Harder-Narasimhan filtrations of persistence modules

Marc Fersztand
Kellogg College
University of Oxford

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Abstract

Multiparameter persistence modules are central objects in Topological Data Analysis. Unlike ordinary persistence modules, they do not admit a complete discrete invariant such as the barcode. This thesis explores the use of Harder-Narasimhan theory as a way to devise discrete invariants of multiparameter persistence modules that are discriminating, computable, stable and interpretable.

Harder-Narasimhan types are a family of discrete invariants of persistence modules over finite posets. We first study their discriminating power in several settings arising in Topological Data Analysis. We then use Harder-Narasimhan types to define the skyscraper invariant, a novel discrete invariant of multiparameter persistence modules. We show that this invariant is strictly more discriminating than the rank invariant and is stable with respect to the interleaving distance.

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Chapter 1

Introduction

1.1 Setting and motivation

1.1.1 Persistent homology

The goal of Topological Data Analysis (TDA) is to extract geometric and topological information from a metric space X to support tasks such as classification or visualisation. Often, X will be a finite subset of a euclidean space equipped with the topology inherited from that space. Such a metric space will be called a *pointcloud*. For instance, if X has been sampled uniformly on an unknown manifold, we aim to recover properties about the underlying manifold such as the number of connected components and holes.

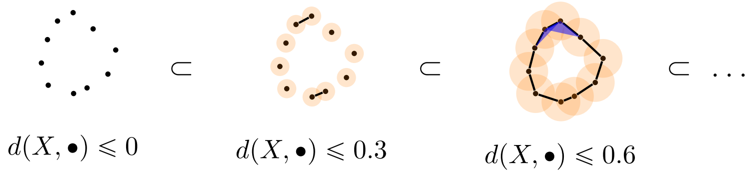
Persistent homology is one of the main tools in TDA. The core idea is to build from a metric space X, a filtration $F(X)$ of topological spaces, *i.e.* a nested sequence of topological spaces indexed by a totally ordered poset P. When $X \subset (\mathbb{R}^m, d)$ is a pointcloud, one can compute the *Čech filtration* of X which is homotopy equivalent to the filtration $(d(X, \bullet) \leqslant t)_{t \in \mathbb{R}}$. Other popular choices for $F(X)$ include the Rips and Alpha filtrations which are typically easier to compute than the Čech filtration.

The homology of the filtration $F(X)$, with coefficients in a fixed field $\mathbb{F}$ defines a functor from P to the category Vect of $\mathbb{F}$-vector spaces. In other words, for $i \geqslant 0$, the i-th homology of $F(X)$ assigns a $\mathbb{F}$-vector space $\mathbf{H}_i(F(X)_x)$ to each element $x \in P$, along with a $\mathbb{F}$-linear map $\mathbf{H}_i(F(X)_x \hookrightarrow F(X)_y)$ for each relation $x \leqslant y$ in P, subject to certain compatibility conditions. This algebraic object is known as an (ordinary) *persistence module*.

In this thesis, we only consider persistence modules whose spaces are finite-dimensional. In this case, every persistence module decomposes [55, 39] into a direct sum of modules of the form

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{F} \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{F} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$

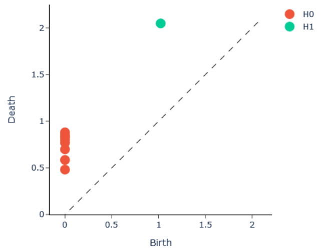
These indecomposable persistence modules are uniquely determined by the intervals in P on which they are supported, and are therefore called *interval (persistence) modules*. Given a persistence module $V : P \to \text{Vect}$, we denote by $\mathbf{Bar}(V)$ the multiset of intervals given by the supports of the indecomposable summands in the decomposition of V. The



(a)

$$\begin{tikzcd}
\mathbf{H}_0 : & \mathbb{F}^{10} \arrow[r, "{[\,I_8 \;\; 0_{8,2}\,]}"] & \mathbb{F}^{8} \arrow[r, "{[\,1 \;\; 0_{1,7}\,]}"] & \mathbb{F} \arrow[r, "1"] & \dots \\
\mathbf{H}_1 : & 0 \arrow[r] & 0 \arrow[r] & \mathbb{F} \arrow[r, "1"] & \dots
\end{tikzcd}$$

(b)



(c)

Figure 1.1: **(a)** Čech filtration Cech(X) of a pointcloud $X \subset (\mathbb{R}^2, d)$ (black and blue) and the corresponding sublevel sets of $d(X,\bullet)$ (orange), **(b)** the zero-th and first homology of Cech(X), and **(c)** the persistence diagrams for the zero-th and first homology of the Rips complex of X (respectively red and green) computed with [58]. The green point far from the diagonal in the persistence diagram corresponds to the hole in the underlying circle.

multiset $\mathbf{Bar}(V)$ is called the *(persistence) barcode* of V. If P is finite, the barcode is a bijection from the set of isomorphism classes of persistence modules to a countable set, and hence characterises the persistence module up to isomorphism. Moreover, if V arises as the homology of a filtered simplicial complex $F(X)$, its barcode $\mathbf{Bar}(V)$ can be computed efficiently using matrix reduction algorithms [46, 117].

A key property of the barcode is its robustness to certain noise in the data. More precisely, for $i \geqslant 0$, let F denote either the Čech or the Rips filtration, and let d_{GH} be the Gromov-Hausdorff distance between metric spaces. Then, one can define two metrics: firstly, the *interleaving distance* d_I between persistence modules, and secondly, the *bottleneck distance* d_b between persistence barcodes. In the following diagram $\mathbf{H}_i \circ F$ is Lipschitz and $\mathbf{Bar}$ is an isometry [38, 34]:

$$(\text{Pointclouds}, d_{GH}) \xrightarrow{\mathbf{H}_i \circ F} (\text{Persistence modules}, d_I) \xrightarrow{\mathbf{Bar}} (\text{Multisets of intervals}, d_b).$$

The barcode of a persistence module V can be visualised by plotting each interval (a, b) in $\mathbf{Bar}(V)$ as a point (a, b) in $\mathbb{R}^2$. This is known as the *persistence diagram* of V (see Figure 1.1**(c)**) and provides an easily interpretable visualisation of the barcode. The barcode can also be used for statistical analysis or can be vectorised and integrated into a machine learning pipeline. As a result, it has found application in a wide range of fields; a comprehensive overview of such applications is available on the DONUT platform [45].

I refer readers to [100, 35] for a more detailed introduction to ordinary persistent homology and its applications.

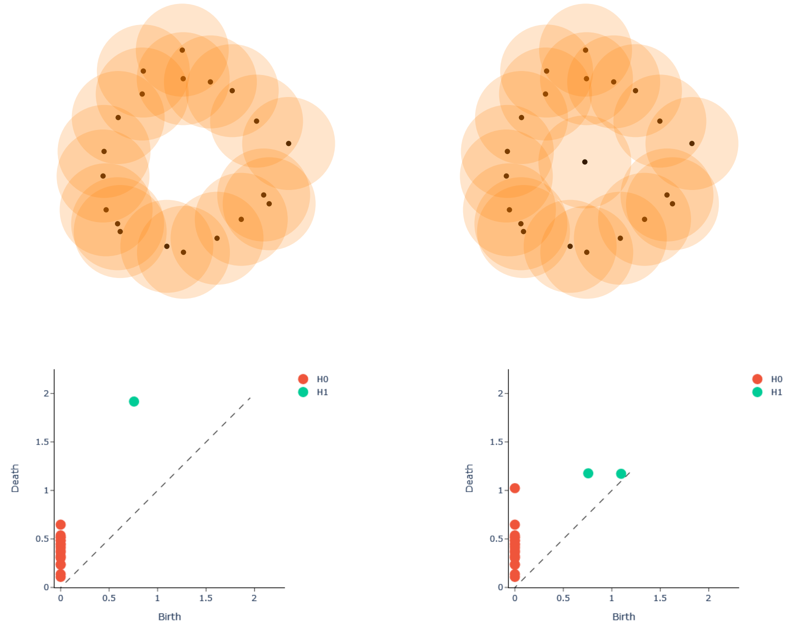


Figure 1.2: Two pointclouds and the persistence diagram of their Rips complex (computed with [58]). Only the pointcloud without outliers has a prominent green dot corresponding to the hole in the underlying circle.

1.1.2 Multiparameter persistent homology

One key limitation of ordinary persistent homology is its lack of robustness to outliers in the data (see Figure 1.2).

One way to mitigate this issue is to filter the pointcloud X by density before computing its persistent homology. More precisely, if we let $f\colon \mathbb{R}^m \to \mathbb{R}_{>0}$ be a density estimator on $\mathbb{R}^m$, we can define a bifiltration $F(X)$ of simplicial complexes indexed by $\mathbb{R}^2$ and homotopy equivalent to the sublevel sets of the function $y \in \mathbb{R}^m \mapsto (d(X,y), 1/f(y)) \in \mathbb{R}^2$. The homology of $F(X)$ is then a functor from the poset $(\mathbb{R}^2, \leqslant)$ to Vect, where $\leqslant$ is the product order on $\mathbb{R}^2$. More generally, one can consider functors from the product of $n \geqslant 2$ totally ordered sets $P := T_1 \times \cdots \times T_n$ to Vect. These are known as *multiparameter persistence modules* and can be used to study pointclouds with outliers or variations in density [112], as well as time-varying data [76] or chromatic pointclouds [95].

The decomposition of ordinary persistence modules into direct sums of interval modules does not generalise to multiparameter persistence modules, even for modules arising from a TDA pipeline [26]. More precisely, given a connected convex subset S of P (in the sense of order theory), we define the *spread module* supported on S as the module $\mathbf{I}[S]$ that assigns $\mathbb{F}$ to each element in S, 0 to each element outside S, and has identity maps between nonzero spaces. In general, multiparameter persistence modules do not decompose into a direct sum of spread modules. Worse, the functors from $\{0,1\} \times \{0,\dots,5\}$ to Vect are already of *wild representation type* [87, 88]. In particular, the decomposition of a multiparameter persistence module is not in general easy to compute and interpret.

As a result, one of the overarching goals of multiparameter persistence has been to assign to a multiparameter persistence module V a quantity $\mathcal{I}(V)$, independent of the isomorphism class of V, that replicates some key properties of the persistence barcode. The assignment $\mathcal{I}$ is usually called an *invariant* (of multiparameter persistence modules).

Goal. Devise an invariant $\mathcal{I}\colon (V\colon P \to \mathrm{Vect}) \mapsto \mathcal{I}(V) \in A$ of multiparameter persistence modules that is

- **Computable**: there should be efficient methods to calculate and store $\mathcal{I}(V)$ for any V coming from a TDA pipeline.
- **Stable**: when $P = \mathbb{R}^m$, the set A should be equipped with a metric d_A so that $\mathcal{I}\colon (\text{Persistence modules}, d_I) \to (A, d_A)$ is Lipschitz, or at least continuous.
- **Discriminating**: the invariant should be as informative as possible about the isomorphism class of V.
- **Interpretable**: there should be a way to either visualise, perform statistical analysis, or integrate $\mathcal{I}(V)$ into a machine learning pipeline.
- **Discrete**: when P is finite, the set A should be countable.

Carlsson and Zomorodian introduced the *rank invariant*, a (necessarily incomplete) discrete invariant ρ which associates to a multiparameter persistence module the rank of all its maps [31]. Many other discrete invariants have since been proposed in the TDA literature. Examples include invariants using ideas from homological and commutative algebra [82, 64, 14], sheaf theory [73, 90] and lattice theory [75, 21, 91, 5]. More details about invariants in multiparameter persistence, including the recent approach using relative homological algebra, can be found in the review articles [20] and [13].

A common idea for many of these invariants is to derive from V a family of less complex persistence modules $(V^i)_{i\in I}$. For instance, the generalised rank invariant [75] considers restrictions of V to certain subposets of P. This idea is also present in the approach using relative homological algebra, where invariants summarise the modules in the projective resolution of V [14]. In this thesis, we follow this line of research and introduce a new family of invariants obtained as the dimension vectors of the subquotients appearing in certain filtrations of V.

The use of filtrations to build invariants of multiparameter persistence modules has been explored recently. Miller and Zhang proposed [94] to filter a multiparameter persistence module by inductively quotienting out a spread submodule with maximal support. The obtained filtration is very interpretable but is not canonical. Another approach is to build [12] a filtration $0 \subset V_\varepsilon \subset V^\varepsilon \subset V$ of a multiparameter persistence module V such that the subquotient $V^\varepsilon/V_\varepsilon$ is more decomposable than V, and the other subquotients in the filtration are ε-small.

1.1.3 Harder-Narasimhan filtrations

Under certain finiteness assumptions, Harder-Narasimhan (HN) filtrations provide a canonical way to filter an object in abelian or triangulated categories [105, 23]. They were originally introduced for vector bundles over a smooth projective curve [63] and were studied in the context of (finite) quiver representations in [78, 103, 65]. A multiparameter persistence module indexed by a finite poset can be viewed as a representation of a quiver with relations. Such modules therefore admit HN filtrations. In this thesis, we investigate whether HN filtrations can be used to define discrete invariants of persistence modules that are computable, stable, discriminating and interpretable.

HN filtrations depend on the choice of a *stability condition*. In the context of persistence modules indexed by a finite poset P, a stability condition Z is an additive map[1] that sends the dimension vector $\underline{\dim}_V \in \mathbb{Z}^P_{\geqslant 0}$ of a nonzero P-persistence module V to a complex number $Z(\underline{\dim}_V)$ in the open right half-plane of $\mathbb{C}$. We now fix the indexing poset P and a stability condition Z. The *Z-slope* of a nonzero $V\colon P \to \mathrm{Vect}$ is then defined as $\mu_Z(V) := \frac{\mathfrak{Im} Z(\underline{\dim}_V)}{\mathfrak{Re} Z(\underline{\dim}_V)}$. A nonzero persistence module V is said to be *Z-semistable* if for every nonzero submodule $W \subset V$, we have $\mu_Z(W) \leqslant \mu_Z(V)$. The *Harder-Narasimhan (HN) filtration* of a persistence module V (along Z) is the unique filtration

$$0 = \mathbf{HN}^0_Z(V) \subsetneq \mathbf{HN}^1_Z(V) \subsetneq \cdots \subsetneq \mathbf{HN}^\ell_Z(V) = V$$

such that each subquotient $\mathbf{HN}^i_Z(V)/\mathbf{HN}^{i-1}_Z(V)$ is Z-semistable and $\mu_Z(\mathbf{HN}^1_Z(V)/\mathbf{HN}^0_Z(V)) > \mu_Z(\mathbf{HN}^2_Z(V)/\mathbf{HN}^1_Z(V)) > \cdots > \mu_Z(\mathbf{HN}^\ell_Z(V)/\mathbf{HN}^{\ell-1}_Z(V))$. HN filtrations, seen as $\mathbb{R}^{\mathrm{opp}}$-filtrations, are functorial. More precisely, if for $\theta \in \mathbb{R}$ we define the submodule

$$\langle V, Z\rangle^\theta := \bigcup_{\substack{\mu_Z(\mathbf{HN}^i_Z(V)/\mathbf{HN}^{i-1}_Z(V)) \geqslant \theta \\ 1 \leqslant i \leqslant \ell}} \mathbf{HN}^i_Z(V) \tag{1.1}$$

of V, then any morphism $f\colon V \to W$ of P-persistence modules satisfies $f(\langle V, Z\rangle^\theta) \subset \langle W, Z\rangle^\theta$ for all $\theta \in \mathbb{R}$ [65].

The *Harder-Narasimhan type* $\mathbf{T}[V; Z]$ of V along Z is the sequence of the dimension vectors of the subquotients in $\mathbf{HN}^\bullet_Z(V)$. HN types define discrete invariants of multiparameter persistence modules. The recent work of Cheng [36] shows that HN filtrations and hence HN types can be computed in polynomial time for multiparameter persistence modules indexed by a finite poset.

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[1] More precisely, for a general poset P, Z is a group homomorphism from the Grothendieck group of (an exact abelian subcategory of) the category of P-persistence modules to the additive group of complex numbers sending the classes of nonzero modules to the right open half-plane.

1.2 Main results

1.2.1 Discriminating power of HN types

One approach to study the discriminating power of HN types is to consider a class $\mathcal{C}$ of persistence modules which decompose into a finite set of well-understood indecomposable modules. In this setting, we say that an invariant is *complete* on $\mathcal{C}$ if it can distinguish any pair of non-isomorphic persistence modules in $\mathcal{C}$. We focus on classes $\mathcal{C}$ that are of interest to the TDA community and address the following question

Question. Given a class of persistence modules $\mathcal{C}$, for which stability conditions Z is the HN type $\mathbf{T}[\bullet; Z]$ complete on $\mathcal{C}$?

In this paragraph, we restrict ourselves to persistence modules indexed by a finite poset P. For simplicity, we further assume that the stability conditions Z are *standard* in the sense that their real part is $\underline{\dim}_V \in \mathbb{Z}^P \mapsto \sum_{p \in P} \dim V_p$. A standard stability condition is therefore determined by the vector $(\mathfrak{Im} Z(\mathbb{1}_p))_{p \in P} \in \mathbb{R}^P$ where $\mathbb{1}_p$ denotes the indicator function of the element $p \in P$.

A zigzag is a poset whose Hasse diagram is of the following form for some $n \geqslant 0$

$$x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} \cdots \overset{e_{n-1}}{\text{———}} x_{n-1} \overset{e_n}{\text{———}} x_n.$$

where each edge e_i points either forward $x_{i-1} \to x_i$ or backward $x_{i-1} \leftarrow x_i$. Persistence modules indexed by zigzags are called *zigzag persistence modules* and appear for instance when studying time-varying data [**29**]. This generalises ordinary persistence modules which correspond to the equioriented case where all the edges in the indexing zigzag point forward. The following theorem is a combination of results by Kinser [**80**] and Reineke [**103**].

Theorem (Theorem 3.4.3)**.** *Let P be a zigzag with $n+1>0$ vertices. The set of standard stability conditions Z for which the HN type along Z is complete is nonempty and can be explicitly described from P. In the case of ordinary persistence modules, the HN type along a standard stability condition Z is complete if and only if*

$$\mathfrak{Im} Z(\mathbb{1}_{x_0}) > \mathfrak{Im} Z(\mathbb{1}_{x_1}) > \cdots > \mathfrak{Im} Z(\mathbb{1}_{x_n}).$$

The characterisation of complete HN types for ordinary persistence modules can be generalised in several directions. For instance, we can consider the class $\mathcal{C}^{\mathrm{rect}}(P)$ of multiparameter persistence modules indexed by a finite poset P that decompose into a direct sum of spread modules whose support is a cube. Such persistence modules are said to be *rectangle-decomposable* (see Figure 1.3).

Theorem (Theorem 3.5.10)**.** *Let P be a product of n totally ordered finite sets and let Z be a standard stability condition. Outside a hyperplane arrangement in the space of standard stability conditions, $\mathbf{T}[\bullet; Z]$ is complete on $\mathcal{C}^{\mathrm{rect}}(P)$ if and only if for $p, q \in P$ we have*

$$p < q \Longrightarrow \mathfrak{Im} Z(\mathbb{1}_p) > \mathfrak{Im} Z(\mathbb{1}_q).$$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u] \arrow[r, "1"] & \mathbb{F} \arrow[u] \arrow[r, "1"] & \mathbb{F} \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{F} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{F} \arrow[u, "1"] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \arrow[r, "1"] & \mathbb{F} \arrow[r, "1"] & \mathbb{F} \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{F} \arrow[u, "1"] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{F} \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

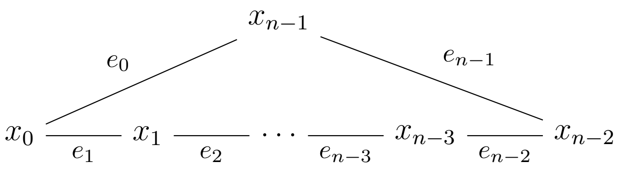
Figure 1.3: Example of an indecomposable module in $\mathcal{C}^{\text{rect}}(P)$ (left) and in $\mathcal{C}^{\text{spread}}(P)$ (right) for $P \coloneqq \{0,1,2,3,4\} \times \{0,1,2,3\}$.

More generally, we can consider the class $\mathcal{C}^{\text{spread}}(P)$ of multiparameter persistence modules that decompose into a direct sum of spread modules. Such persistence modules are said to be *spread-decomposable* (see Figure 1.3). The *sources* of a spread module are the minimal elements in its support seen as a poset with the partial order induced by P. We denote by $\mathcal{C}^{\text{spread-2s}}(P)$ the class of modules in $\mathcal{C}^{\text{spread}}(P)$ whose indecomposable summands have at most two sources.

Theorem (Proposition 3.5.13)**.** *Given a product P of n totally ordered finite sets, there exists a finite number $t \geqslant 1$ of standard stability conditions $Z_1, \ldots, Z_t$ such that the invariant $V \mapsto (\mathbf{T}[V; Z_i])_{1 \leqslant i \leqslant t}$ is complete on $\mathcal{C}^{\text{spread-2s}}(P)$.*

When P is the of the form $\{0,1\} \times \{0,1,\ldots,n\}$ with $n \geqslant 0$, we refer to P-persistence modules as *ladder persistence modules* and $\mathcal{C}^{\text{spread-2s}}(P) = \mathcal{C}^{\text{spread}}(P)$. Ladder persistence modules appear when studying maps between ordinary persistence modules [**47**, **95**]. The above result can be refined in the case of ladder persistence to show that one can obtain an invariant that is complete on $\mathcal{C}^{\text{spread}}(P)$ by using only $t = \mathcal{O}(n^3)$ stability conditions (Theorem 3.6.4).

Finally, we consider a poset P defined by a Hasse diagram of the form



(1.2)

where each edge e_i points either clockwise or counter-clockwise (and the resulting graph is acyclic). Such a poset is called *circular* and P-persistence modules appear when studying the persistence of circle-valued maps [**27**]. For the following result assume that the field $\mathbb{F}$ is algebraically closed. An indecomposable P-persistence module V is said to be *pre-injective* (*resp. pre-projective*) if there exist two edges e_i and e_j with opposite orientations such that, for some choice of bases for V, the maps V_{e_i} and V_{e_j} are given for some $n \geqslant 0$ by the two (*resp.* the transpose of the two) $n \times (n+1)$ matrices $\left[\begin{smallmatrix} I_n & 0_{n,1} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0_{n,1} & I_n \end{smallmatrix}\right]$, and all the other maps of V are given by identity matrices [44, 99]. Building on a result from [65], we show the following

Proposition (Corollary 2.7.5)**.** *Let P be a circular poset. There exists a standard stability condition whose HN type is complete on the class of direct sums of pre-projective and pre-injective P-persistence modules.*

1.2.2 The skyscraper invariant

As we have seen, the discriminating power of HN types depends heavily on the choice of the stability condition. Guided by the results presented in the previous section, we focus on certain stability conditions which produce invariants with good properties.

Definition (Definition 3.3.1)**.** Let P be a finite poset and let $V\colon P \to \mathrm{Vect}$ be a P-persistence module. The *skyscraper invariant* of V is the collection of HN types of V along the standard stability conditions $(Z^p)_{p\in P}$ defined for $p, p' \in P$ by

$$\mathfrak{Im} Z^p(\mathbb{1}_{p'}) = \begin{cases} 1 & \text{if } p = p', \\ 0 & \text{otherwise.} \end{cases} \qquad \triangle$$

The skyscraper invariant is discrete by construction. It can be computed in polynomial time using the algorithm of [36]. In this thesis, we compare the skyscraper invariant to existing invariants in multiparameter persistence. We say that an invariant $\mathcal{I}$ is *stronger* than an invariant $\mathcal{I}'$ if $\mathcal{I}(V) = \mathcal{I}(W)$ implies $\mathcal{I}'(V) = \mathcal{I}'(W)$ for any pair of persistence modules V and W. Two invariants are said to be *equivalent* if each of them is stronger than the other. This notion of relative strength induces a partial order for invariants up to equivalence.

Theorem (Theorem 3.3.5 and Proposition 3.3.9)**.** *The skyscraper invariant is strictly stronger than the rank invariant and incomparable with the generalised rank invariant.*

If we restrict ourselves to ladder persistence modules, the skyscraper invariant can be computed efficiently [69] and provides new information not captured by existing invariants. More precisely, we can compare it to the 6-pack diagram, a collection of six persistence modules associated to an inclusion of filtered topological spaces and studied in [95].

Proposition (Theorem A.2.1)**.** *The skyscraper invariant is strictly stronger than the first four diagrams in the 6-pack diagram and is incomparable with the entirety of the 6-pack diagram.*

In practice, multiparameter persistence modules arising from TDA applications are indexed by $\mathbb{R}^n$ for $n \geqslant 2$ and satisfy certain finiteness conditions. Unless stated otherwise, we will assume in this introduction that every $\mathbb{R}^n$-persistence module V is *finitely presented* (f.p.) in the sense that it can be obtained as the cokernel of a map of free multiparameter persistence modules. Broadly speaking, it means that V can be discretised by a finite subposet of $\mathbb{R}^n$ [**20**, **32**].

Unfortunately, for a general stability condition Z, the existence of HN filtrations of $\mathbb{R}^n$-persistence modules is not guaranteed (see Example 4.3.2). To remedy this issue, we restrict ourselves to a subset $\mathcal{Z}(\mathbb{R}^n)$ of stability conditions that satisfy certain finiteness and positivity assumptions (see Definition 4.3.8). In particular, $\mathcal{Z}(\mathbb{R}^n)$ contains the continuous analogues of the stability conditions used to define the skyscraper invariant. Namely, the stability conditions whose imaginary part are the group homomorphism $\underline{\dim}_V \mapsto \dim V_0$ all belong to $\mathcal{Z}(\mathbb{R}^n)$.

Theorem (Theorem 4.3.11)**.** *Let $V \colon \mathbb{R}^n \to \mathrm{Vect}$ be a f.p. $\mathbb{R}^n$-persistence module and let $Z \in \mathcal{Z}(\mathbb{R}^n)$. Then, V admits a unique HN filtration along Z. Moreover, this HN filtration can be computed in any fine enough discretisation of $\mathbb{R}^n$.*

We now introduce a generalisation of the skyscraper invariant that is defined for $\mathbb{R}^n$-persistence modules and is stable with respect to the interleaving distance. Given $x \in \mathbb{R}^n$, the x-shift of a $\mathbb{R}^n$-persistence module V plays a key role in the definition of the interleaving distance. It is defined as the composition $V \circ T_x$ where T_x is the translation $y \mapsto x + y$ of $\mathbb{R}^n$.

Definition (Definition 4.3.17)**.** Let $V \colon \mathbb{R}^n \to \mathrm{Vect}$ be a f.p. $\mathbb{R}^n$-persistence module and let $Z \in \mathcal{Z}(\mathbb{R}^n)$ be a stability condition. The *HN filtered rank invariant* of V along Z consists, for every $(\theta, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, of the rank

$$s^{\theta}_{Z,V}(\theta, x, y) := \operatorname{rank}\left(\langle V \circ T_x, Z \rangle^{\theta}_{0 \leqslant y - x}\right). \qquad \triangle$$

Given a $\mathbb{R}^n$-persistence module V and a stability condition $Z \in \mathcal{Z}(\mathbb{R}^n)$, the HN filtered rank invariant $(\theta, x, y) \mapsto s^{\theta}_{Z,V}(x, y)$ has a structure of functor from $\mathbb{R} \times (\mathbb{R}^n)^{\mathrm{opp}} \times \mathbb{R}^n$ to $\mathbb{Z}^{\mathrm{opp}}_{\geqslant 0}$ (Theorem 4.4.7), and under some mild assumptions, it is semialgebraically constructible (Proposition 4.6.4). Moreover, for θ small enough, the functor $s^{\theta}_{Z,V}$ coincides with the rank invariant ρ_V (Proposition 4.3.18).

HN filtered rank invariants can be equipped with an erosion distance. More precisely, given two $\mathbb{R}^n$-persistence modules V and W, the *erosion distance between their HN filtered rank invariants* is defined as the supremum over $\theta \in \mathbb{R}$ of $d_E(s^{\theta}_{Z,V}, s^{\theta}_{Z,W})$ where d_E is the erosion distance between rank invariants defined in [**101**]. The functoriality of HN filtrations [**65**] implies that

Theorem (Theorem 4.4.7)**.** *HN filtered rank invariants along any $Z \in \mathcal{Z}(\mathbb{R}^n)$ are Lipschitz continuous with respect to the interleaving distance and the erosion distance between HN filtered rank invariants.*

Example 1.2.1. [Example 4.5.2(ii)] Let $n = 2$, let $b\colon \mathbb{R}^2 \to \mathbb{R}_{>0}$ be an integrable function such that $b_{|[0,2)^2} = 1$. We define the stability condition $Z \in \mathcal{Z}(\mathbb{R}^2)$ by

$$Z\colon \underline{\dim}_V \mapsto \int_{\mathbb{R}^2} b\,\underline{\dim}_V + i\dim V_0. \tag{1.3}$$

Given a bounded spread $S \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, we denote by $L_x(S)$ the area of $\{y \in S \mid y \geqslant x\}$. Let $V := \mathbf{I}[S_1] \oplus \cdots \oplus \mathbf{I}[S_m]$ be the direct sum of $m > 0$ spread modules such that $S_1, \ldots, S_m \subset [0,2)^2$. Using the uniqueness of HN filtrations, one can check that the HN filtration of the x-shift $V \circ T_x$ along Z is given for $\theta > 0$ by

$$\langle V \circ T_x, Z\rangle^{\theta} = \bigoplus_{\substack{1 \leqslant i \leqslant m \\ 0 < L_x(S_i) \leqslant \frac{1}{\theta}}} \mathbf{I}\big[\big\{z \in \mathbb{R}^2_{\geqslant 0} \mid x + z \in S_i\big\}\big].$$

Namely, summands appear in the HN filtration by increasing lifetime as measured by L_x. Figure 1.4 depicts the resulting HN filtered rank invariant for two specific choices of persistence modules V and W. Observe that V and W have the same rank invariants, but the erosion distance between their HN filtered rank invariants is strictly positive.

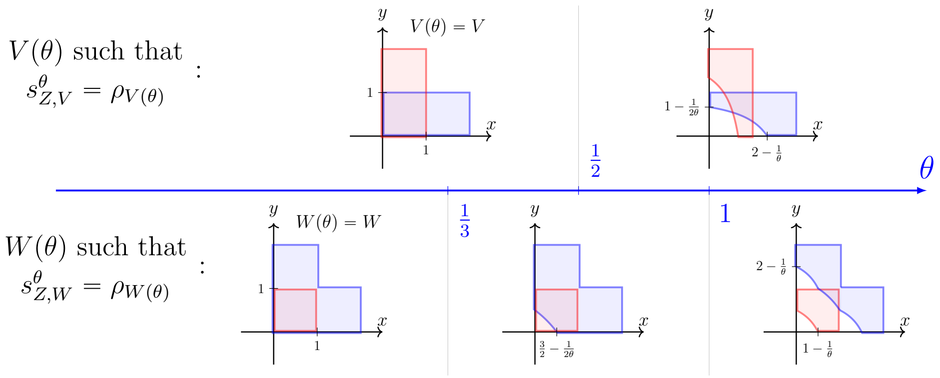


Figure 1.4: Depiction of the HN filtered rank invariants of $V := \mathbf{I}[[0,1) \times [0,2)] \oplus \mathbf{I}[[0,2) \times [0,1)]$ and $W := \mathbf{I}[[0,1)^2] \oplus \mathbf{I}[[0,2)^2 \backslash [1,2)^2]$. Let Z be one of the stability conditions defined in Example 1.2.1. For each $\theta \in \mathbb{R}$, the top and bottom rows depict two spread-decomposable (not necessarily f.p.) persistence modules, respectively $V(\theta)$ and $W(\theta)$, whose rank invariants coincide with respectively $s^{\theta}_{Z,V}$ and $s^{\theta}_{Z,W}$. Spread-decomposable modules are represented by drawing the support of their indecomposable summands.

Let Q be a bounded cube of $\mathbb{R}^n$ and let $Z \in \mathcal{Z}(\mathbb{R}^n)$ be a stability condition such that for every $V\colon \mathbb{R}^n \to \mathrm{Vect}$ supported on Q, we have $Z(\underline{\dim}_V) = \int_Q \underline{\dim}_V + i\dim V_0$. Then, the invariant $V \mapsto s_{V,Z}$ generalises the skyscraper invariant introduced for finite posets (see (4.21)). In a forthcoming work with Jan Jendrysiak, we devise [**71**] and implement in C++ [**72**] an algorithm to compute $s_{Z,V}$ for modules over the field $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. This algorithm relies on the decomposition algorithm AIDA [**41**] and on exhaustive enumerations of submodules. Despite not being polynomial in the worst case, it performs well on multiparameter persistence modules arising from TDA pipelines. We show that it is empirically much faster than the algorithm proposed by Cheng [**36**]. Our algorithm relies on the following observation: for any $\theta > 0$ and $x \in \mathbb{R}^n$, the maps out of 0 in $\langle V \circ T_x, Z \rangle^\theta$ are all surjective (Proposition 3.3.3). The additivity of $\mathbf{T}[\bullet; Z]$ means that, before computing any HN filtrations, we can decompose the largest submodule of $V \circ T_x$ whose maps out of 0 are all surjective. We observed that this submodule tends to have much smaller indecomposable summands than V.

1.3 Readers guide

We give a brief overview of the contents of this thesis. Each chapter includes its own introduction and notations and can be read independently. Chapter 2 provides a detailed example of the interplay between persistence theory and Harder-Narasimhan filtrations. In Chapter 3, we study the discriminating power of HN types for various classes of multiparameter persistence modules, and we introduce the skyscraper invariant. Chapter 4 introduces HN filtered rank invariants and proves their stability. Finally, in Appendix A, we compare the skyscraper invariant to other invariants in the context of chromatic TDA. Some general notations used throughout the thesis are gathered in Appendix B.

Harder-Narasimhan filtrations and zigzag persistence: a motivating example

Chapter 2 is based on the content of an article co-authored with Vidit Nanda and Ulrike Tillmann [**53**]. Only minimal notation changes were made to the original article.

This chapter gives a precise relationship between the HN filtrations of representations of an affine type $\widetilde{\mathbb{A}}$ quiver (see Equation (1.2)) and the barcode of the zigzag persistence module obtained by unwinding the underlying quiver. This preliminary work shows in a simple example that HN types are able to pick up information which is of interest in TDA.

Harder-Narasimhan filtrations of persistence modules: discriminating power

Chapter 3 follows an article written with Emile Jacquard, Vidit Nanda and Ulrike Tillmann [**52**], with only minimal notation changes from the published version. The content of Sections 3.3.1, 3.3.2, and 3.6 is based on slight extensions of results proved by Emile [**69**]. The ideas and results presented in Sections 3.3.3, 3.5.2, and 3.5.3 are my own. The remainder of Chapter 3 was developed and written collaboratively with Emile. Our common supervisors, Vidit Nanda and Ulrike Tillmann, participated in the conceptual development of the results and in the writing and revision of the entire paper.

In this chapter, we investigate the strength and limitations of Harder-Narasimhan types for several families of quiver representations which arise in the study of persistence modules. We introduce the skyscraper invariant, which amalgamates the HN types along stability conditions supported at single vertices, and we generalise the rank invariant from multiparameter persistence modules to arbitrary quiver representations. Our four main results are as follows: (1) we show that the skyscraper invariant is strictly stronger than the rank invariant in full generality and that it is incomparable with the generalised rank invariant, (2) we characterise the set of central charges which produce a complete invariant for zigzag (and hence, ordinary) persistence modules, (3) we extend the preceding characterisation to rectangle-decomposable multiparameter persistence modules of arbitrary dimension; and finally, (4) we show that although no single central charge is complete for interval-decomposable ladder persistence modules, a finite set of central charges is complete.

Harder-Narasimhan filtrations of persistence modules: metric stability

Chapter 4 is broadly based on the preprint [**50**] which is currently under review. The version presented here includes several additional results and examples that were added during the review process.

Building on the previous chapters, we extend the definition of the skyscraper invariant from the finite to the infinite setting and consider multiparameter persistence modules over $\mathbb{Z}^n$ and $\mathbb{R}^n$. We then establish an erosion-type stability result for the skyscraper invariant in this setting.

Invariants for ladder persistence

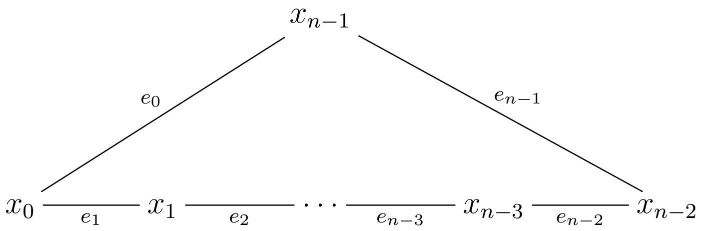
Appendix A focuses on ladder persistence and its application to chromatic TDA. In this context, we compare the discriminating power of several invariants, including the skyscraper invariant, the generalised rank invariant, and the 6-pack diagram.

Chapter 2

Harder-Narasimhan filtrations and zigzag persistence: a motivating example

Introduction

This chapter concerns representations of acyclic quivers of affine type $\widetilde{\mathbb{A}}_n$. The underlying graph of any such quiver is the n-cycle as drawn below, but one does not obtain a directed cycle after the edges have been oriented:



Our goal here is to describe remarkable formulas which relate two discrete quantities that are associated to every finite-dimensional representation V of such a quiver. The first one has its roots in geometric invariant theory, and constitutes a numerical reduction of V's Harder-Narasimhan filtration along a special choice of stability condition. The second quantity of interest arises in the algebraic study of persistent homology. To obtain it, one first lifts V to an n-periodic zigzag persistence module over the infinite quiver

$$\begin{tikzcd}
\cdots \arrow[r, dash, "e_{n-1}"] & x_{n-1} \arrow[r, dash, "e_0"] & x_0 \arrow[r, dash, "e_1"] & x_1 \arrow[r, dash, "e_2"] & \cdots \arrow[r, dash, "e_{n-2}"] & x_{n-2} \arrow[r, dash, "e_{n-1}"] & x_{n-1} \arrow[r, dash, "e_0"] & \cdots ,
\end{tikzcd}$$

and then catalogues the multiplicities of its indecomposable summands. Before outlining the main contributions of our work, we provide brief summaries of both quantities below.

Harder-Narasimhan filtrations

For the purposes of these introductory remarks, a *stability condition* on a finite acyclic quiver Q with vertex set Q_0 is a map $\alpha : Q_0 \to \mathbb{R}$ that assigns a real number α_x to each vertex x. The α-*slope* of a nonzero finite-dimensional representation V of Q is the ratio

$$\mu_\alpha(V) := \frac{\sum_{x\in Q_0} \alpha_x \cdot \dim V_x}{\sum_{x\in Q_0} \dim V_x}.$$

Here V_x denotes the vector space assigned by V to each vertex x. We call V *semistable* if the inequality $\mu_\alpha(U) \leqslant \mu_\alpha(V)$ holds for every nonzero subrepresentation $U \subset V$. Once we fix α, there exists a unique, finite length filtration $V^\bullet$ of V:

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell = V,$$

where the successive quotient representations $S^j := V^j/V^{j-1}$ are semistable and have strictly decreasing α-slopes. This $V^\bullet$ is called the Harder-Narasimhan filtration [63, 103] of V along α. Our first invariant, for a specific choice of α to be described later, is the map $Q_0 \to \mathbb{Z}^\ell$ that sends each x in Q_0 to the vector $\big(\dim(S^1)_x, \ldots, \dim(S^\ell)_x\big)$.

Zigzag persistence

Gabriel's foundational theorems from [55] establish that the set of indecomposable representations of a type $\mathbb{A}_n$ quiver can be canonically identified with the collection of subintervals $[u, v] \subset [0, n-1]$ that have integral endpoints. The study of such representations has enjoyed a substantial recent renaissance, induced largely by their appearance in topological data analysis [100, 30], where they are called *zigzag persistence modules.* Thus, by Gabriel's results, every finite-dimensional zigzag persistence module P decomposes uniquely into a direct sum of the form

$$P \simeq \bigoplus_{[u,v]} \mathbf{I}[u,v]^{d_{u,v}};$$

here $[u, v] \subset [0, n-1]$ ranges over a finite set $\mathbf{Bar}(P)$ called the *barcode* of P. For each such interval, $\mathbf{I}[u, v]$ is the corresponding indecomposable whose *multiplicity*, denoted $d_{u,v}$ above, is a strictly positive integer. A similar interval decomposition theorem also holds for *infinite* zigzag persistence modules [17], where the endpoints of intervals are allowed to attain $\pm\infty$ values. Every representation V of a type $\widetilde{\mathbb{A}}_n$ quiver gives rise to an n-periodic infinite zigzag module $\mathcal{L}V$, and the second invariant of interest is the associated multiplicity function $\mathbf{Bar}(\mathcal{L}V) \to \mathbb{N}_{>0}$.

Main results

We introduce the **Euler** stability condition $\epsilon : Q_0 \to \mathbb{R}$, which is defined on any finite quiver Q as follows. The underlying graph of Q forms a one-dimensional CW complex X; and every representation V of Q functorially induces a cellular sheaf [40] $\mathscr{S}V$ on X. From this sheaf, one can build a two-term cochain complex which computes the cohomology of X with $\mathscr{S}V$ coefficients. The corresponding Euler characteristic has the form

$$\chi(X; \mathscr{S}V) = \sum_{x\in Q_0} (1 - \deg_{\text{in}}(x)) \cdot \dim V_x.$$

Here $\deg_{\text{in}}(x)$ equals the number of edges which point to x. Our stability condition is therefore given by $\epsilon(x) := 1 - \deg_{\text{in}}(x)$.

Harder-Narasimhan filtrations along ϵ enjoy some surprising properties pertaining to barcode decompositions. As a warm-up exercise, we first consider the easiest examples, which are furnished by equioriented quivers of type $\mathbb{A}_n$

$$\begin{tikzcd} x_0 \arrow[r, "e_1"] & x_1 \arrow[r, "e_2"] & \cdots \arrow[r, "e_{n-1}"] & x_{n-1}. \end{tikzcd}$$

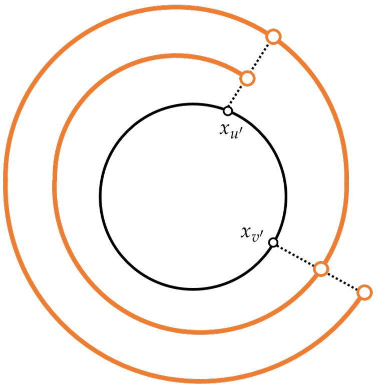
The representations of such quivers are called (ordinary, discrete) persistence modules. For any such representation V, the Harder-Narasimhan filtration $V^\bullet$ along ϵ can be used to directly recover the multiplicities of all intervals which have the form $[0, j]$ in $\mathbf{Bar}(V)$. Here is a simplified version of Theorem 2.5.4.

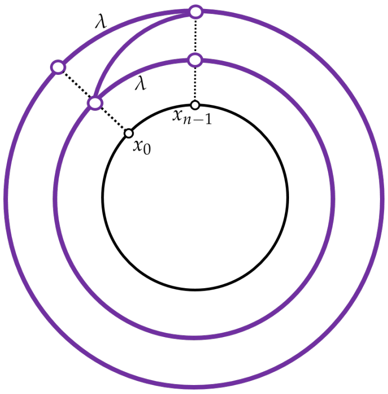
Theorem (A). *Let V be a persistence module and let $j_1 < \ldots < j_\ell$ be the collection of all indices j in $\{0, \ldots, n-1\}$ satisfying $[0, j] \in \mathbf{Bar}(V)$. Then $V^\bullet$ has length either ℓ or $\ell + 1$, and for each integer $0 < k \leqslant \ell$ the quotient $S^k := V^k / V^{k-1}$ satisfies*

$$\dim(S^k)_{x_i} = \begin{cases} d_{0,j_k} & i \in [0, j_k], \\ 0 & \textit{otherwise.} \end{cases}$$

Thus, the Harder-Narasimhan filtration of V along ϵ recovers (the multiplicities of) all those intervals in $\mathbf{Bar}(V)$ which have left endpoint 0. We show that one can also extract similar formulas for multiplicities of all the other intervals in $\mathbf{Bar}(V)$ – and hence, the entire isomorphism class of V – by availing of Harder-Narasimhan filtrations along ϵ of V's restriction to certain truncated subquivers. In future work, we will describe a method for recovering multiplicities of all indecomposables (for representations of a different quiver) by using several different stability conditions at once.

Turning now to the central focus of this chapter, we consider a representation V of a type $\widetilde{\mathbb{A}}_n$ quiver Q over an algebraically closed field. The indecomposable summands of V have two possible forms, illustrated below:





To the left we have $\mathbf{N}[u, v]$, which is obtained by wrapping an interval $[u, v] \subset \mathbb{Z}$ around Q; here u' and v' equal u and v modulo n respectively. To the right lies $\mathbf{J}[\lambda; w]$, where all vector spaces have the same dimension $w \geqslant 1$ and all edge-maps are identities *except* for a Jordan block with diagonal $\lambda \neq 0$ over e_0. As mentioned above, V lifts to a representation $\mathcal{L}V$ of the infinite zigzag quiver $\mathbf{U}Q$ obtained by unwinding Q. This lift operation $\mathcal{L}$ forms a functor from the category of representations of Q to the full subcategory of representations of $\mathbf{U}Q$ spanned by n-periodic objects. Moreover, $\mathcal{L}V$ decomposes as a direct sum of the form

$$\mathcal{L}V \simeq \mathbf{I}[-\infty, \infty]^{d^*_\infty} \oplus \bigoplus_{[u,v]} \left(\bigoplus_{c \in \mathbb{Z}} \mathbf{I}[u + cn, v + cn] \right)^{d^*_{u,v}}. \tag{2.1}$$

The first term on the right consists of infinite intervals corresponding to the $\mathbf{J}[\lambda; w]$ summands of V while the second term consists of (infinitely many) n-shifted copies of $\mathbf{I}[u, v]$ corresponding to the $\mathbf{N}[u, v]$ summands. Here is a condensed description of our main result, Theorem 2.7.4.

Theorem (B). *Let V be a representation of an acyclic type $\widetilde{\mathbb{A}}_n$ quiver Q over an algebraically closed field. Let $V^\bullet$ be the Harder-Narasimhan filtration of V along ϵ, and let $S^j := V^j/V^{j-1}$ be its successive quotients. If $\mu_\epsilon(S^j)$ is nonzero, then for each vertex $x \in Q_0$ we have:*

$$\dim(S^j)_x = \sum_{\mathbf{N}[u,v]} d^*_{u,v} \cdot \dim \mathbf{N}[u, v]_x$$

where the sum is over all $\mathbf{N}[u, v]$ appearing in the decomposition of V which have the same ϵ-slope as S^j. And similarly, if $\mu_\epsilon(S^j)$ equals 0, then for each vertex $x \in Q_0$, we have

$$\dim(S^j)_x = d^*_\infty + \sum_{\mathbf{N}[u,v]} d^*_{u,v} \cdot \dim \mathbf{N}[u, v]_x$$

where the sum is over indecomposable summands $\mathbf{N}[u, v]$ of V that satisfy $\mu_\epsilon(\mathbf{N}[u, v]) = 0$.

Since $\mathcal{L}V$ is periodic, the multiplicities d^*_∞ and $d^*_{u,v}$ of its indecomposable summands can be computed by restricting to a sufficiently long finite zigzag persistence module via the algorithms of [**30**, **29**] — see Corollary 2.7.5 and the subsequent Remark 2.7.7 for complexity estimates. We hope that the results of this chapter will encourage not only the use of tools from geometric invariant theory in the study of persistence modules, but also facilitate an influx of techniques from topological data analysis for efficient computation of Harder-Narasimhan filtrations.

Organisation

Sections 2.1, 2.2 and 2.3 contain notation and preliminary material pertaining to quiver representations, their direct sum decompositions into indecomposable representations, and Harder-Narasimhan filtrations respectively. In Section 2.4 we introduce the Euler stability condition, and in Section 2.5 we prove Theorem (**A**). In Section 2.6 we describe the infinite zigzag persistence modules arising as lifts of $\widetilde{\mathbb{A}}_n$ quiver representations. Finally, Section 2.7 contains the proof of Theorem (**B**) along with some of its consequences.

Related work

The recent work of Kinser [80] classifies *totally stable* conditions on type $\mathbb{A}$ quivers, for which every indecomposable is stable, rather than semistable[1]. Using such a stability condition instead of ϵ in Section 2.5 would allow us to recover the entire barcode at once from the HN filtration (rather than only the intervals with left endpoint 0). We note in passing that total stability conditions have also been described for quivers of type $\mathbb{D}$ and $\mathbb{E}$ by Diaz, Gilbert and Kinser in [43]. For more complicated quivers, however, the set of totally stable conditions is always empty.

In [65], Hille and de la Peña characterise stable representations of tame quivers when the stability condition is in a neighbourhood of a quantity called the *defect*. For $\widetilde{\mathbb{A}}_n$ quivers, the defect happens to coincide with the Euler slope, and hence their work yields a different proof of our Proposition 2.7.3. Unlike their argument, ours does not use any knowledge of tame hereditary algebras (besides the list of indecomposable type $\widetilde{\mathbb{A}}$ quiver representations). More recently, Apruzzese and Igusa [3] have used a geometric model to determine the maximum finite number of stable indecomposables of type $\widetilde{\mathbb{A}}$ quivers as the stability condition is varied.

The idea of relating indecomposables of a quiver to those of its universal cover, which is exploited heavily in Section 2.6, dates back to the work of Riedtmann, [104] Bongartz-Gabriel [16], Gabriel [54] and Green [60]. An algorithmic perspective on covering theory for representations of strictly alternating $\widetilde{\mathbb{A}}_n$ quivers has been employed in the work of Burghelea and Dey on circle-valued persistence [27], where indecomposables of the form $\mathbf{J}[\lambda; w]$ are called *Jordan cells*. Similarly, Cheng [36] presents a polynomial time algorithm to compute Harder-Narasimhan filtrations with respect to any stability condition by relating them to the so-called *discrepancies* of quiver representations. Computing discrepancies requires finding the largest c such that a given space of matrices has a *c-shrunk subspace*. Although this last problem admits a polynomial-time algorithm [68], we are not aware of any practical implementations.

Finally, the work of Henselman and Ghrist [57] seeks to generalise persistence barcodes for functors to non-abelian categories by viewing (ordinary) persistence modules as lattice homomorphisms from a lattice of intervals with respect to a certain partial order. In contrast, the ϵ-slope defines a total preorder on the set of intervals. If the interval modules are all stable (i.e., if the stability condition is totally stable in the sense of [80]), then this preoreder is an order and the HN filtration constitutes a *subsaecular series* in the language of [57].

---

[1] In this context, a Q-representation V is called *stable* with respect to $\alpha : Q_0 \to \mathbb{R}$ if the strict inequality $\mu_\alpha(U) < \mu_\alpha(V)$ holds for every subrepresentation $0 \subsetneq U \subsetneq V$.

2.1 Quiver representation preliminaries

The study of quiver representations is a vast enterprise spanning algebra and geometry, so we will restrict our focus here on the aspects relevant to this work; comprehensive treatments can be found in [81] and [107].

A *quiver* Q consists of two sets Q_0 and Q_1, whose elements are called vertices and edges respectively, equipped with two functions $s, t : Q_1 \to Q_0$ called the source and target map. We typically write $e : x \to y$ when indicating that the edge e has source $s(e) = x$ and target $t(e) = y$. A *path* of Q is any nonempty finite sequence of edges $p = (e_1, \ldots, e_k)$ so that $s(e_j) = t(e_{j-1})$ holds across all $1 < j \leqslant k$. The source and target maps extend to any such p by setting $s(p) = s(e_1)$ and $t(p) = t(e_k)$, and we call p a *cycle* of Q whenever the source and target vertex of p are identical. The quiver Q is called **acyclic** if it does not contain any such cycles.

Let us fix, once and for all, a ground field $\mathbb{F}$ so that all vector spaces and linear maps encountered henceforth are understood to be defined over $\mathbb{F}$. A *representation* V of Q is an assignment of

1. a vector space V_x to each vertex $x \in Q_0$, and
2. a linear map $V_e : V_x \to V_y$ to each edge $e : x \to y$ in Q_1.

Unless stated otherwise, we will assume that both Q_0 and Q_1 are finite, and we will only consider finite-dimensional representations of Q — i.e., each V_x is required to be finite-dimensional over $\mathbb{F}$. The map $\underline{\dim}_V : Q_0 \to \mathbb{Z}$ sending each x to $\dim V_x$ is called the *dimension vector* of V.

A *morphism* of representations $f : V \to V'$ is a collection of vertex-indexed linear maps $\{f_x : V_x \to V'_x \mid x \in Q_0\}$ so that for each edge $e : x \to y$ in Q_1 the evident diagram of vector spaces commutes:

$$\begin{tikzcd}
V_x \arrow[r, "f_x"] \arrow[d, "V_e"'] & V'_x \arrow[d, "V'_e"] \\
V_y \arrow[r, "f_y"'] & V'_y
\end{tikzcd}$$

Equivalently, the identity $f_{t(e)} \circ V_e = V'_e \circ f_{s(e)}$ holds for every edge $e \in Q_1$. With this definition of morphisms in place, the representations of Q define an abelian category $\mathbf{Rep}(Q)$, see [55, Section 1.2]. Injective and surjective morphisms, kernels, images, quotients, direct sums and the zero object are all defined pointwise in $\mathbf{Rep}(Q)$. We say that V is a *subrepresentation* of another representation V' whenever there exists an injective morphism $V \hookrightarrow V'$, in which case we simply write $V \subset V'$.

Remark 2.1.1. The quiver Q is a *subquiver* of another quiver $Q' = (s', t' : Q'_1 \to Q'_0)$ if we have $Q_0 \subset Q'_0$ and $Q_1 \subset Q'_1$ so that s and t are restrictions of s' and t' respectively. Given such a subquiver, we note that each representation V' of Q' automatically induces a representation V of Q by restricting to the available vertices and edges. Namely, $V_x := V'_x$ for all $x \in Q_0$ and $V_e := V'_e$ for all $e \in Q_1$.

2.2 Harder-Narasimhan filtrations

Harder-Narasimhan filtrations were originally introduced for the purpose of studying moduli spaces of vector bundles over algebraic curves [63]; they have since been employed in a plethora of other contexts [78, 23, 61], including geometric invariant theory [106, Chapter 4.2].

The *Grothendieck group* $K(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the abelian group generated by the objects of $\mathcal{A}$ subject to the relation that $V = U + W$ whenever there is an exact sequence $0 \to U \to V \to W \to 0$. By a **stability condition** on $\mathcal{A}$ we mean an abelian group homomorphism $Z : K(\mathcal{A}) \to (\mathbb{C}, +)$ valued in the (additive) complex numbers, so that every nonzero object U is mapped to the right half plane, *i.e.* $\mathfrak{Re} Z(U) > 0$. Given such a stability condition, the Z- *slope* of a nonzero object V in $\mathcal{A}$ is the real number

$$\mu_Z(V) := \frac{\mathfrak{Im} Z(V)}{\mathfrak{Re} Z(V)}.$$

Definition 2.2.1. Let Z be a stability condition on $\mathcal{A}$. A nonzero object V in $\mathcal{A}$ is **Z-semistable** if its nonzero subobjects have smaller or equal slopes. In other words, $\mu_Z(U) \leqslant \mu_Z(V)$ holds whenever there exists an injective morphism $U \hookrightarrow V$ in $\mathcal{A}$ with $U \neq 0$. △

The importance of semistable objects in the study of moduli spaces stems from the fact that every nonzero object admits a unique filtration for which the successive quotients are semistable and have strictly decreasing slopes. A proof of the following result (for the category of representations of a fixed quiver) can be found in [65, Theorem 2.5]; more generally, see [61, Theorem 4.2].

Theorem 2.2.2. *Let $\mathcal{A}$ be a finite length (i.e., Noetherian and Artinian) abelian category equipped with a stability condition $Z : K(\mathcal{A}) \to (\mathbb{C}, +)$. Every object $V \neq 0$ of $\mathcal{A}$ admits a unique filtration $V^\bullet := \mathbf{HN}^\bullet_Z(V)$:*

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell = V$$

whose successive quotients V^j/V^{j-1} are all Z-semistable with strictly decreasing Z-slopes, i.e.,

$$\mu_Z(V^j/V^{j-1}) > \mu_Z(V^{j+1}/V^j).$$

(This unique $V^\bullet$ is called the **Harder-Narasimhan***, or* **HN***, filtration of V along Z.)*

Remark 2.2.3. In particular, it follows from uniqueness that a nonzero object V of $\mathcal{A}$ is Z-semistable if and only if the corresponding HN filtration is the trivial one $0 \subsetneq V$.

Let $\mathrm{Hom}(Q_0, \mathbb{Z})$ be the abelian group consisting of functions $Q_0 \to \mathbb{Z}$ with addition defined vertex-wise. When Q is acyclic, the map $V \mapsto \underline{\dim}_V$ furnishes an isomorphism $K(\mathbf{Rep}(Q)) \simeq \mathrm{Hom}(Q_0, \mathbb{Z})$ — see for instance [81, Theorem 1.15]. As a result, every stability condition on $\mathbf{Rep}(Q)$ amounts to a function $\beta : Q_0 \to \mathbb{C}$ sending vertices x to complex numbers β_x, subject to the requirement that $\mathfrak{Re}\beta_x > 0$. It is customary to assign numbers of the form $\beta_x = 1 + i \cdot \alpha_x$ for arbitrary real numbers α_x [65, 103]. Thus, the stability conditions of interest are precisely the functions $\alpha : Q_0 \to \mathbb{R}$, and for such a function the corresponding α-slope of $V \in \mathbf{Rep}(Q)$ equals

$$\mu_\alpha(V) = \frac{\sum_{x \in Q_0} \alpha_x \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x}.$$

Crucially, the uniqueness result of Theorem 2.2.2 applies only after the stability condition has been fixed — in particular, varying $\alpha : Q_0 \to \mathbb{R}$ is liable to produce a very different HN filtration of the same representation. If α is identically zero for instance, then all quiver representations are semistable with slope 0, and hence have trivial HN filtrations. Our focus here will be on a specific choice of stability condition ϵ arising from cellular sheaf cohomology, which is described in Section 2.4. Throughout the remainder of this section, however, we fix an arbitrary α and all instances of slopes, stability and HN filtrations encountered here are with respect to this fixed α.

The following result was established for vector bundles in [109, §2]; the version stated below is [103, Lemma 2.2].

Lemma 2.2.4. *The following properties hold for every exact sequence*

$$0 \to U \to X \to V \to 0$$

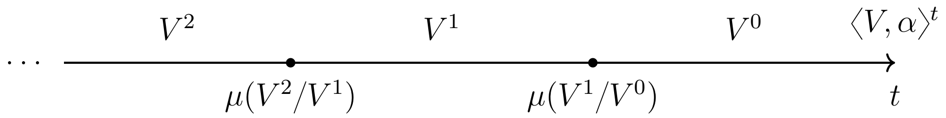
in $\mathbf{Rep}(Q)$ *with* U, V *and* X *all nonzero:*

1. $\min(\mu(U), \mu(V)) \leqslant \mu(X) \leqslant \max(\mu(U), \mu(V))$*; and moreover,*
2. *if* $\mu(U) = \mu(V)$ *with* U *and* V *semistable, then* X *is also semistable.*

It will be convenient for our purposes to describe the HN filtrations of direct sums in terms of the HN filtrations of constituent factors. To this end, given any representation V in $\mathbf{Rep}(Q)$, we consider the HN filtration $V^\bullet$ as being indexed over the real line in the following manner. The *HN* $\mathbb{R}$*-filtration* of V is the assignment $\langle V, \alpha \rangle : t \mapsto \langle V, \alpha \rangle^t$ of quiver representations to real numbers obtained by setting

$$\langle V, \alpha \rangle^t := V^{\lfloor t \rfloor_\mu};$$

here $\lfloor t \rfloor_\mu$ is the smallest integer $i \geqslant 0$ satisfying $t \geqslant \mu(V^{i+1}/V^i)$, as illustrated below:



Given $s \geqslant t$ in $\mathbb{R}$, we have $\lfloor s \rfloor_\mu \leqslant \lfloor t \rfloor_\mu$, so there is an obvious inclusion $\langle V, \alpha \rangle^t \hookrightarrow \langle V, \alpha \rangle^s$. Thus, $\langle V, \alpha \rangle$ constitutes a functor from the $\geqslant$-ordered set of real numbers to the category $\mathbf{Rep}(Q)$. There is a natural direct sum operation on such functors, where $[\langle U, \alpha \rangle \oplus \langle V, \alpha \rangle]$ assigns to each $t \in \mathbb{R}$, the representation $\langle U, \alpha \rangle^t \oplus \langle V, \alpha \rangle^t$.

The following result is well-known to experts, but we were unable to find it in the literature and have included a proof for completeness.

