Topics in multidimensional persistence

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Abstract

A multiparameter persistence module is a representation of the lattice quiver $\mathbb{N}^d$, where maps along all squares commute. Equivalently, it is a functor $V\colon \mathbb{N}^d \mapsto \mathbf{FinVect}$ from the partially ordered set $\mathbb{N}^d$ to the category of finite dimensional vector spaces over an arbitrarily chosen field. When $d = 1$, Gabriel's theorem applies [19] and these modules admit interval decompositions, allowing us to classify one dimensional persistence modules through their associated barcode, a combinatorial invariant first introduced in [53] by Carlsson and Zomorodian. When $d > 1$, no such classification is possible. In this thesis we study these higher dimensional persistence modules, seeking to overcome their lack of simple classification by defining discrete invariants with as much discriminative power as possible. The thesis is composed of three parts.

First, we give an in depth analysis of barcode bases. These are bases of one-dimensional persistence modules that realise the interval decomposition given by Gabriel's theorem. We present a novel algorithm that computes these barcode bases, and give theoretical results that characterise the set of barcode bases of a given persistence module. This allows for a decomposition results of certain types of ladder persistence modules. We generalise all these results to zigzag persistence.

Second, we consider Harder-Narasimhan filtrations for quiver representations and define the skyscraper invariant, a novel discrete invariant for multidimensional persistence which is finer than the rank invariant. We further show the skyscraper invariant can be refined to create a complete invariant on certain families of ladder persistence modules.

Finally, we discuss computation methods for the skyscraper invariant. We exhibit an algorithm that computes the skyscraper invariant for ladder persistence modules. This is done by leveraging the decomposition

result for ladder persistence modules from the first chapter. In doing so, we introduce the ladder invariant, which is computable and more discriminative than the rank invariant. It coincides with the skyscraper invariant on ladder persistence modules and is non-comparable to the skyscraper invariant in general.

Algorithms from the first chapter are given as pseudo-code. These were later implemented as a python package and we give an overview of this package in the appendix.

Acknowledgements

My biggest acknowledgments go to my two exceptional supervisors, Ulrike Tillman and Vidit Nanda. I was fairly anxious coming into my PhD. The young graduate that I was had been left slightly intimidated by Ulrike's intense questioning during our interview. To no one's surprise, these worries were unfounded and I was met with helpful, available and most importantly kind supervisors whose guidance and advice made all of this work possible. Their impact on this thesis cannot be overstated, and I thank them for taking me as their student. Working alongside Marc Fertzsand on Harder-Narasimhan filtrations was a great pleasure, and I'm thankful for his contributions to our work and the helpful discussions that came with it. My dear Anastassia also deserves some praises. Whilst never missing an oppurtunity to reduce my PhD work to connecting dots on a white piece of paper, she also kept me in high spirits during these last 4 years. Her humour and kindness were an invaluable help in producing this work. Finally, whilst of no particular help to this thesis, I feel obliged to thank my parents, sisters and friends as they will undoubtedly be upset if they end up not featuring here[1].

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[1] Names including, but not limited to, Caroline, Benjamin, Adele, Josephine, Lucas, Marius...

Contents

Chapter 1

Introduction

1.1 Setting and motivation

The overarching theme of this thesis is the study of finite representations of a quiver $Q = (Q_0, Q_1)$ over an arbitrary field. These appear in different guises and flavours in a wide range of mathematical fields. The recent developments and interest around topological data analysis (TDA) has led to a growing desire to better our understanding of particular representations of specific quivers, namely the lattice quiver $\mathbb{N}^d$. Whilst our work is motivated by interests from the TDA community, the approach we take is broad and can be read with no prior knowledge of TDA.

1.1.1 Quiver representations for topological data analysis

Topological data analysis is a field which lies at the intersection of algebraic topology, representation theory and data analysis, to name a few. At its core, it seeks to use tools and methods from algebraic topology to gain insight on the underlying shape of a given finite metric space. Carlsson and Zomorodian were the first to outline a pipeline for TDA in [53], using **persistence homology**. Since then, most pipelines have adopted the following framework:

1. Given a finite metric space X (e.g a finite subset of $\mathbb{R}^n$) and a partially ordered set $\mathscr{P}$, we construct a $\mathscr{P}$-indexed filtration of X by topological spaces. This is given as a functor $F(X)\colon \mathscr{P} \mapsto$ **Top** where morphisms in **Top** are given by inclusion. There are several ways to achieve this, but some of the most common methods construct filtered simplicial complexes from the data set X, using either the Čech or Rips complex [28].

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots \\
(2,0) \arrow[u] \arrow[r] & (2,1) \arrow[u] \arrow[r] & (2,2) \arrow[u] \arrow[r] & {} \\
(1,0) \arrow[u] \arrow[r] & (1,1) \arrow[u] \arrow[r] & (1,2) \arrow[u] \arrow[r] & \cdots \\
(0,0) \arrow[u] \arrow[r] & (0,1) \arrow[u] \arrow[r] & (0,2) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Figure 1.1: Lattice quiver $\mathbb{N}^2$

2. We then compose the above filtration with a functor $G\colon \mathbf{Top} \mapsto \mathbf{FinVect}$ where **FinVect** is the category of finite dimensional vector spaces over a field $\mathbb{F}$. The most commonly used is the i-th homology group $H_i(-,\mathbb{F})\colon \mathbf{Top} \mapsto \mathbf{FinVect}$. This yields a functor $G \circ F(X)\colon \mathscr{P} \mapsto \mathbf{FinVect}$, often refered to as a $\mathscr{P}$-indexed *persistence module*. Such functors form a natural abelian category $\mathbf{PersMod}(\mathscr{P})$, with morphisms given by pointwise linear-maps which commute under the partial order, i.e for $p \leq q$ and $\phi\colon V \to W \in \mathbf{PersMod}(\mathscr{P})$ the following diagram commutes

$$\begin{tikzcd}
V_p \arrow[r, "V_{p\leq q}"] \arrow[d, "\phi_p"'] & V_q \arrow[d, "\phi_q"] \\
W_p \arrow[r, "W_{p\leq q}"'] & W_q
\end{tikzcd}$$

3. The goal now is to better our understanding of our given persistence module $V \in \mathbf{PersMod}(\mathscr{P})$ using tools from algebra. The final output should be independent of isomorphism classes, and these are as such referred to in the literature as **invariants**. What can and cannot be done at this step varies dramatically, depending on the chosen partial ordered $\mathscr{P}$. All the work presented in this thesis inserts itself at this step of the pipeline.

In practice, the most used partially ordered sets are $\mathbb{N}$, $\mathbb{N}^2$ or more generally $\mathbb{N}^d$. It is then a good exercise to understand that the category $\mathbf{PersMod}(\mathbb{N})$ is equivalent to $\mathbf{Rep}(Q)$ where Q is a quiver with vertex set $\mathbb{N}$ and arrows $k \mapsto k+1$:

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots
\end{tikzcd}$$

Similarly, the category $\mathbf{PersMod}(\mathbb{N}^2)$ is equivalent to the category of representations of the two dimensional grid quiver Q where all maps between squares commute. It is also customary to take persistence modules of **finite type**. These are elements $V \in \mathbf{PersMod}(\mathbb{N}^d)$ for which there is an upper bound $\ell \in \mathbb{N}^d$ such that $V_\ell \simeq V_q$ whenever $\ell \leq q$. This allows for further simplification, where the analogue quiver representation is now over a quiver with finitely many vertices.

When $d = 1$, we enter the realm of 1-parameter persistence modules. By the above observations this is equivalent to the study of representations of the length $\ell + 1$ path quiver

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow \ell$$

A specific case of Gabriel's foundational result [19] concerning representations of Dynkin quivers is that any representation of the above quiver decomposes in direct sums of interval modules.

The **interval module** corresponding to a pair of non-negative integers $i \leqslant j$ is the persistence module $\mathbf{I}[i, j]$ given by

$$0 \to \cdots \to 0 \to \mathbb{F} \to \cdots \to \mathbb{F} \to 0 \to \cdots \to 0,$$

where the contiguous string of $\mathbb{F}$'s spans $\{i, i+1, \ldots, j-1, j\}$, all intermediate $\mathbb{F} \to \mathbb{F}$ maps are identities, and all other vector spaces are trivial. Carlsson and Zomorodian were the first in [53] to make use of Gabriel's decomposition theorem to introduce the notion of **barcodes** associated to a 1-parameter persistence module.

Theorem 1.1.1. *For each persistence module V of length $\ell + 1$, there exists a finite set of non-negative integer pairs*

$$\mathbf{Bar}(V) := \{i_1 \leqslant j_1, \ldots, i_k \leqslant j_k\},$$

(with $[i_p, j_p] \subset [0, \ell]$ for all $1 \leqslant p \leqslant k$), called the **barcode** *of V, and an integer* **multiplicity** *$d_{i_p j_p} > 0$ so that $(V_\bullet, f_\bullet)$ is isomorphic to a direct sum of interval modules:*

$$(V) \simeq \bigoplus_{p=1}^{k} \mathbf{I}[i_p, j_p]^{d_{i_p j_p}}.$$

Here the i-th summand on the right side is to be interpreted as the $d_{i_p j_p}$-fold direct sum of the interval module $\mathbf{I}[i_p, j_p]$ with itself. The existence of a barcodes is appealing for several reasons.

- It serves as a **complete invariant** in the sense that two persistence modules of the same length are isomorphic if and only if they have the same barcode.

- The barcode is a **discrete** object, determined by the start and end point of finitely many bars.
- It is **computable**. Carlsson and Zomorodian presented in [53] an algorithm which computes the barcode of a persistence module arising as homology groups of a filtered simpicial complex. They later generalised this to computing barcodes of arbitrary persistence modules in [12].
- It is **stable**. Chazal et al. were the first to introduce this notion in [13] by defining the **interleaving distance** on $\mathbf{PersMod}(\mathscr{P})$ and the **bottleneck distance** between two barcodes. They then showed that the assignment of a persistence module V to its barcode $\mathbf{Bar}(V)$ is an isometry for the above metrics.

The above properties have firmly justified the barcode $\mathbf{Bar}(V)$ as the correct invariant to output in any 1-parameter TDA pipeline. The stability condition in particular is key. Given two finite sets X and Y with Gromov-Hausdorff distance ϵ, it can be shown that the corresponding persistence modules obtained using the standard pipeline (filtrations using Čech or Rips complexes, then homology groups) have interleaving distance ϵ and so the associated barcodes also have bottleneck distance ϵ. This yields the desirable and necessary property for any sensible pipeline: close inputs yield close outputs.

When $d > 1$, we enter the realm of multi-parameter persistence. From a quiver representations perspective, this amounts to the study of representations of d-dimensional grid quivers where maps between all natural squares commutes. It is then natural to wonder whether the barcode theory generalises to this context. Carlsson and Zomorodian were the first to address this in [11].

Theorem. *For $d > 1$, there exists no complete discrete invariant on the category $\mathbf{PersMod}(\mathbb{N}^d)$. This still holds when considering finitely generated persistence modules.*

Whilst this kills any hope of generalising the notion of barcodes to higher dimensional persistence, this has introduced a new topic of research which has seen much interest in the TDA community in recent years.

Problem: Can we find a discrete, computable and stable invariant for multiparameter persistence with as much discriminative power as possible? This invariant should also be complete in the 1-parameter case.

The first candidate was introduced by Carlsson and Zomorodian in the same paper. The **rank invariant** for a given module V is the function

$$\begin{aligned} \rho_V \colon \mathscr{P} \times \mathscr{P} &\to \mathbb{N} && (1.1) \\ (x, y) &\to \operatorname{rank}(V_{x \leq y}) && (1.2) \end{aligned}$$

with the convention that $\rho_V(x, y) = 0$ if $x \nleq y$. Whilst the rank invariant has several appealing factors (simple to define, computable and stable as shown in [38]), finding other invariants with similar properties and more discriminative power is a very active area of research, which we seek to make a contribution to in this thesis.

1.1.2 Quiver representations as moduli spaces

King [34] and more recently Reineike [50] have studied the category $\mathbf{Rep}(Q)$ as a moduli space by viewing it as an affine variety and considering the natural change of basis group action. This requires fixing a dimension vector $d \in \mathbb{N}^{Q_0}$ and corresponding d_i-dimensional vector spaces V_i, so that the category $\mathbf{Rep}(Q, d)$ of representations of dimension d can be naturally identified as the affine space

$$X = \bigoplus_{\substack{e \in Q_1 \\ e \colon t(e) \mapsto h(e)}} \operatorname{Hom}(V_{t(e)}, V_{h(e)})$$

upon which the algebraic group $G = \bigoplus_{i \in Q_0} \mathrm{GL}(V_i)$ acts via conjugation. Characterising the isomorphism classes of $\mathbf{Rep}(Q, d)$ then amounts to understanding the quotient X/G, putting us firmly in the setting of geometric invariant theory. Here the quotient variety is taken as the one coming from geometric invariant theory, as presented in Section 3 of [50] (observe orbit spaces coming from the purely topological quotient can be non-Haussdorf).

This quotient is well understood for certain classes of quivers. For instance when Q is the length ℓ path quiver, as seen above the barcode yields a complete discrete invariant on $\mathbf{Rep}(Q)$ so that the quotient X/G is zero-dimensional. For more complex quivers this is rarely the case and as such the quotient set X/G is generally more complex to characterise and understand.

In the context of multiparameter persistence, we focus on specific representations of the d dimensional grid quiver where maps commute on all natural squares. This yields a subcategory $\mathbf{Rep}_{eq}(Q)$ of equalised representations, which corresponds to a subvariety $X_{eq} \subset X$ defined by polynomial relations obtained by multiplying adequate matrices along distinct paths with the same endpoints. Understanding isomorphism classes of equalised representations of Q then still amounts to understanding the quotient variety X_{eq}/G. Carlsson and Zomorodian's result [11] concerning the non-existence of complete discrete invariant for multiparameter persistence shows X_{eq}/G is no longer zero-dimensional. The new quest of finding discrete invariants with strong discriminative power translates in this language to approximating X_{eq}/G by zero dimensional varieties. That is, can we find a map $\pi\colon X_{eq}/G \mapsto Z$ where Z is a zero-dimensional variety and the fibers of π are as small as possible. Here zero-dimensional varieties refers to discrete and countable sets.

Our work here stems from a desire to achieve two things:

Objective 1. For well-understood quivers such as type $\mathbb{A}_\ell$ quivers, we wish to better understand classes $[x] \in X/G$. We achieve this in the first chapter by computing adequate bases of the vectors space V_i called **barcode bases** and studying the automorphism group of their associated matrices in **barcode form**, allowing us to completely characterise the orbit $[x]$ from the direct sum decomposition of x in interval modules.

Objective 2. For more complex quivers, we wish to provide a novel discrete invariant with as much discriminative power as possible. We achieve this by using **Harder-Narasimhan filtrations** of persistence modules, a tool which has previously been used in the study of well-behaved quotient spaces for the action of reductive groups on algebraic varieties [47].

1.2 Readers guide

We give a brief overview of each chapter. They all include their own introduction, literature review and notation.

1.2.1 The space of barcode bases for peristence modules

This part was published as [32]. Some minor remarks and examples were added later on, as well as an additional result on automorphism groups (subsection 2.4.1).

The barcode of a persistence module serves as a complete combinatorial invariant of its isomorphism class. Barcodes are typically extracted by performing changes of basis on a persistence module until the constituent matrices have a special form. Here we describe a new algorithm for computing barcodes which also keeps track of, and outputs, such a change of basis. Our main result is an explicit characterisation of the group of transformations that sends one barcode basis to another. Armed with knowledge of the entire space of barcode bases, we are able to show that any map of persistence modules can be represented via a partial matching between bars provided that neither source nor target admits nested bars in its barcode. We also generalise the algorithm and results described above to work for zizag modules. Accompanying code implementing these algorithms is avalaible as a python package. We give concrete examples of how to use it in the appendix.

1.2.2 Harder-Narasimhan filtration of persistence modules

This featured as a preprint [17]. This was written alongside my colleague Marc Fertzand, and we omit Section 5.2 of that paper here as it was Marc's work. Section 2 and 4 (3.2 and 3.4 in this thesis) were done collaboratively whilst the remaining results of that chapter are my own.

The Harder-Narasimhan type of a quiver representation is a discrete invariant parameterised by a real-valued function (called a central charge) defined on the vertices of the quiver. In this chapter, we investigate the strength and limitations of Harder-Narasimhan types for several families of quiver representations which arise in the study of persistence modules. We introduce the skyscraper invariant, which amalgamates the HN types along central charges supported at single vertices, and generalise the rank invariant from multiparameter persistence modules to arbitrary quiver representations. Our three main results are as follows: (1) we show that the skyscraper invariant is strictly finer than the rank invariant in full generality, (2) we characterise the set of complete central charges for zigzag (and hence, ordinary) persistence modules, (3) we show that although no single central charge is complete for nestfree ladder persistence modules, a finite set of central charges is complete.

1.2.3 Computations methods for Harder-Narasimhan types and the ladder invariant

In this chapter we propose some methods for computing the previously defined skycraper invariant. Using decomposition results from the first chapter, we are able to compute this invariant for equalised ladder persistence modules in $O(n^3\ell)$ time where n is the maximal dimension of the vector space of our given representation and ℓ is the length of our ladder. Whilst this method does not generalise to two dimensional grids due to the non functoriality of the decomposition result, it still allows for the computation of a slight tweak to the skycraper invariant which remains finer than the rank invariant for two dimensional grids. We call this the ladder invariant. It is computable by the above and we show it is non-comparable to the skycraper invariant.

Chapter 2

The Space of barcode bases for persistence modules

2.1 Introduction

Persistence modules appear in different forms and guises across many areas of mathematics. In recent years, particular interest and focus has come from their extensive use in topological data analysis (TDA) in general and persistent homology in particular. In this chapter, we examine persistence modules of the form $(V_\bullet, f_\bullet)$:

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{\ell-1}} V_{\ell-1} \xrightarrow{f_\ell} V_\ell, \tag{2.1}$$

where each V_i is a vector space of finite dimension n_i (over an underlying field $\mathbb{F}$) and each $f_i : V_{i-1} \to V_i$ is a linear map. We study three different aspects.

Computing Barcode Bases

The first is the central question of finding a barcode basis for $(V_\bullet, f_\bullet)$. This amounts to a choice of basis for each V_i with respect to which the linear maps f_i have a particularly nice form — they admit at most a single 1 in each row and column, with all other entries being 0. The existence of such bases and matrix representations is well known [19]. We say that the matrices are in *barcode form* and the corresponding basis of $\bigoplus_{i=0}^{\ell} V_i$ is a *barcode basis*, since the barcodes familiar from TDA can easily be extracted.

Algorithms to compute barcode bases in TDA typically take as input a filtered chain complex as in [53], where one has recourse to matrix representations of the boundary operators. Algorithms for general persistence modules include the well-known [8] and much more recently, [24], [44] and [30]. Here we present a new algorithm that takes as its input a matrix representation $A_\bullet = (A_1, \ldots, A_\ell)$ relative to some initial basis of the persistence module $(V_\bullet, f_\bullet)$ and outputs a sequence $g = (g_0, \ldots, g_\ell)$ of change of basis matrices g_i for each of the V_i so that the new matrix representation $A'_\bullet = (A'_1, \ldots, A'_\ell)$ with $A'_i = g_i \cdot A_i \cdot g_{i-1}^{-1}$ is in barcode form. Our algorithm in Section 2.3.2 is explicit and elementary in the sense that every intermediate step amounts to performing standard (row or column) operations on the constituent A_i's. The key difficulty here is that column operations on A_i often force new matrix operations on A_k for $k < i$, and similarly row operations on A_i often require changes in A_k for $k > i$.

The Space of Barcode Bases

Our second goal is to describe the set of all barcode bases[1] of $(V_\bullet, f_\bullet)$. We show that this set can naturally be identified as the stabiliser of a matrix representation $A_\bullet$ of $(V_\bullet, f_\bullet)$, and hence as a subgroup of the product

$$G := \mathrm{GL}(n_0; \mathbb{F}) \times \cdots \times \mathrm{GL}(n_\ell; \mathbb{F}),$$

where $\mathrm{GL}(d; \mathbb{F})$ indicates the general linear group of invertible $d \times d$ matrices with entries in $\mathbb{F}$. Writing

$$[i_1, j_1] \preceq [i_2, j_2] \quad \text{whenever} \quad i_1 \leqslant i_2 \leqslant j_1 \leqslant j_2$$

and $\mathrm{Mat}(m \times n; \mathbb{F})$ for the set of $m \times n$ matrices with coefficients in $\mathbb{F}$, we show in Theorem 2.4.4 that the set of barcode bases is in one-to-one correspondence with

$$\prod_{0 \leqslant i \leqslant j \leqslant \ell} \mathrm{GL}(d_{ij}; \mathbb{F}) \times \prod_{[i_1, j_1] \precneq [i_2, j_2]} \mathrm{Mat}(d_{i_1 j_1} \times d_{i_2 j_2}; \mathbb{F}).$$

Here d_{ij} is the multiplicity of the interval $[i, j]$ in the barcode of $(V_\bullet, f_\bullet)$. If the matrix representation is already in barcode form, then the elements in $\mathrm{GL}(d_{ij}; \mathbb{F})$ correspond to changes of basis for the sub-vector space of V_i spanned by the basis elements corresponding to the bars $[i, j]$, and the elements in $\mathrm{Mat}(d_{i_1 j_1} \times d_{i_2 j_2}; \mathbb{F})$ represent the changes to basis vectors in V_{i_2} corresponding to intervals $[i_2, j_2]$ obtained by adding vectors from V_{i_2} which correspond to intervals $[i_1, j_1]$.

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[1] This set has a natural topology when working over a field with topology such as $\mathbb{R}$ or $\mathbb{C}$.

Simplifying Maps of Persistence Modules

We now turn attention to our third problem. Given a map of persistence modules

$$\phi : (V_\bullet, f_\bullet) \to (W_\bullet, h_\bullet),$$

we seek barcode bases for source and target in terms of which ϕ assumes its simplest form, in the sense that we now specify. An interval $[i, j]$ represents a submodule canonically isomorphic to the interval module $\mathbf{I}[i, j]_\bullet$. It is an elementary observation that such a module can be mapped non-trivially to another interval module $\mathbf{I}[i', j']_\bullet$ if and only if $[i', j'] \preceq [i, j]$; and in this case the non-zero map is unique up to a non-zero scalar. In the simplest case, ϕ induces a partial matching where each bar in the source is mapped to exactly one in the target or mapped to zero. Surprisingly, we show in Theorem 2.5.3 that such a partial matching exists (after a change of barcode bases) whenever neither source nor target admit a pair of strictly nested intervals in their respective barcode decompositions.[2] In an example we also show that these conditions are necessary.

Zigzag Modules

Finally, we generalise the algorithm and theorems described above to zizag modules of a fixed type τ: the linear maps of the persistence module (2.1) can go either forward $V_{i-1} \xrightarrow{f_i} V_i$ or backward $V_{i-1} \xleftarrow{g_i} V_i$ according to pattern fixed by τ. Such modules are also classified in terms of sums of interval modules. Our algorithm can be adapted to compute barcode bases of zizag modules of any type. Next, we introduce a generalisation of the order $\preceq$ that depends on the type τ. This order takes into account that the order $\preceq$ has to be reversed when all the arrows in (2.1) are reversed. With this order in place, we can once again classify the set of all barcode bases, see Theorem 2.6.11. Similarly when considering maps of zizag modules, we generalise the notion of strictly nested bars, which once again depends on the type τ. Excluding such nested bars, we are able to obtain barcode bases of the source and target zizag modules in terms of which the map is described by a partial matching on the set of bars; see Theorem 2.6.14.

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[2] Two bars $[i, j]$ and $[i', j']$ are strictly nested if $i < i'$ and $j' < j$.

Outline

We take the view that a persistence module $(V_\bullet, f_\bullet)$ is a quiver representation. In Section 2.3, we provide a constructive proof (and concomitant algorithm) of Gabriel's decomposition theorem for persistence modules. In Section 2.4 we identify the set of all barcode bases with a stabiliser of the action of the group G on the set of all possible matrix representations of $(V_\bullet, f_\bullet)$. In Section 2.5 we study maps between persistence modules by viewing them as representations of ladder quivers with relations. Gabriel's theorem no longer applies here; however, using similar arguments as in [1], we are able to prove a finite decomposition when source and target have no nested bars. Finally, in Section 2.6 we extend our results to any quiver of type A, that is zigzag persistence modules.

Related work

As mentioned above, there are several well known algorithms which compute barcodes of persistence modules. Some start with chains on a filtered simplicial complex, some deal with more general persistence modules and in some cases zizag modules.

There are two algorithms that explicitly deal with computing the barcode bases associated to the interval decomposition. In [10] the authors use matrix factorisation techniques to obtain bases in which the matrices are in echelon form. This technique also applies to zizag modules. In [20] the authors inductively compute interval bases using basis completions techniques at each step, but they do not deal with the zizag case. Neither of these papers attempts to compute the set of barcode bases associated with the persistence module, and the algorithm we describe here takes a different approach to reducing matrices in barcode form. Most recently, the authors of [23] use U-match matrix factorisation to reduce computational complexity and memory storage in computing barcodes.

It was shown in [16] that maps between persistence modules of length less than 5 admit a tractable classification in the sense that the associated ladder persistence modules are always of finite type. In contrast, the authors of [7] find an infinite class of indecomposable non-isomorphic ladder persistence module whenever the length is greater then 5. In [1] the authors outline an algorithm which computes the decomposition into a sum of indecomposables for ladder persistence modules of length < 5.

2.2 Persistence Modules and Barcode Bases

A **persistence module**, for the purposes of this chapter, is a finite collection $(V_\bullet, f_\bullet)$ of finite-dimensional vector spaces V_i over a field $\mathbb{F}$ along with $\mathbb{F}$-linear maps f_i arranged as follows:

$$\begin{tikzcd} V_0 \arrow[r, "f_1"] & V_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{\ell-1}"] & V_{\ell-1} \arrow[r, "f_\ell"] & V_\ell. \end{tikzcd}$$

The number $\ell + 1$ is called the *length* of $(V_\bullet, f_\bullet)$. The *direct sum* of $(V_\bullet, f_\bullet)$ with another persistence module $(W_\bullet, h_\bullet)$ of the same length is defined pointwise — in other words, the vector space at its i-th position is $V_i \oplus W_i$ for each admissible index i, and similarly the corresponding linear map is given by $f_i \oplus h_i$. We call $(V_\bullet, f_\bullet)$ *isomorphic* to $(W_\bullet, h_\bullet)$ if there are invertible linear maps $\phi_i : V_i \to W_i$ so that the square

$$\begin{tikzcd} V_{i-1} \arrow[r, "f_i"] \arrow[d, "\phi_{i-1}"', "\sim"] & V_i \arrow[d, "\phi_i", "\sim"'] \\ W_{i-1} \arrow[r, "h_i"'] & W_i \end{tikzcd}$$

commutes for each index i in $\{1, \ldots, \ell\}$. Isomorphisms from $(V_\bullet, f_\bullet)$ to itself are called *automorphisms*, and these evidently form a group under composition. We denote this group by $\mathrm{Aut}(V_\bullet, f_\bullet)$.

The **interval module** corresponding to a pair of non-negative integers $i \leqslant j$ is the persistence module $\mathbf{I}[i, j]_\bullet$ given by

$$0 \to \cdots \to 0 \to \mathbb{F} \to \cdots \to \mathbb{F} \to 0 \to \cdots \to 0,$$

where the contiguous string of $\mathbb{F}$'s spans $\{i, i+1, \ldots, j-1, j\}$, all intermediate $\mathbb{F} \to \mathbb{F}$ maps are identities, and all other vector spaces are trivial. The importance of interval modules stems from the decomposition result 1.1.1, which can be found for instance in [53].

This *interval decomposition* theorem follows from Gabriel's foundational result on the decomposability of quiver representations [19] — since $(V_\bullet, f_\bullet)$ is a representation of a type-$\mathbf{A}_{\ell+1}$ quiver. Our goal here is to provide an explicit algorithm which not only furnishes such the isomorphism (1.1), but can also be readily implemented on a computer.

To this end, fix a persistence module $(V_\bullet, f_\bullet)$ of length $\ell + 1$ and set $n_i := \dim_{\mathbb{F}} V_i$ for each $i \in \{0, \ldots, \ell\}$. Without loss of generality, we may select a *basis* *family*

$$\mathscr{B} := \{B_i \subset V_i \mid 0 \leqslant i \leqslant \ell\},$$

where each B_i forms an ordered basis for the vector space V_i. This choice amounts to fixing an isomorphism $V_i \simeq \mathbb{F}^{n_i}$ for each i. Thus, every linear map $f_i : V_{i-1} \to V_i$ can be represented (in terms of the chosen bases B_{i-1} and B_i from $\mathscr{B}$) as a matrix A_i of size $n_i \times n_{i-1}$ with entries in $\mathbb{F}$; consequently, $(V_\bullet, f_\bullet)$ is isomorphic to

$$\mathbb{F}^{n_0} \xrightarrow{A_1} \mathbb{F}^{n_1} \xrightarrow{A_2} \cdots \xrightarrow{A_{\ell-1}} \mathbb{F}^{n_{\ell-1}} \xrightarrow{A_\ell} \mathbb{F}^{n_\ell}. \tag{2.2}$$

In light of Theorem 1.1.1, we are particularly interested in a special class of basis families.

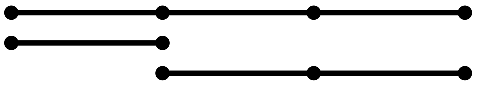
Definition 2.2.1. An $m \times n$ matrix A of rank r is in **barcode form** if there exists a strictly increasing function $c : \{1, \ldots, r\} \to \{1, \ldots, n\}$ so that

$$A_{ij} = \begin{cases} 1 & \text{if } j = c(i), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a matrix is in barcode form whenever its entries lie in $\{0, 1\}$, with at most one non-zero term in each row and column, and the r non-zero terms appear in the first r rows and in strictly increasing column order.

A basis family $\mathscr{B}$ is called an **barcode basis** for $(V_\bullet, f_\bullet)$ if all of the A_i are in barcode form. The natural basis arising from an interval decomposition of a persistence module is a barcode basis.

Example 2.2.2. Consider, for instance, the persistence module of length 4 given by the barcode containing $0 \leqslant 3$ along with $0 \leqslant 1$ and $1 \leqslant 3$, each with multiplicity one:



With respect to the basis family obtained by ordering these intervals from top to bottom, the matrices A_i are given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and all three are evidently in barcode form. Conversely, one can also recover the interval decomposition immediately from these three matrices.

To put our quest for a constructive proof of Theorem 1.1.1 on a firm algebraic footing, let X be the set of all the possible matrix-sequences $A_\bullet$ which can arise in (2.2). It is a (strict) subset of the product of matrices of the appropriate dimensions:

$$X \subset \prod_{i=1}^{\ell} \mathrm{Mat}\left(n_i \times n_{i-1}; \mathbb{F}\right). \tag{2.3}$$

Writing $\mathrm{GL}(n;\mathbb{F})$ for the group of all $n \times n$ invertible matrices over $\mathbb{F}$, consider the product

$$G := \prod_{i=0}^{\ell} \mathrm{GL}(n_i;\mathbb{F}), \tag{2.4}$$

which acts naturally via a change-of-basis action on X: the group element $g := (g_0, \ldots, g_\ell)$ sends each matrix-sequence $A_\bullet$ in X to the new sequence $(gA)_\bullet$ given by

$$(gA)_i := g_i \cdot A_i \cdot g_{i-1}^{-1} \tag{2.5}$$

for each admissible index i. This is equivalent to replacing the original basis family $\mathscr{B} = \{B_i\}$ with the new basis family $g\mathscr{B} = \{g_i B_i\}$. Thus, X is the free orbit of $A_\bullet$ under this G-action. So our first task, solved in Section 2.3, translates to discovering some $g \in G$ that transforms a given basis family $\mathscr{B}$ of $(V_\bullet, f_\bullet)$ to a barcode basis.

2.3 Constructing a Barcode Basis

Throughout this section, we fix a persistence module $(V_\bullet, f_\bullet)$ expressed as a sequence of matrices $A_\bullet$ as in (2.2) with respect to an arbitrary (i.e., not necessarily barcode) basis family $\mathscr{B}$.

2.3.1 Barcode bases via elementary matrix operations

To conveniently describe relevant elements of G, we fix notation for matrices which implement certain fundamental row and column operations.

Definition 2.3.1. For each dimension $n > 0$, distinct indices $1 \leqslant p, q \leqslant n$, and scalar $\lambda \in \mathbb{F}$, let $e^n_{p,q}(\lambda)$ denote the **elementary matrix** in $\mathrm{GL}(n;\mathbb{F})$ which has 1's all along its diagonal, λ in the (p,q)-th position, and zeros everywhere else.

Since the dimension n will be clear from context, we omit it from the superscript and simply write $e_{p,q}(\lambda)$ to indicate the relevant elementary matrix. The following standard facts about such matrices will be freely used in the sequel —

1. multiplying a matrix on the left by $e_{p,q}(\lambda)$ implements the following **elementary row operation**
$$\text{Row}(p) \leftarrow \text{Row}(p) + \lambda \cdot \text{Row}(q)$$
which we denote $\mathbf{R}_{p\leftarrow q}(\lambda)$; similarly,
2. multiplying a matrix on the right by $e_{p,q}(\lambda)$ implements the following **elementary column operation**
$$\text{Col}(q) \leftarrow \text{Col}(q) + \lambda \cdot \text{Col}(p),$$
which we denote $\mathbf{C}_{q\leftarrow p}(\lambda)$; and finally,
3. the inverse of $e_{p,q}(\lambda)$ is $e_{p,q}(-\lambda)$.

Remark 2.3.2. Consider the element $g = (g_0, \dots, g_\ell) \in G$ for which $g_i = e_{p,q}(\lambda)$ and all the other g_j are identity matrices. The action of this g on a given matrix sequence $A_\bullet$ is to simultaneously perform $\mathbf{C}_{p\leftarrow q}(\lambda)$ on A_i and $\mathbf{R}_{q\leftarrow p}(-\lambda)$ on A_{i-1} while leaving all the other A_j's invariant.

