Lipschitz Functions on Unparameterised Rough Paths and the Brownian Motion Associated to the Bilaplacian

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To my parents.

Abstract

This thesis is a tale of two halves. The first introduces the space of unparameterised geometric rough paths and develops a notion of Lipschitz function on this space. The second associates an expected signature to the bilaplacian and culminates in an analogue of the Feynman-Kac formula for high-order parabolic partial differential equations.

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Symbolic Index

We collect some of the symbols used, together with their meaning and the page where they were first introduced.

Symbol	Meaning	Page
$G\Omega_p(V)$	Parameterised geometric rough paths	20
$\mathbb{G}_p(V)$	Space of unparameterised $p-$rough paths	20
$\mathrm{Hist}(g)$	History of the path g	25
$\mathfrak{i}(g)$	Initial point of the path g	21
M_k	k^{th} moment of $p(1,\cdot)$	106
$\Omega(G^n(V))$	Space of $\mathbb{R}-$valued cocyclic one-forms on $G^n(V)$	36
$\Omega(G^n(V) \to W)$	Space of $W-$valued cocyclic one-forms on $G^n(V)$	36
$p(t,x)$	Heat kernel of $-\Delta^2$	106
$\mathbb{P}_D^{[s,t];x}$	Signed measure on rough paths associated to $-\Delta^2$	111
$\mathcal{P}^{(n)}(W \to X)$	Space of polynomials $W \to X$ up to degree n	34
$\mathcal{P}(W \to X)$	Space of polynomials $W \to X$	34
$\mathcal{P}^{(n)}(G)$	Space of polynomials $G \to \mathbb{R}$ up to degree n	145
$\mathcal{P}(G)$	Space of polynomials $G \to \mathbb{R}$	145
$\mathfrak{S}_n$	The n^{th} symmetric group	
$\mathscr{S}(V)$	The Schwartz space on V	81
$\mathscr{S}'(V)$	Tempered distributions on V	82
$S^{(n)}(W)$	Symmetric tensors up to level n	128
$\sqcup\!\sqcup$	Shuffle product	15

Symbol	Meaning	Page
$\mathfrak{t}(g)$	Tail point of the path g	21
$T^N(V)$	Truncated tensor algebra, $T^N(V) \cong \bigoplus_{i=0}^N V^{\otimes i}$	13
$T(V)$	Tensor algebra $T(V) \cong \bigoplus_{i=0}^\infty V^{\otimes i}$	13
$\bar{T}(V)$	Universal topological tensor algebra	97
$T_\lambda(V)$	Tensors in $T((V))$ which decay faster than λ^{-i}	95
$T((V))$	Extended tensor algebra $T((V)) = \prod_{i=0}^\infty V^{\otimes i}$	12
$\upsilon_D^{[s,t];x}$	Signed cylinder set measure associated to $-\Delta^2$	109
V, W, X	Finite-dimensional Banach spaces over $\mathbb{R}$	9
$WG\Omega_p(V)$	Parameterised weakly geometric $p-$rough paths	17
$\mathbb{X}$	Signature	20
$\mathbb{X}^N$	Signature at level N	20
$\mathbb{X}^{\leqslant N}$	Signature up to level N	20

1 Introduction

1.1 Motivation and overview

Part I: Chapters 2 and 3

We begin by interpreting rough path theory ([32], [33]) as a calculus of sufficiently regular *unparameterised* paths with values in a truncated free nilpotent Lie group. This interpretation is possible because many of the fundamental operations in rough paths theory are independent of parameterisation.

More concretely, we shall call any increasing homeomorphism $\tau : [0,1] \to [0,1]$ a reparameterisation. Then if $g : [0,1] \to G^{\lfloor p \rfloor}(V)$ is a geometric $p-$rough path, $g \circ \tau$ is also a geometric $p-$rough path with $\|g\|_{p-\mathrm{var}} = \|g \circ \tau\|_{p-\mathrm{var}}$. Furthermore, if $\alpha : V \to V^*$ is a smooth one-form, then

$$\int_g \alpha = \int_{g \circ \tau} \alpha.$$

Likewise, if $\mathcal{A} : V \xrightarrow{\mathrm{lin}} \mathrm{Vect}_{\mathrm{b}}^{\infty}(W)$ is a linear map into smooth, bounded vector fields on another vector space, W, and Y solves the differential equation

$$\mathrm{d}Y = \mathcal{A}(Y)\mathrm{d}g, \quad Y_0 \in W,$$

then $Y \circ \tau$ solves

$$\mathrm{d}(Y \circ \tau) = \mathcal{A}(Y \circ \tau)\mathrm{d}(g \circ \tau), \quad (Y \circ \tau)(0) = Y_0 \in V'.$$

Our motivation for studying unparameterised rough paths is twofold. The first is mathematical simplicity and elegance. As alluded to above, the main results of rough path theory [32] do not require a parameterisation to be stated nor proved. The additional notation and clutter from fixing a parameterisation is a hindrance to progress. The second is that there are many real-world examples of data that have a natural *order*, but no natural *parameterisation*: think of a stream of text in a book or of a handwritten Chinese character [18]. Equally, ignoring the parameterisation of a data stream – whilst retaining its order – could be efficient: see [28] and [25] for unparameterised path techniques in the context of Econometrics and Mental Health data, respectively. There is value, therefore, not only in studying spaces of unparameterised rough paths, but also *functions* on these spaces.

In Chapter 3, we develop a notion of Lipschitz function (in the sense of E. Stein [41]) on unparameterised rough paths. Unlike Malliavin-Sobolev functions on Wiener space, Lipschitz functions are a measure-free concept. We shall see (Theorem 3.3.4) that these Lipschitz functions satisfy a type of Martingale Representation Theorem: namely, that they can be represented by integrating appropriate path-dependent one-forms. Further, analogously to their Euclidean counterparts, these functions form a Banach algebra (Proposition 3.4.9) and separate points (Lemma 3.5.5), suggesting that they may form a core class of test functions for analysis on path space. Further, a Whitney-type extension property (Theorem 3.5.2) is satisfied: this is of relevance in (theoretical) supervised Machine Learning or Econometrics where one often has a function defined on some subset of data points which one wants to extend to unseen points. We refer to [15] for a discussion relating Whitney extension problems with the interpolation of data.

We have generalised the notion of Lipschitz function by moving away from Euclidean space to unparameterised rough path space, and with co-cyclic one-forms ([35]) playing the role of polynomial functions: locally describing functions on unparameterised rough paths. Still on Euclidean spaces, but in the context of locally subscritical singular stochastic partial differential equations, M. Hairer ([19]) fruitfully generalised the notion of Lipschitz function to so-called *modelled distributions* (or $\mathcal{D}^\gamma$ spaces). In his Theory of Regularity Structures, modelled distributions are locally described by objects tailor-made to the problem at hand and which are far more general than polynomials.

Part II: Chapters 4 and 5

The connection between Brownian motion and the Laplacian is rich and complex. One of the foundations of this relationship is that the heat semi-group of the Laplacian induces measures on path space that converge to Wiener measure. The solving of stochastic differential equations (SDEs) is a complicated operation on path space which allows to associate measures on path space to new differential operators of dissipative type. Indeed, a celebrated family of results in stochastic analysis describe the infinitesimal generator of the solution to SDEs. Formally, the solution to the Stratonovich SDE driven by Brownian motion and vector fields (V_i) has the sum-of-squares generator: $\frac{1}{2}\sum V_i^2$ (well-explained in the classical text of N. Ikeda and S. Watanabe [23]). There are many benefits to this probabilistic perspective of partial differential equations, for example: Markov diffusions can be studied on fractals where classical PDEs have no meaning – see [3] for a construction of "Brownian motion" and "Laplacian" on the Sierpinski carpet which are invariant with respect to the local symmetries of the carpet. Another example is in alternative numerical schemes to heat equations of subelliptic operators (see [30] and the references therein).

However, a significant drawback of this classical probabilistic perspective on PDEs is that it is restricted to partial differential operators of order at most two. The reason for this is that high-order (order above two) elliptic differential operators do not in general satisfy maximum principles. Therefore, although the associated heat kernels integrate to one, they are not positive functions and hence cannot be interpreted as the transition functions of some Markov process.

Using ideas going back at least to V. Krylov ([26]), together with techniques from rough path theory, D. Levin and T. Lyons ([27]) began a programme to overcome this obstruction of signed heat kernels. We give now an overview of their result and our generalisation of it.

Consider the operator $-\Delta^2$ on $\mathbb{R}^d$ – defined through Fourier analysis as having symbol $p(\xi) = -16\pi^4\|\xi\|^4$ – or more generally, a translation-invariant operator whose symbol tends to $-\infty$ off compact sets. For any partition of time, D, the associated heat semigroup induces a family of consistent signed measures on piecewise linear paths, which we denote by $\mathbb{P}_D$. The total variation of these measures grows geometrically with $\#D$ and they do not extend to a countably additive measure on the space of continuous paths. In spite of this, Levin and Lyons showed that expected coordinate signatures of the measures associated to the different partitions do converge as the mesh tends to 0 ($|D| \to 0$) (Theorem 7.4 in [27]). In a nutshell, that the expected values of core functions on unparameterised rough paths have limits against these exploding measures. We mention and address two of their conjectures (Section 8 in [27]):

Conjecture 1) The measures, $\mathbb{P}_D$, converge to a finite signed measure on rough paths when one considers the filtration that ignores parameterisation.

Conjecture 2) Suppose that (A_i) are sufficiently regular vector fields, and Ψ_A is the Itô-Lyons map taking paths in V to somewhere else via

$$dY^x = A(Y^x)d\gamma, \quad Y_0^x = x.$$

For each partition of time, D, denote by $\mathbb{E}_D$ the integral with respect to $\mathbb{P}_D$. Then we can define

$$u^D(t,x) := \mathbb{E}_D[f(Y_t^x)]$$

and ask if

$$\lim_{|D|\to 0} u^D(t,x)$$

exists for regular enough f and if the limiting function, u, solved a PDE semigroup whose generator depends on the A_i in analogy with results from classical stochastic analysis (see [30]).

This first conjecture remains open, however it appears that a distributional limit "$\lim_{|D|\to 0} \mathbb{P}_D = \mathbb{P}$" does exist and we significantly generalise the class of test functions on paths, ϕ, for which $\lim_{|D|\to 0} \mathbb{E}_D[\phi] =: \mathbb{E}[\phi]$ exists. More precisely, in Chapter 4 we consider a topological algebra, $\bar{T}(V)$, containing signatures of all $V-$valued rough paths, where V is a finite-dimensional Banach space. The topology on $\bar{T}(V)$ is the coarsest such that any continuous linear map $A \in \mathbb{L}(V \to B)$ into any Banach algebra, B, uniquely extends to a continuous algebra homomorphism $\bar{T}(V) \to B$ (see Section 4.3 for the precise statements). The main result of Chapter 4, Theorem 4.6.2, shows existence of the limit

$$\lim_{|D|\to 0} \mathbb{E}_D[\mathbb{X}] \in \bar{T}(V),$$

where $\mathbb{X}(g)$ denotes the signature of the rough path g. In particular, if $\phi \in \bar{T}(V)^*$ is in the continuous dual, ϕ induces a function on rough paths by

$$\phi(g) := \langle \phi, \mathbb{X}(g) \rangle_{\bar{T}(V)^* \times \bar{T}(V)}$$

and $\lim_{|D| \to 0} \langle \phi, \mathbb{P}_D \rangle$ exists for all such $\phi \in \bar{T}(V)^*$. From this theorem, we derive several corollaries that Levin and Lyons could not. For example, Chevyrev and Lyons introduced a faithful Fourier transform on Borel measures on signatures ([9]). In Theorem 4.7.11, we show that $\lim_{|D| \to 0} \widehat{\mathbb{P}_D} =: \widehat{\mathbb{P}}$ exists pointwise, where $\widehat{\cdot}$ denotes Fourier transform. We therefore have a candidate for the Fourier transform of this distribution, $\mathbb{P}$, but it does not seem straightforward to determine the existence or otherwise of a measure with such a Fourier transform. Nonetheless, we think of $\mathbb{P}$ as the "Brownian motion" associated to the bilaplacian – the reasons for this will become more apparent when we discuss the contents of Chapter 5.

In Chapter 5 we prove Conjecture 2) for sufficiently regular vector fields and initial data – see Theorem 5.4.37 for the precise statement. Loosely speaking, suppose G is a compact Lie group equipped with the normalised bi-invariant measure and (A_i) are left-invariant vector fields on G then the limit

$$\lim_{|D| \to 0} \mathbb{E}_D[f(Y_t^x)] =: u(t, x)$$

exists for all $f \in \mathcal{P}(G)$ ("polynomials on G": see Definition 5.4.10) and forms a one-parameter semigroup in t. If, furthermore, the operator

$$\mathcal{L} : C^\infty(G) \to C^\infty(G), \quad \mathcal{L} = -\left(\sum_i A_i^2\right)^2$$

satisfies an appropriate L^2-Poincaré inequality (see Corollary 5.4.21) then u solves the PDE semigroup

$$\begin{cases} \partial_t u = \mathscr{L} u, \\ u(0) = f \in \mathcal{P}(G) \subset L^2(G), \end{cases}$$

where $\mathscr{L}$ is an appropriate closed extension of $\mathcal{L}$ to $L^2(G)$. $\mathcal{P}(G)$ is dense in $L^2(G)$ and $e^{t\mathscr{L}}|_{\mathcal{P}(G)}$ uniquely extends to a contractive C_0-semigroup on $L^2(G)$. In fact, $\mathcal{P}(G)$ is a core for $\mathscr{L}$, so we obtain a quasi-probabilistic understanding of this high-order operator through

$$\mathscr{L} f(g) = \lim_{t\to 0} \lim_{|D|\to 0} \frac{\mathbb{E}_D[f(Y_t^g)] - f(g)}{t}.$$

Parts I and II are connected by a conjecture:

Conjecture 3) Let $\gamma > p > 4$ then there exists a constant $C > 0$ such that for all $(f, \nabla f) \in \mathrm{Lip}^\gamma(\mathbb{G}_p(V))$,

$$\sup_D |\mathbb{E}_D[f]| \leqslant C \|(f, \nabla f)\|_{\mathrm{Lip}^\gamma}$$

and the limit

$$\lim_{|D|\to 0} \mathbb{E}_D[f]$$

exists.

1.2 Notation and basic facts

⚠Throughout, V and W shall be real, finite-dimensional Banach spaces.

Definition 1.2.1 (Projective tensor product)**.** *Let $V_i, i = 1, \ldots, k$, be real, finite-dimensional Banach spaces. Define the projective norm on the algebraic tensor product $\|\cdot\|_\pi : V_1 \otimes_a \cdots \otimes_a V_k \to [0, \infty)$ by*

$$\|v\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|v_1^i\| \ldots \|v_k^i\| \;:\; v = \sum_{i=1}^{n} v_1^i \otimes \cdots \otimes v_k^i \right\}.$$

We shall call the Banach space $(V_1 \otimes \cdots \otimes V_k, \|\cdot\|_\pi)$ the projective tensor product *of $V_1, \ldots, V_k$. If $V_i \cong V$ for all i, we shall denote the projective tensor product simply by $V^{\otimes k}$.*

$\|\cdot\|_\pi$ is indeed a norm. We refer to [40] for further details of tensor products of Banach spaces. We collect several properties of projective tensor products required in the sequel.

Definition 1.2.2 (Cross norm)**.** *When V, W are Banach spaces, a* cross norm, *$\|\cdot\|$, on the algebraic tensor product $V \otimes_a W$ is a norm satisfying*

i) $\|v \otimes w\| = \|v\|_V \cdot \|w\|_W$

ii) $\|v^* \otimes w^*\|_* = \|v^*\|_{V^*} \cdot \|w^*\|_{W^*}$

for all $v \in V, w \in W, v^* \in V^*, w^* \in W^*$. Here $\|\cdot\|_*$ is the dual norm of the cross norm $\|\cdot\|$.

Lemma 1.2.3. *The projective tensor norm is a cross norm.*

Proof. See Proposition 2.1 in [40]. □

Definition 1.2.4 (Action of $\mathfrak{S}_n$ on $V^{\otimes n}$)**.** *The action*

$$\mathfrak{S}_n \to \mathrm{GL}(V^{\otimes n}), \qquad \sigma(v_1 \otimes \ldots \otimes v_n) := v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

is a linear isometry for all $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the n^{th} symmetric group.

Definition 1.2.5. *Let $i_1, \ldots, i_k \geqslant 0$ be natural numbers.* $\mathrm{Sh}(i_1, \ldots, i_k) \subset \mathfrak{S}_{i_1+\cdots+i_k}$ *is the subgroup of the $(i_1 + \cdots + i_k)^{\text{th}}$ symmetric group consisting of permutations σ satisfying*

$$\begin{aligned}
&\sigma(1) < \cdots < \sigma(i_1),\\
&\sigma(i_1+1) < \cdots < \sigma(i_1+i_2),\\
&\vdots\\
&\sigma(i_1+\cdots+i_{k-1}+1) < \cdots < \sigma(i_1+\cdots+i_k).
\end{aligned}$$

Any $\sigma \in \mathrm{Sh}(i_1, \ldots, i_k)$ induces a bounded linear map of Banach spaces

$$V^{\otimes(i_1+\cdots+i_k)} \to V^{\otimes i_1} \otimes \cdots \otimes V^{\otimes i_k}$$

given by

$$\begin{aligned}
&\sigma(v_1 \otimes \cdots \otimes v_{i_1+\cdots+i_k})\\
&\qquad = \left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i_1)}\right) \otimes \ldots \otimes \left(v_{\sigma(i_1+\cdots+i_{k-1}+1)} \otimes \cdots \otimes v_{\sigma(i_1+\cdots+i_k)}\right).
\end{aligned}$$

Definition 1.2.6 (Product map)**.** *Let $A \in \mathbb{L}(V \to W)$ be a linear map. Define tensor products of A*

$$A^{\otimes 0} \equiv \mathrm{Id} : \mathbb{R} \to \mathbb{R},$$

and for $k \geqslant 1$,

$$A^{\otimes k} \in \mathbb{L}\left(V^{\otimes k} \to W^{\otimes k}\right), \quad A^{\otimes k} : v_1 \otimes \cdots \otimes v_k \mapsto A(v_1) \otimes \cdots \otimes A(v_k)$$

all $v_i \in V$.

When W is an algebra $A^{\otimes k} \in \mathbb{L}\left(V^{\otimes k} \to W\right)$ shall also denote the linear map

$$A^{\otimes k} : v_1 \otimes \cdots \otimes v_k \mapsto A(v_1) \dots A(v_k).$$

□

A further property of the projective tensor norm is that its associated operator norm is submultiplicative for tensor products of linear maps:

Lemma 1.2.7. *Let V be a Banach space, B a Banach algebra and $A \in \mathbb{L}(V \to B)$. Then for all $k \in \mathbb{N}$,*

$$\|A^{\otimes k}\|_{\mathbb{L}(V^{\otimes k} \to B)} \leqslant \|A\|^k_{\mathbb{L}(V \to B)}.$$

Proof. When $k = 0$, the result is clear so we assume that $k \geqslant 1$. Let $v \in V^{\otimes k}$ with $\|v\|_\pi = 1$. For all $\varepsilon > 0$, we can write

$$v = \sum_{i=1}^{n} v_1^i \otimes \cdots \otimes v_k^i, \text{ with } \sum_{i=1}^{n} \|v_1^i\| \dots \|v_k^i\| \leqslant 1 + \varepsilon.$$

Then

$$\begin{aligned}\|A^{\otimes k}v\| &= \left\|\sum_{i=1}^{n} A(v_1^i)\dots A(v_k^i)\right\| \\ &\leqslant \sum_{i=1}^{n} \|A\|_{\mathbb{L}(V\to B)}^k \|v_1^i\|\dots\|v_k^i\| \\ &\leqslant (1+\varepsilon)\|A\|_{\mathbb{L}(V\to B)}^k.\end{aligned}$$

The result follows by letting ε tend to 0. □

Definition 1.2.8 (Extended tensor algebra)**.** *The extended tensor algebra over V, denoted by $T((V))$, is defined to be the following Cartesian product:*

$$T((V)) = \prod_{k=0}^{\infty} V^{\otimes k}.$$

It is endowed with two binary operations: an addition and a product. Let $\mathbf{a} = (a_0, a_1, \dots)$, $\mathbf{b} = (b_0, b_1, \dots) \in T((V))$ where $a_i, b_i \in V^{\otimes i}$ for all $i \geqslant 0$. Then

$$\mathbf{a} + \mathbf{b} = (a_0 + b_0, a_1 + b_1, \dots)$$

and

$$\mathbf{ab} = (c_0, c_1, \dots),$$

where for all $n \geqslant 0$,

$$c_n = \sum_{k=0}^{n} a_k \otimes b_{n-k}.$$

For all $k \geqslant 0$, we denote by

$$\pi_k : T((V)) \to V^{\otimes k}, \quad \pi_k \colon \mathbf{a} \mapsto a_k$$

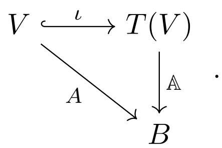
the projection onto $k-$tensors.

The space $T((V))$ endowed with these two binary operations and the action of $\mathbb{R}$ $\lambda\mathbf{a} = (\lambda a_0, \lambda a_1, \dots)$ is a non-commutative ($\dim(V) > 1$) unital algebra with unit $\mathbf{1} = (1, 0, \dots)$.

Definition 1.2.9 (Tensor algebra)**.** *Denote by $T(V) \subset T((V))$ the subalgebra*

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

$T(V)$ is a graded algebra, with, for all $k \in \mathbb{N}$, $V^{\otimes k}$ the grade$-k$ subspace. $\iota : V \to T(V)$ is the canonical inclusion of V into $T(V)$. The tensor algebra satisfies the universal property that any linear map $A : V \to B$ into a real algebra, B, can be uniquely extended to an algebra homomorphism $\mathbb{A} : T(V) \to B$ as indicated by the following commutative diagram:



For each $n \geqslant 0$ denote the linear subspace of $T((V))$

$$B_n = \{\mathbf{a} \in T((V)) \mid a_0 = \cdots = a_n = 0\}.$$

B_n is an ideal of $T((V))$.

Definition 1.2.10 (Truncated tensor algebra)**.** *Let $n \geqslant 1$. The truncated tensor algebra of order n over V is the quotient algebra*

$$T^n(V) := T((V))/B_n.$$

The quotient map $T((V)) \to T^n(V)$ is denoted $\Pi_{0,n}$.

$T^n(V)$ is canonically isomorphic to $\bigoplus_{k=0}^n V^{\otimes k}$ equipped with the product

$$(a_0, a_1, \dots, a_n)(b_0, b_1, \dots, b_n) = (c_0, c_1, \dots, c_n),$$

where $c_k = \sum_{i=0}^k a_i \otimes b_{k-i}$ for all $0 \leqslant k \leqslant n$.

The dual of $T((V))$

Let $(e_1, \dots, e_d)$ be a basis for V and $(e_1^*, \dots, e_d^*)$ the corresponding dual basis for V^*. Let $n \geqslant 1$. The tensors

$$\{e_I = e_{i_1} \otimes \cdots \otimes e_{i_n} \mid I = (i_1, \dots, i_n) \in \{1, \dots, d\}^n\}$$

form a basis of $V^{\otimes n}$. We identify $(V^*)^{\otimes n} \xrightarrow{\cong} (V^{\otimes n})^*$ by identifying the basis $\left\{e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_n}^* \mid I = (i_1, \dots, i_n) \in \{1, \dots, d\}^n\right\}$ of $(V^*)^{\otimes n}$ with the dual basis of $(V^{\otimes n})^*$ as follows:

$$\left\langle e_{i_1}^* \otimes \cdots \otimes e_{i_n}^*, e_{j_1} \otimes \cdots \otimes e_{j_n} \right\rangle = \delta_{i_1, j_1} \dots \delta_{i_n, j_n}.$$

The action of $(V^*)^{\otimes n}$ on $V^{\otimes n}$ extends to an action on $T((V))$ by:

$$e_I^*(\mathbf{a}) = e_I^*(\pi_n(\mathbf{a})).$$

By letting n vary between 0 and ∞, we obtain the action of $T(V^*) = \bigoplus_{n=0}^{\infty}(V^*)^{\otimes n}$ on $T((V))$ and in fact have $T((V))^* \cong T(V^*)$.

Definition 1.2.11 (Shuffle product)**.** *Let $s, r \in \mathbb{N}$, $I = (i_1, \ldots, i_r) \in \{1, \ldots, d\}^r$ and $J = (j_1, \ldots, j_s) \in \{1 \ldots, d\}^s$. Define their shuffle product $e_I^* \sqcup e_J^*$ by:*

$$e_I^* \sqcup e_J^* := \sum_{\sigma \in \mathrm{Sh}(r,s)} e^*_{k_{\sigma(1)}, \ldots, k_{\sigma(r+s)}} \in T(V^*).$$

This extends to a commutative product on $T(V^*)$.

Definition 1.2.12 (Grouplike elements)**.** *Denote by*

$$G^*(V) = \{\mathbf{a} \in T((V)) \backslash \{0\} \mid \mathbf{e}^*(\mathbf{a})\mathbf{f}^*(\mathbf{a}) = \mathbf{e}^* \sqcup \mathbf{f}^*(\mathbf{a}),\ \forall \mathbf{e}^*, \mathbf{f}^* \in T(V^*)\}$$

the set of grouplike elements. For $n \in \mathbb{N}$,

$$\Pi_{0,n}\left(G^*(V)\right) = G^n(V) \subseteq T^n(V)$$

is the step$-n$ free nilpotent group over V.

Definition 1.2.13. *Let $\mathbf{a} \in T((V))$ then*

$$\exp(\mathbf{a}) = \sum_{k=0}^{\infty} \frac{a^k}{k!} \in T((V)).$$

Definition 1.2.14 (Free Lie algebra)**.** *Denote by $[\cdot, \cdot]$ the commutator on the algebra $T((V))$. Consider the sequence $(\mathfrak{L}^{(n)}(V))_{n \geqslant 0}$ of subspaces of $T((V))$ given by $\mathfrak{L}^{(0)}(V) = 0$, $\mathfrak{L}^{(1)}(V) = V$ and*

$$\mathfrak{L}^{(n+1)}(V) = \left[V, \mathfrak{L}^{(n)}(V)\right]$$

for all $n \geqslant 1$.

$\mathfrak{g}^n(V) := \Pi_{0,n}\left(\bigoplus_{i=0}^n \mathfrak{L}^{(i)}(V)\right)$ is the Lie algebra of $G^n(V)$ and

$$\exp_n : \mathfrak{g}^n(V) \to G^n(V), \quad \exp_n : \mathbf{a} \mapsto \sum_{k=0}^{n} \frac{\mathbf{a}^k}{k!}$$

is a global diffeomorphism.

Thus, $G^n(V)$ is a Carnot group and we equip it with a symmetric, subadditive homogeneous norm (Section 3 in [31]), $|\cdot|_{G^n(V)}$, and induced metric.

2 The Space of Unparameterised Geometric Rough Paths

We begin this chapter by describing the metric space of unparameterised $p-$rough paths, $\mathbb{G}_p(V)$, together with various operations on this space. We equip $\mathbb{G}_p(V)$ with a natural partial order: for $g, h \in \mathbb{G}_p(V)$, we will write $g \subseteq h$ if one can decompose $h = g * h_2$, where $h_2 \in \mathbb{G}_p(V)$ and $*$ denotes concatenation. We collect also some technical results required in the sequel, where we shall describe and study Lip^γ functions on unparameterised paths.

Many of the results in this chapter are well-known. In particular, Section 2.2 largely reformulates aspects of classical rough integration theory – as exposed in [32], [33], [35], [43] – to the unparameterised setting.

2.1 Preliminary definitions and notations

Definition 2.1.1. *Let $a < b$ and G be a group. Given a path $g : [a, b] \to G$, and real numbers $s, t \in [a, b]$, we shall denote the increment*

$$g_{s,t} := g_s^{-1} g_t,$$

where $\bullet^{-1}$ indicates the inverse in G.

Definition 2.1.2 (Parameterised weakly geometric $p-$rough paths)**.** *Let $a < b$. Denote by $WG\Omega_p([a, b] \to V)$ the set of continuous functions*

$$g : [a, b] \to G^{\lfloor p \rfloor}(V)$$

of finite p−variation:

$$\|g\|_p^p := \sup_D \sum_{t_i \in D} \left|g_{t_i,t_{i+1}}\right|^p_{G^{\lfloor p \rfloor}(V)} < +\infty,$$

where the supremum is taken over all finite partitions,

$$D = (a = t_0 < \cdots < t_n = b),$$

of $[a, b]$.

This is a metric space under

$$d(g,h)^p := \sup_D \sum_{t_i \in D} \left| \left(g_{t_{i-1},t_i}\right)^{-1} \left(h_{t_{i-1},t_i}\right) \right|^p_{G^{\lfloor p \rfloor}(V)}.$$

Remark 2.1.3. Sometimes in the literature (e.g. [32, 33]), a $p-$rough path is defined as a two-parameter object

$$\mathbf{x} : \{(u, v) : a \leqslant u \leqslant v \leqslant b\} \to G^{\lfloor p \rfloor}(V)$$

satisfying $\mathbf{x}_{s,u}\,\mathbf{x}_{u,t} = \mathbf{x}_{s,t}$ for all $a \leqslant s \leqslant u \leqslant t \leqslant b$ together with a finite $p-$variation condition. This is equivalent to our definition. Indeed, given such a two-parameter object, $\mathbf{x}$, we may define

$$g : [a, b] \to G^{\lfloor p \rfloor}(V), \quad g_t := \mathbf{x}_{a,t}$$

and then g is a $p-$rough path in the sense of Definition 2.1.2. This perspective of viewing a rough path as a Lie-group valued path is not novel (see Section 2.3 in [16] or Definition 4 in [43], for example) and will be more suitable when we come to treating unparameterised rough paths in the sequel.

Definition 2.1.4 (Canonical rough path lift)**.** *Let* $\mathbf{x} : [a, b] \to V$ *be a path of bounded variation. Let* $N \in \mathbb{N}$*. The path* $g^N(\mathbf{x}) : [a, b] \to G^N(V) \subseteq T^N(V)$

$$g^N(\mathbf{x})_t = 1 + \sum_{k=1}^{N} \int_{\triangle^k_{a,t}} \mathrm{d}\mathbf{x}_{u_1} \otimes \cdots \otimes \mathrm{d}\mathbf{x}_{u_k},$$

where $\triangle^k_{a,t} := \{a < u_1 < \cdots < u_k < t\} \subset \mathbb{R}^k$ *is the* $k-$*simplex on* $[a, t]$*, is called the level*$-N$ *canonical rough path lift of* $\mathbf{x}$*. One can show that* $g^N(\mathbf{x})$ *indeed is a* $p-$*rough path for any* $p \in [N, N + 1)$*.*

As is common in the rough path literature, we sometimes denote

$$\mathbf{x}^k_{s,t} := \int_{\triangle^k_{s,t}} \mathrm{d}\mathbf{x}_{u_1} \otimes \cdots \otimes \mathrm{d}\mathbf{x}_{u_k} .$$

Theorem 2.1.5 (*Lyons' Extension theorem:* Theorem 3.7 in [33])**.** *Let* $g \in WG\Omega_p([a, b] \to V)$ *be a rough path. Let* $N \in \mathbb{N}$ *with* $N > \lfloor p \rfloor$*. Then there exists a unique* $N-$*rough path* $\mathbb{X}^{\leqslant N}(g) : [a, b] \to G^N(V)$ *such that*

1. $\Pi_{0,\lfloor p \rfloor}\left(\mathbb{X}^{\leqslant N}(g)_t\right) = g_t$*, for all* $t \in [a, b]$*;*

2. $\mathbb{X}^{\leqslant N}(g)$ *has finite* $p-$*variation.*

Furthermore, there exists a constant $B_p < +\infty$ *independent of* k *such that*

$$\|\pi_k(\mathbb{X}^{\leqslant N}(g)_{s,t})\| \leqslant B_p \frac{\|g|_{[s,t]}\|_p^{k/p}}{(k/p)!}.$$

This leads to the definition

Definition 2.1.6 (Signature)**.** *Let* $g \in WG\Omega_p([a,b] \to V)$ *be a rough path. Let* $N \in \mathbb{N}$*. We shall call* $\mathbb{X}^{\leqslant N}(g)_{a,b} = \mathbb{X}^{\leqslant N}(g)$*, as defined in Theorem 2.1.5, the level–*N *truncated signature of* g*. Let* $k \leqslant N$*, we call* $\mathbb{X}^k(g) = \pi_k\left(\mathbb{X}^{\leqslant N}(g)\right)$ *the level–*k *signature of* g*. And, finally,*

$$\mathbb{X}(g) = 1 + \sum_{k=1}^{\infty} \mathbb{X}^k(g) \in T((V))$$

the signature of g*.*

Definition 2.1.7 (Parameterised geometric p–rough paths)**.** *Let* $a < b$*. Denote by* $G\Omega_p([a,b] \to V)$ *the closure of the set*

$$\{g \in WG\Omega_p([a,b]) \ : \ g \text{ is the canonical lift of a smooth path } [a,b] \to V\}$$

in the metric space $WG\Omega_p([a,b] \to V)$*. When* $[a,b] = [0,1]$*, we shall denote this space simply by* $G\Omega_p(V)$*.*

Definition 2.1.8 (The set of unparameterised geometric p–rough paths, $\mathbb{G}_p(V)$)**.** *Introduce the equivalence relation on* $G\Omega_p(V)$*, writing* $g \sim h$ *if and only if there exists a strictly increasing homeomorphism* $\tau : [0,1] \to [0,1]$ *(a "reparameterisation") such that* $g \circ \tau = h$*. Define the set of unparameterised paths as the quotient set*

$$\mathbb{G}_p(V) := G\Omega_p(V)/\sim .$$

An unparameterised geometric p–rough path is then an equivalence class of parameterised paths, $[g]$*, which we shall often denote by choosing a representative:* g*.*

Remark 2.1.9. Within $\mathbb{G}_p(V)$, two paths differ if one cannot reparameterise one of them to equal the other. This can be quite a subtle distinction. For example consider the two $1-$rough paths $\gamma^1, \gamma^2 : [0,1] \to \mathbb{R}$, $\gamma^1_t := t$ for all t and

$$\gamma^2_t := \begin{cases} 1.5t, & t < \frac{1}{3}, \\ 0.5, & t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1.5t - 0.5, & t > \frac{2}{3}. \end{cases}$$

If one plots the curves $t \mapsto \gamma^1_t$ and $t \mapsto \gamma^2_t$ on the real line, one would not be able to distinguish between the two. Indeed, they both monotonically traverse the interval $[0,1]$, but γ^2 remains constant over a period; whereas, γ^1 does not. The signature fails to distinguish between these two paths: a direct calculation shows that $\mathbb{X}(\gamma^1) = \mathbb{X}(\gamma^2) = \sum_{k=0}^{\infty} \frac{1}{k!} \in T((\mathbb{R}))$. However, the two paths are different in $\mathbb{G}_1(\mathbb{R})$. Indeed, there does not exist an increasing homeomorphism $\tau : [0,1] \to [0,1]$ such that $\gamma^1 = \gamma^2 \circ \tau$.

In Chapter 3 we shall define Lipschitz functions on unparameterised rough paths and we shall see (Lemma 3.5.5) that these functions are, unlike signatures, rich enough to separate points in $\mathbb{G}_p(V)$.

2.1.1 Operations on unparameterised paths.

When one quotients out parameterisations, one loses certain operations on paths. For example, it is meaningless to evaluate an unparameterised path at time $t = 1/2$, or to add two unparameterised paths. We collect below some of the relevant operations that are well-defined on the quotient.

Define the **initial and tail** point of an unparameterised path

$$\mathfrak{i}, \mathfrak{t} : \mathbb{G}_p(V) \to G^{\lfloor p \rfloor}(V), \qquad \mathfrak{i}([g]) := g(0), \;\; \mathfrak{t}([g]) := g(1).$$

Concatenation. If $g, h \in G\Omega_p(V)$ are parameterised paths with $g_1 = h_0$. Define their concatenation $g * h \in G\Omega_p(V)$ by

$$(g * h)_t = \begin{cases} g_{2t}, & t \in [0, 1/2], \\ h_{2t-1}, & t \in [1/2, 1]. \end{cases}$$

If $[g], [h] \in \mathbb{G}_p(V)$ are unparameterised paths with $\mathfrak{t}(g) = \mathfrak{i}(h)$ define their *concatenation*

$$[g] * [h] := [g * h] \in \mathbb{G}_p(V).$$

Reversal of a path. Let $g \in G\Omega_p(V)$. Define its reversal $\overleftarrow{g} \in G\Omega_p(V)$ by

$$\overleftarrow{g_t} := g_{1-t}.$$

For an unparameterised path $[g] \in \mathbb{G}_p(V)$, define

$$\overleftarrow{[g]} := [\overleftarrow{g}].$$

Translation of a path. Let $g \in G\Omega_p(V)$ and $\kappa \in G^{\lfloor p \rfloor}(V)$ define the left and right translations of g by κ as $\kappa g, g\kappa \in G\Omega_p(V)$

$$(\kappa g)_t := \kappa(g_t), \quad (g\kappa)_t := (g_t)\kappa,$$

respectively. And, for $[g] \in \mathbb{G}_p(V)$,

$$\kappa[g] := [\kappa g], \quad [g]\kappa := [g\kappa].$$

$p-$**variation**. Let $[g] \in \mathbb{G}_p(V)$, then define

$$\|[g]\|_p := \|g\|_p.$$

Definition 2.1.10 (Factorisation of an unparameterised path)**.** *Let $g \in \mathbb{G}_p(V)$ be an unparameterised geometric $p-$rough path and $m \geqslant 1$.*
$D_g = \big(g_0, \ldots, g_m\big) \subset \mathbb{G}_p(V)$ is a rough path factorisation of g (or simply: a factorisation of g) provided

- *$g_i \in \mathbb{G}_p(V)$ for all $i = 0, \ldots, m$ with $\mathfrak{t}(g_i) = \mathfrak{i}(g_{i+1})$ for all $i = 0, \ldots, m-1$,*
- *$g_0 * \cdots * g_m = g$,*
- *g_0 is the constant path $g_0 \equiv \mathfrak{i}(g)$.*

Any such factorisation has a mesh:

$$|D_g| := \max_{j=0,\ldots,m} \|g_j\|_p.$$

If $m = 1$, then $D_g = \big(g_0, g\big)$. We call such a D_g the trivial factorisation *of g.*

Definition 2.1.11 (Coarsening a factorisation)**.** *Let $D_g = \big(g_0, \ldots, g_m\big)$ be a rough path factorisation of some $g \in \mathbb{G}_p(V)$. For $i = 1, \ldots, m$, we may* *coarsen* D_g

$$D_g \backslash \{g_i\} := \big(g_0, \ldots, g_{i-2}, g_{i-1} * g_i, g_{i+1}, \ldots, g_m\big).$$

$D_g \backslash \{g_i\}$ is again a factorisation of g and satisfies $|D_g \backslash \{g_i\}| \geqslant |D_g|$.

2.1.2 A partial order on unparameterised paths

Definition 2.1.12. *Let $g, h \in \mathbb{G}_p(V)$. We write $g \subseteq h$ if and only if there exists $h_2 \in \mathbb{G}_p(V)$ such that $h = g * h_2$.*

Lemma 2.1.13. *Let $g, h \in \mathbb{G}_p(V)$. Then*

$$g \subseteq h \implies \|g\|_p \leqslant \|h\|_p.$$

Proof. Immediate from definition of $p-$variation. □

Lemma 2.1.14. *If $g, h \in \mathbb{G}_p(V)$ such that*

$$g \subseteq h \quad \text{and} \quad \|g\|_p = \|h\|_p$$

then $g = h$.