Proposition 2.2.5. *For nonzero U and V in $\mathbf{Rep}(Q)$, we have $\langle U \oplus V, \alpha \rangle = \langle U, \alpha \rangle \oplus \langle V, \alpha \rangle$.*

Proof. Writing ℓ and m for the lengths of $U^\bullet$ and $V^\bullet$ respectively, let

$$\mu := \left\{ \mu(U^i / U^{i-1}) \right\}_{1 \leqslant i \leqslant \ell} \cup \left\{ \mu(V^j / V^{j-1}) \right\}_{1 \leqslant j \leqslant m}$$

be the union of slopes attained by the successive quotients of both filtrations. By construction, $\langle U \oplus V, \alpha \rangle$ is locally constant on $\mathbb{R} - \mu$. Let $\theta_1 > \ldots > \theta_n$ be the slopes in μ arranged in strictly decreasing order, and consider the filtration $W^\bullet$ of $U \oplus V$ given by $W^k := \langle U, \alpha \rangle^{\theta_k} \oplus \langle V, \alpha \rangle^{\theta_k}$ for each $1 \leqslant k \leqslant n$. We now claim that $W^\bullet$ is the HN filtration of $(U \oplus V)$. To see this, note that we have an isomorphism of quotient representations

$$\frac{W^k}{W^{k-1}} \simeq \frac{\langle U, \alpha \rangle^{\theta_k}}{\langle U, \alpha \rangle^{\theta_{k-1}}} \oplus \frac{\langle V, \alpha \rangle^{\theta_k}}{\langle V, \alpha \rangle^{\theta_{k-1}}} \tag{2.2}$$

for each k. We will now show that W^k / W^{k-1} is semistable with slope θ_k, which – when combined with the uniqueness guarantee of Theorem 2.2.2 – produces the desired result. By (2.2), we have a (split) exact sequence of the form

$$0 \to \frac{\langle U, \alpha \rangle^{\theta_k}}{\langle U, \alpha \rangle^{\theta_{k-1}}} \hookrightarrow \frac{W^k}{W^{k-1}} \twoheadrightarrow \frac{\langle V, \alpha \rangle^{\theta_k}}{\langle V, \alpha \rangle^{\theta_{k-1}}} \to 0$$

The two summands on the right side of (2.2) are either trivial or semistable with slope θ_k by construction. In the nontrivial case, the first assertion of Lemma 2.2.4 ensures that the slope of W^k / W^{k-1} is also θ_k, while the second assertion guarantees semistability. $\square$

2.3 Indecomposables and barcodes

A representation $V \neq 0$ of a quiver Q is *indecomposable* if it cannot be written as a direct sum of two nontrivial representations. Since we have assumed that Q is finite and V is finite-dimensional, the Krull-Schmidt theorem [9, 83] applies, so there is a unique pair consisting of a finite set $\text{Ind}(V)$ of indecomposable representations and a function $\text{Ind}(V) \to \mathbb{N}_{>0}$ denoted $I \mapsto d_I$ that satisfies

$$V = \bigoplus_{I \in \text{Ind}(V)} I^{d_I}.$$

It follows from Proposition 2.2.5 that the HN filtrations of all I in $\text{Ind}(V)$ determine the HN filtration of V, so in principle it suffices to restrict attention to indecomposable representations. Unfortunately, the task of decomposing a given $V \in \mathbf{Rep}(Q)$ into indecomposables, and the task of cataloguing all possible indecomposables in **Rep**(Q) are both remarkably difficult problems for arbitrary Q. Two prominent exceptions are the Dynkin quivers of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$, whose indecomposables we describe below.

2.3.1 Quivers of type $\mathbb{A}$

Gabriel's seminal result [55] established that **Rep**(Q) is *representation finite*[2] if and only if the undirected graph obtained from Q by ignoring source and target information is a disjoint union of simply-laced Dynkin diagrams (i.e., of types $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$). In particular, we recall that a quiver Q is said to be of *type* $\mathbb{A}_n$ for some $n \geqslant 0$ whenever its underlying graph is:

$$x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} \cdots \overset{e_{n-1}}{\text{———}} x_{n-1}.$$

Representations of type $\mathbb{A}_n$ quivers are also called **zigzag persistence modules** [30]. If $s(e_i) = x_{i-1}$ and $t(e_i) = x_i$ holds for each i, then Q is called *equioriented*, and its representations are (ordinary) **persistence modules** [117].

Definition 2.3.1. Let Q be a quiver of type $\mathbb{A}_n$ and consider a subinterval $[u, v] \subset [0, n-1]$ with u and v integers. The **interval module** $\mathbf{I}[u, v]$ is the representation of Q defined as follows. The vector spaces assigned to vertices are

$$\mathbf{I}[u, v]_{x_i} = \begin{cases} \mathbb{F} & \text{if } i \in [u, v] \\ 0 & \text{otherwise.} \end{cases}$$

The linear map $\mathbf{I}[u, v]_{e_j}$ assigned to the edge e_j is the identity $\mathrm{id}_{\mathbb{F}}$ whenever both source and target vector spaces are nontrivial, and is necessarily zero otherwise. △

The following result is a direct consequence of Gabriel's theorem [55].

Theorem 2.3.2. *Let Q be a type $\mathbb{A}_n$ quiver. Then up to isomorphism, the indecomposables of* **Rep**(Q) *are precisely the interval modules* $\mathbf{I}[u, v]$ *for* $[u, v] \subset [0, n-1]$.

It follows directly from the above result that for every zigzag persistence module V there exists a unique finite set **Bar**(V) containing subintervals of $[0, n-1]$ and a unique function $\mathbf{Bar}(V) \to \mathbb{N}_{>0}$ denoted $[u, v] \mapsto d_{uv}$ which satisfy the following property: there is an isomorphism

$$V \simeq \bigoplus_{[u,v] \in \mathbf{Bar}(V)} \mathbf{I}[u, v]^{d_{uv}} \tag{2.3}$$

in **Rep**(Q). This set **Bar**(V) is called the **barcode** of V while $[u, v] \mapsto d_{uv}$ is called the **multiplicity** function. There are practical algorithms which can compute both barcodes and multiplicities for zigzag persistence modules — see [30].

---

[2] Representation finiteness of Q means that there are only finitely many indecomposables in **Rep**(Q) which attain a given dimension vector.

In this chapter we will also be interested in representations of type $\mathbb{A}_\infty$ quivers; the underlying graph of any such quiver Q has vertices indexed by $\mathbb{Z}$ as depicted below:

$$\begin{tikzcd}\cdots \arrow[r, dash, "e_{-1}"] & x_{-1} \arrow[r, dash, "e_0"] & x_0 \arrow[r, dash, "e_1"] & x_1 \arrow[r, dash, "e_2"] & \cdots\end{tikzcd}$$

Interval modules $\mathbf{I}[u,v]$ in $\mathbf{Rep}(Q)$ are described as in Definition 2.3.1, except that in addition to the usual subintervals $[u,v] \subset \mathbb{Z}$, one also allows infinite intervals with $u = -\infty$ or $v = \infty$ or both. The following result summarises [17, Theorem 1.6 and Theorem 1.7].

Theorem 2.3.3. *Let Q be a type $\mathbb{A}_\infty$ quiver. Then,*

1. *the only indecomposable objects in $\mathbf{Rep}(Q)$ are the interval modules; and moreover,*
2. *every $V \in \mathbf{Rep}(Q)$ satisfying $\dim V_x < \infty$ for all $x \in Q_0$ admits a unique (and not necessarily finite) set $\mathbf{Bar}(V)$ consisting of subintervals $[u,v] \subset \mathbb{Z} \cup \{\pm\infty\}$ and a multiplicity function $d : \mathbf{Bar}(V) \to \mathbb{Z}_{>0}$ so that the decomposition*

$$V \simeq \bigoplus_{[u,v]\in\mathbf{Bar}(V)} \mathbf{I}[u,v]^{d_{uv}}$$

holds in $\mathbf{Rep}(Q)$.

2.3.2 Affine quivers of type $\widetilde{\mathbb{A}}$

We say that Q is of *type* $\widetilde{\mathbb{A}}_n$ if its underlying graph has the form

$$\begin{tikzcd} & & x_{n-1} \arrow[dll, dash, "e_0"'] \arrow[drr, dash, "e_{n-1}"] & & \\ x_0 \arrow[r, dash, "e_1"'] & x_1 \arrow[r, dash, "e_2"'] & \cdots \arrow[r, dash, "e_{n-3}"'] & x_{n-3} \arrow[r, dash, "e_{n-2}"'] & x_{n-2}. \end{tikzcd} \tag{2.4}$$

for $n > 1$. In this section we assume that the ground field $\mathbb{F}$ is algebraically closed. With this assumption in place, the indecomposables of Q can be classified into two families, which we describe below (with illustrative examples to follow).

Definition 2.3.4. Let Q be a quiver of type $\widetilde{\mathbb{A}}_n$.

1. For each interval $[u,v] \subset \mathbb{Z}$, the representation $\mathbf{N}[u,v]$ of Q is defined as follows. Setting $\ell := \lfloor \frac{v-u}{n} \rfloor$, the vector space assigned to x_j is

$$\mathbf{N}[u,v]_{x_j} = \begin{cases} \mathbb{F}^{\ell+1} & \text{if } j \in [u,v] \bmod n, \text{ and} \\ \mathbb{F}^{\ell} & \text{otherwise.} \end{cases}$$

The linear map over e_j is the identity whenever its source and target vector spaces are equidimensional; the exceptional cases occur when j equals either u or $v+1$ modulo n. In such cases, if e_j has a clockwise orientation, then we have:

$$\mathbf{N}[u,v]_{e_j} = \begin{cases} [0 \ \mathrm{id}_\ell] & \text{if } (u \bmod n) = j \neq (v+1 \bmod n), \\ [\mathrm{id}_\ell \ 0]^T & \text{if } (u \bmod n) \neq j = (v+1 \bmod n), \text{ and} \\ \begin{bmatrix} 0 & \mathrm{id}_\ell \\ 0 & 0 \end{bmatrix} & \text{if } (u \bmod n) = j = (v+1 \bmod n) \end{cases}.$$

Similarly, when e_j has a counterclockwise orientation, $\mathbf{N}[u,v]_{e_j}$ is the transpose of the appropriate matrix above.

2. For each field element $\lambda \neq 0$ in $\mathbb{F}$ and integer $w \geqslant 1$, let $\mathbf{J}[\lambda; w]$ be the Q-representation which assigns to every vertex the vector space $\mathbb{F}^w$, to every edge e_j for $j \neq 0$ the identity map id_w, and to e_0 the Jordan block with λ along its diagonal. △

We observe that $\mathbf{N}[u,v]$ and $\mathbf{N}[u',v']$ are isomorphic whenever both (a) lengths $v-u$ and $v'-u'$ coincide, and (b) the equalities $u = u'$ and $v = v'$ hold modulo n. To remove the ambiguity, we will implicitly assume henceforth that u lies in $\{0,\ldots,n-1\}$ for every $\mathbf{N}[u,v]$.

Example 2.3.5. Here is the representation $\mathbf{N}[1,9]$ of a type $\widetilde{\mathbb{A}}_6$ quiver; note that the dimension is larger between vertices x_1 and x_3, and that unlabelled arrows carry identity maps:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[dll] & & \\
\mathbb{F} \arrow[r, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"'] & \mathbb{F}^2 \arrow[r] & \mathbb{F}^2 & \mathbb{F}^2 \arrow[l] \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[ull]
\end{tikzcd}$$

Similarly, here is the representation $\mathbf{J}[2;3]$ of the same quiver:

$$\begin{tikzcd}
 & & \mathbb{F}^3 \arrow[dll, "{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}}"'] & & \\
\mathbb{F}^3 \arrow[r] & \mathbb{F}^3 \arrow[r] & \mathbb{F}^3 & \mathbb{F}^3 \arrow[l] \arrow[r] & \mathbb{F}^3 \arrow[ull]
\end{tikzcd}$$

A proof of the following result can be found in [44].

Theorem 2.3.6. *A representation of a type $\widetilde{\mathbb{A}}_n$ quiver is indecomposable if and only if it is isomorphic either to $\mathbf{N}[u,v]$ for some interval $[u,v] \subset \mathbb{Z}$ or to $\mathbf{J}[\lambda;w]$ for some $\lambda \neq 0$ in $\mathbb{F}$ and $w \geqslant 1$ in $\mathbb{Z}$.*

2.4 The Euler stability condition

The stability condition that we will use throughout this chapter is simple to define. By the discussion following Remark 2.2.3, we only require a function $Q_0 \to \mathbb{R}$. This function is

$$x \mapsto (1 - \#\{e \in Q_1 \mid t(e) = x\}),$$

with # denoting cardinality. Readers who are satisfied with this definition may safely proceed to the next section; in this section we will only describe the reasons which have motivated our choice.

Consider a CW complex X, write $\sigma \leqslant \tau$ to indicate that the cell σ lies in the boundary of the cell τ in X, and denote the poset of cells ordered by this face relation by $(X, \leqslant)$. A

cellular sheaf F on X is a functor from $(X, \leqslant)$ to the category Vect of $\mathbb{F}$-vector spaces [40]. Thus, F assigns to each cell σ a vector space F_σ and to each face relation $\sigma \leqslant \tau$ a linear map $F_{\sigma\tau} : F_\sigma \to F_\tau$, subject to two axioms:

1. (identity) the map $F_{\sigma\sigma} : F_\sigma \to F_\sigma$ is the identity for each cell σ, and
2. (associativity) across any triple $\sigma \leqslant \tau \leqslant \nu$ of cells, we have $F_{\sigma\nu} = F_{\tau\nu} \circ F_{\sigma\tau}$.

When X is one-dimensional, the associativity axiom holds automatically since there are no triples of the form $\sigma < \tau < \nu$. Cellular sheaves on X form an abelian category $\mathbf{Shf}(X)$ whose morphisms are given by the natural transformations between functors $(X, \leqslant) \to$ Vect.

Let F be a cellular sheaf on a CW complex X, and consider the sequence of vector spaces and linear maps

$$\begin{tikzcd} 0 \arrow[r] & C^0(X;F) \arrow[r, "\delta_F^0"] & C^1(X;F) \arrow[r, "\delta_F^1"] & \cdots \end{tikzcd}, \tag{2.5}$$

where $C^j(X;F) := \prod_{\dim \sigma = j} F_\sigma$ for each dimension j, and the map δ_F^j has the following block structure. Its component $F_\sigma \to F_\tau$ equals

$$\delta_F^j|_{\sigma\tau} := \begin{cases} [\sigma : \tau] \cdot F_{\sigma\tau} & \text{if } \sigma \leqslant \tau \\ 0 & \text{otherwise,} \end{cases}$$

with $[\sigma : \tau]$ denoting the degree of the attaching map of τ's boundary along σ. A simple calculation [40, Lemma 6.2.2] confirms that (2.5) is a cochain complex, and the associated cohomology is denoted $H^j(X;F) := \ker \delta_F^j / \operatorname{img} \delta_F^{j-1}$. If these *sheaf cohomology* groups are finite dimensional for all j and vanish for $j \gg 0$, then there is a well-defined **Euler characteristic** of F:

$$\chi(X;F) := \sum_{j \geqslant 0} (-1)^j \cdot \dim H^j(X;F),$$

which (as usual) also equals the alternating sum $\sum_{j\geqslant 0}(-1)^j \cdot \dim C^j(X;F)$ whenever this sum makes sense.

The graph underlying any quiver may be treated as a one-dimensional regular CW complex — cells of dimension 0 and 1 are Q_0 and Q_1 respectively. In this case, the zeroth sheaf cohomology $H^0(X; \mathscr{S}V)$ coincides with the *space of sections* [108, Section 1] of V.

Definition 2.4.1. Let Q be a quiver with underlying graph X. The **associated sheaf functor** $\mathscr{S} : \mathbf{Rep}(Q) \to \mathbf{Shf}(X)$ sends each V in $\mathbf{Rep}(Q)$ to the cellular sheaf $\mathscr{S}V$ in $\mathbf{Shf}(X)$ which assigns to each cell $\sigma \in Q_0 \cup Q_1$ the vector space:

$$(\mathscr{S}V)_\sigma := \begin{cases} V_\sigma & \text{if } \sigma \in Q_0 \\ V_{t(\sigma)} & \text{if } \sigma \in Q_1, \end{cases}$$

and to each face relation $\sigma \leqslant \tau$, with $\tau \in Q_1$ and $\sigma \in \{s(\tau), t(\tau)\}$, the linear map:

$$(\mathscr{S}V)_{\sigma\tau} := \begin{cases} V_\tau & \text{if } \sigma = s(\tau) \\ \mathrm{id}_{V_\sigma} & \text{if } \sigma = t(\tau). \end{cases}$$

Morphisms $V \to W$ in $\mathbf{Rep}(Q)$ induce morphisms $\mathscr{S}V \to \mathscr{S}W$ in $\mathbf{Shf}(X)$ in a straightforward manner. △

Our next result shows that the Euler characteristic $\chi(X; \mathscr{S}V)$ can form the imaginary part of a stability condition on $\mathbf{Rep}(Q)$.

Proposition 2.4.2. *Let Q be a quiver with underlying graph X. The map $V \mapsto \chi(X; \mathscr{S}V)$ is a well-defined group homomorphism $K_0(\mathbf{Rep}(Q)) \to (\mathbb{R}, +)$.*

Proof. It is not difficult to confirm that the functor $\mathscr{S}$ is exact, so every exact sequence $0 \to U \to V \to W \to 0$ in $\mathbf{Rep}(Q)$ produces a short exact sequence of cochain complexes

$$0 \to C^\bullet(X; \mathscr{S}U) \to C^\bullet(X; \mathscr{S}V) \to C^\bullet(X; \mathscr{S}W) \to 0.$$

Passing to the long exact sequence on sheaf cohomology, we obtain the desired equality $\chi(X; \mathscr{S}V) = \chi(X; \mathscr{S}U) + \chi(X; \mathscr{S}W)$. □

Since there are only two nontrivial cochain groups to consider in (2.5) when X is the underlying graph of a quiver Q, we have for each V in $\mathbf{Rep}(Q)$ the formula

$$\begin{aligned} \chi(X; \mathscr{S}V) &= \sum_{x \in Q_0} \dim V_x - \sum_{e \in Q_1} \dim V_{t(e)} && \text{by (2.5) and Def 2.4.1,} \\ &= \sum_{x \in Q_0} (1 - \deg_{\text{in}}(x)) \cdot \dim V_x && . \end{aligned}$$

Here $\deg_{\text{in}}(x) := \# \{e \in Q_1 \mid t(e) = x\}$. We have arrived at the desired stability condition.

Definition 2.4.3. The **Euler stability condition** for an acyclic quiver Q is the map $\epsilon : Q_0 \to \mathbb{R}$ given by $\epsilon(x) := (1 - \deg_{\text{in}}(x))$, and the associated slope defined for $V \neq 0$ in $\mathbf{Rep}(Q)$ by

$$\mu_\epsilon(V) := \frac{\sum_{x \in Q_0} \epsilon(x) \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x},$$

is called the **Euler slope**. △

2.5 HN filtrations of ordinary persistence modules

Throughout this section, Q will denote the equioriented quiver of type $\mathbb{A}_n$ for some $n \geqslant 1$, and V will denote a fixed nonzero object of $\mathbf{Rep}(Q)$, i.e., an ordinary persistence module:

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_{n-1}\,. \tag{2.6}$$

Our goal here is to link the barcode decomposition of V from (2.3), recalled below:

$$V \simeq \bigoplus_{[u,v] \in \mathbf{Bar}(V)} \mathbf{I}[u, v]^{d_{uv}},$$

to its HN filtration with respect to the Euler stability condition (see Theorem 2.2.2 and Definition 2.4.3). The first step in this direction is to establish some general results which hold for a wider class of stability conditions.

2.5.1 General results for antitone slopes

Let $\preceq$ denote the lexicographical total ordering on subintervals of $[0, n-1]$, where $[u, v] \preceq [u', v']$ holds if we have either $u < u'$ or both $u = u'$ and $v \leqslant v'$. Let $\alpha : Q_0 \to \mathbb{R}$ be a stability condition on Q with associated slope μ_α. We say that our slope μ_α is **antitone** if the inequality $\mu_\alpha(\mathbf{I}[u, v]) \geqslant \mu_\alpha(\mathbf{I}[u', v'])$ holds in $\mathbb{R}$ whenever we have $[u, v] \preceq [u', v']$.

Lemma 2.5.1. *If μ_α is antitone, then every interval module $\mathbf{I}[u, v]$ for $[u, v] \in \mathbf{Bar}(V)$ is α-semistable.*

Proof. Fix an interval $[u, v] \in \mathbf{Bar}(V)$, and consider a nonzero submodule $U \subset \mathbf{I}[u, v]$. Using the barcode decomposition of U from Theorem 2.3.2 followed by Lemma 2.2.4, we obtain the inequality

$$\mu_\alpha(U) \leqslant \max_{[u',v'] \in \mathbf{Bar}(U)} \mu_\alpha(\mathbf{I}[u', v']).$$

Thus, we are required to prove that $\mu_\alpha(\mathbf{I}[u', v']) \leqslant \mu_\alpha(\mathbf{I}[u, v])$ holds whenever $\mathbf{I}[u', v']$ is a submodule of $\mathbf{I}[u, v]$. It is readily checked that this submodule condition holds if and only if $u' \geqslant u$ and $v' = v$. This fact follows from [11, Theorem 4.2(i)], but for completeness let us recall that we need $u' \geqslant u$ for dimension reasons. Moreover, $v' = v$ follows from the fact that if v exceeds v' then the following diagram of vector spaces fails to commute:

$$\begin{tikzcd}
\mathbb{F} = \mathbf{I}[u', v']_{v'} \arrow[r] \arrow[d, hook] & 0 = \mathbf{I}[u', v']_{v'+1} \arrow[d, hook] \\
\mathbb{F} = \mathbf{I}[u, v]_{v'} \arrow[r, "\mathrm{id}"'] & \mathbb{F} = \mathbf{I}[u, v]_{v'+1}
\end{tikzcd} \tag{2.7}$$

Thus, we have $[u, v] \preceq [u', v']$ and the desired conclusion follows because μ_α is antitone. ☐

For antitone μ_α's we can now describe HN filtrations directly in terms of barcodes.

Proposition 2.5.2. *Assume that the slope μ_α is antitone, then for each $t \in \mathbb{R}$ the HN $\mathbb{R}$-filtration of V evaluated at t is given by*

$$\langle V, \alpha \rangle^t \simeq \bigoplus_{[u,v]} \mathbf{I}[u, v]^{d_{uv}}, \tag{2.8}$$

where the sum is over all $[u, v] \in \mathbf{Bar}(V)$ satisfying $\mu_\alpha(\mathbf{I}[u, v]) \geqslant t$. The multiplicities d_{uv} are the ones appearing in (2.3) and the isomorphism can be obtained by restricting the decomposition from (2.3).

Proof. We know from Lemma 2.5.1 that interval modules are semistable for antitone μ_α. By Remark 2.2.3, the HN filtration of such a module is the trivial one ($0 \subsetneq \mathbf{I}[u, v]$), which clearly satisfies (2.8). By Proposition 2.2.5, the left side of (2.8) is (also) additive, whence V satisfies (2.8) as claimed. ☐

Corollary 2.5.3. *Assume that μ_α is antitone and that the HN filtration $V^\bullet$ of V has length ℓ. Then for each j in $\{1, \ldots, \ell\}$, we have an isomorphism*

$$\frac{V^j}{V^{j-1}} \simeq \bigoplus_{[u,v]} \mathbf{I}[u,v]^{d_{uv}},$$

where the sum ranges over all $[u,v] \in \mathbf{Bar}(V)$ satisfying $\mu_\alpha(\mathbf{I}[u,v]) = \mu_\alpha(V^j/V^{j-1})$, and where d indicates the multiplicity function of V from (2.3).

2.5.2 Specific results for Euler slopes

We continue to work with a fixed persistence module V as in (2.6), but now specialise to the Euler stability condition $\epsilon : Q_0 \to \mathbb{R}$ from Definition 2.4.3. It is immediate from this definition that the Euler slope of V is

$$\mu_\epsilon(V) = \frac{\dim V_0}{\sum_{i=0}^{n-1} \dim V_i}. \tag{2.9}$$

Here is the full version of Theorem (**A**) from the Introduction.

Theorem 2.5.4. *Let $[u,v] \mapsto d_{u,v}$ be the multiplicity function of V as described in (2.3), and let J be the (possibly empty) set of all indices $j \in \{0, \ldots, n-1\}$ for which $[0,j]$ lies in* $\mathbf{Bar}(V)$. *If J^* is the set of all intervals $[u,v] \in \mathbf{Bar}(V)$ with $u \neq 0$, then,*

1. *the length ℓ of the (Euler) HN filtration $V^\bullet$ is*

$$\ell := \begin{cases} 1 + (\#J) & \text{if } J^* \neq \varnothing \\ (\#J) & \text{otherwise;} \end{cases}$$

and moreover,

2. *if the elements of J are $j_1 < \ldots < j_{(\#J)}$, then we have for $1 \leqslant k \leqslant \ell$*

$$\frac{V^k}{V^{k-1}} = \begin{cases} \mathbf{I}[0, j_k]^{d_{0,j_k}} & \text{if } k \leqslant (\#J) \\ \bigoplus_{[u,v] \in J^*} \mathbf{I}[u,v]^{d_{u,v}} & \text{if } k = 1 + (\#J). \end{cases}$$

Proof. Replacing V by an indecomposable module $\mathbf{I}[u,v]$ in the formula (2.9) gives

$$\mu_\epsilon(\mathbf{I}[u,v]) = \begin{cases} \frac{1}{v+1}, & \text{if } u = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, μ_ϵ is antitone and we may apply Corollary 2.5.3 to establish both assertions. □

The preceding result establishes a direct relationship between barcode decompositions and (Euler) HN filtrations of ordinary persistence modules — to obtain the HN filtration from the barcode inductively, one lexicographically orders the intervals in $\mathbf{Bar}(V)$ with left endpoint zero as $[0, j_0] \prec [0, j_1] \prec \cdots \prec [0, j_{n-1}]$; then for $k < n$, we have

$$V^k \simeq V^{k-1} \bigoplus \mathbf{I}[0, j_k]^{d_{0,j_k}}.$$

Conversely, it is possible to extract the multiplicity of any interval of the form $[0, v]$ in $\mathbf{Bar}(V)$ by using the HN filtration: there is at most one index k for which the equality $\mu_\epsilon(V^k/V^{k-1}) = 1/(v+1)$ holds. If such a k exists, then we have $d_{0,v} = \dim(V^k/V^{k-1})_{x_0}$; otherwise, $[0, v]$ is not in $\mathbf{Bar}(V)$. There is, however, no separation of intervals $[u, v] \in J^*$ from each other since all of them have slope 0. Two remedies are available for this unfortunate incompleteness — the first of these comes in the form of slopes which separate all intervals in $\mathbf{Bar}(V)$. Any such *totally stable* slope [80], if used instead of ϵ in the proofs above, would allow us to recover the entire barcode of V via successive quotients of the associated HN filtration. The second remedy is described below.

Remark 2.5.5. Let us label the vertices and edges of the underlying quiver Q as follows:

$$\begin{tikzcd} x_0 \arrow[r, "e_1"] & x_1 \arrow[r, "e_2"] & \cdots \arrow[r, "e_{n-1}"] & x_{n-1}. \end{tikzcd}$$

For each integer k in $\{0, \ldots, n-1\}$, define $Q^{\geqslant k}$ to be the type $\mathbb{A}_{n-k}$ subquiver of Q consisting of vertices $\{x_j \mid k \leqslant j \leqslant n-1\}$ and all the edges between them. Let $V^{\geqslant k}$ be the representation of $Q^{\geqslant k}$ obtained by restricting V to $Q^{\geqslant k}$. By the restriction principle [30], the decomposition (2.3) becomes

$$V^{\geqslant k} \simeq \bigoplus_{[u,v]} \mathbf{I}[\max(u-k, 0), v-k]^{d_{uv}},$$

where the sum is over all intervals $[u, v] \in \mathbf{Bar}(V)$ satisfying $v \geqslant k$. For a fixed interval $[k, v]$ in $\mathbf{Bar}(V)$, the above decomposition induces the relation on multiplicities

$$d_{k,v} = d^{\geqslant k}_{0,v-k} - \sum_{[u,v]} d_{u,v}.$$

Here $d^{\geqslant k}_{0,v-k}$ denotes the multiplicity of $[0, v-k]$ in $\mathbf{Bar}(V^{\geqslant k})$, and the sum is over intervals $[u, v]$ in $\mathbf{Bar}(V)$ satisfying $u < k$. By Theorem 2.5.4, we have that $d^{\geqslant k}_{0,v-k}$ can be recovered from the HN filtration of $V^{\geqslant k}$ along the Euler stability condition. Thus, we can inductively reconstruct the entire multiplicity function of V from the HN filtrations of $\left\{V^{\geqslant k} \mid k \geqslant 0\right\}$.

2.6 Unwinding affine type $\widetilde{\mathbb{A}}$ quivers

We say that a quiver is of **type** $\mathbb{A}_\infty$ whenever its underlying graph has the form

$$\begin{tikzcd} \cdots \arrow[r, dash, "a_{-1}"] & y_{-1} \arrow[r, dash, "a_0"] & y_0 \arrow[r, dash, "a_1"] & y_1 \arrow[r, dash, "a_2"] & \cdots \end{tikzcd}$$

Explicitly, the vertex set is identified with the set of integers $\mathbb{Z}$, and there is a unique edge between every adjacent pair of integers. Thus, representations of $\mathbb{A}_\infty$ quivers are *infinite* zigzag persistence modules.

We now fix a quiver $Q = (s, t : Q_1 \to Q_0)$ of type $\widetilde{\mathbb{A}}_n$ labelled as in (2.4).

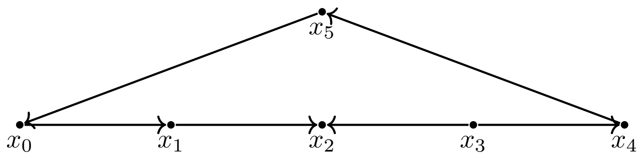
Definition 2.6.1. The **unwinding** of Q is a quiver $\mathbf{U}Q$ of type $\mathbb{A}_\infty$ whose vertex and edge sets are $\mathbf{U}Q_0 := Q_0 \times \mathbb{Z}$ and $\mathbf{U}Q_1 := Q_1 \times \mathbb{Z}$, with source and target maps $\sigma, \tau : \mathbf{U}Q_1 \to \mathbf{U}Q_0$ given as follows: for all $j \in \{1, 2, \ldots, n-1\}$ and $c \in \mathbb{Z}$, we have

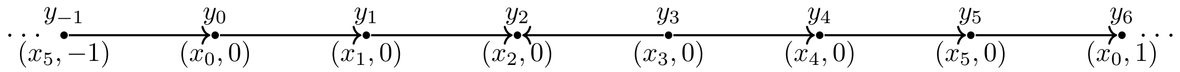
$$\sigma(e_j, c) := (s(e_j), c) \quad \text{and} \quad \tau(e_j, c) := (t(e_j), c).$$

If $s(e_0) = x_0$ then the edge $(e_0, c) \in \mathbf{U}Q_1$ has source (x_0, c) and target $(x_{n-1}, c-1)$ for each $c \in \mathbb{Z}$. Conversely, if $s(e_0) = x_{n-1}$, then (e_0, c) has source $(x_{n-1}, c-1)$ and target (x_0, c). △

Let $\phi : \mathbf{U}Q_0 \to \mathbb{Z}$ be the bijection $(x_i, c) \mapsto i + cn$; for each interval $[p, q] \subset \mathbb{Z}$, let $\mathbf{U}Q^{p,q} \subset \mathbf{U}Q$ denote the *truncated* subquiver spanned by all vertices satisfying $\phi(x_j, c) \in [p, q]$ and the edges between them.

Example 2.6.2. Here is $\widetilde{\mathbb{A}}_6$ quiver from Example 2.3.5 depicted above its unwinding. The labels above and below the vertices of the unwinding are $\phi(\mathbf{U}Q_0)$ and $\mathbf{U}Q_0$ respectively.





As before, we only consider representations whose assigned vector spaces are all finite-dimensional. Such a representation $W \in \mathbf{Rep}(\mathbf{U}Q)$ is said to be *periodic* whenever we have:

$$W_{(x_j,c)} = W_{(x_j,c+1)} \text{ for all } (x_j, c) \in \mathbf{U}Q_0, \text{ and}$$
$$W_{(e_j,c)} = W_{(e_j,c+1)} \text{ for all } (e_j, c) \in \mathbf{U}Q_1.$$

We denote by $\mathbf{PRep}(\mathbf{U}Q)$ the full subcategory of $\mathbf{Rep}(\mathbf{U}Q)$ which consists of all such periodic representations. Crucially, we do *not* require that the morphisms in $\mathbf{PRep}(\mathbf{U}Q)$ be periodic, otherwise the resulting category would be equivalent to $\mathbf{Rep}(Q)$.

Proposition 2.6.3. *Up to isomorphism, the indecomposable representations in* $\mathbf{PRep}(\mathbf{U}Q)$ *are either*

1. *the infinite interval module* $\mathbf{I}[-\infty, \infty]$*, or*
2. *the direct sum* $\mathbf{\Sigma}[u, v] := \bigoplus_{c \in \mathbb{Z}} \mathbf{I}[u + cn, v + cn]$ *for some interval* $[u, v] \subset \mathbb{Z}$.

Proof. Since $\mathbf{PRep}(\mathbf{U}Q)$ is a subcategory of $\mathbf{Rep}(\mathbf{U}Q)$, it follows from Theorem 2.3.3 that every representation W in $\mathbf{PRep}(\mathbf{U}Q)$ admits a decomposition into indecomposables in $\mathbf{Rep}(\mathbf{U}Q)$ of the form

$$W \simeq \bigoplus_{[u,v]} \mathbf{I}[u,v]^{d_{u,v}}. \tag{2.10}$$

Here the sum ranges over intervals $[u,v] \subset \mathbb{Z}$ lying in the barcode $\mathbf{Bar}(W)$, and $d_{u,v}$ is the multiplicity of each such interval as described in (2.3). We first claim that the multiplicity function is n-periodic — namely, for all $[p,q] \subset \mathbb{Z}$, we have $d_{p,q} = d_{p+n,q+n}$. To verify this claim, let $W^{p,q}$ denote the representation of $\mathbf{U}Q^{p,q}$ obtained by restricting W to the available vertices and edges. By (2.10), we have

$$W^{p-1,q+1} \simeq \bigoplus_{[u,v]} \mathbf{I}\big[[u,v] \cap [p-1,q+1]\big]^{d_{u,v}}$$

where the sum is over intervals $[u,v] \in \mathbf{Bar}(W)$ that have nonempty intersection with $[p-1,q+1]$. The desired n-periodicity of d now follows from the fact that the left side remains invariant if p and q are replaced by $p+n$ and $q+n$ respectively. Similarly, $\mathbf{Bar}(W)$ can not contain any intervals of the form $[u,\infty]$ or $[-\infty,v]$ — the inclusion of any $[u,\infty]$ interval would, for instance, force the inclusion of $[u+cn,\infty]$ for all $c \in \mathbb{Z}$, which contradicts the finite dimensionality of W. Thus, (2.10) becomes

$$\begin{aligned} W &\simeq \mathbf{I}[-\infty,\infty]^{d_{-\infty,\infty}} \oplus \bigoplus_{[u,v]} \left(\bigoplus_{c\in\mathbb{Z}} \mathbf{I}[u+cn, v+cn] \right)^{d_{u,v}} \\ &= \mathbf{I}[-\infty,\infty]^{d_{-\infty,\infty}} \oplus \bigoplus_{[u,v]} \mathbf{\Sigma}[u,v]^{d_{u,v}}, \end{aligned}$$

where the sums range over intervals $[u,v] \in \mathbf{Bar}(W)$ with $0 \leqslant u < n$. Moreover, since $\mathbf{PRep}(\mathbf{U}Q)$ is a full subcategory of $\mathbf{Rep}(\mathbf{U}Q)$, the above decomposition holds in $\mathbf{PRep}(\mathbf{U}Q)$, as desired. Finally, we note that $\mathbf{I}[-\infty,\infty]$ is indecomposable in $\mathbf{PRep}(\mathbf{U}Q)$ because it is indecomposable in the larger category $\mathbf{Rep}(\mathbf{U}Q)$ by Theorem 2.3.3, and that each $\mathbf{\Sigma}[u,v]$ is indecomposable in $\mathbf{PRep}(\mathbf{U}Q)$ because any nontrivial periodic decomposition of it would produce a nontrivial decomposition of $\mathbf{I}[u,v]$ in a sufficiently long truncation of $\mathbf{U}Q$. □

Remark 2.6.4. The argument given above also establishes that the multiplicity function $\mathbf{Bar}(W) \to \mathbb{N}_{>0}$ of $W \in \mathbf{PRep}(\mathbf{U}Q)$ can be recovered from the decomposition (into indecomposables) of a sufficiently large but finite zigzag persistence module. More precisely, let $D \in \mathbb{Z}$ be chosen so that for any $[u,v] \in \mathbf{Bar}(W)$ there is a $c \in \mathbb{Z}$ satisfying $[u+cn, v+cn] \subset [1, D-1]$. Then there is a one-to-one correspondence between the non-zero summands of the decompositions of W and the truncation $W^{0,D}$. In fact, $D := (\dim W_0 + 2)n$ always suffices: if $[u,v]$ lies in $\mathbf{Bar}(W)$, then we have

$$\#([u,v] \cap n\mathbb{Z}) \leqslant \dim W_0 \quad \text{and} \quad v - u < (\dim W_0 + 1)n = D - n,$$

so the translation of $[u,v]$ by $n\mathbb{Z}$ whose starting point is in $[1,n]$ is included in $[1, D-1]$.

There is an evident **lift** map that sends objects of $\mathbf{Rep}(Q)$ to those of $\mathbf{Rep}(\mathbf{U}Q)$ defined as follows. The lift $\mathcal{L}V$ of a representation V of Q assigns V_{x_j} to every vertex in $\mathbf{U}Q_0$ of the form (x_j, c) and similarly V_{e_j} to every edge of the form (e_j, c) in $\mathbf{U}Q_1$. We note that $\mathcal{L}V$ is called the *infinite cyclic covering* in [**27**].

Example 2.6.5. Continuing Examples 2.3.5 and 2.6.2, here are the representation $\mathbf{N}[1,9]$ of the $\widetilde{\mathbb{A}}_6$ quiver from example 2.3.5 and its lift $\mathcal{L}(\mathbf{N}[1,9])$. As before, unlabelled edges carry identity maps:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[dll] & & \\
\mathbb{F} \arrow[r, swap, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] & \mathbb{F}^2 \arrow[r] & \mathbb{F}^2 & \mathbb{F}^2 \arrow[l] \arrow[r, swap, "{[1\ \ 0]}"] & \mathbb{F} \arrow[ull]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, swap, "{[1\ \ 0]}"] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r, swap, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] & \mathbb{F}^2 \arrow[r] & \mathbb{F}^2 & \mathbb{F}^2 \arrow[l] \arrow[r, swap, "{[1\ \ 0]}"] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r, swap, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] & \cdots
\end{tikzcd}$$

Given a morphism $f : V \to V'$ in $\mathbf{Rep}(Q)$, the lifted morphism $\mathcal{L}f : \mathcal{L}V \to \mathcal{L}V'$ is given by

$$\mathcal{L}f_{(x_j,c)} = f_{x_j} \text{ for all } c \in \mathbb{Z}.$$

Thus, the morphisms of $\mathbf{PRep}(\mathbf{U}Q)$ lying in the image of $\mathcal{L}$ are always periodic even though – as mentioned above – morphisms in $\mathbf{PRep}(Q)$ are not periodic in general. As a consequence, we can not expect all the indecomposable representations from Definition 2.3.4 to lift to indecomposable representations in $\mathbf{PRep}(\mathbf{U}Q)$ from Proposition 2.6.3. However, we have the following convenient result.

Proposition 2.6.6. *Let* $\mathcal{L} : \mathbf{Rep}(Q) \to \mathbf{PRep}(\mathbf{U}Q)$ *be the lift functor. Then,*

1. *for every interval* $[u, v] \subset \mathbb{Z}$*, we have*

$$\mathcal{L}(\mathbf{N}[u,v]) = \mathbf{\Sigma}[u,v],$$

where the right side is as defined in Proposition 2.6.3. And moreover,

2. *for each field element* $\lambda \neq 0$ *in* $\mathbb{F}$ *and integer* $w \geqslant 1$*, we have*

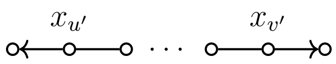
$$\mathcal{L}(\mathbf{J}[\lambda; w]) = \mathbf{I}[-\infty, \infty]^w.$$

Proof. The matrices in the description of $\mathbf{N}[u,v]$ from Definition 2.3.4 are already in barcode form (see [**70**, Definition 2.2]), so the desired interval decomposition of $\mathcal{L}\mathbf{N}[u,v]$ can be directly inferred from these matrices. Moreover, since all the matrices in the description of $\mathbf{J}[\lambda; w]$ are isomorphisms, it follows from Theorem 2.3.3 that there exists a change of basis which turns them all into identities. □

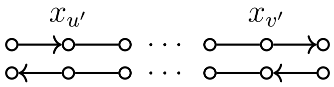
2.7 HN filtrations of affine type $\widetilde{\mathbb{A}}$ quivers

Let Q be a quiver of type $\widetilde{\mathbb{A}}_n$ as defined in (2.4), and fix a representation $V \neq 0$ of Q valued in vector spaces over an algebraically closed field $\mathbb{F}$. Our goal here is to relate the HN filtration of V along the Euler stability condition (from Definition 2.4.3) to the barcode decomposition of the lifted representation $\mathcal{L}V \in \mathbf{Rep}(\mathbf{U}Q)$ which has been described in Proposition 2.6.3.

The first task at hand is to compute the Euler slopes of all indecomposables in $\mathbf{Rep}(Q)$ — we recall from Theorem 2.3.6 that these have form $\mathbf{N}[u,v]$ and $\mathbf{J}[\lambda; w]$, both of which have been described in Definition 2.3.4. There is a single number $p_{u,v} \in \{0,1,2\}$ which determines the sign of the Euler slope of each $\mathbf{N}[u,v]$, and it admits the following graphical description (with $u' := u \bmod n$ and $v' := v \bmod n$):



$p_{u,v} = 0$



$p_{u,v} = 1$

$p_{u,v} = 2$

Explicitly, $p_{u,v}$ counts whether the edge $e_{u'}$ has target u' and whether the edge $e_{v'+1}$ has target v'. The cases $p_{u,v} = 0, 1, 2$ are equivalent [56, Sec 11.3] to $\mathbf{N}[u,v]$ being called respectively pre-injective, regular and pre-projective in standard texts [81, 107].

Proposition 2.7.1. *The Euler slopes of indecomposable representations in* $\mathbf{Rep}(Q)$ *are given as follows.*

1. *For each* $[u,v] \subset \mathbb{Z}$*, we have*

$$\mu_\epsilon(\mathbf{N}[u,v]) = \frac{1 - p_{u,v}}{v - u + 1}.$$

2. *For each nonzero* $\lambda \in \mathbb{F}$ *and integer* $w \geqslant 1$*, we have*

$$\mu_\epsilon(\mathbf{J}[\lambda; w]) = 0.$$

Proof. The second assertion follows from the following elementary calculation: for each λ and w as above, by Definitions 2.3.4 and 2.4.3 we have

$$\mu_\epsilon(\mathbf{J}[\lambda; w]) = \frac{\sum_{x \in Q_0} (1 - \deg_{\text{in}}(x)) \cdot w}{\sum_{x \in Q_0} w}.$$

Since $\sum_{x \in Q_0} \deg_{\text{in}}(x)$ equals the cardinality of the edge set, we obtain

$$\mu_\epsilon(\mathbf{J}[\lambda; w]) = \frac{\#Q_0 - \#Q_1}{\#Q_0}.$$

The numerator on the right side vanishes, as desired, because both cardinalities equal n.

Turning now to $\mathbf{N}[u,v]$, by the discussion preceding Definition 2.4.3 we have

$$\mu_\epsilon(\mathbf{N}[u,v]) = \frac{\chi(X; \mathscr{S}\,\mathbf{N}[u,v])}{\sum_{x \in Q_0} \dim \mathbf{N}[u,v]_x}. \tag{2.11}$$

Here X denotes the underlying graph of Q, while $\mathscr{S} : \mathbf{Rep}(Q) \to \mathbf{Shf}(X)$ is the associated sheaf functor from Definition 2.4.1. This associated sheaf is locally constant on a decomposition of X into two disjoint intervals $X = A \sqcup B$, whose exact nature depends on the value of $p_{u,v}$. In particular, setting $\ell := \lfloor \frac{v-u}{n} \rfloor$, let $A \subset X$ be the union of cells on which $\mathbf{N}[u,v]$ has dimension $\ell + 1$, and let B be the complement of A in X (on which $\mathbf{N}[u,v]$ necessarily has dimension ℓ). Whether A is closed, clopen or open in X depends on $p_{u,v}$ being $0, 1$ or 2. Therefore, A's Euler characteristic is

$$\chi(A) = \begin{cases} +1 & p_{u,v} = 0, \\ 0 & p_{u,v} = 1, \\ -1 & p_{u,v} = 2. \end{cases}$$

Equivalently, we have $\chi(A) = 1 - p_{u,v}$.

Now the numerator in (2.11) is given by

$$\chi(X; \mathscr{S}\,\mathbf{N}[u,v]) = \chi(A; \underline{\mathbb{F}^{\ell+1}}_A) + \chi(B; \underline{\mathbb{F}^{\ell}}_B),$$

where $\underline{\mathbb{F}^{\ell+1}}_A$ is the constant $\mathbb{F}^{\ell+1}$-valued sheaf on A, etc. Thus,

$$\chi(X; \mathscr{S}\,\mathbf{N}[u,v]) = \chi(A) \cdot (\ell + 1) + \chi(B) \cdot \ell.$$

By additivity of the Euler characteristic, we have $\chi(B) = -\chi(A)$ since X is homeomorphic to a circle and satisfies $\chi(X) = 0$. Therefore, $\chi(X; \mathscr{S}\,\mathbf{N}[u,v])$ equals $\chi(A)$, which in turn equals $1 - p_{u,v}$. Finally, the denominator in (2.11) is

$$\begin{aligned} &(\ell+1) \cdot (v' - u' + 1) + \ell \cdot (n - (v' - u' + 1)) \\ =&(v' - u' + 1) + \ell n, \end{aligned}$$

which, by definition of ℓ, simplifies to $v - u + 1$ as desired. □

Next, we seek to establish that all the indecomposable representations in $\mathbf{Rep}(Q)$ are ϵ-semistable. In light of Lemma 2.5.1, it suffices to show that the inequality $\mu_\epsilon(I) \leqslant \mu_\epsilon(I')$ holds whenever we have an injective morphism $I \hookrightarrow I'$ between a pair of indecomposables in $\mathbf{Rep}(Q)$. There are only three nontrivial cases of $I \hookrightarrow I'$ to consider, which – based on the slope computations from Proposition 2.7.1 – we call $(++)$, $(0-)$ and $(--)$. Explicitly,

$(++)$ has $\mathbf{N}[u,v] \hookrightarrow \mathbf{N}[\bar{u},\bar{v}]$ with $p_{u,v} = 0 = p_{\bar{u},\bar{v}}$,

$(0-)$ has $\mathbf{J}[\lambda; w] \hookrightarrow \mathbf{N}[u,v]$ with $p_{u,v} = 2$,

$(--)$ has $\mathbf{N}[u,v] \hookrightarrow \mathbf{N}[\bar{u},\bar{v}]$ with $p_{u,v} = 2 = p_{\bar{u},\bar{v}}$.

Remark 2.7.2. If we have an injective morphism $U \hookrightarrow V$ in $\mathbf{Rep}(Q)$, then for every edge $e \in Q_1$ the linear map U_e is precisely the restriction of V_e. It follows that if V_e is injective, then so is U_e. We note that all the edges are assigned injective maps in $\mathbf{J}[\lambda; w]$; but exactly $2 - p_{u,v}$ edges are assigned a map with a nontrivial kernel by $\mathbf{N}[u, v]$. Therefore, the case $(+0)$ involving $\mathbf{N}[u, v] \hookrightarrow \mathbf{J}[\lambda; w]$ for $p_{u,v} = 0$ never arises. And similarly, all the cases involving $\mathbf{N}[u, v] \hookrightarrow \mathbf{N}[\bar{u}, \bar{v}]$ with $p_{u,v} < p_{\bar{u},\bar{v}}$ are impossible.

The following proposition can be seen as a corollary of [**65**, Prop 5.2]. We prove it here without using any property of tame hereditary algebras except the decomposition from definition 2.3.4.

Proposition 2.7.3. *The indecomposable representations of Q are all ϵ-semistable.*

Proof. We treat each of the three cases separately.

Case $(++)$: The injectivity requirement from Remark 2.7.2 forces both $u = \bar{u} \bmod n$ and $v = \bar{v} \bmod n$, and (translating if necessary) we may as well assume that $\bar{u} = u$ and $\bar{v} = v + cn$ for some $c \in \mathbb{Z}$. If c is negative, then $(v - u - 1)$ exceeds $(\bar{v} - u - 1)$, which implies $\mu_\epsilon(\mathbf{N}[u, v]) < \mu_\epsilon(\mathbf{N}[u, \bar{v}])$ by Proposition 2.7.1. Thus, we assume that c is non-negative and seek to verify that $c = 0$. Consider the lifts

$$W := \mathcal{L}(\mathbf{N}[u, v]) \quad \text{and} \quad W' := \mathcal{L}(\mathbf{N}[u, \bar{v}])$$

which are defined on the unwinding $\mathbf{U}Q$. By Proposition 2.6.6, we have $W = \mathbf{\Sigma}[u, v]$ and $W' = \mathbf{\Sigma}[u, \bar{v}]$. The injective map $\mathbf{N}[u, v] \hookrightarrow \mathbf{N}[u, \bar{v}]$ in $\mathbf{Rep}(Q)$ lifts to an injective map $\iota : W \hookrightarrow W'$ in $\mathbf{PRep}(\mathbf{U}Q)$. Restricting to the *finite* type $\mathbb{A}$ subquiver obtained by truncating $\mathbf{U}Q$ to $[\phi^{-1}(u) - 1, \phi^{-1}(\bar{v}) + 1]$, we obtain a distinguished interval $[u, v]$ in $\mathbf{Bar}(W)$ that is generated by the kernel of $W_{\phi^{-1}(u)}$. Similarly, there is an interval $[u, \bar{v}]$ in $\mathbf{Bar}(W')$ generated by the kernel of $W'_{\phi^{-1}(u)}$. The fact that ι induces an injective map $\mathbf{I}[u, v] \hookrightarrow \mathbf{I}[u, \bar{v}]$ in $\mathbf{Rep}(Q')$ now forces $v = \bar{v}$ by (2.7). Thus, $\mathbf{N}[u, v]$ and $\mathbf{N}[\bar{u}, \bar{v}]$ have equal slopes by Proposition 2.7.1.

Case $(0-)$: Proceeding as in the previous case, we define the relevant lifts W of $\mathbf{J}[\lambda; w]$ and W' of $\mathbf{N}[u, v]$. By Proposition 2.6.6, we have

$$W = \mathbf{I}[-\infty, \infty]^w \quad \text{and} \quad W' = \mathbf{\Sigma}[u, v].$$

We now claim that there is in fact no injective map $\mathbf{J}[\lambda; w] \hookrightarrow \mathbf{N}[u, v]$ in $\mathbf{Rep}(Q)$; if such a map existed, it would induce an injective map $W \hookrightarrow W'$ in $\mathbf{PRep}(\mathbf{U}Q)$. But by the same reasoning as in (2.7), no infinite interval can map injectively to a direct sum of finite intervals.

Case $(--)$: Since $\mathbf{N}[u, v]$ is a sub-representation of $\mathbf{N}[\bar{u}, \bar{v}]$ by assumption, we have the inequality $\dim \mathbf{N}[u, v]_x \leqslant \dim \mathbf{N}[\bar{u}, \bar{v}]_x$ for every vertex $x \in Q_0$. By Definition 2.3.4, these inequalities force $(\bar{v} - \bar{u} + 1) \leqslant (v - u + 1)$, whence Proposition 2.7.1 guarantees that we have $\mu_\epsilon(\mathbf{N}[u, v]) \leqslant \mu_\epsilon(\mathbf{N}[\bar{u}, \bar{v}])$ as desired. ☐

Here is our main result, which is Theorem (**B**) from the Introduction.

Theorem 2.7.4. *Let Q be an acyclic quiver of type $\widetilde{\mathbb{A}}_n$ and let V be a representation of Q over an algebraically closed field. Let $d^* : \mathbf{Bar}(\mathcal{L}V) \to \mathbb{N}_{>0}$ be the multiplicity function of the lift of V, and let $S^j = V^j/V^{j-1}$ be the successive quotients in the HN filtration of V along ϵ.*

If $\mu_\epsilon(S^j)$ is nonzero, then it is of the form $\mu_\epsilon(S^j) = \frac{1}{k}$ for some nonzero integer $k \in \mathbb{Z}_{\neq 0}$ and for each vertex $x \in Q_0$ we have:

$$\dim(S^j)_x = \sum_{\mathbf{N}[u,v]} d^*_{u,v} \cdot \dim \mathbf{N}[u,v]_x \tag{2.12}$$

where the sum is over all $\mathbf{N}[u,v]$ appearing in the decomposition of V such that $p_{u,v} = 1 - \operatorname{sign}(k)$ and $v - u + 1 = |k|$. Similarly, if $\mu_\epsilon(S^j)$ equals 0, then for each vertex $x \in Q_0$, we have:

$$\dim(S^j)_x = d^*_{-\infty,\infty} + \sum_{\mathbf{N}[u,v]} d^*_{u,v} \cdot \dim \mathbf{N}[u,v]_x \tag{2.13}$$

where the sum is over indecomposable summands $\mathbf{N}[u,v]$ of V that satisfy $p_{u,v} = 1$.

Proof. From Theorem 2.3.6, we know that there exist integers $d_{u,v} > 0$ and $d_{\lambda;w} > 0$ so that V decomposes as

$$V \simeq \bigoplus_{[u,v]} \mathbf{N}[u,v]^{d_{u,v}} \oplus \bigoplus_{\lambda,w} \mathbf{J}[\lambda; w]^{d_{\lambda;w}},$$

where the first sum is over $[u,v] \subset \mathbb{Z}$ with $u \in [0, n-1]$ such that $\mathbf{N}[u,v] \in \operatorname{Ind}(V)$ and the second sum is over $(\lambda, w) \in (\mathbb{F} - \{0\}) \times \mathbb{N}_{>0}$ such that $\mathbf{J}[\lambda; w] \in \operatorname{Ind}(V)$. Similarly, by Proposition 2.6.3 we have integers $d^*_{u,v} > 0$ and $d^*_\infty > 0$ satisfying

$$\mathcal{L}V \simeq \bigoplus_{[u,v]} \mathbf{\Sigma}[u,v]^{d^*_{u,v}} \oplus \bigoplus_{(\lambda,w)} \mathbf{I}[-\infty, \infty]^{d^*_\infty}$$

By Proposition 2.6.6 and the additivity of the lift functor, we obtain the following relations[3] between the multiplicities of indecomposables in V and in $\mathcal{L}V$:

$$\begin{cases} d_{u,v} & = d^*_{u,v} \\ \sum_{\lambda,w} w \cdot d_{\lambda;w} & = d^*_\infty \end{cases} \tag{2.14}$$

where $[u,v] \subset \mathbb{Z}$ and $(\lambda, w) \in (\mathbb{F} - \{0\}) \times \mathbb{N}_{>0}$. We use the convention $d_I = 0$ when the indecomposable I is not in $\operatorname{Ind}(V)$, and similarly $d^*_I = 0$ when I is not in $\operatorname{Ind}(\mathcal{L}V)$. Consider a successive quotient $S^j := V^j/V^{j-1}$ of the HN filtration of V. By Propositions 2.2.5 and 2.7.3, the quotient S^j can be written as the direct sum of the indecomposable summands of V with the same slopes as S^j. By combining this result with (2.14) and Proposition 2.7.1, we obtain that when $\mu_\epsilon(S^j)$ is nonzero, it is of the form $1/k$ with $k \in \mathbb{Z}_{\neq 0}$. In that case,

---

[3] These formulae were established by Burghelea and Haller[28, Lemma 2.1(a)] in the case of alternating type $\widetilde{\mathbb{A}}_n$ quivers.

the dimension vector of S^j can be decomposed as a sum over indecomposables $\mathbf{N}[u,v]$ with $p_{u,v} = 1 - \mathrm{sign}(k)$ and $v - u + 1 = |k|$:

$$\underline{\dim}_{S^j} = \sum_{\mathbf{N}[u,v]} d^*_{u,v} \cdot \underline{\dim}_{\mathbf{N}[u,v]} .$$

When $\mu_\epsilon(S^j) = 0$, there is an extra term induced by the summands of the form $\mathbf{J}[\lambda; w]$:

$$\underline{\dim}_{S^j} = d^*_\infty \cdot \underline{\dim}_{\mathbf{J}[1;1]} + \sum_{\mathbf{N}[u,v]} d^*_{u,v} \cdot \underline{\dim}_{\mathbf{N}[u,v]},$$

and the sum is over indecomposables $\mathbf{N}[u,v]$ such that $p_{u,v} = 1$. This last formula simplifies further since $\underline{\dim}_{\mathbf{J}[1;1]}$ equals one at every vertex of Q. □

Let $\mathbf{T}[V;\epsilon] := (\underline{\dim}_{S^j})_j$ be the dimension vectors of the successive quotients $S^j = V^j/V^{j-1}$ in the HN filtration of V along ϵ. The assignment $V \mapsto \mathbf{T}[V;\epsilon]$ is a discrete invariant under isomorphisms in $\mathbf{Rep}(Q)$. Theorem 2.7.4 implies the following two results respectively about the discriminating power of $\mathbf{T}[V;\epsilon]$ and about its computability:

Corollary 2.7.5. *For all $[u,v] \subset \mathbb{Z}$ satisfying $p_{u,v} \neq 1$, the multiplicity of $\mathbf{N}[u,v]$ in the decomposition of V into indecomposables can be retrieved from $\mathbf{T}[V;\epsilon]$.*

Proof. Consider an interval $[u,v] \subset \mathbb{Z}$ such that $p_{u,v} \neq 1$, and let $\mathcal{A}_{u,v}$ be the set of indecomposables with the same Euler slope as $\mathbf{N}[u,v]$ that occur in the decomposition of V into indecomposable summands. By Proposition 2.7.1, the slope of $\mathbf{N}[u,v]$ is nonzero and the intervals $[u',v'] \subset \mathbb{Z}$ satisfying $\mathbf{N}[u',v'] \in \mathcal{A}_{u,v}$ have the same value of p and the same length as $[u,v]$.

We claim that the membership of $\mathbf{N}[u,v]$ in $\mathcal{A}_{u,v}$ and its multiplicity in the decomposition of V can be obtained via the relation

$$\dim S^j_{u_*} - \dim S^j_{(u-1)_*} = \begin{cases} d_{u,v} & \text{if } \mathbf{N}[u,v] \in \mathcal{A}_{u,v} \\ 0 & \text{otherwise} \end{cases}$$

where a_* denotes the residue modulo n of an integer a. Indeed, by equation (2.12), the difference $\dim S^j_{u_*} - \dim S^j_{(u-1)_*}$ can be decomposed as

$$\sum_{\mathbf{N}[u',v'] \in \mathcal{A}_{u,v}} d_{u',v'} \cdot \big[\#([u',v'] \cap (n\mathbb{Z} + u)) - \#([u',v'] \cap (n\mathbb{Z} + u - 1))\big].$$

The summands in the above sum are null unless either $u' = u \bmod n$ or $v' + 1 = u \bmod n$. The latter case does not happen as $p_{u',v'} = p_{u,v}$ forces $e_{u \bmod n}$ and $e_{(v'+1) \bmod n}$ to have different orientations. The former case, $u' = u \bmod n$, only happens when $[u',v']$ is a translation by $n\mathbb{Z}$ of $[u,v]$ since $[u',v']$ and $[u,v]$ are required to have the same length. □

We will represent the invariant $\mathbf{T}[V;\epsilon] = (\underline{\dim}\, S^j)_j$ by a collection of lists indexed by the vertices $x \in Q_0$:

$$\mathbf{T}[V;\epsilon]_{x,\bullet} := \left\{ (j, \dim S^j_x) \mid 1 \leqslant j \leqslant \ell,\ \text{ and } S^j_x \neq 0 \right\}.$$

Each of these n lists $\mathbf{T}[V;\epsilon]_{x,\bullet}$ has size at most $\dim V_x$ and are sorted by their first coordinate. One can for instance retrieve the dimension of S^j_x in time $\mathcal{O}(\log(\dim V_x))$ by searching for a pair $(j,d) \in \mathbf{T}[V;\epsilon]_{x,\bullet}$. Let V and W be two representations of Q such that each of their spaces V_x and W_x has dimension at most Δ. Then deciding whether $\mathbf{T}[V;\epsilon] = \mathbf{T}[W;\epsilon]$ from the lists $\mathbf{T}[V;\epsilon]_{x,\bullet}$ and $\mathbf{T}[W;\epsilon]_{x,\bullet}$ only takes time $\mathcal{O}(n\Delta)$.

