinite κ. T is *uncountably categorical* if T is categorical for all uncountable κ.

EXAMPLE 1. Consider the theory of algebraically closed fields: ACF. We use ACF_0 to denote the theory of algebraically closed fields of characteristic 0. Thanks to the existence of transcendence bases, ACF_0 is κ-categorical for any uncountable κ (so it is uncountably categorical), but it is not $\aleph_0$-categorical as $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}(X)}$ are non-isomorphic countable models of ACF_0.

PROPOSITION 1.1.1 (see [**57**, §3.3.4]). *Let $K \models \mathrm{ACF}$ and $A \subseteq K$. Let k be the subfield of K generated by A. Then the n-types over A are determined by prime ideals in the poynomial ring $k[X_1, \ldots, X_n]$.*

This means that if $m_1, \ldots, m_n \in K$, then $\mathrm{tp}(m_1, \ldots, m_n/A)$ is determined by the smallest algebraic subvariety $W \subseteq K^n$ defined over k which contains the point $(m_1, \ldots, m_n)$. Any other point of W which is generic in W over k, will also realise this type.

DEFINITION. An $\mathcal{L}$-structure M is *strongly minimal* if every definable subset of M is either finite or cofinite.

EXAMPLE 2. ACF_0 is strongly minimal (see [**57**, §5.7]).

1.1.1. Quantifier Elimination. We recall a few standard definitions and results about theories with quantifier elimination. The exposition is taken from [**46**].

DEFINITION. Let M and N be $\mathcal{L}$-structures. A *finite partial isomorphism* between M and N is a pair $(\overline{a}, \overline{b})$ with $\overline{a} \in M$ and $\overline{b} \in N$, such that $\mathrm{qftp}_M(\overline{a}) = \mathrm{qftp}_N(\overline{b})$.

In other words, a finite partial isomorphism between M and N is the same thing as an isomorphism of $\mathcal{L}$-structures between a finitely generated substructure of M and a finitely generated substructure of N.

DEFINITION. Let M and N be $\mathcal{L}$-structures. Let I be a set of partial isomorphisms between M and N. I has *back-and-forth* if for every partial isomorphism $\left(\overline{a},\overline{b}\right) \in I$ and every $c \in M$, there is $d \in N$ such that $\left(\overline{a}c,\overline{b}d\right) \in I$ and dually.

DEFINITION. An $\mathcal{L}$-structure M is $\aleph_0$-*homogeneous* if for every finite subset $A \subseteq M$ and every $m \in M$, any elementary map $A \to M$ can be extended to an elementary map $A \cup \{m\} \to M$.

When talking about the class of models of a theory T, the previous concept will appear in the following two variants:

(1) $\aleph_0$-*homogeneity over the empty set.* This will mean that, given any two models of T, the set I of all finite partial isomorphism between them has back-and-forth.
(2) $\aleph_0$-*homogeneity over countable models.* For this we need to adapt the definition of finite partial isomorphism. Given $M, M', N \models T$, where N is a countable submodel common to M and M', a *partial isomorphism over* N is a pair $\left(\overline{a},\overline{b}\right)$ with $\overline{a} \in M$ and $\overline{b} \in M'$, such that $\mathrm{qftp}_M\left(\overline{a}/N\right) = \mathrm{qftp}_{M'}\left(\overline{b}/N\right)$. Now we can define $\aleph_0$-homogeneity over countable models in an analogous way to $\aleph_0$-homogeneity over the empty set.

The following proposition gives us the strategy we will use later for proving quantifier elimination for a given theory.

PROPOSITION 1.1.2 (see [**46**, Proposition 2.29]). *Let T be an $\mathcal{L}$-theory. The following are equivalent:*

(a) T has quantifier elimination.
(b) If M and N are ω-saturated models of T and I is the set of all finite partial isomorphism between M and N, then I has back-and-forth.

A similar strategy can also be used to prove completeness of the theory.

Proposition 1.1.3 (see [**46**, Proposition 2.30]). *Suppose that for any two ω-saturated models M and N of T, the set of finite partial isomorphisms between M and N has back-and-forth and is non-empty. Then T is complete.*

1.1.2. Pregeometries. This notion will be used both in our o-minimal study of the j function, and in our quasiminimal approach to Shimura varieties.

Definition. Let X be a set. A function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ (here $\mathcal{P}(X)$ denotes the power set of X) is called a *pregeometry* on X if it satisfies the following properties for every $A, B \in \mathcal{P}(X)$:

(a) $A \subseteq \mathrm{cl}(A)$,

(b) If $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,

(c) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,

(d) Finite character: if $a \in \mathrm{cl}(A)$, then there is a finite subset $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0)$,

(e) Exchange: if $a, b \in X$ are such that $a \in \mathrm{cl}(A \cup \{b\})$ and $a \notin \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A \cup \{a\})$.

A crucial property of pregeometries is that they allow us to have well-defined notions of independence and dimension. If cl is a pregeometry on X, then a set $A \subseteq X$ is cl-*independent* if for every $a \in A$ we have that $a \notin \mathrm{cl}(A \smallsetminus \{a\})$. The *dimension* of a set $B \subseteq X$ with respect to cl, denoted $\dim^{\mathrm{cl}}(B)$, is the cardinality of any cl-independent set $A \subseteq B$ satisfying $\mathrm{cl}(A) = B$. For more properties of pregeometries, see [**57**, Appendix C].

Example 3. We give two well-known examples of pregeometries.

(1) Let K be a field. If X is a K-vector space, we can take $\mathrm{cl}(A)$ to be the linear span generated by A: $\mathrm{cl}(A) := \mathrm{lin.span}_K(A)$. In this case, the dimension of A is just the linear dimension: $\dim A = \mathrm{l.dim.}_K(\mathrm{cl}(A))$.

(2) If X is a field, we can take $\mathrm{cl}(A)$ to be the set of elements of X which are algebraic over A. In this case $\dim A = \mathrm{t.d.}(A)$. This pregeometry is usually denoted (and this is the name we will use): acl. This example is an instance of a more general result, namely that the model-theoretic notion of algebraic closure defines a pregeometry on strongly minimal sets (see [**57**, Theorem 5.7.5]).

1.1.3. Quasiminimality. Here we will introduce the terminology related to quasiminimal structures as presented in [**6**] and [**29**]. The final theorem of this subsection provides the strategy we will follow to obtain one of our main results: Theorem 4.2.10. As always, we assume $\mathcal{L}$ is countable.

DEFINITION. An $\mathcal{L}$-structure M is *quasiminimal* is every definable subset of M is countable or co-countable.

DEFINITION. Let M be an $\mathcal{L}$-structure and let cl be a pregeometry on M. The pair (M, cl) is called a *quasiminimal pregeometry structure* if it satisfies the following axioms:

- **(QM1):** cl is determined by the language: given $a, a', b_1, \ldots, b_n, b_1', \ldots, b_n', \in M$ such that $\mathrm{tp}\left(a, \overline{b}\right) = \mathrm{tp}\left(a', \overline{b}'\right)$, then $a \in \mathrm{cl}\left(\overline{b}\right)$ if and only if $a' \in \mathrm{cl}\left(\overline{b}'\right)$.
- **(QM2):** M is infinite dimensional with respect to cl.
- **(QM3):** *Countable Closure Property.* If $A \subseteq M$ is finite, then $\mathrm{cl}(A)$ is countable.
- **(QM4):** *Uniqueness of the generic type.* Suppose $C, C' \subseteq M$ are countable closed subsets, enumerated such that $\mathrm{tp}(C) = \mathrm{tp}(C')$. If $a \in M \setminus C$ and $a' \in M \setminus C'$, then $\mathrm{tp}(C, a) = \mathrm{tp}(C', a')$ (with respect to the same enumerations for C and C').
- **(QM5):** $\aleph_0$*-homogeneity over closed sets and the empty set.* Let $C, C' \subseteq M$ be countable closed subsets or empty, enumerated such that $\mathrm{tp}(C) = \mathrm{tp}(C')$, and let $\overline{b}, \overline{b}'$ be finite tuples from M such that $\mathrm{tp}\left(C, \overline{b}\right) = \mathrm{tp}\left(C', \overline{b}'\right)$, and let $a \in \mathrm{cl}\left(H, \overline{b}\right)$. Then there is $a' \in M$ such that $\mathrm{tp}\left(C, \overline{b}, a\right) = \mathrm{tp}\left(C', \overline{b}', a'\right)$.

(M, cl) is said to be a *weakly quasiminimal pregeometry structure* if it satisfies all the axioms except maybe (QM2).

DEFINITION. Let (M, cl) be a quasiminimal pregeometry structure. Let $\mathscr{K}^-(M)$ be the smallest class of $\mathcal{L}$-structures that contains M, all the cl-closed substructures of M, and that is closed under isomorphisms. Let $\mathscr{K}(M)$ be the smallest class containing $\mathscr{K}^-(M)$ and that is closed under unions of chains of closed embeddings. Any class of structures of the form $\mathscr{K}(M)$ is called a *quasiminimal class.*

The following definition come from infinitary logic.

Definition. Let ω denote the first infinite ordinal and ω_1 the first uncountable ordinal. In the infinitary logic $\mathcal{L}_{\omega_1,\omega}$, one builds formulas over the language $\mathcal{L}$ with the negation symbol $\neg$, the connectives $\bigwedge$, $\bigvee$, the quantifiers $\forall, \exists$, and variables $\{v_\alpha\}_{\alpha<\omega_1}$. Formulas are defined recursively, keeping the same definitions of terms, atomic formulas, negation and quantifiers as in first-order logic, and the following rules for $\bigwedge$ and $\bigvee$: if Φ is a set of $\mathcal{L}_{\omega_1,\omega}$-formulas with $|\Phi| < \omega_1$, then

$$\bigvee_{\phi\in\Phi} \phi \text{ and } \bigwedge_{\phi\in\Phi} \phi$$

are also $\mathcal{L}_{\omega_1,\omega}$-formulas. The notions of *free* and *bound* variables are as in first-order logic, and thus we define an *$\mathcal{L}_{\omega_1,\omega}$-sentence* to be an $\mathcal{L}_{\omega_1,\omega}$-formula with no free variables.

Let Q denote a new quantifier symbol and let $\mathcal{L}_{\omega_1,\omega}(Q)$ denote the extension of the logic $\mathcal{L}_{\omega_1,\omega}$. We now define an *$\mathcal{L}_{\omega_1,\omega}(Q)$-sentence* in the same way as we defined $\mathcal{L}_{\omega_1,\omega}$-sentence. We interpret $Qx\phi(x)$, where $\phi(x)$ is an $\mathcal{L}_{\omega_1,\omega}(Q)$-formula, as "there exist uncountably many x such that $\phi(x)$".

Now we present the main result about categoricity and quasiminimal classes.

Theorem 1.1.4 (see [**6**, Theorem 2.3]). *If $\mathscr{K}$ is a quasiminimal class, then every structure $A \in \mathscr{K}$ is a weakly quasiminimal pregeometry structure and, up to isomorphism, there is exactly one structure in $\mathscr{K}$ of each cardinal dimension. In particular, $\mathscr{K}$ is uncountably categorical. Furthermore, $\mathscr{K}$ is the class of models of an $\mathcal{L}_{\omega_1,\omega}(Q)$-sentence.*

It will be important for us that this theorem can be applied to structures defined by $\mathcal{L}_{\omega_1,\omega}$-sentences because the Standard Fibres condition (see §3.1.5) will be such a sentence. The following proposition describes the isomorphism classes of models of cardinality at most $\aleph_1$ in a quasiminimal class.

Theorem 1.1.5 (see [**29**, Theorem 2.1 and Corollary 2.2]). *Let $\mathcal{L}$ be a fixed language. Let $\mathscr{K}$ be a class of pairs (M, cl_M), where M is an $\mathcal{L}$-structure and (M, cl_M) is a quasiminimal pregeometry structure. Suppose that if $(M, \mathrm{cl}_M) \in \mathscr{K}$ and $A \subseteq M$, then $(A, \mathrm{cl}_M|_A) \in \mathscr{K}$. Let $(M, \mathrm{cl}_M), (M', \mathrm{cl}_{M'}) \in \mathscr{K}$ be such that $\dim M \leq \aleph_1$. Let $G \subseteq M$ be empty or closed and countable, and let $f_0 : G \to M'$ be a closed embedding (or the empty* *map if* $G = \varnothing$*). Let* B *be a basis of* M *over* G *and suppose* $\psi_B : M \to M'$ *is an injective partial map with preimage* B *and image an independent set over* $\mathrm{Im} f_0$*. Then* $\psi = f_0 \cup \psi_B$ *extends to a closed map* $\widetilde{\psi} : M \to M'$*. In particular, if* $\mathrm{Im}(\psi)$ *spans* M'*, then* $\widetilde{\psi}$ *is an isomorphism, and so the models of* $\mathscr{K}$ *of dimension less than or equal to* $\aleph_1$ *are determined up to isomorphism by their dimension.*

1.1.4. ℓ-Isolation. Definitions and results presented here can be found in [**7**, §3].

Definition. Given a type p with parameters in a set B and a formula $\phi(x, y)$, the *complete ϕ-type implied by* p, denoted p_ϕ, is the set of all formulas of the form $\phi\left(x, \bar{b}\right)$ that are in p, where $\bar{b}$ is a tuple from B.

A type p is *ℓ-isolated* (which is short for *locally isolated*) if for every formula $\phi(x, y)$, there exists $\psi(x) \in p$ such that ψ implies the complete ϕ-type implied by p, i.e. $\psi \vDash p_\phi$.

A set A is said to be *atomic* over a set B if $\mathrm{tp}(a/B)$ is isolated for each tuple $a \in A$. A is *constructible* over B if there is an enumeration $(a_i)_{i \leq \lambda}$ of A such that $\mathrm{tp}\left(a_i / B a_{<i}\right)$ is isolated for each $i \leq \lambda$, where $B a_{<i} \coloneqq B \cup \{a_j : j \leq i\}$. We say A is *ℓ-atomic* and *ℓ-constructible* over B by replacing 'isolated' in the previous definitions with 'ℓ-isolated'.

Remark 1.1.6. It is easy to see that isolated types are ℓ-isolated, so atomicity implies ℓ-atomicity and constructibility implies ℓ-constructibility. Also constructibility implies atomicity and ℓ-constructibility implies ℓ-atomicity (see [**52**, Theorem IV.3.2]). The converse statements hold for countable sets (see [**52**, Lemma IV.3.16]).

Lemma 1.1.7 (see [**7**, Lemma 3.6-a)]). *Let* $\mathbb{M}$ *be a monster model of a complete stable theory* T*.*

(a) For any $A \subseteq \mathbb{M}^{\mathrm{eq}}$*, there exists* $M \prec \mathbb{M}$ *such that* $A \subseteq M^{\mathrm{eq}}$ *and* M^{eq} *is ℓ-constructible over* A*. In other words, ℓ-constructible models exist over arbitrary sets.*

(b) Let $M \prec \mathbb{M}$*. Suppose* ϕ *is a formula over* M *such that* $\phi(M) \subseteq A \subseteq \mathbb{M}^{\mathrm{eq}}$ *and* $\mathrm{dcl}^{\mathrm{eq}}(A) \cap \mathrm{dcl}^{\mathrm{eq}}(\phi(\mathbb{M})) \subseteq M^{\mathrm{eq}}$*. If* $b \in \mathbb{M}$ *is ℓ-isolated over* A *and* $\vDash \phi(b)$*, then* $b \in \phi(M)$*.*

1.1.5. Independent Systems. Here we give some definitions and results from [**7**, §3.5] necessary for the classification of Shimura covers in §4.3. Let T be a stable theory and $\mathbb{M}$ be a monster model of T.

Definition. Let I be a downward-closed set of sets. An *I-system* in $\mathbb{M}$ is a collection $\mathscr{M} = (M_s)_{s \in I}$ of elementary submodels of $\mathbb{M}$ such that for every $s, t \in I$ satisfying $s \subseteq t$, M_s is an elementary submodel of M_t. Given $J \subseteq I$, define the set $\mathscr{M}_J := \bigcup_{s \in J} M_s \subseteq \mathbb{M}$.

For $s \in I$, define $< s := \mathcal{P}(s) \setminus \{s\}$ and $\not\geq s := I \setminus \{t : s \subseteq t\}$.

$\mathscr{M}$ is called *constructible* (resp. *atomic*, *ℓ-constructible*, *ℓ-atomic*) if for all $s \in I$ with $|s| > 1$, M_s is constructible (resp. atomic, ℓ-constructible, ℓ-atomic) over $\mathscr{M}_{<s}$.

The I-system is called *independent* (or *stable*) if $M_s \downarrow_{\mathscr{M}_{<s}} \mathscr{M}_{\not\geq s}$ for all $s \in I$.

The I-system is *Noetherian* if each $s \in I$ is finite.

An *enumeration* of I is a sequence $(s_i)_{i \leq \lambda}$, where λ is an ordinal, such that $I = \{s_i : i \in \lambda\}$ and for all $i, j \in \lambda$, $s_i \subseteq s_j \implies i \leq j$. We write $s_{<i}$ for $\{s_j : j < i\}$.

Lemma 1.1.8 (see [**52**, Lemma XII.2.3(1)]). *Let $\mathscr{M} = (M_s)_{s \in I}$ be an I-system, and let $(s_i)_{i \in \lambda}$ be an enumeration of I. Then the I-system is independent if and only if $M_{s_i} \downarrow_{\mathscr{M}_{<s_i}} \mathscr{M}_{s_{<i}}$ holds for all $i \in \lambda$.*

Definition. Let M be a (possibly multi-sorted) structure. If $B \subseteq C \subseteq M$, we say that B is *Tarski-Vaught* in C, written $B \subseteq_{TV} C$, if every formula over B which is realised in C, has a realisation in B.

Lemma 1.1.9 (see [**7**, Lemma 3.18]). *Suppose $B \subseteq_{TV} C \subseteq M$.*

(1) Let $a \in M$. If $\operatorname{tp}(a/B)$ is ℓ-isolated, then $\operatorname{tp}(a/B) \vDash \operatorname{tp}(a/C)$ and $Ba \subseteq_{TV} Ca$.

(2) If $D \subseteq M$ is constructible over B, then D is contructible over C.

(3) If $D \subseteq M$ is ℓ-atomic over B, then $D \downarrow_B C$.

Lemma 1.1.10 (see [**7**, Lemma 3.19]). *Suppose M is a model, and $B \downarrow_M C$. Then $MB \subseteq_{TV} MBC$.*

Lemma 1.1.11 (see [**52**, Lemma XII.2.3(2)]). *Suppose $\mathscr{M} = (M_s)_{s \in I}$ is an independent I-system, $J \subseteq I$, and $\forall s \in I\, (s \subseteq \bigcup J \implies s \in J)$. Then $\mathscr{M}_J \subseteq_{TV} \mathscr{M}_I$.*

Lemma 1.1.12 (see [**7**, Lemma 3.21]). *Let $\mathscr{M} = (M_s)_{s \in I}$ be a constructible Noetherian independent I-system. Suppose that for each $p \in \bigcup I$, B_p is a subset of $M_{\{p\}}$ for which $M_{\{p\}}$ is constructible over $M_\varnothing B_p$. Then $\mathscr{M}_I$ is constructible over $C := M_\varnothing \cup \bigcup_{p \in \bigcup I} B_p$.*

LEMMA 1.1.13 (see [**7**, Lemma 3.22]). *Suppose $\bigcup I$ is finite. For an ℓ-atomic I-system $\mathscr{M} = (M_s)_{s\in I}$ to be independent, it suffices to show that for each $p \in \bigcup I$*

$$M_{\{p\}} \underset{M_\varnothing}{\downarrow} \mathscr{M}_{\not\supseteq\{p\}}.$$

1.2. The j function

We will now give a very brief review of complex elliptic curves to have enough context for the set up of the j function. For the general theory of elliptic curves, and most of the results stated here, see e.g. [**56**] and [**55**].

A *lattice* in $\mathbb{C}$ is a subgroup Λ of the additive group of $\mathbb{C}$ generated by two elements $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$. This means that Λ is isomorphic to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The quotient space $E_\Lambda = \mathbb{C}/\Lambda$ has a natural complex structure induced by that of $\mathbb{C}$. In fact E_Λ is a compact Riemann surface of genus 1 and it can be realised as a plane complex projective curve by means of the Weierstrass $\wp$ associated to Λ as follows.

Given the lattice Λ, consider the Weierstrass $\wp$ function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda,\lambda\neq 0}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

Using the Weierstrass M-test for convergence, one can see that $\wp(z)$ is a meromorphic function on $\mathbb{C}$ that is Λ-invariant. It also satisfies the differential equation: $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, where:

$$g_2 = 60\sum_{\lambda\in\Lambda,\lambda\neq 0}\frac{1}{\lambda^4}, \qquad g_3 = 140\sum_{\lambda\in\Lambda,\lambda\neq 0}\frac{1}{\lambda^6}.$$

So, the map $[z] \mapsto [\wp(z) : \wp'(z) : 1]$ defines an isomorphism between E_Λ and the plane projective curve $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$.

This construction motivates the following definition. An *elliptic curve* is a plane projective curve given by an equation of the form:

$$Y^2Z = 4X^3 - aXZ^2 - bZ^3, \tag{1.2.1}$$

such that the discriminant $\Delta := a^3 - 27b^2 \neq 0$. Any lattice Λ in $\mathbb{C}$ produces an elliptic curve E_Λ by the above construction. In fact, all elliptic curves can be obtained in this way. Also notice that the analytic construction we gave of E_Λ (as a quotient space $\mathbb{C}/\Lambda$) immediately shows that E_Λ has a group structure (as a quotient group). In fact the group operation is a rational map, and so it is compatible with the algebraic structure of E_Λ as a plane projective curve. For this reason we will use the name E_Λ to refer both to the analytic and algebraic realisations of the elliptic curve.

If E is an elliptic curve defined by equation (1.2.1), define the *j-invariant of E* as:

$$j(E) \coloneqq 1728\frac{a^3}{\Delta}.$$

A *morphism of elliptic curves* is a morphism of algebraic varieties that is also a group homomorphism. For any elliptic curve E' we have that E is isomorphic to E' if and only if $j(E) = j(E')$. Furthermore, given two lattices, Λ and Λ', E_Λ is isomorphic to $E_{\Lambda'}$ if and only if there is $c \in \mathbb{C}^*$ such that $c\Lambda = \Lambda'$. So, all isomorphism classes of elliptic curves are obtained by considering only lattices of the form $\mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}^+$ (the upper-half plane). We can now formulate the definition we have been waiting for.

Definition. The *j function* is the holomorphic map $j : \mathbb{H}^+ \to \mathbb{C}$ given by $j(\tau) = j(E_\tau)$, where E_τ is the elliptic curve defined by the lattice $\Lambda(\tau) \coloneqq \mathbb{Z} + \mathbb{Z}\tau$. As stated in the Introduction, we will extend j to the lower half-plane, so that $j : \mathbb{H}^+ \cup \mathbb{H}^- \to \mathbb{C}$, where $j(\tau)$ for $\tau \in \mathbb{H}^-$ is defined as $j(\tau) \coloneqq \overline{j(\overline{\tau})}$ (here $\overline{z}$ denotes the complex conjugate of z).

We will now review a few properties of the j function. As it can be easily verified, given $\tau, \tau' \in \mathbb{H}^+$, we have that $\Lambda(\tau) = \Lambda(\tau')$ if and only if there is $g \in \mathrm{SL}_2(\mathbb{Z})$ such that:

$$g\tau \coloneqq \frac{a\tau + b}{c\tau + d} = \tau', \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This means that j is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. In fact, a lot more is known. The first thing is to note that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}^+$ through Möbius transformations, in fact $\mathrm{SL}_2(\mathbb{R})$ is the group of biholomorphisms of $\mathbb{H}^+$. Given $g \in \mathrm{GL}_2^+(\mathbb{Q})$ (the + denotes positive determinant), one can rescale g by a positive rational number $N(g)$ such that $N(g)g \in \mathrm{GL}_2(\mathbb{Z})^+$ and the entries of $N(g)g$ are relatively prime. There exists a family of polynomials $\{\Phi_N\}_{N\geq 1} \subseteq \mathbb{Z}[X,Y]$, usually referred to as the *modular polynomials*, that satisfy $\Phi_N(j(x), j(y)) = 0$ if and only if there is $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $\det(N(g)g) = N$ and $x = gy$. The first modular polynomial is $\Phi_1(X,Y) = X - Y$, and $\det(N(g)g) = 1$ if and only if $g \in \mathrm{SL}_2(\mathbb{Z})$, from where we obtain the fact mentioned earlier that j is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. Also $\Phi_N(X,Y)$ is symmetric for $N \geq 2$, but the coefficients become very large very quickly, and so it is not so enlightening to know them explicitly. For a precise construction of these polynomials, see e.g. [**63**, §1.6].

As we have just seen, the modular polynomials give us algebraic properties of the j function. On the analytic side, j is a holomorphic function onto $\mathbb{C}$ that satisfies the following differential equation:

$$\frac{j'''}{j'} - \frac{3}{2}\left(\frac{j''}{j'}\right)^2 + \left(\frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}\right)(j')^2 = 0. \tag{1.2.2}$$

This can be proven using the definition of j given in [**63**, page 22] plus [**63**, Proposition 15]. So, for the purposes of obtaining transcendence results for j and its derivatives, it suffices to consider only j, j' and j''.

Thanks to the Modular Ax-Lindemann-Weierstrass (ALW) theorem below, we can also say that the only algebraic relations that the functions $j(z)$ and $j(gz)$ can satisfy are those given by the modular polynomials. On an algebraic function field K of characteristic zero, $x_1, \ldots, x_n \in K$ are said to be *geodesically independent* if the x_i are nonconstant and there are no relations of the form $x_i = gx_k$, where $i \neq k$ and $g \in \mathrm{GL}_2(\mathbb{Q})^+$.

Theorem 1.2.1 (Modular ALW, [**43**, Theorem 1.1]). *Let $\mathbb{C}(W)$ be an algebraic function field, where $W \subseteq \mathbb{C}^n$ is an irreducible algebraic variety. Suppose that $a_1, \ldots, a_n \in \mathbb{C}(W)$ take values in $\mathbb{H}$ at some $P \in W$. If $a_1, \ldots, a_n$ are geodesically independent, then the $3n$ functions:*

$$j(a_1), \ldots, j(a_n), j'(a_1), \ldots, j'(a_n), j''(a_1), \ldots, j''(a_n)$$

(considered as functions on W locally near P) are algebraically independent over $\mathbb{C}(W)$.

For the more general Ax-Schanuel theorem for j see Theorem 2.4.1.

1.3. Introduction to Shimura Varieties

Shimura varieties are algebraic varieties which parameterise isomorphism classes of Hodge structures. In particular, moduli spaces of principally polarised abelian varieties of a given dimension can be realised as Shimura varieties. The current theory of Shimura varieties is based largely on the work of Deligne: [**18**] and [**19**], which reformulates work of Shimura, Taniyama and Weil on abelian varieties with complex multiplication (see [**53**], [**54**] and [**61**]) in the language of reductive groups and, among other things, extended results of Shimura on the existence of canonical models of these objects. Shimura varieties obtained further notoriety when Langlands, who also coined the term 'Shimura variety', incorporated them as central objects to his program.

A standard introduction to this subject is [**35**], which provides many details and references on the theory of Shimura varieties. On the other hand, as we will only be dealing with connected varieties, [**48**] is also a good starting point. We will use both sources in this work, as well as [**16**], which provides a general overview of the construction of Shimura varieties. An important observation though is that in [**48**] the author deals with "connected mixed Shimura varieties", whereas [**35**] mainly refers to "pure Shimura varieties". In this work we will work in the setting of pure Shimura varieties, only because there are a few results which hold for the pure case for which we could not find a reference in the mixed case. However, we expect that the methods of this thesis can be easily generalised to the mixed case. Whenever possible, we will point out if a certain result can be extended from the pure case to the mixed case.

1.3.1. Congruence Subgroups. Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a linear algebraic group over $\mathbb{Q}$ and let $G(\mathbb{Z})$ denote $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

Definition. A subgroup Γ of $G(\mathbb{Q})$ is *arithmetic* if $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$, in other words, Γ and $G(\mathbb{Z})$ are commensurable.

Definition. Given an embedding of G into GL_n, the *principal congruence subgroup of level N in G* is:

$$\Gamma(N) := \{g \in G(\mathbb{Z}) : g \equiv \mathrm{id} \bmod N\}.$$

This definition depends on the embedding, and to address this issue, we define a *congruence subgroup* of G to be a subgroup Γ of G such that for some N, $\Gamma(N)$ is a finite-index subgroup of Γ. This definition does not depend on the embedding.

DEFINITION. An element $g \in G(\mathbb{Q})$ is called *neat* if for some (for all) faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$, the eigenvalues of $\rho(g)$ in $\mathbb{C}$ generate a torsion-free subgroup of $\mathbb{C}^\times$. A subgroup of $G(\mathbb{Q})$ is *neat*, if all its elements are neat.

PROPOSITION 1.3.1 (see [**35**, Proposition 3.5]). *Let G be an algebraic group over $\mathbb{Q}$, and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. Then Γ contains a neat subgroup Γ' of finite index. Moreover, Γ' can be defined by congruence relations (i.e. for some embedding $G \hookrightarrow \mathrm{GL}_n$ and integer N, $\Gamma' = \{g \in \Gamma : g \equiv \mathrm{id} \bmod N\}$).*

1.3.2. Connected Shimura Varieties. Let $\mathbb{S}$ be the *Deligne torus*, that is, the linear algebraic group over $\mathbb{R}$ such that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$.

A *Shimura datum* is a pair (G, X) consisting of a linear algebraic group G over $\mathbb{Q}$ and the $G(\mathbb{R})$-conjugacy class X of a morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying some conditions on G, the complex structure of X, and the action of G on X: see axioms SV1–SV3 in [**35**, p. 302] for the "pure" axioms, or [**48**, Definition 2.1] for the "mixed" ones. More important than the precise statement of these axioms is that they guarantee that quotients of connected components of X by certain arithmetic subgroups will have an algebraic variety structure. In this thesis, we will use the definitions and notation of [**35**] for pure Shimura varieties.

REMARK 1.3.2. The fact that we do not seem to need the precise statement of the axioms of Shimura varieties could be interpreted as saying that the work we will do here may be generalised to other algebraic varieties obtained as the quotient of some complex domain by the action of a discrete group. However, as it will become apparent, what really makes the model-theory work are the nice geometric properties of the "special" subvarieties (see §1.3.4). Therefore, the present work, as well as [**7**] and [**17**], should be seen as giving insight to the essential properties of the "special structure" of these algebraic varieties, that is, algebraic properties of the special subvarieties which do not depend on the analytic structure of $\mathbb{C}$. Finding an abstract definition of "special structure" is an important question in Diophantine geometry and some attempts at this definition have been made, see for instance [**58**, §2].

Let G^{ad} denote G modulo its centre, let $G^{\mathrm{ad}}(\mathbb{R})^+$ be the connected component of the identity in $G^{\mathrm{ad}}(\mathbb{R})$, and let $G^{\mathrm{ad}}(\mathbb{Q})^+ := G^{\mathrm{ad}}(\mathbb{Q}) \cap G^{\mathrm{ad}}(\mathbb{R})^+$. Let X^+ denote a connected component of X; then its stabiliser in $G^{\mathrm{ad}}(\mathbb{R})$ is in fact $G^{\mathrm{ad}}(\mathbb{R})^+$. We call (G, X^+) a *connected Shimura datum.*

Let K be a compact open subgroup of $G(\mathbb{A}_f)$ (where $\mathbb{A}_f$ denotes the ring of finite rational adèles). Define the double coset space:

$$\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash (X \times (G(\mathbb{A}_f)/K)), \tag{1.3.1}$$

where $G(\mathbb{Q})$ acts diagonally. We will denote the elements of this space by $[x, gK]$. The following lemma will show that $\mathrm{Sh}_K(G,X)$ can be seen as a finite disjoint union of connected components. To that end, let $G(\mathbb{Q})_+$ denote the stabiliser of X^+ in $G(\mathbb{Q})$, and let $\mathscr{C}$ be a set of representatives for the finite set $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$. Given $g \in \mathscr{C}$, let $\Gamma_g := G(\mathbb{Q})_+ \cap gKg^{-1}$ (which is a congruence subgroup of G), and let $[x]_{\Gamma_g}$ denote the class of an element $x \in X^+$ in the quotient $\Gamma_g\backslash X^+$.

Lemma 1.3.3 (see [**35**, Lemma 5.13]). *If X^+ is a connected component of X and we endow $G(\mathbb{A}_f)$ with the adèlic topology, then there is a homeomorphism*

$$\coprod_{g\in\mathscr{C}} \Gamma_g\backslash X^+ \cong G(\mathbb{Q})\backslash (X \times (G(\mathbb{A}_f)/K)) \tag{1.3.2}$$

which is given by

$$\begin{aligned} \Gamma_g\backslash X^+ &\to G(\mathbb{Q})\backslash (X \times (G(\mathbb{A}_f)/K)) \\ [x]_{\Gamma_g} &\mapsto [x, gK]. \end{aligned}$$

A fundamental result of Bailey and Borel (see [**3**]) ensures that, when the Γ_g are neat arithmetic subgroups of the group of holomorphic automorphisms of X^+, the members of the disjoint union in (1.3.2) have canonical realisations as quasi-projective algebraic varieties over $\mathbb{C}$ (see also [**35**, Theorem 3.12] and [**48**, Fact 2.3]). **Unless stated otherwise, we will assume from now on that Γ is neat.**

EXAMPLE 4 (The j function). Take $G = \mathrm{GL}_2$. Let X be the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of the morphism $h : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$, which on real points is given by

$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Then we can see X as $X = \mathbb{H}^+ \cup \mathbb{H}^-$, the union of the upper and lower half-planes. Let $K = \mathrm{GL}_2\left(\widehat{\mathbb{Z}}\right)$, so that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then the map $p : X^+ \to S(\mathbb{C})$ becomes the j function: $j : \mathbb{H}^+ \to \mathbb{C}$ (see §1.2 for the usual construction of j). Note that this is a case where Γ is not neat, but, because it is so well-known, it is a good example to keep in mind.

REMARK 1.3.4. Assuming Γ is neat is not restrictive for our purposes. As we show in Appendix B, a straight-forward generalisation of [**17**, §5.3] shows that one can obtain a categoricity in the case that Γ is not neat, from the categoricity result for the neat case.

EXAMPLE 5 (Siegel moduli spaces). Here we generalise the previous example to construct the moduli space of principally polarised abelian varieties of dimension g (following well-established tradition, in this example we reserve the letter g to be the dimension of the abelian varieties, <u>not</u> an element of the group G). Let:

$$J_g := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix. Consider:

$$\mathrm{GSp}_{2g}(\mathbb{R}) := \left\{ M \in \mathrm{GL}_{2g}(\mathbb{R}) : M^t J_g M = v(M) J_g \right\},$$

where $v(M) \in \mathbb{R}^\times$. Then $v : \mathrm{GSp}_{2g} \to \mathbb{G}_m$ is a homomorphism of linear algebraic groups. So we can define Sp_{2g} as the kernel of v, that is: $\mathrm{Sp}_{2g} := \{M \in \mathrm{GL}_{2g} : M^t J_g M = J_g\}$. Just like in the previous example, we define:

$$\begin{aligned} h : \mathbb{C}^\times &\longrightarrow \mathrm{GSp}_{2g}(\mathbb{R}) \\ a + ib &\mapsto a I_{2g} + b J_g \end{aligned}$$

In this case, $X := \mathrm{GSp}_{2g}(\mathbb{R}) \cdot h$ can be identified with the union of the upper and lower *Siegel half-space*, that is $X = \mathbb{H}_g := \mathbb{H}_g^+ \cup \mathbb{H}_g^-$, where:

$$\mathbb{H}_g^+ := \{Z = A + iB \in M_{g\times g}(\mathbb{C}) : Z = Z^t, B > 0\},$$

and $\mathbb{H}_g^-$ is defined in the same way, but changing $B > 0$ for $B < 0$. The action of $\mathrm{GSp}_{2g}(\mathbb{R})$ on $\mathbb{H}_g$ is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z := (AZ + B)(CZ + D)^{-1}.$$

Now we take $K = \mathrm{GSp}_{2g}\left(\widehat{\mathbb{Z}}\right)$, so $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$. The quotient $\mathcal{A}_g(\mathbb{C}) := \Gamma\backslash\mathbb{H}_g^+$ is the (coarse) moduli space of principally polarised abelian varieties of dimension g. These spaces are also known as *Siegel moduli spaces*. This is yet another case in which Γ is not neat, but the quotient space still has the structure of an algebraic variety.

$\mathrm{Sh}_K(G,X)$ possesses a what is known as a canonical model, that is, $\mathrm{Sh}_K(G,X)$ can be realised canonically as an algebraic variety (this is due to a combination of works by Borovoi, Deligne, Milne and Shimura). The field of definition of this model is known as the *reflex field* $E := E(G,X)$, which is independent of the choice of K (see [**16**, §19] and [**35**, §12, 13, 14] for definitions, details and references). We point out that for all Shimura varities, its reflex field E is a number field (see [**35**, Remark 12.3-a)]).

In the notation of (1.3.1), let $\Gamma = \Gamma_e = G(\mathbb{Q})_+ \cap K$ and let $S(\mathbb{C}) := \Gamma\backslash X^+$, which is a connected component of $\mathrm{Sh}_K(G,X)$. By the theory of Galois actions on connected components of the canonical model, see [**35**, §13], S is defined as an algebraic variety over a finite abelian extension of E, which we will call E^S. Let $p : X^+ \to S(\mathbb{C})$ denote the surjective holomorphic map given by automorphic forms (i.e. p complex holomorphic function obtained by composing the quotient map with the embedding into projective space). In [**35**] and [**48**], $\mathrm{Sh}_K(G,X)$ is called a *Shimura variety*, while the connected component S is called a *connected Shimura variety*. As our work is only concerned with the connected case, we will use the term "Shimura variety" to refer to connected components.

We will indulge in a slight abuse of notation: we will use the same name for Γ and for the image of Γ in $G^{\mathrm{ad}}(\mathbb{Q})^+$. This could be avoided if we only consider what are known as *adjoint Shimura datum.* Given a Shimura datum (G, X) we can form its adjoint datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, by setting G^{ad} as the adjoint of G, and X^{ad} is the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \to G^{\mathrm{ad}}_{\mathbb{R}}$, where h is the homomorphism defining X. Because the centre of G acts trivially on X, this distinction will not be made manifest in the model theory, and so we feel that having this ambiguity about the name Γ will not lead to confusion.

Remark 1.3.5 (Products of Shimura data). Let (G_1, X_1) and (G_2, X_2) be Shimura data, with associated connected Shimura varieties $S_1(\mathbb{C}) = \Gamma_1 \backslash X_1^+$ and $S_2(\mathbb{C}) = \Gamma_2 \backslash X_2^+$. Then $(G_1 \times G_2, X_1 \times X_2)$ is a Shimura datum, $X_1^+ \times X_2^+$ is a connected component of $X_1 \times X_2$ whose stabiliser in $(G_1 \times G_2)^{\mathrm{ad}}(\mathbb{R}) = (G_1^{\mathrm{ad}} \times G_2^{\mathrm{ad}})(\mathbb{R})$ is $G_1^{\mathrm{ad}}(\mathbb{R})^+ \times G_2^{\mathrm{ad}}(\mathbb{R})^+$. And so we can see $S_1 \times S_2$ as the Shimura variety with datum $(G_1 \times G_2, X_1 \times X_2)$ whose complex points are given by $(\Gamma_1 \backslash X_1^+) \times (\Gamma_2 \backslash X_2^+)$.

1.3.3. Classifying Points. Let (G, X) be a Shimura datum. By its definition, a point $x \in X$ corresponds to a homomorphism $x : \mathbb{S} \to G_{\mathbb{R}}$. We can then classify the points of X in terms of the image of the homomorphism they represent in the following way: the *Mumford-Tate group* of x, denoted $\mathrm{MT}(x)$, is the smallest algebraic subgroup H of G defined over $\mathbb{Q}$ such that x factors through $H_{\mathbb{R}}$.

Definition. Let $x \in X^+$ and $p(x) = z \in S(\mathbb{C})$. The *Hecke-orbit* of z is the set of points $z' \in S(\mathbb{C})$ for which there exists $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $p(gx) = z'$. We denote this set by $\mathrm{He}(z)$.

Definition. A point $x \in X$ is said to be *special* if $\mathrm{MT}(x)$ is a subtorus of G. On the other hand, x is called *Hodge-generic* if $\mathrm{MT}(x)$ is G.

If we want to develop a model theory for Shimura varieties, we should use a language that is capable of recognising when the points of X^+ have different Mumford-Tate group (this is done in Remark 3.1.4). We will show some results in §1.3.4 on special subvarieties that will eventually allow us to do that. In the mean time, we recall two results that show that the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on X^+ can already distinguish between special points and Hodge-generic points in the case of pure Shimura varieties. The next theorem can be interpreted as a generalisation of Schneider's theorem ([**50**]).

Theorem 1.3.6 (see [**17**, Theorem 2.3]). *Let (G, X) be a pure Shimura datum and let X^+ be a connected component of X. For any $x \in X^+$ the following are equivalent:*

(1) x is a special point.

(2) There exists $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ such that x is the unique fixed point of g in X^+.

Proposition 1.3.7 (see [**60**, Lemma 2.2]). *Let (G, X) be a pure Shimura datum and let X^+ be a connected component of X. Let $x \in X^+$ be Hodge-generic and suppose $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ fixes x. Then g is the identity.*

Example 6 (Special points of $\mathcal{A}_g$). Using Theorem 1.3.6 we know that the points of $\mathbb{H}_g^+$ which are special are those which are fixed by some element of $\mathrm{GSp}_{2g}(\mathbb{Q})^+$. These are precisely the points that correspond to principally polarised abelian varieties with complex multiplication (CM for short).

1.3.4. Special Subvarieties. Let (G, X) and (H, Y) be Shimura data. Let $\phi : H \to G$ be a homomorphism of algebraic groups over $\mathbb{Q}$ that induces a map $Y^+ \to X^+$ given by $y \mapsto \phi \circ y$. In this case we say that $\phi : (H, Y^+) \to (G, X^+)$ is a *morphism of connected Shimura data*. If furthermore ϕ is an inclusion, then we say that (H, Y^+) is a *connected Shimura subdatum* of (G, X^+).

Let $\Gamma^G \leq G$ and $\Gamma^H \leq H$ be congruence subgroups and let $S_1 = \Gamma^H \backslash Y^+$ and $S_2 = \Gamma^G \backslash X^+$ be corresponding connected Shimura varieties. If in addition to being a morphism of Shimura data, $\phi(\Gamma^H) \subseteq \Gamma^G$, then ϕ induces a map in the quotients $[\phi] : \Gamma^H \backslash Y^+ \to \Gamma^G \backslash X^+$ given by $[y] \mapsto [\phi \circ y]$ which is well-defined, holomorphic, and algebraic with respect to the algebraic structures of S_1 and S_2 (see [**48**, Fact 2.6] for references). The morphisms $S_1 \to S_2$ which are obtained in this way are called *morphisms of Shimura varieties*.

Definition. A *special subvariety* of a Shimura variety S is an irreducible component of the image of a morphism of Shimura varieties $T \to S$. A *special set* in S is a finite union of special subvarieties.

This definition agrees with the definition of special point we gave in §1.3.3. The Mumford-Tate group of a point $x \in X^+$ is a torus if and only if $p(x) \in S(\mathbb{C})$ is a zero-dimensional special subvariety (see [**16**, §18], [**35**, Remark 12.6], or [**48**, §4]).

EXAMPLE 7 (Special subvarieties of $\mathbb{C}^2$). Let us start by looking at $j : \mathbb{H}^+ \to \mathbb{C}$ as a Shimura variety. Given that is it one-dimensional, $\mathbb{C}$ does not have special subvarieties other than the special points (which, as we mentioned before, are exactly those points which correspond to elliptic curves with complex multiplication). We can instead consider $j : (\mathbb{H}^+)^2 \to \mathbb{C}^2$, where j is now defined on each component, which gives us a Shimura surface, and so we can look at its one-dimensional special subvarieties. These consist of the subvarieties cut out by the family $\{\Phi_N(X,Y)\}_{N\in\mathbb{N}}$ of modular polynomials described in §1.2, plus the subvarieties of the form $\{\tau\} \times \mathbb{C}$ or $\mathbb{C} \times \{\tau\}$, where τ is a special point. This result will be generalised in Lemma 1.3.24.