Proof. Pick parameterisations of g and h,

$$g : [0,1] \to G^{\lfloor p \rfloor}(V), \quad h : [0,2] \to G^{\lfloor p \rfloor}(V)$$

such that $g = h|_{[0,1]}$ and write $h_2 := h|_{[1,2]}$. The claim is equivalent to showing that h_2 is a constant path.

Suppose for a contradiction that h_2 is not a constant path, then $\exists s < t \in [1,2]$ such that

$$|h_{s,t}|^p =: \varepsilon > 0.$$

Since $\varepsilon/2 > 0$, there exists a partition $D = D(\varepsilon/2) \subset [0,1]$ such that

$$\sum_{t_i \in D} |g_{t_i,t_{i+1}}|^p \geqslant \|g\|_p^p - \varepsilon/2.$$

Define the partition of $[0,2]$, $\mathscr{D} := D \cup \{s,t,2\}$ then

$$\begin{aligned}
\|h\|_p^p \geqslant \sum_{t_i \in \mathscr{D}} |h_{t_i,t_{i+1}}|^p &\geqslant \sum_{t_i \in D} |g_{t_i,t_{i+1}}|^p + |h_{s,t}|^p \\
&\geqslant \|g\|_p^p - \varepsilon/2 + \varepsilon \\
&= \|g\|_p^p + \varepsilon/2.
\end{aligned}$$

This contradicts that $\|h\|_p = \|g\|_p$, completing the proof. □

Corollary 2.1.15. $(\mathbb{G}_p(V), \subseteq)$ *is a partially ordered set (poset).*

Definition 2.1.16 (History of a path)**.** *Let $g \in \mathbb{G}_p(V)$ define the history of g,* $\mathrm{Hist}(g) \subset \mathbb{G}_p(V)$*, by*

$$\mathrm{Hist}(g) := \{\xi \in \mathbb{G}_p(V) : \ \xi \subseteq g\}.$$

Definition 2.1.17 (Metric on $\mathrm{Hist}(g)$)**.** *Let $g \in \mathbb{G}_p(V)$ and choose a parameterisation $g : [0,1] \to G^{\lfloor p \rfloor}(V)$. For $a_1, a_2 \in \mathrm{Hist}(g)$, there exists $s,t \in [0,1]$ such that $a_1 = g|_{[0,s]}$ and $a_2 = g|_{[0,t]}$. Define the metric, d, on S by*

$$d(a_1,a_2) := \|g|_{[s\wedge t, s\vee t]}\|_p.$$

By parameter independence of $p-$variation, this metric is independent of the choice of parameterisation for g.

Lemma 2.1.18. *Let $g \in \mathbb{G}_p(V)$ then the set $\{a \in \mathbb{G}_p(V) : a \subseteq g\}$ is homeomorphic to a (possibly degenerate) closed, bounded interval in $\mathbb{R}$. Further, the homeomorphism can be taken to be increasing.*

Proof. See Lemma A.1.2 in the Appendix for the proof. □

Definition 2.1.19. *Let $\mathcal{K} \subseteq \mathbb{G}_p(V)$ be a nonempty set of paths. We say that $\mathcal{K}$ is bounded above (resp., below) if there exists $g \in \mathbb{G}_p(V)$ such that, $k \subseteq g$ (resp., $g \subseteq k$) for all $k \in \mathcal{K}$. We call such a g an upper (resp., lower) bound for $\mathcal{K}$.*

Lemma 2.1.20 (Supremum)**.** *Let $S \subseteq \mathbb{G}_p(V)$ be nonempty and bounded above. Then $\exists! \sup(S) \in \mathbb{G}_p(V)$ satisfying*

1. *For all $g \in S$, $g \subseteq \sup(S)$,*
2. *If $\exists \bar{g} \in \mathbb{G}_p(V)$ such that $\forall g \in S$, $g \subseteq \bar{g}$ then $\sup(S) \subseteq \bar{g}$.*

Proof. Uniqueness follows from $(S, \subseteq)$ being a poset. As for existence, let $\bar{g} \in \mathbb{G}_p(V)$ be an upper bound for S, $I \subset \mathbb{R}$ a closed, bounded interval and $\varphi : \{a \in \mathbb{G}_p(V) : a \subseteq \bar{g}\} \to I$ an increasing homeomorphism. By the completeness axiom in $\mathbb{R}$, $\varphi(S) \subseteq I$ has a least upper bound, which we denote $\mathbf{u} = \sup(\varphi(S)) \in I$. Then, $\sup(S) := \varphi^{-1}(\mathbf{u}) \in \mathbb{G}_p(V)$ satisfies the desired properties. □

Definition/Lemma 2.1.21 (The youngest common ancestor)**.** *Let* $g, h \in \mathbb{G}_p(V)$ *with* $\mathfrak{i}(g) = \mathfrak{i}(h)$*. Define the* youngest common ancestor *of* g *and* h

$$g \wedge h = h \wedge g := \sup \{\xi \in \mathbb{G}_p(V) : \xi \subseteq g,\ \xi \subseteq h\}.$$

Note that $g \wedge h \subseteq g, h$*. We verify the operation* $\wedge$ *is well-defined.*

Proof. $S = \{\xi \in \mathbb{G}_p(V) : \xi \subseteq g,\ \xi \subseteq h\} \subseteq \mathbb{G}_p(V)$ is nonempty and bounded above so $\exists! \sup(S) \in \mathbb{G}_p(V)$. □

Remark 2.1.22. For $g, h \in \mathbb{G}_p(V)$ with $\mathfrak{i}(g) = \mathfrak{i}(h)$, $g \wedge h \in \mathbb{G}_p(V)$ is the unique path of maximal $p-$variation ("youngest") such that $g \wedge h \subseteq g$ and $g \wedge h \subseteq h$ "common ancestor").

Definition/Corollary 2.1.23. *Let* $g, h \in \mathbb{G}_p(V)$ *with* $\mathfrak{i}(g) = \mathfrak{i}(h)$*.*

$$g_2(h) \in \mathbb{G}_p(V)$$

is the unique path given implicitly by the equation

$$g = (g \wedge h) * g_2(h).$$

2.1.3 A metric on unparameterised paths

Lemma 2.1.24 ($p-$variation inequalities)**.** *Suppose* $x, y \in \mathbb{G}_p(V)$ *with* $\mathfrak{i}(y) = \mathfrak{t}(x)$ *then the following hold:*

1) $\|x * y\|_p \leqslant \|x\|_p + \|y\|_p$, *2)* $\|x * y\|_p^p \geqslant \|x\|_p^p + \|y\|_p^p$,

3) $\|x * y\|_p \geqslant 2^{\frac{1-p}{p}} \left(\|x\|_p + \|y\|_p\right)$.

Proof. See Lemma A.1.5 for the proof. □

The following metric reflects the type of path perturbations that we shall be considering for previsible Lipschitz functions on paths. Loosely, two paths $x, y \in \mathbb{G}_p(V)$ will be close if they share a significant common history: namely, $x = \xi * x_2$ and $y = \xi * y_2$ with $\|x_2\|_p, \|y_2\|_p$ small. If $x = y * x_2$, we consider x a perturbation of y. This is a natural notion of perturbation for paths and is very different from Cameron-Martin-type perturbations, which make little sense in our parameter-independent, measure-free framework.

Definition 2.1.25 (The metric space of unparameterised geometric $p-$rough paths)**.** *Let $x, y \in \mathbb{G}_p(V)$. Recall the decompositions from Definition 2.1.23 valid when $\mathfrak{i}(x) = \mathfrak{i}(y)$*

$$x = (x \wedge y) * x_2(y), \qquad y = (x \wedge y) * y_2(x).$$

Define the function $d_p : \mathbb{G}_p(V) \times \mathbb{G}_p(V) \to [0, \infty)$ by

$$d_p(x, y) := \begin{cases} \left\| \overleftarrow{x_2(y)} * y_2(x) \right\|_p, & \mathfrak{i}(x) = \mathfrak{i}(y), \\ \left\| \overleftarrow{x} * \left(\mathfrak{i}(x)\, \mathfrak{i}(y)^{-1} y\right) \right\|_p + d_{G^{\lfloor p \rfloor}(V)}\left(\mathfrak{i}(x), \mathfrak{i}(y)\right), & \mathfrak{i}(x) \neq \mathfrak{i}(y). \end{cases}$$

See Lemma A.1.4 in the Appendix for the proof that d_p is a metric.

In order to show that $(\mathbb{G}_p(V), d_p)$ is complete, we first collect some basic technical results.

Definition 2.1.26. *We say a sequence $g_n \in \mathbb{G}_p(V)$ is increasing (resp., decreasing) if for all $n \in \mathbb{N}$ $g_n \subseteq g_{n+1}$ (resp., $g_{n+1} \subseteq g_n$).*

Lemma 2.1.27. *Let $g_n \in \mathbb{G}_p(V)$ be a bounded, increasing sequence. Then g_n converges.*

Proof. Let $g \in \mathbb{G}_p(V)$ be an upper bound for $\{g_n : n \in \mathbb{N}\}$, $I \subset \mathbb{R}$ a closed, bounded interval and $\varphi : \{a \in \mathbb{G}_p(V) : a \subseteq g\} \to I$ an increasing homeomorphism. Then, since $\varphi(g_n) \in \mathbb{R}$ is bounded and increasing, $\varphi(g_n) \xrightarrow{n\to\infty} \varphi(\bar{g})$ for some $\bar{g} \in \mathbb{G}_p(V)$ and hence $g_n \xrightarrow{n\to\infty} \bar{g}$. □

Similarly, one has

Lemma 2.1.28. *Let $g_n \in \mathbb{G}_p(V)$ be a decreasing sequence. Then g_n converges.*

Proof. Let $I \subset \mathbb{R}$ be a closed, bounded interval and $\varphi : \{a \in \mathbb{G}_p(V) : a \subseteq g_1\} \to I$ be an increasing homeomorphism. Then, since $\varphi(g_n) \in \mathbb{R}$ is bounded and decreasing, $\varphi(g_n) \xrightarrow{n\to\infty} \varphi(\bar{g})$ for some $\bar{g} \in \mathbb{G}_p(V)$ and hence $g_n \xrightarrow{n\to\infty} \bar{g}$. □

Lemma 2.1.29. *Let $g_n \in \mathbb{G}_p(V)$ be an increasing sequence with*

$$U := \sup_n \|g_n\|_p < +\infty.$$

Then g_n converges.

Proof. Choose a parameterised path $\mathbf{g} : [0, 1) \to G^{\lfloor p \rfloor}(V)$ such that

$$\left[\mathbf{g}|_{[0,1-1/n]}\right] = g_n, \quad \forall n \geqslant 2.$$

Since $p-$variation of parameterised, continuous paths is lower semi-continuous with respect to the topology of pointwise convergence (Proposition 1.8 in [33]),

$$\left\|\mathbf{g}|_{[0,1)}\right\|_p \leqslant \liminf_{n\to\infty} \|g_n\|_p \leqslant U < +\infty$$

so that $\mathbf{g}$ extends uniquely to a geometric $p-$rough path $\mathbf{g} : [0, 1] \to G^{\lfloor p \rfloor}(V)$.

Denote $g = [\mathbf{g}] \in \mathbb{G}_p(V)$. We hope to show that $g_n \xrightarrow{n\to\infty} g$ in $\mathbb{G}_p(V)$. To this end, note that for all n we may factorise $g = g_n * (g_n)_2(g)$, where $g_n = \left[\mathbf{g}|_{[0,1-1/n]}\right]$ and $(g_n)_2(g) = \left[\mathbf{g}|_{[1-1/n,1]}\right]$. Then,

$$d_p(g_n, g) = \|(g_n)_2(g)\|_p = \left\|\mathbf{g}|_{[1-1/n,1]}\right\|_p \xrightarrow{n\to\infty} 0,$$

as desired. □

Proposition 2.1.30. *The metric space $(\mathbb{G}_p(V), d_p)$ is complete.*

Proof. Let $g_n \in \mathbb{G}_p(V)$ be a Cauchy sequence. We intend to show that g_n converges. Consider the two cases.

Case 1) *Suppose exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $\mathfrak{i}(g_n) = \mathfrak{i}(g_m)$.* Fix $M \in \mathbb{N}$ large so that $\sup_{n\geqslant M} d_p(g_M, g_n) \leqslant 1$ then note the factorisation

$$g_n = (g_M \wedge g_n) * (g_M)_2(g_n)$$

hence

$$\|g_n\|_p \leqslant \|g_M\|_p + d_p(g_M, g_n) \leqslant \|g_M\|_p + 1$$

and therefore $U := \sup_{n\in\mathbb{N}} \|g_n\|_p < +\infty$. Denote for all large n

$$S_n = \{g \in \mathbb{G}_p(V) \mid g \subseteq g_k \ \forall k \geqslant n\} \subseteq \mathbb{G}_p(V).$$

Note that $\mathfrak{i}(g_n) \in S_n \neq \varnothing$ and that S_n is bounded above (by g_n). Therefore, we may set

$$\underline{g_n} := \sup\{g \in \mathbb{G}_p(V) \mid g \subseteq g_k,\ \forall k \geqslant n\} \in \mathbb{G}_p(V).$$

Note that $\underline{g_n} \subseteq \underline{g_{n+1}}$ and that $\|\underline{g_n}\|_p \leqslant \|g_n\|_p \leqslant U$ so that, by Lemma 2.1.29, there exists $\underline{g} \in \mathbb{G}_p(V)$ such that $\underline{g_n} \to \underline{g}$.

We claim that $d_p(\underline{g_n}, g_n) \xrightarrow{n\to\infty} 0$. Assume for a contradiction that the claim is false: there exists $\varepsilon > 0$ such that $d(g_n, \underline{g_n}) \geqslant \varepsilon$ for infinitely many n. Then, by the definition of $\underline{g_n}$ and properties of sup, for each such n there exists $m \geqslant n$ such that $\underline{g_n} \subseteq g_m \subseteq g_n$ and $d(g_m, g_n) \geqslant \varepsilon/2$, contradicting the Cauchy property.

Finally, by the triangle inequality,

$$d_p(g_n, \underline{g}) \leqslant d_p\left(g_n, \underline{g_n}\right) + d_p\left(\underline{g_n}, \underline{g}\right) \xrightarrow{n\to\infty} .$$

So that $g_n \xrightarrow{n\to\infty} \underline{g}$, as desired.

Case 2) *Suppose for all $N \in \mathbb{N}$ there exist $n, m \geqslant N$ such that $\mathfrak{i}(g_n) \neq \mathfrak{i}(g_m)$.*

$\mathfrak{i}(g_n) \in G^{\lfloor p \rfloor}(V)$ is a Cauchy sequence in a complete space. Denote $\underline{g} = \lim_{n\to\infty} \mathfrak{i}(g_n) \in G^{\lfloor p \rfloor}(V)$. Then g_n converges to the constant path $\underline{g}$, completing the proof. □

2.2 Rough Integration

We briefly examine aspects of the recently developed rough integration theory ([34], [35], [43]), making appropriate modifications so as to dovetail with the space of unparameterised $p-$rough paths, $\mathbb{G}_p(V)$. The reader ought keep in mind that our eventual goal is to develop a class of scalar-valued functions on paths analogous to Lipschitz functions on Euclidean spaces. The emphasis here, therefore, is on viewing rough integration as an operation from *paths to scalars*, as opposed to *from paths to paths*.

2.2.1 Polynomial and cocyclic integration

The integration of polynomial one-forms against rough paths is well understood, see for example [32]. Cocyclic one-forms generalise polynomial one-forms and were introduced in [35]. They satisfy the property that the integral of a cocyclic one-form against a rough path depends linearly on the signature of said rough path. We review the integration of both polynomial and cocyclic one-forms against rough paths. Cocyclic one-forms will be one of the building blocks when defining the forthcoming Lipschitz functions on $\mathbb{G}_p(V)$.

Definition 2.2.1 (Fréchet derivatives)**.** *A map between Banach spaces* $f : V \to W$ *is called Fréchet differentiable if for all* $x \in V$*, there exists a (bounded) linear map* $\nabla f(x) \in \mathbb{L}(V \to W)$ *such that*

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - \nabla f(x)h\|}{\|h\|} = 0.$$

The map

$$\nabla f : V \to \mathbb{L}(V \to W), \quad x \mapsto \nabla f(x)$$

is then called the Fréchet derivative (or simply: derivative) of f*. For* $n \geqslant 2$*, we define recursively*

$$\nabla^n f := \nabla(\nabla^{n-1} f)$$

whenever $\nabla^{n-1} f$ *is Fréchet differentiable. We say that* f *is* $n-$*times Fréchet differentiable if* $\nabla^{n-1} f$ *is Fréchet differentiable. If* f *is* $n-$*times Fréchet differentiable for all* n*, we say that* f *is Fréchet smooth.*

Remark 2.2.2. A well-known and useful feature of tensor product spaces is their ability to linearise multilinear maps. Indeed for finite-dimensional Banach spaces $V_1, \ldots, V_n$:

$$\mathbb{L}(V_1 \to \mathbb{L}(V_2 \to W)) \cong \mathbb{L}(V_1 \otimes V_2 \to W)$$

and more generally

$$\mathbb{L}(V_1 \to \mathbb{L}(V_2 \to \ldots \mathbb{L}(V_n \to W) \ldots) \cong \mathbb{L}(V_1 \otimes \cdots \otimes V_n \to W).$$

⚠We shall make implicit use of such identifications in the sequel. For example if $f : V \to W$ is $n-$times Fréchet differentiable, we shall view $\nabla^n f$ as taking values in *linear* maps $V^{\otimes n} \to W$:

$$\nabla^n f : V \to \mathbb{L}(V^{\otimes n} \to W).$$

Definition 2.2.3 (Polynomial functions)**.** *Let $n \in \mathbb{N}$. $\mathcal{P}^{(n)}(V \to W)$ denotes the vector space of degree at most n polynomial functions $V \to W$. Namely, $q \in \mathcal{P}^{(n)}(V \to W)$ if and only if $q : V \to W$ is Fréchet smooth with $\nabla^{n+1} q = q^{n+1} \equiv 0$. If $W = \mathbb{L}(V \to U)$ for some Banach space U, we call q a $U-$*valued polynomial one-form *(if $U \cong \mathbb{R}$, simply: polynomial one-form).*

We further denote

$$\mathcal{P}(W \to X) = \mathcal{P}^{(\infty)}(W \to X) := \bigcup_{n \geqslant 0} \mathcal{P}^{(n)}(W \to X).$$

If $X \cong \mathbb{R}$, we simply write $\mathcal{P}^{(n)}(W)$ and $\mathcal{P}(W) = \mathcal{P}^{(\infty)}(W)$ for $\mathcal{P}^{(n)}(W \to X)$ and $\mathcal{P}(W \to X)$, respectively.

Let $x_0 \in V$ then by multidimensional Taylor's theorem (equation 1.7 in [33])

$$q(x) = \sum_{k=0}^{n} q^k(x_0) \frac{(x - x_0)^{\otimes k}}{k!}.$$

Integrating polynomial one-forms against rough paths

Suppose that $g(\mathbf{x}) = 1 + \sum_{k=1}^{n} \mathbf{x}^k$ is the canonical rough path lift onto $G^n(V)$ of a piecewise smooth path $\mathbf{x}_t \in V$. Suppose further that $q \in \mathcal{P}^{(n-1)}(V \to$

V^*) is a polynomial one-form. We may compute the line integral

$$\int_{\mathbf{x}|_{[s,t]}} q = \int_s^t q(\mathbf{x}_u)\mathrm{d}\mathbf{x}_u = \sum_{k=0}^{n-1} q^k(\mathbf{x}_s)\int_s^t \frac{(\mathbf{x}_u - \mathbf{x}_s)^{\otimes k}}{k!} \otimes \mathrm{d}\mathbf{x}_u$$
$$\stackrel{(\star)}{=} \sum_{k=0}^{n-1} q^k(\mathbf{x}_s)\int_s^t \mathbf{x}_{s,u}^k \otimes \mathrm{d}\mathbf{x}_u$$
$$= \sum_{k=0}^{n-1} q^k(\pi_1(g(\mathbf{x})_s))\pi_{k+1}(g(\mathbf{x})_{s,t}) =: \mathcal{Q}(g(\mathbf{x})_s)g(\mathbf{x})_{s,t}, \tag{2.2.1}$$

where for the equality $(\star)$ we used that $q^k(\mathbf{x}_s)$ is symmetric on $k-$tensors and that $\mathrm{Sym}(\mathbf{x}_{s,u}^k) = (\mathbf{x}_u - \mathbf{x}_s)^{\otimes k}/k!$.

Remark 2.2.4. The simple calculation (2.2.1) highlights several ideas, which we informally discuss. A polynomial one-form, q, is not in general a closed one-form. However, upon lifting the path $\mathbf{x}$ to a rough path $g(\mathbf{x})$, and the one-form q to

$$\mathcal{Q} : G^n(V) \to \mathbb{L}(T^n(V) \to \mathbb{R}),$$

satisfying for all $a, b \in G^n(V)$

$$\mathcal{Q}(a)b := \sum_{k=0}^{n-1} q^k(\pi_1(a))\pi_{k+1}(b),$$

we shall see (Definition 2.2.7) that $\mathcal{Q}$ *is* a closed one-form, integrating rough paths such that

$$\int_{g(\mathbf{x})} \mathcal{Q} = \int_{\mathbf{x}} q.$$

The final observation we make from (2.2.1) is that additivity of integration for concatenation of paths: $\int_g \mathcal{Q} + \int_h \mathcal{Q} = \int_{g*h} \mathcal{Q}$ together with Chow's theorem (Theorem 7.28 in [17]) imply

$$\mathcal{Q}(a)b + \mathcal{Q}(ab)c = \mathcal{Q}(a)bc$$

for all $a, b, c \in G^n(V)$. The so-called cocyclic condition ([35]). □

Integrating cocyclic one-forms against rough paths

Definition 2.2.5 (Cocyclic one-forms, [35]). *We say β is a $W-$valued cocyclic one-form on $G^n(V)$ if*

$$\beta : G^n(V) \to \mathbb{L}(T^n(V) \to W)$$

is a continuous function satisfying

$$\beta(a)b + \beta(ab)c = \beta(a)bc \tag{2.2.2}$$

for all $a, b, c \in G^n(V)$. We write $\beta \in \Omega(G^n(V) \to W)$.

Remark 2.2.6. A cocyclic one-form $\beta : G^n(V) \to \mathbb{L}(T^n(V) \to W)$ is determined by its value at one point in space. Namely, if $a \in G^n(V)$ then the linear map $\beta(a)$ determines β (cf. a polynomial is determined by its value and derivatives at a single point). Indeed, if $\beta \in \Omega(G^n(V) \to W)$, then the algebraic condition (2.2.2) ensures

$$\beta(ab)(c) = \beta(a)\Big(b(c - 1_G)\Big)$$

for all $a, b, c \in G^n(V)$. Further, $\mathrm{Span}_{\mathbb{R}}(G^n(V)) = T^n(V)$ (see Section 3.3.1. in [35]) so that $\beta(a)|_{G^n(V)}$ determines $\beta(a)$. From this we readily see that if $\beta(a) = 0$ for some a, then in fact $\beta \equiv 0$.

Conversely, given a covector $\mathbf{v}^* \in T^n(V)$ and $a \in G^n(V)$, there exists a unique cocyclic one-form $\beta \in \Omega(G^n(V))$ such that $\beta(g) = \mathbf{v}^*$ (see Proposition 7 in [35].

Definition 2.2.7 (Cocyclic integral)**.** *Let $\beta \in \Omega(G^n(V) \to W)$ be a cocyclic one-form and $g \in \mathbb{G}_p(V)$ an unparameterised p-rough path with $\lfloor p \rfloor = n$. Define the integral of β against g*

$$\int_g \beta := \beta(\mathfrak{i}(g))\Big(\mathfrak{i}(g)^{-1}\mathfrak{t}(g)\Big).$$

Example 2.2.8. A simple consequence of Remark 2.2.6 and Definition 2.2.7 is that one may represent linear functions on the signature of a rough path through a cocyclic integral. Namely, if $g \in \mathbb{G}_p(V)$ and $\mathbf{v}^* \in T^{\lfloor p \rfloor}(V)$, then we may define $\beta \in \Omega(G^{\lfloor p \rfloor}(V))$ by $\beta(\mathfrak{i}(g)) := \mathbf{v}^*$ and we have that

$$\int_g \beta = \beta(\mathfrak{i}(g))\left(\mathfrak{i}(g)^{-1}\,\mathfrak{t}(g)\right) = \mathbf{v}^*\left(\mathfrak{i}(g)^{-1}\,\mathfrak{t}(g)\right).$$

For the reader interested in the applications of linear functions on the signature, we refer to [37, 36]. [37] shows that one may well-approximate certain exotic financial derivatives by linear functions on the signature of the underlying. This insight combined with tools from expected signatures leads to novel methods of computing risk-neutral prices of derivatives. The forthcoming work [36] builds on these ideas and shows that one may well-approximate dynamic hedges for exotic derivatives using certain linear functions on the signature of the underlying.

Lemma 2.2.9. *Let* $n \in \mathbb{N}_{\geqslant 1}$. *Let* $q \in \mathcal{P}^{(n-1)}(V \to \mathbb{L}(V \to W))$ *be a* $W-$*valued polynomial one-form. Then there exists a unique cocyclic one-form* $\mathcal{Q} \in \Omega(G^n(V) \to W)$ *such that for all piecewise smooth paths* $t \mapsto \mathbf{x}_t \in V$,

$$\int_{\mathbf{x}} q = \int_{g(\mathbf{x})} \mathcal{Q},$$

where $g(\mathbf{x})$ *is the canonical rough path lift of* $\mathbf{x}$ *to* $G^n(V)$.

Proof.

Existence. Given $q \in \mathcal{P}^{(n-1)}(V \to \mathbb{L}(V \to W))$, define $\mathcal{Q} \in \Omega(G^n(V) \to W)$ by

$$\mathcal{Q}(a)b := \sum_{k=0}^{n-1} q^k(\pi_1(a))\pi_{k+1}(b), \quad \text{for all} \quad a, b \in G^n(V). \tag{2.2.3}$$

Remark 2.2.6 ensures that $\mathcal{Q}$ is well-defined. The calculation (2.2.1) shows that $\int_{\mathbf{x}} q = \int_{g(\mathbf{x})} \mathcal{Q}$ for all piecewise smooth paths, $\mathbf{x}$, in V, as desired.

Uniqueness. Chow's theorem (Theorem 7.28 in [17]) ensures that for all $b \in G^n(V)$ there exists a piecewise smooth path $\mathbf{x} : [0, 1] \to V$ whose canonical rough path lift, $g(\mathbf{x})$, onto $G^n(V)$ satisfies $g(\mathbf{x})_1 = b$. This, combined with Remark 2.2.6, ensures that $\mathcal{Q}$ is uniquely determined by (2.2.3). □

This Lemma suggests

Definition 2.2.10 (Cocyclic lift of a polynomial one-form). *Let* $q \in \mathcal{P}^{(n-1)}(V \to \mathbb{L}(V \to W))$ *be a polynomial one-form. We shall call* $\mathcal{Q} \in \Omega(G^n(V) \to W)$, *given by*

$$\mathcal{Q}(a)b := \sum_{k=0}^{n-1} q^k(\pi_1(a))\pi_{k+1}(b), \quad \textit{for all} \quad a, b \in G^n(V)$$

the cocyclic lift *of* q.

2.2.2 Lipschitz and $\gamma-$cocyclic integration

The $W-$valued cocyclic one-forms on $G^n(V)$ form a finite-dimensional vector space and are quite restricted. There is a precise sense in which one can allow a cocyclic one-form to vary slowly along the trajectory of a path thereby arriving at a richer class of one-forms along a given unparameterised rough path. We call these $\gamma-$*cocyclic one-forms* – they are greatly influenced by *time-varying cocyclic one-forms* developed in [35] and build upon the notion of Lip^γ function on Euclidean spaces, which we recall below. See Definition 2 and the subsequent remarks in [34], and Chapter VI, Section 2.3 in [41] for further details of the Euclidean theory.

Lipschitz functions on Euclidean spaces

We begin with a more classical definition of Lipschitz function on a subset of a Euclidean space, used in the context of rough path theory. Following this, we give our definition, which is much more in line with that given in Definition 2 of the recent work [34]. Our definition will better dovetail with the forthcoming definition of Lipschitz function on a subset of unparameterised rough path space in Chapter 3.

Definition 2.2.11 (Definition 1.21, [33])**.** *Let* $n \geqslant 0$ *be an integer and* $\gamma \in (n, n+1]$ *be a real number. Let* $\mathcal{K} \subseteq V$ *be a closed set. For each integer* $j = 0, \dots n$ *let* $f^j : \mathcal{K} \to \mathbb{L}(V^{\otimes j} \to W)$ *be a function which takes its values in the space of symmetric* $j-$*linear mappings* $V^{\otimes j} \to W$*. The collection* $(f_0, f_1, \dots, f_n)$ *is an element of* $\mathrm{Lip}^{\gamma}(\mathcal{K} \to W)$ *if the following conditions hold: there exists a constant* M *such that, for each* $j = 0, \dots, n$,

$$\sup_{x \in \mathcal{K}} \|f^j(x)\| \leqslant M$$

and, for each $x, y \in \mathcal{K}$, $j = 0, \dots, n$,

$$\left\| f^j(y) - \sum_{l=0}^{n-j} \frac{1}{l!} f^{j+l}(x)(y-x)^{\otimes l} \right\| \leqslant M \|y - x\|^{\gamma - j}.$$

The smallest constant M *for which the inequalities hold is called the* $\mathrm{Lip}^{\gamma}(F)-$*norm of* $(f_0, \dots, f_n)$.

⚠The above perspective on Lipschitz functions is *not* the one we shall be taking in this work. The below shall be our working definition of Lipschitz function – the subsequent remarks will involve a discussion about the relationship between the two.

Definition 2.2.12 (Lip^{γ} functions on Euclidean spaces)**.** *Let* $\gamma \geqslant 1$*, and* $\mathcal{K} \subseteq V$ *a closed set. We say* $q \in \mathrm{Lip}^{\gamma}(\mathcal{K} \to W)$ *is a* $W-$*valued Lipschitz*$-\gamma$

(Lip$^\gamma$) *function on* $\mathcal{K}$ *if*

$$q : \mathcal{K} \to \mathcal{P}^{\lfloor \gamma \rfloor}(V \to W)$$

is a function satisfying

$$\begin{aligned}\|q\|_{\mathrm{Lip}^\gamma} := &\sup_{\substack{x, x+h \in \mathcal{K} \\ 0 < \|h\| \leqslant 1}} \max_{k=0,\dots,\lfloor \gamma \rfloor} \frac{\|q^k_{x+h}(x+h) - q^k_x(x+h)\|}{\|h\|^{\gamma-k}} \\ &+ \sup_{x \in \mathcal{K}} \max_{k=0,\dots,\lfloor \gamma \rfloor} \|q^k_x(x)\| < +\infty.\end{aligned} \tag{2.2.4}$$

If $W = \mathbb{L}(V \to U)$*, we call* q *a* $U-$*valued* Lip$^\gamma$ *one-form over* $\mathcal{K} \subset V$*.*

We supplement this definition with some remarks:

1. If $x \in \mathcal{K}$, $y \in V$ and $k \in \mathbb{N}$, $q_x \in \mathcal{P}^{\lfloor \gamma \rfloor}$ is a *globally defined* polynomial and

   $$q^k_x(y) := \nabla^k(q_x)(y)$$

   denotes the k^{th} derivative of q_x at y.

2. Define $f : \mathcal{K} \to W$, $f : x \mapsto q_x(x)$. For all $x \in \mathrm{Int}(\mathcal{K})$, q_x is the degree$-\lfloor \gamma \rfloor$ Taylor expansion of f about x. Indeed, f determines q on open sets.

3. Our definition of Lip$^\gamma$ functions, in particular the perspective of said functions taking values in polynomial functions, is almost identical to that given in Definition 2, [34]. The difference being that our norm (2.2.4) includes the additional terms involving suprema of derivatives: $\sup_{x \in \mathcal{K}} \|q^k_x(x)\|$ for $k = 1, \dots, \lfloor \gamma \rfloor$.

4. It is possible to go between the two given definitions (2.2.11, 2.2.12) of Lipschitz function. Namely, given $n \in \mathbb{N}$, $\gamma \in \mathbb{R}$ with $\lfloor \gamma \rfloor = n$ and $(f_0, \ldots, f_n)$ a $\mathrm{Lip}^\gamma(\mathcal{K} \to W)$ function in the sense of Definition 2.2.11, define

$$q : \mathcal{K} \to \mathcal{P}^{\lfloor \gamma \rfloor}(V \to W), \quad q_x(y) := \sum_{k=0}^{\lfloor \gamma \rfloor} f^k(x) \frac{(y-x)^{\otimes k}}{k!},$$

then q is a $\mathrm{Lip}^\gamma(\mathcal{K} \to W)$ function in the sense of Definition 2.2.12.

Conversely, if q is a $\mathrm{Lip}^\gamma(\mathcal{K} \to W)$ function in the sense of Definition 2.2.12, define for $j = 0, 1, \ldots \lfloor \gamma \rfloor = n$,

$$f^j : \mathcal{K} \to \mathbb{L}(V^{\otimes j} \to W), \quad f^j(x) := q_x^j(x),$$

then $(f_0, \ldots, f_n)$ is a $\mathrm{Lip}^\gamma(\mathcal{K} \to W)$ function in the sense of Definition 2.2.11.

⚠We shall use the terms Lipschitz and Lip^γ interchangeably.

Integrating Lip^γ one-forms against rough paths

Definition 2.2.13 (Corollary 43, [35])**.** *Let* $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$ *and* $\mathcal{K} \subset V$ *a closed set. Let*

$$q : \mathcal{K} \longrightarrow \mathcal{P}^{\lfloor \gamma \rfloor - 1}(V \to \mathbb{L}(V \to W))$$

be a $W-$*valued* $\mathrm{Lip}^{\gamma-1}$ *one-form on* $\mathcal{K}$*. Appealing to Definition 2.2.10, we lift each polynomial one-form,* q_x*, to a cocyclic one-form,* $\mathcal{Q}_x$*. Yielding*

$$\mathcal{Q} : \mathcal{K} \to \Omega(G^{\lfloor p \rfloor}(V) \to W), \quad \mathcal{Q} : x \mapsto \mathcal{Q}_x.$$

Let $g \in \mathbb{G}_p(V)$ such $\pi_1(g) \in \mathcal{K}$ and recall the notion of factorisation of an unparameterised path from Definition 2.1.10. Define the (level-1) integral of $\mathcal{Q}$ against g

$$\int_g \mathcal{Q} := \lim_{|D_g| \to 0} \sum_{g_j \in D_g} \int_{g_{j+1}} \mathcal{Q}_{\pi_1(\mathfrak{t}(g_j))}, \tag{2.2.5}$$

whenever the limit exists.

Lemma 2.2.14. *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$, $\mathcal{K} \subset V$ a closed set and q a $W-$valued $\operatorname{Lip}^{\gamma-1}$ one-form on $\mathcal{K}$. Let $g \in \mathbb{G}_p(V)$ such that $\pi_1(g) \in \mathcal{K}$ then the limit* (2.2.5) *exists and coincides with the rough path integral $\int q(g_u)\mathrm{d}g_u$ as defined, for example, in Section 4.3 of [33].*

Proof. See Corollary 43 in [35]. □

Definition 2.2.15 ($\gamma-$cocyclic one-form)**.** *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$ and let $g \in \mathbb{G}_p(V)$. We say that*

$$\beta : \operatorname{Hist}(g) \longrightarrow \Omega(G^{\lfloor p \rfloor}(V) \to W)$$

is a $W-$valued, $\gamma-$cocyclic one-form along g *if and only if the following two conditions hold:*

i)

$$\|\beta\|_\infty := \sup_{\xi \in \operatorname{Hist}(g)} \|\beta(\xi)(\mathfrak{t}(\xi))\| < +\infty.$$

ii)

$$|\beta|_\gamma := \sup_{\substack{\xi = g_1 * g_2 * g_3 \in \operatorname{Hist}(g) \\ 0 < \|g_2 * g_3\|_p \leqslant 1}} \frac{\left| \int_{g_2} \beta(g_1) + \int_{g_3} \beta(g_1 * g_2) - \int_{g_2 * g_3} \beta(g_1) \right|}{\|g_2 * g_3\|_p^\gamma} < +\infty.$$

Proposition 2.2.16 (Integration of $\gamma-$cocyclic one-forms along rough paths)**.** *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$. Let $g \in \mathbb{G}_p(V)$ and β a $\gamma-$cocyclic one-form along g. For each factorisation, $D_g = (g_0, \dots, g_m)$, of g denote*

$$\int_{D_g} \beta := \sum_{i=0}^{m-1} \int_{g_{i+1}} \beta(g_0 * \cdots * g_i).$$

Then the following limit exists

$$\lim_{|D_g| \to 0} \int_{D_g} \beta =: \int_g \beta$$

and is called the integral of β along g. Furthermore,

$$\left| \int_g \beta - \int_{\{g_0, g\}} \beta \right| \leqslant C_{\gamma,p} |\beta|_\gamma \|g\|_p^\gamma$$

and consequently

$$\left| \int_g \beta \right| \leqslant \|\beta\|_\infty \|g\|_p + C_{\gamma,p} |\beta|_\gamma \|g\|_p^\gamma,$$

where $C_{\gamma,p} < +\infty$ depends only on γ and p.

Proof. The proof of this proposition follows a commonly-used strategy in rough path theory which goes back to L.C. Young ([44]) – see for example Theorem 1.16 in [33]. The main ingredient of which is bounding $\left| \int_{D_g} \beta \right|$ independently of D_g. See Proposition A.1.6 for the proof. □

Definition 2.2.17 (Lift of $\mathrm{Lip}^{\gamma-1}$ one-form to $\gamma-$cocyclic one-form)**.** *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$ and $\mathcal{K} \subset V$ a closed set. Let q be a $W-$valued* $\mathrm{Lip}^{\gamma-1}$ *one-form on $\mathcal{K}$ and $g \in \mathbb{G}_p(V)$. Appealing to Definition 2.2.10, we define*

$$\beta(q) : \mathrm{Hist}(g) \to \Omega\left(G^{\lfloor p \rfloor}(V) \to W\right), \quad \beta(q) : \xi \mapsto \mathcal{Q}_{\pi_1(\mathfrak{t}(\xi))}.$$

We call $\beta(q)$ the lift of q to a $\gamma-$cocyclic one-form.

Remark 2.2.18. $\gamma-$cocyclic one-forms are more general than lifts of $\mathrm{Lip}^{\gamma-1}$ one-forms. Indeed, $\beta(q)$ depends only on the underlying path, ξ, only through its end point. General $\gamma-$cocyclic one-forms can fully depend on the trajectory of the underlying path. This path dependence will be crucial for the forthcoming calculus theorems in Section 3.4.

The following shows that a cocyclic one-form is a *constant* $\gamma-$cocyclic one-form (cf. a polynomial function is a constant Lip^{γ} function).

Lemma 2.2.19 (Consistency with cocyclic integration)**.** *Let $\alpha \in \Omega(G^{\lfloor p \rfloor}(V))$ and $g \in \mathbb{G}_p(V)$. Define the constant function*

$$\beta : \mathrm{Hist}(g) \to \Omega\left(G^{\lfloor p \rfloor}(V)\right), \quad \beta(g) := \alpha \quad \forall g \in \mathbb{G}_p(V).$$

Then for all $\gamma \in (p, \lfloor p \rfloor + 1]$, β is a $\gamma-$cocyclic one-form and further

$$\int_g \beta = \int_g \alpha.$$

Proof. Let $D_g = (g_0, \ldots, g_m)$ be a factorisation of g. Then

$$\begin{aligned}\int_{D_g} \beta &= \sum_{i=0}^{m-1} \int_{g_{i+1}} \beta(g_0 * \cdots * g_i) \\ &= \sum_{i=0}^{m-1} \int_{g_{i+1}} \alpha \\ &= \int_g \alpha.\end{aligned}$$

Hence $\int_g \beta = \lim_{|D_g| \to 0} \int_{D_g} \beta = \int_g \alpha$, as claimed. □

Remark 2.2.20. The definition of $\gamma-$cocyclic one-form should be compared to that of *time-varying cocyclic one-form* in [35]. The two are closely related – but different. Firstly, $\gamma-$cocyclic one-forms make no reference to time nor parameterisation. $\gamma-$cocyclic one-forms also make explicit the previsible dependence of β on the underlying rough path. This will be relevant in the sequel when we study Lipschitz functions, which assign cocyclic one-forms to *collections* of rough paths. We shall see also that $\gamma-$cocyclic one-forms are compatible with the metric, d_p, on unparameterised path space.