Corollary 2.7.6. *The invariant* $\mathbf{T}[V;\epsilon]$ *can be computed from* d^* *in time* $\mathcal{O}(n\Delta\log\Delta)$ *where* $\Delta := \max_{x\in Q_0} \dim V_x$.

Proof. Using its periodicity, we represent the multiplicity function d^* as the list of all the pairs $([u,v], d^*_{u,v})$ such that $d^*_{u,v} > 0$ and $u \in \{-\infty\} \cup \{0,1,\ldots,n-1\}$. By equation (2.14), each such pair $([u,v], d^*_{u,v})$ contributes $d^*_{u,v} \times (v-u+1)$ to $\sum_{x\in Q_0} \dim V_x$; unless $u = -\infty$, in which case the contribution is $d^*_\infty \times n$. As a consequence, d^* has at most $n\Delta$ elements. Moreover, there are $\mathcal{O}(\sqrt{n\Delta})$ different lengths of intervals in $\mathbf{Bar}(\mathcal{L}V)$ and those lengths take values in $[1, n\Delta] \cup \{\infty\}$. By Theorem 2.7.4, there are $\ell = \mathcal{O}(\sqrt{n\Delta})$ steps in the HN filtration of V. One can thus associate to each element in d^*, the index j of the quotient S^j to which it contributes in total time $\mathcal{O}(n\Delta)$.

We initialise the lists $\mathbf{T}[V;\epsilon]_{x,\bullet}$ as empty and we iterate over the $\mathcal{O}(n\Delta)$ elements of d^*. For each pair $([u,v], d^*_{u,v})$, we update $\mathbf{T}[V;\epsilon]$ by the contribution of $d^*_{u,v}$ in equations (2.12) and (2.13). The lists $\mathbf{T}[V;\epsilon]_{\bullet,x}$ that need to be updated can be determined in constant time from $[u,v]$. Hence, we perform at most $n\Delta$ insertions in total and the time complexity is dominated by the sorting time $\mathcal{O}(n\Delta\log\Delta)$ of the lists $\mathbf{T}[V;\epsilon]_{x,\bullet}$. ☐

From Remark 2.6.4 we know that d^* can be deduced from the decomposition (into indecomposables) of a zigzag persistence module of length $\mathcal{O}(n\Delta)$ with all spaces of dimension at most Δ. We assume that V is given as a sequence of matrices indexed by edges of Q and that field operations in $\mathbb{F}$ require constant time. Then, since the algorithm [30] involves performing one Gaussian elimination per map of the zigzag persistence module, it gives a worst-case complexity for the computation of $\mathbf{T}[V;\epsilon]$ of $\mathcal{O}(n\Delta^4)$ where $\Delta := \max_{x\in Q_0} \dim V_x$.

Remark 2.7.7. Consider the special case where we are given as input a circular zigzag diagram of simplicial complexes and inclusion maps, e.g.,

$$\begin{tikzcd}
 & & K_{n-1} \arrow[dll, hook] & & \\
K_0 \arrow[r, hook] & K_1 \arrow[r, hook] & K_2 & \dots \arrow[l, hook'] \arrow[r, hook] & K_{n-2} \arrow[ull, hook']
\end{tikzcd}$$

where each arrow denotes an inclusion of simplicial complexes which only differ by one simplex. We wish to compute $\mathbf{T}[H_i(K_\bullet);\epsilon]$ where $H_i(K_\bullet)$ is obtained by taking the homology (with $\mathbb{F}$ coeffients) in dimension i of the given diagram. We denote $m \in \mathbb{N}$, the number of simplices in the largest simplicial complex among $K_0, K_1, \dots, K_{n-1}$. Since $\dim H_i(K_x) \leqslant m$ for all $x \in Q_0$, the dimension function d^* can be computed by using the algorithm [**29**, Section 4] directly on a zigzag of at most mn simplicial complexes with at most m simplices each. As a consequence, the worst-case complexity for computing d^* and $\mathbf{T}[V;\epsilon]$ is $\mathcal{O}(nm^3)$. Such a framework arises for instance when considering fibres of certain circle-valued maps on a finite simplicial complex [**27**].

Finally, we note that although the results obtained above require $\mathbb{F}$ to be algebraically closed, one can always fix bases for the vector spaces of a given $V \in \mathbf{Rep}(Q)$ and consider a new representation V^+ with the same matrix description but now over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. The isomorphism class of V^+ only depends on the isomorphism class of V, and as such, $\mathbf{T}[V^+;\epsilon]$ still constitutes an (incomplete) invariant of V. By Theorem 2.7.4 above, this invariant can be related to the barcode of $\mathcal{L}V^+$ in the category of $\mathbf{U}Q$-representations valued in vector spaces over $\overline{\mathbb{F}}$.

Chapter 3

Harder-Narasimhan filtrations of persistence modules: discriminating power

3.1 Introduction

More than sixty years ago, the notion of *semistability* was introduced by Mumford [96, 97] to construct well-behaved quotient spaces for the actions of reductive groups on algebraic varieties. Subsequently, Harder and Narasimhan described a stratification of the moduli space of finite rank vector bundles over a complex curve for the purpose of computing its cohomology groups [63]. The top stratum consists of semistable bundles, and every bundle lying in a lower stratum admits a canonical filtration of finite length whose associated graded components are semistable with strictly decreasing slopes (i.e., ratios of degree to rank). This *Harder-Narasimhan filtration* continues to play a vital role in moduli problems involving vector bundles and coherent sheaves [7, 66]; its existence and uniqueness have been established more generally for objects of certain abelian categories by Rudakov [105] to triangulated categories by Bridgeland [23] and to modular lattices by Haiden et al. [61]. The work of King [78] and Reineke [103] has extended the Harder-Narasimhan formalism to categories of quiver representations.

Our efforts here stem from the desire to use Harder-Narasimhan theory to build isomorphism invariants for multiparameter persistence modules. In general, such modules are representations of wild type quivers, so there cannot be a complete[1] discrete invariant. Nevertheless, the quest for discriminative invariants has been a central theme within topological data analysis. The earliest work in this direction was by Carlsson and Zomorodian [31], who proposed the *rank invariant*. Subsequent efforts to study multiparameter persistence modules have involved a plethora of tools sourced from diverse subject areas — these include sheaf theory [73, 90], commutative and homological algebra [82, 64, 93, 14], lattice theory [57, 91, 21] and beyond [86].

---

[1] An invariant is *complete* if two persistence modules are isomorphic whenever their invariants are equal.

Outline and summary of results

Fix a finite quiver Q and let $\mathbf{Rep}(Q)$ denote the category of finite-dimensional representations of Q valued in vector spaces. By a *central charge* on Q, we mean any real-valued function α defined on the set Q_0 of vertices[2]. Consider a nontrivial representation $V \in \mathbf{Rep}(Q)$, which assigns vector spaces V_x to vertices $x \in Q_0$. The α-slope of V is the ratio

$$\mu_\alpha(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x},$$

and V is said to be α-semistable whenever the inequality $\mu_\alpha(V) \geqslant \mu_\alpha(V')$ holds for every nontrivial subrepresentation $V' \subset V$.

Once we fix a central charge α on Q, every nonzero representation V admits a unique *Harder-Narasimhan* filtration [103, 62] of finite length

$$0 = \mathbf{HN}^0_\alpha(V) \subsetneq \mathbf{HN}^1_\alpha(V) \subsetneq \cdots \subsetneq \mathbf{HN}^{n-1}_\alpha(V) \subsetneq \mathbf{HN}^n_\alpha(V) = V$$

whose successive quotients $S^i := \mathbf{HN}^i_\alpha(V)/\mathbf{HN}^{i-1}_\alpha(V)$ are α-semistable and satisfy $\mu_\alpha(S^i) < \mu_\alpha(S^{i-1})$ for all i. The Harder-Narasimhan filtration thus provides a canonical method for understanding arbitrary quiver representations through the α-semistable ones. The **HN type** of V along α is the n-tuple of functions

$$\mathbf{T}[V;\alpha] = \left(\underline{\dim}_{S^1}, \underline{\dim}_{S^2} \ldots, \underline{\dim}_{S^n}\right),$$

where $\underline{\dim}_{S^i} : Q_0 \to \mathbb{N}$ assigns the natural number $\dim S^i_x$ to each vertex $x \in Q_0$.

An important role in our paper is played by the *spanning* subrepresentation of V at a vertex x — this is defined as the smallest subrepresentation $\langle V_x \rangle \subset V$ containing V_x. The function $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ that sends each (x, y) to the dimension of $\langle V_x \rangle_y$ vastly generalises the rank invariant of Carlsson and Zomorodian. Consider, for each vertex x, the central charge $\delta_x : Q_0 \to \mathbb{R}$ which maps x to 1 and all other vertices to 0. The skyscraper invariant δ_V is defined on $\mathbf{Rep}(Q)$ as the collection of HN types $\mathbf{T}[V;\delta_x]$ indexed by the vertices of Q. Our first main result is presented in Section 3.3, and may be stated as follows.

Theorem (C). *The skyscraper invariant is finer than the rank invariant on* $\mathbf{Rep}(Q)$ *for any finite quiver* Q.

The full statement may be found in Theorem 3.3.5, where we provide a precise formula for recovering $\rho_V(x, y)$ in terms of $\mathbf{T}[V;\delta_x]$. We also prove in Proposition 3.3.9 that the skyscraper invariant is incomparable to the generalised rank invariant from [75].

We then specialise in Section 3.4 to the setting of zigzag persistence modules — these are finite-dimensional representations of type $\mathbb{A}$ quivers of arbitrary (but finite) length $\ell \geqslant 0$:

$$x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} \cdots \overset{e_{\ell-1}}{\text{———}} x_{\ell-1} \overset{e_\ell}{\text{———}} x_\ell.$$

---

[2] This is the imaginary part of an abelian group homomorphism $K(\mathbf{Rep}(Q)) \to \mathbb{C}$; see Section 3.2.2.

Here each edge e_i points either forward $x_{i-1} \to x_i$ or backward $x_{i-1} \leftarrow x_i$, and ordinary persistence modules correspond to the equioriented case where all of the e_i point forward. It is well known from Gabriel's theorem [55] that any representation V of such a quiver decomposes uniquely as a direct sum of *interval representations*, which are supported on sub-intervals $[a, b] \subset [0, \ell]$. The intervals which appear with nonzero multiplicity comprise the *barcode* of V — see [30]. We exploit knowledge of these indecomposables as well as some recent work of Kinser [80] to prove the following result, which is Theorem 3.4.3 below.

Theorem (D). *There is a classification of all complete central charges on the category of zigzag persistence modules; in the special case of ordinary persistence modules, a central charge* $\alpha : Q_0 \to \mathbb{R}$ *is complete iff* $\alpha(x_i) > \alpha(x_{i+1})$ *holds for all* i*.*

In Section 3.5, we study the natural d-dimensional analogues of ordinary persistence modules [85], for $d > 1$. These are certain representations of *grid quivers*, whose vertices are parameterised by integer points in the product $[0, \ell_1] \times [0, \ell_2] \times \cdots \times [0, \ell_d]$, with each $\ell_i \geqslant 1$. The $d = 2$ case is illustrated below:

$$\begin{tikzcd}
x_{0,\ell_2} \arrow[r] & x_{1,\ell_2} \arrow[r] & \cdots \arrow[r] & x_{\ell_1,\ell_2} \\
\vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] \\
x_{0,1} \arrow[u] \arrow[r] & x_{1,1} \arrow[u] \arrow[r] & \cdots \arrow[r] & x_{\ell_1,1} \arrow[u] \\
x_{0,0} \arrow[u] \arrow[r] & x_{1,0} \arrow[u] \arrow[r] & \cdots \arrow[r] & x_{\ell_1,0} \arrow[u]
\end{tikzcd}$$

The representations of interest are those for which every rectangle in sight commutes as a diagram of vector spaces. Unlike the $d = 1$ case, these quivers are of wild representation type; as mentioned above, much effort has been invested towards finding good discrete invariants for multiparameter persistence modules. A far more tractable subcategory is spanned by the *rectangle-decomposable* representations, which admit direct sum decompositions into the obvious d-dimensional analogue of interval modules. Here we establish the following result, consisting of Corollary 3.5.4 and Theorem 3.5.10.

Theorem (E). *The skyscraper invariant is strictly finer than the rank invariant on the category of multiparameter persistence modules. Moreover, for the subcategory of rectangle decomposable modules, there is a classification of complete central charges which lie outside of a hyperplane arrangement in the space of maps* $Q_0 \to \mathbb{R}$*.*

We further establish in Section 3.5.3 that no single central charge is complete on the larger category of *interval-decomposable* persistence modules [4]. Nevertheless, we show (in Proposition 3.5.13 below) that a finite set of central charges is complete on the subcategory of persistence modules that decompose into intervals with at most two sources.

In Section 3.6, we focus on the special case of 2-parameter persistence modules arising from representations of *ladder quivers* of length $\ell \geqslant 1$:

$$\begin{tikzcd}
x_0^+ \arrow[r] \arrow[d] & x_1^+ \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{\ell-1}^+ \arrow[r] \arrow[d] & x_\ell^+ \arrow[d] \\
x_0^- \arrow[r] & x_1^- \arrow[r] & \cdots \arrow[r] & x_{\ell-1}^- \arrow[r] & x_\ell^-
\end{tikzcd}$$

Again, every rectangle is required to commute. These *ladder persistence modules* are important because they arise from morphisms of (ordinary) persistence modules. Ladder quivers are known [47, 26] to be of finite representation type only for $\ell \leqslant 3$. Here we obtain the following results (Proposition 3.6.1 and Theorem 3.6.4).

Theorem (F). *There is no complete central charge on the category of interval-decomposable ladder persistence modules of length $\ell \geqslant 4$; however, for all ℓ there exists a finite set $A = A(\ell)$ of central charges which is complete on this category.*

The set $A(\ell)$ is explicitly described in Definition 3.6.8, and its cardinality grows cubically with ℓ; every constituent element of this set is expressible as an $\mathbb{R}$-linear combination of at most two skyscraper central charges.

Stability and computability

Besides high discriminative power, the most important properties that an invariant of multiparameter persistence modules should possess are computability and metric stability (with respect to the interleaving distance, see [84]). The recent work of Cheng [36] introduces a deterministic algorithm to compute Harder-Narasimhan filtrations of quiver representations along integer-valued central charges. For persistence modules, this algorithm runs in polynomial time (with respect to the number of vertices, dimensions of vector spaces and values of the central charge). We expect that various optimisations and simplifications will become available when one focuses exclusively on the skyscraper invariant, and we hope to explore these computational issues thoroughly in future work. Turning to metric stability, the desired result for skyscraper invariants is established in Chapter 4.

3.2 HN types of quiver representations

In this Section, we establish notation and recall pertinent aspects of quiver representations [81, 107], their Harder-Narasimhan filtrations, and the associated Harder-Narasimhan types [103, 62].

3.2.1 Quiver representations

A quiver Q consists of the following data: a set Q_0 whose elements are called vertices, a set Q_1 whose elements are called edges, and a pair of functions $s, t : Q_1 \to Q_0$ called the source and target map respectively. One often depicts each edge $e \in Q_1$ as an arrow $s(e) \longrightarrow t(e)$. A path in the quiver $Q = (s, t : Q_1 \to Q_0)$ is any finite sequence of edges $p = (e_1, \ldots, e_k)$ satisfying $t(e_i) = s(e_{i+1})$ for all i, and Q is called *acyclic* if it admits no such paths with $s(e_1) = t(e_k)$. In this paper, all quivers are assumed to have only finitely many vertices and edges, and moreover, we will primarily be interested in acyclic quivers. We call Q *connected* if for every pair of distinct vertices x and y, there is a finite sequence of vertices $(x = x_0, x_1, \ldots, x_k = y)$ and edges e_i satisfying $\{s(e_i), t(e_i)\} = \{x_i, x_{i+1}\}$ for all i.

Definition 3.2.1. By a **subquiver** of $Q = (s, t : Q_1 \to Q_0)$ we mean any quiver $Q' = (s', t' : Q'_1 \to Q'_0)$ with $Q'_0 \subset Q_0$ and $Q'_1 \subset Q_1$ such that s' and t' are the restrictions of s and t respectively. We call Q'

- *full* if Q'_1 contains all edges of Q_1 between pairs of vertices of Q'_0, and
- *convex* if every path of Q between vertices of Q' is also a path of Q'. △

We work over a field $\mathbb{F}$ which remains fixed throughout (and hence suppressed from the notation), so that all vector spaces encountered henceforth are defined over $\mathbb{F}$. Let us recall that a **representation** V of Q constitutes assignments of vector spaces V_x to vertices $x \in Q_0$ and linear maps $V_e : V_{s(e)} \to V_{t(e)}$ to edges $e \in Q_1$. All representations considered here are finite-dimensional in the sense that $\dim V_x < \infty$ holds for each vertex $x \in Q_0$; the *dimension vector* of any such V is the function $\underline{\dim}_V : Q_0 \to \mathbb{N}$ given by $x \mapsto \dim V_x$. The set $\mathbf{Rep}(Q)$ of all finite-dimensional representations of Q is readily upgraded to a category as follows. A morphism $\phi : V \to W$ consists of linear maps $\{\phi_x : V_x \to W_x \mid x \in Q_0\}$ which make the following diagram commute for each edge $e \in Q_1$:

$$\begin{tikzcd}
V_{s(e)} \arrow[r, "\phi_{s(e)}"] \arrow[d, "V_e"'] & W_{s(e)} \arrow[d, "W_e"] \\
V_{t(e)} \arrow[r, "\phi_{t(e)}"'] & W_{t(e)}
\end{tikzcd}$$

We call ϕ an (epi, iso, mono)-morphism if each ϕ_x is (sur, bi, in)-jective; V is called a *subrepresentation* of W, denoted $V \subset W$, whenever there exists a monomorphism $\phi : V \to W$. The category $\mathbf{Rep}(Q)$ is known to be abelian, with composition, (co)kernels and (co)products being defined vertex-wise [107, Section 1.3].

Remark 3.2.2. To any quiver Q we may associate a *free category* $\mathcal{Q}$ whose objects are vertices and whose morphisms are paths in Q. With this point of view, a representation $V \in \mathbf{Rep}(Q)$ induces a functor from $\mathcal{Q}$ to the category Vect of finite-dimensional $\mathbb{F}$-vector spaces. Indeed, $\mathbf{Rep}(Q)$ is equivalent via adjunction to the functor category $\mathcal{Q} \to$ Vect (for details, see [89, p. II.7]).

A representation of Q is called **indecomposable** if it does not admit any nontrivial direct sum decompositions in $\mathbf{Rep}(Q)$. It is known [107, Theorem 1.2] that for every finite-dimensional representation V of a finite quiver Q, there exists a unique set $\mathrm{Ind}_Q(V)$ containing isomorphism classes of indecomposables and a unique multiplicity function $d_V : \mathrm{Ind}_Q(V) \to \mathbb{Z}_{>0}$ for which there is an isomorphism

$$V \simeq \bigoplus_I I^{d_V(I)}, \tag{3.1}$$

with I ranging over $\mathrm{Ind}_Q(V)$. The seminal work of Gabriel [55] established that the set of indecomposables in $\mathbf{Rep}(Q)$ with a fixed dimension vector is finite if and only if the undirected graph associated to Q is a finite union of simply laced Dynkin diagrams.

Definition 3.2.3. Given a subquiver Q' of Q, the **identity representation** $\mathbf{I}[Q'] \in \mathbf{Rep}(Q)$ of Q' is defined for $x \in Q_0$ and $e \in Q_1$ by

$$\mathbf{I}[Q']_x = \begin{cases} \mathbb{F} & \text{if } x \in Q'_0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{I}[Q']_e = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } e \in Q'_1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

When Q' is a full subquiver, the representation $\mathbf{I}[Q']$ is uniquely determined by the vertices Q'_0 of Q' and we write $\mathbf{I}[Q'_0]$ rather than $\mathbf{I}[Q']$. If Q' is moreover full and convex, then $\mathbf{I}[Q']$ is called [4, Definition 10] an **interval representation**. △

These identity representations $\mathbf{I}[Q']$ provide examples of indecomposable representations whenever Q' is connected.

3.2.2 Harder-Narasimhan filtrations in abelian categories

The *Grothendieck group* of an abelian category $\mathcal{A}$ is the abelian group $K(\mathcal{A})$ freely generated by the isomorphism classes $[V]$ of objects V in $\mathcal{A}$ modulo a relation of the form $[V] = [U] + [W]$ for each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathcal{A}$. Let $(\mathbb{C}, +)$ denote the abelian group of complex numbers under addition; we recall that a **stability condition**[3] [23] on $\mathcal{A}$ is any group homomorphism

$$Z : K(\mathcal{A}) \to (\mathbb{C}, +)$$

that sends nonzero objects to the open half-plane $\{z \in \mathbb{C} \mid \mathfrak{Re}(z) > 0\}$. The Z-**slope** of an object $V \neq 0$ is the real number

$$\mu_Z(V) := \frac{\mathfrak{Im} Z(V)}{\mathfrak{Re} Z(V)}.$$

We say that V is Z-**semistable** if the inequality $\mu_Z(U) \leqslant \mu_Z(V)$ holds for every nonzero subobject $U \subset V$. If this inequality is strict for all subobjects $U \notin \{0, V\}$, then V is said to be Z-**stable**.

---

[3] Such a homomorphism is sometimes called a *linear stability function* [80] or a *central charge* [23].

The following result is a direct consequence of [23, Proposition 2.4]. We recall that an abelian category is said to satisfy the Noetherian (respectively, Artinian) hypothesis if every sequence $\cdots \subset a_1 \subset a_0$ of subobjects (respectively, every sequence $b_0 \twoheadrightarrow b_1 \twoheadrightarrow \cdots$ of quotient objects) eventually stabilises up to isomorphism.

Theorem 3.2.4. *Let $\mathcal{A}$ be any abelian category which satisfies the Noetherian and Artinian hypotheses. Fix a stability condition Z on $\mathcal{A}$. Every nonzero object V of $\mathcal{A}$ admits a unique filtration $V^\bullet$ of finite length $n \geqslant 1$:*

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^n = V, \tag{3.3}$$

whose successive quotients $S^i := V^i/V^{i-1}$ are Z-semistable and have strictly decreasing slopes:

$$\mu_Z(S^1) > \mu_Z(S^2) > \cdots > \mu_Z(S^n).$$

*This $V^\bullet$ is called the **Harder-Narasimhan** (or **HN**) filtration of V along Z.*

Crucially, the category of finite dimensional representations of a finite quiver satisfies the hypotheses of the above result. It follows from uniqueness that the HN filtration of a Z-semistable object V always has length one, i.e., $0 \subsetneq V$.

3.2.3 HN types of quiver representations

Let $Q = (s, t : Q_1 \to Q_0)$ be a quiver which remains fixed throughout this subsection. The assignment $V \mapsto \underline{\dim}_V$ defines a group homomorphism from the Grothendieck group of $\mathbf{Rep}(Q)$ to the group of functions from Q_0 to $\mathbb{Z}$:

$$\underline{\dim} : K(\mathbf{Rep}(Q)) \longrightarrow \mathbb{Z}^{Q_0}$$

(For acyclic Q, this is an isomorphism [81, Theorem 1.15]). We note that any stability condition Z on $\mathbf{Rep}(Q)$ which factors through $\underline{\dim}$ amounts to a choice of two maps $\alpha : Q_0 \to \mathbb{R}$ and $\beta : Q_0 \to \mathbb{R}_{>0}$. Explicitly, for nonzero $V \in \mathbf{Rep}(Q)$ we have

$$Z(V) = \sum_{x \in Q_0} \big(\beta(x) + \sqrt{-1} \cdot \alpha(x)\big) \cdot \dim V_x,$$

and the corresponding slope is the ratio

$$\mu_Z(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \beta(x) \cdot \dim V_x}.$$

In this paper, we will only work with stability conditions Z which factor through $\underline{\dim}$. Furthermore, as in [103, 62], we further restrict attention to those stability conditions for which β is the constant map sending all vertices to 1. These are called *standard* stability conditions; and any such stability condition Z depends on a single function $\alpha : Q_0 \to \mathbb{R}$

that we will henceforth call the **central charge** of Z. In light of these simplifications, we will denote the slope of any nonzero V by

$$\mu_\alpha(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x}. \tag{3.4}$$

Similarly, the Harder-Narasimhan filtration of any nonzero $V \in \mathbf{Rep}(Q)$ along Z is indicated by $\mathbf{HN}^\bullet_\alpha(V)$. The following *seesaw lemma* is stated in [**103**, Lemma 2.2]; we include a proof here for completeness.

Lemma 3.2.5. *Let $\alpha : Q_0 \to \mathbb{R}$ be a central charge for Q. Given any three nonzero objects which fit into a short exact sequence in $\mathbf{Rep}(Q)$:*

$$0 \to U \to V \to W \to 0,$$

their α-slopes must satisfy one of the following chains of (in)equalities. Either

1. *$\mu_\alpha(U) > \mu_\alpha(V) > \mu_\alpha(W)$, or*
2. *$\mu_\alpha(U) = \mu_\alpha(V) = \mu_\alpha(W)$, or*
3. *$\mu_\alpha(U) < \mu_\alpha(V) < \mu_\alpha(W)$.*

In Case (2), V is α-semistable if and only if both U and W are α-semistable.

Proof. Since $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is a strictly monotone increasing function, it suffices to establish the desired inequalities for the composite $\theta := \arctan \circ \mu_\alpha$ rather than for μ_α. By definition, the (standard) stability condition Z induced by α is a group homomorphism $K(\mathbf{Rep}(Q)) \to (\mathbb{C}, +)$, so we have $Z(U) + Z(W) = Z(V)$. The desired results now follow from examining the parallelogram in $\mathbb{C}$ determined by the origin, $Z(U)$ and $Z(W)$ whose fourth point must be $Z(V)$. The angle $\theta(V)$ lies between the angles $\theta(U)$ and $\theta(W)$, with equality of all three angles occurring only in the degenerate case where $Z(U)$ is an $\mathbb{R}$-multiple of $Z(W)$.

Let us now consider the case (2) where U, V and W share a common α-slope μ. If U is not α-semistable, then it admits a subrepresentation U' with $\mu_\alpha(U') > \mu$; but any such U' is automatically a subrepresentation of V which violates its α-semistability. Similarly, any quotient W' of W with $\mu_\alpha(W') < \mu$ violates the semistability of V. Thus, the semistability of V forces semistability of both U and W. Conversely, assume that U and W are α-semistable and label the maps in the short exact sequence as $\iota : U \to V$ and $\pi : V \to W$. Given any subrepresentation $V' \subset V$, we have a short exact sequence

$$0 \to \iota^{-1}(V') \to V' \to \pi(V') \to 0.$$

In the nontrivial case, $\iota^{-1}(V')$ and $\pi(V')$ are nonzero subrepresentations of U and W respectively, so by semistability both must have α-slopes no larger than μ. By the first part of this Lemma, we therefore obtain $\mu_\alpha(V') \leqslant \mu$, which confirms the α-semistability of V. ☐

Here is an immediate (but important) consequence of the preceding result.

Corollary 3.2.6. *If U and W are α-semistable objects of* $\mathbf{Rep}(Q)$ *with the same α-slope μ, then their direct sum $U \oplus W$ is also α-semistable with slope μ.*

Definition 3.2.7. The **Harder-Narasimhan type**, or (**HN type**) of a representation $V \neq 0$ in $\mathbf{Rep}(Q)$ along a central charge $\alpha : Q_0 \to \mathbb{R}$ is denoted $\mathbf{T}[V;\alpha]$ and defined as follows. Let n be the length of the Harder-Narasimhan filtration $\mathbf{HN}^\bullet_\alpha(V)$, and let $S^i = \mathbf{HN}^i_\alpha(V)/\mathbf{HN}^{i-1}_\alpha(V)$ denote its successive quotients for $1 \leqslant i \leqslant n$. Then,

$$\mathbf{T}[V;\alpha] := \left(\underline{\dim}_{S^1}, \underline{\dim}_{S^2}, \ldots, \underline{\dim}_{S^n} \right). \qquad \triangle$$

In the context of the preceding definition, it is often useful to view $\mathbf{T}[V;\alpha]$ as a function $\mathbb{R} \to \mathbb{N}^{Q_0}$ in the following manner:

$$\mathbf{T}[V;\alpha](\lambda) = \begin{cases} \underline{\dim}_{S^i} & \text{if } \lambda = \mu_\alpha(S^i) \text{ for some } i, \\ (0,0,\ldots,0) & \text{otherwise.} \end{cases} \tag{3.5}$$

This function is well-defined since the successive quotients S^i have strictly decreasing slopes by Theorem 3.2.4. The uniqueness promised by this theorem further guarantees that $\mathbf{T}[\bullet;\alpha]$ is invariant under isomorphisms in $\mathbf{Rep}(Q)$. It is evident from Definition 3.2.7 that this invariant is discrete, since it only produces finite sequences of (non-negative) integer values. Moreover, this invariant is additive under direct sums; a proof of the following folklore result may be found in Proposition 2.2.5.

Proposition 3.2.8. *Let V and W be two representations of a quiver Q, and let $\alpha : Q_0 \to \mathbb{R}$ be a central charge. Adopting the notation of* (3.5)*, we have*

$$\mathbf{T}[V \oplus W;\alpha] = \mathbf{T}[V;\alpha] + \mathbf{T}[W;\alpha]$$

as functions $\mathbb{R} \to \mathbb{N}^{Q_0}$.

3.2.4 Complete central charges

The main theme of this paper is to measure the strength of the HN type as an invariant of certain quiver representations across various choices of central charge. The definition below corresponds to the best-case scenario.

Definition 3.2.9. Let $Q = (s,t : Q_1 \to Q_0)$ be an acyclic quiver and $\mathcal{A}$ any subcategory of $\mathbf{Rep}(Q)$. A collection of central charges A is said to be **complete** on $\mathcal{A}$ if $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$ for all $\alpha \in A$ implies that V and W are isomorphic in $\mathcal{A}$. $\triangle$

If a collection of central charges A is complete on all of $\mathbf{Rep}(Q)$, then we simplify terminology by saying that A is complete for Q. For the simplest quivers, one hopes to find a single complete central charge; we will appeal to the following result frequently in our quest for such central charges.

Lemma 3.2.10. *If $\alpha : Q_0 \to \mathbb{R}$ is a complete central charge for the acyclic quiver Q, then every indecomposable representation in* $\mathbf{Rep}(Q)$ *is α-stable.*

Proof. Assume that α is a complete central charge and consider an indecomposable I in $\mathbf{Rep}(Q)$. If I is not α-semistable, then its HN filtration

$$0 \subsetneq \mathbf{HN}^1_\alpha(I) \subsetneq \mathbf{HN}^2_\alpha(I) \subsetneq \cdots \subsetneq \mathbf{HN}^n_\alpha(I) = I$$

has length $n > 1$. Abbreviating $I^i := \mathbf{HN}^i_\alpha(I)$, in particular we have $I^1 \subsetneq I$. Now consider the filtration of $I^1 \oplus (I/I^1)$ given by:

$$0 \subsetneq I^1 \subsetneq I^1 \oplus (I^2/I^1) \subsetneq \cdots \subsetneq I^1 \oplus (I^n/I^1).$$

Since the successive quotients of this filtration are identical to those of $\mathbf{HN}^\bullet_\alpha(I)$, it follows (from uniqueness) that this new filtration is precisely $\mathbf{HN}^\bullet_\alpha(I^1 \oplus (I/I^1))$. Moreover, since the Harder-Narasimhan type depends only on these successive quotients, we have $\mathbf{T}[I;\alpha] = \mathbf{T}[I^1 \oplus (I/I^1);\alpha]$. But since I is indecomposable, it can not be isomorphic to $I^1 \oplus (I/I^1)$ for $I^1 \neq I$, so the completeness of α forces α-semistability of I.

Given this α-semistability, if I is not α-stable, then there exists a nonzero subrepresentation $J \subsetneq I$ with $\mu_\alpha(J) = \mu_\alpha(I)$. Using the exact sequence $0 \to J \to I \to I/J \to 0$ along with Lemma 3.2.5, we know that both J and I/J are α-semistable with slope $\mu_\alpha(I)$. Another appeal to the same Lemma establishes that the direct sum $J \oplus (I/J)$ is also α-semistable with slope $\mu_\alpha(I)$. For dimension reasons, $\mathbf{T}[I;\alpha]$ equals $\mathbf{T}[J \oplus (I/J);\alpha]$. But once again, since I is indecomposable, it is not isomorphic to $J \oplus (I/J)$ for $J \neq I$. Thus, if I is not α-stable, then α is not a complete central charge for Q. ☐

Let $\alpha\colon Q_0 \to \mathbb{R}$ be a central charge and let $\mathcal{A}$ be a subcategory of $\mathbf{Rep}(Q)$. We stress that for a nonzero $V \in \mathcal{A}$, the HN filtration $\mathbf{HN}^\bullet_\alpha(V)$ is computed in $\mathbf{Rep}(Q)$ — in particular, we do not require $\mathcal{A}$ to be abelian. The next result describes sufficient conditions under which one can determine $\mathbf{HN}^\bullet_\alpha(V)$ by only considering objects in $\mathcal{A}$.

Lemma 3.2.11. *Let $\mathcal{I}$ be a family of indecomposable representations of Q and let $\mathcal{A} = \mathcal{A}(\mathcal{I})$ be the full subcategory of $\mathbf{Rep}(Q)$ spanned by those representations which decompose as a direct sum of elements in $\mathcal{I}$. Further assume that quotients and subrepresentations of indecomposables in $\mathcal{I}$ all lie in $\mathcal{A}$. The following hold:*

1. *if α is complete on $\mathcal{A}$, then each indecomposable in $\mathcal{I}$ is α-stable; and*
2. *for every nonzero $V \in \mathcal{A}$, the representations and quotients in $\mathbf{HN}^\bullet_\alpha(V)$ are objects in $\mathcal{A}$.*

Proof. Statement (1) follows from the fact that the proof of Lemma 3.2.10 only compares pairs of HN types of the form $(\mathbf{T}[I;\alpha], \mathbf{T}[J \bigoplus I/J;\alpha])$ with $I \in \mathcal{I}$ and $0 \subsetneq J \subsetneq I$. Indeed, the assumptions on $\mathcal{A}$ ensure that $J \bigoplus I/J$ lies in $\mathcal{A}$ and that it is not isomorphic to I in $\mathcal{A}$.

We prove Statement (2) by induction on the length ℓ of the HN filtration $V^\bullet$ of $V \in \mathcal{A}$ along α. When $\ell = 1$, the result is trivial as $V^\bullet$ is the one-step filtration $0 \subsetneq V$. Assume that $\ell > 1$ and that the result holds for all lengths up to $\ell - 1$. Let $V \simeq \bigoplus_{I \in \mathcal{I}} I^{d_V(I)}$ be the decomposition of V into indecomposables. By Proposition 2.2.5, the filtrations $\mathbf{HN}^\bullet_\alpha(I)$ have length at most ℓ and they determine $V^\bullet$. We can thus restrict ourselves to the case where $V \in \mathcal{I}$. By assumption, the subrepresentations $V^1, \dots, V^\ell$ all lie in $\mathcal{A}$, and similarly, the quotient V/V^1 is an object of $\mathcal{A}$. Finally, by the inductive hypothesis applied to V/V^1, the representations V^i and V^i/V^{i-1} are objects of $\mathcal{A}$ for all i, as desired. ☐

Remark 3.2.12. If we assume that $\mathcal{A}$ is an abelian exact subcategory of $\mathbf{Rep}(Q)$ in the setting of Lemma 3.2.11, then Theorem 3.2.4 guarantees that HN filtrations are well-defined in $\mathcal{A}$. If we further assume that the group homomorphism $F\colon K(\mathcal{A}) \to K(\mathbf{Rep}(Q)) \simeq \mathbb{Z}^{Q_0}$ naturally induced by the inclusion $\mathcal{A} \subset \mathbf{Rep}(Q)$ is injective, then α can be pulled back to $K(\mathcal{A})$. More precisely, the standard stability condition $Z_\alpha : \mathbb{Z}^{Q_0} \to \mathbb{C}$ on $\mathbf{Rep}(Q)$ associated to α induces a stability condition $F^*\alpha$ on $\mathcal{A}$ given by the composite $Z_\alpha \circ F$:

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r, hook, "F"] \arrow[dr, dashed, "F^*\alpha"'] & \mathbb{Z}^{Q_0} \arrow[d, "Z_\alpha"] & . \\
 & \mathbb{C} &
\end{tikzcd}$$

By the naturality of F, for every nonzero $V \in \mathcal{A}$, we have $\mu_{F^*\alpha}(V) = \mu_\alpha(V)$, and the α-(semi)stability of V in $\mathbf{Rep}(Q)$ implies its $F^*\alpha$-(semi)stability in $\mathcal{A}$. By Lemma 3.2.11(2) and by uniqueness of HN filtrations, we have

$$\mathbf{HN}^\bullet_{F^*\alpha}(V) = \mathbf{HN}^\bullet_\alpha(V).$$

The above remark applies whenever $\mathcal{A}$ is the full subcategory of representations of Q subject to some commutativity relations. Indeed, in this case, $\mathcal{A}$ is stable under subrepresentations and quotients; moreover, $F\colon K(\mathcal{A}) \to \mathbb{Z}^{Q_0}$ is an isomorphism — see [**6**, p. III.3.5]. We will therefore treat the representations of quivers with and without relations on an equal footing throughout the rest of this paper.

3.3 The skyscraper and rank invariants

Here we study the HN types arising from central charges defined by delta functions on the vertices of a given quiver Q; we call this the *skyscraper invariant*. We also extend the rank invariant of [**31**] to the setting of arbitrary quiver representations, and show that the skyscraper invariant is finer than the rank invariant. Moreover, we prove that the skyscraper invariant is neither finer nor coarser than the generalised rank invariant of [**75**].

3.3.1 Skyscraper invariant

Let $Q = (s, t : Q_1 \to Q_0)$ be an arbitrary (i.e., finite, but not necessarily acyclic) quiver. In the absence of specific knowledge regarding the structure of Q or its indecomposable representations, it is not immediately obvious how one might identify interesting classes of central charges for Q à la Theorem 3.4.2. Among the simplest nontrivial central charges which may be defined on any quiver are the ones supported on a single vertex.

Definition 3.3.1. The **skyscraper** central charge at a vertex $x \in Q_0$ is the map $\delta_x : Q_0 \to \mathbb{R}$ given by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

And the **skyscraper invariant** $\delta_\bullet$ on $\mathbf{Rep}(Q)$ assigns to each representation V the collection of HN types $\delta_V = \{\mathbf{T}[V; \delta_x] \mid x \in Q_0\}$ along skyscraper central charges at all of the vertices. △

Definition 3.3.2. Let $S \subset Q_0$ be a nonempty subset of vertices. The **spanning subrepresentation** of V at S, denoted $\langle V_S \rangle$, is the intersection of all subrepresentations $W \subset V$ for which W_x is isomorphic to V_x whenever x lies in S. △

We will simply write $\langle V_x \rangle$ when S is the singleton $\{x\}$. Spanning representations at singletons determine the HN filtrations along skyscraper central charges.

Proposition 3.3.3. *Given a vertex x of Q, let $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ be the HN filtration of V along δ_x. If j is the smallest index for which V^j_x equals V_x, then:*

1. *either $j = n$ or $j = n - 1$, and*
2. *for every $1 \leqslant k \leqslant j$, we have $V^k = \langle V^k_x \rangle$.*

Proof. We note that the δ_x-slope of a nonzero representation W of Q is given by

$$\mu_{\delta_x}(W) = \frac{\dim W_x}{\sum_{y \in Q_0} \dim W_y}, \tag{3.6}$$

which is evidently non-negative. Assuming that $V^j_x = V_x$ holds for some j in $\{0, \dots, n\}$, we have $V^k_x = V_x$ for all $k \geqslant j$, whence the successive quotients $S^k := V^k / V^{k-1}$ satisfy $S^k_x = 0$ for all $k > j$. By (3.6), we obtain equalities of slopes:

$$0 = \mu_{\delta_x}(S^{j+1}) = \mu_{\delta_x}(S^{j+2}) = \cdots = \mu_{\delta_x}(S^{n-1}) = \mu_{\delta_x}(S^n).$$

Since these slopes are required to strictly decrease in the HN filtration, there are only two possible options. Either $j = n - 1$, in which case only the last slope is 0; or $j = n$, in which case all slopes are non-zero. Thus, we have established assertion (1).

We now prove assertion (2) by induction on $k \in \{1, \dots, j\}$.

Base case: Since V^1 is δ_x-semistable and $\langle V^1_x \rangle$ is its subrepresentation, we must have $\mu_{\delta_x}(V^1) \geqslant \mu_{\delta_x}(\langle V^1_x \rangle)$, whence

$$\frac{\dim V^1_x}{\sum_{y \in Q_0} \dim V^1_y} \geqslant \frac{\dim V^1_x}{\sum_{y \in Q_0} \dim \langle V^1_x \rangle_y}.$$

If $\dim V^1_x = 0$ then there is nothing to check, so we assume that this dimension is nonzero. Since each $\langle V^1_x \rangle_y$ is a subspace of the corresponding V^1_y for $y \in Q_0$, we obtain $V^1 = \langle V^1_x \rangle$.

Inductive step: Assume that $V^k = \langle V^k_x \rangle$ holds for some $k < j$. Now S^{k+1} is δ_x-semistable by definition of the HN filtration; and by the argument which established assertion (1), it has a strictly positive δ_x-slope. Thus, we have $\dim S^{k+1}_x > 0$, and applying the base case (to S^{k+1} instead of V^1) yields $S^{k+1} = \langle S_x^{k+1} \rangle$. Given any subrepresentation $W \subset V^{k+1}$ such that $W_x = V_x^{k+1}$, we show that $W = V^{k+1}$. By the inductive hypothesis, since $(W \cap V^k)_x = V_x^k$, we have $V^k = W \cap V^k \subset W$. Similarly, since $S^{k+1} = \langle S_x^{k+1} \rangle$ and $(W/V^k)_x = S_x^k$, we have $W/V^k = S^{k+1}$. By the 5-lemma [114, Ex 1.3.3] applied to the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & V^k \cap W \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, hook] & W/(V^k \cap W) \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & V^k \arrow[r] & V^{k+1} \arrow[r] & S^{k+1} \arrow[r] & 0
\end{tikzcd}$$

we obtain $W = V^{k+1}$ and thus $V^{k+1} = \langle V_x^{k+1} \rangle$, as desired. □

3.3.2 Rank invariant

The following notion constitutes a substantial generalisation of the *rank invariant*, which was introduced in [31] for multi-parameter persistence modules.

Definition 3.3.4. The **rank invariant** of a representation $V \in \mathbf{Rep}(Q)$ is the function $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ given by:

$$\rho_V(x, y) := \dim \langle V_x \rangle_y. \qquad \triangle$$

It follows from the above definition that $\rho_\bullet$ is a discrete isomorphism invariant for $\mathbf{Rep}(Q)$. The following result gives a formula for the rank invariant in terms of the skyscraper invariant $\delta_\bullet$ — in fact, we show that for any vertex x, the rank $\rho_V(x, y)$ can be recovered from the single Harder-Narasimhan type $\mathbf{T}[V; \delta_x]$.

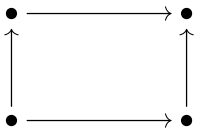
Theorem 3.3.5. *Let Q be a finite quiver. The skyscraper invariant is strictly more discriminative than the rank invariant on $\mathbf{Rep}(Q)$ in the following sense.*

1. *Consider $V \in \mathbf{Rep}(Q)$ and any vertex x in Q_0. If $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ is the HN filtration of V along δ_x, then for any vertex y of Q we have*

$$\rho_V(x, y) = \sum_{k=1}^{j} \dim S_y^k,$$

where $S^k := V^k/V^{k-1}$ and j is the smallest index for which V_x^j equals V_x.

2. *There exist two representations W and W' of the quiver*



for which $\rho_W = \rho_{W'}$ whereas $\delta_W \neq \delta_{W'}$.

Proof. By Proposition 3.3.3, we have $j \in \{n, n-1\}$ and $V^j = \langle V_x \rangle$. So by Definition 3.3.4, the value of $\rho_V(x, y)$ equals $\dim V^j_y$. Since the $S^\bullet$ are successive quotients of the HN filtration $V^\bullet$, we have

$$\dim V^j_y = \sum_{k=1}^{j} \dim S^k_y,$$

which establishes the first assertion. Turning now to the second assertion, let us consider the representations W (left) and W' (right) depicted below:

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "{[0\ 1]}"] \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "{[1\ 0]}"] \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[u]
\end{tikzcd}$$

Both evidently have the same rank invariant. Let x be the $\leqslant$-minimal vertex of this quiver (i.e., the vertex on the bottom-left). By examining (the slopes of) sub-representations, one readily checks that W is δ_x-semistable, so that its HN filtration is just the trivial one $0 \subsetneq W$. On the other hand, W' has a two-step HN filtration $0 \subset U \subset W'$, where U is given by

$$\begin{tikzcd}
0 \arrow[r] & 0 \\
\mathbb{F} \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Since W and W' have different HN types along δ_x, the skyscraper invariants δ_W and $\delta_{W'}$ are distinct as claimed above. ☐

3.3.3 Generalised rank invariant

The authors of [75] introduced a generalised version of the rank invariant for multiparameter persistence modules by studying their restrictions to certain subposets. A similar approach was considered in [79] for representations of rooted tree quivers. Here we extend both constructions to the setting of general quiver representations. To this end, let Q be an arbitrary finite quiver, and denote by $\mathrm{SubQuiv}(Q)$ the set of its subquivers. Given a representation $V \in \mathbf{Rep}(Q)$ and a subquiver $Q' \in \mathrm{SubQuiv}(Q)$, the *restriction* $V_{|Q'}$ of V to Q' is the representation defined by $(V_{|Q'})_x = V_x$ for $x \in Q'_0$ and $(V_{|Q'})_e = V_e$ for $e \in Q'_1$.

As mentioned in Remark 3.2.2, a representation $V \in \mathbf{Rep}(Q)$ may be viewed as a functor $\mathcal{Q} \to \mathrm{Vect}$, where $\mathcal{Q}$ is the category induced by Q. One may therefore examine its categorical limit (written $\varprojlim V$) and colimit (written $\varinjlim V'$) — see [89, Ch III and Ch V]. Explicitly, the limit may be viewed as a subspace of the product $\prod_{x \in Q_0} V_x$:

$$\varprojlim V \simeq \left\{ v \in \prod_{x \in Q_0} V_x \;\middle|\; v_{t(e)} = V_e(v_{s(e)}) \text{ for all } e \in Q_1 \right\}. \tag{3.7}$$

For each vertex $x \in Q$, the desired map $\pi_x : \varprojlim V \to V_x$ is the restriction of the projection $\prod_{y \in Q_0} V_y \twoheadrightarrow V_x$. We note that the limit of V is the *space of global sections* of V and may be computed efficiently using the algorithm from [108], which is implemented in [51]. Conversely, the colimit of V is isomorphic to the space of its global cosections. This is the quotient of the sum $\bigoplus_{x \in Q_0} V_x$ given by

$$\varinjlim V \simeq \bigoplus\nolimits_{x \in Q_0} V_x \Bigg/ \left\{ v - V_e(v) \;\middle|\; v \in V_{s(e)} \text{ for some } e \in Q_1 \right\}. \tag{3.8}$$

Given any vertex $x \in Q_0$, the desired map $\iota_x : V_x \to \varinjlim V$ now arises as the quotient of the canonical inclusion of V_x into the direct sum $\bigoplus_{x \in Q_0} V_x$.

By definition, we have the following commutative diagram for every $e \in Q_1$:

$$\begin{tikzcd}
& \varprojlim V \arrow[dl, "\pi_{s(e)}"'] \arrow[dr, "\pi_{t(e)}"] & \\
V_{s(e)} \arrow[rr, "V_e"] \arrow[dr, "\iota_{s(e)}"'] & & V_{t(e)} \arrow[dl, "\iota_{t(e)}"] \\
& \varinjlim V &
\end{tikzcd}$$

Thus, when Q is connected, the composite $\iota_x \circ \pi_x : \varprojlim V \to \varinjlim V$ does not depend on $x \in Q_0$ and one may unambiguously define the *limit-to-colimit* map as $\iota_x \circ \pi_x$ for any $x \in Q_0$. The limit-to-colimit map is still defined in general, as $\varprojlim V$ and $\varinjlim V$ can be obtained as the direct sums of $\varprojlim V_{|Q'}$ and $\varinjlim V_{|Q'}$ over the maximal connected subquivers Q' of Q.

Definition 3.3.6. The **generalised rank invariant** of $V \in \mathbf{Rep}(Q)$ is the map

$$\mathrm{GRI}_V : \mathrm{SubQuiv}(Q) \to \mathbb{N}$$

that sends each $Q' \in \mathrm{SubQuiv}(Q)$ to the rank $\mathrm{GRI}_V(Q') \in \mathbb{N}$ of the limit-to-colimit map of the restriction $V_{|Q'}$. △

It follows that the assignment $V \mapsto \mathrm{GRI}_V$ constitutes an additive invariant on $\mathbf{Rep}(Q)$ (in the sense that $V \oplus W$ is sent to $\mathrm{GRI}_V + \mathrm{GRI}_W$). We note that Definition 3.3.6 yields an alternate characterisation of the rank invariant for multiparameter persistence modules. Explicitly, given vertices $x, y \in Q_0$, let $Q_{x,y}$ be the smallest full subquiver of Q which contains all the paths from x to y in Q. It is now readily checked that the assignment

$$\widetilde{\rho}_V(x, y) := \mathrm{GRI}_V(Q_{x,y}) \tag{3.9}$$

coincides with the classical rank invariant.

Remark 3.3.7. The restriction of GRI_V to a subset $\mathcal{S}$ of $\mathrm{SubQuiv}(Q)$ will be denoted $\mathrm{GRI}_V^{\mathcal{S}}$. In the multiparameter persistence setting which was the focus of [75], the family $\mathcal{S}$ usually consists of full and connected subquivers; see [75, Definition 3.5].

We denote by $\mathbf{Rep}_{\mathrm{id}}(Q)$ the full subcategory of representations $V \in \mathbf{Rep}(Q)$ which admit a decomposition into a direct sum of identity representations, as in (3.2). Adapting the proof of [75, Theorem 3.14], we arrive at the following result.

Proposition 3.3.8. *The generalised rank invariant* GRI_V *is complete on* $\mathbf{Rep}_{\mathrm{id}}(Q)$.

Proof. Given $Q' \in \mathrm{SubQuiv}(Q)$, we compute $\mathrm{GRI}_{\mathbf{I}[Q']}(Q)$. If $Q = Q'$, then one can check using (3.7) and (3.8) that $\varprojlim \mathbf{I}[Q'] \simeq \varprojlim \mathbf{I}[Q'] \simeq \mathbb{F}$ as vector spaces, and that the maps $(\pi_x)_{Q_0}$ and $(\iota_x)_{Q_0}$ can be chosen to be all identities, whence $\mathrm{GRI}_{\mathbf{I}[Q']}(Q) = 1$. Otherwise, if there exists a vertex $x \in Q_0 \backslash Q_0'$, then $\iota_x \circ \pi_x = 0$ and so $\mathrm{GRI}_{\mathbf{I}[Q']}(Q) = 0$. Similarly, if there is an edge $e \in Q_1 \backslash Q_1'$, then $\pi_{t(e)} = V_e \circ \pi_{s(e)} = 0$ so once again $\mathrm{GRI}_{\mathbf{I}[Q']}(Q) = 0$. Given $\tilde{Q}, Q' \in \mathrm{SubQuiv}(Q)$, by the above (with $\tilde{Q}$ instead of Q), we have

$$\mathrm{GRI}_{\mathbf{I}[Q']}(\tilde{Q}) = \mathrm{GRI}_{\mathbf{I}[Q' \cap \tilde{Q}]}(\tilde{Q}) = \begin{cases} 1 & \text{if } Q' \supset \tilde{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Finally, let $V \in \mathbf{Rep}_{\mathrm{id}}(Q)$ and let $d_V(\mathbf{I}[Q'])_{Q' \in \mathrm{SubQuiv}(Q)}$ be the multiplicities in decomposition (3.1). By additivity of GRI, we have

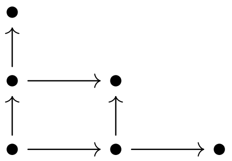
$$\mathrm{GRI}_V(\tilde{Q}) = \sum_{Q' \supset \tilde{Q}} d_V(\mathbf{I}[Q']).$$

Hence, one can prove by descending induction over $Q' \in (\mathrm{SubQuiv}(Q), \subset)$ that the multiplicity $d_V(\mathbf{I}[Q'])$ can be obtained from GRI_V. ☐

We now establish the main result of this subsection.

Proposition 3.3.9. *The skyscraper and the generalised rank invariant are not comparable. More precisely,*

1. *there exist two representations* V *and* W *of the quiver*



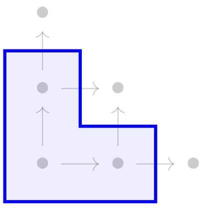
   such that $\delta_V \neq \delta_W$ *whereas* $\mathrm{GRI}_V = \mathrm{GRI}_W$*; and*

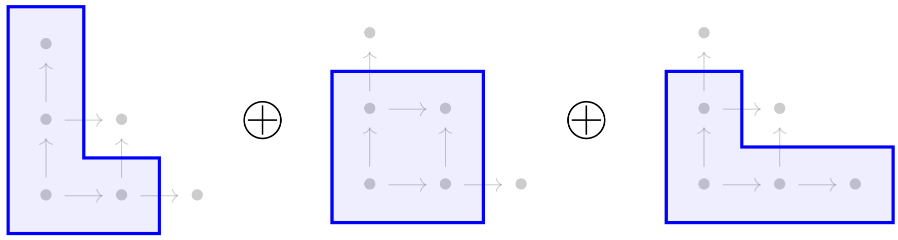
2. *there exist two representations* V' *and* W' *of the quiver*

   such that $\delta_{V'} = \delta_{W'}$ *whereas* $\mathrm{GRI}_{V'} \neq \mathrm{GRI}_{W'}$.

Proof. The representations V and W are given below:

$$V := \begin{tikzcd} \mathbb{F} & & \\ \mathbb{F}^2 \arrow[u, "{[1\ 0]}"] \arrow[r, "{[0\ 1]}"] & \mathbb{F} & \\ \mathbb{F}^2 \arrow[u, equal] \arrow[r, equal] & \mathbb{F}^2 \arrow[u, "{[0\ 1]}"] \arrow[r, "{[1\ 1]}"] & \mathbb{F} \end{tikzcd} \quad \bigoplus$$



$W :=$ 

Here, following the discussion around (3.2), we depict a representation $\mathbf{I}[Q']$ where Q' is a full subquiver of Q by the contour (in blue) of $Q'_0 \subset Q_0$. Let $x \in Q_0$ be the bottom left vertex of Q and let $Q_{3\uparrow}$ (*resp.* $Q_{3\rightarrow}$) be the full subquiver of Q induced by its three top (*resp.* rightmost) vertices.

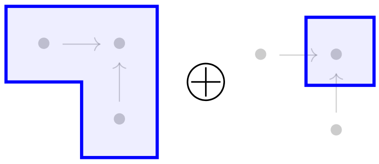
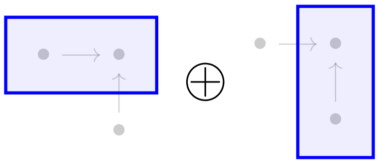
On the one hand, W is δ_x-semistable because each of its summands is δ_x-stable of slope $1/4$. On the other hand, the δ_x-slope of the second summand of V (on the right) exceeds the δ_x-slope of V; whence V is not δ_x-semistable. We thus have $\mathbf{T}[V;\delta_x] \neq \mathbf{T}[W;\delta_x]$, and in particular, $\delta_V \neq \delta_W$.

We show that given $Q' \in \mathrm{SubQuiv}(Q)$, we have $\mathrm{GRI}_V(Q') = \mathrm{GRI}_W(Q')$. When $Q_{3\uparrow} \subset Q'$, the limit-to-colimit map of both $V_{|Q'}$ and $W_{|Q'}$ factors through the following commutative diagram

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[dr] & \\
\varprojlim \arrow[r] & \mathbb{F}^3 \arrow[ur, "{[1\ 0\ 0]}"] \arrow[dr, "{[0\ 1\ 0]}"'] & & \varinjlim \\
 & & \mathbb{F} \arrow[ur] &
\end{tikzcd}$$

whence the map $\mathbb{F}^3 \to \varinjlim$ is null and $\mathrm{GRI}_V(Q') = \mathrm{GRI}_W(Q') = 0$. By symmetry, we also have $\mathrm{GRI}_V(Q') = \mathrm{GRI}_W(Q') = 0$ whenever $Q_{3\rightarrow} \subset Q'$. When $Q'_0 \neq Q_0$, we claim that both $V_{|Q'}$ and $W_{|Q'}$ lie in $\mathbf{Rep}_{\mathrm{id}}(Q')$ and are isomorphic. Indeed, one can check this by hand for each of the 6 maximal proper subsets of Q_0. Finally, assume that $Q'_1 \neq Q_1$ but $Q'_0 = Q_0$ and that Q' contains neither $Q_{3\uparrow}$ nor $Q_{3\rightarrow}$. Then, Q' cannot be connected and by the previous case applied to each connected component of Q', we have $V_{|Q'} \simeq W_{|Q'}$.

For the second assumption, consider the following two representations in $\mathbf{Rep}_{\mathrm{id}}(Q)$:

$V' :=$  and $W' :=$ 

By Proposition 3.3.8, we have $\mathrm{GRI}_{V'} \neq \mathrm{GRI}_{W'}$. Since $\langle V'_x \rangle \simeq \langle W'_x \rangle$ for all $x \in Q_0$ and $\underline{\dim}_V = \underline{\dim}_W$, we have $\delta_{V'} = \delta_{W'}$. □

The remainder of this subsection will be occupied by a comparison of the four invariants introduced so far:

- δ, the skyscraper invariant from Definition 3.3.1,
- GRI, the generalised rank invariant from Definition 3.3.6,
- ρ, the version of the rank invariant from Definition 3.3.4, and

- $\widetilde{\rho}$, the version of the rank invariant from Equation (3.9).

Let us denote by $\mathcal{R}$ the set of pairs (Q, V) where Q is a finite quiver and $V \in \mathbf{Rep}(Q)$. The discriminative power of invariants on a subset $\mathcal{R}'$ of $\mathcal{R}$ defines a preorder $\preceq_{\mathcal{R}'}$ given by $I \preceq_{\mathcal{R}'} I'$ whenever I' is more discriminative than I, or equivalently if the implication

$$I'(V) = I'(W) \implies I(V) = I(W)$$

holds for all $(Q, V), (Q, W) \in \mathcal{R}'$. Let $\mathcal{R}_{\mathrm{MP}} \subset \mathcal{R}$ be the subset of pairs (Q, V) where Q is a grid quiver and V is an equalised representation of Q (see Section 3.5.1 for details).

Proposition 3.3.10. *The Hasse diagrams of the partial orders induced by* $\preceq_{\mathcal{R}_{MP}}$ *and* $\preceq_{\mathcal{R}}$ *are given below:*

$$\begin{tikzcd} \mathrm{GRI} & & \delta \\ & \widetilde{\rho} = \rho \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Multiparameter persistence setting ($\preceq_{\mathcal{R}_{MP}}$)

$$\begin{tikzcd} \mathrm{GRI} & \delta \\ \widetilde{\rho} \arrow[u] & \rho \arrow[u] \end{tikzcd}$$

Quiver setting ($\preceq_{\mathcal{R}}$)

Proof. We start with the quiver setting. We have already proved that $\delta \npreceq_{\mathcal{R}} \rho \preceq_{\mathcal{R}} \delta$ (Theorem 3.3.5) and that $\mathrm{GRI} \npreceq_{\mathcal{R}} \delta \npreceq_{\mathcal{R}} \mathrm{GRI}$ (Proposition 3.3.9). Moreover, by construction, we have $\widetilde{\rho} \preceq_{\mathcal{R}} \mathrm{GRI}$. By transitivity, showing the following three statements:

1. $\mathrm{GRI} \npreceq_{\mathcal{R}} \widetilde{\rho}$
2. $\rho \npreceq_{\mathcal{R}} \mathrm{GRI}$
3. $\widetilde{\rho} \npreceq_{\mathcal{R}} \delta$

is enough to determine the whole Hasse diagram induced by $\preceq_{\mathcal{R}}$.

1. Consider W and W' like in the proof of Theorem 3.3.5(2). We have $\widetilde{\rho}_W = \widetilde{\rho}_{W'}$ whereas Proposition 3.3.8 implies that $\mathrm{GRI}_W \neq \mathrm{GRI}_{W'}$; whence $\mathrm{GRI} \npreceq_{\mathcal{R}} \widetilde{\rho}$.