The following result plays an essential part in our constructive proof of Theorem 1.1.1. In its statement and beyond, we will use $A(p,q)$ to indicate the entry in the p-th row and q-th column of a given matrix A.

Lemma 2.3.3. *Assume that the first $\ell - 1$ matrices $\{A_i \mid 1 \leqslant i < \ell\}$ of (2.2) are in barcode form, and that the last matrix A_ℓ has a pivot in the (r,q) position, i.e., $A_\ell(r,q) = 1$ and all other entries in the q-th column are zero. If there is a nonzero entry $\alpha := A_\ell(r,p)$ in the same row r but some other column $p > q$, then there exists $g \in G$ with $g_\ell = \text{Id}$ so that $(gA)_\bullet$ equals $A_\bullet$ except A_ℓ where the α entry is replaced by zero.*

Proof. We proceed by induction on ℓ, noting that the case $\ell = 1$ is immediately true since there is only one matrix in sight. Assume that the statement holds up to $\ell - 1$. The r-th row of A_ℓ contains a pivot 1 in the q-th column and some $\alpha \neq 0$ in the p-th column. To eliminate this offending α, we perform $\mathbf{C}_{p\leftarrow q}(-\alpha)$ on A_ℓ by performing the basis change $e_{p,q}(-\alpha)$ on $V_{\ell-1}$. Since $A_\ell(r,q)$ is assumed to be a pivot, the only resulting difference in A_ℓ is that the (r,p)-th entry changes from α to 0. But by

Remark 2.3.2, we are also compelled to perform $\mathbf{R}_{q\leftarrow p}(\alpha)$ on the preceding matrix $A_{\ell-1}$. This results in a new matrix $A'_{\ell-1}$, and there are now 2 cases to consider, of which only the second requires the inductive hypothesis:

Case 1: if the p-th row of $A_{\ell-1}$ is identically zero, then our row operation has had no effect whatsoever; thus, $A'_{\ell-1} = A_{\ell-1}$ is still in barcode form and we have arrived at the desired result.

Case 2: If the p-th row of $A_{\ell-1}$ is nonzero, then since $q < p$ and $A_{\ell-1}$ is in barcode form, we see that the q-th row of $A'_{\ell-1}$ must also be non-zero. Then by Definition 2.2.1 they must have pivot ones in distinct columns, say c and d respectively, and furthermore $c < d$. Thus, after we have performed $\mathbf{R}_{q\leftarrow p}(\alpha)$ on $A_{\ell-1}$, the resulting matrix $A'_{\ell-1}$ has the form

$$A'_{\ell-1} = \begin{array}{cc} \begin{array}{ccccccc} & c & & & & d & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & \cdots & 0 & \alpha & 0 \\ & 0 & & & & 0 & \\ & \vdots & & & & \vdots & \\ & 0 & & & & 0 & \\ 0 & 0 & 0 & \cdots & & 1 & 0 \end{array}\right] & \begin{array}{c} q \\ \\ \\ \\ p \end{array} \end{array}$$

By induction, there exists a $g \in \prod_{i=0}^{\ell-1} \mathrm{GL}_{n_i}(\mathbb{F})$ with $g_{\ell-1} = \mathrm{Id}$ so that $g_i A_i g_{i-1}^{-1}$ is still in barcode form for $1 \leqslant i \leqslant \ell - 2$, and

$$g_{\ell-1} A_{\ell-1} g_{\ell-2}^{-1} = \begin{array}{cc} \begin{array}{ccccccc} & c & & & & d & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & 0 & & & & 0 & \\ & \vdots & & & & \vdots & \\ & 0 & & & & 0 & \\ 0 & 0 & 0 & \cdots & & 1 & 0 \end{array}\right] & \begin{array}{c} q \\ \\ \\ \\ p \end{array} \end{array}$$

is again in barcode form. Furthermore, since $g_{\ell-1} = \mathrm{Id}$, the matrix A_ℓ is left unchanged by this change of basis. The desired basis change is $(g_0, g_1, \ldots, g_{\ell-2}, e_{p,q}(-\alpha), \mathrm{Id})$. □

Proposition 2.3.4. *Given the sequence of matrices $A_\bullet$ as in (6), there is a $g \in G$ such that $(gA)_\bullet$ has all its matrices in barcode form.*

Proof. When $\ell = 1$, we may diagonalise the matrix A_1 via standard row and column operations. Proceeding by induction for $\ell > 1$, assume the existence of some group element

$$g' = (g_0, \ldots, g_{\ell-1}) \in \prod_{i=0}^{\ell-1} \mathrm{GL}_{n_i}(\mathbb{F})$$

satisfying the following property: the matrices $g_i A_i g_{i-1}^{-1}$ are in barcode form for $1 \leqslant i \leqslant \ell - 1$.

Consider $g = (g', \mathrm{Id}_{n_\ell})$, which evidently lies in G. Replacing $A_\bullet$ by $(gA)_\bullet$ if necessary, we may assume that $A_\bullet$ has its first $\ell - 1$ matrices in barcode form. Performing row operations on A_ℓ has no impact on the previous matrices, as it corresponds to multiplying A_ℓ on the left by some g_ℓ. Thus, we may assume without loss of generality that all previous matrices are in barcode form while A_ℓ itself is in reduced row echelon form. By Lemma 2.3.3, there is a basis change $g \in G$ which zeroes out each non-pivot entry whilst maintaining the barcode form of the previous matrices. Applying these basis changes gets us to the desired barcode basis. □

Remark 2.3.5. If $\mathscr{B}$ is the basis family with respect to which $(V_\bullet, f_\bullet)$ has matrix form $A_\bullet$, then $g\mathscr{B}$ is a barcode basis where $g \in G$ is as in Proposition 2.3.4. We may therefore regard it as a constructive analogue of Theorem 1.1.1.

2.3.2 Algorithms

Here we describe algorithms which implement the constructions of Lemma 2.3.3 and Proposition 2.3.4. In particular, the main algorithm CompPers described below accepts as input an initial sequence of matrices $A_\bullet$ as in (2.2) and puts them in barcode form. The sub-computations which we require frequently have been isolated into concomitant subroutines, described as follows.

1. The first subroutine **ColOp** implements the inductive strategy underlying our proof of Lemma 2.3.3; in particular, this algorithm acts as step k of the inductive procedure described in the proof of that lemma.

2. The second subroutine **Reduce** takes as input a sequence $A_\bullet$ for which the first $\ell - 1$ matrices are in barcode form together with an invertible matrix $g \in \prod_{i=0}^{\ell-1} \mathrm{GL}(n_i; \mathbb{F})$. It then reduces the final matrix A_ℓ until it is in barcode form, *while maintaining the barcode form of all previous matrices* and suitably updating the basis change g.

3. Finally, the main algorithm **CompPers**$(A_\bullet)$ takes as input an arbitrary sequence of matrices $A_\bullet$ and produces as output $g \in G$ together with $(gA)_\bullet$ in barcode form. From these matrices we can directly access all intervals in barcode of $(V_\bullet, f_\bullet)$.

Algorithm 1: ColOp

```
Input: $A_\bullet, g, k, r, q, p$
Output: Updated $A_\bullet$ and basis change $g$, zeroing out $A_k(r,p)$
C_{p←q}(−α) on A_k: $\mathbf{C}_{p\leftarrow q}(-\alpha)$ on $A_k$
$\mathbf{R}_{q\leftarrow p}(\alpha)$ on $A_{k-1}$
$g_k = e_{p,q}(-\alpha)$
if $k = 0$ or p-th row of $A_{k-1} = 0$ then
    return $(A_\bullet, g)$
end
else
    Find pivot columns $c < d$ of the pivot rows $q < p$ of $A_{k-1}$
    return (ColOp $(A_\bullet, g, k-1, q, c, d)$ )
end
```

Algorithm 2: Reduce

```
Input: $A_\bullet, g$, where $A_\bullet$ has its first $\ell - 1$ matrices in reduced form
Output: $A_\bullet$ in reduced barcode form and updated $g$
row reduce $(A_\ell)$
Append $g$ with corresponding $g_\ell$
while there are $A_\ell(r,p) \neq 0$ terms with $A_\ell(r,q) = 1$ a pivot do
    $A_\bullet, h =$ ColOp$(A_\bullet, g, \ell, r, q, p)$
    $g = hg$
end
return $(A_\bullet, g)$
```

Remark 2.3.6. The computational complexity of **CompPers**$(A_\bullet)$ can be expressed in terms of $n = \max_{0 \leqslant i \leqslant \ell} n_i$ and ℓ. The cost of placing all the A_i in reduced row echelon form via Gaussian elimination is $O(n^3\ell)$. Furthermore, performing column operations to further reduce these matrices requires at most $O(n^2)$ operations on each matrix. And column operations on A_i will, in the worst case, require downstream column operations on $A_{i-1} \dots A_1$. Thus, for column operations, we have a $O(n^2 \sum_{i=1}^{\ell} i) = O(n^2 \frac{\ell(\ell-1)}{2}) = O(n^2\ell^2)$ complexity. Combining these factors, the total complexity of the algorithm is

$$O(n^3\ell + n^2\ell^2).$$

Algorithm 3: CompPers

```
Input: $A_\bullet$
Output: Reduced $A_\bullet$ with corresponding change of basis $g$
$A'_\bullet = \{A_1\}$
$g = (\mathrm{Id}_{n_0})$
for $1 \leqslant i \leqslant \ell - 1$ do
    $A'_\bullet, g =$ Reduce($A'_\bullet, g$)
    $A'_\bullet$.append($A_{i+1} g_i^{-1}$)
end
return (Reduce($A'_\bullet, g$))
```

At each step of the algorithm, we perform an elementary basis change on a single vector space V_i, which amounts to multiplying a matrix $g_i \in \mathrm{GL}(V_i)$ by an elementary matrix., This incurs an $O(n)$ cost; thus, if we also wish to keep track of the basis changes, then the total complexity of **CompPers** becomes

$$O(n^4 \ell + n^3 \ell^2).$$

We conclude with an illustrative example of how **CompPers** acts on a sequence of input matrices.

Example 2.3.7. Consider

$$A_\bullet = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

To put A_3 in barcode form, we must zero out the terms $A_3(1,3)$ and $A_3(2,3)$ using column operations. At each step, we will be performing row and column operations on matrices of $A_\bullet$, amounting to basis changes on the vectors spaces V_0, V_1 and V_2. For conciseness sake, we will not keep track of the basis changes done along the way, and will simply be performing operations on the matrices to put them in barcode form.

We begin by zeroing out the $A_3(2,3)$ term.

1: $\mathbf{C}_{3 \leftarrow 2}(-1)$ on A_3, inducing $\mathbf{R}_{2 \leftarrow 3}(1)$ on A_2, giving us matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2: $\mathbf{C}_{3 \leftarrow 2}(-1)$ on A_2, inducing $\mathbf{R}_{2 \leftarrow 3}(1)$. We see here that the third row of A_1 is zero, so we are in **Case 1** of Lemma 2.3.3, and so we are done.

We have achieved our goal of zeroing out $A_3(2,3)$ whilst keeping the previous matrices in barcode form, making no other changes to A_3. It remains to zero out the $A_3(1,3)$ term.

1: $\mathbf{C}_{3\leftarrow 1}(-1)$ on A_3, inducing $\mathbf{R}_{1\leftarrow 3}(1)$ on A_2 giving us matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2: $\mathbf{C}_{3\leftarrow 1}(-1)$ on A_2, inducing $\mathbf{R}_{1\leftarrow 3}(1)$ on A_1. Since the third row of A_1 is zero, this operations has no impact on A_1, giving us matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so we are done.

2.4 The Space of Barcode Bases

Let $A_\bullet \in X$ be a sequence of matrices as in (2.2) arising from an arbitrary choice of basis for some persistence module $(V_\bullet, f_\bullet)$. Consider the group G from (2.4), recalling that G acts on X via change of basis. Our quest to describe all possible barcode bases for $(V_\bullet, f_\bullet)$ begins with a formula for the stabiliser of the chosen matrices $A_\bullet$ under this G-action. Namely, we seek the subgroup of G given by

$$\mathrm{Stab}(A_\bullet) := \left\{ g \in G \mid g_i \cdot A_i \cdot g_{i-1}^{-1} = A_i \text{ for all } 1 \leqslant i \leqslant \ell \right\}. \tag{2.6}$$

To describe $\mathrm{Stab}(A_\bullet)$, we employ two binary relations on the set of all intervals which might possibly arise in the barcode decomposition of $(V_\bullet, f_\bullet)$ à la Theorem 1.1.1.

Definition 2.4.1. Let $\preceq$ be the binary relation on $\left\{ [i,j] \in \mathbb{Z}^2 \mid 0 \leqslant i \leqslant j \leqslant \ell \right\}$ given by

$$[a,b] \preceq [c,d] \text{ whenever } a \leqslant c \leqslant b \leqslant d.$$

(Although this relation $\preceq$ is reflexive and anti-symmetric on its domain, it is not transitive and hence does not form a partial order.)

The second binary relation is the standard lexicographic order.

Definition 2.4.2. Let $\trianglelefteq$ be the lexicographic ordering on $\{[i,j] \in \mathbb{Z}^2 \mid 0 \leqslant i \leqslant j \leqslant \ell\}$, given by

$$[a,b] \trianglelefteq [c,d] \quad \Longleftrightarrow \quad a < c \text{ or } a = c \text{ and } b \leqslant d.$$

This yields a total order on the set of all possible bars in the interval decomposition of our persistence module $(V_\bullet, f_\bullet)$.

Remark 2.4.3. The binary relation $\preceq$ is compatible with the lexicographical order $\trianglelefteq$ in the sense that $[a,b] \preceq [c,d]$ implies $[a,b] \trianglelefteq [c,d]$.

Given a barcode basis $\mathscr{B}$, we may totally order its bars using the lexicographic order, arbitrarily ordering bars with the same start and end point. This in turn yields a natural ordering of the bases B_i. The matrix representation of such bases is unique. Indeed, the first d_{0i} basis vectors of B_i are part of $[0,i]$ bars, then the next $d_{0,i+1}$ basis vectors are those part an $[0,i+1]$ bar, and so on following the lexicographic ordering until finally the $d_{i,\ell}$ basis vectors part of an $[i,\ell]$ bar. This yields a matrix representation for which A_i is of the form

$$A_i = \begin{bmatrix} M_0 & & & \\ & M_1 & & \\ & & \ddots & \\ & & & M_i \end{bmatrix}, \tag{2.7}$$

where

$$M_j = \begin{bmatrix} & \mathrm{Id}_{d_{j,i}} & & & \\ 0 & & \mathrm{Id}_{d_{j,i+1}} & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{d_{j,\ell}} \end{bmatrix}$$

is a matrix of dimension $(\sum_{k=i}^{\ell} d_{jk}) \times (\sum_{k=i-1}^{\ell} d_{jk})$. As such, bases that have been ordered in the above way are barcode bases in the usual sense; we call these **ordered** barcode bases of $(V_\bullet, f_\bullet)$ and devote the remainder of this section to completely characterising them.

Theorem 2.4.4. *For each pair $[i,j]$ in $\{0,1,\ldots,\ell\}$ with $i \leqslant j$, let d_{ij} equal the multiplicity of $i \leqslant j$ in the barcode of $(V_\bullet, f_\bullet)$, with the understanding that $d_{ij} = 0$ whenever $[i,j]$ is not in $\mathbf{Bar}(V_\bullet, f_\bullet)$. Then there is a bijection of sets:*

$$\mathrm{Stab}(A_\bullet) \;\cong\; \prod_{[i,j]} \mathrm{GL}(d_{ij};\mathbb{F}) \times \prod_{[i_1,j_1] \precneq [i_2,j_2]} \mathrm{Mat}(d_{i_1j_1} \times d_{i_2j_2};\mathbb{F}).$$

(The induced group structure on the right side is given in Corollary 2.4.5 below)

Proof. Elements in the same orbit have isomorphic stabilisers, so without loss of generality we may assume $A_\bullet$ is given by the matrix representation of the linear maps in a ordered barcode basis. An element $g = (g_0, \dots, g_\ell)$ of G lies in $\mathrm{Stab}(A_\bullet)$ if and only if we have an equality of matrix products

$$g_i \cdot A_i = A_i \cdot g_{i-1}$$

for each $i \in \{1, \dots, \ell\}$. Set $k_i := \operatorname{rank} A_i$ and note that since A_i is in barcode form, there is a strictly increasing function $c_i : \{1, \dots, k_i\} \to \{1, \dots, n_{i-1}\}$ so that the unique nonzero entry in the p-th row of A_i lies in column $c_i(p)$. The product $g_i \cdot A_i$ on the left side of our equality has as its q-th column either the $c_i^{-1}(q)$-th column of g_i (if q lies in the image of c_i), or is identically zero otherwise. Conversely, for $p \leqslant k_i$ the matrix $A_i \cdot g_{i-1}$ on the right side has as its p-th row the $c(p)$-th row of g_{i-1}, and its rows corresponding to $p > k_i$ are identically zero.

Therefore, requiring these two products to be equal amounts to imposing three types of constraints on the entries of g_{i-1} and g_i:

1. $g_i(p, q) = 0$ whenever $p > k_i \geqslant q$.
2. $g_{i-1}(p, q) = 0$ whenever $p \in \mathrm{Img}(c_i)$ and $q \notin \mathrm{Img}(c_i)$.
3. $g_{i-1}(c_i(p), c_i(q)) = g_i(p, q)$ whenever both p and q are $\leqslant k_i$.

Recalling that $(V_\bullet, f_\bullet)$ is the persistence module represented by $A_\bullet$, we have a bijection

$$\begin{bmatrix} \text{intervals } [i,j] \text{ in the} \\ \text{barcode of } (V_\bullet, f_\bullet) \end{bmatrix} \overset{\simeq}{\longleftrightarrow} \begin{bmatrix} \text{sequences } \{p_k \mid i \leqslant k \leqslant j\} \text{ with} \\ c_k(p_k) = p_{k-1} \text{ for } i+1 \leqslant k \leqslant j \end{bmatrix}$$

Let $[i_1, j_1]$ and $[i_2, j_2]$ be two intervals in the barcode decomposition of $(V_\bullet, f_\bullet)$, and denote their corresponding sequences by $\{p_\bullet\}$ and $\{q_\bullet\}$. It follows from constraint (3) above that $g_k(p_k, q_k)$ remains constant whenever k ranges over the indices in $[i, j] := [i_1, j_1] \cap [i_2, j_2]$. In other words, we have

$$g_k(p_k, q_k) = g_{k'}(p_{k'}, q_{k'}) \text{ for all } k, k' \in [i, j]. \tag{2.8}$$

The following observation is crucial.

Claim: The entry $g_k(p_k, q_k)$ is zero for all $k \in [i, j]$ whenever $[i_1, j_1] \not\preceq [i_2, j_2]$.

To prove this claim, note that if $i_2 < i_1$ holds then $p_{i_1} > k_{i_1} \geqslant q_{i_1}$, so $g_{i_1}(p_{i_1}, q_{i_1}) = 0$ by constraint (1) above. Thus the claim extends to all k in $[i, j]$ by (2.8). Similarly, if

$j_2 < j_1$ then $p_{j_2} \in \mathrm{Img}(c_{j_2})$ but $q_{j_2} \notin \mathrm{Img}(c_{j_2})$, whence $g_{j_2}(p_{j_2}, q_{j_2}) = 0$ by constraint (2). Once again, this extends to all $k \in [i,j]$ by (2.8), and so the claim is proved.

Returning to the main argument, for each pair $[i_1, j_1] \preceq [i_2, j_2]$ of intervals in the barcode of $(V_\bullet, f_\bullet)$, we may select some $k \in [i_1, j_1] \cap [i_2, j_2]$. We denote by $g^{[i_2,j_2]}_{[i_1,j_1]}$ the submatrix of g_k spanned by all entries $g_k(p_k, q_k)$ for which $\{p_\bullet\}$ and $\{q_\bullet\}$ are sequences corresponding to intervals of type $[i_1, j_1]$ and $[i_2, j_2]$ respectively. Thus, $g^{[i_2,j_2]}_{[i_1,j_1]}$ has exactly $d_{i_1 j_1}$ rows and $d_{i_2 j_2}$ columns; and from (2.8) we know that it forms a submatrix of g_k for all k in $[i_1, j_1] \cap [i_2, j_2]$. It follows from our claim that each g_k is block upper-triangular:

$$g_k = \begin{bmatrix} g^{[0,k]}_{[0,k]} & g^{[0,k+1]}_{[0,k]} & \cdots & \cdots & g^{[0,\ell]}_{[0,k]} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & g^{[0,k+1]}_{[0,k+1]} & \cdots & \cdots & g^{[0,\ell]}_{[0,k+1]} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & g^{[k,\ell-1]}_{[k,\ell-1]} & g^{[k,\ell]}_{[k,\ell-1]} \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & g^{[k,\ell]}_{[k,\ell]} \end{bmatrix} \tag{2.9}$$

The fact that g_k must be invertible forces the diagonal blocks to be invertible, while the off-diagonal blocks remain entirely unconstrained. The map

$$\mathrm{Stab}(A_\bullet) \to \prod_{[i,j]} \mathrm{GL}(d_{ij}; \mathbb{F}) \times \prod_{[i_1,j_1] \precsim [i_2,j_2]} \mathrm{Mat}(d_{i_1 j_1} \times d_{i_2 j_2}; \mathbb{F}),$$

which sends each g to this distinguished collection of invertible $g^{[i,j]}_{[i,j]}$ and arbitrary $g^{[i_2,j_2]}_{[i_1,j_1]}$ furnishes the desired bijection. ☐

Using the block upper triangular form of the matrices g_k described in the argument above, we may immediately obtain the group structure of $\mathrm{Stab}(A_\bullet)$.

Corollary 2.4.5. *For $g, h \in \mathrm{Stab}(A_\bullet)$, we have*

$$(gh)^{[i_2,j_2]}_{[i_1,j_1]} = \sum_{[a,b]} g^{[a,b]}_{[i_1,j_1]} h^{[i_2,j_2]}_{[a,b]},$$

with the sum being indexed over intervals that satisfy $[i_1, j_1] \preceq [a,b] \preceq [i_2, j_2]$.

If $\mathbb{F}$ is the field of real or complex numbers, as a subgroup of $\prod_{i=0}^{\ell} \mathrm{GL}_{n_i}(\mathbb{F})$, the group $\mathrm{Stab}(A_\bullet)$ is a Lie group. Theorem 2.4.4 immediately allows us to obtain its dimension.

Corollary 2.4.6. *If* $\mathbb{F}$ *is the field of real or complex numbers,* $\mathrm{Stab}(A_\bullet)$ *is a Lie group of dimension*

$$\sum_{[i_1,j_1]\preceq[i_2,j_2]} d_{i_1j_1}d_{i_2j_2}$$

As stated at the start of this section, our task here is to determine the space of all possible ordered barcode bases for a given persistence module $(V_\bullet, f_\bullet)$.

We denote by $\mathbb{B} = \{\mathscr{B} = (B_i)_{0\leqslant i\leqslant \ell} \mid B_i \subset V_i \text{ is an ordered basis}\}$ the set of all possible ordered bases of $(V_\bullet, f_\bullet)$. Having fixed an initial basis $\mathscr{B} \in \mathbb{B}$, we know the group G from (2.4) acts freely and transitively on the set $\mathbb{B}$, so that any element of $\mathbb{B}$ may be expressed as $g\mathscr{B}$ for some unique $g \in G$. Thus, once we have fixed an initial basis $\mathscr{B}$, the set $\mathbb{B}$ may be identified with G. Then as subset of $\mathbb{B}$, the set of all possible ordered barcode bases of $(V_\bullet, f_\bullet)$ can be identified as a subset of the group G.

Recall, X is the set of all possible matrix-sequences as defined in (2.3). For each possible basis $\mathscr{B} \in \mathbb{B}$, we define $A(\mathscr{B})_\bullet \in X$ to be the matrix representation of the linear maps $f_\bullet$ in the chosen basis $\mathscr{B}$. This assignment prescribes the *matrix representation map*

$$A()_\bullet : \mathbb{B} \mapsto X,$$

and finding all possible ordered barcode bases for $(V_\bullet, f_\bullet)$ amounts to determining all bases $\mathscr{B} \in \mathbb{B}$ for which $A(\mathscr{B})_\bullet$ is as in (2.7). Furthermore, this map is equivariant in the sense that $A(g\mathscr{B})_\bullet = (gA(\mathscr{B}))_\bullet$ for each $g \in G$, where g acts via the basis action defined in (2.5).

The following result makes the link between the stabiliser of $A_\bullet$ and the set of ordered barcode bases of a persistence module $(V_\bullet, f_\bullet)$.

Proposition 2.4.7. *Given a persistence module* $(V_\bullet, f_\bullet)$ *together with an ordered barcode basis* $\mathscr{B}$ *with matrix representation* $A_\bullet$*, the set of all ordered barcode bases is given by the orbit* $\mathrm{Stab}(A_\bullet)\mathscr{B}$*.*

Proof. Let $\mathscr{B}'$ be another ordered barcode basis. As seen above, two ordered barcode bases have the same matrix representations (2.7), so that $A(\mathscr{B}')_\bullet = A_\bullet$. As previously stated, G acts freely and transitively on $\mathbb{B}$ so that there exists a unique $g \in G$ for which $\mathscr{B}' = g\mathscr{B}$. We then have $A_\bullet = A(\mathscr{B})_\bullet = A(g\mathscr{B}')_\bullet = gA(\mathscr{B}')_\bullet = gA_\bullet$, so that $g \in \mathrm{Stab}(A_\bullet)$ which implies $\mathscr{B}' \in \mathrm{Stab}(A_\bullet)\mathscr{B}$. ☐

As such, we may identify the set of all ordered barcode bases of a persistence module $(V_\bullet, f_\bullet)$ with $\mathrm{Stab}(A_\bullet)$, which was fully characterised in Theorem 2.4.4.

Remark 2.4.8. The automorphism group $\mathrm{Aut}_Q(M)$ of a representation M of a general quiver Q has been described, for instance in [5, Section 2.2]. It is known that $\mathrm{Aut}_Q(M)$ is a semi-direct product of the form

$$U \rtimes \prod_{i=1}^{r} \mathrm{GL}(m_i, \mathbb{F}).$$

Here $M = \bigoplus_{i=1}^{r} M_i^{m_i}$ is a decomposition of M into indecomposable summands M_i, while U is unipotent normal subgroup of $\mathrm{Aut}_Q(M)$ (see [5, Prop 2.2.1]). Viewed from this context, the main content of Theorem 2.4.4 is an explicit description of U in the special case where Q is a type-A quiver. In particular, U is generated by matrices which have the form (2.9), but with identity blocks along the diagonal. This explicit description of U in the type-A case plays a crucial role in subsequent results which appear in this chapter.

2.4.1 Automorphism groups of persistence modules

We wish to show that under certain conditions, isomorphic automorphism groups yield isomorphic persistence modules. We will require the automorphism groups to be Lie groups, and so we assume throughout this subsection that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We skip here some of the introductory definitions and results concerning Lie groups and Lie algebras for conciseness's sake. All tools we use in this subsection can be found in standard textbooks, see for e.g. [22].

Given a persistence module $(V_\bullet, f_\bullet)$, we fix bases $\mathscr{B}$ yielding matrix representation $A_\bullet$. The automorphism group $\mathrm{Aut}(V_\bullet, f_\bullet)$ may then be indentified as $\mathrm{Stab}(A_\bullet)$, which was fully characterised in Theorem 2.4.4. As seen in the proof of that theorem, $\mathrm{Aut}(V_\bullet, f_\bullet)$ can be identified as a subgroup of $\mathrm{GL}(\sum_{[i,j]} d_{ij}, \mathbb{F})$ where the d_{ij}'s are multiplicity of the bars in $(V_\bullet, f_\bullet)$'s decomposition as in 1.1.1. Assuming $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, this gives $\mathrm{Aut}(V_\bullet, f_\bullet)$ a Lie group structure.

An isomorphism $\phi_\bullet : (V_\bullet, f_\bullet) \mapsto (W\bullet, q_\bullet)$ clearly yields an isomorphism of automorphism groups given by

$$\begin{aligned} \Phi \colon \mathrm{Aut}(V_\bullet, f_\bullet) &\mapsto \mathrm{Aut}(W_\bullet, q_\bullet) \\ g_\bullet &\mapsto \phi_\bullet g_\bullet \phi_\bullet^{-1} \end{aligned}$$

Observe that $\mathrm{Aut}(V_\bullet, f_\bullet) \subset \prod_{k=0}^{\ell} \mathrm{GL}(V_k)$, giving us projection maps $p_k \colon \mathrm{Aut}(V_\bullet, f_\bullet) \to \mathrm{GL}(V_k)$.

Setting $\mathrm{Aut}(V_\bullet, f_\bullet)_k = p_k(\mathrm{Aut}(V_\bullet, f_\bullet))$ (and similarly for $\mathrm{Aut}(W_\bullet, q_\bullet)_k$), we see that $\Phi = (\Phi_k)_{0 \leqslant k \leqslant \ell}$ where Φ_k is the isomorphism

$$\begin{aligned} \Phi_k \colon \mathrm{Aut}(V_\bullet, f_\bullet)_k &\mapsto \mathrm{Aut}(W_\bullet, q_\bullet)_k \\ g_k &\mapsto \phi_k g_k \phi_k^{-1} \end{aligned}$$

Remark 2.4.9. By Theorem 2.4.4, any persistence module with a single bar has its automorphism group isomorphic to $\mathrm{GL}(1)$. Then in the stricly group theoretic sense, we may have isomorphic automorphism groups coming from non-isomorphic persistence modules.

Definition 2.4.10. A graded isomorphism between automorphism groups of persistence modules $(V_\bullet, f_\bullet)$ and $(W_\bullet, q_\bullet)$ is an isomorphism Φ such that there are isomorphisms $\Phi_k \colon \mathrm{Aut}(V_\bullet, f_\bullet)_k \simeq \mathrm{Aut}(W_\bullet, q_\bullet)_k$ for $0 \leqslant k \leqslant \ell$ such that $\Phi = (\Phi_k)_{0 \leqslant k \leqslant \ell}$.

Remark 2.4.11. The induced map Φ coming from an isomorphism $\phi_\bullet : (V_\bullet, f_\bullet) \mapsto (W\bullet, q_\bullet)$ is a graded isomorphism.

Proposition 2.4.12. *Persistence modules $(V_\bullet, f_\bullet)$ and $(W_\bullet, q_\bullet)$ are isomorphic if and only if there is a graded isomorphism between their automorphism groups.*

To prove proposition 2.4.12, we will require the following lemma.

Lemma 2.4.13. *Let $(V_\bullet, f_\bullet)$ be a persitence module that decomposes as in 3.7. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $Aut(V_\bullet, f_\bullet)$ and $\mathfrak{g}_k$ the Lie algebra of $Aut(V_\bullet, f_\bullet)_k$. Then*

$$\begin{aligned} Rad(\mathfrak{g}) &= \prod_{[i,j]} \lambda I_{d_{ij}^V} \times \prod_{[i_1,j_1] \lnsim [i_2,j_2]} \mathrm{Mat}(d_{i_1 j_1}^V \times d_{i_2 j_2}^V; \mathbb{F}) \\ Rad(\mathfrak{g}_k) &= \prod_{k \in [i,j]} \lambda I_{d_{ij}^V} \times \prod_{\substack{[i_1,j_1] \lnsim [i_2,j_2] \\ k \in [i_1,j_1] \cap [i_2,j_2]}} \mathrm{Mat}(d_{i_1 j_1}^V \times d_{i_2 j_2}^V; \mathbb{F}) = \mathfrak{p}_k(Rad(\mathfrak{g})) \end{aligned}$$

Proof. By Theorem 2.4.4,

$$G = \mathrm{Aut}(V_\bullet, f_\bullet) \simeq \prod_{[i,j]} \mathrm{GL}(d_{ij}; \mathbb{F}) \times \prod_{[i_1,j_1] \lnsim [i_2,j_2]} \mathrm{Mat}(d_{i_1 j_1} \times d_{i_2 j_2}; \mathbb{F}),$$

where each element $g \in G$ is represented as a single block upper diagonal matrix

$$g = \begin{bmatrix} g_{[0,0]}^{[0,0]} & g_{[0,0]}^{[0,1]} & \cdots & \cdots & 0 \\ 0 & g_{[0,1]}^{[0,1]} & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & g_{[\ell-1,\ell]}^{[\ell-1,\ell]} & g_{[\ell-1,\ell]}^{[\ell,\ell]} \\ 0 & 0 & \cdots & 0 & g_{[\ell,\ell]}^{[\ell,\ell]} \end{bmatrix}$$

We also know

$$G_k \simeq \prod_{k\in[i,j]} \mathrm{GL}(d_{ij};\mathbb{F}) \times \prod_{\substack{[i_1,j_1]\precsim[i_2,j_2]\\ k\in[i_1,j_1]\cap[i_2,j_2]}} \mathrm{Mat}(d_{i_1j_1}\times d_{i_2j_2};\mathbb{F})$$

where each element $g_k \in G_k$ is represented as a single block upper diagonal matrix

$$g_k = \begin{bmatrix} g^{[0,k]}_{[0,k]} & g^{[0,k+1]}_{[0,k]} & \cdots & \cdots & 0 \\ 0 & g^{[0,k+1]}_{[0,k+1]} & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & g^{[k,\ell-1]}_{[k,\ell-1]} & g^{[k,\ell]}_{[k,\ell-1]} \\ 0 & 0 & \cdots & 0 & g^{[k,\ell]}_{[k,\ell]} \end{bmatrix}$$

with the maps $p_k\colon G \mapsto G_k$ being the natural projections.