REMARK 1.3.8. Let S be a Shimura variety and $V \subseteq S$ be a special subvariety. By definition this means that it can be realised as the image of a morphism of Shimura varieties $[\phi] : S' \to S$. If (G, X^+) is the connected Shimura datum of S, let H be the image of ϕ in G. Therefore H is a $\mathbb{Q}$-subgroup of G. Also, let X_V^+ be the image of ϕ in X^+. Thus (H, X_V^+) is a connected Shimura datum and, if Γ is the congruence subgroup defining S, then $\Gamma^H := H(\mathbb{Q})_+ \cap \Gamma$ is a congruence subgroup of H such that the inclusion $i : H \hookrightarrow G$ defines a morphism between the connected Shimura varieties $\Gamma^H \backslash X_V^+$ and S with image V. In other words, we can always find a subgroup of G that will define a Shimura datum for V.

We hope that the following commutative diagram becomes a useful visual aid to help the reader with the notation we have introduced so far:

$$\begin{tikzcd} & X_V^+ \subseteq X^+ \arrow[dl, "p^H"'] \arrow[dr, "p"] & \\ \Gamma^H \backslash X_V^+ \arrow[rr, "{[i]}"] & & V \subseteq S(\mathbb{C}) \end{tikzcd}$$

DEFINITION. Let $W \subseteq S$ be a subvariety. A point $x \in X^+$ is said to be *Hodge-generic in W* (or also $p(x)$ is said to be Hodge-generic in W) if $p(x) \in W$ and for every special subvariety V of S for which $V \cap W \subsetneq W$, $p(x) \notin V$. The *Mumford-Tate group of W* is defined to be the Mumford-Tate group of any Hodge-generic point in W.

Given a subset $A \subseteq S(F)$ (where F is some algebraically closed field of characteristic zero), the *special closure* of A, denoted $\mathrm{spcl}(A)$, is the smallest special set of $S(F)$ that contains A. So a point is always Hodge-generic in its special closure.

REMARK 1.3.9. This definition of 'Hodge-generic', agrees with the one given in §1.3.3. This is because the special closure of a point $x \in X^+$ is given by its Mumford-Tate group. This is because $(\mathrm{MT}(x), \mathrm{MT}(x)(\mathbb{R}) \cdot x)$ is a Shimura subdatum of (G, X), and so the image of $(\mathrm{MT}(x)(\mathbb{R})^+ \cdot x) \times \{g\}$ in $\mathrm{Sh}_K(G,X)(\mathbb{C})$ defines the smallest special subvariety containing $[x, g]_K$.

In what remains of this section, fix a Shimura variety $p : X^+ \to S(\mathbb{C})$, let Σ be the set of coordinates of special points of $S(\mathbb{C})$ and let E^S be the minimal abelian extension of the reflex field associated to the Shimura datum of S over which $S(\mathbb{C})$ is defined.

REMARK 1.3.10 (see [**48**, Proposition 4.14]). Every special subvariety contains a dense set of special points, so every special subvariety is defined as an algebraic variety over the field $E^S(\Sigma)$.

DEFINITION. Let $V \subseteq S^n$ be a special subvariety and choose a $\mathbb{Q}$-subgroup H of G^n and a point $x \in p^{-1}(V)$. So $(H, H(\mathbb{R}) \cdot x)$ is a Shimura subdatum of (G^n, X^n) for V. This can be done more canonically (but not completely) if we choose $x \in p^{-1}(V)$ to be Hodge-generic in V and then H can be taken to be the Mumford-Tate group of x. In this case we call the set $X_V^+ := H(\mathbb{R})^+ \cdot x \subseteq X^+$ a *special domain* for V. A set constructed like X_V^+ is called a *special domain* in X^+.

Note that if we choose a point $y \in p^{-1}(V)$ different from x that is Hodge-generic in V, then $H(\mathbb{R})^+ \cdot y$ is another special domain for V which is either equal to $H(\mathbb{R})^+ \cdot x$ or disjoint from it. In fact, for every special domain $X_V^+ \subseteq (X^+)^m$ of V, there is $\gamma \in \Gamma^m$ such that $\gamma X_V^+ = (H(\mathbb{R})^+ \cdot x)^+$. Therefore, we can find at most countably many different special domains for V, so we will index them and denote them by $X_{V,i}^+$, where i is a natural number between 1 and the number of special domains for V (usually ω). This way, if $i \neq j$, then $X_{V,i}^+ \cap X_{V,j}^+ = \varnothing$.

REMARK 1.3.11. Let $X_{V,i}^+$ be a special domain in X^+, and suppose $\gamma \in \Gamma$ is such that $\gamma X_{V,i}^+ = X_{V,i}^+$. If H is the Mumford-Tate group of V, then $\gamma \in H(\mathbb{Q})^+$ (here H is seen as a subgroup of G), because $H(\mathbb{R})^+$ maps surjectively to the group of biholomorphisms of $X_{V,i}^+$ (see [**35**, Proposition 4.8]).

Usually, the order in which we index the special domains will not be important, but we will make one distinction when we define the special subvarieties $Z_{\overline{g}}^{V,i}$ in §1.3.6 (see Remark 1.3.25). One could define a more intrinsic form of indexing the special domains, but as we will not use this extra information, we have opted for a simpler form of notation.

DEFINITION. A *special set* in $(X^{+})^{m}$ is a finite union of special domains. The *special closure* of $B \subseteq (X^{+})^{m}$ is the smallest special set containing B, which again will be denoted by $\mathrm{spcl}(B)$.

EXAMPLE 8 (Trivial special closures). If $x \in X^{+}$ is Hodge-generic, then $\mathrm{spcl}(x) = X^{+}$ and $\mathrm{spcl}(p(x)) = S(\mathbb{C})$. If $x \in X^{+}$ is special, then $\mathrm{MT}(x)$ is a torus, and as it is commutative, we have that $\mathrm{MT}(x)(\mathbb{R})^{+} \cdot x = \{x\}$. So, for some i, $X^{+}_{p(x),i} = \{x\} = \mathrm{spcl}(x)$ and $\mathrm{spcl}(p(x)) = \{p(x)\}$.

REMARK 1.3.12 (A Word on Reflex Fields). We conclude this section with an observation about reflex fields. Given a special subvariety $V \subseteq S$ and a corresponding special domain $X^{+}_{V,i} \subseteq X^{+}$, we can view V as $S_H = \Gamma^{H} \backslash X^{+}_{V,i}$, where $\Gamma^{H} := \Gamma \cap H(\mathbb{Q})_{+}$ and H is the Mumford-Tate group of V. Let E^{S_H} be the (finite abelian extension of the) reflex field of S_H. By [**35**, Remark 12.3], $E^{S} \subseteq E^{S_H}$, and by the general theory of canonical models (see [**35**, §12]), we have that $E^{S_H} \subseteq E^{S}(\Sigma)$.

1.3.5. Galois Representations I. Following [**60**, §2] and [**48**, §6], we describe Galois representations attached to points in Shimura varieties, and we will end by stating Pink's conjecture, which will later influence the way we state conditions FIC 1 & 2.

Given a congruence subgroup Γ' of Γ we have a morphism of connected Shimura varieties $[\mathrm{id}] : \Gamma' \backslash X^{+} \to \Gamma \backslash X^{+}$ induced by the identity map $\mathrm{id} : G \hookrightarrow G$. So as to distinguish elements in the quotients of X^{+} with respect to the different congruence groups, let us denote by $[x]_{\Gamma}$ the class of $x \in X^{+}$ in the quotient $\Gamma \backslash X^{+}$. Recall that Γ is neat.

LEMMA 1.3.13. *For every $\gamma \in \Gamma \smallsetminus \{e\}$ and every $x \in X^{+}$, $\gamma x \neq x$.*

PROOF. Suppose that $\gamma x = x$ for some $x \in X^{+}$ and some $\gamma \in \Gamma$. By definition $\Gamma = G(\mathbb{Q})_{+} \cap K$, but if $\gamma x = x$, then this also means that γ belongs to $Z_{G_{\mathbb{R}}}(x(\mathbb{S}))(\mathbb{R})$. As

Γ is neat, we can use [**60**, Lemma 2.2] and [**60**, Lemma 2.3] to conclude that γ acts trivially. $\square$

Fix a point $z \in S(\mathbb{C}) = \Gamma\backslash X^+$ and choose $x \in X^+$ so that $[x]_\Gamma = z$. Lemma 1.3.13 shows that the fibres of $[\mathrm{id}] : \Gamma'\backslash X^+ \to \Gamma\backslash X^+$ carry a transitive Γ-action. If Γ' is normal in Γ, then this action factors through Γ/Γ', and can be described as follows: for $y \in X^+$ such that $[y]_\Gamma = [x]_\Gamma$ and for every $\gamma \in \Gamma$, define $\gamma[y]_{\Gamma'} := [\gamma y]_{\Gamma'}$. Thus we obtain:

COROLLARY 1.3.14. *The action of* Γ/Γ' *on* $[\mathrm{id}]^{-1}(z)$ *is transitive and free.*[1]

The varieties of the form $\Gamma'\backslash X^+$, where Γ' is a normal congruence subgroup of Γ, form an inverse system of varieties where the morphisms are the maps $[\mathrm{id}]$ described above. Let $\mathbb{X}^+$ be the inverse limit of the system. Any $\tilde{x} \in \mathbb{X}^+$ can be denoted as $([x_i]_{\Gamma_i})_{\Gamma_i}$, so that if $\Gamma_i \subseteq \Gamma_j$, then $[x_i]_{\Gamma_j} = [x_j]_{\Gamma_j}$. Also, there are natural maps

$$\begin{aligned} \pi_{\Gamma_i} : \quad \mathbb{X}^+ \quad & \longrightarrow \Gamma'\backslash X^+ \\ \big([x_i]_{\Gamma_i}\big)_{\Gamma_i} \quad & \mapsto [x_i]_{\Gamma_i}\,, \end{aligned}$$

compatible with the maps of the system. As we mentioned before, the Shimura variety $\mathrm{Sh}_K(G, X)$ is canonically defined over its reflex field E, which is independent of the choice of K. But when we look at a connected component of this variety like $S(\mathbb{C}) = \Gamma\backslash X^+$, it is defined over E^S, which is a finite abelian extension of E. The map $[\mathrm{id}] : S'(\mathbb{C}) = \Gamma'\backslash X^+ \to S(\mathbb{C})$ is defined over the compositum of the fields $E^{S'}$ and E^S. And so, the whole inverse system which gives $\mathbb{X}^+$ is defined over E^{ab}, the maximal abelian extension of E.

Let $L \subseteq \mathbb{C}$ be a finitely generated field extension of E^{ab} such that z (the same z we fixed earlier) is defined over L. Let $\overline{L}$ denote the algebraic closure of L in $\mathbb{C}$. Then the Galois group $\mathrm{Gal}\big(\overline{L}/L\big)$ also acts on the fibre over z of the morphism $[\mathrm{id}] : \Gamma'\backslash X^+ \to \Gamma\backslash X^+$, for every normal subgroup Γ' of Γ. We would like to compare the respective actions of $\mathrm{Gal}\big(\overline{L}/L\big)$ and Γ/Γ' on $[\mathrm{id}]^{-1}(z)$. For this we need some elementary results.

LEMMA 1.3.15 (see [**60**, Lemma 2.4]). *Let H and G be groups, and let X be a set with a left G-action and a compatible right H-action. If the action of H is transitive and*

[1] In the terminology of group actions, 'transitive and free' is the same thing as saying 'simply transitive'.

free, then, fixing any point $x_0 \in X$, there is a unique group homomorphism $\rho_{x_0} : G \to H$ such that for every $g \in G$ $gx_0 = x_0\rho_{x_0}(g)$. Moreover, if $x_1 = x_0 h$, then the associated homomorphism ρ_{x_1} satisfies $\rho_{x_1} = h^{-1}\rho_{x_0}h$.

PROOF. As H acts transitively on X, for every $g \in G$ there is $h \in H$ such that $gx_0 = x_0 h$. If $h_1, h_2 \in H$ are such that $x_0 h_1 = x_0 h_2$, then x_0 is a fixed point of $h_1 h_2^{-1}$, but as the action of H is free, then $h_1 = h_2$. Therefore we have a well-defined map $\rho_{x_0} : G \to H$ given by $\rho_{x_0}(g) = h$, where h is chosen so that $gx_0 = x_0 h$.

Now we show that ρ_{x_0} is a group homomorphism. If $g_1 x_0 = x_0 h_1$ and $g_2 x_0 = x_0 h_2$, then $g_1 g_2 x_0 = g_1 x_0 h_2 = x_0 h_1 h_2$, and so $\rho_{x_0}(g_1 g_2) = h_1 h_2 = \rho_{x_0}(g_1)\rho_{x_0}(g_2)$.

Finally, if x_1 is any other point in X, then, by transitivity, there is $h \in H$ such that $x_1 = x_0 h$. Then $x_1 h^{-1}\rho_{x_0}(g)h = x_0\rho_{x_0}(g)h = gx_0 h = gx_1$, and so $\rho_{x_1} = h^{-1}\rho_{x_0}h$. □

Although in the notation we are using Γ acts on the left on X^+, we can define a corresponding right action of Γ on X^+ in the usual way by $x\gamma := \gamma^{-1}x$, for all $x \in X^+$ and $\gamma \in \Gamma$. Alternatively, Γ is a subgroup of K, and we will see that K has a right action on some subsets of $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$.

LEMMA 1.3.16. *Assuming that Γ' is a normal congruence subgroup of Γ, the actions of $\mathrm{Gal}\left(\overline{L}/L\right)$ and Γ/Γ' on $[\mathrm{id}]^{-1}(z)$ commute.*

PROOF. Every $\gamma \in \Gamma$ defines an automorphism $G \to G$ given by $g \mapsto \gamma g\gamma^{-1}$, which itself induces a morphism of Shimura data $(G, X) \to (G, X)$ which on X is given by $x \mapsto \gamma x$. As Γ' is normal in Γ, this in turn induces a morphism of Shimura varieties $[\gamma] : S'(\mathbb{C}) = \Gamma'\backslash X^+ \to S(\mathbb{C}) = \Gamma\backslash X^+$. This morphism is defined over the compositum $E^S E^{S'}$. As L contains $E^S E^{S'}$, then $[\gamma]$ is fixed under the action of any element of $\mathrm{Gal}\left(\overline{L}/L\right)$, which proves the statement. □

With these results we can now compare the actions of $\mathrm{Gal}\left(\overline{L}/L\right)$ and Γ/Γ' on $[\mathrm{id}]^{-1}(z)$. Choosing $[y]_{\Gamma'} \in [\mathrm{id}]^{-1}(z)$, by Lemma 1.3.15 (whose conditions are ensured by Lemma 1.3.16) we obtain a homomorphism

$$\rho_{[y]_{\Gamma'}} : \mathrm{Gal}\left(\overline{L}/L\right) \to \Gamma/\Gamma'.$$

We remark that for this step we only need L to contain E^S, $E^{S'}$ and the coordinates of z; it is not necessary that L contain all of E^{ab}.

Let $\hat{\Gamma} := \varprojlim \Gamma/\Gamma'$, where the limit is taken over the normal congruence subgroups Γ' of Γ. Both $\mathrm{Gal}\left(\overline{L}/L\right)$ and $\hat{\Gamma}$ act on $\pi_\Gamma^{-1}(z)$, where $\pi_\Gamma : \mathbb{X}^+ \to \Gamma\backslash X^+$ is as above. In fact, the action of $\hat{\Gamma}$ can be understood more naturally in the following way. For the Shimura datum (G, X) define the quotient space

$$\mathrm{Sh}(G,X) := G(\mathbb{Q})\backslash\left(X \times G(\mathbb{A}_f)\right),$$

whose elements we denote as $[x, g]$, and for K a compact open subgroup of $G(\mathbb{A}_f)$, let

$$\begin{aligned} \pi_K : \mathrm{Sh}(G,X) &\to \mathrm{Sh}_K(G,X) \\ [x,g] &\mapsto [x,gK]. \end{aligned}$$

When K is neat, the fibre over any point of $\mathrm{Sh}_K(G,X)$ under π_K has a simply transitive right action by K (see [**60**, Lemma 2.1]) given by $[x,g]k := [x,gk]$. On the other hand, there is an injective map $X^+ \to \mathrm{Sh}(G,X)$ given by $x \mapsto [x,1]$. Observe that for all $\gamma \in \Gamma$ we have that $[\gamma x, 1] = [x, \gamma^{-1}]$. The action of $\hat{\Gamma}$ on $\pi_\Gamma^{-1}(z)$ is then just the restriction of the action of K, and so $\hat{\Gamma}$ acts simply transitively on $\pi_\Gamma^{-1}(z)$. Choosing a point $\tilde{y} \in \pi_\Gamma^{-1}(z)$, we get a continuous homomorphism:

$$\rho_{\tilde{y}} : \mathrm{Gal}\left(\overline{L}/L\right) \to \hat{\Gamma}.$$

We call such a homomorphism a *Galois representation attached to z*.

Most of our future discussion on Galois representations, and on conditions FIC 1 & 2 of the main categoricity theorems, will be dealing with representations of absolute Galois groups on profinite groups. We will therefore need to use some standard results of the topological properties of profinite groups as well as the Krull topology of the absolute Galois group of a field. There are many sources of this, one of which is [**23**, Chapter 1]. One such result which will be used repeatedly, is that a subgroup of $\hat{\Gamma}$ is open if and only if it is closed and of finite-index in $\hat{\Gamma}$ (see [**23**, Remar 1.2.1-(a)]).

REMARK 1.3.17. Keeping the notation we have used (i.e. $S(\mathbb{C}) = \Gamma\backslash X^+$, $z \in S(\mathbb{C})$, $x \in X^+$ and $[x]_\Gamma = z$), let M be the Mumford-Tate group of x. We now form the corresponding connected Shimura datum (M, Y^+), where $Y^+ = M^{\mathrm{ad}}(\mathbb{R})^+ \cdot x$. Let $\Gamma' = \Gamma \cap M(\mathbb{Q})_+$. Defining $S' := \Gamma'\backslash Y^+$ we get a morphism of connected Shimura varieties $S' \to S$, which, by the above discussion and Remark 1.3.12, is defined over E'^{ab}, where E' is the reflex field associated with (M, Y). Let $z' = [x]'_\Gamma$, and so the morphism $S' \to S$ maps $z' \mapsto z$. Let L be a finitely generated extension of E'^{ab} over which z' is defined; hence z is defined over L. We can see Y^+ as a subset of X^+, and in this sense, we can choose the same base point $\widetilde{y}$ for the corresponding homomorphisms $\rho_{\tilde{y}} : \mathrm{Gal}\left(\overline{L}/L\right) \to \hat{\Gamma}$ and $\rho'_{\tilde{y}} : \mathrm{Gal}\left(\overline{L}/L\right) \to \hat{\Gamma}'$. As $\hat{\Gamma}' \subseteq M(\mathbb{A}_f)$, $\hat{\Gamma} \subseteq G(\mathbb{A}_f)$, and $M \leq G$, then the image of $\rho_{\tilde{y}}$ and the image of $\rho'_{\tilde{y}}$ coincide in $G(\mathbb{A}_f)$. In particular, the image of $\rho_{\tilde{y}}$ is contained in $M(\mathbb{A}_f)$. For more details on the functoriality of Galois representations, see [**12**, §4.1].

REMARK 1.3.18. In the case that z is a special point, the homomorphism $\rho_{\tilde{y}}$ can be described using the reciprocity map coming from the theory of complex multiplication (see [**60**, Remark 2.7]).

DEFINITION. With the above notation, $z \in S(\mathbb{C})$ is called *Galois-generic* if the image of $\rho_{\tilde{y}}$ is open in $\hat{\Gamma}$ (observe that openness of the image does not depend on the choice of $\tilde{y}$). The point z is called *strictly Galois-generic* if the image of $\rho_{\tilde{y}}$ equals $\hat{\Gamma}$.

That this definition is well-defined (which in this case means that being Galois-generic does not depend on the different choices of fields we made along the way) is given by [**48**, Proposition 6.4].

PROPOSITION 1.3.19 (see [**48**, Proposition 6.7]). *Every Galois-generic point on $S(\mathbb{C})$ is Hodge-generic.*

REMARK 1.3.20. Combining Proposition 1.3.19 with [**12**, Proposition 6.2.1] and [**12**, 3.3.1.1. Fact] (see also [**51**, §10.6]), we get that any Shimura variety has Hodge-generic points defined over $\overline{\mathbb{Q}}$.

That Galois-generic points exist is given by the following result.

PROPOSITION 1.3.21 (see [**48**, Proposition 6.9]). *Let z be any $\mathbb{C}$-valued point over the generic point of S over E^{ab}. Then z is Galois-generic.*

With all these results as motivation, we are now in a position to state Pink's conjecture.

CONJECTURE 1 (see [**48**, Conjecture 6.8]). Every Hodge-generic point on $S(\mathbb{C})$ is Galois-generic.

This conjecture is closely related to another important conjecture in number theory which we now state. Given a prime number ℓ, let $\mathbb{Q}_\ell$ denote the field of ℓ-adic numbers.

CONJECTURE 2 (Mumford-Tate, see [**37**, §4] and [**60**, Conjecture 4.5]). Let $(G, \mathbb{H}_g)$ be the Shimura datum of a Siegel moduli space (see Example 5). Given $z \in \mathcal{A}_g$, let $\rho_{\tilde{y}} : \mathrm{Gal}\left(\overline{L}/L\right) \to \hat{\Gamma}$ be a corresponding Galois representation. Let $\hat{\Gamma}_\ell$ be the image of $\hat{\Gamma}$ under the projection $\pi_\ell : G(\mathbb{A}_f) \to G(\mathbb{Q}_\ell)$. If z is Hodge-generic, then the image of $\mathrm{Gal}\left(\overline{L}/L\right)$ under $\pi_\ell \circ \rho_{\tilde{y}}$ is open for every prime number ℓ.

In the case of Siegel moduli spaces, [**48**, Remark 6.10] and [**60**, p. 265] explain the relation between Conjecture 1 and the Mumford-Tate conjecture. We will explore this relation in more detail in §4.4 when we study the categoricity of $\mathcal{A}_g$. In the meantime we recall the following result which is a direct consequence of a theorem of J. P. Serre.

THEOREM 1.3.22 (see [**48**, Theorem 6.13]). *Let* $(\mathrm{GSp}_{2g}, \mathbb{H}_g)$ *be the Shimura data of Example 5. Assume that g is odd or equal to 2 or 6. Then for any suitable congruence subgroup Γ, the Hodge-generic points of the corresponding Shimura variety $S(\mathbb{C})$ are Galois-generic.*

1.3.6. Galois Representations II. Here we will describe the aspects of Galois representations described in §1.3.5 that can be "seen" by the model theory. Essentially, the difference with the previous treatment is that we need to restrict the normal subgroups Γ' of Γ to be of a specific form. Other than that, the purpose of this section is mostly to introduce specific objects and notation.

Fix a Shimura variety S with Shimura datum (G, X^+), and let $S(\mathbb{C}) = \Gamma\backslash X^+$. The normal subgroups of Γ we are interested in come from intersecting Γ with a group of the form $g\Gamma g^{-1}$, where $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$. We first recall the following result.

PROPOSITION 1.3.23 (see [**49**, §4.1 Corollary 1]). *Let Γ be an arithmetic subgroup of an algebraic group G defined over $\mathbb{Q}$. Then for any $g \in G(\mathbb{Q})$, $g\Gamma g^{-1}$ is also an arithmetic subgroup of $G(\mathbb{Z})$.*

In particular $\Gamma \cap g\Gamma g^{-1}$ has finite index in Γ, and so it is a congruence subgroup of $G(\mathbb{Z})$. In the following lemma we define the notation for the special subvariety $Z_{\overline{g}}^{V,i}$.

LEMMA 1.3.24 (cf. [**17**, Lemma 2.6]). *Let $\overline{g} = (g_1, \dots, g_n)$ be a tuple of elements belonging to $G^{\mathrm{ad}}(\mathbb{Q})^+$ and let $X_{V,i}^+$ be a special domain in $(X^+)^m$. The image of the map*

$$\begin{aligned} f: \quad X_{V,i}^+ \quad &\longrightarrow \quad S(\mathbb{C})^{mn} \\ (x_1, \dots, x_m) &\mapsto (p(\overline{g}x_1), \dots, p(\overline{g}x_m)) \end{aligned}$$

is a special subvariety which we will denote $Z_{\overline{g}}^{V,i}$.

PROOF. The Shimura datum of $S(\mathbb{C})^{mn}$ is $(G^{mn}, (X^+)^{mn})$. Let H be the Mumford-Tate group of V. Consider the morphism of algebraic groups

$$\begin{aligned} \phi: \quad H \quad &\longrightarrow \quad G^{mn} \\ (h_1, \dots, h_m) &\mapsto \left(g_1 h_1 g_1^{-1}, \dots, g_n h_1 g_n^{-1}, g_1 h_2, g_1^{-1}, \dots, g_n h_m g_n^{-1}\right). \end{aligned}$$

This induces a morphism of connected Shimura data $\phi : (H, X_{V,i}^+) \to (G^{mn}, (X^+)^{mn})$. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ such that if $\Gamma = G(\mathbb{Q})_+ \cap K$, then $S(\mathbb{C}) = \Gamma \backslash X^+$. Then $S(\mathbb{C})^{mn} = \Gamma^{mn} \backslash (X^+)^{mn}$.

Let $\Gamma' = (g_1^{-1}\Gamma g_1 \cap \dots \cap g_n^{-1}\Gamma g_n)^m \cap H(\mathbb{A}_f)$. Notice that $\phi(\Gamma') \subseteq \Gamma^{mn}$. Therefore, ϕ induces a morphism of connected Shimura varieties $\Gamma' \backslash X_{V,i}^+ \to \Gamma^{mn} \backslash (X^+)^{mn}$ whose image is precisely:

$$Z_{\overline{g}}^{V,i} := \left\{ (p(\overline{g}x_1), \dots, p(\overline{g}x_m)) : (x_1, \dots, x_m) \in X_{V,i}^+ \right\}.$$

By construction, $Z_{\overline{g}}^{V,i}$ is a special subvariety. $\square$

REMARK 1.3.25 (Indexing of the special domains of $Z_{\overline{g}}^{V,i}$). It usually does not matter how we index special domains, but for $Z_{\overline{g}}^{V,i}$ we will make one natural exception. We will reserve the index i for the special domain of $Z_{\overline{g}}^{V,i}$ which is parametrised by $X_{V,i}^+$, that is,

$X^+_{Z^{V,i}_{\overline{g}},i} := \overline{g}X^+_{V,i}$. Also, S has only one special domain, which is X^+, so we will simply say $X^+ = X^+_{S,1}$, $Z_{\overline{g}} := Z^{S,1}_{\overline{g}}$, and thus $\overline{g}X^+ = X^+_{Z_{\overline{g}},1}$.

The special subvarieties $Z^{V,i}_{\overline{g}}$ encode information on the Galois representations attached to points in S, and so they will become essential in our model theoretic approach. It is rather crucial understanding how they relate to each other, so we will now recall the treatment done in [**17**, §2.5] and extend it to the subvarieties $Z^{V,i}_{\overline{g}}$.

Let $\overline{g} = (e, g_1, \dots, g_n)$ be a tuple of distinct elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$. The corresponding variety $Z_{\overline{g}}$ is biholomorphic to $\Gamma_{\overline{g}} \backslash X^+$, where:

$$\Gamma_{\overline{g}} := \Gamma \cap g_1^{-1}\Gamma g_1 \cap \cdots \cap g_n^{-1}\Gamma g_n.$$

Suppose $\{h_1, \dots, h_m\}$ is a subset of $G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $\{g_1, \dots, g_n\} \subseteq \{h_1, \dots, h_m\}$. Let $\overline{h} := (e, h_1, \dots, h_m)$. Then there is a morphism of finite type

$$\psi_{\overline{h},\overline{g}} : Z_{\overline{h}} \to Z_{\overline{g}}$$

induced by the natural map $\Gamma_{\overline{h}} \backslash X^+ \to \Gamma_{\overline{g}} \backslash X^+$. If $\Gamma_{\overline{h}}$ is normal in $\Gamma_{\overline{g}}$, then the fibres carry a simply transitive action of $\Gamma_{\overline{g}}/\Gamma_{\overline{h}}$. **For the remainder of this section, we will only consider the case when $\Gamma_{\overline{h}}$ is normal in $\Gamma_{\overline{g}}$.**

The action of $\Gamma_{\overline{g}}/\Gamma_{\overline{h}}$ on the fibers of $\psi_{\overline{h},\overline{g}}$ is given by regular maps defined over $E^S(\Sigma)$. Choose $z \in Z_{\overline{g}}$ and let L be a finitely generated field extension of $E^S(\Sigma)$ such that z is defined on L. Then $\mathrm{Aut}(\mathbb{C}/L)$ acts on the fibre of $\psi_{\overline{h},\overline{g}}$ over z and, because the action of $\Gamma_{\overline{g}}/\Gamma_{\overline{h}}$ is given by regular maps defined over $E^S(\Sigma)$, then this action commutes with the action of $\Gamma_{\overline{g}}/\Gamma_{\overline{h}}$. The action of $\Gamma_{\overline{g}}/\Gamma_{\overline{h}}$ on the fibres of $\psi_{\overline{h},\overline{g}}$ is transitive and (as Γ is neat) free. Therefore, just as in §1.3.5, we obtain a homomorphism corresponding to a given point in the fibre above $z \in Z_{\overline{g}}$

$$\mathrm{Aut}(\mathbb{C}/L) \to \Gamma_{\overline{g}}/\Gamma_{\overline{h}}.$$

If $\overline{z} = (z_1, \dots, z_m) \in Z^{V,i}_{\overline{g}} \subseteq (Z_{\overline{g}})^m$ for some special subvariety $V \subseteq (S(\mathbb{C}))^m$, then the map $\psi_{\overline{h},\overline{g}}$ induces a map

$$\psi^{V,i}_{\overline{h},\overline{g}} : Z^{V,i}_{\overline{h}} \to Z^{V,i}_{\overline{g}}.$$

The stabilizer of the fiber of $\psi^{V,i}_{\overline{h},\overline{g}}$ over $\overline{z}$ is a subgroup of $(\Gamma_{\overline{g}}/\Gamma_{\overline{h}})^m$ which we will denote $\Gamma^{V,i}_{\overline{g},\overline{h}}$. Notice that in this case, if L also contains the coordinates of special points in $S(\mathbb{C})$, the homomorphism $\mathrm{Aut}(\mathbb{C}/L) \to (\Gamma_{\overline{g}}/\Gamma_{\overline{h}})^m$ factors through $\mathrm{Aut}(\mathbb{C}/L) \to \Gamma^{V,i}_{\overline{g},\overline{h}}$.

For every special subvariety $V \subseteq S^m$, we can construct then an inverse system of varieties $Z^{V,i}_{\overline{g}}$, where the tuples $\overline{g}$ are taken so that $\Gamma_{\overline{g}} \triangleleft \Gamma$. This induces an inverse system of groups $\Gamma^{V,i}_{\overline{g},\overline{h}}$. We can therefore define the inverse limits

$$\overline{\Gamma} := \varprojlim_{\overline{g}} \Gamma/\Gamma_{\overline{g}} \text{ and } \overline{\Gamma^{V,i}} := \varprojlim_{\overline{g}} \Gamma^{V,i}_{(e),\overline{g}},$$

where the limits are taken over all tuples of distinct elements $\overline{g} = (e, g_1, \ldots, g_n)$ such that $\Gamma_{\overline{g}}$ is normal in Γ. If $\overline{z}$ has coordinates in an extension L of $E^S(\Sigma)$, then, by our discussion in §1.3.5, we get the conjugacy class of a continuous homomorphism $\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$. We will also refer to this map as a *Galois representation attached to z*. The notation will show whether the Galois representation is on $\hat{\Gamma}$ (like in §1.3.5), or $\overline{\Gamma}$ like above.

1.3.7. Normal Subgroups in Γ. We now show that there are normal subgroups of Γ of the form $\Gamma_{\overline{g}}$. We first recall an important result about arithmetic groups.

THEOREM 1.3.26 (see [**49**, §4.1 Theorem 4.2]). *Let Γ be an arithmetic subgroup of an algebraic group G defined over $\mathbb{Q}$. Then Γ is finitely presented as an abstract group.*

In particular, Γ is finitely generated. Therefore, for any $n \in \mathbb{N}$, there are only finitely many subgroups of Γ of index n (see e.g. [**21**, Proposition 5.11]). Let $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$. Then $\Gamma' := \Gamma \cap g\Gamma g^{-1}$ has finite index in Γ by Proposition 1.3.23. For every $\gamma \in \Gamma$, $\gamma\Gamma'\gamma^{-1}$ has the same index as Γ' in Γ. Given that there are only finitely many subgroups of Γ of this index, the intersection

$$\Gamma'' := \bigcap_{\gamma\in\Gamma} \gamma\Gamma'\gamma^{-1}$$

is a finite intersection. On the other hand, it is clear that Γ'' is normal in Γ. Also $\gamma\Gamma'\gamma^{-1} = \Gamma \cap \gamma g\Gamma g^{-1}\gamma^{-1}$. And so $\Gamma'' = \Gamma_{\overline{g}}$ for some tuple $\overline{g}$ from $G^{\mathrm{ad}}(\mathbb{Q})^+$.

Observe that this argument can be replicated for any subgroup of Γ of the form $\Gamma_{\overline{g}}$ to obtain a normal subgroup of Γ of the form $\Gamma_{\overline{h}}$ such that $\overline{h}$ contains the tuple $\overline{g}$.

Part 2

Transcendence Properties of j-fields

2.1. j-Fields

Given a field K, for any subfield F of K there is a natural action of $\mathrm{GL}_2(F)$ on $\mathbb{P}^1(K) = K \cup \{\infty\}$ given by Möbius tranformations as follows:

$$gx := \frac{ax+b}{cx+d}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $g \in \mathrm{GL}_2(F)$. Whenever we say that $\mathrm{GL}_2(F)$ acts on K, it will be in this manner. Throughout, let $G = \mathrm{GL}_2(\mathbb{Q})$.

Definition. A *j-field* is a two-sorted structue $\langle \mathbb{K}, \mathbb{D}, \alpha, j, j', j'', j''' \rangle$, where:

- $\mathbb{K} = \langle K, +, \cdot, 0, 1 \rangle$ is a field of characteristic zero,
- $\mathbb{D} = \left\langle D, (g)_{g \in G} \right\rangle$ is a G-set (i.e. a set with an action of G),

and $\alpha, j, j', j'', j''' : D \to K$ are maps that satisfy:

(1) α is injective.

(2) For every $z \in D$ and $g \in G$, $\alpha(gz) = g\alpha(z)$.

(3) For every $z \in D$,

$$(j(z) \neq 0 \wedge j(z) \neq 1728 \wedge j'(z) \neq 0) \implies \urcorner(j(z), j'(z), j''(z), j'''(z)) = 0,$$

where $\urcorner \in \mathbb{Q}(X, Y, Z, W)$ is given by:

$$\urcorner(X, Y, Z, W) := \frac{W}{Y} - \frac{3}{2}\left(\frac{Z}{Y}\right)^2 + \left(\frac{X^2 - 1968X + 2654208}{2X^2(X - 1728)^2}\right) Y^2. \tag{2.1.1}$$

Thus $\urcorner(j(z), j'(z), j''(z), j'''(z)) = 0$ corresponds to equation (1.2.2).

(4) The axiom scheme: for every $z_1, z_2 \in D$, if $z_1 = gz_2$, then $\Phi_N(j(z_1), j(z_2)) = 0$ (note that given g we can obtain the value of N as $N = \det(N(g)g)$, so this axiom scheme is first-order expressible). We also include here the expressions obtained by differentiating the modular relations. This means the following. Choosing g and N as before, we have that for every $z \in D$, $\Phi_N(j(z), j(gz)) = 0$. If we interpret this expression in $\mathbb{C}$ and take the derivative with respect to z,

then we get:

$$\Phi_{N1}(j(z),j(gz))j'(z)+\Phi_{N2}(j(z),j(gz))j'(gz)\frac{ad-bc}{(cz+d)^2}=0, \tag{2.1.2}$$

where Φ_{N1} and Φ_{N2} are the derivatives of $\Phi_N(X,Y)$ with respect to the variables X and Y respectively. So, for each $g \in G$, we also include the axiom that says: if $z_1 = gz_2$, then:

$$\Phi_{N1}(j(z_1),j(z_2))j'(z_1)+\Phi_{N2}(j(z_1),j(z_2))j'(z_2)\frac{ad-bc}{(c\alpha(z_1)+d)^2}=0.$$

Deriving equation (2.1.2) again with respect to z again, we get another equation, this time involving $j''(z)$ and $j''(gz)$, that we restate as a first-order axiom as we just did with (2.1.2). Deriving (2.1.2) twice with respect to z, we get an equation involving $j'''(z)$ and $j'''(gz)$, which we likewise restate as an axiom.

(5) The axiom scheme (one $\mathscr{L}_{\omega_1,\omega}$-statement for each $N \in \mathbb{N}$): for all $z_1, z_2 \in D$,

$$\Phi_N(j(z_1),j(z_2))=0 \implies \bigvee_{g\in G,\det(N(g)g)=N}(gz_2=z_1),$$

so this axiom is a converse to part of axiom 4.

(6) The $\mathscr{L}_{\omega_1,\omega}$-statement: let $X \subseteq G$ be the set of non-scalar matrices. Then for every $z \in D$,

$$(j(z)=0 \vee j(z)=1728 \vee j'(z)=0) \implies \bigvee_{g\in X}(gz=z).$$

Note that the values chosen are the same as those which appear in the denominators in axiom 3, and correspond to the points z where the expression $\urcorner(j(z),j'(z),j''(z),j'''(z))$ is not defined.

Unless it is necessary to specify the functions, we will normally denote j-fields just as (K,D).

The name "j-field" and part of this definition come from work of Jonathan Kirby and Adam Harris privately communicated to the author.

Note that from axiom 2 we get that $\alpha(D)\cap\mathbb{Q}=\varnothing$ because the image of α is contained in K and for every element $x \in \mathbb{Q}$ there is $g \in G$ such that $gx = \infty$. Also, the axioms for α

say that D is embedded in K, and this allows us to identify D with $\alpha(D)$. Therefore we will normally avoid mentioning α in the future. For the case of the modular j-function, we take $\alpha : \mathbb{H}^+ \cup \mathbb{H}^- \to \mathbb{C}$ to be the inclusion, where we view $\mathbb{C}$ as $\mathbb{R}(i)$.

REMARK 2.1.1. The set of axioms we have given does not define a complete theory, we have only given the axioms that will be needed for our transcendence results. The most natural way to define a complete set of axioms would be to take the full theory of the j-field $\langle \mathbb{C}, \mathbb{H}^+ \cup \mathbb{H}^-, j, j', j'', j''' \rangle$. Call this theory $\mathrm{Th}(j)$. From a model-theoretic perspective, we should only focus on models of $\mathrm{Th}(j)$, and maybe the name "j-fields" should be reserved for these structures. Even if the theory $\mathrm{Th}(j)$ is not the subject of this thesis, we will point out a couple of things.

(a) If $(K, D) \models \mathrm{Th}(j)$, then any point of D which is fixed by some non-scalar $g \in G$ is $\varnothing$-definable (using terminology we will introduce shortly, this says that the special points are $\varnothing$-definable), and so its image under j is also $\varnothing$-definable. As it turns out, points fixed by non-scalar elements of G are the only points x such that both x and $j(x)$ are algebraic over $\mathbb{Q}$ (this is a famous theorem of Schneider, see [**50**]). Using some known values of j, one can prove that numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are definable in K.

(b) Note that $\mathbb{C} \smallsetminus (\mathbb{H}^+ \cup \mathbb{H}^-)$ defines the real closed field $\mathbb{R}$. So if $(K, D) \models \mathrm{Th}(j)$, then $K \smallsetminus D$ is a real closed field. As real closed fields are neither categorical nor ω-stable, it should not be expected that $\mathrm{Th}(j)$ has these properties. On a different note, it is not obvious how to associate to a given real closed field a corresponding j function on its algebraic closure so as to construct a model of $\mathrm{Th}(j)$. In [**41**] it is proven that the analytic construction of j we showed in §1.2 using genus one tori, need not work in general.

Because of this last point, $\mathrm{Th}(j)$ is not a very nice theory to work with from a model-theoretic point of view. But our aim is to show that, even so, one can obtain interesting number-theoretic results.

DEFINITION. Let (K, D) be a j-field. Let F be a subfield of K. We define G^F to be the subgroup of $\mathrm{GL}_2(F)$ defined as $G^F := \{g \in \mathrm{GL}_2(F) : gD \subseteq D\}$. Note that $G^{\mathbb{Q}} = G$. We say that F is an *active subfield* of K if $G^F = \mathrm{GL}_2(F)$. For example, in the case of $K = \mathbb{C}$, the active subfields of $\mathbb{C}$ are exactly the subfields of $\mathbb{R}$.

A *morphism of j-fields* $\sigma : (K_1, D_1) \to (K_2, D_2)$ is a field morphism $\sigma : K_1 \to K_2$ such that the map $\sigma \circ \alpha_1$ is contained in the image of α_2 and $\widetilde{\sigma} := \alpha_2^{-1} \circ \sigma \circ \alpha_1 : D_1 \to D_2$ satisfies: for every $f \in \{j, j', j'', j'''\}$ and for every $z \in D$ we have $\sigma(f_1(z)) = f_2(\widetilde{\sigma}(z))$. Note that field morphisms respect the action of the group G^F on K, so we clearly have $\widetilde{\sigma}(gz) = g\widetilde{\sigma}(z)$ for every $g \in G$ and $z \in D$.

Let (K, D) be a j-field and let F be a subfield of K. We say that F is a *j-subfield* of K if the pair (F, D_F), where $D_F := \alpha^{-1}(D \cap F)$, satisfies the axioms of j-field with the functions of (K, D) restricted to (F, D_F). Note that if $(K_i)_{i \in I}$ is a family of j-subfields of K, then $\bigcap_{i \in I} K_i$ is a j-subfield of K.

2.2. Geodesic Closures

Fix a j-field (K, D). In this section we study the pregeometry gcl_F defined by the action of G^F on K.

Definition. Let A be a subset of K and F a subfield of K. We define the *F-geodesic closure of A*, denoted $\mathrm{gcl}_F(A)$, as the set of $x \in K$ such that there exist $a \in A$ and $g \in G^F$ such that $x = ga$ (which, save for the exclusion of the point at infinity, is the union of the G^F-orbits of points in A). When $F = \mathbb{Q}$ we will simply write $\mathrm{gcl}(A)$.

It is straightforward to check that for every subfield F of K the operator gcl_F is a pregeometry of trivial type. So given $A, B \subseteq K$, let $\dim^g_F(A/B)$ be the dimension defined by the pregeometry gcl_F, i.e. the number of distinct orbits of elements in A that do not contain elements of B. If $B = \varnothing$ then we write simply $\dim^g_F(A)$.

Let E be a subfield of K. A point $x \in K$ is *E-special* (or *special over E*) if there is a non-scalar $g \in G^E$ such that $gx = x$ (when $E = \mathbb{Q}$ we will simply say that x is *special*).

It is easy to see that if E is an active subfield of K, then $\dim^g_E(E) = 1$. Note that if $L \subseteq F \subseteq K$ are subfields of K, then for every $A \subseteq K$ we have that $\dim^g_F(A) \leq \dim^g_L(A)$.

2.2.1. Geodesic Disjointness.

Definition. Let E, F, L be subfields of K such that $E \subseteq F \cap L$.

(a) F is *linearly disjoint from L over E*, denoted $F \perp_E^l L$, if every finite tuple of elements in L that is linearly independent over E is also linearly independent over F. Equivalently, $F \perp_E^l L$ if and only if for any tuple $\overline{\ell}$ from L, $\mathrm{l.dim}_F\left(\overline{\ell}\right) = \mathrm{l.dim}_E\left(\overline{\ell}\right)$.

(b) F and L are *E-free*, denoted $F \perp_E^f L$, if every finite set of elements of L which is algebraically independent over E is also algebraically independent over F. Equivalently, $F \perp_E^f L$ if and only if for any tuple $\overline{\ell}$ from L, $\mathrm{t.d.}\left(\overline{\ell}/F\right) = \mathrm{t.d.}\left(\overline{\ell}/E\right)$.

(c) We say that F is *geodesically disjoint from L over E*, denoted $F \perp_E^g L$, if for every tuple $\overline{\ell}$ of elements of L that is gcl_E-independent is also gcl_F-independent. Equivalently, $F \perp_E^g L$ if and only if for any tuple $\overline{\ell}$ from L, $\dim_F^g\left(\overline{\ell}\right) = \dim_E^g\left(\overline{\ell}\right)$.