2. Consider the following representations of the quiver $Q := \begin{tikzcd} x \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] & y \end{tikzcd}$

$$\begin{tikzcd} V := \mathbb{F} \arrow[r, bend left=50, "{[1\ 0\ 0]^T}"] \arrow[r, "{[0\ 1\ 0]^T}"] \arrow[r, bend right=50, "{[0\ 0\ 1]^T}"'] & \mathbb{F}^3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V' := \mathbb{F} \arrow[r, bend left=50, "{[1\ 0\ 0]^T}"] \arrow[r, "{[0\ 1\ 0]^T}"] \arrow[r, bend right=50, "{[1\ 1\ 0]^T}"'] & \mathbb{F}^3 \end{tikzcd}.$$

We have $\rho_V(x, y) = 3 \neq 2 = \rho_{V'}(x, y)$ whereas $\mathrm{GRI}_V = \mathrm{GRI}_{V'}$. Indeed, for any strict subquiver $Q' \subsetneq Q$, we have $V_{|Q'} \simeq V'_{|Q'}$ and $\dim \varprojlim V = 0 = \dim \varprojlim V'$.

3. Consider the following two representations of the quiver $Q = \begin{tikzcd} x \arrow[r, shift left] \arrow[r, shift right] & y \end{tikzcd}$

$$\begin{tikzcd} V := \mathbb{F}^2 \arrow[r, bend left, "\mathrm{id}_{\mathbb{F}^2}"] \arrow[r, bend right, "{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}"'] & \mathbb{F}^2 \end{tikzcd} \quad \text{and} \quad V' := \mathbf{I}[Q]^2$$

On the one hand, $\widetilde{\rho}_V(x, y) \leqslant \dim \varprojlim V = 1$ whereas $\widetilde{\rho}_{V'}(x, y) = 2 \times \mathrm{GRI}_{\mathbf{I}[Q]}(Q) = 2$. On the other hand, both V and V' are δ_x-semistable because all the maps are injective. Hence, $\delta_V = \delta_{V'}$.

In the multiparameter persistence setting, both ρ and $\widetilde{\rho}$ coincide with the classical rank invariant. Moreover, since positive statements of the form $I \preceq_{\mathcal{R}} I'$ still hold when restricting from $\mathcal{R}$ to $\mathcal{R}_{\mathrm{MP}}$, we have GRI $\succeq_{\mathcal{R}_{\mathrm{MP}}} \rho \preceq_{\mathcal{R}_{\mathrm{MP}}} \delta$. Finally, since the example from the proof of Theorem 3.3.5(2) lies in $\mathcal{R}_{\mathrm{MP}}$, we have GRI $\succnsim_{\mathcal{R}_{\mathrm{MP}}} \rho \precnsim_{\mathcal{R}_{\mathrm{MP}}} \delta$, as desired. ☐

Remark 3.3.11. Let $\mathcal{S} \subset \mathrm{SubQuiv}(Q)$ be a subset containing all the full connected convex subquivers of Q. Since the result of Proposition 3.3.10 only relies on examples in $\mathcal{S}$, it holds with $\mathrm{GRI}_V^{\mathcal{S}}$ instead of GRI_V. Moreover, the proof of Proposition 3.3.8 can be adapted to show that $\mathrm{GRI}_V^{\mathcal{S}}$ is complete on those representations which admit a decomposition into indecomposables of the form $\mathbf{I}[Q']$ with $Q' \in \mathcal{S}$.

3.4 HN types of zigzag persistence modules

The goal of this section is to characterise complete central charges for type $\mathbb{A}_\ell$ quivers.

3.4.1 Zigzag persistence modules

Fix an integer $\ell \geqslant 0$. A quiver Q is said to be of **type** $\mathbb{A}_\ell$ whenever its underlying undirected graph has the form

$$\begin{tikzcd} x_0 \arrow[r, dash, "e_1"] & x_1 \arrow[r, dash, "e_2"] & \cdots \arrow[r, dash, "e_{\ell-1}"] & x_{\ell-1} \arrow[r, dash, "e_\ell"] & x_\ell. \end{tikzcd}$$

We describe the direction of edges via a boolean string τ of length ℓ, called the *orientation* of Q: its i-th entry τ_i indicates whether e_i points forward (1) from x_{i-1} to x_i or backward (0) from x_i to x_{i-1}. For instance, when $\ell = 3$, the sequence $\tau = 100$ implicates the following quiver:

$$\begin{tikzcd} x_0 \arrow[r, "e_1"] & x_1 & x_2 \arrow[l, "e_2"'] & x_3. \arrow[l, "e_3"'] \end{tikzcd}$$

We say that Q is *equioriented* if every τ_i equals 1 (i.e., all edges point forward). Our goal here is to describe all complete central charges α for representations of type $\mathbb{A}_\ell$ quivers.

Representations of type $\mathbb{A}_\ell$ quivers are called *zigzag persistence modules* [**30**], and these specialise in the equioriented case to *ordinary persistence modules* [**100**]. It follows from Gabriel's theorem [**55**] that the indecomposable summands which appear in the decomposition (3.1) of a nonzero $V \in \mathbf{Rep}(Q)$ have a particularly simple form when Q is of type $\mathbb{A}_\ell$. Each such indecomposable corresponds to a subinterval $[a, b] \subset [0, \ell]$ with integral endpoints. Recalling that $\mathbb{F}$ is the ground field over which all of our vector spaces are defined, the indecomposable $\mathbf{I}_\tau[a, b] \in \mathbf{Rep}(Q)$ associated to $[a, b]$ has the form

$$0 \longleftrightarrow \cdots \longleftrightarrow 0 \longleftrightarrow \mathbb{F} \longleftrightarrow \cdots \longleftrightarrow \mathbb{F} \longleftrightarrow 0 \longleftrightarrow \cdots \longleftrightarrow 0. \tag{3.10}$$

Here the arrows point in accordance with the orientation τ of Q, the contiguous string of $\mathbb{F}$'s spans vertex indices $\{a, a+1, \ldots, b-1, b\}$, all maps with source and target $\mathbb{F}$ are identities, and all other vector spaces are trivial. These indecomposables are often called *interval* modules.

Explicitly, if Q is a type $\mathbb{A}_\ell$ quiver with orientation τ, then associated to each representation $V \in \mathbf{Rep}(Q)$ there exists a unique finite set $\mathbf{Bar}(V)$ consisting of subintervals $[a,b] \subset [0,\ell]$ with $a \leqslant b$ integers, and a unique function $\mathbf{Bar}(V) \to \mathbb{N}$ sending each $[a,b]$ to its multiplicity d_{ab}, so that there is an isomorphism

$$V \simeq \bigoplus_{[a,b]} (\mathbf{I}_\tau[a,b])^{d_{ab}}\,. \tag{3.11}$$

Here the direct sum ranges over $[a,b] \in \mathbf{Bar}(V)$. Thus, a central charge $\alpha : Q_0 \to \mathbb{R}$ is complete for Q in the sense of Definition 3.2.9 if and only if the multiplicity function $[a,b] \mapsto d_{ab}$ of every $V \in \mathbf{Rep}(Q)$ can be recovered from the HN type $\mathbf{T}[V;\alpha]$.

3.4.2 Characterising complete central charges

The first step in our quest to describe all complete central charges of Q is a converse to Lemma 3.2.10. This result is mentioned (without a detailed proof) in the discussion after [**103**, Conjecture 7.1]. Throughout, we fix a quiver $Q = (s,t : Q_1 \to Q_0)$ of type $\mathbb{A}_\ell$ and denote its orientation by τ.

Proposition 3.4.1. *A central charge $\alpha : Q_0 \to \mathbb{R}$ is complete for Q if every indecomposable $\mathbf{I}_\tau[a,b]$ in $\mathbf{Rep}(Q)$ is α-stable.*

Proof. Let $V \in \mathbf{Rep}(Q)$ have the decomposition (3.11); we seek to establish that the multiplicities d_{ab} which appear in this decomposition can be recovered from $\mathbf{T}[V;\alpha]$. To this end, let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the collection of all slopes contained in the set

$$\{\mu_\alpha(\mathbf{I}_\tau[a,b]) \mid [a,b] \in \mathbf{Bar}(V)\}\,.$$

For each i in $\{1,\ldots,k\}$ we denote by $B_i \subset \mathbf{Bar}(V)$ the subset consisting of all $[a,b]$ for which $\mu_\alpha(\mathbf{I}_\tau[a,b]) \geqslant \lambda_i$. Consider the filtration $V^\bullet$ of V given by

$$V^i := \bigoplus_{[a,b]\in B_i} \mathbf{I}_\tau[a,b]^{d_{ab}}.$$

By construction, the quotient V^i/V^{i-1} is a direct sum of stable representations with α-slope equal to λ_i. Now Corollary 3.2.6 and uniqueness (described in Theorem 3.2.4) ensure that $V^\bullet$ is the HN filtration of V along α.

For each $b \in \{0,1,\ldots,\ell\}$, define $\phi_b : \{0,\ldots,b\} \to \mathbb{R}$ as $\phi_b(a) := \mu_\alpha(\mathbf{I}_\tau[a,b])$. We claim that these maps are injective: given $a' < a$, there are two cases to consider, depending on the orientation of $e_a \in Q_1$ (or equivalently, on the value of $\tau_a \in \{0,1\}$). If $t(e_a) = x_a$, then there exists a monomorphism $\mathbf{I}_\tau[a,b] \subset \mathbf{I}_\tau[a',b]$ and the stability of the latter representation guarantees $\phi_b(a) < \phi_b(a')$. On the other hand, if $t(e_a) = x_{a-1}$ then $\mathbf{I}_\tau[a,b]$ is a quotient of $\mathbf{I}_\tau[a',b]$: we have a short exact sequence in $\mathbf{Rep}(Q)$ of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{I}_\tau[a',a-1] \arrow[r] & \mathbf{I}_\tau[a',b] \arrow[r] & \mathbf{I}_\tau[a,b] \arrow[r] & 0. \end{tikzcd}$$

An appeal to Lemma 3.2.5 along with the α-stability of $\mathbf{I}_\tau[a',b]$ gives $\phi_b(a) > \phi_b(a')$. In both cases we obtain $\phi_b(a) \neq \phi_b(a')$ for $a' < a$, whence ϕ_b is injective as claimed.

Given this injectivity, for each fixed i we may order the elements of B_i as

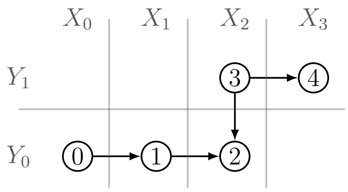
$$B_i = \{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\},$$

where $b_1 > \cdots > b_n$. Now the multiplicity $d_{a_1 b_1}$ is precisely the dimension of V^i/V^{i-1} at any vertex $x_j \in Q_0$ where $b_2 < j \leqslant b_1$. Proceeding inductively, we similarly recover the multiplicities $d_{a_k b_k}$ for all $1 < k \leqslant n$. □

The preceding result, combined with Lemma 3.2.10, confirms that a central charge α for Q is complete if and only if every indecomposable $\mathbf{I}_\tau[a, b] \in \mathbf{Rep}(Q)$ is α-stable. The main result of [**80**] is a complete characterisation of such central charges in terms of two functions: let $\chi, \eta \colon \{0, 1, \ldots, \ell\} \mapsto \mathbb{N}$ be defined inductively as follows. Beginning with $\chi(0) = 0$ and $\eta(0) = 0$, for each $i > 0$ we set

$$\chi(i+1) = \begin{cases} \chi(i) + 1 & \text{if } \tau_i = 1 \\ \chi(i) & \text{if } \tau_i = 0 \end{cases} \quad \text{and} \quad \eta(i+1) = \begin{cases} \eta(i) + 1 & \text{if } \tau_i = 0 \\ \eta(i) & \text{if } \tau_i = 1 \end{cases}$$

For instance, when $\tau = 1101$, the function χ takes on values $(0, 1, 2, 2, 3)$ while the function η takes on values $(0, 0, 0, 1, 1)$ for inputs $(0, 1, 2, 3, 4)$:



For each $k \in \mathbb{N}$ we have the level sets $X_k := \{i \mid \chi(i) = k\}$ and $Y_k := \{i \mid \eta(i) = k\}$, both of which are subintervals of $\{0, \ldots, \ell\}$ as χ and η are monotone. Writing $X_k = [a_k, b_k]$ and $Y_k = [a'_k, b'_k]$ for each k, we are able to state [**80**, Theorem 1.13].

Theorem 3.4.2. *All indecomposables* $\mathbf{I}_\tau[a, b]$ *in* $\mathbf{Rep}(Q)$ *are* α*-stable for a given central charge* $\alpha : Q_0 \to \mathbb{R}$ *if and only if the following inequalities hold:*

$$\mu_\alpha(\mathbf{I}_\tau[a_0, b_0]) > \mu_\alpha(\mathbf{I}_\tau[a_1, b_1]) > \cdots > \mu_\alpha(\mathbf{I}_\tau[a_{\chi(\ell)}, b_{\chi(\ell)}]),$$
$$\mu_\alpha(\mathbf{I}_\tau[a'_0, b'_0]) < \mu_\alpha(\mathbf{I}_\tau[a'_1, b'_1]) < \cdots < \mu_\alpha(\mathbf{I}_\tau[a'_{\eta(\ell)}, b'_{\eta(\ell)}]).$$

It is also shown in [**80**] that, for every possible orientation τ, (**a**) this is a *minimal* set of inequalities for characterising those central charges along which all indecomposables are stable, and (**b**) the set of all such central charges defines a non-empty open subset in $\mathbb{R}^{Q_0}$ which is linearly equivalent to $\mathbb{R} \times \mathbb{R}^{Q_1}_{>0}$. These results, when combined with our Proposition 3.4.1 and Lemma 3.2.10, completely describe all complete central charges for $\mathbb{A}_\ell$ quivers.

Theorem 3.4.3. *Given an integer* $\ell \geqslant 0$*, let* Q *be a quiver of type* $\mathbb{A}_\ell$ *and orientation* τ*. The set of complete central charges for* Q *is nonempty, and consists precisely of those* $\alpha : Q_0 \to \mathbb{R}$ *which satisfy the inequalities from Theorem 3.4.2.*

We note here that the set of complete central charges admits a particularly appealing description in the case where Q is equioriented.

Corollary 3.4.4. *For ordinary persistence modules, a central charge α is complete if and only if the inequality $\alpha(x_i) > \alpha(x_{i+1})$ holds for all $i \in \{0, 1, \ldots, \ell - 1\}$.*

Proof. If $\tau = 11\ldots1$, then the function $\chi : \{0, 1, \ldots, \ell\} \to \mathbb{N}$ is given by $\chi(i) = i$, whereas the function η is identically zero. We therefore seek any $\alpha : Q_0 \to \mathbb{R}$ which satisfies $\mu_\alpha(\mathbf{I}_\tau[i, i]) > \mu_\alpha(\mathbf{I}_\tau[i + 1, i + 1])$ for all i. By (3.4) and (3.10), this string of inequalities reduces to

$$\alpha(x_0) > \alpha(x_1) > \cdots > \alpha(x_{\ell-1}) > \alpha(x_\ell), \tag{3.12}$$

as desired. □

3.5 HN types of multiparameter persistence modules

Finding discriminative and computable invariants for multiparameter persistence modules remains a central challenge in topological data analysis. In this setting, we prove a generalisation of Corollary 3.4.4 for rectangle-decomposable representations. We then establish another completeness result for a larger class of interval-decomposable representations.

3.5.1 Multiparameter persistence modules as equalised representations

Let $Q = (s, t : Q_1 \to Q_0)$ be an acyclic quiver. Its source and target maps may be extended from edges to paths $\gamma = (e_1, \ldots, e_k)$ by setting $s(\gamma) := s(e_1)$ and $t(\gamma) := t(e_k)$. Given a representation $V \in \mathbf{Rep}(Q)$, there is a distinguished linear map $V_\gamma : V_{s(\gamma)} \to V_{t(\gamma)}$ induced by the composite

$$V_\gamma := V_{e_k} \circ V_{e_{k-1}} \circ \cdots \circ V_{e_2} \circ V_{e_1}.$$

Definition 3.5.1. We say that $V \in \mathbf{Rep}(Q)$ is **equalised** if the following property holds: for any pair of vertices $x, y \in Q_0$ and any pair of paths γ, γ' with common source x and common target y, the composite maps V_γ and $V_{\gamma'}$ are identical. Let $\mathbf{Rep}_{\mathrm{eq}}(Q) \subset \mathbf{Rep}(Q)$ be the full subcategory spanned by equalised representations. △

We recall from Remark 3.2.12 that the HN filtrations of an equalised representation coincide in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ and in $\mathbf{Rep}(Q)$.

Example 3.5.2. A large class of interesting equalised representations arises in the study of *cellular sheaves* [**40**]. Every such sheaf $\mathcal{F}$ on a regular CW complex X is a functor from the face-ordered poset of cells $(X, <)$ to the category $\mathbf{Vec}(\mathbb{F})$ of $\mathbb{F}$-vector spaces. The *Hasse graph* of X is the quiver $Q(X)$ whose vertices correspond bijectively to the cells of X, with a unique edge $\sigma \to \tau$ being present whenever σ is a face of τ of codimension one. Any given sheaf $\mathcal{F} : (X, <) \to \mathbf{Vec}(\mathbb{F})$ on X induces a representation $V(\mathcal{F})$ of $Q(X)$ as follows: every vertex σ is assigned the vector space $\mathcal{F}(\sigma)$ and every edge $\sigma \to \tau$ is assigned the linear map $\mathcal{F}(\sigma < \tau)$. The fact that $\mathcal{F}$ is a functor directly implies that $V(\mathcal{F})$ is equalised.

Definition 3.5.3. The **flow partial order** on vertices Q_0 of the acyclic quiver Q is defined as follows: given x and y in Q_0, we have $x \leqslant y$ if either $x = y$ or if there exists a path γ in Q with $s(\gamma) = x$ and $t(\gamma) = y$. △

Given $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ and a pair of vertices $x \leqslant y$ in Q_0, we write $V_{x \leqslant y} : V_x \to V_y$ to indicate the map defined by any path γ from x to y, with the understanding that this map is the identity for $y = x$. Since V is equalised, this is well-defined and it follows that the image of $V_{x\leqslant y}$ is isomorphic to $\langle V_x \rangle_y$ (from Definition 3.3.2). Thus, the value of $\rho_V(x, y)$ is precisely the rank of $V_{x\leqslant y}$ when V is equalised. This is the genesis of the terminology of the rank invariant, which was introduced in [**31**] to study certain equalised representations of grid quivers, described below.

Let us fix a vector $L = (\ell_1, \ell_2, \ldots, \ell_d)$ of $d \geqslant 2$ integers, with each $\ell_i \geqslant 1$. Here we consider the case where $Q = (s, t : Q_1 \to Q_0)$ is the d-dimensional *grid quiver* of shape L, defined as follows. Its vertices x_p are indexed by all p lying in the product

$$\Lambda(L) := \prod_{i=1}^{d} \{0, 1, \ldots, \ell_i\},$$

and there exists a unique edge $x_p \to x_q$ whenever $q - p$ is a standard basis vector of $\mathbb{R}^d$. Here, for instance, is the quiver of shape $L = (\ell, 2)$ for arbitrary $\ell \geqslant 1$:

$$\begin{tikzcd}
x_{0,2} \arrow[r] & x_{1,2} \arrow[r] & \cdots \arrow[r] & x_{\ell-1,2} \arrow[r] & x_{\ell,2} \\
x_{0,1} \arrow[r] \arrow[u] & x_{1,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell-1,1} \arrow[r] \arrow[u] & x_{\ell,1} \arrow[u] \\
x_{0,0} \arrow[r] \arrow[u] & x_{1,0} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell-1,0} \arrow[r] \arrow[u] & x_{\ell,0} \arrow[u]
\end{tikzcd}$$

Equalised representations of d-dimensional grid quivers are also referred to as **d-parameter persistence modules** [**31**]. We note that every grid quiver Q contains an embedded copy of the grid quiver of shape $L = (1, 1)$, and that both the representations W and W' which appeared in the proof of Theorem 3.3.5 are equalised. Thus, we obtain the following consequence.

Corollary 3.5.4. *Given any integer $d \geqslant 2$, let Q be the grid quiver corresponding to some integer vector $L = (\ell_1, \ldots, \ell_d)$ with each $\ell_i \geqslant 1$.*

1. *The skyscraper invariant δ_V of $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ determines its rank invariant (via the formula in Theorem 3.3.5).*

2. *There exist representations W and W' in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ which have identical rank invariant and satisfy $\delta_W \neq \delta_{W'}$.*

3.5.2 Rectangle-decomposable representations

In general, grid quivers are of wild representation type and one cannot expect to obtain a tractable classification of all indecomposable objects in $\mathbf{Rep}_{\text{eq}}(Q)$. One can, however, impose a higher-dimensional analogue of (3.11) by passing to the subset of *rectangle-decomposable* representations, which we describe below.

As before, let $Q = (s, t : Q_1 \to Q_0)$ be the grid quiver of shape $L = (\ell_1, \ell_2, \ldots, \ell_d)$ for $d \geqslant 2$ and all $\ell_i \geqslant 1$. Given any subset $P \subset \Lambda(L)$, we recall from Definition 3.2.3 that $\mathbf{I}[P]$ denotes the representation of Q which assigns vector spaces

$$\mathbf{I}[P]_{x_p} = \begin{cases} \mathbb{F} & \text{if } p \in P, \\ 0 & \text{otherwise;} \end{cases}$$

the linear map associated to each edge is the identity whenever both source and target spaces are $\mathbb{F}$, and it must necessarily equal zero otherwise. By a **rectangle representation** we mean $\mathbf{I}[R]$, where $R := [a_1, b_1] \times \cdots \times [a_d, b_d]$ for some $[a_i, b_i] \subset [0, \ell_i]$ is a rectangle inside $\Lambda(L)$. We note that $\mathbf{I}[R]$ is always equalised when R is a rectangle. We write $\mathbf{Rep}_{\text{rec}}(Q)$ for the full subcategory of $\mathbf{Rep}_{\text{eq}}(Q)$ spanned by objects which are (isomorphic to) direct sums of rectangle representations. The rank invariant is complete when restricted to this subcategory [**19**, **37**]; and by Corollary 3.5.4, so is the skyscraper invariant.

Our goal in this subsection is to prove a much sharper result — we extend Corollary 3.4.4 to the category of rectangle-decomposable representations of arbitrary dimension d by classifying the set of complete central charges. For this purpose, it is necessary to exclude from consideration a finite union of hyperplanes in the vector space of central charges:

$$\mathcal{H} := \bigcup_{R \neq R'} \{\alpha : Q_0 \to \mathbb{R} \mid \mu_\alpha(\mathbf{I}[R]) = \mu_\alpha(\mathbf{I}[R'])\}, \tag{3.13}$$

where R and R' range over distinct rectangles in $\Lambda(L)$. The following result serves to justify this exclusion.

Proposition 3.5.5. *If $\alpha \notin \mathcal{H}$ is a central charge for which each rectangle representation $\mathbf{I}[R] \in \mathbf{Rep}_{\text{eq}}(Q)$ is α-stable, then $\mathbf{T}[\bullet; \alpha]$ is complete on $\mathbf{Rep}_{\text{rec}}(Q)$.*

Proof. Given $V \in \mathbf{Rep}_{\text{rec}}(Q)$, consider its decomposition

$$V \simeq \bigoplus_R \mathbf{I}[R]^{m_R},$$

where the direct sum is indexed over all subrectangles $R \subset \Lambda(L)$, of which only finitely many have multiplicity $m_R > 0$. It suffices to recover these multiplicities from the HN type of V along α. By Proposition 3.2.8, for each real number $c \in \mathbb{R}$ we have

$$\mathbf{T}[V; \alpha](c) = \sum_{\mu_\alpha(R) = c} m_R \cdot \underline{\dim}_{\mathbf{I}[R]}$$

Since $\alpha \notin \mathcal{H}$ by assumption, the multiplicity $m_{R'}$ of any rectangle R' can be obtained by letting $c = \mu_\alpha(R')$, so the sum simplifies to $\mathbf{T}[V; \alpha](c) = m_{R'} \cdot \underline{\dim}_{\mathbf{I}[R']}$. ☐

The following is a multiparameter generalisation of Corollary 3.4.4.

Theorem 3.5.6. *Let Q be a grid quiver of shape $L = (\ell_1, \ell_2, \ldots, \ell_d)$ for any $\ell_i \geqslant 1$. A central charge $\alpha \notin \mathcal{H}$ is complete on $\mathbf{Rep}_{\mathrm{rec}}(Q)$ if and only if it satisfies the inequality $\alpha \circ s(e) > \alpha \circ t(e)$ for each edge $e \in Q_1$.*

The remainder of this subsection will be occupied by the proof.

We first establish a technical result in lattice theory. Let us recall the flow partial order on Q_0 from Definition 3.5.3. A subset of vertices $U \subset Q_0$ is said to be **up-closed** if $x \in U$ and $y \geqslant x$ implies $y \in U$. Given an arbitrary nonempty subset $A \subset Q_0$, we denote by A^+ the smallest up-closed subset of Q_0 containing A. The poset $(Q_0, \leqslant)$ has a structure of finite lattice with $\wedge$ and $\vee$ given by applying respectively min and max coordinate-wise. One can check that this lattice is distributive (the distributive law is true in each coordinate), and hence that it satisfies the following standard inequality [**1**, Corollary 6.1.3].

Proposition 3.5.7. *Let $(L, \wedge, \vee)$ be a finite distributive lattice. For any subsets $X, Y \subset L$, we have the inequality*

$$|X| \cdot |Y| \leqslant |X \wedge Y| \cdot |X \vee Y|, \text{ where:}$$

1. *$|\bullet|$ denotes cardinality,*
2. *$X \vee Y := \{x \vee y \mid x \in X \text{ and } y \in Y\}$, and*
3. *$X \wedge Y := \{x \wedge y \mid x \in X \text{ and } y \in Y\}$.*

We will use this combinatorial inequality to establish the following result about up-closed subsets of Q_0.

Lemma 3.5.8. *Let $U \subset Q_0$ be an up-closed subset with complement $D := Q_0 \backslash U$. Then, for all subsets $A \subset D$ we have*

$$|A| \cdot |U| \leqslant |D| \cdot |U \cap A^+|$$

where $|\bullet|$ denotes cardinality.

Proof. Since, $(A^+ \cap D)^+ = A^+$, it suffices to establish the desired inequality with A replaced by the larger set $A^+ \cap D$. Since $A^+ \cap D$ equals $A^+ \backslash (U \cap A^+)$, this inequality becomes

$$\big(|A^+| - |U \cap A^+|\big) \cdot |U| \leqslant (|Q_0| - |U|) \cdot |U \cap A^+|,$$

which is equivalent to

$$|A^+| \cdot |U| \leqslant |Q_0| \cdot |U \cap A^+|.$$

Since U and A^+ are up-closed, we have $U \cap A^+ = U \vee A^+$. Applying Proposition 3.5.7 with subsets $X = U$ and $Y = A^+$ of $L = Q_0$, we obtain

$$|A^+| \cdot |U| \leqslant |U \wedge A^+| \cdot |U \vee A^+| \leqslant |Q_0| \cdot |U \cap A^+|,$$

as desired. ☐

The main tool in the proof of our generalisation of Corollary 3.4.4 to $\mathbf{Rep}_{\mathrm{rec}}(Q)$ is the following **max-flow/min-cut theorem** [15, Chapter III.1].

Theorem 3.5.9. *Let* $\Phi = (\sigma, \tau : \Phi_1 \to \Phi_0)$ *be a quiver whose vertex set contains a distinguished source* $s_* \notin \tau(\Phi_1)$ *and target* $t_* \notin \sigma(\Phi_1)$*, and let* $\kappa : \Phi_1 \to [0, \infty]$ *be a function defined on edges. The maximum value attained by a* κ*-flow equals the minimum* κ*-capacity of a cut separating* s_* *from* t_**.*

Recall that a κ-flow on Φ is a map $f : \Phi_1 \to [0, \infty]$ satisfying two constraints:

1. $f(\epsilon) \leqslant \kappa(\epsilon)$ for all $\epsilon \in \Phi_1$, and
2. $\sum_{\sigma(\epsilon)=x} f(\epsilon) = \sum_{\tau(\epsilon)=x} f(\epsilon)$ for all $x \in \Phi_0 \setminus \{s_*, t_*\}$.

The *value* of f is the sum $\nu(f) := \sum_{\sigma(\epsilon)=s_*} f(\epsilon)$; by the second constraint above, $\nu(f)$ also equals $\sum_{\tau(\epsilon)=t_*} f(\epsilon)$. On the other hand, an (s_*, t_*)-cut is any subset $E \subset \Phi_1$ whose removal disconnects s_* from t_*; the κ*-capacity* of such a cut is $c(E) := \sum_{\epsilon \in E} \kappa(\epsilon)$. We are now able to characterise complete central charges for Q which do not lie in the union of hyperplanes $\mathcal{H}$ from (3.13).

Theorem 3.5.10. *Let* Q *be a grid quiver of shape* $L = (\ell_1, \ell_2, \ldots, \ell_d)$ *for any* $\ell_i \geqslant 1$*. A central charge* $\alpha \notin \mathcal{H}$ *is complete on* $\mathbf{Rep}_{\mathrm{rec}}(Q)$ *if and only if it satisfies the inequality* $\alpha \circ s(e) > \alpha \circ t(e)$ *for each edge* $e \in Q_1$*.*

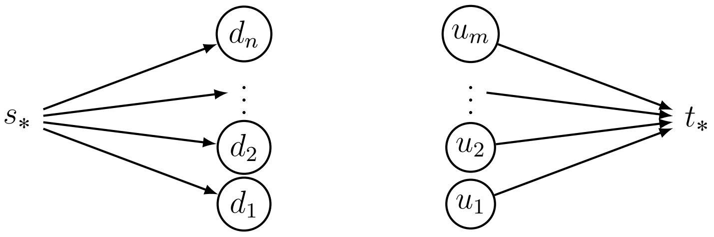
Proof. If the inequality $\alpha \circ s(e) \leqslant \alpha \circ t(e)$ holds for some edge e, then we obtain

$$\mu_\alpha(\mathbf{I}[\{t(e)\}]) \leqslant \mu_\alpha(\mathbf{I}[\{s(e), t(e)\}]).$$

We now have from Lemma 3.2.10 that the restriction of α to the $\mathbb{A}_2$ quiver $s(e) \to t(e)$ is not complete. It is straightforward to confirm that as a consequence α is not complete on $\mathbf{Rep}_{\mathrm{rec}}(Q)$. The remainder of the argument will be devoted to the converse implication — assuming that $\alpha \circ s(e) > \alpha \circ t(e)$ holds for all $e \in Q_1$, we will show that α is complete.

By Proposition 3.5.5, it is enough to prove that each rectangle representation is α-stable. By passing to the full subquiver induced by vertices lying within any such rectangle, it suffices to assume that the rectangle representation at hand is $V = \mathbf{I}[\Lambda(L)]$. Consider a subrepresentation $V' \subsetneq \mathbf{I}[\Lambda(L)]$; this assigns vector spaces of dimension at most 1 with all nontrivial maps being isomorphisms. Thus, V' has the form $\mathbf{I}[U]$ for some up-closed proper subset $U \subsetneq Q_0$. Let $\{u_1, u_2, \ldots, u_m\}$ be the vertices lying in U, and similarly let $\{d_1, d_2, \ldots, d_n\}$ be the vertices lying in its complement $D := Q_0 \setminus U$.

Construct a new quiver $\Phi = (\sigma, \tau : \Phi_1 \to \Phi_0)$ as follows: its vertex set Φ_0 consists of Q_0 along with two additional vertices s_* and t_*. The edge set Φ_1 is built in two stages as follows: first we insert a unique edge from s_* to each vertex of D, and similarly a unique edge from each vertex of U to t_*, as depicted below:



In the second step, we add a unique edge $d_i \to u_j$ whenever $d_i \leqslant u_j$ holds in Q_0. Define $\kappa : \Phi_1 \to [0, \infty]$ by:

$$\kappa(\epsilon) = \begin{cases} 1/n & \text{if } \sigma(\epsilon) = s_* \\ 1/m & \text{if } \tau(\epsilon) = t_* \\ +\infty & \text{otherwise.} \end{cases}$$

Claim: Every (s_*, t_*)-cut $E \subset \Phi_1$ has κ-capacity $c(E) \geqslant 1$.
Given such a cut, let S and T denote the vertices lying in the component of s_* and t_* respectively after the edges of E have been removed. We may safely assume that E contains no edges of the form $d_i \to u_j$ since that would immediately force $c(E) = \infty$. Therefore, writing $A := S \cap D$ and $B := T \cap U$, we know that $A^+ \cap B$ is empty because the removal of E must separate s_* from t_*. As a result, we have

$$c(E) = \frac{|D \backslash A|}{|D|} + \frac{|U \backslash B|}{|U|}.$$

Using the fact that $U \backslash B$ contains $A^+ \cap U$ followed by Lemma 3.5.8, we have

$$\frac{|U \backslash B|}{|U|} \geqslant \frac{|U \cap A^+|}{|U|} \geqslant \frac{|A|}{|D|} = 1 - \frac{|D \backslash A|}{|D|}.$$

Using this bound in our expression for $c(E)$ given above establishes the claim.

Returning to the main argument, we have by Theorem 3.5.9 that Φ admits a κ-flow $f : \Phi_1 \to [0, \infty]$ of value $\nu(f) \geqslant 1$. Select any such f and note that it must evaluate to $1/n$ on each edge $s_* \to d_i$ and to $1/m$ on each edge $u_j \to t_*$, whence its value $\nu(f)$ is exactly 1. Define $F : D \times U \to \mathbb{R}_{\geqslant 0}$ by

$$F(d_i, u_j) = \begin{cases} f(d_i \to u_j) & \text{if } d_i \leqslant u_j \text{ holds in } Q_0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha \circ s(e) > \alpha \circ t(e)$ holds for each edge $e \in Q_1$, and since F takes strictly positive values on at least some (d_i, u_j) pairs, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F(d_i, u_j) \cdot \alpha(d_i) > \sum_{i=1}^{n} \sum_{j=1}^{m} F(d_i, u_j) \cdot \alpha(u_j).$$

Using the fact that f is a κ-flow, for each i we have $\sum_{j=1}^{m} F(d_i, u_j) = 1/n$, and similarly for each j we have $\sum_{i=1}^{n} F(d_i, u_j) = 1/m$. Thus, the inequality above is $\phi_\alpha(\mathbf{I}[D]) > \phi_\alpha(\mathbf{I}[U])$. Finally, the desired inequality $\phi_\alpha(V') < \phi_\alpha(V)$ follows from Lemma 3.2.5 applied to the short exact sequence $0 \to V' \to V \to \mathbf{I}[D] \to 0$. □

3.5.3 Interval-decomposable representations

The class of interval-decomposable representations (see Definition 3.2.3) generalises the class of rectangle-decomposable representations and plays an important role in multiparameter persistence. Let Q be a finite acyclic quiver, and let $\mathbf{Rep}_{\text{int}}(Q)$ be the full subcategory of $\mathbf{Rep}(Q)$ spanned by the objects which admit a direct sum decomposition into interval representations. The subcategories of $\mathbf{Rep}(Q)$ introduced in this document are related by the following inclusions

$$\begin{tikzcd}
 & & \mathbf{Rep}_{\mathrm{id}}(Q) \arrow[dr, hook] & \\
\mathbf{Rep}_{\mathrm{rec}}(Q) \arrow[r, hook] & \mathbf{Rep}_{\mathrm{int}}(Q) \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{Rep}(Q) \\
 & & \mathbf{Rep}_{\mathrm{eq}}(Q) \arrow[ur, hook] &
\end{tikzcd}$$

where $\mathbf{Rep}_{\mathrm{id}}(Q)$ is defined in (3.2). The results from [4] establish that the equality

$$\mathbf{Rep}_{\mathrm{int}}(Q) = \mathbf{Rep}_{\mathrm{id}}(Q) \cap \mathbf{Rep}_{\mathrm{eq}}(Q)$$

holds when Q is a 2-dimensional grid quiver, but not in general.

Let $\mathcal{I}$ denote the set of (isomorphism classes of) indecomposables in $\mathbf{Rep}_{\mathrm{int}}(Q)$. The **support** of a representation $V \in \mathbf{Rep}(Q)$ is the subset $\mathrm{supp}(V) \subset Q_0$ consisting of all vertices x for which the vector space V_x is nontrivial (or equivalently, those vertices x where $\underline{\dim}_V(x)$ is nonzero). Note that the representations in $\mathcal{I}$ are uniquely determined by their support. We recall the flow partial order $\leqslant$ on Q_0 from Definition 3.5.3, and say that a subset $S \subset Q_0$ is convex if the full subquiver induced by S is convex. Following the proof of [5, Lemma 4.4], we show that $\mathbf{Rep}_{\mathrm{int}}(Q)$ satisfies the conditions of Lemma 3.2.11.

Lemma 3.5.11. *The subrepresentations and the quotients of an interval representation are interval-decomposable.*

Proof. Let $I \in \mathcal{I}$ and let $W \subset I$ be a nonzero subrepresentation. We prove that W and I/W both lie in $\mathbf{Rep}_{\mathrm{int}}(Q)$. For each $x \in \mathrm{supp}(I)$, let 1_x be the unit of $I_x = \mathbb{F}$. Since $\underline{\dim}_I$ takes values in $\{0, 1\}$, the vectors (1_x) constitute bases of the nonzero spaces of W and I/W. Hence, W and W/I are isomorphic to identity representations.

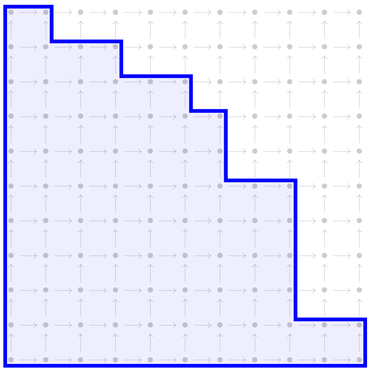
We now prove that $\mathrm{supp}(W)$ and $\mathrm{supp}(I/W)$ are convex. Let p be a path in Q from a vertex in $\mathrm{supp}(W)$ to a vertex in $\mathrm{supp}(I)$. By convexity of $\mathrm{supp}(I)$, the composite map I_p is an isomorphism; whence, W_p is injective and $t(p) \in \mathrm{supp}(W)$. So the subset $\mathrm{supp}(W) \subset \mathrm{supp}(I)$ is upward-closed with respect to $\leqslant$. In particular, $\mathrm{supp}(W)$ is convex in the full subquiver induced by $\mathrm{supp}(I)$ and hence in Q. Moreover, the complement $\mathrm{supp}(I/W) = \mathrm{supp}(I) \backslash \mathrm{supp}(W)$ is a downward-closed subset of $\mathrm{supp}(I)$ and is also convex in Q.

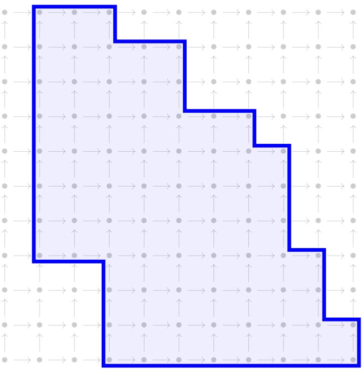
We have proven that W and I/W are isomorphic to some identity representation of the form $\mathbf{I}[P]$ where $P \subset Q$ is a full convex subquiver. Let $P^1, \ldots, P^k$ be the connected components of P, we have $\mathbf{I}[P] \simeq \bigoplus_{i=1}^k \mathbf{I}[P^i]$, and by definition each $\mathbf{I}[P^i]$ lies in $\mathcal{I}$. $\square$

Given an integer $k \geqslant 1$, we denote by $\mathcal{I}_{\leqslant k}$ the family of interval representations with at most k sources. Namely,

$$\mathcal{I}_{\leqslant k} := \{ I \in \mathcal{I}, \ |\min \mathrm{supp}(I)| \leqslant k \},$$

where $\min(I)$ denotes the set of minimal vertices lying in the subposet $(\mathrm{supp}(I), \leqslant)$. The supports of two indecomposables of $\mathcal{I}_{\leqslant 2}$ inside a grid of shape $(10, 10)$ are shown below in blue:





A natural question to ask is whether there is a family A of central charges which is complete on $\mathbf{Rep}_{\mathrm{int}}(Q)$. Let us first state a negative result.

Corollary 3.5.12. *Assume that the underlying field $\mathbb{F}$ is different from $\mathbb{Z}/2\mathbb{Z}$. If Q contains a grid of shape $(4, 1)$, then no single central charge is complete on the class of $\mathcal{I}_{\leqslant 2}$-decomposable modules and hence on $\mathbf{Rep}_{\mathrm{int}}(Q)$.*

This follows from Corollary 3.6.2, which we will prove in the next Section. The good news takes the form of the following Proposition, which is the main result of this Subsection.

Proposition 3.5.13. *There is a finite family of central charges A which is complete on $\mathcal{I}_{\leqslant 2}$-decomposable representations.*

The first step in our proof is the following Lemma.

Lemma 3.5.14. *Let $\mathcal{J} \subset \mathcal{I}$ be a family of interval representations and assume that for all $I \in \mathcal{J}$, there is a central charge $\alpha^I : Q_0 \to \mathbb{R}$ and a real number $\lambda^I \in \mathbb{R}$ such that I is the only α^I-semistable interval representation in $\mathcal{I}$ of slope λ^I. Then, the invariant $\mathbf{T}[V; (\alpha^I)_{I \in \mathcal{J}}]$ is complete on $\mathcal{J}$-decomposable representations.*

Proof. By Lemmas 3.2.11(2) and 3.5.11, the HN type $\mathbf{T}[J; \alpha^I](\lambda^I)$ of a nonzero interval representation $J \in \mathcal{J}$ is the dimension vector of some α^I-semistable $W \in \mathbf{Rep}_{\mathrm{int}}(Q)$. By Lemma 3.2.5, the indecomposable summands in the decomposition of W must have slope λ^I. Since, $\underline{\dim}_W \leqslant \underline{\dim}_J$, the dimension vector $\mathbf{T}[J; \alpha^I](\lambda^I)$ is either 0 or $\underline{\dim}_I$. Given the decomposition $V \simeq \bigoplus_{J \in \mathcal{J}} J^{d_V(J)}$, we have

$$\mathbf{T}[V; \alpha^I](\lambda^I) = \sum_{J \in \mathcal{U}^I} d_V(J)\, \underline{\dim}_I\,.$$

Since the support of every $J \in \mathcal{U}^I$ contains $\mathrm{supp}(I)$, the multiplicities $(d_V(J))_{J \in \mathcal{J}}$ can be obtained from $(\mathbf{T}[V; \alpha^J])_{J \in \mathcal{J}}$ by descending induction on $\mathcal{J}$ equipped with the partial order induced by the inclusion of supports. $\square$

We now prove the main result of this subsection by applying Lemma 3.5.14 with the family of interval representations $\mathcal{J} = \mathcal{I}_{\leqslant 2}$.

Proof of Proposition 3.5.13. Let $I \in \mathcal{I}_{\leqslant 2}$, we first prove the existence of a central charge along which I is stable. If the set $\min(\operatorname{supp}(I))$ of minimal vertices of the subposet $\operatorname{supp}(I)$ is a singleton $\{x\} \subset Q_0$, then I is δ_x-stable. If $\min \operatorname{supp}(I)$ contains two vertices $x, x' \in Q_0$, we consider a nonzero central charge $\alpha\colon Q_0 \to \mathbb{R}$ whose support is $\{x, x'\}$. I has two proper submodules of nonzero α-slope $\langle I_x \rangle$ and $\langle I_{x'} \rangle$. Hence, I is α-stable if and only if

$$\begin{cases} \frac{\alpha(x)}{|\operatorname{supp}(\langle I_x \rangle)|} & < \frac{\alpha(x)+\alpha(x')}{|\operatorname{supp}(I)|} \\ \frac{\alpha(x')}{|\operatorname{supp}(\langle I_{x'} \rangle)|} & < \frac{\alpha(x)+\alpha(x')}{|\operatorname{supp}(I)|} \end{cases}$$

which can be rewritten as

$$\frac{|\operatorname{supp}(I)| - |\operatorname{supp}(\langle I_{x'} \rangle)|}{|\operatorname{supp}(\langle I_{x'} \rangle)|} < \frac{\alpha(x)}{\alpha(x')} < \frac{|\operatorname{supp}(\langle I_x \rangle)|}{|\operatorname{supp}(I)| - |\operatorname{supp}(\langle I_x \rangle)|}$$

There exist $\alpha(x), \alpha(x') \in \mathbb{R}_{>0}$ satisfying the above inequalities if and only if

$$\Big(|\operatorname{supp}(I)| - |\operatorname{supp}(\langle I_x \rangle)|\Big) \cdot \Big(|\operatorname{supp}(I)| - |\operatorname{supp}(\langle I_{x'} \rangle)|\Big) < |\operatorname{supp}(\langle I_x \rangle)| \cdot |\operatorname{supp}(\langle I_{x'} \rangle)|$$

or equivalently,

$$|\operatorname{supp}(I)| < |\operatorname{supp}(\langle I_x \rangle)| + |\operatorname{supp}(\langle I_{x'} \rangle)|.$$

This last inequality holds because $\operatorname{supp}(I) = \operatorname{supp}(\langle I_x \rangle) \cup \operatorname{supp}(\langle I_{x'} \rangle)$ and it is strict because the full subquiver induced by $\operatorname{supp}(I)$ is connected.

Let $\alpha\colon Q_0 \to \mathbb{R}_{\geqslant 0}$ be a central charge such that I is α-stable and let

$$m := \min_{J \subsetneq I} \mu_\alpha(I) - \mu_\alpha(J).$$

One can define a central charge α^I whose values are linearly independent over $\mathbb{Q}$ and such that for all $x \in Q_0$, we have $\alpha^I(x) \in [\alpha(x), \alpha(x) + m)$. Then for an interval representation $J \subsetneq I$, we have $\mu_{\alpha^I}(J) < \mu_\alpha(J) + m \leqslant \mu_\alpha(I) \leqslant \mu_{\alpha^I}(I)$. So I is also α^I-stable and moreover, by linear independence over $\mathbb{Q}$, each interval representation in $\mathcal{I}$ has a different α^I-slope. By Lemma 3.5.14, the invariant $\mathbf{T}[V; (\alpha^I)_{I \in \mathcal{I}_{\leqslant 2}}]$ is complete on $\mathcal{I}_{\leqslant 2}$-decomposable modules. ☐

It remains open whether the above result generalises to any family $\mathcal{J} \subset \mathcal{I}$ of interval representations. The main bottleneck for this generalisation is the existence for each $I \in \mathcal{J}$ of a central charge along which I is stable. This last problem only depends on the shapes of the supports of interval representations in $\mathcal{J}$. Moreover, the complete family A of central charges from the proof of Proposition 3.5.13 has the same cardinality as $\mathcal{I}_{\leqslant 2}$. It would be interesting to prove a stronger version of Proposition 3.5.13 with a smaller family A of central charges. Theorem 3.6.4 in the next Section gives a partial result in this direction in the case of ladder persistence modules.

3.6 HN types of interval-decomposable ladder persistence modules

Fix an integer $\ell \geqslant 1$. Here we will be concerned with certain equalised representations of the following *ladder* quiver $Q = (s, t : Q_1 \to Q_0)$ of length ℓ:

$$\begin{tikzcd}
x_0^+ \arrow[r, "e_1^+"] \arrow[d, swap, "e_0"] & x_1^+ \arrow[r, "e_2^+"] \arrow[d, swap, "e_1"] & \cdots \arrow[r, "e_{\ell-1}^+"] & x_{\ell-1}^+ \arrow[r, "e_\ell^+"] \arrow[d, swap, "e_{\ell-1}"] & x_\ell^+ \arrow[d, swap, "e_\ell"] \\
x_0^- \arrow[r, swap, "e_1^-"] & x_1^- \arrow[r, swap, "e_2^-"] & \cdots \arrow[r, swap, "e_{\ell-1}^-"] & x_{\ell-1}^- \arrow[r, swap, "e_\ell^-"] & x_\ell^-
\end{tikzcd}$$

Equalised representations of such quivers are sometimes called *ladder persistence modules*; these are precisely 2-parameter persistence modules of shape $L = (\ell, 1)$, but it is customary to represent them with vertical arrows pointing down instead of up. In particular, they arise when studying the morphisms of ordinary persistence modules. The authors of [**47**] established that $\mathbf{Rep}_{\mathrm{eq}}(Q)$ is of finite type for $\ell \leqslant 3$ and completely classified its indecomposable objects via Auslander-Reiten theory. In contrast, in this section, we focus on interval-decomposable ladder persistence modules. The last three authors introduced in [**70**, Section 5] the nestfree condition which guarantees the interval-decomposability of a ladder persistence module. This condition was later generalised in [**110**].

3.6.1 Interval-decomposable representations of ladders

Given $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$, we let V^+ and V^- denote its restrictions to the top and bottom rows of Q respectively. Since both $V^\pm$ are representations of the (equioriented) $\mathbb{A}_\ell$ quiver, they admit direct sum decompositions into interval modules as described in (3.11). Up to isomorphism, the indecomposable objects of $\mathbf{Rep}_{\mathrm{int}}(Q)$ have one of three possible forms:

1. Given subintervals $[a, b]$ and $[c, d]$ of $[0, \ell]$ whose endpoints satisfy $c \leqslant a \leqslant d \leqslant b$, let $\mathbf{R}^{a,b}_{c,d}$ be the representation V for which V^+ is the interval module $\mathbf{I}[a, b]$ while V^- is the interval module $\mathbf{I}[c, d]$, and all vertical maps are $1'$s whenever possible and 0 otherwise:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \\
\mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

2. Given an interval $[a, b] \subset [0, \ell]$, let $\mathbf{R}^{a,b}$ be the ladder representation V for which V^+ equals $\mathbf{I}[a, b]$ and V^- is trivial, with all vertical maps necessarily being 0:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[d] & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

3. Finally, given an interval $[c, d] \subset [0, \ell]$, let $\mathbf{R}_{c,d}$ be representation V for which V^+ is trivial while V^- is $\mathbf{I}[c, d]$, so once again all vertical maps are 0:

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \\
& & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

It will be convenient in the sequel to unify notation by adopting the convention

$$\mathbf{R}^{a,b}_{\infty,\infty} := \mathbf{R}^{a,b} \qquad \text{and} \qquad \mathbf{R}^{\infty,\infty}_{c,d} := \mathbf{R}_{c,d}, \tag{3.14}$$

so that for every $V \in \mathbf{Rep}_{\text{int}}(Q)$ there exist multiplicities $r^{a,b}_{c,d} \in \mathbb{N}$ satisfying:

$$V \simeq \bigoplus_{a,b,c,d} \left(\mathbf{R}^{a,b}_{c,d}\right)^{r^{a,b}_{c,d}}, \tag{3.15}$$

with (a, b, c, d) ranging over admissible subsets of $\{0, 1, \ldots, \ell, \infty\}^4$ as per (3.14). Our goal throughout the remainder of this section is to quantify the extent to which these multiplicities can be recovered from HN types. The first result in this direction is negative: for $\ell \geqslant 4$, there is no complete central charge $\alpha : Q_0 \to \mathbb{R}$.

Proposition 3.6.1. *If Q has length $\ell \geqslant 4$, then for every central charge $\alpha : Q_0 \to \mathbb{R}$ there exists at least one indecomposable in $\mathbf{Rep}_{\text{int}}(Q)$ which is not α-stable.*

Proof. It suffices to consider $\ell = 4$ which embeds into all larger ladder quivers:

$$\begin{tikzcd}
x_0^+ \arrow[r] \arrow[d] & x_1^+ \arrow[r] \arrow[d] & x_2^+ \arrow[r] \arrow[d] & x_3^+ \arrow[r] \arrow[d] & x_4^+ \arrow[d] \\
x_0^- \arrow[r] & x_1^- \arrow[r] & x_2^- \arrow[r] & x_3^- \arrow[r] & x_4^-
\end{tikzcd}$$

Let $\alpha_j^\pm$ be the valued assigned by a given central charge α to the vertex $x_j^\pm$ for $0 \leqslant j \leqslant 4$. Assume, for the purposes of contradiction, that all the indecomposables in $\mathbf{Rep}_{\text{int}}(Q)$ are α-stable. Consider the inequalities of slopes arising from the following inclusions of indecomposables:

$$\begin{array}{rcll}
3(\alpha_1^- + \alpha_1^+) & < & 2(\alpha_0^- + \alpha_1^- + \alpha_1^+) & \qquad \mathbf{R}^{1,1}_{1,1} \subset \mathbf{R}^{1,1}_{0,1} \\
3(\alpha_0^- + \alpha_1^-) & < & (\alpha_0^- + \alpha_1^- + \alpha_1^+ + \alpha_2^+ + \alpha_3^+ + \alpha_4^+) & \qquad \mathbf{R}_{0,1} \subset \mathbf{R}^{1,4}_{0,1} \\
4(\alpha_3^- + \alpha_3^+ + \alpha_4^+) & < & 3(\alpha_2^- + \alpha_3^- + \alpha_3^+ + \alpha_4^+) & \qquad \mathbf{R}^{3,4}_{3,3} \subset \mathbf{R}^{3,4}_{2,3} \\
4(\alpha_2^- + \alpha_2^+ + \alpha_3^+) & < & 3(\alpha_1^- + \alpha_2^- + \alpha_2^+ + \alpha_3^+) & \qquad \mathbf{R}^{2,3}_{2,2} \subset \mathbf{R}^{2,3}_{1,2} \\
4\alpha_4^+ & < & (\alpha_2^- + \alpha_3^- + \alpha_3^+ + \alpha_4^+) & \qquad \mathbf{R}^{4,4} \subset \mathbf{R}^{3,4}_{2,3}
\end{array}$$

Labelling these five inequalities as $(a), (b), \ldots, (e)$ respectively, we obtain

$$4(a) + 4(b) + (c) + 4(d) + (e) \text{ holds if and only if } 0 < 0,$$

which provides the desired contradiction. □

When combined with Lemmas 3.2.11(1) and 3.5.11, the calculation in Proposition 3.6.1 yields the following consequence.

Corollary 3.6.2. *For $\ell \geqslant 4$, there is no central charge $\alpha : Q_0 \to \mathbb{R}$ that is complete on $\mathbf{Rep}_{\mathrm{int}}(Q)$.*

Before remedying this defect by considering a larger collection of central charges, we describe another negative result which highlights the necessity of restricting our focus to interval-decomposable representations.

Proposition 3.6.3. *Let $\mathbb{F}$ be any field other than $\mathbb{Z}/2\mathbb{Z}$, and consider a ladder quiver Q of length $\ell \geqslant 4$. There exist non-isomorphic representations $V \not\simeq W$ in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ such that $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$ for every central charge $\alpha : Q_0 \to \mathbb{R}$.*

Proof. For each scalar λ in $\mathbb{F}$, consider $V(\lambda) \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{F} \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"'] & \mathbb{F}^2 \arrow[r, "\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix}1&\lambda\\1&1\end{smallmatrix}\right]"'] & \mathbb{F}^2 \arrow[r, "{[1\ 0]}"] \arrow[d, "{[1\ \lambda]}"'] & \mathbb{F} \arrow[d] \\
\mathbb{F} \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] & \mathbb{F}^2 \arrow[r, "\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]"'] & \mathbb{F}^2 \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

(See also [**26**, Definition 4(2)].) Since we have assumed $\mathbb{F} \neq \mathbb{Z}/2\mathbb{Z}$, there exist distinct nonzero scalars $\lambda \neq \mu$ in $\mathbb{F}$; we set $V := V(\lambda)$ and $W := V(\mu)$. Any isomorphism $\phi : V \to W$ must restrict to automorphisms $\phi^{\pm} : V^{\pm} \to W^{\pm}$ of the top and bottom rows. By [**70**, Theorem 4.4], both $\phi^{\pm}$ are forced to be trivial in the chosen bases, i.e., represented by the identity matrix on each vertex. It is readily checked (along the middle vertical edge) that this collection of identity matrices does not constitute a morphism $V \to W$ in $\mathbf{Rep}(Q)$, whence V and W are non-isomorphic in the subcategory $\mathbf{Rep}_{\mathrm{eq}}(Q)$. On the other hand, there is an evident bijection between the subrepresentations of V and those of W that preserves dimension vectors, and hence, α-semistability for any choice of central charge $\alpha : Q_0 \to \mathbb{R}$. This yields $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$, as claimed. ☐

3.6.2 A complete set of central charges

Fix a ladder quiver Q of length $\ell \geqslant 1$. Our goal in this subsection is to prove a stronger version of Proposition 3.5.13 in the case of ladder persistence modules. More precisely, by performing a more refined analysis, we are able to reduce the size of the collection of central charges and to make the relationship between multiplicities of interval representations and HN types more explicit.

Theorem 3.6.4. *There exists a finite collection of central charges A whose cardinality grows cubically with ℓ and for which the HN type $\mathbf{T}[\bullet; A]$ is complete on $\mathbf{Rep}_{\mathrm{int}}(Q)$.*

We let $\mathcal{I}$ denote the set of (isomorphism classes of) indecomposable objects which were defined at the beginning of this Section. We will also adopt the convention (3.14) for describing these indecomposables as $\mathbf{R}^{a,b}_{c,d}$ for certain a, b, c, d in the set $[\ell]_\infty := \{0, 1, \ldots, \ell, \infty\}$.

Remark 3.6.5. Here are all the possible strict inclusions $I' \subsetneq I$ in $\mathcal{I}$:

(1) $\mathbf{R}^{a',b}_{\infty,\infty} \subsetneq \mathbf{R}^{a,b}_{\infty,\infty}$ if $a < a'$

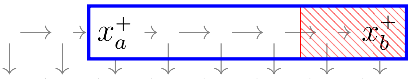
(2) $\mathbf{R}^{\infty,\infty}_{c',d} \subsetneq \mathbf{R}^{\infty,\infty}_{c,d}$ if $c < c'$

(3) $\mathbf{R}^{a',b}_{\infty,\infty} \subsetneq \mathbf{R}^{a,b}_{c,d}$ if $d < a'$

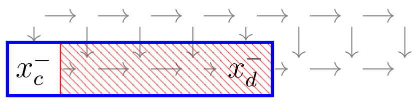
(4) $\mathbf{R}^{\infty,\infty}_{c',d} \subsetneq \mathbf{R}^{a,b}_{c,d}$ if $c \leqslant c'$

(5) $\mathbf{R}^{a',b}_{c',d} \subsetneq \mathbf{R}^{a,b}_{c,d}$ if $a \leqslant a'$ and $c \leqslant c'$.

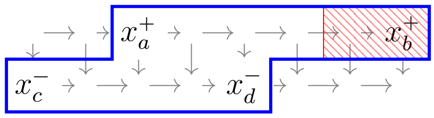
We recall from Section 3.5.3 that the support of an indecomposable $I \in \mathcal{I}$ is the subset $\operatorname{supp}(I) \subset Q_0$ consisting of all vertices x for which the vector space I_x is nontrivial. Since the indecomposables of $\mathbf{Rep}_{\mathrm{int}}(Q)$ can be uniquely identified by their supports, we may illustrate these five containments $I' \subsetneq I$ from Remark 3.6.5 by depicting the supports of I' (shaded red) and I (outlined blue).



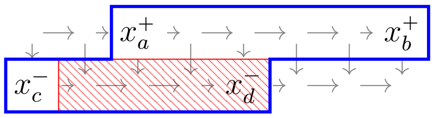
(1)



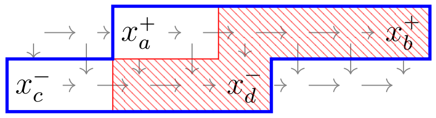
(2)



(3)



(4)



(5)

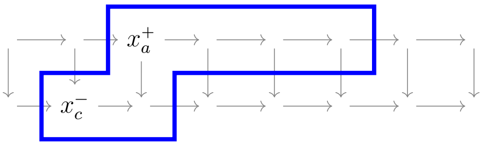
Definition 3.6.6. For every integer $k \geqslant 0$ and subset $S \subset Q_0$, define the set

$$\mathcal{I}^k_S := \left\{ I \in \mathcal{I} \mid \sum_{x \in Q_0} \dim I_x = k \text{ and } \min(I) = S \right\}. \qquad \triangle$$

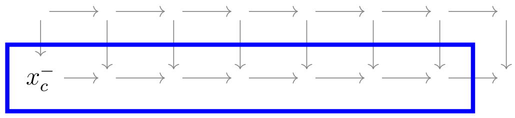
There is a partition $\mathcal{I} = \coprod_{k,S} \mathcal{I}^k_S$ where k ranges over $\mathbb{Z}_{>0}$ while S ranges over subsets of Q_0. The constituent part $\mathcal{I}^k_S$ is empty unless S has one of two possible forms:

- $\{x\}$ for any vertex $x = x_a^+$ or $x = x_c^-$,
- $\{x_a^+, x_c^-\}$ with $0 \leqslant c < a \leqslant \ell$.

