



Generalised Graph Laplacians and Canonical Feynman Integrals with Kinematics

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Abstract: To any graph with external half-edges and internal masses, we associate canonical integrals which depend non-trivially on particle masses and momenta, and are always finite. They are generalised Feynman integrals which satisfy graphical relations obtained from contracting edges in graphs, and a coproduct involving both ultra-violet and infra-red subgraphs. Their integrands are defined by evaluating bi-invariant forms, which represent stable classes in the cohomology of the general linear group, on a generalised graph Laplacian matrix which depends on the external kinematics of a graph.

1. Introduction

In the paper [Bro21] we introduced canonical differential forms on moduli spaces of metric graphs, and showed how they provide a connection between the cohomology of the commutative graph complex, the algebraic K -theory of the integers, and Feynman integrals. In this paper, we extend this theory to the case of graphs with external momenta and masses, with an emphasis on physical aspects.

Let us first recall some background on Feynman integrals. Consider a connected graph G , with n external legs (or half-edges), h_G loops, and e_G internal edges. Every external leg represents an incoming particle with momentum $q_i \in \mathbb{R}^d$ subject to overall momentum conservation $\sum_{i=1}^n q_i = 0$. The number of spacetime dimensions d will be $d = 2$ or $d = 4$ in this paper. To each edge e one additionally associates one of a finite set of particle masses m_e , which are arbitrary real numbers.

For scalar theories, the parametric Feynman integral is the projective integral

$$I_G^{\text{Feyn}}(q, m) = \Gamma\left(e_G - \frac{h_G D}{2}\right) \int_{\sigma_G} \frac{1}{\Psi_G^{D/2}} \left(\frac{\Psi_G}{\Xi_G(q, m)} \right)^{e_G - h_G D/2} \Omega_G \quad (1.1)$$

where the edges of G are numbered from 1 to e_G , the form Ω_G is defined by

$$\Omega_G = \sum_{i=1}^{e_G} (-1)^i \alpha_i d\alpha_1 \wedge \cdots \wedge \widehat{d\alpha_i} \wedge \cdots \wedge d\alpha_{e_G},$$

and $\sigma_G = \{(\alpha_1 : \cdots : \alpha_{e_G}) \in \mathbb{P}^{e_G-1}(\mathbb{R}) : \alpha_i \geq 0\}$ is the coordinate simplex in projective space. The quantity D is typically an even integer, or, in the setting of dimensional regularisation, $D = 2k - \varepsilon$, for a small positive ε .

The integrand involves the ‘second Symanzik’ polynomial

$$\Xi_G(q, m) = \Phi_G(q) + \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Psi_G$$

which is expressed in terms of the two more basic polynomials $\Psi_G, \Phi_G(q)$, which are homogeneous in the $\alpha_e, e \in E_G$ of degrees $h_G, h_G + 1$ respectively. They are defined as sums over spanning forests in the graph G with 1 or 2 connected components (see §6.1). The integral (1.1) diverges in general and can be regularised in a variety of manners including, for example, Laurent expansion in the parameter ε .

For most quantum field theories of relevance for collider physics, one is led to consider a wider class of Feynman integrals, which in parametric form (see, e.g. [Gol19]) have the following general shape (omitting Γ -factors for simplicity):

$$\int_{\sigma_G} \frac{N_G}{\Psi_G^a \Xi_G^b(q, m)} \Omega_G \quad (1.2)$$

where $a, b \in \mathbb{Z}$ (or $a, b \in \mathbb{Z} + \varepsilon\mathbb{Z}$), and the numerator N_G is a polynomial in the parameters α_i with typically complicated coefficients.

Faced with the considerable difficulty in computing integrals (1.1) or (1.2), a common recent theme of research is to seek alternative theoretical frameworks in which the corresponding amplitudes are simpler and more highly structured, with a long term view to unearthing mathematical properties which are valid for general quantum field theories. Notable examples in this direction include the amplituhedron programme [AHT14], which organises certain amplitudes in $N = 4$ SYM according to geometric principles; string perturbation theory, which studies scattering amplitudes defined on punctured Riemann surfaces; or integrable ‘fishnet’ models [Zam80, GK16] which reduce to a small number of Feynman graphs.

In this paper, we introduce a special class of geometrically-defined integrals of the form (1.2), which are always finite irrespective of the graph G . They generalize [Bro21] to incorporate masses and momenta, and satisfy symmetry properties including a family of graphically-encoded relations. The latter involve both contraction of internal edges and also the ‘motivic’ coproduct of [Bro17] which has applications to the study of both UV and IR divergences [BBH20, AHHM22].

1.1. Canonical forms and their integrals. Our starting point is the moduli space $\mathcal{M}_g^{\text{trop}}$ of stable tropical curves of genus g , or more precisely the open locus

$$\left(\mathcal{M}_g^{\text{trop}} \right)_{w=0} \subset \mathcal{M}_g^{\text{trop}}$$

consisting of graphs whose vertices have weight zero. The quotient $(\mathcal{M}_g^{\text{trop}})_{w=0}/\mathbb{R}_{>0}^\times$ is isomorphic to the quotient of Culler–Vogtmann’s Outer space $\mathcal{O}_g$ [CV86] by the group of outer automorphisms $\text{Out}(F_g)$ of the free group F_g on g elements. Points in this space are equivalence classes of connected metric graphs (with no external legs), where a metric on a graph is an assignment of a length $\ell_e > 0$ to every internal edge, normalised so that the total length of all edges is 1.

The tropical Torelli map [BMV11] on $\mathcal{M}_g^{\text{trop}}$ restricts to a map

$$(\mathcal{M}_g^{\text{trop}})_{w=0} \longrightarrow \mathcal{P}_g/\text{GL}_g(\mathbb{Z}) \quad (1.3)$$

where $\mathcal{P}_g$ is the space of positive definite $g \times g$ symmetric matrices X , upon which $P \in \text{GL}_g(\mathbb{Z})$ acts via $X \mapsto P^T X P$. The map (1.3) assigns to a connected metric graph G the $\text{GL}_g(\mathbb{Z})$ -equivalence class of any choice of graph Laplacian matrix Λ_G . The latter is a $g \times g$ symmetric matrix whose determinant

$$\det \Lambda_G = \Psi_G$$

equals the first Symanzik, or Kirchhoff, graph polynomial which arises in the integrals (1.1). The graph Laplacian may be interpreted as a tropical version of the Riemann polarization form on a compact Riemann surface of genus g . Canonical forms on $(\mathcal{M}_g^{\text{trop}})_{w=0}$ are defined as follows. The forms $\text{tr}((X^{-1}dX)^{4k+1})$, for $k \geq 1$, are bi-invariant under left and right multiplication by $\text{GL}_g(\mathbb{R})$ and were shown by Borel to generate the stable cohomology of the symmetric space $\mathcal{P}_g/\text{GL}_g(\mathbb{Z})$. Their pull-backs $\text{tr}((\Lambda_G^{-1}d\Lambda_G)^{4k+1})$ along the map (1.3) defines a distinguished family of differential forms on the images of cells in $(\mathcal{M}_g^{\text{trop}})_{w=0}/\mathbb{R}_{>0}^\times$. The integrals of these forms are called canonical since they do not depend on any choices, and were studied in [Bro21]. In that paper it was shown that interesting examples of Feynman residues for vacuum diagrams (see [Sch10] for a survey) of relevance to quantum field theory can arise as canonical integrals. A natural question is whether general Feynman integrals with non-trivial kinematics are also amenable to a ‘canonical’ geometrical interpretation of this kind. This is addressed in this paper.

1.1.1. Canonical forms with kinematic dependence We consider a generalised moduli space $\mathcal{M}_{g,n,M}^{\text{trop}}$, first considered in [BM19], of weighted metric graphs which have n external legs labelled from $1, \dots, n$, and where each internal edge e is assigned one of M possible particle masses. When $M = 0$, this space coincides with the moduli space $\mathcal{M}_{g,n}^{\text{trop}}$ of n -marked tropical curves of genus g . Denote the open locus of graphs whose vertex weights are all zero by:

$$(\mathcal{M}_{g,n,M}^{\text{trop}})_{w=0} \subset \mathcal{M}_{g,n,M}^{\text{trop}}. \quad (1.4)$$

Consider in the first instance external particle momenta $q_1, \dots, q_n \in \mathbb{R}^2$ in two-dimensional Euclidean space, which we identify with the complex numbers, and any choice of M internal particle masses. For any choice of routing μ_e of the external momenta through the internal edges e of G , we consider a generalised graph Laplacian matrix (see (5.4)) whose determinant satisfies

$$\det \tilde{\Lambda}_G(\mu, m) = \Xi_G(q, m). \quad (1.5)$$

The generalised Laplacian $\tilde{\Lambda}_G(\mu, m)$ is a Hermitian $(g+1) \times (g+1)$ matrix (note that $g+1$ is the degree of the polynomial $\Xi_G(q, m)$). It is the tropical version of a regularised Hermitian polarization form on the cohomology of a compact Riemann surface with n punctures. From this perspective, the external particle momenta are interpreted as tropical versions of the residues of a differential form of the third kind. The case when the external momenta lie in $\mathbb{R}^4$ may be treated by replacing complex momenta with momenta in the ring of quaternions $\mathcal{Q} \cong \mathbb{R}^4$, leading to a quaternionic Hermitian graph Laplacian which satisfies

$$\text{Det } \tilde{\Lambda}_G(\mu, m) = (\Xi_G(q, m))^2, \quad (1.6)$$

where Det of a quaternionic matrix is the determinant of its complex adjoint. One of our key results (§9) is that the generalised graph Laplacian admits an asymptotic decomposition into block matrices as one approaches the boundary of the moduli space $(\mathcal{M}_{g,n,M}^{\text{trop}})_{w=0}$. This provides a geometric interpretation of asymptotic factorisation formulae for graph polynomials which are important for the study of infra-red and ultra-violet singularities of Feynman integrals [BBH20, AHHM22].

Define canonical forms¹ of the ‘first’ and ‘second’ kinds for all $k \geq 1$, by

$$\begin{aligned} \omega_G^{4k+1} &= \text{tr} \left(\left(\Lambda_G^{-1} d\Lambda_G \right)^{4k+1} \right) \\ \varpi_G^{2k+1} &= \text{tr} \left(\left(\tilde{\Lambda}_G(\mu, m)^{-1} d\tilde{\Lambda}_G(\mu, m) \right)^{2k+1} \right), \end{aligned} \quad (1.7)$$

when the external momenta lie in $\mathbb{R}^2 \cong \mathbb{C}$. In the case of quaternionic momenta in $\mathbb{R}^4 \cong \mathcal{Q}$, the definition of canonical forms of the first kind is unchanged, but the canonical forms of the second kind are now defined for all $k \geq 1$ by:

$$\varpi_G^{\mathcal{Q}, 4k+1} = \text{Tr} \left(\left(\tilde{\Lambda}_G(\mu, m)^{-1} d\tilde{\Lambda}_G(\mu, m) \right)^{4k+1} \right). \quad (1.8)$$

where the quaternionic trace Tr is the trace of the complex adjoint. A crucial property of canonical forms (Theorem 9.3) is that they factorise as one approaches infinity along the boundary faces of Feynman polytopes.

When the external momenta lie in $\mathbb{R}^2 \cong \mathbb{C}$, we define a *canonical form* ω_G to be any homogeneous polynomial in the forms (1.7). When the momenta lie in $\mathbb{R}^4 \cong \mathcal{Q}$, a (quaternionic) canonical form, denoted $\omega_G^{\mathcal{Q}}$, is a homogeneous polynomial in ω_G^{4k+1} and $\varpi_G^{\mathcal{Q}, 4k+1}$. For any connected graph G , and any such form ω_G of degree $e_G + 1$, we may consider the integral

$$I_G(\omega, q, m) = \int_{\sigma_G} \omega_G. \quad (1.9)$$

Since it is independent of all choices which go into defining generalised graph Laplacian matrices, we call it a *canonical integral*.

We prove the following facts about the integrals (1.9):

¹ The reader familiar with Chern–Simons theory may recognise canonical forms as a special case: for example, the Chern–Simons 3-form $\text{tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A)$ reduces to $-\frac{1}{3} \text{tr}((\Lambda^{-1} d\Lambda)^3)$ upon substituting $A = \Lambda^{-1} d\Lambda$. The higher degree forms are similar.

- (1) They are always finite.
- (2) They are generalised Feynman integrals of the form (1.2).
- (3) They satisfy graphical relations involving both the contraction of internal edges and a generalisation of the Connes–Kreimer coproduct which encodes not only ultraviolet, but also infra-red phenomena.

Thus the integrals (1.9), which may be loosely interpreted as certain ‘volumes’ of cells on a moduli space of tropical curves, pick out a distinguished class of generalised Feynman integrals with special properties. The reader may wish to turn to Sects. 11–13 for examples of canonical integrals and their relations.

Remark 1.1. The canonical forms (1.7), (1.8) arise from the graded exterior algebras of invariant differential forms I_{H_n} on the symmetric spaces associated to the following sequences of classical non-compact Lie groups H_n . The following table is extracted from [Bor74], page 265, and based on results of H. Cartan:

H_n	$\lim I_{H_n}$
$\mathrm{SL}_n(\mathbb{R})$	$E[x_{4k+1}, k \geq 1]$
$\mathrm{SL}_n(\mathbb{C})$	$E[x_{2k+1}, k \geq 1]$
$\mathrm{SL}_n(\mathbb{Q})$	$E[x_{4k+1}, k \geq 1]$

where $E[\{x_i\}]$ denotes the graded exterior algebra generated by elements x_i in odd degrees i , and the inverse limit of the spaces of forms I_{H_n} is taken in the category of graded exterior algebras. The generators are given by β_X^n in the case of real and complex matrices, and by $\mathrm{Tr}((X^{-1}dX)^n)$ in the quaternionic case. They always vanish when n is even, and additionally for $n \equiv 3 \pmod{4}$ in the case of $\mathbb{R}$ and $\mathbb{Q}$.

1.2. Discussion and questions for further research. It would be very interesting to find a momentum or position space formulation for the integrals (1.9), and to relate them more closely to the theory of graphical functions [GPS17, BS21] in the case when all masses vanish. The generalised graph Laplacian is a tropical version of ‘single-valued’ period integrals on a punctured Riemann sphere, which are prevalent in closed string perturbation theory, and suggests a possible connection with string perturbation theory. In a different direction, it was shown in [Bro21] that the canonical integrals for graphs without kinematics are closely connected to the homology of the commutative, even, graph complex. It would be interesting to find such a relation for the integrals (1.9) based on Corollary 10.4, for example. See also [BK20] for related interpretations of Feynman integrals.

We also expect the work in this paper to have applications to the motivic study of Feynman integrals. For example, the existence of the generalised graph Laplacian implies that the graph motives at a fixed loop order have a universal family, and the canonical forms provide universally-defined classes in the relative cohomology of graph hypersurface complements and the de Rham realisation of graph motives.

In particular, since canonical integrals are distinguished elements of period matrices associated to ordinary Feynman integrals, they should be related to the latter via differential equations and also via the motivic coaction. The Stokes’ relations proven here should thus enable one to transfer information about differential equations, periods, and infra-red singularities between graphs with different topologies.