3 Lipschitz Functions on Unparameterised Geometric Rough Paths

In this chapter, $\gamma-$cocyclic one-forms are used as building blocks in developing a Lipschitz function theory on unparameterised rough path space. We shall see (Theorem 3.3.4) that these Lipschitz functions can be realised as previsible integrals of $\gamma-$cocyclic one-forms. Further, analogously to their Euclidean counterparts, these functions form a Banach algebra (Proposition 3.4.9), satisfy a Whitney-type extension property (Theorem 3.5.2) and separate points (Lemma 3.5.5).

Influence is drawn from Lipschitz function theory on Euclidean spaces (as outlined for example in Chapter VI, Section 2.3 in [41]) as well as the recently-developed rough integration theory of cocyclic one-forms developed in [35] and [43]. In [35], Lyons and Yang define cocyclic one-forms and their integration against rough paths. This integration theory generalises that of the seminal work [32] and, further, comes with a natural product rule. Subsection 3.4 in particular directly builds upon this preprint. Indeed, a cocyclic one-form can only act one *one* rough path. Lipschitz functions on unparameterised paths, however, have no such restriction and may act on a family of rough paths. Further, they maintain the product rule property of cocyclic one-forms, whilst enjoying various additional properties, as outlined in the previous paragraph, namely: a fundamental theorem of calculus, a Whitney-type extension property and they separate points in path space.

3.1 Lipschitz functions on Euclidean spaces

When doing analysis or statistics, it is judicious to find a core of real functions on the space one is working on. Ideally this core should be an algebra which separates points. We shall see that Lipschitz functions satisfy these properties – both on Euclidean spaces and on unparameterised rough path space (see Proposition 3.4.9 and Lemma 3.5.5).

Further, an interesting and well-studied property of Lipschitz functions on Euclidean spaces is the possibility of extending them to the whole space in the spirit of the classical Whitney extension theorem. Such a property is of particular relevance in (theoretical) supervised Machine Learning or Econometrics where one often has a function defined on some subset of data points which one wants to extend to unseen points. See [15] for a discussion relating Whitney extension problems with the interpolation of data. We shall see that Lipschitz functions on unparameterised paths satisfy such extension properties (see Theorem 3.5.2).

Making these remarks precise,

Proposition 3.1.1. *Let B be a real Banach algebra, $\mathcal{K} \hookrightarrow V$ a closed set and $\gamma \geqslant 1$. Then $\mathrm{Lip}^\gamma(\mathcal{K} \to B)$ is a Banach algebra.*

Proof. See [8] for the definition of product and in particular Proposition 1.10 for the proof. □

The following is an extension theorem for scalar-valued Lipschitz functions on Euclidean spaces and has its analogue to unparameterised path space in Theorem 3.5.2.

Theorem 3.1.2. *Let $\mathcal{K} \hookrightarrow V$ a closed set and $\gamma \geqslant 1$. Then there exists a continuous extension map*

$$\mathcal{E} : \mathrm{Lip}^{\gamma}(\mathcal{K}) \to \mathrm{Lip}^{\gamma}(V).$$

Proof. See Theorem 4, Section 2, Chapter VI in [41]. □

Remark 3.1.3. In anticipation of the forthcoming definition of Lipschitz function on unparameterised path space, we provide an alternative description of Lipschitz functions on Euclidean spaces.

Let $\mathcal{K} \hookrightarrow V$ be path-connected, $\gamma \geqslant 1$ and $q : \mathcal{K} \to \mathcal{P}^{\lfloor\gamma\rfloor}(V \to W)$ be a measurable function. For $j \leqslant \lfloor\gamma\rfloor$ and $x \in \mathcal{K}$ denote $f^j(x) := q_x^j(x)$. Then $q \in \mathrm{Lip}^{\gamma}(\mathcal{K} \to W)$ if and only if

$$\sup_{\substack{x, x+h \in \mathcal{K} \\ 0 < \|h\| \leqslant 1}} \max_{k=0,\ldots,\lfloor\gamma\rfloor} \frac{\|f^k(x+h) - f^k(x) - \int_{\mathbf{h_x}} (q_x)^{k+1}\|}{\|h\|^{\gamma-k}}$$
$$+ \sup_{x \in \mathcal{K}} \max_{k=0,\ldots,\lfloor\gamma\rfloor} \|f^k(x)\| \leqslant M < +\infty,$$

where $\mathbf{h_x}$ is any continuous path in $\mathcal{K}$ joining x to $x+h$. As $(q_x)^{k+1}$ is an exact one-form, the above expression is independent of the choice of $\mathbf{h_x}$. Furthermore, by choosing the optimal such M, one arrives at an equivalent norm on Lip^{γ} functions.

3.2 Lipschitz functions on unparameterised path space

Definition 3.2.1 (Closed under histories)**.** *We say that a set* $\mathcal{K} \subseteq \mathbb{G}_p(V)$ *is* closed under histories *if for all* $g \in \mathcal{K}$*,* $\mathrm{Hist}(g) \subseteq \mathcal{K}$*.*

⚠Henceforth, fix $\mathcal{K} \subseteq \mathbb{G}_p(V)$ a nonempty set of paths closed under histories. $p, \gamma \in \mathbb{R}$ shall be constants satisfying $p \geqslant 1$ and $\gamma \in (p, \lfloor p \rfloor + 1]$, unless otherwise stated.

Recall that $\Omega(G^n(V))$ is the linear space of $\mathbb{R}-$valued cocyclic one forms on $G^n(V)$ (Definition 2.2.5).

Definition 3.2.2. *Let* $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$*. Denote by* $\mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$ *the* $\mathbb{R}-$*linear space of pairs of functions* $(f, \nabla f)$

$$f : \mathcal{K} \to \mathbb{R}, \qquad \nabla f : \mathcal{K} \to \Omega\left(G^{\lfloor p \rfloor}(V)\right)$$

under pointwise addition and scalar multiplication, satisfying

1.
$$|(f, \nabla f)|_\gamma := \sup_{\left\{\substack{g, g*h \in \mathcal{K} \\ 0 < \|h\|_p \leqslant 1 \\ \mathfrak{t}(g) = \mathfrak{i}(h)}\right\}} \frac{\left|f(g * h) - f(g) - \int_h \nabla f(g)\right|}{\|h\|_p^\gamma} < +\infty,$$

2.
$$\|f\|_\infty := \sup_{g \in \mathcal{K}} |f(g)| < +\infty,$$

3.
$$\|\nabla f\|_\infty := \sup_{g \in \mathcal{K}} \|\nabla f(g)(\mathfrak{t}(g), \cdot)\| < +\infty.$$

Define $\|(f,\mathbb{V}f)\|_{\mathrm{Lip}^\gamma} := |(f,\mathbb{V}f)|_\gamma + \|f\|_\infty + \|\mathbb{V}f\|_\infty$.

We emphasise that for $g \in \mathcal{K}$, *it holds that* $\mathbb{V}f(g) \in \Omega(G^n(V))$ *and hence* $\mathbb{V}f(g)(\mathfrak{t}(g),\cdot) \in T^n(V)^*$ *by the definition of cocyclic one-form.*

Example 3.2.3. Let $p \in [1,2)$ and $\mathbf{x} \in \mathbb{G}_p(V)$. Let $\varphi : V \to \mathbb{R}$ be a $C^{1+\alpha}$ function for some $\alpha \in (p-1,1]$. Namely, φ is continuously differentiable and $\nabla\varphi$ is uniformly $\alpha-$Hölder continuous. Define

$$\begin{cases} f : \mathrm{Hist}(\mathbf{x}) \to \mathbb{R}, & f(\mathbf{x}) := \varphi(\mathfrak{t}(\mathbf{x})), \\ \mathbb{V}f : \mathrm{Hist}(\mathbf{x}) \to \Omega(G^1(V)), & \mathbb{V}f(\mathbf{x}) := \nabla\varphi(\mathfrak{t}(\mathbf{x})). \end{cases}$$

Then the pair $(f,\mathbb{V}f)$ is a $\mathrm{Lip}^{1+\alpha}-$function. Indeed, boundedness of f and $\mathbb{V}f$ respectively follow from continuity of φ and $\nabla\varphi$ together with the fact that $\mathbf{x}$ has compact image in V. Further, if $g, g*h \in \mathrm{Hist}(x)$ with $\mathfrak{t}(g) = \mathfrak{i}(h)$,

$$\begin{aligned}
&\left| f(g*h) - f(g) - \int_h \mathbb{V}f(g) \right| \\
&\quad = \left| \varphi(\mathfrak{t}(h)) - \varphi(\mathfrak{t}(g)) - \int_h \nabla\varphi(\mathfrak{t}(g)) \right| \\
&\quad \overset{(\star)}{=} \Big| \nabla\varphi\big(\,\mathfrak{t}(g) + \lambda(\mathfrak{t}(h)-\mathfrak{t}(g))\big)\cdot(\mathfrak{t}(h)-\mathfrak{t}(g)) \\
&\qquad\quad - \nabla\varphi(\mathfrak{t}(g))\cdot(\mathfrak{t}(h)-\mathfrak{t}(g)) \Big| \\
&\quad \leqslant \|\nabla\varphi\|_\alpha \cdot \|\lambda(\mathfrak{t}(h)-\mathfrak{t}(g))\|^\alpha \cdot \|\,\mathfrak{t}(h)-\mathfrak{t}(g)\| \\
&\quad = \lambda^\alpha \|\nabla\varphi\|_\alpha \cdot \|\,\mathfrak{t}(h)-\mathfrak{t}(g)\|^{1+\alpha} \\
&\quad \leqslant \lambda^\alpha \|\nabla\varphi\|_\alpha \cdot \|h\|_p^{1+\alpha},
\end{aligned}$$

where $\lambda \in (0,1)$ arising in $(\star)$ is from the mean value theorem. Hence, $|(f, \mathbb{V}f)|_{1+\alpha} \leqslant |\nabla\varphi\|_\alpha < +\infty$ as desired.

See the forthcoming Example 3.4.17 for a more subtle example of a Lip^γ function.

Remark 3.2.4. Condition *1.* may be equivalently written

$$\sup_{\substack{g \in \mathcal{K} \\ x,y \in \mathbf{B}(g,1) \\ \mathfrak{i}(x)=\mathfrak{i}(y)}} \frac{\left| f(y) - \int_{y_2(x)} \mathbb{V} f(x \wedge y) - f(x) + \int_{x_2(y)} \mathbb{V} f(x \wedge y) \right|}{d_p(x,y)^\gamma} < +\infty,$$

where $\mathbf{B}(g,1)$ denotes the ball of radius 1 about g in $(\mathbb{G}_p(V), d_p)$.

Condition 1 in Definition 3.2.2 can be thought of as Hölder$-\gamma$ uniformly on balls of radius 1 – this together with boundedness of f and $\mathbb{V}f$ imply a global Hölder$-\gamma$ condition:

Proposition 3.2.5. *Let* $(f, \mathbb{V}f) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$ *then*

$$\sup_{\left\{\substack{g, g*h \in \mathcal{K} \\ \|h\|_p > 0 \\ \mathfrak{t}(g)=\mathfrak{i}(h)}\right\}} \frac{\left| f(g*h) - f(g) - \int_h \mathbb{V} f(g) \right|}{\|h\|_p^\gamma} \leqslant \max\left(|(f, \mathbb{V}f)|_\gamma,\ 2\|f\|_\infty + \|\mathbb{V}f\|_\infty\right).$$

Proof. When $\|h\|_p \leqslant 1$, the supremum is bounded by $|(f, \mathbb{V}f)|_\gamma$. Suppose that $\|h\|_p > 1$, then

$$\begin{aligned} \frac{\left| f(g*h) - f(g) - \int_h \mathbb{V} f(g) \right|}{\|h\|_p^\gamma} &\leqslant 2\|f\|_\infty + \frac{\|\mathbb{V}f\|_\infty \|h\|_p}{\|h\|_p^\gamma} \\ &\leqslant 2\|f\|_\infty + \|\mathbb{V}f\|_\infty \|h\|_p^{1-\gamma} \\ &\leqslant 2\|f\|_\infty + \|\mathbb{V}f\|_\infty \end{aligned}$$

since $1-\gamma \leqslant 0$ and $\|h\|_p > 1$. $\square$

We prove now that the $|\cdot|_\gamma$ seminorm of Lip functions is lower semi-continuous with respect to the topology of pointwise convergence.

Lemma 3.2.6 (Lower semi-continuity). *Let $(f_n, \nabla f_n)_{n\geqslant 0}$ be a sequence in $\mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$ which converges pointwise to $(f, \nabla f)$. Then*

$$\liminf_{n\to\infty} |(f_n, \nabla f_n)|_\gamma \geqslant |(f, \nabla f)|_\gamma.$$

Proof. If $\liminf_{n\to\infty} |(f_n, \nabla f_n)|_\gamma = +\infty$, there is nothing to show. Else $|(f, \nabla f)|_\gamma < +\infty$: choose $\varepsilon > 0$ and let $g, g*h \in \mathcal{K}$ such that

$$\frac{\left|f(g*h) - f(g) - \int_h \nabla f(g)\right|}{\|h\|_p^\gamma} \geqslant |(f, \nabla f)|_\gamma - \varepsilon.$$

By pointwise convergence,

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|f_n(g*h) - f_n(g) - \int_h \nabla f_n(g)\right|}{\|h\|_p^\gamma} &= \frac{\left|f(g*h) - f(g) - \int_h \nabla f(g)\right|}{\|h\|_p^\gamma} \\ &\geqslant |(f, \nabla f)|_\gamma - \varepsilon.\end{aligned}$$

The result follows by letting $\varepsilon \to 0$. $\square$

Equipped with this lower semi-continuity result, we show completeness of the space $\mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$.

Proposition 3.2.7. $\mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$ *is a Banach space.*

Proof. Let $(f_n, \nabla f_n)_{n\geqslant 0} \in \mathrm{Lip}^\gamma(\mathcal{K})$ be a Cauchy sequence. Then f_n and ∇f_n are uniformly Cauchy and hence converge uniformly to bounded functions $f : \mathcal{K} \to \mathbb{R}$, $\nabla f : \mathcal{K} \to \Omega(G^{\lfloor p\rfloor}(V))$

$$f_n \xrightarrow{\|\cdot\|_\infty} f, \quad \nabla f_n \xrightarrow{\|\cdot\|_\infty} \nabla f.$$

By the lower semi-continuity result in Lemma 3.2.6

$$|(f_n, \nabla f_n) - (f, \nabla f)|_\gamma \leqslant \liminf_{m\to\infty} |(f_n, \nabla f_n) - (f_m, \nabla f_m)|_\gamma,$$

which can be made arbitrarily small by taking n large. Therefore $\|(f_n, \nabla f_n) - (f, \nabla f)\|_{\mathrm{Lip}^\gamma} \to 0$, as desired. □

Proposition 3.2.8 (Embeddings of Lip^γ spaces)**.** *Let $p' \geqslant p \geqslant 1$ and $\gamma' \in (p', \lfloor p'\rfloor + 1], \gamma \in (p, \lfloor p\rfloor + 1]$ with $\gamma' \geqslant \gamma$. Let $\mathcal{K} \subseteq \mathbb{G}_p(V)$, $\mathcal{K}' \subseteq \mathbb{G}_{p'}(V)$ with $\mathcal{K} \subseteq \mathcal{K}'$. The restriction map $\mathcal{R} : \mathrm{Lip}^{\gamma'}(\mathcal{K}') \to \mathrm{Lip}^\gamma(\mathcal{K})$ is linear and bounded.*

Proof. Let $(f, \nabla f) \in \mathrm{Lip}^{\gamma'}(\mathcal{K}')$. Since $\mathcal{K} \subseteq \mathcal{K}'$, $\|f\|_{\infty,\mathcal{K}} \leqslant \|f\|_{\infty,\mathcal{K}'}$ and $\|\nabla f\|_{\infty,\mathcal{K}} \leqslant \|\nabla f\|_{\infty,\mathcal{K}'}$. Let $g \in \mathcal{K}$ and $h \in \mathcal{K}$ with $\mathfrak{t}(g) = \mathfrak{i}(h)$ and $\|h\|_{p'} \leqslant 1$ then

$$\begin{aligned}\left| f(g * h) - f(g) - \int_h \nabla f(g) \right| &\leqslant \|(f, \nabla f)\|_{\gamma'} \|h\|_{p'}^{\gamma'} \\ &\leqslant \|(f, \nabla f)\|_{\gamma'} \|h\|_p^{\gamma},\end{aligned}$$

where for the last inequality we've used that $\|h\|_{p'} \leqslant \|h\|_p$ if ever $p' \geqslant p$. Hence $\|(f, \nabla f)\|_{\mathrm{Lip}^\gamma} \leqslant \|(f, \nabla f)\|_{\mathrm{Lip}^{\gamma'}}$.

□

3.3 Fundamental theorem of calculus

The main result in this section is Theorem 3.3.4. Here, we demonstrate that Lip^γ functions on unparameterised path space can be represented as integrals against appropriate $\gamma-$cocyclic one-forms. Namely, if $f \in \mathrm{Lip}^\gamma(\mathcal{K})$ and $g \in \mathcal{K} \subseteq \mathbb{G}_p(V)$ then there exists $C = C(g_0)$ and a $\gamma-$cocyclic one-form, β, such that

$$f(g) = C + \int_g \beta.$$

This result can be interpreted as a measure-free analogue of the celebrated Martingale Representation Theorem to unparameterised rough paths. Indeed, it demonstrates that one can represent $f(g)$ by integrating a suitable $\gamma-$cocyclic one-form along g.

The following technical lemma will allow to connect Lip^γ functions with $\gamma-$cocyclic one-forms.

Lemma 3.3.1. *Let* $(f, \mathbb{\nabla} f) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$. *Let* $g = g_1 * g_2 * g_3 \in \mathcal{K}$ *with* $g_1, g_2, g_3 \in \mathbb{G}_p(V)$ *and* $\|g_2 * g_3\|_p \leqslant 1$. *Then*

$$\left| \int_{g_3} \mathbb{\nabla} f(g_1 * g_2) - \int_{g_3} \mathbb{\nabla} f(g_1) \right| \leqslant 3 |(f, \mathbb{\nabla} f)|_\gamma \|g_2 * g_3\|_p^\gamma.$$

Proof. Recall $\left| f(g * h) - f(g) - \int_h \mathbb{\nabla} f(g) \right| \leqslant |(f, \mathbb{\nabla} f)|_\gamma \|h\|_p^\gamma$ and that cocyclic integration is additive for concatenation of paths. Then

$$\begin{aligned}
&\left|\int_{g_3} \mathbb{V} f(g_1 * g_2) - \int_{g_3} \mathbb{V} f(g_1)\right| \\
&= \left|\int_{g_3} \mathbb{V} f(g_1 * g_2) + \int_{g_2} \mathbb{V} f(g_1) - \int_{g_2 * g_3} \mathbb{V} f(g_1)\right| \\
&\leqslant \left|\int_{g_3} \mathbb{V} f(g_1 * g_2) - (f(g_1 * g_2 * g_3) - f(g_1 * g_2))\right| \\
&\qquad + \left|\int_{g_2} \mathbb{V} f(g_1) - (f(g_1 * g_2) - f(g_1))\right| \\
&\qquad + \left|(f(g_1 * g_2 * g_3) - f(g_1)) - \int_{g_2 * g_3} \mathbb{V} f(g_1)\right| \\
&\leqslant |(f, \mathbb{V} f)|_\gamma \cdot \left(\|g_3\|_p^\gamma + \|g_2\|_p^\gamma + \|g_2 * g_3\|_p^\gamma\right) \\
&\leqslant 3|(f, \mathbb{V} f)|_\gamma \cdot \|g_2 * g_3\|_p^\gamma,
\end{aligned}$$

as desired. □

Let $g \in \mathbb{G}_p(V)$. Recall the history of g

$$\mathrm{Hist}(g) := \{\xi \in \mathbb{G}_p(V) : \xi \subseteq g\}.$$

Corollary 3.3.2. *Let* $(f, \mathbb{V} f) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$ *and* $g \in \mathcal{K}$. *Then*

$$\mathbb{V} f|_{\mathrm{Hist}(g)} : \mathrm{Hist}(g) \to \Omega\left(G^{\lfloor p \rfloor}(V)\right)$$

is a $\gamma-$*cocyclic one-form along* g.

In light of this Corollary,

Definition 3.3.3. *Let* $(f, \mathbb{V} f) \in \operatorname{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$ *and* $g \in \mathcal{K}$. *Appealing to Proposition 2.2.16, we define*

$$\int_g \mathbb{V} f := \int_g \mathbb{V} f|_{\operatorname{Hist}(g)}.$$

Theorem 3.3.4. *Let* $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$, $\mathcal{K} \subseteq \mathbb{G}_p(V)$ *be closed under histories and* $(f, \mathbb{V} f) \in \operatorname{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$. *Then for all* $g \in \mathcal{K}$

$$f(g) = f(g_0) + \int_g \mathbb{V} f.$$

Proof. Introduce the notation $\operatorname{Er}_f(g, g * h) := f(g * h) - f(g) - \int_h \mathbb{V} f(g)$ for the error term. And, note the bound $|\operatorname{Er}_f(g, g * h)| \leqslant |(f, \mathbb{V} f)|_\gamma \, \|h\|_p^\gamma$.

Let $g \in \mathcal{K}$ and $D^g = (g_0, \ldots, g_m)$ a $p-$rough path factorisation of g with $g_0 \equiv \mathfrak{i}(g)$. Then

$$\begin{aligned} f(g) &= f(g_0) + \sum_{i=1}^{m} \big(f(g_0 * \cdots * g_i) - f(g_0 * \cdots * g_{i-1}) \big) \\ &= f(g_0) + \sum_{i=1}^{m} \int_{g_i} \mathbb{V} f(g_0 * \cdots * g_{i-1}) + \sum_{i=1}^{m} \operatorname{Er}_f(g_0 * \cdots * g_{i-1}, g_0 * \cdots * g_i). \end{aligned}$$

Since $\gamma > p$, the error terms vanish as one refines the partition:

$$\left| \sum_{i=1}^{m} \operatorname{Er}_f(g_0 * \cdots * g_{i-1}, g_0 * \cdots * g_i) \right| \leqslant \sum_{i=1}^{m} |(f, \mathbb{V} f)|_\gamma \cdot \|g_i\|_p^\gamma \xrightarrow{|D_g| \to 0} 0.$$

And, from Definition 3.3.3

$$\sum_{i=1}^{m} \int_{g_i} \mathbb{V} f(g_0 * \cdots * g_{i-1}) \xrightarrow{|D_g| \to 0} \int_g \mathbb{V} f.$$

Combining these

$$\begin{aligned} f(g) = \lim_{|D_g|\to 0} \Big(f(g_0) &+ \sum_{i=1}^{m} \int_{g_i} \mathbb{V} f(g_0 * \cdots * g_{i-1}) \\ &+ \sum_{i=1}^{m} \mathrm{Er}_f(g_0 * \cdots * g_{i-1}, g_0 * \cdots * g_i) \Big) \\ = f(g_0) &+ \int_g \mathbb{V} f, \end{aligned}$$

as claimed. □

Example 3.3.5. Taking $\varphi : V \to \mathbb{R}$ a $C^{1+\alpha}$ function, $\mathbf{x} \in \mathbb{G}_p(V)$ and $(f, \mathbb{V} f) \in \mathrm{Lip}^{1+\alpha}$ as in Example 3.2.3, the above theorem reads like the usual first fundamental theorem of calculus on Euclidean space:

$$\begin{aligned} \varphi(\mathfrak{t}(\mathbf{x})) - \varphi(\mathfrak{t}(\mathbf{x}_0)) &= f(\mathbf{x}) - f(\mathbf{x}_0) \\ &= \int_{\mathbf{x}} \mathbb{V} f \\ &= \int_{\mathbf{x}} \nabla \varphi. \end{aligned}$$

The following is a partial converse to the above Integral Representation Theorem 3.3.4

Lemma 3.3.6. *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$, $\mathcal{K} \subseteq \mathbb{G}_p(V)$ be closed under histories and*

$$\mathbb{V} f : \mathcal{K} \to \Omega(G^{\lfloor p \rfloor}(V))$$

a function satisfying

1.

$$\|\mathbb{\nabla} f\|_\infty := \sup_{g\in\mathcal{K}} \|\mathbb{\nabla} f(g)(\mathfrak{t}(g),\cdot)\| < +\infty,$$

2.

$$\sup_{\left\{\substack{g=g_1*g_2*g_3\in\mathcal{K}\\ 0<\|g_2*g_3\|_p\leqslant 1}\right\}} \frac{\left|\int_{g_3}\mathbb{\nabla} f(g_1*g_2)-\int_{g_3}\mathbb{\nabla} f(g_1)\right|}{\|g_2*g_3\|_p^\gamma} < +\infty.$$

Appealing to Proposition 2.2.16, define the scalar function

$$f:\mathcal{K}\to\mathbb{R},\quad f(g) := \lim_{|D_g|\to 0}\sum_{g_i\in D_g}\int_{g_{i+1}}\mathbb{\nabla} f(g_0*\cdots*g_i) = \int_g \mathbb{\nabla} f.$$

Then, as a pair, $|(f,\mathbb{\nabla} f)|_\gamma < +\infty$.

Remark 3.3.7. A pair $(f,\mathbb{\nabla} f)$ generated as above will not always be Lip^γ as f may be unbounded.

Proof. Given $g\in\mathcal{K}$ define

$$\beta:\mathrm{Hist}(g)\to\Omega(G^{\lfloor p\rfloor}(V)),\quad \beta(\xi) := \mathbb{\nabla} f(\xi)$$

so that

$$f(\xi) = \int_\xi \beta.$$

Conditions 1. and 2. ensure that β is a $\gamma-$cocyclic one-form. Hence by Proposition 2.2.16,

$$\begin{aligned}\left|f(x*y)-f(x)-\int_y \mathbb{\nabla} f(x)\right| &\leqslant \left|\int_y\beta-\int_y\beta(x)\right|\\ &\leqslant C_{\gamma,p}|\beta|_\gamma\|y\|_p^\gamma,\end{aligned}$$

so that $|(f,\mathbb{\nabla} f)|_\gamma < +\infty$, as desired. □

3.4 Differentiation theorems and algebraic properties

3.4.1 Algebraic preliminaries

We show that the Lipschitz functions on paths form a Banach *algebra* under pointwise products and a perturbation ($p \geqslant 2$) of the Leibniz rule. We first discuss some of the product structure of rough paths. Recall shuffles and the shuffle product – Definitions 1.2.5 and 1.2.11.

Definition 3.4.1 (Shuffle product of projections)**.** *Given $i \in \mathbb{N}_{\geqslant 0}$, recall $\pi_i : T((V)) \to V^{\otimes i}$ is the projection onto i-tensors. The shuffle product of the projections $\pi_{i_1}, \ldots, \pi_{i_k}$ is a commutative, associative product yielding a linear map*

$$\pi_{i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \pi_{i_k} : T((V)) \to V^{\otimes i_1} \otimes \cdots \otimes V^{\otimes i_k},$$

$$\pi_{i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \pi_{i_k}(\cdot) := \sum_{\sigma \in \mathrm{Sh}(i_1,\ldots,i_k)} \sigma \circ \pi_{i_1+\cdots+i_k}(\cdot).$$

The following Lemma 3.4.3 tells us that we may *linearly* approximate polynomials of the increment of a rough path up to high order. The forthcoming maps $\mathscr{A}_k^n$ will play an instrumental role in the product and chain rules for Lip^γ functions when $\gamma \geqslant 2$.

Definition 3.4.2 (The linear maps $\mathscr{A}_k^n$)**.** *For $k, n \in \mathbb{N}_{\geqslant 1}$ define the linear map*

$$\mathscr{A}_k^n : T^n(V) \to T^n(V)^{\otimes k}, \qquad \mathscr{A}_k^n : v \longmapsto \sum_{\substack{i_1+\cdots+i_k \leqslant n, \\ i_1,\ldots,i_k \geqslant 1}} \pi_{i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \pi_{i_k}(v).$$

Lemma 3.4.3 (High-order linear approximations to products)**.** *Let $p \geqslant 1$ if $a \in G^{\lfloor p \rfloor}(V) \subset T^{\lfloor p \rfloor}(V)$ then*

$$(a-1)^{\otimes k} - \mathscr{A}_k^{\lfloor p \rfloor} a = \sum_{\substack{i_1+\cdots+i_k>\lfloor p \rfloor \\ i_1,\ldots,i_k \leqslant \lfloor p \rfloor}} \pi_{i_1}(a) \otimes \ldots \otimes \pi_{i_k}(a) \in \bigoplus_{\substack{i_1+\cdots+i_k>\lfloor p \rfloor \\ i_1,\ldots,i_k \leqslant \lfloor p \rfloor}} V^{\otimes i_1} \otimes \cdots \otimes V^{\otimes i_k},$$

where $1 \in G^{\lfloor p \rfloor}(V)$ denotes the unit in the group.

Proof. Let $a \in G^{\lfloor p \rfloor}(V) \subset T^{\lfloor p \rfloor}(V)$. If the covectors $\mathbf{v}_j^* \in (V^{\otimes i_j})^*$, $j = 1, \ldots, k$ with $i_1 + \cdots + i_k \leqslant \lfloor p \rfloor$ and $i_j \geqslant 1$ for all j, then by the grouplike property

$$\mathbf{v}_1^*(a) \ldots \mathbf{v}_k^*(a) = (\mathbf{v}_1^* \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathbf{v}_k^*)(a).$$

Consequently,

$$\pi_{i_1}(a) \otimes \cdots \otimes \pi_{i_k}(a) = (\pi_{i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \pi_{i_k})(a).$$

Therefore

$$\begin{aligned}
(a-1)^{\otimes k} &= \sum_{i_1,\ldots,i_k=1}^{\lfloor p \rfloor} \pi_{i_1}(g) \otimes \cdots \otimes \pi_{i_k}(a) \\
&= \sum_{\substack{i_1+\cdots+i_k \leqslant \lfloor p \rfloor, \\ i_1,\ldots,i_k \geqslant 1}} (\pi_{i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \pi_{i_k})(a) + \sum_{\substack{i_1+\cdots+i_k>\lfloor p \rfloor \\ i_1,\ldots,i_k \leqslant \lfloor p \rfloor}} \pi_{i_1}(a) \otimes \ldots \otimes \pi_{i_k}(a) \\
&= \mathscr{A}_k^{\lfloor p \rfloor} a + \sum_{i_1+\cdots+i_k>\lfloor p \rfloor} \pi_{i_1}(a) \otimes \ldots \otimes \pi_{i_k}(a).
\end{aligned}$$

Rearranging this, we see

$$(a-1)^{\otimes k} - \mathscr{A}_k^{\lfloor p \rfloor} a = \sum_{\substack{i_1+\cdots+i_k>\lfloor p \rfloor \\ i_1,\ldots,i_k \leqslant \lfloor p \rfloor}} \pi_{i_1}(a) \otimes \ldots \otimes \pi_{i_k}(a) \in \bigoplus_{\substack{i_1+\cdots+i_k>\lfloor p \rfloor \\ i_1,\ldots,i_k \leqslant \lfloor p \rfloor}} V^{\otimes i_1} \otimes \cdots \otimes V^{\otimes i_k},$$

as desired. ∎

Definition 3.4.4. *Set for $k, n \geqslant 1$*

$$C_{k,n} := \#\left\{(i_1, \dots, i_k) \in \mathbb{N}^k : i_1 + \cdots + i_k > n, \;\; i_1, \dots, i_k \leqslant n\right\} < +\infty.$$

Corollary 3.4.5. *Let $h \in \mathbb{G}_p(V)$ with $\|h\|_p \leqslant 1$ and denote $a := \mathfrak{i}(h)^{-1}\,\mathfrak{t}(h) \in G^{\lfloor p \rfloor}(V)$. Then,*

$$\left\| \mathscr{A}_k^{\lfloor p \rfloor} a - (a-1)^{\otimes k} \right\| \leqslant C_{k,\lfloor p \rfloor} \cdot B_p^k \cdot \|h\|_p^{\lfloor p \rfloor + 1},$$

where $C_{k,\lfloor p \rfloor}$ is given as in Definition 3.4.4 and $B_p < +\infty$ is a constant dependent only on p.

Proof. Recall by Lyons' Extension Theorem (Theorem 2.1.5) that

$$\|\pi_i(a)\|_{V^{\otimes i}} \leqslant B_p \frac{\|h\|_p^i}{(i/p)!} \leqslant B_p \|h\|_p^i,$$

where $B_p < +\infty$ is a constant dependent only on p. Then, by Lemma 3.4.3,

$$\begin{aligned}
\left\| \mathscr{A}_k^{\lfloor p \rfloor} a - (a-1)^{\otimes k} \right\| &= \left\| \sum_{\substack{i_1 + \cdots + i_k > \lfloor p \rfloor \\ i_1, \dots, i_k \leqslant \lfloor p \rfloor}} \pi_{i_1}(a) \otimes \ldots \otimes \pi_{i_k}(a) \right\| \\
&\leqslant \sum_{\substack{i_1 + \cdots + i_k > \lfloor p \rfloor \\ i_1, \dots, i_k \leqslant \lfloor p \rfloor}} B_p^k \cdot \|h\|_p^{i_1 + \cdots + i_k} \\
&\leqslant C_{k,\lfloor p \rfloor} \cdot B_p^k \cdot \|h\|_p^{\lfloor p \rfloor + 1},
\end{aligned}$$

as claimed. ∎

The following is another technical lemma required for the forthcoming product and chain rules for Lip^γ with $\gamma \geqslant 2$.

Lemma 3.4.6 (Linearly approximating products of integrals)**.** *Let* $\lfloor p\rfloor+1 \geqslant \gamma > p \geqslant 1$*,* $k \in \mathbb{N}$ *and* $\beta_1,\dots,\beta_k \in \Omega(G^{\lfloor p\rfloor}(V))$ $\mathbb{R}-$*valued cocyclic one-forms. Let* $h \in \mathbb{G}_p(V)$ *with* $\|h\|_p \leqslant 1$*. Denote*

$$a := \mathfrak{i}(h)^{-1}\,\mathfrak{t}(h) \in G^{\lfloor p\rfloor}(V)$$

then

$$\left|\left(\prod_{i=1}^{k}\int_h \beta_i\right) - \left(\bigotimes_{i=1}^{k}\beta_i(\mathfrak{i}(h))\right)\left(\mathscr{A}_k^{\lfloor p\rfloor}a\right)\right| \leqslant \|\beta_1\|\dots\|\beta_k\|\cdot B_p\cdot C_{k,\lfloor p\rfloor}\|h\|_p^{\gamma},$$

where $\|\beta_i\| := \|\beta_i(\mathfrak{i}(h))\|_{\mathbb{L}\left(T^{\lfloor p\rfloor}(V)\to\mathbb{R}\right)}$ *and the constants* B_p*,* $C_{k,\lfloor p\rfloor}$ *are given as in Corollary 3.4.5.*

Proof. First observe that if $\beta \in \Omega(G^{\lfloor p\rfloor}(V))$ then $\beta(a)1 \equiv 0$ for all $a \in G^{\lfloor p\rfloor}(V)$. Then, directly calculating

$$\begin{aligned}\left(\prod_{i=1}^{k}\int_h \beta_i\right) &= \left[\beta_1(\mathfrak{i}(h))\otimes\cdots\otimes\beta_k(\mathfrak{i}(h))\right](a-1)^{\otimes k}\\ &= \left(\bigotimes_{i=1}^{k}\beta_i(\mathfrak{i}(h))\right)(a-1)^{\otimes k}.\end{aligned}$$

Hence, appealing to Corollary 3.4.5,

$$\begin{aligned}\Big|\left(\prod_{i=1}^{k}\int_h \beta_i\right) - \left(\bigotimes_{i=1}^{k}\beta_i(\mathfrak{i}(h))\right)\left(\mathscr{A}_k^{\lfloor p\rfloor}a\right)\Big| &= \left|\left(\bigotimes_{i=1}^{k}\beta_i(\mathfrak{i}(h))\right)\left(\mathscr{A}_k^{\lfloor p\rfloor}a - (a-1)^{\otimes k}\right)\right| \\ &\leqslant \|\beta_1\|\dots\|\beta_k\|\cdot B_p\cdot C_{k,\lfloor p\rfloor}\|h\|_p^{\lfloor p\rfloor+1} \\ &\leqslant \|\beta_1\|\dots\|\beta_k\|\cdot B_p\cdot C_{k,\lfloor p\rfloor}\|h\|_p^{\gamma},\end{aligned}$$

as claimed. □

3.4.2 The product rule

Definition 3.4.7 (Products of Lip^γ functions)**.** *Let $\lfloor p\rfloor+1 \geqslant \gamma > p \geqslant 1$ and $\mathcal{K} \subseteq \mathbb{G}_p(V)$ be closed under histories. Let $(f_1, \nabla f_1), (f_2, \nabla f_2) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$. Define their product: $(f_1 \cdot f_2, \nabla(f_1 \cdot f_2))$*

$$f_1 \cdot f_2 : \mathcal{K} \to \mathbb{R}, \quad \nabla(f_1 \cdot f_2) : \mathcal{K} \to \Omega(G^{\lfloor p\rfloor}(V))$$

by for all $g \in \mathcal{K}$ and $\mathbf{v} \in T^{\lfloor p\rfloor}(V)$

$$(f_1 f_2)(g) := f_1(g) f_2(g)$$

and

$$\begin{aligned}
&\mathbb{V}(f_1 f_2)(g)(\mathfrak{t}(g), \mathbf{v}) \\
&\qquad = f_1(g) \cdot \mathbb{V} f_2(g)\,(\mathfrak{t}(g), \mathbf{v}) \\
&\qquad + f_2(g) \cdot \mathbb{V} f_1(g)\,(\mathfrak{t}(g), \mathbf{v}) \\
&\qquad + [\mathbb{V} f_1(g)(\mathfrak{t}(g), \cdot) \otimes \mathbb{V} f_2(g)(\mathfrak{t}(g), \cdot)] \left(\mathscr{A}_2^{\lfloor p \rfloor} \mathbf{v}\right),
\end{aligned}$$

where $\mathscr{A}_2^{\lfloor p \rfloor} : T^{\lfloor p \rfloor}(V) \to T^{\lfloor p \rfloor}(V)^{\otimes 2}$ *is the linear map defined in Definition 3.4.2.*

Remark 3.4.8. If $1 \leqslant p < 2$, then $\lfloor p \rfloor = 1$, $\mathscr{A}_2^1 \equiv 0$ and the above product rule reads like the usual Leibniz rule:

$$\mathbb{V}(f_1 f_2) = f_1 \mathbb{V} f_2 + f_2 \mathbb{V} f_1.$$

Proposition 3.4.9. *Under the definition of product from Definition 3.4.7, the Banach space* $\left(\mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R}),\ \| \cdot \|_{\mathrm{Lip}^\gamma}\right)$ *is a Banach algebra. Namely, if*

$$(f_1, \mathbb{V} f_1),\ (f_2, \mathbb{V} f_2) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$$

then

$$\left(f_1 \cdot f_2, \mathbb{V}(f_1 \cdot f_2)\right) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R}).$$

Furthermore, there exists a constant $C_p > 0$ *dependent only on* p *such that*

$$\|(f_1 \cdot f_2, \mathbb{V}(f_1 \cdot f_2))\|_{\mathrm{Lip}^\gamma} \leqslant C_p \|(f_1, \mathbb{V} f_1)\|_{\mathrm{Lip}^\gamma} \cdot \|(f_2, \mathbb{V} f_2)\|_{\mathrm{Lip}^\gamma}.$$

Proof. Denote $f := f_1 \cdot f_2$ and $\mathbb{V} f := \mathbb{V}(f_1 \cdot f_2)$. Note that

$$\|f\|_\infty = \sup_{g \in \mathcal{K}} |f_1(g) f_2(g)| \leqslant \|f_1\|_\infty \|f_2\|_\infty$$

and

$$\begin{aligned} \|\mathbb{V} f\|_\infty &= \sup_{g \in \mathcal{K}} \|\mathbb{V} f(g)(\mathfrak{t}(g), \cdot)\| \\ &\leqslant \|f_1\|_\infty \|\mathbb{V} f_2\|_\infty + \|f_2\|_\infty \|\mathbb{V} f_1\|_\infty + \left\| \mathscr{A}_2^{[p]} \right\| \|\mathbb{V} f_1\|_\infty \|\mathbb{V} f_2\|_\infty. \end{aligned}$$

It remains to control $|(f, \mathbb{V} f)|_\gamma$. To this end, let $g, g * h \in \mathcal{K}$ with $\|h\|_p \leqslant 1$ then appealing to Lemma 3.4.6 (with $k = 2$) and the definition of Lip^γ functions

$$\begin{aligned} &\Big| f(g * h) - f(g) \\ &\quad - f_1(g) \cdot (f_2(g * h) - f_2(g)) \\ &\quad - f_2(g) \cdot (f_1(g * h) - f_1(g)) \\ &\quad - \int_h \mathbb{V} f_1(g) \cdot \int_h \mathbb{V} f_2(g) \Big| \leqslant C_p \|(f_1, \mathbb{V} f_1)\|_{\mathrm{Lip}^\gamma} \cdot \|(f_2, \mathbb{V} f_2)\|_{\mathrm{Lip}^\gamma} \|h\|_p^\gamma \end{aligned}$$

hence

$$\left| f(g * h) - f(g) - \int_h \mathbb{V} f(g) \right| \leqslant C_p \|(f_1, \mathbb{V} f_1)\|_{\mathrm{Lip}^\gamma} \cdot \|(f_2, \mathbb{V} f_2)\|_{\mathrm{Lip}^\gamma} \|h\|_p^\gamma$$

and therefore $|(f, \mathbb{V} f)|_\gamma \leqslant C_p \|(f_1, \mathbb{V} f_1)\|_{\mathrm{Lip}^\gamma} \cdot \|(f_2, \mathbb{V} f_2)\|_{\mathrm{Lip}^\gamma}$, as desired. □

Lemma 3.4.10. *Let B be an algebra with $\varrho : B \to [0, \infty)$ a norm such that for all $x, y \in B$*

$$\varrho(xy) \leqslant C\varrho(x)\varrho(y),$$

where $C > 0$ is independent of x, y. Then there exists an equivalent norm, ϱ', such that

$$\varrho'(xy) \leqslant \varrho'(x)\varrho'(y).$$

Proof. Define $\varrho' := C\varrho$. □

Consequently, there exists an equivalent norm on $\mathrm{Lip}^\gamma(\mathcal{K})$ making it a Banach algebra.