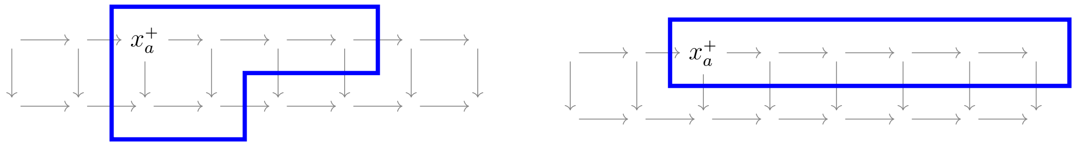
Here are the supports of certain $I \in \mathcal{I}^6_S$ for nontrivial choices of $S \subset Q_0$:



$S = \{x_a^+, x_c^-\}$



$S = \{x_c^-\}$



$S = \{x_a^+\}$

Lemma 3.6.7. *Given any integer $k > 0$ and subset $S \subset Q_0$ of vertices, the set of dimension vectors $\left\{\underline{\dim}_I \mid I \in \mathcal{I}_S^k\right\}$ is linearly independent in the vector space of maps $Q_0 \to \mathbb{R}$.*

Proof. It suffices to consider the nontrivial cases where $|\mathcal{I}_S^k| > 1$, which can only occur when either $S = \{x_i^+\}$ for some i or when $S = \{x_i^+, x_j^-\}$ for some $i > j$. We claim that every indecomposable $\mathbf{R}_{c,d}^{a,b} \in \mathcal{I}_S^k$ is uniquely determined by b. To verify this claim, note first that if $S = \{x_i^+\}$ for some $i \in \{0, 1 \ldots, \ell\}$ then we must have $a = i$ and $c \in \{i, \infty\}$, with $c = \infty$ occurring if and only if $b = i + k - 1$. Otherwise, if $S = \{x_i^+, x_j^-\}$ for $i > j$, then $a = i$ and $c = j$. In both cases, d is determined by the formula

$$k = (b - a) + (d - c) + 2.$$

Thus, the map $\iota : \mathcal{I}_S^k \to [\ell]_\infty$ sending each $\mathbf{R}_{c,d}^{a,b}$ to b is injective, as claimed above. Now consider an $\mathbb{R}$-linear combination of the form

$$v := \sum_{I \in \mathcal{I}_S^k} r_I \, \underline{\dim}_I \, .$$

A brief examination of the supports of indecomposables lying in $\mathcal{I}_S^k$ reveals that the value $v(x_j^+)$ depends only on those I which satisfy $\iota(I) \geqslant j$. Thus, the real numbers r_I may be recovered by descending induction on $\iota(I)$. □

Let us fix a collection of irrational numbers $\{t_{p,q}\}$ indexed by pairs of integers $0 \leqslant q < p$ so that the following inequalities hold:

$$\frac{q}{p - q} < t_{p,q} < \frac{q + 1}{p - (q + 1)}. \tag{3.16}$$

Definition 3.6.8. For each integer $k > 0$ and subset $S \subset Q_0$ for which $\mathcal{I}_S^k$ is nonempty, define the central charge $\alpha_S^k : Q_0 \to \mathbb{R}$ as

$$\alpha_S^k := \begin{cases} \delta_x & \text{if } S = \{x\}. \\ \delta_{x_a^+} + t_{k,a-c} \cdot \delta_{x_c^-} & \text{if } S = \{x_a^+, x_c^-\}. \end{cases}$$

Here δ_x is the skyscraper central charge at vertex x (from Definition 3.3.1) while the $t_{\bullet,\bullet}$ are irrational numbers chosen to satisfy (3.16). △

We now compute the HN type of every indecomposable $I \in \mathcal{I}$ along these central charges. The calculation below makes essential use of spanning subrepresentations $\langle I_S \rangle$, which were introduced in Definition 3.3.2.

Lemma 3.6.9. *Fix any $k > 0$ and $S \subset Q_0$ for which $\mathcal{I}_S^k$ is nonempty, and define*

$$\lambda_S^k := \begin{cases} 1/k & \text{if } |S| = 1, \\ (1 + t_{k,a-c})/k & \text{if } S = \{x_a^+, x_c^-\}. \end{cases}$$

The following assertions are equivalent for every indecomposable $I \in \mathcal{I}_{S'}^{k'}$:

1. *The α_S^k-slope of I equals λ_S^k.*
2. *Both $k = k'$ and $S \subset \operatorname{supp}(I)$ hold.*

If either assertion is true, then we also have that I is α_S^k-semistable if and only if $S = S'$.

Proof. We abbreviate α_S^k to α and λ_S^k to λ. Recall from Definition 3.6.6 that either $S = \{x\}$ or $S = \{x_a^+, x_c^-\}$ with $a > c$. In the first case, the desired equivalence follows from computing

$$\mu_\alpha(I) = \begin{cases} 1/k' & \text{if } x \in \operatorname{supp}(I) \\ 0 & \text{otherwise} \end{cases}. \tag{3.17}$$

In the case $S = \{x_a^+, x_c^-\}$ with $a > c$, we similarly have:

$$\mu_\alpha(I) = \begin{cases} 1/k' & \text{if } S \cap \operatorname{supp}(I) = \{x_a^+\}, \\ t_{k,a-c}/k' & \text{if } S \cap \operatorname{supp}(I) = \{x_c^-\}, \\ (1 + t_{k,a-c})/k' & \text{if } S \subset \operatorname{supp}(I), \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

Since $t_{k,a-c}$ is irrational by assumption, $\mu_\alpha(I)$ determines $S \cap \operatorname{supp}(I)$, as desired.

We now assume that I is α-semistable (in addition to satisfying $\mu_\alpha(I) = \lambda$), and seek to show that $S = S'$. Since $\langle I_S \rangle$ is a subrepresentation of I with $\mu_\alpha(\langle I_S \rangle) \geqslant \lambda$, we must have $I = \langle I_S \rangle$. This forces I to lie in $\mathcal{I}_S^k$, thus ensuring $S' = S$ as desired. Conversely, if $S = S'$, then I lies in $\mathcal{I}_S^k$. By Lemma 3.2.5, it suffices to show that indecomposable subrepresentations $I' \subsetneq I$ have smaller α-slope than I. It is readily checked that any subrepresentation $I' \subset I$ must be equalised, and by Lemma 3.5.11, I' is also interval-decomposable. Therefore, it suffices to consider only those I' which have been listed in Remark 3.6.5. Of these, the I' with nonzero α-slope all have the form $\langle I_{S'} \rangle$ for $S' \subsetneq S$. When $|S| = 1$, there are no such I' to consider; and when $S = \{x_a^+, x_c^-\}$ for $a > c$, the only relevant I' are $\langle I_{x_a^+} \rangle$ and $\langle I_{x_c^-} \rangle$. Thus, the desired semistability of I reduces to verifying two inequalities:

$$\frac{1}{|\{x \geqslant x_a^+\} \cap \operatorname{supp}(I)|} \leqslant \frac{1 + t_{k,a-c}}{k} \geqslant \frac{t_{k,a-c}}{|\{x \geqslant x_c^-\} \cap \operatorname{supp}(I)|}.$$

Both hold by the constraints imposed on $t_{k,a-c}$ in (3.16). ☐

We recall from (3.5) that given a central charge $\alpha : Q_0 \to \mathbb{R}$, the HN type $\mathbf{T}[V; \alpha]$ of any $V \in \mathbf{Rep}_{\text{int}}(Q)$ may be viewed as a function $\mathbb{R} \to \mathbb{N}^{Q_0}$. The following result describes the values of the function associated to each central charge α_S^k from Definition 3.6.8 at the corresponding slope λ_S^k from Lemma 3.6.9.

Proposition 3.6.10. *Consider any (k, S) such that $\mathcal{I}_S^k$ is nonempty, and let $I \in \mathcal{I}$. Then for $\alpha := \alpha_S^k$ and $\lambda := \lambda_S^k$, we have*

$$\mathbf{T}[I;\alpha](\lambda) = \begin{cases} \underline{\dim}_{\langle I_S \rangle} & \textit{if } \langle I_S \rangle \in \mathcal{I}_S^k \\ 0 & \textit{otherwise.} \end{cases} \tag{3.19}$$

Proof. Letting $I^\bullet$ be the HN filtration $\mathbf{HN}_\alpha^\bullet(I)$, we consider two possible cases:

Case 1: $I = \langle I_S \rangle$. By Lemma 3.6.9, we have the equivalence $I \in \mathcal{I}_S^k \Leftrightarrow \mu_\alpha(I) = \lambda$. Hence, if $\langle I_S \rangle$ is α-semistable, then $\mathbf{T}[V;\alpha]$ equals $\underline{\dim}_{\langle I_S \rangle}$ whenever $\langle I_S \rangle$ lies in $\mathcal{I}_S^k$, and is trivial otherwise. Assume now that $\langle I_S \rangle$ is α-unstable. In particular, $S \cap \operatorname{supp}(I)$ is of the form $\{x_a^+, x_c^-\}$ because otherwise $\langle I_S \rangle$ would not have any proper subrepresentations of strictly positive α-slope. In this case, we claim that $I^\bullet$ is the 3-step filtration $0 \subsetneq \langle I_s \rangle \subsetneq \langle I_S \rangle \subsetneq I$ for some $s \in S$. Indeed, $\langle I_{x_a^+} \rangle$ and $\langle I_{x_c^-} \rangle$ are the only proper subrepresentations of I of nonzero slope and they are α-semistable. Then from Definition 3.6.8, the slopes $\mu_\alpha(I^i/I^{i-1})$ lie in either $\mathbb{Q}$ or in $t_{k,a-c} \cdot \mathbb{Q}$, neither of which contain $\lambda = (1 + t_{k,a-c})/k$, whence $\mathbf{T}[I;\alpha](\lambda) = 0$.

Case 2: $I \neq \langle I_S \rangle$. We claim that $\mathbf{T}[I;\alpha](\lambda) = \mathbf{T}[\langle I_S \rangle;\alpha](\lambda)$. Indeed, by the argument that established Proposition 3.3.3, we have $I^{\ell-1} = \langle I_S \rangle$ and $\mu_\alpha(I^\ell/I^{\ell-1}) = 0 < \lambda$ where ℓ is the length of $I^\bullet$. Finally, the result follows from the first case. ☐

To complete the proof of Theorem 3.6.4, recall the collection of central charges $A = \{\alpha_S^k\}$ from Definition 3.6.8, and consider some $V \in \mathbf{Rep}_{\text{int}}(Q)$ with decomposition $V \simeq \bigoplus_{I \in \mathcal{I}} I^{r_I}$. We show that the multiplicities $(r_I)_{I \in \mathcal{I}}$ can be recovered from $\mathbf{T}[V;A]$ by descending strong induction on the partial order $\subset$ from Remark 3.6.5. To this end, note that for indecomposables $I \in \mathcal{I}_S^k$ and $J \in \mathcal{I}$, we have $\langle J_S \rangle = I$ if and only if I is a subrepresentation of J. By combining Propositions 3.2.8 and 3.6.10, we get

$$\mathbf{T}[V;\alpha_S^k](\lambda_S^k) = \sum_{I \in \mathcal{I}_S^k} r_I^+ \cdot \underline{\dim}_I,$$

where $r_I^+ := \sum_J r_J$ for J ranging over representations in $\mathcal{I}$ which satisfy $J \supseteq I$. By Lemma 3.6.7, the integer r_I^+ can be obtained from $\mathbf{T}[V;A]$ for each $I \in \mathcal{I}_S^k$. Finally, assume by the inductive hypothesis that r_J is known for all $J \supsetneq I$ in $\mathcal{I}$. Now r_I can be recovered from $\mathbf{T}[V;A]$ as the difference $r_I = r_I^+ - \sum_{J \supsetneq I} r_J$.

Finally, it follows from Definition 3.6.8 that the set of complete central charges A has cardinality

$$c(\ell) := \frac{2\ell^3 + 3\ell^2 + 13\ell + 12}{6},$$

and hence grows cubically with the length ℓ of the underlying ladder quiver.

Remark 3.6.11. One can choose different irrational numbers $t_{p,q}$ in (3.16) to generate uncountably many other such complete sets of the same cardinality $c(\ell)$. It is not clear to us (for large ℓ) whether *all* complete sets of central charges for interval-decomposable representations of ladder quivers can be obtained in this fashion; nor is it clear whether $c(\ell)$ is the minimal size for such a complete set.

Chapter 4

Harder-Narasimhan filtrations of persistence modules: metric stability

Introduction

Motivation

Persistent homology is one of the main tools in Topological Data Analysis (TDA). It provides a compact, computable and easy-to-read summary measuring how certain geometric and topological features of the data persist across multiple scales. The underlying mathematical structure here are sequences of vector spaces connected by linear maps, usually called *persistence modules*. These modules arise naturally when one computes the homology of a scale-indexed filtration of topological spaces built on an underlying dataset. Most persistent homology techniques output a discrete summary, called a *barcode*, which is complete in the sense that it characterises the given persistence module up to isomorphism [46, 117]. This barcode is known to be stable under certain natural perturbations of the persistence modules or of the underlying dataset [34].

For some applications, scale is not the only parameter of interest. For instance, adding density as a second parameter makes persistent homology techniques more robust to outliers in the data. It therefore becomes necessary to study more general notions of persistence modules, which are given by functors from arbitrary posets to the category of vector spaces. However, Carlsson and Zomorodian showed [31] that there are no discrete complete invariants for such persistence modules with two or more parameters. They also introduced the *rank invariant*, a (necessarily incomplete) discrete invariant which associates to a multiparameter persistence module the rank of all its maps. Many other discrete invariants have since been proposed in the TDA literature. Examples of those include invariants using ideas from homological and commutative algebra [82, 64, 14], sheaf theory [73, 90] and lattice theory [75, 21, 91, 5]. More details about invariants in multiparameter persistence, including the recent approach using relative homological algebra, can be found in the review articles [20] and [13].

In Chapter 3, we studied a family of invariants based on ideas from Geometric Invariant Theory applied to the representation theory of posets. Indeed, for every pointwise finite-dimensional (p.f.d.) persistence module over a finite poset, one can define *Harder-Narasimhan*(HN) *filtrations* [78, 65]. These filtrations are parameterised by a *stability condition* which can be described by two real-valued functions on the elements of the poset. The collection of dimensions of the vector spaces appearing in an HN filtration are called the *HN type* of the persistence module for the given stability condition. A particular elegant choice of stability conditions is based on the Kronecker delta function that picks out individual elements of the poset. These can be used to build the *skyscraper invariant* which is discrete, computable in polynomial time [36] and strictly stronger than the rank invariant (see Chapter 3).

Stability

Multiparameter persistence modules indexed over $\mathbb{Z}^n$ or $\mathbb{R}^n$ are equipped with the *interleaving distance* [34, 85] which makes standard TDA pipelines robust to small variations in the input data. When introducing an invariant of persistence modules, it is desirable to replicate the stability property enjoyed by barcodes of one parameter persistence modules. Explicitly, one seeks a metric on the output space which makes the invariant Lipschitz continuous with respect to the interleaving distance. For instance, the rank invariant of a $\mathbb{R}^n$-indexed persistence module is a functor from $(\mathbb{R}^n)^{\text{opp}} \times \mathbb{R}^n$ to $(\mathbb{Z}_{\geqslant 0} \cup \{\infty\})^{\text{opp}}$ and the *erosion distance* between such functors makes the rank invariant stable [101, 102, 76]. A similar statement holds for the generalised rank invariant introduced by Kim and Mémoli [37]. Moreover, rank exact decompositions, which are an example of the signed decompositions considered in [14, 13], are stable when equipped with a generalisation of the bottleneck distance [22].

In this work, we establish a stability result for the skyscraper invariant. In order to do so, we extend the definition of this invariant from the case of p.f.d. persistence modules over finite posets to the setting of finitely presentable persistence modules over $\mathbb{Z}^n$ or $\mathbb{R}^n$. Using the erosion distance, we define a distance, $d_{\mathbf{HN}}$, on the output of this version of the skyscraper invariant so that the rank invariant factors continuously through the skyscraper invariant

$$\begin{tikzcd}
\text{interleaving distance} \arrow[rr, "\text{rank invariant}"] \arrow[dr, "\text{skyscraper invariant}"'] & & \text{erosion distance} \\
& d_{\mathbf{HN}} \arrow[ur, "\text{top stratum}"'] &
\end{tikzcd}$$

The Harder-Narasimhan filtrations were introduced as stratifications of finite-rank vector bundles over a complex curve and were used to compute the cohomology groups of certain moduli spaces [63]. They were later adapted to the case of finite quiver representations and hence of persistence modules over a finite poset [78, 65]. HN filtrations have also been studied in categories which are not necessarily abelian [**23**, **2**]. However, when applied to the case of finitely presentable $\mathbb{R}^n$-persistence modules, these frameworks make restrictive assumptions about the stability condition.

Outline and main results

Given a poset Q, the *dimension vector* of a pointwise finite-dimensional Q-persistence module $V \in \mathrm{Vect}^Q$ is the function $\underline{\dim}_V : x \in Q \mapsto \dim V_x \in \mathbb{Z}$. A *stability condition* Z over Q is an additive function sending the dimension vector of a nonzero Q-persistence module to a complex number with positive real part[1]. The *Z-slope* of a nonzero Q-persistence module $V \in \mathrm{Vect}^Q$ is the real number

$$\mu_Z(V) := \frac{\mathfrak{Im}(Z(\underline{\dim}_V))}{\mathfrak{Re}(Z(\underline{\dim}_V))}$$

where $\mathfrak{Re}$ and $\mathfrak{Im}$ denote the real and imaginary parts. Moreover, V is said to be *Z-semistable* if the Z-slopes of its nonzero submodules do not exceed $\mu_Z(V)$.

Assume that Q is finite. There is a unique finite-length filtration:

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell = V \tag{4.1}$$

called the *HN filtration* [**63**] of V along Z such that the successive quotients $S^i := V^i/V^{i-1}$ are Z-semistable and have strictly decreasing Z-slopes. The *HN type* of V along Z is the discrete invariant given by the dimension vectors of the quotients $S^1, \ldots, S^\ell$. Given $q \in Q$, a particular choice of stability condition over Q is given by $Z_q : \underline{\dim}_V \in \mathbb{Z}^Q \mapsto \sqrt{-1}\dim V_q + \sum_{q' \in Q} \dim V_{q'} \in \mathbb{C}$. Broadly speaking, the HN type of V along Z_q can be interpreted as sorting the features alive at q by increasing size of their remaining lifespan (see Remark 4.2.5). The *skyscraper invariant* of V amalgamates for every $q \in Q$ the HN type of V along Z_q. We compute the skyscraper invariant for spread-decomposable modules (Proposition 4.2.8) and show that it is complete on the class of persistence modules over a 3×3 grid whose inner maps are all surjective (Proposition 4.2.11).

We now assume that Q is infinite and we work in the category $\mathrm{Pers}_{\mathrm{fp}}(Q)$ of finitely presentable Q-persistence modules. Depending on the choice of stability condition Z over Q, HN filtrations along Z may not exist for every $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ (Example 4.3.2). In Section 4.3, we introduce two notions of *discretisable* stability condition which guarantee the existence of HN filtrations. The first notion (Definition 4.3.3), inspired by the construction of the skyscraper invariant over a finite poset, requires that $\mathfrak{Im}Z$ is a nonnegative finite linear combination of the *evaluation forms* $(\delta_q : \underline{\dim}_V \mapsto \dim V_q)_{q \in Q}$. For the second notion (Definition 4.3.6), defined for $Q = \mathbb{R}^n$, we ask that Z is induced by a complex-valued step function. More precisely, let $f : \mathbb{R}^n \to \mathbb{C}$ be an integrable function with a positive real part.

---

[1] Z is a group homomorphism from the Grothendieck group of (an exact abelian subcategory of) Vect^Q to the additive group of complex numbers sending the classes of nonzero modules to the right open half-plane.

We say that f is a *step function* if it is constant on the parts of a locally finite partition of Q into hyperrectangles. The stability condition Z_f induced by f is given for $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ by

$$Z_f \colon \underline{\dim}_V \mapsto \int_Q f\, \underline{\dim}_V\,.$$

Theorem 4.3.11 gives a more general version of the following statement:

Theorem (G). *Let Z be a stability condition over an upper semilattice Q. If either*

(i) $\mathfrak{Im} Z$ is a nonnegative finite linear combination of the functions $(\underline{\dim}_V \mapsto \dim V_q)_{q\in Q}$; or,

(ii) $Q = \mathbb{R}^n$ and $Z = Z_f$ is induced by an integrable step function $f \colon \mathbb{R}^n \to \mathbb{C}$ such that $\mathfrak{Re} f$ is positive and $\mathfrak{Im} f$ is compactly supported,

then HN filtrations along Z of finitely presentable Q-persistence modules exist, are unique and can be computed over a finite poset.

For the rest of this summary, Z is a stability condition over $Q = \mathbb{R}^n$ which satisfies the hypotheses of Theorem (G). Let $x \in \mathbb{R}^n$ and let T_x be the translation $y \in \mathbb{R}^n \mapsto x + y \in \mathbb{R}^n$. Given $V \in \mathrm{Pers}_{\mathrm{fp}}(\mathbb{R}^n)$, temporarily denote by $V^{x,\bullet}$ the HN filtration of V along the stability condition $T^x_* Z \colon \underline{\dim}_V \mapsto Z(\underline{\dim}_{V\circ T_x})$. Let $\theta \in \mathbb{R}$ and let $i(\theta)$ be the largest index such that either $i(\theta) = 0$ or $\mu_{T^x_* Z}(V^{x,i(\theta)}/V^{x,i(\theta)-1}) \geqslant \theta$. We define (Definition 4.3.17) the function $s^\theta_{Z,V}$ which sends $x \leqslant y \in \mathbb{R}^n$ to the integer[2]

$$s^\theta_{Z,V}(x, y) := \mathrm{rank}(V^{x,i(\theta)})_{x\leqslant y}.$$

The function $\theta \in \mathbb{R} \mapsto s^\theta_{Z,V}$ is nonincreasing and for θ small enough $s^\theta_{Z,V}$ coincide with the rank invariant – see Equation (4.19) and Proposition 4.3.18. For this reason, we call the family $(s^\theta_{Z,V})_\theta$, the *HN filtered rank invariant* of V along Z.

In Section 4.4, we establish a stability result for HN filtered rank invariants. Let $V, W \in \mathrm{Pers}_{\mathrm{fp}}(\mathbb{R}^n)$ and denote by $\vec{\varepsilon}$ the vector $(\varepsilon, \dots, \varepsilon) \in \mathbb{R}^n$. The *interleaving distance* $d_I(V, W)$ between V and W [85] is the infimum of all $\varepsilon \geqslant 0$ such that there exist maps of persistence modules

$$\phi_\bullet \colon V \to W \circ T_{\vec{\varepsilon}} \qquad \text{and} \qquad \psi_\bullet \colon W \to V \circ T_{\vec{\varepsilon}}$$

whose compositions $\psi_{\bullet+\vec{\varepsilon}} \circ \phi_\bullet$ and $\phi_{\bullet+\vec{\varepsilon}} \circ \psi_\bullet$ are given by the internal maps of V and W. We prove that for each $\theta \in \mathbb{R}$, the functions $s^\theta_{Z,V}$ and $s^\theta_{Z,W}$ can be seen as functors from $(\mathbb{R}^n)^{\mathrm{opp}} \times \mathbb{R}^n$ to $(\mathbb{Z}_{\geqslant 0} \cup \{\infty\})^{\mathrm{opp}}$. The *erosion distance* $d_E(s, r)$ between two such functors r and s [101] is the infimum of all $\varepsilon \geqslant 0$ such that for each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$s(x - \vec{\varepsilon}, y + \vec{\varepsilon}) \leqslant r(x, y) \qquad \text{and} \qquad r(x - \vec{\varepsilon}, y + \vec{\varepsilon}) \leqslant s(x, y).$$

The following result, stated in full detail in Theorem 4.4.7, shows that HN filtered rank invariants are stable:

---

[2] The following expression corresponds to Equation (4.20) with $V^{x,i(\theta)} := \langle V, T^x_* Z\rangle^\theta$.

Theorem (H). *For every pair of finitely presentable $\mathbb{R}^n$-persistence modules V and W, we have*

$$\sup_{\theta\in\mathbb{R}} d_E(s^\theta_{Z,V}, s^\theta_{Z,W}) \leqslant d_I(V,W).$$

The above theorem may be used to produce a stability result for the HN filtered version of the persistence landscapes [25, 111] – see Corollary 4.4.13.

Given a spread (see Subsection 4.1.1) $I \subset \mathbb{R}^n$, an important example is the module $\mathbb{F}_I \in \mathrm{Vect}^{\mathbb{R}^n}$ whose dimension vector is the indicator function of I and whose inner maps between nonzero spaces are identities. In Section 4.5, we compute the HN filtered rank invariants of direct sums of such modules along certain stability conditions (Proposition 4.5.1) and give examples illustrating Theorem (H).

Example (A). Consider the $\mathbb{R}^2$-persistence modules $V := \mathbb{F}_{[0,2)\times[0,1)} \oplus \mathbb{F}_{[0,1)\times[0,2)}$ and $W := \mathbb{F}_{[0,2)^2\setminus[1,2)^2} \oplus \mathbb{F}_{[0,1)^2}$. Let Z be the stability condition over $\mathbb{R}^2$ given by $\underline{\dim}_V \mapsto \sqrt{-1}\dim V_{(0,0)} + \int_{x\in\mathbb{R}^2}\min(1, e^{2-\|x\|_\infty})\dim V_x \mathrm{d}x$. The modules V and W have the same rank invariant but we have $0.5 \geqslant d_I(V,W) \geqslant d_E(s^1_{Z,V}, s^1_{Z,W}) > 0.2$ – see Figure 4.1 and Example 4.5.2(2).

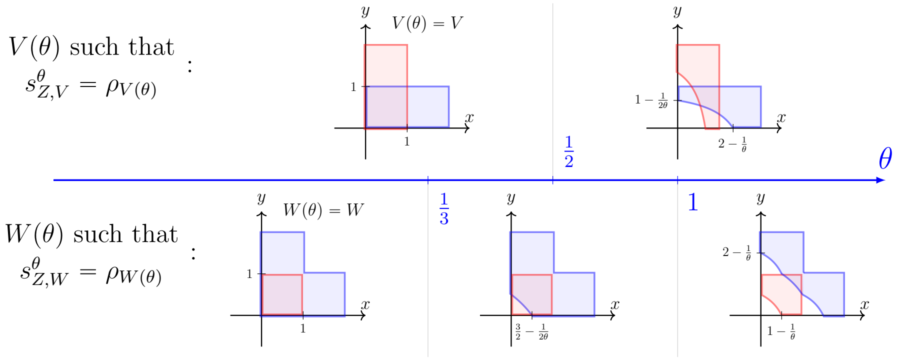


Figure 4.1: HN filtered rank invariants $(s^\theta_{Z,V})_\theta$ and $(s^\theta_{Z,W})_\theta$ from Example (A). In the top (*resp.* bottom) row, we depict, for $\theta \in \mathbb{R}$, two areas B^θ and R^θ such that $s^\theta_{Z,V}$ (*resp.* $s^\theta_{Z,W}$) coincides with the rank invariant of $\mathbb{F}_{B^\theta} \oplus \mathbb{F}_{R^\theta}$.

Finally, in Section 4.6, we prove in the setting of Theorem (G)(ii) that if $\mathfrak{Im} f$ is nonnegative, then the HN filtered rank invariants along Z_f are semialgebraically constructible functions in the sense of [92]. This result ensures that the HN filtered rank invariants over $\mathbb{R}^n$ can be computed deterministically by calling Cheng's algorithm [36] finitely many times.

4.1 Preliminaries

We first introduce discretisable persistence modules over general posets [18, 32] and over finite products of totally ordered sets [86]. After defining the notion of discretisable functions, we briefly go through the concepts of stability conditions, slope stability, HN filtrations and HN types in abelian categories [23]. Finally, we define the pullback of a stability condition.

4.1.1 Persistence modules

Given a field $\mathbb{F}$ and a poset P, a *P-persistence module* is a functor $V : P \to \mathrm{Vect}_{\mathbb{F}}$ from P to the category of $\mathbb{F}$-vector spaces and linear maps. Let V be a P-persistence module, the *dimension vector* of V is the function $\underline{\dim}_V : P \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ which sends $p \in P$ to $\dim V_p$. The *support* of V is the subset $\{p \in P \mid V_p \neq 0\}$ of P. The *rank invariant* of a persistence module V is the function $\rho_V : P \times P \to \mathbb{Z} \cup \{+\infty\}$ given by

$$\rho_V(p, p') := \begin{cases} \operatorname{rank} V_{p \leqslant p'} & \text{if } p \leqslant p' \text{ in } P \\ +\infty & \text{otherwise} \end{cases} \tag{4.2}$$

We fix once and for all the field $\mathbb{F}$ and will henceforth omit it from the notation. We further ask P-persistence modules to be pointwise finite-dimensional – meaning that the dimension vectors are integer-valued functions. The P-persistence modules define an abelian category $\mathrm{Pers}(P)$ where morphisms are natural transformations.

A subset I of P is called a *spread* (or sometimes an interval) if it is:

- <u>connected</u>: for all $u, v \in I$, there is a sequence $(u = u_0, u_1, \dots, u_\ell = v)$ in I such that u_i and u_{i+1} are comparable for each i; and
- <u>convex</u>: I contains the set $\{p \in P \mid u \leqslant p \leqslant v\}$ for each $u \leqslant v$ in I.

For instance, given $p \in P$, the subset $\langle p \rangle := \{p' \in P \mid p' \geqslant p\}$ is a spread. Given a spread I of P, the *spread module* supported on I is the indecomposable module $\mathbb{F}_I$ whose spaces $(\mathbb{F}_I)_p$ are $\mathbb{F}$ if $p \in I$ and 0 otherwise and whose maps between nonzero spaces are identities. A P-persistence module is called *spread-decomposable* if it is isomorphic to a direct sum of spread modules.

4.1.2 Discretisable persistence modules

Let P and Q be two posets. A *map of posets* $f \colon P \to Q$ is an increasing map from P to Q. The *pullback* of a Q-persistence module V by f is the P-persistence module

$$f^* V := V \circ f$$

where $\circ$ is the composition between functors. Let U be a P-persistence module and let $\mathrm{Lan}_f(U)$ denote the left Kan extension (see [89, Chapter X]) of U along f. The *pushforward* of U by f is the Q-persistence module

$$f_* U := \mathrm{Lan}_f(U).$$

More concretely, for $q \in Q$, we have $(f_* U)_q := \varinjlim_{p \mid f(p) \leqslant q} U_p$. The functors $f^* \colon \mathrm{Pers}(Q) \to \mathrm{Pers}(P)$ and $f_* \colon \mathrm{Pers}(P) \to \mathrm{Pers}(Q)$ are respectively called the restriction (sometimes pullback) and induction (sometimes extension [13] or pushforward) functors. They form [18, Lemma 2.15] an adjoint pair (f_*, f^*).

Definition 4.1.1. A Q-persistence module V is said to be *f-discretisable* if it is isomorphic in $\mathrm{Pers}(Q)$ to the pushforward by f of some P-persistence module U

$$V \simeq f_* U.$$

In that case, we say that f is *adapted* to V. A Q-persistence module is said to be *discretisable* (sometimes tame [32]) if it is f-discretisable for some map of posets f from a finite poset P to Q. We denote by $\mathrm{Pers}_{\mathrm{fp}}(Q) \subset \mathrm{Pers}(Q)$ the full subcategory of discretisable Q-persistence modules. △

Observe that the induction functor of a map of (not necessarily finite) posets $g\colon Q \to Q'$ sends discretisable Q-persistence modules to discretisable Q'-persistence modules.

Example 4.1.2.
- When Q is finite, we have $\mathrm{Pers}_{\mathrm{fp}}(Q) = \mathrm{Pers}(Q)$
- Discretisable $\mathbb{R}$-persistence modules are the ones whose barcode consists of a finite number of intervals of the form $[a, b)$ with $(a, b) \in \mathbb{R} \times \mathbb{R} \cup \{\infty\}$

A Q-persistence module is *free and finitely generated* if it decomposes as a direct sum of a finite number of spread modules of the form $\mathbb{F}_{\langle q \rangle}$ with $q \in Q$. A Q-persistence module V is said to be *finitely presentable* if there is a map f between two free and finitely generated Q-persistence modules such that $V \simeq \operatorname{coker} f$. Our interest in finitely presentable persistence modules is justified by the fact that a Q-persistence module lies in $\mathrm{Pers}_{\mathrm{fp}}(Q)$ if and only if it is finitely presentable [32, 10.3(2)].

A *refinement* of $f\colon P \to Q$ is a map of posets $f'\colon P' \to Q$ such that f factors through f'. Namely, there is a map of posets g fitting into the commutative diagram

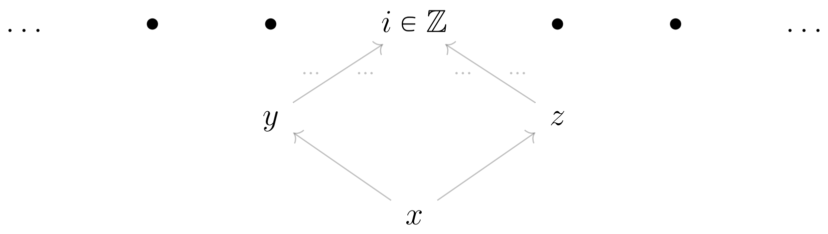
$$\begin{tikzcd} P \arrow[r, dotted, "g"] \arrow[d, "f"'] & P' \arrow[dl, "f'"] \\ Q & \end{tikzcd}$$

A map of posets $f\colon P \to Q$ is an *inclusion of posets* if whenever $f(p) \leqslant f(p')$ in Q we also have $p \leqslant p'$ in P. A subset S of the poset $(Q, \leqslant)$ induces an inclusion of posets $(S, \leqslant_{|S\times S}) \hookrightarrow (Q, \leqslant)$ from S equipped with the partial order induced by Q. If f is an inclusion of posets, then the unit $\mathrm{id}_{\mathrm{Pers}(P)} \Rightarrow f^* f_*$ is a natural isomorphism [18, Lemma 2.19].

Remark 4.1.3. Let $f\colon P \to Q$ be a map of posets and let $S \subset Q$ be a subset containing $f(P)$, the inclusion of posets induced by $S \subset Q$ is a refinement of f. Hence, a pair of maps of posets $f^1\colon P^1 \to Q$ and $f^2\colon P^2 \to Q$ has a common refinement induced by the subset $f^1(P^1) \cup f^2(P^2) \subset Q$.

The existence of common refinements implies that the subcategory $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is additive. Moreover, $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is closed under finite colimits [32, 8.2(5)]; however, in general, it is not closed under finite limits:

Example 4.1.4. Let Q be the set $\mathbb{Z} \sqcup \{x, y, z\}$ equipped with the partial order induced by the relations $z \geqslant x \leqslant y$ and for $i \in \mathbb{Z}$, $y \leqslant i \geqslant z$.



Let $\eta \colon \mathbb{F}_{\langle y \rangle} \oplus \mathbb{F}_{\langle z \rangle} \to \mathbb{F}_{\langle x \rangle}$ be the map of Q-persistence modules such that $\eta_y = \mathrm{id}_{\mathbb{F}}$ and $\eta_z = \mathrm{id}_{\mathbb{F}}$. The domain and codomain of η are discretisable because they are free and finitely generated. But the kernel of η, which is isomorphic to $\mathbb{F}_{\mathbb{Z}}$, is not finitely generated and hence not discretisable. In this example, the full subcategory $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is not closed under kernels which are finite limits[3].

In order to ensure that $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is an exact abelian subcategory, we will make assumptions about the poset Q. Chachólski, Jin and Tombari proved that if Q is an upper semilattice, then $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is closed under finite limits [**32**, 8.4]. A more general statement can be found in [**33**, Corollary 4.8].

We recall that a poset Q is an *upper semilattice* if every pair of elements $(x, y) \in Q^2$ admits a least upper bound $x \vee y$. A map (*resp.* inclusion) of posets $f \colon P \to Q$ is called a *map (resp. inclusion) of semilattices* if P and Q are upper semilattices, and for all $(x, y) \in P^2$, we have $f(x \vee y) = f(x) \vee f(y)$.

We now assume that $f \colon P \hookrightarrow Q$ is an inclusion of semilattices with P finite. For $q \in Q$, the finite subset $\{p \in P \mid f(p) \leqslant q\} \subset P$ is either empty or has a maximum in P:

$$\lfloor q \rfloor_f := \max \{p \in P \mid f(p) \leqslant q\}.$$

The function $q \in Q \mapsto \lfloor q \rfloor_f \in P \cup \{-\infty\}$ is called the *f-floor function*. By definition, the pushforward of some $U \in \mathrm{Pers}(P)$ by f can be written pointwise as $(f_* U)_q = U_{\lfloor q \rfloor_f}$ with the convention $U_{-\infty} = 0$. Given $p \in P \cup \{-\infty\}$, we denote by

$$\mathrm{cub}_f(p) := \{q \in Q \mid \lfloor q \rfloor_f = p\}$$

the fiber of p by $\lfloor \ \rfloor_f$.

Lemma 4.1.5. *Assume that Q is an upper semilattice, then,*

(i) $\mathrm{Pers}_{fp}(Q)$ *is an exact abelian subcategory of* $\mathrm{Pers}(Q)$ *[**32**, last paragraph of §8],*

(ii) any map of semilattices $f \colon P \to Q$ from a finite upper semilattice P has an exact induction functor $f_* \colon$ $\mathrm{Pers}(P) \to \mathrm{Pers}(Q)$ *[**32**, 10.5], and*

*(iii) a discretisable Q-persistence module can be discretised by an inclusion of semilattices with a finite domain [**32**, 10.3(1)].*

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[3] This example was suggested by Luis Scoccola.

Proof. (i) We have seen that $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is full, additive, closed under finite colimits (and hence under cokernels) and closed under finite limits (and hence under kernels). This implies [**114**, Lemma 1.6.2(2)] that $\mathrm{Pers}_{\mathrm{fp}}(Q)$ is an exact abelian subcategory of $\mathrm{Pers}(Q)$.

(ii) For each $q \in Q$, the functor $(f_* \ )_q \colon U \in \mathrm{Pers}(P) \mapsto (f_* U)_q = U_{\lfloor q \rfloor_f} \in \mathrm{Vect}$ is exact. Since exactness in $\mathrm{Pers}(P)$ and $\mathrm{Pers}(Q)$ are determined pointwise, f_* is exact.

(iii) Let $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$. There is a map of posets $f \colon P \to Q$ with P finite which is adapted to V. The subset $f(P)$ generates a finite sub-semilattice of Q

$$\langle f(P) \rangle := f(P) \cup \left\{ \bigvee S \mid S \subset f(P) \text{ finite and nonempty} \right\}.$$

By remark 4.1.3, the inclusion of semilattices $\langle f(P) \rangle \subset Q$ is adapted to V. □

4.1.3 Grid functions

In this subsection, the poset Q is the product of n totally ordered posets equipped with the product partial order. This setting contains our two main cases of interest where Q is either $\mathbb{Z}^n$ or $\mathbb{R}^n$. We will give a more concrete definition of discretisable persistence modules.

An *interval* in a totally ordered poset T is a nonempty subset $I \subset T$ that has a supremum $\sup I$ and an infimum $\inf I$ in $T \cup \{\pm\infty\}$ and such that for $z \in I$, we have $\inf I < z < \sup I \implies z \in I$. The *length* of an interval I of $\mathbb{R}$ or $\mathbb{Z}$ is the number $\operatorname{len} I := \sup I - \inf I$.

A *cube* of $Q = Q_1 \times \cdots \times Q_n$ is a subset $I_1 \times \cdots \times I_n \subset Q$ such that each I_i is an interval of Q_i. A *face* of a cube $I_1 \times \cdots \times I_n \subset Q$ is another cube $J_1 \times \cdots \times J_n \subset Q$ such that for each $i \in \{1, \ldots, n\}$, the interval J_i is either I_i, $\{\inf I_i\}$ or $\{\sup I_i\}$.

Definition 4.1.6 ([**86**, Section 2.5])**.** A *grid function* $G \colon P \hookrightarrow Q$ is an embedding of a cube $P = P_1 \times \cdots \times P_n$ of $\mathbb{Z}^n$ into Q which is defined coordinate-wise by strictly increasing functions $G_i \colon P_i \hookrightarrow Q_i$ and such that $\lim_{t \to \pm\infty} G_i(t) = \pm\infty$ – if those limits exist. When P is finite, G is said to be a finite grid function. △

Observe that by Remark 4.1.3, a Q-persistence module is discretisable if and only if it is discretisable by a finite grid function.

The interest of working with a finite grid function $G \colon P \hookrightarrow Q$ is that the floor function $\lfloor \ \rfloor_G$ can be computed coordinate-wise and that for every $p \in P$ the fibers $\mathrm{cub}_G(p)$ are cubes of the form $\prod_{i=1}^n [a_i, b_i)$ where $(a_i, b_i) \in Q_i \times (Q_i \cup \{\infty\})$.

Notation 4.1.7. Given a grid function $G \colon P \hookrightarrow Q$, let $\mathcal{C}(G)$ denote the partition delimited by G. Namely, $\mathcal{C}(G) := \{\mathrm{cub}_G(x) \mid x \in P\}$. When Q is a cube of $\mathbb{R}^n$ or $\mathbb{Z}^n$, we also define $\mathcal{C}_{\mathrm{b}}(G) := \{c \in \mathcal{C}(G) \mid c \text{ is bounded}\}$ and $\mathcal{C}_{\mathrm{bf}}(G) := \{c' \subset Q \mid \exists c \in \mathcal{C}_{\mathrm{b}}(G), c' \text{ is a face of } c\}$.

Most of the above notions can easily be adapted to the case where $G \colon P \hookrightarrow Q$ is an infinite grid function. Indeed, for $q \in Q$, the subset $\{p \in P \mid G(p) \leqslant q\}$ is a cube of $\mathbb{Z}^n$ which is bounded above in each direction so that its maximum $\lfloor q \rfloor_G$ is well-defined. Hence, the argument of Lemma 4.1.5(ii) still holds for G and the induction functor G_* is exact.

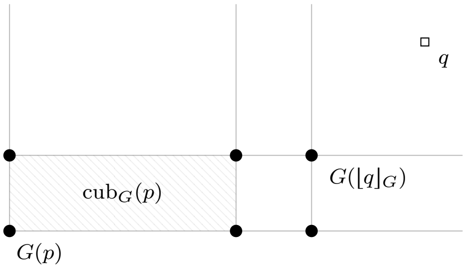


Figure 4.2: The black disks are the image of a grid function $G\colon \{0,1,2\} \times \{0,1\} \hookrightarrow \mathbb{R}^2$, the dashed area is a fibre of $p := (0,0)$ under $\lfloor\ \rfloor_G$ and the square node is some $q \in Q$

Let $G\colon P \hookrightarrow Q$ be a grid function, a *refinement* of G is a grid function $G'\colon P' \hookrightarrow Q$ such that there exists a grid function $\tau\colon P \hookrightarrow P'$ satisfying $G = G' \circ \tau$. A refinement G' of G such that $\operatorname{Im} G = \operatorname{Im} G' \cap c$ for some cube c of Q is called an *extension* of G. If G' is an extension of a grid function $G\colon P \hookrightarrow \mathbb{R}^n$, then $\mathcal{C}_{\mathrm{b}}(G) \subset \mathcal{C}_{\mathrm{b}}(G')$. Among the two refinements of G depicted in Figure 4.3, only the right-hand side one is an extension.

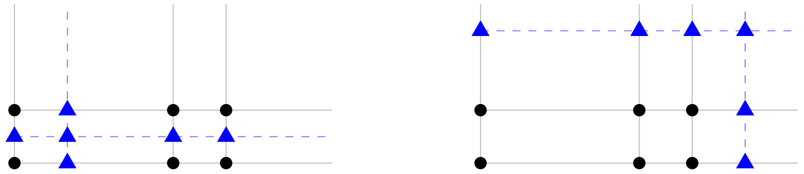


Figure 4.3: These two figures depict two refinements G' of the finite grid function G from Figure 4.2. The set $\operatorname{Im} G$ is given by the black disks and the set $\operatorname{Im} G' \backslash \operatorname{Im} G$ is represented by the blue triangles. The full gray lines delimit $\mathcal{C}(G)$ and $\mathcal{C}(G')$ is delimited by all the lines.

There is a more general notion of tameness for $\mathbb{R}^n$-persistence modules [**93**] which defines, under some auxiliary assumptions, an exact abelian subcategory of $\mathrm{Pers}(\mathbb{R}^n)$ [**113**]. This article will only consider the notion of discretisable persistence modules defined in Definition 4.1.1.

4.1.4 Discretisable functions

We now introduce the necessary notation and vocabulary to study the dimension vectors of discretisable persistence modules. Fix an upper semilattice Q and a set Y. Let $f\colon P \hookrightarrow Q$ be an inclusion of upper semilattices.

The *pullback* of a function $b\colon Q \to Y$ by f is the function $f^*b := b \circ f$. The *pushforward* of a function $a\colon P \to Y$ by f is the function $f_*a\colon Q \to Y$ defined for $q \in Q$ by

$$f_*a(q) := a(\lfloor q \rfloor_f)$$

with the convention $a(-\infty) = 0$. A function $b\colon Q \to Y$ is said to be *discretisable* by f if there exist $a\colon P \to Q$ such that $b = f_*a$. Let $\mathcal{F}(Q, Y)$ denote the set of functions $b\colon Q \to Y$ which are discretisable by some inclusion of semilattices with a finite domain.

Note that for $U \in \mathrm{Pers}(P)$ and $V \in \mathrm{Pers}(Q)$, we have

$$\underline{\dim}_{f^*V} = f^* \underline{\dim}_V \qquad \text{and} \qquad \underline{\dim}_{f_*U} = f_* \underline{\dim}_U \,. \tag{4.3}$$

In particular, if $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$, then $\underline{\dim}_V$ lies in $\mathcal{F}(Q, \mathbb{Z}_{\geqslant 0})$. We denote by $\mathcal{F}(Q, \mathbb{R})^*$ the set of $\mathbb{R}$-linear forms on $\mathcal{F}(Q, \mathbb{R})$ and we say that $L \in \mathcal{F}(Q, \mathbb{R})^*$ is strictly positive (*resp.* nonnegative) if $L(\mathcal{F}(Q, \mathbb{R}_{>0})) = \mathbb{R}_{>0}$ (*resp.* $L(\mathcal{F}(Q, \mathbb{R}_{\geqslant 0})) = \mathbb{R}_{\geqslant 0}$).

Given $S \subset T$ be two sets, the *indicator function* of S (in T) is the function $\mathbb{1}_S \colon T \to \{0, 1\}$ sending $t \in T$ to

$$\mathbb{1}_S(t) := \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Example 4.1.8. Given an element $q \in Q$, the function $\mathbb{1}_{\langle q \rangle} = \underline{\dim}_{\mathbb{F}_{\langle q \rangle}}$ lies in $\mathcal{F}(Q, \mathbb{Z}_{\geqslant 0})$. Indeed, if $f \colon \{0\} \hookrightarrow Q$ sends 0 to q, we have $\mathbb{1}_{\langle q \rangle} = f_* \mathbb{1}_{\{0\}}$.

Assume now that Q is a product of n totally ordered posets. A *step function* on Q is a function on Q which is discretisable by a (potentially infinite) grid function. A discretisable function $b \in \mathcal{F}(Q, \mathbb{R})$ is a step function since it can be discretised by a finite grid function. However, the floor function $\lfloor\ \rfloor \colon x \in \mathbb{R} \mapsto \lfloor x \rfloor := \max\{p \in \mathbb{Z} \mid p \leqslant x\} \in \mathbb{Z}$ is a step function but does not lie in $\mathcal{F}(\mathbb{R}, \mathbb{Z})$.

Example 4.1.9.

- when $Q = \mathbb{Z}^n$, every function on Q is a step function.
- when $Q = \mathbb{R}^n$, a function $b \colon Q \to \mathbb{R}$ is discretisable if and only if it is a finite linear combination of the family

$$\left\{ \mathbb{1}_{\prod_{i=1}^n [a_i, b_i)} \mid \forall i \in \{1, \ldots, n\}, a_i < b_i \in \mathbb{R} \cup \{+\infty\} \right\}. \tag{4.5}$$

  Moreover, a function $b \colon Q \to \mathbb{R}$ is a step function if and only if it is a countable linear combination of family (4.5).

Let Q be either $\mathbb{R}$ or $\mathbb{Z}$, let I be an interval of Q, and let $b \colon I \to Q$ be a function. We say that b is *affine* if there are constants $b_0, b_1 \in Q$ such that for all $x \in I$, we have $b(x) = b_1 x + b_0$. Let C be a cube of $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $\int_C$ denote either the Lebesgue integral if C is a cube of $\mathbb{R}^n$ or the countable sum over C if C is a cube of $\mathbb{Z}^n$. A function $b \colon C \to \mathbb{C}$ is said to be *integrable* if the integral $\int_C |b|$ is well defined and finite.

4.1.5 HN filtrations in abelian categories

Let $\mathcal{A}$ be an abelian category. The *Grothendieck group* of $\mathcal{A}$ is the abelian group $K(\mathcal{A})$ generated by the isomorphism classes $[V]$ of objects subject to the relations $[V'] = [V] + [V'']$ for every short exact sequence $0 \to V \to V' \to V'' \to 0$ in $\mathcal{A}$ [115, Definition II.6.1.1]. A *stability condition* Z over $\mathcal{A}$ is a group morphism between the Grothendieck group of $\mathcal{A}$ and $(\mathbb{C}, +)$ which sends classes of nonzero objects to the right open half-plane $\{z \in \mathbb{C} \mid \mathfrak{Re}\, z > 0\}$. The *$Z$-slope* of a nonzero object $V \in \mathcal{A}$ is the real number $\mu_Z(V) :=$

$\frac{\Im\mathfrak{m} Z([V])}{\Re\mathfrak{e} Z([V])}$. Moreover, V is said to be *Z-stable* if all its proper subobjects have a strictly smaller slope:

$$0 \subsetneq W \subsetneq V \quad \Longrightarrow \quad \mu_Z(W) < \mu_Z(V).$$

If instead $0 \subsetneq W \subsetneq V \implies \mu_Z(W) \leqslant \mu_Z(V)$, then V is said to be *Z-semistable.*

Definition 4.1.10 ([63, 65, 105])**.** Given an object V in $\mathcal{A}$, a HN filtration of V along Z is a sequence of subobjects $0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell = V$ of finite length $\ell \in \mathbb{Z}_{\geqslant 0}$ such that

- for each $0 \leqslant j < \ell$, the quotient V^{j+1}/V^j is Z-semistable, and
- $\mu_Z(V^1/V^0) > \mu_Z(V^2/V^1) > \cdots > \mu_Z(V^\ell/V^{\ell-1})$. △

Example 4.1.11. Let P be the poset $\{0 < 1\}$ and let $\mathcal{A} := \mathrm{Pers}(P)$. For every short exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathrm{Pers}(P)$, we have $\underline{\dim}_V = \underline{\dim}_{V'} + \underline{\dim}_{V''}$. As a consequence, the assignment $V \in \mathrm{Pers}(P) \mapsto i\dim V_1 + \dim V_0 + \dim V_1 \in \mathbb{C}$ induces a stability condition Z_1 over $\mathrm{Pers}(P)$. The Z_1-slope of a nonzero $V \in \mathrm{Pers}(P)$ is $\mu_{Z_1}(V) = \frac{\dim V_1}{\dim V_0 + \dim V_1}$ and $0 \subsetneq (\mathbb{F}_{\{1\}})^{\dim V_1} \subsetneq V$ is a HN filtration of V along Z_1.

Proposition 4.1.12. *Let $\mathcal{A}$ be an abelian category and let Z be a stability condition over $\mathcal{A}$.*

(1) [23, paragraph above Proposition 2.4] If an object V of $\mathcal{A}$ has a HN filtration $V^\bullet$ along Z, then $V^\bullet$ is unique.

(2) [23, Proposition 2.4] If the following two conditions are satisfied

- *there are no infinite sequences of subobjects $\cdots \subset V_{j+1} \subset V_j \subset \cdots \subset V_2 \subset V_1$ in $\mathcal{A}$ with $\mu_Z(V_{j+1}) > \mu_Z(V_j)$ for all j,*
- *there are no infinite sequences of quotients $V_1 \twoheadrightarrow V_2 \twoheadrightarrow \cdots \twoheadrightarrow V_j \twoheadrightarrow V_{j+1} \twoheadrightarrow \dots$ in $\mathcal{A}$ with $\mu_Z(V_j) > \mu_Z(V_{j+1})$ for all j,*

then every nonzero object in $\mathcal{A}$ has a HN filtration along Z.

A detailed proof of Proposition 4.1.12(1) can be found in [61, Theorem 4.2] (the proof in [61] is written in the setting of modular lattices, which include the lattice of subobjects of a nonzero object V of $\mathcal{A}$).

Notation 4.1.13. Let $0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell = V$ be the HN filtration of $V \in \mathcal{A}$ along Z. For $\theta \in \mathbb{R}$ and $0 \leqslant i \leqslant \ell$, we write $\langle V, Z\rangle^\theta := V^i$ whenever $\mu_Z(\frac{V^i}{V^{i-1}}) \geqslant \theta > \mu_Z(\frac{V^{i+1}}{V^i})$ with the conventions $\mu_Z(\frac{V^0}{V^{-1}}) = +\infty$ and $\mu_Z(\frac{V^{\ell+1}}{V^\ell}) = -\infty$.

Let $\mathrm{Filtr}(\mathcal{A})$ be the category of $\mathbb{R}^{\mathrm{opp}}$-indexed filtrations of objects of $\mathcal{A}$. Given a stability condition Z, we refer to the assignment $\langle -, Z\rangle \colon V \in \mathcal{A} \mapsto (\langle V, Z\rangle^\theta)_{\theta \in \mathbb{R}^{\mathrm{opp}}} \in \mathrm{Filtr}(\mathcal{A})$ as the *HN functor* along Z. Indeed, it has been observed in many settings that under some finiteness assumptions on Z and $\mathcal{A}$, the assignment $\langle -, Z\rangle$ is well-defined and functorial – see [49], [65, Theorem 2.8] and [2, 3.1].

Definition 4.1.14 ([**103**, Definition 2.6])**.** Let Q be a poset and let $\mathcal{A}$ be an exact abelian subcategory of $\operatorname{Pers}(Q)$. Assume that an object $V \in \mathcal{A}$ has a HN filtration along Z. Then its *HN type* $\mathbf{T}[V;Z]$ along Z is the data of the dimension vectors appearing in its HN filtration. Using Notation 4.1.13, we write it as a family $(\mathbf{T}[V;Z]^\theta)_{\theta\in\mathbb{R}}$ given for $\theta \in \mathbb{R}$ by

$$\mathbf{T}[V;Z]^\theta := \underline{\dim}_{\langle V,Z\rangle^\theta}\,. \qquad \triangle$$

Let Q be an upper semilattice (*resp.* a finite poset) and let $\mathcal{A} := \operatorname{Pers}_{\mathrm{fp}}(Q)$. We will see in Lemma 4.3.1 (*resp.* Equation (4.7)) that for $V \in \mathcal{A}$, the class $[V] \in K(\operatorname{Pers}_{\mathrm{fp}}(Q))$ can be represented by the dimension vector $\underline{\dim}_V$. This justifies that the definitions of HN filtrations in Equation (4.1) and in Definition 4.1.10 coincide.

4.1.6 Pullback of stability conditions

Let $F\colon \mathcal{A} \to \mathcal{A}'$ be an additive exact functor between abelian categories and let $Z\colon K(\mathcal{A}') \to \mathbb{C}$ be a stability condition over $\mathcal{A}'$. By naturality of the Grothendieck group construction [**115**, §II.6.1.5], F induces a group homomorphism $K(F)\colon K(\mathcal{A}) \to K(\mathcal{A}')$. We further assume that $K(F)$ is injective. Then, one can define the *pullback* $F^*Z\colon K(\mathcal{A}) \to \mathbb{C}$ of Z by F to be the stability condition over $\mathcal{A}$ which fits into the following commutative diagram:

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, hook, "K(F)"] \arrow[dr, dotted, "F^*Z"'] & K(\mathcal{A}') \arrow[d, "Z"] \\ & \mathbb{C} \end{tikzcd} \tag{4.6}$$

Lemma 4.1.15. *For every nonzero object U of $\mathcal{A}$, one has $\mu_{F^*Z}(U) = \mu_Z(F(U))$ and moreover,*

$$F(U) \text{ is } Z\text{-semistable} \implies U \text{ is } F^*Z\text{-semistable.}$$

Proof. The equality $\mu_{F^*Z}(U) = \mu_Z(F(U))$ follows from the commutativity of (4.6). Assume now that $F(U)$ is Z-semistable and let $0 \subsetneq U' \subsetneq U$. By exactness of F and injectivity of $K(F)$, we have $0 \subsetneq F(U') \subset F(U)$ and by semistability $\mu_{F^*Z}(U') = \mu_Z(F(U')) \leqslant \mu_Z(F(U)) = \mu_{F^*Z}(U)$. □

The converse does not always hold, but if it does, F commutes with the HN functor:

Lemma 4.1.16. *Let $F\colon \mathcal{A} \to \mathcal{A}'$ be an additive exact functor between abelian categories such that $K(F)$ is injective. Let Z be a stability condition over $\mathcal{A}'$. Assume that every nonzero $U \in \mathcal{A}$ has a HN filtration along F^*Z and satisfies:*

$$U \text{ is } F^*Z\text{-semistable} \implies F(U) \text{ is } Z\text{-semistable.}$$

Then, for every nonzero object $U \in \mathcal{A}$, $F(U)$ has a HN filtration along Z and we have:

$$\langle F(U), Z\rangle = F(\langle U, F^*Z\rangle).$$

Proof. Let $U \in \mathcal{A}\setminus\{0\}$ and let $0 = U^0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^\ell = U$ be its HN filtration along F^*Z. By definition, the modules $(U^i/U^{i-1})_{1\leqslant i\leqslant \ell}$ have decreasing F^*Z-slopes and are F^*Z-semistable. By Lemma 4.1.15, for $1 \leqslant i \leqslant \ell$, we have $\mu_Z(F(U^i/U^{i-1})) = \mu_{F^*Z}(U^i/U^{i-1})$ so the modules $(F(U^i/U^{i-1}))$ have decreasing Z-slopes. Moreover, by assumption, these modules are Z-semistable.

By exactness of F, the inclusions in the filtration $0 = F(U^0) \subsetneq F(U^1) \subsetneq \cdots \subsetneq F(U^\ell) = F(U)$ are well defined and for $1 \leqslant i \leqslant \ell$, the quotient $F(U^i)/F(U^{i-1})$ is isomorphic to $F(U^i/U^{i-1})$.

The uniqueness of HN filtrations (Proposition 4.1.12(1)) ensures that $0 = F(U^0) \subsetneq F(U^1) \subsetneq \cdots \subsetneq F(U^\ell) = F(U)$ is the HN filtration of $F(U)$ along Z. ☐

For our purposes, the most important application of the above results is the following setting:

Example 4.1.17. Let $f\colon P \hookrightarrow Q$ be an inclusion of semilattices with P finite. The induction functor f_* is exact and induces an injective group morphism from $K(\mathrm{Pers}_{\mathrm{fp}}(P))$ to $K(\mathrm{Pers}_{\mathrm{fp}}(Q))$. Indeed, Lemma 4.1.5(ii) guarantees the exactness of f_*. Moreover, by naturality of K, we have $\mathrm{id}_{K(\mathrm{Pers}_{\mathrm{fp}}(P))} = K(f^*f_*) = K(f^*)K(f_*)$ and the morphism $K(f_*)$ is injective. Lemma 4.1.16 applied to $F := f_*$ will be used in Section 4.3 to prove existence results for HN filtrations in $\mathrm{Pers}_{\mathrm{fp}}(Q)$.

4.2 HN filtrations of persistence modules over finite posets

In Subsection 4.1.5, we defined HN filtrations in the context of abelian categories. We now specialise to the case of persistence modules over a finite poset [65]. We then introduce the skyscraper invariant which relies on HN filtrations along specific choices of stability conditions (see Chapter 3). Finally, we compute the skyscraper invariant for several examples of persistence modules over finite small grid posets.

4.2.1 Definition and properties

We fix a finite poset P and consider the abelian category $\mathrm{Pers}(P)$. Since every stability condition over $\mathrm{Pers}(P)$ satisfies the assumptions of Proposition 4.1.12(2), HN filtrations always exist in $\mathrm{Pers}(P)$.