Finally, it would also be interesting to study integrals of the form

$$\int_{\sigma_G} \left(\frac{\Psi_G^{h_G+1}}{\Xi_G^{h_G}(q, m)} \right)^\varepsilon \omega_G$$

in dimensional regularisation, where ω_G is canonical. They too will satisfy relations via Stokes' formula, but of a slightly different form to those considered here.

1.3. Contents. In §2 we recall some notations and conventions as well as some background on quaternionic matrices. In §3 we describe a moduli space of marked metric graphs with additional edge colourings, which encode particle masses. This space was previously introduced by Berghoff and Mühlbauer in [BM19]. After a brief discussion in §4 of momentum routings, and complex and quaternionic momenta, we define the generalised Laplacians in §5. In the case when momenta are quaternionic and all masses are zero, they coincide with matrices previously considered by Bloch and Kreimer in [BK10]. The way we arrived at the definition is explained in §14; namely by taking the tropical analogue of the single-valued integration pairing on the cohomology of a punctured Riemann surface. In §6 we recall the definitions of Symanzik polynomials, and prove a formula for Φ_G in terms of spanning forest polynomials, which is used to deduce (1.5) and (1.6). Starting in §8, we recall the definitions of bi-invariant differential forms. A key result is a formula (§9) for the asymptotic behaviour at infinity of graph Laplacian matrices, which is used to prove the convergence of canonical integrals in §10, along with the Stokes relations. It could also be used to prove (1.5) and (1.6). Sections 11 and 13 provide examples of canonical integrals for some well-studied Feynman diagrams, and §12 illustrates the general Stokes relations by deducing a graphical 5-term functional equation for the massive box diagram in 4 spacetime dimensions.

2. Notations and Conventions

Graphs in this paper are finite, and usually connected. They have a finite number n of external half-edges. The set of internal edges is denoted by $E(G)$, and the set of vertices by $V(G)$. We write $e_G = |E(G)|$ for the number of internal edges, and h_G for the number of loops of G . When an internal edge is directed, we denote by $s(e)$ its source, and $t(e)$ its target. An edge with $s(e) = t(e)$ is called a self-edge or tadpole.

2.1. Reminders on quaternions. Denote the ring of quaternions by

$$\mathcal{Q} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k,$$

where 1 is the unit, $i^2 = j^2 = k^2 = -1$, and $ij = -ji = k$, $ki = -ik = j$, $jk = -kj = i$. It admits a representation $\chi : \mathcal{Q} \rightarrow M_2(\mathbb{C})$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The ring $\mathcal{Q}$ has an anti-involution, denoted $x \mapsto \bar{x}$, which is induced by Hermitian conjugation $V \mapsto (\bar{V})^T$ on $M_2(\mathbb{C})$. The conjugate of $x = x_1 1 + x_2 i + x_3 j + x_4 k$ is $\bar{x} = x_1 1 - x_2 i - x_3 j - x_4 k$, and the square of the quaternion norm

$$\|x\|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2)1$$

coincides with the square of the Euclidean norm.

2.1.1. Quaternionic matrices Matrices with entries in the ring of quaternions share many properties with rings of matrices over a commutative ring (see [Zha97] for a survey). A convenient method for studying them is via the complex adjoint representation, which is the ring homomorphism

$$\chi : M_n(\mathcal{Q}) \longrightarrow M_n(M_2(\mathbb{C})), \quad (2.1)$$

induced by the representation χ defined above. One shows that a matrix $M \in M_n(\mathcal{Q})$ has a unique left inverse (which is necessarily also a right inverse) if and only if its image χ_M is invertible. However, the determinant of a quaternionic matrix is not defined in general, and its trace is not a similarity invariant, meaning that $\text{tr}(P^{-1}MP) \neq \text{tr}(M)$ for general P, M . For this reason it is customary to define the determinant and trace of $M \in M_n(\mathcal{Q})$ to be $\text{Det}(M) = \det \chi_M$ and $\text{Tr}(M) = \text{tr} \chi_M$ respectively. They take values in the complex numbers.

Two particular representations of quaternionic matrices are convenient:

- (i) If one identifies $M_n(M_2(\mathbb{C}))$ with $M_{2n}(\mathbb{C})$, then the image of χ is the set of matrices $\{V \in M_{2n}(\mathbb{C}) : \overline{V} = J_n V J_n^{-1}\}$, where J_n is the block diagonal matrix consisting of n copies of χ_j along the diagonal.
- (ii) If one identifies $M_n(M_2(\mathbb{C}))$ with $M_2(M_n(\mathbb{C}))$, then the image of χ consists of block matrices of the form:

$$\chi_{A+jB} = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}, \quad A, B \in M_n(\mathbb{C}), \quad (2.2)$$

since any quaternionic matrix may be uniquely written $A + jB$, with $A, B \in M_n(\mathbb{C})$. Here, and later, we identify the complex numbers with $\mathbb{R}1 + \mathbb{R}i \subset \mathcal{Q}$.

Either representation may be used to compute the quaternionic trace or determinant. We shall use (i) in the examples.

2.1.2. Moore determinant The first part of the following proposition implies the existence of the reduced Pfaffian norm Nrp [DR22, Tig99].

Proposition 2.1. *Let χ be the map (2.1).*

- (i) *Let $M \in M_n(\mathcal{Q})$ be a quaternionic Hermitian matrix, i.e., $M^T = \overline{M}$, where $\overline{M}$ denotes its quaternionic conjugate. Then χ_M is Hermitian and*

$$\text{Det}(M) = \det(\chi_M) = (F(\chi_M))^2, \quad (2.3)$$

where F is a polynomial in the entries of χ_M which is invariant under complex conjugation.

- (ii) *In the case when $M \in M_n(\mathbb{C}) \subset M_n(\mathcal{Q})$ has complex entries*

$$\text{Det}(M) = \det(\chi_M) = \det(M)^2. \quad (2.4)$$

Although we shall not use it, the Moore determinant $F = \text{Nrp}(M)$ is a canonical solution to (2.3) which satisfies $\text{Nrp}(M) = \det(M)$ for all $M \in M_n(\mathbb{C})$.

3. Moduli of Marked Metric Graphs with Masses and Momenta

We consider a moduli space of metric graphs with external legs and masses. It was previously studied in [BM19].

3.1. Graphs with external legs and massive edges.

3.1.1. Combinatorial masses Let $M \geq 0$ be a fixed integer. For a connected graph G , the assignment of a mass to each internal edge $e \in E(G)$ is encoded by

$$\underline{M} : E(G) \longrightarrow \{0, 1, \dots, M\}.$$

The labelling 0 denotes the zero mass.

3.1.2. Physical masses We fix a finite set of non-zero masses $m_i \in \mathbb{R}^\times$, for $i = 1, \dots, M$, and set $m_0 = 0$ to be the zero mass, i.e., $m : \{0, 1, \dots, M\} \rightarrow \mathbb{R}$. Only the mass squares m_i^2 will play a role in the theory. The mass m_i denotes the mass of a particle of type i , and there is no need for the m_i to be distinct. When drawing Feynman diagrams, our convention is to depict massless edges with a single line, and massive edges with a doubled line.

Two cases are of particular interest, namely the ‘massless case’ $M = 0$, when all edges have mass zero; and the ‘generic mass case’, when $M = e_G$, and $\underline{M} : E(G) \xrightarrow{\sim} \{1, \dots, e_G\}$ is a bijection.

3.1.3. Combinatorial momenta Let $n \geq 0$ be a fixed integer. If $n \geq 1$, the data of n external legs or ‘markings’ is given by a map

$$\underline{n} : \{1, \dots, n\} \longrightarrow V(G)$$

which can be realised combinatorially by attaching a half-edge, labelled i , to the vertex $\underline{n}(i)$. These external half-edges will always be oriented towards their endpoint, i.e., all external momenta are incoming. The map $\underline{n}$ does not need to be injective or surjective. In the case when $n = 0$, or when the image of $\underline{n}$ reduces to a single vertex, the associated graph is called a vacuum diagram, or momentumless.

3.1.4. Physical momenta Let $d \geq 0$. Consider n vectors in Euclidean space

$$Q = \{q_i \in \mathbb{R}^d \text{ for } i = 1, \dots, n\}$$

which satisfy momentum conservation

$$\sum_{i=1}^n q_i = 0. \quad (3.1)$$

The external half-edge i represents an incoming particle carrying momentum q_i . Given such a collection of momenta, we obtain a vector at every vertex

$$\underline{q} = (q_v)_v \in (\mathbb{R}^d)^{V(G)} \quad \text{where} \quad q_v = \sum_{i: \underline{n}(i)=v} q_i, \quad (3.2)$$

which is defined to be 0 if vertex v has no external edges attached to it, and otherwise is the sum of all incoming momenta at v .

3.2. *A category of graphs.* Let us fix integers $g, n, M \geq 0$.

Definition 3.1. A *marked, weighted metric graph with masses and momenta* is a tuple $(G, w, \underline{n}, \underline{M})$ where G is a connected graph, and

$$\begin{aligned} w &: V(G) \longrightarrow \mathbb{N} \cup \{0\} \\ \underline{n} &: \{1, \dots, n\} \longrightarrow V(G) \\ \underline{M} &: E(G) \longrightarrow \{0, \dots, M\} \end{aligned} \quad (3.3)$$

are the data of: a weighting w for every vertex; n external half lines $\underline{n}$; and an assignment $\underline{M}$ of a mass label to each internal edge of G . Its genus is

$$g(G) = h_G + \sum_{v \in V(G)} w(v)$$

where h_G is the first Betti (loop) number of G . Such a graph is called *stable* if, for every vertex $v \in V(G)$, one has the inequality $2w(v) - 2 + d(v) \geq 0$ where $d(v)$ is the total degree at v (including both internal and external half-edges).

An isomorphism of graphs $(G, w, \underline{n}, \underline{M})$ is an isomorphism of the underlying graphs which respects the additional data (3.3).

For every edge $e \in E(G)$, the contraction

$$(G, w, \underline{n}, \underline{M})/e = (G/e, w', \underline{n}', \underline{M}'),$$

of e is defined as follows. If e has distinct endpoints, then G/e is the graph with edge e removed and its endpoints $\{s(e), t(e)\}$ identified. As usual, the weight of the new vertex is the sum of weights $w(s(e)) + w(t(e))$, and $\underline{n}'$ is the composite of $\underline{n}$ with the quotient map $V(G) \rightarrow V(G') = V(G)/\{s(e) = t(e)\}$. The map $\underline{M}'$ is the restriction of $\underline{M}$ to the subset $E(G/e) = E(G) \setminus \{e\}$.

When e is a tadpole or self-edge, and its endpoints coincide $v = s(e) = t(e)$, the contraction G/e is simply the deletion of edge e . The weight of the vertex v in G/e is defined to be one more than its weight in G : $w'(v) = w(v) + 1$. The map $\underline{n}$ is unchanged, and $\underline{M}'$ is defined by restriction of $\underline{M}$ to $E(G) \setminus \{e\}$ as before.

Edge contraction preserves both the genus and the property of stability.

Definition 3.2. For fixed g, n, M , let $I_{g,n,M}$ denote the category whose objects $(G, w, \underline{n}, \underline{M})$ are connected stable graphs of genus g , and whose morphisms are generated by isomorphisms and edge contractions.

When $M = 0$, it is nothing other than the category of marked stable metric graphs of genus g with n external markings. The category $I_{g,n,M}$ has a final object given by the graph consisting of a single vertex of weight g , and n external legs.

3.3. *Associated moduli space.* Based on ideas in [ACP22, BBC+20], one can efficiently define the moduli space of marked metric graphs as follows.

Let $\mathcal{T}op$ denote the category whose objects are topological spaces and whose morphisms are continuous maps. Consider a functor $\mathbf{s} : I \rightarrow \mathcal{T}op$, where I is a finite (diagram) category. Its realisation $X_{\mathbf{s}}$ is the topological space

$$X_{\mathbf{s}} = \varinjlim_{i \in I} \mathbf{s}(i).$$

A morphism from $\mathbf{s} : I \rightarrow \mathcal{T}op$ to $\mathbf{s}' : J \rightarrow \mathcal{T}op$ is the data of a functor $j : I \rightarrow J$, together with a natural transformation from $\mathbf{s}$ to $\mathbf{s}' \circ j$. Such a morphism induces a continuous map $X_{\mathbf{s}} \rightarrow X_{\mathbf{s}'}$ between the associated topological realisations.

Definition 3.3. Define a functor

$$\mathbf{s} : I_{g,n,M}^{\text{opp}} \longrightarrow \mathcal{T}op \quad (3.4)$$

as follows. To the object $(G, w, \underline{n}, \underline{M})$ it associates the closed cell

$$\overline{C}(G) = \overline{C}(G, w, \underline{n}, \underline{M}) = \mathbb{R}_{\geq 0}^{E(G)},$$

given by the space of non-negative edge lengths ℓ_e , for $e \in E(G)$. It only depends on the underlying graph G . An isomorphism $(G, w, \underline{n}, \underline{M}) \cong (G', w', \underline{n}', \underline{M}')$ induces a bijection $E(G) \cong E(G')$, and a linear isomorphism on the corresponding cells. Contraction of an edge $e \in E(G)$ induces a natural inclusion of cells

$$\iota_e : \overline{C}(G/e) \hookrightarrow \overline{C}(G)$$

upon identifying $\overline{C}(G/e)$ with the subset $\ell_e = 0$.

Definition 3.4. Define the tropical moduli space to be its topological realisation

$$\mathcal{M}_{g,n,M}^{\text{trop}} = \varinjlim_{G \in I_{g,n,M}^{\text{opp}}} \overline{C}(G).$$

Although the cases when $g = 0$ are topologically interesting, they will play no role in this paper: in the absence of internal edges the canonical differential forms we shall consider are identically zero and only interesting for $g > 0$.

Similarly, one can consider the functor $L\mathbf{s} : I_{g,n,M}^{\text{opp}} \rightarrow \mathcal{T}op$ which assigns to every object $(G, w, \underline{n}, \underline{M})$ the closed simplex

$$\overline{\sigma}_G = \{(\ell_e)_e \in \mathbb{R}_{\geq 0}^{E(G)} \mid \sum_{e \in E(G)} \ell_e = 1\}$$

where the edge lengths are normalised to sum to 1. If $E(G)$ is empty, then $\overline{\sigma}_G$ is defined to be the empty set.

It defines a functor for the same reasons as above. The tropical moduli space $\mathcal{M}_{g,n,M}^{\text{trop}}$ is a cone whose cone point p is the cell associated to the final object in $I_{g,n,M}$. The link $\mathcal{M}_{g,n,M}^{\text{trop}} \setminus \{p\} / \mathbb{R}_{>0}^\times$ is isomorphic to the topological space

$$L\mathcal{M}_{g,n,M}^{\text{trop}} = \varinjlim_{G \in I_{g,n,M}^{\text{opp}}} \overline{\sigma}_G.$$

Remark 3.5. In the case when $M = 0$, one retrieves the definition of the moduli space $\mathcal{M}_{g,n}^{\text{trop}}$ of curves with n marked points. In general, the data of masses $\underline{M}$ is not completely anodine, since assigning distinct masses to the edges of a graph will in general reduce the size of its automorphism group.