REMARK 2.2.1. Although the definitions are not stated in a symmetric way, both $\perp^l$ and $\perp^f$ are symmetric relations (see [**31**, p. 360] and [**31**, p. 362]).

We now state some simple properties about geodesic disjointness.

LEMMA 2.2.2. *Let E, F, L be subfields of K such that $E \subseteq F \cap L$. Let $E' = F \cap L$.*

(a) If $F \perp_E^g L$, then $F \perp_{E'}^g L$.

(b) If $G^E = G^F$, then $F \perp_E^g L$.

(c) If $F \perp_{E'}^l L$ and E' is active, then $F \perp_{E'}^g L$.

(d) If $F \perp_{E'}^g L$ and $G^E = G^{E'}$, then $F \perp_E^g L$.

PROOF. (a) As $E \subseteq E' \subseteq F$, then for any tuple $\overline{\ell}$ from L we have that $\dim_E^g\left(\overline{\ell}\right) \geq \dim_{E'}^g\left(\overline{\ell}\right) \geq \dim_F^g\left(\overline{\ell}\right)$. $F \perp_E^g L$ means that $\dim_E^g\left(\overline{\ell}\right) = \dim_F^g\left(\overline{\ell}\right)$, so $\dim_{E'}^g\left(\overline{\ell}\right) = \dim_F^g\left(\overline{\ell}\right)$, which means that $F \perp_{E'}^g L$.

(b) Immediate from definition.

(c) Let $\ell_1, \ell_2 \in L$, $g \in G^F$, and suppose that $g\ell_1 = \ell_2$. This equation can be restated as:

$$a\ell_1 + b = c\ell_1\ell_2 + d\ell_2,$$

where $a, b, c, d \in F$ are the entries of g is the usual way. This is an F-linear combination of the elements $1, \ell_1, \ell_2, \ell_1\ell_2 \in L$, so by linear disjointness there are coefficients $a', b', c', d' \in E'$ (not all zero) such that

$$a'\ell_1 + b' = c'\ell_1\ell_2 + d'\ell_2.$$

If $c'\ell_1 + d' = 0$ or $a'\ell_1 + b' = 0$, then $\ell_1 \in E'$ (because the coefficients are not all zero). As $g\ell_1 = \ell_2$, then $\ell_2 \in F$, and so $\ell_2 \in E'$. As $G^{E'}$ acts transitively on E', then there is $g' \in G^{E'}$ such that $g\ell_1 = \ell_2$.

So now we assume that $c'\ell_1 + d' \neq 0$ and $a'\ell_1 + b' \neq 0$. If the vectors (a, b) and (c, d) are E'-linearly dependent, then that would mean $\ell_2 \in E'$, which in turn would mean that $\ell_1 \in E'$, bringing us back to the situation of the previous paragraph. Otherwise, $a'd' \neq b'c'$, and so a', b', c', d' define an element $h \in G^{E'}$ satisfying $h\ell_1 = \ell_2$.

(d) Immediate from definition.

□

Lemma 2.2.3. *Let E, L be subfields of K such that $E \subseteq L$ and E is active. Suppose that $\overline{x}$ is a finite tuple from K which is algebraically independent over L and that there are non-scalar elements in $G^{E(\overline{x})} \setminus G^E$. Then:*

(a) $E(\overline{x}) \perp^g_E L$.

(b) If $\ell \in L$ is special over $E(\overline{x})$, then it is special over E.

Proof. Let $\ell_1, \ell_2 \in L$ (not necessarily distinct) and let $g \in G^{E(\overline{x})} \setminus G^E$ be non-scalar. Suppose that $\ell_1 = g\ell_2$. First observe that either $\ell_1, \ell_2 \in E$ or $\ell_1, \ell_2 \in L \setminus E$. To see this, suppose first that $\ell_2 \in E$. Then, as $\ell_1 = g\ell_2$ and $g \in G^{E(\overline{x})}$, we get that $\ell_1 \in E(\overline{x})$. But as $E(\overline{x}) \cap L = E$, then $\ell_1 \in E$. Reversing the roles of ℓ_1 and ℓ_2, we also get that if $\ell_2 \notin E$, then $\ell_1 \notin E$.

If $\ell_1, \ell_2 \in E$, then, as E is active, there is $h \in G^E$ such that $h\ell_2 = \ell_1$, so we get that $\dim^g_E(\ell_1, \ell_2) = 1$, proving (a). On the other hand, it is a simple exercise to see that, because E is active, every point in E is E-special, proving (b).

So from now on we assume that $\ell_1, \ell_2 \in L \setminus E$. Note that we can take the entries of g to be in $E[\overline{x}]$. Write:

$$g = \begin{pmatrix} a(\overline{x}) & b(\overline{x}) \\ c(\overline{x}) & d(\overline{x}) \end{pmatrix},$$

where $a(\overline{x}), b(\overline{x}), c(\overline{x}), d(\overline{x}) \in E[\overline{x}]$, they are not all constant polynomials, and there is no common non-constant factor to all four polynomials. To set notation, write:

$$a(\overline{x}) = \textstyle\sum_i a_i \overline{x}^i, \quad b(\overline{x}) = \sum_i b_i \overline{x}^i, \quad c(\overline{x}) = \sum_i c_i \overline{x}^i, \quad d(\overline{x}) = \sum_i d_i \overline{x}^i;$$

here we are using multi-index notation, so that i is a tuple of non-negative integers. From the equation $\ell_1 = g\ell_2$, we obtain:

$$\sum_i (c_i\ell_1\ell_2 + d_i\ell_1)\overline{x}^i = \sum_i (a_i\ell_2 + b_i)\overline{x}^i.$$

Given that $a_i, b_i, c_i, d_i \in E \subseteq L$ and that $\overline{x}$ is algebraically independent over L, we deduce that the coefficient of $\overline{x}^i$ is zero for every i. So we obtain that for all i:

$$\ell_1(c_i\ell_2 + d_i) = a_i\ell_2 + b_i.$$

Let $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. For some i we must have $g_i \neq 0$, say this happens for $i = n$. If $\det(g_n) = 0$, this would mean that the rows of g_n are linearly dependent over E. Suppose that $(c_n, d_n) \neq (0,0)$ (a similar argument can be made if instead we assume $(a_n, b_n) \neq (0,0)$). If $(c_n\ell_2 + d_n) = 0$, then $\ell_2 \in E$, which is a contradiction. So let us assume $(c_n\ell_2 + d_n) \neq 0$. As $\det(g_n) = 0$, we may assume that the rows of g_n are linearly dependent over E, i.e. there exists $\lambda_n \in E$ such that $a_n = \lambda_n c_n$ and $b_n = \lambda_n d_n$. But this would mean that $\ell_1 \in E$, again arriving at a contradiction.

So for all i either $g_i = 0$ or $\det(g_i) \neq 0$. If every matrix g_i is scalar, then so would g be. Therefore there is some i for which g_i is non-scalar, $\det(g_i) \neq 0$ (so $g_i \in G^E$), and $g_i\ell_2 = \ell_1$. So ℓ_1 and ℓ_2 are gcl_E-dependent, proving (a). Finally, if $\ell_1 = \ell_2$, then ℓ_1 is special over E, proving (b). □

LEMMA 2.2.4. *Let E, F, L be subfields of K such that $E \subseteq F \cap L$. Suppose $F \perp^g_E L$. Then for any tuple $\overline{x}$ from K and any $A \subseteq L$, we have:*

$$\dim^g_F(\overline{x}/L) - \dim^g_E(\overline{x}/L) \leq \dim^g_F(\overline{x}/A) - \dim^g_E(\overline{x}/A).$$

PROOF. Let $\overline{\ell} \in L$ be a tuple such that $\dim^g_F\left(\overline{x}/\overline{\ell}A\right) = \dim^g_F(\overline{x}/L)$ and $\dim^g_E\left(\overline{x}/\overline{\ell}A\right) = \dim^g_E(\overline{x}/L)$. The addition formula gives us two ways to calculate $\dim^g_F\left(\overline{x}, \overline{\ell}/A\right)$:

$$\begin{aligned} \dim^g_F\left(\overline{x}, \overline{\ell}/A\right) &= \dim^g_F(\overline{x}/A) + \dim^g_F\left(\overline{\ell}/\overline{x}A\right) \\ &= \dim^g_F\left(\overline{\ell}/A\right) + \dim^g_F\left(\overline{x}/\overline{\ell}A\right). \end{aligned}$$

Therefore:

$$\begin{aligned}\dim_F^g(\overline{x}/A)-\dim_F^g\left(\overline{x}/\overline{\ell}A\right) &= \dim_F^g\left(\overline{\ell}/A\right)-\dim_F^g\left(\overline{\ell}/\overline{x}A\right)\\ &= \dim_E^g\left(\overline{\ell}/A\right)-\dim_F^g\left(\overline{\ell}/\overline{x}A\right)\\ &\geq \dim_E^g\left(\overline{\ell}/A\right)-\dim_E^g\left(\overline{\ell}/\overline{x}A\right)\\ &= \dim_E^g(\overline{x}/A)-\dim_E^g\left(\overline{x}/\overline{\ell}A\right).\end{aligned}$$

□

Another result about geodesic disjointness can be found in Appendix A.

2.3. j-Derivations

In this section we introduce j-derivations and the pregeometry $j$cl they define. The exposition is analogous to [**28**, §4].

Definition. Let K be a field and M a K-vector space. A map $\partial : K \to M$ is a called a *derivation* if it satisfies for every $a, b \in K$:

(1) $\partial(a+b) = \partial(a) + \partial(b)$.

(2) $\partial(ab) = a\partial(b) + b\partial(a)$.

Let (K, D) be a j-field. The map $\partial : K \to M$ is called a *j-derivation* if it is a derivation and it satisfies for every $z \in D$:

(3) $\partial(j(z)) = j'(z)\partial(z), \quad \partial(j'(z)) = j''(z)\partial(z), \quad \partial(j''(z)) = j'''(z)\partial(z)$.

For $C \subseteq K$, let $\mathrm{Der}(K/C, M)$ denote the set of derivations $\partial : K \to M$ such that $\partial(c) = 0$ for every $c \in C$. Let $j\mathrm{Der}(K/C, M)$ be the set of j-derivations $\partial : K \to M$ satisfying $\partial(c) = 0$ for every $c \in C$. For convenience we write $\mathrm{Der}(K/C) := \mathrm{Der}(K/C, K)$ and $j\mathrm{Der}(K/C) := j\mathrm{Der}(K/C, K)$. Note that all these spaces are K-vector spaces.

Let C be a subset of K. Define $\Omega(K/C)$ as the K-vector space generated by formal symbols of the form dr, where $r \in K$, quotiented by the relations given by the axioms of derivations plus that for every $c \in C$, $dc = 0$. Denote by $d : K \to \Omega(K/C)$ the map $r \mapsto dr$. The map d is called the *universal derivation on C*. Let $\Xi(K/C)$ be obtained from $\Omega(K/C)$ by taking the quotient with the axioms for j-derivations. This induces a map $d_j : K \to \Xi(K/C)$ which is called the *universal j-derivation.*

PROPOSITION 2.3.1. *Let $C \subseteq K$. For every j-derivation $\partial : K \to M$ which vanishes on C, there exists a unique K-linear map $\partial^* : \Xi(K/C) \to M$ such that $\partial^* \circ d_j = \partial$.*

PROOF. For every $r \in K$, set $\partial^*(d_j r) = \partial(r)$ and then extend linearly. $\square$

Clearly, a similar proposition exists characterising the universal property of $\Omega(K/C)$. These universal properties in fact show that $\mathrm{Der}(K/C)$ and $j\mathrm{Der}(K/C)$ are the dual spaces of $\Omega(K/C)$ and $\Xi(K/C)$ respectively.

DEFINITION. Let $C \subseteq K$, and $a \in K$. We say that a belongs to the *j-closure* of C, denoted $a \in j\mathrm{cl}(C)$, if for every $\partial \in j\mathrm{Der}(K/C)$ we have that $\partial(a) = 0$. That is:

$$j\mathrm{cl}(C) = \bigcap_{\partial \in j\mathrm{Der}(K/C)} \ker \partial.$$

In particular this means that for any $A \subseteq K$ and any $\partial \in j\mathrm{Der}(K/A)$ we have that $\ker \partial$ is $j\mathrm{cl}$-closed. For any $C \subseteq K$, $F = j\mathrm{cl}(C)$ defines a j-subfield of K. Indeed, as every j-derivation is an additive homomorphism, $j\mathrm{cl}(C)$ is a group under addition. The second axiom of j-derivations shows that it is closed under multiplication. Finally, as 1 is in the kernel of any j-derivation, then the multiplicative inverse of every nonzero element of F is in F. So F is a field. The other axioms of j-derivations show that we can restrict j to F and still obtain a j-field. For this, first define $D_F = D \cap F$. Then we can define $\alpha_F : D_F \to F$ as the restriction of α to D_F. Let $z \in D_F$ and let $\partial \in j\mathrm{Der}(K/C)$. Then as $\partial(j(z)) = j'(z)\partial(\alpha(z))$ we get that $\partial(j(z)) = 0$, and so $j(z) \in F$. Similarly $j'(z), j''(z) \in F$. The fact that $j'''(z) \in F$ comes from axiom 3 of j-fields and the fact that F is relatively algebraically closed in K. The final thing to check is that D_F is a G-set. But this follows because for every $g \in G$, $\alpha_F(gz)$ is algebraic over F (because $\alpha_F(z) \in F$), and so $\partial(\alpha_F(gz)) = 0$. Therefore (F, D_F) is a j-subfield of K.

REMARK 2.3.2. Note that $a \in j\mathrm{cl}(C)$ if and only if $d_j a = 0$ in $\Xi(K/C)$. Clearly if $d_j a = 0$, then $a \in j\mathrm{cl}(C)$. Conversely suppose that $d_j a \neq 0$. Assuming $\Xi(K/C) \neq 0$, we can choose a linear transformation $\eta : \Xi(K/C) \to K$ such that $\eta(d_j a) \neq 0$. It is a straightforward exercise to show that $\eta \circ d_j \in j\mathrm{Der}(K/C)$. Therefore, there exists $\partial \in j\mathrm{Der}(K/C)$ such that $\partial(a) \neq 0$. From this we can conclude that $j\mathrm{Der}(K/C) = j\mathrm{Der}(K/j\mathrm{cl}(C))$.

Recall the following standard results about derivations.

LEMMA 2.3.3 (see [**31**, Chapter VIII, §5]). *Let K be a field and $\partial : K \to K$ be a derivation.*

(1) Let $C \subseteq K$ and let $a \in K$ be algebraic over $\mathbb{Q}(C)$. Then every derivation which vanishes on C, also vanishes on a.

(2) Let X be transcendental over K and let $a \in K(X)$. Then ∂ can be extended to a derivation $\partial' : K(X) \to K(X)$ such that $\partial'(X) = a$.

(3) If L is a separable algebraic extension of K, then ∂ extends to a derivation on L.

LEMMA 2.3.4 (see [**34**, Lemma 6.7]). *Let $K_1 \subseteq K_2$ be a field extension. Then*

$$\mathrm{l.dim.}_{K_2}\Omega\left(K_2/K_1\right) = \mathrm{t.d.}\left(K_2/K_1\right).$$

LEMMA 2.3.5 (see [**2**, Theorem 3]). *Let K be a field and Δ a set of derivations on K. Let $x_1, \ldots, x_n \in K$ and set $r = \mathrm{rk}(\partial x_i)_{\partial\in\Delta, i=1,\ldots,n}$. Then there is a set of derivations $\widetilde{\Delta}$ and $\partial'_1, \ldots, \partial'_n \in \widetilde{\Delta}$ such that: $\partial'_i(x_k) = \delta_{ik}$ (the Kronecker delta), for every $\partial \in \widetilde{\Delta} \setminus \{\partial'_1, \ldots, \partial'_n\}$ and every $i = 1, \ldots, n$ we have that $\partial(x_i) = 0$; and the elements of $\widetilde{\Delta}$ are K-linear combinations of the elements of Δ. Furthermore, $\bigcap_{\partial\in\Delta} \ker\partial = \bigcap_{\partial\in\widetilde{\Delta}} \ker\partial$. In particular, $\mathrm{rk}(\partial x_i)_{\partial\in\widetilde{\Delta}, i=1\ldots,n} = r$.*

PROOF. Assume that $\partial_1, \ldots, \partial_r \in \Delta$ and $x_1, \ldots, x_r$ are such that: $\det(\partial_i x_k)_{i,k=1,\ldots,r} \neq 0$. Let $A = (a_{ik}) = (\partial_i x_k)^{-1}_{i,k=1,\ldots,r}$ and set $\partial'_i = \sum_{k=1}^r a_{ik}\partial_k \in \mathrm{Der}(F/C)$. In this way $\partial'_i(x_k) = \delta_{ik}$.

For each $\partial \in \Delta$ there exist unique $b_i(\partial) \in F$ such that: $\partial(x_k) = \sum_{i=1}^r b_i(\partial)\partial_i(x_k)$ for $k = 1, \ldots, r$. Set $\widetilde{\partial} = \partial - \sum_{i=1}^r b_i(\partial)\partial_i$ and $\widetilde{\Delta} = \left\{\widetilde{\partial} : \partial \in \Delta\right\} \cup \{\partial'_1, \ldots, \partial'_r\}$.

Let $C = \bigcap_{\partial\in\Delta} \ker\partial$. Let $x \in K$ be such that $\partial(x) = 0$ for every $\partial \in \widetilde{\Delta}$. In particular $\partial'_i(x) = 0$, for $i = 1, \ldots, r$. Given that A is invertible, we get that $\partial_i(x) = 0$ for $i = 1, \ldots, r$. Therefore, for each $\partial \in \Delta$ we have that $0 = \widetilde{\partial}(x) = \partial(x) - \sum_{i=1}^r b_i(\partial)\partial_i(x) = \partial(x)$, which means that $x \in C$.

Conversely, it is immediate that if $x \in C$, then $\partial(x) = 0$ for every $\partial \in \widetilde{\Delta}$. □

Now we will check that $j\mathrm{cl}$ satisfies the axioms of pregeometries. The following lemma is straightforward.

LEMMA 2.3.6 (see [**62**, Theorem 1.10]). *Given subsets $B, C \subseteq K$ we have that:*

(a) $C \subseteq j\mathrm{cl}(C)$.

(b) $B \subseteq C \implies j\mathrm{cl}(B) \subseteq j\mathrm{cl}(C)$.

(c) $j\mathrm{cl}(j\mathrm{cl}(C)) = j\mathrm{cl}(C)$.

LEMMA 2.3.7 (Finite character of $j\mathrm{cl}$). *Let (K, D) be a j-field, $C \subseteq K$. If $a \in j\mathrm{cl}(C)$, then there is a finite set $C_0 \subseteq C$ such that $a \in j\mathrm{cl}(C_0)$. Furthermore, there is a finitely generated j-subfield K_0 of K such that $C_0 \subseteq K_0$ and $a \in j\mathrm{cl}_{K_0}(C_0)$.*

PROOF. Let $\mathcal{L}$ be the language that consists purely of constant symbols, one for each element of $\Xi(K/\varnothing)$. Let T be the $\mathcal{L}$-theory which says that these constant symbols satisfy the axioms of K-vector spaces, the axioms saying that d_j is a j-derivation, and also $d_j c = 0$ for each $c \in C$. Then $\Xi(K/C) \vDash T$ and $\Xi(K/C) \vDash d_j a = 0$ by Remark 2.3.2. By the universal property of $\Xi(K/C)$, every other j-derivation $\partial : K \to M$ which vanishes over C also satisfies $\partial(a) = 0$. Also, every such M is a model of T. Therefore $T \vDash d_j a = 0$. By the Completeness Theorem $T \vdash d_j a = 0$. There are only finitely many elements of C used in the proof of $d_j a = 0$. Let T_0 be the set of formulas used in the proof of $d_j a = 0$ (thus, T_0 is finite). Let C_0 consist of the elements $c \in C$ such that the formula $d_j c = 0 \in T_0$. Let K_0 be the j-subfield of K generated by all the $x \in K$ which appear in some axiom of T_0. Note that $C_0 \subseteq K_0$. Then $d_j a = 0$ in $\Xi(K_0/C_0)$ and in $\Xi(K/C_0)$. Therefore $a \in j\mathrm{cl}_{K_0}(C_0)$ and $a \in j\mathrm{cl}_K(C_0)$. ☐

LEMMA 2.3.8 (Exchange Principle for $j\mathrm{cl}$). *Let $C \subseteq K$, $a, b \in K$. If $a \in j\mathrm{cl}(Cb)$ and $a \notin j\mathrm{cl}(C)$, then $b \in j\mathrm{cl}(Ca)$.*

PROOF. Suppose $a \in j\mathrm{cl}(Cb)$ and $b \notin j\mathrm{cl}(Ca)$. Then there is $\partial \in j\mathrm{Der}(K/C)$ such that $\partial(a) = 0$ and $\partial(b) = 1$. Let $\partial' \in j\mathrm{Der}(K/C)$ and let $\partial'' = \partial' - (\partial'(b))\partial$. Then $\partial''(a) = \partial'(a) - (\partial'(b))\partial(a) = \partial'(a)$. Also $\partial''(b) = \partial'(b) - (\partial'(b))\partial(b) = 0$. So $\partial'' \in j\mathrm{Der}(K/Cb)$, which means that $\partial''(a) = 0$, therefore $\partial'(a) = 0$. Therefore $a \in j\mathrm{cl}(C)$. ☐

Combining these lemmas we get that $j\mathrm{cl}$ defines a pregeometry on any j-field.

DEFINITION. $C \subseteq K$ is said to be $j$cl-*closed* if $C = j\mathrm{cl}(C)$. C is said to be $j$cl-*independent* if it is an independent set with respect to $j$cl. If C is $j$cl-independent and consists of only one element $C = \{\tau\}$, then we will say τ is j-*generic*. For $A, C \subseteq K$, let $\dim^j(A/C)$ denote the dimension of A over C with respect to $j$cl.

DEFINITION. Let (K, D) be a j-field and let $\tau_1, \ldots, \tau_m \in K$ be $j$cl-independent. A *corresponding system of j-derivations for $\overline{\tau}$* is a set of j-derivations $\partial_1, \ldots, \partial_m \in j\mathrm{Der}(K)$ such that $\partial_i(\tau_k) = \delta_{ik}$.

REMARK 2.3.9. Lemma 2.3.5 shows that (if one restricts the proof so that it only uses j-derivations) for any $j$cl-independent set, there is a corresponding system of j-derivations.

PROPOSITION 2.3.10. *Let $C \subseteq K$ be $j$cl-closed, and let $z \in D$.*

(a) $z \in C \implies \{j(z), j'(z), j''(z)\} \subseteq C$

(b) Suppose $j'''(z) \neq 0$. If $j^{(t)}(z) \in C$ for some $t \in \{0, 1, 2\}$, then $z \in C$.

PROOF. (a) Let $\partial \in j\mathrm{Der}(K/C)$. If $\partial(z) = 0$, then by axioms of j-derivations we get $\partial(j^{(t)}(z)) = j^{(t+1)}(z)\partial(z) = 0$, for $t \in \{0, 1, 2\}$. As ∂ was arbitrary in $j\mathrm{Der}(K/C)$, we are done.

(b) Let $\partial \in j\mathrm{Der}(K/C)$. Suppose first that $j''(z) \in C$. Then as $0 = \partial(j''(z)) = j'''(z)\partial(z)$, we conclude $\partial(z) = 0$. Assume now that $j'(z) \in C$. Then $0 = \partial(j'(z)) = j''(z)\partial(z)$. If $\partial(z) \neq 0$, then $j''(z) = 0 \in C$, but then we are in the previous case and we get a contradiction. So $\partial(z) = 0$. Finally assume that $j(z) \in C$. Then $0 = \partial(j(z)) = j'(z)\partial(z)$. Reasoning as before, if $\partial(z) \neq 0$, then $j'(z) = 0 \in C$, and so we are in the previous case, again arriving at a contradiction. Therefore, in all cases we arrive at $\partial(z) = 0$. As ∂ was arbitrary in $j\mathrm{Der}(K/C)$, we are done.

□

REMARK 2.3.11. The special points over $\mathbb{Q}$ are in $j\mathrm{cl}(\varnothing)$. This is because the equation $gx = x$ is either linear or quadratic over $\mathbb{Q}$, so special points have degree 2 over $\mathbb{Q}$.

PROPOSITION 2.3.12. *Let $z_1, \ldots, z_n \in D$ be such that $z_1, \ldots, z_n$ are $j$cl-independent over $C \subseteq K$. Then we have that $\dim^g(z_1, \ldots, z_n/C) = n$.*

Proof. Clearly $\dim^g(z_1,\dots,z_n/C) \le n$. Suppose that $\dim^g(z_1,\dots,z_n/C) < n$. Then, without loss of generality, we may assume that $z_1 \in \mathrm{gcl}(C, z_2,\dots,z_n)$. This means that there is $x \in C \cup \{z_2,\dots,z_n\}$ and $g \in G$ such that $z_1 = gx$. Every j-derivation which vanishes on x vanishes also on z_1. This means that $z_1 \in j\mathrm{cl}(x)$, which contradicts that $z_1,\dots,z_n$ are $j$cl-independent over C. □

Lemma 2.3.13. *Let (K,D) be a j-field and $C \subseteq K$. Then*

$$\dim^j(K/C) \le \mathrm{l.dim}_K \Xi(K/C).$$

Proof. Suppose $t_1,\dots,t_n$ are $j$cl-independent over C. Then, as in Lemma 2.3.5, there are j-derivations $\partial_i : K \to K$ such that $\partial_i(t_k) = \delta_{ik}$. Consider the universal j-derivation $d_j : K \to \Xi(K/C)$ and for each i choose $\chi_i : \Xi(K/C) \to K$ such that $\partial_i = \chi_i \circ d_j$. Suppose that $a_1,\dots,a_n \in K$ satisfy:

$$\sum_{i=1}^n a_i d_j t_i = 0.$$

Then, applying χ_k to this equation we get $\sum a_i \partial_k t_i = 0$, that is to say $a_k = 0$. Therefore $d_j t_1,\dots,d_j t_n$ are K-linearly independent. □

To summarise the results we have shown about dimensions of universal derivations, we may write:

$$\begin{aligned}\dim^j(K/C) \le \mathrm{l.dim}_K \Xi(K/C) &= \mathrm{l.dim}_K \Xi(K/j\mathrm{cl}(C)) \\ &\le \mathrm{l.dim}_K \Omega(K/j\mathrm{cl}(C)) = \mathrm{t.d.}(K/j\mathrm{cl}(C)).\end{aligned}$$

2.3.1. j-Derivations on $\mathbb{C}$. Now we prove that there are non-trivial j-derivations on $\mathbb{C}$. It is well known that j has a standard fundamental domain:

$$\mathcal{F} := \left\{ z = x+iy \in \mathbb{H}^+ : -\frac{1}{2} \le x \le 0, |z| \ge 1 \right\} \cup \left\{ z = x+iy \in \mathbb{H}^+ : 0 < x < \frac{1}{2}, |z| > 1 \right\}.$$

For every $z \in \mathbb{H}^+$ there exists $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $gz \in \mathcal{F}$. Any set of the form $g\mathcal{F}$, with $g \in \mathrm{SL}_2(\mathbb{Z})$ is called a *fundamental domain of j*. Now note that the differential equation (1.2.2) of j is defined when $j'(z) \ne 0$ and $j(z)(j(z)-1728) \ne 0$. If we restrict z to $\mathcal{F}$, then it is well known that:

- if $j'(z) = 0$, then $z = i$ or $z = \rho$, where $\rho = e^{2\pi i/3}$,

- if $j(z) = 0$, then $z = \rho$,
- if $j(z) = 1728$, then $z = i$.

PROPOSITION 2.3.14. *Let ∂ be a j-derivation on $\mathbb{C}$. Let $z \in \mathbb{H}^+ \cup \mathbb{H}^-$ not be in the $\mathrm{SL}_2(\mathbb{Z})$-orbits of i, $-i$, ρ and $\overline{\rho}$. Then for every $n \in \mathbb{N}$ we get that:*

$$\partial\left(j^{(n)}(z)\right) = j^{(n+1)}(z)\partial(z).$$

PROOF. To prove the result for $n \geq 3$, we express equation (1.2.2) in the following form:

$$j''' = \frac{3(j'')^2}{2j'} - \frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}(j')^3. \tag{2.3.1}$$

Now we operate this equation in two ways: on one hand we differentiate it (as holomorphic functions), and on the other, apply ∂ to it. Compare both results to get that $\partial(j''') = j^{(4)}(z)\partial(z)$. By induction, we obtain the desired result. □

Given that i and ρ are algebraic over $\mathbb{Q}$, we get the following corollary.

COROLLARY 2.3.15. *On $\mathbb{C}$, for every $n \in \mathbb{N}$, if $j^{(n)}(z) = 0$, then $z \in j\mathrm{cl}(\varnothing)$.*

PROOF. Suppose not, and let ∂ be a j-derivation such that $\partial(z) \neq 0$. The order of a zero of a holomorphic function is finite, so after applying ∂ to $j^{(n)}(z)$ enough consecutive times, we will get $\partial(z) = 0$, a contradiction. □

REMARK 2.3.16. Recall that Schneider's theorem (see [**50**]) says that, for $\tau \in \mathbb{H}$, both τ and $j(\tau)$ are algebraic over $\mathbb{Q}$ if and only if τ is special. This means then that if τ is algebraic over $\mathbb{Q}$ but not special, then $j(\tau)$ is transcendental over $\mathbb{Q}$. Then, in this case $j(\tau)$ is a transcendental number which is not j-generic. Indeed, if ∂ is a j-derivation on $\mathbb{C}$, then $\partial(j(\tau)) = j'(\tau)\partial(\tau) = 0$.

Similarly, if we choose τ such that $j(\tau)$ is algebraic over $\mathbb{Q}$ but not the image of a special point, then necessarily τ must be transcendental over $\mathbb{Q}$. By Corollary 2.3.15 we get that τ is a transcendental number which is not j-generic. Both arguments give us families of complex numbers which are transcendental over $\mathbb{Q}$, but which are not j-generic.

It is also known (see [**20**, Theorem 9 (2-c)]) that if $\tau \in \mathbb{H}$ is special and such that $j(\tau) \notin \{0, 1728\}$, then $j'(\tau)$ is transcendental over $\mathbb{Q}$. So this gives yet another family.

In the case of the exponential function, there is an explicit equation relating e and π (i.e. $e^{\pi i} = -1$), which implies that π is not exponentially transcendental. However, such a formula is not known for the j function. It is therefore not clear whether or not πi is j-generic. Results like [**39**, Corollary 1.3] may be seen as evidence that j, j', j'' and π are not related in a way that could allow us to prove that π is not j-generic.

Now we will look for non-trivial j-derivations on $\mathbb{C}$. The key idea here is that, to get j-derivations, we should look for derivations on $\mathbb{R}$ that respect the real and imaginary parts of j on its fundamental domain. Let us set the following notation. If $F : U \to \mathbb{C}$ is a holomorphic function, where $U \subseteq \mathbb{C}$ is an open domain, then the real and imaginary parts of F, say F_1 and F_2 respectively, are the real analytic functions with domain:

$$U_{\mathbb{R}} := \left\{(x, y) \in \mathbb{R}^2 : x + iy \in U\right\}$$

satisfying $F(x + iy) = F_1(x, y) + iF_2(x, y)$, for $(x, y) \in U_{\mathbb{R}}$.

Given two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, define the function (for $x, y \in \mathbb{R}$)

$$\begin{aligned} [f_1 : f_2] : \quad \mathbb{C} &\longrightarrow \mathbb{C} \\ (x + iy) &\mapsto (f_1(x) - f_2(y)) + i(f_1(y) + f_2(x)). \end{aligned}$$

Definition. Let ∂ be a derivation on $\mathbb{C}$ and $F : U \to \mathbb{C}$ be a holomorphic function. We say that ∂ *respects* F *at a point* $z \in U$ if:

$$\partial(F(z)) = \frac{dF}{dz}(z)\partial(z).$$

If F_0 is the real or imaginary part of F and ∂' is a derivation on $\mathbb{R}$, then we say that ∂' *respects* F_0 *at* $z = x + iy \in U$ if:

$$\partial'(F_0(x, y)) = \partial_1(F_0(x, y))\partial'(x) + \partial_2(F_0(x, y))\partial'(y),$$

where ∂_1, ∂_2 are the partial derivatives of F_0 with respect to the first and second variables fo F_0 respectively.

The next lemma tells us that to obtain j-derivations on $\mathbb{C}$ we can start with certain derivations on $\mathbb{R}$.

LEMMA 2.3.17 (see [**62**, Lemma 4.2]). *If ∂_1, ∂_2 are derivations on $\mathbb{R}$, then $[\partial_1 : \partial_2]$ is a derivation on $\mathbb{C}$. Further, if F is a holomorphic function with real and imaginary parts F_1, F_2 and domain $U \subseteq \mathbb{C}$, and if ∂_1, ∂_2 respect F_1 and F_2 at a point $(x, y) \in U_{\mathbb{R}}$, then $[\partial_1 : \partial_2]$ respects F at the point $x + iy$.*

Let $F : U \to \mathbb{C}$ be a holomorphic function. The *Schwarz reflection of F* is the holomorphic function $F^{SR} : U' \to \mathbb{C}$ given by $F^{SR}(z) := \overline{F(\overline{z})}$, where $U' = \{z : \overline{z} \in U\}$ and the bar denotes complex conjugation.

The following theorem determines the kind of j-derivations we could hope to find on $\mathbb{C}$. Given a collection $\mathcal{C}$ of holomorphic functions, let $\mathrm{Der}_{\mathbb{C}}(\mathcal{C})$ denote the set of derivations on $\mathbb{C}$ which respect every function in $\mathcal{C}$, and let $\mathrm{Der}_{\mathbb{R}}(\mathcal{C}_{\mathrm{real}})$ be the set of derivations on $\mathbb{R}$ which respect the real and imaginary parts of the functions in $\mathcal{C}$.

THEOREM 2.3.18 (see [**62**, Theorem 4.3]). *Let $\mathcal{C}$ be any collection of holomorphic functions closed under Schwarz reflection and holomorphic derivation. Then the elements of $\mathrm{Der}_{\mathbb{C}}(\mathcal{C})$ are precisely the maps of the form $[\lambda : \mu] : \mathbb{C} \to \mathbb{C}$, for $\lambda, \mu \in \mathrm{Der}_{\mathbb{R}}(\mathcal{C}_{\mathrm{real}})$.*

Let $\overline{\mathbb{R}}$ be the expansion of $\mathbb{R}$ with the real and imaginary parts of j, j' and j'', with all these functions restricted to $\mathcal{F}$. It is noted in [**41**] that the function $j : \mathcal{F} \to \mathbb{C}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. So the structure $\overline{\mathbb{R}}$ is o-minimal (for definitions and basic results of o-minimality see e.g. [**33**]). The next lemma is a basic result from complex analysis.

LEMMA 2.3.19. *Let $f : U \to \mathbb{C}$ be a holomorphic function. Write $f(x + iy) = u(x, y) + iv(x, y)$, for $(x, y) \in U_{\mathbb{R}}$. Then:*

$$\mathcal{R}e\left(\frac{df}{dz}\right) = u_x, \qquad \mathcal{I}m\left(\frac{df}{dz}\right) = v_x,$$

where u_x and v_x denote the partial derivatives of u and v with respect to the first variable.

PROPOSITION 2.3.20. *The structure $\overline{\mathbb{R}}$ has non-trivial derivations that respect the real and imaginary parts of j, j' and j'' restricted to the interior of $\mathcal{F}$.*

PROOF. Let dcl denote the definable closure operator of the o-minimal structure $\overline{\mathbb{R}}$ (see [**33**, §2.2] for definitions). Choose $\tau \in \mathbb{R}$ such that $\tau \notin \mathrm{dcl}(\varnothing)$. Let $B \subseteq \mathbb{R}$ be such that $\{\tau\} \cup B$ is a basis of $\overline{\mathbb{R}}$ with respect to dcl, so in particular $\mathrm{dcl}(\tau, B) = \mathbb{R}$. Given $a \in \mathbb{R}$ there is a tuple $\bar{b}$ of B and a formula $\phi_a(\tau, \bar{b}, x)$ such that a is the only point satisfying ϕ_a. By o-minimality, the formula $\psi(y) \equiv \exists! x(\phi_a(y, \bar{b}, x))$ partitions $\mathbb{R}$ into finitely many intervals. As τ satisfies $\psi(y)$, but $\tau \notin \mathrm{dcl}(B)$, then τ is not an end-point of any of these intervals. Then there exists a function $f_a(t) : \mathbb{R} \to \mathbb{R}$ such that around τ, $f_a(y) = x \iff \phi_a(y, \bar{b}, x)$. In particular, $f_a(\tau) = a$. Using the $\epsilon - \delta$ definitions of continuity and derivation one can also prove that $f_a(t)$ is differentiable in an interval around τ (this is a standard o-minimal result, for reference see [**47**, Theorem 4.2]).

Define $\partial : \mathbb{R} \to \mathbb{R}$ as $\partial(a) = \frac{df_a}{dt}(\tau)$. Of course, one should ask if this is well defined. It may be that there exist many different functions satisfying the properties of f_a. But if $g(t)$ is another such function, then using o-minimality and looking at the set $\{t \in \mathbb{R} : f_a(t) = g(t)\}$, one concludes that there is an interval around τ on which both functions agree (because $\tau \notin \mathrm{dcl}(B)$). Therefore ∂ is well defined.

Now we show that ∂ is a derivation which respects j, j' and j'' on the interior of $\mathcal{F}$. Let $a_1, a_2 \in \mathbb{R}$. Note that if we have functions f_{a_1} and f_{a_2}, then we can choose $f_{a_1+a_2}$ to be $f_{a_1} + f_{a_2}$. Similarly, we can choose $f_{a_1 a_2}$ as $f_{a_1} f_{a_2}$. This proves that ∂ is a derivation.

Let j_1, j_2 be the real and imaginary parts of j respectively. Let z be in the interior of $\mathcal{F}$, and set x and y as the real and imaginary parts of z. To calculate $\partial(j_1(z))$ we consider the function $g(t) = j_1(f_x(t), f_y(t))$, where we have changed our notation as we now consider j_1 as a function $j_1 : \mathbb{R}^2 \to \mathbb{R}$. These changes between $\mathbb{R}^2$ and $\mathbb{C}$ will be used many times, but they are very natural, so no confusion should arise. Now

$$\frac{dg}{dt}(\tau) = \partial_1(j_1)\frac{df_x}{dt}(\tau) + \partial_2(j_1)\frac{df_y}{dt}(\tau),$$

where ∂_1 and ∂_2 denote the real analytic partial derivatives of j_1 with respect to each of its variables. By Lemma 2.3.19 we get that $\partial_1(j_1) = j_1'$, where j_1' is the real part of j'. By the Cauchy-Riemann equations we also get $\partial_2(j_1) = -\partial_1(j_2)$, and again by Lemma 2.3.19

we get $\partial_1(j_2) = j_2'$, where j_2 is the imaginary part of j'. Therefore:

$$\partial(j_1(z)) = j_1'(z)\partial(x) - j_2'(z)\partial(y).$$

Note that this condition is precisely what you get when you look at the real part of the equation $\partial(j(z)) = j'(z)\partial(z)$. Similarly one shows that $\partial(j_2(z)) = j_2'(z)\partial(x) + j_1(z)\partial(y)$. An analogous argument shows that ∂ respects the real and imaginary parts of j' and j'' restricted to $\mathcal{F}$. $\square$

THEOREM 2.3.21. *There is a non-trivial j-derivation on $\mathbb{C}$.*

PROOF. Let ∂ be a non-trivial derivation on $\mathbb{R}$ given by Proposition 2.3.20. We now consider the derivation $[\partial : 0]$ on $\mathbb{C}$. As this derivation extends ∂, we will keep the same name, so that $\partial = [\partial : 0]$. We show that ∂ is in fact a j-derivation. It is clear that ∂ satisfies the axioms of j-derivation for every z in the interior of the standard fundamental domain $\mathcal{F}$. Choose now any $z \in \mathbb{H}$.

Choose $g \in G$ such that gz is in the interior of $\mathcal{F}$ and $(j(z), j(gz))$ is not a singular point of $\Phi_N(X,Y) = 0$ (where $N = \det(N(g)g)$). When z is not in the boundary of a fundamental domain, we can choose $g \in \mathrm{SL}_2(\mathbb{Z})$. If z is in the boundary of a fundamental domain, then we can choose g so that $N = 2$, as the curve $\Phi_2(X,Y) = 0$ also does not have singularities in $\mathbb{C}^2$ (this can be seen as a consequence of [**30**, Proposition 3.1 and Corollary 3.1], or can be proven directly using the explicit form of Φ_2 given, for example, in [**63**, page 70]).

Now we consider the equation $\Phi_N(j(z), j(gz)) = 0$. Deriving with respect to z gives:

$$\Phi_{N1}(j(z), j(gz))j'(z) + \Phi_{N2}(j(z), j(gz))j'(gz)\frac{ad-bc}{(cz+d)^2} = 0, \tag{2.3.2}$$

where a, b, c, d are the entries of g in the usual way, and Φ_{N1} and Φ_{N2} are the partial derivatives of $\Phi_N(X,Y)$ with respect to X and Y respectively. On the other hand, applying ∂ to the equation $\Phi_N(j(z), j(gz)) = 0$ (and using that ∂ respects j and its derivatives inside of $\mathcal{F}$) gives us:

$$\Phi_{N1}(j(z), j(gz))\partial(j(z)) + \Phi_{N2}(j(z), j(gz))j'(gz)\frac{ad-bc}{(cz+d)^2}\partial(z) = 0. \tag{2.3.3}$$

Observe now that if $\Phi_{N1}(j(z), j(gz)) = 0$, then using the fact that $ad - bc = \det(g) \neq 0$, that gz is in the interior of $\mathcal{F}$, and that the zeros of j' are in the boundary of $\mathcal{F}$, we get from equation (2.3.2) that $\Phi_{N2}(j(z), j(gz)) = 0$, i.e. $(j(z), j(gz))$ is a singular point of $\Phi_N(X, Y)$, which contradicts our assumptions. So $\Phi_{N1}(j(z), j(gz)) \neq 0$.

Consider now equation (2.3.3). If $\partial(z) = 0$, then, by the previous paragraph, $\partial(j(z)) = 0$. If $\partial(z) \neq 0$, then comparing equations (2.3.2) and (2.3.3), we get that $j'(z) = \partial(j(z))/\partial(z)$. In both cases we obtain $\partial(j(z)) = j'(z)\partial(z)$.

To verify the other axioms of j-derivations, we proceed analogously. We take equation (2.3.2), we differentiate it with respect to z and we also apply ∂ to it. We compare both results. Using that $\partial(j(z)) = j'(z)\partial(z)$ and that $(j(z), j(gz))$ is not a singularity of Φ_N, one concludes that $\partial(j'(z)) = j''(z)\partial(z)$. A similar argument will show that $\partial(j''(z)) = j'''(z)\partial(z)$.

Finally, if we have $z \in \mathbb{H}^-$, we have to work with $-z$ and proceed as above. So ∂ is a non-trivial j-derivation on $\mathbb{C}$. □

REMARK 2.3.22. As i is algebraic over $\mathbb{Q}$, then for any derivation ∂ on $\mathbb{C}$ we have that $\partial(x + iy) = \partial(x) + i\partial(y)$. So, given $z = x + iy \in \mathbb{C}$ such that x or y is not in $\operatorname{dcl}(\varnothing)$ (where dcl is taken in $\overline{\mathbb{R}}$), there is a j-derivation which does not vanish on z. Therefore $j\operatorname{cl}(\varnothing)$ is contained in the set of complex numbers whose real and imaginary parts are both in $\operatorname{dcl}(\varnothing)$. Since this set is countable, we have that $j\operatorname{cl}(\varnothing)$ is countable.

A more general study of derivations that respect given holomorphic functions can be found in [**26**, §2, §3 and §4].

2.4. A Schanuel Property for j

In this section we are interested in results relating the transcendence degree of values of j and its derivatives with respect to the dimensions $\dim^j$ and $\dim^g_F$. The exposition is inspired by [**9**]. First, let us introduce a "modular version" of Schanuel's conjecture. This is a special case of the *generalised period conjecture* of Grothendieck-André (see [**1**, §23.4.4], [**11**, §1 Conjecture modulaire], and [**44**, Conjecture 8.3]). Recall that in our notation, gcl-independence means independence with respect to the pregeometry $\operatorname{gcl}_{\mathbb{Q}}$.