Moreover, the Grothendieck group $K(\mathrm{Pers}(P))$ is isomorphic to $\mathbb{Z}^P$ and stability conditions over $\mathrm{Pers}(P)$ can be explicitly described by two functions on P. More precisely, the family $\{[\mathbb{F}_{\{p\}}]\}_{p\in P}$ freely generates $K(\mathrm{Pers}(P))$ and the assignment $\underline{\dim}\colon V \in \mathrm{Pers}(P) \mapsto \underline{\dim}_V \in \mathbb{Z}^P$ induces a group isomorphism:

$$\underline{\dim}\colon K(\mathrm{Pers}(P)) \xrightarrow{\sim} \mathbb{Z}^P \tag{4.7}$$

(see [6, Theorem III.3.5] and [14, §3.1]). Given two functions $a\colon P \to \mathbb{R}$ and $b\colon P \to \mathbb{R}_{>0}$, let $Z_{a,b}\colon K(\mathrm{Pers}(P)) \to \mathbb{C}$ denote the stability condition given for $V \in \mathrm{Pers}(P)$ by

$$Z_{a,b}([V]) = \sum_{p \in P} (b(p) + \sqrt{-1}a(p)) \dim V_p. \tag{4.8}$$

Let Z be a stability condition over $\mathrm{Pers}(P)$ and let $a\colon P \to \mathbb{R}$ and $b\colon P \to \mathbb{R}_{>0}$ be the imaginary and the real parts of the function $p \in P \mapsto Z([\mathbb{F}_{\{p\}}]) \in \mathbb{C}$. Then, we have $Z = Z_{a,b}$ since Z and $Z_{a,b}$ coincide on the basis $\{[\mathbb{F}_{\{p\}}]\}_{p \in P}$ of $K(\mathrm{Pers}(P))$.

Remark 4.2.1. In the literature [65, 67], HN filtration are usually studied for quiver representations instead of persistence modules. We now detail how these two approaches relate.

The finite poset P can be seen as a quiver $\mathrm{Hasse}(P)$ whose vertices are the elements of P and whose edges are given by the covering relation in P. Namely, $\mathrm{Hasse}(P)$ is the directed simple graph such that given $p, p' \in P$, there is an edge from p to p' if and only if $p < p'$ and there is no $p'' \in P$ such that $p < p'' < p'$.

Let V be a quiver representation. For a path γ in the quiver, let V_γ denote the composition of the maps of V along γ. The representation V is said to be *equalised* if for every pair of paths γ, γ' with the same sources and targets, we have $V_\gamma = V_{\gamma'}$.

The P-persistence modules can be identified with the equalised finite-dimensional representations of $\mathrm{Hasse}(P)$. Let F be the inclusion functor of $\mathrm{Pers}(P)$ into the category of finite-dimensional representations of $\mathrm{Hasse}(P)$. F is exact and induces a group isomorphism at the level of Grothendieck groups. Moreover, subrepresentations of equalised representations are also equalised. So F satisfies the assumptions of Lemma 4.1.16. It is thus equivalent to compute the HN filtrations of $V \in \mathrm{Pers}(P)$ as a P-persistence module and as a representation of the quiver without relation $\mathrm{Hasse}(P)$.

One of the key tools for computing HN filtrations in small examples is the fact that HN filtrations are additive. Namely, given two P-persistence modules V and W and a stability condition Z on $\mathrm{Pers}(P)$, we have Proposition 2.2.5 an isomorphism of filtrations.

$$\langle V \oplus W, Z \rangle \simeq \langle V, Z \rangle \oplus \langle W, Z \rangle. \tag{4.9}$$

4.2.2 The skyscraper invariant over finite posets

Let P be a finite poset and let $p \in P$. Recall from (4.4) that $\mathbb{1}_{\{p\}}\colon P \to \mathbb{R}$ denotes the indicator function of $\{p\}$ in P. Given a function $b\colon P \to \mathbb{R}_{>0}$, the *skyscraper invariant* of $V \in \mathrm{Pers}(P)$ along b is the data of the HN types (see Definition 4.1.14):

$$(\mathbf{T}[V; Z_{\mathbb{1}_{\{p\}},b}])_{p \in P}. \tag{4.10}$$

Definition 4.2.2. Let $V \in \mathrm{Pers}(P)$ and let $p \in P$, the *spanning submodule* of V at p is the submodule $\mathrm{Span}_p(V)$ defined for $p' \in P$ by

$$\mathrm{Span}_p(V)_{p'} = \begin{cases} \mathrm{Im}\, V_{p \leqslant p'} & \text{if } p \leqslant p' \\ 0 & \text{otherwise.} \end{cases}$$

The *remaining lifespan* of V at p is the function $(\underline{\dim}_{\mathrm{Span}_p(V)})/(\dim V_p)$. △

Example 4.2.3. Assume that b is the constant 1 function. Let S be a spread of P and let $p \in S$. The remaining lifetime of $\mathbb{F}_S$ at p is $\mathbb{1}_{S\cap\langle p\rangle}$ and $\mu_{Z_{\mathbb{1}_{\{p\}},b}}(\mathrm{Span}_p(\mathbb{F}_S)) = \frac{1}{|S\cap\langle p\rangle|}$ where $|\;|$ denotes the cardinal of a set.

HN filtrations along $Z_{\mathbb{1}_{\{p\}},b}$ enjoy certain properties that were first proven in Proposition 3.3.3 for a specific choice of b.

Lemma 4.2.4. *Let $V \in \mathrm{Pers}(P)$, $p \in P$ and $b\colon P \to \mathbb{R}_{>0}$*

(i) for all $\theta \leqslant 0$, we have $\langle V, Z_{\mathbb{1}_{\{p\}},b}\rangle^{\theta} = V$.

(ii) for all $\theta > 0$, we have $\langle V, Z_{\mathbb{1}_{\{p\}},b}\rangle^{\theta} = \mathrm{Span}_p(\langle V, Z_{\mathbb{1}_{\{p\}},b}\rangle^{\theta})$.

(iii) there exists $\theta_{min} > 0$ such that for all $\theta \in (0, \theta_{min}]$, we have $\langle V, Z_{\mathbb{1}_{\{p\}},b}\rangle^{\theta} = \mathrm{Span}_p(V)$.

Proof. Let Z_p denote the stability condition $Z_{\mathbb{1}_{\{p\}},b}$ and let $V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^{\ell} = V$ be the HN filtration of V along Z_p.

(i) The result follows from the fact that $\mu_{Z_p}(V^{\ell}/V^{\ell-1}) \geqslant 0$.

(ii) Given $W \in \mathrm{Pers}(P)$, let $\widetilde{W} := \mathrm{Span}_p(W)$. Given $1 \leqslant i \leqslant \ell$, let S^i denote the quotient $S^i := V^i/V^{i-1}$. We show by induction on i in the interval $\{0\} \cup \big\{1 \leqslant i \leqslant \ell \mid \mu_{Z_p}(S^i) > 0\big\} \subset \mathbb{Z}$ that $V^i = \widetilde{V^i}$.

<u>Base case</u>: when $i = 0$, we have $V^0 = 0 = \widetilde{V^0}$.

<u>induction step</u>: let $i > 0$ such that $\mu_Z(S^i) > 0$ and assume that $V^{i-1} = \widetilde{V^{i-1}}$. We first show that $S^i = \widetilde{S^i}$. By definition, S^i is Z_p-semistable and $\mathfrak{Im} Z_p([\widetilde{S^i}]) = \mathfrak{Im} Z_p([S^i]) = \dim S^i_p > 0$. Hence, $0 \subsetneq \widetilde{S^i} \subset S^i$ and by semistability $\mu_{Z_p}(\widetilde{S^i}) \leqslant \mu_{Z_p}(S^i)$. But since $\mathfrak{Im} Z_p([\widetilde{S^i}]) = \mathfrak{Im} Z_p([S^i])$, we must have

$$\sum_{p' \in P} (b\, \underline{\dim}_{\widetilde{S^i}})(p') = \mathfrak{Re} Z_p([\widetilde{S^i}]) \geqslant \mathfrak{Re} Z_p([S^i]) = \sum_{p' \in P} (b\, \underline{\dim}_{S^i})(p').$$

Since $\underline{\dim}_{\widetilde{S^i}} \leqslant \underline{\dim}_{S^i}$, we obtain by positivity of b that $\underline{\dim}_{\widetilde{S^i}} = \underline{\dim}_{S^i}$, whence $S^i = \widetilde{S^i}$. As a consequence, $\underline{\dim}_{V^i} = \underline{\dim}_{\widetilde{V^{i-1}}} + \underline{\dim}_{\widetilde{S^i}}$ and V^i is supported on $\langle p\rangle$. Moreover, for $p' \geqslant p$, since $V^{i-1}_{p \leqslant p'}$ is surjective, $\dim V^i_{p'} = \operatorname{rank} V^{i-1}_{p \leqslant p'} + \operatorname{rank} S^i_{p \leqslant p'} = \operatorname{rank} V^i_{p \leqslant p'} = \dim \widetilde{V^i_{p'}}$. Finally, since $\underline{\dim}_{\widetilde{V^i}} = \underline{\dim}_{V^i}$, we conclude that $V^i = \widetilde{V^i}$.

(iii) If $V_p = 0$, then V is Z_p-semistable of null Z_p-slope. In particular, for all $\theta > 0$, we have $\langle V, Z_p\rangle^{\theta} = 0 = \mathrm{Span}_p(V)$. We now assume that $V_p \neq 0$. Let $0 < j \leqslant \ell$ be the first index such that $V^j_p = V_p$. Since $V^{j-1}_p \neq V^j_p$, the number $\theta_{\min} := \mu_{Z_p}(V^j/V^{j-1})$ is positive, and by (ii), $\langle V, Z_p\rangle^{\theta_{\min}} = V^j = \mathrm{Span}_p(V)$. Moreover, either $j = \ell$ or $\mu_Z(V^{j+1}/V^j) = 0$, so for all θ in $(0, \theta_{\min}]$, we have $\langle V, Z_p\rangle^{\theta} = V^j = \mathrm{Span}_p(V)$.

□

Remark 4.2.5. Let $p \in P$ and let Z_p be the stability condition $Z_p := Z_{\mathbb{1}_{\{p\}},b}$ (see (4.8)). Lemma 4.2.4(ii) means that one can interpret the HN type $\mathbf{T}[V; Z_p]$ of $V \in \mathrm{Pers}(P)$ as recovering information about how the features of V "alive" at p "die". More precisely, letting $0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell$ be the HN filtration of V along Z_p, the inequalities $\mu_{Z_p}(V^1/V^0) > \cdots > \mu_{Z_p}(V^\ell/V^{\ell-1})$ can be understood as having decomposed the space V_p into subspaces $(V^1/V^0)_p, \ldots, (V^\ell/V^{\ell-1})_p$ corresponding to persistence modules $V^1/V^0, \ldots, V^\ell/V^{\ell-1}$ with remaining lifespans at p (see Definition 4.2.2) of increasing size. The HN type $\mathbf{T}[V; Z_p]$ recovers these remaining lifespans at p and the skyscraper invariant amalgamates this information for every $p \in P$.

We recall that ρ denotes the rank invariant defined (4.2). Let $(s^\theta_{b,V} \colon P \times P \to \mathbb{Z}_{\geqslant 0} \cup \{+\infty\})_{\theta \in \mathbb{R}}$ be the family of functions given for $\theta \in \mathbb{R}$ and $p, p' \in P$ by

$$s^\theta_{b,V}(p, p') := \rho_{\langle V, Z_{\mathbb{1}_{\{p\}}, b}\rangle^\theta}(p, p'). \tag{4.11}$$

A consequence of Lemma 4.2.4 is the following.

Corollary 4.2.6. *Fix a function $b \colon P \to \mathbb{R}_{>0}$ and let $V \in \mathrm{Pers}(P)$.*

(i) the rank invariant of V is determined by the skyscraper invariant of V along b; and

(ii) Let $p \in P$, the HN type $\mathbf{T}[V; Z_{\mathbb{1}_{\{p\}}, b}]$ and the pair $(\mathbf{T}[\mathrm{Span}_p(V); Z_{\mathbb{1}_{\{p\}}, b}], \underline{\dim}_V)$ are determined by each other.

(iii) the skyscraper invariant of V along b and the family of functions $(s^\theta_{b,V})_\theta$ are determined by each other.

Proof. To simplify the notation, let $Z_p := Z_{\mathbb{1}_{\{p\}}, b}$ and let $(s^\theta_V)_\theta := (s^\theta_{b,V})_\theta$.

(i) Let $p \leqslant p' \in P$. By Lemma 4.2.4(iii) there exists $\theta_{\min} > 0$ such that $\langle V, Z_p\rangle^{\theta_{\min}}_p = V_p$. Moreover, by Lemma 4.2.4(ii), the map $\langle V, Z_p\rangle^{\theta_{\min}}_{p \leqslant p'}$ is surjective and we have

$$\rho_V(p, p') = \dim \mathrm{Im}\, V_{p \leqslant p'} = \mathbf{T}[V; Z_p]^{\theta_{\min}}_{p'}.$$

(ii) Let $0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^\ell$ be the HN filtration of V along Z_p and let $\widetilde{V} := \mathrm{Span}_p(V)$. By Lemma 4.2.4(iii), there exists $j \in \{0, \ldots, \ell\}$, such that $V^j = \widetilde{V}$. One can check that $0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^j$ is the HN filtration of $\widetilde{V}$ along Z_p. It follows from Lemma 4.2.4(i) and (iii) that for $\theta \in \mathbb{R}$, we have

$$\mathbf{T}[V; Z_p]^\theta = \begin{cases} \mathbf{T}[\widetilde{V}; Z_p]^\theta & \text{if } \theta > 0 \\ \underline{\dim}_V & \text{otherwise.} \end{cases}$$

(iii) Let $\theta > 0$. By Lemma 4.2.4(ii), for $p, p' \in P$ the integers $\mathbf{T}[V; Z_p]^\theta_{p'}$ and $s^\theta_V(p, p')$ coincide if $p \leqslant p'$ and are respectively 0 and $+\infty$ if $p \nleqslant p'$. Hence, the invariants $(\mathbf{T}[V; Z_p]^\theta)_{p \in P, \theta > 0}$ and $(s^\theta_V)_{\theta > 0}$ determine each other.

By Lemma 4.2.4(i), for $\theta \leqslant 0$ and $p \in P$ we have $\mathbf{T}[V; Z_p]^\theta = \underline{\dim}_V$ while $s^\theta_V = \rho_V$. But $\underline{\dim}_V$ is determined by ρ_V which, by Lemma 4.2.4(iii), is itself determined by $(s^\theta_V)_{\theta > 0}$. Hence, $(\mathbf{T}[V; Z_p]^\theta)_{p \in P, \theta \in \mathbb{R}}$ and $(s^\theta_V)_{\theta \in \mathbb{R}}$ determine each other. □

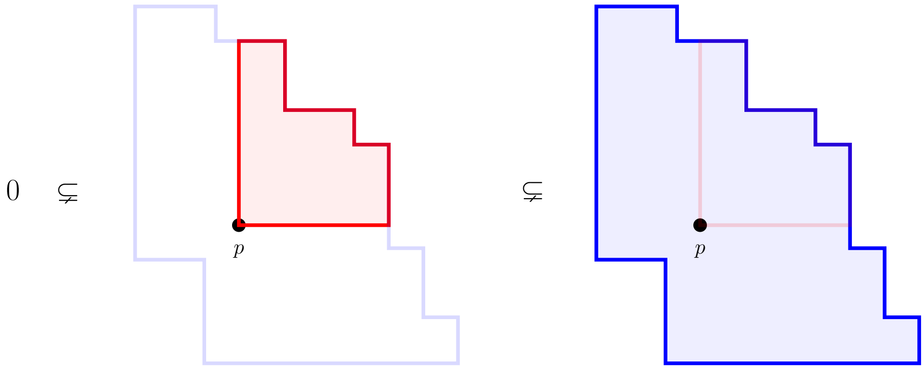
The functions $(s^\theta_{b,V} \colon P \times P \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\})_\theta$ constitute a $\mathbb{R}^{\mathrm{opp}}$-indexed filtration of the rank invariant in the sense that for $x \leqslant y \in P$, the function $s^\bullet_{b,V}(x,y) \colon \theta \in \mathbb{R} \mapsto s^\theta_V(x,y) \in \mathbb{Z}_{\geqslant 0}$ is nonincreasing and its maximum is $\rho_V(x,y)$. Given two nonzero P-persistence modules V and W, instead of comparing ρ_V and ρ_W, one could compare for every $\theta \in \mathbb{R}$, the two functions $s^\theta_{b,V}$ and $s^\theta_{b,W}$. This observation is the inspiration for the distance defined in Notation 4.4.9.

We now compute the skyscraper invariant of spread-decomposable modules. Let S be a spread of P, the spread module $\mathbb{F}_S$ is determined by its support S and will be depicted by filling S in the Hasse quiver of P.

Lemma 4.2.7. *Let $b \colon P \to \mathbb{R}_{>0}$ be a function and let S be a spread of P. The skyscraper invariant of $\mathbb{F}_S$ along b is determined for $\theta > 0$ and $p \in P$ by*

$$s^\theta_{b,\mathbb{F}_S}(p,\bullet) = \begin{cases} \mathbb{1}_{S \cap \langle p \rangle} & \text{if } 0 < \sum_{p' \in S \cap \langle p \rangle} b(p') \leqslant \frac{1}{\theta} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $p \in S$ and let Z_p denote the stability condition $Z_{\mathbb{1}_{\{p\}},b}$. We show that the HN filtration of $\mathbb{F}_S$ along Z_p is $0 \subsetneq \mathbb{F}_{S \cap \langle p \rangle} \subsetneq \mathbb{F}_S$.



Indeed, the two successive quotients $\mathbb{F}_{S \cap \langle p \rangle}$ and $\mathbb{F}_S / \mathbb{F}_{S \cap \langle p \rangle}$ have respective Z_p-slopes $\frac{1}{\sum_{p' \in S \cap \langle p \rangle} b(p')}$ and 0. Moreover, all the strict submodules of $\mathbb{F}_{S \cap \langle p \rangle}$ and $\mathbb{F}_S / \mathbb{F}_{S \cap \langle p \rangle}$ have null Z_p-slopes. Hence, these two successive quotients have decreasing Z_p-slopes and are Z_p-semistable. By Proposition 4.1.12(1), $0 \subsetneq \mathbb{F}_{S \cap \langle p \rangle} \subsetneq \mathbb{F}_S$ is the HN filtration of $\mathbb{F}_S$ along Z_p.

Assume now that $p \notin S$, then $\mathbb{F}_S$ and all its nonzero submodules have null Z_p-slope. The spread module $\mathbb{F}_S$ is hence Z_p-semistable. Now that we have computed for all $p \in P$ the HN filtration of $\mathbb{F}_S$ along Z_p, the expression of $(s^\theta_{b,\mathbb{F}_S})_\theta$ follows from the definitions (see Notation 4.1.13 and Equation (4.11)). ☐

Proposition 4.2.8. *Let $b \colon P \to \mathbb{R}_{>0}$ be a function, let $S_1, \ldots, S_k$ be spreads of P and let V denote the module $V := \bigoplus_{i=1}^k \mathbb{F}_{S_i}$. Let $\theta > 0$, for $1 \leqslant i \leqslant k$, we define the spread $S_i^{b,\theta} := \left\{ p \in S_i \mid \sum_{q \in S_i \cap \langle p \rangle} b(q) \leqslant \frac{1}{\theta} \right\}$. If $V(\theta)$ denotes the module $V(\theta) := \bigoplus_{i=1}^k \mathbb{F}_{S_i^{b,\theta}}$, we have*

$$s^\theta_{b,V} = \rho_{V(\theta)}.$$

Proof. By (4.9), one can assume that $k = 1$. Let $\theta > 0$, let S be a spread of P and let $S^{b,\theta} := \left\{ p \in S \mid \sum_{q \in S \cap \langle p \rangle} b(q) \leqslant \frac{1}{\theta} \right\}$. By Lemma 4.2.7, we have $s^\theta_{b,V} = \rho_{\mathbb{F}_{S^{b,\theta}}}$. ☐

The construction of the skyscraper invariant is also motivated by computational considerations. Firstly, Corollary 4.2.6(ii) allows to speed up the computations of the HN types appearing in the skyscraper invariant. Secondly, the complexity of the algorithm developed by Cheng [36] to compute HN filtrations along a stability condition $Z_{a,b}$ determined by $a\colon P \to \mathbb{Z}$ and $b\colon P \to \mathbb{Z}_{>0}$ depends polynomially on the absolute values of a and b. Thus, I expect the computation of HN filtrations along $Z_{a,b}$ to be less challenging when $a = \mathbb{1}_{\{p\}}$.

4.2.3 The skyscraper invariant over finite grids

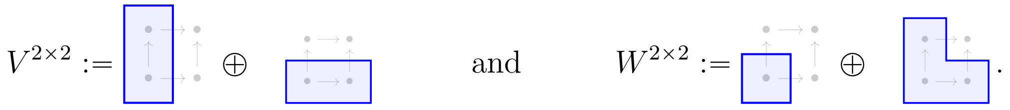
The remainder of this Subsection is devoted to computing the skyscraper invariant in three small examples which are inspired by Chapter 3. Example 4.2.9 describes two persistence modules with the same rank invariant but different skyscraper invariants. We then compute in Example 4.2.10 the skyscraper invariant of a persistence module that is not spread-decomposable. Finally, we consider the category of persistence modules with surjective inner maps over the 3×3 grid and we prove that the skyscraper invariant is complete on this category whereas the generalised rank invariant is not complete.

Let P be a finite poset. For this subsection, b will be the constant function $p \in P \mapsto 1 \in \mathbb{R}_{>0}$. To simplify the notation, for $p \in P$, we denote by Z_p the stability condition $Z_{\mathbb{1}_{\{p\}},b}$ and by $(s^\theta_V)_\theta$ the invariant $(s^\theta_{b,V})_\theta$.

Given two integers $m, n \geqslant 1$, the *grid poset* $P^{m\times n}$ is the finite cube of $\mathbb{Z}^2$

$$P^{m\times n} := \{0, 1, \dots m-1\} \times \{0, 1, \dots n-1\}$$

Example 4.2.9. The two nonisomorphic $P^{2\times 2}$-persistence modules depicted below have the same rank invariant



Observe in Figure 4.4 that for $\theta \in (\frac{1}{3}, 1]$, one has $s^\theta_{V^{2\times 2}} \neq s^\theta_{W^{2\times 2}}$ so the skyscraper invariant can distinguish $V^{2\times 2}$ and $W^{2\times 2}$.

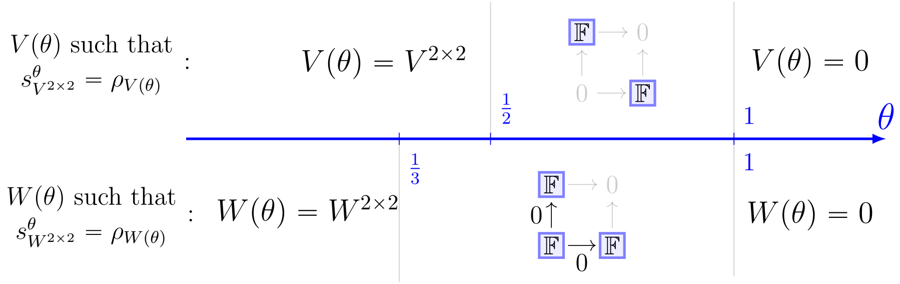


Figure 4.4: The top (*resp.* bottom) row depicts for $\theta > 0$ the module $V(\theta)$ (*resp.* $W(\theta)$) defined in Proposition 4.2.8.

We now compute the skyscraper invariant of a persistence module which is not spread-decomposable:

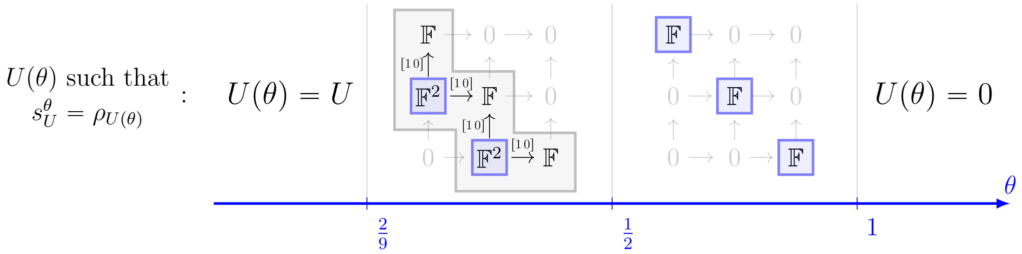
Example 4.2.10. Consider the following indecomposable $P^{3\times 3}$-persistence module:

$$U := \begin{tikzcd} \mathbb{F} \arrow[r] & 0 \arrow[r] & 0 \\ \mathbb{F}^2 \arrow[u, "{[1\ 0]}"] \arrow[r, "{[1\ 1]}"] & \mathbb{F} \arrow[u] \arrow[r] & 0 \arrow[u] \\ \mathbb{F}^2 \arrow[u, equal] \arrow[r, equal] & \mathbb{F}^2 \arrow[u, "{[1\ 1]}"] \arrow[r, "{[0\ 1]}"] & \mathbb{F} \arrow[u] \end{tikzcd} . \tag{4.12}$$

Choose $p \in P^{3\times 3}\backslash\{(0,0)\}$. The spanning submodule $\mathrm{Span}_p(V)$ is spread-decomposable. By Proposition 4.2.8, we can compute its skyscraper invariant which by Corollary 4.2.6(2) determines the HN type $\mathbf{T}[U; Z_p]$.

Assume now that $p := (0,0)$, we claim that the U is Z_p-stable and hence that its HN filtration along Z_p is the trivial one-step filtration $0 \subsetneq U$. Let $U' \subsetneq U$ be a nonzero submodule, we show that $\mu_{Z_p}(U')$ is less than $\mu_{Z_p}(U) = \frac{2}{9}$. We can assume that $\dim U'_p = 1$ because otherwise $\mu_{Z_p}(U') = 0$. Let S be the support of U'. Since the kernels of the maps $(U_{p\leqslant p'})_{p'>p}$ have pairwise null intersections, S has at least 5 elements and $\mu_{Z_p}(U') \leqslant \frac{1}{5} < \mu_{Z_p}(U)$. Hence, U is Z_p-stable.

The family $(s_U^\theta)_\theta$ is described below by depicting for every $\theta \in \mathbb{R}$ a module $U(\theta) \in \mathrm{Pers}(P^{3\times 3})$ such that $s_U^\theta = \rho_{U(\theta)}$.



Let $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$ be the full subcategory of modules $V \in \mathrm{Pers}(P^{3\times 3})$ whose inner maps $(V_{p\leqslant p'})_{p\leqslant p'\in P}$ are all surjective. For instance, the module U defined in (4.12) lies in $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$. The generalised rank invariant is not complete on $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$ (see Proposition 3.3.9(1)). However,

Proposition 4.2.11. *Let b be the constant 1 function on P and let $Z_{(0,0)}$ be the stability condition $Z_{\mathbb{1}_{\{(0,0)\}},b}$ – see (4.8). The HN type $V \mapsto \mathbf{T}[V; Z_{(0,0)}]$ and hence the skyscraper invariant along b is complete on* $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$.

Before proving Proposition 4.2.11, we use a counting argument to check that the 20 non-isomorphic indecomposable modules described in Figure 4.5 constitute an exhaustive list of all the indecomposable modules (up to isomorphism) in $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$.

Indeed, if $\mathbb{C}_{\mathrm{e,e}} \subset \mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$ denotes the additive subcategory generated by the spread modules $\left\{\mathbb{F}_{P^{3\times 3}\backslash\langle p\rangle} \mid p \in P^{3\times 3}\backslash\{(0,0)\}\right\} \cup \{\mathbb{F}_P\}$, we have [**10**, Theorem 1.7] an equivalence of categories

$$\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})/\mathbb{C}_{\mathrm{e,e}} \simeq \mathrm{Pers}(P^{2\times 2}).$$

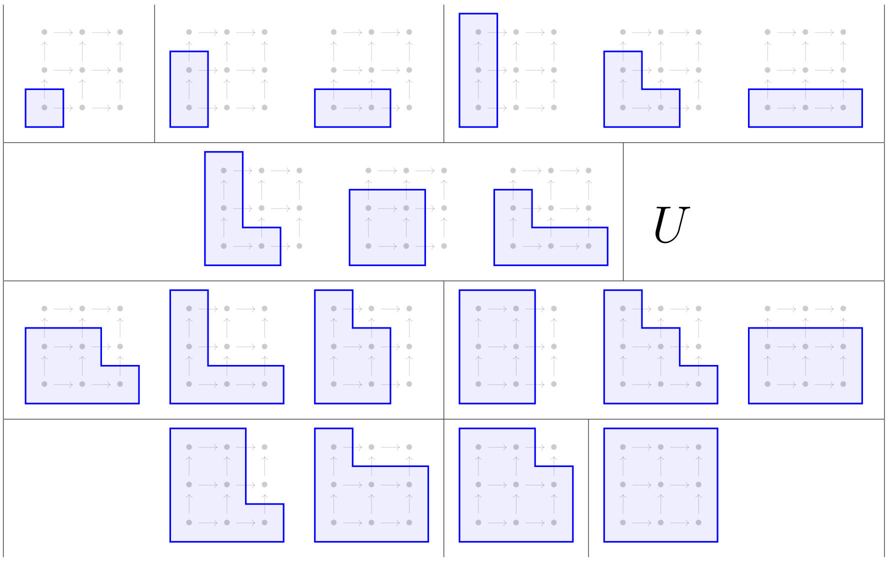


Figure 4.5: Indecomposables modules in $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$ ordered by decreasing $Z_{(0,0)}$-slope. U is defined in Equation (4.12)

Indecomposable $P^{2\times 2}$-persistence modules are known to be isomorphic to one of the 11 spread modules over $P^{2\times 2}$ [**47**, Figure 13]. As desired, there are exactly 11+9=20 isomorphism classes of indecomposable modules in $\mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$.

Proof of Proposition 4.2.11. Let $\mathcal{I}$ be the family of indecomposable modules described in Figure 4.5. Let $V \in \mathrm{Pers}^{\mathrm{s}}(P^{3\times 3})$ be a nonzero module, there are nonnegative integers $(d_I)_{I\in\mathcal{I}}$ such that

$$V \simeq \bigoplus_{I\in\mathcal{I}} I^{d_I}.$$

Let $p := (0,0)$, by Lemma 4.2.7 and Example 4.2.10, every indecomposable module $I \in \mathcal{I}$ is Z_p-stable. Hence, by (4.9), we have for $\theta \in \mathbb{R}$,

$$\mathbf{T}[V; Z_p]^{\theta} = \sum_{I\in\mathcal{I}} d_I \mathbf{T}[I; Z_p]^{\theta} = \sum_{I\in\mathcal{I}} d_I \mathbb{1}_{(-\infty,\mu_{Z_p}(I)]}(\theta)\, \underline{\dim}_I\,.$$

Let $J \in \mathcal{I}$, we show that $\mathbf{T}[V; Z_p]$ determines d_J. Let $\mathcal{I}^J$ denote the set $\{I \in \mathcal{I} \mid \mu_{Z_p}(I) = \mu_{Z_p}(J)\}$. By the above equation, one can recover the vector $\sum_{I\in\mathcal{I}^J} d_I\, \underline{\dim}_I$ from $\mathbf{T}[V; Z_p]$. Moreover, observe in Figure 4.5 that the family of vectors $\{\underline{\dim}_I \mid I \in \mathcal{I}^J\}$ is linearly independent. Thus, the multiplicity d_J can be recovered from $\mathbf{T}[V; Z_p]$. □

4.3 HN filtrations of discretisable persistence modules

The two main goals of this Section are, firstly, to find a framework where HN filtrations of discretisable $\mathbb{R}^n$-persistence modules exist, and secondly, to define a continuous version of the skyscraper invariant.

We first compute the Grothendieck group of discretisable persistence modules over an upper semilattice. We then give examples of stability conditions and discretisable persistence modules for which HN filtrations do not exist (Example 4.3.2). In Subsections 4.3.2

and 4.3.3, we state and prove Theorem (**G**). Finally, we define the HN filtered rank invariants which generalise and give a continuous version of the skyscraper invariant defined in (4.11).

4.3.1 Stability conditions for discretisable persistence modules

Let Q be an upper semilattice. When Q is finite, remember from (4.7) that the assignment $\underline{\dim}\colon V \in \mathrm{Pers}(Q) \mapsto \underline{\dim}_V \in \mathbb{Z}^Q_{\geqslant 0}$ induces an isomorphism

$$\underline{\dim}\colon K(\mathrm{Pers}_{\mathrm{fp}}(Q)) = K(\mathrm{Pers}(Q)) \simeq \mathbb{Z}^Q = \mathcal{F}(Q, \mathbb{Z}).$$

We now generalise this result to the case where Q is infinite.

Lemma 4.3.1. *Let Q be an upper semilattice. The assignment $\underline{\dim}\colon V \in \mathrm{Pers}_{fp}(Q) \mapsto \underline{\dim}_V \in \mathcal{F}(Q,\mathbb{Z})$ induces a group isomorphism*

$$\underline{\dim}\colon K(\mathrm{Pers}_{fp}(Q)) \xrightarrow{\sim} \mathcal{F}(Q, \mathbb{Z}).$$

Proof. If $0 \to V \to V' \to V'' \to 0$ is exact in $\mathrm{Pers}_{\mathrm{fp}}(Q)$, then $\underline{\dim}_{V'} - \underline{\dim}_V - \underline{\dim}_{V''} = 0$ so there is a well-defined group homomorphism $\underline{\dim}\colon K(\mathrm{Pers}_{\mathrm{fp}}(Q)) \to \mathcal{F}(Q,\mathbb{Z})$.

Let $f\colon P \hookrightarrow Q$ be an inclusion of upper semilattices with a finite domain, the following diagram of abelian groups

$$\begin{tikzcd}
K(\mathrm{Pers}(P)) \arrow[r, "f_*"] \arrow[d, "\underline{\dim}", "\wr"'] & K(\mathrm{Pers}_{\mathrm{fp}}(Q)) \arrow[d, "\underline{\dim}"] \\
\mathbb{Z}^P \arrow[r, hook, "f_*"] & \mathcal{F}(Q,\mathbb{Z})
\end{tikzcd} \tag{4.13}$$

is well defined – see Lemma 4.1.5(ii), and by (4.3), it commutes. Moreover, by (4.7), the vertical arrow on the left is an isomorphism. If $f^*\colon \mathcal{F}(Q,\mathbb{Z}) \to \mathbb{Z}^P$ denotes the pullback by f, then the pushforward $f_*\colon \mathbb{Z}^P \to \mathcal{F}(Q,\mathbb{Z})$ satisfies $f^* f_* = \mathrm{id}_{\mathbb{Z}^P}$ and is hence injective.

<u>surjectivity</u>: let $b \in \mathcal{F}(Q,\mathbb{Z})$. There exists an inclusion of upper semilattices $f\colon P \hookrightarrow Q$ with a finite domain and a function $a \in \mathbb{Z}^P$ such that $b = f_* a$. By (4.7), there exists a class $\widehat{U} \in K(\mathrm{Pers}(P))$ such that $a = \underline{\dim}\,\widehat{U}$. Finally, by commutativity of (4.13), we have $b = f_* a = (f_* \circ \underline{\dim})\widehat{U} = (\underline{\dim} \circ f_*)\widehat{U}$.

<u>injectivity</u>: choose $\widehat{V} \in K(\mathrm{Pers}_{\mathrm{fp}}(Q))$ in the kernel of $\underline{\dim}$. By definition, there exist integers $\lambda_1, \dots, \lambda_\ell \in \mathbb{Z}$ and modules $V_1, \dots, V_\ell \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ such that $\widehat{V} = \sum_i \lambda_i [V_i]$. Then, by Remark 4.1.3, there is an inclusion of upper semilattices $f\colon P \hookrightarrow Q$ with P finite which is simultaneously adapted to all the $(V_i)_{1 \leqslant i \leqslant \ell}$. More precisely, there exist $U_1, \dots, U_\ell \in \mathrm{Pers}(P)$ such that $\widehat{V} = \sum_i \lambda_i [f_* U_i]$. Let $\widehat{U}$ denote the class $\widehat{U} := \sum_i \lambda_i [U_i] \in K(\mathrm{Pers}(P))$ so that $\widehat{V} = f_* \widehat{U}$. By commutativity of (4.13), we have

$$0 = \underline{\dim}\,\widehat{V} = (f_* \circ \underline{\dim})\widehat{U}$$

Finally, by injectivity of $\underline{\dim}\colon K(\mathrm{Pers}(P)) \to \mathbb{Z}^P$ and $f_*\colon \mathbb{Z}^P \to \mathcal{F}(Q,\mathbb{Z})$, we have $\widehat{U} = 0$ and $\widehat{V} = 0$. □

Two $\mathbb{R}$-linear forms $\alpha, \beta \in \mathcal{F}(Q, \mathbb{R})^*$, such that β is strictly positive, induce a stability condition $\mathrm{StabCond}_{\alpha,\beta}$ over $\mathrm{Pers}_{\mathrm{fp}}(Q)$ defined for $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ by

$$\mathrm{StabCond}_{\alpha,\beta}([V]) := \beta(\underline{\dim}_V) + \sqrt{-1}\alpha(\underline{\dim}_V). \tag{4.14}$$

Let $Z\colon K(\mathrm{Pers}_{\mathrm{fp}}(Q)) \to \mathbb{C}$ be a stability condition. By Lemma 4.3.1, Z factors through the group isomorphism $\underline{\dim}\colon K(\mathrm{Pers}_{\mathrm{fp}}(Q)) \to \mathcal{F}(Q, \mathbb{Z})$. More precisely, let $\alpha^Z\colon \mathcal{F}(Q, \mathbb{Z}) \to \mathbb{R}$ and $\beta^Z\colon \mathcal{F}(Q, \mathbb{Z}) \to \mathbb{R}$ be the imaginary and real parts of $Z \circ \underline{\dim}^{-1}\colon \mathcal{F}(Q, \mathbb{Z}) \to \mathbb{C}$. The group morphisms α^Z and β^Z can be seen as linear forms from the real vector space $\mathcal{F}(Q, \mathbb{Z}) \otimes \mathbb{R} \cong \mathcal{F}(Q, \mathbb{R})$. One can check that $\mathrm{StabCond}_{\alpha^Z,\beta^Z}$ is well defined and coincide with Z.

When the cardinality $|Q|$ of Q is infinite, working in $\mathrm{Pers}_{\mathrm{fp}}(Q)$ rather than $\mathrm{Pers}(Q)$ is essential to guarantee the existence of stability conditions. Indeed, given a stability condition Z over $\mathrm{Pers}(Q)$, the module $W := \bigoplus_{x\in Q} \mathbb{F}_{\{x\}}^{\lfloor(\mathfrak{Re}Z([\mathbb{F}_{\{x\}}]))^{-1}\rfloor+1} \in \mathrm{Pers}(Q)$ satisfies $\mathfrak{Re}Z([W]) \geqslant |Q|$.

Let $f\colon P \hookrightarrow Q$ be an inclusion of upper semilattices with a finite domain. Given a stability condition Z over $K(\mathrm{Pers}_{\mathrm{fp}}(Q))$, we denote by f^*Z the pullback of Z by f_* which is defined by diagram (4.6) with $F = f_*$ – see Example 4.1.17. Let $\alpha, \beta \in \mathcal{F}(Q, \mathbb{R})^*$ with β strictly positive. We write the pullback $f^*\mathrm{StabCond}_{\alpha,\beta}$ in the form of (4.8) as $Z_{a,b}$. The functions $a, b\colon P \to \mathbb{R}$ are defined for $p \in P$ by

$$a(p) := \alpha(\mathbb{1}_{\mathrm{cub}_f(p)}) \qquad \text{and} \qquad b(p) := \beta(\mathbb{1}_{\mathrm{cub}_f(p)}).$$

The notions of slope, (semi)stability and HN filtrations are defined in $\mathcal{A} := \mathrm{Pers}_{\mathrm{fp}}(Q)$ as per Subsection 4.1.5. Importantly, if we restrict ourselves to f-discretisable Q-persistence modules where f is the inclusion of a finite upper semilattice P into Q, then all the slopes can be computed in $\mathrm{Pers}(P)$ using the same framework as in Subsection 4.2.1.

4.3.2 Existence of HN filtrations of discretisable persistence modules

The main result of this Subsection (Theorem 4.3.11) is an existence result for HN filtrations of discretisable persistence modules along certain stability conditions. The following examples in the single-parameter setting demonstrate that discretisable persistence modules do not always admit a HN filtration, depending on the choice of stability condition.

Example 4.3.2. (1) Let $Q := [0, 1)$ and let $a, b\colon [0, 1) \to \mathbb{R}_{>0}$ be integrable functions such that a is increasing and b is constant. Consider the stability condition Z over $\mathrm{Pers}_{\mathrm{fp}}([0, 1))$, given for every $V \in \mathrm{Pers}_{\mathrm{fp}}([0, 1))$ by

$$Z([V]) = \int_{[0,1)} (b + \sqrt{-1}a)\, \underline{\dim}_V \in \mathbb{C}.$$

Let $V \in \mathrm{Pers}_{\mathrm{fp}}([0,1))$ be the spread module $V := \mathbb{F}_{[0,1)}$. Observe that the nonzero discretisable submodules of V are given by $\{\mathbb{F}_{[t,1)} \mid t \in [0,1)\}$ and their Z-slopes are given for $t \in [0,1)$ by

$$\mu_Z(\mathbb{F}_{[t,1)}) = \frac{1}{(1-t)b(0)} \int_t^1 a.$$

Since a is increasing, the slopes $\mu_Z(\mathbb{F}_{[t,1)})$ increase with $t \in [0,1)$.

Assume by contradiction that V admits a HN filtration $0 \subsetneq V^1 \subsetneq V^2 \subsetneq \cdots \subsetneq V^\ell$ along Z. Let $t \in [0,1)$ such that $V^1 = \mathbb{F}_{[t,1)}$. By definition of $V^\bullet$, the module $\mathbb{F}_{[t,1)}$ is Z-semistable but it has a submodule $\mathbb{F}_{[\frac{t+1}{2},1)}$ of strictly higher Z-slope. So $\mathbb{F}_{[0,1)}$ does not have a HN filtration along Z.

(1') A similar phenomenon occurs with a stability condition $Z = \mathrm{StabCond}_{\alpha,\beta}$ over $\mathrm{Pers}_{\mathrm{fp}}([0,1))$ whose imaginary part is induced by $\alpha\colon b \in \mathcal{F}([0,1),\mathbb{R}) \mapsto -b(0) \in \mathbb{R}$. Assume by contradiction that $V := \mathbb{F}_{[0,1)}$ has a HN filtration $0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^{\ell-1} \subsetneq V^\ell$ along Z. There is $t \in [0,1)$, such that $V^{\ell-1} = \mathbb{F}_{[t,1)}$. The quotient $V^\ell/V^{\ell-1} \simeq \mathbb{F}_{[0,t)}$ is Z-semistable but it has a submodule $\mathbb{F}_{[\frac{t}{2},t)}$ satisfying $\mu_Z(\mathbb{F}_{[\frac{t}{2},t)}) = 0 > -\frac{1}{\beta(\mathbb{1}_{[0,t)})} = \mu_Z(V^\ell/V^{\ell-1})$. By contradiction, the filtration $\langle \mathbb{F}_{[0,1)}, Z\rangle$ does not exist.

(2) Let $Q := \mathbb{Z}$ and let $Z := \mathrm{StabCond}_{\alpha,\beta}$ be a stability condition over $\mathrm{Pers}_{\mathrm{fp}}(\mathbb{Z})$ whose imaginary part is induced by $\alpha\colon b \in \mathcal{F}(\mathbb{Z},\mathbb{R}) \mapsto \lim_{+\infty} b \in \mathbb{R}$. If the spread module $V := \mathbb{F}_{[0,+\infty)} \in \mathrm{Pers}_{\mathrm{fp}}(\mathbb{Z})$ had a HN filtration $V^\bullet$ along Z, then the first nonzero submodule in $V^\bullet$ would be of the form $\mathbb{F}_{[t,+\infty)}$ for some $t \in \mathbb{Z}_{\geqslant 0}$. The Z-semistability of $\mathbb{F}_{[t,+\infty)}$ would be contradicted by the existence of a submodule $\mathbb{F}_{[t+1,+\infty)}$ of strictly higher Z-slope.

We will now consider certain stability conditions such that the induction functor preserves semistability. We will then use Lemma 4.1.16 to guarantee the existence of HN filtrations.

Let Q be an upper semilattice. The *evaluation form* at $q \in Q$ is the linear form

$$\delta_q\colon b \in \mathcal{F}(Q,\mathbb{R}) \longmapsto b(q) \in \mathbb{R}. \tag{4.15}$$

Definition 4.3.3. Given an inclusion $f\colon P \hookrightarrow Q$ of upper semilattices, we denote by $\mathcal{Z}^{\mathrm{eval}}(f)$ the set of stability conditions $Z := \mathrm{StabCond}_{\alpha,\beta}$ over $\mathrm{Pers}_{\mathrm{fp}}(Q)$ such that α is a nonnegative linear combination of the evaluation forms $(\delta_q)_{q\in\mathrm{Im} f}$. If $Z \in \mathcal{Z}^{\mathrm{eval}}(f)$, we say that f is *adapted* to Z or equivalently that Z is *f-discretisable*. We denote by $\mathcal{Z}^{\mathrm{eval}}(Q)$ the union of $\mathcal{Z}^{\mathrm{eval}}(f)$ over all inclusions f of a finite semilattice into Q. △

Proposition 4.3.4. *Let $f\colon P \hookrightarrow Q$ be an inclusion of upper semilattices with a finite domain (in particular P and Q are upper semilattices) and assume that $Z \in \mathcal{Z}^{eval}(f)$. Then, for any nonzero P-persistence module U, we have*

$$U \text{ is } f^*Z\text{-semistable} \implies f_*U \text{ is } Z\text{-semistable in } \mathrm{Pers}_{fp}(Q).$$

Proof. Let W be a submodule of f_*U, we first prove that the pixelisation f_*f^*W is a submodule of W. For $q \in Q$, consider the linear map $\phi_q := W_{f(\lfloor q\rfloor_f)\leqslant q}$. Its domain is $(f^*W)_{\lfloor q \rfloor_f} = (f_*f^*W)_q$, and it is injective as it is the restriction of the isomorphism $(f_*U)_{f(\lfloor q \rfloor_f) \leqslant q}$. The maps $(\phi_q)_q$ define a map of Q-persistence modules $\phi_\bullet \colon f_*f^*W \hookrightarrow W$ because for $q \leqslant q'$ in Q, we have,

$$\phi_{q'} \circ (f_*f^*W)_{q \leqslant q'} = \phi_{q'} \circ W_{f(\lfloor q \rfloor_f) \leqslant f(\lfloor q' \rfloor_f)} = W_{f(\lfloor q \rfloor_f) \leqslant q'} = W_{q \leqslant q'} \circ \phi_q .$$

Assume now that $U \in \mathrm{Pers_{fp}}(P)$ is f^*Z-semistable. Note that since P is finite, we have $\mathrm{Pers_{fp}}(P) = \mathrm{Pers}(P)$. Let $0 \neq W \subset f_*U$, we prove that $\mu_Z(W) \leqslant \mu_Z(U)$. Let $\alpha, \beta \colon \mathcal{F}(Q, \mathbb{R}) \to \mathbb{R}$ be the linear forms such that $Z = \mathrm{StabCond}_{\alpha,\beta}$. By definition, α is a nonnegative linear combination of $(\delta_{f(p)})_{p \in P}$. If $\alpha(\underline{\dim}_W) = 0$, then $\mu_Z(W) = \frac{\alpha(\underline{\dim}_W)}{\beta(\underline{\dim}_W)} = 0 \leqslant \frac{\alpha(\underline{\dim}_U)}{\beta(\underline{\dim}_U)} = \mu_Z(U)$. We now assume that $\alpha(\underline{\dim}_W) > 0$. Given $p \in P$, we have $\delta_{f(p)}(\underline{\dim}_{f_*f^*W}) = \dim(f^*W)_p = \delta_{f(p)}(\underline{\dim}_W)$, whence $\alpha(\underline{\dim}_{f_*f^*W}) = \alpha(\underline{\dim}_W) > 0$. In particular, f_*f^*W is nonzero. Since f_*f^*W is a submodule of W, we have $\underline{\dim}_{f_*f^*W} \leqslant \underline{\dim}_W$ and $\mu_Z(f_*f^*W) \geqslant \mu_Z(W)$ because β is nonnegative. But, by applying the restriction functor, $f^*W \subset f^*f_*U \simeq U$ so by f^*Z-semistability of U and Lemma 4.1.15, we finally have

$$\mu_Z(W) \leqslant \mu_Z(f_*f^*W) = \mu_{f^*Z}(f^*W) \leqslant \mu_{f^*Z}(U) = \mu_Z(f_*U). \qquad \square$$

We now restrict ourselves to the case where Q is a cube of either $\mathbb{R}^n$ or $\mathbb{Z}^n$ and state an analogue of Proposition 4.3.4 for a larger family of stability conditions.

Notation 4.3.5. Let Q a cube of either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Given two integrable functions $a \colon Q \to \mathbb{R}$ and $b \colon Q \to \mathbb{R}_{>0}$, we denote by $Z_{a,b}$ the stability condition over $\mathrm{Pers_{fp}}(Q)$ given for $V \in \mathrm{Pers_{fp}}(Q)$ by

$$Z_{a,b}([V]) = \int_Q (b + \sqrt{-1}a)\, \underline{\dim}_V .$$

Definition 4.3.6. Let $G \colon P \hookrightarrow Q$ be a grid function. We denote by $\mathcal{Z}^{\mathrm{step}}(G)$ the set of stability conditions of the form $Z_{a,b} \colon K(\mathrm{Pers_{fp}}(Q)) \to \mathbb{C}$ (see Notation 4.3.5) such that the functions $a, b \colon Q \to \mathbb{R}$ are discretisable (see Subsection 4.1.4) by an extension (see Subsection 4.1.3) of G and

- the support of a is contained in $\bigcup_{c \in \mathcal{C}_{\mathrm{b}}(G)} c$ (see Notation 4.1.7), and
- b is positive and integrable on Q. △

Let c be a nonempty bounded cube of $\mathbb{R}^n$. The dimension of c is the number $0 \leqslant \dim c \leqslant n$ of coordinates over which the projection of c has a positive length. We define the linear form δ_c which sends $b \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ to its average over c. Namely,

$$\delta_c(b) := \frac{\int_c b}{\int_c 1} \tag{4.16}$$

where the integral is $\dim c$-dimensional. Similarly, if c is a bounded cube of $\mathbb{Z}^n$, we define $\delta_c \colon b \in \mathcal{F}(\mathbb{Z}^n, \mathbb{R}) \mapsto \sum_c b / \sum_c 1$. If $c = \{x\}$ is a point of $\mathbb{R}^n$ or $\mathbb{Z}^n$, then δ_c is the evaluation form at x – see (4.15). Given a bounded cube c of Q, if ι^Q is the inclusion of Q into either $\mathbb{R}^n$ or $\mathbb{Z}^n$, we denote by δ_c the linear form $b \in \mathcal{F}(Q, \mathbb{R}) \mapsto \delta_c(\iota^Q_* b)$. Definition 4.3.6 can be reformulated as follows:

Remark 4.3.7. Let $G\colon P \hookrightarrow Q$ be a grid function and let $Z \in \mathcal{Z}(G)$. There exists an extension $\widetilde{G}$ of G such that $\mathfrak{Im} Z$ (*resp.* $\mathfrak{Re} Z$) is a finite (*resp.* positive) linear combination of $(\delta_c \circ \underline{\dim})_{c \in \mathcal{C}_b(\widetilde{G})}$ – see Notation 4.1.7.

We now slightly extend the family considered in Definition 4.3.6 to include some of the stability conditions belonging to $\mathcal{Z}^{\text{eval}}(Q)$.

Definition 4.3.8. Let Q be a cube of $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $G\colon P \hookrightarrow Q$ be a grid function. We denote by $\mathcal{Z}(G)$ the set of stability conditions over $\mathrm{Pers}_{\mathrm{fp}}(Q)$ for which there exist $Z^{\text{step}} \in \mathcal{Z}^{\text{step}}(G)$ and $(\lambda_c)_c \in \mathbb{R}_{\geqslant 0}^{\mathcal{C}_{\text{bf}}(G)}$ such that (see Notation 4.1.7 and (4.16)):

$$Z = Z^{\text{step}} + \sqrt{-1} \sum_{c \in \mathcal{C}_{\text{bf}}(G)} \lambda_c \delta_c \circ \underline{\dim}\,. \qquad \triangle$$

We use the same vocabulary and notations with $\mathcal{Z}$ as for $\mathcal{Z}^{\text{eval}}$. Namely, $\mathcal{Z}(Q) := \bigcup_{G \text{ finite}} \mathcal{Z}(G)$, and when a grid function $G\colon P \hookrightarrow Q$ satisfies $Z \in \mathcal{Z}(G)$, we say that G is *adapted* to Z (or that Z is G*-discretisable*). In the case where Q is a cube of $\mathbb{Z}^n$, the formulation of Definition 4.3.8 can be simplified:

Lemma 4.3.9. *(i) Let Q be a cube of $\mathbb{Z}^n$, a stability condition Z over* $\mathrm{Pers}_{fp}(Q)$ *lies in $\mathcal{Z}(Q)$ if and only if there exist $a\colon Q \to \mathbb{R}$ finitely supported and $b\colon Q \to \mathbb{R}_{>0}$ integrable such that $Z = Z_{a,b}$ (see Notation 4.3.5).*

(ii) Let Q be a cube of $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $Z \in \mathcal{Z}(Q)$. Given a (potentially infinite) grid function $G\colon P \hookrightarrow Q$, we have $G^* Z \in \mathcal{Z}(P)$.

Proof. (i) let $Z \in \mathcal{Z}(Q)$. By definition, there exist $b\colon Q \to \mathbb{R}_{>0}$ integrable, such that $\mathfrak{Re} Z$ is the morphism $\widehat{V} \in K(\mathrm{Pers}_{\mathrm{fp}}(Q)) \mapsto \sum_Q b\, \underline{\dim}_{\widehat{V}}$. Let $|\ |$ denote cardinality, by Remark 4.3.7, there exist a finite family $(\lambda_c)_c \in \mathbb{R}^{\mathcal{C}_{\text{bf}}(G)}$ such that $\mathfrak{Im} Z = \sum_{c \in \mathcal{C}_{\text{bf}}(G)} \lambda_c \delta_c \circ \underline{\dim}$. We have $Z = Z_{\sum_{c \in \mathcal{C}_{\text{bf}}(G)} \lambda_c \frac{\mathbb{1}_c}{|c|}, b}$, as desired.

Conversely, let $a\colon Q \to \mathbb{R}$ be compactly supported and let $b\colon Q \to \mathbb{R}_{>0}$ be integrable. Consider a finite grid function $G\colon P \hookrightarrow Q$ such that $P \subset Q$ is a finite cube containing the support of a. The grid function $\mathrm{id}_Q\colon Q \to Q$ is an extension of G and both a and b are id_Q-discretisable. Hence, $Z_{a,b}$ lies in $\mathcal{Z}^{\text{step}}(G) \subset \mathcal{Z}(Q)$.

(ii) By definition, there exist step functions $a, b\colon Q \to \mathbb{R}$ with a having a bounded support and b being positive and integrable such that $Z = Z_{a,b}$. Consider the functions $\widetilde{a}\colon p \in P \mapsto \int_{\mathrm{cub}_G(p)} a \in \mathbb{R}$ and $\widetilde{b}\colon p \in P \mapsto \int_{\mathrm{cub}_G(p)} b \in \mathbb{R}_{>0}$. Since $\mathcal{C}(G)$ is locally finite, $\widetilde{a}$ is finitely supported. Moreover, $\int_P \widetilde{b} = \int_Q b < +\infty$ so $\widetilde{b}$ is integrable. Given $\widehat{U} \in K(\mathrm{Pers}_{\mathrm{fp}}(P))$, we have $G^* Z(\widehat{U}) = \int_Q (b + \sqrt{-1} a) G_* \,\underline{\dim}_{\widehat{U}} = \sum_P (\widetilde{b} + \sqrt{-1}\widetilde{a})\, \underline{\dim}_{\widehat{U}} = Z_{\widetilde{a},\widetilde{b}}(\widehat{U})$; whence, by (i), $G^* Z_{a,b} = Z_{\widetilde{a},\widetilde{b}} \in \mathcal{Z}(P)$. $\square$

Proposition 4.3.10. *Let Q be a cube of either $\mathbb{R}^n$ or $\mathbb{Z}^n$, let $G\colon P \hookrightarrow Q$ be a finite grid function and let $Z \in \mathcal{Z}(G)$. Then for every nonzero P-persistence module U:*

$$U \text{ is } G^*Z\text{-semistable} \implies G_*U \text{ is } Z\text{-semistable in } \mathrm{Pers}_{fp}(Q).$$

We defer the proof of Proposition 4.3.10 to the next Subsection. We now state our existence theorem for HN filtrations of discretisable persistence modules:

Theorem 4.3.11. *Let Q be an upper semilattice. If either*

(i) $Z \in \mathcal{Z}^{eval}(Q)$ *(see Definition 4.3.3); or*

(ii) Q *is a cube of* $\mathbb{R}^n$ *or* $\mathbb{Z}^n$ *and* $Z \in \mathcal{Z}(Q)$ *(see Definition 4.3.8)*

Then, we have

(1) *Every* $V \in \mathrm{Pers}_{fp}(Q)$ *has a unique HN filtration along* Z*.*

(2) *Let* $f\colon P \hookrightarrow Q$ *be adapted to* Z *with* P *finite. Assume in case (i) that* f *is an inclusion of upper semilattices, or in case (ii) that* f *is a grid function. Then the following diagram is well-defined and commutative:*

$$\begin{tikzcd}
\mathrm{Pers}(P) \arrow[r, "\langle -{,}f^*Z\rangle"] \arrow[d, "f_*"] & \mathrm{Filtr}(\mathrm{Pers}(P)) \arrow[d, "f_*"] \\
\mathrm{Pers}_{fp}(Q) \arrow[r, "\langle -{,}Z\rangle"] & \mathrm{Filtr}(\mathrm{Pers}_{fp}(Q))
\end{tikzcd} \tag{4.17}$$

Proof. Let Q be an upper semilattice (*resp.* a cube of $\mathbb{R}^n$ or $\mathbb{Z}^n$) and let $Z \in \mathcal{Z}^{\mathrm{eval}}(Q)$ (*resp.* $Z \in \mathcal{Z}(Q)$).

(2) Let $f\colon P \hookrightarrow Q$ be an inclusion of upper semilattices (*resp.* a grid function) adapted to Z with P finite. Since P is finite, Proposition 4.1.12(2) ensures that HN filtrations along f^*Z exist in $\mathrm{Pers}(P) = \mathrm{Pers}_{\mathrm{fp}}(P)$. By combining Lemma 4.1.16 and Proposition 4.3.4 (*resp.* 4.3.10), we obtain that the diagram (4.17) is well-defined and commutes.

(1) Given $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$, there exists an inclusion of upper semilattices (*resp.* a grid function) $f\colon P \hookrightarrow Q$ with P finite adapted to both Z and V. By **(2)** applied to f, the HN filtration $\langle V, Z\rangle$ is well-defined. ☐

Theorem 4.3.11**(2)** can be extended to the case where f is an infinite grid function.

Corollary 4.3.12. *Let* Q *be a cube of either* $\mathbb{R}^n$ *or* $\mathbb{Z}^n$*. Let* $G\colon P \hookrightarrow Q$ *be a (potentially infinite) grid function and let* $Z \in \mathcal{Z}(G) \cap \mathcal{Z}(Q)$*. Then, the pushforward* G^*Z *of* Z *by* G_* *is well-defined and for* $U \in \mathrm{Pers}_{fp}(P)$*, we have* $\langle G_*U, Z\rangle = G_*\langle U, G^*Z\rangle$*.*

Proof. First recall from Subsection 4.1.3 that $G_*\colon \mathrm{Pers}_{\mathrm{fp}}(P) \to \mathrm{Pers}_{\mathrm{fp}}(Q)$ is exact. By naturality of K (see Example 4.1.17), $K(G_*)$ is injective and G^*Z is well defined.

By Lemma 4.3.9(ii), we have $G^*Z \in \mathcal{Z}(P)$ and by Theorem 4.3.11**(1)**, both of the assignments $\langle -, G^*Z\rangle\colon \mathrm{Pers}_{\mathrm{fp}}(P) \to \mathrm{Filtr}(\mathrm{Pers}_{\mathrm{fp}}(P))$ and $\langle -, Z\rangle\colon \mathrm{Pers}_{\mathrm{fp}}(Q) \to \mathrm{Filtr}(\mathrm{Pers}_{\mathrm{fp}}(Q))$ are well-defined. Let $U \in \mathrm{Pers}_{\mathrm{fp}}(P)$, there is a finite grid function $G'\colon P' \hookrightarrow P$ such that U is G'-discretisable. We can choose G' such that Z lies in $\mathcal{Z}(G \circ G')$. Indeed, since $Z \in \mathcal{Z}(G) \cap \mathcal{Z}(Q)$, there exists a bounded cube B of P such that $Z \in \mathcal{Z}(G_{|B})$. It is enough to refine G' such that $B \subset \operatorname{Im} G'$.