The functor $\mathbf{s}$ defined above factors through a category of rational polyhedral cones as in [ACP22, BBC+20], which can in turn be upgraded to a category built out of affine spaces equipped with certain linear embeddings. There is a variant for the links $L\mathbf{s}$ involving projective spaces. These notions will be pursued elsewhere.

3.4. Homology and graph complexes. The homology of the spaces $\mathcal{M}_{g,n,M}^{\text{trop}}$ can be expressed in terms of complexes of graphs in the category $I_{g,n,M}$ via a variant of the classical result which relates cellular and singular homology (see [BBC+20, Proposition 2.1], [ACP22, Theorem 4.2]). We refer the reader to [BM19] for specific results about the homology of $\mathcal{M}_{g,n,M}^{\text{trop}}$ when $M > 0$.

4. Interpretations of Momenta

4.1. Genericity. Let G be a connected graph with external momenta $q_i \in \mathbb{R}^d$, and internal masses m_e . We shall say that momenta are *generic* if

$$\left(\sum_{i \in I} q_i \right)^2 \neq 0 \quad (4.1)$$

for all strict subsets I of the set of external half edges [Bro17, (1.18)]. It is automatic from our definitions that all non-trivial edge masses m_e are non-zero. Note that the genericity conditions for complex-valued momenta, which will not be considered in this paper, are more general than the Euclidean condition above.

4.2. Momentum routing. Choose an orientation on every edge of G , such that external edges are oriented inwards.

Definition 4.1. A *momentum routing* relative to the edge orientation is the data, for every internal edge $e \in E(G)$, of a momentum vector

$$\mu_e \in \mathbb{R}^d$$

such that momentum conservation holds at every vertex $v \in V(G)$, i.e.,

$$\sum_{s(e)=v} \mu_e - \sum_{t(e)=v} \mu_e = 0, \quad (4.2)$$

where the first sum is over all edges emanating from v , and the second is over all edges terminating at v . In the second sum we include external half edges, for which we write $\mu_e = q_e$ for the incoming external momentum along e .

To see that momentum routings exist, we can reformulate (4.2) as follows. Let us write $R = \mathbb{R}^d$, and consider the exact sequence:

$$0 \longrightarrow H_1(G; R) \longrightarrow R^{E_G} \xrightarrow{\partial} R^{V_G} \longrightarrow R \longrightarrow 0,$$

where $\partial(e) = t(e) - s(e)$. The data of external momenta defines a vector $\underline{q} \in R^{V_G}$ where $\underline{q} = (q_v)_{v \in V_G}$ via (3.2). The condition of momentum conservation (3.1) is precisely the statement that its image under the map $R^{V_G} \rightarrow R$ is zero. Therefore $\underline{q} \in \text{Im}(\partial)$. A choice of momentum routing is any element

$$\underline{\mu} = (\mu_e)_{e \in E_G} \in R^{E_G} \quad \text{such that} \quad \partial \underline{\mu} = \underline{q}.$$

Any two choices of momentum routing differ by an element of $H_1(G; R)$.

Note that one may always assume that $\mu_e = 0$ for every self-edge or tadpole e .

Remark 4.2. One can define a canonical momentum routing by demanding that $\underline{\mu} \in R^{EG}$ be orthogonal to the subspace $H_1(G; R)$ with respect to the inner product on R^{EG} for which the edges form an orthonormal basis. However, the canonical routing is not respected by contraction of edges.

4.3. Complex momenta. Suppose that the number of spacetime dimensions is $d = 2$, and the external momentum associated with the k th external half edge is

$$q_k \in \mathbb{R}^2. \quad (4.3)$$

Having fixed a choice of square root i of -1 , there is an identification

$$\begin{aligned} \mathbb{R}^2 &\cong \mathbb{C} \\ q &= (x, y) \mapsto \mathbf{q} = x + iy \end{aligned}$$

which is compatible with the Euclidean norm. In other words, if $q_k = (x_k, y_k)$ and $\mathbf{q}_k = x_k + iy_k$ for $k = 1, 2$, then the Euclidean inner product is

$$2 q_1 \cdot q_2 = 2 (x_1 x_2 + y_1 y_2) = 2 \operatorname{Re}(\mathbf{q}_1 \bar{\mathbf{q}}_2) = \mathbf{q}_1 \bar{\mathbf{q}}_2 + \bar{\mathbf{q}}_1 \mathbf{q}_2 \quad (4.4)$$

and $\|q_1\|^2 = \mathbf{q}_1 \bar{\mathbf{q}}_1$. We can always assume (4.3) holds when the total number of external momenta is at most 3, since by momentum conservation, the Feynman integral only depends on two external momenta, which lie in a Euclidean plane.

4.4. Quaternionic momenta. Suppose that G is a connected graph as above, but the number of Euclidean dimensions is $d = 4$. The external momenta satisfy

$$q_k = (q_k^{(1)}, \dots, q_k^{(4)}) \in \mathbb{R}^4. \quad (4.5)$$

Let us denote the corresponding quaternions §2.1 by $\mathbf{q}_k \in \mathcal{Q}$, where

$$\mathbf{q}_k = q_k^{(1)} \mathbf{1} + q_k^{(2)} \mathbf{i} + q_k^{(3)} \mathbf{j} + q_k^{(4)} \mathbf{k},$$

which we may identify with their image in $M_2(\mathbb{C})$ under χ (2.1). The inner product will be written interchangeably in either notation via the identity:

$$\mathbf{q}_i \bar{\mathbf{q}}_j + \mathbf{q}_j \bar{\mathbf{q}}_i = (2q_i \cdot q_j) \mathbf{1} = 2 \sum_{k=1}^4 q_i^{(k)} q_j^{(k)}. \quad (4.6)$$

5. Generalised Laplacians for Graphs with External Legs

We define a generalised graph Laplacian of a connected graph G with external momenta subject to momentum conservation, and internal particle masses. In the following, let $\mathcal{C}$ be a ring, which is not necessarily commutative, equipped with an anti-commutative involution $\iota : \mathcal{C} \rightarrow \mathcal{C}$ such that its invariant subspace $\mathcal{R} = \{c \in \mathcal{C} : \iota(c) = c\}$ is commutative.

Examples include:

- (1) $\mathcal{C} = \mathbb{C}$ the complex numbers, equipped with complex conjugation.

- (2) $\mathcal{C} = \mathbb{Q}$ the ring of quaternions, with quaternionic conjugation.
- (3) $\mathcal{C} = \mathbb{Q}[\mu_e, \bar{\mu}_e, m_i]$ an abstract ring of kinematic variables, where ι is the $\mathbb{Q}$ linear map acting trivially on m_i , and such that $\iota(\mu_e) = \bar{\mu}_e$, $\iota(\bar{\mu}_e) = \mu_e$.
- (4) $\mathcal{C} = \mathbb{Q}[\lambda_e, \bar{\lambda}_e, v_e, \bar{v}_e, m_i]$ as in (3), where $\iota(\lambda_e) = \bar{\lambda}_e$, and $\iota(v_e) = -v_e$, $\iota(\bar{v}_e) = -\bar{v}_e$. The multiplication is given by viewing $\mu_e = \lambda_e + jv_e$ as an abstract quaternion, or equivalently as a 2×2 matrix $\begin{pmatrix} \lambda_e & v_e \\ -\bar{v}_e & \bar{\lambda}_e \end{pmatrix}$.

In the first instance, the reader may wish to consider only the case (1) where $\mathcal{C} = \mathbb{C}$. The generalised graph Laplacian requires the data of:

- an external momentum vector $\underline{q} \in \mathcal{C}^{V_G}$ subject to momentum conservation
- a vector of internal masses $\underline{m} \in \mathcal{C}^{E_G}$ which is invariant under ι .

It is defined in two stages: first we define the graph Laplacian for zero masses, and then modify it very slightly to take into account the internal masses.

5.1. Definition of the generalised graph Laplacian. Choose an orientation on the internal edges of G . Consider the exact sequence

$$0 \longrightarrow H_1(G; \mathcal{C}) \xrightarrow{\mathcal{H}_G} \mathcal{C}^{E_G} \xrightarrow{\partial} \mathcal{C}^{V_G} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Since the momentum vector $\underline{q} \in \mathcal{C}^{V_G}$ maps to zero, we obtain an extension

$$0 \longrightarrow H_1(G; \mathcal{C}) \longrightarrow \mathcal{E}_G \longrightarrow \mathcal{C} \longrightarrow 0 \quad (5.1)$$

where $\mathcal{E}_G = \{\underline{\mu} \in \mathcal{C}^{E_G} : \partial(\underline{\mu}) \in \mathcal{C}\underline{q}\}$ and the element $1 \in \mathcal{C}$ in the right-hand term is identified with $\underline{q} \in \mathcal{C}^{V_G}$. We shall call this the *extension by momenta*.

Consider the pairing satisfying

$$\begin{aligned} \mathcal{C}^{E_G} \times \mathcal{C}^{E_G} &\longrightarrow \mathcal{C}[x_e, e \in E_G] \\ \langle \alpha e, \beta e' \rangle_0 &= \alpha \iota(\beta) x_e \delta_{e, e'}. \end{aligned} \quad (5.2)$$

It restricts to a Hermitian form (with respect to the involution ι) on $\mathcal{E}_G$, which we also denote by $\langle \rangle_0$, and which may be interpreted as a $\mathcal{C}$ -linear map $y \mapsto (x \mapsto \langle y, x \rangle_0)$ which we call the ‘graph Laplacian for zero masses’:

$$\tilde{\Lambda}_G(\mu, 0) : \mathcal{E}_G \rightarrow \text{Hom}_{(\mathcal{C}, \iota)}(\mathcal{E}_G, \mathcal{C}[x_e]).$$

Here, $\text{Hom}_{(\mathcal{C}, \iota)}(A, B)$ denotes anti-linear maps $\phi : A \rightarrow B$ satisfying $\phi(\lambda x) = \iota(\lambda)\phi(x)$ for all $\lambda \in \mathcal{C}$. Equivalently, (5.2) may be viewed as a $\mathcal{C}$ -linear map:

$$D_G : \mathcal{C}^{E_G} \longrightarrow \text{Hom}_{(\mathcal{C}, \iota)}(\mathcal{C}^{E_G}, \mathcal{C}[x_e])$$

using which one has the identity $\tilde{\Lambda}_G(\mu, 0) = \tilde{\mathcal{H}}_G^* D_G \tilde{\mathcal{H}}_G$ where $\tilde{\mathcal{H}}_G : \mathcal{E}_G \rightarrow \mathcal{C}^{E_G}$ is inclusion and $X \mapsto X^*$ denotes Hermitian conjugation, i.e., $X^* = \iota(X)^T$.

The subspace $H_1(G; \mathcal{C}) \subset \mathcal{E}_G$ contains the subspace $H_1(G; \mathbb{Z})$, which is invariant under the involution ι . Restricting the Hermitian form $\langle \rangle_0$ to the space $H_1(G; \mathbb{Z})$ therefore defines a symmetric bilinear form which is nothing other than the usual graph Laplacian Λ_G , which depends neither on masses nor momenta.

The masses of G are encoded by a single element

$$\underline{m} = \sum_e m_e e \in \mathcal{C}^{E_G}$$

where $m_e = \iota(m_e)$. The norm of this element with respect to (5.2) is

$$\|\underline{m}\|^2 = \langle \sum_e m_e e, \sum_e m_e e \rangle = \sum_e m_e^2 x_e$$

We define the *mass-correction* Hermitian inner product $\langle \cdot, \cdot \rangle_m$ on $\mathcal{E}_G$ to be the unique inner product vanishing identically on $H_1(G; \mathcal{C})$:

$$\langle H_1(G; \mathcal{C}), \mathcal{E}_G \rangle_m = \langle \mathcal{E}_G, H_1(G; \mathcal{C}) \rangle_m = 0$$

but for which $\langle f, f \rangle_m = \|\underline{m}\|^2$ for any $f \in \mathcal{E}_G$ whose image is $1 \in \mathcal{C}$ under the natural map in (5.1). Consider the sum of the two Hermitian forms

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0 + \langle \cdot, \cdot \rangle_m.$$

The associated $\mathcal{C}$ -linear map defines the generalised graph Laplacian

$$\tilde{\Lambda}_G(\mu, m) : \mathcal{E}_G \longrightarrow \text{Hom}_{(\mathcal{C}, \iota)}(\mathcal{E}_G, \mathcal{C}[x_e]). \quad (5.3)$$

5.2. Generalised graph Laplacian matrix. The generalised graph Laplacian may be computed by splitting (5.1). Number the edges of G from $1, \dots, N$. Choose a basis $c_1, \dots, c_h$ of $H_1(G; \mathbb{Z})$ and a routing of edge momenta $\underline{\mu} \in \mathcal{C}^{E_G}$.

Define a matrix $\tilde{\mathcal{H}}_G$ with N rows, $h+1$ columns and entries in $\mathcal{C}$ as follows. If $1 \leq k \leq h$ then the entries $(\tilde{\mathcal{H}}_G)_{e,k}$ count the number of times (with multiplicity) that the oriented edge e appears in the cycle c_k . Thus the first h columns consists of the usual *edge-cycle incidence matrix* $\mathcal{H}_G$. The final column is defined by

$$(\tilde{\mathcal{H}}_G)_{e,h+1} = \mu_e \quad 1 \leq e \leq N.$$

Now consider the matrix $\tilde{M}_G$ with N rows and $h+1$ columns

$$(\tilde{M}_G)_{e,c} = \begin{cases} m_e & \text{if } c = h+1, \\ 0 & \text{otherwise.} \end{cases}.$$

Thus $\tilde{M}_G$ is zero except for the last column, which is the vector of masses m_e .

The generalised graph Laplacian matrix is defined to be

$$\tilde{\Lambda}_G(q, m) = \tilde{\mathcal{H}}_G^* D_G \tilde{\mathcal{H}}_G + \tilde{M}_G^* D_G \tilde{M}_G,$$

where $X \mapsto X^*$ denotes Hermitian conjugation, i.e., $X^* = \iota(X)^T$. Since $\tilde{M}_G$ has entries which are invariant under ι , we have $\tilde{M}_G^* = \tilde{M}_G^T$. Note that the matrix $\tilde{M}_G^* D_G \tilde{M}_G$ only has a single non-zero entry in the bottom right-hand corner:

$$\tilde{M}_G^* D_G \tilde{M}_G = \sum_{e=1}^N \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & m_e^2 \alpha_e \end{pmatrix}.$$

In general, the matrix $\tilde{\Lambda}_G$ has the following block-matrix form:

$$\tilde{\Lambda}_G = \left(\begin{array}{c|c} \Lambda_G & \begin{matrix} c_1(\alpha, \mu) \\ \vdots \\ c_h(\alpha, \mu) \end{matrix} \\ \hline \begin{matrix} c_1(\alpha, \iota(\mu)) & \dots & c_h(\alpha, \iota(\mu)) \end{matrix} & X_G \end{array} \right), \quad (5.4)$$

where $\Lambda_G = \mathcal{H}_G^T D \mathcal{H}_G$ is the usual graph Laplacian matrix,

$$X_G = \sum_{e=1}^N (\mu_e \iota(\mu_e) + m_e^2) \alpha_e, \quad (5.5)$$

and where for any $\mathbb{Z}$ -linear combination of edges $c = \sum_{e \in E(G)} p_e e$, we write

$$c(\alpha, \mu) = \sum_{e \in E(G)} p_e \mu_e \alpha_e,$$

and $c(\alpha, \iota(\mu))$ for its image under ι .