3.4.3 The chain rule

Definition 3.4.11 (Vector-valued Lip^γ functions)**.** *Let W be a finite-dimensional, real Banach space. $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$. $\mathrm{Lip}^\gamma(\mathcal{K} \to W)$ denotes the Banach space of pairs*

$$f : \mathcal{K} \to W, \qquad \nabla f : \mathcal{K} \to \Omega\left(G^{\lfloor p \rfloor}(V) \to W\right)$$

such that for all $w^* \in W^*$,

$$(w^* \circ f, w^* \circ \nabla f) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R}).$$

The norm on $\mathrm{Lip}^\gamma(\mathcal{K} \to W)$ is given as in Definition 3.2.2, but with all instances of $|\cdot|_{\mathbb{R}}$ replaced by $\|\cdot\|_W$.

Definition 3.4.12 (Post-composition with regular functions)**.** *Let* $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$ *and* V, W, X *be real, finite-dimensional Banach spaces. Let* $u : W \to X$ *be* $\lfloor \gamma \rfloor-$*times differentiable and let*

$$f : \mathcal{K} \to W, \quad \mathbb{\nabla} f : \mathcal{K} \to \Omega\left(G^{\lfloor p \rfloor}(V) \to W\right)$$

be functions. Define the post-composition of $(f, \mathbb{\nabla} f)$ *with* u *as the pair* $(u \circ f, \mathbb{\nabla}(u \circ f))$*, where*

$$u \circ f : \mathcal{K} \to X, \quad u \circ f : g \mapsto u(f(g))$$

and $\mathbb{\nabla}(u \circ f) : \mathcal{K} \to \Omega\left(G^{\lfloor p \rfloor}(V) \to X\right)$ *with for all* $g \in \mathcal{K}$ *and* $\mathbf{v} \in T^{\lfloor p \rfloor}(V)$,

$$\mathbb{\nabla}(u \circ f)(g)(\mathfrak{t}(g))(\mathbf{v}) = \sum_{k=1}^{\lfloor p \rfloor} \nabla^k u(f(g)) \frac{1}{k!} \left(\mathbb{\nabla} f(g)(\mathfrak{t}(g))(\cdot)\right)^{\otimes k} \mathscr{A}_k^{\lfloor p \rfloor} \mathbf{v}, \tag{3.4.1}$$

with $\mathscr{A}_k^{\lfloor p \rfloor}$ *as defined in Definition 3.4.2.*

Remark 3.4.13. If $1 \leqslant p < 2$, then $\lfloor p \rfloor = 1$ and $\mathscr{A}_1^1 \equiv \mathrm{Id}$. Therefore, the above composition rule reads like the usual chain rule on locally Euclidean spaces:

$$\mathbb{\nabla}(u \circ f)(g) = \nabla u(f(g)) \mathbb{\nabla} f(g).$$

Definition 3.4.14. *Denote by* $C^\gamma(W \to X)$ *by the Banach space of* $\lfloor \gamma \rfloor-$*times continuously differentiable with bounded derivatives and* $\lfloor \gamma \rfloor^{\text{th}}$ *derivative* $(\gamma - \lfloor \gamma \rfloor)-$*Hölder continuous. This space has norm*

$$\|u\|_{C^\gamma(W \to X)} := \sum_{k=0}^{\lfloor \gamma \rfloor} \|\nabla^k u\|_\infty + \left|\nabla^{\lfloor \gamma \rfloor} u\right|_{\gamma - \lfloor \gamma \rfloor},$$

where for $\alpha \in (0,1]$, $|\cdot|_\alpha$ *denotes the* $\alpha-$*Hölder seminorm.*

Proposition 3.4.15 (Chain Rule)**.** *Let* $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$ *and* V, W, X *be real, finite-dimensional Banach spaces. Let*

$$f : \mathcal{K} \to W, \quad \mathbb{V} f : \mathcal{K} \to \Omega\left(G^{\lfloor p \rfloor}(V) \to W\right)$$

satisfy $\|\mathbb{V} f\|_\infty$, $|(f, \mathbb{V} f)|_\gamma < +\infty$. *Suppose* $u \in C^\gamma(W \to X)$ *then under the definition of composition (Definition 3.4.12),*

$$(u \circ f, \mathbb{V}(u \circ f)) \in \mathrm{Lip}^\gamma(\mathcal{K} \to X).$$

Furthermore, we have the following bounds

1. $\|u \circ f\|_\infty \leqslant \|u\|_\infty$,
2. $\|\mathbb{V}(u \circ f)\|_\infty \leqslant \sum_{k=1}^{\lfloor \gamma \rfloor} \|\nabla^k u\|_\infty \frac{1}{k!} \|\mathbb{V} f\|_\infty^k \|\mathscr{A}_k^{\lfloor p \rfloor}\|$,
3. $\left|\left(u \circ f, \mathbb{V}(u \circ f)\right)\right|_\gamma \leqslant C_{\lfloor p \rfloor, \lfloor p \rfloor} \cdot B_p \cdot (\|u\|_{C^\gamma} - \|u\|_\infty)) \cdot C(f)$, *where*

$$C(f) := \max\left(\|\mathbb{V} f\|_\infty + |(f, \mathbb{V} f)|_\gamma,\ \|\mathbb{V} f\|_\infty^\gamma + |(f, \mathbb{V} f)|_\gamma^\gamma\right).$$

In particular, there exists a constant $C_p < +\infty$ *dependent only on* p *such that*

$$\|(u \circ f, \mathbb{V}(u \circ f))\|_{\mathrm{Lip}^\gamma} \leqslant C_p \cdot \|u\|_{C^\gamma} \cdot \max\left(\|\mathbb{V} f\|_\infty + |(f, \mathbb{V} f)|_\gamma,\ \|\mathbb{V} f\|_\infty^\gamma + |(f, \mathbb{V} f)|_\gamma^\gamma\right).$$

Remark 3.4.16. In the above, we do not impose $\|f\|_\infty < +\infty$.

Proof. $\|u \circ f\|_\infty \leqslant \|u\|_\infty < +\infty$ is immediate. Likewise

$$\begin{aligned}\|\mathbb{V}(u\circ f)\|_\infty &= \sup_{g\in\mathcal{K}} \left\| \sum_{k=1}^{\lfloor\gamma\rfloor} \nabla^k u(f(g)) \frac{1}{k!} \Big(\mathbb{V} f(g)(\mathfrak{t}(g))(\cdot) \Big)^{\otimes k} \mathscr{A}_k^{\lfloor p\rfloor} \right\| \\ &\leqslant \sum_{k=1}^{\lfloor\gamma\rfloor} \|\nabla^k u\|_\infty \frac{1}{k!} \|\mathbb{V} f\|_\infty^k \|\mathscr{A}_k^{\lfloor p\rfloor}\| < +\infty.\end{aligned}$$

It remains to control $\big|\big(u\circ f, \mathbb{V}(u\circ f)\big)\big|_\gamma$. To this end, recall the notation $\mathrm{Er}_f(g, g*h) := f(g*h) - f(g) - \int_h \mathbb{V} f(g)$ for the error term. And, the bound $\|\mathrm{Er}_f(g, g*h)\| \leqslant |(f, \mathbb{V} f)|_\gamma \, \|h\|_p^\gamma$. Then let $g, g*h \in \mathcal{K}$ and $\|h\|_p \leqslant 1$ then by Taylor's theorem

$$\begin{aligned}&\left\| u(f(g*h)) - u(f(g)) - \sum_{k=1}^{\lfloor\gamma\rfloor} \nabla^k u(f(g)) \frac{(f(g*h) - f(g))^{\otimes k}}{k!} \right\|_X \\ &\leqslant \left\| \nabla^{\lfloor\gamma\rfloor} u \right\|_{C^{\gamma-\lfloor\gamma\rfloor}} \cdot \frac{\|f(g*h) - f(g)\|_W^\gamma}{\lfloor\gamma\rfloor!} \\ &= \left\| \nabla^{\lfloor\gamma\rfloor} u \right\|_{C^{\gamma-\lfloor\gamma\rfloor}} \cdot \frac{1}{\lfloor\gamma\rfloor!} \cdot \left\| \int_h \mathbb{V} f(g) + \mathrm{Er}_f(g, g*h) \right\|_W^\gamma \\ &\leqslant \left\| \nabla^{\lfloor\gamma\rfloor} u \right\|_{C^{\gamma-\lfloor\gamma\rfloor}} \cdot 2^{\gamma-1} \cdot \frac{1}{\lfloor\gamma\rfloor!} \cdot \big(\|\mathbb{V} f\|_\infty^\gamma + |(f, \mathbb{V} f)|_\gamma^\gamma \big) \|h\|_p^\gamma.\end{aligned}$$

From the identity

$$f(g*h) - f(g) = \int_h \mathbb{V} f(g) + \mathrm{Er}_f(g, g*h)$$

and the binomial theorem, we obtain the bound

$$\left\| (f(g*h) - f(g))^{\otimes k} - \left(\int_h \mathbb{V} f(g) \right)^{\otimes k} \right\| \leqslant \sum_{j=0}^{k-1} \binom{k}{j} \|\mathbb{V} f\|_\infty^j \cdot |(f, \mathbb{V} f)|_\gamma^{k-j} \cdot \|h\|_p^\gamma$$

Further, denote $\mathbf{v} = \mathfrak{i}(h)^{-1}\,\mathfrak{t}(g)$, then, by Corollary 3.4.5,

$$\left\| \left(\int_h \mathbb{\nabla} f(g) \right)^{\otimes k} - \mathbb{\nabla} f(g)(\mathfrak{t}(g))^{\otimes k} \mathscr{A}_k^{\lfloor p \rfloor} \mathbf{v} \right\| \leqslant \|\mathbb{\nabla} f\|_\infty^k \cdot C_{k,\lfloor p \rfloor} \cdot B_p \cdot \|h\|_p^\gamma .$$

Then, by the triangle inequality,

$$\begin{aligned} &\left\| (f(g * h) - f(g))^{\otimes k} - \mathbb{\nabla} f(g)(\mathfrak{t}(g), \cdot)^{\otimes k} \mathscr{A}_k^{\lfloor p \rfloor} \mathbf{v} \right\| \\ &\qquad \leqslant C_{k,\lfloor p \rfloor} \cdot B_p \cdot \Big(\|\mathbb{\nabla} f\|_\infty + |(f, \mathbb{\nabla} f)|_\gamma \Big)^k \cdot \|h\|_p^\gamma \end{aligned}$$

From which we see

$$\begin{aligned} &\left\| \sum_{k=1}^{\lfloor \gamma \rfloor} \nabla^k u(f(g)) \frac{(f(g * h) - f(g))^{\otimes k}}{k!} - \int_h \mathbb{\nabla}(u \circ f)(g) \right\|_X \\ &= \left\| \sum_{k=1}^{\lfloor \gamma \rfloor} \nabla^k u(f(g)) \frac{(f(g * h) - f(g))^{\otimes k}}{k!} - \sum_{k=1}^{\lfloor \gamma \rfloor} \nabla^k u(f(g)) \frac{\mathbb{\nabla} f(g)(\mathfrak{t}(g), \cdot)^{\otimes k}}{k!} \mathscr{A}_k^{\lfloor p \rfloor} \mathbf{v} \right\|_X \\ &\leqslant \sum_{k=1}^{\lfloor \gamma \rfloor} C_{k,\lfloor p \rfloor} \cdot B_p \cdot \|\nabla^k u\|_\infty \Big(|(f, \mathbb{\nabla} f)|_\gamma + \|\mathbb{\nabla} f\|_\infty \Big)^k \|h\|_p^\gamma . \end{aligned} \tag{$\star\star$}$$

Combining ($\star$) and ($\star\star$) using the triangle inequality, we find

$$\begin{aligned} &\left\| u(f(g * h) - u(f(g)) - \int_h \mathbb{\nabla}(u \circ f)(g) \right\|_X \\ &\quad \leqslant C_{\lfloor p \rfloor, \lfloor p \rfloor} \cdot B_p \cdot \|u\|_{C^\gamma} \cdot \max \Big(\|\mathbb{\nabla} f\|_\infty + |(f, \mathbb{\nabla} f)|_\gamma,\ \|\mathbb{\nabla} f\|_\infty^\gamma + |(f, \mathbb{\nabla} f)|_\gamma^\gamma \Big) \cdot \|h\|_p^\gamma . \end{aligned}$$

Set $C_p = C_{\lfloor p \rfloor, \lfloor p \rfloor} \cdot B_p$, then we see that

$$\begin{aligned} &\big| \big(u \circ f, \mathbb{\nabla}(u \circ f) \big) \big|_\gamma \\ &\quad \leqslant C_{\lfloor p \rfloor, \lfloor p \rfloor} \cdot B_p \cdot \|u\|_{C^\gamma} \cdot \max \Big(\|\mathbb{\nabla} f\|_\infty + |(f, \mathbb{\nabla} f)|_\gamma,\ \|\mathbb{\nabla} f\|_\infty^\gamma + |(f, \mathbb{\nabla} f)|_\gamma^\gamma \Big), \end{aligned}$$

as desired. □

Example 3.4.17. Let $A : V \to C^\gamma(W \to W)$ be a linear map into C^γ vector fields on W. Let $g \in G\Omega_p([0,1] \to V)$ be a geometric $p-$rough path and $y_t^x \in W$ the solution to the rough differential equation (in the sense of Section 10.3 [17])

$$dy_t^x = A(y_t^x)dg_t, \quad y_0^x = x \in W.$$

Denote by $\pi_{(A)}(\cdot,\cdot)$ the corresponding Itô map, namely $\pi_{(A)}(x, g|_{[0,t]}) = y_t^x$ for all $t \in [0,1]$, $x \in W$. $\pi_{(A)}(x,\cdot)$ is parameter independent and so is well defined on the quotient

$$\pi_{(A)}(x,\cdot) : \mathbb{G}_p(V) \to C^{p-\mathrm{var}}([0,1] \to W)/\sim .$$

We claim that if $u \in C^\gamma(W \to \mathbb{R})$, then $g \mapsto u(\pi_{(A)}(x,g))$ can be made into a $\mathrm{Lip}^\gamma(\mathbb{G}_p(V))$ function.

Friz and Victoir (Corollary 10.15 in [17]) assert that

$$\begin{aligned}&\left\|\pi_{(A)}(x, g|_{[0,t]}) - \pi_{(A)}(x, g|_{[0,s]}) - \mathcal{E}_{(A)}(\pi_{(A)}(x, g|_{[0,s]}), g_{s,t})\right\| \\ &\leqslant C(\gamma, p) \cdot \|A\|^\gamma \cdot \|g_{[s,t]}\|_p^\gamma,\end{aligned} \tag{3.4.2}$$

where the continuous, bounded map $\mathcal{E}_{(A)} : W \to \mathbb{L}(T^{\lfloor p\rfloor}(V) \to W)$ is the high-order Euler scheme operator (Definition 10.1 in [17]).

Now, let $u \in C_b^\gamma(W \to \mathbb{R})$ and define the pair

$$f : \mathbb{G}_p(V) \to \mathbb{R}, \quad \nabla f : \mathbb{G}_p(V) \to \Omega(G^{\lfloor p\rfloor}(V))$$

by

$$f(g) := u(\pi_{(A)}(x,g)),$$
$$\mathbb{V} f(g)(\mathfrak{t}(g), \mathbf{v}) := \sum_{k=1}^{\lfloor \gamma \rfloor} \nabla^k u \left(\pi_{(A)}(x,g)\right) \frac{\mathcal{E}_{(A)} \left(\pi_{(A)}(x,g), \cdot\right)^{\otimes k}}{k!} \mathscr{A}_k^{\lfloor p \rfloor} \mathbf{v}.$$

The estimate (3.4.2) together with boundedness of the high-order Euler scheme operator $\mathcal{E}_{(A)}$ ensure that $(f, \mathbb{V} f) \in \mathrm{Lip}^\gamma(\mathbb{G}_p(V))$.

Since products of Lipschitz functions are again Lipschitz (Proposition 3.4.9), we may build more exotic Lipschitz functions by considering products of solutions to rough differential equations, possibly composed with smooth maps.

3.5 A Whitney-type extension theorem

In this section, we prove a Whitney-type extension property (Theorem 3.5.2) for Lip^γ functions on unparameterised paths. A corollary is that Lip^γ functions separate points (Lemma 3.5.5).

The following technical lemma will be required in the proof of the forthcoming extension theorem.

Lemma 3.5.1. *Let $\mathcal{K} \subseteq \mathbb{G}_p(V)$ be topologically closed and closed under histories. Let $g \in \mathbb{G}_p(V)$ be a path such that the constant path $\mathfrak{i}(g) \in \mathcal{K}$. Then*

$$g \wedge \mathcal{K} := \sup\{g \wedge k \mid k \in \mathcal{K},\ \mathfrak{i}(k) = \mathfrak{i}(g)\}$$

is well-defined and $g \wedge \mathcal{K} \in \mathcal{K}$.

Proof. Denote $S := \{g \wedge k \mid k \in \mathcal{K},\ \mathfrak{i}(k) = \mathfrak{i}(g)\} \subseteq \mathbb{G}_p(V)$. Since $g \wedge k \subseteq k$ for all $k \in \mathcal{K}$ and $\mathcal{K}$ is closed under histories, $S \subseteq \mathcal{K}$. Further, $\mathfrak{i}(g) \in S \neq \varnothing$ and S is bounded above by g. Hence, by Lemma 2.1.20, $\exists \sup S = g \wedge \mathcal{K} \in \mathbb{G}_p(V)$. And, since $\mathcal{K}$ is topologically closed, $g \wedge \mathcal{K} \in \mathcal{K}$, as claimed. □

Theorem 3.5.2 (An Whitney-type extension theorem). *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$, X a finite-dimensional Banach space and $\mathcal{K} \subseteq \mathbb{G}_p(V)$ be topologically closed and closed under histories. Then there exists a map*

$$\mathcal{E} : \mathrm{Lip}^\gamma(\mathcal{K} \to X) \longrightarrow \mathrm{Lip}^\gamma(\mathbb{G}_p(V) \to X), \qquad \mathcal{E} : (f, \mathbb{\nabla} f) \mapsto (\mathcal{E} f, \mathbb{\nabla}(\mathcal{E} f))$$

such that

$$(\mathcal{E} f)|_{\mathcal{K}} = f, \quad \text{and} \quad (\mathbb{\nabla}(\mathcal{E} f))\,|_{\mathcal{K}} = \mathbb{\nabla} f.$$

Furthermore, we have the bound

$$\|(\mathcal{E} f, \mathbb{\nabla}(\mathcal{E} f))\|_{\mathrm{Lip}^\gamma} \leqslant C \cdot \Big(\max\Big(\|(f, \mathbb{\nabla} f)\|_{\mathrm{Lip}^\gamma}, \|(f, \mathbb{\nabla} f)\|^\gamma_{\mathrm{Lip}^\gamma} \Big) + 1 \Big),$$

where $C = C(d, p) < +\infty$ is a constant depending only on $d = \dim(X)$ and p.

Proof. We construct the extension in two stages: first by building an unbounded extension $(\bar{f}, \mathbb{\nabla} \bar{f})$ and then appealing to the Chain Rule (Proposition 3.4.15) to appropriately cut off $\bar{f}$ so as to make it bounded.

Stage I. Define $\mathbb{\nabla} \bar{f} : \mathbb{G}_p(V) \to \Omega(G^{\lfloor p \rfloor}(V) \to X)$ by

$$\mathbb{\nabla} \bar{f}(g) := \begin{cases} \mathbb{\nabla} f(g \wedge \mathcal{K}), & \mathfrak{i}(g) \in \mathcal{K}, \\ 0, & \mathfrak{i}(g) \notin \mathcal{K}. \end{cases}$$

Note that $\left(\mathbb{V}\bar{f}\right)|_{\mathcal{K}} = \mathbb{V}f$, $\|\mathbb{V}\bar{f}\|_\infty = \|\mathbb{V}f\|_\infty$ and that $\mathbb{V}\bar{f}$ satisfies the hypotheses of Lemma 3.3.6. Consequently we may define

$$\bar{f}(g) := \begin{cases} f(g_0) + \int_g \mathbb{V}\bar{f}, & \mathfrak{i}(g) \in \mathcal{K}, \\ 0, & \mathfrak{i}(g) \notin \mathcal{K} \end{cases}$$

and $|(\bar{f}, \mathbb{V}\bar{f})|_\gamma < +\infty$. From Theorem 3.3.4, $\bar{f}|_{\mathcal{K}} = f$.

Stage II. Since $\|f\|_\infty < +\infty$, set $R := \|f\|_\infty + 1/2$ and note that $\mathrm{Ball}(0, R) \supseteq \mathrm{Image}(f)$. Then choose some $u \in C^\infty(X \to X)$ such that

- $u|_{\mathrm{Ball}(0,R)} \equiv \mathrm{Id}_{\mathrm{Ball}(0,R)}$,
- $\|u\|_\infty \leqslant \|f\|_\infty + 1$,
- $\|u\|_{C^{\lfloor\gamma\rfloor+1}} - \|u\|_\infty = \sum_{k=1}^{\lfloor\gamma\rfloor+1} \|\nabla^k u\|_\infty \leqslant C_d$, where the constant $C_d < +\infty$ depends only on $d = \dim(X)$.

Appealing to Proposition 3.4.15, we define

$$\left(\mathcal{E}f, \mathbb{V}(\mathcal{E}f)\right) := \left(u \circ \bar{f}, \mathbb{V}(u \circ \bar{f})\right) \in \mathrm{Lip}^\gamma(\mathbb{G}_p(V) \to X)$$

and obtain the bounds

1. $\|u \circ \bar{f}\|_\infty \leqslant \|f\|_\infty + 1$,
2. $\|\mathbb{V}(u \circ \bar{f})\|_\infty \leqslant C_d \cdot C_{\lfloor p\rfloor,\lfloor p\rfloor} \cdot \max\left(\|\mathbb{V}f\|_\infty, \|\mathbb{V}f\|_\infty^{\lfloor\gamma\rfloor}\right)$,
3. $\left|(u \circ \bar{f}, \mathbb{V}(u \circ \bar{f})\right|_\gamma$
$\leqslant C_{\lfloor p\rfloor,\lfloor p\rfloor} \cdot B_p \cdot C_d \max\left(\|\mathbb{V}f\|_\infty + |(f, \mathbb{V}f)|_\gamma, \|\mathbb{V}f\|_\infty^\gamma + |(f, \mathbb{V}f)|_\gamma^\gamma\right)$.

And, since

$$u|_{\mathrm{Im}(f)} \equiv \mathrm{Id} \quad \& \quad \nabla u|_{\mathrm{Im}(f)} \equiv \mathrm{Id},$$

we have that

$$(\mathcal{E}f)|_{\mathcal{K}} = f \quad \& \quad (\mathbb{V}(\mathcal{E}f))|_{\mathcal{K}} = \mathbb{V}f,$$

so that $(\mathcal{E}f, \mathbb{V}(\mathcal{E}f))$ indeed extends $(f, \mathbb{V}f)$. This together with the bounds 1., 2., 3. yields the claimed result. □

Next, we show that Lip^γ functions separate points, but we first present two technical lemmas. Recall from Definition 2.1.6 that $\mathbb{X}(g) \in T((V))$ denotes the signature of the rough path $g \in \mathbb{G}_p(V)$.

Lemma 3.5.3. *Let $g \in \mathbb{G}_p(V)$ with $\mathbb{X}(g) = 1$ and $\|g\|_p > 0$ then there exists a decomposition $g = g_1 * g_2$ with $g_1, g_2 \in \mathbb{G}_p(V)$ such that $\mathbb{X}(g_1), \mathbb{X}(g_2) \neq 1$.*

Proof. Pick a parameterisation of g. Then, since $\|g\|_p > 0$, there exists $t \in (0,1)$ such that $g_t \neq g_0$. Set $\xi_1 = \big[g|_{[0,t]}\big]$, $\xi_2 = \big[g|_{[t,1]}\big]$; we claim that the factorisation $g = \xi_1 * \xi_2$ satisfies the desired properties. Indeed, since $g_t \neq g_0$, $\mathbb{X}(\xi_1) \neq 1$. Further, since $\mathbb{X}(\xi_1)\,\mathbb{X}(\xi_2) = \mathbb{X}(g) = 1$, it holds that $\mathbb{X}(\xi_2) \neq 1$, completing the proof. □

Lemma 3.5.4. *Let $N \in \mathbb{N}_{\geqslant 1}$, $1 \leqslant p \leqslant N$ and $\alpha \in \Omega(G^N(V))$ a cocyclic one-form. For all $g \in \mathbb{G}_p(V)$, define*

$$\beta : \mathrm{Hist}(g) \to \Omega(G^{\lfloor p \rfloor}(V)), \quad \beta(\xi)(\mathfrak{t}(\xi)) := \alpha(\mathfrak{t}(\xi^N))|_{T^{\lfloor p \rfloor}(V)}.$$

Then β is a $(\lfloor p \rfloor + 1)-$cocyclic one-form along g and furthermore

$$\int_g \beta = \int_{g^N} \alpha.$$

Proof. See Lemma A.1.7 in the Appendix. □

The following lemma demonstrates that Lipschitz functions separate points in $\mathbb{G}_p(V)$. The significance of this lies in the fact that the fundamental transform of rough path theory, the signature, fails to separate points which differ by so-called tree-like pieces (see Theorem 1.1 in [6]). Indeed, consider the $1-$rough path

$$\mathbf{x} : [0, 2] \to \mathbb{R}, \quad \mathbf{x}_t = \begin{cases} t, & t \in [0, 1], \\ 2 - t, & t \in [1, 2]. \end{cases}$$

A direct calculation shows that $\mathbf{x}$ has trivial signature, namely $\mathbb{X}(\mathbf{x}) = 1$. Therefore, the signature does not separate $\mathbf{x}$ from a constant path.

Lemma 3.5.5 (Lipschitz functions separate points)**.** *For all $x, y \in \mathbb{G}_p(V)$ with $x \neq y$, there exists $\gamma > p$ and $(f, \nabla f) \in \mathrm{Lip}^\gamma(\mathbb{G}_p(V) \to \mathbb{R})$ such that $f(x) \neq f(y)$.*

Proof. Suppose $x, y \in \mathbb{G}_p(V)$ with $x \neq y$. By Theorem 3.5.2, it suffices to construct a Lipschitz function $(f, \nabla f) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$, with $\mathcal{K} = \mathrm{Hist}(x) \cup \mathrm{Hist}(y)$, such that $f(x) \neq f(y)$.

If $\mathfrak{i}(x) \neq \mathfrak{i}(y)$, define $\nabla f \equiv 0$, $f(\mathfrak{i}(x)) = 0, f(\mathfrak{i}(y)) = 1$. Then $f(x) \neq f(y)$ and we are done.

If $\mathfrak{i}(x) = \mathfrak{i}(y)$ then without loss $y \supsetneq (x \wedge y)$ and we can (Lemma 3.5.3) factorise $y = y_1 * y_2$ such that $y_1 \supseteq x \wedge y$ and $\mathbb{X}(y_2) \neq 1$. Thus for sufficiently large $N \in \mathbb{N}$, $\mathbb{X}^{\leqslant N}(y_2) - \mathbb{X}^{\leqslant \lfloor p \rfloor}(y_2) \neq 0$ so we may by the Hahn-Banach Theorem choose a covector $\mathbf{v}^* \in T^N(V)^*$ such that $\mathbf{v}^*|_{T^{\lfloor p \rfloor}(V)} \equiv 0$ and $\mathbf{v}^* \left(\mathbb{X}^{\leqslant N}(y_2)\right) = 1$. Define the cocyclic one-form $\alpha \in \Omega(G^N(V))$ by $\alpha(\mathfrak{i}(y_2)) := \mathbf{v}^*$ so that

$$\int_{y_2^N} \alpha = \mathbf{v}^* \left(\mathbb{X}^{\leqslant N}(y_2)\right) = 1.$$

Appealing to Lemma 3.5.4, choose a $(\lfloor p \rfloor + 1)-$cocyclic one-form along y_2, β, such that

$$\int_{y_2^N} \alpha = \int_{y_2} \beta.$$

[1] Define $\nabla f|_{\mathrm{Hist}(x)} = \nabla f|_{\mathrm{Hist}(y_1)} \equiv 0$, $\nabla f(y_1 * \xi) = \beta(\xi)$ for all ξ with $y_1 * \xi \subseteq y$ and $f(g) := \int_g \nabla f$ for all $g \in \mathcal{K}$. Then, by Lemma 3.3.6 with $\gamma = \lfloor p \rfloor + 1$, the pair $(f, \nabla f) \in \mathrm{Lip}^\gamma(\mathcal{K} \to \mathbb{R})$. Further, $f(x) = 0$ and

$$f(y) = \int_y \nabla f = \int_{y_1} 0 + \int_{y_2} \beta = \alpha\left(\mathbb{X}^N(y_2)\right) = 1,$$

so that $f(x) \neq f(y)$, as desired.

□

This final corollary of the chapter ties together several of the above

[1]This is well-defined at y_1. Indeed, $\nabla f(y_1)(\cdot) = \alpha(\cdot)|_{T^{\lfloor p \rfloor}(V)} \equiv 0$, since $\alpha(\mathfrak{t}(y_1))|_{T^{\lfloor p \rfloor}(V)} = \mathbf{v}^*|_{T^{\lfloor p \rfloor}(V)} \equiv 0$ and a cocyclic one-form which is zero at a point is zero everywhere – Remark 2.2.6.

results to show that Lipschitz functions are uniformly dense in continuous functions on compact sets of unparameterised rough paths.

Corollary 3.5.6. *Assume further that $\mathcal{K} \subseteq \mathbb{G}_p(V)$ is compact. Then $\mathrm{Lip}^{\gamma}(\mathcal{K})$ is uniformly dense in the space of continuous functions $\mathcal{C}(\mathcal{K} \to \mathbb{R})$ in the sense that for all $\varepsilon > 0$ and $F \in \mathcal{C}(\mathcal{K} \to \mathbb{R})$, there exists $(f, \mathbb{\nabla} f) \in \mathrm{Lip}^{\gamma}(\mathcal{K})$ such that $\|F - f\|_{\infty} \leqslant \varepsilon$.*

Proof. $\mathrm{Lip}^{\gamma}(\mathcal{K})$ contains constants, is an algebra (Proposition 3.4.9) and separates points (Lemma 3.5.5). Appealing to the Stone-Weierstrass theorem, the result follows. □

4 The Expected Signature of the Bilaplacian

Given a constant coefficient, elliptic operator on a finite-dimensional Banach space, V, Levin and Lyons ([27]) obtained the associated expected signature through a convergent net of expected signatures in $T((V)) = \prod_{k\in\mathbb{N}} V^{\otimes k}$ under the product topology. We specialise to the operator $L = -\Delta^2$ and show convergence of the same net of expected signatures in a stronger topology: see Theorem 4.6.2. The forthcoming Chapter 5 is devoted to understanding some of the implication of this result, culminating in quasi-probabilistic representations of some high-order parabolic semigroups.

The reader shall see that the specialisation to $-\Delta^2$ is for simplicity. Indeed, the key properties of the constant coefficient operator elliptic operator that are used in the proof of Theorem 4.6.2 is the existence of a Schwartz class translation invariant heat kernel, $p(\cdot,\cdot)$, integrating to 1, with subexponential decay and the consequent sub-factorial moment bounds on $\int x^{\otimes k} p(1,x)\mathrm{d}x$ – see Lemma 4.5.4. Heat kernel bounds for high order elliptic operators is a delicate topic, but simplifies in the constant coefficient case. If L is a self-adjoint, constant coefficient elliptic operator of order $2m$ on a $d-$dimensional Hilbert space with heat kernel $p(\cdot,\cdot)$, one expects small time ($t \leqslant 1$) bounds of the form

$$|p(t,x)| \leqslant c_1 \exp\left(-c_2^2 \cdot \|x\|^{2m/(2m-1)} \cdot t^{-1/(2m-1)}\right).$$

See Theorem 2 ($d < 2m$) in [11] and Theorem 1 ($d \geqslant 2m$) in [2] for the precise statements.

4.1 Prerequisite material

4.1.1 Real harmonic analysis

Henceforth, we suppose further that V **is a (finite-dimensional) Hilbert space** with inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$, $d := \dim(V) < +\infty$ and shall make implicit use of the standard Hilbert space identification $V \cong V^*$. As before, we equip tensor powers of V with the projective tensor norm.

We refer to [13] for further details on real harmonic analysis.

Definition 4.1.1. *If $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$ is a multi-index and $\varphi : V \to \mathbb{C}$, then we define $x^a := x_1^{a_1} \ldots x_d^{a_d}$ and*

$$D^a \varphi := \frac{\partial^{|a|} \varphi}{\partial x_1^{a_1} \ldots \partial x_d^{a_d}},$$

where $|a| = a_1 + \cdots + a_d$.

Definition 4.1.2. *Let $e_1, \ldots, e_d$ be a basis for V. The Schwartz space on V, $\mathscr{S}(V)$, is the Fréchet space*

$$\mathscr{S}(V) = \{\varphi \in C^\infty(V \to \mathbb{C}) : \|\varphi\|_{\alpha,\beta} < +\infty \ \forall \ \alpha, \beta \in \mathbb{N}^d\},$$

where $\|\varphi\|_{\alpha,\beta} := \sup_{x \in V} \left| x^\alpha D^\beta \varphi(x) \right|$.

Definition 4.1.3. *We denote by* $\mathscr{S}'(V)$ *the space of* tempered distributions. *A linear map* $f : \mathscr{S}(V) \to \mathbb{C}$ *is in* $\mathscr{S}'(V)$ *if an only if*

$$\lim_{k\to\infty} f(\eta_k) = 0 \qquad \textit{whenever} \qquad \lim_{k\to\infty} \eta_k = 0 \quad \textit{in } \mathscr{S}(V).$$

We denote the action of $f \in \mathscr{S}'(V)$ *on* $\eta \in \mathscr{S}(V)$ *in one of three ways:*

$$f(\eta) = \langle f, \eta \rangle = \int_V f(x)\eta(x)\mathrm{d}x.$$

Definition 4.1.4. *Denote by* $C^\infty_{\text{poly}}(V)$ *the algebra of smooth functions* $f : V \to \mathbb{C}$ *for which there exist* $C, N > 0$ *such that* $|f(x)| \leqslant C(1 + \|x\|^N)$.

Remark 4.1.5. Let $\mathrm{d}x$ denote the Lebesgue measure on V. We shall implicitly embed various function spaces into $\mathscr{S}'(V)$.

$$\begin{aligned} &X \hookrightarrow \mathscr{S}'(V) \\ &\varphi \mapsto \left(\phi \mapsto \int_V \varphi(x)\phi(x)\mathrm{d}x\right), \quad \forall \phi \in \mathscr{S}(V), \end{aligned}$$

where for example $X = \mathscr{S}(V)$, $W^{k,p}(V, dx), C^\infty_{\text{poly}}$ or subspaces thereof.

Properties of $\mathscr{S}(V)$ and $\mathscr{S}'(V)$

1. $\mathscr{S}(V)$ is an ideal in C^∞_{poly}.

2. Let $f \in \mathscr{S}'(V)$ be a tempered distribution and $p \in C^\infty_{\text{poly}}(V)$, we define the product $fp \in \mathscr{S}'(V)$ by

$$fp(\phi) = f(p\phi), \quad \forall \phi \in \mathscr{S}(V).$$

3. Let $f \in \mathscr{S}'(V)$ and $\eta \in \mathscr{S}(V)$ their convolution is the smooth function

$$f * \eta : V \to \mathbb{C}, \quad f * \eta(x) = \eta * f(x) = f\left(\tau_x \tilde{\eta}\right),$$

where $\tilde{\eta}(y) := \eta(-y)$ and $\tau_x \eta(y) := \eta(x+y)$.

Definition 4.1.6. *The Fourier transform $\mathcal{F} : \mathscr{S}(V) \to \mathscr{S}(V)$ is a Fréchet space isomorphism given by for all $\xi \in V$*

$$\mathcal{F}\varphi(\xi) := \int_V \varphi(x) e^{-2\pi \mathrm{i}\langle \xi, x\rangle} \mathrm{d}x,$$

with $\mathrm{d}x$ indicating integration with respect to the Lebesgue measure on V. The Fourier transform extends to tempered distributions:

$$\mathcal{F} : \mathscr{S}'(V) \to \mathscr{S}'(V)$$

by for $f \in \mathscr{S}'(V)$ and $\varphi \in \mathscr{S}(V)$

$$\langle \mathcal{F} f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle.$$

The inverse Fourier transform $\mathcal{F}^{-1} : \mathscr{S}(V) \to \mathscr{S}(V)$ is given by for $\xi \in V$, $(\mathcal{F}^{-1}\varphi)(\xi) = (\mathcal{F}\varphi)(-\xi)$ and similarly extends to tempered distributions.