As such we have

$$\begin{aligned} \mathfrak{g} &\simeq \prod_{[i_1,j_1]\preceq[i_2,j_2]} \mathrm{Mat}(d_{i_1j_1}\times d_{i_2j_2};\mathbb{F}) \\ \mathfrak{g}_k &\simeq \prod_{\substack{[i_1,j_1]\preceq[i_2,j_2]\\ k\in[i_1,j_1]\cap[i_2,j_2]}} \mathrm{Mat}(d_{i_1j_1}\times d_{i_2j_2};\mathbb{F}) \end{aligned}$$

with elements being represented as in 2.9 in block upper triangular form, with the usual matrix lie bracket $[A,B] = AB - BA$. The projection map p_k yield maps of lie algebras $\mathfrak{p}_k\colon \mathfrak{g} \mapsto \mathfrak{g}_k$, being the natural projection map. The result then follows from similar proof to that of computing the radical of the lie algebra of upper triangular matrices. □

Proof of Propositon 2.4.12. By Theorem 1.1.1,

$$\begin{aligned} (V_\bullet, f_\bullet) &\simeq \bigoplus_{0\leqslant i\leqslant j\leqslant \ell} \mathbf{I}[i,j]_\bullet^{d^V_{ij}} \\ (W_\bullet, q_\bullet) &\simeq \bigoplus_{0\leqslant i\leqslant j\leqslant \ell} \mathbf{I}[i,j]_\bullet^{d^W_{ij}} \end{aligned}$$

for integers d^V_{ij}, d^W_{ij}. To show $(V_\bullet, f_\bullet) \simeq (W_\bullet, q_\bullet)$, we need to prove $d^V_{ij} = d^W_{ij}$ for all intervals $[i,j]$. Even given isomorphisms $\Phi_k\colon G_k \simeq H_k$, this is not obvious. There is no reason for Φ_k to induce $\mathrm{GL}(d^V_{ij}) \simeq \mathrm{GL}(d^W_{ij})$, which would give us the desired result.

Take $\mathfrak{g}$ and $\mathfrak{h}$ to be the lie algebras of the lie groups G and H. As seen in the proof of Lemma 2.4.13 we have

$$\begin{aligned}
\mathfrak{g} &\simeq \prod_{[i_1,j_1]\preceq[i_2,j_2]} \mathrm{Mat}(d^V_{i_1 j_1} \times d^V_{i_2 j_2}; \mathbb{F}) \\
\mathfrak{h} &\simeq \prod_{[i_1,j_1]\preceq[i_2,j_2]} \mathrm{Mat}(d^W_{i_1 j_1} \times d^W_{i_2 j_2}; \mathbb{F})
\end{aligned}$$

with the projection map $p_k \colon G \mapsto G_k$ inducing a map of lie algebra $\mathfrak{p}_k \colon \mathfrak{g} \mapsto \mathfrak{g}_k$ where

$$\mathfrak{g}_k = \prod_{\substack{[i_1,j_1]\preceq[i_2,j_2] \\ k\in[i_1,j_1]\cap[i_2,j_2]}} \mathrm{Mat}(d^V_{i_1 j_1} \times d^V_{i_2 j_2}; \mathbb{F})$$

is the lie algebra of $G_k = p_k(G)$.

We have isomorphisms $\psi \colon \mathfrak{g} \mapsto \mathfrak{h}$ and $\psi_k \colon \mathfrak{g}_k \mapsto \mathfrak{h}_k$ induced by the isomorphism Φ and Φ_k's, satisfying

$$\mathfrak{p}_k(\psi(x)) = \psi_k(\mathfrak{p}_k(x))$$

where the projection maps $\mathfrak{p}_k$ are taken on $\mathfrak{g}$ or $\mathfrak{h}$ accordingly.

We wish to decompose $\mathfrak{g}$ and $\mathfrak{h}$ into direct sums of simple lie algebras. To do so, we must first quotient out by the radicals.

As $\mathrm{Mat}(n, \mathbb{F})/\lambda I_n = \mathfrak{sl}(n, \mathbb{F})$, by Lemma 2.4.13 we have

$$\begin{aligned}
\mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) &= \prod_{[i,j]} \mathfrak{sl}(d^V_{ij}, \mathbb{F}) \\
\mathfrak{g}_k/\mathrm{Rad}(\mathfrak{g}) &= \prod_{k\in[i,j]} \mathfrak{sl}(d^V_{ij}, \mathbb{F})
\end{aligned}$$

with the projection maps $\mathfrak{p}_k$ naturally extending to maps $\overline{\mathfrak{p}_k} \colon \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \mapsto \mathfrak{g}_k/\mathrm{Rad}(\mathfrak{g}_k)$ since $\mathfrak{p}_k(\mathrm{Rad}(\mathfrak{g})) = \mathrm{Rad}(\mathfrak{g}_k)$. Similarly, as $\psi(\mathrm{Rad}(\mathfrak{g})) = \mathrm{Rad}(\mathfrak{h})$ and $\psi_k(\mathrm{Rad}(\mathfrak{g}_k)) = \mathrm{Rad}(\mathfrak{h}_k)$, we have natural isomorphisms

$$\begin{aligned}
\overline{\psi} &\colon \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \simeq \mathfrak{g}/\mathrm{Rad}(\mathfrak{h}) \\
\overline{\psi_k} &\colon \mathfrak{g}_k/\mathrm{Rad}(\mathfrak{g}_k) \simeq \mathfrak{h}_k/\mathrm{Rad}(\mathfrak{h}_k)
\end{aligned}$$

which as before commutes with the projection maps

$$\overline{\mathfrak{p}_k}(\overline{\psi}(x)) = \overline{\psi_k}(\overline{\mathfrak{p}_k}(x)) \tag{2.10}$$

Identifying $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ as $\prod_{[i,j]} \mathfrak{sl}(d_{ij}^V, \mathbb{F})$, we see that

$$\mathfrak{p}_k(\mathfrak{sl}(d_{ij}^V, \mathbb{F})) = \begin{cases} \mathfrak{sl}(d_{ij}^V, \mathbb{F}) \subset \mathfrak{g}_k/\mathrm{Rad}(\mathfrak{g}_k) & \text{for } i \leqslant k \leqslant j \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\prod_{[i,j]} \mathfrak{sl}(d_{ij}^V, \mathbb{F}) = \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \simeq \mathfrak{h}/\mathrm{Rad}(\mathfrak{h}) = \prod_{[i,j]} \mathfrak{sl}(d_{ij}^W, \mathbb{F})$$

Since $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, the lie algebras $\mathfrak{sl}(n, \mathbb{F})$ are simple. Then we know that

$$\overline{\psi}(\mathfrak{sl}(d_{ij}^V, \mathbb{F})) = \mathfrak{sl}(d_{i'j'}^W, \mathbb{F})$$

for some $0 \leqslant i' \leqslant j' \leqslant k$, giving us $d_{ij}^V = d_{i'j'}^W$. It remains to show $[i,j] = [i',j']$.

We have

$$\overline{\mathfrak{p}_k}(\overline{\psi}(\mathfrak{sl}(d_{ij}^V, \mathbb{F})) = \begin{cases} \mathfrak{sl}(d_{i'j'}^W, \mathbb{F}) \subset \mathfrak{h}_k/\mathrm{Rad}(\mathfrak{h}_k) & \text{for } i' \leqslant k \leqslant j' \\ 0 & \text{otherwise} \end{cases}$$

By 2.10, $\overline{\mathfrak{p}_k}(\overline{\psi}(\mathfrak{sl}(d_{ij}^V, \mathbb{F}))) = \overline{\psi_k}(\overline{\mathfrak{p}_k}(\mathfrak{sl}(d_{ij}^V, \mathbb{F})))$, which is none-zero if and only if $i \leqslant k \leqslant j$ since $\overline{\psi}_k$ is an isormorphism. Then we must have $[i,j] = [i',j']$. ☐

Remark 2.4.14. It is natural to wonder whether the above is not a simple consequence of more general results concerning automorphism groups of quiver representations. We address this in a later section through Remark 2.5.7.

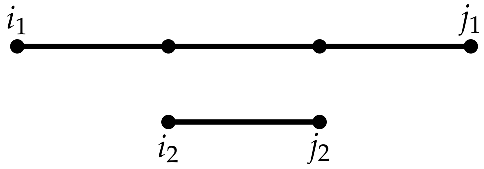
2.5 Simplifying Maps of Persistence Modules

We now shift our interest to maps of persistence modules $\phi_\bullet : (V_\bullet, f_\bullet) \mapsto (W_\bullet, h_\bullet)$, where $V_\bullet$ and $W_\bullet$ are persistence modules of length $\ell + 1$. We recall that each such $\phi_\bullet$ is a collection of linear maps $\phi_i : V_i \mapsto W_i$ satisfying $\phi_i \circ f_i = h_i \circ \phi_{i-1}$. In other words, the following diagram commutes:

$$\begin{tikzcd}
V_0 \arrow[r, "f_1"] \arrow[d, "\phi_0"'] & V_1 \arrow[r, "f_2"] \arrow[d, "\phi_1"'] & \cdots \arrow[r, "f_{\ell-1}"] & V_{\ell-1} \arrow[r, "f_\ell"] \arrow[d, "\phi_{\ell-1}"'] & V_\ell \arrow[d, "\phi_\ell"'] \\
W_0 \arrow[r, "h_1"'] & W_1 \arrow[r, "h_2"'] & \cdots \arrow[r, "h_{\ell-1}"'] & W_{\ell-1} \arrow[r, "h_\ell"'] & W_\ell
\end{tikzcd}$$

Our objective here is to show that if neither $V_\bullet$ nor $W_\bullet$ admits a pair of strictly nested bars in its barcode, then there exist barcode bases of $V_\bullet$ and $W_\bullet$ in which ϕ induces a *partial matching* of the bars. To make this precise, we first describe the nestedness condition.

Definition 2.5.1. A bar $[i_2, j_2]$ is strictly nested in a bar $[i_1, j_1]$, denoted $[i_2, j_2] \subset [i_1, j_1]$ if $i_1 < i_2 \leqslant j_2 < j_1$. This is best represented as



Note that two intersecting bars are strictly nested if and only if they are not related by $\preceq$.

Before stating the main result, we remark that the data of our map $\phi_\bullet$ can be interpreted as a representation of the *rectangle* quiver of length $\ell + 1$:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Such representations have been called **ladder** persistence modules in the literature [16]. When treating ϕ as a ladder persistence module, we will denote it $(V_\bullet, W_\bullet, \phi_\bullet)$. We may therefore seek to decompose $\phi_\bullet$ into a direct sum of indecomposable ladder persistence modules.

Definition 2.5.2. Three families of ladder persistence modules are defined below:

1. Given intervals $[i_1, j_1] \preceq [i_2, j_2]$, denote by $\mathbf{R}^{[i_2,j_2]}_{[i_1,j_1]\bullet}$ the ladder persistence module where $V_\bullet$ is the interval module $\mathbf{I}[i_2, j_2]_\bullet$ while $W_\bullet$ is the interval module $\mathbf{I}[i_1, j_1]_\bullet$; all vertical maps are 1's whenever possible and 0 otherwise:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \\
\mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

2. Given an interval $[i, j]$, let $\mathbf{I}^+[i, j]_\bullet$ denote the ladder persistence module for which $V_\bullet$ is $\mathbf{I}[i, j]_\bullet$ and $W_\bullet$ is 0, with all vertical maps necessarily being 0:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[d] & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

3. And finally, given an interval $[i,j]$, let $\mathbf{I}^-[i,j]_\bullet$ be the ladder persistence module for which $V_\bullet$ is trivial while $W_\bullet$ is $\mathbf{I}[i,j]_\bullet$, so once again all vertical maps are 0:

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \\
 & & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

It is readily seen that these three families of ladder persistence modules are mutually non-isomorphic and indecomposable. Therefore, by the Krull-Schmidt theorem, if $\phi_\bullet$ were to decompose as a direct sum of modules sourced from these three families, then such a decomposition would be unique. From such a decomposition, we can obtain the desired partial matching of the source and target bars: the presence of each $\mathbf{R}^{[i_2,j_2]}_{[i_1,j_1]_\bullet}$ summand matches a bar $[i_2,j_2]$ of $V_\bullet$ to a bar $[i_1,j_1]$ of $W_\bullet$, whilst the existence of $\mathbf{I}^+[i,j]_\bullet$ (or $\mathbf{I}^-[i,j]_\bullet$) summands reveals bars $[i,j]$ in $V_\bullet$ (or $W_\bullet$) that are matched to 0. With this in mind, here is the main result of this section.

Theorem 2.5.3. *Let $(V_\bullet, W_\bullet, \phi_\bullet)$ be a ladder persistence module of length $\ell+1$ where neither $V_\bullet$ nor $W_\bullet$ admit a pair of strictly nested bars. Then there are integers $r^{[i_2,j_2]}_{[i_1,j_1]}$ and $d^{\pm}_{ij} \in \mathbb{N}$ for which:*

$$(V_\bullet, W_\bullet, \phi_\bullet) \simeq \bigoplus_{[i_1,j_1] \preceq [i_2,j_2]} \left(\boldsymbol{R}^{[i_2,j_2]}_{[i_1,j_1]_\bullet}\right)^{r^{[i_2,j_2]}_{[i_1,j_1]}} \oplus \bigoplus_{i \leqslant j} \left(\mathbf{I}^+[i,j]_\bullet\right)^{d^+_{ij}} \oplus \bigoplus_{i \leqslant j} \left(\mathbf{I}^-[i,j]_\bullet\right)^{d^-_{ij}}$$

(In Example 2.5.4 below we show that the assumption precluding nested bars is necessary.)

Proof. The argument proceeds along three basic steps.

Step 1: Representing ϕ. Consider ordered barcode bases

$$\mathscr{B}_V := \left\{B_{V,i} \subset V_i \mid 0 \leqslant i \leqslant \ell\right\} \text{ and } \mathscr{B}_W := \left\{B_{W,i} \subset W_i \mid 0 \leqslant i \leqslant \ell\right\}$$

of $V_\bullet$ and $W_\bullet$. Let $d^V_{i,j}$ and $d^W_{i,j}$ be the multiplicity of $[i,j]$ bars in the barcodes of $V_\bullet$ and $W_\bullet$. We denote by $b_\bullet$ the matrix representations of the maps $\phi_\bullet$ in these chosen bases. As in the proof Theorem 2.4.4, let $[i_1,j_1]$ and $[i_2,j_2]$ be two intervals in the barcode decomposition of $(V_\bullet, f_\bullet)$, and denote their corresponding sequences by $\{p_\bullet\}$ and $\{q_\bullet\}$. Using the commuting relations $\phi_k \circ f_k = h_k \circ \phi_{k-1}$, we have $\phi_k(p_k, q_k) = \phi_{k'}(p_{k'}, q_{k'})$, for all $k, k' \in [i,j] = [i_1,j_1] \cap [i_2,j_2]$, with this coefficient being zero unless $[i_1,j_1] \preceq [i_2,j_2]$. Assuming this order relation holds, define $X^{[i_2,j_2]}_{[i_1,j_1]}$, to be the submatrix of b_i obtained by taking the $d^V_{i_2,j_2}$ columns corresponding to basis vectors part of an $[i_1, j_1]$ bar of V, and the $d^W_{i_1,j_1}$ rows corresponding to basis to basis vectors part of an $[i_2, j_2]$ bar of W. From the above observation, $X^{[i_2,j_2]}_{[i_1,j_1]}$ is a submatrix of $b_i, b_{i+1}, \dots b_j$, and is of dimension $d^W_{i_1,j_1} \times d^V_{i_2,j_2}$. Thus, the matrices $b_\bullet$ are completely determined by the matrices $X^{[i_2,j_2]}_{[i_1,j_1]}$ and may therefore be represented as a single block matrix

$$\begin{bmatrix} X^{[0,0]}_{[0,0]} & X^{[0,1]}_{[0,0]} & \dots & X^{[0,\ell]}_{[0,0]} & 0 & \dots & 0 \\ 0 & X^{[0,1]}_{[0,1]} & \dots & \dots & X^{[1,\ell]}_{[0,1]} & \dots & 0 \\ \vdots & \vdots & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & X^{[\ell,\ell-1]}_{[\ell,\ell-1]} & X^{[\ell,\ell]}_{[\ell,\ell-1]} \\ 0 & 0 & & & & 0 & X^{[\ell,\ell]}_{[\ell,\ell]} \end{bmatrix}$$

This is a matrix of size $(\sum d^W_{i,j}) \times (\sum d^V_{i,j})$.

Step 2: Admissible Operations. Given two ordered barcode bases $\mathscr{B}_V$ and $\mathscr{B}_w$ we may define a block matrix as above, which we denote $b(\mathscr{B}_V, \mathscr{B}_W)$. By Proposition 2.4.7, the set of ordered all barcode bases of $(V_\bullet, f_\bullet)$ coincides precisely with the orbit $\text{Stab}(A(\mathscr{B}_1)_\bullet)\mathscr{B}_1$, where $\mathscr{B}_1$ is an ordered barcode basis. Then given $(h, k) \in \text{Stab}(A(\mathscr{B}_V)_\bullet) \times \text{Stab}(A(\mathscr{B}_W)_\bullet)$, we may consider $b(h\mathscr{B}_V, k\mathscr{B}_W)$. As seen in the proof of Theorem 2.4.4, we are able to completely characterise an element $h \in \text{Stab}(A(\mathscr{B}_V)_\bullet)$ in terms of the submatrices $h^{[i_2,j_2]}_{[i_1,j_1]}$, so that h may be represented as a single block matrix

$$\begin{bmatrix} h^{[0,0]}_{[0,0]} & h^{[0,1]}_{[0,0]} & \dots & h^{[0,\ell]}_{[0,0]} & 0 & \dots & 0 \\ 0 & h^{[0,1]}_{[0,1]} & \dots & \dots & h^{[1,\ell]}_{[0,1]} & \dots & 0 \\ \vdots & \vdots & \ddots & & & & \\ & & & \ddots & & & \\ & & & & & h^{[\ell,\ell-1]}_{[\ell,\ell-1]} & h^{[\ell,\ell]}_{[\ell,\ell-1]} \\ 0 & 0 & & & & 0 & h^{[\ell,\ell]}_{[\ell,\ell]} \end{bmatrix}$$

of size $(\sum d^V_{i,j}) \times (\sum d^V_{i,j})$. The same is true for $k \in \text{Stab}(A(\mathscr{B}_W)_\bullet)$, whence $b(h\mathscr{B}_V, k\mathscr{B}_W)$ equals the (matrix) product $k \cdot b(\mathscr{B}_V, \mathscr{B}_W) \cdot h^{-1}$. Thus, we are only allowed to perform the following legal operations on $b(\mathscr{B}_V, \mathscr{B}_W)$:

1. Using invertible block diagonal elements $h^{[i_1,j_1]}_{[i_1,j_1]}$ and $k^{[i_1,j_1]}_{[i_1,j_1]}$, we may perform any operations between column and rows corresponding to $[i_1, j_1]$ bars in our block matrix.

2. Using the block matrices $h^{[i_2,j_2]}_{[i_1,j_1]}$, we see we may modify columns corresponding to $[i_2, j_2]$ bars using columns corresponding to $[i_1, j_1]$ in our block matrix, whenever $[i_1, j_1] \preceq [i_2, j_2]$.

3. Using the block matrices $k^{[i_2,j_2]}_{[i_1,j_1]}$, we see we may modify rows corresponding to $[i_1, j_1]$ bars using columns corresponding to $[i_2, j_2]$ in our block matrix, whenever $[i_1, j_1] \preceq [i_2, j_2]$.

Step 3: Matrix Reduction. We wish to find ordered barcode bases for which the corresponding matrix $b(\mathscr{B}_V, \mathscr{B}_W)$ admits at most one non-zero term 1 in each row and column. From this form, the desired decomposition can be easily extracted: every 1 in a row corresponding to an $[i_1, j_1]$ bar and column corresponding to an $[i_2, j_2]$ bar (with $[i_1, j_1] \preceq [i_2, j_2]$) corresponds to an $\mathbf{R}^{[i_2,j_2]}_{[i_1,j_1]\bullet}$ summand. Similarly, zero rows and columns then yield $\mathbf{I}^-[i, j]_\bullet$ and $\mathbf{I}^+[i, j]_\bullet$ summands respectively.

Let $\mathscr{B}_V$, $\mathscr{B}_W$ be ordered barcode bases of $V_\bullet$, $W_\bullet$ so that

$$b(\mathscr{B}_V, \mathscr{B}_W) = \begin{bmatrix} X^{[0,0]}_{[0,0]} & X^{[0,1]}_{[0,0]} & \cdots & X^{[0,\ell]}_{[0,0]} & 0 & \cdots & 0 \\ 0 & X^{[0,1]}_{[0,1]} & \cdots & \cdots & X^{[1,\ell]}_{[0,1]} & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & X^{[\ell,\ell-1]}_{[\ell,\ell-1]} & X^{[\ell,\ell]}_{[\ell,\ell-1]} \\ 0 & 0 & & & & 0 & X^{[\ell,\ell]}_{[\ell,\ell]} \end{bmatrix}$$

We seek stabiliser elements (h, k) so that $B(h\mathscr{B}_V, k\mathscr{B}_W) = kB(\mathscr{B}_V, \mathscr{B}_W)h^{-1}$ is in barcode form. To this end, we perform basis changes using legal operations of type (1), (2) and (3). We process the column-blocks of this matrix from left to right, in each case starting from the diagonal block and working our way upwards. We will denote each treated matrix that has been put in adequate form by $P^{[i_2,j_2]}_{[i_1,j_1]}$.

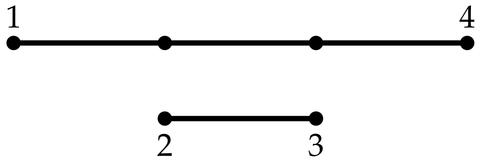
That is, we start with $X^{[0,0]}_{[0,0]}$, putting it in Smith normal form using basis changes $h^{[0,0]}_{[0,0]}$ and $k^{[0,0]}_{[0,0]}$. Now assume we wish to treat $X^{[i_2,j_2]}_{[i_1,j_1]}$, where all matrices below it and to its left have been treated. That is, we have

$$\begin{bmatrix} 0 & \cdots & P^{[i_1,j_1]}_{[i_1,j_1]} & \cdots & X^{[i_2,j_2]}_{[i_1,j_1]} \\ & & & & \vdots \\ & & & & P^{[i_2,j_2]}_{[i_2,\ell]} \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{bmatrix}$$

Given non-intersecting bars $[a,b] \trianglelefteq [c,d]$, we have either $[a,b] \preceq [c,d]$ or $[a,b] \subset [c,d]$. Then by hypothesis, given $P^{[a,b]}_{[i_1,j_1]}$ to the left of $X^{[i_2,j_2]}_{[i_1,j_1]}$, we have $[a,b] \preceq [i_2,j_2]$. So we may zero out rows of $X^{[i_2,j_2]}_{[i_1,j_1]}$ in which $P^{[a,b]}_{[i_1,j_1]}$ has 1's using operations of type (2). Similarly, given $P^{[i_2,j_2]}_{[a,b]}$ below $X^{[i_2,j_2]}_{[i_1,j_1]}$, we may zero out corresponding columns using operations of type (3). The non-zero columns of the resulting matrix $\tilde{X}^{[i_2,j_2]}_{[i_1,j_1]}$ have 0's below them, and non-zero rows have 0's to their left.

Then using basis changes $h^{[i_2,j_2]}_{[i_2,j_2]}$ and $k^{[i_1,j_1]}_{[i_1,j_1]}$, we may put A in Smith normal form, without adding non-zero terms in any rows below and column to its left, preserving the desired structure. □

Example 2.5.4. We illustrate the difficulties imposed by strictly nested bars in the context of Theorem 2.5.3. Consider the map $\phi_\bullet : V_\bullet \mapsto W_\bullet$ where $V_\bullet$ has barcode



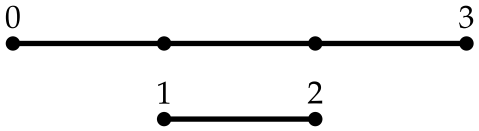
and $W_\bullet$ has barcode

with associated ordered barcode bases $\mathscr{B}_V, \mathscr{B}_W$, and $\phi_\bullet$ is given by the block matrix representation

$$b(\mathscr{B}_V, \mathscr{B}_W) = \begin{array}{cc} \begin{array}{cc} [1,4] & [2,3] \end{array} & \\ \begin{bmatrix} 1 & \quad\quad 1 \end{bmatrix} & [0,3] \end{array}$$

Since $[2,3] \subset [1,4]$, stabiliser changes of basis for $\mathscr{B}_V$ are invertible diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, and stabiliser changes of basis for $\mathscr{B}_W$ are invertible matrices $\begin{bmatrix} c \end{bmatrix}$. As such, $(V_\bullet, W_\bullet, \phi_\bullet)$ which will never be expressible as a direct sum of modules as in Theorem 2.5.3, since there is no change of basis which will allow us to transform this matrix into either $\begin{bmatrix} 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \end{bmatrix}$. The same is true if $V_\bullet$ has barcode

and $W_\bullet$ has barcode



with associated ordered barcode bases $\mathscr{B}_V$, $\mathscr{B}_W$, and $\phi_\bullet$ is given by the block matrix representation

$$\begin{array}{cc} [1,4] & \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{matrix} [0,3] \\ [1,2] \end{matrix} \end{array}$$

Thus, if the nested condition is violated on either $V_\bullet$ or $W_\bullet$, a decomposition as in Theorem 2.5.3 is not always possible.

Remark 2.5.5. The non-nestedness hypothesis on the bars of $V_\bullet$ and $W_\bullet$ from Theorem 2.5.3 is quite restrictive. There are, however, several scenarios of interest where it is satisfied:

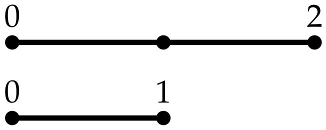
1. The 0-th persistent homology of a point cloud satisfies the hypothesis because all bars have left endpoint 0.
2. Similarly, the 1st persistent homology of a filtered graph admits no strictly nested bars because all bars have right endpoint ∞.
3. More generally, the n-th persistent homology of an n-dimensional filtered complex satisfies the non-nestedness criterion. One may consider, for instance, the *Linial-Meshulam* model of random simplicial complexes [41]. A random simplicial complex chosen from this model on a given vertex set consists of every possible simplex of dimension $< n$, with candidate n-simplices being included independently with uniform probability $p \in [0,1]$. The n-th persistent homology of any filtration of such a random complex satisfies the hypothesis.

Theorem 2.5.3 applies in all such cases.

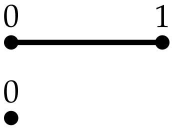
Remark 2.5.6. Theorem 2.5.3 does not generalise to wider rectangles, i.e. rectangles with more than two horizontal rows. Consider for instance the representation of the rectangle

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

given by persistence modules $V_\bullet$ with barcode



$W_\bullet$ with barcode



and $Z_\bullet$ with barcode

with horizontals maps $\phi_\bullet \colon V_\bullet \mapsto W_\bullet$ and $\psi_\bullet \colon W_\bullet \mapsto Z_\bullet$ given by block matrix representations

$$\begin{array}{cc} [0,1] & [0,2] \\ \end{array} \qquad \begin{array}{c} \end{array}$$
$$\begin{array}{ccc} [0,1] & [0,2] & \\ \left[\begin{matrix} 0 \\ 1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 0 \end{matrix}\right] & \begin{matrix} [0,0] \\ [0,1] \end{matrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} [0,0] & [0,1] & \\ [\ 1 & 1\ ] & [0,0] \end{array}$$

in the corresponding barcode bases $\mathscr{B}_V$, $\mathscr{B}_W$ and $\mathscr{B}_W$. Observing that the stabiliser changes of bases for $\mathscr{B}_V$ and $\mathscr{B}_W$ are invertible upper triangular matrices $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ and stabiliser changes of basis for $\mathscr{B}_W$ are invertible matrices $\begin{bmatrix} d \end{bmatrix}$ we are able to compute all possible matrix representations of the vertical maps in barcode bases. Doing so, we see that we will never be able to simultaneously have these matrices in reduced form. This shows we cannot expect decompositions a la Theorem 2.5.3

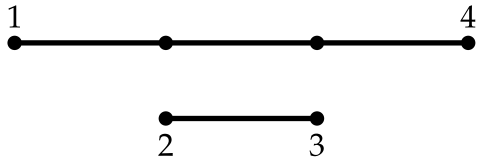
for representations of rectangles with more than 1 row, even with the no nestedness assumption satisfied.

The above result also illustrates the non-functoriality of Theorem 2.5.3 when looking at the induced partial matchings on barcodes. Indeed, ϕ induces a partial matching $[0,2]_{V_\bullet} \mapsto [0,0]_{W_\bullet}, [0,1]_{V_\bullet} \mapsto [0,1]_{W_\bullet}$ and ψ induces a partial matching $[0,0]_{W_\bullet} \mapsto [0,0]_{W_\bullet}$. The matrix representation of $\psi \circ \phi$ in the above barcode bases gives

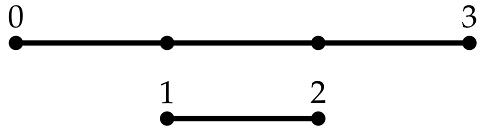
$$\begin{array}{cc} [0,1] & [0,2] & \\ \left[\ 1\right. & \left. 1\ \right] & [0,0] \end{array}$$

which induces a partial matching $[0,1]_{V_\bullet} \mapsto [0,0]_{W_\bullet}$. This differs from the composition of the partial matchings induced by ϕ and ψ.

Remark 2.5.7. We address here Remark 2.4.14 and show Proposition 2.4.12 concerning automorphism groups of persistence does not generalise to arbitrary quivers. Consider the length 4 ladder persistence module L_α given by persistence modules $V_\bullet^\alpha$ with barcode



and $W_\bullet^\alpha$ with barcode



with horizontal map $\phi_\bullet^\alpha \colon V_\bullet^\alpha \mapsto W_\bullet^\alpha$ given by block matrix representation

$$\begin{array}{c} \begin{array}{cc} [1,4] & [2,3] \end{array} \\ \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix} \begin{array}{c} [0,3] \\ [1,2] \end{array} \end{array} .$$

As seen in Example 2.5.4, due to the nestedness of both barcodes, we have stabliser changes of basis for both $V_\bullet^\alpha$ and $W_\bullet^\alpha$ given by invertible diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then observe that the possible matrix representations of $\phi_\bullet^\alpha$ over all barcode bases is given by

$$\left(\begin{array}{ccc} [1,4] & [2,3] & \\ a & 0 & [1,4] \\ 0 & b & [2,3] \end{array}\right) \cdot \left(\begin{array}{ccc} [1,4] & [2,3] & \\ 1 & \alpha & [0,3] \\ 1 & 1 & [1,2] \end{array}\right) \cdot \left(\begin{array}{ccc} [0,3] & [1,2] & \\ c & 0 & [0,3] \\ 0 & d & [1,2] \end{array}\right)^{-1}$$

$$= \begin{array}{ccc} [1,4] & [2,3] & \\ \frac{c}{d} & \frac{a}{d}\alpha & [0,3] \\ \frac{b}{c} & \frac{b}{d} & [1,2] \end{array},$$

for all non-zero $a, b, c, d \in \mathbb{F}$. Then if $L_\alpha \simeq L_\beta$ there are stabiliser changes of basis for $V^\alpha_\bullet$ and $W^\alpha_\bullet$ such that

$$\begin{array}{ccc} [1,4] & [2,3] & \\ \frac{c}{d} & \frac{a}{d}\alpha & [0,3] \\ \frac{b}{c} & \frac{b}{d} & [1,2] \end{array} = \begin{array}{ccc} [1,4] & [2,3] & \\ 1 & \beta & [0,3] \\ 1 & 1 & [1,2] \end{array} \tag{2.11}$$

which implies $\alpha = \beta$. Then for $\alpha \neq \beta$, we have $L_\alpha \not\simeq L_\beta$ which shows that over infinite fields $\mathbb{F}$ there are infinitely many non-isomorphic ladder persistence modules, when the length of the ladder is greater than 5. Note that this is a similar example to those exhibited in [7].

By equation 2.11, the automorphism group of L_α can be characterised by stabiliser changes of bases $\begin{array}{ccc} [1,4] & [2,3] & \\ a & 0 & [1,4] \\ 0 & a & [2,3] \end{array}$ for $V^\alpha_\bullet$ and $\begin{array}{ccc} [1,4] & [2,3] & \\ a & 0 & [1,4] \\ 0 & a & [2,3] \end{array}$ for $W^\alpha_\bullet$ with $a \neq 0 \in \mathbb{F}$, so that for $\alpha \neq \beta$, L_α and L_β are non-isomorphic modules with graded isomorphic automorphism groups (Definition 2.4.10), which show Proposition 2.4.12 does not generalise to all quivers.

2.6 Zigzag Modules

In this section, we wish to generalise the previous work to representations of any type A quiver, in other words zizag persistence modules. The nomenclature we adpot here is extracted from [8]. A zizag module is given by a sequence

$$V_0 \xleftrightarrow{p_1} V_1 \xleftrightarrow{p_2} \cdots \xleftrightarrow{p_{\ell-1}} V_{\ell-1} \xleftrightarrow{p_\ell} V_\ell$$

where each $\xleftrightarrow{p_i}$ is either a forward map $\xrightarrow{f_i}$ or a backwards map $\xleftarrow{q_i}$. The direction of the arrows define the **type** τ of the zizag module, which is the direction of the arrows of the underlying type A quiver. For example,

$$\begin{tikzcd} V_0 \arrow[r, "f_1"] & V_1 & V_2 \arrow[l, "q_2"'] \end{tikzcd}$$

has type $\tau = fq$. We denote zizag modules as $(V_\bullet, p_\bullet, \tau)$.

The **interval module** of type τ corresponding to a pair of non-negative integers $i \leqslant j$ is the zizag module $\mathbf{I}_\tau[i,j]_\bullet$ given by

$$0 \leftrightarrow \cdots \leftrightarrow 0 \leftrightarrow \mathbb{F} \leftrightarrow \cdots \leftrightarrow \mathbb{F} \leftrightarrow 0 \leftrightarrow \cdots \leftrightarrow 0,$$

where the contiguous string of $\mathbb{F}$'s spans $\{i, i+1, \ldots, j-1, j\}$, all intermediate $\mathbb{F} \longleftrightarrow \mathbb{F}$ maps are identities in the direction depending on τ, and all other vector spaces are trivial. Zizag modules also decompose into interval modules: this follows from the main result of [19], and is established more directly in [8]. In particular, every zigzag module $(V_\bullet, p_\bullet, \tau)$ is isomorphic to a direct sum

$$(V_\bullet, p_\bullet, \tau) \cong \bigoplus_{0 \leqslant i \leqslant j \leqslant \ell} (\mathbf{I}_\tau[i,j])^{d_{ij}},$$

for some uniquely determined $d_{ij} \in \mathbb{N}$.