5.2.1. Change of edge orientations Changing the orientation of an edge does not modify the matrix Λ_G . To see this, observe that switching the orientation of any number of edges amounts to multiplying the matrix $\tilde{\mathcal{H}}_G$ on the left by a diagonal matrix E with entries in $\{1, -1\}$: reversing the orientation of edge e results in a change of sign for the associated momentum μ_e . The claim follows since $E^* D_G E = D_G$ and since M_G does not depend on the edge orientations.

5.2.2. Change of basis The matrix $\tilde{\Lambda}_G$ depends on the choice of basis for the homology $H_1(G; \mathbb{Z})$. Changing basis modifies the matrix $\tilde{\Lambda}_G$ by

$$\tilde{\Lambda}'_G = \tilde{P}^T \tilde{\Lambda}_G \tilde{P} \quad (5.6)$$

where $\tilde{P}$ is an invertible block matrix of the form

$$\tilde{P} = \left(\begin{array}{c|c} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline P & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array} \right), \quad (5.7)$$

where $P \in \text{GL}(H_1(G; \mathbb{Z}))$ is a change of basis matrix with integer entries. In particular, one has $\det(P)^2 = \det(\tilde{P})^2 = 1$.

5.2.3. Change of momentum routing Two momentum routings $\bar{\mu}, \bar{\mu}'$ differ by an element of $H_1(G; \mathbb{C})$. Therefore changing momentum routing (or changing a choice of splitting of (5.1)) is equivalent to modifying the matrix $\tilde{\Lambda}_G$ by

$$\tilde{\Lambda}'_G = \tilde{S}^* \tilde{\Lambda}_G \tilde{S} \quad (5.8)$$

where $\tilde{S}$ is an invertible block matrix of the form

$$\tilde{S} = \left(\begin{array}{c|c} & \begin{matrix} s_1 \\ \vdots \\ s_h \end{matrix} \\ \hline I & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array} \right), \quad (5.9)$$

where I denotes the $h_G \times h_G$ identity matrix and $s_1, \dots, s_h \in \mathbb{C}$.

5.2.4. Group of indeterminacy The graph Laplacian matrix $\tilde{\Lambda}_G$ is therefore ambiguous up to the action of the semi-direct product

$$\mathcal{C}^g \rtimes \mathrm{GL}_g(\mathbb{Z}) \quad (5.10)$$

where $P \in \mathrm{GL}_g(\mathbb{Z})$ acts on $S \in \mathcal{C}^g$ via $S \mapsto PSP^{-1}$. Note that it only involves the additive structure on $\mathcal{C}$, which is commutative, and not the multiplicative structure, which may not be. The group (5.10) can be viewed as the group of automorphisms of the extension $\mathcal{E}_G$ (5.1) which respects the integral structure $H_1(G; \mathbb{Z}) \subset H_1(G; \mathcal{C})$. Recall that $\mathcal{C}^g \rtimes \mathrm{GL}_g(\mathbb{Z})$ is the set $(S, P) \in \mathcal{C}^g \times \mathrm{GL}_g(\mathbb{Z})$, which can be identified with the set of matrices

$$SP = \left(\begin{array}{c|c} & \begin{matrix} s_1 \\ \vdots \\ s_h \end{matrix} \\ \hline P & \\ \hline 0 \dots 0 & 1 \end{array} \right), \quad (5.11)$$

equipped with the group law $(S_1, P_1)(S_2, P_2) = (S_1 P_1 S_2 P_1^{-1}, P_1 P_2)$.

5.3. Complex adjoint of the generalised graph Laplacian. Let G be a connected graph as before, and suppose that the external momenta lie in $\mathcal{C} = \mathcal{Q}$, the ring of quaternions. The generalised graph Laplacian $\tilde{\Lambda}_G(\mu, m)$ is a quaternionic Hermitian form of rank $g + 1$, and may be represented by a $(g + 1) \times (g + 1)$ Hermitian matrix with entries in $\mathcal{Q}$. Its image under the complex adjoint map (2.1) is a Hermitian complex matrix of rank $2g + 2$.

In block matrix form §2.1.1(i), one may represent a choice of complex adjoint graph Laplacian in the form $M_{2n}(\mathbb{C}) \cong M_n(M_2(\mathbb{C}))$ as follows:

$$\chi_{\tilde{\Lambda}_G(\mu, m)} = \left(\begin{array}{c|cc} & c_1(\lambda, \alpha) & c_1(v, \alpha) \\ & -c_1(\bar{v}, \alpha) & c_1(\bar{\lambda}, \alpha) \\ & & \vdots \\ \hline c_1(\bar{\lambda}, \alpha) & -c_1(v, \alpha) & \dots \\ c_1(\bar{v}, \alpha) & c_1(\lambda, \alpha) & \dots \\ \hline & X & 0 \\ & 0 & X \end{array} \right) \quad (5.12)$$

where the quaternionic momenta are $\mu_e = \lambda_e + \mathbf{j} v_e$, for $\lambda_e, v_e \in \mathbb{C}$, where

$$\chi_{\Lambda_G} = \left(\begin{array}{cccc} \Lambda_{1,1} & 0 & \Lambda_{1,2} & 0 & \dots \\ 0 & \Lambda_{1,1} & 0 & \Lambda_{1,2} & \\ \Lambda_{2,1} & 0 & & & \\ 0 & \Lambda_{2,1} & \ddots & & \vdots \\ \vdots & & & \Lambda_{g,g} & 0 \\ & & \dots & 0 & \Lambda_{g,g} \end{array} \right) \quad (5.13)$$

is the complex adjoint of the ordinary graph Laplacian $\Lambda_G = (\Lambda_{ij})_{ij}$, and

$$X = \sum_{e=1}^{e_G} \left(m_e^2 + \lambda_e \bar{\lambda}_e + v_e \bar{v}_e \right) \alpha_e.$$

In the massless case $m_e = 0$, these matrices were considered in [BK10].

the reader may prefer to use the equivalent matrix representation §2.1.1(ii).

6. Symanzik and Spanning Forest Polynomials

We recall the definition of Symanzik polynomials and relate the polynomial $\phi_G(q)$ to a sum of spanning forest polynomials relative to a choice of momentum routing. For further background on Symanzik polynomials and some of their basic properties, we refer the reader to [Bro17, §1], for example.

6.1. Definition of Symanzik polynomials. We recall the definition of the graph polynomials which arise in Feynman integrals in parametric form, since conventions can vary slightly in the literature. Let $d \geq 0$ be any non-negative integer. In this section, G is any connected graph with external edges and external momenta $q_i \in \mathbb{R}^d$. Every internal edge e is assigned a mass $m_e \in \mathbb{R}$.

6.1.1. 1st Symanzik polynomial Recall that the first Symanzik polynomial (also known as the ‘Kirchhoff’ or graph polynomial), is defined by

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e \quad (6.1)$$

where the sum is over all spanning trees of G . It does not depend on the external half-edges of G . It is equal to the determinant of the ordinary graph Laplacian:

$$\det \Lambda_G = \Psi_G. \quad (6.2)$$

It is homogeneous of degree h_G and is not identically zero.

6.1.2. 2nd Symanzik polynomial The second Symanzik polynomial depends on external momenta. It is the homogeneous polynomial of degree $h_G + 1$ defined by

$$\Phi_G(q) = - \sum_{T_1, T_2 \subset G} (q^{T_1} \cdot q^{T_2}) \prod_{e \notin T_1 \cup T_2} \alpha_e \quad (6.3)$$

where the sum is over all spanning 2-trees of G (or spanning forests with exactly two connected components), and q_{T_i} denotes the total momentum entering T_i . By momentum conservation $q^{T_1} + q^{T_2} = 0$.

6.1.3. Dependence on masses The ‘2nd Symanzik’ polynomial which occurs in the (Euclidean) parametric representation of Feynman integrals is

$$\Xi_G(q, m) = \Phi_G(q) + \left(\sum_{e \in E_G} \alpha_e m_e^2 \right) \Psi_G \quad (6.4)$$

which is also homogenous of degree $h_G + 1$.

6.2. Vanishing of Symanzik polynomials.

Definition 6.1. [Bro17, §1.4] A subgraph $\gamma \subset E_G$ is *momentum-spanning* if all non-zero external momenta q_v of G meet vertices v which lie in a single connected component of γ . It is called *mass-spanning* if it contains all massive edges of G .

It is called *mass-momentum spanning* (or *m.m.* for short) if both hold.

It is shown in [Bro17, §1.6] that if $\gamma \subset E_G$ and genericity (4.1) holds, then

$$\Xi_G(q, m) \Big|_{\alpha_e=0, e \in \gamma} = \Xi_{G/\gamma}(q, m) \quad (6.5)$$

is identically zero if and only if γ is an *m.m.* subgraph. In this paper we work in the Euclidean region (*loc. cit.* §1.7) which implies that the coefficients of every monomial in the variables α_i in Ξ_G are non-negative (this would not be true in Minkowski space). If γ is not *m.m.* then G/γ has a scale and one shows that

$$\Xi_{G/\gamma}(q, m) > 0$$

in the region $\alpha_e > 0$, for all $e \in E_{G/\gamma} = E_G \setminus E_\gamma$.

6.3. Spanning forest and Dodgson polynomials. The second Symanzik can be expressed using spanning forest polynomials, which were defined in [BY11].

6.3.1. Spanning forest polynomials

Definition 6.2. Let $P = P_1 \cup \dots \cup P_r$ be any partition of a subset of vertices of G into disjoint sets. The associated spanning forest polynomial is defined by

$$\phi_G^P = \sum_{\mathcal{F}} \prod_{e \in E(G) \setminus \mathcal{F}} \alpha_e$$

where the sum is over spanning forests $\mathcal{F} = T_1 \cup \dots \cup T_r$ of G with exactly r connected components T_i such that

$$V(T_i) \cap P = P_i \quad \text{for } 1 \leq i \leq r.$$

In other words, each tree T_i contains the vertices in P_i and no other vertices of P . Trees consisting of a single vertex are permitted. We set $\phi_G^P = 0$ if the sets P_i are not all disjoint.

6.3.2. An identity for the 2nd Symanzik polynomial

Proposition 6.3. Let G be as in the previous paragraph. For any choice of momentum routing, the second Symanzik polynomial satisfies:

$$\begin{aligned} \Phi_G(q) &= \sum_{e \in E(G)} \mu_e \cdot \mu_e \alpha_e \Psi_{G//e} + \sum_{\substack{e, f \in E(G) \\ e \neq f}} \mu_e \cdot \mu_f \alpha_e \alpha_f \\ &\quad \left(\phi_G^{\{s(e), s(f)\}, \{t(e), t(f)\}} - \phi_G^{\{s(e), t(f)\}, \{s(f), t(e)\}} \right), \end{aligned} \quad (6.6)$$

where we recall that $\phi_G^{P_1, P_2}$ is zero if $P_1 \cap P_2$ is non-empty. Note that the second summand is symmetric in e and f and therefore each term is counted twice.

Proof. Let $T = T_1 \cup T_2$ be a spanning forest in G with two connected components T_1, T_2 . By momentum conservation (4.2), the momentum flowing into T_1 equals

$$q^{T_1} = -q^{T_2} = \sum_{t(e) \in T_1} \mu_e - \sum_{s(e) \in T_1} \mu_e \quad (6.7)$$

where each sum is over all internal edges $e \in E(G)$ such that $e \cup T$ has no loops, and is hence a spanning tree of G . In particular, e has one endpoint in $V(T_1)$ and the other in $V(T_2)$.

Suppose that e is an edge with distinct endpoints. It follows from (6.7) that the coefficient of $\mu_e \cdot \mu_e$ in $-q^{T_1} \cdot q^{T_2}$ is 1 if $e \cup T_1 \cup T_2$ is a tree, and 0 otherwise. By (6.3), the coefficient of $\mu_e \cdot \mu_e$ in $\Phi_G(q)$ is therefore equal to

$$\sum_T \prod_{e' \notin T} \alpha_{e'}$$

where the sum is over the set of spanning forests T with two connected components such that $T \cup e$ is a tree. The map $T \mapsto T \cup e$ is a bijection between the latter and the set of spanning trees which contain e , which in turn are in one-to-one correspondence with the set of spanning trees in G/e . We conclude that the coefficient of $\mu_e \cdot \mu_e$ in the formula (6.3) is precisely $\alpha_e \Psi_{G/e}$ (which also equals $\alpha_e \Psi_{G//e}$).

In the case when e is a loop, $\Psi_{G//e}$ vanishes. By (6.7), μ_e does not contribute to q^{T_1} and therefore does not appear in $\Phi_G(q)$. This establishes the first line of (6.6).

Now consider two distinct edges e, f and assume first of all that they have in total four distinct endpoints. Formula (6.7) implies that $\mu_e \cdot \mu_f$ does not arise in $-q^{T_1} \cdot q^{T_2}$ unless both $T \cup e$ and $T \cup f$ are trees. In this situation the coefficient is 1 if $s(e), s(f)$ lie in the same connected component of T , and -1 if they lie in different components. In the former case, this means precisely that $T = T_1 \cup T_2$ is a spanning forest with 2 components such that the vertices of one component $V(T_1)$ contains $\{s(e), s(f)\}$ and the other, $V(T_2)$, contains $\{t(e), t(f)\}$. In the latter case, $V(T_1)$ contains $\{s(e), t(f)\}$ and $V(T_2)$ contains $\{s(f), t(e)\}$, or vice-versa. From (6.3), it follows that the coefficient of $\mu_e \cdot \mu_f$ in $\Phi_G(q)$ is exactly

$$2\alpha_e \alpha_f \left(\phi_G^{\{s(e), s(f)\}, \{t(e), t(f)\}} - \phi_G^{\{s(e), t(f)\}, \{s(f), t(e)\}} \right). \quad (6.8)$$

It remains to check that (6.8) remains true in the cases when $\{s(e), s(f), t(e), t(f)\}$ has 3 or fewer elements. The reader may verify this in every situation (e.g., $s(e) = s(f)$, $s(e) = t(f)$, etc), using a similar argument and the fact that $\phi_G^{P_1, P_2}$ vanishes if P_1 and P_2 are not disjoint. Note that if e or f is a loop (i.e., $s(e) = t(e)$) then μ_e does not occur in (6.7), and indeed, both of the spanning forest polynomials in (6.8) are defined to be zero in this case. $\square$

6.3.3. Spanning forests and minors of graph Laplacians The difference of spanning forest polynomials occurring in Proposition 6.3 can be interpreted as a ‘Dodgson polynomial’ $\Psi_G^{e, f}$. We will make frequent use of the following well-known lemma.