CONJECTURE 3 (Modular Schanuel Conjecture: MSC). Suppose $z_1, \dots, z_n \in \mathbb{H}^+$ are gcl-independent and non-special. Then:

$$\text{t.d.}(\overline{z}, j(\overline{z}), j'(\overline{z}), j''(\overline{z})) \geq 3n. \tag{2.4.1}$$

Following the notation we have been using so far, whenever we write t.d.(A), we mean the transcendence degree of the field $\mathbb{Q}(A)$ over $\mathbb{Q}$. The main ingredient for our transcendence results is the following.

THEOREM 2.4.1 (Ax-Schanuel for j, see [**45**, Theorem 1.3]). *Let K be a characterstic zero differential field with m commuting derivations ∂_k. Let $C = \bigcap_k \ker \partial_k$. Let $z_i, j_i, j_i', j_i'', j_i''' \in K^*$, $i = 1, \dots, n$, be such that*

$$\partial_k j_i = j_i' \partial_k z_i, \qquad \partial_k j_i' = j_i'' \partial_k z_i, \qquad \partial_k j_i'' = j_i''' \partial_k z_i$$

for all i and k. Suppose further that $\urcorner(j_i, j_i', j_i'', j_i''') = 0$ for $i = 1, \dots, n$, where $\urcorner$ is given by equation (2.1.1). Suppose also that for every modular polynomial Φ_N we have $\Phi_N(j_i, j_k) \neq 0$ for all i, k, N and $j_i \notin C$ for all i. Then:

$$\text{t.d.}(z_1, j_1, j_1', j_1'', \dots, z_n, j_n, j_n', j_n''/C) \geq 3n + \text{rank}(\partial_k z_i)_{i,k}.$$

This theorem is the first instance where we actually need the explicit shape of the differential equation (1.2.2) and that of the modular polynomials. Up until now, we have only used that they exist, some smoothness properties, and that they are defined over $\mathbb{Q}$.

2.4.1. Consequences of Ax-Schanuel for j.

PROPOSITION 2.4.2. *Let (K, D) be a j-field. Let $z_1, \dots, z_n \in D$ and let $C \subseteq K$ be* $j$cl*-closed. Then:*

$$\text{t.d.}(\overline{z}, j(\overline{z}), j'(\overline{z}), j''(\overline{z})/C) \geq 3 \dim^g(\overline{z}/C) + \dim^j(\overline{z}/C).$$

PROOF. The strategy of the proof is first to reduce the problem to a (possibly) different j-field, and then check conditions on the new j-field to see that we can use the Ax-Schanuel Theorem for j. Let $m = \dim^j(\overline{z}/C)$. By renumbering, we can assume that $z_1, \dots, z_m$ are

$j$cl-independent over C. Also, by renumbering and using the fact that $j$cl-independence implies gcl-independence, we can assume that $z_1, \ldots, z_r$ are gcl-independent over C, with $r = \dim^g(\overline{z}/C)$. Note that $m \leq r$. As in Lemma 2.3.5, we can find a corresponding system of j-derivations $\partial_1, \ldots, \partial_m \in j\mathrm{Der}(K/C)$. Thus $m = \mathrm{rank}(\partial_k(z_i))_{i,k=1,\ldots,r}$.

For $m < i \leq r$ we have that $z_i \in j\mathrm{cl}(z_1, \ldots, z_m, C)$, but $z_i \notin C$ (because C is $j$cl-closed). By the exchange property $z_1 \in j\mathrm{cl}(z_2, \ldots, z_m, z_i, C)$. As $\partial_1(z_1) = 1$, then we must necessarily have $\partial_1(z_i) \neq 0$. So in fact, $\partial_k(z_i) \neq 0$ for every $k = 1, \ldots, m$ and $i = m+1, \ldots, r$.

Let $F = j\mathrm{cl}(z_1, \ldots, z_n, C)$. We know that F is a j-field. We want to show that the j-derivations $\partial_1, \ldots, \partial_m$ can be restricted to F. Consider ∂_1. Clearly ∂_1 can be restricted to C, because $\partial_1(C) = 0$ and $0 \in C$. As a field, $C(z_1, \ldots, z_m)$ is isomorphic to the field of rational functions on C with m variables. Then ∂_1 acts on $C(z_1, \ldots, z_m)$ as partial derivation with respect to z_1, so it can also be restricted to $C(z_1, \ldots, z_m)$. Let $x \in F$ be algebraic over $C(z_1, \ldots, z_m)$, and let $p(x) = 0$ be an algebraic equation with coefficients in $C(z_1, \ldots, z_m)$ satisfied by x. Applying ∂_1 to $p(x) = 0$ shows that $\partial_1(x) \in C(z_1, \ldots, z_m, x)$. Finally, suppose that $z \in D_F$ is such that $\partial_1(z) \in F$. Then, as F is a j-field, the axioms of j-derivations say that $\partial_1(j(z)), \partial_1(j'(z)), \partial_1(j''(z)) \in F$. Therefore, ∂_1 can be restricted to a j-derivation on F. The most important thing about this is that, as $z_i \in F$, for $m < i \leq r$, then $\partial_1(z_i) \in F$. This means that $z_1, \ldots, z_m$ are still $j$cl-independent over C in F. Also, $z_1, \ldots, z_r$ are clearly gcl-independent over C in F.

So, in what remains of the proof, we will work in the j-field F while preserving the notation we have already set. We now show that in F we have the conditions necessary for applying the Ax-Schanuel Theorem for j.

Let $B = \bigcap_{k=1}^{m} \ker(\partial_k)$ (remember that the kernels are taken inside F). Then $z_i \notin B$, for $i = 1, \ldots, r$. Note that $C \subseteq B$ and that $z_1, \ldots, z_m$ are still $j$cl-independent over B. Given i and k, consider the j-derivation $\partial = \partial_i\partial_k - \partial_k\partial_i$. Consider ∂ as a j-derivation on K. As ∂ vanishes on $C \cup \{z_1, \ldots, z_m\}$, then for any $x \in K$ such that $x \in j\mathrm{cl}(z_1, \ldots, z_m, C)$, we must have that $\partial(x) = 0$. In other words, $\partial \equiv 0$ on F. This means that the j-derivations $\partial_1, \ldots, \partial_m$ commute on F.

Let $j_i = j(z_i)$, $j_i' = j'(z_i)$, $j_i'' = j''(z_i)$ and $j_i''' = j'''(z_i)$, for $i = 1, \ldots, r$. Now we verify the conditions on Theorem 2.4.1. Note that the system of differential equations $\partial_k(j_i) =$

$j_i'\partial_k(z_i)$, $\partial_k(j_i') = j_i''\partial_k(z_i)$, $\partial_k(j_i'') = j_i'''\partial_k(z_i)$, for all $i = 1,\dots,r$, $k = 1,\dots,m$ is satisfied by the axioms of j-derivations. By the axioms of j-fields we have that $\urcorner(j_i, j_i', j_i'', j_i''') = 0$, $i = 1,\dots,r$ (where $\urcorner$ is given by equation (2.1.1)). Note that B contains all special points, so z_i is not special and $\urcorner$ is well defined at every z_i, $i = 1,\dots,r$. By gcl-independence we have that $\Phi_N(j_i, j_k) \neq 0$ for all $1 \leq i, k \leq r$,and all N.

By Proposition 2.3.10, if $j_i''' \neq 0$ for some i, then $j_i \notin B$ and $z_i, j_i, j_i', j_i'', j_i''' \in F^*$ (because $0 \in B$). So now we show $j_i''' \neq 0$ for $i = 1,\dots,r$. If $j_k''' = 0$ for some $1 \leq k \leq r$, then for every j-derivation ∂ we have that $\partial(j_k'') = 0$. So $j_k'' \in B$. As $\partial(j_k') = j_k''\partial(z_k)$, then $j_k' = j_k'' z_k + b_0$, for some $b_0 \in B$. Therefore $\partial(j_k) = (j_k'' z_k + b_0)\partial(z_k) = \partial(j_k'' z_k^2/2 + b_0 z_k)$, which means there is $b_1 \in B$ such that $j_k = j_k'' z_k^2/2 + b_0 z_k + b_1$. Plugging this values into the equation $\urcorner(j_k, j_k', j_k'', j_k''') = 0$, we conclude that z_k is algebraic over B, but as B is algebraically closed, this would imply that $z_k \in B$, which is impossible for $1 \leq k \leq r$.

We can therefore apply Theorem 2.4.1 to obtain:

$$\text{t.d.}(z_1, j_1, j_1', j_1'', \dots, z_r, j_r, j_r', j_r''/B) \geq 3r + m.$$

Now use the general fact that if $A \subseteq A'$ and $C \subseteq B$, then $\text{t.d.}(A'/C) \geq \text{t.d.}(A/B)$. $\square$

The following lemma will allow us to obtain a weaker version of our main result, but the proof uses a different method. The argument presented here is based on ideas of Alex Wilkie for treating the exponential function, communicated to the author.

LEMMA 2.4.3. *Let (K, D) be a j-field and ∂ be a j-derivation on K. Let $m \geq 1$, $d \geq 0$ and set $C = \ker\partial$. Suppose that:*

(1) $z_1 \dots, z_m \in D$ are gcl*-independent,*

(2) There is $\tau \in K$ such that $\partial(\tau) = 1$,

(3) $d = \dim_F^g(z_1, \dots, z_m)$, where $F = \mathbb{Q}(\tau)$.

Then:

$$\text{t.d.}(j(\overline{z}), j'(\overline{z}), j''(\overline{z})) \geq 3m - 3d.$$

PROOF. If $d = m$ then the result is trivial, so we assume that $d < m$. Renumbering if necessary, let $0 \leq p \leq m$ be maximal so that $z_1, \dots, z_p \in C$. Note that by Lemma 2.2.3 $F \perp_{\mathbb{Q}}^g C$, so $z_1, \dots, z_p$ are gcl_F-independent. Therefore $p \leq d$.

Now, by renumbering again, we can assume that $z_1, \ldots, z_p, z_{p+1}, \ldots, z_d$ is an gcl_F-basis for $z_1, \ldots, z_m$. Then:

$$\begin{aligned} \text{t.d.}(z_{p+1}, \ldots, z_m / C) &\leq \text{t.d.}(z_{p+1}, \ldots, z_m, \tau / C) \\ &= \text{t.d.}(\tau / C) + \text{t.d.}(z_{p+1}, \ldots, z_m / C, \tau) \\ &\leq 1 + d - p. \end{aligned}$$

Let $\widetilde{z} = (z_{p+1}, \ldots, z_m)$. Suppose that the Lemma is false: $\text{t.d.}(j(\widetilde{z}), j'(\widetilde{z}), j''(\widetilde{z})/C) \leq 3m - 3d - 1$. Then it is also true that: $\text{t.d.}(j(\widetilde{z}), j'(\widetilde{z}), j''(\widetilde{z})/C, \widetilde{z}) \leq 3m - 3d - 1$. Therefore:

$$\begin{aligned} \text{t.d.}(\widetilde{z}, j(\widetilde{z}), j'(\widetilde{z}), j''(\widetilde{z})/C) &= \text{t.d.}(\widetilde{z}/C) + \text{t.d.}(j(\widetilde{z}), j'(\widetilde{z}), j''(\widetilde{z})/C, \widetilde{z}) \\ &\leq (1 + d - p) + (3m - 3d - 1) = 3m - p - 2d. \end{aligned}$$

However, under the hypothesis of the Lemma, we should have by the Ax-Schanuel theorem for j (or by Proposition 2.4.2) that:

$$\text{t.d.}(\widetilde{z}, j(\widetilde{z}), j'(\widetilde{z}), j''(\widetilde{z})/C) \geq 3m - 3p + 1,$$

which would mean that $3m - 3p + 1 \leq 3m - p - 2d$, or equivalently $2d + 1 \leq 2p$. This contradicts that $p \leq d$ (which was already established). $\square$

For the following Corollary recall Remark 2.3.22, which ensures the existence of τ.

Corollary 2.4.4 (Weak version of the main result). *Choose $\tau \in \mathbb{R}$ so that $\tau \notin j\mathrm{cl}(\varnothing)$. Suppose $z_1, \ldots, z_n \in \mathbb{H}$ and $g \in G(\mathbb{Q}(\tau))$ are such that $z_1, \ldots, z_n, gz_1, \ldots, gz_n$ are* gcl-*independent. Then:*

$$\text{t.d.}(j(\overline{z}), j(g\overline{z}), j'(\overline{z}), j'(g\overline{z}), j''(\overline{z}), j''(g\overline{z})) \geq 3n.$$

Proof. Use Lemma 2.4.3 with $m = 2n$. Proposition 2.3.20 constructs a j-derivation ∂ satisfying $\partial(\tau) = 1$. Note also that in this case $d \leq n$. So $3m - 3d \geq 6n - 3n = 3n$. $\square$

2.4.2. Transcendence Results. We start this section with a corollary which is just a rephrasing of Proposition 2.4.2 in a more explicit way.

COROLLARY 2.4.5. *Let (K,D) be a j-field and $C \subseteq K$ be $j$cl-closed. Let $\tau_1,\dots,\tau_m \in D$ be such that they are $j$cl-independent over C. Then for any $z_1,\dots,z_n \in D$:*

$$\text{t.d.}(\overline{\tau},\overline{z},j(\overline{\tau}),j(\overline{z}),j'(\overline{\tau}),j'(\overline{z}),j''(\overline{\tau}),j''(\overline{z})/C) \geq 3\dim^g(\overline{\tau},\overline{z}/C)+m.$$

Next, we need a couple of technical results.

PROPOSITION 2.4.6. *Let (K,D) be a j-field, $C \subseteq K$ $j$cl-closed. Suppose $\tau_1,\dots,\tau_m \in D$ are $j$cl-independent over C. Then for any $z_1,\dots,z_n \in D$:*

(a) $\text{t.d.}(j(\overline{\tau}),j'(\overline{\tau}),j''(\overline{\tau})/C,\overline{\tau},\overline{z},j(\overline{z}),j'(\overline{z}),j''(\overline{z})) \leq 3\dim^g(\overline{\tau}/C,\overline{z})$,

(b) $\text{t.d.}(\overline{z}/C,\overline{\tau}) \leq \dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/C)$.

PROOF. (a) Suppose that $\tau_1 \in \text{gcl}(\tau_2,\dots,\tau_m,\overline{z},C)$. Then, by $j$cl-independence, we have that $\tau_1 = gz_i$, for some z_i and some $g \in G$. This says that $\tau_1 \in \mathbb{Q}(z_i)$. Also, for some N, we have that $\Phi_N(j(\tau_1),j(z_i)) = 0$, which says that $j(\tau_1)$ is algebraic over $\mathbb{Q}(j(z_i))$.

Now we consider j-derivations. Given that C contains the field of constants and that $z_i \notin C$ (because $\tau_1 = gz_i$), we can assume that there is a j-derivation ∂ such that $\partial(z_i) \neq 0$. So we can assume that $\partial(z_i) = 1$. From the equality:

$$\partial(\tau_1) = \frac{ad-bc}{(cz_i+d)^2}\partial(z_i) \tag{2.4.2}$$

(where a, b, c and d are the parameters of g in the usual way), we get that $\partial(\tau_1)$ is algebraic over $\mathbb{Q}(z_i)$. Also:

$$\begin{aligned} 0 &= \partial\Phi_N(j(\tau_1),j(z_i)) \\ &= \Phi_{N,1}(j(\tau_1),j(z_i))j'(\tau_1)\partial(\tau_1)+\Phi_{N,2}(j(\tau_1),j(z_i))j'(z_i)\partial(z_i), \end{aligned} \tag{2.4.3}$$

where $\Phi_{N,1}$ and $\Phi_{N,2}$ are the derivatives of Φ_N with respect to the first and second coordinates respectively. So $j'(\tau_1)$ is algebraic over $\mathbb{Q}(j(\tau_1),j(z_i),j'(z_i),\tau_1,z_i)$.

Finally, if we apply ∂ once more to equations (2.4.2) and (2.4.3), we obtain that $j''(\tau_i)$ is algebraic over $\mathbb{Q}(j(\tau_1),j(z_i),j'(\tau_1),j'(z_i),j''(z_i),\tau_1,z_i)$.

(b) If $z_1 \in \text{gcl}_{\mathbb{Q}(\overline{\tau})}(z_2,\dots,z_n,C)$, then suppose that for some $g \in G(\mathbb{Q}(\overline{\tau}))$ and some $x \in \{z_2,\dots,z_n\} \cup C$ we have that $z_1 = gx$. This equality can be translated into a polynomial relation of the form: $(cx+d)z_1 = ax+b$ (where a, b, c, and d are the usual parameters of g), meaning that z_1 is algebraic over $\mathbb{Q}(a,b,c,d,x)$. Given that a, b, c, and d are in $\mathbb{Q}(\overline{\tau})$, we obtain the desired result.

$\square$

THEOREM 2.4.7. *Let (K,D) be a j-field and $C \subseteq K$ be* jcl*-closed. Let $\tau_1,\dots,\tau_m \in D$ be such that they are* jcl*-independent over C. Then for any $z_1,\dots,z_n \in D$:*

$$\text{t.d.}(j(\overline{z}),j'(\overline{z}),j''(\overline{z})/C,\overline{\tau},\overline{z}) + \dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/C) - 3\dim^g(\overline{z}/C) \geq 0.$$

PROOF. From Corollary 2.4.5 we get:

$$\text{t.d.}(\overline{\tau},\overline{z},j(\overline{\tau}),j(\overline{z}),j'(\overline{\tau}),j'(\overline{z}),j''(\overline{\tau}),j''(\overline{z})/C) \geq 3\dim^g(\overline{\tau},\overline{z}/C) + m.$$

Expanding using the addition formula we get:

$$\begin{aligned}\text{t.d.}(\overline{\tau}/C) + \text{t.d.}(\overline{z}/C,\overline{\tau}) + \text{t.d.}(j(\overline{\tau}),j'(\overline{\tau}),j''(\overline{\tau})/C,\overline{\tau},\overline{z},j(\overline{z}),j'(\overline{z}),j''(\overline{z}))\\ +\text{t.d.}(j(\overline{z}),j'(\overline{z}),j''(\overline{z})/C,\overline{\tau},\overline{z}) - 3\dim^g(\overline{\tau}/C,\overline{z}) - 3\dim^g(\overline{z}/C) \geq m.\end{aligned}$$

Using that $\text{t.d.}(\overline{\tau}/C) = m$ and both inequalities from Proposition 2.4.6, we obtain the statement of the theorem. $\square$

COROLLARY 2.4.8. *Let (K,D) be a j-field and $C \subseteq K$ be* jcl*-closed. Let $\tau_1,\dots,\tau_m \in D$ be such that they are* jcl*-independent over C. Let A be a subset of C. Then for any $z_1,\dots,z_n \in D$:*

$$\text{t.d.}(j(\overline{z}),j'(\overline{z}),j''(\overline{z})/C,\overline{\tau},\overline{z}) + 3\dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/A) - 3\dim^g(\overline{z}/A) \geq 2\dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/C).$$

PROOF. By the previous theorem we get:

$$\text{t.d.}(j(\overline{z}),j'(\overline{z}),j''(\overline{z})/C,\overline{\tau},\overline{z}) + \dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/C) - 3\dim^g(\overline{z}/C) \geq 0.$$

So we also get:

$$\text{t.d.}(j(\overline{z}),j'(\overline{z}),j''(\overline{z})/C,\overline{\tau},\overline{z}) + 3\dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/C) - 3\dim^g(\overline{z}/C) \geq 2\dim^g_{\mathbb{Q}(\overline{\tau})}(\overline{z}/C).$$

If $\overline{\tau}$ is $j$cl-independent over C, then it is algebraically independent over C. So, by Lemma 2.2.3, we get that $\mathbb{Q}(\overline{\tau})$ is geodesically disjoint from C over $\mathbb{Q}$. Now we can use Lemma 2.2.4. $\square$

Now we prove our main transcendence property for j-fields.

Theorem 2.4.9 (Schanuel Property for j-fields). *Let (K,D) be a j-field. Suppose $\tau_1,\dots,\tau_k,\ z_1,\dots,z_n \in D$ are such that $\overline{\tau}$ is $j$cl-independent and $\overline{z}$ is gcl-independent. Let $F=\mathbb{Q}(\overline{\tau})$ and suppose that there are non-central elements in $G^F \smallsetminus G$. Let m be a positive integer, and for each $1 \leq i \leq m$, choose $g_i \in G^F$ so that it is not of the form $g_i = ah$, where $a \in F$ and $h \in G$. Let $\partial_1,\dots,\partial_k$ be a corresponding system of j-derivations for $\overline{\tau}$, and let C be $j$cl-closed and such that $C \subseteq \bigcap_i \ker\partial_i$. Suppose any of the following conditions is satisfied:*

(a) The set $\{\bar{z}, g_1\bar{z},\dots,g_m\bar{z}\}$ is gcl-independent.

(b) $\bar{z} \subseteq C$ and each z_i is non-special.

Then:

$$\text{t.d.}\left(j(\bar{z}),j'(\bar{z}),j''(\bar{z}),j(\bar{g}\bar{z}),j'(\bar{g}\bar{z}),j''(\bar{g}\bar{z})/F,C,\bar{z}\right) \geq 3nm. \tag{2.4.4}$$

Proof. Using that $\dim_F^g(\bar{z},\bar{g}\bar{z}) = \dim_F^g(\bar{z})$ and Corollary 2.4.8 (with $A=\varnothing$), we get:

$$\text{t.d.}\left(j(\bar{z}),j'(\bar{z}),j''(\bar{z}),j(\bar{g}\bar{z}),j'(\bar{g}\bar{z}),j''(\bar{g}\bar{z})/F,C,\bar{z}\right) \geq 3\dim^g(\bar{z},\bar{g}\bar{z}) - 3\dim_F^g(\bar{z}) + 2\dim_F^g(\bar{z}/C).$$

(a) The result follows immediately as

$$3\dim^g(\bar{z},\bar{g}\bar{z}) - 3\dim_F^g(\bar{z}) + 2\dim_F^g(\bar{z}/C) \geq 3n(m+1) - 3n = 3nm.$$

(b) By Lemma 2.2.3 we know that $F \perp_{\mathbb{Q}}^g C$, and so $\dim_F^g(\bar{z}) = n$. Even more, unless some z_i is F-special, the points $\bar{z},\bar{g}\bar{z}$ are gcl-independent. However if any z_i is F-special, then by Lemma 2.2.3 it is also $\mathbb{Q}$-special. So we get that $\dim^g(\bar{z},\bar{g}\bar{z}) = n(m+1)$, and now we conclude as in (a).

$\square$

Observe that in case (b) we have that $j(\bar{z}), j'(\bar{z}), j''(\bar{z}) \in C$ (as C is $j$cl-closed), and so we obtain equality in (2.4.4).

2.4.3. Essential Counterexamples to MSC. Now we relate our transcendence results to the modular Schanuel conjecture.

Definition. Let $\overline{z}$ be a tuple in D and let $B \subseteq K$. We define the *predimension of* $\overline{z}$ *over* B to be:

$$\delta(\overline{z}/B) = \text{t.d.}(\overline{z}, j(\overline{z}), j'(\overline{z}), j''(\overline{z})/B, j(A), j'(A), j''(A)) - 3\dim^g(\overline{z}/B),$$

where $A = B \cap D$. When $B = \varnothing$ we write simply $\delta(\overline{z})$. By Proposition 2.4.2, $\delta(\overline{z}/C) \geq \dim^j(\overline{z}/C)$.

Remark 2.4.10. It is a simple exercise to see that for any pregeometry cl on a set X with associated dimension $\dim^{\text{cl}}$, given A, B, C three subsets of X such that $\dim^{\text{cl}}(A, C/B)$ is finite, then we have that $\dim^{\text{cl}}(A, C/B) = \dim^{\text{cl}}(A/B, C) + \dim^{\text{cl}}(C/B)$. Therefore, on a j-field (K, D), for any finite $C \subseteq D$ we get that $\delta(\overline{z}/B, C) = \delta(\overline{z}, C/B) - \delta(C/B)$. So, if we choose $B = \varnothing$, we get that $\delta(A, C) = \delta(A/C) + \delta(C)$. Also note that for any finite $A \subseteq D$ we have that $\delta(A) = \delta(\text{gcl}(A))$.

Definition. A tuple $\overline{z}$ from D is called an *essential counterexample* to the modular Schanuel conjecture if it does not satisfy inequality (2.4.1) and for every tuple $\overline{c}$ from $\text{gcl}(\overline{z})$ we have that $\delta(\overline{z}) \leq \delta(\overline{c})$.

A counterexample of MSC is a tuple $\overline{z}$ of D such that $\delta(\overline{z}) < 0$. On $\mathbb{C}$, using the relation given by the modular polynomials and their derivatives, we get that for any z_0 and any tuple $\overline{z}$ from $\mathbb{H}^+ \cup \mathbb{H}^-$, $\delta(\overline{z}, z_0) \leq \delta(\overline{z}) + 1$. This means that if we had a counterexample $\overline{z}$ such that $\delta(\overline{z}) < -1$, then we can produce $|\mathbb{C}|$ many counterexamples. Essential counterexamples can be used to classify families of counterexamples of MSC. Clearly, every counterexample of MSC has an essential counterexample in its gcl-closure.

Proposition 2.4.11. *Let* (K, D) *be a* j*-field. Let* $\overline{z} \in D$ *be such that for some* i*,* $z_i \notin j\text{cl}(\varnothing)$*. Then* $\overline{z}$ *is not an essential counterexample.*

PROOF. Let $A = \mathrm{gcl}(\overline{z}) \cap j\mathrm{cl}(\varnothing)$. By hypothesis, $A \neq \mathrm{gcl}(\overline{z})$. As $A \subseteq j\mathrm{cl}(\varnothing)$, then $j(A), j'(A), j''(A) \subseteq j\mathrm{cl}(\varnothing)$ by Proposition 2.3.10, so:

$$\mathrm{t.d.}(\overline{z}, j(\overline{z}), j'(\overline{z}), j''(\overline{z})/A, j(A), j'(A), j''(A)) \geq \mathrm{t.d.}(\overline{z}, j(\overline{z}), j'(\overline{z}), j''(\overline{z})/j\mathrm{cl}(\varnothing)).$$

Also $\dim^g(\overline{z}/A) = \dim^g(\overline{z}/j\mathrm{cl}(\varnothing))$. Using Proposition 2.4.2 we get:

$$\delta(\overline{z}/A) \geq \delta(\overline{z}/j\mathrm{cl}(\varnothing)) \geq \dim^j(\overline{z}/j\mathrm{cl}(\varnothing)) \geq 1.$$

Let $\overline{b} \in A$ be such that $\delta(\overline{z}/\overline{b}) = \delta(\overline{z}/A)$. Then $\delta(\overline{b}) = \delta(\overline{z}, \overline{b}) - \delta(\overline{z}/\overline{b}) = \delta(\overline{z}) - \delta(\overline{z}/A) < \delta(\overline{z})$, so $\overline{z}$ is not an essential counterexample. □

COROLLARY 2.4.12. *By Remark 2.3.22 we get that on $\mathbb{C}$ there are at most countably many essential counterexamples to the modular Schanuel conjecture.*

2.4.4. Final Remarks. We have succeded in producing a transcendence result for the j function in the spirit of the main theorem of [**9**]. The key ingredient to make everything work is the Ax-Schanuel theorem for j, whereas the general framework can be adapted for any other modular function.

For instance, consider the modular λ function (see [**56**, Chapter III.1] for details). On $\mathbb{C}$, every elliptic curve is isomorphic to an elliptic curve in the Legendre form $E_\lambda : Y^2 = X(X-1)(X-\lambda)$, where $\lambda \neq 0, 1$. The j-invariant of such an elliptic curve is:

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

We can then define the function: $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0, 1\}$ where $\lambda(\tau)$ is chosen so that the elliptic curves $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ and $Y^2 = X(X-1)(X-\lambda(\tau))$ are isomorphic. This function is algebraic over j and so we will have a family of polynomials for λ like the modular polynomials for j, and a differential equation for λ, and with them we can define "λ-fields". We can also obtain an Ax-Schanuel type statement for λ from the Ax-Schanuel for j, and then, with the method we have described, we can obtain a generic transcendence result for λ like Theorem 2.4.9.

Considering the results for the exponential function of [**9**] and our work for j, there seems to be a general philosophy regarding what is needed to repeat this strategy with a given holomorphic function f to obtain a transcendence result like in Theorem 2.4.7. Of course, one needs an Ax-Schanuel statement for f, i.e. an inequality of the form:

$$\text{t.d.}(\overline{z}, f(\overline{z}), \dots, f^{(k)}(\overline{z})) \geq k \dim(\overline{z}) + \dim^f(\overline{z}).$$

Here $\dim^f$ is the dimension corresponding to the pregeometry defined by f-derivations (i.e. derivations that respect the derivatives of f in the same sense as j-derivations). One can count on f-derivations defining a pregeometry, but if one wants to construct non-trivial f-derivations on $\mathbb{C}$, then one also has to ask that f (or part of f, if f has some symmetry properties) be definable in $\mathbb{R}_{\text{an,exp}}$ (see **[26]** and **[62]** for a general study of this topic). The dimension dim on the other hand is more interesting. This dimension has to be given by a pregeometry cl with (at least) the following characteristics:

(i) cl has to be definable in the field structure of $\mathbb{Q}$, so that we have a family of pregeometries for every subfield, like we have with gcl_F.

(ii) cl has to identify the algebraic relations between the functions f and its derivatives (at least the derivatives considered in the Ax-Schanuel statement).

By the main result of **[36]**, the universal covering map of a Shimura curve has the properties required to obtain a corresponding transcendence result.

2.4.5. Open Problems. There is one important aspect about the results of **[28]** that we were not able to replicate: is there a more explicit description of the pregeometry $j$cl? In the case of exponential fields it turns out that the pregeometry defined by exponential derivations agrees with the closure opertator defined by Khovanskii systems, which are defined as non-degenerate systems of exponential polynomial equations (see [**28**, Propositions 4.7 and 7.1]). A natural generalisation of Khovanskii systems for j-fields can be defined, and one can ask if they define the pregeometry $j$cl. However, this is not yet known as the strategy used in **[28]** for exponential fields relies too much on the properties of the exponential map (like the fact that it is a group homomorphism, and the shape of its differential equation). An equivalent formulation of this problem was already mentioned, in its general form, at the end of [**26**, §4].

Part 3

Models of Shimura Varieties

3.1. Shimura Structures

Now we will present our first model-theoretic treatment of Shimura varieties (the second approach will be given in §3.2). For every Shimura variety S we will define a corresponding Shimura structure. This first treatment is quite similar to the one in [**17**], but differs in that we enhance the language to include names for the special domains of special subvarieties.

Definition. A *Shimura structure for* $p : X^+ \to S(\mathbb{C})$ is a two-sorted structure $\mathbf{q} := \langle \mathbf{D}, \mathbf{S}, q \rangle$ where:

(1) $\mathbf{D} := \left\langle D; \{g\}_{g \in G^{\mathrm{ad}}(\mathbb{Q})^+}, \mathscr{R}_D \right\rangle$ is a $G^{\mathrm{ad}}(\mathbb{Q})^+$-*covering structure*. This means that D is a set with an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ (so the elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$ denote function symbols), and $\mathscr{R}_D$ is a countable set of subsets of powers of D. The elements of $\mathscr{R}_D$ are named $D_{V,i}$, where the indexing comes from the indexing of special domains of X^+.

(2) $\mathbf{S} := \langle S(F); \mathscr{R}_S \rangle$ is a *variety structure*. This means that $S(F)$ is the set of F-points of S, where F is an algebraically closed field. Variety structures will be interpreted in the field sort $\mathbf{F} := \langle F; +, \cdot, F_0 \rangle$, where F_0 is a countable subfield of F, which is algebraic over $\mathbb{Q}$, contains $E^S(\Sigma)$, and which is interpreted as a set of constants in $\mathbf{F}$. $\mathscr{R}_S$ is the set of all Zariski closed subsets of $S(F)^n$ defined over F_0 for all $n \in \mathbb{N}$.

(3) $q : D \to S(F)$ is a function.

Every Shimura variety $p : X^+ \to S(\mathbb{C})$ determines (up to the choice of a field of constants F_0) a natural Shimura structure which we will denote $\mathbf{p}$, where we take $F = \mathbb{C}$, $D = X^+$, $q = p$, and $D_{V,i} = X^+_{V,i}$. For F_0 we will normally use $E^S(\Sigma)$.

We have implicitly defined the language we will use for Shimura structures (which depends on S). Denote by $\mathcal{L}_D$ the language we have used in the $G^{\mathrm{ad}}(\mathbb{Q})^+$-covering sort, by $\mathcal{L}_F$ the language of the field sort, and by $\mathcal{L}$ the two-sorted language of Shimura structures. F_0 is countable, so $\mathcal{L}$ is countable. Finally, let $\mathrm{Th}(\mathbf{p})$ be the complete first-order theory of the two-sorted structure $\mathbf{p}$ in the language $\mathcal{L}$.

We will not prove in full detail that $\mathcal{L}$ is more expressive than the language used in [**17**] as it would be too much of a detour for us, but we will give the main idea in Remark 3.2.4.

3.1.1. Theory of the Variety Sort. Let $T := \mathrm{Th}(S(\mathbb{C}))$, which is the complete first-order theory of $S(\mathbb{C})$ in the language $\mathcal{L}_F$. We make a few observations, which are standard results from model theory. The models of T are bi-interpretable with F, and so any model of T is of the form $S(F)$, for F an algebraically closed field. Also T has quantifier elimination, finite Morley rank, it has a prime model (in fact, it is $S\left(\overline{\mathbb{Q}}\right)$), it is uncountably categorical and almost strongly minimal (see [**57**, p.170] for definition).

Remark 3.1.1 (Projections of special sets). Let $V \subseteq S^n$ be a special subvariety. The Shimura datum of S^n is (G^n, X^n) where G^n acts coordinate-wise on X^n. A coordinate projection $\mathrm{pr} : S(F)^n \to S(F)^m$ (where $m < n$) is a Shimura morphism as it is induced by the corresponding coordinate projections $G(\mathbb{R})^n \to G(\mathbb{R})^m$ and $X^n \to X^m$. Therefore, $\mathrm{pr}(V)$ is special. Similarly, we can obtain an analogous result for the projections of special sets in X^+, that is, the coordinate projection of a special set is again a special set. Similarly, we can obtain an analogous result for the projections of special sets in X^+, that is, the coordinate projection of a special set is again a special set.

Remark 3.1.2. Our model-theoretic approach of Shimura structures has been purely algebraic, effectively forgetting the analytic structure coming from $\mathbb{C}$. But one can instead take the opposite approach and study Shimura varieties from the perspective of o-minimality (so both X^+ and S are now defined over real closed fields), giving the two sorts part of their analytic structure. This has been done by B. Zilber in [**66**].

In his paper, Zilber studies the structure of special subvarieties in a very general class of arithmetic varieties using o-minimality. Most results are framed in terms of "weakly special subvarieties" and "weakly special domains", but one can obtain results about special subvarieties using the fact that a weakly special subvariety is special if and only if it contains a special point. The paper proves that weakly special subvarietes define a Noetherian Zariski structure (see [**66**, Theorem 4.11]) and it also gives a characterisation of special subvarieties (see [**66**, Example 6.4]). The paper also proves that the special structure of a Shimura curve has trivial geometry in the model-theoretic sense. This result can be interpreted as saying that if we can only see the special subvarieties on the powers of a Shimura curve and we are given a subset A of the curve, then the only points that are algebraic (in the model-theoretic sense) over A are those that belong to the Hecke-orbit of some point in A.

At the end of [**66**, §3.2], the author asks if mixed Shimura varieties are in the class of arithmetic varieties he is considering. But [**24**, §10.1] shows that mixed Shimura varieties do in fact satisfy the axioms of [**66**, §3.1].

3.1.2. Axiomatisation. Here we will define the special subvarieties condition (SS), and after that we will give an axiomatisation of $\mathrm{Th}(\mathbf{p})$. As usual, if $\overline{g} = (g_1, \dots, g_n)$ is a tuple from $G^{\mathrm{ad}}(\mathbb{Q})^+$, then $\overline{g}x = (g_1 x, \dots, g_n x)$ and $q(\overline{g}x) = (q(g_1 x), \dots, q(g_n x))$.

Definition. Given a special subvariety V of S^n, define the $\mathcal{L}$-sentences:

(1) $\mathrm{SS}^1_{V,i} := \forall \overline{x} \in D_{V,i}\, (q(\overline{x}) \in V)$.
(2) $\mathrm{SS}^2_{V,i} := \forall z \in V \exists \overline{x} \in D_{V,i}\, (q(\overline{x}) = z)$.

We define the *special subvarieties condition* as the axiom scheme $\mathrm{SS} := \bigcup_{V,i} \left\{\mathrm{SS}^1_{V,i} \wedge \mathrm{SS}^2_{V,i}\right\}$, where the union ranges over all indices (V,i) of all special domains in powers of X^+.

Definition. Given a tuple $\overline{g} = (g_1, \dots, g_n)$ of elements in $G^{\mathrm{ad}}(\mathbb{Q})^+$, define the $\mathcal{L}$-sentences:

(1) $\mathrm{MOD}^1_{\overline{g},V,i} := \forall x \in D_{V,i} \left((q(g_1 x), \dots, q(g_n x)) \in Z^{V,i}_{\overline{g}}\right)$.
(2) $\mathrm{MOD}^2_{\overline{g},V,i} := \forall z \in Z^{V,i}_{\overline{g}} \exists x \in D_{V,i}\, ((q(g_1 x), \dots, q(g_n x)) = z)$.

Define the *modularity condition* as $\mathrm{MOD} := \bigcup_{\overline{g},V,i} \left\{\mathrm{MOD}^1_{\overline{g},V,i} \wedge \mathrm{MOD}^2_{\overline{g},V,i}\right\}$, where the union ranges over all indices (V,i) of special domains, and all tuples $\overline{g}$ of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$.

Remark 3.1.3. Observe that MOD follows from SS. To see this recall Remark 1.3.25, and observe that whenever we have a definable set $A \subseteq D$, we have a corresponding definable set $Z^{q(A)}_{\overline{g}}$ (for any tuple $\overline{g}$ of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$) whose elements are of the form $q(\overline{g}a)$ for some $a \in A$. There is also a definable set $B \subseteq D^n$ (where n is the length of $\overline{g}$) such that $q(B) = Z^{q(A)}_{\overline{g}}$, with a definable parametrisation from A (i.e. the map

$A \to B$ given by $a \mapsto \overline{g}a$ is definable). This is because we can write in a first-order way the parametrisation of $D_{Z_{\overline{g}}^{V,i},i}$ in terms of $D_{V,i}$: $\overline{y} \in D_{Z_{\overline{g}}^{V,i},i} \iff \exists x \in D_{V,i}(\overline{y} = \overline{g}x)$.

Set $\mathrm{T}(\mathbf{p}) \coloneqq \mathrm{Th}\,(X^+) \cup T \cup \mathrm{SS}$, where $\mathrm{Th}\,(X^+)$ is the complete first-order theory of X^+ in the language $\mathcal{L}_D$. We will show in the next subsection that $\mathrm{T}(\mathbf{p})$ axiomatises $\mathrm{Th}(\mathbf{p})$.

Remark 3.1.4. For $x \in X^+$, note that $\mathrm{qftp}(x)$ "knows" about the Mumford-Tate group of x. Recall that the Mumford-Tate group determines the smallest special subvariety of $S(\mathbb{C})$ which contains $p(x)$ (see Remark 1.3.9). So if $\mathrm{qftp}(x_1) = \mathrm{qftp}(x_2)$, then $x_1, x_2 \in X^+$ have the same Mumford-Tate groups. However, if $\mathrm{qftp}(x_1) \neq \mathrm{qftp}(x_2)$, it could still happen that the Mumford-Tate groups are the same. This happens precisely when $\mathrm{spcl}(p(x_1)) = \mathrm{spcl}(p(x_2)) = V$, but x_1 and x_2 lie in different special domains, that is, $x_1 \in X^+_{V,i}$ and $x_2 \in X^+_{V,k}$, with $k \neq i$.

Remark 3.1.5. Suppose that S is a pure Shimura variety. As special points of S are zero-dimensional special subvarieties, then SS implies that if $x \in X^+$ is a special point and $g_x \in G^{\mathrm{ad}}(\mathbb{Q})^+$ fixes x and only x, then

$$\mathrm{T}(\mathbf{p}) \models \forall y \in D\,(g_x y = y \implies q(y) = p(x))\,,$$

where we view $p(x)$ as a constant symbol (because the coordinates of $p(x)$ are in F_0). In **[17]** this is called the *special points condition.* Recall that the existence of g_x is given by Theorem 1.3.6.

3.1.3. Quantifier Elimination.

Let $p : X^+ \to S(\mathbb{C})$ be a Shimura variety and $\mathbf{p}$ its corresponding two-sorted structure. In this section we will show that $\mathrm{Th}(\mathbf{p})$ has quantifier elimination. To do that, we will show that $\mathrm{T}(\mathbf{p})$ has quantifier elimination and is complete. The strategy for this is to use Propositions 1.1.2 and 1.1.3.

Proposition 3.1.6. $\mathrm{T}(\mathbf{p})$ *has quantifier elimination and is complete. In particular,* $\mathrm{T}(\mathbf{p}) = \mathrm{Th}(\mathbf{p})$.

Proof. Let $\mathbf{q} \coloneqq \langle \mathbf{D}, \mathbf{S}, q\rangle$ and $\mathbf{q}' \coloneqq \langle \mathbf{D}', \mathbf{S}', q'\rangle$ be ω-saturated models of $\mathrm{T}(\mathbf{p})$, and suppose $x_1, \dots, x_m \in D$ and $x_1', \dots, x_m' \in D'$ satisfy $\mathrm{qftp}(x_1, \dots, x_m) = \mathrm{qftp}(x_1', \dots, x_m')$. By

Propositions 1.1.2 and 1.1.3, it suffices to show that given $y \in D$, there is $y' \in D'$ such that $\mathrm{qftp}(x_1, \ldots, x_m, y) = \mathrm{qftp}(x'_1, \ldots, x'_m, y')$.

Let L be the field generated over $E^S(\Sigma)$ by the coordinates of the $q(x_i)$. Let $D_{V,i}$ be the special closure of $(x_1, \ldots, x_m, y)$. Using quantifier elimination of T and ω-saturation of $\mathbf{q}'$, we will first show that we can find $y' \in D'$ so that $(x'_1, \ldots, x'_m, y') \in D'_{U,i}$ and

$$(3.1.1)\quad \bigcup_{\overline{g}} \mathrm{qftp}\left(q(\overline{g}x_1), \ldots, q(\overline{g}x_m), q(\overline{g}y)/L\right) = \bigcup_{\overline{g}} \mathrm{qftp}\left(q'(\overline{g}x'_1), \ldots, q'(\overline{g}x'_m), q'(\overline{g}y')/L\right),$$

where the unions are taken over all finite tuples $\overline{g}$ of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$. To see the equality of (3.1.1) note that for a tuple $\overline{g}$, $\mathrm{qftp}\left(q(\overline{g}x_1), \ldots, q(\overline{g}x_m), q(\overline{g}y)/L\right)$ is determined by the smallest algebraic variety W defined over L which contains the point $(q(\overline{g}x_1), \ldots, q(\overline{g}x_m), q(\overline{g}y))$. Notice that $W \subseteq Z^{V,i}_{\overline{g}}$.