Definition 2.6.1. An $m \times n$ matrix A of rank r is in **reversed barcode form** if there exists a strictly increasing function $c : \{m-r+1, \ldots, m\} \to \{1, \ldots n\}$ so that

$$A_{ij} = \begin{cases} 1 & \text{if } j = c(i), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a matrix is in reversed barcode form whenever its entries lie in $\{0,1\}$, with at most one non-zero term in each row and column, *and* satisfies the following additional requirement: the r non-zero terms appear in the last r columns, with increasing row order. (We warn the reader that if a matrix A is in barcode form, then its transpose will not in general be in reversed barcode form; however, the off-diagonal transpose of A will be in reversed barcode form).

Definition 2.6.2. A basis family $\mathscr{B}$ for a zizag module is called an **barcode basis** for $(V_\bullet, p_\bullet, \tau)$ if all of the A_i for which $V_{i-1} \xrightarrow{f_i} V_i$ are in barcode form, and all the A_i for which $V_{i-1} \xleftarrow{q_i} V_i$ are in reversed barcode form. The corresponding matrices $A_\bullet$ are then said to be in **zizag barcode form**.

We note that this notion coincides with Definition 2.2.1 whenever all maps in sight are forward. Barcode bases for zizag modules are the natural bases arising from their decomposition into interval modules.

2.6.1 Algorithm for Zizag Modules

We wish to generalise our algorithm from Section 2.3 to treat the case of zizag modules. Instead of presenting concrete algorithms, we will adapt our proof of Proposition 2.3.4 in Section 2.3 to the more general setting. Implementing the algorithm is then achieved in similar fashion to what was done in Section 2.3 for classical persistence modules. To this end, we fix a zizag module $(V_\bullet, p_\bullet, \tau)$ expressed as a sequence of matrices $A_\bullet$ with respect to an arbitrary (i.e., not necessarily barcode) basis family $\mathscr{B}$:

$$\mathbb{F}^{n_0} \xleftrightarrow{A_1} \mathbb{F}^{n_1} \xleftrightarrow{A_2} \cdots \xleftrightarrow{A_{\ell-1}} \mathbb{F}^{n_{\ell-1}} \xleftrightarrow{A_\ell} \mathbb{F}^{n_\ell} \tag{2.12}$$

The matrices A_i are of dimension either $n_i \times n_{i-1}$ or $n_{i-1} \times n_i$, depending on the type τ of $V_\bullet$. Analogously to Section 2.2, we may define X_τ to be the set of all the possible matrix-sequences $A_\bullet$ which can arise in (2.12). It is a (strict) subset

$$X_\tau \subset \prod_{i=1}^{\ell} Y_i, \quad \text{where} \quad Y_i = \begin{cases} \mathrm{Mat}\,(n_i \times n_{i-1}; \mathbb{F}) & \text{if} \quad i-1 \longrightarrow i \\ \mathrm{Mat}\,(n_{i-1} \times n_i; \mathbb{F}) & \text{if} \quad i-1 \longleftarrow i \end{cases}$$

Changes of bases for zigzag modules are obtained by a new action of the group G from (2.4) on the set X_τ; the major difference between this action and the one treated in Section 2.3 is that the (direction of) conjugation now depends on the type τ. Explicitly, if $V_{i-1} \xrightarrow{f_i} V_i$ points forward, then it gets sent to $g_i \circ f_i \circ g_{i-1}^{-1}$ as before; and conversely, if $V_{i-1} \xleftarrow{q_i} V_i$ points backwards, then it is sent to $g_{i-1} \circ q_i \circ g_i^{-1}$. For this action, we obtain the following analogue of Lemma 2.3.3.

Lemma 2.6.3. *Assume that the first $\ell - 1$ matrices of $\{A_i \mid 1 \leqslant i < \ell\}$ from (2.12) are in zizag barcode form.*

- *If $f_\ell : V_{\ell-1} \to V_\ell$, and the last matrix A_ℓ has a pivot in the (r, q) position*[3]*, together with a nonzero entry $\alpha := A_\ell(r, p)$ in the same row r but some other column $p > q$, then there exists $g \in G$ with $g_\ell = \mathrm{Id}$ so that $(gA)_\bullet$ equals $A_\bullet$ except A_ℓ where the α entry is replaced by zero.*

- *If $q_\ell : V_{\ell-1} \leftarrow V_\ell$, and the last matrix A_ℓ has a pivot in the (r, q) position, together with a nonzero entry $\alpha := A_\ell(s, q)$ in the same column q but some other row $s < r$, then there exists $g \in G$ with $g_\ell = \mathrm{Id}$ so that $(gA)_\bullet$ equals $A_\bullet$ except A_ℓ where the α entry is replaced by zero.*

---

[3] i.e., we have $A_\ell(r, q) = 1$ while all other entries in the q-th column of A_ℓ are zero.

Proof. The proof is done by induction, in similar fashion to that of Lemma 2.3.3. The case $\ell = 1$ remains trivial. To prove the induction, 4 cases should now be considered.

Case 1 :$V_{\ell-2} \to V_{\ell-1} \to V_{\ell}$

This case is precisely that of Lemma 2.3.3, and the proof remains, the same.

Case 2: $V_{\ell-2} \leftarrow V_{\ell-1} \to V_{\ell}$

Performing $\mathbf{C}_{p\leftarrow q}(-\alpha)$ on A_{ℓ} induces the same operation on $A_{\ell-1}$. If the q-th column of $A_{\ell-1}$ is identically zero, this operation leaves $A_{\ell-1}$ unchanged. Otherwise, since $p > q$ and $A_{\ell-1}$ is in reversed barcode form, if the q-th column isn't zero then the p-th column is also non-zero. Then the resulting matrix $A'_{\ell-1}$ has the form

$$A'_{\ell-1} = \begin{array}{cc} \begin{array}{ccccccc} & q & & & & p & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & \cdots & 0 & -\alpha & 0 \\ & 0 & & & & 0 & \\ & \vdots & & & & \vdots & \\ & 0 & & & & 0 & \\ 0 & 0 & 0 & \cdots & & 1 & 0 \end{array}\right] & \begin{array}{c} c \\ \\ \\ \\ d \end{array} \end{array}$$

where we may again use induction.

Case 3: $V_{\ell-2} \to V_{\ell-1} \leftarrow V_{\ell}$

We must perform the row operation $\mathbf{R}_{s\leftarrow r}(-\alpha)$ on A_{ℓ}, inducing the same row operation on $A_{\ell-1}$. Again, if the r-th row of $A_{\ell-1}$ is identically zero, we are done. Otherwise, since A_{ℓ} is in barcode form and $s < r$, if the r-th row isn't zero then the s-th row isn't zero, and so the resulting matrix $A'_{\ell-1}$ has the form $A'_{\ell-1}$ has the form

$$A'_{\ell-1} = \begin{array}{cc} \begin{array}{ccccccc} & c & & & & d & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & \cdots & 0 & -\alpha & 0 \\ & 0 & & & & 0 & \\ & \vdots & & & & \vdots & \\ & 0 & & & & 0 & \\ 0 & 0 & 0 & \cdots & & 1 & 0 \end{array}\right] & \begin{array}{c} s \\ \\ \\ \\ r \end{array} \end{array}$$

where we may again use induction.

Case 4: $V_{\ell-2} \leftarrow V_{\ell-1} \leftarrow V_{\ell}$

We perform the row operation $\mathbf{R}_{s\leftarrow r}(-\alpha)$ on A_{ℓ}, inducing the column operation $\mathbf{C}_{r\leftarrow s}(\alpha)$ on $A_{\ell-1}$. Again, if the s-th column of $A_{\ell-1}$ is zero, we are done. Otherwise, since $A_{\ell-1}$ is in barcode form and $s < r$, the r-th column is also non-zero so that the resulting matrix $A'_{\ell-1}$ has the form

$$A'_{\ell-1} = \begin{array}{cccccccc} & s & & & & r & & \\ 0 & 1 & 0 & \cdots & 0 & \alpha & 0 & c \\ & 0 & & & & 0 & & \\ & \vdots & & & & \vdots & & \\ & 0 & & & & 0 & & \\ 0 & 0 & 0 & \cdots & & 1 & 0 & d \end{array}$$

where we may again use induction. ☐

The zigzag-compatible avatar of Proposition 2.3.4 is given as follows. As before, we regard this result as the 'matrix version' of the decomposition theorem for zizag persistence modules.

Proposition 2.6.4. *Given the sequence of matrices* $A_\bullet$ *as in* (2.12)*, there is a* $g \in G$ *such that* $(gA)_\bullet$ *has all its matrices in zizag barcode form.*

Proof. We proceed by induction. The case $\ell = 1$ is trivial. We can reduce ourselves by induction hypothesis to the case where the first $\ell - 1$ matrices are in zizag barcode form. If $V_{\ell-1} \to V_\ell$ we perform row operations on A_ℓ to put it in reduced row echelon form. Otherwise, if $V_{\ell-1} \leftarrow V_\ell$, we perform column operations to put A_ℓ in reversed reduced column echelon form. That is, we put A_ℓ in the form

$$\left[\begin{array}{c|ccccc} & \star & \star & \cdots & \cdots & \star \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline \multicolumn{6}{c}{\ddots} \\ \hline \multicolumn{3}{c|}{} & \star & \star & \star \\ \multicolumn{3}{c|}{0} & 1 & 0 & 0 \\ \hline \multicolumn{4}{c|}{} & \star & \star \\ \multicolumn{4}{c|}{0} & 1 & 0 \\ \hline \multicolumn{5}{c|}{} & \star \\ \multicolumn{5}{c|}{0} & 1 \\ \hline \multicolumn{6}{c}{0} \end{array}\right]$$

This is achieved through a slight tweak to the standard Gaussian algorithm for placing matrices in column echelon form, where one starts with the last row and works upwards. Finally, we may apply Lemma 2.6.3 to the non-zero $\star$ term of A_ℓ obtain the desired result. ☐

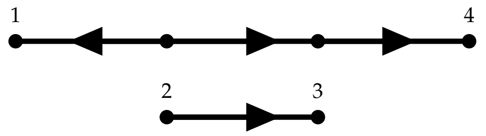
2.6.2 Barcode bases of zizag modules

In order to characterise the set of barcode bases of a zizag module, we must again attempt to characterise the set $\mathrm{Stab}(A_\bullet)$ corresponding to linear maps of a zizag module in a barcode basis. In Section 2.4, we defined an order $\preceq$ on the set of intervals that allowed us to classify the stabiliser as

$$\mathrm{Stab}(A_\bullet) \;\cong\; \prod_{[i,j]} \mathrm{GL}(d_{ij};\mathbb{F}) \times \prod_{[i_1,j_1]\precneq[i_2,j_2]} \mathrm{Mat}(d_{i_1j_1} \times d_{i_2j_2};\mathbb{F}).$$

The $\mathrm{Mat}(d_{i_1j_1} \times d_{i_2j_2};\mathbb{F})$ came from the fact that in standard persistence, the interval module $\mathbf{I}[i_1,j_1]$ can be mapped non-trivially to the interval $\mathbf{I}[i_2,j_2]$ if and only if $[i_1,j_1] \preceq [i_2,j_2]$. In zizag persistence, this is no longer true.

Example 2.6.5. One checks that in the following scenario



the interval $[1,4]$ may be non-trivially mapped to the interval $[2,3]$.

To then obtain a similar classification result for zizag modules, we see we must adapt our partial order $\preceq$ to the type τ of our zizag module.

Definition 2.6.6. Let τ be a type of zizag module, defining an orientation on the standard length ℓ quiver. Let $\preceq_\tau$ be the binary relation on $\{[i,j] \in \mathbb{Z}^2 \mid 0 \leqslant i \leqslant j \leqslant \ell\}$ given by

$$[i_1,j_1] \preceq_\tau [i_2,j_2] \Leftrightarrow \begin{cases} [i_1,j_1] \cap [i_2,j_2] = [i,j] \neq \varnothing \quad \text{and} \\ i_1 \leqslant i_2 \quad \text{if} \quad i-1 \to i, \quad i_2 \leqslant i_1 \quad \text{if} \quad i-1 \leftarrow i \quad \text{and} \\ j_1 \leqslant j_2 \quad \text{if} \quad j \to j+1, \quad j_2 \leqslant j_1 \quad \text{if} \quad j \leftarrow j+1 \end{cases}$$

We observe that when all maps point forward, i.e., when $\tau = ff\ldots f$, we have $\preceq_\tau = \preceq$. And if all maps point backwards, ie $\tau = qq\ldots q$, then $\preceq_\tau$ is the reverse of $\preceq$ in the sense that

$$[i_1,j_1] \preceq_\tau [i_2,j_2] \Leftrightarrow [i_2,j_2] \preceq [i_1,j_1].$$

Our description of the stabiliser from Theorem 2.4.4 relied on the compatibility of $\preceq$ with the lexicographical total order $\trianglelefteq$ on the bars. Having produced a τ-analogue of $\preceq$, we must now construct the zigzag version of $\trianglelefteq$. To this end, we define two new auxiliary total orders on the set of all possible endpoints $\{0,\ldots,\ell\}$.

Note that any such order $<$ amounts to a choice of element σ_ℓ lying in the permutation group $S_{\ell+1}$ via the identification

$$\sigma(0) < \sigma(1) < \cdots < \sigma(\ell).$$

Definition 2.6.7. Let $\leqslant_\tau$ be the total order on $\{0, 1 \ldots \ell\}$ given by the permutation σ_ℓ defined inductively as follows. Assuming we have ordered $\{0, \ldots, i\}$ with corresponding permutation σ_i, we order $\{0, \ldots, i+1\}$ with permutation σ_{i+1}, setting

$$\sigma_{i+1}(k) = \begin{cases} \sigma_i(k) & \text{if} \quad i \longrightarrow i+1 \quad \text{for} \quad k \in \{0, \ldots i\} \\ i+1 & \text{if} \quad i \longrightarrow i+1 \quad \text{for} \quad k = i+1 \\ \sigma_i(k)+1 & \text{if} \quad i \longleftarrow i+1 \quad \text{for} \quad k \in \{0. \ldots i\} \\ 0 & \text{if} \quad i \longleftarrow i+1 \quad \text{for} \quad k = i+1 \end{cases}$$

Definition 2.6.8. Let $\leqslant_\tau^*$ be the total order on $\{0, 1 \ldots, \ell\}$ given by the permutation σ_ℓ^* defined inductively as follows. Assuming we have ordered $\{j+1, \ldots, \ell\}$ with corresponding permutation σ_{j+1}^*, we order $\{j, \ldots, \ell\}$ with permutation σ_j^*, setting

$$\sigma_j^*(k) = \begin{cases} \sigma_{j+1}^*(k) & \text{if} \quad j \longrightarrow j+1 \quad \text{for} \quad k \in \{j+1. \ldots \ell\} \\ j & \text{if} \quad j \longrightarrow j+1 \quad \text{for} \quad k = j \\ \sigma_{j+1}^*(k)-1 & \text{if} \quad j \longleftarrow j+1 \quad \text{for} \quad k \in \{j+1. \ldots \ell\} \\ \ell & \text{if} \quad j \longleftarrow j+1 \quad \text{for} \quad k = j \end{cases}$$

If $\tau = ff \ldots f$, both of the above total orders coincide with the standard ordering $\leqslant$ on $\{0, 1 \ldots \ell\}$.

Example 2.6.9. Consider zigzag modules of length 4 with type $0 \longleftarrow 1 \longrightarrow 2 \longleftarrow 3$. For such modules, the two total orders defined above are

$$3 \leqslant_\tau 1 \leqslant_\tau 0 \leqslant_\tau 2, \text{ and}$$
$$1 \leqslant_\tau^* 3 \leqslant_\tau^* 2 \leqslant_\tau^* 0.$$

Definition 2.6.10. Let τ be a type of length-ℓ zigzag module. The corresponding total order $\trianglelefteq_\tau$ on $\left\{[i,j] \in \mathbb{Z}^2 \mid 0 \leqslant i \leqslant j \leqslant \ell\right\}$ is given by

$$[a,b] \trianglelefteq_\tau [c,d] \Leftrightarrow a <_\tau c \text{ or } a = c \text{ and } b \leqslant_\tau^* d.$$

By construction, if $\tau = ff \ldots f$, this coincides with the lexicographic order $\trianglelefteq$ from Section 2.4, and $[i_1, j_1] \preceq_\tau [i_2, j_2] \Rightarrow [i_1, j_1] \trianglelefteq_\tau [i_2, j_2]$. Using this total order on the bars, we may generalise our notion of **ordered barcode bases** to zizag modules, by ordering the bars with the order $\trianglelefteq_\tau$. We now have all the necessary tools to adapt our results from Section 2.4 to the case of zizag modules.

Theorem 2.6.11. *For each pair $[i,j]$ in $\{0,1,\dots,\ell\}$ with $i \leqslant j$, let d_{ij} be the multiplicity in the barcode of $(V_\bullet, p_\bullet, \tau)$, with the understanding that $d_{ij} = 0$ whenever $[i,j]$ is not in* **Bar**$(V_\bullet, p_\bullet, \tau)$. *Then there is a bijection of sets:*

$$\mathrm{Stab}(A_\bullet) \;\cong\; \prod_{[i,j]} \mathrm{GL}(d_{ij};\mathbb{F}) \times \prod_{[i_1,j_1] \lnsim_\tau [i_2,j_2]} \mathrm{Mat}(d_{i_1 j_1} \times d_{i_2 j_2};\mathbb{F}).$$

Proposition 2.6.12. *Given a zizag module $(V_\bullet, p_\bullet, \tau)$ together with an ordered barcode basis $\mathscr{B}$ with matrix representation $A_\bullet$, the set of all ordered barcode bases is given by the orbit* $\mathrm{Stab}(A_\bullet)\mathscr{B}$.

2.6.3 Maps of zizag modules

We now turn our attention to morphisms $(V_\bullet, p_\bullet, \tau) \to (W_\bullet, \tilde{p}_\bullet, \tau)$ between zigzag persistence modules of the same type τ. Each such morphism is determined by linear maps $\phi_i : V_i \mapsto W_i$ along with the requirement that the evident squares commute. That is, if $p_i = f_i$ (and so $\tilde{p}_i = \tilde{f}_i$), we have $\phi_i \circ f_i = \tilde{f}_i \circ \phi_{i-1}$, and if $p_i = q_i$ (and so $\tilde{p}_i = \tilde{q}_i$), we have $\phi_{i-1} \circ q_i = \tilde{q}_i \circ \phi_i$. This is best represented through the following diagram

$$\begin{tikzcd}
V_0 \arrow[r, leftrightarrow, "p_1"] \arrow[d, "\phi_0"'] & V_1 \arrow[r, leftrightarrow, "p_2"] \arrow[d, "\phi_1"'] & \cdots \arrow[r, leftrightarrow, "p_{\ell-1}"] & V_{\ell-1} \arrow[r, leftrightarrow, "p_\ell"] \arrow[d, "\phi_{\ell-1}"'] & V_\ell \arrow[d, "\phi_\ell"'] \\
W_0 \arrow[r, leftrightarrow, "\tilde{p}_1"'] & W_1 \arrow[r, leftrightarrow, "\tilde{p}_2"'] & \cdots \arrow[r, leftrightarrow, "\tilde{p}_{\ell-1}"'] & W_{\ell-1} \arrow[r, leftrightarrow, "\tilde{p}_\ell"'] & W_\ell
\end{tikzcd}$$

where each natural square commutes.

As in Section 2.5, such maps may be identified as ladder persistence module on the rectangle quiver with orientation τ. We denote such modules $(V_\bullet, W_\bullet, \phi_\bullet, \tau)$. We may analogously define a special class of of modules, namely $\mathbf{R}^{[i_2,j_2]}_{\tau_{[i_1,j_1]_\bullet}}$, $\mathbf{I}^{+}_{\tau}[i,j]_\bullet$ and $\mathbf{I}^{-}_{\tau}[i,j]_\bullet$. Here is the zigzag analogue of Definition 2.5.1

Definition 2.6.13. Given a type τ of zizag module, we say a bar $[i_2,j_2]$ is strictly nested in a bar $[i_1,j_1]$ with regards to τ, denoted $[i_2,j_2] \subset_\tau [i_1,j_1]$, if they are non-intersecting with $[i_1,j_1] \trianglelefteq_\tau [i_2,j_2]$ but $[i_1,j_1] \nleqslant_\tau [i_2,j_2]$.

As in Section 2.5, maps of zizag module may be compactly represented as block upper triangular matrices using ordered barcode basis for the source and target and zizag modules. Having excluded strictly nested bars, we are free to perform operations of type (1), (2) and (3) à la Theorem 2.5.3 and obtain the following result.

Theorem 2.6.14. *Let $(V_\bullet, W_\bullet, \phi_\bullet, \tau)$ be a ladder persistence module of length $\ell + 1$ and type τ, where neither $V_\bullet$ nor $W_\bullet$ admit a pair of strictly nested bars with regards to τ. Then there are integers $r_{[i_1,j_1]}^{[i_2,j_2]}$, $d_{ij}^{\pm} > 0$ for which :*

$$(V_\bullet, W_\bullet, \phi_\bullet, \tau) \simeq \bigoplus_{[i_1,j_1] \preceq_\tau [i_2,j_2]} (\mathbf{R}_{\tau_{[i_1,j_1]_\bullet}}^{[i_2,j_2]})^{r_{[i_1,j_1]}^{[i_2,j_2]}} \oplus \bigoplus_{i \leqslant j} (\mathbf{I}_\tau^+[i,j]_\bullet)^{d_{ij}^+} \oplus \bigoplus_{i \leqslant j} (\mathbf{I}_\tau^-[i,j]_\bullet)^{d_{ij}^-}$$

Chapter 3

Harder-Narasimhan Filtration of Persistence Modules

This chapter functions as a standalone part of this thesis and can be read independently from our previous work. It contains its own introduction and literature review. We will now be dealing with general quiver representations as opposed to (ladder) persistence modules. As such, notation will be slightly different than that of the previous chapter and this new notation is introduced below.

3.1 Introduction

More than sixty years ago, the notion of *semistability* was introduced by Mumford [46, 47] to construct well-behaved quotient spaces for the actions of reductive groups on algebraic varieties. Subsequently, Harder and Narasimhan described a stratification of the moduli space of finite rank vector bundles over a complex curve for the purpose of computing its cohomology groups [26]. The top stratum consists of semistable bundles, and every bundle lying in a lower stratum admits a canonical filtration of finite length whose associated graded components are semistable with strictly decreasing slopes (i.e., ratios of degree to rank). This *Harder-Narasimhan filtration* continues to play a vital role in moduli problems involving vector bundles and coherent sheaves [2, 31]; its existence and uniqueness have been established more generally for objects of certain abelian categories by Rudakov [51] to triangulated categories by Bridgeland [4] and to modular lattices by Haiden et al [21]. The work of King [34] and Reineke [49] has extended the Harder-Narasimhan formalism to categories of quiver representations.

Our efforts here stem from the desire to use Harder-Narasimhan theory to build isomorphism invariants for multiparameter persistence modules. In general, such modules are representations of wild type quivers, so there is no hope of obtaining a complete[1] discrete invariant. Nevertheless, the quest for discriminative invariants has been a central theme within topological data analysis. The earliest work in this direction was by Carlsson and Zomorodian [11], who proposed the *rank invariant*. Subsequent efforts to study multiparameter persistence modules have involved a plethora of tools sourced from diverse locales — these include sheaf theory [42, 33], commutative and homological algebra [45, 27, 37], lattice theory [43, 29, 3] and beyond [40].

Outline and summary of results

Fix a finite quiver Q and let $\mathbf{Rep}(Q)$ denote the category of finite-dimensional representations of Q valued in vector spaces. By a *central charge* on Q, we mean any real-valued function α defined on the set Q_0 of vertices[2]. Consider a nontrivial representation $V \in \mathbf{Rep}(Q)$, which assigns vector spaces V_x to vertices $x \in Q_0$. The α-slope of V is the ratio

$$\mu_\alpha(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x},$$

and V is said to be α-semistable whenever the inequality $\mu_\alpha(V) \geqslant \mu_\alpha(V')$ holds for every nontrivial subrepresentation $V' \subset V$.

Once we fix a central charge α on Q, every nonzero representation V admits a unique *Harder-Narasimhan* filtration [49, 25] of finite length

$$0 = \mathbf{HN}^0_\alpha(V) \subsetneq \mathbf{HN}^1_\alpha(V) \subsetneq \cdots \subsetneq \mathbf{HN}^{n-1}_\alpha(V) \subsetneq \mathbf{HN}^n_\alpha(V) = V$$

whose successive quotients $S^i := \mathbf{HN}^i_\alpha(V)/\mathbf{HN}^{i-1}_\alpha(V)$ are α-semistable and satisfy $\mu_\alpha(S^i) > \mu_\alpha(S^{i-1})$ for all i. The Harder-Narasimhan filtration thus provides a canonical method for building arbitrary quiver representations out of the α-semistable ones. The **HN type** of V along α is the n-tuple of functions

$$\mathbf{T}[V;\alpha] = \left(\underline{\dim}_{S^1}, \underline{\dim}_{S^2} \ldots, \underline{\dim}_{S^n}\right),$$

where $\underline{\dim}_{S^i} : Q_0 \to \mathbb{N}$ assigns the natural number $\dim S^i_x$ to each vertex $x \in Q_0$.

---

[1] An invariant is *complete* if two persistence modules are isomorphic whenever their invariants are equal.

[2] This is the imaginary part of an abelian group homomorphism $K(\mathbf{Rep}(Q)) \to \mathbb{C}$; see Section 3.2.2.

An important role in this chapter is played by the *spanning* subrepresentation of V at a vertex x — this is defined up to isomorphism as the smallest subrepresentation $\langle V_x \rangle \subset V$ containing V_x. The function $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ that sends each (x, y) to the dimension of $\langle V_x \rangle_y$ vastly generalises the rank invariant of Carlsson and Zomorodian. Consider, for each vertex x, the central charge $\delta_x : Q_0 \to \mathbb{R}$ which maps x to 1 and all other vertices to 0. The skyscraper invariant δ_V is defined on $\mathbf{Rep}(Q)$ as the collection of HN types $\mathbf{T}[V; \delta_x]$ indexed by the vertices of Q.

Our first main result is presented in Section 3.3, and may be stated as follows.

Theorem (A). *The skyscraper invariant is finer than the rank invariant on* $\mathbf{Rep}(Q)$ *for any finite quiver* Q.

The full statement may be found in Theorem 3.3.5, where we provide a precise formula for recovering $\rho_V(x, y)$ in terms of $\mathbf{T}[V; \delta_x]$. In forthcoming work, we will address the computability and stability (with respect to the interleaving distance [39]) of the skyscraper invariant.

We then specialise in Section 3.4 to the setting of zigzag persistence modules — these are finite-dimensional representations of type $\mathbb{A}$ quivers of arbitrary (but finite) length $\ell \geqslant 0$:

$$\begin{tikzcd} x_0 \arrow[r, dash, "e_1"] & x_1 \arrow[r, dash, "e_2"] & \cdots \arrow[r, dash, "e_{\ell-1}"] & x_{\ell-1} \arrow[r, dash, "e_\ell"] & x_\ell . \end{tikzcd}$$

Here each edge e_i points either forward $x_{i-1} \to x_i$ or backward $x_{i-1} \leftarrow x_i$, and ordinary persistence modules correspond to the equioriented case where all of the e_i point forward. It is well known from Gabriel's theorem [19] that any representation V of such a quiver decomposes uniquely as a direct sum of *interval representations*, which are supported on sub-intervals $[a, b] \subset [0, \ell]$. The intervals which appear with nonzero multiplicity comprise the *barcode* of V — see [9]. We exploit knowledge of these indecomposables as well as some recent work of Kinser [35] to prove the following result, which is Theorem 3.4.3 below.

Theorem (B). *There is a classification of all complete central charges on the category of zigzag persistence modules; in the special case of ordinary persistence modules, a central charge* $\alpha : Q_0 \to \mathbb{R}$ *is complete*[3] *iff* $\alpha(x_i) > \alpha(x_{i+1})$ *holds for all* i.

---

[3] A central charge α is complete if its associated invariant $\mathbf{T}[\bullet; \alpha]$ is complete.

The natural d-dimensional analogues of ordinary persistence modules, for $d > 1$, are certain representations of *grid quivers* whose vertices are parameterised by integer points in the product $[0, \ell_1] \times [0, \ell_2] \times \cdots \times [0, \ell_d]$, with each $\ell_i \geqslant 1$. The $d = 2$ case is illustrated below:

$$\begin{tikzcd}
x_{0,\ell_2} \arrow[r] & x_{1,\ell_2} \arrow[r] & \cdots \arrow[r] & x_{\ell_1,\ell_2} \\
\vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] \\
x_{0,1} \arrow[u] \arrow[r] & x_{1,1} \arrow[u] \arrow[r] & \cdots \arrow[r] & x_{\ell_1,1} \arrow[u] \\
x_{0,0} \arrow[u] \arrow[r] & x_{1,0} \arrow[u] \arrow[r] & \cdots \arrow[r] & x_{\ell_1,0} \arrow[u]
\end{tikzcd}$$

The representations of interest are those for which every rectangle in sight commutes as a diagram of vector spaces. Unlike the $d = 1$ case, these quivers are of wild representation type; as mentioned above, much effort has been invested towards finding good discrete invariants for multiparameter persistence modules.

Theorem (C). *The skyscraper invariant is strictly finer than the rank invariant on the category of multiparameter persistence modules.*

In Section 3.6, we focus on to the special case of 2-parameter persistence modules arising from representations of *ladder quivers* of length $\ell \geqslant 1$:

$$\begin{tikzcd}
x_0^+ \arrow[r] \arrow[d] & x_1^+ \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{\ell-1}^+ \arrow[r] \arrow[d] & x_\ell^+ \arrow[d] \\
x_0^- \arrow[r] & x_1^- \arrow[r] & \cdots \arrow[r] & x_{\ell-1}^- \arrow[r] & x_\ell^-
\end{tikzcd}$$

Again, every rectangle is required to commute. These *ladder persistence modules* arise from morphisms of (ordinary) persistence modules. Ladder quivers are known [16, 6] to be of finite representation type only for $\ell \leqslant 3$. We restrict attention to the subcategory spanned by those representations whose top and bottom rows, when viewed as ordinary persistence modules, do not admit a pair of strictly nested intervals in their barcode decompositions[4]. It was shown in via Theorem 2.5.3 that this subcategory is representation-finite for all ℓ. Here we obtain the following results (Proposition 3.6.3 and Theorem 3.6.6) for such *nestfree* ladder persistence modules.

---

[4] Two intervals $[i, j]$ and $[i', j']$ are strictly nested if $i < i'$ and $j' < j$.

Theorem (D). *There is no complete central charge on the category of nestfree ladder persistence modules of length* $\ell \geqslant 4$*; however, for all* ℓ *there exists a finite set* $A = A(\ell)$ *of central charges which is complete on this category.*

The set $A(\ell)$ is explicitly described in Definition 3.6.10, and its cardinality grows cubically with ℓ; every constituent element of this set is expressible an $\mathbb{R}$-linear combination of at most two skyscraper central charges.

3.2 HN types of quiver representations

In this Section, we establish notation and recall pertinent aspects of quiver representations [36, 52], their Harder-Narasimhan filtrations, and the associated Harder-Narasimhan types [49, 25].

3.2.1 Quiver representations

A quiver Q consists of the following data: a set Q_0 whose elements are called vertices, a set Q_1 whose elements are called edges, and a pair of functions $s, t : Q_1 \to Q_0$ called the source and target map respectively. One often depicts each edge $e \in Q_1$ as an arrow $s(e) \longrightarrow t(e)$. A path in the quiver $Q = (s, t : Q_1 \to Q_0)$ is any finite sequence of edges $p = (e_1, \ldots, e_k)$ satisfying $t(e_i) = s(e_{i+1})$ for all i, and Q is called *acyclic* if it admits no such paths with $s(e_1) = t(e_k)$. In this chapter, all quivers are assumed to have only finitely many vertices and edges, and moreover, we will primarily be interested in acyclic quivers.

We work over a field $\mathbb{F}$ which remains fixed throughout (and hence suppressed from the notation), so that all vector spaces encountered henceforth are defined over $\mathbb{F}$. We recall that a **representation** V of Q constitutes assignments of vector spaces V_x to vertices $x \in Q_0$ and linear maps $V_e : V_{s(e)} \to V_{t(e)}$ to edges $e \in Q_1$. All representations considered here are finite-dimensional in the sense that $\dim V_x < \infty$ holds for each vertex $x \in Q_0$; the *dimension vector* of any such V is the function $\underline{\dim}_V : Q_0 \to \mathbb{N}$ given by $x \mapsto \dim V_x$. The set $\mathbf{Rep}(Q)$ of all finite-dimensional representations of Q is readily upgraded to a category as follows. A morphism $\phi : V \to W$ consists of linear maps $\{\phi_x : V_x \to W_x \mid x \in Q_0\}$ which make the following diagram commute for each edge $e \in Q_1$:

$$\begin{tikzcd}
V_{s(e)} \arrow[r, "\phi_{s(e)}"] \arrow[d, "V_e"'] & W_{s(e)} \arrow[d, "W_e"] \\
V_{t(e)} \arrow[r, "\phi_{t(e)}"'] & W_{t(e)}
\end{tikzcd}$$

We call ϕ an (epi, iso, mono)-morphism if each ϕ_x is (sur, bi, in)-jective; V is called a *subrepresentation* of W, denoted $V \subset W$, whenever there exists a monomorphism $\phi : V \to W$. The category $\mathbf{Rep}(Q)$ is known to be abelian, with composition, (co)kernels and (co)products being defined vertex-wise [52, Section 1.3].