Lemma 6.4. *Let G be a connected graph with directed edges. Suppose that $e_1, \dots, e_n$ are distinct edges such that $G \setminus \{e_1, \dots, e_n\}$ is connected. Then $h = h_G \geq n$ and we may choose cycles $c_1, \dots, c_h \in \mathbb{Z}^{E_G}$ representing a basis for $H_1(G; \mathbb{Z})$ such that:*

$$(\mathcal{H}_G)_{e_i, c_j} = \delta_{i, j} \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq h_G.$$

In other words, for each $1 \leq i \leq n$, the edge e_i appears in cycle c_i with coefficient 1 and occurs in no other cycle.

For a matrix M , let $M^{i,j}$ denote the matrix with row i and column j removed.

Lemma 6.5. *Let G be a connected graph with directed edges.*

(i) *Let $e \in E(G)$ be an edge such that $G \setminus \{e\}$ is connected (i.e., e is not a bridge). Choose cycles $c_1, \dots, c_h$ as in the previous lemma and let Λ_G be the graph Laplacian matrix with respect to this basis. Then we have*

$$\det \Lambda_G^{1,1} = \Psi_{G \setminus e}.$$

(ii) *Let $e_1, e_2 \in E(G)$ be two edges such that $G \setminus \{e_1, e_2\}$ is connected, and choose cycles $c_1, \dots, c_h$ as in the previous lemma. Then*

$$\det \Lambda_G^{1,2} = \pm \left(\phi_G^{\{s_1, s_2\}, \{t_1, t_2\}} - \phi_G^{\{s_1, t_2\}, \{s_2, t_1\}} \right).$$

where s_i is the source of the directed edge e_i , and t_i its target, for $i = 1, 2$.

Proof. (i) The variable α_e only occurs in the matrix Λ_G in a single place, namely the top left entry $(\Lambda_G)_{1,1}$. A row expansion shows that the coefficient of α_1 in $\det(\Lambda_G)$ equals $\det(\Lambda_G^{1,1})$. On the other hand, the coefficient of α_1 in Ψ_G is precisely $\Psi_{G \setminus e_1}$ by the contraction-deletion identity $\Psi_G = \alpha_1 \Psi_{G \setminus e} + \Psi_{G // e}$.

(ii) By the same argument as in (i), we have $\det(\Lambda_G^{i,i}) = \Psi_{G \setminus e_i}$ for $i = 1, 2$ and $\det(\Lambda_G^{12,12}) = \Psi_{G \setminus \{e_1, e_2\}}$. Now the ‘Dodgson identity’

$$\left(\det(\Lambda_G^{1,2}) \right)^2 = \Psi_{G \setminus e_1} \Psi_{G \setminus e_2} - \Psi_{G \setminus \{e_1, e_2\}} \Psi_G$$

implies that $\det(\Lambda_G^{1,2})$ is equal, up to a possible sign, to the ‘Dodgson polynomial’ $\Psi_G^{1,2}$ which satisfies the identical identity (but is defined using the graph matrix, and not the Laplacian). An expression for the latter in terms of the requisite forest polynomials follows from [BY11, §2]. $\square$

7. Determinants of Generalised Graph Laplacians

Let G be a connected graph as in §5 with internal particle masses and external momenta lying in 2-dimensional Euclidean space $\mathbb{R}^2 \cong \mathbb{C}$.

Theorem 7.1. *Let $\tilde{\Lambda}_G(\mu, m)$ be a generalised graph Laplacian for any choice of momentum routing $\mu_e \in \mathbb{C}^{E_G}$. It satisfies*

$$\det \tilde{\Lambda}_G(\mu, m) = \Xi_G(q, m). \quad (7.1)$$

Now let G be as above with external momenta in $\mathbb{R}^4 \cong \mathcal{Q}$.

Theorem 7.2. *Let $\tilde{\Lambda}_G(\mu, m)$ be a generalised quaternionic graph Laplacian with respect to a choice of momentum routing $\mu \in \mathcal{Q}^{E_G}$. It satisfies*

$$\text{Det } \tilde{\Lambda}_G(\mu, m) = (\Xi_G(q, m))^2. \quad (7.2)$$

Remark 7.3. The right-hand side of (7.1) or (7.2) only depends on external kinematics. It follows that the determinant of the generalised graph Laplacian with generic momentum variables μ_e only depends on the quantities

$$\mu_v = \sum_{s(e)=v} \mu_e - \sum_{t(e)=v} \mu_e$$

where the sum is over all internal edges incident to a vertex $v \in V(G)$.

We first prove Theorem 7.1 and use it to deduce Theorem 7.2. It is instructive to prove Theorem 7.1 by direct computation of the determinant of the left-hand side, making use of Proposition 6.3. An alternative approach could exploit the asymptotic factorisation identities proven in §9, which were shown in [Bro17, §4] to determine $\Xi_G(q, m)$ essentially uniquely.

7.1. Preliminary reductions. It follows from the presentation (5.4) of $\tilde{\Lambda}_G(\mu, m)$ in block matrix form that

$$\det \tilde{\Lambda}_G(\mu, m) = \left(\sum_{i=1}^N \alpha_i m_i^2 \right) \det \Lambda_G + \det \tilde{\Lambda}_G(\mu, 0)$$

Since $\det(\Lambda_G) = \Psi_G$, it is enough by definition of $\Xi_G(q, m)$ to prove (7.1) in the case when all masses are zero. Thus, Theorem 7.1 is equivalent to:

$$\det \tilde{\Lambda}_G(\mu, 0) = \Phi_G(q). \quad (7.3)$$

Now recall that $\tilde{\Lambda}_G(\mu, 0)$ takes the form:

$$\tilde{\Lambda}_G(\mu, 0) = \left(\begin{array}{c|c} \Lambda_G & \begin{matrix} c_1(\alpha, \mu) \\ c_2(\alpha, \mu) \\ \vdots \end{matrix} \\ \hline \begin{matrix} c_1(\alpha, \bar{\mu}) & c_2(\alpha, \bar{\mu}) & \dots \end{matrix} & \sum_{i=1}^N \alpha_i \mu_i \bar{\mu}_i \end{array} \right).$$

By §5.2.2, its determinant does not depend on the choices of cycles c_i . To verify (7.3), we use Proposition 6.3 and compare the coefficients of $\mu_{e_1} \bar{\mu}_{e_2}$ on both sides of (7.3). First we treat the case $e_1 = e_2 = e$.

Lemma 7.4. *Let e be an edge of G , then the coefficients of $\mu_e \bar{\mu}_e$ on the left and right hand sides of (7.3) agree.*

Proof. Consider the coefficient of $\mu_e \bar{\mu}_e$ in $\det \tilde{\Lambda}_G(\mu, 0)$. If e is a bridge, then we can find cycles $c_1, \dots, c_h \in \mathbb{Z}^{E_G}$ representing a basis of $H_1(G; \mathbb{Z})$ which do not involve e at all. In this case, μ_e only appears in $\tilde{\Lambda}_G(\mu, 0)$ in the bottom right hand corner, and the coefficient of $\mu_e \bar{\mu}_e$ is

$$\det(\Lambda_G) \alpha_e = \Psi_G \alpha_e = \Psi_{G//e} \alpha_e.$$

If e is not a bridge, then choose cycles $c_1, \dots, c_h$ as in Lemma 6.4. There are two possible types of terms in an expansion of the determinant which can involve $\mu_e \bar{\mu}_e$,

namely those involving the top right and bottom left corners $c_1(\alpha, \mu)$ and $c_1(\alpha, \bar{\mu})$, and those involving the bottom right hand corner as before. In total they contribute

$$\alpha_e \det(\Lambda_G) - \alpha_e^2 \det(\Lambda_G^{1,1}) = \alpha_e (\Psi_G - \alpha_e \Psi_{G \setminus e}) = \alpha_e \Psi_{G // e}.$$

The minus sign on the left comes from $(-1)^h(-1)^{h-1} = -1$; the first equality is a consequence of Lemma 6.5(i); and the second equality is the contraction-deletion identity: $\Psi_G = \alpha_e \Psi_{G \setminus e} + \Psi_{G // e}$. In all cases, we conclude that the coefficient of $\mu_e \bar{\mu}_e$ in $\det \tilde{\Lambda}_G(\mu, 0)$ is $\alpha_e \Psi_{G // e}$, in agreement with (6.6). $\square$

Before proceeding to the case e_1, e_2 distinct, we consider some boundary cases.

Lemma 7.5. *Let e be a tadpole in G . Then*

$$\det \tilde{\Lambda}_G(\mu, 0) = \alpha_e \det \tilde{\Lambda}_{G \setminus e}(\mu, 0)$$

Proof. By Lemma 6.4, choose the first cycle in a basis $c_1, \dots, c_h$ of $H_1(G)$ to be $c_1 = e$, and all other c_i to be independent of e . The matrix $\tilde{\Lambda}_G(\mu, 0)$ has the form

$$\tilde{\Lambda}_G(\mu, 0) = \left(\begin{array}{c|ccc|c} \alpha_e & 0 & \dots & 0 & \alpha_e \mu_e \\ 0 & & & & c_2(\alpha, \mu) \\ \vdots & & \Lambda_{G \setminus e} & & \vdots \\ 0 & & & & c_h(\alpha, \mu) \\ \hline \alpha_e \bar{\mu}_e & c_2(\alpha, \bar{\mu}) & \dots & c_h(\alpha, \bar{\mu}) & \sum_{i=1}^N \alpha_i \mu_i \bar{\mu}_i \end{array} \right).$$

The result follows by subtracting $\bar{\mu}_e$ times the first row from the final row, and performing an expansion of the determinant along the first column. $\square$

Next we verify that the theorem is true in the massless case for ‘banana’ graphs, i.e., graphs with exactly two vertices joined by ≥ 3 internal edges.

Example 7.6. Let G have two vertices v_1, v_2 , and $n \geq 3$ edges. Orient them so that they all have source v_1 and target v_2 , except for e_n , which has source v_2 and target v_1 . Consider the basis of cycles:

$$c_1 = e_1 + e_n \quad c_2 = e_2 + e_n \quad \dots, \quad c_h = e_{n-1} + e_n$$

where $h = n - 1$. With these conventions, the graph Laplacian is

$$\Lambda_G = \begin{pmatrix} \alpha_1 + \alpha_n & \alpha_n & \dots & \alpha_n \\ \alpha_n & \alpha_2 + \alpha_n & \dots & \alpha_n \\ \vdots & & \ddots & \vdots \\ \alpha_n & \dots & \alpha_n & \alpha_{n-1} + \alpha_n \end{pmatrix}.$$

It follows from the explicit description of $\tilde{\Lambda}_G(\mu, m)$ in the form (5.4) that the coefficient of $\mu_1 \bar{\mu}_2$ in $\det \tilde{\Lambda}_G(\mu, 0)$ is $\alpha_1 \alpha_2 \det \Lambda_G^{1,2}$, where

$$\Lambda_G^{1,2} = \begin{pmatrix} \alpha_n & \alpha_n & \dots & \alpha_n \\ \alpha_n & \alpha_3 + \alpha_n & \dots & \alpha_n \\ \vdots & & \ddots & \vdots \\ \alpha_n & \dots & \alpha_n & \alpha_{n-1} + \alpha_n \end{pmatrix}.$$

By subtracting the first row from all other rows, we see that $\det \Lambda_G^{1,2} = \alpha_3 \cdots \alpha_n$, and therefore the coefficient of $\mu_1 \bar{\mu}_2$ in $\det(\Lambda_G(\mu, 0))$ is $\alpha_1 \cdots \alpha_n$. By symmetry, the same holds for any $\mu_i \bar{\mu}_j$, where $i \neq j$. This agrees with the coefficient of $\mu_i \bar{\mu}_j$ in $\Phi_G(q)$, by its definition (there is only one spanning two-tree). The case when $i = j$ follows from Lemma 7.4, and it follows that $\det \tilde{\Lambda}_G(\mu, 0) = \Phi_G(q)$ as claimed.

7.2. Proof of Theorem 7.1. Now let $e_1, e_2 \in E(G)$ be two distinct edges and consider the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ in $\det(\tilde{\Lambda}_G(\mu, 0))$. Suppose first of all that one of the edges, say e_1 , is a bridge. By the argument given above, no terms in the matrix $\tilde{\Lambda}_G(\mu, 0)$ depend on μ_{e_1} except for the bottom right-hand corner, and so the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ in $\det(\tilde{\Lambda}_G(\mu, 0))$ is zero. Now since e_1 is a bridge, it follows that e_2 lies in just one of the two connected components of $G \setminus e_1$, and therefore:

$$\phi_G^{\{s(e_1), s(e_2)\}, \{t(e_1), t(e_2)\}} = \phi_G^{\{s(e_1), t(e_2)\}, \{t(e_1), s(e_2)\}} = 0.$$

Now suppose that neither e_1 nor e_2 is a bridge, yet $G \setminus \{e_1, e_2\}$ has two connected components. In particular, e_2 is a bridge in $G \setminus \{e_1\}$. In this case we may choose the cycles $c_1, \dots, c_h$ such that $c_2, \dots, c_h$ are cycles in $G \setminus \{e_1\}$ which do not depend on e_2 . Thus we can assume that only cycle c_1 involves either e_1 or e_2 , and furthermore we can direct the edges e_1, e_2 in such a way that their coefficient in c_1 are both 1. In this case, the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ in $\det(\tilde{\Lambda}_G(\mu, 0))$ is

$$(-1)^h (-1)^{h-1} \alpha_{e_1} \alpha_{e_2} \det \tilde{\Lambda}_G^{1,1}(\mu, 0) = -\alpha_{e_1} \alpha_{e_2} \Psi_{G \setminus \{e_1\}}$$

Since the sources of e_1, e_2 lie in different components of $G \setminus \{e_1, e_2\}$, we have

$$\phi_G^{\{s(e_1), s(e_2)\}, \{t(e_1), t(e_2)\}} = 0.$$

There is a one-to-one correspondence between the set of spanning forests with two components $T_1 \cup T_2$ such that $\{s(e_1), t(e_2)\} \subset V(T_1)$ and $\{t(e_1), s(e_2)\} \subset V(T_2)$, and the set spanning trees $T = T_1 \cup T_2 \cup e_2$ of $G \setminus \{e_1\}$. This implies that

$$\phi_G^{\{s(e_1), t(e_2)\}, \{s(e_1), t(e_2)\}} = \alpha_{e_1} \alpha_{e_2} \Psi_{G \setminus \{e_1\}}$$

and again the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ in $\det(\tilde{\Lambda}_G(\mu, 0))$ and $\Phi_G(q)$ are in agreement.

Finally, consider the case when $G \setminus \{e_1, e_2\}$ is connected. Choose cycles $c_1, \dots, c_h$ as in Lemma 6.4. By expanding $\det \tilde{\Lambda}_G(\mu, 0)$, its coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ is

$$\alpha_{e_1} \alpha_{e_2} \det(\Lambda_G^{1,2}) = \pm \alpha_{e_1} \alpha_{e_2} \left(\phi_G^{\{s(e_1), s(e_2)\}, \{t(e_1), t(e_2)\}} - \phi_G^{\{s(e_1), s(e_2)\}, \{t(e_1), t(e_2)\}} \right),$$

by Lemma 6.5. By Proposition 6.3, the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ in $\det \tilde{\Lambda}_G(\mu, 0)$ is equal to that of $\mu_{e_1} \cdot \mu_{e_2}$ in $\Phi_G(q)$ up to a possible sign. It remains to determine the sign.