Let pr be the coordinate projection that sends $(\overline{g}x_1, \ldots, \overline{g}x_m, \overline{g}y) \mapsto (\overline{g}x_1, \ldots, \overline{g}x_m)$. Applying pr to $\overline{g}D_{V,i}$ will give us finitely many special domains, say $D_{U_1,j_1}, \ldots, D_{U_r,j_r}$ (by Remark 3.1.1). The formula which says that the projection of $\overline{g}D_{V,i}$ consists exactly of the union of $D_{U_1,j_1}, \ldots, D_{U_r,j_r}$ is in $T(\mathbf{p})$. On the other hand, by quantifier elimination of T, $\mathrm{pr}(W)$ is quantifier-free definable over L. Therefore, the formula

$$\exists u\left((v_1, \ldots, v_m, u) \in D_{V,i} \wedge (q(\overline{g}v_1), \ldots, q(\overline{g}v_m), q(\overline{g}u)) \in W\right)$$

is equivalent (in $T(\mathbf{p})$) to the quantifier-free formula

$$(\overline{g}v_1, \ldots, \overline{g}v_m) \in \left(D_{U_1,j_1} \cup \cdots \cup D_{U_r,j_r}\right) \cap q^{-1}\left(\mathrm{pr}(W)\right).$$

As this formula is quantifier-free and it is satisfied by $(x_1, \ldots, x_m)$, it is also satisfied by $(x'_1, \ldots, x'_m)$, and therefore there is $y' \in D'$ such that $(x'_1, \ldots, x'_m, y') \in D'_{V,i}$ and

$$\mathrm{qftp}\left(q(\overline{g}x_1), \ldots, q(\overline{g}x_m), q(\overline{g}y)/L\right) = \mathrm{qftp}\left(q'(\overline{g}x'_1), \ldots, q'(\overline{g}x'_m), q'(\overline{g}y')/L\right).$$

Not only that, but $\overline{g}D_{V,i} = \mathrm{spcl}(\overline{g}x_1, \ldots, \overline{g}x_m, \overline{g}y)$, and so $(\overline{g}x'_1, \ldots, \overline{g}x'_m, \overline{g}y') \in D'_{V,i}$. This means that the left-hand-side of (3.1.1) plus the set of formulas

$$\{(\overline{g}v_1, \ldots, \overline{g}v_m, \overline{g}u) \in \overline{g}D_{V,i}\}_{\overline{g}}$$

form a finitely satisfiable set in $\mathbf{q}'$, so ω-saturation gives us the y' we wanted.

Now we show that ω-saturation allows us to choose y' so that, in addition, $D'_{V,j}$ is the special closure of $(\overline{g}x'_1, \ldots, \overline{g}x'_m, \overline{g}y')$. A special domain cannot be covered by finitely many proper special subdomains, and so, when choosing y' above, we can avoid finitely many proper special subdomains of $D'_{V,i}$. By saturation we get a y' avoiding all of them. This finishes the proof. □

3.1.4. Special Locus.

DEFINITION. For $D_{V,i} \subseteq D^m$ and $a_1, \ldots, a_n \in D$, where $n < m$, let $D_{V,i}(\overline{a})$ denote the set of tuples $\overline{y} = (y_1, \ldots, y_{m-n})$ such that $(\overline{y}, \overline{a}) \in D_{V,i}$. Let $y_1, \ldots, y_m \in D$ and $A \subseteq D$. We define the *special locus* of the tuple $\overline{y}$ over A, denoted $\mathrm{sploc}(\overline{y}/A)$, to be the smallest set of the form $D_{V,i}(\overline{a})$ containing $\overline{y}$, where $\overline{a}$ ranges through all finite tuples of A.

The following lemma shows that the special locus is well-defined.

LEMMA 3.1.7. *The special locus always exists and, if $C \subseteq D^m$ is the special locus of $\overline{y}$ over A, then C is definable over A and $q(C)$ is a subvariety of S^m defined over $E^S(\Sigma, q(A))$.*

PROOF. We will assume $m = 1$; the same proof goes through for higher dimensions. We will preserve the notation from the definition of special locus. To see that the special locus exists, consider first a set of the form $D_{V,i}(\overline{a})$. We know that $V = q(D_{V,i})$ is a special subvariety of S^{n+1}. Intersecting V with the hyperplanes $X_k = q(a_k)$, $k = 2, \ldots, n+1$, is still a subvariety (although probably not special), and then if we project onto the first coordinate we get that the set $q(D_{V,i}(\overline{a}))$ is a (possibly reducible) subvariety of S. So an arbitrary intersection of sets of the form $D_{V,i}(\overline{a})$ is actually a finite intersection. The finite intersection of special domains can be written as a finite union of some other special domains. For this reason, $C = \mathrm{sploc}(y/A)$ is of the form $D_{V,i}(\overline{a})$ for some finite tuple $\overline{a}$ of A, and further, C is definable over A as a set. Also, by the above construction, $q(C)$ is definable over $E^S(\Sigma, q(A))$. □

The next proposition follows from the proof of Proposition 3.1.6. We will need this independent result later on, and so we state it for future reference. We remark that this proposition does not require that the models be ω-saturated. First we introduce the following notation.

Let $\mathbf{q} := \langle \mathbf{D}, \mathbf{S}, q\rangle$ and $\mathbf{q}' := \langle \mathbf{D}', \mathbf{S}', q'\rangle$ be models of $\mathrm{Th}(\mathbf{p})$. Suppose that there is a partial isomorphism between $\mathbf{q}$ and $\mathbf{q}'$, represented by the partial isomorphism $\xi : D \dashrightarrow D'$ with finitely generated domain $U \subseteq D$, and an embedding $\sigma : L \to F'$, where L is the field generated over F_0 by the coordinates of the images of the points in U. Observe that a set of the form $D_{V,i}(\overline{a})$ is a subset of D^n (for some n) which is definable with parameters (given by $\overline{a}$). In what follows, we write $\xi(\mathrm{sploc}(\overline{x}/U))$ for $D'_{V,i}(\xi(\overline{a}))$, where $D_{V,i}(\overline{a}) = \mathrm{sploc}(\overline{x}/U)$.

Proposition 3.1.8. *Let $x \in D_{V,i} \subseteq D$. Then for every tuple $\overline{g} = (g_1, \dots, g_n)$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$ there exists $x' \in D'_{V,i} \subseteq D'$ such that $(g_1x', \dots, g_nx') \in \xi\left(\mathrm{sploc}(g_1x, \dots, g_nx/U)\right)$ and $(q'(g_1x'), \dots, q'(g_nx'))$ realises $\sigma\left(\mathrm{qftp}\left((q(g_1x), \dots, q(g_nx))/L\right)\right)$.*

The following lemmas preserve the notation used in the definition of special locus.

Lemma 3.1.9. *If $\overline{\alpha}$ is a subtuple of $\overline{g}$ and $y \in X^+$ is such that $\overline{g}y \in \mathrm{sploc}(\overline{g}x/A)$, for some $x \in X^+$ and $A \subseteq X^+$, then $\overline{\alpha}y \in \mathrm{sploc}(\overline{\alpha}x/A)$.*

Proof. Let $\overline{\beta}$ be the complement subtuple of $\overline{\alpha}$ in $\overline{g}$, so that $\overline{g} = (\overline{\alpha}, \overline{\beta})$. The result follows from noting that $\mathrm{sploc}(\overline{g}x/A) \subseteq \mathrm{sploc}(\overline{\alpha}x/A) \times \mathrm{sploc}(\overline{\beta}x/A)$. $\square$

Lemma 3.1.10. *Let $\overline{g}_1, \dots, \overline{g}_n$ be tuples of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$. Let $y_1, \dots, y_n \in D$, $A \subseteq D$ and let $C := \mathrm{sploc}(y_1, \dots, y_n/A)$. Let $\epsilon : C \to D^r$ be given by $(x_1, \dots, x_n) \mapsto (\overline{g}_1x_1, \dots, \overline{g}_nx_n)$, where r is the sum of the lengths of the $\overline{g}_i$. Then the image of ϵ equals $\mathrm{sploc}(\overline{g}_1y_1 \dots, \overline{g}_ny_n/A)$.*

Proof. First we make a quick observation about X^+. If $X^+_{V,i} \subseteq (X^+)^n$ is a special domain and $g_1, \dots, g_n \in G^{\mathrm{ad}}(\mathbb{Q})^+$, then the image of the map:

$$\begin{array}{ccc} X^+_{V,i} & \longrightarrow & (X^+)^r \\ & (x_1, \dots, x_n) \mapsto (\overline{g}_1x_1, \dots, \overline{g}_nx_n) & \end{array}$$

is again a special domain. This may be shown by a straightforward adaptation of the proof of Lemma 1.3.24.

Now we return to the setting of the Lemma. We know that the special locus has the form $C = D_{V,i}(\overline{a})$ for some tuple of elements of A. Let m be the length of $\overline{a}$. Before tackling the full version of the Lemma, let us focus on the following special case. Let $g_1, \dots, g_n$ be elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$. We will prove that the map $\epsilon_0 : C \to D^n$ given by $(x_1, \dots, x_n) \mapsto (g_1 x_1, \dots, g_n x_n)$ is a bijection between C and $C' := \mathrm{sploc}((g_1 x_1, \dots, g_n x_n)/A)$. Clearly ϵ_0 is injective. Let

$$\begin{aligned} \epsilon_0' : \qquad D_{V,i} \qquad &\longrightarrow \qquad D^{n+m} \\ (x_1, \dots, x_{n+m}) &\mapsto (g_1 x_1, \dots, g_n x_n, x_{n+1}, \dots, x_{n+m}). \end{aligned}$$

By the above observation on X^+, we know that the image of ϵ_0' is a special domain, which shows that the image of ϵ_0 contains C'. But now we can repeat the argument backwards to show that C is contained in $\epsilon_0^{-1}(C')$. And so $\epsilon(C) = C'$.

Now we prove the general case. Consider the map

$$\begin{aligned} \epsilon' : \qquad D_{V,i} \qquad &\longrightarrow \qquad D^{r+m} \\ (x_1, \dots, x_{n+m}) &\mapsto (\overline{g}_1 x_1, \dots, \overline{g}_n x_n, x_{n+1}, \dots, x_{n+m}). \end{aligned}$$

The image of ϵ' is a special domain of D^{r+m} (because of our above observation on X^+). Let $D_{W,j}$ be the image of ϵ'. This shows that the image of ϵ is of the form $D_{W,j}(\overline{a})$, and therefore $\mathrm{sploc}(\overline{g}_1 y_1 \dots, \overline{g}_n y_n/A)$ is contained in the image of ϵ.

Say $D_{W',j}(\overline{a}') := \mathrm{sploc}(\overline{g}_1 y_1 \dots, \overline{g}_n y_n/A)$. Observe that the previous paragraph shows that every point of $D_{W',j}(\overline{a}')$ is of the form $(\overline{g}_1 x_1 \dots, \overline{g}_n x_n)$, for some $(x_1, \dots, x_n) \in C$. Let pr be the coordinate projection such that $(\overline{g}_1 y_1 \dots, \overline{g}_n y_n, \overline{a}) \in D_{W',j}$ gets mapped to $(g_1 y_1, \dots, g_n y_n, \overline{a}')$. Let $D_{V',i}$ be the special domain contained in $\mathrm{pr}(D_{W',j})$ such that $(g_1 y_1, \dots, g_n y_n, \overline{a}') \in D_{V',i}$, with g_i being the first coordinate of the tuple $\overline{g}_i$. Thanks to the special case, we now know that there is an explicit bijection between C and $\mathrm{sploc}((g_1 y_1, \dots, g_n y_n)/A)$, which finishes the proof. □

Corollary 3.1.11. *For any tuple* $\overline{g}$ *of* $G^{\mathrm{ad}}(\mathbb{Q})^+$ *and any* $x \in D$, $\mathrm{sploc}(\overline{g}x/x) = \{\overline{g}x\}$

LEMMA 3.1.12. *Given $x_1, \dots, x_n \in D$ and $A \subseteq D$ we have that*

$$\operatorname{sploc}(\overline{x}/A) = \operatorname{sploc}\left(\overline{x}/G^{\mathrm{ad}}(\mathbb{Q})^+ A\right),$$

where $G^{\mathrm{ad}}(\mathbb{Q})^+ A$ denotes the union of orbits of elements in A.

PROOF. Let $D_{V,i}(\overline{a}) = \operatorname{sploc}(\overline{x}/A)$. If m is the length of the tuple $\overline{a}$, then consider the tuple $(e, \dots, e, g_1, \dots, g_m)$. The map $\epsilon : D_{V,i} \to D_{W,j}$ defined by that element is a bijection (because the statement that says that it is a bijection between the corresponding special domains of X^+ is a first-order sentence in the theory of X^+). $\square$

We finish this section by describing the types in models of $\mathrm{Th}(\mathbf{p})$. By Proposition 3.1.6, it is enough to describe the quantifier-free types. First let us set some notation. Let $\mathbf{q} \models \mathrm{Th}(\mathbf{p})$, $x_1, \dots, x_m \in D$ and $A \subseteq D$. If $v_1, \dots, v_m$ are first-order variables in $\mathcal{L}_D$, we will write

$$\operatorname{sploc}((\overline{g}v_1, \dots, \overline{g}v_m)/A) = \operatorname{sploc}((\overline{g}x_1, \dots, \overline{g}x_m)/A)$$

to mean that $\operatorname{sploc}((\overline{g}v_1, \dots, \overline{g}v_m)/A)$ is the set of all the formulas in $\operatorname{qftp}(\overline{x}/A)$ which are either of the form $(\overline{g}v_1, \dots, \overline{g}v_m) \in D_{V,i}(\overline{a})$ or of the form $\neg((\overline{g}v_1, \dots, \overline{g}v_m) \in D_{V,i}(\overline{a}))$.

PROPOSITION 3.1.13. *Let $\mathbf{q} \models \mathrm{Th}(\mathbf{p})$, $x_1, \dots, x_m \in D$, and let $A \subseteq D$ be closed under the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$. Then $\operatorname{qftp}(\overline{x}/A)$ is determined by the partial type $Q(\overline{v})$ defined as:*

$$\bigcup_{\overline{g}} \left[\operatorname{qftp}(q(\overline{g}x_1, \dots, \overline{g}x_m)/L) \cup \{\operatorname{sploc}((\overline{g}v_1, \dots, \overline{g}v_m)/A) = \operatorname{sploc}((\overline{g}x_1, \dots, \overline{g}x_m)/A)\}\right],$$

where the union is taken over all tuples $\overline{g} = (g_1, \dots, g_n)$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$, and L is the field generated over F_0 by the coordinates of the points in $q(A)$.

PROOF. Let $y_1, \dots, y_m$ be a realisation of $Q(\overline{v})$ in some elementary extension of $\mathbf{q}$ that also satisfies SF. We need to show that $\operatorname{qftp}(\overline{x}/U_{\mathbf{D}}) = \operatorname{qftp}(\overline{y}/U_{\mathbf{D}})$.

The non-trivial atomic formulas in $\operatorname{qftp}(\overline{x}/A)$ that can be written in the domain language $\mathcal{L}_D$ are of the form: $x_i = a$, $x_i = gx_j$, $(\overline{g}x_1, \dots, \overline{g}x_m) \in D_{V,i}(\overline{a})$, for all parameters a and $\overline{a}$ from A, and all g, $\overline{g} = (g_1, \dots, g_n)$ from $G^{\mathrm{ad}}(\mathbb{Q})^+$. The quantifier-free formulas in the language $\mathcal{L}_F$ in $\operatorname{qftp}(\overline{x}/A)$ are already contained in $Q(\overline{v})$.

Observe that formulas of the form $x_i = a$ or $x_i = gx_j$ are in fact equivalent to formulas of the form $(\overline{g}x_1, \ldots, \overline{g}x_m) \in D_{V,i}(\overline{a})$, using the MOD conditions of the special subvarieties $Z_{(e,e)}$ and $Z_{(e,g)}$. So we only need to focus on this last case.

Note that $(\overline{g}x_1, \ldots, \overline{g}x_m) \in D_{V,i}(\overline{a})$, for some $\overline{g} = (g_1, \ldots, g_n)$ from $G^{\mathrm{ad}}(\mathbb{Q})^+$ and some tuple $\overline{a}$ from A, if and only if $\mathrm{sploc}((\overline{g}x_1, \ldots, \overline{g}x_m)/A) \subseteq D_{V,i}(\overline{a})$. So $(\overline{g}x_1, \ldots, \overline{g}x_m) \in D_{V,i}(\overline{a})$ implies $(\overline{g}y_1, \ldots, \overline{g}y_m) \in D_{V,i}(\overline{a})$. The converse works the same way. □

3.1.5. Models With Standard Fibres. By construction, we know that each fibre of $p : X^+ \to S(\mathbb{C})$ consists of exactly one Γ-orbit. As Γ is infinite, it seems unlikely at first sight that this property about the fibres of p can be expressed with a first-order sentence in $\mathcal{L}$. This suspicion is confirmed once we consider the following partial types:

(1) Let $z \in S(\mathbb{C})$ be a non-Hodge-generic point, and let $V \subsetneq S(\mathbb{C})$ be its special closure. V has (countably) infintely many corresponding special domains $X^+_{V,i} \subset X^+$. As different special domains are disjoint, the set of formulas (in the variable v with parameter z):

$$\{(q(v) = z) \wedge (v \notin D_{V,i})\}_{i\in\mathbb{N}}$$

is finitely satisfiable, and so it is part of a complete type. Therefore it is realised in a saturated model of $\mathrm{Th}(\mathbf{p})$. Observe that if x is a realisation of such a type, then we will see discrepancies between special closures on the domain and special closure on the variety, specifically: $\mathrm{spcl}(q(x)) \subsetneq q(\mathrm{spcl}(x))$. In this case, we will say that x is a *non-standard point*. For example, a model in which the fibre over a special point is saturated, will necessarily have non-standard points.

(2) Even if the domain of a Shimura structure consists only of standard points, we can still have that the fibres of q consists of many Γ-orbits. To see this, choose $x \in X^+$ non-special, let $D_{V,i} = \mathrm{spcl}(x)$, and consider the set of formulas (with variable v and parameter x):

$$\{(q(v) = q(x)) \wedge (v \in D_{V,i}) \wedge (\gamma x \neq v)\}_{i\in\mathbb{N},\gamma\in\Gamma}.$$

That this set is finitely satisfiable is a consequence of the fact that, for every $z \in V$, $p^{-1}(z) \cap X^+_{V,i}$ is made up of exactly one Γ^H-orbit, where $\Gamma^H = \Gamma \cap H(\mathbb{Q})_+$ and $H = \mathrm{MT}(x)$. For example, if a model has saturated fibres over Hodge-generic points, then these fibres consist of only of standard points.

We will deal with the presence of non-standard points and new standard points when we study Shimura covers. In this section we will instead omit these types as we will restrict to the class of models of $\mathrm{Th}(\mathbf{p})$ that satisfy the *standard fibres condition*, which is given by the $\mathcal{L}_{\omega_1,\omega}$-sentence:

$$\mathrm{SF} := \forall x, y \in D\left((q(x) = q(y)) \to \bigvee_{\gamma\in\Gamma}(x = \gamma y)\right).$$

Definition. Given $\mathbf{q} \models \mathrm{Th}(\mathbf{p})$, a point $x \in D$ is called a *standard point* if $q\left(\mathrm{spcl}(x)\right) = \mathrm{spcl}(q(x))$. A *section* of q is a subset $A \subset D$ which contains exactly one element for every fibre of q. A *standard section* of q is a section $A \subseteq D$ that only contains standard points and such that if $z \in A$, then for every $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ there is $h \in G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $hz \in A$ and $q(hz) = q(gz)$.

Remark 3.1.14. We point out that by $\mathrm{MOD}^2_{\bar{g},V,i}$, if $q : D \to S(F) \models \mathrm{Th}(\mathbf{p})$, then every $z \in S(F)$ has a standard point in its q-fibre. Also, by the properties of special loci, if $x \in D$ is standard, then gx is standard for every $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$.

Let $\mathrm{Th}_{\mathrm{SF}}(\mathbf{p}) := \mathrm{Th}(\mathbf{p}) \cup \mathrm{SF}$. As the next proposition shows, we can construct models of $\mathrm{Th}_{\mathrm{SF}}(\mathbf{p})$ from models of $\mathrm{Th}(\mathbf{p})$.

Lemma 3.1.15. *Given any model* $\mathbf{q} \models \mathrm{Th}(\mathbf{p})$*, there is* $\mathbf{q}' \prec \mathbf{q}$ *such that* $S(F) = S(F')$ *and* $\mathbf{q}' \models \mathrm{SF}$*.*

Proof. First observe that if $x \in D$ is standard, then gx is standard for any $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$. Now choose a standard section of q, call it A. Standard sections exist because we can look at the space of orbits of D modulo $G^{\mathrm{ad}}(\mathbb{Q})^+$ and first choose orbits which contain standard points, and then from each of these orbits, choose representatives modulo Γ. Let $D' = \Gamma A$ and let q' be the restriction of q to D'. Define $\mathbf{q}'$ as the structure given by $q' : D' \to S(F)$, where $D'_{V,i} = D_{V,i} \cap (D')^n$.

The action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ restricts to an action on D' because of the definition of standard section. No $D'_{V,i}$ can be empty because the special points of D are in D'.

It is clear that $\mathbf{q}'$ satisfies SS and, because the $D'_{V,i}$ are just restrictions of the $D_{V,i}$, we have that $\mathbf{q}'$ satisfies $\mathrm{T}(\mathbf{p})$. So $\mathbf{q}' \vDash \mathrm{Th}_{\mathrm{SF}}(\mathbf{p})$ by Proposition 3.1.6. □

When our models have standard fibres, we can strengthen the result from Proposition 3.1.13 to give a better description of the types.

PROPOSITION 3.1.16. *Let $\mathbf{q} \vDash \mathrm{Th}_{\mathrm{SF}}(\mathbf{p})$, $x_1, \ldots, x_m \in D$, and let $A \subseteq D$ be closed under the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$. Then $\mathrm{qftp}(\overline{x}/A)$ is determined (in $\mathrm{Th}_{\mathrm{SF}}(\mathbf{p})$) by:*

$$Q(\overline{v}) := \bigcup_{\overline{g}} \left[\mathrm{qftp}\left(q(\overline{g}x_1, \ldots, \overline{g}x_m)/L\right) \cup \left\{(\overline{g}v_1, \ldots, \overline{g}v_m) \in \mathrm{sploc}\left((\overline{g}x_1, \ldots, \overline{g}x_m)/A\right)\right\}\right],$$

where the union is taken over all tuples $\overline{g} = (g_1, \ldots, g_n)$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$, and L is the field generated over F_0 by the coordinates of the points in $q(A)$.

PROOF. Let $y_1, \ldots, y_m$ be a realisation of $Q(\overline{v})$ in some elementary extension of $\mathbf{q}$ that also satisfies SF. By Proposition 3.1.13 is suffices to show that $\mathrm{sploc}\left((\overline{g}y_1, \ldots, \overline{g}y_m)/A\right) = \mathrm{sploc}\left((\overline{g}x_1, \ldots, \overline{g}x_m)/A\right)$.

Note that $(\overline{g}x_1, \ldots, \overline{g}x_m) \in D_{V,i}(\overline{a})$, for some $\overline{g} = (g_1, \ldots, g_n)$ from $G^{\mathrm{ad}}(\mathbb{Q})^+$ and some tuple $\overline{a}$ from A, if and only if $\mathrm{sploc}\left((\overline{g}x_1, \ldots, \overline{g}x_m)/A\right) \subseteq D_{V,i}(\overline{a})$. So $(\overline{g}x_1, \ldots, \overline{g}x_m) \in D_{V,i}(\overline{a})$ implies $(\overline{g}y_1, \ldots, \overline{g}y_m) \in D_{V,i}(\overline{a})$. Conversely, suppose $(\overline{g}y_1, \ldots, \overline{g}y_m) \in D_{V,i}(\overline{a})$. By hypothesis and SF, there is $\overline{\gamma} \in \Gamma^{\ell m + d}$, where ℓ is the length of $\overline{g}$ and d is the length of $\overline{a}$, such that $\overline{\gamma}(\overline{g}x_1, \ldots, \overline{g}x_m, \overline{a}) = (\overline{g}y_1, \ldots, \overline{g}y_m, \overline{a})$. So $(\overline{g}x_1, \ldots, \overline{g}x_m, \overline{a}) \in D_{V,k}$, for some k. But that means that $\mathrm{sploc}\left((\overline{g}x_1, \ldots, \overline{g}x_m)/A\right) \subseteq D_{V,k}(\overline{a})$. As special domains of the same special subvariety are either the same or disjoint, we conclude that $D_{V,i} = D_{V,k}$. □

REMARK 3.1.17. Let $x_1, \ldots, x_m \in D$ be in different $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits, and let $\overline{g} = (e, g_1, \ldots, g_n)$ be a tuple of distinct elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$. Let L be the field generated over F_0 with the coordinates of the $q(x_i)$, and let $V = \mathrm{spcl}\left(q(x_1), \ldots, q(x_m)\right)$. As we have said before, $\mathrm{qftp}\left(\left(q(\overline{g}x_1), \ldots, q(\overline{g}x_n)\right)/L\right)$ is equivalent to the minimal algebraic subset of $Z_{\overline{g}}^{V,i}$ defined over L that contains the tuple. This is a subset of the fibre over $(q(x_1), \ldots, q(x_m))$

of the morphism

$$\psi^{V,i}_{\overline{g},(e)} : Z^{V,i}_{\overline{g}} \to S(\mathbb{C})^m.$$

In fact, it is the $\mathrm{Aut}(\mathbb{C}/L)$-orbit in this fibre containing this tuple.

3.2. Shimura Covers

Here we will address the following natural question: given $S(F) \vDash T$, is it possible to construct explicitly an $\mathcal{L}_D$-structure D and a map $q : D \to S(F)$ so that this becomes a model of $\mathrm{Th}(\mathbf{p})$? To answer this question, we will first take a seemingly different approach to defining Shimura structures.

3.2.1. The Covering Sort. Given $\overline{g} = (g_1, \dots, g_n)$ a tuple of $G^{\mathrm{ad}}(\mathbb{Q})^+$, set $\Gamma_{\overline{g}} := g_1^{-1}\Gamma g_1 \cap \dots \cap g_n^{-1}\Gamma g_n$. As we discussed in §1.3.6, the quotients of X^+ by the groups $\Gamma_{\overline{g}}$ can be realised as special subvarieties $Z_{\overline{g}}$ which form an inverse system of varieties, where the morphisms are those induced by the inclusion of groups. Let $\widehat{S}(\mathbb{C}) := \varprojlim Z_{\overline{g}}$ denote the inverse limit of this system. If $\overline{g}$, $\overline{h}$ and $\overline{\alpha}$ are tuples of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $\overline{g} \subseteq \overline{h}$ and $\overline{\alpha}$ is distinct to the other two, then part of this inverse system looks like:

$$\begin{tikzcd}
& \widehat{S}(\mathbb{C}) \arrow[ddl] \arrow[dddd] \arrow[dr] & \\
& & Z_{\overline{h}} \arrow[dddl] \arrow[dd] \\
Z_{\overline{\alpha}} \arrow[ddr] & & \\
& & Z_{\overline{g}} \arrow[dl] \\
& S(\mathbb{C}) &
\end{tikzcd}$$

Denote an equivalence class of $\Gamma_{\overline{g}}\backslash X^+$ by $[\cdot]_{\overline{g}}$, with the implicit understanding that if $\Gamma_{\overline{g}} = \Gamma_{\overline{h}}$, then $[\cdot]_{\overline{g}} = [\cdot]_{\overline{h}}$. So a point in $\widehat{S}(\mathbb{C})$ can be thought of as a collection of points indexed by tuples $\overline{g}$, such that $[x_{\overline{g}}]_{\overline{g}} \in \Gamma_{\overline{g}}\backslash X^+$ and satisfying the compatibility condition that if $\Gamma_{\overline{g}} \subseteq \Gamma_{\overline{h}}$, then $\psi_{\overline{g},\overline{h}}\left([x_{\overline{g}}]_{\overline{g}}\right) = [x_{\overline{h}}]_{\overline{h}}$.

The action of $\alpha \in G^{\mathrm{ad}}(\mathbb{Q})^+$ on X^+ induces a map

$$\Gamma_{\overline{g}}\backslash X^+ \to \alpha\Gamma_{\overline{g}}\alpha^{-1}\backslash X^+$$

which, as a map of algebraic varieties, is defined over E^{ab} and $E(\Sigma)$. So we can define an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\widehat{S}(\mathbb{C})$ where, if we let $\overline{g}\alpha^{-1} := (g_1\alpha^{-1}, \ldots, g_n\alpha^{-1})$, then we can describe the action on components by:

$$\alpha\,[x_{\overline{g}}]_{\overline{g}} := [\alpha x_{\overline{g}}]_{\overline{g}\alpha^{-1}}\,.$$

We recall the following lemma with the observation that its proof works for the case of mixed Shimura varieties.

Lemma 3.2.1 (see [**17**, Lemma 4.3]). *The group $\Gamma_\infty := \bigcap_{\overline{g}} \Gamma_{\overline{g}}$, where the intersection ranges over all finite tuples of distinct elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$, is contained in $Z_G(\mathbb{Q})$, where Z_G stands for the centre of G.*

Lemma 3.2.2. *There is an embedding of X^+ into $\widehat{S}(\mathbb{C})$ given by:*

$$\begin{array}{rccc} \iota: & X^+ & \longrightarrow & \widehat{S}(\mathbb{C}) \\ & x & \mapsto & ([x]_{\overline{g}})_{\overline{g}} \end{array},$$

which is $G^{\mathrm{ad}}(\mathbb{Q})^+$-equivariant.

Proof. Equivariance is immediate from the definition of the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\widehat{S}(\mathbb{C})$. Let $x, y \in X^+$ be such that $\iota(x) = \iota(y)$. Let $\overline{g} = (e, g_1, \ldots, g_n)$. We know from §1.3.6 that there is an isomorphism:

$$\Gamma_{\overline{g}} \backslash X^+ \xrightarrow{\sim} Z_{\overline{g}}$$

$$[x]_{\overline{g}} \mapsto (p(x), p(g_1 x), \ldots, p(g_n x)).$$

Therefore we have $(p(x), p(g_1 x), \ldots, p(g_n x)) = (p(y), p(g_1 y), \ldots, p(g_n y))$, which means that $y \in \Gamma_{\overline{g}} x$. As this happens for every tuple of the form $\overline{g}$, we get that $y \in \Gamma_\infty x$, so that $y = x$ by Lemma 3.2.1. $\square$

Let $\widehat{p} : \widehat{S}(\mathbb{C}) \to S(\mathbb{C})$ be given by $\widehat{p}\Big(\big([x_{\overline{g}}]_{\overline{g}}\big)_{\overline{g}}\Big) = p(x_e)$. Thus we get a commutative diagram:

$$\begin{tikzcd} X^+ \arrow[rr, "\iota"] \arrow[dr, "p"'] & & \widehat{S}(\mathbb{C}) \arrow[dl, "\widehat{p}"] \\ & S(\mathbb{C}) & \end{tikzcd}$$

where ι is the embedding given in Lemma 3.2.2.

From its construction, we can see $\widehat{S}(\mathbb{C})$ as the inverse limit of the system of special subvarieties $Z_{\overline{g}}$, for all tuples $\overline{g}$ from $G^{\mathrm{ad}}(\mathbb{Q})^{+}$. So a point $\widetilde{x} \in \widehat{S}(\mathbb{C})$ can be written as $\widetilde{x} = (z_{\overline{g}})_{\overline{g}}$, where $z_{\overline{g}} \in Z_{\overline{g}}$, and whenever $\overline{g} \subseteq \overline{h}$,

$$\psi_{\overline{h},\overline{g}}\left(z_{\overline{h}}\right) = z_{\overline{g}}. \tag{3.2.1}$$

This structure comes equipped with natural maps $\widehat{p_{\overline{g}}} : \widehat{S}(\mathbb{C}) \to Z_{\overline{g}}$, which are determined by the natural map $\widehat{p} : \widehat{S}(\mathbb{C}) \to S(\mathbb{C})$ given by $\widehat{p_{\overline{g}}}\left(\widetilde{x}\right) = z_{\overline{g}}$.

LEMMA 3.2.3. *For every* $\widetilde{x} \in \widehat{S}(\mathbb{C})$ *and any tuple* $\overline{g}$ *of* $G^{\mathrm{ad}}(\mathbb{Q})^{+}$*, we have that:*

$$\widehat{p_{\overline{g}}}\left(\widetilde{x}\right) = \widehat{p}\left(\overline{g}\widetilde{x}\right).$$

PROOF. In order to avoid messy notation, we will prove the statement for an arbitrary tuple from $G^{\mathrm{ad}}(\mathbb{Q})^{+}$ of the form (α_1, α_2). The general case can be proven using the same argument. Choose $\widetilde{x} \in \widehat{S}(\mathbb{C})$. We denote its components using both systems of notation, i.e. $\widetilde{x} = (z_{\overline{g}})_{\overline{g}}$ (with $z_{\overline{g}} \in Z_{\overline{g}}$), and $\widetilde{x} = \left([x_{\overline{g}}]_{\overline{g}}\right)_{\overline{g}}$, where $z_{\overline{g}} = p\left(\overline{g}x_{\overline{g}}\right)$.

On one hand $\widehat{p}_{(\alpha_1,\alpha_2)}\left(\widetilde{x}\right) = z_{(\alpha_1,\alpha_2)}$, but we also know that $\alpha_1[x_{\overline{g}}]_{\overline{g}} = [\alpha_1 x_{\overline{g}}]_{\overline{g}\alpha_1^{-1}}$, so in particular $\alpha_1\left[x_{\alpha_1}\right]_{\alpha_1} = \left[\alpha_1 x_{\alpha_1}\right]_e$. Therefore $\widehat{p}(\alpha_1\widetilde{x}, \alpha_2\widetilde{x}) = (\left[\alpha_1 x_{\alpha_1}\right]_e, \left[\alpha_2 x_{\alpha_2}\right]_e)$.

By compatibility of components, we have that $\left[x_{(\alpha_1,\alpha_1)}\right]_{\alpha_i} = \left[x_{\alpha_i}\right]_{\alpha_i}$ for $i = 1, 2$. This means that there are $\gamma_1 \in \alpha_1^{-1}\Gamma\alpha_1$ and $\gamma_2 \in \alpha_2^{-1}\Gamma\alpha_2$ such that $x_{\alpha_1} = \gamma_1 x_{(\alpha_1,\alpha_2)}$ and $x_{\alpha_2} = \gamma_2 x_{(\alpha_1,\alpha_2)}$. So there are $\gamma_1', \gamma_2' \in \Gamma$ such that $\alpha_1 x_{\alpha_1} = \gamma_1'\alpha_1 x_{(\alpha_1,\alpha_2)}$ and $\alpha_2 x_{\alpha_2} = \gamma_2'\alpha_2 x_{(\alpha_1,\alpha_2)}$, so that $\widehat{p}(\alpha_1\widetilde{x}, \alpha_2\widetilde{x}) = p\left(\alpha_1 x_{(\alpha_1,\alpha_2)}, \alpha_2 x_{(\alpha_1,\alpha_2)}\right) = z_{(\alpha_1,\alpha_2)}$. □

REMARK 3.2.4. Let us call $\mathcal{L}_{\mathrm{curves}}$ the language used in [**17**]. As we already hinted at the end of §3.1, $\mathcal{L}$ is more expressive than $\mathcal{L}_{\mathrm{curves}}$. Specifically, $\mathcal{L}_{\mathrm{curves}}$ cannot define the special domains $D_{V,i}$ corresponding to special subvarieties V which are not of the form $Z_{\overline{g}}$. For example, suppose $p : X^{+} \to S(\mathbb{C})$ is a Shimura variety of dimension 2 and that $V \subset S$ is a special subvariety of dimension 1. If the only structure on D is the one coming from the action of $G^{\mathrm{ad}}(\mathbb{Q})^{+}$, then the set $X^{+}_{V,i} = \left\{x \in X^{+} : p(\overline{g}x) \in Z^{V,i}_{\overline{g}}\right\}$ is type-definable, but not definable. To see that it is not definable, one can proceed by contradiction. If it were definable, then the type defining $X^{+}_{V,i}$ would be isolated. But one can find a sub

$\mathcal{L}_{\text{curves}}$-structure of $\widehat{S}(\mathbb{C})$ omitting this type. For this, first consider $\iota(X^+) \subset \widehat{S}(\mathbb{C})$, and now replace every $\widetilde{x} \in \iota\left(X^+_{V,i}\right)$ by a non-standard point in the same p-fibre. That this can be done in a compatible way which omits the type, is left as an exercise.

DEFINITION. If $\widetilde{x}_1, \ldots, \widetilde{x}_m \in \widehat{S}(\mathbb{C})$, we define the map

$$\widehat{p_{\overline{g}}}(\widetilde{x}_1, \ldots, \widetilde{x}_m) := (\widehat{p_{\overline{g}}}(\widetilde{x}_1), \ldots, \widehat{p_{\overline{g}}}(\widetilde{x}_m)).$$

Given a special subvariety $V \subseteq S(\mathbb{C})^m$, we define the set $\widehat{S}(\mathbb{C})_{V,i}$ as the set of points $\widetilde{x} \in \widehat{S}(\mathbb{C})^m$ such that for every tuple $\overline{g}$ from $G^{\text{ad}}(\mathbb{Q})^+$, $\widehat{p_{\overline{g}}}(\widetilde{x}) \in Z^{V,i}_{\overline{g}}$. We call a set of the form $\widehat{S}(\mathbb{C})_{V,i}$ a *special domain* of $\widehat{S}(\mathbb{C})$.

REMARK 3.2.5. Let $x \in X^+$ be special. Any point $\widetilde{x} \in \widehat{S}(\mathbb{C})$ satisfying $\widehat{p}(\widetilde{x}) = p(x)$ and $\widetilde{x} \notin \iota(X^+)$ is a non-standard point. This is because by Lemmas 3.2.1 and 3.2.2, any special domain in $\widehat{S}(\mathbb{C})$ for the special subvariety $\{p(x)\}$ is a singleton, i.e. for every i there is some $\gamma \in \Gamma$ such that $\widehat{S}(\mathbb{C})_{\{p(x)\},i} = \{\iota(\gamma x)\}$.

Thus we get a two-sorted structure $\widehat{\mathbf{p}} := \left\langle \widehat{S}(\mathbb{C}), S(\mathbb{C}), \{\widehat{p_{\overline{g}}}\}_{\overline{g}} \right\rangle$, with each $\widehat{p_{\overline{g}}}$ denoting a function from $\widehat{S}(\mathbb{C})$ to a corresponding power of $S(\mathbb{C})$. The language for $S(\mathbb{C})$ is the same language $\mathcal{L}_F$ from before, but now the language on $\widehat{S}(\mathbb{C})$ consists only of relation symbols for each $\widehat{S}(\mathbb{C})_{V,i}$, without an explicit action from $G^{\text{ad}}(\mathbb{Q})^+$.

DEFINITION. A *covering structure* for the Shimura variety $p : X^+ \to S(\mathbb{C})$ is a two-sorted structure $\widetilde{\mathbf{q}} := \left\langle \widetilde{M}, M, \{q_{\overline{g}}\} \right\rangle$ defined in the following way.

(a) Let $\mathcal{L}_{\text{cov}}$ be the language consisting of relation symbols for every $\widehat{S}(\mathbb{C})_{V,i}$, which can be seen as a reduct of the language $\mathcal{L}_D$. Let $\widetilde{\mathcal{L}}$ be the language $\widetilde{\mathcal{L}} := \mathcal{L}_{\text{cov}} \cup \mathcal{L}_F \cup \{q_{\overline{g}}\}$, where every $q_{\overline{g}}$ is a function symbol, indexed by a tuple $\overline{g}$ from $G^{\text{ad}}(\mathbb{Q})^+$.

(b) Let $T = \text{Th}(S(\mathbb{C}))$ as before, and let $\widetilde{T}(\widehat{\mathbf{p}})$ be the complete first-order theory of of the two-sorted structure $\widehat{\mathbf{p}}$ in the language $\widetilde{\mathcal{L}}$. Models of $\widetilde{T}$ will be called *Shimura covers.*

(c) Models of T will usually be denoted simply by M, but sometimes we will be more explicit by saying that $M = S(F)$ for some algebraically closed field F. Models of $\widetilde{T}(\widehat{\mathbf{p}})$ will be denoted $\widetilde{\mathbf{q}} := \left\langle \widetilde{M}, M, \{q_{\overline{g}}\} \right\rangle$. Given a special subvariety $V \subseteq M^n$, let $\widetilde{M}_{V,i}$ denote a subset of $\widetilde{M}^n$. These sets will still be called *special domains.*

(d) Given an $\widetilde{\mathcal{L}}$-structure $\langle \widetilde{M}, M, \{q_{\overline{g}}\} \rangle$ and $B \subseteq \widetilde{M}$, let $\mathrm{He}(B)$ denote the set of points in M that are the coordinates of the points of the form $q_{\overline{g}}(b)$, for all $b \in B$ and all tuples $\overline{g}$ from $G^{\mathrm{ad}}(\mathbb{Q})^{+}$.

The expectation now is that given $S(F) \models T$, we can construct $\widehat{S}(F)$ as the inverse limit of the $Z_{\overline{g}}(F)$, and then the corresponding structure $\langle \widehat{S}(F), S(F), \{q_{\overline{g}}\} \rangle$ will be a model of $\widetilde{T}(\widehat{\mathbf{p}})$. There is, however, an important detail that we need to consider before proceeding.

As we explained in Remark 3.2.5, for every special point $z \in S(\mathbb{C})$, the corresponding special domains $\widehat{S}(\mathbb{C})_{z,i}$ consist each of exactly one point, something which is also true of the special domains $X^{+}_{z,i}$. If $p(x) = z$, then Theorem 1.3.6 says that there is some $\alpha \in G^{\mathrm{ad}}(\mathbb{Q})^{+}$ such that x is the unique fixed point of α. This means then that the intersection of special domains $X^{+}_{Z_{(e,\alpha)},1} \cap X^{+}_{Z_{(e,e)},1}$ consists of exactly one point: (x,x). But the intersection $\widehat{S}(\mathbb{C})_{Z_{(e,\alpha)},1} \cap \widehat{S}(\mathbb{C})_{Z_{(e,e)},1}$ may have more than one point if α fixes more elements of $\widehat{S}(\mathbb{C})$. Indeed, let $x_0 \in X^{+}$ be the unique point fixed by α. Now considering the following set of formulas in the language $\widetilde{\mathcal{L}}$:

$$\{p_{\overline{g}}(v) = p_{\overline{g}}(\alpha v) \wedge v \neq x_0\}_{\overline{g}},$$

where $\overline{g}$ is any tuple from $G^{\mathrm{ad}}(\mathbb{Q})^{+}$. Given $\overline{g}$, choose $\gamma \in \Gamma_{\overline{g}}$ non-trivial and suppose $x \in X^{+}$ satisfies $\alpha x = \gamma x$; this shows that the set of formulas can be finitely satisfiable, and thus it defines a partial type. We see then that α has a new fixed point in $\widehat{S}(\mathbb{C})$. We summarise this in the following lemma.

Lemma 3.2.6. *If we view $\widehat{S}(\mathbb{C})$ as an $\mathcal{L}_D$-structure, then it is not elementarily equivalent to X^{+}.*

It makes sense then to consider the following $\mathcal{L}_{\mathrm{cov}}$-substructure of $\widehat{S}(\mathbb{C})$.

Definition. Let $\widetilde{S}(\mathbb{C})$ denote the subset of $\widehat{S}(\mathbb{C})$ which consists exactly of the standard points. Given a tuple $\overline{g}$ of $G^{\mathrm{ad}}(\mathbb{Q})^{+}$, let $\widetilde{p_{\overline{g}}} : \widetilde{S}(\mathbb{C}) \to S(\mathbb{C})^{n}$ be the restriction of $\widehat{p_{\overline{g}}}$. We therefore get an $\widetilde{\mathcal{L}}$-structure $\widetilde{\mathbf{p}} := \langle \widetilde{S}(\mathbb{C}), S(\mathbb{C}), \{\widetilde{p_{\overline{g}}}\} \rangle$. In fact, $\widetilde{S}(\mathbb{C})$ also inherits a $G^{\mathrm{ad}}(\mathbb{Q})^{+}$-action from $\widehat{S}(\mathbb{C})$.

LEMMA 3.2.7. *$\widetilde{x} \in \widehat{S}(\mathbb{C})$ is non-standard if and only if there is a tuple $\overline{g}$ of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$ and a standard point $\widetilde{y} \in \operatorname{spcl}(\widetilde{x})$ such that $\widehat{p}(\widetilde{x}) = \widehat{p}(\widetilde{y})$ and $\widehat{p_{\overline{g}}}(\widetilde{x}) \notin \operatorname{spcl}(\widehat{p_{\overline{g}}}(\widetilde{y}))$.*

PROOF. Suppose first that $\widetilde{x}$ is non-standard. Let $V = \operatorname{spcl}(\widehat{p}(\widetilde{x}))$ and $\widehat{S}(\mathbb{C})_{W,j} = \operatorname{spcl}(\widetilde{x})$. Then $V \subsetneq W$. For some $i \in \mathbb{N}$ we can find $\widetilde{y} \in \widehat{S}(\mathbb{C})_{V,i} \subsetneq \widehat{S}(\mathbb{C})_{W,j}$ such that $\widehat{p}(\widetilde{y}) = \widehat{p}(\widetilde{x})$. As $\widetilde{x} \notin \widehat{S}(\mathbb{C})_{V,i}$, then by definition there must be a tuple $\overline{g}$ such that $\widehat{p_{\overline{g}}}(\widetilde{x}) \notin Z_{\overline{g}}^{V,i}$. But by Lemma 3.1.10, $Z_{\overline{g}}^{V,i} = \operatorname{spcl}(\widehat{p_{\overline{g}}}(\widetilde{y}))$.