A representation of Q is called **indecomposable** if it does not admit any non-trivial direct sum decompositions in $\mathbf{Rep}(Q)$. It is known [52, Theorem 1.2] that for every finite-dimensional representation V of a finite quiver Q, there exists a unique set $\mathrm{Ind}_Q(V)$ containing isomorphism classes of indecomposables and a unique multiplicity function $d_V : \mathrm{Ind}_Q(V) \to \mathbb{Z}_{>0}$ for which there is an isomorphism

$$V \simeq \bigoplus_I I^{d_V(I)}, \tag{3.1}$$

with I ranging over $\mathrm{Ind}_Q(V)$. The seminal work of Gabriel [19] established that the set of indecomposables in $\mathbf{Rep}(Q)$ with a fixed dimension vector is finite if and only if the undirected graph associated to Q is a finite union of simply laced Dynkin diagrams.

3.2.2 Harder-Narasimhan filtrations in abelian categories

The *Grothendieck group* of an abelian category $\mathscr{C}$ is the abelian group $K(\mathscr{C})$ freely generated by the isomorphism classes $[V]$ of objects V in $\mathscr{C}$ modulo a relation of the form $[V] = [U] + [W]$ for each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathscr{C}$. Let $(\mathbb{C}, +)$ denote the abelian group of complex numbers under addition; we recall that a **stability condition**[5] [4] on $\mathscr{C}$ is any group homomorphism

$$Z : K(\mathscr{C}) \to (\mathbb{C}, +)$$

that sends nonzero objects to the open half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. The Z-**slope** of an object $V \neq 0$ is the real number

$$\mu_Z(V) := \frac{\mathrm{Im}\, Z(V)}{\mathrm{Re}\, Z(V)}.$$

---

[5] Such a homomorphism is sometimes called a *linear stability function* [35] or a *central charge* [4].

We say that V is Z-**semistable** if the inequality $\mu_Z(U) \leqslant \mu_Z(V)$ holds for every nonzero subobject $U \subset V$. If this inequality is strict for all subobjects $U \notin \{0, V\}$, then V is said to be Z-**stable**.

The following result is a direct consequence of [4, Proposition 2.4]. We recall that an abelian category is said to satisfy the Noetherian (respectively, Artinian) hypothesis if every sequence $\cdots \subset a_1 \subset a_0$ of subobjects (respectively, every sequence $b_0 \twoheadrightarrow b_1 \twoheadrightarrow \cdots$ of quotient objects) eventually stabilises up to isomorphism.

Theorem 3.2.1. *Let $\mathscr{C}$ be any abelian category which satisfies the Noetherian and Artinian hypotheses. Fix a stability condition Z on $\mathscr{C}$. Every nonzero object V of $\mathscr{C}$ admits a unique filtration $V^\bullet$ of finite length $n \geqslant 1$:*

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^n = V, \tag{3.2}$$

whose successive quotients $S^i := V^i/V^{i-1}$ are Z-semistable and have strictly decreasing slopes:

$$\mu_Z(S^1) > \mu_Z(S^2) > \cdots > \mu_Z(S^n).$$

*This $V^\bullet$ is called the **Harder-Narasimhan** (or **HN**) filtration of V along Z.*

Crucially, the category of finite dimensional representations of a finite quiver satisfies the hypotheses of the above result. It follows from uniqueness that the HN filtration of a Z-semistable object V always has length one, i.e., $0 \subsetneq V$.

3.2.3 HN types of quiver representations

Let $Q = (s, t : Q_1 \to Q_0)$ be a quiver which remains fixed throughout this subsection. The assignment $V \mapsto \underline{\dim}_V$ defines a group homomorphism from the Grothendieck group of $\mathbf{Rep}(Q)$ to the group of functions from Q_0 to $\mathbb{Z}$:

$$\underline{\dim} : K(\mathbf{Rep}(Q)) \longrightarrow \mathbb{Z}^{Q_0}$$

(For acyclic Q, this is an isomorphism [36, Theorem 1.15]). We note that any stability condition Z on $\mathbf{Rep}(Q)$ which factors through $\underline{\dim}$ amounts to a choice of two maps $\alpha : Q_0 \to \mathbb{R}$ and $\beta : Q_0 \to \mathbb{R}_{>0}$. Explicitly, for nonzero $V \in \mathbf{Rep}(Q)$ we have

$$Z(V) = \sum_{x \in Q_0} \left(\beta(x) + \sqrt{-1} \cdot \alpha(x)\right) \cdot \dim V_x,$$

and the corresponding slope is the ratio

$$\mu_Z(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \beta(x) \cdot \dim V_x}.$$

In this chapter, we will only work with stability conditions Z which factor through $\underline{\dim}$. Furthermore, as in [49, 25], we further restrict attention to those stability conditions for which β is the constant map sending all vertices to 1. These are called *standard* stability conditions; and any such stability condition Z depends on a single function $\alpha : Q_0 \to \mathbb{R}$ that we will henceforth call the **central charge** of Z. In light of these simplifications, we will denote the slope of any nonzero V by

$$\mu_\alpha(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x}. \tag{3.3}$$

Similarly, the Harder-Narasimhan filtration of any nonzero $V \in \mathbf{Rep}(Q)$ along Z is indicated by $\mathbf{HN}^\bullet_\alpha(V)$. The following *seesaw lemma* is stated in [49, Lemma 2.2]; we include a proof here for completeness.

Lemma 3.2.2. *Let $\alpha : Q_0 \to \mathbb{R}$ be a central charge for Q. Given any three nonzero objects which fit into a short exact sequence in $\mathbf{Rep}(Q)$:*

$$0 \to U \to V \to W \to 0,$$

their α-slopes must satisfy one of the following chains of (in)equalities. Either

1. *$\mu_\alpha(U) > \mu_\alpha(V) > \mu_\alpha(W)$, or*
2. *$\mu_\alpha(U) = \mu_\alpha(V) = \mu_\alpha(W)$, or*
3. *$\mu_\alpha(U) < \mu_\alpha(V) < \mu_\alpha(W)$.*

In Case (2), V is α-semistable if and only if both U and W are α-semistable.

Proof. Since $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is a strictly monotone increasing function, it suffices to establish the desired inequalities for the composite $\theta := \arctan \circ \mu_\alpha$ rather than for μ_α. By definition, the (standard) stability condition Z induced by α is a group homomorphism $K(\mathbf{Rep}(Q)) \to (\mathbb{C}, +)$, so we have $Z(U) + Z(W) = Z(V)$. The desired results now follow from examining the parallelogram in $\mathbb{C}$ determined by the origin, $Z(U)$ and $Z(W)$ whose fourth point must be $Z(V)$. The angle $\theta(V)$ lies between the angles $\theta(U)$ and $\theta(W)$, with equality of all three angles occurring only in the degenerate case where $Z(U)$ is an $\mathbb{R}$-multiple of $Z(W)$.

Let us now consider the case (2) where U, V and W share a common α-slope μ. If U is not α-semistable, then it admits a subrepresentation U' with $\mu_\alpha(U') > \mu$; but any such U' is automatically a subrepresentation of V which violates its α-semistability. Similarly, any quotient W' of W with $\mu_\alpha(W') < \mu$ violates the semistability of V. Thus, the semistability of V forces semistability of both U and W. Conversely, assume that U and W are α-semistable and label the maps in the short exact sequence as $\iota : U \to V$ and $\pi : V \to W$. Given any subrepresentation $V' \subset V$, we have a short exact sequence

$$0 \to \iota^{-1}(V') \to V' \to \pi(V') \to 0.$$

In the nontrivial case, $\iota^{-1}(V')$ and $\pi(V')$ are nonzero subrepresentations of U and W respectively, so by semistability both must have α-slopes no larger than μ. By the first part of this Lemma, we therefore obtain $\mu_\alpha(V') \leqslant \mu$, which confirms the α-semistability of V. □

Here is an immediate (but important) consequence of the preceding result.

Corollary 3.2.3. *If U and W are α-semistable objects of $\mathbf{Rep}(Q)$ with the same α-slope μ, then their direct sum $U \oplus W$ is also α-semistable with slope μ.*

Definition 3.2.4. The **Harder-Narasimhan type**, or (**HN type**) of a representation $V \neq 0$ in $\mathbf{Rep}(Q)$ along a central charge $\alpha : Q_0 \to \mathbb{R}$ is denoted $\mathbf{T}[V;\alpha]$ and defined as follows. Let n be the length of the Harder-Narasimhan filtration $\mathbf{HN}^\bullet_\alpha(V)$, and let $S^i = \mathbf{HN}^i_\alpha(V)/\mathbf{HN}^{i-1}_\alpha(V)$ denote its successive quotients for $1 \leqslant i \leqslant n$. Then,

$$\mathbf{T}[V;\alpha] := \left(\underline{\dim}_{S^1}, \underline{\dim}_{S^2}, \ldots, \underline{\dim}_{S^n}\right).$$

In the context of the preceding definition, it is often useful to view $\mathbf{T}[V;\alpha]$ as a function $\mathbb{R} \to \mathbb{N}^{Q_0}$ in the following manner:

$$\mathbf{T}[V;\alpha](\lambda) = \begin{cases} \underline{\dim}_{S^i} & \text{if } \lambda = \mu_\alpha(S^i) \text{ for some } i, \\ (0,0,\ldots,0) & \text{otherwise.} \end{cases} \tag{3.4}$$

This function is well-defined since the successive quotients S^i have strictly decreasing slopes by Theorem 3.2.1. The uniqueness promised by this theorem further guarantees that $\mathbf{T}[\bullet;\alpha]$ is invariant under isomorphisms in $\mathbf{Rep}(Q)$. It is evident from Definition 3.2.4 that this invariant is discrete, since it only produces finite sequences of (non-negative) integer values. Moreover, this invariant is additive under direct sums; a proof of the following folklore result may be found in [18, Proposition 2.5].

Proposition 3.2.5. *Let V and W be two representations of a quiver Q, and let $\alpha : Q_0 \to \mathbb{R}$ be a central charge. Adopting the notation of (3.4), we have*

$$\mathbf{T}[V \oplus W;\alpha] = \mathbf{T}[V;\alpha] + \mathbf{T}[W;\alpha]$$

as functions $\mathbb{R} \to \mathbb{N}^{Q_0}$.

3.2.4 Complete central charges

The main theme of this chapter is to measure the strength of the HN type as an invariant of certain quiver representations across various choices of central charge. The definition below corresponds to the best-case scenario.

Definition 3.2.6. Let $Q = (s, t : Q_1 \to Q_0)$ be an acyclic quiver and $\mathscr{C}$ any subcategory of $\mathbf{Rep}(Q)$. A collection of central charges A is said to be **complete** on $\mathscr{C}$ if $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$ for all $\alpha \in A$ implies that V and W are isomorphic in $\mathscr{C}$.

If a collection of central charges A is complete on all of $\mathbf{Rep}(Q)$, then we simplify terminology by saying that A is complete for Q. For the simplest quivers, one hopes to find a single complete central charge; we will appeal to the following result frequently in our quest for such central charges.

Lemma 3.2.7. *If $\alpha : Q_0 \to \mathbb{R}$ is a complete central charge for the acyclic quiver Q, then every indecomposable representation in $\mathbf{Rep}(Q)$ is α-stable.*

Proof. Assume that α is a complete central charge and consider an indecomposable I in $\mathbf{Rep}(Q)$. If I is not α-semistable, then its HN filtration

$$0 \subsetneq \mathbf{HN}^1_\alpha(I) \subsetneq \mathbf{HN}^2_\alpha(I) \subsetneq \cdots \subsetneq \mathbf{HN}^n_\alpha(I) = I$$

has length $n > 1$. Abbreviating $I^i := \mathbf{HN}^i_\alpha(I)$, in particular we have $I^1 \subsetneq I$. Now consider the filtration of $I^1 \oplus (I/I^1)$ given by:

$$0 \subsetneq I^1 \subsetneq I^1 \oplus (I^2/I^1) \subsetneq \cdots \subsetneq I^1 \oplus (I^n/I^1).$$

Since the successive quotients of this filtration are identical to those of $\mathbf{T}[I;\alpha]$, it follows (from uniqueness) that this new filtration is precisely $\mathbf{HN}^\bullet_\alpha(I^1 \oplus (I/I^1))$. Moreover, since the Harder-Narasimhan type depends only on these successive quotients, we have $\mathbf{T}[I;\alpha] = \mathbf{T}[I^1 \oplus (I/I^1);\alpha]$. But since I is indecomposable, it can not be isomorphic to $I^1 \oplus (I/I^1)$ for $I^1 \neq I$, so the completeness of α forces α-semistability of I.

Given this α-semistability, if I is not α-stable, then there exists a nonzero subrepresentation $J \subsetneq I$ with $\mu_\alpha(J) = \mu_\alpha(I)$. Using the exact sequence $0 \to J \to I \to I/J \to 0$ along with Lemma 3.2.2, we know that both J and I/J are α-semistable with slope $\mu_\alpha(I)$. Another appeal to the same Lemma establishes that the direct sum $J \oplus (I/J)$ is also α-semistable with slope $\mu_\alpha(I)$. For dimension reasons, $\mathbf{T}[I;\alpha]$ equals $\mathbf{T}[J \oplus (I/J);\alpha]$. But once again, since I is indecomposable, it is not isomorphic to $J \oplus (I/J)$ for $J \neq I$. Thus, if I is not α-stable, then α is not a complete central charge for Q. ☐

3.3 The skyscraper and rank invariants

We study the invariants defined by delta functions on vertices of Q, the so-called skyscrapers, and show that they provide a finer invariant than the rank invariant which we define here for representations of any Q.

3.3.1 Skyscraper invariant

Let $Q = (s, t : Q_1 \to Q_0)$ be an arbitrary (i.e., finite, but not necessarily acyclic) quiver. In the absence of specific knowledge regarding the structure of Q or its indecomposable representations, it is not immediately obvious how one might identify interesting classes of central charges for Q à la Theorem 3.4.2. Among the simplest nontrivial central charges which may be defined on any quiver are the ones supported on a single vertex.

Definition 3.3.1. The **skyscraper** central charge at a vertex $x \in Q_0$ is the map $\delta_x : Q_0 \to \mathbb{R}$ given by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

And the **skyscraper invariant** $\delta_\bullet$ on $\mathbf{Rep}(Q)$ assigns to each representation V the collection of HN types $\delta_V = \{\mathbf{T}[V; \delta_x] \mid x \in Q_0\}$ along skyscraper central charges at all of the vertices.

Definition 3.3.2. Let $S \subset Q_0$ be a nonempty subset of vertices. The **spanning subrepresentation** of V at S, denoted $\langle V_S \rangle$, is the intersection of all subrepresentations $W \subset V$ for which W_x is isomorphic to V_x whenever x lies in S.

We will simply write $\langle V_x \rangle$ when S is the singleton $\{x\}$. Spanning representations at singletons determine the HN filtrations along skyscraper central charges.

Proposition 3.3.3. *Given a vertex x of Q, let $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ be the HN filtration of V along δ_x. If j is the smallest index for which V^j_x equals V_x, then:*

1. *either $j = n$ or $j = n - 1$, and*
2. *for every $1 \leqslant k \leqslant j$, we have $V^k = \langle V^k_x \rangle$.*

Proof. We note that the δ_x-slope of a nonzero representation W of Q is given by

$$\mu_{\delta_x}(W) = \frac{\dim W_x}{\sum_{y \in Q_0} \dim W_y}, \tag{3.5}$$

which is evidently non-negative. Assuming that $V_x^j = V_x$ holds for some j in $\{0, \dots, n\}$, we have $V_x^k = V_x$ for all $k \geqslant j$, whence the successive quotients $S^k := V^k/V^{k-1}$ satisfy $S_x^k = 0$ for all $k > j$. By (3.5), we obtain equalities of slopes:

$$0 = \mu_{\delta_x}(S^{j+1}) = \mu_{\delta_x}(S^{j+2}) = \cdots = \mu_{\delta_x}(S^{n-1}) = \mu_{\delta_x}(S^n).$$

Since these slopes are required to strictly decrease in the HN filtration, there are only two possible options. Either $j = n-1$, in which case only the last slope is 0; or $j = n$, in which case all slopes are non-zero. Thus, we have established assertion (1). We now prove assertion (2) by induction on $k \in \{1, \dots, j\}$.

Base case: Since V^1 is δ_x-semistable and $\langle V_x^1 \rangle$ is its subrepresentation, we must have $\mu_{\delta_x}(V^1) \geqslant \mu_{\delta_x}(\langle V_x^1 \rangle)$, whence

$$\frac{\dim V_x^1}{\sum_{y \in Q_0} \dim V_y^1} \geqslant \frac{\dim V_x^1}{\sum_{y \in Q_0} \dim \langle V_x^1 \rangle_y}.$$

If $\dim V_x^1 = 0$ then there is nothing to check, so we assume that this dimension is nonzero. Since each $\langle V_x^1 \rangle_y$ is a subspace of the corresponding V_y^1 for $y \in Q_0$, we obtain $V^1 = \langle V_x^1 \rangle$.

Inductive step: Assume that $V^k = \langle V_x^k \rangle$ holds for some $k < j$. Now S^{k+1} is δ_x-semistable by definition of the HN filtration; and by the argument which established assertion (1), it has a strictly positive δ_x-slope. Thus, we have $\dim S_x^{k+1} > 0$, and applying the base case (to S^{k+1} instead of V^1) yields $S^{k+1} = \langle S_x^{k+1} \rangle$. Consequently, given any vertex $y \geqslant x$ and vector $\eta \in V_y^{k+1}$, there exists a vector $\xi \in V_x^{k+1}$ for which the difference $\eta' := V_{x \leqslant y}(\xi) - \eta$ lies in V_y^k. By the inductive hypothesis, this η' must equal $V_{x \leqslant y}(\xi')$ for some $\xi' \in V_x^k$. Therefore, we have $V_{x \leqslant y}(\xi - \xi') = \eta$, whence V^{k+1} equals $\langle V_x^{k+1} \rangle$ as desired. □

3.3.2 Rank invariant

The following notion constitutes a substantial generalisation of the *rank invariant*, which was introduced in [11] for multi-parameter persistence modules.

Definition 3.3.4. The **rank invariant** of $V \in \mathbf{Rep}(Q)$ is the map $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ given by:

$$\rho_V(x, y) := \dim \langle V_x \rangle_y.$$

It follows from the above definition that $\rho_\bullet$ is a discrete isomorphism invariant for $\mathbf{Rep}(Q)$. The following result gives a formula for the rank invariant in terms of the skyscraper invariant $\delta_\bullet$ — in fact, we show that for any vertex x, the rank $\rho_V(x, y)$ can be recovered from the single Harder-Narasimhan type $\mathbf{T}[V; \delta_x]$.

Theorem 3.3.5. *Let Q be a finite quiver. The skyscraper invariant is strictly more discriminative than the rank invariant on* $\mathbf{Rep}(Q)$ *in the following sense.*

1. *Consider $V \in \mathbf{Rep}(Q)$ and any vertex x in Q_0. If $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ is the HN filtration of V along δ_x, then for any vertex $y \geqslant x$ of Q we have*

$$\rho_V(x,y) = \sum_{k=1}^{j} \dim S^k_y,$$

where $S^k := V^k / V^{k-1}$ and j is the smallest index for which V^j_x equals V_x.

2. *There exist two representations W and W' of the quiver*

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u]
\end{tikzcd}$$

for which $\rho_W = \rho_{W'}$ whereas $\delta_W \neq \delta_{W'}$.

Proof. By Proposition 3.3.3, we have $j \in \{n, n-1\}$ and $V^j = \langle V_x \rangle$. So by Definition 3.3.4, the value of $\rho_V(x,y)$ equals $\dim V^j_y$. Since the $S^\bullet$ are successive quotients of the HN filtration $V^\bullet$, we have

$$\dim V^j_y = \sum_{k=1}^{j} \dim S^k_y,$$

which establishes the first assertion. Turning now to the second assertion, let us consider the representations W (left) and W' (right) depicted below:

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "{[0,1]}"] \arrow[r, "{[1,0]}"'] & \mathbb{F} \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "{[1,0]}"] \arrow[r, "{[1,0]}"'] & \mathbb{F} \arrow[u]
\end{tikzcd}$$

Both evidently have the same rank invariant. Let x be the $\leqslant$-minimal vertex of this quiver (i.e., the vertex on the bottom-left). By examining (the slopes of) subrepresentations, one readily checks that W is δ_x-semistable, so that its HN filtration is just the trivial one $0 \subsetneq W$. On the other hand, W' has a two-step HN filtration $0 \subset U \subset W'$, where U is given by

$$\begin{tikzcd}
0 \arrow[r] & 0 \\
\mathbb{F} \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Since W and W' have different HN types along δ_x, the skyscraper invariants δ_W and $\delta_{W'}$ are distinct as claimed above. □

3.4 HN types of zigzag persistence modules

The goal of this section is to characterise complete central charges for type $\mathbb{A}_\ell$ quivers.

3.4.1 Zigzag persistence modules

Fix an integer $\ell \geqslant 0$. A quiver Q is said to be of **type** $\mathbb{A}_\ell$ whenever its underlying undirected graph has the form

$$\begin{tikzcd} x_0 \arrow[r, dash, "e_1"] & x_1 \arrow[r, dash, "e_2"] & \cdots \arrow[r, dash, "e_{\ell-1}"] & x_{\ell-1} \arrow[r, dash, "e_\ell"] & x_\ell. \end{tikzcd}$$

We describe the direction of edges via a boolean string τ of length ℓ, called the *orientation* of Q: its i-th entry τ_i indicates whether e_i points forward (1) from e_{i-1} to e_i or backward (0) from e_i to e_{i-1}. For instance, when $\ell = 3$, the sequence $\tau = 100$ implicates the following quiver:

$$\begin{tikzcd} x_0 \arrow[r, "e_1"] & x_1 & x_2 \arrow[l, "e_2"'] & x_3. \arrow[l, "e_3"'] \end{tikzcd}$$

We say that Q is *equioriented* if every τ_i equals 1 (i.e., all edges point forward). Our goal here is to describe all complete central charges α for representations of type $\mathbb{A}_\ell$ quivers.

Representations of type $\mathbb{A}_\ell$ quivers are called *zigzag persistence modules* [9], and these specialise in the equioriented case to *ordinary persistence modules* [48]. It follows from Gabriel's theorem [19] that the indecomposable summands which appear in the decomposition (3.1) of a nonzero $V \in \mathbf{Rep}(Q)$ have a particularly simple form when Q is of type $\mathbb{A}_\ell$. Each such indecomposable corresponds to a subinterval $[a, b] \subset [0, \ell]$ with integral endpoints. Recalling that $\mathbb{F}$ is the ground field over which all of our vector spaces are defined, the indecomposable $\mathbf{I}_\tau[a, b] \in \mathbf{Rep}(Q)$ associated to $[a, b]$ has the form

$$0 \longleftrightarrow \cdots \longleftrightarrow 0 \longleftrightarrow \mathbb{F} \longleftrightarrow \cdots \longleftrightarrow \mathbb{F} \longleftrightarrow 0 \longleftrightarrow \cdots \longleftrightarrow 0. \tag{3.6}$$

Here the arrows point in accordance with the orientation τ of Q, the contiguous string of $\mathbb{F}$'s spans vertex indices $\{a, a+1, \ldots, b-1, b\}$, all maps with source and target $\mathbb{F}$ are identities, and all other vector spaces are trivial. These indecomposables are often called *interval* modules.

Explicitly, if Q is a type $\mathbb{A}_\ell$ quiver with orientation τ, then associated to each representation $V \in \mathbf{Rep}(Q)$ there exists a unique finite set $\mathbf{Bar}(V)$ consisting of subintervals $[a,b] \subset [0,\ell]$ with $a \leqslant b$ integers, and a unique function $\mathbf{Bar}(V) \to \mathbb{N}$ sending each $[a,b]$ to its multiplicity d_{ab}, so that there is an isomorphism

$$V \simeq \bigoplus_{[a,b]} (\mathbf{I}_\tau[a,b])^{d_{ab}}. \tag{3.7}$$

Here the direct sum ranges over $[a,b] \in \mathbf{Bar}(V)$. Thus, a central charge $\alpha : Q_0 \to \mathbb{R}$ is complete for Q in the sense of Definition 3.2.6 if and only if the multiplicity function $[a,b] \mapsto d_{ab}$ of every $V \in \mathbf{Rep}(Q)$ can be recovered from the HN type $\mathbf{T}[V;\alpha]$.

3.4.2 Characterising complete central charges

The first step in our quest to describe all complete central charges of Q is a converse to Lemma 3.2.7. Throughout, we fix a quiver $Q = (s,t : Q_1 \to Q_0)$ of type $\mathbb{A}_\ell$ and denote its orientation by τ.

Proposition 3.4.1. *A central charge $\alpha : Q_0 \to \mathbb{R}$ is complete for Q if every indecomposable $\mathbf{I}_\tau[a,b]$ in $\mathbf{Rep}(Q)$ is α-stable.*

Proof. Let $V \in \mathbf{Rep}(Q)$ have the decomposition (3.7); we seek to establish that the multiplicities d_{ab} which appear in this decomposition can be recovered from $\mathbf{T}[V;\alpha]$. To this end, let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the collection of all slopes contained in the set

$$\{\mu_\alpha(\mathbf{I}_\tau[a,b]) \mid [a,b] \in \mathbf{Bar}(V)\}.$$

For each i in $\{1,\ldots,k\}$ we denote by $B_i \subset \mathbf{Bar}(V)$ the subset consisting of all $[a,b]$ for which $\mu_\alpha(\mathbf{I}_\tau[a,b]) \geqslant \lambda_i$. Consider the filtration $V^\bullet$ of V given by

$$V^i := \bigoplus_{[a,b]\in B_i} \mathbf{I}_\tau[a,b]^{d_{ab}}.$$

By construction, the quotient V^i/V^{i-1} is a direct sum of stable representations with α-slope equal to λ_i. Now Corollary 3.2.3 and uniqueness (described in Theorem 3.2.1) ensure that $V^\bullet$ is the HN filtration of V along α.

For each $b \in \{0,1,\ldots,\ell\}$, define $\phi_b : \{0,\ldots,b\} \to \mathbb{R}$ as $\phi_b(a) := \mu_\alpha(\mathbf{I}_\tau[a,b])$. We claim that these maps are injective: given $a' < a$, there are two cases to consider, depending on the orientation of $e_a \in Q_1$ (or equivalently, on the value of $\tau_a \in \{0,1\}$). If $t(e_a) = x_a$, then there exists a monomorphism $\mathbf{I}_\tau[a,b] \subset \mathbf{I}_\tau[a',b]$ and the stability of the latter representation guarantees $\phi_b(a) < \phi_b(a')$. On the other hand, if $t(e_a) = x_{a-1}$ then $\mathbf{I}_\tau[a,b]$ is a quotient of $\mathbf{I}_\tau[a',b]$: we have a short exact sequence in $\mathbf{Rep}(Q)$ of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{I}_\tau[a',a-1] \arrow[r] & \mathbf{I}_\tau[a',b] \arrow[r] & \mathbf{I}_\tau[a,b] \arrow[r] & 0. \end{tikzcd}$$

An appeal to Lemma 3.2.2 along with the α-stability of $\mathbf{I}_\tau[a',b]$ gives $\phi_b(a) > \phi_b(a')$. In both cases we obtain $\phi_b(a) \neq \phi_b(a')$ for $a' < a$, whence ϕ_b is injective as claimed.

Given this injectivity, for each fixed i we may order the elements of B_i as

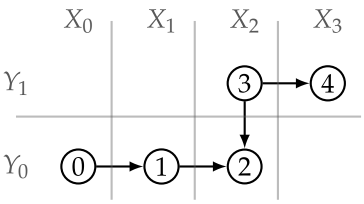
$$B_i = \{[a_1,b_1],[a_2,b_2],\dots,[a_n,b_n]\},$$

where $b_1 > \cdots > b_n$. Now the multiplicity $d_{a_1b_1}$ is precisely the dimension of V^i/V^{i-1} at any vertex $x_j \in Q_0$ where $b_2 < j \leqslant b_1$. Proceeding inductively, we similarly recover the multiplicities $d_{a_kb_k}$ for all $1 < k \leqslant n$. ☐

The preceding result, combined with Lemma 3.2.7, confirms that a central charge α for Q is complete if and only if every indecomposable $\mathbf{I}_\tau[a,b] \in \mathbf{Rep}(Q)$ is α-stable. The main result of [35] is a complete characterisation of such central charges in terms of two functions: let $\chi,\eta\colon \{0,1,\dots,\ell\} \mapsto \mathbb{N}$ be defined inductively as follows. Beginning with $\chi(0) = 0$ and $\eta(0) = 0$, for each $i > 0$ we set

$$\chi(i+1) = \begin{cases} \chi(i)+1 & \text{if } \tau_i = 1 \\ \chi(i) & \text{if } \tau_i = 0 \end{cases} \quad \text{and} \quad \eta(i+1) = \begin{cases} \eta(i)+1 & \text{if } \tau_i = 0 \\ \eta(i) & \text{if } \tau_i = 1 \end{cases}$$

For instance, when $\tau = 1101$, the function χ takes on values $(0,1,2,2,3)$ while the function η takes on values $(0,0,0,1,1)$ for inputs $(0,1,2,3,4)$:



For each $k \in \mathbb{N}$ we have the level sets $X_k := \{i \mid \chi(i) = k\}$ and $Y_k := \{i \mid \eta(i) = k\}$, both of which are subintervals of $\{0,\dots,\ell\}$ as χ and η are monotone. Writing $X_k = [a_k,b_k]$ and $Y_k = [a'_k,b'_k]$ for each k, we are able to state [35, Theorem 1.13].

Theorem 3.4.2. *All indecomposables $\mathbf{I}_\tau[a,b]$ in* $\mathbf{Rep}(Q)$ *are α-stable for a given central charge $\alpha : Q_0 \to \mathbb{R}$ if and only if the following inequalities hold:*

$$\mu_\alpha(\mathbf{I}_\tau[a_0, b_0]) > \mu_\alpha(\mathbf{I}_\tau[a_1, b_1]) > \cdots > \mu_\alpha(\mathbf{I}_\tau[a_{\chi(\ell)}, b_{\chi(\ell)}]),$$
$$\mu_\alpha(\mathbf{I}_\tau[a'_0, b'_0]) < \mu_\alpha(\mathbf{I}_\tau[a'_1, b'_1]) < \cdots < \mu_\alpha(\mathbf{I}_\tau[a'_{\eta(\ell)}, b'_{\eta(\ell)}]).$$

It is also shown in [35] that, for every possible orientation τ, (**a**) this is a *minimal* set of inequalities for characterising those central charges along which all indecomposables are stable, and (**b**) the set of all such central charges defines an nonempty open subset in $\mathbb{R}^{Q_0}$ which is linearly equivalent to $\mathbb{R} \times \mathbb{R}^{Q_1}_{>0}$. These results, when combined with our Proposition 3.4.1 and Lemma 3.2.7, completely describe all complete central charges for $\mathbb{A}_\ell$ quivers.

Theorem 3.4.3. *Given an integer $\ell \geqslant 0$, let Q be a quiver of type $\mathbb{A}_\ell$ and orientation τ. The set of complete central charges for Q is nonempty, and consists precisely of those $\alpha : Q_0 \to \mathbb{R}$ which satisfy the inequalities from Theorem 3.4.2.*

We note here that the set of complete central charges admits a particularly appealing description in the case where Q is equioriented.

Corollary 3.4.4. *For ordinary persistence modules, a central charge α is complete if and only if the inequality $\alpha(x_i) > \alpha(x_{i+1})$ holds for all $i \in \{0, 1, \ldots, \ell - 1\}$.*

Proof. If $\tau = 11\ldots1$, then the function $\chi : \{0, 1, \ldots, \ell\} \to \mathbb{N}$ is given by $\chi(i) = i$, whereas the function η is identically zero. We therefore seek any $\alpha : Q_0 \to \mathbb{R}$ which satisfies $\mu_\alpha(\mathbf{I}_\tau[i,i]) > \mu_\alpha(\mathbf{I}_\tau[i+1, i+1])$ for all i. By (3.3) and (3.6), this string of inequalities reduces to

$$\alpha(x_0) > \alpha(x_1) > \cdots > \alpha(x_{\ell-1}) > \alpha(x_\ell), \tag{3.8}$$

as desired. □

3.5 HN types of multiparameter persistence modules

Finding good invariants for multiparameter persistence modules is a central challenge in topological data analysis. In this section we prove a generalisation of Corollary 3.4.4 to the multiparameter setting.

3.5.1 Multiparameter persistence modules as equalised representations

Let $Q = (s, t : Q_1 \to Q_0)$ be an acyclic quiver. Its source and target maps may be extended from edges to paths $\gamma = (e_1, \ldots, e_k)$ by setting $s(\gamma) := s(e_1)$ and $t(\gamma) := t(e_k)$. Given a representation $V \in \mathbf{Rep}(Q)$, there is a distinguished linear map $V_\gamma : V_{s(\gamma)} \to V_{t(\gamma)}$ induced by the composite

$$V_\gamma := V_{e_k} \circ V_{e_{k-1}} \circ \cdots \circ V_{e_2} \circ V_{e_1}.$$

Definition 3.5.1. We say that $V \in \mathbf{Rep}(Q)$ is **equalised** if the following property holds: for any pair of vertices $x, y \in Q_0$ and any pair of paths γ, γ' with common source x and common target y, the composite maps V_γ and $V_{\gamma'}$ are identical. Let $\mathbf{Rep}_{\mathrm{eq}}(Q) \subset \mathbf{Rep}(Q)$ be the full subcategory spanned by equalised representations.