If the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ vanishes in $\det \tilde{\Lambda}_G(q, 0)$ then there is nothing to prove. Otherwise, it is enough to select a single non-zero monomial m in the coefficient of $\mu_{e_1} \bar{\mu}_{e_2}$ in $\det \tilde{\Lambda}_G(\mu, 0)$, and compare with the corresponding coefficient in $\Phi_G(q)$. Such a monomial corresponds to a unique spanning forest $\mathcal{F} = T_1 \cup T_2$ via the formula $m = \prod_{e \notin \mathcal{F}} \alpha_e$. The sign is determined by setting all α_e to 0 for $e \in \mathcal{F}$, which corresponds to contracting all edges in $\mathcal{F}$. Since

$$\tilde{\Lambda}_{G/\mathcal{F}}(\mu, 0) = \tilde{\Lambda}_G(\mu, 0) \Big|_{\alpha_e=0, e \in \mathcal{F}},$$

it therefore suffices to prove the identity for the graph $G' = G/\mathcal{F}$. Because $\mathcal{F}$ has two components, G' has exactly two vertices. We can assume that $G' \setminus \{e_1, e_2\}$ is connected since we have already considered the disconnected case. By Lemma 7.5, we can remove all tadpoles from G' , and thus assume that G' is a banana graph with two vertices. The result then follows from Example 7.6.

7.3. Quaternionic case (proof of Theorem 7.2). It follows from Proposition 2.1 that there exists a well-defined polynomial F in the α_i, m_i and the components λ_e, ν_e of the momenta $\mu_e = \lambda_e + j\nu_e$, and their conjugates, such that

$$\text{Det}(\tilde{\Lambda}_G(\mu, m)) = \det(\chi_{\tilde{\Lambda}_G(\mu, m)}) = F^2.$$

By inspection of (5.12), the middle term is at most quadratic in the mass squares m_e^2 , and therefore F may be written in the form

$$F = F_0 + \sum_{e \in E_G} m_e^2 F_e.$$

More precisely, by expanding $\det(\chi_{\tilde{\Lambda}_G(\mu, m)})$ using the representation (5.12), one sees that the quadratic terms in the mass squares all arise from the term

$$X^2 \det \chi_{\Lambda_G} = (X \det(\Lambda_G))^2$$

using Proposition 2.1(ii). By comparing terms in m_e^4 , it follows from this that

$$\sum_{e \in E_G} m_e^2 F_e = \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \det(\Lambda_G) = \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Phi_G$$

and therefore

$$\text{Det}(\tilde{\Lambda}_G(\mu, 0)) = F_0^2.$$

We are thus reduced to proving the case when all masses are zero.

By inspection of the matrix (5.12), the determinant is of degree at most four in the momentum components $\lambda_e, \nu_e, \bar{\lambda}_e, \bar{\nu}_e$, and therefore F is of degree at most two in them. It suffices to check that for any two edges e, f , the coefficient of $\lambda_e \bar{\lambda}_f$ (or $\bar{\nu}_e \nu_f$, etc) in F coincides with that of $\Phi_G(q)$. Since F does not depend on the choice of momentum routing, we can assume that the momenta through edges e, f are purely complex, since they lie in a copy of $\mathbb{R}^2 \cong \mathbb{C}$ inside $\mathbb{R}^4 \cong \mathcal{Q}$. In other words, we may assume that $\nu_e = \nu_f = 0$ is zero. Since the coefficient of $\lambda_e \bar{\lambda}_f$ in F does not depend on the other momentum variables, we can assume that the latter are complex as well. Now if $\nu_e = 0$ for all edges e , the complex adjoint Laplacian $\chi_{\tilde{\Lambda}_G(\mu, m)}$ is simply the image under χ of the generalised Laplacian with complex momenta. Thus Theorem 7.2 follows from Theorem 7.1 and Proposition 2.1(ii).

8. Invariant Forms Associated to Graphs

We define invariant differential forms associated to generalised graph Laplacians.

8.1. Invariant forms. Let us define a graded exterior algebra

$$B_{\text{can}} = \bigwedge \bigoplus_{k \geq 1} \mathbb{Q} \beta^{2k+1}$$

in which each element β^{2k+1} has degree $2k+1$. It is a graded-commutative Hopf algebra with respect to the coproduct

$$\Delta_{\text{can}} : B_{\text{can}} \longrightarrow B_{\text{can}} \otimes B_{\text{can}} \quad (8.1)$$

for which the generators β^{2k+1} are primitive:

$$\Delta_{\text{can}}(\beta^{2k+1}) = 1 \otimes \beta^{2k+1} + \beta^{2k+1} \otimes 1.$$

We shall call a primitive element *odd* or *even* depending on whether k is odd or even. The Hopf subalgebra generated by the even forms β^{4k+1} was denoted by $\Omega_{\text{can}} \subset B_{\text{can}}$ in [Bro21]. Let $R = \bigoplus_{n \geq 0} R^n$ be a graded-commutative, unitary, differential graded algebra with differential $d : R^n \rightarrow R^{n+1}$ of degree 1.

Definition 8.1. For any invertible matrix $X \in \text{GL}_h(R_0)$, let us denote by

$$\beta_X^{2k+1} = \text{tr} \left(\left(X^{-1} dX \right)^{2k+1} \right) \in R^{2k+1}.$$

By taking exterior products, we can define β_X for any $\beta \in B_{\text{can}}$.

We briefly recall properties of these forms. Self-contained proofs of these facts, which are classical, may be found in [Bro21].

- (1) The forms β_X^{2k+1} are closed: $d\beta_X^{2k+1} = 0$
- (2) One has $\beta_{X^T}^{2k+1} = (-1)^k \beta_X^{2k+1}$. In particular, the odd primitive forms vanish on the space of symmetric matrices.
- (3) The forms β_X^{2k+1} are bi-invariant:

$$\beta_{AXB}^{2k+1} = \beta_X^{2k+1}$$

for any matrices $A, B \in \text{GL}_h(R_0)$ which are constant, i.e., $dA = dB = 0$.

- (4) The primitive forms are additive: $\beta_{X_1 \oplus X_2}^{2k+1} = \beta_{X_1}^{2k+1} + \beta_{X_2}^{2k+1}$ with respect to direct sums. It follows for general $\beta \in B_{\text{can}}$ that

$$\beta_{X_1 \oplus X_2} = \sum \beta'_{X_1} \wedge \beta''_{X_2}$$

where we write $\Delta_{\text{can}} \beta = \sum \beta' \otimes \beta''$.

- (5) Let $\lambda \in (R^0)^\times$ be invertible. Then

$$\beta_{\lambda X} = \beta_X \quad \text{for any } \beta \in B_{\text{can}}$$

- (6) If X has rank h , then

$$\beta_X^{2k+1} = 0 \quad \text{if } k \geq h.$$

8.2. Canonical forms associated to graphs. The canonical forms introduced in [Bro21], which do not depend on kinematics, will henceforth be called ‘of the first kind’ since their denominators involve the first Symanzik polynomial.

Definition 8.2. Let $\omega \in B_{\text{can}}$. For any connected graph G with internal particle masses and external momenta in $\mathbb{R}^2 \cong \mathbb{C}$, define:

(i) The *first canonical form* to be

$$\omega_G = \omega_{\Lambda_G}, \quad (8.2)$$

where Λ_G is any choice of graph Laplacian matrix.

(ii) Define the *second canonical form* to be

$$\varpi_G(\mu, m) = \omega_{\tilde{\Lambda}_G(\mu, m)} \quad (8.3)$$

where $\tilde{\Lambda}_G(\mu, m)$ is any choice of generalised graph Laplacian matrix. It depends *a priori* on the external momenta μ_e and particle masses m_e .

(iii) Define a *mixed canonical form*, or simply *canonical form* to be any polynomial in the canonical forms ω_G, ϖ_G of the first or second kinds.

The case when momenta lie in $\mathbb{R}^4$ is discussed below.

Since the graph Laplacian Λ_G is symmetric, odd canonical forms of the first kind are identically zero. This is not the case for the forms of the second kind. A mixed canonical form is abstractly represented, therefore, by an element of

$$\mathcal{U}_{\text{can}} = B_{\text{can}} \otimes \Omega_{\text{can}} \quad (8.4)$$

The following theorem summarizes some basic properties of these forms.

Theorem 8.3. *The primitive canonical graph forms are well-defined, projectively invariant closed differential forms of the shape:*

$$\begin{aligned} \omega_G^{4k+1} &= \beta_{\Lambda_G}^{4k+1} = \frac{N_1(\alpha, d\alpha)}{\Psi_G^{k+1}} \\ \varpi_G^{2k+1}(q, m) &= \beta_{\tilde{\Lambda}_G(\mu, m)}^{2k+1} = \frac{N_2(\alpha, d\alpha, m_e, \mathbf{q}_v, \bar{\mathbf{q}}_v)}{\Xi_G(q, m)^{k+1}} \end{aligned}$$

where N_1, N_2 are polynomials with rational coefficients. Let

$$X'_G = V(\Psi_G) \cup V(\Xi_G(q, m)) \subset \mathbb{P}^{E_G}$$

denote the generalised graph hypersurface. For any canonical form $\omega \in \mathcal{U}_{\text{can}}$ of degree n , ω_G is a closed, projectively invariant form of degree n :

$$\omega_G \in \Omega^n(\mathbb{P}^{E_G} \setminus X'_G)$$

which, under the assumption of generic momenta (§4.1), is smooth on the open coordinate simplex $\sigma_G \subset \mathbb{P}^{E_G}(\mathbb{R})$ defined by $\alpha_1, \dots, \alpha_{e_G} > 0$.

Proof. The fact that the forms are closed and well-defined follows from bi-invariance §8.1, as does projective invariance. It is clear from the definition that their denominators only involve a power of the determinant: the order of this power follows from [Bro21, Theorem 5.2], and the expression for the determinant as a graph polynomial follows from Theorem 7.1. The numerator N_2 is a priori only a polynomial in the momentum routing μ_e , but again by bi-invariance, it is independent of the choice of momentum routing, and hence only depends on external momenta $\mathbf{q}_v, \bar{\mathbf{q}}_v$. The fact that ω is smooth on the open simplex follows from the assumption that kinematics are generic, and hence $\Xi_G(q, m)$ and Ψ_G are strictly positive on σ_G . $\square$

Remark 8.4. For any Hermitian matrix X , one has

$$\overline{\beta_X^{2k+1}} = (-1)^k \beta_X^{2k+1}. \quad (8.5)$$

As a consequence, ϖ_G^{2k+1} is real if k is even, and imaginary if k is odd. In the case when k is even, ϖ_G will be a rational function of the inner products $q_i \cdot q_j$ of external momenta. In the case when k is odd, ϖ_G^{2k+1} may also depend on the quantities

$$\lambda_{ij} = \mathbf{q}_i \bar{\mathbf{q}}_j - \mathbf{q}_j \bar{\mathbf{q}}_i = \pm 2 \sqrt{(q_i \cdot q_j)^2 - q_i^2 q_j^2} \quad (8.6)$$

Remark 8.5. If G has only two external momenta $q_1 = -q_2$, $\tilde{\Lambda}_G(\mu, m)$ is equivalent to a symmetric matrix because we can find a momentum routing so that the μ_e lie on the real line spanned by q_1 . In this case, ϖ_G^{2k+1} vanishes when k is odd.

8.3. Further properties of canonical forms. The following properties of canonical forms are almost immediate from the definitions.

Lemma 8.6. (i) *Let π be an automorphism of G . It acts on R^0 by permutation of the edge variables x_e . For any canonical form of mixed type $\omega \in \mathcal{U}_{\text{can}}$,*

$$\pi^* \omega_G = \omega_G$$

(ii) *Let $\gamma \subset G$ be any subset of edges such that γ neither contains a loop, nor is mass-momentum spanning. Then*

$$\omega_G \Big|_{L_\gamma} = \omega_{G/\gamma}$$

where $L_\gamma \subset \mathbb{P}^{EG}$ denotes the linear subspace defined by $\alpha_e = 0$ for $e \in \gamma$.

Proof. The proof for a general canonical form are essentially the same as for the first canonical form [Bro21, §6.2]. $\square$

8.4. Additional form in degree 1. The bi-invariant form

$$\beta_X^1 = \text{tr}(X^{-1} dX) = d \log \det(X)$$

is excluded because it is poorly behaved. In particular, the forms

$$\omega_G^1 = d \log \det(\Lambda_G) = \frac{d \Psi_G}{\Psi_G}, \quad \text{and} \quad \varpi_G^1 = d \log \det(\tilde{\Lambda}_G(\mu, m)) = \frac{d \Xi_G(q, m)}{\Xi_G(q, m)}$$

are not projectively invariant since $\partial\omega_G^1 = h_G$ and $\partial\varpi_G^1 = h_G + 1$, where ∂ denotes the Euler vector field. Nonetheless, the combination

$$\phi_G^1 = \partial(\omega_G^1 \wedge \varpi_G^1) = h_G \varpi_G^1 - (h_G + 1) \omega_G^1 \quad (8.7)$$

is projectively invariant since $\partial^2 = 0$, which provides an additional form in degree one. However, it has poles at infinity and so products of canonical forms involving ϕ_G^1 often give rise to divergent integrals, and will be excluded on this account. See however, §11.1, for an example of a convergent integral associated to ϕ_G^1 .

Remark 8.7. The significance of the form ϕ_G^1 is that it appears in the connection which defines the twisted cohomology underlying dimensional regularisation. For example, one may consider the Mellin transform

$$\int_{\sigma_G} \left(\frac{\Psi_G^{h+1}}{\Xi_G(q, m)^h} \right)^s \omega_G$$

where the form ω_G in the integrand is to be viewed as a cohomology class with respect to the twisted connection $\nabla(\omega) = -s\phi_G^1 \wedge \omega + d\omega$.

8.5. Quaternionic forms. Let G be a connected graph with external momenta in $\mathbb{R}^4$, which we identify with quaternions via §4.4.

Definition 8.8. For any $\omega \in \Omega^{\text{can}}$, define the *quaternionic canonical form of the second kind* associated to G to be

$$\varpi_G^Q(\mu, m) = \omega_{\chi_{\tilde{\Lambda}_G(\mu, m)}}$$

where $\chi_{\tilde{\Lambda}_G(\mu, m)}$ is the image under (2.1) of any choice of quaternionic Laplacian.

Remark 8.9. Since the quaternionic graph Laplacian is Hermitian, its complex adjoint $X = \chi_{\tilde{\Lambda}_G(\mu, m)}$ satisfies the following identity, where J_n was defined in §2.2,

$$X = J_n^{-1} X^T J_n$$

and is therefore similar to its transpose. For any such matrix $\beta_X^{2k+1} = 0$ if k is odd. For this reason only the even canonical forms are relevant here.

The quaternionic forms of the second kind have very similar properties to the complex case. We briefly summarise these properties in the following theorem.

Theorem 8.10. *Every quaternionic canonical form of degree n of the second kind is a well-defined, closed, projectively-invariant differential form*

$$\varpi_G^Q(\mu, m) \in \Omega^n(\mathbb{P}^{E_G} \setminus X'_G)$$

which is smooth on the open simplex σ_G when momenta are generic (4.1). It is invariant with respect to automorphisms of the graph G and compatible with contraction of subgraphs which have no loops and are not mass-momentum spanning.