Assume that U is G^*Z-semistable and let U' be a G'-discretisation of U. By Lemma 4.1.15, U' is $(G \circ G')^*Z$-semistable, and by Proposition 4.3.10, the Q-persistence module $G_*U \simeq (G \circ G')_*U'$ is Z-semistable. Finally, with Lemma 4.1.16 applied to G, we have $\langle G_*U, Z\rangle = G_*\langle U, G^*Z\rangle$. $\square$

4.3.3 Proof of Proposition 4.3.10

Let Q be a cube of either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Fix a finite grid function $G\colon P \hookrightarrow Q$, a stability condition $Z \in \mathcal{Z}(G)$ and a nonzero P-persistence module U which is G^*Z-semistable. Like in the proof of Proposition 4.3.4, the key to show that G_*U is Z-semistable will be to pixelate the discretisable submodules of G_*U while controlling their slope. More precisely, given a discretisable submodule $W \subset G_*U$ and a finite grid function H adapted to both W and G_*U, we set $X := H^*W$ and we want to modify the finite grid function H so that H_*X becomes G-discretisable while enforcing $H_*X \subset G_*U$ and $\mu_Z(W) \leqslant \mu_Z(H_*X)$.

We thus need to study how the Z-slope of the submodule H_*X of G_*U evolves when the grid function H is perturbed. In order to simplify the computations, we introduce and restrict ourselves to certain elementary perturbations of the grid function H (see Definition 4.3.14). By Remark 4.3.7, there is a countable family $\mathcal{C}^Z$ of bounded cubes of Q such that Z is a complex linear combination of $(\delta_c \circ \underline{\dim})_{c \in \mathcal{C}^Z}$. Lemma 4.3.16 will control the numbers $(\delta_c(\underline{\dim}_{H_*X}))_{c \in C^Z}$ as H varies.

Remark 4.3.13. For convenience, we will sometimes consider products of n maps of posets $G\colon P \to Q$ which are not necessarily injective. We call these grid functions *improper* grid functions.

If Q is a cube of $\mathbb{R}^n$ (*resp.* $\mathbb{Z}^n$), let C denote an interval of $\mathbb{R}$ (*resp.* $\mathbb{Z}$) of positive length. Otherwise, if Q is a cube of $\mathbb{Z}^n$, let C denote an interval of $\mathbb{Z}$. Let $H^\bullet\colon R \to Q$ be a family of grid functions indexed by C. Namely, the (potentially improper) grid functions $(H^x)_{x \in C}$ all have the same (co)domain, but their values are parametrised by x. Let $1 \leqslant i \leqslant n$ and $p \in R_i$.

Definition 4.3.14. A family $H^\bullet\colon R \to Q$ of grid functions is said to be *affine at* (i, p) if $x \mapsto H^x_i(p) \in Q_i$ is affine and for each other $1 \leqslant i' \leqslant n$ and $p' \in R_{i'}$, the function $x \mapsto H^x_{i'}(p')$ is constant. △

Lemma 4.3.15. *Let* $(H^x\colon R \to Q)_{x \in C}$ *be a family of grid functions affine at* (i, p) *and let*

$$R' := \left(\prod_{j=1}^{i-1} R_j\right) \times (R_i \backslash \{p\}) \times \left(\prod_{j=i+1}^{n} R_j\right). \tag{4.18}$$

There exists a finite grid function $\widehat{H}\colon \widehat{R} \to Q$ *and an isomorphism of posets* $\widehat{\tau}\colon R' \xrightarrow{\sim} \widehat{R}$ *such that for all* $x \in C$*, we have* $H^x_{|R'} = \widehat{H} \circ \tau$*.*

Proof. Choose any isomorphism of posets τ from R' to a cube $\widehat{R}$ of $\mathbb{Z}$. By definition, the functions $(H^x_{|R'})_{x \in C}$ are independent of x. We can thus choose $\widehat{H} := H^x_{|R'} \circ \tau^{-1}$. $\square$

We denote respectively by π_i and e_i the i-th orthogonal projection and the i-th canonical basis vector in a cube of either $\mathbb{R}^n$ or $\mathbb{Z}^n$.

Lemma 4.3.16. *Let* $(H^x \colon R \to Q)_{x \in C}$ *be affine at* (i, p) *for some* $1 \leqslant i \leqslant n$ *and* $p \in R_i \setminus \{\min R_i, \max R_i\}$ *and assume that* $x \in C \mapsto H^x_i(p)$ *is increasing. Let* $\widetilde{H}$ *be an extension of* $\widehat{H}$ *(see Lemma 4.3.15). Given* $c \in \mathcal{C}_{bf}(\widetilde{H})$ *(see Notation 4.1.7) and* $X \in \operatorname{Pers}(R)$*, let* $a \colon C \to \mathbb{R}$ *be the function*

$$a \colon x \in C \longmapsto \delta_c(\underline{\dim}_{H^x_* X}) \in \mathbb{R}.$$

1. *If* $\operatorname{len} \pi_i(c) > 0$*, then the function* a *is affine.*
2. *Otherwise,* $a_{|C \setminus \{\inf C\}}$ *is constant. If, in addition, for all* $r \in \pi_i^{-1}(p)$*, we have* $\dim X_r \geqslant \dim X_{r-e_i}$*, then for* $x \leqslant x' \in C$*, we have* $a(x') \leqslant a(x)$*.*

Proof. The function a decomposes as

$$a \colon x \mapsto \sum_{r \in R} \dim X_r \prod_{j=1}^{n} \delta_{\pi_j(c)}(\mathbb{1}_{\operatorname{cub}_{H^x_j}(\pi_j(r))})$$

Observe that the function $x \mapsto \mathbb{1}_{\operatorname{cub}_{H^x_j}(\pi_j(r))}$ is constant unless $i = j$ and $\pi_j(r) \in \{p-1, p\}$. Hence, we only need to compute the functions $b_s \colon x \mapsto \delta_{\pi_i(c)}(\mathbb{1}_{\operatorname{cub}_{H^x_i}(s)})$ for $s \in \{p-1, p\}$.

Let (h^-, h^+) be the constants $(H^x_i(p-1), H^x_i(p+1))$. If $\pi_i(c) \cap [h^-, h^+) = \varnothing$ then $b_p = b_{p-1} = 0$ and a is constant. Otherwise, since $c \in \mathcal{C}_{\mathrm{bf}}(\widetilde{H})$, we are in one of the following two cases:

1. $\underline{\pi_i(c) = [h^-, h^+)}$: we have $\operatorname{cub}_{H^x_i}(s) \subset \pi_i(c)$ for all $x \in C$ so

   $$b_s(x) = \operatorname{len} \operatorname{cub}_{H^x_i}(s) = \begin{cases} H^x_i(p) - h^- & \text{if } s = p-1 \\ h^+ - H^x_i(p) & \text{if } s = p \end{cases}$$

   which is affine. a is a linear combination of affine functions and is hence affine.

2. $\underline{\pi_i(c) = \{h^-\}}$: Let $L \colon \mathbb{R} \to \mathbb{R}$ be the affine function such that for all $x \in C$, we have $L(x) = H^x_i(p)$. There exists a unique $x^- \in \mathbb{R}$ such that $L(x^-) = h^-$. Since for all $x \in C$, the function H^x_i is increasing, one has $x^- \leqslant \inf C$. And for $x \in C$,

   $$b_s(x) = \mathbb{1}_{H^x_i(p-1) \in \operatorname{cub}_{H^x_i}(s)} = \begin{cases} 1 - \mathbb{1}_{x^-} & \text{if } s = p-1 \\ \mathbb{1}_{x^-} & \text{if } s = p \end{cases}$$

   which is constant on $C \setminus \{x^-\}$. We rewrite the function a as

   $$a(x) = A + \sum_{r \in \pi_i^{-1}(p)} a_r \times (\dim X_r b_p + \dim X_{r-e_i} b_{p-1})$$

   where $A = \sum_{\pi_i(r) \notin \{p-1, p\}} \dim X_r \delta_c(\mathbb{1}_{\operatorname{cub}_{H^x}(r)})$ and $a_r = \prod_{j \neq i} \delta_{\pi_j(c)}(\mathbb{1}_{\operatorname{cub}_{H^x_j}(\pi_j(r))})$. For all $x \in C \setminus \{x^-\}$, we have $a(x^-) - a(x) = \sum_{r \in \pi_i^{-1}(p)} a_r \times (\dim X_r - \dim X_{r-e_i})$. The result follows from the nonnegativity of the constants $(a_r)_r$. □

Proposition (Restatement of Proposition 4.3.10)**.** *Let Q be a cube of either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Given a finite grid function $G\colon P \hookrightarrow Q$ and a stability condition $Z \in \mathcal{Z}(G)$, the pushforward G_*U of a G^*Z-semistable nonzero P-persistence module U is Z-semistable in* $\mathrm{Pers}_{fp}(Q)$.

Proof. Let U be a G^*Z-semistable nonzero P-persistence module and let $H\colon R \hookrightarrow Q$ be a finite refinement of G. We prove by induction over the cardinal

$$N_H := \left| \bigsqcup_{i=1}^{n} \operatorname{Im} H_i \backslash \operatorname{Im} G_i \right|$$

that every nonzero H-discretisable submodule $W \subset G_*U$ satisfies $\mu_Z(W) \leqslant \mu_Z(G_*U)$. We denote by X the R-persistence module H^*W. Without loss of generality, we can translate R in $\mathbb{Z}^n$ so that $\min R$ and $\min P$ coincide.

<u>Base Case</u>: when $N_H = 0$, we have $G = H$ and by semistability $\mu_Z(W) = \mu_Z(G_*X) = \mu_{G^*Z}(X) \leqslant \mu_{G^*Z}(U) = \mu_Z(G_*U)$.

<u>Induction Case</u>: assume that $N_H > 0$ and that the result holds for all refinements H' of G such that $N_{H'} = N_H - 1$. Let $1 \leqslant i \leqslant n$ and let $p \in \operatorname{Im} H_i \backslash \operatorname{Im} G_i$. We define

$$x^{\pm} := \begin{cases} H_i(p \pm 1) & \text{if } p \pm 1 \in R_i \\ \pm\infty & \text{otherwise.} \end{cases}$$

For $x \in [x^-, x^+] \backslash \{\pm\infty\}$, consider the finite grid function $H^x\colon R \to Q$ defined for $1 \leqslant i' \leqslant n$ and $p' \in R_{i'}$ by

$$H^x_{i'}(p') := \begin{cases} x & \text{if } (i', p') = (i, p) \\ H_{i'}(p') & \text{otherwise.} \end{cases}$$

The family $(H^x)_{x \in [x^-, x^+] \backslash \{\pm\infty\}}$ is affine at (i, p) and we have $H = H^{H_i(p)}$. Assume first that $p \notin \{\min R_i, \max R_i\}$. In particular, the family $(H^x)_x$ is indexed by $[x^-, x^+]$.

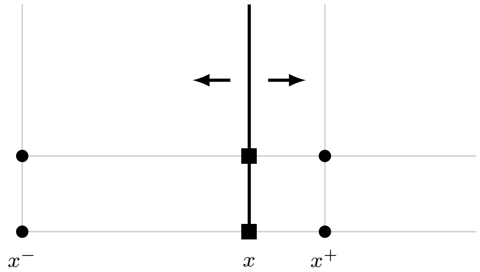


Figure 4.6: Let $H\colon \{0,1,2\} \times \{0,1\} \hookrightarrow \mathbb{R}^2$ be the grid function depicted in Figure 4.2 and let $(i, p) := (1, 1)$. Let $x \in [x^-, x^+]$ be the horizontal coordinate of the square nodes, the lines in the figure delimit $\mathcal{C}(H^x)$. The round nodes represent the set $H^x(R') = \operatorname{Im} \widehat{H}$ which does not depend on x.

We consider the two Q-persistence modules $W^{\pm} := (H^{x^{\pm}})_* X$ and we prove that the following partial function admits a maximum which is reached either at x^- or at x^+

$$m\colon \begin{cases} [x^-, x^+] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \mu_Z(H^x_* X) \end{cases}.$$

$\underline{m\text{ is monotonic on }(x^-,x^+)}$: since $W \subset G_*U$ and $p \notin \operatorname{Im} G_i$, we have that for every $r \in \pi_i^{-1}(p)$, the linear map $X_{r-e_i} \to X_r$ is a restriction of the isomorphism $(G_*U)_{H(r-e_i)} \to (G_*U)_{H(r)}$ and is hence injective. By Remark 4.3.7 and Lemma 4.3.16(i), the nonnegative function $x \in [x^-,x^+] \mapsto \beta(\underline{\dim}_{H^x_* X})$ is affine. Moreover, since the family $(\dim X_r - \dim X_{r-e_i})_{r\in\pi_i^{-1}(p)}$ is nonnegative, by Remark 4.3.7 and Lemma 4.3.16(ii), $x \in [x^-,x^+] \mapsto \alpha(\underline{\dim}_{H^x_* X})$ is the sum of an affine function and of $\lambda \mathbb{1}_{x^-}$ with $\lambda \geqslant 0$. Hence, the function $m_{|(x^-,x^+)}$ is well-defined and monotonic as the quotient of an affine function by a positive affine function.

$\underline{m\text{ has a maximum and it is reached at }x^-\text{ or }x^+}$: since $R\backslash \operatorname{Im}(\lfloor\ \rfloor_{H^{x^-}}) = \pi_i^{-1}(p-1)$ and $W \neq 0$, the injectivity of $X_{r-e_i} \hookrightarrow X_r$ for $r \in \pi_i^{-1}(p)$ ensures that W^- is never null. Thus, at x^-, the function m is well-defined and by the above it is upper semicontinuous. If $W^+ \neq 0$, m is also well-defined at x^+ and monotonic on $(x^-,x^+]$. It thus admits a maximum which is reached at x^- or x^+. Otherwise, if $W^+ = 0$, we then have that both $x \mapsto \alpha(\underline{\dim}_{H^x_* X})$ and $x \mapsto \beta(\underline{\dim}_{H^x_* X})$ are affine on $(x^-,x^+]$ and null at x^+. Whence, m is constant on (x^-,x^+) and we have $m(x^-) \geqslant m(x)$ for all $x \in [x^-,x^+)$.

$\underline{\text{Case }p = \max R_i}$: we write $W^+ := (H_{R'})_* X$. Since $p > \max P_i$, the function $x \in (x^-,x^+) \mapsto \alpha(\underline{\dim}_{H^x_* X})$ is constant while $x \in (x^-,x^+) \mapsto \beta(\underline{\dim}_{H^x_* X})$ is decreasing and positive. Hence, m is monotonic on (x^-,x^+), upper semicontinuous at x^- and – if $W^+ \neq 0$ – continuous at $x^+ = +\infty$. Moreover, as before, $W^+ = 0$ implies that m is constant on (x^-,x^+).

$\underline{\text{Case }p = \min R_i}$: By definition, $p < \min P_i$ so for every $r \in \pi_i^{-1}(p)$, we have $X_r = 0$ and for all $x \in (x^-,x^+]$, we have $H^x_* X \simeq W$. As a consequence, m is constant over $(x^-,x^+]$.

In any case, we found a nonzero $W' \in \{W^-,W^+\}$ such that $\mu_Z(W') \geqslant \mu_Z(W)$.

$\underline{W'\text{ is submodule of }G_*U}$: since $\widehat{H}$ (see Lemma 4.3.15) is a refinement of G, by Lemma 4.3.15, there exists a finite grid function $\tau\colon P \hookrightarrow R$, independent of x, such that $G = H^x \circ \tau$ for all $x \in [x^-,x^+]$. By definition, $W' = H^{x^\pm}_* H^* W$, and by Lemma 4.1.5(ii),

$$W' \subset H^{x^\pm}_* H^* G_* U \simeq H^{x^\pm} H^* H_* \tau_* U \simeq H^{x^\pm}_* \tau_* U \simeq G_* U.$$

Finally, $0 \neq W' \subset G_*U$ is $\widehat{H}$-discretisable and since $N_{\widehat{H}} = N_H - 1$, by induction hypothesis, $\mu_Z(W) \leqslant \mu_Z(W') \leqslant \mu_Z(G_*U)$. □

4.3.4 HN filtered rank invariants

For this subsection Q is either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Let $x \in Q$, the map of posets

$$T_x\colon y \in Q \mapsto x + y \in Q$$

is called the *x-shift*. Its pullback $T_x^*\colon V \in \operatorname{Pers}_{\mathrm{fp}}(Q) \mapsto V \circ T_x \in \operatorname{Pers}_{\mathrm{fp}}(Q)$ induces an endofunctor of $\operatorname{Pers}_{\mathrm{fp}}(Q)$ which we call the *x-shift functor*.

We fix a stability condition $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{\mathrm{eval}}(Q)$ – see Theorem 4.3.11. The following invariant of discretisable Q-persistence modules is inspired by the skyscraper invariant (4.11) defined over finite posets.

Definition 4.3.17. The *HN filtered rank invariant* of $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ along $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{\mathrm{eval}}(Q)$ is the filtration $(s^{\theta}_{Z,V})_{\theta \in \mathbb{R}^{\mathrm{opp}}}$ of integer-valued functions on $Q \times Q$ given by

$$s^{\theta}_{Z,V} \colon \begin{cases} Q \times Q & \to & \mathbb{Z}_{\geqslant 0} \cup \{+\infty\} \\ (x,y) & \mapsto & \begin{cases} \mathrm{rank}(\langle T^*_x V, Z \rangle^{\theta}_{0 \leqslant y-x}) & \text{if } x \leqslant y \\ +\infty & \text{otherwise.} \end{cases} \end{cases} \qquad \triangle$$

Theorem 4.3.11 ensures that $s^{\theta}_{Z,V}$ is well-defined and that for $(x,y) \in Q \times Q$, the integer $s^{\theta}_{Z,V}(x,y)$ can be computed in any suitable discretisation of V. Moreover, given $x \in Q$ and $\theta \geqslant \theta' \in \mathbb{R}$, we have by definition $\langle T^*_x V, Z \rangle^{\theta} \subset \langle T^*_x V, Z \rangle^{\theta'}$, whence $s^{\theta}_{Z,V}(x, \bullet) \leqslant s^{\theta'}_{Z,V}(x, \bullet)$. The use of the term filtration is justified by the implication

$$\theta \geqslant \theta' \Longrightarrow s^{\theta}_{Z,V} \leqslant s^{\theta'}_{Z,V}. \tag{4.19}$$

Note that, we can translate Z instead of V. Namely, for $x \in Q$ and $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$, let $T^x_* Z := (T^*_x)^* Z$ be the pullback of Z by T^*_x (see Equation (4.6)). By Lemma 4.1.16 applied to $F := T^*_x$, we have for $\theta \in \mathbb{R}$ and $y \geqslant x$ in Q

$$s^{\theta}_{Z,V}(x,y) = \mathrm{rank}(\langle V, T^x_* Z \rangle^{\theta}_{x \leqslant y}). \tag{4.20}$$

Examples of computation of $(s^{\theta}_{Z,V})_{\theta}$ can be found in Subsection 4.5.2.

Proposition 4.3.18. *Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{eval}(Q)$. There is $\theta_{min} \in \mathbb{R}$ such that for all $\theta \leqslant \theta_{min} \in \mathbb{R}$ and for all $V \in \mathrm{Pers}_{fp}(Q)$, we have*

$$s^{\theta}_{Z,V} = \rho_V.$$

Proof. Write Z in the form $\mathrm{StabCond}_{\alpha,\beta}$ – see (4.14). If $Z \in \mathcal{Z}^{\mathrm{eval}}(Q)$, the result follows from the nonnegativity of α. Assume now that there exists a finite grid function $G \colon P \hookrightarrow Q$ such that $Z \in \mathcal{Z}(G)$. Given $x \in Q$, by definition, $s^{\theta}_{Z,V}(x, \bullet) = \rho_V(x, \bullet)$ for θ small enough. We now find a uniform lower bound for θ. Following Remark 4.3.7, there exists numbers $(a_c) \in \mathbb{R}^{\mathcal{C}_b(G)}$ and $(b_c) \in \mathbb{R}^{\mathcal{C}_b(G)}_{>0}$ such that

$$\alpha \geqslant \sum_{c \in \mathcal{C}_b(G)} a_c \delta_c \qquad \text{and} \qquad \beta \geqslant \sum_{c \in \mathcal{C}_b(G)} b_c \delta_c.$$

Given $W \in \mathrm{Pers}_{\mathrm{fp}}(Q)$, we have the inequalities

$$\alpha(\underline{\dim}_W) \geqslant \min(0, \min_{c \in \mathcal{C}_b(G)} a_c) \sum_{c \in \mathcal{C}_b(G)} \delta_c(\underline{\dim}_W)$$

$$\beta(\underline{\dim}_W) \geqslant \min_{c \in \mathcal{C}_b(G)} (b_c) \sum_{c \in \mathcal{C}_b(G)} \delta_c(\underline{\dim}_W).$$

Let $\theta_{\min}$ be the real number $\min(0, \min_c a_c)/\min_c(b_c)$. If $V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^{\ell-1} \subsetneq V^{\ell} = T^*_x V$ is the HN filtration of $T^*_x V$ along Z, we have $\mu_Z(V^{\ell}/V^{\ell-1}) \geqslant \theta_{\min}$. Then for $\theta \leqslant \theta_{\min}$, we have $\langle T^*_x V, Z \rangle^{\theta} = T^*_x V$, and given $y \in Q$, $s^{\theta}_{Z,V}(x,y) = \mathrm{rank}(T^*_x V)_{0 \leqslant y-x} = \rho_V(x,y)$. $\square$

Definition 4.3.19. Let $\beta \in \mathcal{F}(Q, \mathbb{R})^*$ be a strictly positive linear form. The HN filtered rank invariant along $\mathrm{StabCond}_{\delta_0,\beta}$ will be called the *continuous skyscraper invariant* along β. △

Continuous skyscraper invariants are closely related to the finite version of the skyscraper invariant defined in (4.11).

Fix a strictly positive linear form $\beta \in \mathcal{F}(Q, \mathbb{R})^*$ and let Z_0 denote the stability condition $\mathrm{StabCond}_{\delta_0,\beta}$. Let $x, y \in Q$ and $\theta \in \mathbb{R}$, we claim that the integer $s^{\theta}_{Z_0,V}(x, y)$ can be recovered from a skyscraper computation over a finite poset. Indeed, choose a finite grid function $G^x \colon P_x \hookrightarrow Q$ adapted to both V and $T^x_* Z_0$. Let $b_x \colon P_x \to \mathbb{R}_{>0}$ be the function $p \mapsto \beta(\mathbb{1}_{\mathrm{cub}_{T_{-x} \circ G^x}(p)})$ and remember the definition of $s_{b_x,(G^x)^*V}$ from (4.11). Since $G^*_x T^x_* Z_0 = (T_{-x} \circ G^x)^* Z_0 = Z_{\mathbb{1}_{\lfloor x \rfloor_{G^x}}, b_x}$, by Theorem 4.3.11, we have

$$s^{\theta}_{Z_0,V}(x, y) = s^{\theta}_{b_x,(G^x)^*V}(\lfloor x \rfloor_{G^x}, \lfloor y \rfloor_{G^x}). \tag{4.21}$$

We will see in Proposition 4.6.4, that if β is induced by a step function, then one can choose the grid functions $(G^x)_{x \in Q}$ so that the set $\{(P_x, b_x) \mid x \in Q\}$ is finite.

4.4 Stability theorems

The rank invariant, equipped with the erosion distance introduced by Patel [**101**] has been shown to be stable [**76**] with respect to the interleaving distance. We extend this result to the HN filtered rank invariants obtained in Definition 4.3.17. The HN filtrations also induce filtered persistence landscapes which are a generalisation of the multiparameter persistence landscapes introduced by Vipond [**111**]. As a consequence of the above result, this last invariant is also stable. For this section, the poset Q will be either $\mathbb{R}^n$ or $\mathbb{Z}^n$.

4.4.1 Functoriality of HN filtrations

The key to adapt the existing stability theorems to the case of HN filtered rank invariants will be the functoriality of HN filtrations. We restate [**65**, Theorem 2.8] using Remark 4.2.1:

Proposition 4.4.1. *Let P be a finite poset and let Z be a stability condition over* $\mathrm{Pers}(P)$*. Then for every map $f \colon U \to U'$ in* $\mathrm{Pers}(P)$ *and every $\theta \in \mathbb{R}$*

$$f(\langle U, Z \rangle^{\theta}) \subset \langle U', Z \rangle^{\theta}.$$

Corollary 4.4.2. *Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Let $f \colon V \to V'$ be a map in* $\mathrm{Pers}_{\mathrm{fp}}(Q)$ *and let $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{eval}(Q)$. Then for every $\theta \in \mathbb{R}$*

$$f(\langle V, Z \rangle^{\theta}) \subset \langle V', Z \rangle^{\theta}.$$

Proof. Assume that V and V' are in $\mathrm{Pers}_{\mathrm{fp}}(Q)$ and choose a finite grid function $G \colon P \hookrightarrow Q$ such that $Z \in \mathcal{Z}(G) \cup \mathcal{Z}^{\mathrm{eval}}(G)$ and V and V' both have a G-discretisation, respectively U and U'. By Proposition 4.4.1, $G^* f$ induces a map of P-persistence modules $\widetilde{f} \colon \langle U, G^* Z \rangle^{\theta} \to$

$\langle U', G^*Z\rangle^\theta$ where $\widetilde{f} = G^*(G_*U \simeq V \xrightarrow{f} V' \simeq G_*U')$. By Theorem 4.3.11, $\widetilde{f}$ extends to a map

$$G_*\widetilde{f}\colon \langle G_*U, Z\rangle^\theta \to \langle G_*U', Z\rangle^\theta.$$

Finally, by exactness of G_*, the map $\langle V, Z\rangle^\theta \simeq \langle G_*U, Z\rangle^\theta \xrightarrow{G_*\widetilde{f}} \langle G_*U', Z\rangle^\theta \simeq \langle V', Z\rangle^\theta$ is the restriction of f. Hence, $f(\langle V, Z\rangle^\theta) \subset \langle V', Z\rangle^\theta$ in $\mathrm{Pers}_{\mathrm{fp}}(Q)$. □

4.4.2 Stability of HN filtered rank invariants

The *x-shift map* of V is the map of Q-persistence modules $\mathrm{sh}_V^x\colon V \to T_x^*V$ defined by $(\mathrm{sh}_V^x)_y := V_{y\leqslant x+y}$.

For $\varepsilon > 0$, $\vec{\varepsilon}$ denotes the vector $(\varepsilon, \dots, \varepsilon) \in Q$ and we see Q and $\mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ as posetal categories (associated with the standard partial order). We recall the definition of the following two extended pseudometrics:

Definition 4.4.3 ([85])**.** Given $\varepsilon > 0$, an *ε-interleaving* between two Q-persistence modules V and W is a pair of maps $f\colon V \to T_{\vec{\varepsilon}}^*W$ and $g\colon W \to T_{\vec{\varepsilon}}^*V$ in $\mathrm{Pers}(Q)$ such that

$$T_{\vec{\varepsilon}}^* g \circ f = \mathrm{sh}_V^{2\vec{\varepsilon}} \qquad \text{and} \qquad T_{\vec{\varepsilon}}^* f \circ g = \mathrm{sh}_W^{2\vec{\varepsilon}}.$$

We will write $f\colon V \overset{\varepsilon}{\leftrightarrow} W\colon g$ to signify that (f, g) is an ε-interleaving between V and W. The *interleaving distance* between V and W, denoted by $d_I(V, W)$, is the infimum over $\varepsilon \geqslant 0$ such that there is an ε-interleaving between V and W. △

Definition 4.4.4 ([101, 76])**.** Consider two functors

$$F, G\colon Q^{\mathrm{opp}} \times Q \to (\mathbb{Z}_{\geqslant 0} \cup \{+\infty\})^{\mathrm{opp}}.$$

For $\varepsilon > 0$, we say that there is an *ε-erosion* between F and G if for all $(a, b) \in Q^{\mathrm{opp}} \times Q$

$$F(a - \vec{\varepsilon}, b + \vec{\varepsilon}) \leqslant G(a, b) \qquad \text{and} \qquad G(a - \vec{\varepsilon}, b + \vec{\varepsilon}) \leqslant F(a, b).$$

The *erosion distance* between F and G, denoted by $d_E(F, G)$, is the infimum over $\varepsilon \geqslant 0$ such that there is an ε-erosion between F and G. △

An example of such functors is the rank invariant ρ_V of some $V \in \mathrm{Pers}(Q)$. ρ_V is known to be stable with respect to the erosion and interleaving distances:

Theorem 4.4.5 ([76, Theorem 6.2][102, Theorem 3.11])**.** *Let Q be $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $V, W \in \mathrm{Pers}(Q)$, we have $d_E(\rho_V, \rho_W) \leqslant d_I(V, W)$.*

We fix a stability condition $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{\mathrm{eval}}(Q)$ and two persistence modules V and W in $\mathrm{Pers}_{\mathrm{fp}}(Q)$. For convenience, we drop Z from the $s_{Z,V}^\theta$ notation. The main argument of the proof of Theorem 4.4.5 can be adapted to our setting and results in the following lemma:

Lemma 4.4.6. *Let $x \leqslant x' \leqslant y' \leqslant y$ in Q and assume the existence of maps*

$$f\colon V \to T_{x'-x}^*W \quad \textit{and} \quad g\colon W \to T_{y-y'}^*V$$

such that $g \circ f = \mathrm{sh}_V^{(y-x)-(y'-x')}$. Then for all $\theta \in \mathbb{R}$, one has $s_V^\theta(x, y) \leqslant s_W^\theta(x', y')$.

Proof. The properties of f and g make the following diagram well-defined and commutative

$$\begin{tikzcd}
\langle T_x^*V, Z\rangle_0^\theta \arrow[rrr] \arrow[dr, "f_x"] & & & \langle T_x^*V, Z\rangle_{y-x}^\theta \\
& (T_x^*f)(\langle T_x^*V, Z\rangle^\theta)_0 \arrow[r] & (T_x^*f)(\langle T_x^*V, Z\rangle^\theta)_{y'-x'} \arrow[ur, "g_{y'}"] &
\end{tikzcd}$$

Here, the top horizontal map is the restriction of $V_{x\leqslant y} = (T_x^*V)_{0\leqslant y-x}$ to the submodule $\langle T_x^*V, Z\rangle^\theta \subset T_x^*V$. By definition, its rank is $s_V^\theta(x,y)$. The bottom horizontal map is the restriction of $W_{x'\leqslant y'} = (T_{x'}^*W)_{0\leqslant y'-x'}$ to the submodule $(T_x^*f)(\langle T_x^*V, Z\rangle^\theta) \subset T_{x'}^*W$. We denote its rank as $s(x',y')$.

Since the above diagram commutes, we have $s_V^\theta(x,y) \leqslant s(x',y')$. Moreover, Corollary 4.4.2 implies that $(T_x^*f)(\langle T_x^*V, Z\rangle^\theta) \subset \langle T_{x'}^*W, Z\rangle^\theta$. Finally, $s_V^\theta(x,y) \leqslant s(x',y') \leqslant s_W^\theta(x',y')$. ☐

Theorem 4.4.7. *Let Q be $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{eval}(Q)$ (see Definitions 4.3.3 and 4.3.8). Let $V, W \in \mathrm{Pers}_{fp}(Q)$ be two Q-persistence modules, then for each $\theta \in \mathbb{R}$, their HN filtered rank invariant $s_{Z,V}^\theta$ and $s_{Z,W}^\theta$ are functors $Q^{opp} \times Q \to (\mathbb{Z}_{\geqslant 0} \cup \{+\infty\})^{opp}$. Moreover,*

$$\sup_{\theta\in\mathbb{R}} d_E(s_{Z,V}^\theta, s_{Z,W}^\theta) \leqslant d_I(V,W).$$

Proof. Let $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ and let $x \leqslant x' \leqslant y' \leqslant y$, the shift maps $\mathrm{sh}_V^{x'-x}$ and $\mathrm{sh}_V^{y-y'}$ satisfy the conditions of Lemma 4.4.6. Hence, $s_V^\theta(x,y) \leqslant s_V^\theta(x',y')$ or in other words $s_V^\theta : Q^{\mathrm{opp}} \times Q \to (\mathbb{Z}_{\geqslant 0} \cup \{+\infty\})^{\mathrm{opp}}$ is a functor.

Let $V, W \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ and assume that there is an ε-interleaving $f\colon V \overset{\varepsilon}{\leftrightarrow} W : g$. Let $\theta \in \mathbb{R}$, we show that $d_E(s_V^\theta, s_W^\theta) \leqslant \varepsilon$. For $a \leqslant b$ in Q, the maps f and g satisfy the conditions of Lemma 4.4.6 with $(x \leqslant x' \leqslant y' \leqslant y) := (a - \vec{\varepsilon} \leqslant a \leqslant b \leqslant b + \vec{\varepsilon})$. Hence, $s_V^\theta(a-\vec{\varepsilon}, b+\vec{\varepsilon}) \leqslant s_W^\theta(a,b)$ and by symmetry $s_W^\theta(a-\vec{\varepsilon}, b+\vec{\varepsilon}) \leqslant s_V^\theta(a,b)$. ☐

Remark 4.4.8. Theorem 4.4.5 for discretisable persistence modules is a special case of Theorem 4.4.7 obtained when the image of $Z\colon K(\mathrm{Pers}_{\mathrm{fp}}(Q)) \to \mathbb{C}$ is included in $\mathbb{R}$.

Notation 4.4.9. Given two Q-persistence modules $V, V' \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ and a stability condition $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{\mathrm{eval}}(Q)$, we write

$$d_{\mathbf{HN}}^Z(V,V') := \sup_{\theta\in\mathbb{R}} d_E(s_{Z,V}^\theta, s_{Z,W}^\theta)$$

and we call this extended pseudometric the *erosion distance between HN filtered rank invariants* along Z.

4.4.3 Stability of HN filtered landscapes

The HN filtered rank invariants induce filtered persistence landscapes. We show that those are also stable.

Definition 4.4.10. Given a functor $s\colon Q^{\mathrm{opp}} \times Q \to (\mathbb{Z}_{\geqslant 0} \cup \{+\infty\})^{\mathrm{opp}}$, the *persistence landscape* of s is the function $\lambda_s \in \mathrm{L}^\infty(\mathbb{Z}_{>0} \times Q)$ given for $(k,x) \in \mathbb{Z}_{>0} \times Q$ by

$$\lambda_s(k,x) := \sup\left\{\varepsilon > 0 \mid s(x-h, x+h) \geqslant k, \forall \|h\|_\infty \leqslant \varepsilon\right\}. \qquad \triangle$$

The persistence landscape introduced in [25, 111] is the invariant $V \in \mathrm{Pers}(Q) \mapsto \lambda_{\rho_V} \in \mathrm{L}^\infty(\mathbb{Z}_{>0} \times Q)$. They are stable with respect to the interleaving distance [111, Theorem 30].

Proposition 4.4.11. *Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Given two functors $r, s \colon Q^{opp} \times Q \to (\mathbb{Z}_{\geqslant 0} \cup \{+\infty\})^{opp}$ we have*

$$\|\lambda_r - \lambda_s\|_\infty \leqslant d_E(r, s). \tag{4.22}$$

Proof. Let $\varepsilon > d_E(r, s)$ and let $(k, x) \in \mathbb{Z}_{>0} \times Q$. We show that $|\lambda_r(k, x) - \lambda_s(k, x)| \leqslant \varepsilon$. Without loss of generality, we restrict ourselves to the case where $\varepsilon \leqslant \lambda_r(k, x) \geqslant \lambda_s(k, x)$. Let $h \in Q$ such that $\|h\|_\infty \leqslant \lambda_r(k, x) - \varepsilon$. By definition,

$$s(x - h, x + h) \geqslant r(x - h - \vec{\varepsilon}, x + h + \vec{\varepsilon}) \geqslant k.$$

Hence, $\lambda_s(k, x) \geqslant \lambda_r(k, x) - \varepsilon$ and $\lambda_r(k, x) - \lambda_s(k, x) \leqslant d_E(r, s)$. ☐

Remark 4.4.12. Assume that Q is either $\mathbb{R}$ or $\mathbb{Z}$. The persistence landscape map is known to be an isometry; that is, Equation (4.22) holds with equality [77, Theorem 4.3]. However, even in the one-parameter setting, the erosion distance generally differs from the interleaving distance [116, 8].

As before, let $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{\mathrm{eval}}(Q)$ and let $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$. The *HN filtered persistence landscape* of V along Z is the filtration $(\lambda_{s^\theta_{Z,V}})_{\theta \in \mathbb{R}^{\mathrm{opp}}}$ of λ_{ρ_V} in $\mathrm{L}^\infty(\mathbb{Z}_{>0} \times Q)$. The stability of this last invariant is a direct corollary of Theorem 4.4.7 and Proposition 4.4.11.

Corollary 4.4.13. *Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $Z \in \mathcal{Z}(Q) \cup \mathcal{Z}^{eval}(Q)$. Given two discretisable Q-persistence modules V and W, their HN filtered persistence landscapes satisfy*

$$\sup_{\theta \in \mathbb{R}} \|\lambda_{s^\theta_{Z,V}} - \lambda_{s^\theta_{Z,W}}\|_\infty \leqslant d_I(V, W).$$

Remark 4.4.14. Let $i \colon Q' \hookrightarrow Q$ be the inclusion of a bounded cube of Q containing the origin and let $Z \in \mathcal{Z}(Q')$. One could adapt the stability results of this section to the invariant $V \mapsto s_{Z,Q',V}$ defined by the function

$$(x, y) \in Q^2 \mapsto \begin{cases} \mathrm{rank}(\langle i^* T_x^* V, Z \rangle_{0 \leqslant y - x}) & \text{if } 0 \leqslant y - x \in Q' \\ 0 & \text{otherwise.} \end{cases}$$

Even though $s_{Z,Q',V}$ is not stronger than the rank invariant in general, the stability condition Z has the advantage of being a finite linear combination of the $\delta_c \circ \underline{\dim}$ – see (4.16).

4.5 Examples and computations

We compute the HN filtered rank invariant of spread-decomposable persistence modules (Proposition 4.5.1) and we give several examples of computations of the erosion distance between the HN filtered rank invariants of 2-parameter persistence modules.

4.5.1 HN filtered rank invariants between spread-decomposable persistence modules

Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Generalising Proposition 4.2.8, we compute the skyscraper invariant of spread-decomposable modules.

Let $\beta \in \mathcal{F}(Q, \mathbb{R})^*$ be a strictly positive linear form. Given $\theta > 0$ and a spread S of Q, the (θ, β)-*erosion* of S is the spread

$$S^{\beta,\theta} := \left\{ x \in S \mid \beta(T_x^* \mathbb{1}_{S \cap \langle x \rangle}) \leqslant \frac{1}{\theta} \right\}. \tag{4.23}$$

The family of subsets $(S^{\beta,\theta})_{\theta > 0}$ of S is decreasing. Examples of (β, θ)-erosion are depicted in Figure 4.1. Observe that the spreads are eroded from their minimal elements.

Proposition 4.5.1. *Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $S_1, \ldots, S_k$ be spreads of Q such that the module $V := \bigoplus_{i=1}^k \mathbb{F}_{S_i}$ is discretisable. Given a strictly positive $\beta \in \mathcal{F}(Q, \mathbb{R})^*$, we denote by Z_0 the stability condition $Z_0 := \mathrm{StabCond}_{\delta_0, \beta}$ – see (4.14) and (4.15). For $\theta > 0$, the Q-persistence module $V(\theta) := \bigoplus_{i=1}^k \mathbb{F}_{S_i^{\beta,\theta}}$ satisfies*

$$s^\theta_{Z_0, V} = \rho_{V(\theta)}.$$

Proof. By (4.9) and Theorem 4.3.11, $V \mapsto s^\theta_{Z_0,V}$ is additive and we can assume that $k = 1$. Let S be a spread of Q such that $\mathbb{F}_S \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ and let $\theta > 0$. To simplify the notation, for $\theta > 0$, we write $s^\theta := s^\theta_{Z_0, \mathbb{F}_S}$ and $r^\theta := \rho_{S^{\beta,\theta}}$. Since $0 \leqslant s^\theta, r^\theta \leqslant \rho_{\mathbb{F}_S}$, we know that s^θ and r^θ coincide outside of the subset $S_+ := \{(x, y) \in S^2 \mid x \leqslant y\}$ and that $s^\theta_{|S_+}$ and $r^\theta_{|S_+}$ take values in $\{0, 1\}$.

We first show that s^θ and r^θ coincide on $\{(x, x) \mid x \in S\}$. Let $x \in S$, let $G \colon P \hookrightarrow Q$ be a grid function adapted to $\mathbb{F}_S$ and $T_*^x Z_{\delta_{\{0\}}, \beta}$ and let $b_x \colon q \in P_x \mapsto \beta(T_x^* \mathbb{1}_{\mathrm{cub}_{G^x}(q)})$. We set $T := G^{-1}(S)$ and $p := G^{-1}(x)$. Following Proposition 4.2.8, we write $T^{b_x,\theta} := \left\{ q \in T \mid \sum_{q' \in T \cap \langle q \rangle} b_x(q') \leqslant \frac{1}{\theta} \right\}$. Since $\mathbb{1}_{S \cap \langle x \rangle} = G_* \mathbb{1}_{T \cap \langle p \rangle}$, we have

$$\beta(T_x^* \mathbb{1}_{S \cap \langle x \rangle}) = \sum_{q \in T \cap \langle p \rangle} \beta(T_x^* G_* \mathbb{1}_{\{q\}}) = \sum_{q \in T \cap \langle p \rangle} b_x(q),$$

whence,

$$s^\theta(x, x) = 1 \overset{(4.21)}{\Leftrightarrow} s^\theta_{b_x, \mathbb{F}_T}(p, p) = 1 \overset{4.2.8}{\Leftrightarrow} p \in T^{b_x,\theta} \Leftrightarrow x \in S^{\beta,\theta} \Leftrightarrow r^\theta(x, x) = 1.$$

Let $(x, y) \in S_+$. On the one hand, $\langle V, T_*^x Z_0 \rangle^\theta_{x \leqslant y}$, as a restriction of the isomorphism $(\mathbb{F}_S)_{x \leqslant y}$, is injective and $s^\theta(x, y) = s^\theta(x, x)$. On the other hand, we observe that $x \in S^{\theta,\beta} \implies y \in S^{\theta,\beta}$. Indeed, by convexity $T_{-y}(S) \cap \langle 0 \rangle \subset T_{-x}(S) \cap \langle 0 \rangle$ and since β is nonnegative, $\beta(T_y^* \mathbb{1}_{S \cap \langle y \rangle}) \leqslant \beta(T_x^* \mathbb{1}_{S \cap \langle x \rangle})$. Hence, we have $s^\theta(x, y) = s^\theta(x, x) = r^\theta(x, x) = r^\theta(x, y)$. $\square$

4.5.2 Examples with two-parameter persistence modules

Generalising Example 4.2.9, we now give examples of computations of the distance defined in Notation 4.4.9.

Example 4.5.2. Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Fix $M \in \mathbb{Z}_{>0}$ and denote by P^M the cube $[0, M)^n$ of Q. In order to study Q-persistence modules whose support is contained in P^M, one can consider the strictly positive linear form

$$\beta^M : a \in \mathcal{F}(Q, \mathbb{R}) \mapsto \int_Q \min(1, e^{M - \|x\|_\infty}) a dx.$$

The stability condition $Z^M := \mathrm{StabCond}_{\delta_0, \beta^M}$ over $\mathrm{Pers}_{\mathrm{fp}}(Q)$ has been chosen so as to simplify the skyscraper invariant computations. In particular, over $\mathbb{Z}^2$, these computations will coincide with the computations from Subsection 4.2.3. The stability condition Z^M over $\mathbb{R}^2$ has been used to produce Figure 4.1.

(i) over $Q = \mathbb{Z}^2$: let G^M be the inclusion $P^M \subset Q$, choose $V^M \in \mathrm{Pers}(P^M)$ and consider the module $V \in \mathrm{Pers}_{\mathrm{fp}}(Q)$ supported on P^M and such that $(G^M)^* V = V^M$. Let b be the constant 1 function on P^M. Using Notation (4.7) one can check that, for $x \in P^M$, $T^x_* Z^M([V]) = Z_{\mathbb{1}_{\{x\}}, b}([V^M])$. As a consequence, the continuous skyscraper invariant $s^\bullet_{Z^M, V}$ of V can be obtained by extending the skyscraper invariant $s^\bullet_{b, V^M}$ of V^M by either 0 or $+\infty$ outside of $(P^M)^2$.

Continuing Example 4.2.9, we choose $M = 2$ and we compare $V := \mathbb{F}_{\{0,1\} \times \{0\}} \oplus \mathbb{F}_{\{0\} \times \{0,1\}}$ and $W := \mathbb{F}_{\{0,1\}^2 \setminus \{(1,1)\}} \oplus \mathbb{F}_{\{(0,0)\}}$. We obtain

$$d^{Z^2}_{\mathbf{HN}}(V, W) = 1 > 0 = d_E(\rho_V, \rho_W).$$

Similarly, taking $M = 3$, Proposition 3.3.9(1) provides an example of two modules in $V, W \in \mathrm{Pers}_{\mathrm{fp}}(\mathbb{Z}^2)$ with the same generalised rank invariant (see [**75**]) but such that $d^{Z^3}_{\mathbf{HN}}(V, W) > 0$.

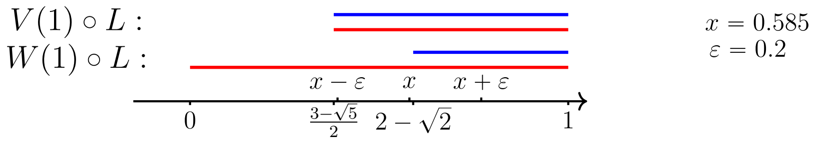
(ii) over $Q = \mathbb{R}^2$: Given a spread S of P^M and $\theta > 0$, if vol denotes the volume, then the (β^M, θ)-erosion (see (4.23)) of S is

$$S^{\beta^M, \theta} = \left\{ x \in S, \mathrm{vol}(S \cap \langle x \rangle) \leqslant \frac{1}{\theta} \right\}.$$

Fixing $M = 2$, we compare $V := \mathbb{F}_{[0,1) \times [0,2)} \oplus \mathbb{F}_{[0,2) \times [0,1)}$ and $W := \mathbb{F}_{[0,2)^2 \setminus [1,2)^2} \oplus \mathbb{F}_{[0,1)^2}$. For each support S of the spread modules in the decomposition of V and W, and for every $\theta > 0$, we depict in Figure 4.1 the spread $S^{\beta^2, \theta}$. We show that

$$0.5 \geqslant d_I(V, W) \geqslant d^{Z^2}_{\mathbf{HN}}(V, W) > 0.2 > 0 = d_E(\rho_V, \rho_W).$$

Indeed, the null maps provide a 0.5-interleaving between V and W, so $d_I(V, W) \leqslant 0.5$. Moreover, let $L\colon x \in \mathbb{R} \mapsto (x, x) \in \mathbb{R}^2$, the barcode of the modules $V(1) \circ L$ and $W(1) \circ L$ (see Proposition 4.5.1) are given below



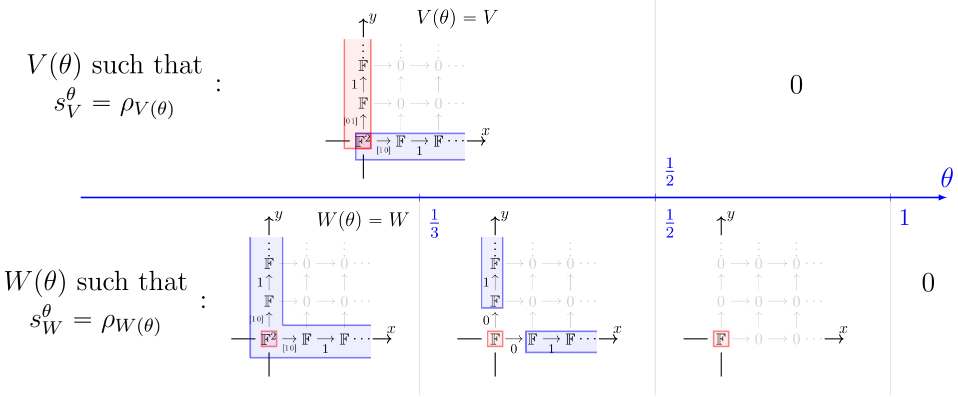
Taking $(x, \varepsilon) := (0.585, 0.2)$, we have $s^1_{Z^2, V}(L(x - \varepsilon), L(x + \varepsilon)) = 2 > s^1_{Z^2, W}(L(x), L(x))$. Thus, by Theorem 4.4.5, $d_I(V, W) \geqslant d^{Z^2}_{\mathbf{HN}}(V, W) \geqslant d_E(s^1_{Z^2, V}, s^1_{Z^2, W}) > 0.2$.

We now give an example of erosion distance between HN filtered rank invariants of persistence modules that are not compactly supported.

Example 4.5.3. Let β be the strictly positive linear form

$$\beta\colon a \in \mathcal{F}(\mathbb{Z}^2, \mathbb{R}) \mapsto \sum_{(x,y)\in\mathbb{Z}^2} \frac{a(x,y)}{2^{|x|+|y|}}.$$

and let Z denote the stability condition $\mathrm{StabCond}_{\delta_0,\beta}$ over $\mathrm{Pers}_{\mathrm{fp}}(\mathbb{Z}^2)$. Consider the $\mathbb{Z}^2$-persistence modules $V := \mathbb{F}_{\mathbb{Z}_{\geqslant 0}\times\{0\}} \oplus \mathbb{F}_{\{0\}\times\mathbb{Z}_{\geqslant 0}}$ and $W := \mathbb{F}_{\mathbb{Z}^2_{\geqslant 0}\backslash\mathbb{Z}^2_{>0}} \oplus \mathbb{F}_{\{(0,0)\}}$. Let S be the support of one of the spread modules in the decomposition of V (*resp.* W). Z has been chosen so that for all $x \in S\backslash\{(0,0)\}$, we have $\mathfrak{Re}Z[T_x^*\mathbb{F}_{S\cap\langle x\rangle}] = 2$ and $\mathfrak{Re}Z[\mathbb{F}_{S\cap\langle(0,0)\rangle}] = |S\cap\{0,1\}^2|$ where $|\ |$ denotes cardinality. In the first (*resp.* second) row of the diagram below, we depict for each $\theta > 0$, the spread $S^{\beta,\theta}$ (see Proposition 4.5.1):



We thus obtain that $d^Z_{\mathbf{HN}}(V,W) = 1 > 0 = d_E(\rho_V, \rho_W)$.

4.6 Discreteness of HN filtered rank invariants

Let Q be either $\mathbb{Z}^n$ or $\mathbb{R}^n$, let V be a discretisable Q-persistence module and let $Z \in \mathcal{Z}(Q)$ be a discretisable stability condition over Q. Figure 4.1 shows that, in general, the HN filtered rank invariants over Q are not discretisable functions in the sense of Subsection 4.1.5. We show instead that when $\mathfrak{Im}Z$ is nonnegative, the HN filtered rank invariants over Q are constant on the parts of a subdivision of its domain into a finite number of semialgebraic sets.

The wall-chamber structure of the space of stability conditions has been well-studied for representations of finite quivers [**65**] and more generally for modules over a finite-dimensional algebra [**24**]. For the purpose of this paper, we use the following definitions inspired by [**65**, Section 3]. Let P be a finite poset and let $U \in \mathrm{Pers}(P)$. Following equation (4.8), we see a stability condition $\widehat{Z}$ over P as a vector of $\mathbb{R}^P \times \mathbb{R}^P_{>0}$. A *wall system* $\{W^i\}_I$ for U is a set of codimension 1 algebraic subvarieties of $\mathbb{R}^P \times \mathbb{R}^P$ such that for each $I' \subset I$, and each connected component C of $(\bigcap_{I'} W^i)\backslash(\bigcup_{I\backslash I'} W^i)$, the HN filtration $0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^\ell = U$ of U along $\widehat{Z}$ does not depend on $\widehat{Z} \in C \cap (\mathbb{R}^P \times \mathbb{R}^P_{>0})$. In that case, the semialgebraic sets $C \cap (\mathbb{R}^P \times \mathbb{R}^P_{>0})$ are called *chambers* and each W^i is called a *wall*. A key observation is that the codimension 1 algebraic subvarieties amongst

$$\left(\left\{ \widehat{Z} \in \mathbb{R}^P \times \mathbb{R}^P_{>0} \mid \frac{\mathfrak{Im}\widehat{Z}(d)}{\mathfrak{Re}\widehat{Z}(d)} = \frac{\mathfrak{Im}\widehat{Z}(d')}{\mathfrak{Re}\widehat{Z}(d')} \right\}\right)_{0 \leqslant d, d' \leqslant \underline{\dim}_U} \tag{4.24}$$

provide a finite wall system for U.

In order to extend the wall-chamber structure to the infinite case, we first study common refinements between two families of grid functions. Let $(G^{1,x} \colon P^1 \hookrightarrow Q)_x$ and $(G^{2,x} \colon P^2 \hookrightarrow Q)_x$ be two families of grid functions whose values (but not their domains) depend on a parameter $x \in C$. An *independent common refinement* of $(G^{1,x})_x$ and $(G^{2,x})_x$ is a choice for each x of a common refinement $G^x \colon P \hookrightarrow Q$ of $G^{1,x}$ and $G^{2,x}$ such that there are two grid functions $\tau^1 \colon P^1 \hookrightarrow P$ and $\tau^2 \colon P^2 \hookrightarrow P$ independent of x which fit for all x into the following commutative diagram:

$$\begin{tikzcd}
P^1 \arrow[r, hook, "\tau^1"] \arrow[dr, hook, "G^{1,x}"'] & P \arrow[d, hook, "G^x"] & P^2 \arrow[l, hook', "\tau^2"'] \arrow[dl, hook', "G^{2,x}"] \\
& Q &
\end{tikzcd} \tag{4.25}$$

Lemma 4.6.1. *Let $(G^{1,x} \colon P^1 \hookrightarrow Q)_{x \in C}$ and $(G^{2,x} \colon P^2 \hookrightarrow Q)_{x \in C}$ be two families of grid functions. $(G^{1,x})_x$ and $(G^{2,x})_x$ have an independent common refinement $(G^x)_{x \in C}$ if and only if for $1 \leqslant i \leqslant n$ and $(p^1, p^2) \in P^1_i \times P^2_i$, the functions*

$$x \in C \mapsto G^{1,x}_i(p^1) - G^{2,x}_i(p^2) \in Q_i \tag{4.26}$$

all have constant sign (in $\{-1, 0, 1\}$).

Proof. We first assume that $n = 1$. Assume that there exist $(G^x)_{x \in C}$, τ^1 and τ^2 fitting diagram (4.25). Let $(p^1, p^2) \in P^1 \times P^2$, since, for $x \in C$, G^x is increasing, the function $x \in C \mapsto \operatorname{sign}(G^{1,x}_i(p^1) - G^{2,x}_i(p^2)) = \operatorname{sign}(G^x \circ \tau^1(p^1) - G^x \circ \tau^2(p^2))$ is the constant $\operatorname{sign}(\tau^1(p^1) - \tau^2(p^2))$.

Conversely, assume that the functions in (4.26) have a constant sign and fix $y \in C$. By Remark 4.1.3, there exists a common refinement $G^y \colon P \hookrightarrow Q$ of $G^{1,y}$ and $G^{2,y}$ such that $\operatorname{Im} G^y = \operatorname{Im} G^{1,y} \cup \operatorname{Im} G^{2,y}$. For $k \in \{1, 2\}$, let $\tau^k \colon P^k \hookrightarrow P$ be a grid function such that $G^{y,k} = G^y \circ \tau^k$. By injectivity of G^y, we have $P = \operatorname{Im} \tau^1 \cup \operatorname{Im} \tau^2$. For $x \in C$ we define the function $G^x \colon P \hookrightarrow Q$ such that for $k \in \{1, 2\}$, we have $G^x \circ \tau^k = G^{k,x}$. We only need to prove that G^x is increasing. Indeed, given $(p, p') \in P \times P$, there exist $(k, \ell) \in \{1, 2\}^2$ and $(p^k, p^\ell) \in P^k \times P^\ell$ such that $(p, p') = (\tau^k(p^k), \tau^\ell(p^\ell))$. If $k = \ell$, since $G^{k,x}$ is increasing for all $x \in C$, the sign of $G^x(p) - G^x(p') = G^k(p^k) - G^\ell(p^\ell)$ is given by the sign of $p^k - p^\ell$. Otherwise, $x \mapsto G^x(p) - G^x(p')$ is one of the functions in (4.26) and since G^y is increasing its sign is the constant $\operatorname{sign}(p - p')$.

Assume now that $n > 1$. The result is obtained by applying the Lemma, for each $1 \leqslant i \leqslant n$, to the families $(G^{1,x}_i)_x$ and $(G^{2,x}_i)_x$. $\square$

A family of grid functions $G^x \colon P \hookrightarrow Q$ indexed by x is said to be *affine* if $x \in C \mapsto G^x(p) \in Q$ is affine in x for each $p \in P$. Let

$$G^V \colon P^V \hookrightarrow Q \qquad \text{and} \qquad G^Z \colon P^Z \hookrightarrow Q \tag{4.27}$$

be finite grid functions adapted to, respectively, V and Z. Without loss of generality, assume that $0 \in \operatorname{Im} G^Z$. Since $Z \in \mathcal{Z}(Q)$, we can choose an extension $\widetilde{G}^Z \colon \mathbb{Z}^n \hookrightarrow Q$ of G^Z such that Z is a complex linear combination of $(\delta_c \circ \underline{\dim})_{c \in \mathcal{C}_{\mathrm{b}}(\widetilde{G}^Z)}$.

Lemma 4.6.2. *There exists a partition $\mathcal{P}$ of Q into a finite number of cubes such that for each part $C \in \mathcal{P}$, there is a finite affine independent common refinement $(G^x \colon P \to Q)_{x \in C}$ of $(T_{-x} \circ G^V)_x$ and $(G^Z)_x$ (see (4.27)) so that the function*

$$x \in C \mapsto (G^x)^* Z \in \mathbb{R}^P \times \mathbb{R}^P_{>0}$$

is a polynomial in x whenever C is bounded.

Proof.

Construction of $\mathcal{P}$ and G^x: The functions in (4.26) with for $x \in Q$, $G^{1,x} := T_{-x} \circ G^V$ and $G^{2,x} := G^Z$ delimit a finite partition $\widehat{\mathcal{P}}$ of Q where inside each part, the sign (in $\{-1, 0, 1\}$) of each of those functions is constant. Moreover, since G^Z is independent of x and each $(T_{-x} \circ G^V)_i$ is affine in x_i and independent of $(x_j)_{j \neq i}$, the parts of this partition are cubes.

Let C be a cube in $\widehat{\mathcal{P}}$, by Lemma 4.6.1, there is a finite independent common refinement $G^x \colon P \hookrightarrow Q$ of $(T_{-x} \circ G^V)$ and G^Z indexed by $x \in C$. And we can choose $(G^x)_x$ to be affine since $(T_{-x} \circ G^V)$ and (G^Z) are both affine.

Assume now that $C \in \widehat{\mathcal{P}}$ is bounded. Since (G^x) is affine, there is a bounded cube B of Q such that for each $x \in C$, we have $\operatorname{Im} G^x \subset B$. Hence, $\operatorname{Im} \widetilde{G}^Z_i \cap B_i$ is finite for all i, and by Lemma 4.6.1, we can further partition C into a finite number of cubes C' inside which (G^x) and $\widetilde{G}^Z$ have an affine independent common refinement $(\widetilde{G}^x \colon \mathbb{Z}^n \to Q)$. Let $\mathcal{P}$ be the refinement of $\widehat{\mathcal{P}}$ induced by this partition of each bounded $C \in \widehat{\mathcal{P}}$.