Properties of the Fourier transform

1. Let $f \in \mathscr{S}'(V)$ and $\eta \in \mathscr{S}(V)$ then $\mathcal{F}(f * \eta) = \mathcal{F}(f) \cdot \mathcal{F}(\eta)$.

2. If $\eta_\lambda(x) = \lambda^{-d}\eta(\lambda^{-1}x)$ then $\mathcal{F}(\eta_\lambda)(\xi) = \mathcal{F}\eta(\lambda\xi)$.

3. Denote the Gauss kernel $g(x) = \exp(-\pi\|x\|^2)$ then $g \in \mathscr{S}(V)$ and $\mathcal{F}g(\xi) = \exp(-\pi\|\xi\|^2)$.

4.1.2 Measure theory

Definition 4.1.7 (Pushforward measure). *Let* $X : (\Omega_1, F_1, \mu) \to (\Omega_2, F_2)$ *be a measurable map from a signed measure space into a measurable space.* X *induces a pushfoward measure on* (Ω_2, F_2)*: for all* $\mathcal{U} \in F_2$

$$\mu_X(\mathcal{U}) = \mu \circ X^{-1}(\mathcal{U}) := \mu\{\omega \in \Omega_1 : X(\omega) \in \mathcal{U}\}.$$

We refer to the measure $\mu \circ X^{-1}$ *as the* law of X.

Definition 4.1.8 (Independence). *Let* $X, Y : (\Omega_1, F_1, \mu) \to (\Omega_2, F_2)$ *be measurable maps from a signed measure space into a measurable space. We say* X *and* Y *are* independent *if*

$$\mu \circ (X, Y)^{-1} = (\mu \circ X^{-1}) \otimes (\mu \circ Y^{-1}),$$

where $(X, Y) : \omega \mapsto (X(\omega), Y(\omega)) \in \Omega_2 \times \Omega_2$.

Let I be a nonempty set. Suppose we are given nonempty measurable spaces $(\Omega_i, \mathscr{F}_i)$, $i \in I$. For every nonempty $F \subset I$, denote by $\Omega_F := \prod_{i \in F} \Omega_i$ equipped with the product $\sigma-$field $\mathscr{F}_F := \otimes_{i \in F} \mathscr{F}_i$.

Theorem (Kolmogorov extension). *Suppose that for every finite set* $F \subset I$*, we are given a signed Radon measure* μ_F *on* $(\Omega_F, \mathscr{F}_F)$ *with uniformly bounded total variation*

$$\sup_{F \subset I,\, |F| < +\infty} \|\mu_F\|_{\mathrm{TV}} < +\infty$$

and such that the measures are consistent. Namely, if $F_1 \subset F_2 \subset I$ *and* $\pi_{F_2 \to F_1} : \Omega_{F_2} \to \Omega_{F_1}$ *is the natural projection, then* $\mu_{F_1} = \mu_{F_2} \circ \pi^{-1}_{F_2,F_1}$. *Then, there is a signed measure* μ *on the measurable space*

$$\left(\Omega := \prod_{i \in I} \Omega_i, \mathscr{F} := \otimes_{i \in I} \mathscr{F}_i\right)$$

such that if $F \subset I$ *is a finite set and* $\pi_{I \to F} : \Omega \to \Omega_F$ *the natural projection, then* $\mu_F = \mu \circ \pi^{-1}_{I \to F}$.

Proof. See Theorem 7.7.1. in [7]. □

4.1.3 The Bochner integral

Throughout, B will be a Banach space equipped with its Borel $\sigma-$field and (X, Σ, μ) a complete, signed, finite measure space unless stated otherwise. With the exception of Proposition 4.1.12 about independence, the material in this subsection can be found in [12].

Definition 4.1.9. *A function* $h : X \to B$ *is called* simple *if there exist* $b_1, \ldots, b_n \in B$ *and* $E_1, \ldots, E_n \in \Sigma$ *such that* $h = \sum_{i=1}^n b_i \mathbb{I}_{E_i}$, *where* $\mathbb{I}_{E_i}(x) = 1$ *if* $x \in E_i$ *and* $\mathbb{I}_{E_i}(x) = 0$ *if* $x \notin E_i$.

A function $f : X \to B$ *is called strongly* $\mu-$*measurable there exist a sequence of simple functions* (h_n) *with* $\lim_n \|h_n - f\|_B = 0$ μ*-almost everywhere.*

A function $f : X \to B$ *is called weakly* μ*-measurable if for each* $b^* \in B^*$ *the scalar function* $b^* f$ *is strongly* $\mu-$*measurable.*

Reference to the measure μ may be suppressed when there is no chance of ambiguity.

The results of the following proposition are well-known in the folklore of vector measures; we include the proofs of those for which we lack precise references.

Proposition 4.1.10. *Suppose that a function $f : X \to B$ is $\mu-$essentially separably valued, i.e. there exists $N \subseteq X$ with $\mu(N) = 0$ such that $f(X \backslash N)$ is a separable subset of B, then the following are equivalent:*

i) f is weakly $\mu-$measurable.

ii) f is strongly $\mu-$measurable.

iii) f is measurable.

Proof. $i) \iff ii)$: is the Pettis Measurability Theorem (Theorem 2, pp 42 [12]).

$ii) \implies iii)$: if f is strongly $\mu-$measurable, there exist simple functions f_n such that $f = \lim_{n\to\infty} f_n$ $\mu-$almost everywhere. Hence f is measurable as simple functions are measurable.

$iii) \implies i)$: suppose that f is measurable and let $b^* \in B^*$. For $k \in \mathbb{Z}$, $n \in \mathbb{N}$, define $A_k^n \in \Sigma$, $A_k^n := (b^* \circ f)^{-1}([k/n, (k+1)/n))$. Then $f_n : X \to \mathbb{R}$

$$f_n := \sum_{k=-n^2}^{n^2} \frac{k}{n} \mathbb{I}_{A_k^n}$$

are simple functions and $\lim_{n\to\infty} f_n = \langle f, b^* \rangle$, μ-a.e. □

Definition 4.1.11 (Bochner integrability and the Bochner integral)**.** *A strongly measurable function* $f : (X, F, \mu) \to B$ *is called* Bochner integrable *if there exists a sequence of simple functions* (s_n) *such that*

$$\lim_{n\to\infty} \int_X \|f - s_n\| \mathrm{d}\mu = 0.$$

For such an f*,* $\int_E f \mathrm{d}\mu$ *is defined for each* $E \in F$ *by*

$$\int_E f \mathrm{d}\mu = \lim_{n\to\infty} \int_E s_n \mathrm{d}\mu.$$

Theorem. *A strongly measurable* $f : (X, F, \mu) \to B$ *is Bochner integrable if and only if* $\int_X \|f\| \mathrm{d}\mu < +\infty$.

Proof. See Theorem 2 page 45 in [12]. □

Proposition 4.1.12. *Suppose* B *is a separable Banach algebra. Let* $f, g, fg : (X, F, \mu) \to B$ *be Bochner integrable, with* fg *denoting the point-wise product and suppose that* f *and* g *are independent (Definition 4.1.8). Then*

$$\int_X fg \mathrm{d}\mu = \int_X f \mathrm{d}\mu \int_X g \mathrm{d}\mu.$$

Proof. Suppose first that f and g are simple

$$f = \sum_{i=1}^{n} \mathbb{I}_{A_i} a_i, \quad g = \sum_{j=1}^{m} \mathbb{I}_{B_j} b_j,$$

and so

$$fg = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{I}_{A_i \cap B_j} a_i b_j$$

which is also simple. Integrating

$$\int_X fgd\mu = \sum_{i=1}^{n}\sum_{j=1}^{m}\mu(A_i \cap B_j)a_ib_j,$$

$$\left(\int_X fd\mu\right)\left(\int_X gd\mu\right) = \left(\sum_{i=1}^{n}\mu(A_i)a_i\right)\left(\sum_{j=1}^{m}\mu(B_j)b_j\right)$$

and so the result will follow if we have $\mu(A_i \cap B_j) = \mu(A_i)\mu(B_j)$ for all i, j. But $A_i = f^{-1}\{a_i\}$ and $B_j = g^{-1}\{b_j\}$, and so by independence

$$\mu(A_i \cap B_j) = \mu_{(f,g)}(\{a_i\} \times \{b_j\}) = \mu_f\{a_i\}\mu_g\{b_j\} = \mu(A_i)\mu(B_j).$$

For the general case, since B is separable, f is strongly $\sigma(f)-$measurable (Lemma 4.1.10) so there exist $\sigma(f)-$simple functions f_n with $\lim_{n\to\infty} f_n = f$, μ-a.e. and $\int f = \lim_n \int f_n$. Multiplying by $\mathbb{I}_{\{\|f_n\|\leqslant 2\|f\|\}}$, we may assume that $\|f_n\| \leqslant 2\|f\|$. Likewise, $g = \lim_{n\to\infty} g_n$ $\mu-$a.e. for $\sigma(g)-$simple g_n with $\|g_n\| \leqslant 2\|g\|$.

Note that $f_ng_n \to fg$ $\mu-$a.e. since multiplication is jointly continuous, and that $\|f_ng_n - fg\| \leqslant 4\|fg\|$. Since $\int_X \|f\|\|g\|\mathrm{d}\mu = \int_X \|f\|\mathrm{d}\mu \int_X \|g\|\mathrm{d}\mu < +\infty$, the scalar function $\|f\|\|g\|$ is integrable, whence by dominated convergence

$$\begin{aligned}\int fg\mathrm{d}\mu &= \lim_{n\to\infty}\int f_ng_n\mathrm{d}\mu \\ &= \lim_{n\to\infty}\int f_n\mathrm{d}\mu\int g_n\mathrm{d}\mu \\ &= \int f\mathrm{d}\mu\int g\mathrm{d}\mu.\end{aligned}$$

□

4.1.4 Fréchet spaces

Definition 4.1.13. *A topological vector space, $\mathcal{X}$, is a* Fréchet space *if and only if the following hold:*

1. *$\mathcal{X}$ is Hausdorff;*

2. *there exists a countable family of seminorms on $\mathcal{X}$, $(\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}}$, so that the topology on $\mathcal{X}$ is the initial topology with respect to the functions $\{\|\cdot - x\|_\lambda : X \to [0,\infty) \mid x \in X, \lambda \in \mathbb{N}\}$;*

3. *$\mathcal{X}$ is complete for $(\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}}$. Namely, if (x_n) is $\|\cdot\|_\lambda-$Cauchy for all $\lambda \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$ for some $x \in \mathcal{X}$.*

Remark 4.1.14. There is no loss in generality in assuming that $\|\cdot\|_\lambda \leqslant \|\cdot\|_{\lambda+1}$ for all $\lambda \in \mathbb{N}$. Indeed, setting $\|\cdot\|'_\lambda := \sum_{k=1}^{\lambda} \|\cdot\|_k$, we see that $\|\cdot\|'_\lambda \leqslant \|\cdot\|'_{\lambda+1}$ for all $\lambda \in \mathbb{N}$ and that $(\|\cdot\|'_\lambda)_{\lambda\in\mathbb{N}}$ induces the same topology on $\mathcal{X}$.

Definition 4.1.15. *Let $\mathcal{X}$ be a vector space together with an ordered countable family of seminorms $(\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}})$. We will define the associated metric on $\mathcal{X}$ by, for all $x, y \in X$,*

$$d(x,y) := \sum_{\lambda=1}^{\infty} 2^{-\lambda} \frac{\|x-y\|_\lambda}{1+\|x-y\|_\lambda}.$$

Remark 4.1.16. Suppose that $(\mathcal{X}, (\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}})$ is a Fréchet space together with a choice of an ordered countable family of seminorms as in Definition 4.1.13. Then $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$, as defined in Definition 4.1.15, is a complete, translation-invariant metric inducing the same topology on $\mathcal{X}$.

Lemma 4.1.17. *Suppose that $(\mathcal{X}, (\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}})$ is a Fréchet space and the metric, d, is given as in Definition 4.1.15. Then, for all $\lambda \geqslant 1$ and $x, y \in \mathcal{X}$ with $\|x-y\|_\lambda \leqslant 1$, it holds that*

$$\|x-y\|_\lambda \leqslant 2^{\lambda+1} d(x,y).$$

Proof. Recall the inequality $\frac{1}{2}r \leqslant \frac{r}{(1+r)}$ valid for $r \in [0,1]$. Then, if $\|x-y\|_\lambda \leqslant 1$, we have that

$$\frac{1}{2}\|x-y\|_\lambda \leqslant \frac{\|x-y\|_\lambda}{1+\|x-y\|_\lambda} \leqslant 2^\lambda d(x,y)$$

and hence

$$\|x-y\|_\lambda \leqslant 2^{\lambda+1} d(x,y),$$

as desired. □

Lemma 4.1.18. *Let B be a real Banach space. Then a linear map $\mathscr{A} : (\mathcal{X}, (\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}}) \to B$ is continuous if and only if $\mathscr{A}$ is $\|\cdot\|_\lambda$−bounded for some $\lambda \in \mathbb{N}$.*

Proof. The "if" direction is clear. As for the "only if" direction, suppose for a contradiction that $\mathscr{A}$ is unbounded for each $\|\cdot\|_\lambda$ then there exists $(v_{n,\lambda})_{n,\lambda\in\mathbb{N}} \in \mathcal{X}$ such that

$$\|v_{n,\lambda}\|_\lambda = \frac{1}{n}, \quad \|\mathscr{A} v_{n,\lambda}\|_B \geqslant 1,$$

but then the diagonal sequence $v_{n,n} \to 0$ in $\mathcal{X}$ yet $\mathscr{A} v_{n,n} \nrightarrow 0$ in B, contradicting continuity. □

4.2 Historical remarks

4.2.1 A probability measure on continuous paths associated to the Laplacian

We review some classical results in stochastic analysis associating Markovian measures to second-order elliptic operators. In the sequel we explore whether analogous associations exist for higher-order elliptic operators.

Definition 4.2.1. *Define the Laplacian $\Delta : \mathscr{S}'(V) \to \mathscr{S}'(V)$ and its associated heat semigroup $e^{t\Delta} : \mathscr{S}'(V) \to \mathscr{S}'(V)$, $t \geqslant 0$, through their multipliers*

$$\mathcal{F}(\Delta f) := -4\pi^2 \|\cdot\|^2 \mathcal{F} f,$$
$$\mathcal{F}\left(e^{t\Delta} f\right) := e^{-4\pi^2 t \|\cdot\|^2} \mathcal{F} f.$$

Definition 4.2.2 (The Gauss kernel)**.** *Through properties of the Fourier transform we have for $t > 0$ the identity*

$$\begin{aligned} e^{t\Delta} f &= \mathcal{F}^{-1}\left(e^{-4\pi^2\|\cdot\|^2 t} \cdot (\mathcal{F} f)\right) \\ &= g_t * f \end{aligned}$$

for all $f \in \mathscr{S}'(V)$, where $g_t \in \mathscr{S}(V)$ is the Gauss kernel

$$g_t(x) := \left(\mathcal{F}^{-1} e^{-4\pi^2\|\cdot\|^2 t}\right)(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\|x\|^2/4t}.$$

Note that for all $t > 0$, $g_t \geqslant 0$ (in particular: real-valued) and

$$\int_V g_t(x)\mathrm{d}x = \mathcal{F}^{-1}(g_t(\cdot))(0)$$
$$= e^{-4\pi^2 t\|\xi\|^2}|_{\xi=0} = 1,$$

so that g_t is a probability density function.

Denote by $\mathscr{B}(V)$ the Borel sigma-field on V. For a collection of sigma-fields $(\mathscr{F}_i)_{i\in I}$ on sets $(X_i)_{i\in I}$, denote by $\otimes_I \mathscr{F}_i$ the product sigma-field on $\prod_{i\in I} X_i$.

Definition 4.2.3. *Given a measurable space, $(\Omega, \mathscr{F})$, we shall denote by* $\mathrm{Meas}(\Omega)$ *the space of signed measures on $(\Omega, \mathscr{F})$.*

Definition 4.2.4. *For $0 \leqslant s \leqslant t < +\infty$ and $x \in V$, define the cylinder set measure on $\left(V^{[s,t]}, \bigotimes_{[s,t]} \mathscr{B}(V)\right)$ by, for finite partitions $D = (s = t_0 < \cdots < t_n = t)$ of $[s,t]$, $\mu_D^{[s,t];x} \in \mathrm{Meas}(V^{n+1})$*

$$\mu_D^{[s,t];x}(U_0 \times \cdots \times U_n) := \int_{U_0} \cdots \int_{U_n} \prod_{i=0}^{n} g_{t_i - t_{i-1}}(x_i - x_{i-1})\mathrm{d}x_i,$$

where $U_i \in \mathscr{B}(V)$ are Borel sets, $x_{-1} \equiv x$ and $t_{-1} = t_0$. $g_0(y-x)\mathrm{d}y = \delta_x(\mathrm{d}y)$ denotes the Dirac measure centred at x.

The semigroup property $g_t * g_s = g_{t+s}$ ensures that for fixed s, t, x, the measures $\mu_D^{[s,t];x}$ are consistent as one enlarges D and thus they induce an a priori *finitely additive measure, $\mu^{[s,t];x}$, on $V^{[s,t]}$*

$$\mu^{[s,t];x}(X_{t_i} \in U_i \text{ for } i = 0, \ldots, n) = \mu_{\{t_0,\ldots,t_n\}}^{[s,t];x}(U_0 \times \cdots \times U_n),$$

where for $u \in [s,t]$, $X_u : V^{[s,t]} \to V$ is the coordinate map.

Lemma 4.2.5. *For $0 \leqslant s < t$ and $x \in V$, $\mu^{[s,t];x}$ is a* countably additive *measure supported on the subspace of continuous paths $\mathcal{C}([s,t] \to V)$.*

Proof. Immediate from Kolmogorov extension and continuity theorems – see Corollary 2.6 in [24]. □

Remark 4.2.6. The above construction of $\mu^{[s,t];x}$ is classical. Indeed, $\mu^{[s,t];x}$ is the law of $V-$valued Brownian motion (or Wiener process) starting at x at time s.

4.2.2 The Cauchy problem and a Feynman-Kac representation

That there was a relationship between Brownian motion and the Laplacian was known in the early days of stochastic analysis. Below, we outline a simple demonstrative result in the context of evolution equations on the whole space.

Let $t \in [0,T]$, denote the coordinate process

$$B_t : C([0,T] \to V), \quad B_t : \omega \mapsto \omega(t).$$

Then, under $\mu^{[0,T];x}$, $(B_t)_{t\in[0,T]}$ is a $V-$valued Brownian motion starting at x at time $t = 0$. Let $f \in L^2(V)$, then the following identity holds:

$$\int_{\omega\in C([0,T]\to V)} f(B_t)\mu^{[0,T];x}(\mathrm{d}\omega) = e^{t\Delta_{L^2}} f(x).$$

This line of reasoning can be generalised in several directions (see for example Theorems 6.12 and 6.15 in [5]). We consider only one such direction. Namely, let $A_1, \ldots, A_d \in C_b^\infty(V \to V)$ be smooth bounded vector fields with $d = \dim(V)$. Denote by Y_t^x the solution to the Stratonovich SDE

$$dY_t^x = \sum_i A_i(Y_t^x) \circ \mathrm{d}B_t^i, \quad Y_0^x = x$$

and by

$$\Psi_A^x : C([0,T] \to V) \longrightarrow C([0,T] \to V)$$

the associated Itô map. Ψ_A^x is a measurable map and so induces a pushforward measure $\mu^{[0,T]} \circ (\Psi_A^x)^{-1} = (\Psi_A^x)_* \mu^{[0,T]}$.

Theorem. *Let $f \in C_c^\infty(V)$, then the following identity holds:*

$$\int_{\omega \in C([0,T] \to V)} f(B_t)(\Psi_A^x)_* \mu^{[0,T]}(\mathrm{d}\omega) = e^{t \frac{1}{2} \sum_i A_i^2} f(x).$$

Proof. This is precisely equation (3.2) in [4]. □

In a nutshell, given the Laplace operator, one can build Brownian motion. Then, through solving SDEs driven by Brownian motion, one can build new, related PDE semigroups. In subsection 5.4.5, we show an analogous result for high-order PDE.

4.3 A universal topological algebra containing signatures

Recall that the tensor algebra, $T(V)$, satisfies the universal property that any linear map into an algebra $A : V \to B$ uniquely extends to an algebra homomorphism $\mathbb{A} : T(V) \to B$:

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & T(V) \\ & {\scriptstyle A}\searrow & \downarrow{\scriptstyle \mathbb{A}} \\ & & B \end{array} .$$

In this section we introduce a topological algebra, $\bar{T}(V)$, which satisfies the property that any continuous map $A : V \to B$ into a *Banach* algebra extends uniquely to a *continuous* algebra homomorphism $\bar{\mathbb{A}} : \bar{T}(V) \to B$:

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \bar{T}(V) \\ & {\scriptstyle A}\searrow & \downarrow{\scriptstyle \bar{\mathbb{A}}} \\ & & B \end{array} .$$

We shall see that one can take $\bar{T}(V)$ as a subalgebra of the extended tensor algebra, $T((V))$, and that $\bar{T}(V)$ contains signatures of $p-$rough paths.

A similar construction also in the context of rough path theory can be found in Section 2 of [9].

Definition 4.3.1. *For all $\lambda \geqslant 1$, define*

$$\| \cdot \|_\lambda : T((V)) \to [0, \infty], \quad \|v\|_\lambda := \sum_0^\infty \|v^k\|_{V^{\otimes k}} \lambda^k$$

and

$$T_\lambda(V) = \{v \in T((V)) : \|v\|_\lambda < +\infty\}.$$

Lemma 4.3.2. *The following hold:*

1) $T_\lambda(V)$ is a subalgebra of $T((V))$.

2) $(T_\lambda(V), \|\cdot\|_\lambda)$ *is a separable Banach algebra.*

Proof. 1) It is clear that $\|\cdot\|_\lambda$ is a norm on $T_\lambda(V)$. Thus $T_\lambda(V)$ being a subalgebra of $T((V))$ follows from submultiplicativity of $\|\cdot\|_\lambda$, which we now demonstrate. Recall the definition of product: if $a, b \in T((V))$ then $\pi_k(ab) = \sum_{i=0}^k \pi_i(a)\pi_{k-i}(b)$. For $a, b \in T_\lambda(V) \subset T((V))$,

$$\begin{aligned}
\|ab\|_\lambda &= \sum_{k=0}^{\infty} \lambda^k \|(ab)^k\|_{V^{\otimes k}} \\
&\leqslant \sum_{k=0}^{\infty} \lambda^k \sum_{i=0}^{k} \|a^i\|_{V^{\otimes i}} \|b^{k-i}\|_{V^{\otimes (k-i)}} \\
&= \|a\|_\lambda \|b\|_\lambda,
\end{aligned}$$

so that $\|\cdot\|_\lambda$ is submultiplicative, as claimed.

2) Suppose now that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $T_\lambda(V)$, then for all $k \in \mathbb{N}$, $(a_n^k)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $V^{\otimes k}$ and so $a_n^k \to a^k \in V^{\otimes k}$ as $n \to \infty$. Given $\varepsilon > 0$, choose N so that $n, m \geqslant N \implies \|a_n - a_m\|_\lambda < \varepsilon$. Set $a := \sum_{k=0}^\infty a^k$ and let $n \geqslant N$, then

$$\begin{aligned}
\|a_n - a\|_\lambda &= \sum_{k=0}^{\infty} \lambda^k \|a_n^k - a^k\| \\
&= \sum_{k=0}^{\infty} \liminf_{m\to\infty} \lambda^k \|a_n^k - a_m^k\| \\
&\leqslant \liminf_{m\to\infty} \sum_{k=0}^{\infty} \lambda^k \|a_n^k - a_m^k\| \leqslant \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we see that $a_n \to a$ in $T_\lambda(V)$ and thus completeness holds.

For separability note first that for $x \in T_\lambda(V)$, $\lim_n \Pi_{0,n}(x) = x$, where $\Pi_{0,n}(x)$ denotes the truncation to tensors of degree $\leqslant n$. Since $V^{\otimes k}$ is separable for each k, there exists $\mathcal{Q}_k \subset V^{\otimes k}$ countable and dense. Note then that the countable set

$$\mathcal{Q} := \left\{ \sum_{k=0}^{n} d_k : n \in \mathbb{N},\ d_k \in \mathcal{Q}_k \right\} \subset T(V) \tag{4.3.1}$$

is dense in $T_\lambda(V)$. □

We shall see that the aforementioned universal topological algebra, $\bar{T}(V)$, is given by a countable intersection of Banach algebras

$$\bar{T}(V) = \bigcap_{\lambda \in \mathbb{N}} T_\lambda(V).$$

$\bar{T}(V)$ will have the structure of a Fréchet space (see Section 4.1.4).

Definition 4.3.3. *Define*

$$\bar{T}(V) := \left\{ v \in T((V)) : \|v\|_\lambda := \sum_{k=0}^{\infty} \lambda^k \|v^k\|_{V^{\otimes k}} < +\infty \textit{ for all } \lambda \in \mathbb{N} \right\}.$$

Lemma 4.3.4. *The following hold:*

1) $\bar{T}(V)$ *is a subalgebra of* $T((V))$.

2) $\left(\bar{T}(V), (\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}}\right)$ *is a separable Fréchet space with jointly continuous multiplication.*

Proof. 1) Noting that $\bar{T}(V)$ is an intersection of subalgebras of $T((V))$:

$$\bar{T}(V) = \bigcap_{\lambda \in \mathbb{N}} T_\lambda(V),$$

we see that $\bar{T}(V)$ is a subalgebra of $T((V))$.

2) Completeness of $\big(\bar{T}(V), (\|\cdot\|_\lambda)_{\lambda\in\mathbb{N}}\big)$ follows immediately from completeness of each $(T_\lambda(V), \|\cdot\|_\lambda)$.

Separability: recall that for all $\lambda \in \mathbb{N}$ and $x \in T_\lambda(V)$, $\lim_{n\to\infty} \|\Pi_{0,n}(x) - x\|_\lambda = 0$, where $\Pi_{0,n}(x)$ denotes the truncation to tensors of degree $\leqslant n$. Since $V^{\otimes k}$ is separable for each k, there exists $\mathcal{Q}_k \subset V^{\otimes k}$ countable and dense. The countable set, first mentioned in equation (4.3.1),

$$\mathcal{Q} = \left\{ \sum_{k=0}^{n} d_k : n \in \mathbb{N},\ d_k \in \mathcal{Q}_k \right\} \subset T(V)$$

is dense in $\bar{T}(V)$.

Joint continuity of multiplication: since the topology on $\bar{T}(V)$ is first countable, joint continuity of multiplication is equivalent to joint sequential continuity of multiplication. Let $a_n \to a$ and $b_n \to b$ as $n \to \infty$ in $\bar{T}(V)$. Then for all $\lambda \in \mathbb{N}$, by submultiplicativity of $\|\cdot\|_\lambda$,

$$\|a_n b_n - ab\|_\lambda \leqslant \|a_n - a\|_\lambda \|b_n\|_\lambda + \|a\|_\lambda \|b_n - b\|_\lambda \longrightarrow 0,$$

so that $a_n b_n \to ab$ in $\bar{T}(V)$, as desired.

□

Remark 4.3.5. We note the strict inclusions

$$T(V) \subset \bar{T}(V) \subset T_\lambda(V) \subset T_\mu(V) \subset T((V))$$

whenever $\lambda > \mu$. $\square$

Recall that any linear map $A : V \to B$ from V to an algebra, B, uniquely extends to an algebra homomorphism on the tensor algebra $\mathbb{A} : T(V) \to B$. The following proposition states that if B is further a *Banach* algebra, then A uniquely extends to a *continuous* algebra homomorphism $\bar{T}(V) \to B$. We shall denote this extension by $\bar{\mathbb{A}}$ or simply by $\mathbb{A}$, when there is no risk of confusion.

We refer to [42], [9] and the references therein for more information about universal extensions to topological algebras.

Proposition 4.3.6 (Universal property of $\bar{T}(V)$)**.** *Let B be a Banach algebra and $A \in \mathbb{L}(V \to B)$ a continuous linear map. Then A uniquely extends to a continuous algebra homomorphism $\bar{\mathbb{A}} : \bar{T}(V) \to B$.*

Proof. Fix $\lambda \in \mathbb{N}$ with $\lambda \geqslant \|A\|_{\mathbb{L}(V\to B)}$. Then for any $v \in T(V)$ with $\|v\|_\lambda = 1$

$$\begin{aligned}\|\mathbb{A}\, v\|_B &= \left\|\sum_{k=0}^{\infty} A^{\otimes k} v^k\right\|_B \\ &\leqslant \sum_{k=0}^{\infty} \|A^{\otimes k}\|_{\mathbb{L}(V^{\otimes k}\to B)} \|v^k\|_{V^{\otimes k}} \\ &\leqslant \sum_{k=0}^{\infty} \lambda^k \|v^k\|_{V^{\otimes k}} = \|v\|_\lambda = 1\end{aligned}$$

so that $\mathbb{A} : (T(V), \|\cdot\|_\lambda)) \to B$ is bounded. Recall that for all $v, w \in T(V)$ with $\|v - w\|_\lambda \leqslant 1$, we have that $\|v - w\|_\lambda \leqslant 2^{\lambda+1} d(v, w)$. Therefore,

$\mathbb{A} : (T(V), d) \to B$ is Cauchy continuous and thus extends uniquely to a continuous map on the closure $\overline{T(V)}^d = \bar{T}(V)$. □

Proposition 4.3.7. *Let X be a Banach space and $\mathscr{A} : T(V) \to X$ a linear map. For $k \in \mathbb{N}$, denote $\mathscr{A}^k := \mathscr{A}|_{V^{\otimes k}} : V^{\otimes k} \to X$ the restricted function. Suppose there exists $C > 0$ such that for all $k \in \mathbb{N}$,*

$$\|\mathscr{A}^k\|_{\mathbb{L}(V^{\otimes k} \to X)} \leqslant C^k.$$

Then $\mathscr{A}$ has a unique continuous extension to $\bar{T}(V)$.

Proof. Let $\lambda \geqslant C$ and $v \in \bar{T}(V)$ satisfy $\|v\|_\lambda = 1$. Then

$$\begin{aligned}\|\mathscr{A}v\|_X &= \left\|\sum_{k=0}^{\infty} \mathscr{A}^k v^k\right\|_X \\ &\leqslant \sum_{k=0}^{\infty} C^k \|v^k\|_{V^{\otimes k}} \\ &\leqslant \|v\|_\lambda,\end{aligned}$$

so that $\mathscr{A}$ is $\|\cdot\|_\lambda$−bounded and hence Cauchy continuous for d. □

The following proposition shows that $\bar{T}(V) \subset T((V))$ contains signatures of p−rough paths.

Proposition 4.3.8. *Let $p \geqslant 1$, $s < t$ and recall*

$$\mathbb{X} : G\Omega_p([s,t] \to V) \longrightarrow T((V))$$

the signature map from Definition 2.1.6. Then Image($\mathbb{X}$) $\subset \bar{T}(V)$ *and, further, $\mathbb{X}$ is continuous as map into $\bar{T}(V)$.*

Proof. *Lyons' extension theorem* (Theorem 2.1.5) asserts that the signature of $g \in G\Omega_p([s,t] \to V)$, $\mathbb{X}(g) \in T((V))$, satisfies for all $k \in \mathbb{N}$ the bound

$$\left\|\mathbb{X}^k(g)\right\|_{V^{\otimes k}} \leqslant \frac{C^{k/p}}{\beta(k/p)!},$$

for some $\beta = \beta(p)$, $C = C(\|g\|_{p-\text{var}}) < +\infty$. From this bound, we see that for all $\lambda \geqslant 1$

$$\begin{aligned}\|\mathbb{X}(g)\|_\lambda &= \sum_{k=0}^{\infty} \left\|\mathbb{X}^k(g)\right\|_{V^{\otimes k}} \cdot \lambda^k \\ &\leqslant \sum_{k=0}^{\infty} \frac{(C\lambda^p)^{(k/p)}}{\beta(k/p)!} < +\infty.\end{aligned}$$

Hence, $\mathbb{X}(g) \in \bar{T}(V)$, as claimed.

Similarly, appealing to Theorem 2.2.2. [32], one may show that $\mathbb{X}$ is continuous. □

4.4 An approximation to the exponential in Banach algebras

The results in this section are purely technical and straightforward. They will only be required in the proof of convergence of expected signatures in Section 4.6.

Definition 4.4.1. *Let B be a real unital Banach algebra with unit* 1. *Write*

$$\exp(x) := \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

The series is absolutely convergent for all $x \in B$*, and it holds that* $\|\exp(x)\| \leqslant \exp(\|x\|)$*. For* $x, y \in B$ *with* $xy = yx$

$$\exp(x+y) = \exp(x)\exp(y).$$

It follows that $\exp(-x)\exp(x) = 1$*.*

Define, for $\|x\| < 1$*,*

$$\log(1+x) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k.$$

The series absolutely convergent if $\|x\| < 1$*. For* $x \in B, r > 0$*, denote by* $\mathbf{B}(x,r) \subset B$ *the open ball of radius* r *centred at* x*.* $\log(\cdot) : \mathbf{B}(1,1) \to \log(\mathbf{B}(1,1))$ *is a topological homeomorphism with inverse* exp*. Consequently*

$$\log(xy) = \log(x) + \log(y)$$

if $xy, x, y \in \mathbf{B}(1,1)$ *with* $xy = yx$*.*

Lemma 4.4.2 (An approximation to the exponential function in Banach algebras)**.** *Let* B *be a Banach algebra,* $x \in B$ *and* $D = (s = t_0 < \cdots < t_n = t)$ *a partition of some interval* $[s,t]$ *denote* $\tau_i := t_{i+1} - t_i$ *then*

$$\lim_{|D| \to 0} \prod_{i=1}^{n} (1 + \tau_i x) = \exp((t-s)x).$$

Proof. First note the bound uniform in D

$$\left\| \prod_{i=1}^{n} (1+\tau_i x) \right\| \leqslant \prod_{i=1}^{n} (1+\tau_i \|x\|) \leqslant \prod_{i=1}^{n} \exp(\tau_i \|x\|) = \exp\left((t-s)\|x\|\right).$$

Introduce the notation $\exp_D(y) := \prod_{i=1}^n (1+\tau_i y)$. Given $x \in B$ choose $\eta > 0$ sufficiently small ($\eta = \frac{1}{2\|x\|} \wedge 1$ would do) so that for all $0 < \delta \leqslant \eta$

i) the power series $\log(1+\delta x) := \sum_1^\infty \frac{(-1)^{k+1}}{k}(\delta x)^k$ converges absolutely;

ii) $\|\log(1+\delta x) - \delta x\| \leqslant 2\delta^2\|x\|^2$.

Assume for now that $t - s \leqslant \eta$ then $\tau_i \leqslant \eta$ for all i and

$$\begin{aligned}\|\log(\exp_D(x)) - (t-s)x\| &= \left\|\sum_{i=1}^n \log(1+\tau_i x) - \sum_{i=1}^n \tau_i x\right\| \\ &\leqslant \sum_{i=1}^n 2|\tau_i|^2\|x\|^2 \leqslant 2\cdot|D|\cdot|t-s|\cdot\|x\|^2 \xrightarrow{|D|\to 0} 0.\end{aligned}$$

Since $\log(1+\cdot)|_{B(0,1)}$ is a homeomorphism onto its image, this proves the result for small intervals.

Assume now that $\eta < t - s \leqslant 2\eta$, recall the identity valid in noncommutative unital rings

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{k=1}^n \left(\left(\prod_{i=1}^{k-1} a_i\right)(a_k - b_k)\left(\prod_{i=k+1}^n b_i\right)\right)$$

then

$$\begin{aligned}&\prod_{i=1}^n \left(1+\frac{\tau_i}{2}x\right)^2 - \prod_{i=1}^n (1+\tau_i x) \\ &\quad= \prod_{i=1}^n \left(1+\tau_i x + \frac{\tau_i^2}{4}x^2\right) - \prod_{i=1}^n (1+\tau_i x) \\ &\quad= \sum_{k=1}^n \left(\left(\prod_{i=1}^{k-1} 1+\tau_i x+\frac{\tau_i^2}{4}x^2\right)\left(\frac{\tau_k^2}{4}x^2\right)\left(\prod_{i=k+1}^n 1+\tau_i x\right)\right)\end{aligned}$$

and hence

$$\left\| \prod_{i=1}^{n} \left(1 + \frac{\tau_i}{2} x\right)^2 - \prod_{i=1}^{n} (1 + \tau_i x) \right\| \leqslant \sum_{k=1}^{n} \exp(\eta \|x\|) \|x\|^2 \frac{\tau_k^2}{4}$$
$$\leqslant \exp(\eta\|x\|)\|x\|^2 \cdot |D| \cdot \frac{\eta}{4} \xrightarrow{|D| \to 0} 0.$$

However, by the above result for intervals smaller than η and continuity of the product,

$$\prod_{i=1}^{n} \left(1 + \frac{\tau_i}{2} x\right)^2 = \prod_{i=1}^{n} \left(1 + \frac{\tau_i}{2} x\right) \cdot \prod_{i=1}^{n} \left(1 + \frac{\tau_i}{2} x\right)$$
$$\xrightarrow{|D| \to 0} \exp\left(\frac{t-s}{2} x\right) \cdot \exp\left(\frac{t-s}{2} x\right) = \exp\left((t-s)x\right),$$

proving the result for intervals smaller than $2 \cdot \eta$. Iterating this procedure for intervals smaller than $4 \cdot \eta, 8 \cdot \eta, \ldots$ yields the result for all intervals. □

Corollary 4.4.3. *Let B be a Banach algebra, $x \in B$ and $D = (s = t_0 < \cdots < t_n = t)$ a partition of some interval $[s,t]$ denote $\tau_i := t_{i+1} - t_i$. Suppose $Y_i^{(n)} \in B$, $n \geqslant 1, i = 1, \ldots, n$ with $M := \sup_{i,n} \|Y_i^{(n)}\| < +\infty$ then*

$$\lim_{|D| \to 0} \prod_{i=1}^{n} \left(1 + \tau_i x + \tau_i^2 Y_i^{(n)}\right) = \exp((t-s)x).$$

Proof. Set $U := \max(\|x\|, 2M)$ then

$$\|1 + \tau_i x\|,\ \|1 + \tau_i x + \tau_i^2 Y_i^{(n)}\| \leqslant \exp(U\tau_i)$$

yielding the bound uniform in D

$$\left\| \prod_{i=1}^{k-1} \left(1 + \tau_i x + \tau_i^2 Y_i^{(n)}\right) \right\| \leqslant \exp\left(U \cdot \sum_{i=1}^{k-1} \tau_i\right).$$

Recalling the identity valid in noncommutative, unital rings

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{k=1}^{n} \left(\left(\prod_{i=1}^{k-1} a_i \right) (a_k - b_k) \left(\prod_{i=k+1}^{n} b_i \right) \right)$$

and the fact that $\lim_{|D|\to 0} \prod_{i=1}^{n} (1 + \tau_i x) = \exp((t-s)x)$ (Lemma 4.4.2), we see that

$$\left\| \prod_{i=1}^{n} \left(1 + \tau_i x + \tau_i^2 Y_i^{(n)}\right) - \prod_{i=1}^{n} (1 + \tau_i x) \right\| \leqslant \exp((t-s)U) \cdot M \cdot \sum_{i=1}^{n} \tau_i^2 \xrightarrow{|D|\to 0} 0$$

and therefore

$$\lim_{|D|\to 0} \prod_{i=1}^{n} \left(1 + \tau_i x + \tau_i^2 Y_i^{(n)}\right) = \exp((t-s)x),$$

as desired. □

4.5 Signed measures on rough paths associated to the bi-laplacian

Definition 4.5.1. *Define the linear operator $-\Delta^2 : \mathscr{S}'(V) \to \mathscr{S}'(V)$ and its associated heat semigroup $e^{-t\Delta^2} : \mathscr{S}'(V) \to \mathscr{S}'(V)$, $t \geqslant 0$, through their Fourier multipliers*

$$\mathcal{F}(-\Delta^2 f) = -16\pi^4 \| \cdot \|^4 \mathcal{F} f,$$
$$\mathcal{F}(e^{-t\Delta^2} f) = e^{-16\pi^4 t \|\cdot\|^4} \mathcal{F} f.$$

Definition 4.5.2 (The heat kernel, $p(t,x)$, for $-\Delta^2$)**.** *For $t > 0$, we have the identity*

$$\begin{aligned} e^{-t\Delta^2} f &= \mathcal{F}^{-1}\left(e^{-16\pi^4\|\cdot\|^4 t} \cdot (\mathcal{F}f)\right) \\ &= p_t * f \end{aligned}$$

for all $f \in \mathscr{S}'(V)$, where $p_t \in \mathscr{S}(V)$ is the heat kernel for $-\Delta^2$:

$$p(t,x) = p_t(x) := \left(\mathcal{F}^{-1} e^{-16\pi^4\|\cdot\|^4 t}\right)(x).$$

Emulating the preceding probabilistic approach to the $\Delta-$heat equation (4.2), we attempt to interpret p as the transition densities of some Markov process. However, there are several obstructions to doing so.