Remark 8.11. The primitive quaternionic forms may be defined directly using the quaternionic trace instead of passing via complex adjoints, using the identity:

$$\text{Tr}((M^{-1}dM)^n) = \text{tr}((\chi_M^{-1}d\chi_M)^n) = \beta_{\chi_M}^n$$

This uses the fact that $M \in M_n(\mathbb{Q})$ is invertible if and only if $\chi_M \in M_{2n}(\mathbb{C})$ is invertible §2.2.

9. Canonical Forms at Infinity

This section consists of technical results concerning the asymptotic behaviour of canonical forms in the limit when the generalised graph Laplacians become singular.

9.1. Asymptotic behaviour of generalised Laplacian matrices. Let $I \subset E_G$ denote any subset of the set of edges of G . Consider the map

$$\begin{aligned} s_I^* : \mathbb{Q}[\alpha_e, e \in E_G] &\longrightarrow \mathbb{Q}[z, \alpha_e, e \in E_G] \\ s_I^*(\alpha_e) &= z\alpha_e \quad \text{if } e \in I, \\ s_I^*(\alpha_e) &= \alpha_e \quad \text{otherwise.} \end{aligned} \quad (9.1)$$

Suppose that G has internal masses and external momenta in $\mathbb{R}^2 \cong \mathbb{C}$, for the time being. Recall that $\mathcal{E}_G$ was the extension defined in (5.1).

Lemma 9.1. *i). Let $\gamma \subset E_G$ be a subgraph. Then, writing matrices in block matrix form with respect to a choice of splitting*

$$\mathcal{E}_G \cong H_1(\gamma; \mathbb{C}) \oplus \mathcal{E}_{G/\gamma}$$

the rescaled generalised graph Laplacian satisfies:

$$s_\gamma^* \tilde{\Lambda}_G(\mu, m) = \begin{pmatrix} z\Lambda_\gamma & zB \\ zC & D \end{pmatrix} \quad (9.2)$$

where B, C, D are matrices with entries in $\mathbb{C}[z, \alpha_e, e \in E_G]$ and

$$D \equiv \tilde{\Lambda}_{G/\gamma}(\mu, m) \pmod{z}.$$

ii). Now let $\gamma \subset E_G$ be a mass-momentum spanning subgraph. If we write matrices in block matrix form with respect to a splitting

$$\mathcal{E}_G \cong H_1(G/\gamma; \mathbb{C}) \oplus \mathcal{E}_\gamma,$$

then the rescaled generalised graph Laplacian satisfies:

$$s_\gamma^* \tilde{\Lambda}_G(\mu, m) = \begin{pmatrix} A & zB \\ zC & z\tilde{\Lambda}_\gamma(\mu, m) \end{pmatrix} \quad \text{where } A \equiv \Lambda_{G/\gamma} \pmod{z} \quad (9.3)$$

for some matrices A, B, C with entries in $\mathbb{C}[z, \alpha_e, e \in E_G]$.

Proof. i). The short exact sequence

$$0 \longrightarrow H_1(\gamma; \mathbb{Z}) \longrightarrow H_1(G; \mathbb{Z}) \longrightarrow H_1(G/\gamma; \mathbb{Z}) \longrightarrow 0$$

defines a filtration $0 \subset H_1(\gamma; \mathbb{Z}) \subset H_1(G; \mathbb{Z})$. We may choose representatives $c_1, \dots, c_g \in \mathbb{Z}^{E_G}$ for a basis of $H_1(G; \mathbb{Z})$ with the property that $c_1, \dots, c_h \subset \mathbb{Z}^{E_\gamma}$ represent a basis for $H_1(\gamma; \mathbb{Z})$. A choice of momentum routing μ defines a splitting $\mathcal{E}_G = H_1(G; \mathbb{C}) \oplus \mathbb{C}$ of (5.1), and hence we may write

$$\mathcal{E}_G = H_1(G; \mathbb{C}) \oplus \mathbb{C} \cong H_1(\gamma; \mathbb{C}) \oplus H_1(G/\gamma; \mathbb{C}) \oplus \mathbb{C} \cong H_1(\gamma; \mathbb{C}) \oplus \mathcal{E}_{G/\gamma}.$$

The statement (9.2) follows from the general form (5.4) of the matrix $\tilde{\Lambda}_G(\mu, m)$ using the fact that $s_\gamma^* c_i(\alpha, \mu) = z c_i(\alpha, \mu)$ for $1 \leq i \leq h$, and the fact that

$$s_\gamma^* \tilde{\Lambda}_G(\mu, m) \Big|_{z=0} = \tilde{\Lambda}_G(\mu, m) \Big|_{\alpha_e=0, e \in \gamma} = \tilde{\Lambda}_{G/\gamma}(\mu, m)$$

ii). Now let γ be mass-momentum spanning. As above, a choice of momentum routing μ defines a splitting $\mathcal{E}_G \cong H_1(G; \mathbb{C}) \oplus \mathbb{C}$, and we may write

$$\mathcal{E}_G \cong H_1(G/\gamma; \mathbb{C}) \oplus \mathcal{E}_\gamma.$$

Let $c_1, \dots, c_g$ be a system of representative cycles for a basis of $H_1(G; \mathbb{Z})$, adapted to the filtration $H_1(\gamma; \mathbb{Z}) \subset H_1(G; \mathbb{Z})$. Consider the form (5.4) of the matrix $\tilde{\Lambda}_G(\mu, m)$ with respect to this basis. Since γ is momentum spanning, one has by definition $s_\gamma^* c_i(\alpha, \mu) = z c_i(\alpha, \mu)$ for all $1 \leq i \leq g$. Since γ is mass and momentum spanning, one has $s_\gamma^* X_G = z X_\gamma$, where $X_\bullet$ is defined by formula (5.5). The statement therefore follows from $s_\gamma^* \tilde{\Lambda}_G(\mu, m) \Big|_{z=0} = \Lambda_{G/\gamma}$. $\square$

Remark 9.2. By taking the determinant and using Theorem 7.1, the previous lemma gives an interpretation of the ‘asymptotic’ factorisation formulae

$$\begin{aligned} \Xi_G(q, m) &\sim \Phi_\gamma \Xi_{G/\gamma}(q, m) \quad \text{if } \gamma \subset E_G \text{ not m.m.} \\ \Xi_G(q, m) &\sim \Xi_\gamma(q, m) \Phi_{G/\gamma} \quad \text{if } \gamma \subset E_G \text{ m.m.} \end{aligned}$$

stated in [Bro10, Theorem 2.7] for generic kinematics. This observation leads to another proof of Theorem 7.1 since the above factorisation formulae determine $\Xi_G(q, m)$ essentially uniquely [Bro10, Proposition 4.8].

Note that in the case when γ is mass-momentum spanning, both Eqs. (9.2) and (9.3) are simultaneously valid. Since in this case G/γ is scaleless, $\Lambda_{G/\gamma}(\mu, m)$ has vanishing determinant and (9.2) is not relevant.

9.2. Reminders on motives for graphs with generic kinematics. We assume generic kinematics (4.1) throughout this section. Let us denote by

$$X_G = V(\Psi_G) \cup V(\Xi_G(q, m)) \subset \mathbb{P}^{E_G}$$

the union of graph hypersurfaces. By blowing up linear subspaces as in [BEK06] corresponding to motic subgraphs (the notion of a motic subgraph is defined in [Bro17, Definition 3.1]), one obtains a space [Bro10, Definition 6.3]

$$\pi_G : P^G \longrightarrow \mathbb{P}^{E_G} \tag{9.4}$$

which contains a distinguished simple normal crossing divisor D whose image under π_G is the union of coordinate hyperplanes. The irreducible components of D are of two types: exceptional divisors D_γ corresponding to γ a motic subgraph of G , and divisors D_e where e is an edge of E_G , which are strict transforms of the coordinate hyperplanes $\alpha_e = 0$. We denote the strict transform of X_G by $Y_G \subset P^G$.

Every exceptional irreducible component D_γ of D admits a canonical isomorphism $D_\gamma \cong P^\gamma \times P^{G/\gamma}$ which induces an isomorphism

$$D_\gamma \setminus (D_\gamma \cap Y_G) \cong (P^\gamma \setminus Y_\gamma) \times (P^{G/\gamma} \setminus Y_{G/\gamma}). \tag{9.5}$$

9.3. Canonical forms along exceptional divisors. For any canonical form η , denote its pull-back to $P^G \setminus (Y_G \cup \mathcal{E})$ along (9.4) by

$$\tilde{\eta}_G = \pi_G^*(\eta_G),$$

where $\mathcal{E}$ denotes the exceptional divisor of π_G . It could *a priori* have poles along $\mathcal{E}$, but the following theorem shows that in fact it does not.

Theorem 9.3. *Let $\eta_G \in \mathcal{U}_{\text{can}}^n$ be any canonical form. Then $\tilde{\eta}_G$ has no poles along D and extends to a smooth form on $P^G \setminus Y_G$, i.e.,*

$$\tilde{\eta}_G \in \Omega^n(P^G \setminus Y_G).$$

Its restriction to an irreducible boundary component D_e of D satisfies:

$$\tilde{\eta}_G|_{D_e} = \tilde{\eta}_{G/e}$$

if D_e is the strict transform of the hyperplane $L_e = V(\alpha_e)$ corresponding to a single edge e of G . The restriction of η_G to exceptional divisors depend on its kind. For simplicity, assume that $\eta_G \in \{\omega_G, \varpi_G\}$ is primitive (the general case follows by multiplying primitive forms together), and that $\gamma \subset G$ is motic and $e_\gamma > 1$.

(i) *Suppose that $\eta_G = \omega_G$ is of the first kind. Then*

$$\tilde{\omega}_G|_{D_\gamma} = \tilde{\omega}_\gamma \wedge 1 + 1 \wedge \tilde{\omega}_{G/\gamma} \quad (9.6)$$

if γ is a core (bridgeless but not necessarily connected) subgraph, and $\tilde{\omega}_G|_{D_\gamma}$ vanishes if D_γ is the exceptional divisor corresponding to a m.m. subgraph which is not core.

(ii) *Suppose that $\eta_G = \varpi_G$ is of the second kind. Then*

$$\begin{aligned} \tilde{\varpi}_G|_{D_\gamma} &= \tilde{\varpi}_\gamma \wedge 1 + 1 \wedge \tilde{\varpi}_{G/\gamma} \quad \text{if } \gamma \text{ core, not m.m.} \\ \tilde{\varpi}_G|_{D_\gamma} &= \tilde{\varpi}_\gamma \wedge 1 + 1 \wedge \tilde{\varpi}_{G/\gamma} \quad \text{if } \gamma \text{ is m.m.} \end{aligned} \quad (9.7)$$

Proof. The argument in [Bro21, Theorem 7.4] proves the case when η is of the first kind and γ is a core subgraph. The general result follows in an identical manner from a local description of the blow-ups in affine coordinates, combined with the asymptotic formulae of Lemma 9.1. The essential part of the argument is the calculation (*loc. cit.*) of the behaviour of the bi-invariant form β_X^n where X is a matrix which is asymptotically block-diagonal as in (9.2) or (9.3). $\square$

Remark 9.4. An interesting feature of Eq. (9.7) is that the right-hand side involves canonical forms of both the first and second kinds. In particular, if ω is primitive of degree $n \equiv 3 \pmod{4}$ then the corresponding form of the first kind ω_G vanishes, and the right-hand sides of (9.7) reduce to a single term.

The case of a non-primitive form is easily deduced from Theorem 9.3 by taking products. For example, for a general form ω , Eqs. (9.6), (9.7) take the form

$$\begin{aligned} \tilde{\omega}_G|_{D_\gamma} &= \sum \tilde{\omega}'_\gamma \wedge \tilde{\omega}''_{G/\gamma} \quad \text{if } \gamma \text{ is core,} \\ \tilde{\omega}_G|_{D_\gamma} &= \sum \tilde{\omega}'_\gamma \wedge \tilde{\omega}''_{G/\gamma} \quad \text{if } \gamma \text{ is core and not m.m.,} \\ \tilde{\omega}_G|_{D_\gamma} &= \sum \tilde{\omega}'_\gamma \wedge \tilde{\omega}''_{G/\gamma} \quad \text{if } \gamma \text{ is m.m.,} \end{aligned} \quad (9.8)$$

where $\Delta_{\text{can}}\omega = \sum \omega' \otimes \omega''$ is the coproduct of ω .

Remark 9.5. The differential form ϕ_G^1 in general has poles along boundary divisors. Let G be as above, and consider a core subgraph $\gamma \subset E_G$. Then by the partial factorisations for the first Symanzik polynomial (or by Lemma 9.1(i)), one has:

$$\begin{aligned} s_\gamma^* \omega_G^1 &= s_\gamma^* d \log \Psi_G = h_\gamma \frac{dz}{z} + d \log \Psi_\gamma + d \log \Psi_{G/\gamma} + O(z) \\ s_\gamma^* \varpi_G^1 &= s_\gamma^* d \log \Xi_G = h_\gamma \frac{dz}{z} + d \log \Psi_\gamma + d \log \Xi_{G/\gamma} + O(z). \end{aligned}$$

It follows that

$$s_\gamma^* \phi_G^1 = -h_\gamma \frac{dz}{z} - \omega_\gamma^1 + \phi_{G/\gamma}^1 + O(z)$$

and so ϕ_G^1 has a simple pole along the exceptional boundary divisor D_γ (which is given by $z = 0$ in local coordinates) and its residue depends only on the loop number of γ . A similar argument shows that ϕ_G^1 also has poles along exceptional divisors D_γ , when γ is mass-momentum spanning. Differential forms of the type $\phi_G^1 \wedge \eta$, where η is canonical, will therefore in general have poles along the divisor D , and will not be considered here. They naturally arise, however, when considering Stokes' formula for canonical forms in dimensional-regularisation.

9.4. Quaternionic case. The above arguments similarly apply to the case of a canonical form of the second kind ϖ_G^Q associated to a quaternionic generalised Laplacian, by replacing $\tilde{\Lambda}_G(\mu, m)$ with the complex adjoint matrix $\chi_{\tilde{\Lambda}_G(\mu, m)}$. Theorem 9.3 holds verbatim (with the only difference that odd canonical forms of the second kind vanish in the quaternionic setting). In particular, ϖ_G^Q has no poles along D and extends to a smooth differential form on $P^G \setminus Y^G$. Its restriction to irreducible components of D are given by (9.8).

10. Canonical Integrals and Relations from Stokes Theorem

Throughout this section we assume generic (Euclidean) kinematics (4.1).

10.1. Canonical integrals. Let G be a connected graph with an orientation $e_1 \wedge \cdots \wedge e_N \in (\bigwedge \mathbb{Z}^{E(G)})^\times$. It induces an orientation on the simplex σ_G .

10.1.1. Complex momenta Suppose that external momenta lie in $\mathbb{R}^2 \cong \mathbb{C}$.

Theorem 10.1. *Let $\omega \in \mathcal{U}_{\text{can}}$ be a canonical form of degree $N + 1$ and let G be a connected graph with N edges. Then under assumption (4.1), the integral*

$$I_G(\omega, q, m) = \int_{\sigma_G} \omega_G \tag{10.1}$$

is finite and defines an analytic function of the external kinematics.