Semialgebraicity of $(G^x)^* Z$: Assume that $C \in \mathcal{P}$ is bounded. For $x \in C$, we denote by $Z^x = (\mathfrak{Im} Z^x, \mathfrak{Re} Z^x) \in \mathbb{R}^P \times \mathbb{R}^P_{>0}$ the stability condition $(G^x)^* Z$. Let $\tau \colon P \to \mathbb{Z}^n$ be a grid function such that $\widetilde{G}^x \circ \tau = G^x$ for all $x \in C$. Fix $p \in P$, one can check that for $z \in \mathbb{Z}^n$,

$$\operatorname{cub}_{\widetilde{G}^x}(z) \cap \operatorname{cub}_{G^x}(p) = \begin{cases} \operatorname{cub}_{\widetilde{G}^x}(z) & \text{if } z \in \operatorname{cub}_\tau(p) \\ \varnothing & \text{otherwise} \end{cases}$$

So for $c \in \mathcal{C}_{\mathrm{bf}}(\widetilde{G}^x)$, the map $x \in C \mapsto \delta_c(\mathbb{1}_{\operatorname{cub}_{G^x}(p)})$ is polynomial of degree at most n. Since $\mathfrak{Im} Z^x_p$ is a linear combination of finitely many such maps, it is also polynomial in $x \in C$. By independence of $\widetilde{G}^x$, there is $b \colon \mathbb{Z}^n \to \mathbb{R}_{>0}$ such that for every $x \in C$ we have $\mathfrak{Re} Z^x \colon \widehat{V} \in \mathrm{Pers}_{\mathrm{fp}}(Q) \mapsto \int_Q (\widetilde{G}^x_* b) \, \underline{\dim}_{\widehat{V}}$. The function $x \in C \mapsto \mathfrak{Re} Z^x_p$ can now be expressed as the pointwise limit when $N \to +\infty$ of the polynomials

$$R_N \colon x \in C \mapsto \sum_{z \in [-N,N]^n \cap \operatorname{cub}_\tau(p)} b_z |\operatorname{cub}_{\widetilde{G}^x}(z)|.$$

whose variables are the x_i for $1 \leqslant i \leqslant n$ such that $\operatorname{len} C_i > 0$. Here, $|\ |$ denotes the volume if $Q = \mathbb{R}^n$ or the cardinality if $Q = \mathbb{Z}^n$. Finally, since the degree of R_N is bounded by n, by Lagrangian approximation, the coefficients of R_N are also bounded and the limit $\mathfrak{Re} Z^x_p$ is itself polynomial. □

Definition 4.6.3 ([92, Section 1])**.** A function $\phi\colon X \to \mathbb{Z}$ from a real algebraic set X is called *semialgebraically constructible* if it can be written as a finite sum

$$\phi = \sum_i m_i \mathbb{1}_{X_i}$$

where for each i, $m_i \in \mathbb{Z}$ and X_i is a semialgebraic subset of X. △

We denote by Q_+ the half-space $\{(x, y) \in Q^2 \mid x \leqslant y\}$ of Q^2.

Proposition 4.6.4. *Let Q be either $\mathbb{R}^n$ or $\mathbb{Z}^n$ and let $Z \in \mathcal{Z}(Q)$ such that $\mathfrak{Im} Z \circ \underline{\dim}^{-1}$ is nonnegative. Given $V \in \mathrm{Pers}_{fp}(Q)$, the HN filtered rank invariant*

$$s^{\bullet}_{Z,V} \left\{ \begin{array}{ccc} Q_+ \times \mathbb{R} & \longrightarrow & \mathbb{Z} \\ (x, y, \theta) & \longmapsto & s^{\theta}_{Z,V}(x, y) \end{array} \right.$$

is a semialgebraically constructible function.

Proof. Recall the finite grid functions $G^V\colon P^V \hookrightarrow Q$ and $G^Z\colon P^Z \hookrightarrow Q$ from (4.27) and let $\mathcal{P}$ be a partition of Q given by Lemma 4.6.2. Let $C = C_1 \times \cdots \times C_n \in \mathcal{P}$ and let $\widetilde{U} \in \mathrm{Pers}(P^V)$ be a G^V-discretisation of V.

Unbounded below: If one of the components C_i is unbounded below, then for all $x \in C$, we have

$$s^{\theta}_{Z,V}(x, y) = \mathrm{rank}\langle T^*_x V, Z\rangle^{\theta}_{0 \leqslant y-x} = \begin{cases} \mathrm{rank}\, V_{x\leqslant y} & \text{if } \theta \leqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, by Lemma 4.6.1, for all $(p^V, p^Z) \in P^V_i \times P^Z_i$, the function $x_i \in C_i \mapsto G^V_i(p^V) - x_i - G^Z_i(p^Z) \in Q_i$ is of constant sign and is hence positive – meaning that the support of $T^*_x V$ does not intersect $\bigcup_{c \in \mathcal{C}_{\mathrm{b}}(G^Z)} c$; whence $\mathfrak{Im} Z(T^*_x \widehat{V}) = 0$.

Bounded: If C is bounded, then using the result and notations of Lemma 4.6.2, there is $U \in \mathrm{Pers}(P)$ such that $T^*_x V \simeq G^x_* U$ for all $x \in C$. Indeed, one can choose $U = \tau_* \widetilde{U}$ where $G^x \circ \tau = G^V$. By Theorem 4.3.11, we obtain $\langle G^x_* U, Z\rangle^{\theta} = G^x_* \langle U, (G^x)^* Z\rangle^{\theta}$ for every $\theta \in \mathbb{R}$, and in particular,

$$s^{\theta}_{Z,V}(x, y) = \mathrm{rank}\langle U, (G^x)^* Z\rangle^{\theta}_{\lfloor 0\rfloor_{G^x} \leqslant \lfloor y-x\rfloor_{G^x}}.$$

The inverse image of the chambers defined in (4.24) by the polynomial $x \in C \mapsto (G^x)^* Z \in \mathbb{R}^P \times \mathbb{R}^P_{>0}$ are semialgebraic. Fix one such semialgebraic set C', the HN filtration $0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^{\ell} = U$ of U along $(G^x)^* Z$ does not depend on $x \in C'$. Let $p \in P$ and $0 \leqslant i \leqslant \ell$, the set $S_{C',p,i}$ of all $(x, y, \theta) \in C' \times Q \times \mathbb{R}$ such that

$$y - x \in \mathrm{cub}_{G^x}(p) \qquad \text{and} \qquad \mu_{(G^x)^* Z}(U^i/U^{i-1}) \geqslant \theta > \mu_{(G^x)^* Z}(U^{i+1}/U^i) \tag{4.28}$$

is semialgebraic. There are finitely many $S_{C',p,i}$ and inside each of them, $s^{\theta}_{Z,V}(x, y)$ is the constant given by $\mathrm{rank}(U^i)_{0 \leqslant p}$.

Unbounded above: Finally, assume that each component of $C = C_1 \times \cdots \times C_n$ is either bounded or unbounded above. Let I_b and I_u denote the set of indices $1 \leqslant i \leqslant n$ where C_i is respectively bounded and unbounded above. Let π_b and π_u be the projection of C onto respectively $\prod_{i \in I_b} C_i$ and $\prod_{i \in I_u} C_i$. Given $\theta \in \mathbb{R}$, we show that $(x, y) \in (C \times Q) \cap Q_+ \mapsto$

$s^{\theta}_{Z,V}(x, y)$ only depends on $\pi_b(x)$, $\pi_b(y)$ and $\pi_u(y - x)$. For $W \in \mathrm{Pers}_{\mathrm{fp}}(Q)$, define the discretisable submodule $W^{\geqslant}$ of W determined by the subspaces

$$W^{\geqslant}_x := \begin{cases} W_x & \text{if } x \geqslant G^Z(\min P^Z) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathfrak{Re} Z$ is strictly positive and $\mathfrak{Im} Z([W^{\geqslant}]) = \mathfrak{Im} Z([W]) \geqslant 0$, if $0 \subsetneq W^{\geqslant} \subsetneq W$, then $\mu_Z(W^{\geqslant}) > \mu_Z(W)$. As a consequence, every nonzero Z-semistable W satisfies either $\mu_Z(W) = 0$ or $W = W^{\geqslant}$, whence $\langle V, Z\rangle^{\geqslant} = \langle V^{\geqslant}, Z\rangle$.

By Lemma 4.6.1, for $i \in I_u$ and $p^V \in P^V_i$, the expression $x_i \in C_i \mapsto G^V_i(p^V) - x_i - G^Z_i(\min P^Z_i)$ has constant sign and is hence negative. As a consequence, given $x, x' \in C$ such that $\pi_b(x) = \pi_b(x')$, we have $(T^*_x V)^{\geqslant} \simeq (T^*_{x'} V)^{\geqslant}$, and since $0 \in \mathrm{Im}\, G^Z$, for every $z \geqslant 0$ in Q, we obtain

$$\begin{aligned} s^{\bullet}_{Z,V}(x, x + z) = \mathrm{rank}[(\langle T^*_x V, Z\rangle^{\bullet})^{\geqslant}]_{0 \leqslant z} &= \mathrm{rank}\langle (T^*_x V)^{\geqslant}, Z\rangle^{\bullet}_{0 \leqslant z} \\ &= \mathrm{rank}\langle (T^*_{x'} V)^{\geqslant}, Z\rangle^{\bullet}_{0 \leqslant z} \\ &= s^{\bullet}_{Z,V}(x', x' + z). \end{aligned}$$

Fix $x_0 \in C$, by the previous case the bounded cube $\pi_u^{-1}(\{x_0\}) \times Q \times \mathbb{R}$ of $C \times Q \times \mathbb{R}$ can be partitioned into finitely many algebraic sets $(S_j)_j$ such that each $(s^{\bullet}_{Z,V})_{|S^j}$ is constant. Since the map $\pi_0 \colon (x, y, \theta) \in C \times Q \times \mathbb{R} \mapsto (x_0, x_0 + y - x, \theta) \in \pi_u^{-1}(\{x_0\}) \times Q \times \mathbb{R}$ is affine, the set $C \times Q \times \mathbb{R}$ can be partitioned into finitely many semialgebraic sets $(\pi_0^{-1}(S^j))_j$ inside which $s^{\bullet}_{Z,V}$ is constant. □

The polynomial inequalities that define the semialgebraic sets used in the proof of Proposition 4.6.4 are given by (4.24), (4.26) and (4.28). Using the result from [**36**], they can be computed algorithmically. In general, this set of inequalities is not minimal and its size is not polynomial in the cardinal of P^V and in $\max \underline{\dim}_V$.

4.7 Conclusions

Theorem 4.3.11 shows that HN filtrations of discretisable $\mathbb{R}^n$-persistence modules exist along discretisable stability conditions. These filtrations can be used to define the HN filtered rank invariants (Definition 4.3.17) which are stable with respect to the interleaving distance (Theorem 4.4.7). Several open questions remain:

1. This article only considers discretisable (*i.e.* finitely presentable) persistence modules. There is a more general notion of tameness for $\mathbb{R}^n$-persistence modules [**93**] which defines, under some auxiliary assumptions, a full abelian subcategory of $\mathrm{Pers}(\mathbb{R}^n)$ [**113**]. Under which hypotheses can the HN formalism be extended to this more inclusive notion of tameness?

2. Section 4.3 gives examples and non-examples of stability conditions over $\mathrm{Pers}_{\mathrm{fp}}(\mathbb{R}^n)$ for which HN filtrations are well-defined. Can one characterise such stability conditions?

3. Given integrable functions $a\colon \mathbb{R}^n \to \mathbb{R}$ and $b\colon \mathbb{R}^n \to \mathbb{R}_{>0}$, is it possible to define the HN filtered rank invariant along $Z_{a,b}$ by approximating a and b by step functions?

In order to use HN filtered rank invariants in practice, it is necessary to develop efficient tools to compute them.

(i) In the finite setting (over a finite poset), Cheng proved [**36**] that the HN filtration of a persistence module along an integral stability condition can be computed in polynomial time (in the size of the poset, the dimensions of the persistence module and the values of the stability condition) but to the best of my knowledge there is no implementation available.

(ii) In the infinite setting (over $\mathbb{R}^n$), Section 4.6 shows that the HN filtered rank invariants can be recovered from a finite number of computations in the finite setting. However, in the description of Section 4.6, this number of computations is not necessarily polynomial. A possible workaround to obtain a polynomial-time algorithm could be to compute approximations of the HN filtered rank invariants by discretising their domain.

Finally, once HN filtered rank invariants are computed, the next step is to compute the erosion distance between them. In the finite setting, there exists a polynomial-time algorithm to compute the classical erosion distance [**76**, Section 5]. One possible approach would be to adapt this algorithm to HN filtered rank invariants indexed over $\mathbb{R}^n$.

Appendix A

Invariants for ladder persistence

Introduction

Chromatic topological data analysis adapts techniques like persistent homology to the case of coloured pointclouds. One of the motivations for developing this approach is the study of the relative spatial arrangement of cells of different types in a tissue. By applying the chromatic variants of the Alpha filtration to a pointcloud partitioned into two colour classes, one can efficiently produce an inclusion of filtered simplicial complexes $\iota\colon K_\bullet \hookrightarrow L_\bullet$ that contains information about the spatial interaction of the two colour classes [95, 98]. The persistent homology in degree $p \geqslant 0$ of this inclusion is a ladder persistence module $\mathbf{H}_p(\iota)$, or in other words, a functor from the poset $\{0,1\} \times \mathbb{R}$ to the category Vect of vector spaces over a fixed field.

Invariants Many invariants have been proposed to study the inclusion $\iota\colon K_\bullet \hookrightarrow L_\bullet$. using persistent homology. In the literature, most of these invariants rely on computing barcodes or matchings between barcodes. A few examples of these invariants are

- the *six-pack diagram* [95]: it is obtained by extracting 6 different ordinary persistence modules from $\mathbf{H}_p(\iota)$ and computing their persistence diagram. More precisely, consider the following ordinary persistence modules: the domain, the codomain, the image, the kernel, and the cokernel of $\mathbf{H}_p(\iota)$, plus the relative homology $\mathbf{H}_p(L_\bullet, K_\bullet)$. For $k \in \{3, \ldots, 6\}$, we call *k-pack diagram*, the persistence diagrams of the k first persistence modules in this list.
- induced matchings [110, 59]: the idea is to build from ι a partial matchings between the barcodes of $\mathbf{H}_p(K_\bullet)$ and $\mathbf{H}_p(L_\bullet)$

Another venue is to directly apply invariants to the functor $\mathbf{H}_p(\iota)\colon \{0,1\} \times \mathbb{R} \to \text{Vect}$. Indeed, the functor $\mathbf{H}_p(\iota)$ is an instance of a persistence module, and many invariants of persistence modules exist in the literature. Examples of those include

- the *rank invariant* [31]: this invariant is given by the rank of the maps $\mathbf{H}_p(\iota)_{x\leqslant y}$ for every $x \leqslant y$ in $\{0,1\} \times \mathbb{R}$.

- the *generalised rank invariant* [75]: the main idea is to compute the rank of the limit-to-colimit map of restrictions of the functor $\mathbf{H}_p(\iota)\colon \{0,1\} \times \mathbb{R} \to \mathrm{Vect}$ to different subposets. If the subposets used to restrict $\{0,1\} \times \mathbb{R}$ are given by the spreads (see Subsection A.1.1) with one source (*resp.* either one source or one target), we denote the obtained invariant by $\mathrm{GRI}^{(1,2)}$ (*resp.* $\mathrm{GRI}^{(1,2),(2,1)}$).

- the *skyscraper invariant* [52, 69]: when applied to $\mathbf{H}_p(\iota)$, this invariant is equivalent to the decomposition of the spread-decomposable ladder persistence modules $(\operatorname{Im}\mathbf{H}_p(\iota)_{x\leqslant y})_{y\geqslant x}$ for every $x \in \{0,1\} \times \mathbb{R}$.

Computability Another important question is to determine which of these invariants can be computed efficiently. The existing software to compute the barcode of a filtered simplicial complex can be adapted to the case of the six-pack diagram, leading to algorithms which are in theory cubic in the number of simplices $|L|$ of L, but are much faster in practice [95, 98]. In the case of ladder persistence, Jacquard developed [69] and implemented a matrix reduction algorithm to compute the skyscraper invariant in time $\mathcal{O}(|L|^4)$. This algorithm operates at the level of the module $\mathbf{H}_p(\iota)$, represented by matrices in fixed bases. It is possible to obtain a quartic-time algorithm that works directly on the simplices of L by decomposing, for every $x \in \mathbb{R}$, the zigzag persistence module which arises as the homology of the sequence of simplicial complexes indexed by the zigzag

$$\begin{tikzcd}
(1,x) \arrow[r] & {} \\
(0,x) \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

In general, the generalised rank invariant is hard to compute [74]. To the best of my knowledge, there is no efficient implementation of the generalised rank invariant for the case of ladder persistence.

Main result In this Appendix, we compare the discriminating power of certain invariants in chromatic TDA.

Given an invariant I of inclusions of filtered simplicial complexes, we denote by $\mathrm{undist}(I)$ the set of pairs of inclusions $(\iota\colon K_\bullet \hookrightarrow L_\bullet, \iota'\colon K'_\bullet \hookrightarrow L'_\bullet)$ such that $I(\iota) = I(\iota')$. We say that two invariants I and I' are *equivalent* if $\mathrm{undist}(I) = \mathrm{undist}(I')$. We say that I' is *stronger* than I if $\mathrm{undist}(I') \subseteq \mathrm{undist}(I)$. This notion of strength defines a partial order $\preceq$ for invariants up to equivalence.

Theorem (I). *The partial order $\preceq$ induced by the relative strength of the invariants introduced in this Appendix is summarized by the following Hasse diagram.*

$$\begin{tikzcd}
\mathrm{GRI}^{(1,2),(2,1)} & \textit{6-pack diagram} & \\
\mathrm{GRI}^{(1,2)} \arrow[u] & \textit{5-pack diagram} \arrow[ul] \arrow[u] & \textit{skyscraper invariant} \\
 & \textit{4-pack diagram} \arrow[ul] \arrow[u] \arrow[ur] & \\
 & \textit{3-pack diagram} = \textit{rank invariant} \arrow[u] &
\end{tikzcd}$$

Here, there is a path from invariant I to invariant I' if and only if I' is stronger than I.

A.1 Preliminaries

A.1.1 Ladder persistence modules

Let P be a poset, a *P-persistence module* is a functor V from P to the category Vect of finite-dimensional vector spaces over a fixed field $\mathbb{F}$. We denote by Vect^P the category of P-persistence modules.

A subset I of P is called a *spread* (or sometimes an *interval*) if it is:

- connected: for all $u, v \in I$, there is a sequence $(u = u_0, u_1, \ldots, u_\ell = v)$ in I such that u_i and u_{i+1} are comparable in P for each i; and
- convex: I contains the set $\{p \in P \mid u \leqslant p \leqslant v\}$ for each $u \leqslant v$ in I.

We denote by $\mathcal{I}(P)$ the set of spreads of P. We will often see a spread $I \in \mathcal{I}(P)$ as a subposet of P defined for $x, y \in I$ by $x \leqslant y$ in I whenever $x \leqslant y$ in P. The sources and targets of I are the number of respectively minimal and maximal elements in I, seen as a poset. Given two integers $s, t \geqslant 0$, we denote by $\mathcal{I}^{(s,t)}(P)$, the set of spreads of P with at most s sources and t targets. Given integers $s_1, \ldots, s_n \geqslant 0$ and $t_1, \ldots, t_n \geqslant 0$, we set $\mathcal{I}^{(s_1,t_1),\ldots,(s_n,t_n)}(P) := \bigcup_i \mathcal{I}^{(s_i,t_i)}(P)$.

The *spread module* of support I is the P-persistence $\mathbf{I}[I]$ whose spaces are given for $x \in P$ by $\mathbf{I}[I]_x = \mathbb{F}$ if $x \in I$ and $\mathbf{I}[I]_x = 0$ otherwise; and whose maps between nonzero spaces are identities. The *restriction* $V_{|I}$ of a P-persistence module $V \in \mathrm{Vect}^P$ to a spread $I \in \mathcal{I}(P)$ is the composition $V \circ \theta$ where $\theta \colon I \hookrightarrow P$ is the inclusion of the subposets induced by I.

If P is totally ordered, we will refer to P-persistence modules as *(ordinary) persistence modules.* An ordinary persistence module V can be decomposed into a direct sum of spread modules [55]. The multiset of spreads in the decomposition V is called the *persistence diagram* (or the *barcode*) of V, and will be denoted by $\mathbf{Bar}(V)$ [46, 117].

When there is a totally ordered poset T such that $P \simeq \{0,1\} \times T$, P-persistence modules are usually called *ladder persistence modules* [47]. In this Appendix, we restrict ourselves to ladder persistence modules obtained as the homology of an inclusion of filtered simplicial complexes. More precisely, the input for the invariants we study will be real numbers $r_0 < \cdots < r_n$ and inclusions of simplicial complexes

$$\begin{array}{ccccccc} L_{r_0} & \subset & L_{r_1} & \subset & \dots & \subset & L_{r_n} \\ \cup & & \cup & & & & \cup \\ K_{r_0} & \subset & K_{r_1} & \subset & \dots & \subset & K_{r_n} \end{array} \tag{A.1}$$

We denote by InclFiltSimp the set of such inclusions of filtered simplicial complexes. The p-th homology of the inclusion $K_\bullet \subset L_\bullet$ in (A.1) is determined by the poset $P = \{0,1\} \times \{0,1,\dots,n\}$, the inclusion $i \in \{0,1,\dots,n\} \mapsto r_i \in \mathbb{R}$, and the P-persistence module

$$\begin{tikzcd} \mathbf{H}_p(L_{r_0}) \arrow[r] & \mathbf{H}_p(L_{r_1}) \arrow[r] & \dots \arrow[r] & \mathbf{H}_p(L_{r_n}) \\ \mathbf{H}_p(K_{r_0}) \arrow[r] \arrow[u] & \mathbf{H}_p(K_{r_1}) \arrow[r] \arrow[u] & \dots \arrow[r] & \mathbf{H}_p(K_{r_n}) \arrow[u] \end{tikzcd}$$

For convenience, we will relabel the elements of the poset P as follows

$$\begin{tikzcd} x_0^+ \arrow[r] & x_1^+ \arrow[r] & \cdots \arrow[r] & x_{n-1}^+ \arrow[r] & x_n^+ \\ x_0^- \arrow[r] \arrow[u] & x_1^- \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{n-1}^- \arrow[r] \arrow[u] & x_n^- \arrow[u] \end{tikzcd} \tag{A.2}$$

Note that the results and algorithms in [95] are only defined for inclusions $\iota\colon K_\bullet \hookrightarrow L_\bullet$ such that simplices in K appear at the time in $K_\bullet$ and in $L_\bullet$. If this assumption is not satisfied, we can consider $\tilde{\iota} \in$ InclFiltSimp obtained from triangulating the inclusion $K_\bullet \times \{0\} \hookrightarrow (L_\bullet \times \{1\}) \cup (K_\bullet \times [0,1))$. The inclusions ι and ι' define homotopy equivalent pairs, and hence the invariants defined in the introduction will give the same output for ι and ι'. Moreover, the simplices in $K_\bullet \times \{0\}$ appear at the same time in the domain and codomain of $\tilde{\iota}$.

A.1.2 Invariants

For the purpose of this appendix, an *invariant* is a map from InclFiltSimp to a countable set $\mathcal{P}$. When studying the discriminating power of an invariant I, it is enough to consider the set of pairs of inputs that cannot be distinguished by I

$$\operatorname{undist}(I) := \left\{(\iota, \iota') \in \text{InclFiltSimp}^2 \mid I(\iota) = I(\iota')\right\}.$$

An invariant I is said to be *equivalent* to (*resp.* *stronger* than) another invariant I' if $\operatorname{undist}(I) = \operatorname{undist}(I')$ (*resp.* $\operatorname{undist}(I) \subseteq \operatorname{undist}(I')$).

In this Subsection, we define the invariants appearing in Theorem (I). Apart from the relative persistent homology, all these invariants factor through the homology functor. For convenience, when an invariant is of the form

$$I\colon \mathrm{InclFiltSimp} \xrightarrow{\mathbf{H}_p} \bigcup_{n>0} \mathrm{Vect}^{\{0,1\}\times\{0,1,\dots,n\}} \xrightarrow{\tilde{I}} \mathcal{P}$$

we will refer to $\tilde{I}$ as the invariant instead of $\tilde{I}\circ\mathbf{H}_p$. For this appendix, $\tilde{I}$ will be either (a restriction of) the GRI, the domain, codomain, image, kernel or cokernel persistent homology, or the skyscraper invariant. We will see that all these invariants are *additive*. Namely, the codomain $\mathcal{P}$ of I has a structure of abelian group; and given $n \geqslant 0$ and $V, W \in \mathrm{Vect}^{\{0,1\}\times\{0,\dots,n\}}$, we have $\tilde{I}(V\oplus W) = \tilde{I}(V) + \tilde{I}(W)$. The discriminating power of additive invariants of persistence modules has previously been studied in [14, 13, 48].

The Generalised Rank Invariant Let P be a poset. Kim and Mémoli [75] introduced a generalised version of the rank invariant for P-persistence modules by studying their restriction to spreads of P.

Definition A.1.1. The *generalised rank invariant* (GRI) of $V \in \mathrm{Vect}^P$ if the assignment for each spread $I \in \mathcal{I}(P)$ of the rank of the limit-to-colimit map of $V_{|I}$:

$$\mathrm{GRI}_V \begin{cases} \mathcal{I}(P) & \longrightarrow & \mathbb{Z} \\ I & \longmapsto & \mathrm{rank}(\varprojlim V_{|I} \to \varinjlim V_{|I}) \end{cases}. \qquad \triangle$$

One can restrict the GRI to subsets of $\mathcal{I}(P)$. In this work, given integers $s_1, \dots, s_n \geqslant 0$ and $t_1, \dots, t_n \geqslant 0$, the restriction of GRI to $\mathcal{I}^{(s_1,t_1),\dots,(s_n,t_n)}(P)$ is denoted $\mathrm{GRI}^{(s_1,t_1),\dots,(s_n,t_n)}$.

When P is the product of two totally ordered posets, $\mathrm{GRI}^{(s,t)}$ can be computed using zigzag persistence. More precisely, given $V \in \mathrm{Vect}^P$ and $I \in \mathcal{I}(P)$, the integer $\mathrm{GRI}_V(I)$ is the multiplicity of the contour zigzag ∂I of I in $\mathbf{Bar}(V_{|\partial I})$ [75, 42]. Moreover, $\mathrm{GRI}^{(1,1)}$ is called the *rank invariant* [31] and is equivalent to the invariant $V \in \mathrm{Vect}^P \mapsto (\mathrm{rank}\, V_{x\leqslant y})_{x\leqslant y\in P}$.

The skyscraper invariant For a general poset P, the *skyscraper invariant* of a P-persistence module V is defined as the dimension vectors of certain filtrations of V [52]. In the case of ladder persistence, it has a much simpler expression. Let $P := \{0,1\}\times\{0,\dots,n\}$ and let $V \in \mathrm{Vect}^P$ be a ladder persistence module. Fix $x \in P$, the *spanning submodule* of V at x is given for $y \in P$ by the subspace

$$\mathrm{Span}_x(V)_y := \begin{cases} \mathrm{Im}\, V_{x\leqslant y} & \text{if } x \leqslant y \\ 0 & \text{otherwise} \end{cases}.$$

Since all the maps of $V_{|\{y\in P | y\geqslant x\}}$ are surjective, each spanning submodule $\mathrm{Span}_x(V)$ is spread-decomposable [10].

Definition A.1.2. The *skyscraper invariant* δ assigns to a ladder persistence module V indexed by P the spread decomposition of $\mathrm{Span}_x(V)$ for each $x \in P$. $\triangle$

Six-pack diagram Let P be the poset defined by the Hasse diagram (A.2) and let $V \in \mathrm{Vect}^P$ be a ladder persistence module. By restricting V to the top and bottom rows, we obtain two ordinary persistence modules, respectively $V^+ := (V_{x_i^+})_i$ and $V^- := (V_{x_i^-})_i$. In addition, one can build the following three ordinary persistence modules from the vertical maps

$$\ker_V := (\ker V_{x_i^- \leqslant x_i^+})_i \qquad \mathrm{Im}_V := (\mathrm{Im}\, V_{x_i^- \leqslant x_i^+})_i \qquad \mathrm{coker}_V := (\mathrm{coker}\, V_{x_i^- \leqslant x_i^+})_i.$$

Moreover, if V is induced by $\iota\colon K_\bullet \hookrightarrow L_\bullet$ in InclFiltSimp, one can define the ordinary persistence module $\mathbf{H}_p(L_\bullet, K_\bullet)$ given by the persistent relative homology of $(L_\bullet, K_\bullet)$.

Definition A.1.3 ([95])**.** For $k \in \{3,4,5,6\}$, the *k-pack diagram* of an inclusion $\iota\colon K_\bullet \hookrightarrow L_\bullet$ in InclFiltSimp is the data of the persistence diagrams of the k first persistence modules among

$$\mathbf{H}_p(K_\bullet),\ \mathbf{H}_p(L_\bullet), \ker_{\mathbf{H}_p\iota},\ \mathrm{Im}_{\mathbf{H}_p\iota}, \mathrm{coker}_{\mathbf{H}_p\iota}\text{, and } \mathbf{H}_p(L_\bullet, K_\bullet). \qquad \triangle$$

A.2 Comparing invariants

We now demonstrate the main theorem of this Appendix.

Theorem A.2.1. *The partial order $\preceq$ induced by the relative strength of the skyscraper invariant, the k-pack diagrams for $k \in \{3,4,5,6\}$, the rank invariant, the* $\mathrm{GRI}^{(1,2)}$ *and the* $\mathrm{GRI}^{(1,2),(2,1)}$ *is summarized by the following Hasse diagram.*

$$\begin{tikzcd}
\mathrm{GRI}^{(1,2),(2,1)} & \textit{6-pack diagram} & \\
\mathrm{GRI}^{(1,2)} \arrow[u] & \textit{5-pack diagram} \arrow[u] \arrow[ul] & \textit{skyscraper invariant} \\
& \textit{4-pack diagram} \arrow[u] \arrow[ul] \arrow[ur] & \\
& \textit{3-pack diagram} = \textit{rank invariant} \arrow[u] &
\end{tikzcd}$$

where there is a path from invariant I to invariant I' if and only if I' is stronger than I.

First, we prove that the claimed strength relations (*i.e.* we show the existence of the arrows in the Hasse diagram). Then, we show that there are no other strength relations between the invariants we consider. More precisely, we show the following strength relations

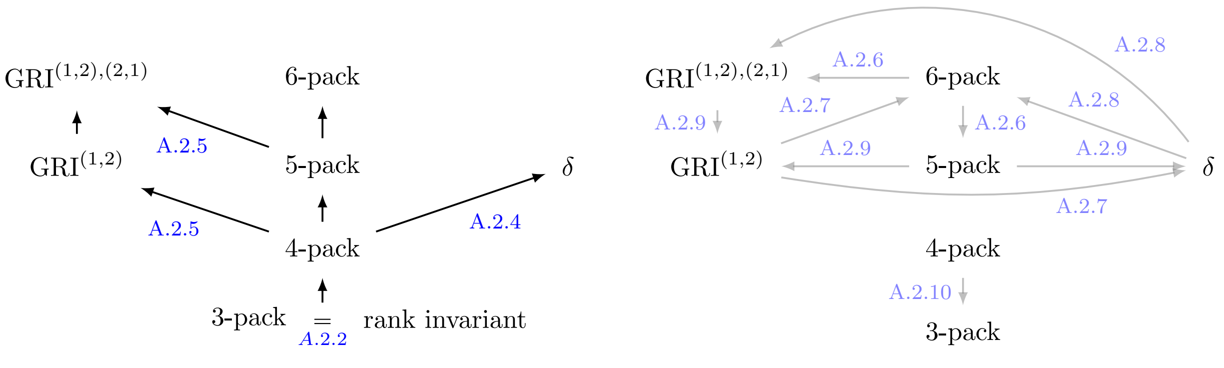
- Proposition A.2.2: the rank invariant is equivalent to the 3-pack diagram
- Proposition A.2.4: the skyscraper invariant is stronger than the 4-pack diagram

- Proposition A.2.5: the $\mathrm{GRI}^{(1,2),(2,1)}$ is stronger than the 5-pack diagram and the $\mathrm{GRI}^{(1,2)}$ is stronger than the 4-pack diagram,

and the following absence of strength relations

- Example A.2.7: the 6-pack diagram and the skyscraper invariant are not stronger than the $\mathrm{GRI}^{(1,2)}$
- Example A.2.8: the 6-pack diagram and the GRI are not stronger than the skyscraper invariant.
- Example A.2.9: the skyscraper invariant and the $\mathrm{GRI}^{(1,2)}$ are not stronger than the 5-pack diagram or the $\mathrm{GRI}^{(1,2),(2,1)}$
- Example A.2.6: the 5-pack diagram and the GRI are not stronger than the 6-pack diagram
- Example A.2.10: the 3-pack diagram is not stronger than the 4-pack diagram

The different components of the proof of Theorem A.2.1 are summarized in Figure A.1. One can check that the partial order $\preceq$ induced by the relative strength of the invariants we consider is entirely determined by the arrows depicted in Figure A.1.



(a) Strength relations **(b)** Absence of strength relations

Figure A.1: Summary of the proof of Theorem A.2.1. **(a)** shows the strength relations established in the proof, while **(b)** indicates the absence of strength relations. For $k \in \{3, 4, 5, 6\}$, "k-pack" refers to the "k-pack diagram", and δ denotes the skyscraper invariant.

A.2.1 Strength relations

Since we are only comparing invariants that factor through the homology functor, we will work directly with ladder persistence modules. Fix $n \geqslant 0$ and let P be the poset defined by the Hasse diagram (A.2). Let V be a P-persistence module.

Proposition A.2.2. *The 3-pack diagram is equivalent to the rank invariant.*

Proof. Let $n \geqslant 0$ and let V be a ladder persistence module indexed by the poset described in (A.2). The persistence diagrams $\mathbf{Bar}(V^-)$ and $\mathbf{Bar}(V^+)$ are equivalent to the ranks of the maps of V along horizontal arrows. Given $x \leqslant y$, we have

$$\begin{aligned}\operatorname{rank} V_{x^-\leqslant y^+} &= \dim(V_{x^+\leqslant y^+}(V_{x^-\leqslant x^+}(V_{x^-})) \\ &= \dim(V_{x^+\leqslant y^+}((\mathrm{Im}_V)_x)) \\ &= \operatorname{rank}(\mathrm{Im}_V)_{x\leqslant y}.\end{aligned}$$

So $\mathbf{Bar}(\mathrm{Im}_V)$ is equivalent to the rank of non-horizontal maps in V. □

The following dual statements will be useful to prove the remaining strength relations.

Lemma A.2.3. *Let $x \leqslant y$ in $\{x_0, \ldots, x_n\}$. Then,*

1. $\operatorname{rank}(\ker_V)_{x\leqslant y}$ *is given by the multiplicity of the interval module $\mathbf{I}[x^-, y^-]$ in the decomposition of the restriction of V to the zigzag*

$$x^+ \longleftarrow x^- \longrightarrow \cdots \longrightarrow y^- \tag{$Z_{x,y}$}$$

2. $\operatorname{rank}(\mathrm{coker}_V)_{x\leqslant y}$ *is given by the multiplicity of the interval module $\mathbf{I}[x^+, y^+]$ in the decomposition of the restriction of V to the zigzag*

$$x^+ \longrightarrow \cdots \longrightarrow y^+ \longleftarrow y^- \tag{$Z'_{x,y}$}$$

Proof. 1. Let $V_{|Z_{x,y}} \overset{\phi}{\simeq} \bigoplus_{I\in\mathcal{I}(Z_{x,y})} \mathbf{I}[I]^{d_I}$ be the decomposition of the restriction of V to ($Z_{x,y}$). We have as subspaces of $\phi_{x^-}(V_{x^-})$

$$\phi_{x^-}(\ker V_{x^-\leqslant x^+}) = \left(\bigoplus_{w\in[x,y]} \mathbf{I}[x^-, w^-]^{d_{[x^-,w^-]}}\right)_{x^-}$$

Hence we have,

$$\begin{aligned}\operatorname{rank}(\ker_V)_{x\leqslant y} &= \dim V_{x^-\leqslant y^-}(\ker V_{x^-\leqslant x^+}) \\ &= \dim \phi(V)_{x^-\leqslant y^-}(\phi_{x^-}(\ker V_{x^-\leqslant x^+})) \\ &= d_{[x^-,y^-]}.\end{aligned}$$

2. Let $V_{|Z'_{x,y}} \overset{\phi}{\simeq} \bigoplus_{I\in\mathcal{I}(Z'_{x,y})} \mathbf{I}[I]^{d_I}$ be the decomposition of the restriction of V to ($Z'_{x,y}$). We have,

$$\begin{aligned}\operatorname{rank}(\mathrm{coker}_V)_{x\leqslant y} &= \dim \phi_{y^+}(\mathrm{Im}\, V_{x^+\leqslant y^+}) - \dim \phi_{y^+}(\mathrm{Im}\, V_{y^-\leqslant y^+} \cap \mathrm{Im}\, V_{x^+\leqslant y^+}) \\ &= (d_{[x^+,y^+]} + d_{[x^+,y^+]\cup\{y^-\}}) - d_{[x^+,y^+]\cup\{y^-\}} \\ &= d_{[x^+,y^+]}\end{aligned}$$

□

Proposition A.2.4. *The skyscraper invariant is stronger than the 4-pack persistence diagram.*

Proof. By Theorem 3.3.5, the skyscraper invariant is stronger than the rank invariant which by Proposition A.2.2 is equivalent to $(\mathbf{Bar}(V^-), \mathbf{Bar}(V^+), \mathbf{Bar}(\mathrm{Im}_V))$. Let $x \leqslant y$ in $\{x_0, \dots, x_n\}$, it is enough to show that the integer $\mathrm{rank}(\ker_V)_{x\leqslant y}$ is determined by the skyscraper invariant of V.

By Lemma A.2.3.1, $\mathrm{rank}(\ker_V)_{x\leqslant y}$ is given by the multiplicity of the interval $[x^-, y^-]$ in $\mathbf{Bar}(V_{|(Z_{x,y})})$, which is also the multiplicity of $[x^-, y^-]$ in $\mathbf{Bar}(\mathrm{Span}_x(V)_{|(Z_{x,y})})$. The skyscraper invariant of V at x is equivalent to the multiplicities in the decomposition $\mathrm{Span}_x(V) = \bigoplus_{I\in\mathcal{I}(P)} \mathbf{I}[I]^{d_I}$. Restricting this decomposition to $(Z_{x,y})$, we obtain

$$\mathrm{rank}(\ker_V)_{x\leqslant y} = \sum_{I\cap Z_{x,y} = [x^-, y^-]} d_I.$$

Finally, $\mathrm{rank}(\ker_V)_{x\leqslant y}$ and hence the 4-pack diagram can be recovered from the skyscraper invariant. ☐

Proposition A.2.5. *As invariants of ladder persistence modules*

1. $\mathrm{GRI}^{(1,2)}$ *is stronger than the 4-pack persistence diagram; and*
2. $\mathrm{GRI}^{(1,2),(2,1)}$ *is stronger than the 5-pack persistence diagram.*

Proof. Let $V \in \mathrm{Vect}^P$. By Proposition A.2.2, the triple $(\mathbf{Bar}(V^-), \mathbf{Bar}(V^+), \mathbf{Bar}\,\mathrm{Im}_V)$ is equivalent to the rank invariant which is equivalent to $\mathrm{GRI}^{(1,1)}$. By Lemma A.2.3, for $x \leqslant y$, the ranks $\mathrm{rank}(\ker_V)_{x\leqslant y}$ and $\mathrm{rank}(\mathrm{coker}_V)_{x\leqslant y}$ are given by the multiplicities of respectively the interval $[x^-, y^-]$ in $\mathbf{Bar}(V_{|(Z_{x,y})})$ and the interval $[x^+, y^+]$ in $[x^-, y^-]$ in $\mathbf{Bar}(V_{|(Z'_{x,y})})$. Hence,

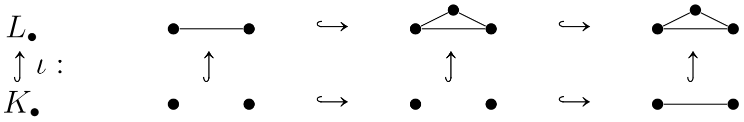
$$\begin{aligned}\mathrm{rank}(\ker_V)_{x\leqslant y} &= \mathrm{GRI}_V([x^-, y^-]) - \mathrm{GRI}_V(Z_{x,y})\\ \mathrm{rank}(\mathrm{coker}_V)_{x\leqslant y} &= \mathrm{GRI}_V([x^+, y^+]) - \mathrm{GRI}_V(Z'_{x,y})\end{aligned}$$

Since $[x^-, y^-]$ and $Z_{x,y}$ lie in $\mathcal{I}^{(1,2)}(P)$, the invariant $\mathrm{GRI}_V^{(1,2)}$ is stronger than $\mathbf{Bar}(\ker_V)$, and hence stronger than the 4-pack diagram. And since $[x^+, y^+]$ and $Z'_{x,y}$ all belong to $\mathcal{I}^{(2,1)}(P)$, the invariant $\mathrm{GRI}_V^{(2,1),(1,2)}$ is stronger than $(\mathbf{Bar}(\ker_V), \mathbf{Bar}(\mathrm{coker}_V))$, and hence stronger than the 5-pack diagram. ☐

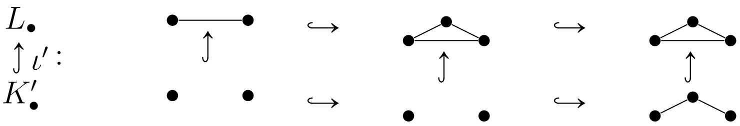
A.2.2 Absence of strength relations

The following example shows that no invariant that factors through $\iota \in \mathrm{InclFiltSimp} \mapsto \mathbf{H}_p(\iota)$ can be stronger than the relative persistent diagram $(\iota\colon K_\bullet \hookrightarrow L_\bullet) \mapsto \mathbf{H}_p(L_\bullet, K_\bullet)$.

Example A.2.6. Consider the following two inclusions in InclFiltSimp:



and,



For each $p \geqslant 0$, the ladder persistence modules $\mathbf{H}_p(\iota)$ and $\mathbf{H}_p(\iota')$ are isomorphic. However, the ordinary persistence modules $\mathbf{H}_1(L_\bullet, K_\bullet)$ and $\mathbf{H}_1(L_\bullet, K'_\bullet)$ have a different persistence diagram: unlike $\mathbf{H}_1(L_\bullet, K_\bullet)$, $\mathbf{H}_1(L_\bullet, K'_\bullet)$ has a 1-cycle that is born at the first step of the filtration and never dies.

Example A.2.7. The example in Figure A.2 shows that the 6-pack diagram and the skyscraper invariant are not stronger than the generalised rank invariant $\mathrm{GRI}^{(1,2)}$. Indeed, we will prove that the inclusions $\iota \colon K_\bullet \hookrightarrow L_\bullet$ and $\iota' \colon K'_\bullet \hookrightarrow L'_\bullet$ have the same six-pack diagrams and skyscraper invariants, but different $\mathrm{GRI}^{(1,2)}$.

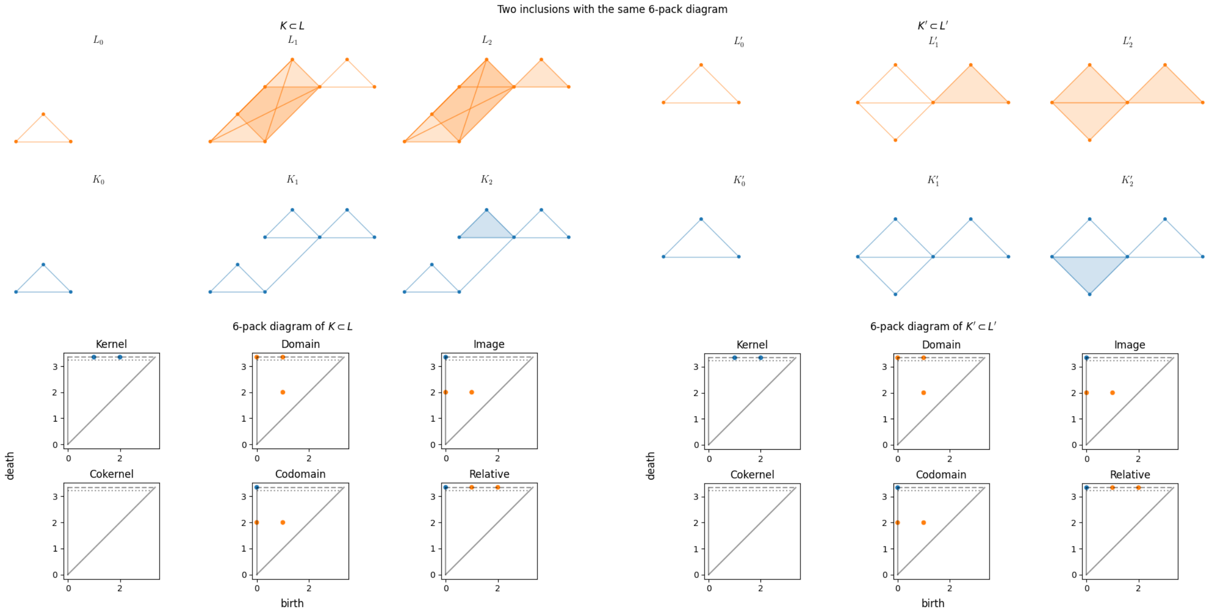


Figure A.2: Two inclusions with the same six-pack diagrams but different $\mathrm{GRI}^{(1,2)}$. The triangular prism is hollow, its 3 side faces are full, its front face is hollow, its back face is hollow at step 1 and full at step 2. The six-pack computations use the algorithm from [98]

The underlying poset for the homology ι and ι' is $P := \{0,1\} \times \{0,1,2\}$, which we label as per (A.2). Observe that for all $p \neq 1$, we have $\mathbf{H}_p(\iota) \simeq \mathbf{H}_p(\iota')$. Let S be the spread $P \backslash \left\{x_2^+\right\}$, we have

$$\mathbf{H}_1(\iota) = \begin{tikzcd} \mathbb{F} \arrow[r, "1"] & \mathbb{F} \arrow[r] & 0 \\ \mathbb{F} \arrow[u, "1"] \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] & \mathbb{F}^2 \arrow[u, "\left[\begin{smallmatrix}1 & 1\end{smallmatrix}\right]"'] \arrow[r, "\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]"'] & \mathbb{F} \arrow[u] \end{tikzcd} \bigoplus \mathbf{I}[\{x_1^\pm, x_2^-\}] \tag{A.3}$$

$$\mathbf{H}_1(\iota') = \mathbf{I}[S] \bigoplus \mathbf{I}[\{x_1^\pm\}] \bigoplus \mathbf{I}[\{x_1^-, x_2^-\}] \tag{A.4}$$

We first observe that $\mathbf{H}_1(\iota)$ and $\mathbf{H}_1(\iota')$ have the same skyscraper invariants. Indeed, these two ladder persistence modules have the same spanning submodules. We have, at x_0^-,

$\mathrm{Span}_{x_0^-}(\mathbf{H}_1(\iota)) \simeq \mathbf{I}[S] \simeq \mathrm{Span}_{x_0^-}(\mathbf{H}_1(\iota'))$, and at x_1^-,

$$\mathrm{Span}_{x_1^-}(\mathbf{H}_1(\iota)) \simeq \mathbf{I}[x_1^\pm] \bigoplus \mathbf{I}[x_1^-, x_2^-] \bigoplus \mathbf{I}[x_1^\pm, x_2^-] \simeq \mathrm{Span}_{x_1^-}(\mathbf{H}_1(\iota')).$$

We now check that the six-pack diagrams are the same for ι and ι'. By Proposition A.2.4, we only need to check the cokernel and relative persistence diagrams. From (A.3), we have $\mathrm{coker}_{\mathbf{H}_1(\iota)} = 0 = \mathrm{coker}_{\mathbf{H}_1(\iota')}$. One can check that the quotient induced by ι and ι' are both homotopic to the sequence

$$X_\bullet : \qquad * \hookrightarrow S^2 \hookrightarrow S^2 \vee S^2$$

where the second $\hookrightarrow$ includes the sphere into one of the two spheres of $S^2 \vee S^2$.

We finally check that $\mathrm{GRI}_{\mathbf{H}_1(\iota)}(S) \neq \mathrm{GRI}_{\mathbf{H}_1(\iota')}(S)$. The contour zigzag of S contains the zigzag $x_0^+ \to x_1^+ \leftarrow x_1^- \to x_2^-$. The restriction of $\mathbf{H}_1(\iota)$ to this zigzag is

$$\left(\mathbb{F} \xrightarrow{1} \mathbb{F} \overset{[1\ 1]}{\longleftarrow} \mathbb{F}^2 \overset{[1\ 0]}{\longrightarrow} \mathbb{F}\right) \oplus \mathbf{I}[\{x_1^\pm, x_2^-\}]$$

and the multiplicity of the full-length interval $\{x_0^+, x_1^\pm, x_2^-\}$ is zero, whence, $\mathrm{GRI}_{\mathbf{H}_1(\iota)}(S) = 0$. Moreover, by additivity of GRI, we have

$$\mathrm{GRI}_{\mathbf{H}_1(\iota')}(S) = \mathrm{GRI}_{\mathbf{I}[S]}(S) + \mathrm{GRI}_{\mathbf{I}[\{x_1^-, x_2^-\}]}(S) + \mathrm{GRI}_{\mathbf{I}[\{x_1^\pm\}]}(S) = 1 + 0 + 0 = 1.$$

Example A.2.8 ([14, Proposition 7.8]). The example in Figure A.3 shows that the 6-pack diagram and the GRI are not stronger than the skyscraper invariant. For $p \neq 1$, the inclusions $\iota\colon K_\bullet \hookrightarrow L_\bullet$ and $\iota'\colon K'_\bullet \hookrightarrow L'_\bullet$ have the same p-th homology. For $p = 1$, we have

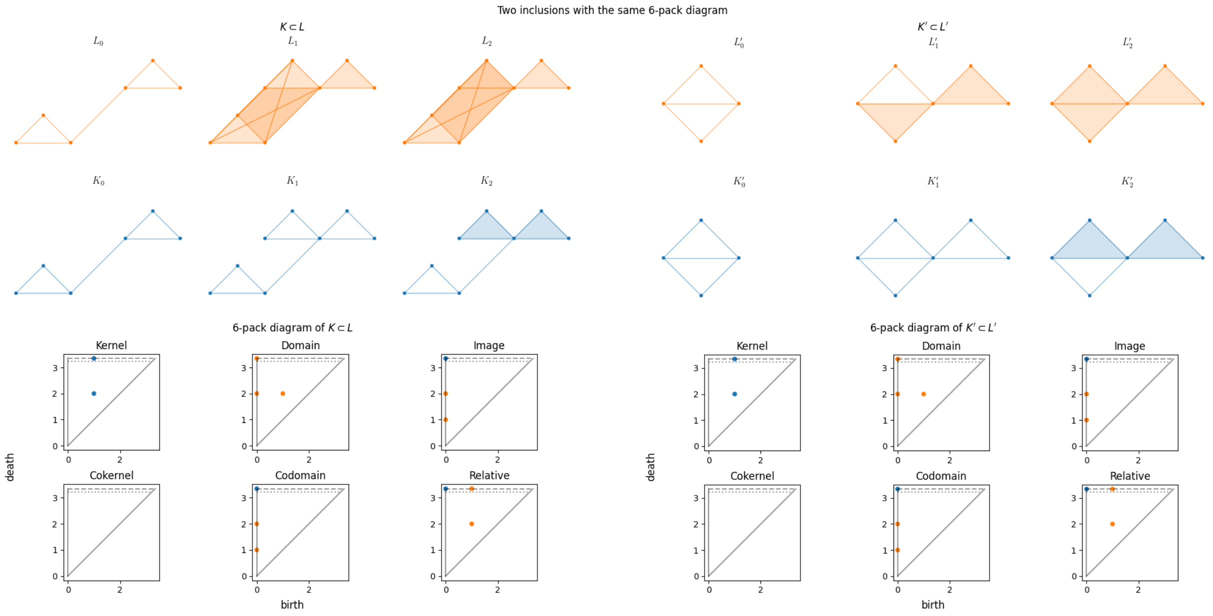


Figure A.3: Two inclusions with the same six-pack diagrams but different skyscraper invariants. The triangular prism is hollow, its 3 side faces are full, its front face is hollow, its back face is hollow at step 1 and full at step 2.

$$\mathbf{H}_1(\iota) = \begin{tikzcd} \mathbb{F} \arrow[r, "1"] & \mathbb{F} \arrow[r] & 0 \\ \mathbb{F} \arrow[u, "1"] \arrow[r, "{[1\ 0]^T}"'] & \mathbb{F}^2 \arrow[u, "{[1\ 1]}"] \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[u] \end{tikzcd} \bigoplus \mathbf{I}[\{x_0^\pm, x_1^-\}]$$

and

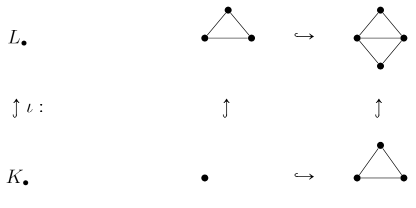
$$\mathbf{H}_1(\iota') \simeq \mathbf{I}[\{x_0^{\pm}, x_1^{\pm}\}] \bigoplus \mathbf{I}[\{x_0^{\pm}, x_1^{-}, x_2^{-}\}] \bigoplus \mathbf{I}[\{x_1^{-}\}].$$

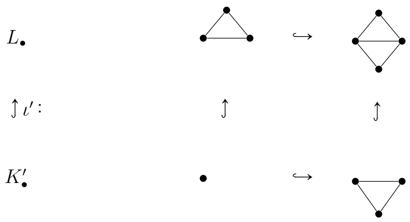
We first show that $\mathrm{GRI}_{\mathbf{H}_1(\iota)} = \mathrm{GRI}_{\mathbf{H}_1(\iota')}$. Let $S := P \setminus \{x_2^+\}$, one can check that for every spread $I \subsetneq S$ of P, the modules $\mathbf{H}_1(\iota)$ and $\mathbf{H}_1(\iota')$ are spread-decomposable and isomorphic. Using the computations of A.2.7 and the additivity of GRI, we have $\mathrm{GRI}_{\mathbf{H}_1(\iota)}(S) = 0 = \mathrm{GRI}_{\mathbf{H}_1(\iota')}(S)$. Hence, $\mathrm{GRI}_{\mathbf{H}_1(\iota)} = \mathrm{GRI}_{\mathbf{H}_1(\iota')}$.

Furthermore, by Proposition A.2.5.2, the 5-pack diagrams of ι and ι' coincide. The quotient induced by ι and ι' are both homotopic to the sequence $* \to S^2 \vee S^2 \to S^2$ where the second map sends the first sphere to S^2 and the second sphere to a point. As a consequence, ι and ι' have the same 6-pack diagram.

Finally, we check that $\mathbf{H}_1(\iota)$ and $\mathbf{H}_1(\iota')$ have different skyscraper invariants. We have $\mathrm{Span}_{x_0^-}(\mathbf{H}_1(\iota)) \simeq \mathbf{I}[S] \oplus \mathbf{I}[\{x_0^{\pm}, x_1^{-}\}]$ whereas $\mathrm{Span}_{x_0^-}(\mathbf{H}_1(\iota')) \simeq \mathbf{I}[\{x_0^{\pm}, x_1^{\pm}\}] \bigoplus \mathbf{I}[\{x_0^{\pm}, x_1^{-}, x_2^{-}\}]$.

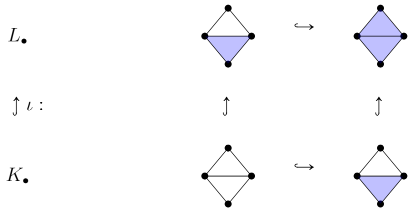
Example A.2.9. The inclusions ι and ι' depicted below have the same skyscraper invariant and the same $\mathrm{GRI}^{(1,2)}$, but different 5-pack diagrams and $\mathrm{GRI}^{(1,2),(2,1)}$.

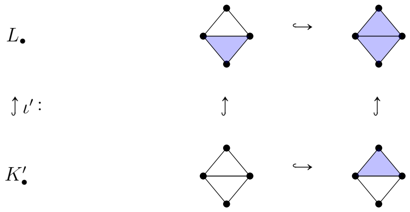




Indeed, for $p \neq 1$, we have $\mathbf{H}_p(\iota) \simeq \mathbf{H}_p(\iota')$. And using the labels of A.2 for the underlying poset P of ι and ι', we have $\mathbf{H}_1(\iota) = \mathbf{I}[P \setminus \{x_0^-\}] \oplus \mathbf{I}[\{x_1^+\}]$ whereas $\mathbf{H}_1(\iota') = \mathbf{I}[\{x_0^+, x_1^+\}] \oplus \mathbf{I}[\{x_1^-, x_1^+\}]$. One can check that these two modules have the same spanning submodules and the same $\mathrm{GRI}^{(1,2)}$, but that $\mathbf{Bar}(\mathrm{coker}_{\mathbf{H}_1(\iota)}) \neq \mathbf{Bar}(\mathrm{coker}_{\mathbf{H}_1(\iota')})$. Finally, observe that $\mathrm{GRI}_{\mathbf{H}_1(\iota)}(\{x_0^+, x_1^-, x_1^+\}) = 1 \neq 0 = \mathrm{GRI}_{\mathbf{H}_1(\iota')}(\{x_0^+, x_1^-, x_1^+\})$. So $\mathbf{H}_1(\iota)$ and $\mathbf{H}_1(\iota')$ have different 5-pack diagrams and $\mathrm{GRI}^{(1,2),(2,1)}$.

Example A.2.10. The inclusions ι and ι' depicted below have the same 3-pack diagrams but different 4-pack diagrams.





Indeed, for $p \neq 1$, we have $\mathbf{H}_p(\iota) \simeq \mathbf{H}_p(\iota')$. And using the labels of A.2 for the underlying poset P of ι and ι', we have $\mathbf{H}_1(\iota) = \mathbf{I}[P \setminus \{x_1^+\}] \oplus \mathbf{I}[\{x_0^-\}]$ whereas $\mathbf{H}_1(\iota') = \mathbf{I}[\{x_0^-, x_1^-\}] \oplus \mathbf{I}[\{x_0^-, x_0^+\}]$. One can check that these two modules have the same 3-pack diagrams., but that $\mathbf{Bar}(\ker_{\mathbf{H}_1(\iota)}) \neq \mathbf{Bar}(\ker_{\mathbf{H}_1(\iota')})$.

Appendix B

Glossary of Notations

This glossary provides a list of notations used throughout the document, organized by theme.

General notations

Symbol	**Description**
$\mathfrak{Re}$, $\mathfrak{Im}$	Real and imaginary parts of a complex number.
$\lceil\ \rceil$, $\lfloor\ \rfloor$	The ceiling and floor functions
$\sqcup$	disjoint union
$\simeq$ or $\xrightarrow{\sim}$	an isomorphism
$\mathrm{sign}(x)$	the sign in $\{-1, 0, 1\}$ of a number $x \in \mathbb{R}$
$\mathbb{1}_S$	The indicator function of a subset S
$\mathbb{1}_p$	The indicator function of an element p
$\mathbb{F}$	A field
$\\|\ \\|_\infty$	The supremum norm
Vect	The category of finite-dimensional vector spaces over $\mathbb{F}$
rank, Im	The rank and image of a linear map
ker, coker	The kernel and cokernel of a linear map
I_n, $0_{n,m}$	The $n \times n$ identity matrix and the $n \times m$ zero matrix
$Q = (s, t \colon Q_1 \to Q_0)$	A finite quiver defined by its source and target maps s and t
$\mathbf{Rep}(Q)$	The category of representations of a quiver Q
$\leq$	A partial order on a set
$\vee$, $\wedge$	The join and meet from order theory
Vect^P or $\mathrm{Pers}(P)$	The category pointwise finite-dimensional P-persistence modules for a poset P
$\underline{\dim}_V$	The dimension vector of a quiver representation or a persistence module V
T_x	The translation by some element x in an euclidean space
$\mathbb{R}^P$	Functions from P — seen as a set — to $\mathbb{R}$

$\varprojlim$, $\varinjlim$	The categorical limit and colimit
$\mathcal{C}^{\text{opp}}$	The opposite category of a category $\mathcal{C}$
$\mathcal{A}$	An abelian category
$K(\mathcal{A})$	The Grothendieck group of an abelian category $\mathcal{A}$
$[V]$	The Grothendieck class of an object $V \in \mathcal{A}$
$\text{Lan}_f(V)$	The left Kan extension of $V \in \mathcal{A}$ along the functor f
$\mathcal{O}$	Worst-time complexity of an algorithm
$\square$, $\triangle$	end of a proof and of a definition

Persistence

Let P be a poset and let $V \in \text{Vect}^P$ be a P-persistence module.

Symbol	**Description**
$\mathbf{H}_i$	The homology functor in degree i
$\mathbf{Bar}(V)$	The barcode of V (if P is totally ordered)
$\mathbf{I}[u,v]$	The interval module supported on the interval $[u,v]$
$\mathbf{I}[S]$ or $\mathbb{F}_S$	The spread (or interval) module supported on S
ρ_V	The rank invariant of V
GRI_V	The generalised rank invariant of V
d_I	The interleaving distance
d_E	The erosion distance

Harder-Narasimhan Filtrations and Stability

Let Z be a stability condition. If Z is standard and is determined by a weight vector α, we replace Z by α in the following notations.

Symbol	**Description**
$\mu_Z(V)$	The Z-slope of V
$\mathbf{HN}_Z^\bullet(V)$	The HN filtration of V along Z seen as a finite-length filtration
$\langle V, Z\rangle^\bullet$	The HN filtration of V along Z seen as a $\mathbb{R}^{\text{opp}}$-indexed filtration
$\mathbf{T}[V;Z]$	The HN type of V along Z
δ_V	The skyscraper invariant of V
$s_{Z,V}^\bullet$	The HN filtered rank invariant of V along Z

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