Example 3.5.2. A large class of interesting equalised representations arises in the study of *cellular sheaves* [15]. Every such sheaf $\mathscr{F}$ on a regular CW complex X is a functor from the face-ordered poset of cells $(X, <)$ to the category $\mathbf{Vec}(\mathbb{F})$ of $\mathbb{F}$-vector spaces. The *Hasse graph* of X is the quiver $Q(X)$ whose vertices correspond bijectively to the cells of X, with a unique edge $\sigma \to \tau$ being present whenever σ is a face of τ of codimension one. Any given sheaf $\mathscr{F} : (X, <) \to \mathbf{Vec}(\mathbb{F})$ on X induces a representation $V(\mathscr{F})$ of $Q(X)$ as follows: every vertex σ is assigned the vector space $\mathscr{F}(\sigma)$ and every edge $\sigma \to \tau$ is assigned the linear map $\mathscr{F}(\sigma < \tau)$. The fact that $\mathscr{F}$ is a functor directly implies that $V(\mathscr{F})$ is equalised.

Definition 3.5.3. The **flow partial order** on vertices Q_0 of the acyclic quiver Q is defined as follows: given x and y in Q_0, we have $x \leqslant y$ if either $x = y$ or if there exists a path γ in Q with $s(\gamma) = x$ and $t(\gamma) = y$.

Given $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ and a pair of vertices $x \leqslant y$ in Q_0, we write $V_{x \leqslant y} : V_x \to V_y$ to indicate the map defined by any path γ from x to y, with the understanding that this map is the identity for $y = x$. Since V is equalised, this is well-defined and it follows that the image of $V_{x \leqslant y}$ is isomorphic to $\langle V_x \rangle_y$ (from Definition 3.3.2). Thus, the value of $\rho_V(x, y)$ is precisely the rank of $V_{x \leqslant y}$ when V is equalised. This is the genesis of the terminology of the rank invariant, which was introduced in [11] to study certain equalised representations of grid quivers, described below.

Let us fix a vector $L = (\ell_1, \ell_2, \dots, \ell_d)$ of $d \geqslant 2$ integers, with each $\ell_i \geqslant 1$. Here we consider the case where $Q = (s, t : Q_1 \to Q_0)$ is the d-dimensional *grid quiver* of shape L, defined as follows. Its vertices x_p are indexed by all p lying in the product

$$\Lambda(L) := \prod_{i=1}^{d} \{0, 1, \dots, \ell_i\},$$

and there exists a unique edge $x_p \to x_q$ whenever $q - p$ is a standard basis vector of $\mathbb{R}^d$. Here, for instance, is the quiver of shape $L = (\ell, 2)$ for arbitrary $\ell \geqslant 1$:

$$\begin{tikzcd}
x_{0,2} \arrow[r] & x_{1,2} \arrow[r] & \cdots \arrow[r] & x_{\ell-1,2} \arrow[r] & x_{\ell,2} \\
x_{0,1} \arrow[r] \arrow[u] & x_{1,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell-1,1} \arrow[r] \arrow[u] & x_{\ell,1} \arrow[u] \\
x_{0,0} \arrow[r] \arrow[u] & x_{1,0} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell-1,0} \arrow[r] \arrow[u] & x_{\ell,0} \arrow[u]
\end{tikzcd}$$

Equalised representations of d-dimensional grid quivers are also referred to as **d-parameter persistence modules** [11]. We note that every grid quiver Q contains an embedded copy of the grid quiver of shape $L = (1, 1)$, and that both the representations W and W' which appeared in the proof of Theorem 3.3.5 are equalised. Thus, we obtain the following consequence.

Corollary 3.5.4. *Given any integer $d \geqslant 2$, let Q be the grid quiver corresponding to some integer vector $L = (\ell_1, \dots, \ell_d)$ with each $\ell_i \geqslant 1$.*

1. *The skyscraper invariant δ_V of $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ determines its rank invariant (via the formula in Theorem 3.3.5).*

2. *There exist representations W and W' in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ which have identical rank invariant and satisfy $\delta_W \neq \delta_{W'}$.*

3.6 HN types of nestfree ladder persistence modules

Fix an integer $\ell \geqslant 1$. Here we will be concerned with certain equalised representations of the following *ladder* quiver $Q = (s, t : Q_1 \to Q_0)$ of length ℓ:

$$\begin{tikzcd}
x_0^+ \arrow[r, "e_1^+"] \arrow[d, "e_0"'] & x_1^+ \arrow[r, "e_2^+"] \arrow[d, "e_1"'] & \cdots \arrow[r, "e_{\ell-1}^+"] & x_{\ell-1}^+ \arrow[r, "e_\ell^+"] \arrow[d, "e_{\ell-1}"'] & x_\ell^+ \arrow[d, "e_\ell"'] \\
x_0^- \arrow[r, "e_1^-"'] & x_1^- \arrow[r, "e_2^-"'] & \cdots \arrow[r, "e_{\ell-1}^-"'] & x_{\ell-1}^- \arrow[r, "e_\ell^-"'] & x_\ell^-
\end{tikzcd}$$

Equalised representations of such quivers are sometimes called *ladder persistence modules*; these are precisely 2-parameter persistence modules of shape $L = (\ell, 1)$, but it is customary to represent them with vertical arrows pointing down instead of up. In particular, they arise when studying the morphisms of ordinary persistence modules. The authors of [16] established that $\mathbf{Rep}_{\text{eq}}(Q)$ is of finite type for $\ell \leqslant 3$ and completely classified its indecomposable objects via Auslander-Reiten theory. In contrast, in Chapter 2 we studied a full subcategory of $\mathbf{Rep}_{\text{eq}}(Q)$ which turns out to be representation finite regardless of ℓ. These are the nest-free representations.

3.6.1 Nestfree representations

Given $V \in \mathbf{Rep}_{\text{eq}}(Q)$, we let V^+ and V^- denote its restrictions to the top and bottom rows of Q respectively. Since both $V^\pm$ are representations of the (equioriented) $\mathbb{A}_\ell$ quiver, they admit direct sum decompositions into interval modules as described in (3.7). We say that such an $\mathbb{A}_\ell$ representation W admits a pair of *strictly nested intervals* whenever there exist $[a, b]$ and $[c, d]$ in $\mathbf{Bar}(W)$ satisfying $a < c \leqslant d < b$ — in other words, the interval $[c, d]$ lies within the interior of the interval $[a, b]$.

Definition 3.6.1. A representation $V \in \mathbf{Rep}_{\text{eq}}(Q)$ is said to be **nestfree** if neither V^+ nor V^- admits a pair of strictly nested intervals.

We will examine the full subcategory of $\mathbf{Rep}_{\text{eq}}(Q)$ spanned by nestfree representations, which is denoted $\mathbf{Rep}_{\text{nf}}(Q)$ henceforth. Its indecomposable objects were completely classified in Theorem 2.5.3, which we state below.

Theorem 3.6.2. *Up to isomorphism, the indecomposable objects of* $\mathbf{Rep}_{\mathrm{nf}}(Q)$ *have one of three possible forms:*

1. *Given subintervals* $[a,b]$ *and* $[c,d]$ *of* $[0,\ell]$ *whose endpoints satisfy* $c \leqslant a \leqslant d \leqslant b$*, let* $\mathbf{R}^{a,b}_{c,d}$ *be the representation* V *for which* V^+ *is the interval module* $\mathbf{I}[a,b]$ *while* V^- *is the interval module* $\mathbf{I}[c,d]$*, and all vertical maps are* $1'$*s whenever possible and* 0 *otherwise:*

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \\
\mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

2. *Given an interval* $[a,b] \subset [0,\ell]$*, let* $\mathbf{R}^{a,b}$ *be the ladder representation* V *for which* V^+ *equals* $\mathbf{I}[a,b]$ *and* V^- *is trivial, with all vertical maps necessarily being* 0*:*

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[d] & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

3. *Finally, given an interval* $[c,d] \subset [0,\ell]$*, let* $\mathbf{R}_{c,d}$ *be representation* V *for which* V^+ *is trivial while* V^- *is* $\mathbf{I}[c,d]$*, so once again all vertical maps are* 0*:*

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \\
 & & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

It will be convenient in the sequel to unify notation by adopting the convention

$$\mathbf{R}^{a,b}_{\infty,\infty} := \mathbf{R}^{a,b} \qquad \text{and} \qquad \mathbf{R}^{\infty,\infty}_{c,d} := \mathbf{R}_{c,d}, \tag{3.9}$$

so that for every $V \in \mathbf{Rep}_{\mathrm{nf}}(Q)$ there exist multiplicities $r^{a,b}_{c,d} \in \mathbb{N}$ satisfying:

$$V \simeq \bigoplus_{a,b,c,d} \left(\mathbf{R}^{a,b}_{c,d}\right)^{r^{a,b}_{c,d}}, \tag{3.10}$$

with (a,b,c,d) ranging over admissible subsets of $\{0,1,\ldots,\ell,\infty\}^4$ as per (3.9). Our goal throughout the remainder of this section is to quantify the extent to which these multiplicities can be recovered from HN types. The first result in this direction is negative: for $\ell \geqslant 4$, there is no complete central charge $\alpha : Q_0 \to \mathbb{R}$.

Proposition 3.6.3. *If* Q *has length* $\ell \geqslant 4$*, then for every central charge* $\alpha : Q_0 \to \mathbb{R}$ *there exists at least one indecomposable in* $\mathbf{Rep}_{\mathrm{nf}}(Q)$ *which is not* α*-stable.*

Proof. It suffices to consider $\ell = 4$ which embeds into all larger ladder quivers:

$$\begin{tikzcd}
x_0^+ \arrow[r] \arrow[d] & x_1^+ \arrow[r] \arrow[d] & x_2^+ \arrow[r] \arrow[d] & x_3^+ \arrow[r] \arrow[d] & x_4^+ \arrow[d] \\
x_0^- \arrow[r] & x_1^- \arrow[r] & x_2^- \arrow[r] & x_3^- \arrow[r] & x_4^-
\end{tikzcd}$$

Let $\alpha_j^\pm$ be the valued assigned by a given central charge α to the vertex $x_j^\pm$ for $0 \leqslant j \leqslant 4$. Assume, for the purposes of contradiction, that all the indecomposables in $\mathbf{Rep}_{\mathrm{nf}}(Q)$ are α-stable. Consider the inequalities of slopes arising from the following inclusions of indecomposables:

$$\begin{aligned}
3(\alpha_1^- + \alpha_1^+) &< 2(\alpha_0^- + \alpha_1^- + \alpha_1^+) && \mathbf{R}_{1,1}^{1,1} \subset \mathbf{R}_{0,1}^{1,1} \\
3(\alpha_0^- + \alpha_1^-) &< (\alpha_0^- + \alpha_1^- + \alpha_1^+ + \alpha_2^+ + \alpha_3^+ + \alpha_4^+) && \mathbf{R}_{0,1} \subset \mathbf{R}_{0,1}^{1,4} \\
4(\alpha_3^- + \alpha_3^+ + \alpha_4^+) &< 3(\alpha_2^- + \alpha_3^- + \alpha_3^+ + \alpha_4^+) && \mathbf{R}_{3,3}^{3,4} \subset \mathbf{R}_{2,3}^{3,4} \\
4(\alpha_2^- + \alpha_2^+ + \alpha_3^+) &< 3(\alpha_1^- + \alpha_2^- + \alpha_2^+ + \alpha_3^+) && \mathbf{R}_{2,2}^{2,3} \subset \mathbf{R}_{1,2}^{2,3} \\
4\alpha_4^+ &< (\alpha_2^- + \alpha_3^- + \alpha_3^+ + \alpha_4^+) && \mathbf{R}^{4,4} \subset \mathbf{R}_{2,3}^{3,4}
\end{aligned}$$

Labelling these five inequalities as $(a), (b), \ldots, (e)$ respectively, we obtain

$$4(a) + 4(b) + (c) + 4(d) + (e) \text{ holds if and only if } 0 < 0,$$

which provides the desired contradiction. □

Note that Lemma 3.2.7 also holds with $\mathbf{Rep}(Q)$ replaced by $\mathbf{Rep}_{\mathrm{nf}}(Q)$ since all the representations involved in its proof remain nestfree for indecomposable I. When combined with this Lemma, the calculation in Proposition 3.6.3 yields the following consequence.

Corollary 3.6.4. *For $\ell \geqslant 4$, there is no central charge $\alpha : Q_0 \to \mathbb{R}$ that is complete on $\mathbf{Rep}_{\mathrm{nf}}(Q)$.*

Before remedying this defect by considering a larger collection of central charges, we describe another negative result which highlights the necessity of restricting our focus to nestfree representations.

Proposition 3.6.5. *Let $\mathbb{F}$ be any field other than $\mathbb{Z}/2$, and consider a ladder quiver Q of length $\ell \geqslant 4$. There exist non-isomorphic representations $V \not\simeq W$ in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ such that $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$ for every central charge $\alpha : Q_0 \to \mathbb{R}$.*

Proof. For each scalar λ in $\mathbb{F}$, consider $V(\lambda) \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{F} \arrow[r, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \arrow[d, swap, "\begin{bmatrix} 1 \\ 1 \end{bmatrix}"] & \mathbb{F}^2 \arrow[r, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[d, swap, "\begin{bmatrix} 1 & \lambda \\ 1 & 1 \end{bmatrix}"] & \mathbb{F}^2 \arrow[r, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] \arrow[d, swap, "\begin{bmatrix} 1 & \lambda \end{bmatrix}"] & \mathbb{F} \arrow[d] \\
\mathbb{F} \arrow[r, swap, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & \mathbb{F}^2 \arrow[r, swap, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & \mathbb{F}^2 \arrow[r, swap, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] & \mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

(See also [6, Definition 4(2)].) We note that regardless of the chosen λ, the upper row $V(\lambda)^+$ is $\mathbf{I}[1,4] \oplus \mathbf{I}[2,3]$ whereas the lower row $V(\lambda)^-$ is $\mathbf{I}[0,3] \oplus \mathbf{I}[1,2]$, so both admit nested intervals in their barcodes. Since we have assumed $\mathbb{F} \neq \mathbb{Z}/2$, there exist distinct nonzero scalars $\lambda \neq \mu$ in $\mathbb{F}$; we set $V := V(\lambda)$ and $W := V(\mu)$. Any isomorphism $\phi : V \to W$ must restrict to automorphisms $\phi^\pm : V^\pm \to W^\pm$ of the top and bottom rows. By Theorem 3.6.2, both $\phi^\pm$ are forced to be trivial in the chosen bases, i.e., represented by the identity matrix on each vertex. It is readily checked (along the middle vertical edge) that this collection of identity matrices does not constitute a morphism $V \to W$ in $\mathbf{Rep}(Q)$, whence V and W are non-isomorphic in the subcategory $\mathbf{Rep}_{\mathrm{eq}}(Q)$. On the other hand, there is an evident bijection between the subrepresentations of V and those of W that preserves dimension vectors, and hence, α-semistability for any choice of central charge $\alpha : Q_0 \to \mathbb{R}$. This yields $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$, as claimed. $\square$

3.6.2 A complete set of central charges

Fix a ladder quiver Q of length $\ell \geqslant 1$. Our goal in this subsection is to prove the following result.

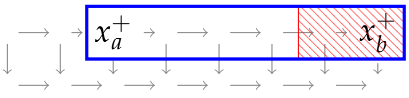
Theorem 3.6.6. *There exists a finite collection of central charges A for which the HN type $\mathbf{T}[\bullet; A]$ is complete on $\mathbf{Rep}_{\mathrm{nf}}(Q)$.*

We let $\mathscr{I}$ denote the set of (isomorphism classes of) indecomposable objects of $\mathbf{Rep}_{\mathrm{nf}}(Q)$ from Theorem 3.6.2; we will also adopt the convention (3.9) for describing these indecomposables as $\mathbf{R}^{a,b}_{c,d}$ for certain a, b, c, d in the set $[\ell]_\infty := \{0, 1, \ldots, \ell, \infty\}$.

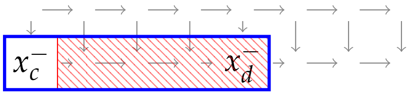
Remark 3.6.7. Here are all the possible strict inclusions $I' \subsetneq I$ in $\mathscr{I}$:

$$\begin{array}{lll}
(1) & \mathbf{R}^{a',b}_{\infty,\infty} \subsetneq \mathbf{R}^{a,b}_{\infty,\infty} & \text{if } a < a' \\
(2) & \mathbf{R}^{\infty,\infty}_{c',d} \subsetneq \mathbf{R}^{\infty,\infty}_{c,d} & \text{if } c < c' \\
(3) & \mathbf{R}^{a',b}_{\infty,\infty} \subsetneq \mathbf{R}^{a,b}_{c,d} & \text{if } d < a' \\
(4) & \mathbf{R}^{\infty,\infty}_{c',d} \subsetneq \mathbf{R}^{a,b}_{c,d} & \text{if } c \leqslant c' \\
(5) & \mathbf{R}^{a',b}_{c',d} \subsetneq \mathbf{R}^{a,b}_{c,d} & \text{if } a \leqslant a' \text{ and } c \leqslant c.
\end{array}$$

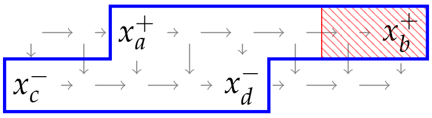
The **support** of an indecomposable $I \in \mathscr{I}$ is the subset $\operatorname{supp}(I) \subset Q_0$ consisting of all vertices x for which the vector space I_x is nontrivial (or equivalently, those vertices x where $\underline{\dim}_I(x)$ is nonzero). Since the indecomposables of $\mathbf{Rep}_{\mathrm{nf}}(Q)$ can be uniquely identified by their supports, we illustrate these five containments $I' \subsetneq I$ from Remark 3.6.7 by depicting the supports of I' (shaded red) and I (outlined blue).



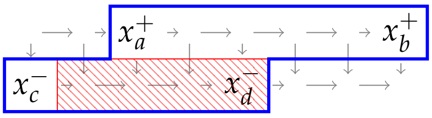
(1)



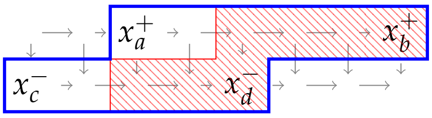
(2)



(3)



(4)



(5)

Let $\leqslant$ be the flow partial order on Q_0 from Definition 3.5.3. For each indecomposable $I \in \mathscr{I}$, we let $\min(I)$ denote the set of minimal vertices lying in the subposet $(\operatorname{supp}(I), \leqslant)$.

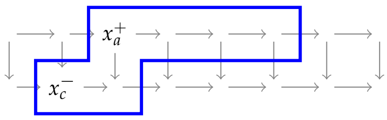
Definition 3.6.8. For every integer $k \geqslant 0$ and subset $S \subset Q_0$, define the set

$$\mathscr{I}^k_S := \left\{ I \in \mathscr{I} \mid \sum_{x \in Q_0} \dim I_x = k \text{ and } \min(I) = S \right\}.$$

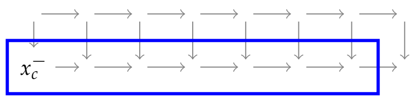
There is a partition $\mathscr{I} = \coprod_{k,S} \mathscr{I}^k_S$ where k ranges over $\mathbb{Z}_{>0}$ while S ranges over subsets of Q_0. The constituent part $\mathscr{I}^k_S$ is empty unless S has one of two possible forms:

- $\{x\}$ for any vertex $x = x_a^+$ or $x = x_c^-$,
- $\{x_a^+, x_c^-\}$ with $0 \leqslant c < a \leqslant \ell$.

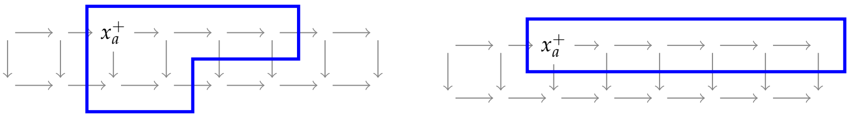
Here are the supports of certain $I \in \mathscr{I}_S^6$ for nontrivial choices of $S \subset Q_0$:



$S = \{x_a^+, x_c^-\}$



$S = \{x_c^-\}$



$S = \{x_a^+\}$

Lemma 3.6.9. *Given any integer $k > 0$ and subset $S \subset Q_0$ of vertices, the set of dimension vectors $\{\underline{\dim}_I \mid I \in \mathscr{I}_S^k\}$ is linearly independent in the vector space of maps $Q_0 \to \mathbb{R}$.*

Proof. It suffices to consider the nontrivial cases where $|\mathscr{I}_S^k| > 1$, which can only occur when either $S = \{x_i^+\}$ for some i or when $S = \{x_i^+, x_j^-\}$ for some $i > j$. We claim that every indecomposable $\mathbf{R}_{c,d}^{a,b} \in \mathscr{I}_S^k$ is uniquely determined by b. To verify this claim, note first that if $S = \{x_i^+\}$ for some $i \in \{0, 1 \ldots, \ell\}$ then we must have $a = i$ and $c \in \{i, \infty\}$, with $c = \infty$ occurring if and only if $b = i + k - 1$. Otherwise, if $S = \{x_i^+, x_j^-\}$ for $i > j$, then $a = i$ and $c = j$. In both cases, d is determined by the formula

$$k = (b - a) + (d - c) + 2.$$

Thus, the map $\iota : \mathscr{I}_S^k \to [\ell]_\infty$ sending each $\mathbf{R}_{c,d}^{a,b}$ to b is injective, as claimed above. Now consider an $\mathbb{R}$-linear combination of the form

$$v := \sum_{I \in \mathscr{I}_S^k} r_I \, \underline{\dim}_I \, .$$

A brief examination of the supports of indecomposables lying in $\mathscr{I}_S^k$ reveals that the value $v(x_j^+)$ depends only on those I which satisfy $\iota(I) \geqslant j$. Thus, the real numbers r_I may be recovered by descending induction on $\iota(I)$. ☐

Let us fix a collection of irrational numbers $\{t_{p,q}\}$ indexed by pairs of integers $0 \leqslant q < p$ so that the following inequalities hold:

$$\frac{q}{p-q} < t_{p,q} < \frac{q+1}{p-(q+1)}. \tag{3.11}$$

Definition 3.6.10. For each integer $k > 0$ and subset $S \subset Q_0$ for which $\mathscr{I}_S^k$ is nonempty, define the central charge $\alpha_S^k : Q_0 \to \mathbb{R}$ as

$$\alpha_S^k := \begin{cases} \delta_x & \text{if } S = \{x\}. \\ \delta_{x_a^+} + t_{k,a-c} \cdot \delta_{x_c^-} & \text{if } S = \{x_a^+, x_c^-\}. \end{cases}$$

Here δ_x is the skyscraper central charge at vertex x (from Definition 3.3.1) while the $t_{\bullet,\bullet}$ are irrational numbers chosen to satisfy (3.11).

We now compute the HN type of every indecomposable $I \in \mathscr{I}$ along these central charges. The calculation below makes essential use of spanning subrepresentations $\langle I_S \rangle$, which were introduced in Definition 3.3.2.

Lemma 3.6.11. *Fix any $k > 0$ and $S \subset Q_0$ for which $\mathscr{I}_S^k$ is nonempty, and define*

$$\lambda_S^k := \begin{cases} 1/k & \textit{if } |S| = 1, \\ (1 + t_{k,a-c})/k & \textit{if } S = \left\{x_a^+, x_c^-\right\}. \end{cases}$$

The following assertions are equivalent for every indecomposable $I \in \mathscr{I}_{S'}^{k'}$:

1. *The α_S^k-slope of I equals λ_S^k.*
2. *Both $k = k'$ and $S \subset S'$ hold.*

If either assertion is true, then we also have that I is α_S^k-semistable if and only if $S = S'$.

Proof. We abbreviate α_S^k to α and λ_S^k to λ. Recall from Definition 3.6.8 that either $S = \{x\}$ or $S = \left\{x_a^+, x_c^-\right\}$ with $a > c$. In the first case, the desired equivalence follows from computing

$$\mu_\alpha(I) = \begin{cases} 1/k' & \text{if } x \in \operatorname{supp}(I) \\ 0 & \text{otherwise} \end{cases}. \tag{3.12}$$

In the case $S = \left\{x_a^+, x_c^-\right\}$ with $a > c$, we similarly have:

$$\mu_\alpha(I) = \begin{cases} 1/k' & \text{if } S \cap \operatorname{supp}(I) = \left\{x_a^+\right\}, \\ t_{k,a-c}/k' & \text{if } S \cap \operatorname{supp}(I) = \left\{x_c^-\right\}, \\ (1 + t_{k,a-c})/k' & \text{if } S \subset \operatorname{supp}(I), \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Since $t_{k,a-c}$ is irrational by assumption, $\mu_\alpha(I)$ determines $S \cap \operatorname{supp}(I)$, as desired.

We now assume that I is α-semistable (in addition to satisfying $\mu_\alpha(I) = \lambda$), and seek to show that $S = S'$. Since $\langle I_S \rangle$ is a subrepresentation of I with $\mu_\alpha(\langle I_S \rangle) \geqslant \lambda$, we must have $I = \langle I_S \rangle$. This forces I to lie in $\mathscr{I}_S^k$, thus ensuring $S' = S$ as desired. Conversely, if $S = S'$, then I lies in $\mathscr{I}_S^k$. By Lemma 3.2.2, it suffices to show that indecomposable subrepresentations $I' \subsetneq I$ have smaller α-slope than I. It is readily checked that any subrepresentation $I' \subset I$ must be both equalised and nestfree, so it suffices to consider only those I' which have been listed in Remark 3.6.7. Of these, the I' with nonzero α-slope all have the form $\langle I_{S'} \rangle$ for $S' \subsetneq S$. When $|S| = 1$, there are no such I' to consider; and when $S = \{x_a^+, x_c^-\}$ for $a > c$, the only relevant I' are $\langle I_{x_a^+} \rangle$ and $\langle I_{x_c^-} \rangle$. Thus, the desired semistability of I reduces to verifying two inequalities:

$$\frac{1}{|\{x \geqslant x_a^+\} \cap \operatorname{supp}(I)|} \leqslant \frac{1 + t_{k,a-c}}{k} \geqslant \frac{t_{k,a-c}}{|\{x \geqslant x_c^-\} \cap \operatorname{supp}(I)|}.$$

Both hold by the constraints imposed on $t_{k,a-c}$ in (3.11). ☐

We recall from (3.4) that given a central charge $\alpha : Q_0 \to \mathbb{R}$, the HN type $\mathbf{T}[V;\alpha]$ of any $V \in \mathbf{Rep}_{\mathrm{nf}}(Q)$ may be viewed as a function $\mathbb{R} \to \mathbb{N}^{Q_0}$. The following result describes the values of the function associated to each central charge α_S^k from Definition 3.6.10 at the corresponding slope λ_S^k from Lemma 3.6.11.

Proposition 3.6.12. *Consider any (k, S) such that $\mathscr{I}_S^k$ is nonempty, and let $I \in \mathscr{I}$. Then for $\alpha := \alpha_S^k$ and $\lambda := \lambda_S^k$, we have*

$$\mathbf{T}[I;\alpha](\lambda) = \begin{cases} \underline{\dim}_{\langle I_S \rangle} & \text{if } \langle I_S \rangle \in \mathscr{I}_S^k \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Proof. Letting $I^\bullet$ be the HN filtration $\mathbf{HN}_\alpha^\bullet(I)$, we consider three possible cases:

Case 1: $I \in \mathscr{I}_S^k$. In this case, $I = \langle I_S \rangle$ and we know by Lemma 3.6.11 that I is α-semistable of α-slope λ. Hence $I^\bullet$ is trivial and each side of (3.14) is equal to $\underline{\dim}_I$.

Case 2: $I \notin \mathscr{I}_S^k$ and $S \not\subset \operatorname{supp}(I)$. In particular, $\langle I_S \rangle$ does not lie in $\mathscr{I}_S^k$ so that the right-hand side of (3.14) is 0. Here we have $S \not\subset \operatorname{supp}(I^i/I^{i-1})$ for all i. If $S = \{x\}$, then $S \cap \operatorname{supp}(I) = \varnothing$ so that from Definition 3.6.10, $\mu_\alpha(I^i/I^{i-1}) = 0 \neq \lambda$ for any step i of the filtration, whence $\mathbf{T}[I;\alpha](\lambda) = 0$. On the other hand, if $S = \{x_a^+, x_c^-\}$, then $S \cap \operatorname{supp}(I)$ can be $\varnothing$, $\{x_a^+\}$ or $\{x_c^-\}$. Then from Definition 3.6.10, the slopes $\mu_\alpha(I^i/I^{i-1})$ lie in either $\mathbb{Q}$ or in $t_{k,a-c} \cdot \mathbb{Q}$, neither of which contain $\lambda = (1 + t_{k,a-c})/k$, whence $\mathbf{T}[I;\alpha](\lambda) = 0$.

Case 3: $I \notin \mathscr{I}_S^k$ and $S \subset \operatorname{supp}(I)$. In this case, we claim that $I^\bullet$ is the one-step filtration $0 \subsetneq \langle I_S \rangle \subsetneq I$. By uniqueness of HN filtrations, it suffices to show that $\langle I_S \rangle$ and $I/\langle I_S \rangle$ are α-semistable, with $\mu_\alpha(\langle I_S \rangle) > \mu_\alpha(I/\langle I_S \rangle)$. We know from Lemma 3.6.11 that $\langle I_S \rangle$ is α-semistable with strictly positive slope. And since $S \cap \operatorname{supp}(I/\langle I_S \rangle)$ is empty by construction, all the subrepresentations of $I/\langle I_S \rangle$ have α-slope 0. Thus, we obtain the desired result: $\mathbf{T}[V;\alpha]$ equals $\underline{\dim}_{\langle I_S \rangle}$ whenever $\langle I_S \rangle$ lies in $\mathscr{I}_S^k$, and is trivial otherwise. ☐

To complete the proof of Theorem 3.6.6, recall the collection of central charges $A = \{\alpha_S^k\}$ from Definition 3.6.10, and consider some $V \in \mathbf{Rep}_{\mathrm{nf}}(Q)$ with decomposition $V \simeq \bigoplus_{I \in \mathscr{I}} I^{r_I}$. We show that the multiplicities $(r_I)_{I \in \mathscr{I}}$ can be recovered from $\mathbf{T}[V;A]$ by descending strong induction on the partial order $\subset$ from Remark 3.6.7. To this end, note that for indecomposables $I \in \mathscr{I}_S^k$ and $J \in \mathscr{I}$, we have $\langle J_S \rangle = I$ if and only if I is a subrepresentation of J. By combining Propositions 3.2.5 and 3.6.12, we get

$$\mathbf{T}[V;\alpha_S^k](\lambda_S^k) = \sum_{I \in \mathscr{I}_S^k} r_I^+ \cdot \underline{\dim}_I,$$

where $r_I^+ := \sum_J r_J$ for J ranging over representations in $\mathscr{I}$ which satisfy $J \supseteq I$. By Lemma 3.6.9, the integer r_I^+ can be obtained from $\mathbf{T}[V;A]$ for each $I \in \mathscr{I}_S^k$. Finally, assume by the inductive hypothesis that r_J is known for all $J \supsetneq I$ in $\mathscr{I}$. Now r_I can be recovered from $\mathbf{T}[V;A]$ as the difference $r_I = r_I^+ - \sum_{J \supsetneq I} r_J$.

Remark 3.6.13. It follows from Definition 3.6.10 that the set of complete central charges A has cardinality

$$c(\ell) := \frac{2\ell^3 + 3\ell^2 + 13\ell + 12}{6},$$

and hence grows cubically with the length ℓ of the underlying ladder quiver. Moreover, one can choose different irrational numbers $t_{p,q}$ in (3.11) to generate uncountably many other such complete sets of the same cardinality $c(\ell)$. It is not clear to us (for large ℓ) whether *all* complete sets of central charges for nestfree representations of ladder quivers can be obtained in this fashion; nor is it clear whether $c(\ell)$ is the minimal size for such a complete set.

Chapter 4

Computational methods for Harder-Narasimhan types and the ladder invariant

Whilst the previous two chapters were independant, here we leverage Theorem 2.5.3 for decomposing well-behaved ladders to help us compute Harder-Narasimhan types of equalised ladder persistence modules. In doing so, we introduce the ladder invariant. It functions as a slight modification of the skyscraper invariant, which happens to be practically computable while remaining finer than the rank invariant.

4.1 Computing the skyscaper invariant for ladder persistence modules

Whilst [14] provides a theoretical polynomial-time algorithm to compute Harder-Narasimhan filtrations in acyclic quivers; there is, to the best of our knowledge, no available implementation for such an algorithm. With this in mind, we restrict ourselves to ladder persistence module and describe an algorithm to compute the skyscraper invariant (see Definition 3.3.1).

4.1.1 Equivalent characterisation of the skyscraper invariant for ladders

Let V be a ladder persistence module of length ℓ, or in the language of Section 3.5, a representation of the quiver $Q = \Lambda(\ell, 1)$

$$\begin{tikzcd}
x_0^+ \arrow[r, "e_1^+"] \arrow[d, "e_0"'] & x_1^+ \arrow[r, "e_2^+"] \arrow[d, "e_1"'] & \cdots \arrow[r, "e_{\ell-1}^+"] & x_{\ell-1}^+ \arrow[r, "e_\ell^+"] \arrow[d, "e_{\ell-1}"'] & x_\ell^+ \arrow[d, "e_\ell"'] \\
x_0^- \arrow[r, "e_1^-"'] & x_1^- \arrow[r, "e_2^-"'] & \cdots \arrow[r, "e_{\ell-1}^-"'] & x_{\ell-1}^- \arrow[r, "e_\ell^-"'] & x_\ell^-
\end{tikzcd}$$

The goal of this section is to compute the skyscraper invariant δ_V. We first show a theoretical result that relates δ_V to the indecomposable decompositions of subrepresentations $\langle V_x \rangle$ for $x \in Q_0$. In this chapter we employ notation used in the rephrasing of Theorem 2.5.3 (Theorem 3.6.2), so that indecomposables appearing in the decomposition of a nestfree ladder persistence modules are form $\mathbf{R}_{c,d}^{a,b}$ with (a, b, c, d) in admissible subsets of $\{0, 1, \cdots, \ell, \infty\}^4$.