Conversely, assume that there exist $\overline{g}$ and $\widetilde{y}$ satisfying the conditions for $\widetilde{x}$. Then $\operatorname{spcl}(\widehat{p}(\widetilde{x})) = \operatorname{spcl}(\widehat{p}(\widetilde{y})) = V$, and as $\widetilde{y}$ is standard, there is some $i \in \mathbb{N}$ such that $\widetilde{y} \in \widehat{S}(\mathbb{C})_{V,i}$. As $\widehat{p}(\widetilde{x}) \notin \operatorname{spcl}(\widehat{p_{\overline{g}}}(\widetilde{y})) = Z_{\overline{g}}^{V,i}$, we get that $\widetilde{x} \notin \widehat{S}(\mathbb{C})_{V,i}$. But because $\widetilde{y} \in \operatorname{spcl}(\widetilde{x})$, it must be that $\widehat{S}(\mathbb{C})_{V,i} \subsetneq \operatorname{spcl}(\widetilde{x})$, showing that $\widetilde{x}$ is non-standard. □

In what remains of Part 3, we will consider the two theories $\widetilde{T}(\widehat{\mathbf{p}})$ and $\widetilde{T}(\widetilde{\mathbf{p}})$, the latter being defined as the complete first-order theory of $\widetilde{\mathbf{p}}$ in the language $\widetilde{\mathcal{L}}$. Many of the results that follow hold in both theories, but we will see that, crucially, $\widetilde{T}(\widetilde{\mathbf{p}})$ is modeled by $\langle X^+, S(\mathbb{C}), \{p_{\overline{g}}\}\rangle$, whereas $\widetilde{T}(\widehat{\mathbf{p}})$ is not (see Corollary 3.2.14).

3.2.2. Axiomatisation and Quantifier Elimination. Given $p : X^+ \to S(\mathbb{C})$, we now give axiomatisations for the theories $\widetilde{T}(\widehat{\mathbf{p}})$ and $\widetilde{T}(\widetilde{\mathbf{p}})$. Define:

- **(A1):** $M \vDash T$.
- **(A2):** For every $\widetilde{M}_{V,i}$ and every tuple $\overline{g}$ from $G^{\mathrm{ad}}(\mathbb{Q})^+$, $q_{\overline{g}}\left(\widetilde{M}_{V,i}\right) = Z_{\overline{g}}^{V,i}$.
- **(A3):** For every special subvariety V, whenever we have two tuples of $G^{\mathrm{ad}}(\mathbb{Q})^+$ satisfying $\overline{g} \subseteq \overline{h}$, then the function $\psi_{\overline{h},\overline{g}}^{V,i} \circ q_{\overline{h}}$ restricted to $\widetilde{M}_{V,i}$ equals the function $q_{\overline{g}}$ restricted to $\widetilde{M}_{V,i}$ for all i.
- **(A4):** For any three special subvarieties V_1, V_2 and W and for every triple of indices i, j, k, we will have that $\widetilde{M}_{W,k} \subseteq \widetilde{M}_{V_1,i} \cap \widetilde{M}_{V_2,j}$ if and only if $\widehat{S}(\mathbb{C})_{W,k} \subseteq \widehat{S}(\mathbb{C})_{V_1,i} \cap \widehat{S}(\mathbb{C})_{V_2,j}$. Furthermore, given indices $k_1, \dots, k_r$ of the special domains of W, we will have that $\widetilde{M}_{V_1,i} \cap \widetilde{M}_{V_2,j} \subseteq \widetilde{M}_{W_1,k_1} \cup \cdots \cup \widetilde{M}_{W_r,k_r}$ if and only if $\widehat{S}(\mathbb{C})_{V_1,i} \cap \widehat{S}(\mathbb{C})_{V_2,j} \subseteq \widehat{S}(\mathbb{C})_{W_1,k_1} \cup \cdots \cup \widehat{S}(\mathbb{C})_{W_r,k_r}$.
- **(A5):** Let V and W denote two special subvarieties and fix a coordinate projection pr. Then for every pair of indices i, j we will have that $\widetilde{M}_{W,j}$ is contained in the coordinate projection $\operatorname{pr}\left(\widetilde{M}_{V,i}\right)$ if and only if $\widehat{S}(\mathbb{C})_{W,j}$ is contained in the

corresponding coordinate projection of $\mathrm{pr}\big(\widehat{S}(\mathbb{C})_{V,i}\big)$. Furthermore, a coordinate projection of $\widetilde{M}_{V,i}$ is contained in $\widetilde{M}_{W,j}$, if and only if the corresponding coordinate projection of $\widehat{S}(\mathbb{C})_{V,i}$ is contained in $\widehat{S}(\mathbb{C})_{W,j}$.

(A6): If $z \in S(\mathbb{C})$ is a special point, then $\widetilde{M}_{z,i}$ is a singleton for every i.

Let $\widetilde{T}'(\widehat{\mathbf{p}})$ be the first-order theory axiomatised by (A1)–(A6) and let $\widetilde{T}'(\widetilde{\mathbf{p}})$ be the first-order theory axiomatised by the corresponding axioms, where every appearance of $\widehat{S}(\mathbb{C})$ is replaced by $\widetilde{S}(\mathbb{C})$.

LEMMA 3.2.8. $\widehat{\mathbf{p}} \models \widetilde{T}'(\widehat{\mathbf{p}})$, *so* $\widetilde{T}'(\widehat{\mathbf{p}}) \subseteq \widetilde{T}(\widehat{\mathbf{p}})$. *Similarly,* $\widetilde{T}'(\widetilde{\mathbf{p}}) \subseteq \widetilde{T}(\widetilde{\mathbf{p}})$.

PROOF. Clearly $\widehat{\mathbf{p}}$ satisfies (A1). For (A2) first observe that by definition, any $x \in \widehat{S}(\mathbb{C})_{V,i}$ and any tuple $\overline{g}$ satisfy $\widehat{p_{\overline{g}}}(x) \in Z_{\overline{g}}^{V,i}$. Also, any $z \in Z_{\overline{g}}^{V,i}$ is part of some compatible system $(z_{\overline{g}})_{\overline{g}} \in \widehat{S}(\mathbb{C})$. (A3) is satisfied by the definition of $\widehat{S}(\mathbb{C})_{V,i}$ and the compatibility condition (3.2.1). (A4) and (A5) are immediate. Finally, as we discussed earlier when we defined $\widetilde{T}(\widetilde{\mathbf{p}})$, (A6) is satisfied. ☐

PROPOSITION 3.2.9. *The theories* $\widetilde{T}'(\widehat{\mathbf{p}})$ *and* $\widetilde{T}'(\widetilde{\mathbf{p}})$ *are complete and have quantifier elimination. Therefore* $\widetilde{T}(\widehat{\mathbf{p}}) = \widetilde{T}'(\widehat{\mathbf{p}})$ *and* $\widetilde{T}'(\widetilde{\mathbf{p}}) = \widetilde{T}(\widetilde{\mathbf{p}})$.

PROOF. Let $\big\langle \widetilde{M}, M, \{q_{\overline{g}}\}\big\rangle$ and $\big\langle \widetilde{M}', M', \{q'_{\overline{g}}\}\big\rangle$ be two ω-saturated models of $\widetilde{T}'$, and suppose $x_1, \ldots, x_m \in \widetilde{M}$ and $x'_1, \ldots, x'_m \in \widetilde{M}'$ satisfy $\mathrm{qftp}(x_1, \ldots, x_m) = \mathrm{qftp}(x'_1, \ldots, x'_m)$. Choose $y \in \widetilde{M}$ and let $\mathrm{spcl}(x_1, \ldots, x_m, y) = \widetilde{M}_{V,i}$. Let L be the field generated over F_0 by the coordinates of $q(x_1), \ldots, q(x_m)$. Given a tuple $\overline{g}$, the type $\mathrm{qftp}(q_{\overline{g}}(x_1, \ldots, x_m, y)/L)$ is determined by the smallest algebraic subvariety defined over L which contains the point $q_{\overline{g}}(x_1, \ldots, x_m, y)$, call it W. Clearly $W \subseteq Z_{\overline{g}}^{V,i}$.

If $M = S(F)$ and $M' = S(F')$, then there is an embedding $\sigma : L \hookrightarrow F'$ defined by the finite partial isomorphism. Let pr denote the coordinate projection sending $(x_1 \ldots, x_m, y) \mapsto (x_1, \ldots, x_m)$. We will use the same name to denote the corresponding coordinate projections

$$\mathrm{pr} : Z^{n(m+1)}_{(g_1, \ldots, g_n)} \longrightarrow Z^{nm}_{(g_1, \ldots, g_n)}.$$

Observe that the argument we will use now will be slightly more subtle than the one used in Proposition 3.1.6 because although Remark 3.1.1 still holds in $\widetilde{\mathbf{p}}$, it does not hold in $\widehat{\mathbf{p}}$. But we can still get through with the same strategy. The set $\mathrm{pr}(W)$ is quantifier-free definable over L. The projection $\mathrm{pr}\left(\widetilde{M}_{V,i}\right)$ will equal a finite union $\widetilde{M}_{U_1,j_1} \cup \cdots \cup \widetilde{M}_{U_r,j_r}$ of special domains in the case of $\widetilde{T}'(\widetilde{\mathbf{p}})$, but in the case of $\widetilde{T}'(\widehat{\mathbf{p}})$ we can only say that $\mathrm{pr}\left(\widetilde{M}_{V,i}\right)$ contains this finite union. What is still true in both theories is the formula:

$$\forall \overline{v}\left[\left(\overline{v} \in \bigcup_{s=1}^{r} \widetilde{M}_{U_s,j_s} \wedge q_{\overline{g}}(\overline{v}) \in \mathrm{pr}(W)\right) \to \exists u\left((\overline{v},u) \in \widetilde{M}_{V,i} \wedge q_{\overline{g}}(\overline{v},u) \in W\right)\right]. \tag{3.2.2}$$

By (A2), (A5) and ω-saturation, we can choose $y' \in \widetilde{M}'$ so that $(x_1',\ldots,x_m',y') \in \widetilde{M}'_{V,i}$, $q_{\overline{g}}(x_1',\ldots,x_m',y') \in W(F')$ and $q_{\overline{g}}(x_1',\ldots,x_m',y')$ does not belong to a smaller algebraic subset defined over L. Indeed, by (A5) we know that the projection of $\widetilde{M}_{V,i}$ onto the first m coordinates contains $\mathrm{spcl}(x_1,\ldots,x_m)$. By (A3) we know that $\widetilde{M}'_{V,i}$ is mapped surjectively onto $Z_{\overline{g}}^{V,i}(F')$. Thus, using (3.2.2) we can find y' so that $(x_1',\ldots,x_m',y') \in \widetilde{M}'_{V,i}$ and

$$\mathrm{qfpt}\left(q_{\overline{g}}\left(x_1,\ldots,x_m,y\right)/L\right) = \mathrm{qfpt}\left(q_{\overline{g}}\left(x_1',\ldots,x_m',y'\right)/L\right)$$

By ω-saturation we can now find y'' so that

$$\bigcup_{\overline{g}} \mathrm{qfpt}\left(q_{\overline{g}}\left(x_1,\ldots,x_m,y\right)/L\right) = \bigcup_{\overline{g}} \mathrm{qftp}\left(q_{\overline{g}}\left(x_1',\ldots,x_m',y''\right)/L\right)$$

and, as $\widetilde{M}_{V,i}$ is not covered by finitely many proper special subdomains, ω-saturation also gives us that $\mathrm{spcl}(x_1',\ldots,x_m',y'') = \widetilde{M}'_{V,i}$. □

Now given a model $S(F) \models T$ we can obtain two covering structures for it: $\widehat{S}(F)$ which is defined as the inverse limit of the varieties $Z_{\overline{g}}$, and $\widetilde{S}(F)$, defined to be the subset consisting of standard points of $\widehat{S}(F)$. The maps $\widehat{q_{\overline{g}}} : \widehat{S}(F) \to Z_{\overline{g}}$ are the natural maps, and $\widetilde{q_{\overline{g}}}$ is the restriction of $\widehat{q_{\overline{g}}}$ to $\widetilde{S}(F)$.

Corollary 3.2.10. *$\left\langle \widehat{S}(F), S(F), \{\widehat{q_{\overline{g}}}\}\right\rangle \models \widetilde{T}(\widehat{\mathbf{p}})$, and $\left\langle \widetilde{S}(F), S(F), \{\widetilde{q_{\overline{g}}}\}\right\rangle \models \widetilde{T}(\widetilde{\mathbf{p}})$.*

Corollary 3.2.11 (see Proposition 3.1.13). *Let $\widetilde{\mathbf{q}}$ be a model of $\widehat{T}(\widehat{\mathbf{p}})$ or $\widetilde{T}(\widetilde{\mathbf{p}})$, and choose $x_1,\ldots,x_m \in \widetilde{M}$ and a set $A \subseteq \widetilde{M}$. Then $\mathrm{qftp}(x_1,\ldots,x_m/A)$ is determined by:*

$$Q(\overline{v}) \coloneqq \bigcup_{\overline{g}} \mathrm{qfpt}\left(q_{\overline{g}}(x_1,\ldots,x_m)/L\right) \cup \left\{\mathrm{sploc}((v_1,\ldots,v_m)/A) = \mathrm{sploc}((x_1,\ldots,x_m)/A)\right\},$$

where L is the field generated over F_0 by the elements of $\mathrm{He}(A)$.

Proposition 3.2.12. *$\langle X^+, S(\mathbb{C}), \{p_{\overline{g}}\}\rangle \models \widetilde{T}(\widetilde{\mathbf{p}})$, where the structure of X^+ consists only of the names of the special domains $X^+_{V,i}$, and we define $p_{\overline{g}}(x) := p(\overline{g}x)$, for all $x \in X^+$ and all tuples $\overline{g}$ from $G^+(\mathbb{Q})^+$.*

Proof. By Proposition 3.2.9, we need to verify that $\langle X^+, S(\mathbb{C}), \{p_{\overline{g}}\}\rangle$ satisfies (A1)–(A6). (A1), (A2), (A3) and (A6) are straightforward. Observe that the axiom scheme (A2) is essentially a combination of the axiom schemes SS and MOD from §3.1.2.

To verify (A4) and (A5), first recall the map ι from Lemma 3.2.2, and notice that its image is completely contained in $\widetilde{S}(\mathbb{C})$. From the definition of special domains in $\widetilde{S}(\mathbb{C})$, we get then that $x \in X^+_{V,i}$ if and only if $\iota(x) \in \widetilde{S}(\mathbb{C})_{V,i}$. This takes care of some of the implications needed to show (A4) and (A5). For example, if $\widetilde{S}(\mathbb{C})_{W,k} \subseteq \widetilde{S}(\mathbb{C})_{V_1,i} \cap \widetilde{S}(\mathbb{C})_{V_2,j}$, then we get that $X^+_{W,k} \subseteq X^+_{V_1,i} \cap X^+_{V_2,j}$. Conversely, if $X^+_{W,k} \subseteq X^+_{V_1,i} \cap X^+_{V_2,j}$, then $Z^{W,k}_{\overline{g}} \subseteq Z^{V_1,i}_{\overline{g}} \cap Z^{V_2,j}_{\overline{g}}$ for all tuples $\overline{g}$. Similarly, $\widetilde{S}(\mathbb{C})_{V,i} \subseteq \mathrm{pr}\left(\widetilde{S}_{W,j}\right)$ if and only if $X^+_{V,i} \subseteq \mathrm{pr}\left(X^+_{W,j}\right)$.

Suppose now that $\widetilde{S}(\mathbb{C})_{V_1,i} \cap \widetilde{S}(\mathbb{C})_{V_2,j} \subseteq \widetilde{S}(\mathbb{C})_{W_1,k_1} \cup \cdots \cup \widetilde{S}(\mathbb{C})_{W_r,k_r}$. As before, we get that $X^{V_1,i} \cap X^+_{V_2,j} \subseteq X^+_{W_1,k_1} \cup \cdots \cup X^+_{W_r,k_r}$. Conversely, suppose $X^{V_1,i} \cap X^+_{V_2,j} \subseteq X^+_{W_1,k_1} \cup \cdots \cup X^+_{W_r,k_r}$. If the corresponding statement is not true in $\widetilde{S}(\mathbb{C})$, then there must be a point $\widetilde{x} \in \widetilde{S}(\mathbb{C})_{V_1,i} \cap \widetilde{S}(\mathbb{C})_{V_2,j}$ such that $\widetilde{x} \notin \widetilde{S}(\mathbb{C})_{W_1,k_1} \cup \cdots \cup \widetilde{S}(\mathbb{C})_{W_r,k_r}$. Choose $y \in X^+_{V_1,i} \cap X^+_{V_2,j}$ such that $p(y) = \widetilde{p}(\widetilde{x})$. By Lemma 3.2.7, we get that $\widetilde{x}$ is non-standard, but $\widetilde{S}(\mathbb{C})$ by definition only has standard points. This contradiction shows that (A4) is satisfied.

To verify what remains of (A5), observe that $\mathrm{pr}\left(X^+_{W,j}\right) \subseteq X^+_{V,i}$ implies that the corresponding projection of $Z^{W,j}_{\overline{g}}$ is contained in $Z^{V,i}_{\overline{g}}$, for all $\overline{g}$. On the other hand, it is immediate that if $\mathrm{pr}\left(\widetilde{S}(\mathbb{C})_{W,j}\right) \subseteq \widetilde{S}(\mathbb{C})_{V,i}$, then $\mathrm{pr}\left(X^+_{W,j}\right) \subseteq X^+_{V,i}$. □

Corollary 3.2.13. *If $\widetilde{\mathbf{q}} \models \widetilde{T}(\widetilde{\mathbf{p}})$, then $\widetilde{M}$ carries an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$.*

Proof. Every element $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ defines a special subvariety $Z_{(e,g)}$, and it has corresponding special domains $\widetilde{M}_{Z_{(e,g)},i}$. Because $\langle X^+, S(\mathbb{C}), \{p_{\overline{g}}\}\rangle \models \widetilde{T}(\widetilde{\mathbf{p}})$ by Proposition 3.2.12, then one of the special domains $X^+_{Z_{(e,g)},i}$ corresponds to the graph of the action of g on X^+, say that it is $X^+_{Z_{(e,g)},1}$. And as $\widetilde{T}$ is complete (by Proposition 3.2.9), then the formula which says that $\widetilde{M}_{Z_{(e,g)},1}$ is the graph of a function and that this function is a bijection from $\widetilde{M}$ to itself, must hold in all models of $\widetilde{T}(\widetilde{\mathbf{p}})$. And so we can define the action of g as: $y = gx \iff (x,y) \in \widetilde{M}_{Z_{(e,g)},1}$. □

Using Lemma 3.2.6 we then get the following corollary.

COROLLARY 3.2.14. *$\widetilde{T}(\widetilde{\mathbf{p}})$ and $\widetilde{T}(\widehat{\mathbf{p}})$ are different theories. Also, $\widetilde{T}(\widetilde{\mathbf{p}})$ and* $\mathrm{Th}(\mathbf{p})$ *are bi-interpretable and have the same 0-definable sets. In this sense, $\widetilde{T}(\widetilde{\mathbf{p}})$ and* $\mathrm{Th}(\mathbf{p})$ *are the "same" theory.*

Due to this, **from now on we will treat $\widetilde{T}(\widetilde{\mathbf{p}})$ and $\mathrm{Th}(\mathbf{p})$ as indistinct**. The following proposition then is an immediate consequence of Lemma 3.1.15.

PROPOSITION 3.2.15 (Minimal models). *Suppose $\widetilde{\mathbf{q}} \models \widetilde{T}(\widetilde{\mathbf{p}})$, and let $B \subseteq M$. Let $S(F) \models T$ be prime over B. Then there is $\widetilde{\mathbf{q}}' \models \widetilde{T}(\widetilde{\mathbf{p}})$ such that $\widetilde{\mathbf{q}}' \prec \widetilde{\mathbf{q}}$, $M' = S(F)$, and $\widetilde{\mathbf{q}}'$ is minimal over $q^{-1}(B) \cup S(F)$.*

COROLLARY 3.2.16. *Let $\widetilde{\mathbf{q}} \models \widetilde{T}(\widetilde{\mathbf{p}})$ and suppose $M' \prec M$, where M is the variety sort of $\widetilde{\mathbf{q}}$. Set $\widetilde{M}' = q^{-1}(M')$ and let $q'_{\overline{g}}$ be the restriction of $q_{\overline{g}}$ to $\widetilde{M}'$. Then $\left\langle \widetilde{M}', M', \left\{ q'_{\overline{g}} \right\} \right\rangle \models \widetilde{T}(\widetilde{\mathbf{p}})$.*

REMARK 3.2.17. For $\widetilde{\mathbf{q}} \models \widetilde{T}(\widetilde{\mathbf{p}})$ with $M = S(F)$, we can define $\iota : \widetilde{M} \to \widehat{S}(F)$ as: $\iota(x) = (z_{\overline{g}})_{\overline{g}}$ if and only if for all tuples $\overline{g}$, $q_{\overline{g}}(x) = z_{\overline{g}}$. If $x, y \in \widetilde{M}$ are distinct and satisfy $x = \gamma y$ for some $\gamma \in \Gamma$, then by Lemma 1.3.13 this is the only possible element of Γ sending y to x (notice that for each $\gamma \in \Gamma$, the statement of Lemma 1.3.13 can be expressed by a first-order sentence). By Lemma 3.2.1, there is a tuple $\overline{g}$ such that $\gamma \notin \Gamma_{\overline{g}}$ (as $x \neq y$, γ cannot be central). The following sentence is in $\mathrm{Th}(\mathbf{p})$:

$$\forall x, y \in D\left((x = \gamma y) \to q\left(\overline{g}x\right) \neq q\left(\overline{g}y\right)\right).$$

This shows that $\iota(x) \neq \iota(y)$. In particular, if $\widetilde{\mathbf{q}} \models \mathrm{SF}$, then ι is injective. Furthermore, this shows that if $\mathbf{q} \models \mathrm{Th}_{\mathrm{SF}}(\mathbf{p})$, then the image of $\iota : D \to \widehat{S}(F)$ is contained in $\widetilde{S}(F)$.

3.2.3. Stability Aspects of $\mathrm{Th}(\mathbf{p})$. We will now discuss some results about the stability properties of $\mathrm{Th}(\mathbf{p})$. We remark that the same results and proofs hold for $\widetilde{T}(\widehat{\mathbf{p}})$. We start with a result which follows immediately from quantifier elimination.

COROLLARY 3.2.18. *Let $\widetilde{\mathbf{q}} \models \mathrm{Th}(\mathbf{p})$ and let $B \subseteq \widetilde{M}$. Suppose $A \subseteq \widetilde{M}^n$ is definable over B. Then there exist a tuple $\overline{g}$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$, special domains $\widetilde{M}_{V_j,i_j}$, tuples $\overline{b}^j$ of B, and*

$\varnothing \neq Y_j \subseteq V_j\left(q_{\overline{g}}\left(\overline{b}^j\right)\right)$, with j ranging over a finite set and with each Y_j $\mathcal{L}_F$-definable over $q_{\overline{g}}(B)$, such that:

$$\bigcup_j\left(\widetilde{M}_{V_j,i_j}\left(\overline{b}^j\right)\cap q_{\overline{g}}^{-1}\left(Y_j\right)\right)\subseteq A\subseteq\bigcup_j\widetilde{M}_{V_j,i_j}\left(\overline{b}^j\right).$$

PROOF. By Proposition 3.2.9, we know that $\mathrm{Th}(\mathbf{p})$ has quantifier elimination. Using the disjunctive normal form of quantifier-free formulas, the fact that for two distinct special domains in the power of $\widetilde{M}$ either one is contained in the other or they are disjoint, and (A2), we get the desired result. □

PROPOSITION 3.2.19. $\mathrm{Th}(\mathbf{p})$ *is superstable.*

PROOF. We will use the classification of the stability spectrum of complete countable theories ([**57**, Theorem 8.6.5]). By quantifier elimination and Corollary 3.2.11, $\mathrm{tp}(x/A)$ is determined by:

$$\bigcup_{\overline{g}}\mathrm{qftp}(q(\overline{g}x)/\overline{g}A)\cup\{\mathrm{sploc}(v/A)=\mathrm{sploc}(x/A)\}.$$

As we have said before, $\mathrm{qftp}(q_{\overline{g}(x)}/q_{\overline{g}}(A))$ is determined by the smallest algebraic subvariety defined over $F_0\left(q_{\overline{g}}(A)\right)$, this will be a subvariety of $Z_{\overline{g}}$. The fibres of the maps $Z_{\overline{h}}\to Z_{\overline{g}}$ are finite, so once we have a realisation of $\mathrm{qftp}\left(q_{\overline{g}}(x)/q_{\overline{g}}(A)\right)$, there are only finitely many possibilities for $\mathrm{qftp}\left(q_{\overline{h}}(x)/q_{\overline{g}}(A)\right)$. On the other hand, given a special subvariety V, there are at most countably many corresponding special domains $\widetilde{M}_{V,i}$. Therefore, if $A\subseteq\widetilde{M}$ is such that $|A|\geq 2^{\aleph_0}$, then the space of 1-types over A has cardinality at most $|A|$, which implies (by the stability spectrum theorem) that $\mathrm{Th}(\mathbf{p})$ is superstable. □

PROPOSITION 3.2.20. *Let $\widetilde{\mathbf{q}}\models\mathrm{Th}(\mathbf{p})$, $x\in\widetilde{M}$ and $B\subseteq\widetilde{M}$. Then $\mathrm{tp}(x/B)$ forks over $A\subseteq B$ if and only if $\mathrm{tp}\left(q(x)/\mathrm{He}(B)\right)$ forks over $\mathrm{He}(A)$ or $\mathrm{sploc}(x/B)$ is not definable over A.*

PROOF. Suppose first that $\mathrm{tp}(x/B)$ forks over A and that $C=\mathrm{sploc}(x/B)$ is defined over A; say that $\varphi\left(v,\overline{b}\right)\in\mathrm{tp}(x/B)$ divides over A . We can assume that $\varphi\left(v,\overline{b}\right)\models v\in C$ (because if φ and ψ are consistent and φ divides over A, then using that $\models\varphi\wedge\psi\to\varphi$ we get $\varphi\wedge\psi$ divides over A). Then, for any $\overline{b}'\equiv_A\overline{b}$, we have $\varphi\left(v,\overline{b}'\right)\models v\in C$. By Corollary

3.2.18, $\varphi\left(v, \bar{b}\right)$ is implied by a formula in $\mathrm{tp}(x/B)$ of the form $v \in C \wedge \psi\left(q_{\bar{g}}(v)\right)$, where ψ is a formula in $\mathcal{L}_{\mathrm{cov}}(B)$ implying $q_{\bar{g}}(v) \in q_{\bar{g}}(C)$. As φ divides over A, ψ must divide over A (we are assuming C is defined over A, so $v \in C$ does not fork over A), which means that $\mathrm{tp}\left(q_{\bar{g}}(x)/\mathrm{He}(B)\right)$ divides over $\mathrm{He}(A)$. Since $q(x)$ is algebraic over $q_{\bar{g}}(x)$,we conclude that $\mathrm{tp}\left(q(x)/\mathrm{He}(B)\right)$ forks over A.

Conversely, suppose C is not defined over A. Then $v \in C$ forks over A. □

LEMMA 3.2.21. $\mathrm{Th}(\mathbf{p})$ *has finite* U*-rank.*

PROOF. We know that T has finite Morley rank, which bounds the length of possible chains of types in the language $\mathcal{L}_F$. On the other hand, the sploc is defined by finitely many parameters. By Proposition 3.2.20, this shows that $\mathrm{Th}(\mathbf{p})$ has finite SU-rank, and as SU-rank and U-rank coincide in stable theories (see [**57**, Exercise 8.6.2]), by Proposition 3.2.19 we are done. □

Part 4

Categoricity Results

4.1. Necessary Conditions

We now come to the question of categoricity of the theory of a Shimura variety. Let $p : X^+ \to S$ be fixed throughout. To begin with, $\mathrm{Th}(\mathbf{p})$ is not categorical because there is no first-order restriction on the size of the fibres of p. Also, from the properties of ACF_0, we know that if $S(F)$ is a countable model of T, then it determined up to isomorphism by the transcendence degree of F over $\mathbb{Q}$. One way of addressing this is to add two (infinitely long) conditions to the theory. Let $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ be the union of $\mathrm{Th}(\mathbf{p})$, SF, and the $\mathcal{L}_{\omega_1,\omega}$-sentences saying that the transcendence degree of F over $\mathbb{Q}$ is infinite. So our question now is whether $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ is categorical.

We will show that if the class of models of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ is uncountably categorical, then we must have open image for every Galois representation attached to points as explained in §1.3.6. Our exposition is influenced by the main argument of [**17**, Theorem 4.1]. We start by recalling a theorem of Keisler, which is central to all the known categoricity results regarding arithmetic varieties.

Theorem 4.1.1 (see [**27**, Corollary 5.6]). *If an $\mathcal{L}_{\omega_1,\omega}$ sentence is $\aleph_1$-categorical, then the set of m-types over the empty set in models of this sentence is at most countable.*

Consider the inverse system of special subvarieties of the form $Z_{\overline{g}}$, where $\overline{g}$ is chosen so that $\Gamma_{\overline{g}}$ is normal in Γ, with the corresponding maps $\psi_{\overline{g},(e)} : Z_{\overline{g}} \to S(\mathbb{C})$. As explained in §1.3.6, for every $z \in S(\mathbb{C})$ the $\psi_{\overline{g},(e)}$-fibre over z carries a simply transitive action of $\Gamma/\Gamma_{\overline{g}}$ and a compatible action of $\mathrm{Aut}(\mathbb{C}/L)$, where L is a finitely generated field containing F_0 and the coordinates of z. If $p(x) = z$, then $\mathrm{tp}(x)$ knows in which of the $\mathrm{Aut}(\mathbb{C}/L)$-orbits in $\psi^{-1}_{\overline{g},(e)}(z)$ lies $p_{\overline{g}}(x)$.

Example 9. Let us look at an example. In the following diagram suppose that $z \in S(\mathbb{C})$ is Hodge-generic, that $\{z_1, z_2\} = \psi^{-1}_{\overline{g},e}(z)$ and that $\{z_{11}, z_{12}, z_{21}, z_{22}\} = \psi^{-1}_{\overline{h},e}(z)$. Suppose also that $\psi_{\overline{h},\overline{g}}(z_{11}) = \psi_{\overline{h},\overline{g}}(z_{12}) = z_1$ and that $\psi_{\overline{h},\overline{g}}(z_{21}) = \psi_{\overline{h},\overline{g}}(z_{22}) = z_2$.

Let us assume that the action of $\mathrm{Aut}(\mathbb{C}/L)$ on $\psi^{-1}_{\overline{g},e}(z)$ is trivial, and so this action partitions the fibre into the orbits $\{z_1\}$ and $\{z_2\}$, and that the action on $\psi^{-1}_{\overline{h},e}(z)$ gives us two orbits: $\{z_{11}, z_{12}\}$ and $\{z_{21}, z_{22}\}$. Each of these sets is then definable over L. The sets of formulas $\{p_{\overline{g}}(v) = z_1, p_{\overline{h}}(v) \in \{z_{11}, z_{12}\}\}$ and $\{p_{\overline{g}}(v) = z_2, p_{\overline{h}}(v) \in \{z_{21}, z_{22}\}\}$ (which have parameters from L) both can be realised in X^+, and the realisations are distinct.

$$\begin{tikzcd}
z_{11}, z_{12}, z_{21}, z_{22} \in Z_{\overline{h}} \arrow[dr, "\psi_{\overline{h},\overline{g}}"] \arrow[dd, "\psi_{\overline{h},(e)}"] & \\
& Z_{\overline{g}} \ni z_1, z_2 \arrow[dl, "\psi_{\overline{g},(e)}"] \\
z \in S(\mathbb{C}) &
\end{tikzcd}$$

In general, the action of $\mathrm{Aut}(\mathbb{C}/L)$ may not be transitive on $\psi_{\overline{g},e}^{-1}(z)$, and so it partitions $\psi_{\overline{g},e}^{-1}(z)$ into finitely many $\mathrm{Aut}(\mathbb{C}/L)$-orbits. By choosing different $\mathrm{Aut}(\mathbb{C}/L)$-orbits in $\psi_{\overline{g},e}^{-1}(z)$ for every $\overline{g}$ in a compatible way, we can construct new types from $\mathrm{tp}(x)$. Indeed, as we saw in Example 9, given a tuple $\overline{g}$ as before, we can find $y \in X^+$ such that $p_{\overline{g}}(y)$ is in a specific $\mathrm{Aut}(\mathbb{C}/L)$-orbit of $\psi_{\overline{g},(e)}^{-1}(z)$ (this is just by the construction of $Z_{\overline{g}}$). We thus get an infinite tree of possible choices of compatible $\mathrm{Aut}(\mathbb{C}/L)$-orbits. Choosing finitely many compatible $\mathrm{Aut}(\mathbb{C}/L)$-orbits is satisfiable. The types that are constructed in this way are satisfied by elements in $\widehat{S}(\mathbb{C})$.

If the action of $\mathrm{Aut}(\mathbb{C}/L)$ is never transitive on the fibres over z (or if there are infintely many tuples $\overline{g}$ for which it is not transitive), then the tree of compatible choices of orbits would branch infinitely many times, and so we would be able to define uncountably many 1-types, with each branch of the tree describing a different type. Keisler's theorem would then say that $\mathrm{Th}_{\mathrm{SF}}^{\infty}(\mathbf{p})$ is not $\aleph_1$-categorical (even if we restrict the size of the fibres and the transcendence degree of the field).

On the other hand, if a Galois representation $\rho_{\overline{y}} : \mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma}$ (following the notation of §1.3.5) has open image, then there is only a finite number of tuples for which the action of $\mathrm{Aut}(\mathbb{C}/L)$ is non-transitive, and so the tree will have only finitely many branches. This shows then that the open image condition of Galois representations attached to points in the Shimura variety is necessary for $\mathrm{Th}_{\mathrm{SF}}^{\infty}(\mathbf{p})$ to be uncountably categorical.

Now, given the observations of Remark 1.3.17, $\rho_{\overline{y}}$ will definitely not have an open image in $\overline{\Gamma}$ if z is not Hodge-generic. If $\mathrm{spcl}(z) = V \subsetneq S(\mathbb{C})$, then we need to restrict the inverse system of varieties V in order to get the correct open image statement, meaning that we need to consider the inverse system of varieties of the form $Z_{\overline{g}}^{V,i}$ (where we choose an index i and fix it). Aside from this, the rest of the argument proceeds in the same way. And so what we require is that the representations $\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$ have open image.

Corollary 4.1.2. *Let $p : X^+ \to S(\mathbb{C})$ be a Shimura variety. Let $\overline{z} \in S(\mathbb{C})^n$ and let $V = \mathrm{spcl}(\overline{z})$. If $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ is $\aleph_1$-categorical, then the image of a homomorphism*

$$\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$$

associated with $\overline{z}$ has finite index (for all i).

4.2. Categoricity Through Quasiminimality

We will now focus on showing that, under the open image conditions we discussed in §4.1, $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ is categorical. The Shimura structure $\mathbf{p}$ is not quasiminimal in general when the dimension of S is at least 2 (even if we restrict to the class of models satisfying SF), and therefore we cannot immediately apply Theorem 1.1.4. The specific issue in our case is that, for higher dimensional Shimura varieties, (QM4) does not hold because (in the notation of (QM4)) a and a' may have different special closures.

So the strategy will be to first find a sufficiently general curve C which, when considered with the structure induced from $\mathbf{p}$, will be quasiminimal (assuming the open image conditions). The categoricity of the theory defined by C will then allow us to obtain categoricity of $\mathbf{p}$.

4.2.1. Hecke-generic Curves.

Definition. Let $\overline{g}$ be a tuple from $G^{\mathrm{ad}}(\mathbb{Q})+$ such that $\Gamma_{\overline{g}}$ is normal in Γ. We say that an irreducible subvariety $C \subseteq S$ is *$\overline{g}$-Hecke-generic* if $\psi^{-1}_{\overline{g},(e)}(C)$ is irreducible in $Z_{\overline{g}}$. We say that C is *Hecke-generic* if $\psi^{-1}_{\overline{g},(e)}(C)$ is irreducible in $Z_{\overline{g}}$ whenever $\Gamma_{\overline{g}}$ is normal in Γ (i.e. it is $\overline{g}$-Hecke-generic for all $\overline{g}$).

For the existence of such curves, we first observe the following.

Lemma 4.2.1. *Let $C \subseteq S$ be an irreducible subvariety defined over a countable algebraically closed field L. Let $b \in C(\mathbb{C})$ be a generic point over L. Let $\overline{g}$ be such that $\Gamma_{\overline{g}}$ is normal in Γ. Then C is $\overline{g}$-Hecke-generic if and only if some (all) corresponding homomorphism* $\mathrm{Aut}\left(\mathbb{C}/L(b)\right) \to \Gamma/\Gamma_{\overline{g}}$ *is surjective.*

PROOF. The map $\psi_{\overline{g},(e)} : Z_{\overline{g}} \to S(\mathbb{C})$, being finite, is closed. So for some irreducible component $Y \subseteq \psi^{-1}_{\overline{g},(e)}(C)$ we must have that $\psi_{\overline{g},(e)}(Y) = C$. If Y' is another irreducible component, then $Y = \gamma Y'$ for some $\gamma \in \Gamma/\Gamma_{\overline{g}}$ (Γ acts transitively in analytic components, cf [**59**, Theorem 4.1]). Choose $b_{\overline{g}} \in Y$ such that $\psi_{\overline{g},(e)}(b_{\overline{g}}) = b$. Thus $b_{\overline{g}}$ is generic in Y over L. Therefore, by considering the Galois representation of $\mathrm{Aut}(\mathbb{C}/L(b))$ on $b_{\overline{g}}$, the homomorphism $\mathrm{Aut}(\mathbb{C}/L(b)) \to \Gamma/\Gamma_{\overline{g}}$ is surjective if and only if $Y = \gamma Y'$ for all $\gamma \in \Gamma/\Gamma_{\overline{g}}$. In other words, $\psi^{-1}_{\overline{g},(e)}(C)$ is irreducible. $\square$

Take C to be a curve in S defined over F_0 whose generic point is Hodge-generic. We can use [**4**, Theorem 2.6] to conclude that C is what is known in [**4**] as "without isotrivial components". After that, we can use [**4**, Theorem 1.3] to conclude that the ℓ-adic monodromy of C is as large as possible for all ℓ big enough. Due to [**12**, §6.1] we can deduce that, in terms of Lemma 4.2.1, almost all (possibly except for finitely many tuples $\overline{g}$) the associated Galois representations are surjective. But that means that there is a tuple $\overline{h}$ such that a connected component of $\psi^{-1}_{\overline{h},(e)}(C)$ is Hecke-generic, seen as a curve in the Shimura variety $Z_{\overline{h}}$. As we are interested in categoricity, the argument of Appendix B shows that categoricity of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ follows from categoricity of $Z_{\overline{h}}$. Therefore, we can assume that S has a Hecke-generic curve.

REMARK 4.2.2. The argument in [**4**] requires G to be reductive. This is because if G is reductive, then G^{ad} is semi-simple (due to [**49**, §2.1.10 Theorem 2.4]) and of adjoint type because the centre of G^{ad} is trivial.[2] This allows us to use the other results mentioned. So the argument we have used to show existence of Hecke-generic curves is restricted to the axioms of pure Shimura varieties.

So now let $C \subseteq S$ be a Hecke-generic curve defined over F_0. Let $X^+_C := p^{-1}(C(\mathbb{C}))$. Consider the two-sorted structure $\mathbf{r} := \left\langle X^+_C, C(\mathbb{C}), \{r_{\overline{g}}\}_{\overline{g}}\right\rangle$, where the structure on X^+_C is inherited from the $\mathcal{L}_{\mathrm{cov}}$-structure on X^+, the structure on $C(\mathbb{C})$ is inherited from the $\mathcal{L}_F$-structure on $S(\mathbb{C})$, and the maps $r_{\overline{g}}$ are just the restrictions of $p_{\overline{g}}$ to X^+_C. Thus $\mathbf{r}$ is a definable (without parameters) $\widetilde{\mathcal{L}}$-substructure of $\mathbf{p}$. In particular, X^+_C has a definable

[2]See Theorem 2.1 in http://math.stanford.edu/~conrad/249BW16Page/handouts/zgstr.pdf for a proof of this.

Γ-action. More generally, we repeat this construction with a given model $\mathbf{q} \vDash \widetilde{T}$ or $\mathrm{Th}(\mathbf{p})$: let $D_C = q^{-1}(C(F))$, and denote by $\mathbf{r_q}$ the two-sorted structure $\left\langle D_C, C(F), \{r_{\overline{g}}\}_{\overline{g}}\right\rangle$.

REMARK 4.2.3. Let $\mathrm{Th}(\mathbf{r})$ be the complete first-order theory of $\mathbf{r}$ in the language $\widetilde{\mathcal{L}}$. Given $\mathbf{q} \vDash \widetilde{T}$, then the corresponding $\widetilde{\mathcal{L}}$-structure $\mathbf{r_q}$ is a model of $\mathrm{Th}(\mathbf{r})$. Also, $\mathrm{Th}(\mathbf{r})$ has quantifier elimination because $\widetilde{T}$ has quantifier elimination and $\mathbf{r}$ is a $\varnothing$-definable substructure of a model of $\widetilde{T}$. The models of $\mathrm{Th}(\mathbf{r})$ have a definable Γ-action, so it makes sense to consider the class $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{r})$ of models which satisfy SF and where the field F has infinite transcendence degree over $\mathbb{Q}$.

LEMMA 4.2.4. *Let* $\mathbf{q} \vDash \mathrm{Th}(\mathbf{p})$. *Then* $\mathbf{r_q}$ *determines* $\mathbf{q}$ *i.e.* $\mathbf{q} = \mathrm{dcl}\,(D_C \cup C(F))$, *where* dcl *denotes the definable closure operator in* $\mathbf{q}$.

PROOF. The domain sorts $C(F)$ and $S(F)$ are both defined over the field sort F (with the same set of constants), so we know that at the level of the variety structures, $S(F)$ is definable over $C(F)$.

Let $x \in D$. As $S(F)$ is definable, then the fibre $q^{-1}(q(x))$ is definable. Now consider $\mathrm{sploc}\,(x/D_C)$, which we know is definable by Lemma 3.1.7. It remains to show that for all non-central $\gamma \in \Gamma$, $\gamma x \notin \mathrm{sploc}\,(x/D_C)$. By Remark 3.2.17, for every non-central $\gamma \in \Gamma$ there is $g \in G^{\mathrm{ad}}(\mathbb{Q})^{+}$ such that $q(gx) \neq q(g\gamma x)$. As $\mathrm{sploc}\,(gx/D_C) = g\mathrm{sploc}\,(x/D_C)$ by Lemma 3.1.10 and $q\,(\mathrm{sploc}\,(gx/D_C)) = q(gx)$ by Lemma 3.1.7, we conclude that $\gamma x \notin \mathrm{sploc}\,(x/D_C)$. $\square$

By Lemma 4.2.4, to prove categoricity of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ it will be enough to prove categoricity of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{r})$. Let $\mathscr{K}_{\mathbf{r}}$ be the class of models of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{r})$. Given a model $\mathbf{t} \in \mathscr{K}_{\mathbf{r}}$, let us denote its sorts by $\mathbf{t} = \left\langle \mathbf{R}, \mathbf{C}, \{t_{\overline{g}}\}_{\overline{g}}\right\rangle$. We define a pregeometry $\mathrm{cl}_{\mathbf{t}}$ on $\mathbf{t}$ as follows. Given a set $B \subseteq C(F)$, let $\mathrm{acl}(B)$ denote the set of all points in $C(F)$ with coordinates that are algebraic over the coordinates of the points in B (this is just the model-theoretic definition of acl). The operator acl is always a pregeometry in strongly minimal structures (such as the curve $C(F)$). Given $A \subseteq R \cup C(F)$ we write $A = A_R \cup A_C$, where $A_R = (A \cap R)$ and $A_C = A \cap C(F)$, and then set $A_1 = A_R \cup r^{-1}(A_R)$. Then we define $\mathrm{cl}_{\mathbf{t}}(A) := \left\langle r^{-1} \circ \mathrm{acl} \circ r(A_1), \mathrm{acl} \circ q(A_1), \{r_{\overline{g}}\}_{\overline{g}}\right\rangle$, where we take the induced structure on each sort. For example, the sort $\mathrm{acl} \circ r(A_1)$ is simply the $\mathcal{L}_F$-structure $C(K)$, where K is the algebraic closure of the field generated over F_0 by the coordinates of $r(A_1)$.