Definition 4.5.3. *Let $k \in \mathbb{N}$. Define M_k to be the k^{th} moment of $p(1,\cdot)$, namely:*

$$M_k := \int_V x^{\otimes k} p(1,x) \mathrm{d}x.$$

Lemma 4.5.4 (Properties of the heat kernel $p(t,x)$)**.** *For all $t > 0$, $x \in V$, $d = \dim(V)$,*

1. $p(t,x) = t^{-d/4} p(1, xt^{-1/4})$.

2. $\int_V p(t,x)\mathrm{d}x = 1$.

3. $M_n = 0$, *for* $n = 1, 2, 3$.

4. $M_4 = -4!\left(\sum_{j=1}^{d} e_j \otimes e_j\right)^{\otimes 2}$, $(e_1, \ldots, e_d)$ *any orthonormal coordinates for* V*. This expression is invariant under orthonormal change of coordinates.*

5. $|p(1,x)| \leqslant C \exp(-\|x\|^{4/3})$ *some* $C > 0$.

6. *For all* $k \in \mathbb{N}$,

$$\|M_k\|_{V^{\otimes k}} \leqslant C_d \Gamma\left(\frac{3}{4}(k+d)\right),$$

where Γ *denotes the classical* Γ *function.*

Proof.

1. Making the change of variables $\xi \mapsto t^{-1/4}\xi$,

$$\begin{aligned} p(t,x) &= \int_V e^{-16\pi^4\|\xi\|^4 t} e^{2\pi i\langle \xi, x\rangle} \mathrm{d}\xi \\ &= \int_V e^{-16\pi^4\|\xi\|^4} e^{2\pi \mathrm{i}\langle \xi, xt^{-1/4}\rangle} t^{-d/4} \mathrm{d}\xi = t^{-d/4} p(1, xt^{-1/4}). \end{aligned}$$

2. By inverse Fourier transform

$$\int_V p(t,x)\mathrm{d}x = \left(\int_V p(t,x) e^{2\pi \mathrm{i}\langle \xi, x\rangle} \mathrm{d}x\right)\Big|_{\xi=0} = e^{-16\pi^4\|\xi\|^4 t}|_{\xi=0} = 1.$$

3&4. Direct calculations similar to 1. and 2.

5. This is a special case of Theorem 2 ($d < 4$) in [11] and Theorem 1 ($d \geqslant 4$) in [2].

6. Using the heat kernel bound from 5. and the coarea formula:

$$\begin{aligned}
\left\| \int_V x^{\otimes k} p(1,x) \mathrm{d}x \right\| &\leqslant C_d \int_V \|x\|^k \exp\left(-\|x\|^{4/3}\right) \mathrm{d}x \\
&= C_d \int_0^\infty r^{k+d-1} \exp\left(-r^{4/3}\right) \mathrm{d}r \\
&= C_d \int_0^\infty r^{\frac{3}{4}(k+d)-1} \exp(-r) \mathrm{d}r \\
&= C_d \Gamma\left(\frac{3}{4}(k+d)\right),
\end{aligned}$$

as claimed.

□

Corollary 4.5.5. *Let p be the heat kernel of $-\Delta^2$ from Definition 4.5.2, then $\int_V |p(t,x)| \, \mathrm{d}x > 1$.*

Proof. From Lemma 4.5.4 parts 2. and 3., we see that p takes positive and negative values on sets of nonzero Lebesgue measure hence

$$\int_V |p(t,x)| \, \mathrm{d}x > \int_V p(t,x) \mathrm{d}x = 1.$$

□

We have established that $p(t,\cdot)$ is not a probability density function as it takes negative values – this is unsurprising as generic high-order elliptic operators do not satisfy maximum principles. Nevertheless, one may still define, as in the Kolmogorov construction of Brownian motion (Definition 4.2.4), the *signed* cylinder set measure associated to p. This same construction occurs in Section 5 of [27].

Definition 4.5.6 (Definition 5.1, [27]). *For $0 \leqslant s \leqslant t < +\infty$, $x \in V$ and a partition $D = (s = t_0 < \cdots < t_n = t)$ of $[s,t]$ define the measure $\upsilon_D^{[s,t];x}$ on $\prod_{t_i \in D} V \cong V^{n+1}$ by*

$$\upsilon_D^{[s,t];x}(A_0 \times \cdots \times A_n) := \int_{A_0} \cdots \int_{A_n} \prod_{i=0}^{n} p_{t_i - t_{i-1}}(x_i - x_{i-1}) \mathrm{d}x_i,$$

where $A_i \in \mathscr{B}(V)$ are Borel, $x_{-1} \equiv x$ and $t_{-1} = t_0$. $p_0(y-x)\mathrm{d}y = \delta_x(\mathrm{d}y)$ denotes the Dirac measure centred at x.

The semigroup property $p_t * p_s = p_{t+s}$ ensures that $\left(\upsilon_D^{[s,t];x}\right)_{D \subset [s,t]}$ is a cylinder set measure on $\left(V^{[s,t]}, \bigotimes_{[s,t]} \mathscr{B}(V)\right)$. We denote the induced finitely additive measure on $\left(V^{[s,t]}, \bigotimes_{[s,t]} \mathscr{B}(V)\right)$ by $\upsilon^{[s,t];x}$. Namely,

$$\upsilon^{[s,t];x}\left(\mathbf{x}_{t_i} \in U_i, \ \ i = 0, \ldots, n\right) = \upsilon_{\{t_0,\ldots,t_n\}}^{[s,t];x}(U_0 \times \cdots \times U_n),$$

where, for $u \in [s,t]$, $\mathbf{x}_u : V^{[s,t]} \to V$ is the coordinate map.

One would hope that $\upsilon^{[s,t];x}$ is countably additive. However,

Lemma 4.5.7. *Let $0 \leqslant s < t < +\infty$, $x \in V$ and define the partitions $D_n := \left(s + \frac{i}{n}(t-s) : i = 0, \ldots, n\right)$ of $[s,t]$. Then*

$$\left\| \upsilon_{D_n}^{[s,t];x} \right\|_{\mathrm{TV}} \to \infty \quad \text{as} \quad n \to \infty,$$

where $\| \cdot \|_{\mathrm{TV}}$ denotes the total variation norm on signed measures.

Proof. By 1. (scaling) of Lemma 4.5.4, $\int_V |p(t,x)|\mathrm{d}x = \int_V |p(1,x)|\mathrm{d}x$ for all $t > 0$. Hence

$$\left\| \upsilon_{D_n}^{[s,t];x} \right\|_{\mathrm{TV}} = \int_V \cdots \int_V \prod_{i=0}^{n} \left| p\left(\frac{t-s}{n}, x_i - x_{i-1}\right) \right| \mathrm{d}x_i$$
$$= \left(\int_V |p(1,x)|\mathrm{d}x \right)^{n+1} \to \infty \quad \text{as} \quad n \to \infty.$$

□

Kolmogorov's extension theorem therefore does not apply. In fact, V. Yu. Krylov asserts that $\nu^{I;x}$ is not a countably additive measure on V^I ([26]) for nondegenerate intervals, I. See also [22] and [27] for further details.

We hope to find convergence of the υ_D in a weaker topology. Namely we seek convergence $\lim_{|D|\to 0}\langle \upsilon_D, \phi\rangle$, for appropriate test functions, ϕ, on path space. Following [27], we first view each υ_D as a measure on continuous, piecewise linear paths $\mathrm{PL}([0,T] \to \mathbb{R}^d)$ thus

Definition 4.5.8 (Continuous piecewise linear paths)**.** *Let $s < t$, $\mathrm{PL}([s,t] \to V) \subseteq C([s,t] \to V)$ denotes the subspace of continuous, piecewise linear paths. $\gamma \in \mathrm{PL}([s,t] \to V)$ iff $\gamma : [s,t] \to V$ is continuous and there exists a finite partition $\mathcal{D} = (s = t_0 < \cdots < t_m = t)$ of $[s,t]$ such that for all $i = 1, \ldots, m$ the restriction $\gamma|_{[t_{i-1},t_i]}$ is an affine path. Such a path will be called $\mathcal{D}-$piecewise linear when we need to emphasise a particular partition.*

Definition 4.5.9. *Let $D = (s = t_0 < \cdots < t_n = t)$ a finite partition of $[s,t]$*

and $(v_1,\ldots,v_n)\in V^n$*. Define* $\gamma_D(v_1,\ldots,v_n)\in \mathrm{PL}[s,t]$ *to be the continuous* $D-$*piecewise linear path interpolating* $(v_1,\ldots,v_n)$*:*

$$\gamma_D(v_1,\ldots,v_n)_t := \frac{(t_j-t)v_{j-1}+(t-t_{j-1})v_j}{t_j-t_{j-1}}, \quad t\in[t_{j-1},t_j].$$

We may implicitly extend $\gamma_D(v_1,\ldots,v_n)$ *to a larger interval* $[S,T]\supseteq[s,t]$ *by for* $t\in[S,s]$*,* $\gamma_D(v_1,\ldots,v_n)_t\equiv v_1$ *and for* $t\in[t_n,T]$*,* $\gamma_D(v_1,\ldots,v_n)_t\equiv v_n$*.*

Definition 4.5.10 (Definition 5.3 [27])**.** *Let* $0\leqslant s<t$*,* $x\in V$ *and* $D=(s=t_0<\cdots<t_n=t)$ *a finite partition of* $[s,t]$*. Recall the measure* $\upsilon_D^{[s,t];x}$ *on* $\prod_{t_i\in D}V$ *from Definition 4.5.6. Define* $\mathbb{P}_D^{[s,t];x}$ *as the push-forward of* $\upsilon_D^{[s,t];x}$ *by* $\gamma_D:\prod_{t_i\in D}V\to \mathrm{PL}[s,t]$*. Namely,*

$$\mathbb{P}_D^{[s,t];x}\in \mathrm{Meas}(\mathrm{PL}[s,t]),\quad \mathbb{P}_D^{[s,t];x}:=\upsilon_D^{[s,t];x}\circ\gamma_D^{-1}.$$

Remark 4.5.11. Recall for $u\in[s,t]$, $\mathbf{x}_u:\mathrm{PL}([s,t]\to V)\to V$ the coordinate map. Then $\mathbb{P}_D^{[s,t];x}$ satisfies

1. $\mathbb{P}_D^{[s,t];x}\circ\mathbf{x}_s^{-1}(\mathrm{d}y)=\delta_x(\mathrm{d}y)$;

2. for all $i=1,\ldots,n$

$$\mathbb{P}_D^{[s,t];x}\circ\mathbf{x}_{t_{i-1},t_i}^{-1}(\mathrm{d}y)=p_{t_i-t_{i-1}}\mathrm{d}y;$$

3. for all $1\leqslant i<j\leqslant n$, $\mathbf{x}_{t_{i-1},t_i}$ and $\mathbf{x}_{t_{j-1},t_j}$ are $\mathbb{P}_D^{[s,t];x}-$independent.

This is analogous to the construction of Brownian motion from finite dimensional distributions outlined in Definition 4.2.4. Indeed, items 2. and 3. state that the increments of the quasi-process with law $\mathbb{P}_D^{[s,t];x}$ are independent and have distribution given by the heat kernel of $-\Delta^2$. This is in parallel with the aforementioned construction from finite-dimensional distributions of Brownian motion as the process which has independent increments with distribution given by the heat kernel of Δ. $\square$

Continuous piecewise linear paths are of bounded variation and therefore admit canonical rough path lifts. Recall that for a piecewise linear path $\gamma \in \mathrm{PL}([s,t] \to V)$ and $p \geqslant 1$, we denote by $g^{\lfloor p \rfloor}(\gamma)$ the canonical lift of γ to a $p-$rough path. The inclusion

$$g^{\lfloor p \rfloor} : \mathrm{PL}([s,t] \to V) \hookrightarrow G\Omega_p([s,t] \to V)$$

is a measurable map and therefore we define

Definition 4.5.12 (Measures on $p-$rough paths)**.** *Let* $0 \leqslant s \leqslant t < +\infty$ *and* $x \in V$*. We shall use the symbol* $\mathbb{P}_D^{[s,t];x}$ *to also denote the pushforward measure on* $G\Omega_p([s,t] \to V)$*,* $\mathbb{P}_D^{[s,t];x} \circ (g^{\lfloor p \rfloor})^{-1}$.

4.6 Convergence of expected signatures

The following Theorem 4.6.2 is the main result of this chapter. A consequence is a weak convergence result for the measures $\left(\mathbb{P}_D^{[S,T];x}\right)_{D \in \mathcal{P}([S,T])}$

on $G\Omega_p([S,T] \to V)$. Namely, $\ell^* \in \bar{T}(V)^*$ induces a scalar-valued function on $p-$rough paths, f_{ℓ^*}, given by

$$f_{\ell^*} : g \mapsto \langle \ell^*, \mathbb{X}(g) \rangle_{\bar{T}(V)^* \times \bar{T}(V)}.$$

Then

$$\lim_{|D| \to 0} \int f_{\ell^*}(g) \mathbb{P}_D^{[S,T];x}(\mathrm{d}g) = \left\langle \ell^*, \exp\left(\frac{M_4}{4!}(T-S) \right) \right\rangle.$$

This class of functions contains in particular coordinate signatures, thereby extending the main result in [27]. We shall see (Corollary 5.1.8) that this class of functions also contains coordinate signatures of solutions to linear equations. Chapter 5 is devoted to showing that Theorem 4.6.2 is strong enough to allow for quasi-probabilistic representations of solutions to high-order parabolic semigroups.

Remark 4.6.1. To help with intuition as to what functionals are contained in the class $\bar{T}(V)^*$, let $\ell^k \in (V^{\otimes k})^*$ be linear functionals for which there exists $C > 0$ such that

$$\|\ell^k\| \leqslant C^k.$$

Then $\ell^* = \sum_{k=0}^{\infty} \ell^k$ is contained in $\bar{T}(V)^*$ (see the forthcoming Proposition 4.3.7 for the proof). Namely, $\bar{T}(V)^*$ contains linear functionals of arbitrary exponential growth – in particular, the above mentioned coordinate signatures of solutions to linear equations.

⚠To ease notation, we shall sometimes write $\mathbb{E}_D[f] := \int f(g) \mathbb{P}_D^{[S,T];x}(\mathrm{d}g)$ for Bochner integrable f. Recall the Banach algebra $T_\lambda(V)$ from Definition

4.3.1 and M_4 the 4^{th} moment of $p(1,\cdot)$ from Definition 4.5.3.

Theorem 4.6.2. *Let $x \in V$, $0 \leqslant S < T < +\infty$ then the following limit exists in $T_\lambda(V)$ for all $\lambda \geqslant 1$*

$$\lim_{|D|\to 0} \int_{g\in G\Omega_p} \mathbb{X}_{S,T}(g)\, \mathbb{P}_D^{[S,T];x}(\mathrm{d}g) = \exp\left(\frac{M_4}{4!}(T-S)\right), \tag{4.6.1}$$

where the integrals are $T_\lambda(V)-$valued Bochner integrals, taken over $g \in G\Omega_p([S,T] \to V)$.

Remark 4.6.3. It was shown in [14] that the expected signature of $V-$valued Brownian motion over $[S,T]$ is:

$$\exp\left(\frac{1}{2}(T-S)\int_V x^{\otimes 2} q(1,x)\mathrm{d}x\right),$$

where q denotes the heat kernel of Δ. So that equation 4.6.1 is indeed what one would naïvely expect for the "expected signature" associated to $-\Delta^2$.

Proof. We first verify Bochner integrability. $\mathbb{X}_{S,T} : G\Omega_p([S,T];V) \to T_\lambda(V)$ is continuous (Proposition 4.3.8) and separably valued. Thus, for each partition $D = (S = t_0 < \cdots < t_n = T)$, $\mathbb{X}_{S,T}$ is $\mathbb{P}_D^{[S,T];x}-$strongly measurable (Proposition 4.1.10). As for summability,

$$\begin{aligned}
&\int_{g\in G\Omega_p(V)} \|\,\mathbb{X}_{S,T}(g)\| \cdot \left|\mathbb{P}_D^{[S,T];x}\right|(\mathrm{d}g) \\
&\qquad = \int_{(v_1,\ldots,v_n)\in V^n} \|\,\mathbb{X}_{S,T}(g\circ\gamma(v_1,\ldots,v_n))\| \cdot |v_D^{[S,T];x}|(\mathrm{d}v_1,\ldots,\mathrm{d}v_n) \\
&\qquad \leqslant \int_{(v_1,\ldots,v_n)\in V^n} \exp(\|v_1\|)\ldots\exp(\|v_n\|) \prod_{i=0}^{n} |p_{\tau_i}(v_i - v_{i-1})|\mathrm{d}v_i < +\infty,
\end{aligned}$$

the final inequality following from better-than-exponential-decay heat kernel bounds (Lemma 4.5.4). Therefore $\mathbb{X}_{S,T}$ is $\mathbb{P}_D^{[S,T];x}-$Bochner integrable for all D, as claimed.

By Chen's relation we may factorise $\mathbb{X}_{S,T} = \mathbb{X}_{S,t_1} \ldots \mathbb{X}_{t_{n-1},T}$. Note that for $s \leqslant t \leqslant u \leqslant v \in [S,T]$, the $T_\lambda(V)-$valued functions $\mathbb{X}_{s,t}$ and $\mathbb{X}_{u,v}$ are $\mathbb{P}_D^{[S,T];x}-$independent provided there exists $r \in D$ with $t \leqslant r \leqslant u$. Whence, by iterating Proposition 4.1.12, we have the identity $\mathbb{E}_D[\mathbb{X}_{S,T}] = \prod_{t_i \in D} \mathbb{E}_D[\mathbb{X}_{t_i,t_{i+1}}]$. Recall (Definition 4.5.10) that the coordinate process, $\mathbf{x}$, is $\mathbb{P}_D^{[S,T];x}-$almost everywhere affine on the interval (t_i, t_{i+1}) with increment distributed like $\mathbb{P}_D^{[S,T];x} \circ \mathbf{x}_{t_i,t_{i+1}}^{-1}(\mathrm{d}y) = p(t_{i+1}-t_i, \mathrm{d}y)$ and therefore has signature $\mathbb{X}(g(\mathbf{x}))_{t_i,t_{i+1}} = \exp(\mathbf{x}_{t_i,t_{i+1}}) \in \bar{T}(V)$, $\mathbb{P}_D^{[S,T];x}-$almost everywhere. Appealing to the heat kernel bound from Lemma 4.5.4 and the dominated convergence theorem for Bochner integrals, writing $\tau_i := t_{i+1} - t_i$ and recalling $M_k := \int_V v^{\otimes k} p(1,v)\mathrm{d}v$,

$$\begin{aligned}
\mathbb{E}_D[\mathbb{X}_{t_i,t_{i+1}}] &= \mathbb{E}_D[\exp(\mathbf{x}_{t_i,t_{i+1}})] \\
&= \sum_{k=0}^{\infty} \int_V \frac{v^{\otimes k}}{k!} p(\tau_i, v)\mathrm{d}v \\
&= \sum_{k=0}^{\infty} \int_V \frac{v^{\otimes k}}{k!} \tau_i^{k/4} p(1, v)\mathrm{d}v \\
&= \sum_{k=0}^{\infty} M_k \frac{\tau_i^{k/4}}{k!} \\
&= 1 + \frac{M_4}{4!} \cdot \tau_i + \sum_{k=8}^{\infty} M_k \frac{\tau_i^{k/4}}{k!}.
\end{aligned}$$

We intend to appeal to Corollary 4.4.3 with Banach algebra $T_\lambda(V)$, $D = (S = t_0 < \cdots < t_n = T)$ a partition of $[S,T]$ and

$$x = \frac{M_4}{4!}, \qquad Y_i^{(n)} = \sum_{k=8}^{\infty} M_k \frac{\tau_i^{(k/4-2)}}{k!}.$$

To do so, we need only obtain the uniform bound $\sup_{i,n} \left\| Y_i^{(n)} \right\|_\lambda < +\infty$. To this end, recall the moment bounds $\|M_k\|_{V^{\otimes k}} \leqslant C_d \Gamma\left(\frac{3}{4}(k+d)\right)$ from Lemma 4.5.4, then

$$\begin{aligned}
\left\| Y_i^{(n)} \right\|_\lambda &= \left\| \sum_{k=8}^{\infty} M_k \frac{\tau_i^{(k/4-2)}}{k!} \right\|_\lambda \\
&= \sum_{k=8}^{\infty} \|M_k\|_{V^{\otimes k}} \cdot \lambda^k \cdot \frac{\tau_i^{(k/4-2)}}{k!} \\
&\leqslant \sum_{k=8}^{\infty} C_d \frac{\Gamma\left(\frac{3}{4}(k+d)\right)}{k!} \cdot \lambda^k \cdot \tau_i^{(k/4-2)} \\
&\leqslant \sum_{k=8}^{\infty} C_d \frac{\Gamma\left(\frac{3}{4}(k+d)\right)}{k!} \cdot \lambda^k \cdot (T-S)^{(k/4-2)} < +\infty,
\end{aligned}$$

noting this final bound is independent of i and n, we obtain $\sup_{i,n} \|Y_i^{(n)}\|_\lambda < +\infty$, as desired. Therefore by Corollary 4.4.3

$$\begin{aligned}
\lim_{|D|\to 0} \mathbb{E}_D[\mathbb{X}_{S,T}] &= \lim_{n\to\infty} \prod_{i=1}^{n} \left(1 + \frac{M_4}{4!}\tau_i + Y_i^{(n)} \tau_i^2\right) \\
&= \exp\left(\frac{M_4}{4!}(T-S)\right),
\end{aligned}$$

with the convergence taking place with respect to $\|\cdot\|_\lambda$. Noting that $\lambda \geqslant 1$ was arbitrary, we see that the convergence takes place in $\bar{T}(V)$. □

We shall denote

$$\mathbb{E}[\mathbb{X}_{S,T}] := \lim_{|D|\to 0} \mathbb{E}_D[\mathbb{X}_{S,T}].$$

Recall that for $n \in \mathbb{N}$, $\Pi_{0,n} : \bar{T}(V) \to T^n(V)$ denotes the projection onto tensors of degree $\leqslant n$.

Proposition 4.6.4. $(S,T) \mapsto \Pi_{0,4}\,\mathbb{E}[\mathbb{X}_{S,T}]$ *is a* $4-$*rough path.*

Proof. Note the multiplicative property for $S \leqslant U \leqslant T$

$$\begin{aligned}
\mathbb{E}[\mathbb{X}_{S,U}]\,\mathbb{E}[\mathbb{X}_{U,T}] &= \exp\left(\frac{M_4}{4!}(U-S)\right)\exp\left(\frac{M_4}{4!}(T-U)\right) \\
&= \exp\left(\frac{M_4}{4!}(T-S)\right) \\
&= \mathbb{E}[\mathbb{X}_{S,T}].
\end{aligned}$$

Note also that $(S,T) \mapsto \Pi_{0,4}(\mathbb{E}[\mathbb{X}_{S,T}]) = 1 + \frac{T-S}{4!}M_4 \in T^{(4)}(V)$ has finite $4-$variation. □

The following corollary is a consequence of pulling continuous linear operators out of integrals and the convergence result of Theorem 4.6.2.

Corollary 4.6.5. *Let* X *be a finite-dimensional, real Banach space and* $\mathscr{A} \in \mathbb{L}(\bar{T}(V) \to X)$ *a continuous linear map. Then* $\mathscr{A}\,\mathbb{X}_{S,T}$ *is* $\mathbb{P}_D^{[S,T]}-$*integrable and*

$$\lim_{|D|\to 0} \mathbb{E}_D[\mathscr{A}\,\mathbb{X}_{S,T}] = \mathscr{A}\,\mathbb{E}[\mathbb{X}_{S,T}] = \mathscr{A}\exp\left(M_4\frac{(T-S)}{4!}\right).$$

Proof. Choose $\lambda \geqslant 1$ so that $\mathscr{A}$ is $\|\cdot\|_\lambda$–bounded then

$$\int \|\mathscr{A}\,\mathbb{X}_{S,T}\|_\lambda \, \mathrm{d}|\mathbb{P}_D| \leqslant C \int \|\mathbb{X}_{S,T}\|_\lambda \, \mathrm{d}|\mathbb{P}_D| < +\infty$$

so that $\mathscr{A}\,\mathbb{X}_{S,T}$ is integrable. Then

$$\mathbb{E}_D[\mathscr{A}\,\mathbb{X}_{S,T}] \overset{(\star)}{=} \mathscr{A}\,\mathbb{E}_D[\mathbb{X}_{S,T}] \xrightarrow{|D|\to 0} \mathscr{A}\,\mathbb{E}[\mathbb{X}_{S,T}],$$

as claimed. The equality $(\star)$ – pulling the continuous linear map, $\mathscr{A}$, out of the Bochner integral – is a consequence of Theorem 6, pp. 47 [12]. The convergence is due to Theorem 4.6.2 and continuity of $\mathscr{A}$. □

Remark 4.6.6. By taking $X = \mathbb{R}$ and $\mathscr{A} \in (V^{\otimes n})^*$ in the above corollary, we recover Theorem 7.4 in [27] which states that expectations of coordinate signatures converge as $|D| \to 0$.

4.7 Convergence of Fourier transforms

This section presents an application of Theorem 4.6.2, which is not possible using only the results in [27]. In [9], Chevyrev and Lyons study a generalisation of the Fourier transform to finite Borel measures on signatures and show that the transform is faithful. We provide a brief overview of their result. The only new result in this section is Theorem 4.7.11, which asserts that the Fourier transforms of the measures $\left(\mathbb{P}_D^{[S,T];x}\right)_{D\in\mathcal{P}([S,T])}$ converge pointwise as $|D| \to 0$ and identifies the limit. The notation and definitions of this section closely follow [9], to which we refer for further details.

Definition 4.7.1. *$T(V)$ is a Hopf algebra with coproduct $\triangle v = v \otimes v$ for all $v \in V$ and antipode $\alpha(v_1 \dots v_k) = (-1)^k v_k \dots v_1$ for all $v_1 \dots v_k \in V^{\otimes k}$ ([39] Chapter 1).*

Chevyrev and Lyons ([9] Section 3) assert that $\triangle$ extends to a continuous algebra homomorphism

$$\triangle : \bar{T}(V) \to \bar{T}(V) \hat{\otimes} \bar{T}(V).$$

Here $\mathscr{A} \hat{\otimes} \mathscr{B}$ denotes the completion of the projective tensor product of the locally convex spaces $\mathscr{A}, \mathscr{B}$.

Definition 4.7.2 (Grouplike elements)**.** *Define*

$$\bar{G}(V) = \left\{ \mathbb{X} \in \bar{T}(V) \backslash \{0\} \mid \triangle \mathbb{X} = \mathbb{X} \otimes \mathbb{X} \right\}.$$

Lemma 4.7.3. *$\bar{G}(V)$ is a topological group with the subspace topology and is closed in $\bar{T}(V)$.*

Proof. See Section 3 in [9]. □

Lemma 4.7.4 (Signatures are grouplike)**.** *Let $p \geqslant 1$, $s < t$ and recall (Proposition 4.3.8) the signature map*

$$\mathbb{X} : G\Omega_p([s,t] \to V) \longrightarrow \bar{T}(V).$$

It holds that $\mathrm{Image}(\mathbb{X}) \subseteq \bar{G}(V)$*.*

Proof. See Subsection 5.1 in [9]. □

Unitary representations

Let H be a finite-dimensional Hilbert space over $\mathbb{C}$. Denote by

$$\mathfrak{u}(H),\ U(H) \subset \mathbb{L}(H)$$

the unitary Lie algebra and Lie group of linear operators on H respectively. Proposition 4.3.6 asserts that any $A \in \mathbb{L}(V \to \mathfrak{u}(H))$ uniquely extends to a continuous representation

$$\mathbb{A} : \bar{T}(V) \to \mathbb{L}(H)$$

of the topological algebra $\bar{T}(V)$. Furthermore,

Lemma 4.7.5. *Let H be a finite-dimensional, complex Hilbert space and $A \in \mathbb{L}(V \to \mathfrak{u}(H))$. Then the restriction*

$$\mathbb{A}\,|_{\bar{G}(V)} : \bar{G}(V) \to U(H)$$

is a continuous, unitary representation of the topological group $\bar{G}(V)$.

Proof. See Section 4 in [9]. □

Definition 4.7.6. *Denote by $\mathcal{A}(V)$ the family of continuous, unitary representations of $\bar{G}(V)$ arising as extensions of bounded linear maps $A \in \mathbb{L}(V \to \mathfrak{u}(H))$, where H ranges over all finite-dimensional Hilbert spaces.*

Definition 4.7.7 (Page 13, Section 4, [9])**.** *Let* $\mu \in \operatorname{Meas}\big(\bar{G}(V)\big)$ *be a finite Borel measure. Denote the Fourier transform of* μ *by* $\widehat{\mu}$*, where*

$$\widehat{\mu}(\mathbb{A}) := \int_{\mathbb{X} \in \bar{G}(V)} \mathbb{A}(\mathbb{X}) \mu(\mathrm{d}\, \mathbb{X})$$

for all $\mathbb{A} \in \mathcal{A}(V)$*.*

Remark 4.7.8. Since $U(H)$ is compact, $\widehat{\mu}$ is well-defined.

Remark 4.7.9. Let μ, ν be Borel measures on $\bar{G}(V)$. Corollary 4.12. in [9] asserts that $\widehat{\mu} = \widehat{\nu}$ if and only if $\mu = \nu$.

Definition 4.7.10 (Measures on signatures associated to $-\Delta^2$)**.** *Let* $0 \leqslant S < T < +\infty$*,* $p \geqslant 1$ *and recall the measures* $\left(\mathbb{P}_D^{[S,T]}\right)_D$ *on* $G\Omega_p([S,T] \to V)$*. For each partition,* D*, of* $[S,T]$*, the signature map*

$$\mathbb{X} : \left(G\Omega_p([S,T] \to V),\ \mathbb{P}_D^{[S,T]}\right) \longrightarrow \bar{G}(V)$$

induces a measure on $\bar{G}(V)$*. We shall denote this pushforward measure also by* $\mathbb{P}_D^{[S,T]}$*.*

The following theorem states that the Fourier transforms of the measures $\left(\mathbb{P}_D^{[S,T]}\right)_D$ on $\bar{G}(V)$ converges pointwise as $|D| \to 0$.

Theorem 4.7.11. *Let* H *be a finite-dimensional Hilbert space and* $A \in \mathbb{L}(V \to \mathfrak{u}(H))$ *with extension* $\mathbb{A} \in \mathcal{A}(V)$*. Then*

$$\lim_{|D| \to 0} \widehat{\mathbb{P}_D^{[S,T]}}(\mathbb{A}) = \exp\left(A^{\otimes 4}(M_4) \frac{T-S}{4!}\right).$$

Proof. Recall from Theorem 4.6.2 that $\mathbb{E}_D[\mathbb{X}_{S,T}] \xrightarrow{|D|\to 0} \exp\left(M_4 \frac{T-S}{4!}\right)$ in $\bar{T}(V)$. Then since $\mathbb{A} : \bar{T}(V) \to \mathbb{L}(H)$ is continuous and linear,

$$\begin{aligned}
&\widehat{\mathbb{P}_D^{[S,T]}}(\mathbb{A}) \\
&= \int_{g \in G\Omega_p(V)} \mathbb{A}(\mathbb{X}(g))\mathbb{P}_D^{[S,T]}(\mathrm{d}g) \\
&= \mathbb{A}\,\mathbb{E}_D[\mathbb{X}_{S,T}] \xrightarrow{|D|\to 0} \mathbb{A}\exp\left(M_4 \frac{T-S}{4!}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbb{A}\exp\left(M_4 \frac{T-S}{4!}\right) &= \sum_{k=0}^{\infty} A^{\otimes 4k}\left(M_4^{\otimes k}\right)\frac{(T-S)^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(A^{\otimes 4}(M_4)\right)^{\otimes k}\frac{(T-S)^k}{k!} \\
&= \exp\left(A^{\otimes 4}(M_4)\frac{T-S}{4!}\right),
\end{aligned}$$

as claimed. □

Remark 4.7.12. An interesting question is whether there exists a measure $\mathbb{P}^{[S,T]} \in \mathrm{Meas}(\bar{G}(V))$ such that

$$\lim_{|D|\to 0} \widehat{\mathbb{P}_D^{[S,T]}} = \widehat{\mathbb{P}^{[S,T]}}$$

in an appropriate sense. This is currently unknown. One would expect that if such a measure did exist, it would be supported on signatures of $p-$rough paths with $p \in [4,5)$.

5 Linear Differential Equations and High-Order Parabolic Semigroups

5.1 Expected signatures of solutions to random linear equations

We recall aspects of linear rough differential equations and collect some continuity results that will be required in the sequel. Theorem 5.1.11 is a further generalisation of the main theorem in [27]; we prove convergence of the $\mathbb{P}_D-$expected signatures of solutions to linear differential equations and identify the limits.

We treat rough differential equations in the sense of Lyons and refer to [33], in particular Chapter 4, for further details and terminology.

Definition 5.1.1. *Let $A \in \mathbb{L}(V \to \mathbb{L}(W))$ and $[0,1] \to V$, $t \mapsto \gamma_t$ a piecewise C^1 path. Then the equation*

$$dY_t^x(\gamma) = A(Y_t^x(\gamma))d\gamma_t, \quad Y_0^x(\gamma) \equiv x \in W \tag{5.1.1}$$

is called a linear equation.

Recall that $\mathbb{X}\left(\gamma|_{[0,t]}\right) = \mathbb{X}(\gamma)_{0,t} \in \bar{T}(V)$ denotes the signature of the path $\gamma|_{[0,t]}$ and that $\mathbb{A} : \bar{T}(V) \to \mathbb{L}(W)$ denotes the continuous, algebra extension of $A \in \mathbb{L}(V \to \mathbb{L}(W))$.

Lemma 5.1.2. *Let $A \in \mathbb{L}(V \to \mathbb{L}(W))$ and $[0,1] \ni t \mapsto \gamma_t \in V$ a piecewise*

C^1 *path. Let* $Y^x_t(\gamma)$ *solve* (5.1.1) *then for all* $s \leqslant t$

$$Y^x_t(\gamma) = \mathbb{A}\,\mathbb{X}(\gamma|_{[s,t]})Y^x_s(\gamma) = \sum_{k=0}^{\infty} A^{\otimes k}(\mathbb{X}^k(\gamma|_{[s,t]}))Y^x_s(\gamma). \tag{5.1.2}$$

Remark 5.1.3. From (5.1.2) with $s = 0$, it is immediate that $Y^x_t(\gamma)$ is a continuous linear function of $\mathbb{X}(\gamma|_{[0,t]})$.

Recall the symmetric group acts isometrically $\mathfrak{S}_K \curvearrowright V^{\otimes K}$ by

$$\sigma : v_1 \otimes \cdots \otimes v_K \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(K)}.$$

Definition 5.1.4 (Ordered shuffles)**.** *Let* $k_i \in \mathbb{N}^*$, $i = 1, \ldots, n$, *with* $K = \sum_{i=1}^n k_i$. *We denote by* $\mathrm{OS}(k_1, \ldots, k_n)$ *the subgroup of elements* $\sigma \in \mathfrak{S}_K$ *satisfying*

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k_1), \quad \sigma(k_1+1) < \cdots < \sigma(k_1+k_2), \ldots,$$
$$\sigma(K-k_n+1) < \cdots < \sigma(K), \quad \textit{and} \quad \sigma(k_1) < \sigma(k_1+k_2) < \cdots < \sigma(K).$$

Definition 5.1.5. *Let* $A \in \mathbb{L}(V \to \mathbb{L}(W))$ *and* $m \in \mathbb{N}_{\geqslant 1}$. *Define the linear map*

$$\mathbb{A}_{(m)} : T(V) \to \mathbb{L}\left(W^{\otimes m}\right), \quad \mathbb{A}_{(m)} : \mathbb{X} \mapsto \left(\sum_{\mathbf{k} \in (\mathbb{N}^*)^m} A^{\otimes |\mathbf{k}|} \sum_{\sigma \in \mathrm{OS}(\mathbf{k})} \sigma\right)(\mathbb{X}).$$

Proposition 5.1.6. $\mathbb{A}_{(m)}$ *extends uniquely to a continuous map on* $\bar{T}(V)$*.*

We shall denote this extension also by $\mathbb{A}_{(m)}$.

Proof. Each ordered shuffle $\sigma \in \mathrm{OS}(\mathbf{k})$ is a linear isometry

$$V^{\otimes|\mathbf{k}|} \to V^{\otimes k_1} \otimes \cdots \otimes V^{\otimes k_m}$$

and by a straightforward combinatorial argument

$$\#\{\mathrm{OS}(\mathbf{k}) : \ \mathbf{k} \in (\mathbb{N}^*)^m, \ |\mathbf{k}| = K\} \leqslant m^K.$$

Consequently, the restriction of $\mathbb{A}_{(m)}$ to $V^{\otimes K} \subseteq T(V)$ has at most exponential growth:

$$\left\|(\mathbb{A}_{(m)})^K\right\|_{\mathbb{L}(V^{\otimes K} \to \mathbb{L}(W^{\otimes m}))} \leqslant m^K \cdot \|A\|^K_{\mathbb{L}(V \to \mathbb{L}(W))}$$

and therefore, by Proposition 4.3.7, $\mathbb{A}_{(m)}$ uniquely extends to a continuous map on $\bar{T}(V)$, as desired. □

Lemma 5.1.7. *The solution, $Y_t^x(\gamma)$, of* (5.1.1) *has level–m signature given by the formula*

$$Y_t(\gamma)^{x,m} = \left(\mathbb{A}_{(m)}\, \mathbb{X}(\gamma|_{[0,t]})\right) x^{\otimes m}.$$

Proof. This formula is equation (4.10) in [33]. □

The following corollary combined Lemma 5.1.7 shows that the map taking the full signature of a path $\gamma|_{[0,1]}$ to the level–m signature of the solution to (5.1.1) is the restriction of a continuous and linear map.

Corollary 5.1.8. *Let $A \in \mathbb{L}(V \to \mathbb{L}(W))$, $x \in W$ and $m \in \mathbb{N}$. The map*

$$\bar{T}(V) \to W^{\otimes m}, \quad \mathbb{X} \mapsto \left(\mathbb{A}_{(m)}\, \mathbb{X}\right) x^{\otimes m} \qquad (\star)$$

is continuous and linear.