Two cases should be considered:

Case 1. If $x = x_a^+$, then $\langle V_x \rangle$ can be viewed as a representation of the $\Lambda(\ell - a, 1)$ quiver. Since $\langle V_x \rangle$ is the subrepresentation generated by the vector space V_x, all bars in both $\langle V_x \rangle^+$ and $\langle V_x \rangle^-$ must start at x_a^+ or x_a^- so that $\langle V_x \rangle$ is a nest-free representation. By Theorem 3.6.2 this yields a decomposition

$$\langle V_x \rangle \simeq \bigoplus_{a,b,c,d} \left(\mathbf{R}_{c,d}^{a,b}\right)^{r_{c,d}^{a,b}},$$

for a, b, c, d ranging over admissible subsets of $\{0, 1, \ldots, \ell, \infty\}^4$. The only indecomposables generated by a vector in V_x are those of the form $\mathbf{R}_{a,d}^{a,b}$ for $d \leqslant b < \infty$ and $\mathbf{R}_{\infty,\infty}^{a,b}$ so that

$$\langle V_x \rangle = \langle V_{x_a^+} \rangle \simeq \bigoplus_{a \leqslant d \leqslant b \leqslant \ell} \left(\mathbf{R}_{a,d}^{a,b}\right)^{r_{a,d}^{a,b}} \bigoplus_{a \leqslant b \leqslant \ell} \left(\mathbf{R}_{\infty,\infty}^{a,b}\right)^{r_{\infty,\infty}^{a,b}} \tag{4.1}$$

Case 2. If $x = x_c^-$, then $\langle V_x \rangle$ can be viewed as a represensation of the $\mathbb{A}_\ell$ quiver with all bars starting at c. Viewed as such, $\langle V_x \rangle$ admits an interval decomposition $\bigoplus_{c \leqslant d \leqslant \ell} \mathbf{I}[c, d]^{d_{cd}}$ which gives

$$\langle V_x \rangle = \langle V_{x_c^-} \rangle \cong \bigoplus_{c \leqslant d \leqslant \ell} \left(\mathbf{R}_{c,d}^{\infty,\infty}\right)^{r_{c,d}^{\infty,\infty}}, \tag{4.2}$$

where $r_{c,d}^{\infty,\infty} = d_{cd}$.

Remark 4.1.1. Observe that given $(a,b,c,d) \in \{0,1,\dots,\ell,\infty\}^4$, there exists at most one vertex $x \in Q_0$ for which r^{ab}_{cd} can be non-zero in the indecomposable decomposition of $\langle V_x \rangle$. Consider the following cases:

- If $0 \leqslant c = a \leqslant b \leqslant d \leqslant \ell$ then r^{ab}_{ad} can only be non-zero for $x = x^+_a$, in the decomposition of $\langle V_{x^+_a} \rangle$ as in 4.1.
- If $0 \leqslant a \leqslant b \leqslant \ell$ and $c = d = \infty$ then $r^{ab}_{\infty,\infty}$ can only be non-zero for $x = x^+_a$, in the decomposition of $\langle V_{x^+_a} \rangle$ as in 4.1.
- If $0 \leqslant c \leqslant d \leqslant \ell$ and $a = b = \infty$ then $r^{\infty,\infty}_{cd}$ can only be non-zero for $x = x^-_c$, in the decomposition of $\langle V_{x^-_c} \rangle$ as in 4.2.

For any other tuple (a,b,c,d), the indecomposable $\mathbf{R}^{a,b}_{c,d}$ does not appear in the direct sum decomposition of $\langle V_x \rangle$ for any $x \in Q_0$. This allows for the following definition.

Definition 4.1.2. Given an equalised representation V of a the length ℓ rectangle $\Lambda(\ell,1)$, let $r^{ab}_{cd}, r^{ab}_{\infty\infty}$ and $r^{\infty\infty}_{ab}$ be the mutliplicities arising from the indecomposable decompositions of $\langle V_{x^+_a} \rangle$ and $\langle V_{x^-_c} \rangle$ as in 4.1 and 4.2. We define the **ladder multiplicity function** (LMF) of V to be the function f_V given by

$$\begin{aligned} f_V \colon \{0,1,\dots,\ell,\infty\}^4 &\mapsto \mathbb{N} \\ (a,b,c,d) &\mapsto \begin{cases} r^{ab}_{ad} & \text{if } 0 \leqslant c = a \leqslant b \leqslant d \leqslant \ell. \\ r^{ab}_{\infty\infty} & \text{if } 0 \leqslant a \leqslant b \leqslant \ell \text{ and } c = d = \infty. \\ r^{\infty\infty}_{cd} & \text{if } 0 \leqslant c \leqslant d \leqslant \ell \text{ and } a = b = \infty. \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This function is well defined by the above remark 4.1.1.

Remark 4.1.3. The ladder multiplicity function is a discrete invariant, defined exclusively for equalised representations of the rectangle quiver $\Lambda(\ell,1)$. The following proposition shows it is equivalent to the skyscraper invariant for equalised representations.

Proposition 4.1.4. *Let V be an equalised representation rectangle quiver $Q = \Lambda(\ell,1)$. Then the skyscraper invariant δ_V can be obtained from the ladder multiplicity function f_V of V and vice-versa.*

Proof. It suffices to show that the H-N type $\mathbf{T}[V;\alpha_x]$ can be obtained from the multiplicities of the indecomposables in the direct sum decomposition of $\langle V_x\rangle$ and vice versa. By Proposition 3.3.3, $\mathbf{T}[V;\alpha_x]$ can be deduced from $\mathbf{T}[\langle V_x\rangle;\alpha_x]$ via:

$$\mathbf{T}[V;\alpha_x](\lambda) = \begin{cases} \underline{\dim}\, V - \sum_{\lambda'} \mathbf{T}[\langle V_x\rangle;\alpha_x](\lambda') & \text{if } \lambda = 0 \\ \mathbf{T}[\langle V_x\rangle;\alpha_x](\lambda) & \text{otherwise} \end{cases}$$

Evidently $\underline{\dim}\, V$ can be obtained via both the skycraper invariant δ_V and the ladder multiplicity function f_V. Then it suffices to show that $\mathbf{T}[\langle V_x\rangle;\alpha_x]$ can be obtained from the multiplicities of the indecomposables in the direct sum decomposition of $\langle V_x\rangle$ and vice versa.

Whether $x = x_a^+$ or $x = x_c^-$, all indecomposables I appearing in decompositions 4.1 or 4.2 are α_x-stable with $\min(I) = \{x\}$ and α_x-slopes $\frac{1}{\operatorname{rank}(I)}$. Then by Proposition 3.2.5, all non-zero slopes of $\mathbf{T}[\langle V_x\rangle;\alpha_x]$ are of the form $\frac{1}{k}$ with $k = \operatorname{rank}(I)$ for some indecomposable I appearing in the decomposition of $\langle V_x\rangle$, with

$$\mathbf{T}[\langle V_x\rangle;\alpha_x](\frac{1}{k}) = \sum_{\operatorname{rank}(I)=k} r^I \cdot \underline{\dim}_I\,. \tag{4.3}$$

This gives the skyscraper invariant δ_V from the multiplicities r^I in the direct sum decomposition $\langle V_x\rangle$ into indecomposables, i.e. the ladder multiplicity function f_V.

By Lemma 3.6.9, the dimension vectors $\{\underline{\dim}_I \mid \operatorname{rank}(I) = k, \min(I) = \{x\}\}$ are linearly independant as maps $Q_0 \mapsto \mathbb{R}$, which gives us the multiplicites r^I from the skyscraper invariant. ☐

We give below two theoretical results which will be useful for computing the skyscraper invariant δ_V in the following subsection. The following requires no proof.

Proposition 4.1.5. *Given an equalised representation V of the quiver $Q = \Lambda(\ell, 1)$, if*

$$\langle V_{x_0^-}\rangle \simeq \bigoplus_{0\leqslant a\leqslant b\leqslant \ell} \mathbf{R}_{a,b}^{\infty,\infty\, r_{a,b}^{\infty\infty}},$$

then the multiplicity of the indecomposable $\mathbf{R}_{c,d}^{\infty,\infty}$ in $\langle V_{x_c^-}\rangle$ is $\sum_{0\leqslant u\leqslant c} r_{ud}^{\infty,\infty}$ for $c \leqslant d \leqslant \ell$.

We remind the reader that for $S_a = \{x_a^+, x_a^-\}$, $\langle V_{S_a}\rangle$ denotes the smallest subrepresentation of V containing both $V_{x_a^+}$ and $V_{x_a^-}$. In particular, $\langle V_{S_a}\rangle = \langle V_{x_a^+}\rangle + \langle V_{x_a^-}\rangle$ and is nest-free since bars on both top and bottom rows all start at a.

Proposition 4.1.6. *Let V be an equalised representation of the quiver $Q = \Lambda(\ell, 1)$, and $S_a = \{x_a^+, x_a^-\}$ for some $0 \leqslant a \leqslant \ell$. If*

$$\langle V_{S_a} \rangle \simeq \bigoplus_{d \leqslant b} \left(\mathbf{R}_{a,d}^{a,b}\right)^{r_{a,d}^{a,b}} \bigoplus_{b} \left(\mathbf{R}_{\infty,\infty}^{a,b}\right)^{r_{\infty,\infty}^{a,b}} \bigoplus_{d} \left(\mathbf{R}_{a,d}^{\infty,\infty}\right)^{r_{a,d}^{\infty,\infty}}$$

is the direct sum decomposition of $\langle V_{S_a} \rangle$ given by Theorem 3.6.2, then

$$\langle V_{a^+} \rangle \simeq \bigoplus_{d \leqslant b} \left(\mathbf{R}_{a,d}^{a,b}\right)^{r_{a,d}^{a,b}} \bigoplus_{b} \left(\mathbf{R}_{\infty,\infty}^{a,b}\right)^{r_{\infty,\infty}^{a,b}}.$$

Proof. Observe that

$$\langle V_{S_a} \rangle = \langle V_{x_a^+} \rangle + \langle V_{x_a^-} \rangle \simeq \left(\langle V_{a^+} \rangle \bigoplus \langle V_{a^-} \rangle\right) / \left(\langle V_{a^+} \rangle \cap \langle V_{a^-} \rangle\right).$$

All representations $\langle V_{a^+} \rangle$, $\langle V_{a^-} \rangle$ and $\langle V_{a^+} \rangle \cap \langle V_{a^-} \rangle$ are nest-free and so admit decompositions as in Theorem 3.6.2. Of those three, the only representation which can have either $\mathbf{R}_{a,d}^{a,b}$ or $\mathbf{R}_{\infty,\infty}^{a,b}$ as direct summands is $\langle V_{a^+} \rangle$. Furthermore $\langle V_{a^+} \rangle$ has no direct summands of the form $\mathbf{R}_{a,d}^{\infty,\infty}$. □

4.1.2 Implementation and algorithms

By Proposition 4.1.4, computing δ_V amounts to computing f_V, i.e. finding multiplicities of the indecomposable decompositions $\langle V_x \rangle$ for $x \in Q_0$. This subsection is dedicated to finding the ladder multiplicity function f_V algorithmically. We present these algorithms as pseudo-code. Throughout this section, V denotes an equalised representation of the length ℓ rectangle $Q = \Lambda(\ell, 1)$, with V^+ and V^- the restriction of V to the top and bottom row of Q.

Computationally, V is given by matrix representations $\mathbf{M}^+ = M_1^+, \ldots, M_\ell^+$, $\mathbf{M}^- = M_1^-, \ldots, M_\ell^-$ and $\mathbf{M} = M_0, \ldots, M_\ell$ of the maps $V_{e_1^+}, \ldots, V_{e_l^+}$, $V_{e_1^-}, \ldots, V_{e_l^-}$ and $V_{e_0}, \ldots, V_{e_l}$ where we have implicitly fixed bases for our vector spaces.

In Chapter 2, we provide an algorithm **CompPers** (3) that computes the barcode of a persistence module by putting the corresponding matrices in barcode form (at most one non-zero term in each row and column). It does so by performing elementary row and column operations at each step. Here we employ a slight tweak to that algorithm to put the matrices of the rows $\mathbf{M}^+$ and $\mathbf{M}^-$ in barcode form whilst adequately changing the vertical maps $\mathbf{M}$ by keeping track of the basis changes we do along the way. Observe V is given by

$$\begin{tikzcd}
V_0^+ \arrow[r, "M_1^+"] \arrow[d, "M_0"'] & V_1^+ \arrow[r, "M_2^+"] \arrow[d, "M_1"'] & \cdots \arrow[r, "M_{\ell-1}^+"] & V_{\ell-1}^+ \arrow[r, "M_\ell^+"] \arrow[d, "M_{\ell-1}"'] & V_\ell^+ \arrow[d, "M_\ell"'] \\
V_0^- \arrow[r, "M_1^-"'] & V_1^- \arrow[r, "M_2^-"'] & \cdots \arrow[r, "M_{\ell-1}^-"'] & V_{\ell-1}^- \arrow[r, "M_\ell^-"'] & V_\ell^-
\end{tikzcd}$$

Then using Section 2.3's notation for elementary row and column operations:

- Any column operation $\mathbf{C}_{p\leftarrow q}(\lambda)$ on M_i^+ is done via a basis change on V_{i-1}^+ which induces the same column $\mathbf{C}_{p\leftarrow q}(\lambda)$ operation on M_{i-1}.
- Any row operation $\mathbf{R}_{p\leftarrow q}(\lambda)$ on M_i^+ is done via a basis change on V_i^+ which induces the reverse column operation $\mathbf{R}_{q\leftarrow p}(-\lambda)$ on M_i.
- Any column operation $\mathbf{C}_{p\leftarrow q}(\lambda)$ on M_i^- is done via a basis change on V_{i-1}^- which induces the reverse row operation $\mathbf{R}_{q\leftarrow p}(-\lambda)$ on M_{i-1}.
- Any row operation $\mathbf{R}_{p\leftarrow q}(\lambda)$ on M_i^- is done via a basis change on V_i^- which induces the same row operation $\mathbf{R}_{p\leftarrow q}(\lambda)$ on M_i.

This yields the **BarcodeForm** process, which runs in $O(v_{\max}^3\ell)$ time, where $v_{\max}$ is the maximal dimension amongst all vector spaces of the representation V.

Algorithm 4: BarcodeForm

```
Input: matrix representations of $V$: $\mathbf{M}^+, \mathbf{M}^-$ and $\mathbf{M}$
Output: matrix representation of $V$: $\mathbf{A}^+$, $\mathbf{A}^-$ and $\mathbf{X}$ where $\mathbf{A}^+$ and $\mathbf{A}^-$ are
        in barcode form
$\mathbf{M}^+, \mathbf{M} \rightarrow \mathbf{A}^+, \mathbf{M}'$ where $\mathbf{A}^+$ in barcode form via CompPers where every
  time we perform a row (resp column) operation on $M_a^+$, we perform the
  same row (resp column) operation on $M_a$ (resp $M_{a-1}$).
$\mathbf{M}^-, \mathbf{M}' \rightarrow \mathbf{A}^-, \mathbf{X}$ where $\mathbf{A}^-$ in barcode form via CompPers where every
  time we perform a row (resp column) operation on $M_a^-$, we perform the
  opposite column (resp row) operation on $M'_{a-1}$ (resp $M'_a$).
return $A^+$,$A^-$,$X$
```

From now on, $\mathbf{A}^+$,$\mathbf{A}^-$ and $\mathbf{X}$ denotes the output of the algorithm **BarcodeForm**. Strictly speaking, we do not yet have the barcodes of V^+ and V^-, only matrix representations $\mathbf{A}^+$ and $\mathbf{A}^-$ of their linear maps in barcode form.

Observe that by construction, matrices in barcode form have at most one non-zero term in each row and column with entries in $\{0,1\}$. Then by iteratively going through matrices in barcode form, we may output sequences of basis vectors representing intervals in the decomposition, and in doing so giving us the barcode. These are the basis vector which realise an interval decomposition and were studied in the first chapter as barcode bases (Definition 2.2.1).

We may achieve this by for example using Algorithm 3 of [10], whilst keeping track of the index every time we extend a bar.

```
Algorithm 5: ExtractBarcode
    Input: Type $\mathbb{A}_\ell$ quiver representation $W$ given as matrix representations
            $A_1, \dots, A_\ell$ in barcode form
    Output: barcode $\mathbf{Bar}(W)$
```

This gives us $\mathbf{Bar}(V^+)$ and $\mathbf{Bar}(V^-)$ in $O(v_{\max}^2 \ell)$ time. For the purpose of this section the barcode $\mathbf{Bar}(W)$ of a length ℓ persistence module W will denote a function that assigns to each admissible interval $[a,b]$ a set of sequences of basis vectors associated to the intervals $\mathbf{I}_{[ab]}$ appearing in the decomposition of W. We denote $d^+_{[ab]}$ and $d^-_{[ab]}$ the mutliplicities of $\mathbf{I}[a,b]$ in V^+ and V^-.

The convention we use here is that if A is a m by n matrix representing a linear map from V to W, the associated basis of V and W are given by the sets $\{0,1,\dots,n-1\}$ and $\{0,1,\dots m-1\}$.

Example 4.1.7. Given a persistence module W

$$\begin{tikzcd} W_0 \arrow[r, "f_1"] & W_1 \arrow[r, "f_2"] & W_2 \arrow[r, "f_3"] & W_3 \end{tikzcd},$$

with matrices in barcode form

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we may extract the barcode as the function

$$\mathbf{Bar}(W) = \begin{cases} [0,4] \mapsto \{[0,0,0,0]\} \\ [0,1] \mapsto \{[1,1]\} \\ [1,4] \mapsto \{[2,1,1]\} \\ [a,b] \mapsto \varnothing \text{ otherwise} \end{cases}.$$

```
Algorithm 6: LocalBarcode
    Input: A barcode $\mathbf{Bar}(W)$ of a persistence module $W$ and index $0 \leqslant a \leqslant \ell$
    Output: Barcode $\mathbf{Bar}(\langle W_a \rangle)$
```

Then given $\mathbf{Bar}(V^-)$, Proposition 4.1.5 immediatly gives us the decomposition of $\langle V_x \rangle$ for $x = x_c^-$, via the simple subroutine which runs in $O(\mathbf{Bar}(V^-)) \leqslant O(v_{\max}\ell)$ time.

We now turn our attention to $x = x_a^+$, i.e. finding the ladder decomposition of $\langle V_{x_a^+} \rangle$ as in Theorem 2.5.3. Observe that this theorem is proved in a constructive manner, where we explicitly compute the decomposition. To do so, we first construct a single block matrix representing all vertical linear maps in given barcode bases, before performing admissible row and column operations to reduce this matrix and extract the decomposition.

We give a brief reminder of how this matrix is constructed. It is given by the upper triangular block matrix

$$X = \begin{bmatrix} X^{[0,0]}_{[0,0]} & X^{[0,1]}_{[0,0]} & \cdots & X^{[0,\ell]}_{[0,0]} & 0 & \cdots & 0 \\ 0 & X^{[0,1]}_{[0,1]} & \cdots & \cdots & X^{[1,\ell]}_{[0,1]} & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & X^{[\ell,\ell-1]}_{[\ell,\ell-1]} & X^{[\ell,\ell]}_{[\ell,\ell-1]} \\ 0 & 0 & & & & 0 & X^{[\ell,\ell]}_{[\ell,\ell]} \end{bmatrix} \tag{4.4}$$

where each block $X^{[a,b]}_{[c,d]}$ for $0 \leqslant c \leqslant a \leqslant d \leqslant b$ is the submatrix of X_a obtained by taking the d^+_{ab} columns corresponding to basis vectors part of an $[a, b]$ bar of V^+ and the d^-_{cd} rows corresponding to basis to basis vectors part of an $[c, d]$ bar of V^-. Observe that this is also a submatrix of $X_{a+1}, \cdots X_d$. This matrix depends on our chosen barcode bases for V^+ and V^-. Assuming we have lexicographically ordered our barcodes and corresponding barcode bases, we also have with this notation

$$X_a = \begin{bmatrix} X^{[0,a]}_{[0,a]} & X^{[0,a+1]}_{[0,a]} & \cdots & \cdots & 0 \\ 0 & X^{[0,a+1]}_{[0,a+1]} & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & X^{[a,\ell-1]}_{[a,\ell-1]} & X^{[a,\ell]}_{[a,\ell-1]} \\ 0 & 0 & \cdots & 0 & X^{[a,\ell]}_{[a,\ell]} \end{bmatrix}. \tag{4.5}$$

Obtaining this single block matrix representation for $\langle V_{x_a^+} \rangle$ would in theory first require us to compute this subrepresentation, i.e. find the barcodes of $\langle V^+_{x_a^+} \rangle$ and $\langle V^-_{x_a^+} \rangle$ and the matrix representations of the vertical maps in these barcode bases.

Proposition 4.1.8. *Barcode bases for V^+ and V^- naturally induce barcode bases for $\langle V_{S_a} \rangle^+ = \langle V_{x_a^+} \rangle^+$ and $\langle V_{S_a} \rangle^- = \langle V_{x_a^-} \rangle^-$, where $S_a = \{x_a^+, x_a^-\}$. In these bases, the single block matrix for $\langle V_{S_a} \rangle$ as in 4.4 is given by X_a.*

Proof. Denote by X the single block matrix for V in some barcode bases for V^+ and V^- and $\tilde{X}$ the single block matrix for $\langle V_{S_a} \rangle$ in the induced barcode basesd. All bars of $\langle V_{S_a} \rangle^+$ or $\langle V_{S_a} \rangle^+$ start at a so that

$$\tilde{X} = \begin{bmatrix} \tilde{X}^{[a,a]}_{[a,a]} & \tilde{X}^{[a,a+1]}_{[a,a]} & \cdots & & \tilde{X}^{[a,\ell]}_{[a,a]} \\ 0 & \tilde{X}^{[a,a+1]}_{[a,a+1]} & \cdots & & \tilde{X}^{[a,\ell]}_{[a,a+1]} \\ \vdots & \vdots & \ddots & & \\ & & & & \ddots \\ & & & \tilde{X}^{[a,\ell-1]}_{[a,\ell-1]} & \tilde{X}^{[a,\ell]}_{[a,\ell-1]} \\ & & & 0 & \tilde{X}^{[a,\ell]}_{[a,\ell]} \end{bmatrix}.$$

Observe that bars $[a, b]$ in $\langle V^+_{S_a} \rangle$ (or $\langle V^+_{S_a} \rangle$) come precisely from bars $[a', b]$ of V^+ (or V^-) for $0 \leqslant a' \leqslant a$. With this mind, we see that

$$\tilde{X}^{[a,b]}_{[a,d]} = \begin{bmatrix} X^{[0,b]}_{[0,d]} & X^{[1,b]}_{[0,d]} & \cdots & \cdots & X^{[a,b]}_{[0,d]} \\ 0 & X^{[1,d]}_{[1,d]} & \cdots & \cdots & X^{[a,b]}_{[1,d]} \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & X^{[a-1,b]}_{[a-1,d]} & X^{[a,b]}_{[a-1,d]} \\ 0 & 0 & \cdots & 0 & X^{[a,d]}_{[a,b]} \end{bmatrix}$$

which together with 4.5 shows the result. $\square$

For the remainder of this section, given $0 \leqslant a \leqslant \ell$,= we denote $S_a = \{x_a^+, x_a^-\}$, $\mathbf{Bar}(V_a^+) = \mathbf{Bar}(\langle V_{a^+}\rangle^+)$ and $\mathbf{Bar}(V_a^-) = \mathbf{Bar}(\langle V_{a^-}\rangle^-)$. By Proposition 4.1.6, it suffices to find the decomposition of $\langle V_{S_a}\rangle$ to find the decomposition of $\langle V_{x_a^+}\rangle$. By the above Proposition 4.1.8, the single block matrix for $\langle V_{S_a}\rangle$ is given by X_a. Then as seen in the proof of Theorem 2.5.3, extracting the decomposition of $\langle V_{S_a}\rangle$ is then a simple exercise of reducing the matrix X_a by performing legal changes of barcode bases. This requires $O(v_{\max}^3)$ operations, and extracting the corresponding decomposition require a further $O(v_{\max}^2)$ pass through the reduced matrix. This yields the following subroutine:

```
Algorithm 7: LadderDecomp
    Input: Matrix $X_a$ and barcodes $\mathbf{Bar}(V_a^+)$, $\mathbf{Bar}(V_a^-)$
    Output: Decomposition of $\langle V_a^+\rangle$ as an array containing multiplicities of all
            indecomposables
```

Knowing which row and column operations can be done on X_a amounts to knowing the barcodes $\mathbf{Bar}(V_a^+)$ and $\mathbf{Bar}(V_a^-)$, and which row and column each bar represents. This information can be derived from $\mathbf{Bar}(V^+)$ and $\mathbf{Bar}(V^-)$ by simply iterating through the bars.

We combine all the above subroutines to obtain the **LadderMultiplicityFunction** algorithm which takes as input a ladder persistence module V given in matrix form $\mathbf{M}^+$, $\mathbf{M}^-$ and $\mathbf{M}$ and outputs its ladder mutiplicity function f_V (Definition 4.1.2). Denote

- $I_a^+ = \{(a,b,a,d) | 0 \leqslant a \leqslant b \leqslant d \leqslant \ell\} \bigcup \{(a,b,\infty,\infty) | 0 \leqslant a \leqslant b \leqslant \ell\}$ the subset of $\{0,1,\cdots,\infty\}^4$ giving tuples (a,b,c,d) for which r_{cd}^{ab} can be non-zero in the decomposition 4.1 of $\langle V_{x_a^+}\rangle$.

- $I_a^- = \{(\infty,\infty,c,d) | 0 \leqslant c \leqslant d \leqslant\leqslant \ell\}$ the subset of $\{0,1,\cdots,\infty\}^4$ giving tuples (a,b,c,d) for which r_{cd}^{ab} can be non-zero in the decomposition 4.2 of $\langle V_{x_a^-}\rangle$.

Remark 4.1.9. It is natural to wonder whether we can generalise the above approach to compute the skycraper invariant for more general two-dimensional persistence modules. Indeed, if V is an equalised representation of a two dimensional grid quiver $\Lambda(\ell, n)$ for some $n \geqslant 1$ and x is some vertex of $\Lambda(\ell, n)$, it remains the case that $\langle V_x\rangle$ has no pairs of strictly nested bars across all its rows. As seen in remark 2.5.6, Theorem 3.6.2 does not generalise to rectangle quiver with more than

Algorithm 8: LadderMultiplicityFunction

```
Input: matrix representations of $V$: $\mathbf{M}^+$, $\mathbf{M}^-$ and $\mathbf{M}$
Output: ladder multiplicity function $f_V$ of $V$
$\mathbf{A}^+, \mathbf{A}^-, \mathbf{X} = \mathbf{BarcodeForm}(\mathbf{M}^+, \mathbf{M}^-, \mathbf{M})$
$\mathbf{Bar}(V^+), \mathbf{Bar}(V^-) = \mathbf{ExtractBarcode}(\mathbf{A}^+), \mathbf{ExtractBarcode}(\mathbf{A}^-)$
for $a = 0, \dots, \ell$ do
    $\mathbf{Bar}(V_a^+), \mathbf{Bar}(V_a^-)$=LocalBarcode($\mathbf{Bar}(V^+), \mathbf{Bar}(V^-), a$)
    $f_V(I_a^-) = \mathbf{Bar}(V_a^-)$
    $f_V(I_a^+) = \mathbf{LadderDecomp}(X_a, \mathbf{Bar}(V_a^+), \mathbf{Bar}(V_a^-))$
return $(f_V)$
```

two rows. That is, unlike for ladder persistence modules, we cannot expect a decomposition of $\langle V_x \rangle$ into simple indecomposables, allowing for a straightforward computation of Harder-Narasimahn types.

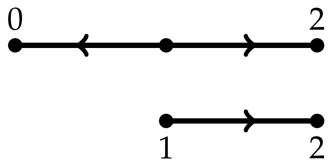
Remark 4.1.10. In Section 2.6, we introduce the notion of nested bars for zigzag persistence modules allowing for a generalisation of Theorem 2.5.3 to maps of zigzag persistence (Theorem 2.6.14). As such, it is natural to wonder whether the above method can be adapted for computing skyscraper invariant of more general zigzag ladders. The key observation for standard ladder persistence was that $\langle V_x \rangle$ was nest-free, allowing for a decomposition via Theorem 2.5.3. For zigzags, this is not the case. Consider the representation V of the zigzag ladder

$$\begin{tikzcd}
x_0^+ \arrow[d, "e_0"'] & x_1^+ \arrow[l, "e_1^+"'] \arrow[r, "e_2^+"] \arrow[d, "e_1"'] & x_2^+ \arrow[d, "e_2"'] \\
x_0^- & x_1^- \arrow[l, "e_1^-"'] \arrow[r, "e_2^-"] & x_2^-
\end{tikzcd}$$

given by

$$\begin{tikzcd}
\mathbb{F} \arrow[d, "0"'] & \mathbb{F}^2 \arrow[l, "{[1]}"'] \arrow[r, "Id"] \arrow[d, "{[1,1]}"'] & \mathbb{F} \arrow[d, "0"'] \\
0 & \mathbb{F} \arrow[l, "0"'] \arrow[r, "0"] & \mathbb{F}
\end{tikzcd}$$

In the language of Section 2.5, this is given by zigzag persistence modules V^+ and V^- with barcodes



and

and single block matrix representing horizontal maps given by $\begin{array}{ccc} [0,2] & [1,2] & \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & \quad 1 \end{bmatrix}} & [1,1] \end{array}$.

In the language of Section 2.6,the orientation of our zigzag is $\tau = qf$. Then the bar $[1,2]$ is strictly nested in the bar $[0,2]$ with regards to τ (Definition 2.6.13). This shows stabiliser changes of basis for V^+ and V^- are trivial (Theorem 2.6.11) and that V is indecomposable. Then we have $V = \langle V_{x_1^+} \rangle$ admitting strictly nested bars with regards to its orientation τ and not admitting a decomposition as in Theorem 2.6.14.

4.2 The ladder invariant

The previous section illustrates the use of the ladder decomposition Theorem 3.6.2 for computing the skyscraper invariant in specific cases. In this section, we leverage this further by defining the ladder invariant.

4.2.1 Definition and initial properties

Definition 4.2.1. A ladder $\mathscr{L}$ of length ℓ of a quiver Q is given by the following data:

- Subsets of vertices $(a_0, \cdots, a_\ell)$ and $(b_0, \cdots, b_\ell)$ of ${Q_0}^\ell$.
- Directed paths $p_i \colon a_{i-1} \mapsto a_i$ and $q_i \colon b_{i-1} \mapsto b_i$ for $1 \leqslant i \leqslant \ell$.
- Directed paths $h_i \colon a_i \mapsto b_i$ for $0 \leqslant i \leqslant \ell$.

Given a representation V of Q, we denote $V^{\mathscr{L}}$ the ladder persistence module for which $V^{\mathscr{L}}_{x_i^+} = V_{a_i}$, $V^{\mathscr{L}}_{x_i^-} = V_{b_i}$, $V^{\mathscr{L}}_{e_i^+} = V_{p_i}$, $V^{\mathscr{L}}_{e_i^-} = V_{q_i}$ and $V^{\mathscr{L}}_{e_i} = V_{h_i}$. That is, $V^{\mathscr{L}}$ may be represented as

$$\begin{tikzcd}
V_{a_0} \arrow[r, "V_{p_1}"] \arrow[d, "V_{h_0}"'] & V_{a_1} \arrow[r, "V_{p_2}"] \arrow[d, "V_{h_1}"'] & \cdots \arrow[r, "V_{p_{\ell-1}}"] & V_{a_{\ell-1}} \arrow[r, "V_{p_\ell}"] \arrow[d, "V_{h_{\ell-1}}"'] & V_{a_\ell} \arrow[d, "V_{h_\ell}"'] \\
V_{b_0} \arrow[r, "V_{q_1}"'] & V_{b_1} \arrow[r, "V_{q_2}"'] & \cdots \arrow[r, "V_{q_{\ell-1}}"'] & V_{b_{\ell-1}} \arrow[r, "V_{q_\ell}"'] & V_{b_\ell}
\end{tikzcd}$$

Remark 4.2.2. Two different ladders $\mathscr{L}_1$ and $\mathscr{L}_2$ of a quiver Q may have the same vertex sets but different paths, yielding different ladder representations $V^{\mathscr{L}_1}$ and $V^{\mathscr{L}_2}$. When V is equalised, we would have $V^{\mathscr{L}_1} = V^{\mathscr{L}_2}$.

Remark 4.2.3. Observe we do not require distinct vertices in our ladders and allow the trivial path $v \to v$ for any vertex $v \in Q_0$.

Definition 4.2.4. Given a ladder $\mathscr{L}$ of length ℓ of a quiver Q, we denote by $\delta_\bullet^{\mathscr{L}}$ the invariant on $\mathbf{Rep}(Q)$ defined by $\delta_V^{\mathscr{L}} = \delta_{V^{\mathscr{L}}}$ where $\delta_\bullet$ denotes the skycraper invariant on the length ℓ rectangle quiver.

We define the **ladder invariant** $\sigma_\bullet$ as the collection of invariants $\delta_\bullet^{\mathscr{L}}$ for all ladders $\mathscr{L}$ of Q. That is, $\sigma_\bullet$ assigns to a representation V of $\mathbf{Rep}(Q)$ the set of invariants $\{\delta_V^{\mathscr{L}} \mid \mathscr{L} \text{ is a ladder of } Q\}$.

By Proposition 4.1.4, the skyscraper invariant for equalised representations of the ladder quiver $\Lambda(\ell, 1)$ is equivalent to the ladder multiplicity function (Definition 4.1.2), i.e. having all indecomposable multiplicities of $\langle V_x \rangle$ as in 4.1 and 4.2 for all vertices x of $\Lambda(\ell, 1)$. This gives the following useful corollary for characterising the ladder invariant of equalised representations.

Corollary 4.2.5. *Given a finite quiver Q, the ladder invariant on an equalised representation V of Q is equivalent to having the multiplicities of the decomposition as in Theorem 2.5.3 of $\langle V_{x_0^+}^{\mathscr{L}} \rangle$ for all possible ladders $\mathscr{L}$ of Q, where x_0^+ is the top left vertex of the ladder* $\Lambda(\ell, 1)$.