As acl is a pregeometry, then it is straightforward to check that $\mathrm{cl}_{\mathbf{t}}$ is also a pregeometry. It is straightforward to adapt the axiomatisation from §3.2.2 to give an axiomatisation for $\mathrm{Th}(\mathbf{r})$, and from this it follows that $\mathrm{cl}_{\mathbf{t}}(A) \vDash \mathrm{Th}_{\mathrm{SF}}(\mathbf{r})$. Lastly, the dimension defined by $\mathrm{cl}_{\mathbf{t}}$ is equal to the transcendence degree of K.

The strategy now is to prove categoricity of $\mathscr{K}_{\mathbf{r}}$ by showing that $(\mathbf{r}, \mathrm{cl}_{\mathbf{r}})$ satisfies the axioms (QM1)–(QM5) (see §1.1.3). The axioms are easy to check except for (QM5), which will be treated in the next section.

4.2.2. Finite-Index Conditions (FIC). Let $\mathscr{K}_{\mathbf{p}}$ denote the class of models of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$. In this section we will show that $(\mathbf{r}, \mathrm{cl}_{\mathbf{r}})$ satisfies (QM5). This will follow by verifying the homogeneity conditions for $\mathscr{K}_{\mathbf{p}}$.

DEFINITION. We say that a Shimura variety $p : X^+ \to S(\mathbb{C})$ has the *first finite-index condition* (FIC1) if the following is satisfied. Given any non-special points $x_1, \ldots, x_m \in D$ in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits, let $D_{V,i} = \mathrm{spcl}(x_1, \ldots, x_m)$ and let L be the field finitely generated over F_0 by the coordinates of the $p(x_i)$. Then the induced homomorphism $\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$ has finite index.

PROPOSITION 4.2.5 (cf. [**17**, Lemma 4.11]). *Suppose $p : X^+ \to S(\mathbb{C})$ has FIC1. Then $\mathscr{K}_{\mathbf{p}}$ satisfies $\aleph_0$-homogeneity over the empty set.*

PROOF. Suppose $\mathbf{q} := \langle \mathbf{D}, \mathbf{S}, q \rangle$ and $\mathbf{q}' := \langle \mathbf{D}', \mathbf{S}', q' \rangle$ are two models of $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ and suppose that $x_1 \ldots, x_m \in D$ is a collection of non-special points in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits. Suppose that $x'_1, \ldots, x'_m \in D'$ is another collection of non-special points in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits, such that

$$\mathrm{qftp}\,(x_1, \ldots, x_m) = \mathrm{qftp}\,(x'_1, \ldots, x'_m)\,.$$

Let L be field generated over F_0 by the coordinates of the $q(x_i)$. This finite partial isomorphism yields a partial $\mathcal{L}_D$-isomorphism $\xi : D \dashrightarrow D'$ and an embedding $\sigma : L \to F'$ fixing F_0.

Let $\tau \in D$ be a non-special point in a separate $G^{\mathrm{ad}}(\mathbb{Q})^{+}$-orbit from the x_i such that the coordinates of $q(\tau)$ are algebraic over L (so $q(\tau) \in \operatorname{acl}(q(x_1), \ldots, q(x_m))$). Let V be the special closure of $(q(x_1), \ldots, q(x_m), q(\tau))$. Let $\overline{g} = (e, g_1, \ldots, g_n)$ be a tuple of elements of G. Let $L_{\overline{g}}$ denote the field generated over L by the coordinates of the $q(\overline{g}x_i)$ and $q(\overline{g}\tau)$ (so $q(\tau)$ is defined over $L_{(e)}$, which means that L and $L_{(e)}$ may be different). Also, let

$$\overline{\Gamma^{V,i}}_{\overline{g}} := \varprojlim \Gamma^{V,i}_{\overline{g},\overline{h}},$$

where $\overline{h}$ varies over all tuples such that $\Gamma_{\overline{h}}$ is contained in $\Gamma_{\overline{g}}$ and is normal in Γ. By FIC1, the image of the homomorphism $\operatorname{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$, corresponding to the point $(q(x_1), \ldots, q(x_m), q(\tau)) \in S(\mathbb{C})^{m+1}$, has finite index. Therefore the image of the homomorphism is open, and so there exists a tuple $\overline{g}$ such that the homomorphism $\operatorname{Aut}(\mathbb{C}/L_{\overline{g}}) \to \overline{\Gamma^{V,i}}_{\overline{g}}$, corresponding to the point $(q(\overline{g}x_1), \ldots, q(\overline{g}x_m), q(\overline{g}\tau)) \in Z^{V,i}_{\overline{g}}$, is surjective. Now observe that

$$\bigcup_{\overline{g}} \operatorname{qftp}\left(q\left(\overline{g}x_1\right), \ldots, q\left(\overline{g}x_m\right), q\left(\overline{g}\tau\right)\right)$$

is equivalent to

$$\operatorname{qftp}\left(q\left(\overline{g}x_1\right), \ldots, q\left(\overline{g}x_m\right), q\left(\overline{g}\tau\right)\right) \cup \bigcup_{\overline{h}} \operatorname{qftp}\left(q\left(\overline{h}x_1\right), \ldots, q\left(\overline{h}x_m\right), q\left(\overline{h}\tau\right)/L_{\overline{g}}\right),$$

where $\overline{h}$ varies over the tuples such that $\Gamma_{\overline{h}}$ is contained in $\Gamma_{\overline{g}}$ and normal in Γ. Using Proposition 3.1.8, we choose $\tau' \in D'$ satisfying the following two conditions

$$\begin{aligned} \xi\left(\operatorname{sploc}\left(\overline{g}\tau, \overline{g}x_1, \ldots, \overline{g}x_m\right)\right) &= \operatorname{sploc}\left(\overline{g}\tau', \overline{g}x'_1, \ldots, \overline{g}x'_m\right), \\ \operatorname{qftp}\left(q\left(\overline{g}x_1\right), \ldots, q\left(\overline{g}x_m\right), q\left(\overline{g}\tau\right)\right) &= \operatorname{qftp}\left(q'\left(\overline{g}x'_1\right), \ldots, q'\left(\overline{g}x'_m\right), q'\left(\overline{g}\tau'\right)\right). \end{aligned}$$

We now extend $\sigma : L \to F'$ to $\sigma : L_{\overline{g}} \to F'$. By Proposition 3.1.16, to finish the proof we need to verify that

$$\xi\left(\operatorname{sploc}\left(\overline{h}\tau, \overline{h}x_1, \ldots, \overline{h}x_m/\overline{g}x_1, \ldots, \overline{g}x_m, \overline{g}\tau\right)\right) = \operatorname{sploc}\left(\overline{h}\tau', \overline{h}x'_1, \ldots, \overline{h}x'_m/\overline{g}x', \ldots, \overline{g}x'_m, \overline{g}\tau'\right)$$

(but this is already guaranteed by Corollary 3.1.11), and that

(4.2.1)

$$\operatorname{qftp}\left(q\left(\overline{h}x_1\right), \ldots, q\left(\overline{h}x_m\right), q\left(\overline{h}\tau\right)/L_{\overline{g}}\right) = \operatorname{qftp}\left(q\left(\overline{h}x'_1\right), \ldots, q\left(\overline{h}x'_m\right), q\left(\overline{h}\tau'\right)/\sigma\left(L_{\overline{g}}\right)\right),$$

where $\overline{h}$ is any tuple such that $\Gamma_{\overline{h}}$ is contained in $\Gamma_{\overline{g}}$ and normal in Γ. The left-hand side of (4.2.1) is determined by the $\mathrm{Aut}(\mathbb{C}/L_{\overline{g}})$-orbit of $\left(q\left(\overline{h}x_1\right),\ldots,q\left(\overline{h}x_m\right),q\left(\overline{h}\tau\right)\right)\in Z_{\overline{h}}^{V,i}$ in the fibre of $\psi_{\overline{h},\overline{g}}$ over $\left(q\left(\overline{g}x_1\right),\ldots,q\left(\overline{g}x_m\right),q\left(\overline{g}\tau\right)\right)\in Z_{\overline{g}}^{V,i}$. Since we are assuming FIC1, this orbit is the whole fibre of the morphism:

$$\psi_{\overline{h},\overline{g}}^{V,i}:Z_{\overline{h}}^{V,i}\to Z_{\overline{g}}^{V,i}.$$

Similarly, the right-hand side of (4.2.1) is determined by the $\mathrm{Aut}(\mathbb{C}/\sigma(L_{\overline{g}}))$-orbit containing $\left(q\left(\overline{h}x_1'\right),\ldots,q\left(\overline{h}x_m'\right),q\left(\overline{h}\tau'\right)\right)$ in the fibre of $\psi_{\overline{h},\overline{g}}^{V,i}$ over $\left(q\left(\overline{g}x_1'\right),\ldots,q\left(\overline{g}x_m'\right),q\left(\overline{g}\tau'\right)\right)$. We need to show that this orbit equals the fibre of $\psi_{\overline{h},\overline{g}}^{V,i}$.

Extend σ to an automorphism of $\mathbb{C}$ and choose a point y in the fibre of $\psi_{\overline{h},\overline{g}}^{V,i}$ over $\left(q\left(\overline{g}x_1\right),\ldots,q\left(\overline{g}x_m\right),q\left(\overline{g}\tau\right)\right)$, and let $y'=\sigma(y)$. Let $\rho\in\mathrm{Aut}\left(\mathbb{C}/\sigma\left(L_{\overline{g}}\right)\right)$, then $\sigma^{-1}\rho\sigma\in\mathrm{Aut}\left(\mathbb{C}/L_{\overline{g}}\right)$ and

$$\rho\left(y'\right)=\rho\sigma(y)=\sigma\sigma^{-1}\rho\sigma(y)=\sigma\left(\phi\left(\sigma^{-1}\rho\sigma\right)\cdot y\right)=\phi\left(\sigma^{-1}\rho\sigma\right)\cdot\sigma(y)=\phi\left(\sigma^{-1}\rho\sigma\right)\cdot y'$$

(recall that by Lemma 1.3.16 the action of the automorphism commutes with that of the elements of the group). Therefore the homomorphism $\mathrm{Aut}\left(\mathbb{C}/\sigma\left(L_{\overline{g}}\right)\right)\to\Gamma_{\overline{g},\overline{h}}^{V,i}$, obtained from the composition of $\rho\mapsto\sigma^{-1}\rho\sigma$ and ϕ, is surjective, completing the proof of (4.2.1). □

To emphasize the main point of the proof of Proposition 4.2.5, we will state it as a separate Corollary.

Corollary 4.2.6. *Suppose $p:X^+\to S(\mathbb{C})$ has FIC1. Then for every $\mathbf{q}\models\mathrm{Th}(\mathbf{p})$ and every $x_1,\ldots,x_n\in D$, there is a finite tuple $\overline{c}$ of elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$ such that*

$$\mathrm{tp}\left(q\left(\overline{c}x_1\right),\ldots,q\left(\overline{c}x_n\right)\right)\models\bigcup_{\overline{g}}\mathrm{tp}\left(q\left(\overline{g}x_1\right),\ldots,q\left(\overline{g}x_n\right)\right).$$

The next definition is a translation of the condition on $\aleph_0$-homogeneity over countable models. We do not specify that F be countable because, as we are only working with finitely many points, we will be working with a countable subfield of F anyway.

Definition. We say that a Shimura variety $p:X^+\to S(\mathbb{C})$ has the *second finite-index condition* (FIC2) if the following is satisfied. Let $F\subseteq\mathbb{C}$ be an algebraically closed field, and let $x_1, \ldots, x_m \in D$ be non-special points in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits. Let $D_{V,i} = \operatorname{sploc}(x_1, \ldots, x_m)$ and suppose that:

(1) $p(x_i) \notin S(F)$ for all $i \in \{1, \ldots, m\}$

(2) the field L generated over F by the coordinates of the $p(x_i)$ has transcendence degree 1 over F

Then the image of the induced homomorphism $\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$ has finite index.

PROPOSITION 4.2.7. *If $p : X^+ \to S(\mathbb{C})$ satisfies FIC2, then $\mathscr{K}_{\mathbf{p}}$ satisfies $\aleph_0$-homogeneity over countable models.*

PROOF. The proof is the same as that of Proposition 4.2.5, now assuming that there is a common countable model embedded in both $\mathbf{q}$ and $\mathbf{q}'$, and working over this countable subset using FIC2. □

REMARK 4.2.8. With Proposition 4.2.7 we get a Corollary analogous to Corollary 4.2.6 in which the types are now allowed to have a countable set of parameters.

4.2.3. Conclusions. In the proofs that follow we will preserve the notation of Hecke-generic curves introduced in §4.2.1. In particular, we fix a Hecke-generic curve C in S.

COROLLARY 4.2.9. *If $p : X^+ \to S(\mathbb{C})$ satisfies FIC1 and FIC2, then $\mathscr{K}_{\mathbf{p}}$ is κ-categorical for $\kappa = \aleph_0$ and $\aleph_1$.*

PROOF. Every field of characteristic 0 and cardinality at most $2^{\aleph_0}$ can be embedded in $\mathbb{C}$. Given $\mathbf{q} \in \mathscr{K}_{\mathbf{p}}$ of cardinality at most $\aleph_1$, the corresponding structure $\mathbf{r}_{\mathbf{q}}$ will define a model in $\mathscr{K}_{\mathbf{r}}$. Following the proofs of Propositions 4.2.5 and 4.2.7, we get that $(\mathbf{r}, \mathrm{cl}_{\mathbf{r}})$ is a quasiminimal pregeometry structure. The corollary now follows from Proposition 1.1.5 applied to $\mathscr{K}_{\mathbf{r}}$, and Lemma 4.2.4. □

THEOREM 4.2.10 (First Main Categoricity Result). *The Shimura variety $p : X^+ \to S$ satisfies FIC1 and FIC2 if and only if $\mathscr{K}_{\mathbf{p}}$ is κ-categorical for all infinite cardinalities κ.*

PROOF. That the conditions are necessary is given by Corollary 4.1.2. So now we focus on the sufficiency. As we mentioned in Corollary 4.2.9, $(\mathbf{r}, \mathrm{cl}_{\mathbf{r}})$ is a quasiminimal pregeometry structure. Using the notation from §1.1.3, the models of $\mathscr{K}(\mathbf{r})$ are determined (up to isomorphism) by their dimension by Theorem 1.1.4. As we saw at the beginning of this section, the dimension of a model of $\mathscr{K}(\mathbf{r})$ is the transcendence degree (over $\mathbb{Q}$) of the field corresponding to the variety of that model. As algebraically closed fields are isomorphic if and only if they have the same characteristic, cardinality and transcendence degree over the prime field, then there is a unique model $\mathbf{r}_0 \in \mathscr{K}(\mathbf{r})$ of cardinality $\aleph_0$ and infinite dimension.

In order to prove categoricity of $\mathscr{K}_{\mathbf{r}}$, we need to show that $\mathscr{K}_{\mathbf{r}} = \mathscr{K}(\mathbf{r})$. It is clear that $\mathscr{K}(\mathbf{r}) \subseteq \mathscr{K}_{\mathbf{r}}$, since $\mathbf{r} \in \mathscr{K}_{\mathbf{r}}$. For the converse, first note that the class $\mathscr{K}(\mathbf{r})$ is quasiminimal under our hypothesis. By Corollary 4.2.9, $\mathscr{K}_{\mathbf{r}}$ has a unique model $\mathbf{t}_0$ of cardinality $\aleph_0$. But as $\mathscr{K}(\mathbf{r}) \subseteq \mathscr{K}_{\mathbf{r}}$, then $\mathbf{t}_0$ is isomorphic to $\mathbf{r}_0$, and so $\mathscr{K}(\mathbf{t_0}) = \mathscr{K}(\mathbf{r_0})$. Let $\mathbf{t} \in \mathscr{K}_{\mathbf{r}}$, and let B be a $\mathrm{cl}_{\mathbf{t}}$-basis of $\mathbf{t}$. Then $\mathbf{t} = \bigcup\{\mathrm{cl}_{\mathbf{t}}(B') : B' \subseteq B, |B'| = \aleph_0\}$. Also, for every $B' \subseteq B$ countable, we have that $\mathrm{cl}_{\mathbf{t}}(B') \in \mathscr{K}_{\mathbf{r}}$ is isomorphic to $\mathbf{t}_0$. As $\mathscr{K}(\mathbf{t_0})$ is closed under unions of chains of closed embeddings (all embeddings in $\mathscr{K}_{\mathbf{r}}$ are closed with respect to the pregeometries $\mathrm{cl}_{\mathbf{t}}$), then it is also closed under taking unions of directed systems, which shows that $\mathbf{t} \in \mathscr{K}(\mathbf{t_0})$. Therefore $\mathscr{K}_{\mathbf{r}} = \mathscr{K}(\mathbf{t_0}) = \mathscr{K}(\mathbf{r_0}) = \mathscr{K}(\mathbf{r})$. The statement now follows for $\mathscr{K}_{\mathbf{p}}$ using Lemma 4.2.4. $\square$

REMARK 4.2.11. We have worked in a theory where we assume that the field has infinite transcendence degree. This is so that there is only one countable model. But the theory of quasiminimal classes also says that countable models will be determined by their dimension, i.e. their transcendence degree over $\mathbb{Q}$. So, assuming FIC1 and FIC2, two countable models of $\mathscr{K}_{\mathbf{p}}$ with respective fields F_1 and F_2 will be isomorphic if and only if $\mathrm{t.d.}_{\mathbb{Q}}(F_1) = \mathrm{t.d.}_{\mathbb{Q}}(F_2)$.

Categoricity of Shimura curves was already proven in the main theorem of [**17**], although in a slightly different language, as the authors did not include predicates for the special domains. However, this difference does not play a role in the case of curves as the only special subvarieties that appear in the powers of a Shimura curve are the ones of the form $Z_{\overline{g}}$ (cf [**66**, 6.3 Proposition]). This is why in [**17**, §5], the authors can prove FIC1 and FIC2 for Shimura curves without having to worry about special closures.

4.3. Classification of Shimura Covers

In this section we will be focused on classifying the models of $\mathrm{Th}_{\mathrm{SF}}(\mathbf{p})$. For this we will use the notation of the language $\widetilde{\mathcal{L}}$ (as we discussed at the end of Corollary 3.2.14). We will use the techniques of independent systems and local isolation as described in [**7**, §3] to classify models with standard fibres. As explained in that paper, these methods provide a more robust approach to the categoricity question than quasiminimality.

Intuitively, the method is the following. We understand all the isomorphism classes of the models of T. Let $M_0 = S\left(\overline{\mathbb{Q}}\right)$ be the prime model of T, and let $\widetilde{\mathbf{q}} := \left\langle \widetilde{M}, M, \{q_{\overline{g}}\}_{\overline{g}}\right\rangle \vDash \mathrm{Th}(\mathbf{p})$ have standard fibres. Then $M_0 \prec M = S(F)$. Now let $\widetilde{M_0} = q^{-1}(M_0) \subseteq \widetilde{M}$. By restricting the maps $q_{\overline{g}}$ to $\widetilde{M_0}$ we obtain a model $\widetilde{\mathbf{q}}_0 := \left\langle \widetilde{M}_0, M_0, \left\{q_{\overline{g}}\restriction_{\widetilde{M_0}}\right\}_{\overline{g}}\right\rangle$ of $\mathrm{Th}(\mathbf{p})$ with standard fibres. Let B be a transcendence basis for F over $\mathbb{Q}$. The strategy is to show that we can reconstruct $\widetilde{\mathbf{q}}$ just out of $\widetilde{\mathbf{q}}_0$ and B, thus effectively showing that there is essentially only one way in which to extend $\widetilde{\mathbf{q}}_0$ with the parameters of B while keeping all fibres standard (see Lemma 4.3.13). This will be done by induction on the size of B via independent systems.

Fix a monster model $\widetilde{\mathfrak{q}} := \left\langle \widetilde{\mathbb{M}}, \mathbb{M}, \{\mathfrak{q}_{\overline{g}}\}_{\overline{g}}\right\rangle$ of $\mathrm{Th}(\mathbf{p})$.

4.3.1. Preliminaries.

Given $\widetilde{\mathbf{q}} := \left\langle \widetilde{M}, M, \{q_{\overline{g}}\}_{\overline{g}}\right\rangle$ and $\widetilde{\mathbf{q}}' := \left\langle \widetilde{M}', M', \left\{q'_{\overline{g}}\right\}_{\overline{g}}\right\rangle$, two models of $\mathrm{Th}(\mathbf{p})$, we write $\widetilde{\mathbf{q}} \prec^* \widetilde{\mathbf{q}}'$ to mean that $\widetilde{\mathbf{q}} \prec \widetilde{\mathbf{q}}'$ and that for every $z \in M$, $q'^{-1}(z) = q^{-1}(z)$. Such elementary extensions are called *fibre-preserving*.

Remark 4.3.1. Suppose $\widetilde{\mathbf{q}} \prec \widetilde{\mathbf{q}}'$. Then $\widetilde{\mathbf{q}} \prec^* \widetilde{\mathbf{q}}'$ if and only if in $\widetilde{M}'$ we have that $\widetilde{M} = q'^{-1}(M)$.

Lemma 4.3.2. *Let $\widetilde{\mathbf{q}} \vDash \mathrm{Th}(\mathbf{p})$ and $M' \prec M$. Let $\widetilde{M}' := q^{-1}(M')$. Then by taking the structure on $\widetilde{M}'$ induced by $\widetilde{M}$ and letting $q'_{\overline{g}}$ denote the restriction of $q_{\overline{g}}$ to $\widetilde{M}'$, we get that $\widetilde{\mathbf{q}}' := \left\langle \widetilde{M}', M', \left\{q'_{\overline{g}}\right\}_{\overline{g}}\right\rangle \prec^* \widetilde{\mathbf{q}}$.*

Proof. By Remark 4.3.1 and quantifier elimination, it suffices to show that $\widetilde{\mathbf{q}} \vDash \mathrm{Th}(\mathbf{p})$. But this is immediate as it satisfies axioms (A1)–(A6). ☐

Lemma 4.3.3. *Let $\widetilde{\mathbf{q}}, \widetilde{\mathbf{q}}' \vDash \widetilde{T}(\widetilde{\mathbf{p}})$. Suppose we have a model $M_1 \vDash T$ contained in $\mathbb{M}$ such that $M \prec M_1$. Let $A := \widetilde{M} \cup M_1$, and suppose that $\widetilde{\mathbf{q}}'$ is ℓ-atomic over A. Then $\widetilde{\mathbf{q}} \prec^* \widetilde{\mathbf{q}}'$ and $M' = M_1$.*

Proof. First we show that fibres are preserved. For this we use part (b) of Lemma 1.1.7. Fix $z \in M$ and let $\phi(v)$ be the $\mathcal{L}^{\mathrm{eq}}$-formula corresponding to the $\mathcal{L}$-formula $q(v) = z$. Let $\beta \in \mathrm{dcl}^{\mathrm{eq}}(A) \cap \mathrm{dcl}^{\mathrm{eq}}\left(\phi\left(\widetilde{\mathbb{M}}\right)\right)$ and choose a finite tuple $\overline{x}$ of $\phi\left(\widetilde{\mathbb{M}}\right)$ such that $\beta \in \mathrm{dcl}^{\mathrm{eq}}(\overline{x})$. Observe that the components of $\overline{x}$ are all in the fibre above z, and as $z \in M$, we get by Corollary 3.2.11 that $\mathrm{tp}(\overline{x}/A)$ is determined by $\mathrm{tp}\left(\overline{x}/\widetilde{M}\right)$. We conclude that $\beta \in \widetilde{\mathbf{q}}^{\mathrm{eq}}$. As $\widetilde{\mathbf{q}}'$ is ℓ-atomic over A, then for any $x \in \widetilde{M}'$ such that $q'(x) = z$, $\mathrm{tp}(x/A)$ is ℓ-isolated, and so by part (b) of Lemma 1.1.7 we conclude that $x \in \widetilde{M}$, thus finishing the proof that fibres are preserved.

Now we prove that $M' = M_1$. Let $\psi(w)$ be the formula $\exists v(q(v) = w)$. Let $\beta \in \mathrm{dcl}^{\mathrm{eq}}(A) \cap \mathrm{dcl}^{\mathrm{eq}}(\psi(\mathbb{M}))$ and let $\overline{z}$ be a finite tuple of $\psi(\mathbb{M})$ such that $\beta \in \mathrm{dcl}^{\mathrm{eq}}(\overline{z})$. As before, by quantifier elimination $\mathrm{tp}(\overline{z}/A)$ is determined by $\mathrm{tp}(\overline{z}/M_1)$. Set $\widetilde{M_1} = \widetilde{S}(F)$, where $M_1 = S(F)$. We know from Corollary 3.2.10 that the natural maps $q^1_{\overline{g}} : \widetilde{M_1} \to M_1^n$ define an elementary substructure $\widetilde{\mathbf{q}}_1 := \left\langle \widetilde{M_1}, M_1, \left\{q^1_{\overline{g}}\right\}_{\overline{g}} \right\rangle$ of $\widetilde{\mathbf{q}}$, so $\mathrm{tp}(\overline{z}/A)$ is determined by $\mathrm{tp}\left(\overline{z}/\widetilde{M_1}\right)$, and therefore $\beta \in \widetilde{\mathbf{q}}_1^{\mathrm{eq}}$ just as before. By ℓ-atomicity and part (b) of Lemma 1.1.7, we conclude that if $z \in M'$ satisfies $\psi(z)$, then $z \in M_1$. Therefore $M_1 = M'$. ☐

Lemma 4.3.4. *Let $\widetilde{\mathbf{q}}_1$, $\widetilde{\mathbf{q}}_2$ and $\widetilde{\mathbf{q}}_3$ be elementary submodels of $\widetilde{\mathbf{q}}$, with $\widetilde{\mathbf{q}}_1 \prec^* \widetilde{\mathbf{q}}_2$, $\widetilde{\mathbf{q}}_1 \prec^* \widetilde{\mathbf{q}}_3$, and $\widetilde{\mathbf{q}}_2 \downarrow_{\widetilde{\mathbf{q}}_1} \widetilde{\mathbf{q}}_3$. Let $\widetilde{\mathbf{q}}_4$ be an ℓ-atomic model over $\widetilde{\mathbf{q}}_2 \cup \widetilde{\mathbf{q}}_3$. Then $\widetilde{\mathbf{q}}_i \prec^* \widetilde{\mathbf{q}}_4$, for $i = 1, 2, 3$.*

Proof. By Proposition 3.2.9 it remains to check that fibres are preserved. To that end, suppose that they are not preserved, say that there is $z \in M_3$ for which there is $x \in \widetilde{M}_4$ such that $q_4(x) = z$ but $x \notin q_3^{-1}(z)$. Let $\psi(v, u)$ be the formula

$$\psi(v, u) := (q(v) = q(u)) \wedge (v \neq u)$$

For $y \in \widetilde{M}_3$ such that $q_3(y) = z$, we have that $\psi(v, y) \in \mathrm{tp}\left(x/\widetilde{M}_2 \cup \widetilde{M}_3\right)$, in particular $\psi(v, y) \in \mathrm{tp}\left(x/\widetilde{M}_2 \cup \widetilde{M}_3\right)_{\psi}$. By ℓ-atomicity there is $\phi(v, \overline{a_2}, \overline{a_3})$, with $\overline{a_i}$ a tuple of $\widetilde{M}_i$, such that $\phi(v, \overline{a_2}, \overline{a_3}) \vDash \mathrm{tp}\left(x/\widetilde{M}_2 \cup \widetilde{M}_3\right)_{\psi}$. Observe that for any $y \in \widetilde{M}_2$ such that $q_2(y) = z$, we will have $\psi(v, y) \in \mathrm{tp}\left(x/\widetilde{M}_2 \cup \widetilde{M}_3\right)_{\psi}$.

By Corollary 3.2.11 we may assume that $\phi(v,\overline{a_2},\overline{a_3})$ is of the form

$$\left(v \in \widetilde{M}_{V,i}(\overline{a_2},\overline{a_3})\right) \wedge \left(q_{\overline{g}}(v) \in q_{\overline{g}}\left(\widetilde{M}_{V,i}(\overline{a_2},\overline{a_3})\right)\right).$$

By independence, $\operatorname{tp}\left(\overline{a_2}/\widetilde{M}_3\right)$ is finitely satisfiable in $\widetilde{M}_1$, and using the form we have chosen for ϕ, we can choose $\overline{a_1}$ in $\widetilde{M}_1$ such that

$$\widetilde{\mathbf{q}}_3 \vDash \exists v \left(\phi\left(v,\overline{a_1},\overline{a_3}\right) \wedge q(v) = z\right),$$

which is witnessed by some $x' \in q_3^{-1}(z)$. But:

$$\forall v \left(\phi\left(v,\overline{u},\overline{a_3}\right) \to \psi\left(v,x'\right)\right) \in \operatorname{tp}\left(\overline{a_2}/\widetilde{M}_3\right),$$

which leads to a contradiction, as we would then have that $\widetilde{\mathbf{q}}_3 \vDash \psi\left(x',x'\right)$. □

LEMMA 4.3.5. *Let* $\widetilde{\mathbf{q}} \vDash \operatorname{Th}(\mathbf{p})$. *Suppose* $B \subseteq \widetilde{M}$ *and* $a \in \widetilde{M}$ *are such that for any tuple* $\overline{g}$ *of elements of* $G^{\mathrm{ad}}(\mathbb{Q})^+$ *we have that* $\operatorname{tp}\left(q_{\overline{g}}(a)/\operatorname{He}(B)\right)$ *is isolated. Then* $\operatorname{tp}(a/B)$ *is* ℓ*-isolated.*

PROOF. By quantifier elimination, it suffices to prove that $\operatorname{tp}(a/B)_\phi$ is isolated for an atomic formula $\phi(x,y)$. If ϕ is of the form $\psi\left(q_{\overline{g}}(x),q_{\overline{g}}(y)\right)$, then this follows by isolation of $\operatorname{tp}\left(q_{\overline{g}}(a)/\operatorname{He}(B)\right)$. For ϕ of the form $(x,y) \in \widetilde{M}_{V,i}$, it follows from the fact that for $b \in B$, $(a,b) \in \widetilde{M}_{V,i}$ if and only if $\operatorname{sploc}(a/B) \subseteq \widetilde{M}_{V,i}(b)$. □

DEFINITION. An $\widetilde{I}$*-system* is an I-system $\mathscr{Q} = \left(\widetilde{\mathbf{q}}_s\right)_{s\in I}$ in $\widetilde{\mathfrak{q}}$ such that:

- $\mathscr{M} := \left(M_s\right)_{s\in I}$ is an independent atomic I-system in $\mathbb{M}$.
- If $s \subseteq t$, then $\widetilde{\mathbf{q}}_s \prec^* \widetilde{\mathbf{q}}_t$.

We will also use the notation $\widetilde{\mathscr{M}} := \left(\widetilde{M}_s\right)_{s\in I}$, where, for every $s \in I$, $\widetilde{M}_s$ is the covering sort of the model $\widetilde{\mathbf{q}}_s$.

PROPOSITION 4.3.6. *An* $\widetilde{I}$*-system* $\mathscr{Q} = \left(\widetilde{\mathbf{q}}_s\right)_{s\in I}$ *is an independent* I*-system.*

PROOF. Let $I = (s_i)_{i\in\lambda}$ be an enumeration of I. By Lemma 1.1.8 it suffices to show that, given $i \in \lambda$, $\widetilde{\mathbf{q}}_{s_i} \downarrow_{\mathscr{Q}_{<s_i}} \mathscr{Q}_{s_{<i}}$, where we may assume inductively that the restriction of $\left(\widetilde{\mathbf{q}}_s\right)_{s\in I}$ to $s_{<i}$ is and independent system. By Proposition 3.2.20 and the independence of $\mathscr{M}$, it suffices to show that for every $a \in \widetilde{M}_{s_i}$, $\operatorname{sploc}\left(a/\widetilde{\mathscr{M}}_{s_{<i}}\right)$ is defined over $\widetilde{\mathscr{M}}_{<s_i}$.

Say that $\mathrm{sploc}\left(a/\widetilde{\mathscr{M}_{s_{<i}}}\right) = \widetilde{M}_{V,j}(\overline{c})$, where $\overline{c}$ is a tuple from $\widetilde{\mathscr{M}_{s_{<i}}}$. So $q(a) \in V\left(q\left(\overline{c}\right)\right)$. In ACF_0, model-theoretic independence ($\downarrow$) can be understood as algebraic independence (see [**57**, Corollary 8.5.13]) in the following sense: if $F, K, L \vDash \mathrm{ACF}_0$, then $K \downarrow_F L$ if and only if for every finite tuple $\overline{k}$ from K, we have that $\mathrm{tr.deg}_F\left(\overline{k}\right) = \mathrm{tr.deg}_L\left(\overline{k}\right)$. This means then that there is a tuple $\overline{z}$ in $\mathscr{M}_{<s_i}$ such that $V\left(q\left(\overline{c}\right)\right) = V\left(\overline{z}\right)$, and so we can assume that the coordinates of $q\left(\overline{c}\right)$ are in $\mathscr{M}_{<s_i}$. As $\widetilde{I}$-systems preserve fibres, then $\overline{c} \in \widetilde{\mathscr{M}_{<s_i}}$, so $\mathrm{sploc}\left(a/\widetilde{\mathscr{M}_{s_{<i}}}\right)$ is defined over $\widetilde{\mathscr{M}_{<s_i}}$. □

4.3.2. ω-Stability Over Models. We will say that $\mathrm{Th}(\mathbf{p})$ *satisfies FIC* if $p : X^+ \to S(\mathbb{C})$ satisfies FIC1 and FIC2.

PROPOSITION 4.3.7. *Assume* $\mathrm{Th}(\mathbf{p})$ *satisfies FIC. Let* $\widetilde{\mathbf{q}} \prec \widetilde{\mathfrak{q}}$ *and* $b \in \mathbb{M}$. *Let* $M(b) \vDash T$ *be prime over* Mb. *Suppose* $\widetilde{\mathbf{q}}'$ *is a model such that* $\widetilde{\mathbf{q}} \prec^* \widetilde{\mathbf{q}}' \prec \widetilde{\mathfrak{q}}$ *and* $M' = M(b)$. *Then* $\widetilde{\mathbf{q}}'$ *is atomic over* $\widetilde{\mathbf{q}}b$. *If* M *is also countable and* $\widetilde{\mathbf{q}}'$ *satisfies SF, then* $\widetilde{\mathbf{q}}'$ *is constructible over* $\widetilde{\mathbf{q}}b$. *Furthermore, if* $\widetilde{\mathbf{q}}$ *satisfies SF, then such a* $\widetilde{\mathbf{q}}'$ *exists.*

PROOF. First we show atomicity. Let $c \in \widetilde{M}'$, we need to show that $\mathrm{tp}\left(c/\widetilde{M}b\right)$ is isolated. Let $\widetilde{M}'_{V,i}(\overline{x}) := \mathrm{sploc}\left(c/\widetilde{M}\right)$, where $\overline{x}$ is a tuple from $\widetilde{M}$. By Corollary 4.2.6 and Remark 4.2.8 (which assume FIC), we have that there is a tuple $\overline{g}$ satisfying:

$$\mathrm{tp}\left(q_{\overline{g}}(c)/M\right) \vDash \mathrm{tp}\left(\mathrm{He}(c)/M\right).$$

As T has finite Morley rank and assuming b is transcendental over M (as otherwise $b \in M$), $\mathrm{tp}\left(b/Mq(c)\right)$ has finite multiplicity (see [**57**, Exercise 8.5.7]). Therefore, $\mathrm{tp}\left(\mathrm{He}(c)/M\right) \cup \mathrm{tp}\left(q(c)/Mb\right)$ has finitely many extensions to Mb. So again, for some tuple $\overline{g}$ we have:

$$\mathrm{tp}\left(q_{\overline{g}}(c)/Mz\right) \vDash \mathrm{tp}\left(\mathrm{He}(c)/Mb\right). \tag{4.3.1}$$

Combining (4.3.1) with Corollary 3.2.11, we get:

$$\mathrm{tp}\left(q_{\overline{g}}(c)/Mb\right) \cup \left\{v \in \widetilde{M}'_{V,i}\right\} \vDash \mathrm{tp}\left(c/\widetilde{M}b\right).$$

As $q_{\overline{g}}(c) \in (M')^n$ and M' is prime over Mb, $\mathrm{tp}\left(q_{\overline{g}}(c)/Mb\right)$ is isolated. So $\mathrm{tp}\left(c/\widetilde{M}b\right)$ is isolated, which finishes the proof of atomicity.

Constructibility follows immediately from countability and SF, because under those conditions the models are countable, and we have just shown atomicity.

It remains to prove that such a $\widetilde{\mathbf{q}}'$ exists. By Lemma 1.1.7(a), there exists a model which is ℓ-constructible over $\widetilde{M} \cup M'$. By Lemma 4.3.3 we get that $\widetilde{\mathbf{q}} \prec^* \widetilde{\mathbf{q}}'$ and $M' = M(b)$. That $\widetilde{\mathbf{q}}'$ satisfies SF can be assumed by Proposition 3.2.15. $\square$

4.3.3. Atomicity Over Independent Systems. We first fix some notation. Let $M := S(F) \models T$, and let $K_0 = \overline{F_0}$. Then $M_0 = S(K_0)$ is the prime model of T. If B is a transcendence basis for F over K_0, then M is constructible and prime over $M_0 B$.

Let $\mathcal{P}^{\text{fin}}(B)$ be the set of finite subsets of B. Let $M_\varnothing := M_0$, and for $s \in \mathcal{P}^{\text{fin}}(B)$ define inductively $M_s \prec M(F)$ to be prime over $M_{<s} \cup s$, that is $M_s = S\left(\overline{F(s)}\right)$.

Lemma 4.3.8. *$\mathscr{M} = (M_s)_{s \in \mathcal{P}^{\text{fin}}(B)}$ is a constructible independent $\mathcal{P}^{\text{fin}}(B)$-system, and*

$$\bigcup_{s \in \mathcal{P}^{\text{fin}}(B)} M_s = S(F).$$

Proof. $\bigcup_{s \in \mathcal{P}^{\text{fin}}(B)} M_s$ is an elementary submodel of M containing $M_0 B$, and as M is minimal over $M_0 B$, then we must have $\bigcup_{s \in \mathcal{P}^{\text{fin}}(B)} M_s = S(F)$.

T is totally transcendental, so every subset of a model has a constructible prime extension (see [**57**, Theorem 5.3.3]), and thus we get constructibility of the system. It remains to show that the system is independent. Also, in totally transcendental theories, prime extensions are atomic (and, hence, ℓ-atomic). Using the finite character of $\downarrow$ and Lemma 1.1.13, it will suffice to show that, for $b \in B$, $M_{\{b\}} \downarrow_{M_\varnothing} M_s$ when $b \notin s \in \mathcal{P}^{\text{fin}}(B)$.

So let $s \in \mathcal{P}^{\text{fin}}(B)$ be such that $b \notin s$. By induction, we can assume that the restriction of the system to s is independent. By Lemma 1.1.12, M_s is constructible over $M_0 s$. Observe that $b \notin M_s$ and $M_{\{b\}} \downarrow_{M_\varnothing} M_s$, which finishes the proof. $\square$

The following Proposition is a straightforward restatement of [**7**, Proposition 3.26]. With what we have shown so far, the proof of [**7**, Proposition 3.26] shows that Proposition 4.3.9 is a formal consequence of previous results. We give the proof for completeness.

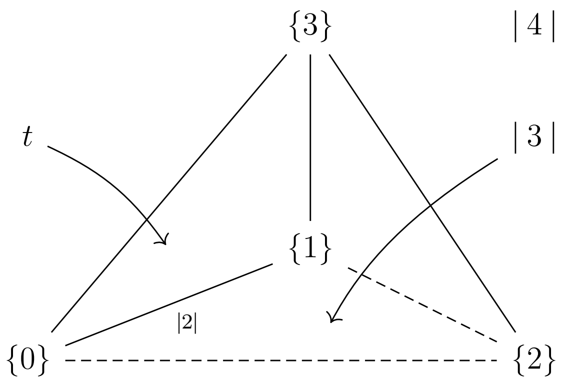
Proposition 4.3.9 (see [**7**, Proposition 3.26]). *Assume* $\mathrm{Th}(\mathbf{p})$ *satisfies FIC. Let $\mathscr{Q} = (\widetilde{\mathbf{q}}_s)_{s \in I}$ be an $\widetilde{I}$-system, with I Noetherian. Then the system is atomic. Also, if each M_s is countable and each $\widetilde{\mathbf{q}}_t$ satisfies SF, then the system is constructible.*

PROOF. Given $n \in \mathbb{N}$, we will denote $|n| := \{0, \ldots, n-1\}$. It suffices to prove the proposition for $I = \mathcal{P}(|n|), n > 1$, as, by Noetherianity, the system below any $s \in I$ is of this form, so for the rest of the proof we will assume that I is of this form. We inductively assume the result for $1 < n' < n$.

We will show that $\widetilde{\mathbf{q}}_{|n|}$ is atomic over $\mathscr{Q}_{<|n|}$, which will be done in several steps. Once this is done, constructibility follows from countability and SF because, under those conditions, the models are countable.

CLAIM 4.3.10. *The $\widetilde{I}$-system $\mathscr{Q} = (\widetilde{\mathbf{q}}_s)_{s\in I}$ can be extended to a $\mathcal{P}\widetilde{(|n+1|)}$-system such that there is an isomorphism $\sigma : \widetilde{\mathbf{q}}_{|n|} \longrightarrow_{\widetilde{\mathbf{q}}_{|n-1|}} \widetilde{\mathbf{q}}_{|n-1|\cup\{n\}}$ which also satisfies $\sigma(\widetilde{\mathbf{q}}_s) = \widetilde{\mathbf{q}}_{(s\setminus\{n-1\})\cup\{n\}}$ for $s \subseteq |n|$.*

PROOF. Let $t := |n-1| \cup \{n\}$. The following diagram describes the situation for $n = 3$, and can be used as a visual aid for the notation we are using.



Let $\widetilde{\mathbf{q}}_t$ be a realisation of $\operatorname{tp}\left(\widetilde{\mathbf{q}}_{|n|}/\widetilde{\mathbf{q}}_{|n-1|}\right)$ independent from $\widetilde{\mathbf{q}}_{|n|}$, and let $\sigma : \widetilde{\mathbf{q}}_{|n|} \longrightarrow_{\widetilde{\mathbf{q}}_{|n-1|}} \widetilde{\mathbf{q}}_t$ be an isomorphism witnessing the equality of types. Let $\widetilde{\mathbf{q}}$ be ℓ-atomic over $\widetilde{\mathbf{q}}_{|n|} \cup \widetilde{\mathbf{q}}_t$ (which exists by Lemma 1.1.7). By Lemma 4.3.4, $\widetilde{\mathbf{q}}$ is a fibre-preserving extension of $\widetilde{\mathbf{q}}_{|n|} \cup \widetilde{\mathbf{q}}_t$.

To prove the claim, we will consider an enumeration $(s_i)_{i\leq\lambda}$ of $\mathcal{P}(|n+1|)$ and recursively define $\widetilde{\mathbf{q}}_{s_i} \prec^* \widetilde{\mathbf{q}}$ such that $M_{s_i} \downarrow_{\mathscr{M}_{<s_i}} \mathscr{M}_{s_{<i}}$ and M_{s_i} is atomic over $\mathscr{M}_{<s_i}$. Using Lemma 1.1.8, this will give us a $\mathcal{P}\widetilde{(|n+1|)}$-system.