Proof. From Proposition 5.1.6, we know the map

$$\bar{T}(V) \to \mathbb{L}(W^{\otimes m}), \quad \mathbb{X} \mapsto \mathbb{A}_{(m)}\,\mathbb{X},$$

is continuous and linear. Also, for a fixed $x \in W$, the evaluation map

$$\mathbb{L}(W^{\otimes m}) \to W^{\otimes m}, \quad \mathbb{B} \mapsto \mathbb{B}x^{\otimes m},$$

is continuous and linear. Whence the map $(\star)$ is a composition of continuous linear maps and is therefore continuous and linear, as desired. □

Definition 5.1.9. *Let $f : (X, F, \mu) \to T((W))$ be a tensor series-valued function such that $\pi_m(f)$ is integrable for all $m \in \mathbb{N}$. Define $\int f \mathrm{d}\mu \in T((W))$ by*

$$\int f \mathrm{d}\mu := \left(\int \pi_0 f \mathrm{d}\mu, \int \pi_1 f \mathrm{d}\mu, \ldots, \int \pi_m f \mathrm{d}\mu, \ldots \right).$$

For a tensor series-valued sequence, $(w_n) \in T((W))$, we write $w_n \to w$ if and only if

$$\|\pi_m(w_n - w)\|_{W^{\otimes m}} \to 0 \quad \forall\, m \in \mathbb{N}.$$

Remark 5.1.10. The above topology is the product topology on

$$T((W)) = \prod_{k \in \mathbb{N}} W^{\otimes k}.$$

As a countable product of metric spaces, the space is metrisable.

The following is a further generalisation of the main theorem in [27]. We prove convergence of the $\mathbb{P}_D-$expected signatures of solutions to linear differential equations and identify the limits.

Theorem 5.1.11. *Let* $[0,1] \ni t \mapsto \gamma_t \in V$ *be a piecewise* C^1 *path,* $A \in \mathbb{L}(V \to \mathbb{L}(W))$, $x \in W$. *Denote*

$$\mathbb{Y}_1^x(\mathbb{X}(\gamma|_{[0,1]})) := \sum_{m=0}^{\infty} \mathbb{A}_{(m)}(\mathbb{X}(\gamma|_{[0,1]}))x^{\otimes m} \in T((W))$$

the full signature of the solution to the linear equation (5.1.1). *Then*

$$\lim_{|D|\to 0} \mathbb{E}_D[\mathbb{Y}_1^x(\mathbb{X})] = \sum_{m=0}^{\infty} \mathbb{A}^{(m)}\,\mathbb{E}[\mathbb{X}_{0,1}]x^{\otimes m}. \tag{5.1.3}$$

Remark 5.1.12. We stress that, unlike in Theorem 4.6.2, the integration here is weak.

Remark 5.1.13. The substance of the above theorem is about interchanging limits. Indeed, $\mathbb{Y}_1^x(\mathbb{X}(\gamma|_{[0,1]}))$ is given by the infinite series

$$\mathbb{Y}_1^x(\mathbb{X}(\gamma|_{[0,1]})) := \lim_{N\to\infty} \sum_{m=0}^{N} \mathbb{A}_{(m)}(\mathbb{X}(\gamma|_{[0,1]}))x^{\otimes m}.$$

The theorem shows that one may interchange $\lim_{N\to\infty}$ and $\lim_{|D|\to 0}$ in equation (5.1.3).

Proof. By our definition of tensor series-valued integration, it suffices to prove the convergence for each level$-m$ projection. To this end, fix $m \in \mathbb{N}$. Then, appealing to Corollaries 5.1.8 & 4.6.5 and Theorem 4.6.2,

$$\begin{aligned}\mathbb{E}_D[\pi_m(\mathbb{Y}_1^x(\mathbb{X}_{0,1}))] &= \mathbb{E}_D[Y_1^{x,m}(\mathbb{X}_{0,1})] \\ &= \mathbb{A}_{(m)}\,\mathbb{E}_D[\mathbb{X}_{0,1}]x^{\otimes m} \xrightarrow{|D|\to 0} \mathbb{A}_{(m)}\,\mathbb{E}[\mathbb{X}_{0,1}]x^{\otimes m},\end{aligned}$$

as desired. □

5.2 Polynomials in solutions to differential equations

5.2.1 Symmetric tensors and polynomial functions

Symmetric tensors are well-studied objects. See for example Section 3.2 in [39]. We collect some basic facts that will be required in the sequel.

Definition 5.2.1. *Let $k \in \mathbb{N}$. Define the symmetrisation map on $W^{\otimes k}$ as the projection*

$$W^{\otimes k} \to W^{\otimes k}, \quad w \mapsto \frac{1}{|\mathfrak{S}_k|} \sum_{\sigma \in \mathfrak{S}_k} \sigma w.$$

Definition 5.2.2 (Symmetric $k-$tensors)**.** *Let $k \in \mathbb{N}$. Denote by $\mathrm{Sym}^k(W) \subseteq W^{\otimes k}$ the $1-$eigenspace of the symmetrisation map. Equivalently, $w \in \mathrm{Sym}^k(W)$ if and only if $\sigma w = w$ for all $\sigma \in \mathfrak{S}_k$.*
For $n \in \mathbb{N}$, denote

$$S^{(n)}(W) = \bigoplus_{k=0}^{n} \mathrm{Sym}^k(W).$$

Remark 5.2.3. $S^{(n)}(W) \subseteq T^n(W)$ is a linear subspace not to be confused with the symmetric algebra.

Definition 5.2.4 (Exponential map)**.** *Let $n \in \mathbb{N}$. Define*

$$\exp_n \colon W \to S^{(n)}(W), \quad \exp_n \colon w \mapsto \sum_{j=0}^{n} \frac{1}{j!} w^{\otimes j}.$$

Remark 5.2.5. Let $n \in \mathbb{N}$ and $p \in \mathcal{P}^{(n)}(W)$. By multidimensional Taylor's theorem about 0 (equation 1.7 in [33]) we may represent

$$\begin{aligned} p(y) &= \sum_{j=0}^{n} p^j(0)\frac{y^{\otimes j}}{j!} \\ &= \left\langle \sum_{j=0}^{n} p^j(0), \exp_n(y) \right\rangle_{S^{(n)}(W^*)\times S^{(n)}(W)} \end{aligned}$$

for all $y \in W$.

Definition 5.2.6. *The maps*

$$\mathcal{P}^{(n)}(W) \xrightarrow{\simeq} S^{(n)}(W^*), \quad p \mapsto \sum_{j=0}^{n} p^j(0)$$

are linear isomorphisms for all $n \in \mathbb{N}$. More explicitly, for $w = (w_0, \dots, w_n) \in S^{(n)}(W)$ and $p \in \mathcal{P}^{(n)}(W)$, we have the dual pairing

$$\langle p, w \rangle = \sum_{j=0}^{n} p^j(0)(w_j).$$

We will implicitly make use of these isomorphisms.[2]

5.2.2 Polynomials in solutions of linear equations

Definition 5.2.7. *Let $A \in \mathbb{L}(V \to \mathbb{L}(W))$ and $j \in \mathbb{N}$. Define the linear vector fields on symmetric $j-$tensors*

$$\mathscr{A}^{(j)} \in \mathbb{L}\left(V \to \mathbb{L}(\mathrm{Sym}^j(W)\right), \quad \mathscr{A}^{(j)} : \mathbf{v} \mapsto \left(\sum_{i=0}^{j-1} \mathrm{Id}_W^{\otimes i} \otimes A(\mathbf{v}) \otimes \mathrm{Id}_W^{\otimes (j-i-1)} \right),$$

[2] This choice of isomorphism is arbitrary. Its usefulness for our purposes stems from the fact that if Y_t^x solves a linear equation driven by γ, then $(Y_t^x)^{\otimes j}$ *also* solves a linear equation driven by γ. $(Y_t^x - y_0)^{\otimes j}$, however, does not.

for $j \geqslant 1$ *and* $\mathscr{A}^{(0)} \equiv 0$.

Lemma 5.2.8 (Monomials in solutions of linear equations)**.** *Let* $j \in \mathbb{N}$, $t \mapsto \gamma_t \in V$ *be a piecewise* C^1 *path and* $Y_t^x(\gamma) \in W$ *solve* (5.1.1) *with linear vector fields* $A \in \mathbb{L}(V \to \mathbb{L}(W))$ *then* $(Y_t^x(\gamma))^{\otimes j}$ *solves the linear equation*

$$d(Y_t^x(\gamma))^{\otimes j} = \mathscr{A}^{(j)} \left(Y_t^x(\gamma)^{\otimes j}\right) \mathrm{d}\gamma_t, \quad Y_0^x(\gamma)^{\otimes j} = x^{\otimes j} \in \mathrm{Sym}^j(W).$$

Proof. Write $Y_t = Y_t^x(\gamma)$. Then by the chain rule

$$\begin{aligned}
\mathrm{d}Y_t^{\otimes j} &= \sum_{i=0}^{j-1} Y_t^{\otimes i} \otimes (\mathrm{d}Y_t) \otimes Y_t^{\otimes(j-i-1)} \\
&= \sum_{i=0}^{j-1} Y_t^{\otimes i} \otimes (A(\mathrm{d}\gamma_t)Y_t) \otimes Y_t^{\otimes(j-i-1)} \\
&= \left(\sum_{i=0}^{j-1} \mathrm{Id}_W^{\otimes i} \otimes A(\mathrm{d}\gamma_t) \otimes \mathrm{Id}_W^{\otimes(j-i-1)}\right) (Y_t^{\otimes j}) \\
&= \mathscr{A}^{(j)}(\mathrm{d}\gamma_t)Y_t^{\otimes j},
\end{aligned}$$

as desired. □

Definition 5.2.9. *Let* $n \in \mathbb{N}$, $A \in \mathbb{L}(V \to \mathbb{L}(W))$ *and recall* $\mathscr{A}^{(j)} \in \mathbb{L}(V \to \mathbb{L}(\mathrm{Sym}^j(W))$ *from Definition 5.2.7. Define*

$$\mathscr{E}_n(A) \in \mathbb{L}\left(V \to \bigoplus_{j=0}^{n} \mathbb{L}\left(\mathrm{Sym}^j(W)\right)\right), \quad \mathscr{E}_n(A)(\mathbf{v}) := \bigoplus_{j=0}^{n} \left(\mathscr{A}^{(j)}(\mathbf{v})\right).$$

Denote by

$$\overline{\mathscr{E}_n(A)} : \bar{T}(V) \to \bigoplus_{j=0}^{n} \mathbb{L}\left(\mathrm{Sym}^j(W)\right)$$

the extension of $\mathscr{E}_n(A)$ *to a continuous algebra homomorphism.*

Remark 5.2.10. $\mathscr{E}_n(A)$ is a linear vector field on $S^{(n)}(W)$ which preserves tensor degree. Namely, for all $j \leqslant n$ and $v \in V$, $\mathrm{Sym}^j(W)$ is an invariant subspace for $\mathscr{E}_n(A)(v)$.

Remark 5.2.11. ⚠There is a consistency in n that we will explicitly employ in the sequel. Namely, for all $n \in \mathbb{N}$, $v \in V$

$$\mathscr{E}_{(n+1)}(A)(v)|_{\bigoplus_{j=0}^n \mathbb{L}(\mathrm{Sym}^j(W))} = \mathscr{E}_{(n)}(A)(v).$$

Corollary 5.2.12 (Exponentials in solutions of linear equations)**.** *Let $n \in \mathbb{N}$ and recall*

$$\exp_n \colon W \to S^{(n)}(W), \quad \exp_n \colon w \mapsto \sum_{j=0}^n \frac{1}{j!} w^{\otimes j}.$$

Let $t \mapsto \gamma_t \in V$ be a piecewise C^1 path and $Y_t^x(\gamma) \in W$ solve (5.1.1) *with linear vector fields $A \in \mathbb{L}(V \to \mathbb{L}(W))$ then $\exp_n(Y_t^x)$ solves the linear equation on symmetric tensors*

$$\begin{cases} \mathrm{d}\exp_n(Y_t^x(\gamma)) = \mathscr{E}_n(A)\big(\exp_n(Y_t^x(\gamma))\big)\mathrm{d}\gamma_t, \\ \exp_n(Y_0^x(\gamma)) = \exp_n(x) \in S^{(n)}(W). \end{cases}$$

Consequently,

$$\exp_n\left(Y_t^x(\gamma)\right) = \overline{\mathscr{E}_n(A)}\left[\mathbb{X}(\gamma|_{[0,t]})\right]\exp_n(x).$$

Proof. This follows immediately from Lemma 5.2.8 and the Definition 5.2.9 of $\mathscr{E}_n(A)$. □

Corollary 5.2.13. *Let $x \in W$, $A \in \mathbb{L}(V \to \mathbb{L}(W))$, $\gamma_t \in V$ a piecewise C^1 path and $Y_t^x(\gamma)$ the solution to (5.1.1). Then if $p \in \mathcal{P}^{(n)}(W)$,*

$$p\left(Y_t^x(\gamma)\right) = \left(\sum_{j=0}^{n} p^j(0)\right) \overline{\mathscr{E}_n(A)} \left[\mathbb{X}(\gamma|_{[0,t]})\right] \exp_n(x).$$

Proof. By Taylor's theorem about 0,

$$p(Y_t^x(\gamma)) = \left(\sum_{j=0}^{n} p^j(0)\right) \exp_n(Y_t^x(\gamma)).$$

And, by Corollary 5.2.12,

$$\exp_n\left(Y_t^x(\gamma)\right) = \overline{\mathscr{E}_n(A)} \left[\mathbb{X}(\gamma|_{[0,t]})\right] \exp_n(x).$$

Putting these together yields the desired result. □

Proposition 5.2.14. *Let $x \in W$, $A \in \mathbb{L}(V \to \mathbb{L}(W))$, $\gamma_t \in V$ a piecewise C^1 path and $Y_t^x(\gamma)$ the solution to (5.1.1). Then if $p \in \mathcal{P}^{(n)}(W)$*

$$\lim_{|D|\to 0} \mathbb{E}_D[p(Y_t^x)] = \left(\sum_{j=0}^{n} p^j(0)\right) \overline{\mathscr{E}_n(A)}\, \mathbb{E}[\mathbb{X}_{0,t}] \exp_n(x).$$

Proof. Recall the formula from Corollary 5.2.13

$$p\left(Y_t^x(\gamma)\right) = \left(\sum_{j=0}^{n} p^j(0)\right) \overline{\mathscr{E}_n(A)} \left[\mathbb{X}(\gamma|_{[0,t]})\right] \exp_n(x),$$

from which we see that $p\left(Y_t^x(\gamma)\right)$ is a continuous linear function of the signature of $\gamma|_{[0,t]}$. We may therefore apply Corollary 4.6.5 to obtain

$$\lim_{|D|\to 0} \mathbb{E}_D[p(Y_t^x)] = \left(\sum_{j=0}^{n} p^j(0)\right) \overline{\mathscr{E}_n(A)}\, \mathbb{E}[\mathbb{X}_{0,t}] \exp_n(x), \tag{5.2.1}$$

as desired. □

Remark 5.2.15. By the grouplike property, a polynomial function in the solution to a differential equation may be realised as a linear function of the signature of said solution. The substance of Proposition 5.2.14 is in the formula (5.2.1).

5.3 High-order semigroups through signed measures on paths

We show that the equation (5.2.1) defines a one-parameter semigroup.

Definition 5.3.1. *Let $x \in W$, $A \in \mathbb{L}(V \to \mathbb{L}(W))$, $\gamma_t \in V$ a piecewise C^1 path, $Y_t^x(\gamma)$ the solution to* (5.1.1)*, $t \geqslant 0, n \in \mathbb{N}$ and $p \in \mathcal{P}^{(n)}(W)$. In light of formula* (5.2.1) *in Proposition 5.2.14, define*

$$P_t^{(n)} p(x) := \lim_{|D| \to 0} \mathbb{E}_D[p(Y_t^x)] = \left(\sum_{j=0}^{n} p^j(0) \right) \overline{\mathscr{E}_n(A)}\, \mathbb{E}[\mathbb{X}_{0,t}] \exp_n(x).$$

Observe that for all $n \in \mathbb{N}$ $P_t^{(n+1)}|_{\mathcal{P}^{(n)}(W)} = P_t^{(n)}$.

Theorem 5.3.2. *For all $n \in \mathbb{N}$, $t \geqslant 0$, $P_t^{(n)} : \mathcal{P}^{(n)}(W) \to \mathcal{P}^{(n)}(W)$ is linear. Furthermore*

$$(\mathbb{R}_{\geqslant 0}, +) \longrightarrow \mathbb{L}(\mathcal{P}^{(n)}(W)), \quad t \mapsto P_t^{(n)}$$

is a uniformly continuous one-parameter semigroup. Namely, the following hold:

1) $P_0^{(n)} \equiv \mathrm{Id}$.

2) $P_s^{(n)} \circ P_t^{(n)} \equiv P_{s+t}^{(n)}$ *for all* $s, t \geqslant 0$.

3) $\lim_{t\to 0}\left\|P_t^{(n)} - \mathrm{Id}\right\| = 0.$

Proof. That $P_t^{(n)}$ linearly maps $\mathcal{P}^{(n)}(W)$ to $\mathcal{P}^{(n)}(W)$ is immediate from its definition. It remains to show the semigroup property.

1)

$$\begin{aligned} P_0^{(n)} p &= \left(\sum_{j=0}^{n} p^j(0)\right) \overline{\mathscr{E}_n(A)}\, \mathbb{E}[\mathbb{X}_{0,0}]\, \exp_n \\ &= \left(\sum_{j=0}^{n} p^j(0)\right) \overline{\mathscr{E}_n(A)}\mathbf{1}\, \exp_n \\ &= \left(\sum_{j=0}^{n} p^j(0)\right) \exp_n \equiv p, \end{aligned}$$

where $\mathbf{1} \in \bar{T}(V)$ is the unit and $\overline{\mathscr{E}_n(A)}\mathbf{1} \equiv \mathrm{Id} \in \mathbb{L}(T^n(W))$. So that $P_0^{(n)} \equiv \mathrm{Id}$, as desired.

2) For all $s, t \geqslant 0$, first note the identity

$$\mathbb{E}[\mathbb{X}_{0,s}]\, \mathbb{E}[\mathbb{X}_{0,t}] = \exp\left(\frac{s}{4!}M_4\right) \exp\left(\frac{t}{4!}M_4\right) = \exp\left(\frac{t+s}{4!}M_4\right) = \mathbb{E}[\mathbb{X}_{0,t+s}].$$

Then, since $\overline{\mathscr{E}_n(A)}$ is an algebra homomorphism,

$$\begin{aligned}
P_s^{(n)} P_t^{(n)} p &= P_s^{(n)} \left(\left(\sum_{j=0}^{n} p^j(0) \right) \overline{\mathscr{E}_n(A)} \left(\mathbb{E}[\mathbb{X}_{0,t}] \right) \exp_n \right) \\
&= \left(\sum_{j=0}^{n} p^j(0) \right) \overline{\mathscr{E}_n(A)} \left(\mathbb{E}[\mathbb{X}_{0,t}] \right) \circ \overline{\mathscr{E}_n(A)} \left(\mathbb{E}[\mathbb{X}_{0,s}] \right) \exp_n \\
&= \left(\sum_{j=0}^{n} p^j(0) \right) \overline{\mathscr{E}_n(A)} \left(\mathbb{E}[\mathbb{X}_{0,t}] \, \mathbb{E}[\mathbb{X}_{0,s}] \right) \exp_n \\
&= \left(\sum_{j=0}^{n} p^j(0) \right) \overline{\mathscr{E}_n(A)} \left(\mathbb{E}[\mathbb{X}_{0,t+s}] \right) \exp_n \\
&= P_{t+s}^{(n)} p,
\end{aligned}$$

as desired.

3) Since $\dim \left(\mathcal{P}^{(n)}(W) \right) < +\infty$, the strong operator topology is equivalent to the operator norm topology. Hence, it suffices to show $P_t^{(n)} p \xrightarrow{t \to 0} p$ for all $p \in \mathcal{P}^{(n)}(W)$. To this end, first note that $\mathbb{E}[\mathbb{X}_{0,t}] \xrightarrow{t \to 0} \mathbf{1}$ in $\bar{T}(V)$, then

$$P_t^{(n)} p = \left(\sum_{j=0}^{n} p^j(0) \right) \overline{\mathscr{E}_n(A)} \, \mathbb{E}[\mathbb{X}_{0,t}] \exp_n \xrightarrow{t \to 0} \left(\sum_{j=0}^{n} p^j(0) \right) \exp_n \equiv p,$$

as desired.

□

We intend to show that $P_t^{(n)}$ coincides with a high-order linear partial differential equation semigroup. We begin by studying its infinitesimal generator.

The generator of $P_t^{(n)}$

Recall that $M_k := \int_V x^{\otimes k} p(1,x)\mathrm{d}x$ the k^{th} moment of the heat kernel of $-\Delta^2$ at $t=1$. Recall also the linear isomorphism $\mathcal{P}^{(n)}(W) \xrightarrow{\simeq} S^{(n)}(W^*)$ identifying a polynomial with its derivatives at the origin so that we may view $P_t^{(n)}$ as a semigroup on $S^{(n)}(W^*)$.

Proposition 5.3.3. *Let $n \in \mathbb{N}$. $(P_t^{(n)})_{t\geqslant 0}$ has infinitesimal generator*

$$\mathcal{L}^{(n)} \in \mathbb{L}\left(S^{(n)}(W^*)\right), \quad \mathcal{L}^{(n)} : \alpha \mapsto \alpha \circ \frac{1}{4!}\left(\mathscr{E}_n(A)\right)^{\otimes 4}(M_4).$$

Proof. Denote by $\mathcal{L}^{(n)}$ the generator of $P_t^{(n)}$. Since $S^{(n)}(W^*)$ is finite-dimensional, $\mathrm{Dom}(\mathcal{L}^{(n)}) = S^{(n)}(W^*)$. Let $\alpha \in S^{(n)}(W^*)$ then

$$\begin{aligned} P_t^{(n)}\alpha - \alpha &= \alpha \circ \overline{\mathscr{E}_n(A)}\,\mathbb{E}[\mathbb{X}_{0,t}] - \alpha \\ &= \alpha \circ \exp\left(\frac{t}{4!}(\mathscr{E}_n(A))^{\otimes 4}(M_4)\right) - \alpha \\ &= \alpha \circ \frac{t}{4!}(\mathscr{E}_n(A))^{\otimes 4}(M_4) + \mathcal{O}(t^2). \end{aligned}$$

Hence

$$\mathcal{L}^{(n)}\alpha = \lim_{t\to 0}\frac{P_t^{(n)}\alpha - \alpha}{t} = \alpha \circ \frac{1}{4!}(\mathscr{E}_n(A))^{\otimes 4}(M_4),$$

as claimed. □

Remark 5.3.4. Recall that if $W_0, \ldots, W_n$ are linear spaces then $\bigoplus_{j=0}^n \mathbb{L}(W_j)$ denotes the linear space of maps

$$T : \bigoplus_{j=0}^n W_j \to \bigoplus_{j=0}^n W_j$$

such that $T(W_j) \subseteq W_j$ for all $j = 0, \ldots, n$. Namely, W_j is an invariant subspace of T for all $j = 0, \ldots, n$.

We remind the reader that

$$\mathscr{E}_n(A) : V \to \bigoplus_{j=0}^{n} \mathbb{L}(\mathrm{Sym}^j(W)) \subseteq \mathbb{L}(S^{(n)}(W))$$

so that

$$(\mathscr{E}_n(A))^{\otimes 4} : V^{\otimes 4} \to \bigoplus_{j=0}^{n} \mathbb{L}(\mathrm{Sym}^j(W)) \subseteq \mathbb{L}(S^{(n)}(W)).$$

In particular, for all $j \leqslant n$ and $\mathbf{v} \in V^{\otimes 4}$, $\mathrm{Sym}^j(W)$ is an invariant subspace for $(\mathscr{E}_n(A))^{\otimes 4}(\mathbf{v})$.

Understanding $\mathcal{L}^{(n)}$ as a differential operator

Definition 5.3.5. *Let $A \in \mathbb{L}(V \to \mathbb{L}(W))$ and $(e_1, \ldots, e_d)$ orthonormal coordinates for V. Set $A_i := A(e_i) \in \mathbb{L}(W)$. To each A_i we associate the vector field/derivation on W*

$$D(A_i) : C^\infty(W) \to C^\infty(W), \quad D(A_i)f(x) := \partial_t f(e^{tA_i}x)|_{t=0} = \langle \nabla f(x), A_i x \rangle.$$

We shall denote this derivation simply by A_i when there is no risk of confusion.

Definition 5.3.6. *Let $A_1, \ldots, A_d \in \mathbb{L}(W)$. Denote the differential operator*

$$\mathscr{L} := -\left(\sum_{k=1}^{d} D(A_k)^2\right)^2 : C^\infty(W) \to C^\infty(W).$$

Recall that $\mathcal{L}^{(n)}$ is the generator of $P_t^{(n)}$.

Proposition 5.3.7. *Let* $n \in \mathbb{N}$ *and* $p \in \mathcal{P}^{(n)}(W) \subseteq C^\infty(W)$ *then*

$$\mathscr{L}p = \mathcal{L}^{(n)}p.$$

Proof. Let $p \in \mathcal{P}^{(n)}(W)$ be of the form $p(x) = \alpha_j x^{\otimes j}$, $\alpha_j \in \mathrm{Sym}^j(W^*)$, $j \leqslant n$. We hope to show $\mathscr{L}p = \mathcal{L}^{(n)}p$. To this end, recalling the formula for M_4 (Lemma 4.5.4),

$$\begin{aligned}
\mathcal{L}^{(n)}\alpha_j &= \alpha_j \circ \frac{1}{4!}\left[\mathscr{E}_n(A)^{\otimes 4}(M_4)\right] \circ \pi_j \\
&= -\sum_{k,l=1}^{d} \alpha_j \circ \mathscr{A}_{(j)}^{\otimes 4}(e_k^{\otimes 2} \otimes e_l^{\otimes 2}) \\
&= -\sum_{k,l=1}^{d} \alpha_j \circ \mathscr{A}_{(j)}(e_k)^{\circ 2} \circ \mathscr{A}_{(j)}(e_l)^{\circ 2}.
\end{aligned}$$

On the other hand, since $D(A_k)$ is a linear differential operator, $D(A_k)x = A_k x$. Then by the chain rule

$$\begin{aligned}
D(A_k)x^{\otimes j} &= \sum_{i=0}^{j-1} x^{\otimes i} \otimes A_k x \otimes x^{\otimes (j-i-1)} \\
&= \left(\sum_{i=0}^{j-1} \mathrm{Id}_W^{\otimes i} \otimes A_k \otimes \mathrm{Id}_W^{\otimes (j-i-1)}\right) x^{\otimes j} \\
&= \mathscr{A}_{(j)}(e_k)x^{\otimes j}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{L}\alpha_j x^{\otimes j} &= -\alpha_j \left(\sum_{k,l=1}^{d} D(A_k)^{\circ 2} \circ D(A_l)^{\circ 2} \right) x^{\otimes j} \\
&= -\alpha_j \circ \left(\sum_{k,l=1}^{d} \mathscr{A}_{(j)}(e_k)^{\circ 2} \circ \mathscr{A}_{(j)}(e_l)^{\circ 2} \right) x^{\otimes j} \\
&= (\mathcal{L}^{(n)}\alpha_j) x^{\otimes j},
\end{aligned}$$

as desired. The result for general $p(x) = \sum_{j=0}^{n} \alpha_j x^{\otimes j}$ follows from linearity.

□

Remark 5.3.8. The above proposition allows for a quasi-probabilistic understanding of the operator $\mathscr{L}$ on $\mathcal{P}(W)$ through

$$\mathscr{L}p(x) = \lim_{t \to 0} \lim_{|D| \to 0} \frac{\mathbb{E}_D[p(Y_t^x)] - p(x)}{t}.$$

This is of particular interest when (an appropriate extension of) $\mathscr{L}$ is a closed operator and $\mathcal{P}(W)$ forms a core. In the following section, we replace the linear state space, W, with a compact Lie group, G. And, $\mathcal{P}(W)$ with a notion of polynomial on G, $\mathcal{P}(G)$ (see forthcoming Definition 5.4.10). It will turn out that $\mathcal{P}(G)$ will be a core for appropriate extensions of $\mathscr{L}$ to various standard Sobolev spaces on G (see forthcoming Lemma 5.4.39).

5.4 The bilaplacian on Lie groups

5.4.1 Matrix Lie groups

We treat matrix Lie groups in the sense of [20], to which we refer for further details. Recall that (over $\mathbb{R}$ or $\mathbb{C}$) a compact Lie group (resp., finite-dimensional Lie algebra) always has a faithful, finite-dimensional group (resp., Lie algebra) representation.

⚠The Lie groups considered shall be compact, connected, continuous matrix groups. The compactness assumption is used to prove a spectral gap property of Δ^2 in Corollary 5.4.21 and to prove uniform density of polynomials on Lie groups in Lemma 5.4.36.

Compact Lie groups have a well-developed theory. Basic examples include torii, the special orthogonal and unitary groups, the compact symplectic groups and compact forms of the exceptional Lie groups: G_2, F_4, E_6, E_7, and E_8.

Definition 5.4.1. *A matrix Lie group is a group, G, together with a faithful representation*

$$\sigma : G \hookrightarrow \mathrm{GL}(X)$$

such that $\sigma(G)$ is topologically closed in $\mathrm{GL}(X)$*.*

Definition 5.4.2. *To any matrix Lie group, G, we associate its Lie algebra*

$$\mathfrak{g} = \{A \in \mathbb{L}(X) \mid \exp(tA) \in G \; \forall \; t \in \mathbb{R}\}.$$

The Lie bracket of $\mathfrak{g}$ *is given by the commutator of matrices* $[A, B] := AB - BA$*. The dimension of* G *as a manifold is the dimension of the vector space* $\mathfrak{g}$*.*

We identify elements $A \in \mathfrak{g}$ *with vector fields on* G *by*

$$Af(g) := \partial_t f(ge^{At})|_{t=0},$$

where $g \in G$ *and* $f \in C^\infty(G \to \mathbb{R})$*.*

Definition 5.4.3 (Universal enveloping algebra)**.** *Define the universal enveloping algebra of* $\mathfrak{g}$ *to be the quotient space*

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathscr{I}(\mathfrak{g})$$

where $\mathscr{I}(\mathfrak{g})$ *is the two-sided ideal of the tensor algebra* $T(\mathfrak{g})$ *generated by tensors of the form* $x \otimes y - y \otimes x - [x, y]$ *for all* $x, y \in \mathfrak{g}$*.*

Lemma 5.4.4 (Universal property of $(\mathcal{U}(\mathfrak{g}), \iota)$)**.** *The canonical map* $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ *is injective. Any Lie map* $\phi : \mathfrak{g} \to B$ *into a unital algebra extends uniquely to an algebra homomorphism* $\hat{\phi} : \mathcal{U}(\mathfrak{g}) \to B$

$$\begin{array}{ccc} \mathfrak{g} & \overset{\iota}{\hookrightarrow} & \mathcal{U}(\mathfrak{g}) \\ & {\scriptstyle\phi}\searrow & \downarrow{\scriptstyle\hat{\phi}} \\ & & B \end{array} .$$

Proof. Theorem 9.7 [20]. □

Definition 5.4.5 ($\mathcal{U}(\mathfrak{g})$ as left-invariant differential operators on G)**.** *The map*

$$\mathfrak{g} \to \mathbb{L}(C^\infty(G)), \quad A \mapsto (f \mapsto Af)$$

is a Lie map into a unital algebra. Its extension to $\mathcal{U}(\mathfrak{g})$ identifies the universal enveloping algebra of $\mathfrak{g}$ with left-invariant differential operators on G.

For our purposes a useful characterisation of $\mathcal{U}(\mathfrak{g})$ is

Theorem (Poincaré-Birkhoff-Witt)**.** *Let $(A_1, \ldots, A_d)$ be a basis for $\mathfrak{g}$. Then a basis for $\mathcal{U}(\mathfrak{g})$ is given by the set of all $A_{i_1}^{j_1} \ldots A_{i_k}^{j_k}$ where $i_1 < \cdots < i_k$, $\{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$ and $j_1, \ldots, j_k \in \mathbb{N}$.*

⚠Henceforth, we shall equip $\mathfrak{g} \cong T_eG$ with an inner product making it a Hilbert space. By transporting to all other tangent spaces using left translation, we obtain a left-invariant Riemannian metric, ρ, on G.

Definition 5.4.6. *Let $(A_1, \ldots, A_d)$ be an orthonormal linear basis for $\mathfrak{g}$. We define the Laplacian and bilaplacian on (G, ρ) to be the elements $\Delta, \Delta^2 \in \mathcal{U}(\mathfrak{g})$ given by*

$$\Delta = \sum_{i=1}^{d} A_i^2, \quad \Delta^2 = \Delta\Delta = \left(\sum_{i=1}^{d} A_i^2\right)^2.$$

Δ and Δ^2 are independent of the choice of orthonormal basis.

5.4.2 Random differential equations on matrix Lie groups

Remark 5.4.7. We implicitly embed $\iota : \mathbb{L}(X) \hookrightarrow \mathbb{L}(\mathbb{L}(X))$ via left multiplication:

$$A \mapsto (B \mapsto A \circ B).$$

ι is an algebra homomorphism.

Definition 5.4.8 (Flow equation)**.** *Let $A : V \to \mathfrak{g}$ be a linear map, $g \in G$ and $\gamma_t \in V$ a piecewise C^1 path. Consider*

$$dY_t^g = A(Y_t^g)\mathrm{d}\gamma_t, \quad Y_0^g = g \in G. \tag{5.4.1}$$

Here $A(Y_t^g)\mathrm{d}\gamma_t$ means the composition of operators $A(\mathrm{d}\gamma_t) \circ Y_t^g$. Employing the embeddings $\mathfrak{g} \hookrightarrow \mathbb{L}(\mathbb{L}(X))$ and $G \hookrightarrow \mathbb{L}(\mathbb{L}(X))$, (5.4.1) *is of the form of the linear equation* (5.1.1) *with solution space $W = \mathbb{L}(X)$.*

Lemma 5.4.9. *Let $A \in \mathbb{L}(V \to \mathfrak{g})$, $g \in G$ and Y_t^g solve* (5.4.1)*. Then*

$$Y_t^g \in G \quad \textit{for all } t \geqslant 0.$$

Proof. We build a simple $G-$valued approximation to Y_t^g. Let $t \geqslant 0$. For each partition $D = (0 = t_0 < \cdots < t_m = t)$ of $[0, t]$ set

$$Y_t^{g,D} := \left(\prod_{t_i \in D} \mathbb{A} \exp(\gamma_{t_{i-1}, t_i}) \right) g.$$

To complete the proof, we consider the two claims

1) $Y_t^{g,D} \in G$ for all partitions D.

2) $\lim_{|D| \to 0} Y_t^{g,D} = Y_t^g$.

Notice first that, since G is closed, these two claims suffice to prove the result. Addressing the claims:

1). If $v \in V$ then, since $\mathbb{A}$ is a continuous algebra homomorphism and $Av \in \mathfrak{g}$, it holds that $\mathbb{A}\exp(v) = \exp(Av) \in G$. Thus

$$Y_t^{g,D} = \prod_{t_i \in D} \exp(A\gamma_{t_{i-1},t_i}) \in G,$$

as claimed.

2). Note the bounds uniform in D

$$\left\| \prod_{t_i \in D} \mathbb{A}\,\mathbb{X}(\gamma)_{t_{i-1},t_i} g \right\| = \| \mathbb{A}\,\mathbb{X}(\gamma)_{0,t} g\| = \|Y_t^g\| < +\infty$$

and

$$\left\| \prod_{t_i \in D} \mathbb{A}\exp(\gamma_{t_{i-1},t_i}) \right\| \leqslant \left\| \prod_{t_i \in D} \exp(A\gamma_{t_{i-1},t_i}) \right\| \leqslant \exp(2\|A\|\|\gamma\|_\infty).$$

Recall $\gamma_{s,t}^k$ denotes the level$-k$ signature of $\gamma|_{[s,t]}$. Then note also

$$\begin{aligned} \|\mathbb{A}\,\mathbb{X}(\gamma)_{t_{i-1},t_i} - \mathbb{A}\exp(\gamma_{t_{i-1},t_i})\| &= \left\| \sum_{k=2}^{\infty} A^{\otimes k} \left(\gamma_{t_{i-1},t_i}^k - \frac{\gamma_{t_{i-1},t_i}^{\otimes k}}{k!} \right) \right\| \\ &\leqslant \sum_{k=2}^{\infty} \frac{\|A\|^k}{k!} \cdot 2 \cdot \|\gamma\|_{\mathrm{Lip}^1}^k |t_i - t_{i-1}|^k \\ &= \mathcal{O}(|t_i - t_{i-1}|^2). \end{aligned}$$

Putting these three estimates together,

$$
\begin{aligned}
Y_t^g - Y_t^{g,D} &= \left(\prod_{t_i \in D} \mathbb{A}\, \mathbb{X}(\gamma)_{t_{i-1},t_i} - \prod_{t_i \in D} \mathbb{A} \exp(\gamma_{t_{i-1},t_i}) \right) g \\
&= \sum_{k=1}^{m} \left(\prod_{i=1}^{k-1} \mathbb{A}\, \mathbb{X}(\gamma)_{t_{i-1},t_i} \right) \left(\mathbb{A}\, \mathbb{X}(\gamma)_{t_{k-1},t_k} - \mathbb{A} \exp(\gamma_{t_{k-1},t_k}) \right) \cdot \\
&\qquad \cdot \left(\prod_{i=k+1}^{n} \mathbb{A} \exp(\gamma_{t_{i-1},t_i}) \right) g \\
&\xrightarrow{|D| \to 0} 0,
\end{aligned}
$$

completing the proof. □

5.4.3 Polynomials on Lie groups

In this subsection, we define polynomials on $G \subseteq \mathbb{L}(X)$, $\mathcal{P}(G)$, as a certain quotient of $\mathcal{P}(\mathbb{L}(X))$. Formally, we identify two polynomials on $\mathbb{L}(X)$ if and only if they coincide on G. We shall see that one may view elements of $\mathcal{P}(G)$ as smooth functions $G \to \mathbb{R}$ and further that this space of functions is uniformly dense in $\mathcal{C}(G)$. Further, the semigroup P_t is well-defined on this quotient (Proposition 5.4.17).

Definition 5.4.10 (Polynomials on G)**.** *Let $G \subseteq \mathbb{L}(X)$ be a set and $n \in \mathbb{N}$. Denote*

$$
\mathcal{I}_G^n := \{ f \in \mathcal{P}^{(n)}(\mathbb{L}(X)) : f|_G \equiv 0 \}, \quad \mathcal{I}_G := \{ f \in \mathcal{P}(\mathbb{L}(X)) : f|_G \equiv 0 \}.
$$

Then define the quotient linear space

$$
\mathcal{P}^{(n)}(G) := \mathcal{P}^{(n)}(\mathbb{L}(X)) / \mathcal{I}_G^n
$$

and the quotient algebra

$$\mathcal{P}(G) = \mathcal{P}^{(\infty)}(G) := \mathcal{P}(\mathbb{L}(X))/\mathcal{I}_G.$$

Remark 5.4.11. $\mathcal{I}_G$ is an ideal in $\mathcal{P}(\mathbb{L}(X))$ hence $\mathcal{P}(G)$ is an algebra.

Definition 5.4.12. *Let $n \in \mathbb{N} \cup \{\infty\}$. We view elements of $\mathcal{P}^{(n)}(G)$ as functions $G \to \mathbb{R}$ via*

$$[p] : g \mapsto p(g).$$

This is well-defined since $\forall f \in \mathcal{I}_G^n$ and $g \in G$, $f(g) = 0$.

If G is a Lie group, elements $\mathcal{P}^{(n)}(G)$ are in fact *smooth* functions $G \to \mathbb{R}$.