Proof. By Proposition 4.1.4, the ladder invariant is equivalent to having all ladder multiplicity functions of $V^{\mathscr{L}}$ for all possible ladders $\mathscr{L}$. Now given a vertex x of a $\Lambda(\ell, 1)$, either $x = x_i^+$ for some $0 \leqslant i \leqslant \ell$ in which case we may consider the sub-ladder $\mathscr{L}'$ given by vertices $(a_i, a_{i+1}, \cdots, a_\ell)$ and $(b_i, b_{i+1}, \cdots, b_\ell)$ taking the same directed paths, or $x = x_i^-$ for some $0 \leqslant i \leqslant \ell$ in which case we may consider the ladder $\mathscr{L}'$ given by vertices $(b_i, b_{i+1}, \cdots, b_\ell)$ and $(b_i, b_{i+1}, \cdots, b_\ell)$ with all horizontal paths given by the paths q_j for $i \leqslant j \leqslant \ell$ and all vertical paths h_j being the trivial ones from b_j to b_j. In both these cases, the decomposition of $\langle V_x^{\mathscr{L}} \rangle$ can be obtained from the decomposition of $\langle V_{x_0^+}^{\mathscr{L}'} \rangle$. $\square$

Proposition 4.2.6. *Let Q be a finite acyclic quiver. The ladder invariant is strictly more discriminative than the rank invariant on* $\mathbf{Rep}(Q)$.

Proof. Given an equalised representation V of Q and vertices $x \leqslant y$, we wish to show we can find $\rho_V(x,y) = \dim \langle V_x \rangle_y$ from the ladder invariant. Since Q is finite and acyclic, there are finitely many paths $p_1, p_2, \cdots, p_\ell$ from x to y. Construct the ladder $\mathscr{L}$ of Q with vertex sets $\{x, x, \cdots, x\}$ and $\{y, y, \cdots, y\}$ and vertical maps all the paths p_i. This gives $V^{\mathscr{L}}$

$$\begin{tikzcd}
V_x \arrow[r, "Id"] \arrow[d, "V_{p_0}"'] & V_x \arrow[r, "Id"] \arrow[d, "V_{p_1}"'] & \cdots \arrow[r, "Id"] & V_x \arrow[r, "Id"] \arrow[d, "V_{p_{\ell-1}}"'] & V_x \arrow[d, "V_{p_\ell}"'] \\
V_y \arrow[r, "Id"'] & V_y \arrow[r, "Id"'] & \cdots \arrow[r, "Id"'] & V_y \arrow[r, "Id"'] & V_y
\end{tikzcd}$$

We show $\langle V_x \rangle_y = \langle V^{\mathscr{L}}_{x_0^+} \rangle_{x_\ell^-}$. Any vector a of $\langle V_x \rangle_y$ arises as some $V_{p_i}(v)$ for some $0 \leqslant i \leqslant \ell$ and some $v \in V_x$. Consider the path $q = e_\ell^- \circ \cdots e_{i+1}^- \circ e_i \circ e_i^+ \cdots \circ e_1^+$ of the length ℓ ladder. Then $V_q^{\mathscr{L}} = Id \circ \cdots Id \circ V_{p_i} \circ Id \cdots \circ Id = V_{p_i}$ so that $a = V_{p_i}(v) = V_q^{\mathscr{L}}(v) \in \langle V^{\mathscr{L}}_{x_0^+} \rangle_{x_\ell^-}$. Observe that any path in the length ℓ ladder from x_0^+ to x_ℓ^- is of the form $q_j := e_\ell^- \circ \cdots e_{j+1}^- \circ e_j \circ e_j^+ \cdots \circ e_1^+$ for some $0 \leqslant j \leqslant \ell$, so that any $a \in \langle V^{\mathscr{L}}_{x_0^+} \rangle_{x_\ell^-}$ arise as some $V_{q_i}^{\mathscr{L}}(v)$ for some $v \in V_x$. Then $a = V_{p_i}(v) \in \langle V_x \rangle_y$. By Theorem 3.3.5, we may obtain $\dim(\langle V^{\mathscr{L}}_{x_0^+} \rangle_{x_\ell^-}) = \dim(\langle V_x \rangle_y) = \rho_V(x,y)$ from the skyscraper invariant $\delta_V^{\mathscr{L}}$, which shows the ladder invariant is finer than the rank invariant.

To see it is strictly finer, we may use the same example as in the proof of Theorem 3.3.5. Namely we may consider the representations $\mathbf{R}_{0,0}^{0,0} \bigoplus \mathbf{R}_{\infty,\infty}^{0,1}$ and $\mathbf{R}_{0,0}^{0,1}$ on the square quiver $Q = \Lambda(1,1)$

$$\begin{tikzcd}
\mathbb{F}^2 \arrow[r, "{[1,0]}"'] \arrow[d, "{[0,1]}"] & \mathbb{F} \arrow[d] \\
\mathbb{F} \arrow[r] & 0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{F}^2 \arrow[r, "{[1,0]}"'] \arrow[d, "{[1,0]}"] & \mathbb{F} \arrow[d] \\
\mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

These have the same rank invariant but different ladder invariants, as seen when considering the ladder $\mathscr{L} = Q$. $\square$

4.2.2 Comparing to the skyscraper invariant

Proposition 4.2.7. *The ladder invariant is non comparable to the skyscrapper in the following way:*

- *There exists a quiver Q and representations V and W of Q for which $\delta_V = \delta_W$ but $\sigma_V \neq \sigma_W$.*

- *There exists a quiver Q' and representations V' and W' of Q' for which $\delta_{V'} \neq \delta_{W'}$ but $\sigma_{V'} = \sigma_{W'}$.*

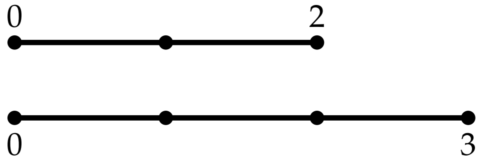
Proof. Consider the quiver $Q = \Lambda(3, 2)$. Consider the representation V given by

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \\
\mathbb{F}^2 \arrow[u, "{[1\ 1]}"'] \arrow[r, "{[0\ 1]}"] & \mathbb{F} \arrow[u, "0"'] \arrow[r, "{[1]}"] & \mathbb{F} \arrow[u, "0"'] \arrow[r, "0"] & 0 \arrow[u, "0"'] \\
\mathbb{F}^2 \arrow[u, "{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}"'] \arrow[r, "\mathrm{Id}"] & \mathbb{F}^2 \arrow[u, "{[1\ 0]}"'] \arrow[r, "\mathrm{Id}"] & \mathbb{F}^2 \arrow[u, "{[1\ 0]}"'] \arrow[r, "{[0\ 1]}"] & \mathbb{F} \arrow[u, "0"']
\end{tikzcd}$$

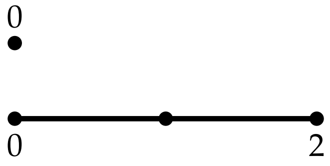
and W given by

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \\
\mathbb{F}^2 \arrow[u, "{[1\ 0]}"'] \arrow[r, "{[0\ 1]}"] & \mathbb{F} \arrow[u, "0"'] \arrow[r, "{[1]}"] & \mathbb{F} \arrow[u, "0"'] \arrow[r, "0"] & 0 \arrow[u, "0"'] \\
\mathbb{F}^2 \arrow[u, "{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}"'] \arrow[r, "\mathrm{Id}"] & \mathbb{F}^2 \arrow[u, "{[1\ 0]}"'] \arrow[r, "\mathrm{Id}"] & \mathbb{F}^2 \arrow[u, "{[1\ 0]}"'] \arrow[r, "{[0\ 1]}"] & \mathbb{F} \arrow[u, "0"']
\end{tikzcd}$$

In the language of Section 2.5, this is given by persistence modules V^0 and W^0 with barcodes



V^1 and W^1 with barcodes



and V^2 and W^2 with barcodes

with horizontal map $\phi_V \colon V^0 \mapsto V^1$ and $\phi_W \colon W^0 \mapsto W^1$ given by identical block matrix representation

$$\begin{array}{cc} \begin{matrix}[0,2] & [0,3]\end{matrix} & \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{matrix}[0,0]\\ [0,2]\end{matrix} \end{array},$$

and horizontal maps $\psi_V \colon V^1 \mapsto V^2$ and $\psi_W \colon W^1 \mapsto W^2$ given by block matrix representations

$$\begin{array}{cc} \begin{matrix}[0,0] & [0,2]\end{matrix} & \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & [0,0] \end{array} \quad \text{and} \quad \begin{array}{cc} \begin{matrix}[0,0] & [0,2]\end{matrix} & \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & [0,0] \end{array}.$$

We first show $\delta_V = \delta_W$. Observe first that given any vertex $x \neq (0,0)$, $\langle V_x \rangle \simeq \langle W_x \rangle$. Indeed, for $x = (x_0, x_1)$, if $x_0 > 1$ then $\langle V_x \rangle^2 = \langle W_x \rangle^2 = 0$ which shows the result since $\phi_V = \phi_W$ in identical barcode bases. Then for $x = (0,1)$, $\langle V_x \rangle$ and $\langle W_x \rangle$ can both be identified as nest-free ladder representations, whose decomposition is given by $\mathbf{R}^{[0,0]}_{[0,0]} \bigoplus \mathbf{R}^{[0,2]}_{[\infty,\infty]}$ (see the proof of Theorem 2.5.3 for obtaining the decomposition of $\langle W_x \rangle$). For $x = (0,2)$ this trivially holds.

Then it suffices to show $\mathbf{T}[V;\alpha_x] = \mathbf{T}[W;\alpha_x]$ for $x = (0,0)$. We show both V and W are α_x-semistable. Both V and W have α_x-slope $\frac{2}{12} = \frac{1}{6}$. Observe that given any subrepresentation V' of V, $\mu_{\alpha_x}(V') = \frac{\dim(V'_x)}{\operatorname{rank}(V')} \leqslant \frac{\dim(V'_x)}{\operatorname{rank}(\langle V_x \rangle)} = \mu_{\alpha_x}(\langle V'_x \rangle)$ so that it suffices to consider subrepresentation for which $V' = \langle V'_x \rangle$. Since V_x is two-dimensional, this amounts to a choice of a single vector $v = \beta_0 a + \beta_0 b$ of V_x, where a and b denote the vectors of V_x forming the $[0,2]$ and $[0,3]$ bars.

For $\beta_0, \beta_1 \in \mathbb{F}$, denote $V(\beta_0, \beta_1)$ the subrepresentation of V with $V(\beta_0, \beta_1) = \langle V(\beta_0, \beta_1)_x \rangle$ and $V(\beta_0, \beta_1)_x = \langle \beta_0 a + \beta_1 b \rangle$. Then observe that $\mu_{\alpha_x}(V(0,1)) = \frac{1}{6}$, $\mu_{\alpha_x}(V(\beta_0, \beta_1)) = \frac{1}{6}$ whenever $\beta_0 + \beta_1 = 0$ and $\mu_{\alpha_x}(V(\beta_0, \beta_1)) = \frac{1}{7}$ otherwise which shows V is α_x-semistable. Employing the same notation for W, we have $\mu_{\alpha_x}(W(1,0)) = \frac{1}{6}$, $\mu_{\alpha_x}(W(0,1)) = \frac{1}{6}$ and $\mu_{\alpha_x}(W(\beta_0, \beta_1)) = \frac{1}{7}$ otherwise which shows W is α_x-semistable.

To see $\sigma_V \neq \sigma_W$, consider the ladder $\mathscr{L}$ of $\Lambda(3,2)$

$$\begin{tikzcd}
(0,2) \arrow[r] & (1,2) \arrow[r] & (2,2) \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[u]
\end{tikzcd}$$

Note that since both V and W are equalised representations we do not need to specify paths between our chosen vertices. Now both $V^{\mathscr{L}}$ and $W^{\mathscr{L}}$ are nest-free ladders with identical barcodes and single block matrix representations

$$\begin{array}{ccc} [0,0] & [0,2] & \\ [\;\; 1 & 1\;\;] & [0,0] \end{array} \quad \text{and} \quad \begin{array}{ccc} [0,0] & [0,2] & \\ [\;\; 0 & 1\;\;] & [0,0] \end{array} .$$

Referring again to the proof of Theorem 2.5.3, this gives

$$V^{\mathscr{L}} \simeq \mathbf{R}_{[0,0]}^{[0,0]} \bigoplus \mathbf{R}_{[\infty,\infty]}^{[0,2]} \not\simeq \mathbf{R}_{[0,0]}^{[0,2]} \bigoplus \mathbf{R}_{[\infty,\infty]}^{[0,0]} \simeq W^{\mathscr{L}}$$

which by Proposition 4.1.4 shows $\sigma_V \neq \sigma_W$.

Next consider the quiver type $\mathbb{A}_2$ quiver Q'

$$\begin{tikzcd} x_0 & x_1 \arrow[l] \arrow[r] & x_2 \end{tikzcd}$$

and representations V' and W'

$$\begin{tikzcd} \mathbb{F} & \mathbb{F}^2 \arrow[l, "{[1,0]}"'] \arrow[r, "{[1,0]}"] & \mathbb{F} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{F} & \mathbb{F}^2 \arrow[l, "{[0,1]}"'] \arrow[r, "{[1,0]}"] & \mathbb{F}. \end{tikzcd}$$

Observe that no ladder $\mathscr{L}$ of Q can have both vertices x_0 and x_2. Indeed, given two vertices a and b of a ladder, there exists some vertex c with directed paths from a to c and b to c. There is no such vertex for x_0 and x_2. Then any ladder of Q is made of vertices from the sets $\{x_0, x_1\}$ or $\{x_1, x_2\}$ which always yield isomorphic ladder persistence modules $V'^{\mathscr{L}} \simeq W'^{\mathscr{L}}$. Then $\sigma_{V'} = \sigma_{W'}$.

For $x = x_1$, observe that V' has a subrepresentation of slope 1 whilst W' is α_x-semistable of slope $\frac{1}{2}$. This shows $\delta_{V'} \neq \delta_{W'}$. □

In our Definition 4.2.1 for ladders, we allow for vertices to appear multiple times. This means for certain, even finite and acyclic quivers, we may construct infinitely many ladders $\mathscr{L}$ of Q. As such it is not clear whether the ladder invariant is a discrete invariant. We adress this for equalised representations.

Proposition 4.2.8. *Let Q be a finite acyclic quiver. Then the ladder invariant $\sigma_\bullet$ is discrete on $\mathbf{Rep}_{\mathrm{eq}}(Q)$ in the following sense: there exists finitely many ladders $\mathscr{L}_1, \mathscr{L}_2, \cdots \mathscr{L}_n$ of Q for which given equalised representations V and W of Q,*

$$\delta_V^{\mathscr{L}_i} = \delta_W^{\mathscr{L}_i} \text{ for all } 1 \leqslant i \leqslant n \Rightarrow \sigma_V = \sigma_W.$$

Proof. We remind the reader that given vertices x and y of Q_0, we say $x \leqslant y$ if there is a directed path from x to y. Denote $\eta(x) := \mid \{y \in Q_0 \mid x \leqslant y\} \mid$, $\nu(x) := \mid \{y \in Q_0 \mid y \leqslant x\} \mid$, and $\ell := \sum_{x \in Q_0} \max(\eta(x), \nu(x))$.

Then to prove the result, it suffices to show that given $V, W \in \mathbf{Rep}_{\mathrm{eq}}(Q)$, if $\delta_V^{\widetilde{\mathscr{L}}} = \delta_W^{\widetilde{\mathscr{L}}}$ for all ladders $\widetilde{\mathscr{L}}$ of Q of length smaller than ℓ then $\sigma_V = \sigma_W$. By Corollary 4.2.5, it suffices to show that given a ladder $\mathscr{L}$ of Q, we can obtain the decomposition of $\langle V_{x_0^+}^{\mathscr{L}} \rangle$ from the decomposition of $\langle V_{x_0^+}^{\widetilde{\mathscr{L}}} \rangle$ for some ladder $\widetilde{\mathscr{L}}$ of Q of length smaller than ℓ.

Assume we have in our ladder $\mathscr{L}$ some j for which $a_j = a_{j+1} := a$ and $b_j = b_{j+1} := b$. Observe that since V is equalised, the path chosen between two vertices is irrelevant, and we denote the linear map from V_a to V_b $V_{a,b}$. Since Q is acyclic, we have $V_{a,a} = Id$. Then the following structure occurs in $V^{\mathscr{L}}$

$$\begin{tikzcd}
\cdots & V_{a_{i-1}} \arrow[r, "V_{a_{i-1},a}"] \arrow[d, "V_{a_{i-1},b_{i-1}}"'] & V_a \arrow[r, "Id"] \arrow[d, "V_{a,b}"'] & V_a \arrow[r, "V_{a,a_{i+1}}"] \arrow[d, "V_{a,b}"'] & V_{a_{i+1}} \arrow[d, "V_{a_{i+1},b_{i+1}}"'] & \cdots \\
\cdots & V_{b_{i-1}} \arrow[r, "V_{b_{i-1},b}"'] & V_b \arrow[r, "Id"'] & V_b \arrow[r, "V_{b,b_{i+1}}"'] & V_{b_{i+1}} & \cdots
\end{tikzcd} \tag{4.6}$$

Consider the ladder $\mathscr{L}^{-1}$ where we remove the redundant vertices a_{i+1} and b_{i+1} so that $V^{\mathscr{L}^{-1}}$

$$\begin{tikzcd}
\cdots & V_{a_{i-1}} \arrow[r, "V_{a_{i-1},a}"] \arrow[d, "V_{a_{i-1},b_{i-1}}"'] & V_a \arrow[r, "V_{a,a_{i+1}}"] \arrow[d, "V_{a,b}"'] & V_{a_{i+1}} \arrow[d, "V_{a_{i+1},b_{i+1}}"'] & \cdots \\
\cdots & V_{b_{i-1}} \arrow[r, "V_{b_{i-1},b}"'] & V_b \arrow[r, "V_{b,b_{i+1}}"'] & V_{b_{i+1}} & \cdots
\end{tikzcd}$$

Then we may obtain the multiplicities of the decomposition of $\langle V_{x_0^+}^{\mathscr{L}} \rangle$ by having those of the decomposition $\langle V_{x_0^+}^{\mathscr{L}^{-1}} \rangle$. This can be seen by observing that the single block matrices constructed in the proof of Theorem 2.5.3 to extract the decomposition is the same for $\langle V_{x_0^+}^{\mathscr{L}} \rangle$ and $\langle V_{x_0^+}^{\mathscr{L}^{-1}} \rangle$. Then we may assume without loss of generality that our ladder $\mathscr{L}$ contains no structure as in 4.6.

Then it suffices to show that any vertex x of Q_0 can appear at most $\eta(x)$ times in the top row and at most $\nu(x)$ times in the bottom row in any such $\mathscr{L}$ to obtain the result. Observe first that since Q is acyclic, any multiple appearances of a vertex x in either row must happen consecutively. Else we have $x \leqslant y \cdots \leqslant x$ given us a cycle in Q.

Now assume we have a consecutive chain of x in the top row of $\mathscr{L}$ that is longer than $\eta(x)$. By the pigeonhole principle, some $b \in \{y \in Q_0 \mid x \leqslant y\}$ must appear at least twice in the bottom row vertices of $\mathscr{L}$ connected to x. Having excluded scenarios as in 4.6, this multiple occurrence of b cannot be consecutive, which would yield a cycle in Q

$$\begin{tikzcd}
\cdots \arrow[r] & x \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & \cdots \arrow[r] & x \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & b \arrow[r] & y \arrow[r] & \cdots \arrow[r] & b \arrow[r] & \cdots
\end{tikzcd}$$

Then we must have a structure as in 4.6, a contradiction. A similar argument shows we cannot have a consecutive chain of x in the bottom row of $\mathscr{L}$ of length more than $\nu(x)$. ☐

Remark 4.2.9. It is as of now unclear whether the above result can be generalised to non-equalised representations. Looking at the quiver Q

$$\begin{tikzcd}
x \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] & y
\end{tikzcd}$$

we may construct infinitely many length ℓ ladders $\mathscr{L}(\ell)$ with top row $(x, \cdots, x)$, bottom row $(y, \cdots, y)$ and horizontal maps from x to y alternating between p_0 and p_1 giving, $V^{\mathscr{L}(\ell)}$

$$\begin{tikzcd}
V_x \arrow[r, "Id"] \arrow[d, "V_{p_0}"'] & V_x \arrow[r, "Id"] \arrow[d, "V_{p_1}"'] & V_x \arrow[r, "Id"] \arrow[d, "V_{p_0}"'] & \cdots \arrow[r, "Id"] & V_x \arrow[r, "Id"] \arrow[d, "V_{p_0}"'] & V_x \arrow[d, "V_{p_1}"'] \\
V_y \arrow[r, "Id"] & V_y \arrow[r, "Id"] & V_y \arrow[r, "Id"] & \cdots \arrow[r, "Id"] & V_y \arrow[r, "Id"] & V_y
\end{tikzcd}$$

Understanding whether there is a finite collection of ladders $\mathscr{L}_1, \mathscr{L}_2, \cdots \mathscr{L}_n$ for which the skyscraper invariant of $V^{\mathscr{L}(\ell)}$ for any length $\ell \geqslant 1$ can be extracted from the skyscraper invariants of $V^{\mathscr{L}_1}, V^{\mathscr{L}_2}, \cdots V^{\mathscr{L}_n}$ is key in understanding whether the ladder invariant is discrete in full generality.

As seen in the proof of Proposition 4.2.8, given a finite acyclic quiver Q there are finitely many ladders of Q that entirely determine the ladder invariant. Considering the standard 2-dimensional grid quiver $\Lambda(\ell_1, \ell_2)$ for $\ell_1, \ell_2 \geqslant 1$, we say a ladder $\mathscr{L}$ of Q is **canonical** if it is obtain by taking two rows or two columns of $\Lambda(\ell_1, \ell_2)$ as the rows of $\mathscr{L}$. For example, for $Q = \Lambda(2, 2)$,

$$\begin{tikzcd}
(0,2) \arrow[r] & (1,2) \arrow[r] & (2,2) \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[u]
\end{tikzcd}$$

is a canonical ladder obtained by taking the first and third row and

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \arrow[r] & (2,1) \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[u]
\end{tikzcd}$$

is a canonical ladder obtained by taking the first and second column. It is natural to wonder whether these canonical ladders are enough to entirely determine the ladder invariant.

Proposition 4.2.10. *There exists equalised representations V and W of $\Lambda(2,2)$ for which $\delta_V^{\mathscr{L}_c} = \delta_W^{\mathscr{L}_c}$ for all canonical ladders $\mathscr{L}_c$ of $\Lambda(2,2)$ but $\sigma_v \neq \sigma_W$.*

Proof. Consider the representation V given by

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \arrow[r] & 0 & & 0 \arrow[r] & 0 \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[r, "{[0\ 1]}"] \arrow[u, "{[1\ 1]}"'] & \mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[u] & \bigoplus & \mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\mathbb{F}^2 \arrow[r, "\mathrm{Id}"] \arrow[u, "{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}"'] & \mathbb{F}^2 \arrow[r, "{[0\ 1]}"] \arrow[u, "{[1\ 0]}"'] & \mathbb{F} \arrow[u] & & \mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

and the representation W given by

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 & & \mathbb{F} \arrow[r] & 0 \arrow[r] & 0 & & 0 \arrow[r] & 0 \arrow[r] & 0 \\
\mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & \bigoplus & \mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & \bigoplus & \mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[u] \\
\mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[u] & & \mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[u] & & \mathbb{F} \arrow[r] \arrow[u] & \mathbb{F} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Observe that if $\mathscr{L}_{c_1}$ is the ladder obtained by taking the first two rows of $\Lambda(2,2)$, then

$$V^{\mathscr{L}_{c_1}} \simeq \mathbf{R}_{[0,0]}^{[0,2]} \bigoplus \mathbf{R}_{[0,0]}^{[0,1]} \bigoplus \mathbf{R}_{[0,1]}^{[0,1]} \simeq W^{\mathscr{L}_{c_1}}.$$

Similarly, if $\mathscr{L}_{c_2}$ is the ladder obtained by taking the first and third row of $\Lambda(2,2)$, then

$$V^{\mathscr{L}_{c_2}} \simeq \mathbf{R}_{[0,0]}^{[0,1]} \bigoplus \mathbf{R}_{[\infty,\infty]}^{[0,2]} \bigoplus \mathbf{R}_{[\infty,\infty]}^{[0,1]} \simeq W^{\mathscr{L}_{c_2}}.$$

Finally if $\mathscr{L}_{c_3}$ is the ladder obtained by taking the second and third row of $\Lambda(2,2)$, then

$$V^{\mathscr{L}_{c_3}} \simeq \mathbf{R}_{[0,0]}^{[0,0]} \bigoplus \mathbf{R}_{[\infty,\infty]}^{[0,1]} \bigoplus \mathbf{R}_{[\infty,\infty]}^{[0,0]} \simeq W^{\mathscr{L}_{c_3}}.$$

To see the same is true for canonical ladders obtained by taking columns of $\Lambda(2,2)$. Observe that if V^T denotes the transposition of the representation V where each row is turned in a column and vice versa, then $V^T \simeq V$ and $W^T \simeq W$.

Now consider the ladder $\mathscr{L}$ given by

$$\begin{tikzcd}
(0,2) \arrow[r] & (1,2) \\
(0,0) \arrow[u] \arrow[r] & (1,1) \arrow[u]
\end{tikzcd}$$

Then we have $V^{\mathscr{L}}$ and $W^{\mathscr{L}}$

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^3 \arrow[u, "{[1\ 1\ 0]}"] \arrow[r, "{[1\ 0\ 0]}"'] & \mathbb{F} \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K \arrow[r] & 0 \\
\mathbb{F}^3 \arrow[u, "{[0\ 1\ 0]}"] \arrow[r, "{[0\ 0\ 1]}"'] & \mathbb{F} \arrow[u]
\end{tikzcd}$$

giving us $V^{\mathscr{L}} \simeq \mathbf{R}^{[0,1]}_{[0,0]} \bigoplus \left(\mathbf{R}^{[0,0]}_{[\infty,\infty]}\right)^2$ and $W^{\mathscr{L}} \simeq \mathbf{R}^{[0,0]}_{[0,0]} \bigoplus \mathbf{R}^{[0,1]}_{[\infty,\infty]} \bigoplus \mathbf{R}^{[0,0]}_{[\infty,\infty]}$. $\square$

Appendix A

bbase Python package overview

We give here an overview of how to use the bbase Python package, available on GitHub. It implements algorithms exposed in Chapter 2 for computing barcode bases for persistence modules, as well as the direct sum decomposition of well behaved ladders.

A.1 Barcode bases for persistence modules

As a reminder, a persistence module V is a representation of a type $\mathbb{A}_\ell$ quiver Q:

$$V_0 \longleftrightarrow V_1 \longleftrightarrow \cdots V_{\ell-1} \longleftrightarrow V_\ell$$

with linears maps $f_i\colon V_i \to V_{i+1}$ or $f_i\colon V_{i+1} \leftarrow V_i$ depending on the directions of the arrows.

The given input should then be a sequence of matrices (A_i) representating the linears maps in a chosen basis, and a parameter $t \in \{0,1\}^\ell$ representating the directions of the arrows, with the convention that $t_i = 0$ if $V_i \to V_{i+1}$ and $t_i = 1$ if $V_{i+1} \leftarrow V_i$. Computations are made over a field $\mathbb{F}$. Supported fields include finite fields $\mathbb{F}_p$ for primes p and $\mathbb{R}$ (with the convention that $p = 0$ in the case of real number).

The function `bbase.bform` takes as input a matrix sequence (A_i), the orientation t of the underlying quiver and the field $\mathbb{F}$ over which computations are made. The field is defined via `F=bbase.Field(p)`. The output is a matrix sequence (X_i) in barcode form and corresponding basis change (B_i) for which $(BA)_\bullet = X_\bullet$. If the default parameter `basis` is set to `False`, `bbase.bform` modifies the underlying matrix sequence (A_i) directly and outputs nothing.

We may check the correct basis change is outputed via `bbase.basis_change(X,t,F,B)` (note that if `B` is not specified this performs a random basis change on X).

Finally, we may extract the barcode of matrices (X_i) in barcode form via `bbase.barcode(X,t)`. The output is a dictionnary with keys pairs (i,j) of admisibble bars $0 \leqslant i \leqslant j \leqslant \ell$ and values the corresponding multiplicity.

Below is an example of how to use the code on the quiver type $\mathbb{A}_3$ quiver Q

$$\begin{tikzcd}
x_0 \arrow[r] & x_1 & x_2 \arrow[l] \arrow[r] & x_3
\end{tikzcd}$$

with representation V over the finite field $\mathbb{Z}_7$ given by

$$\begin{tikzcd}
\mathbb{Z}_7^3 \arrow[r, "\begin{bmatrix} 6 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}"] & \mathbb{Z}_7^2 & \mathbb{Z}_7^4 \arrow[l, "\begin{bmatrix} 0 & 0 & 3 & 2 \\ 1 & 2 & 6 & 4 \end{bmatrix}"'] \arrow[r, "\begin{bmatrix} 1 & 5 & 3 & 6 \\ 1 & 2 & 0 & 4 \end{bmatrix}"] & \mathbb{Z}_7^2
\end{tikzcd}$$

```python
import bbase
import numpy as np

A, t = [np.array([[6,2,5],[1,2,3]]), np.array([[0,0,3,2],[1,2,6,4]])
    , np.array([[1,5,3,6],[1,2,0,4]])], [0,1,0]
F=bbase.Field(7)
X,B=bbase.bform(A,t,F)
Y=bbase.basis_change(X,t,F,B)

for k in range(3):
    if (A[k]==Y[k]).all():
        print('Correct change of basis !')

bbase.bform(A,t,F,basis=False)
#We now have A=X.


barcode=bbase.barcode(X,t)
```

A.2 Indecomposable decomposition for well behaved ladders

This is based on Theorem 2.5.3, which shows well behaved ladders (no nested bars in top and bottom barcodes) admit indecomposable decompositions with summands of the form $\mathbf{R}^{[a,b]}_{[c,d]}$, $\mathbf{R}^{[a,b]}_{[\infty,\infty]}$ and $\mathbf{R}^{[\infty,\infty]}_{[c,d]}$ (see 3.6.2 for notation).

No algorithm is presented in Chapter 2 in the paper but the proof is constructive and this is the approach we take here.

A map of persistence modules (equivalently, a ladder persistence module)

$\phi_\bullet \colon (V_\bullet, f_\bullet) \mapsto (W_\bullet, h_\bullet)$ should be given as a dictionary with keys $0, 1$ and $(0,1)$. The values at 0 and 1 must be a list of ℓ matrices representing the linear maps $f_\bullet$ and $g_\bullet$ in some chosen bases, and the value at $(0,1)$ should be a list of $\ell + 1$ matrices representing the linear maps $\phi_\bullet$ in the same chosen bases. The main function of interest in then `bbase.ladder_decomp`. This decomposes $(V_\bullet, W_\bullet, \phi_\bullet)$ in a direct sum of admissible indecomposables as in Theorem 2.5.3. The output is a dictionary with possible keys :

- (a,b,c,d) for $0 \leqslant c \leqslant a \leqslant d \leqslant b \leqslant \ell$ and value the multiplicity of the indecomposable $\mathbf{R}^{[a,b]}_{[c,d]}$.
- (a,b) for $0 \leqslant a \leqslant b \leqslant \ell$ and value the multiplicity of the indecomposable $\mathbf{R}^{[a,b]}_{[\infty,\infty]}$.
- (c,d) for $0 \leqslant c \leqslant d \leqslant \ell$ and value the multiplicity of the indeomposable $\mathbf{R}^{[\infty,\infty]}_{[c,d]}$.

A warning is raised if either $V_\bullet$ or $W_\bullet$ have nested barcodes.

Below is an example of computations over the finite field $\mathbb{Z}_{11}$ with ladder persistence module given by

$$\begin{tikzcd}
\mathbb{Z}_{11} \arrow[r, "\begin{bmatrix}4\\8\\6\end{bmatrix}"] \arrow[d, "\begin{bmatrix}0\\0\\0\end{bmatrix}"'] & \mathbb{Z}_{11}^3 \arrow[r, "\begin{bmatrix}1&0&8\\1&3&9\\8&10&0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}2&9&5\\9&6&8\\4&0&1\end{bmatrix}"'] & \mathbb{Z}_{11}^3 \arrow[r, "\begin{bmatrix}8&9&6\\8&10&9\end{bmatrix}"] \arrow[d, "\begin{bmatrix}3&7&9\\2&4&4\end{bmatrix}"'] & \mathbb{Z}_{11}^2 \arrow[d, "\begin{bmatrix}5&1\end{bmatrix}"'] \\
\mathbb{Z}_{11}^3 \arrow[r, "\begin{bmatrix}7&0&10\\5&8&5\\5&0&1\end{bmatrix}"'] & \mathbb{Z}_{11}^3 \arrow[r, "\begin{bmatrix}2&10&8\\4&10&7\end{bmatrix}"'] & \mathbb{Z}_{11}^2 \arrow[r, "\begin{bmatrix}3&3\end{bmatrix}"'] & \mathbb{Z}_{11}
\end{tikzcd}$$

```python
import bbase
import numpy as np

l, p = 3, 11
F = bbase.Field(p)
A = {0: [np.array([[4.],
       [8.],
       [6.]]), np.array([[ 1.,  0.,  8.],
       [ 1.,  3.,  9.],
       [ 8., 10.,  0.]]), np.array([[ 8.,  9.,  6.],
       [ 8., 10.,  9.]])],
     1: [np.array([[ 7.,  0., 10.],
       [ 5.,  8.,  5.],
       [ 3.,  0.,  1.]]), np.array([[ 2., 10.,  8.],
       [ 4., 10.,  7.]]), np.array([[3., 3.]])],
     (0, 1): [np.array([[0.],
       [0.],
       [0.]]), np.array([[2., 9., 5.],
       [9., 6., 8.],
       [4., 0., 1.]]), np.array([[3., 7., 9.],
```

```
         [2., 4., 4.]]), np.array([[5., 1.]])]}

mult = bbase.ladder_decomp(A, 1, F)
```

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