To do this, start with an enumeration of $\mathcal{P}(|n|)$; the corresponding $\widetilde{\mathbf{q}}_{s_i} \prec^* \widetilde{\mathbf{q}}$ are already given and satisfy the independence condition. Continue with an enumeration of $\mathcal{P}(t)$, where for $s_i \in \mathcal{P}(t) \setminus \mathcal{P}(|n|)$ we set $\widetilde{\mathbf{q}}_{s_i} := \sigma\left(\widetilde{\mathbf{q}}_{(s_i\setminus\{n\})\cup\{n-1\}}\right) \prec^* \widetilde{\mathbf{q}}_t \prec^* \widetilde{\mathbf{q}}$. To verify the independence condition, observe that $M_{s_i} \downarrow_{\mathscr{M}_{<s_i}} \mathscr{M}_{s_{<i}\cap\mathcal{P}(t)}$ (here, $s_{<i}$ is defined with respect

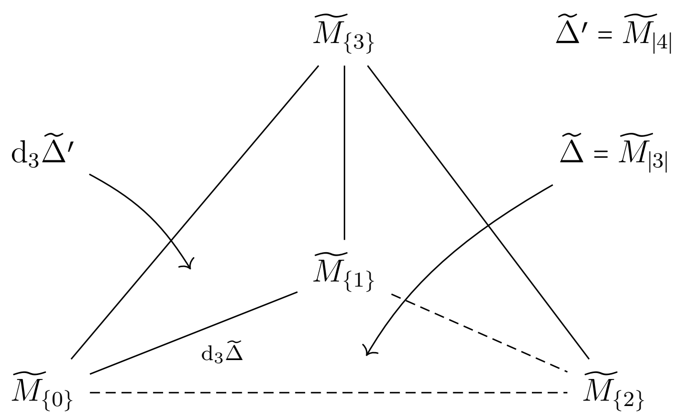
to the enumeration of $\mathcal{P}\widetilde{(|n+1|)})$. We also have that $M_t \downarrow_{M_{|n-1|}} M_{|n|}$ (by construction of $\widetilde{\mathbf{q}}_t$), and so by transitivity $M_{s_i} \downarrow_{\mathscr{M}_{<s_i}} \mathscr{M}_{s_{<i} \cap \mathcal{P}(t)} M_{|n|}$, and hence $M_{s_i} \downarrow_{\mathscr{M}_{<s_i}} \mathscr{M}_{s_{<i}}$.

Now we extend these enumerations; given $s_i \in \mathcal{P}(|n+1|) \setminus (\mathcal{P}(|n|) \cup \mathcal{P}(|t|))$, let $M_{s_i} \prec M$ be a constructible model over $\mathscr{M}_{<s_i} \subseteq M$, and let $\widetilde{M}_{s_i}$ be its inverse image in $\widetilde{M}$. By Lemma 1.1.11, $M_{<s_i} \subseteq_{TV} M_{s_{<i}}$, so by Lemma 1.1.9-(3), $M_{s_i} \downarrow_{\mathscr{M}_{<s_i}} \mathscr{M}_{s_{<i}}$. $\square$

Next, we set up some notation. Define:

$$\begin{array}{rclrcl}
\widetilde{\Delta} & := & \widetilde{M}_{|n|} & \widetilde{\Delta}' & := & \widetilde{M}_{|n+1|} \\
\mathrm{d}_i\widetilde{\Delta} & := & \widetilde{M}_{|n| \setminus \{i-1\}} & \mathrm{d}_i\widetilde{\Delta}' & := & \widetilde{M}_{|n+1| \setminus \{i-1\}} \\
\mathrm{d}\widetilde{\Delta} & := & \bigcup_{1 \leq i \leq n} \mathrm{d}_i\widetilde{\Delta} & \mathrm{d}\widetilde{\Delta}' & := & \bigcup_{1 \leq i \leq n} \mathrm{d}_i\widetilde{\Delta}' \\
\widetilde{\Lambda} & := & \bigcup_{1 \leq i < n} \mathrm{d}_i\widetilde{\Delta} & \widetilde{\Lambda}' & := & \bigcup_{1 \leq i < n} \mathrm{d}_i\widetilde{\Delta}' \\
\mathrm{dd}_i\widetilde{\Delta} & := & \bigcup_{j \in |n| \setminus \{i-1\}} \widetilde{M}_{|n| \setminus \{i-1,j\}}. & & &
\end{array}$$

We also define the corresponding sets in T by applying q, e.g. $\Lambda := q\left(\widetilde{\Lambda}\right)$. With this notation, the isomorphism σ of the previous claim gives isomorphisms: $\widetilde{\Delta} \xrightarrow{\simeq}_{\mathrm{d}_n\Delta} \mathrm{d}_n\widetilde{\Delta}'$ and $\Delta \xrightarrow{\simeq}_{\mathrm{d}_n\Delta} \mathrm{d}_n\Delta'$. When $n = 3$, we can picture these definitions with the following diagram:



Here, the dashed lines indicate $\widetilde{\Lambda}$. The two faces which have a dashed line as one of its sides, make up $\widetilde{\Lambda}'$.

Let $a \in \widetilde{\Delta}$. In this new notation, to prove that $\widetilde{\mathbf{q}}_{|n|}$ is atomic over $\mathscr{Q}_{<|n|}$ means that we need to prove that $\operatorname{tp}\left(a/\mathrm{d}\widetilde{\Delta}\right)$ is isolated. We first need two intermediate steps.

Claim 4.3.11. *There exists $b_0 \in \mathrm{d}_n\Delta$ such that, by setting $A := \operatorname{acl}^{\mathrm{eq}}(\mathrm{dd}_n\Delta b_0) \subseteq \mathrm{d}_n\Delta$, we have that*

$$\operatorname{tp}(\mathrm{He}(a)/A\Lambda) \vDash \operatorname{tp}(\mathrm{He}(a)/\mathrm{d}\Delta).$$

PROOF. Let $b_0 \in \mathrm{d}_n\Delta$ be such that $\mathrm{tp}\,(q(a)/b_0\Lambda) \vDash \mathrm{tp}\,(q(a)/\mathrm{d}\Delta)$. We will first show that every extension of $\mathrm{tp}\,(q_{\overline{g}}(a)/b_0\Lambda)$ to $\mathrm{d}\Delta$ is a non-forking extension. Given how we chose b_0, this holds for $\overline{g} = (e)$ by the uniqueness of the extension (non-forking partial types can always be extended to a non-forking complete type; see [**57**, Lemma 7.1.11]), and hence for any $\overline{g}$ by the interalgebraicity of $q_{\overline{g}}(a)$ with $q(a)$ (see [**57**, Theorem 8.5.6]).

Therefore, to prove the claim, it suffices to show that $\mathrm{tp}\,(q_{\overline{g}}(a)/A\Lambda)$ has a unique non-forking extension to $\mathrm{d}\Delta$, for every finite tuple $\overline{g}$ from $G^{\mathrm{ad}}(\mathbb{Q})^+$. So suppose z_1 and z_2 realise two such extensions. Then $\mathrm{d}_n\Delta \downarrow_{A\Lambda} z_i$. Note that $\mathrm{tp}\,(\mathrm{d}_n\Delta/A)$ is stationary (see [**57**, Corollary 8.5.3]). Since the system is independent, we have that $\mathrm{d}_n\Delta \downarrow_{\mathrm{dd}_n\Delta} \Lambda$, and hence $\mathrm{d}_n\Delta \downarrow_A A\Lambda$ (see [**57**, Corollary 7.2.17]). And so we also have that $\mathrm{tp}\,(\mathrm{d}_n\Delta/A\Lambda)$ is stationary. So $z_1 \equiv_{\mathrm{d}_n\Delta A\Lambda} z_2$, so in particular $z_1 \equiv_{\mathrm{d}\Delta} z_2$. □

CLAIM 4.3.12. $\mathrm{tp}\,(\mathrm{He}(a)/\sigma(\mathrm{He}(a))\Lambda' b_0) \vDash \mathrm{tp}\,(\mathrm{He}(a)/A\Lambda)$.

PROOF. Suppose $\vDash \phi(q_{\overline{g}}(a), b, e)$, where $b \in A$ and $e \in \Lambda$. Let θ be an algebraic formula isolating $\mathrm{tp}\,(b/\mathrm{dd}_n\Delta b_0)$. Let

$$\psi(\overline{v}) \coloneqq \forall u \in \theta\,(\phi(\overline{v}, u, e) \iff \phi\,(\sigma(q_{\overline{g}}(a)), u, \sigma(e)))\,,$$

which is a formula over $\sigma(q_{\overline{g}}(a))\Lambda' b_0$, since $\sigma(e) \in \sigma\Lambda \subseteq \Lambda'$. Since $b \in \mathrm{d}_n\Delta$ and $\sigma : \Delta \xrightarrow{\cong}_{\mathrm{d}_n\Delta} \mathrm{d}_n\Delta'$ is an isomorphism, then $\vDash \phi(\sigma(q_{\overline{g}}(a)), b, \sigma(e))$, and so $\psi(\overline{v}) \vDash \phi(\overline{v}, b, e)$. Therefore $\psi(v) \in \mathrm{tp}\,(q_{\overline{g}}(a)/\sigma(q_{\overline{g}}(a))\Lambda' b_0)$, and so $\mathrm{tp}\,(\mathrm{He}(a)/\sigma(\mathrm{He}(a))\Lambda' b_0) \vDash \phi(q_{\overline{g}}(a), b, e)$. □

Now we return to proving that $\mathrm{tp}\left(a/\mathrm{d}\widetilde{\Delta}\right)$ is isolated. Observe that $\mathrm{d}\widetilde{\Delta} \subseteq_{TV} \mathrm{d}\widetilde{\Delta}'$ by Lemma 1.1.11, and that $\mathrm{tp}\left(a/\mathrm{d}\widetilde{\Delta}\right)$ is ℓ-isolated by Lemma 4.3.5, so by Lemma 1.1.9-(1), $\mathrm{tp}\left(a/\mathrm{d}\widetilde{\Delta}\right) \vDash \mathrm{tp}\left(a/\mathrm{d}\widetilde{\Delta}'\right)$. Let $x \in q^{-1}(b_0) \subseteq \mathrm{d}_n\widetilde{\Delta}$ and let $\overline{y} \in \mathrm{d}_n\widetilde{\Delta}$ be a tuple extending x such that $\mathrm{sploc}\left(a/\mathrm{d}\widetilde{\Delta}\right)$ is defined over $\overline{y}\widetilde{\Lambda}$. Then by Corollary 3.2.11 and the previous two claims, we get:

$$\mathrm{tp}\left(a/\mathrm{d}\widetilde{\Delta}\right) \dashv\vDash \mathrm{tp}\left(a/\sigma(a)\widetilde{\Lambda}'\overline{y}\right),$$

where the $\dashv$-direction is a direct combination of previous results, whereas the $\vDash$-direction is obvious. So, to prove $\mathrm{tp}\left(a/\mathrm{d}\widetilde{\Delta}\right)$ is isolated, it suffices to prove that $\mathrm{tp}\left(a/\sigma(a)\widetilde{\Lambda}'\overline{y}\right)$ is isolated. For this, we will use [**57**, Lemma 5.3.5] and prove instead that $\mathrm{tp}\left(a\sigma(a)\overline{y}/\widetilde{\Lambda}'\right)$ is isolated (which yields the result we want). There are two cases to consider.

If $n > 2$, then $\operatorname{tp}\left(a\sigma(a)\overline{y}/\widetilde{\Lambda}'\right)$ is isolated by the inductive hypothesis applied to the $\mathcal{P}(\widetilde{|n-1|})$-system $(\widetilde{\mathbf{q}}'_s)_s$ defined as $\widetilde{\mathbf{q}}'_s := \widetilde{\mathbf{q}}_{s\cup\{n-1,n\}}$. Observe that in this case $\widetilde{\Lambda}' = \widetilde{\mathscr{M}}'_{<|n-1|}$ and $\widetilde{M}'_{|n-1|} = \widetilde{M}_{|n+1|} = \widetilde{\Delta}'$.

Else, if $n = 2$, then $\widetilde{\Lambda}' = \widetilde{M}_{\{1,2\}}$, $a \in \widetilde{\Delta} = \widetilde{M}_{\{0,1\}}$, $\mathrm{d}_2\widetilde{\Delta}' = \widetilde{M}_{\{0,2\}}$, $\overline{y} \in \mathrm{d}_2\widetilde{\Delta} = \widetilde{M}_{\{0\}}$, and $\sigma : \widetilde{M}_{\{0,1\}} \to_{\widetilde{M}_{\{0\}}} \widetilde{M}_{\{0,2\}}$. We will show that $\operatorname{tp}\left(a\sigma(a)\overline{y}/\widetilde{M}_{\{1,2\}}q(\overline{y})\right)$ is isolated (which is enough because q is a symbol in the language). By Proposition 4.3.7 (which assumes FIC), it suffices to show that $\operatorname{tp}\left(q(a)q(\sigma(a))/M_{\{1,2\}}q(\overline{y})\right)$ is isolated, since then, for an appropriate embedding of the prime model $M_{\{1,2\}}(q(\overline{y}))$ (in the notation of Proposition 4.3.7) into Δ', we have that $q(a), q(\sigma(a)) \in M_{\{1,2\}}(q(\overline{y}))$.

As $q(\overline{y}) \in \mathrm{d}_2\Delta = M_{\{0\}}$, then $\operatorname{tp}\left(q(a)/q(\overline{y})M_{\{1\}}\right) \vDash \operatorname{tp}\left(q(a)/M_{\{0\}}M_{\{1\}}\right)$. As the coordinates of $q(a)$ are algebraic over the field generated by the elements of $q(\overline{y})$ and $M_{\{1\}}$, then $\operatorname{tp}\left(q(a)/q(\overline{y})M_{\{1\}}\right)$ is isolated, and so $\operatorname{tp}\left(q(a)/M_{\{0\}}M_{\{1\}}\right)$ is isolated. As $M_{\{0,1\}} \downarrow_{M_{\{1\}}} M_{\{1,2\}}$, then $\operatorname{tp}\left(q(a)/q(\overline{y})M_{\{1,2\}}\right)$ is also isolated. Applying σ we get that $\operatorname{tp}\left(q(\sigma(a))/q(\overline{y})M_{\{2\}}\right)$ is isolated. Now by Lemma 1.1.11 and Lemma 1.1.9-(1)

$$\begin{aligned}\operatorname{tp}\left(q(\sigma(a))/q(\overline{y})M_{\{2\}}\right) &\vDash \operatorname{tp}\left(q(\sigma(a))/M_{\{0\}}, M_{\{2\}}\right)\\ &\vDash \operatorname{tp}\left(q(\sigma(a))/M_{\{0,1\}}M_{\{1,2\}}\right)\\ &\vDash \operatorname{tp}\left(q(\sigma(a))/q(a)q(\overline{y})M_{\{1,2\}}\right),\end{aligned}$$

and so $\operatorname{tp}\left(q(a), q(\sigma(a))/M_{\{1,2\}}q(\overline{y})\right)$ is isolated. □

4.3.4. Classification of Models with Standard Fibres.

LEMMA 4.3.13. *Assume that* $\mathrm{Th}(\mathbf{p})$ *satisfies FIC. Let* $S(F) = M \vDash T$*, let* $S\left(\overline{\mathbb{Q}}\right) \cong M_0 \prec M$ *be the prime model of* T*, and let* B *be a transcendence basis of* F *over* $\mathbb{Q}$*. Let* $\widetilde{\mathbf{q}}_0 \vDash \mathrm{Th}(\mathbf{p})$ *be such that it satisfies SF and* M_0 *is the variety sort of* $\widetilde{\mathbf{q}}_0$*. Then there exists* $\widetilde{\mathbf{q}} \vDash \mathrm{Th}(\mathbf{p})$ *such that* M *is the variety sort of* $\widetilde{\mathbf{q}}$*,* $\widetilde{\mathbf{q}}_0 \prec^* \widetilde{\mathbf{q}}$*, and* $\widetilde{\mathbf{q}}$ *is constructible, has standard fibres, and so it is minimal over* $B \cup \widetilde{\mathbf{q}}_0$*.*

PROOF. Set $I = \mathcal{P}^{\mathrm{fin}}(B)$. Let $\mathscr{M} = (M_s)_{s\in I}$ be defined as in the beginning of §4.3.3, which is a constructible independent I-system by Lemma 4.3.8. By Lemma 1.1.7(a) there exists $\widetilde{\mathbf{q}} \vDash \mathrm{Th}(\mathbf{p})$ which is ℓ-constructible over $M\widetilde{\mathbf{q}}_0$. By Lemma 4.3.3 we know that

$\widetilde{\mathbf{q}}$ is a fibre-preserving extension of $\widetilde{\mathbf{q}}_0$. By Proposition 3.2.15, we can assume that $\widetilde{\mathbf{q}}$ is also minimal over $M\widetilde{\mathbf{q}}_0$. As $\widetilde{\mathbf{q}}_0$ has standard fibres, then so will $\widetilde{\mathbf{q}}$. For $s \in I$ set $\widetilde{M}_s := q^{-1}(M_s) \subseteq \widetilde{M}$, thus (by Lemma 4.3.2) $\mathscr{Q} = (\widetilde{\mathbf{q}}_s)_{s\in I}$ is an $\widetilde{I}$-system. By Proposition 4.3.9, $\mathscr{Q}$ is a constructible independent $\widetilde{I}$-system. By Proposition 4.3.7, for each $b \in B$ the model $\widetilde{\mathbf{q}}_{\{b\}}$ is constructible over $\widetilde{\mathbf{M}}_{\varnothing}b$, and so by Lemma 1.1.12 $\widetilde{\mathbf{q}} = \mathscr{Q}_I$ is constructible over $\widetilde{\mathbf{q}}_{\varnothing}B = \widetilde{\mathbf{q}}_0 B$. □

THEOREM 4.3.14 (Second Main Categoricity Result). *Assume* $\mathrm{Th}(\mathbf{p})$ *satisfies FIC. Let* $\widetilde{\mathbf{q}}_0, \widetilde{\mathbf{q}}, \widetilde{\mathbf{q}}' \models \mathrm{Th}(\mathbf{p})$ *be such that they satisfy SF. Suppose* M_0 *is the prime model of* T *and that there is an isomorphism* $M \cong M'$ *which is the identity on* M_0*. Then* $\widetilde{\mathbf{q}} \cong_{\widetilde{\mathbf{q}}_0} \widetilde{\mathbf{q}}'$*.*

PROOF. If $S(F) = M \cong M' = S(F')$, let B be a transcendence basis of F over $\overline{\mathbb{Q}}$, and let B' be the image of B in F'. Then $\widetilde{\mathbf{q}}$ and $\widetilde{\mathbf{q}}'$ are constructible and minimal (this is guaranteed by SF) over $\widetilde{\mathbf{q}}_0 B$ by Lemma 4.3.13. By quantifier elimination of $\mathrm{Th}(\mathbf{p})$, $B \cup \widetilde{\mathbf{q}}_0 \equiv B' \cup \widetilde{\mathbf{q}}_0$, and by constructibility of $\widetilde{\mathbf{q}}$ over $B \cup \widetilde{\mathbf{q}}_0$, this extends to an elementary embedding $\widetilde{\mathbf{q}} \prec \widetilde{\mathbf{q}}'$. By minimality of $\widetilde{\mathbf{q}}'$ over $B' \cup \widetilde{\mathbf{q}}_0$, this embedding is an isomorphism. □

4.4. Categoricity of $\mathcal{A}_g$

We will now focus on $\mathcal{A}_g$, the moduli space of principally polarised abelian varieties of dimension g (see Example 5 for the construction). For the terminology and standard results related to the theory of abelian varieties that we will use, see e.g. [**38**] and [**40**]. We recall that in this case $G = \mathrm{GSp}_{2g}$, $X = \mathbb{H}_g := \mathbb{H}_g^+ \cup \mathbb{H}_g-$, and we can view $\mathcal{A}_g$ as the quotient $\Gamma \backslash \mathbb{H}_g^+$, where $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$. Also $E(G, \mathbb{H}_g) = \mathbb{Q}$ (cf [**35**, p. 352]). Because of the set-up in which the Mumford-Tate conjecture is stated (see Conjecture 2), we will use $F_0 = \mathbb{Q}^{\mathrm{ab}}(\Sigma)$.

Unfortunately, in order to stick to standard notation, we will have to use the same symbol, g, for the dimension of the abelian variety and for elements of the group G. However this should not lead to confusion.

We remark first that $\mathrm{Sp}_{2g}(\mathbb{Z})$ is not neat, so we will use the considerations of Appendix B to fix this. Let A be an abelian variety of dimension g defined over a finitely generated field L of characteristic zero. Given an integer N, denote the N-torsion of A by $A[N]$

(here we think of $A[N]$ as a subgroup of $A\left(\overline{L}\right)$). A *level N-structure* on an A is an isomorphism

$$\psi_N : A[N] \to (\mathbb{Z}/N\mathbb{Z})^{2g} \tag{4.4.1}$$

which preserves the standard symplectic form. The *Tate module* of A is defined as the inverse limit of the torsion subgroups: $T_A := \varprojlim A[N]$.

We will now look at the *moduli space of principally polarised abelian varieties of dimension g with a level-N structure*, which is defined as:

$$\mathcal{A}_{g,N} := \Gamma(N)\backslash\mathbb{H}_g^+,$$

where $\Gamma(N)$ is the principal congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ of level N. If $N \geq 4$, then $\mathcal{A}_{g,N}$ is smooth and $\mathbb{H}_g^+ \to \mathcal{A}_{g,N}$ is the topological universal cover. A point $x \in \mathcal{A}_{g,N}$ corresponds to a triple (A, λ, ψ_N), where (A, λ) denotes an abelian variety A of dimension g with a principal polarisation $\lambda : A \to A^\vee$ ($A^\vee$ denotes the dual abelian variety of A), and ψ_N is an isomorphism as in (4.4.1). Let

$$\widehat{\Gamma} := \varprojlim \Gamma/\Gamma(N),$$

where $\Gamma(N)$ is the congruence subgroups of of level N of Γ. Using the notation of §1.3.5, a point $[x, 1] \in \mathrm{Sh}(G, \mathbb{H}_g)$ then corresponds to a triple $\left(A, \lambda, \widetilde{\psi}\right)$, where (A, λ) is a principally polarised abelian variety as before, and

$$\widetilde{\psi} : T_A \to \widehat{\mathbb{Z}}^{2g}$$

is a symplectic isomorphism inducing ψ_N on $A[N]$. Giving an isomorphism $\widetilde{\psi}$ is the same thing as giving a symplectic basis of T_A. Now, T_A has an action of $\mathrm{Gal}\left(\overline{L}/L\right)$ obtained from the action of $\mathrm{Gal}\left(\overline{L}/L\right)$ on every $A[N]$. It is shown in [**60**, p. 265] that, by the uniqueness of Lemma 1.3.15, the Galois representation attached to a point $z \in \mathcal{A}_{g,N}$ as described in §1.3.5 coincides with the action of the Galois group of L on the Tate module of the abelian variety associated to z.

Given a tuple $\overline{g}$ from $G^{\mathrm{ad}}(\mathbb{Q})^{+}$, we can choose N large enough to obtain a short exact sequence of finite groups:

$$1 \to \Gamma_{\overline{g}}/\Gamma(N) \to \Gamma/\Gamma(N) \to \Gamma/\Gamma_{\overline{g}} \to 1$$

which satisfies the Mittag-Leffler conditions. We conclude then that there is surjective homomorphism $\pi : \widehat{\Gamma} \to \overline{\Gamma}$.[3] Observe that $\widehat{\Gamma}$ is open in $\mathrm{GSp}_{2g}(\mathbb{A}_f)$. Given a special domain $D_{V,i} \subseteq \mathbb{H}_g^+$, let $\widehat{\Gamma^{V,i}} := \pi^{-1}\left(\overline{\Gamma^{V,i}}\right)$. With all these observations and [**12**, Theorem A] we can conclude the following.

LEMMA 4.4.1. *The image of* $\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$ *is open if the Mumford-Tate conjecture holds for the abelian variety associated to* z.

The Mumford-Tate conjecture is known to hold for elliptic curves, simple abelian surfaces and simple abelian varieties of dimension 3 (see [**32**, Theorem 6.1]). By [**15**, Theorem 9.5], we know that if the Mumford-Tate conjecture holds for any two abelian varieties, it will also hold for their product, which means then that the Mumford-Tate conjecture holds for all abelian surfaces and all abelian varieties of dimension 3. Therefore, in these cases the corresponding homomorphisms

$$\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{\mathrm{ab}}(p(x_1),\ldots,p(x_m))) \to \widehat{\Gamma^{V,i}} \tag{4.4.2}$$

have open image. This almost takes care of FIC1. What is missing is to extend the field $\mathbb{Q}^{\mathrm{ab}}(p(x_1),\ldots,p(x_m))$ (which is the field under which the Mumford-Tate conjecture is stated) so that it includes the coordinates of special points, which are part of the field of constants we are using in our models. For that, we use the following lemma.

LEMMA 4.4.2. *If* $g=1$, $g=2$ *or* $g=3$, *then* $\mathbb{Q}^{\mathrm{ab}}(\Sigma)$ *is a solvable extension of* $\mathbb{Q}^{\mathrm{ab}}$.

PROOF. Recall that the case $g=1$ was treated in [**17**, §5]. Given $x \in \mathbb{H}_g^+$ special, $p(x)$ represents a CM abelian abelian variety of dimension g. Recall that the reflex field depends only on the Shimura datum, and not on the choice of Γ. Let $E_{p(x)}$ denote the reflex field corresponding to the Shimura datum $(\mathrm{MT}(x),\{x\})$. $E_{p(x)}$ is a CM-field of

[3]See https://stacks.math.columbia.edu/tag/0594 for the relevant details showing the existence of a surjective homomorphism.

degree at most $2g$ over $\mathbb{Q}$, but it may not be Galois over $\mathbb{Q}$. In any case, we can find another special point $x' \in \mathbb{H}_g^+$ such that the compositum $E_{p(x)}E_{p(x')}$ is the Galois closure of $E_{p(x)}$ over $\mathbb{Q}$ (see [**13**, 1.5.3. Example]). This Galois closure is of degree at most 8 over $\mathbb{Q}$ if $g = 2$, and of degree at most 12 over $\mathbb{Q}$ if $g = 3$. In any case, the Galois group will be solvable. So, if we let E_{sp} be the compositum of the reflex fields of all special points, then the extension $\mathbb{Q}^{\mathrm{ab}}E_{\mathrm{sp}}$ is solvable over $\mathbb{Q}^{\mathrm{ab}}$, because the compositum of solvable extensions of bounded length is solvable.

To finish, we use that $\mathbb{Q}(p(x))$ is an abelian extension of $E_{p(x)}$, and that the compositum of abelian extensions is abelian. $\square$

We therefore get (recall that categoricity of $j : \mathbb{H}^+ \to \mathbb{C}$ was already proven in [**17**]):

THEOREM 4.4.3. *$p : \mathbb{H}_2^+ \to \mathcal{A}_2$ and $p : \mathbb{H}_3^+ \to \mathcal{A}_3$ are categorical.*

PROOF. By Theorem 4.2.10 or Theorem 4.3.14, it will be enough to verify conditions FIC1 and FIC2. Throughout this proof we let D denote either $\mathbb{H}_2^+$ or $\mathbb{H}_3^+$. Let $x_1, \ldots, x_n$ be non-special points in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits. Let $L = \mathbb{Q}^{\mathrm{ab}}(p(x_1), \ldots, p(x_n))$ and $L' = \mathbb{Q}^{\mathrm{ab}}(\Sigma, p(x_1), \ldots, p(x_n))$. As $L' = L(\Sigma)$ and every special point has algebraic coordinates, then $\overline{L'} = \overline{L}$. From Lemma 4.4.2, we know that L' is a solvable extension of L, so the quotient $\mathrm{Gal}\left(\overline{L}/L\right)/\mathrm{Gal}\left(\overline{L'}/L'\right)$ is a solvable group.

Let $D_{V,i} = \mathrm{spcl}(x_1, \ldots, x_n)$, let M be the generic Mumford-Tate group of $D_{V,i}$, and let $\Gamma' = \Gamma \cap M(\mathbb{Q})_+$. From §1.3.5 and §1.3.6 we know that there are homomorphisms

$$\mathrm{Gal}\left(\overline{L'}/L'\right) \xrightarrow{\iota} \mathrm{Gal}\left(\overline{L}/L\right) \xrightarrow{\rho} \overline{\Gamma^{V,i}},$$

where ρ is a continuous homomorphism obtained from a Galois representation, ι is the natural map, and $\rho \circ \iota$ is a Galois representation. By Remark 1.3.17, we know that if a Galois representation associated to $(x_1, \ldots, x_n)$ has open image in $\widehat{\Gamma'}$, the corresponding representation will have open image in $\overline{\Gamma^{V,i}}$. Let $\mathcal{U}$ be the image of ρ, and $\mathcal{U}_0$ be the image of $\rho \circ \iota$. By [**15**, Theorem 9.5], $\mathcal{U}$ is open in $\overline{\Gamma^{V,i}}$. Using that L'/L is solvable, we get a nested sequence of subgroups:

$$\mathcal{U}_0 \triangleleft \mathcal{U}_1 \triangleleft \cdots \triangleleft \mathcal{U}_n = \mathcal{U}$$

where every quotient $\mathcal{U}_i/\mathcal{U}_{i-1}$ is abelian. As $\mathcal{U}/\mathcal{U}_{n-1}$ is abelian, this means that $\mathcal{U}_{n-1}$ contains the commutator subgroup of $\mathcal{U}$. We can now resort to the proof of [**48**, Proposition 6.4] or to [**12**, Theorem 5.4], both of which show that the commutator subgroup of any open subgroup of $\overline{\Gamma^{V,i}}$ is open. So $\mathcal{U}_{n-1}$ contains an open subgroup. Inductively, $\mathcal{U}_0$ contains an open subgroup, and so is open. This proves FIC1.

Now we focus on FIC2. Let F be an algebraically closed subfield of $\mathbb{C}$, and assume that $p(x_i) \notin S(F)$ for every $i = 1, \ldots, n$ (with the x_i as before). Let $L = F(p(x_1), \ldots, p(x_n))$ and, in keeping with the conditions of FIC2, assume that the transcendence degree of L over F is 1. This means that there is $t \in L$ transcendental over F such that L is a finite extension of $F(t)$. Observe that L is a normal extension of $\mathbb{Q}(p(x_1), \ldots, p(x_n))$, and so $\mathrm{Aut}(\mathbb{C}/L)$ is a normal subgroup of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{\mathrm{ab}}(p(x_1), \ldots, p(x_n)))$.

Given a positive integer N, we can consider the principal congruence subgroup $\Gamma^{V,i}(N)$ of $\Gamma^{V,i}$. The varieties $\Gamma^{V,i}(N)\backslash D_{V,i}$ and $\Gamma^{V,i}\backslash D_{V,i}$ are both defined over F, as is the natural map $\psi : \Gamma^{V,i}(N)\backslash D_{V,i} \to \Gamma^{V,i}\backslash D_{V,i}$. As we have seen in §1.3.5, there is a representation

$$\rho_N : \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}^{\mathrm{ab}}(p(x_1), \ldots, p(x_n))\right) \to \Gamma^{V,i}/\Gamma^{V,i}(N),$$

which (due to [**32**, Theorem 6.1] and [**15**, Theorem 9.5]) can be assumed to be surjective for large enough N. The action of $\Gamma^{V,i}/\Gamma^{V,i}(N)$ on the ψ-fibre of a point defined over $F(t)$ but not on F, is given by functions which are algebraic over F. As L is a finite extension of $F(t)$, then the restriction of ρ_N to $\mathrm{Aut}(\mathbb{C}/L)$ gives a representation $\mathrm{Aut}(\mathbb{C}/L) \to \Gamma^{V,i}/\Gamma^{V,i}(N)$ which is surjective for large enough N. Therefore the image of $\mathrm{Aut}(\mathbb{C}/L)$ in $\widehat{\Gamma^{V,i}}$ will be open (as it will be a finite-index subgroup of the image of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{\mathrm{ab}}(p(x_1), \ldots, p(x_n)))$ in $\widehat{\Gamma^{V,i}}$), and so its image in $\overline{\Gamma^{V,i}}$ is open, which verifies FIC2. $\square$

APPENDIX A

MSC and Geodesic Disjointness

In [**14**], the authors study two subfields of $\mathbb{C}$, one constructed as an inductive tower of exponential powers, and the other constructed in the same way using the logarithm. The main result of [**14**] is that Schanuel's conjecture implies that these fields are linearly disjoint. A similar result for the exponential map of abelian varieties can be found in [**42**].

Inspired by these results, we will now show that one can construct similar subfields of $\mathbb{C}$ using the j function and that MSC implies that they are geodesically disjoint. We preserve the notation of Part 2, so in particular $G = \mathrm{GL}_2(\mathbb{Q})$. Let $\mathbb{H} := \mathbb{H}^+ \cup \mathbb{H}^-$.

Set $M_0 = N_0 = \overline{\mathbb{Q}}$. For $n \geq 1$ define inductively

$$\begin{aligned} M_n &:= \overline{M_{n-1}\left(j(\tau) : \tau \in M_{n-1} \cap \mathbb{H}\right)} \\ N_n &:= \overline{N_{n-1}\left(\tau \in \mathbb{H} : j(\tau) \in N_{n-1}\right)}. \end{aligned}$$

Finally, let $M = \bigcup_{n\in\mathbb{N}} M_n$, $N = \bigcup_{n\in\mathbb{N}} N_n$.

Remark A.1. Observe that a simple inductive argument shows that

$$\begin{aligned} M_n &= \overline{\mathbb{Q}\left(j(\tau) : \tau \in M_{n-1} \cap \mathbb{H}\right)} \\ N_n &= \overline{\mathbb{Q}\left(\tau \in \mathbb{H} : j(\tau) \in N_{n-1}\right)} \end{aligned}$$

Lemma A.2. *For every $x \in M_n$ there is a finite set $A \subseteq M_{n-1} \cap \mathbb{H}$ such that $A \cup \{x\}$ is algebraic over $\overline{\mathbb{Q}(j(A))}$. Similarly, for all $y \in N_n$, there is a finite set $C \subseteq \mathbb{H}$ with $j(C) \subseteq N_{n-1}$ such that $j(C) \cup \{y\}$ is algebraic over $\mathbb{Q}(C)$.*

Proof. First we find A. Given $x \in M_n$ choose a finite set $A_{n-1} \subseteq M_{n-1} \cap \mathbb{H}^+$ such that x is algebraic over $\mathbb{Q}(j(A))$. We can repeat this process for each of the elements of A_{n-1} and thus choose a finite set $A_{n-2} \subseteq M_{n-2} \cap \mathbb{H}$ such that A_{n-1} is algebraic over $\mathbb{Q}(j(A_{n-2}))$. We continue this process until we reach that A_1 is algebraic over $\mathbb{Q}(j(A_0))$, for some finite set $A_0 \subseteq \overline{\mathbb{Q}} \cap \mathbb{H}$. Now set $A = \bigcup_{m=0}^{n-1} A_m$. Note that $A \subseteq M_{n-1} \cap \mathbb{H}$. It is immediate from construction that $x \in \overline{\mathbb{Q}(j(A))}$, and that A_m is algebraic over $\mathbb{Q}(j(A_{m-1}))$.

To find C, we proceed just as before. Given $y \in N_n$, there is a finite set $C_{n-1} \subseteq N_{n-1} \cap \mathbb{H}$ such that $j(C_{n-1}) \subseteq N_{n-1}$ and y is algebraic over $\mathbb{Q}(C_{n-1})$. Now repeat this process with every element of $j(C_{n-1})$ to form C_{n-2}. Once this process is finished, we let $C = \bigcup_{m=0}^{n-1} C_m$. □

Lemma A.3. *Identify $\mathbb{C} = \mathbb{R}(i)$ (where i is a square root of -1). Let F be an algebraically closed subfield of $\mathbb{C}$. Then for all $z \in \mathbb{C}$, $z \in F$ if and only if the real an imaginary parts of z are in F.*

Proof. Clearly, if the real and imaginary parts of z are in F, the $z \in F$. So assume that $z \in F$ and write $z = a+ib$, with $a, b \in R$. Let ∂ be any derivation of $\mathbb{C}$ such that $F = \ker \partial$ (such a derivation exists because F is algebraically closed). So $0 = \partial(z) = \partial(a) + i\partial(b)$. Using [**62**, §4], we know that there are derivations $\lambda, \mu : \mathbb{R} \to \mathbb{R}$ such that $\partial = [\lambda : \mu]$ (we are following the notation used in §2.3). This means then that

$$\lambda(a) - \mu(b) + i(\lambda(a) + \mu(b)) = 0$$

from where we get $\lambda(a) = \mu(b) = 0$. So $\partial(a) = \lambda(a) + i\lambda(a) = 0$, and so $a \in F$. If follows that $b \in F$. □

Theorem A.4. *MSC implies $M \perp^g_{\overline{\mathbb{Q}}} N$.*

Proof. We will show that $M_m \perp^g_{\overline{\mathbb{Q}}} N_n$ for arbitrary m and n, which is enough to prove the Theorem. By induction on m, we can assume that $M_{m-1} \perp^g_{\overline{\mathbb{Q}}} N_n$.

Firs we will show that $M_{m-1} \cap N_n = \overline{\mathbb{Q}}$. Indeed, let $y \in N_n \setminus \overline{\mathbb{Q}}$. By Lemma A.3, we can assume that y is real. As $y \in N_n \setminus \overline{\mathbb{Q}}$ then $\dim^g_{\overline{\mathbb{Q}}}(1, y) = 2$, so by geodesic disjointness $\dim^g_{M_{m-1}}(1, y) = 2$. If y were in M_{m-1}, then, as y is real, the matrix

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

would be in $G^{M_{m-1}}$, implying that $\dim^g_{M_{m-1}}(1, y) = 1$, which is a contradiction. Therefore $y \notin M_{m-1}$.

Now we will proceed by contradiction, so let $y_1, y_2 \in N_n$ be $\mathrm{gcl}_{\overline{\mathbb{Q}}}$-independent and suppose that there is $g \in G^{M_m}$ such that $gy_1 = y_2$.

By Lemma A.2, there is $A \subseteq M_{m-1} \cap \mathbb{H}$ finite such that A and the components of g are algebraic over $\mathbb{Q}(j(A))$. Similarly, there is $C \subseteq \mathbb{H}$ finite such that $j(C) \subseteq N_n$ and $j(C) \cup \{y_1, y_2\}$ is algebraic over $\mathbb{Q}(C)$.

Choose $B \subseteq A$ so that $j(B)$ is a transcendence basis for $\mathbb{Q}(j(A))$ over $\mathbb{Q}$. Let $D \subseteq C$ be a transcendence basis of $\mathbb{Q}(C)$ over $\mathbb{Q}$. Because D is algebraically independent, then it is also $\mathrm{gcl}_{\overline{\mathbb{Q}}}$-independent. B is $\mathrm{gcl}_{\mathbb{Q}}$-independent because if $gb_1 = b_2$ for some distinct $b_1, b_2 \in B$ and some $g \in G(\mathbb{Q})$, then because of the modular polynomials, $j(b_1)$ would be algebaric over $j(b_2)$. Finally, if $x \in \mathrm{gcl}_{\mathbb{Q}}(B) \cap \mathrm{gcl}_{\mathbb{Q}}(D)$, then as $M_{m-1} \cap N_n = \overline{\mathbb{Q}}$, we would have that x is algebraic, but D is algebraically independent over $\mathbb{Q}$, so $\mathrm{gcl}_{\mathbb{Q}}(B) \cap \mathrm{gcl}_{\mathbb{Q}}(D) = \varnothing$, and so $B \cup D$ is $\mathrm{gcl}_{\mathbb{Q}}$-independent.

Observe that MSC implies the following statement without derivatives: if $z_1, \ldots, z_k \in \mathbb{H}$ are non-special, then

$$\mathrm{t.d.}(z_1, \ldots, z_k, j(z_1), \ldots, j(z_k)) \geq \dim^g(z_1, \ldots, z_k).$$

We then have that MSC implies $\mathrm{t.d.}(B, D, j(B), j(D)) \geq |B| + |D|$. On the other hand

$$\begin{aligned} \mathrm{t.d.}(B, D, j(B), j(D)) &= \mathrm{t.d.}(B, C, j(A), j(D)) \\ &= \mathrm{t.d.}(D, j(B)) \\ &\leq |B| + |D|. \end{aligned}$$

Therefore $\mathrm{t.d.}(D, j(B)) = \mathrm{t.d.}(B, D, j(B), j(D)) = |B| + |D|$, which implies that $j(B)$ is algebraically independent over $\mathbb{Q}(D)$, and D is algebraically independent over $\mathbb{Q}(j(B))$. This can be restated as saying that $\overline{\mathbb{Q}(D)} \perp^f_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}(j(B))}$. By [**31**, VIII Theorem 4.12], $\overline{\mathbb{Q}(j(B))} \perp^l_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}(D)}$. Form this we deduce that $\overline{\mathbb{Q}(D)}$ and $\overline{\mathbb{Q}(j(B))} \cap \mathbb{R}$ are $\overline{\mathbb{Q}} \cap \mathbb{R}$-free, and so $\left(\overline{\mathbb{Q}(j(B))} \cap \mathbb{R}\right) \perp^l_{\overline{\mathbb{Q}} \cap \mathbb{R}} \overline{\mathbb{Q}(D)}$. By Lemma 2.2.2 we get that $\left(\overline{\mathbb{Q}(j(B))} \cap \mathbb{R}\right) \perp^g_{\overline{\mathbb{Q}} \cap \mathbb{R}} \overline{\mathbb{Q}(D)}$. Observe that $y_1, y_2 \in \overline{\mathbb{Q}(D)}$ and that, as $g \in G^{M_m}$, g must be defined over $\mathbb{R}$, so its entries are in $\overline{\mathbb{Q}(j(B))} \cap \mathbb{R}$. As we are assuming that $gy_1 = y_2$, then there would be $h \in G^{\overline{\mathbb{Q}}}$ such that $hy_1 = y_2$, contradicting that y_1 and y_2 are $\mathrm{gcl}_{\overline{\mathbb{Q}}}$-independent. $\square$

APPENDIX B

Reduction to the Neat Case

We now give the argument shown in [**17**, §5.3] to show that if we have categoricity when Γ is neat, then we also have categoricity when we take a group which is commensurable with Γ but not necessarily neat.

Suppose that Γ is a congruence subgroup not necessarily neat, but still gives rise to a locally symmetric variety $\Gamma\backslash X^+$ which has a canonical model S over an abelian extension E^S of E (like it happens with the j function). Then the constructions of Galois representations we did in §1.3.5 and §1.3.6 are no longer allowed. Neatness was not used in other parts of the standard construction of Shimura varieties, and so the other results of §1.3 still hold.

Let $p : X^+ \to S(\mathbb{C})$ be the usual map. We want to know if $\mathrm{Th}^{\infty}_{\mathrm{SF}}(\mathbf{p})$ is categorical. Take two models of the same cardinality: $q : D \to S(F)$ and $q' : D' \to S(F')$. By Proposition 1.3.1, we know that Γ is virtually neat. So for some tuple $\overline{g}$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$ $\Gamma_{\overline{g}}$ is neat.

Let $Z_{\overline{g}}(\mathbb{C}) = \Gamma_{\overline{g}}\backslash X^+$. As we have seen before, we can produce a morphism of Shimura varieties $\pi : Z_{\overline{g}} \to S$ defined over the composite of the respective reflex fields, but as both varieties have the same Shimura datum, it follows that π is defined over $E^S(\Sigma)$. Consider the maps

$$\begin{array}{ccccccc} q_{\overline{g}} : D & \longrightarrow & Z_{\overline{g}}(F) & , & q'_{\overline{g}} : D' & \longrightarrow & Z_{\overline{g}}(F') \\ x & \mapsto & (q(\overline{g}x)) & & x & \mapsto & (q'(\overline{g}x)) \,. \end{array}$$

Observe that $p_{\overline{g}}$ is the uniformisation map of $Z_{\overline{g}}(\mathbb{C})$ and that, becasuse q and q' satisfy the conditions SS and SF determined by p, then $q_{\overline{g}}$ and $q'_{\overline{g}}$ satisfy the conditions SS and SF determined by $p_{\overline{g}}$. Applying Theorem 4.2.10 to $p_{\overline{g}}$ we get that there is an isomorphism of models

$$\begin{tikzcd} D \arrow[r, "\xi"] \arrow[d, "q_{\overline{g}}"] & D' \arrow[d, "q'_{\overline{g}}"] \\ Z_{\overline{g}}(F) \arrow[r, "\sigma"] & Z_{\overline{g}}(F'), \end{tikzcd} .$$

From the work we did in §1.3.6, it is immediate that the special structure induced on X^+ by $Z_{\overline{g}}(\mathbb{C})$ is the same as that induced by $S(\mathbb{C})$. As $p = \pi \circ p_{\overline{g}}$, then $q = \pi \circ q_{\overline{g}}$ and $q' = \pi \circ q_{\overline{g}}$. As π is defined over $E^S(\Sigma)$, we conclude that the automorphism σ descends to give an isomorphism of the original models:

$$\begin{tikzcd} D \arrow[r, "\xi"] \arrow[d, "p"] & D' \arrow[d, "q"] \\ S(F) \arrow[r, "\sigma"] & S(F'). \end{tikzcd}$$

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