Definition 5.4.13. *Let $G \subseteq \mathbb{L}(X)$ be a matrix Lie group and $n \in \mathbb{N}$. Recall the Lie action of vector fields*

$$\mathfrak{g} \rightharpoonup \mathcal{P}^{(n)}(\mathbb{L}(X)), \quad Ap(g) := \partial_t p(e^{tA}g)|_{t=0}.$$

And, the algebra action

$$\mathcal{U}(\mathfrak{g}) \rightharpoonup \mathcal{P}^{(n)}(\mathbb{L}(X)).$$

Lemma 5.4.14. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $G \subset \mathbb{L}(X)$ a matrix group. The action $\mathcal{U}(\mathfrak{g}) \rightharpoonup \mathcal{P}^{(n)}(\mathbb{L}(X))$ is well-defined on the quotient space $\mathcal{P}^{(n)}(G) = \mathcal{P}^{(n)}(\mathbb{L}(X))/\mathcal{I}_G^n$.*

Proof. By the Poincaré-Birkhoff-Witt Theorem, we need only check the action $\mathfrak{g} \rightharpoonup \mathcal{P}^{(n)}(G)$ is well-defined. To this end, let $A \in \mathfrak{g}, g \in G$ and $f \in \mathcal{I}_G^n$ then

$$(Af)(g) = \partial_t f\left(ge^{tA}\right)|_{t=0} \equiv 0$$

since $f|_G \equiv 0$ and $ge^{tA} \in G$ for all t. Hence $A\left(\mathcal{I}_G^n\right) \subset \mathcal{I}_G^n$, as desired. □

The following is a useful characterisation of smoothness for functions on a Lie group.

Lemma 5.4.15. *Let $G \subseteq \mathbb{L}(X)$ be a matrix Lie group. Then a function $f : G \to \mathbb{R}$ is smooth iff for all $m \in \mathbb{N}$ and $X_1, \ldots, X_m \in \mathfrak{g}$,*

$$g \mapsto X_1 \ldots X_m f(g)$$

is well-defined and continuous.

Proof. See Theorem 1.3.5. in [1]. □

Finally, we see that elements of $\mathcal{P}^{(n)}(G)$ are smooth functions:

Corollary 5.4.16. *Let $G \subseteq \mathbb{L}(X)$ be a matrix Lie group. Then $\mathcal{P}^{(n)}(G)$ is a linear subspace of $C^\infty(G)$.*

Proof. Immediate from Lemma 5.4.15. □

Proposition 5.4.17. *Let $G \subset \mathbb{L}(X)$ be a matrix Lie group, $A \in \mathbb{L}(V \to \mathfrak{g})$ and $n \in \mathbb{N} \cup \{\infty\}$. Then the semigroup*

$$P_t^{(n)} : \mathcal{P}^{(n)}(\mathbb{L}(X)) \to \mathcal{P}^{(n)}(\mathbb{L}(X))$$

is well-defined on the quotient space $\mathcal{P}^{(n)}(G)$.

Proof. We need only show $P_t^{(n)}(\mathcal{I}_G^n) \subseteq \mathcal{I}_G^n$. To this end, let $f \in \mathcal{I}_G^n$ and $g \in G$ then by Lemma 5.4.9

$$P_t^{(n)} f(g) = \lim_{|D|\to 0} \mathbb{E}_D[f(Y_t^g)] \equiv 0,$$

so that $P_t^{(n)} f \in \mathcal{I}_G^n$, as desired. □

In light of this, we define

Definition 5.4.18. *Let $G \subset \mathbb{L}(X)$ be a matrix Lie group, $A \in \mathbb{L}(V \to \mathfrak{g})$ and $n \in \mathbb{N} \cup \{\infty\}$. Recall from Definition 5.3.1 the semigroup, $P_t^{(n)}$, on $\mathcal{P}^{(n)}(\mathbb{L}(X))$. From Proposition 5.4.17, there is an induced semigroup on the quotient space $\mathcal{P}^{(n)}(G)$.* We shall use the symbol $P_t^{(n)}$ to also denote this induced semigroup.

5.4.4 Analysis of the bilaplacian

Our intention in this subsection is to build the heat semigroup associated to the closure of the elliptic operator $\Delta_G^2 = \left(\sum_{i=1}^d A_i^2\right)^2$ on $L^2(G)$ via standard techniques from the theory of elliptic/parabolic partial differential equations. Elliptic here meaning that for all $g \in G$, $\mathrm{Span}\big(A_1(g), \ldots, A_d(g)\big) = T_g G$.

In Section 5.4.5, we shall see that this "analyst's semigroup" we build here in fact coincides with the semigroup we quasi-probabilistically build by developing $(\mathbb{P}_D)_D-$random paths onto G using the vector fields $(A_i)_i$ as in Proposition 5.4.17.

The material of this subsection is well-known and we refer to [38] and [10] for more details.

Recall the operators Δ, Δ^2 on $C^\infty(G)$ from Definition 5.4.6.

Lemma 5.4.19. *Let G be compact and connected, and denote by μ the bi-invariant measure on G with $\mu(G) = 1$. Δ and Δ^2 are densely-defined, symmetric operators on $C^\infty(G) \cap L^2_0(G, \mu)$.*

Proof. Left-invariance of the $(A_i)_{i=1}^d$ and right-invariance of μ ensures that Δ and Δ^2 are symmetric operators. □

Lemma 5.4.20 (Spectral gap for Δ)**.** *Let $(A_1, \ldots, A_d) \in \mathfrak{g}$ be an orthonormal linear basis and $\Delta = \sum_{i=1}^d A_i^2$. Then there exists a constant $c > 0$ such that for all $u \in C^\infty(G)$ satisfying $\int_G u(g)\mu(\mathrm{d}g) = 0$,*

$$\|u\|^2_{L^2(G,\mu)} \leqslant -c \langle u, \Delta u \rangle_{L^2(G,\mu)} .$$

Proof. See Lemma 3.8 page 24 in [21]. □

Corollary 5.4.21 (Spectral gap for Δ^2)**.** *Let the constant c be as in Lemma 5.4.20. Then for all $u \in C^\infty(G)$ satisfying $\int_G u(g)\mu(\mathrm{d}g) = 0$ it holds that*

$$\|u\|_2^2 \leqslant c^2 \langle u, \Delta^2 u \rangle.$$

Proof. By the spectral gap for Δ and the Cauchy-Schwartz inequality

$$\begin{aligned}\|u\|_2^2 &\leqslant -c\langle u, \Delta u\rangle \\ &\leqslant c\|u\|_2\|\Delta u\|_2.\end{aligned}$$

Dividing by $\|u\|_2$ and squaring both sides

$$\begin{aligned}\|u\|_2^2 &\leqslant c^2\|\Delta u\|_2^2 \\ &= c^2\langle u, \Delta^2 u\rangle,\end{aligned}$$

as desired. □

Definition 5.4.22 (Hilbert Sobolev spaces)**.** *For each $k \in \mathbb{N}$ define a norm $\|\cdot\|_{k,2}$ on $C^\infty(G)$ by*

$$\|f\|_{k,2} := \langle (I-\Delta)^k f, f\rangle_2^{1/2}.$$

We denote by $H^k(G)$ the completion of $C^\infty(G)$ with the metric associated to $\|\cdot\|_{k,2}$.

We shall further denote the closed linear subspace

$$H_0^k(G) := \left\{ f \in H^k(G) \mid \int f = 0 \right\}$$

and $L_0^2(G) := H_0^0(G)$.

Remark 5.4.23. Note that $H^0(G) = L^2(G)$ and that the spaces $H_0^k(G)$ are unrelated to Dirichlet boundary conditions, in constrast to commonly-used notation.

Proposition 5.4.24. *For all $k \in \mathbb{N}$ the norm $\| \cdot \|_{k,2}$ is equivalent to*

$$\|f\|^2 := \|f\|_2^2 + \sum_{l=1}^{k} \sum_{i_1,\dots,i_l=1}^{d} \|A_{i_1} \dots A_{i_l} f\|_2^2.$$

Proof. See Proposition 3.1.3. in [1]. □

Theorem (Rellich-Kondrachov)**.** *If $k < m$, the inclusions $H^m(G) \hookrightarrow H^k(G)$, $H_0^m(G) \hookrightarrow H_0^k(G)$ are compact.*

Proof. See Theorem 3.1.4. in [1]. □

Definition 5.4.25. *Let $u \in H^2(G)$, then there exists a sequence $u_n \in C^\infty(G)$ such that $\lim_{n\to\infty} \|u_n - u\|_{H^2(G)} = 0$. Δu_n is Cauchy in $L^2(G)$. We extend Δ to $H^2(G)$ via*

$$\Delta u := \lim_{n\to\infty} \Delta u_n \in L^2(G).$$

Thus Δ is a closed, self-adjoint operator on $L^2(G)$ with $\mathrm{Dom}(\Delta) = H^2(G)$.

Remark 5.4.26. Similarly, one can show that Δ^2 is closable with closure having domain $H^4(G)$.

Definition 5.4.27. *For $f \in L_0^2(G, \mu)$, a (weak) solution to the PDE*

$$\begin{cases} \Delta^2 u = f, \\ \int_G u(g)\mu(\mathrm{d}g) = 0, \end{cases}$$

is a $u \in H_0^2(G)$ such that for all $\varphi \in H_0^2(G)$

$$\mathcal{B}(u, \varphi) := \int_G \Delta u(g) \Delta\varphi(g) \mu(\mathrm{d}g) = \int_G f(g)\varphi(g)\mu(\mathrm{d}g). \tag{5.4.2}$$

Proposition 5.4.28. *For all $f \in L_0^2(G)$, there exists a unique solution $u \in H_0^2(G)$ to* (5.4.2)*. Furthermore, there exists a $C > 0$ independent of u and f such that*

$$\|u\|_{H_0^2(G)} \leqslant C\|f\|_2.$$

Proof. $\mathcal{B}$ is a symmetric bilinear form and is bounded above for the $\|\cdot\|_{2,2}$ norm. By the spectral gap (Lemma 5.4.20), we see that $\mathcal{B}$ is coercive (bounded below). Consequently, $\mathcal{B}$ is an inner product equivalent to $\langle\cdot,\cdot\rangle_{H_0^2(G)}$. Note that the RHS of (5.4.2), $\varphi \mapsto \int f\varphi \mathrm{d}\mu$, is a continuous linear functional with norm bounded by $\|f\|_2$. The result then follows from the Riesz representation theorem applied on the Hilbert space $\big(H_0^2(G), \mathcal{B}\big)$. □

Proposition 5.4.29. *For all $f \in L_0^2(G)$, denote by $Kf \in H_0^2(G)$ the solution to* (5.4.2)*. Then K is compact as an operator $L_0^2(G) \to L_0^2(G)$.*

Proof. Recall by the Rellich-Kondrachov theorem that $H_0^2(G)$ embeds compactly in $L_0^2(G)$. By Proposition 5.4.28, $K : L_0^2(G) \to H_0^2(G)$ is continuous. Since the composition of a continuous and a bounded operator is compact, the proof is complete. □

The follow special case of the spectral theorem will allow for a spectral decomposition of K and Δ^2.

Theorem (Hilbert-Schmidt theorem)**.** *Let $K : \mathcal{H} \to \mathcal{H}$ be a symmetric, compact, injective operator on an infinite-dimensional Hilbert space. Then there exists an orthonormal set $(\varphi_k)_{k\in\mathbb{N}}$ consisting of eigenvectors of K with* *real eigenvalues* μ_k *such that* $|\mu_k| \searrow 0$, (φ_k) *is an orthonormal basis for the image of* K *and*

$$Kf = \sum_{k=1}^{\infty} \mu_k \langle f, \varphi_k \rangle \varphi_k$$

for all $f \in \mathcal{H}$.

Lemma 5.4.30. *The solution operator,* K*, to* (5.4.2) *satisfies the hypotheses of the Hilbert-Schmidt theorem and therefore has the spectral decomposition*

$$K = \sum_{k=1}^{\infty} \mu_k \langle \cdot, \varphi_k \rangle \varphi_k.$$

Positivity ensures that $\mu_k > 0$.

Definition 5.4.31 (The closure of Δ^2)**.** *Injectivity of* K *allows to define*

$$K^{-1} : \mathrm{Range}(K) \to L_0^2(G).$$

Note

$$\mathrm{Range}(K) = \left\{ u \in L_0^2(G) : \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} \langle u, \varphi_k \rangle^2 < +\infty \right\}$$

so that $C^\infty(G) \cap L_0^2(G) \subset \mathrm{Range}(K)$ *and in fact* K^{-1} *is the closure of* Δ^2. *We shall use the symbol* Δ^2 *to also denote this closed operator with domain*

$$\mathrm{Dom}(\Delta^2) = \left\{ u \in L_0^2(G) : \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} \langle u, \varphi_k \rangle^2 < +\infty \right\} = H_0^4(G).$$

Set $\lambda_k = \frac{1}{\mu_k}$. *Noting that*

$$K\varphi_k = \mu_k \varphi_k \quad \Longleftrightarrow \quad \Delta^2 \varphi_k = \lambda_k \varphi_k$$

we obtain the spectral decomposition

$$\Delta^2 = \sum_{k=1}^{\infty} \lambda_k \langle \cdot, \varphi_k \rangle \varphi_k.$$

Definition 5.4.32. *For all $t \geqslant 0$, set*

$$e^{-t\Delta^2} : L_0^2(G) \to L_0^2(G), \quad e^{-t\Delta^2} := \sum_{k=1}^{\infty} e^{-t\lambda_k} \langle \cdot, \varphi_k \rangle \varphi_k.$$

Proposition 5.4.33. $\left(e^{-t\Delta^2}\right)_{t\geqslant 0}$ *is a strongly continuous one-parameter semigroup of contractions on $L_0^2(G)$ with generator $\left(-\Delta^2, \operatorname{Dom}\left(\Delta^2\right)\right)$.*

Remark 5.4.34. Note the orthogonal decomposition $L^2(G) = \langle 1 \rangle \bigoplus L_0^2(G)$. We extend $e^{-t\Delta^2}$ to a contractive C_0-semigroup on $L^2(G)$ by

$$e^{-t\Delta^2} 1 \equiv 1, \quad -\Delta^2 1 = 0.$$

Remark 5.4.35. $u : [0, \infty) \to L^2(G)$, $u(t) := e^{-t\Delta^2} u_0$ then solves the PDE

$$\begin{cases} \partial_t u = -\Delta^2 u, \\ u(0) = u_0 \in L^2(G), \end{cases}$$

in the sense that

1. $u \in C^1\big((0, \infty) \to L^2(G)\big) \cap C^0\big([0, \infty) \to L^2(G)\big)$.

2. For all $t > 0$, $u(t) \in \operatorname{Dom}(-\Delta^2)$ and $\partial_t u(t) = -\Delta^2 u(t)$.

3. $\lim_{t\to 0} \|u(t) - u_0\|_{L^2} = 0$.

5.4.5 Quasi-probabilistic representations of high-order PDE semigroups

In this final subsection, we show that the semigroup, P_t, constructed quasi-probabilistically coincides with $e^{-t\Delta^2}$ (see Theorem 5.4.37 for the precise statement). Informally, the solution of

$$\begin{cases} \partial_t u = -\left(\sum_{i=1}^d A_i^2\right)^2 u, \\ u(0,\cdot) = f(\cdot) \in L^2(G) \end{cases}$$

has the representation

$$u(t,g) = \lim_{n\to\infty} \lim_{|D|\to 0} \mathbb{E}_D[p_n(Y_t^g)],$$

where $\mathcal{P}(G) \ni p_n \xrightarrow{n\to\infty} f$ in $L^2(G)$.

Lemma 5.4.36. *$\mathcal{P}(G)$ is dense in $L^2(G)$.*

Proof. Since G is compact, this is immediate from the Stone-Weierstrass theorem. □

Theorem 5.4.37. *Let $G \hookrightarrow \mathbb{L}(X)$ be a compact, connected Lie group with $\dim(G) = \dim(V) = d$ and $A : V \to \mathfrak{g}$ a linear map such that $(A_1, \ldots, A_d)$ is an orthonormal basis for $\mathfrak{g}$. Let Δ^2 denote the closure of $\left(\sum_{i=1}^d A_i^2\right)^2$ in $L^2(G)$ (see Subsection 5.4.4 in particular Definition 5.4.31). Recall P_t the semigroup on $\mathcal{P}(G)$ from Definition 5.4.18. Then for all $t \geqslant 0$*

$$P_t = e^{-t\Delta^2}|_{\mathcal{P}(G)}.$$

Proof. Let $n \in \mathbb{N}$ and $p \in \mathcal{P}^{(n)}(G)$ then

$$P_t p|_{t=0} = p = e^{-t\Delta^2} p|_{t=0}.$$

Also, $\partial_t P_t p = P_t \mathcal{L}^{(n)} p$ and

$$\partial_t e^{-t\Delta^2} p = -e^{-t\Delta^2} \Delta^2 p = e^{-t\Delta^2} \left(\sum_{i=1}^{d} A_i^2 \right)^2 p = e^{-t\Delta^2} \mathcal{L}^{(n)} p.$$

By uniqueness of solutions to linear differential equations, $P_t = e^{-t\Delta^2}|_{\mathcal{P}(G)}$.

□

Corollary 5.4.38. *With assumptions as in Theorem 5.4.37, $P_t : \mathcal{P}(G) \to \mathcal{P}(G)$ uniquely extends to a contractive C_0-semigroup on $L^2(G)$.*

Corollary 5.4.39. *$\mathcal{P}(G)$ is a core for $-\Delta^2$.*

Proof. $\mathcal{P}(G)$ is dense in $L^2(G)$ and for all $t \geqslant 0$, $P_t(\mathcal{P}(G)) \subseteq \mathcal{P}(G)$. By Lemma A.2.1, $\mathcal{P}(G)$ is a core for $-\Delta^2$. □

Corollary 5.4.40. *Let $u_0 \in L^2(G)$ and $\mathcal{P}(G) \ni p_n \to u_0$ in $L^2(G)$. Then the function*

$$u : [0, \infty) \times G \to \mathbb{R}, \quad u(t, g) := \lim_{n\to\infty} \lim_{|D|\to 0} \mathbb{E}_D[p_n(Y_t^g)]$$

solves the PDE

$$\begin{cases} \partial_t u = -\Delta^2 u, \\ u(0) = u_0 \in L^2(G), \end{cases}$$

in the sense that

1. $u \in C^1\big((0,\infty) \to L^2(G)\big) \cap C^0\big([0,\infty) \to L^2(G)\big)$.

2. *For all* $t > 0$, $u(t) \in \mathrm{Dom}(-\Delta^2)$ *and* $\partial_t u(t) = -\Delta^2 u(t)$.

3. $\lim_{t\to 0} \|u(t) - u_0\|_{L^2} = 0$.

A Appendix

A.1

Definition/Lemma A.1.1 (Control functions)**.** *Let $g : [0,1] \to G^{\lfloor p \rfloor}(V)$ be a $p-$rough path. Define the so-called control function*

$$\omega : \{(s,t) \in \mathbb{R}^2 : 0 \leqslant s \leqslant t \leqslant 1\} \to \mathbb{R}, \quad \omega(s,t) := \|g|_{[s,t]}\|_p^p.$$

Then ω is a continuous function with $\omega(t,t) \equiv 0$ for all t, it is non-decreasing in t and non-increasing in s. Further, ω is super-additive:

$$\omega(s,u) + \omega(u,t) \leqslant \omega(s,t) \quad \text{for all } s \leqslant u \leqslant t. \tag{A.1.1}$$

Proof. See Section 1.2.2 in [33]. □

Lemma A.1.2. *Let $g \in \mathbb{G}_p(V)$ then the set $\{a \in \mathbb{G}_p(V) : a \subseteq g\}$ is homeomorphic to a (possibly degenerate) closed, bounded interval in $\mathbb{R}$. Further, the homeomorphism can be taken to be increasing.*

Proof. Define $\varphi : \mathrm{Hist}(g) \to [0, \|g\|_p]$, $\varphi : a \mapsto \|a\|_p$. We claim that φ is a homeomorphism. Choose a parameterisation $g : [0,1] \to G^{\lfloor p \rfloor}(V)$. Surjectivity of φ follows from the intermediate value theorem applied to the map $[0,1] \to [0, \|g\|_p]$, $t \mapsto \|g|_{[0,t]}\|_p$. As for injectivity, let $a_1, a_2 \in \mathrm{Hist}(g)$ with $a_1 \neq a_2$, then there exists $s,t \in [0,1]$ w.l.o.g. $s < t$ such that $a_1 = g|_{[0,s]}$ and $a_2 = g|_{[0,t]}$. Then, since $a_1 \subseteq a_2$ with $a_1 \neq a_2$, by Lemma 2.1.14 it holds that

$$\varphi(a_1) = \|a_1\|_p \neq \|a_2\|_p = \varphi(a_2)$$

hence φ is injective.

Since $\mathrm{Hist}(g)$ is Hausdorff and $[0, \|g\|_p]$ is compact, continuity of φ follows from continuity of φ^{-1}. Thus, it remains only to show that φ^{-1} is continuous. To this end, let $(\|g|_{[0,t_n]}\|)_n \in [0, \|g\|_p]$ be a convergent sequence:

$$\|g|_{[0,t_n]}\|_p \xrightarrow{n\to\infty} \|g|_{[0,t]}\|_p$$

and denote the control function

$$\omega(s,u) := \|g|_{[s,u]}\|_p^p$$

for $0 \leqslant s \leqslant u \leqslant 1$. Suppose for now that $t_n \leqslant t$ for all n. Then, using super-additivity of control functions (equation A.1.1) for inequality $(\star)$,

$$\begin{aligned} d(\varphi^{-1}(\|g|_{[0,t]}\|_p),\ &\varphi^{-1}(\|g|_{[0,t_n]}\|_p)) \\ &= \|g|_{[t_n,t]}\|_p \\ &\overset{(\star)}{\leqslant} \left(\|g|_{[0,t]}\|_p^p - \|g|_{[0,t_n]}\|_p^p\right)^{1/p} \xrightarrow{n\to\infty} 0, \end{aligned}$$

completing the proof for the case $t_n \leqslant t$ for all n. The case where $t < t_n$ follows an identical proof but with the roles of t_n and t exchanged. For the general case, one may simply separately consider two subsequences of t_n corresponding to terms less than t and terms greater than t. $\square$

Lemma A.1.3 (Excursions increase $p-$variation)**.** *Let $x, y, \epsilon \in \mathbb{G}_p(V)$ such that $\mathfrak{t}(x) = \mathfrak{i}(y) = \mathfrak{i}(\epsilon) = \mathfrak{t}(\epsilon)$ then*

$$\|x * y\|_p \leqslant \|x * \epsilon * y\|_p.$$

Proof. Immediate from definition of $p-$variation. □

Lemma A.1.4 (Proof of triangle ineq.). $d_p : \mathbb{G}_p(V) \times \mathbb{G}_p(V) \to [0, \infty)$ *given by*

$$d_p(x,y) := \begin{cases} \left\| \overleftarrow{x_2(y)} * y_2(x) \right\|_p, & \mathfrak{i}(x) = \mathfrak{i}(y), \\ \left\| \overleftarrow{x} * (\mathfrak{i}(x)\,\mathfrak{i}(y)^{-1} y) \right\|_p + d_{G^{\lfloor p \rfloor}(V)}\left(\mathfrak{i}(x), \mathfrak{i}(y)\right), & \mathfrak{i}(x) \neq \mathfrak{i}(y), \end{cases}$$

is a metric.

Proof. Suppose that $x, y, z \in \mathbb{G}_p(V)$ with $\mathfrak{i}(x) = \mathfrak{i}(y) = \mathfrak{i}(z)$. Since $x \wedge y$, $x \wedge z \subseteq x$, by Lemma A.1.2 either $x \wedge y \subseteq x \wedge z$ or $x \wedge z \subseteq x \wedge y$. Assume for now that $x \wedge y \subseteq x \wedge z$. Then note that we may factorise

$$\begin{aligned} &\overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(z)} * z_2(y) \\ &\quad = \overleftarrow{x_2(z)} * \overleftarrow{(x \wedge z)_2(y)} * y_2(x \wedge y) * \overleftarrow{y_2(x)} * (x \wedge z)_2(x \wedge y) * z_2(x), \end{aligned}$$

from which we can see that $\overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(z)} * z_2(y)$ is $\overleftarrow{x_2(z)} * z_2(x)$ with an excursion. Whence by Lemmas A.1.3 and 2.1.24

$$\begin{aligned} d_p(x,z) &= \left\| \overleftarrow{x_2(z)} * z_2(x) \right\|_p \\ &\leqslant \left\| \overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(z)} * z_2(y) \right\|_p \\ &\leqslant \left\| \overleftarrow{x_2(y)} * y_2(x) \right\|_p + \left\| \overleftarrow{y_2(z)} * z_2(y) \right\|_p \\ &= d_p(x,y) + d_p(y,z), \end{aligned}$$

as desired.

Suppose now instead that $x \wedge z \subseteq x \wedge y$. Idea of proof identical: find a clever factorisation then use Lemmas A.1.3 and A.1.5, but here you need to make two factorisations:

$$\overleftarrow{x_2(z)} * z_2(x) = \overleftarrow{x_2(y)} * \overleftarrow{(x \wedge y)_2(z)} * z_2(x)$$

and

$$\begin{aligned}&\overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(z)} * z_2(y)\\ &\qquad = \overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(x)} * \overleftarrow{(x \wedge y)_2(z)} * z_2(x).\end{aligned}$$

Then

$$\begin{aligned}d_p(x,z) &= \left\| \overleftarrow{x_2(z)} * z_2(x) \right\|_p\\ &= \left\| \overleftarrow{x_2(y)} * \overleftarrow{(x \wedge y)_2(z)} * z_2(x) \right\|_p\\ &\leqslant \left\| \overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(x)} * \overleftarrow{(x \wedge y)_2(z)} * z_2(x) \right\|_p\\ &= \left\| \overleftarrow{x_2(y)} * y_2(x) * \overleftarrow{y_2(z)} * z_2(y) \right\|_p\\ &\leqslant \left\| \overleftarrow{x_2(y)} * y_2(x) \right\|_p + \left\| \overleftarrow{y_2(z)} * z_2(y) \right\|_p\\ &= d_p(x,y) + d_p(y,z),\end{aligned}$$

as desired. □

Lemma A.1.5 ($p-$variation inequalities)**.** *Suppose* $x, y \in \mathbb{G}_p(V)$ *with* $\mathfrak{i}(y) = \mathfrak{t}(x)$ *then the following hold:*

1) $\|x * y\|_p \leqslant \|x\|_p + \|y\|_p$,

2) $\|x * y\|_p^p \geqslant \|x\|_p^p + \|y\|_p^p$,

3) $\|x * y\|_p \geqslant 2^{\frac{1-p}{p}} (\|x\|_p + \|y\|_p)$.

Proof. **1)** Pick parameterisations of x and y on $[0,1]$ and $[1,2]$ respectively. Define the continuous paths $X, Y : [0,2] \to T^{\lfloor p \rfloor}(V)$

$$X_t = \begin{cases} x_t, & t \in [0,1] \\ x_1, & t \in [1,2] \end{cases}, \qquad Y_t = \begin{cases} 0, & t \in [0,1] \\ y_t - x_1, & t \in [1,2] \end{cases}.$$

Observe that $X + Y = x * y$ so that

$$\|x * y\|_p = \|X + Y\|_p \leqslant \|X\|_p + \|Y\|_p = \|x\|_p + \|y\|_p$$

by the triangle inequality on the Banach space of continuous $T^{\lfloor p \rfloor}(V)-$valued paths with finite $p-$variation ([32]).

2) This is a well-known result in rough path theory: *superadditivity of control functions*. See for example [29].

3) This is immediately obtained from **2)** and the equivalence of ℓ^p and ℓ^1 norms on $\mathbb{R}^2$. □

Proposition A.1.6 (Integration of $\gamma-$cocyclic one-forms along rough paths)**.** *Let $\lfloor p \rfloor + 1 \geqslant \gamma > p \geqslant 1$. Let $g \in \mathbb{G}_p(V)$ and β a $\gamma-$cocyclic one-form along g. For each factorisation, $D_g = (g_0, \ldots, g_m)$, of g denote*

$$\int_{D_g} \beta := \sum_{i=0}^{m-1} \int_{g_{i+1}} \beta(g_0 * \cdots * g_i).$$

Then the following limit exists

$$\lim_{|D_g|\to 0}\int_{D_g}\beta =: \int_g \beta$$

and is called the integral of β along g. Furthermore,

$$\left|\int_g \beta - \int_{\{g_0,g\}}\beta\right| \leqslant C_{\gamma,p}|\beta|_\gamma\|g\|_p^\gamma$$

and consequently

$$\left|\int_g \beta\right| \leqslant \|\beta\|_\infty\|g\|_p + C_{\gamma,p}|\beta|_\gamma\|g\|_p^\gamma,$$

where $C_{\gamma,p} < +\infty$ depends only on γ and p.

Proof. Suppose that $m \geqslant 2$ and write $D_g = \big(g_0,\ldots,g_m\big)$. By Lemma 2.1.24 part **2)** and the pigeonhole principle, we may choose an $i \in \{1,\ldots,m\}$ such that

$$\|g_{i-1} * g_i\|_p^p \leqslant \frac{2}{m-1}\|g\|_p^p.$$

Then

$$\begin{aligned}
&\left|\int_{D_g}\beta - \int_{D_g\setminus\{g_i\}}\beta\right| \\
&= \left|\int_{g_i}\beta(g_0 * \cdots * g_{i-1}) + \int_{g_{i+1}}\beta(g_0 * \cdots * g_i) - \int_{g_i * g_{i+1}}\beta(g_0 * \cdots * g_{i-1})\right| \\
&\leqslant |\beta|_\gamma\|g_{i-1} * g_i\|_p^\gamma \leqslant |\beta|_\gamma\|g\|_p^\gamma\left(\frac{2}{m-1}\right)^{\gamma/p}.
\end{aligned}$$

Iterating this procedure until the factorisation has only two points left, we see

$$\left|\int_{D_g} \beta - \int_{\{g_0,g\}} \beta\right| \leqslant 2^{\gamma/p}\zeta\left(\frac{\gamma}{p}\right)|\beta|_\gamma \|g\|_p^\gamma$$

and

$$\left|\int_{D_g} \beta\right| \leqslant \|\beta\|_\infty \|g\|_p + 2^{\gamma/p}\zeta\left(\frac{\gamma}{p}\right)|\beta|_\gamma \|g\|_p^\gamma,$$

where ζ denotes the Riemann-zeta function. Set $C_{\gamma,p} = 2^{\gamma/p}\zeta\left(\frac{\gamma}{p}\right) < +\infty$. We obtain the uniform bounds

$$\sup_{D_g}\left|\int_{D_g} \beta - \int_{\{g_0,g\}} \beta\right| \leqslant C_{\gamma,p}|\beta|_\gamma \|g\|_p^\gamma \tag{A.1.2}$$

and

$$\sup_{D_g}\left|\int_{D_g} \beta\right| \leqslant \|\beta\|_\infty \|g\|_p + C_{\gamma,p}|\beta|_\gamma \|g\|_p^\gamma,$$

as desired.

As for existence of the limit, we hope to show that $\int_{D_g} \beta$ is Cauchy in $\mathbb{R}$. Suppose first that $D_g \subseteq D_g'$ in the sense that for all $x_i \in D_g$, there exist $y_j, \ldots, y_{j+m} \in D_g'$ such that $x_i = y_j * \cdots * y_{j+m}$ (we say that D_g' refines D_g). And, we denote $D_g' \cap \{x_i\} = (y_j, \ldots, y_{j+m})$ the rough path partition of x_i. Then,

$$\begin{aligned}\left|\int_{D'_g}\beta-\int_{D_g}\beta\right| &\leqslant \sum_{x_i\in D_g}\left|\int_{D'_g\cap\{x_i\}}\beta-\int_{\{\mathfrak{i}(x_i),x_i\}}\beta\right|\\ &\overset{(\star)}{\leqslant} C_{\gamma,p}|\beta|_\gamma \sum_{x_i\in D_g}\|x_i\|_p^\gamma\\ &\leqslant C_{\gamma,p}|\beta|_\gamma \max_j \|x_j\|_p^{\gamma-p}\sum_{x_i\in D_g}\|x_i\|_p^p\\ &\overset{(\star\star)}{\leqslant} C_{\gamma,p}|\beta|_\gamma \max_j \|x_j\|_p^{\gamma-p}\|g\|_p^p \xrightarrow{|D_g|\to 0} 0\end{aligned}$$

$(\star)$ follows from the maximal inequality (A.1.2) and $(\star\star)$ from item 2 in Lemma A.1.5.

If it does not hold that $D_g \subseteq D'_g$, we set $D''_g := D_g \vee D'_g$ and the result follows by noting

$$\left|\int_{D'_g}\beta-\int_{D_g}\beta\right| \leqslant \left|\int_{D_g}\beta-\int_{D''_g}\beta\right| + \left|\int_{D'_g}\beta-\int_{D''_g}\beta\right|.$$

□

Lemma A.1.7. *Let $N \in \mathbb{N}_{\geqslant 1}$, $1 \leqslant p \leqslant N$ and $\alpha \in \Omega(G^N(V))$ a cocyclic one-form. For all $g \in \mathbb{G}_p(V)$, define*

$$\beta : \mathrm{Hist}(g) \to \Omega(G^{\lfloor p\rfloor}(V)), \quad \beta(\xi)(\mathfrak{t}(\xi)) := \alpha(\mathfrak{t}(\xi^N))|_{T^{\lfloor p\rfloor}(V)}.$$

Then β is a $(\lfloor p\rfloor+1)-$cocyclic one-form along g and furthermore

$$\int_g \beta = \int_{g^N}\alpha.$$

Proof. Define for all $\xi \in \mathrm{Hist}(g)$

$$\beta(\xi)(\mathfrak{t}(\xi)) := \alpha(\mathfrak{t}(\xi^N))|_{T^{\lfloor p \rfloor}(V)}.$$

We show first that β is a $(\lfloor p \rfloor + 1)-$cocyclic one-form. To this end, let $\xi = g_1 * g_2 * g_3 \in \mathrm{Hist}(g)$ then

$$\begin{aligned}
&\int_{g_2} \beta(g_1) + \int_{g_3} \beta(g_1 * g_2) - \int_{g_2 * g_3} \beta(g_1) \\
&\qquad = \beta(g_1)(\mathfrak{t}(g_1))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2)] + \beta(g_1 * g_2)(\mathfrak{t}(g_2))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_3)] \\
&\qquad\qquad - \beta(g_1)(\mathfrak{t}(g_1))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2 * g_3)] \\
&\qquad \overset{(\star)}{=} \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2)] + \alpha(\mathfrak{t}(g_2^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_3)] \\
&\qquad\qquad - \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2 * g_3)] \\
&\qquad = \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant N}(g_2)] + \alpha(\mathfrak{t}(g_2^N))[\mathbb{X}^{\leqslant N}(g_3)] \\
&\qquad\qquad - \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant N}(g_2 * g_3)] \\
&\qquad\qquad + \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2) - \mathbb{X}^{\leqslant N}(g_2)] + \alpha(\mathfrak{t}(g_2^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_3) - \mathbb{X}^{\leqslant N}(g_3)] \\
&\qquad\qquad - \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2 * g_3) - \mathbb{X}^{\leqslant N}(g_2 * g_3)] \\
&\qquad \overset{(\star\star)}{=} \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2) - \mathbb{X}^{\leqslant N}(g_2)] + \alpha(\mathfrak{t}(g_2^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_3) - \mathbb{X}^{\leqslant N}(g_3)] \\
&\qquad\qquad - \alpha(\mathfrak{t}(g_1^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_2 * g_3) - \mathbb{X}^{\leqslant N}(g_2 * g_3)] \\
&\qquad \overset{(\star\star\star)}{=} \mathcal{O}(\|g_2 * g_3\|_p^{\lfloor p \rfloor + 1}),
\end{aligned}$$

so that β is indeed $(\lfloor p \rfloor + 1)-$cocyclic and whence the integral $\int_g \beta$ makes sense by Proposition 2.2.16. $(\star)$ holds by definition of α; $(\star\star)$ is from α satisfying the cocyclic property; and $(\star\star\star)$ follows from the fact that

$\|g_2\|_p, \|g_3\|_p \leqslant \|g_2 * g_3\|_p$ and that $\| \mathbb{X}^m(g)\| \leqslant C_{p,m}\|g\|_p^m$ for a $p-$rough path g and $m \in \mathbb{N}$ by Lyons' Extension theorem for rough paths (Theorem 2.1.5).

Finally, we show that $\int_g \beta = \int_{g^N} \alpha$. Indeed,

$$
\begin{aligned}
\int_g \beta &= \lim_{|D_g|\to 0} \sum_{g_i \in D_g} \int_{g_{i+1}} \beta(g_0 * \cdots * g_i) \\
&= \lim_{|D_g|\to 0} \sum_{g_i \in D_g} \alpha(\mathfrak{t}(g_i^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_{i+1})] \\
&= \lim_{|D_g|\to 0} \Big(\sum_{g_i \in D_g} \alpha(\mathfrak{t}(g_i^N))[\mathbb{X}^{\leqslant N}(g_{i+1})] \\
&\qquad - \sum_{g_i \in D_g} \alpha(\mathfrak{t}(g_i^N))[\mathbb{X}^{\leqslant \lfloor p \rfloor}(g_{i+1}) - \mathbb{X}^{\leqslant N}(g_{i+1})] \Big) \\
&= \int_{g^N} \alpha + \lim_{|D_g|\to 0} \sum_{g_i \in D_g} \mathcal{O}(\|g_{i+1}\|_p^{\lfloor p \rfloor + 1}) \\
&= \int_{g^N} \alpha,
\end{aligned}
$$

since $\lfloor p \rfloor + 1 > p$. □

A.2

Lemma A.2.1. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and $P_t : \mathcal{X} \to \mathcal{X}$ a C_0-semigroup with generator L. Suppose that $\mathcal{P} \subseteq \mathrm{Dom}(L)$ is dense in $\mathcal{X}$ and invariant under P_t, then $\mathcal{P}$ is a core for L.

Proof. Recall that $\mathrm{Dom}(L)$ is complete for the norm $\|f\|_{\mathrm{Dom}(L)} := \|f\| + \|Lf\|$. Denote by $\bar{\mathcal{P}}$ the closure of $\mathcal{P}$ in $\mathrm{Dom}(L)$ with respect to $\|\cdot\|_{\mathrm{Dom}(L)}$. We hope to show $\bar{\mathcal{P}} = \mathrm{Dom}(L)$.

If $f \in \mathrm{Dom}(L)$ then by density of $\mathcal{P}$ in X, there is a sequence $p_n \in \mathcal{P}$ such that $\|f - p_n\| \to 0$. Since $t \mapsto P_t p_n$ is continuous for $\|\cdot\|_{\mathrm{Dom}(L)}$ and $P_u(\mathcal{P}) \subseteq \mathcal{P}$,

$$\int_0^t P_u f_n \mathrm{d}u \in \bar{\mathcal{P}}.$$

And,

$$\begin{aligned}
&\lim_{n\to\infty} \left\| \int_0^t P_u p_n \mathrm{d}u - \int_0^t P_u f \mathrm{d}u \right\|_{\mathrm{Dom}(L)} \\
&\quad = \lim_{n\to\infty} \left\| \int_0^t P_u(p_n - f)\mathrm{d}u \right\| + \lim_{n\to\infty} \|P_t p_n - p_n - P_t f + f\| \\
&\quad = 0,
\end{aligned}$$

so that

$$\int_0^t P_u f \mathrm{d}u \in \bar{\mathcal{P}}.$$

Note also

$$\begin{aligned}
&\lim_{t\to 0} \left\| \frac{1}{t} \int_0^t P_u f \mathrm{d}u - f \right\|_{\mathrm{Dom}(L)} \\
&\quad = \lim_{t\to 0} \left\| \frac{1}{t} \int_0^t P_u f \mathrm{d}u - f \right\| + \lim_{t\to 0} \left\| \frac{1}{t}(P_t f - f) - Lf \right\| \\
&\quad = 0,
\end{aligned}$$

so that $f \in \bar{\mathcal{P}}$ and $\bar{\mathcal{P}} = \mathrm{Dom}(L)$, as required. □

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