Proof. This is a corollary of Theorem 9.3. Let $\tilde{\sigma}_G = \overline{\pi_G^{-1}(\sigma_G)}$ denote the closure in the analytic topology of the pull-back of the open simplex σ_G to P^G . One has

$$I_G(\omega, q, m) = \int_{\sigma_G} \omega_G = \int_{\tilde{\sigma}_G} \tilde{\omega}_G.$$

The right-most integral converges because of the fact [Bro17] that $\tilde{\sigma}_G$, which is compact, does not meet Y_G , which is where the poles of $\tilde{\omega}_G$ are located. $\square$

Reversing the orientation on G reverses the sign of the canonical integral (10.1). Furthermore, if $\tau : G \xrightarrow{\sim} G'$ is an isomorphism of oriented graphs (with external legs and masses, etc), then since $\tau^* \omega_{G'} = \omega_G$, we have:

$$I_G(\omega, q, m) = I_{G'}(\omega, q, m).$$

Thus the integrals (10.1) are invariants of isomorphism classes of oriented graphs.

10.1.2. Quaternionic case The previous theorem, and comments which follow, hold verbatim for graphs with momenta in 4-dimensional Euclidean space with

$$I_G(\omega, q, m) = \int_{\sigma_G} \omega_G^{\mathbb{Q}}.$$

The only difference is that in this case we must assume $\omega \in \Omega_{\text{can}}$ is even, since odd quaternionic canonical forms vanish by Remark 8.9.

10.2. Stokes' formula. In [Bro17], it was shown that one can extend the usual Connes–Kreimer coproduct to an enlarged coproduct

$$\Delta(G) = \sum_{\gamma} \gamma \otimes G/\gamma$$

where the sum is over all motivic subgraphs of G . It has two types of terms: if $\gamma \subset G$ is a core subgraph which is not *m.m.*, then γ is considered to be scaleless, and encodes ultraviolet divergences of the graph. If $\gamma \subset G$ is *m.m.* then G/γ is scaleless, and γ encodes certain kinds of infrared divergences of G .

10.2.1. Complex momenta

Theorem 10.2. *Let $\omega \in \mathcal{U}_{\text{can}}$ be a canonical form of degree k . Write its coproduct in the form $\Delta_{\text{can}} \omega = \sum_i \omega'_i \otimes \omega''_i$. Suppose that G is a connected oriented graph with external kinematics as above with $k + 2$ edges. Then*

$$0 = \sum_{e \in E_G} \int_{\sigma_{G/e}} \omega_{G/e} + \sum_i \sum_{\gamma} \int_{\sigma_{\gamma}} (\omega'_i)_{\gamma} \int_{\sigma_{G/\gamma}} (\omega''_i)_{G/\gamma} \quad (10.2)$$

where the rightmost sum is over motivic subgraphs $\gamma \subsetneq G$ not including G itself (which are not necessarily connected) such that $e_{\gamma} = \deg \omega'_i + 1 > 1$, and the orientations on $\sigma_{G/e}, \sigma_{\gamma} \times \sigma_{G/\gamma}$ are induced from those on σ_G .

Proof. Identical to the proof of [Bro21, Theorem 8.5], using Theorem 9.3. $\square$

Note that in the case when $\omega \in B^{\text{can}}$ is purely of the second kind, one has

$$\begin{aligned} 0 = & \sum_{e \in E_G} \int_{\sigma_{G/e}} \varpi_{G/e} + \sum_i \sum_{\substack{\gamma \text{ core} \\ \text{not } m.m.}} \int_{\sigma_\gamma} (\omega'_i)_\gamma \int_{\sigma_{G/\gamma}} (\varpi''_i)_{G/\gamma} \\ & + \sum_i \sum_{\gamma \text{ } m.m.} \int_{\sigma_\gamma} (\varpi'_i)_\gamma \int_{\sigma_{G/\gamma}} (\omega''_i)_{G/\gamma}, \end{aligned} \quad (10.3)$$

where we have decomposed the set of motic subgraphs γ of G , such that $1 < e_\gamma < e_G$ into two types according to whether γ is *m.m.* or not.

Note that the Stokes' formulae above remain valid by analytic continuation to a larger region of kinematic space wherever the integrals are finite.

10.2.2. Quaternionic case The statement is identical on replacing ω with $\omega^{\mathbb{Q}}$.

Remark 10.3. In [Bro21, Theorem 8.5], the Stokes' formula is not quite correct as stated in the case when a graph G has a tadpole γ . In that case, the corresponding term was counted twice: once in the term G/e , where $e = \gamma$, and again in the form $\gamma \times G/\gamma$. The condition imposed above, that $e_\gamma > 1$, fixes this problem.

Similarly, if γ is an *m.m.* subgraph which has only a single edge, then it only contributes a single term in the formula (10.2), which is counted in the left-most summation over all edges, and not in the right-hand double sum.

10.3. Canonical forms of compact type. Let $h \geq 1$. Let us say that a canonical form $\omega \in B^{\text{can}}$ is of compact type relative to h if it is divisible by the primitive form ω^{2h-1} of degree $2h-1$, i.e., $\omega = \omega^{2h-1} \wedge \eta$ for some $\eta \in B^{\text{can}}$. It follows from the property (6) stated in §8.1, that ω vanishes on all matrices of rank $< h$.

Corollary 10.4. *Let G be as in Theorem 10.2 and let $\varpi \in B^{\text{can}}$ be of the second kind of degree $e_G - 2$, and of compact type relative to $h_G + 1$. Thus $\varpi = \varpi^{2h_G+1} \wedge \eta$ for some $\eta \in B^{\text{can}}$ of the second kind. If G is core, then formula (10.3) reduces to*

$$0 = \sum_{e \in E_G} \int_{\sigma_{G/e}} \varpi_{G/e}. \quad (10.4)$$

Proof. Let $\gamma \subset G$ be a strict motic subgraph with at least two edges. Suppose that γ is not *m.m.* and therefore core. We need to check that a term of the form

$$\int_\gamma \omega'_\gamma \int_{G/\gamma} \varpi''_{G/\gamma}$$

vanishes in the right-hand side of (10.3). Either ω' or ω'' is of compact type relative to $h_G + 1$. If the former, then $\omega'_\gamma = \omega'_{\Lambda_\gamma}$ vanishes since $h_\gamma < h_G + 1$. If ω'' is of compact type, then $\varpi''_{G/\gamma} = \omega''_{\Lambda_{G/\gamma}}$ vanishes if $h_{G/\gamma} + 1 < h_G + 1$. This inequality holds since $h_\gamma + h_{G/\gamma} + 1 = h_G + 1$, and $h_\gamma > 0$ because γ is core.

Now suppose that γ is *m.m.*. We need to check that any term of the form

$$\int_\gamma \varpi'_\gamma \int_{G/\gamma} \omega''_{G/\gamma}$$

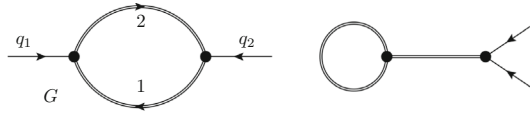


Fig. 1. The two metric graphs of maximal dimension in $\mathcal{M}_{1,2,2}^{\text{trop}}$

vanishes. If ω' is of compact type, then $\varpi'_\gamma = \omega'_{\Lambda_\gamma}$ vanishes if $h_\gamma + 1 < h_G + 1$. Since γ is a strict subgraph, this follows from $h_\gamma < h_G$ since G is core. If ω'' is of compact type, then $\omega''_{G/\gamma} = \omega''_{\Lambda_{G/\gamma}}$ vanishes since $h_{G/\gamma} < h_G + 1$. $\square$

11. Examples of Canonical Amplitudes (Complex Momenta)

For a graph G with edges numbered $1, \dots, N$, we write

$$\Omega_G = \sum_{i=1}^N (-1)^i \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_N.$$

We consider canonical differential forms in low degrees and their integrals. For now, we consider only the case when external momenta lie in $\mathbb{R}^2 \cong \mathbb{C}$.

11.1. Digression: the divergent form ϕ^1 . The integrals involving the differential form ϕ^1 diverge, and for that reason are not considered to be ‘canonical’ forms according to our definitions. However, this form provides some interesting first examples, which may on occasion lead to convergent integrals.

Graphs which pair with ϕ^1 have two edges. Consider the moduli space $\mathcal{M}_{1,2,2}^{\text{trop}}$ of metric graphs with 1 loop, 2 external legs, and 2 distinct non-zero masses. It has two metric graphs of maximal dimension (see Fig. 1). The one on the right has a non-trivial subdivergence and so the integral of ϕ^1 is logarithmically divergent. Therefore, consider the graph G on the left. It is important that its edge masses m_1, m_2 be distinct, for otherwise, this graph has a symmetry which induces an odd permutation of its set of edges, implying that its canonical integral vanishes.

Orient and number the edges as shown in the figure. A representative for a generator of the homology $H_1(G; \mathbb{Z})$ is $c_1 = e_1 + e_2$. Momentum conservation implies that $q_2 = -q_1 \in \mathbb{R}^2 \cong \mathbb{C}$. Any momentum routing is given by a solution to:

$$q_1 = \mu_2 - \mu_1.$$

The generalised graph Laplacian matrix with respect to these choices is:

$$\tilde{\Lambda}_G(\mu, m) = \begin{pmatrix} \alpha_1 + \alpha_2 & \mu_1 \alpha_1 + \mu_2 \alpha_2 \\ \bar{\mu}_1 \alpha_1 + \bar{\mu}_2 \alpha_2 & \alpha_1 (\mu_1 \bar{\mu}_1 + m_1^2) + \alpha_2 (\mu_2 \bar{\mu}_2 + m_2^2) \end{pmatrix}$$

The relevant graph polynomials are given by

$$\Psi_G = \det(\Lambda_G) = \alpha_1 + \alpha_2$$

$$\Phi_G(q) = q_1^2 \alpha_1 \alpha_2$$

$$\Xi_G(q, m) = \det(\tilde{\Lambda}_G(\mu, m)) = q_1^2 \alpha_1 \alpha_2 + (m_1^2 \alpha_1 + m_2^2 \alpha_2)(\alpha_1 + \alpha_2),$$

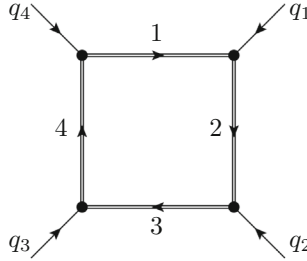


Fig. 2. Massive box graph

where we write q_1^2 for the Euclidean norm, which equals $\bar{\mathbf{q}}_1 \mathbf{q}_1$, where $\mathbf{q}_1$ is q_1 viewed in $\mathbb{C}$. Using (8.7), the exceptional form of degree 1 is

$$\phi_G^1 = \varpi_G^1 - 2\omega_G^1 = d \log \det(\tilde{\Lambda}_G) - 2d \log \det(\Lambda_G) = d \log \left(\frac{\Xi_G(q, m)}{\Psi_G^2} \right).$$

Thus the associated integral is

$$I_G(\phi^1, q, m) = \int_{\sigma_G} \phi_G^1 = \int_0^\infty d \log \left(\frac{\Xi_G(q, m)}{\Psi_G^2} \right) \Big|_{\alpha_2=1} d\alpha_1$$

where the last equality follows by computing the projective integral on the affine chart $\alpha_2 = 1$. The integral reduces to

$$I_G(\phi^1, q, m) = \left[\log \left(\frac{\Xi_G(q, m)}{\Psi_G^2} \right) \right]_{\alpha_1=0, \alpha_2=1}^{\alpha_1=\infty, \alpha_2=1} = \log \left(\frac{m_1^2}{m_2^2} \right)$$

which is an agreement with the renormalised amplitude for a massive bubble. As expected, it vanishes when $m_1 = m_2$ for the symmetry reasons mentioned above.

11.2. Canonical amplitudes associated to ϖ_3 . We now turn to *bona fide* canonical amplitudes, whose integrals are always finite by Theorem 10.1. The canonical form of smallest degree is $\varpi^3 \in B_{\text{can}}$. It pairs with connected graphs which have 4 edges, and vanishes for graphs with fewer than 1 loop. Thus there are 3 cases:

- $h_G = 1, v_G = 4$
- $h_G = 2, v_G = 3$
- $h_G = 3, v_G = 2$

which we consider in turn for the maximal number of masses.

11.2.1. $\mathcal{M}_{1,4,4}^{\text{trop}}$ The massive box diagram (Fig. 2) is the unique graph which gives a cell of maximal dimension in $\mathcal{M}_{1,4,4}^{\text{trop}}$.

Choose any complex numbers μ_e for each edge e such that momentum conservation (4.2) holds: $\mu_{i+1} - \mu_i = \mathbf{q}_i$, where $q_i \in \mathbb{R}^2$, corresponding to $\mathbf{q}_i \in \mathbb{C}$, is the external momentum at vertex i subject to momentum conservation $\mathbf{q}_1 + \cdots + \mathbf{q}_4 = 0$, and

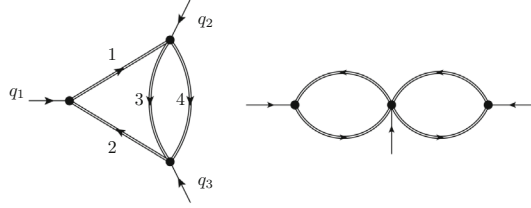


Fig. 3. LEFT: The dunce's cap has a non-trivial divergent subgraph given by the subgraph spanned by edges 3, 4. RIGHT: A double bubble graph, upon which the canonical form vanishes

where we write $\mu_5 = \mu_1$. The generalised graph Laplacian with respect to the cycle $c_1 = e_1 + \dots + e_4$ is

$$\tilde{\Lambda}_G(\mu, m) = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \mu_1\alpha_1 + \mu_2\alpha_2 + \mu_3\alpha_3 + \mu_4\alpha_4 \\ \bar{\mu}_1\alpha_1 + \bar{\mu}_2\alpha_2 + \bar{\mu}_3\alpha_3 + \bar{\mu}_4\alpha_4 & (\mu_1\bar{\mu}_1 + m_1^2)\alpha_1 + \dots + (\mu_4\bar{\mu}_4 + m_4^2)\alpha_4 \end{pmatrix}$$

The associated graph polynomials are:

$$\begin{aligned} \Psi_G &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \Phi_G(q) &= q_1^2\alpha_1\alpha_2 + q_2^2\alpha_2\alpha_3 + q_3^2\alpha_3\alpha_4 + q_4^2\alpha_4\alpha_1 + (q_1 + q_2)^2\alpha_1\alpha_3 + (q_1 + q_4)^2\alpha_2\alpha_4 \\ \Xi_G(q, m) &= \Phi_G(q) + (m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3 + m_4^2\alpha_4)\Psi_G \end{aligned}$$

The canonical form ϖ_G^3 is proportional to the Feynman differential form:

$$\varpi_G^3(q, m) = 3N_G \frac{\Omega_G}{\Xi_G(q, m)^2}$$

where the numerator has the symmetric form:

$$N_G = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{\mu_1}{\bar{\mu}_1} & \frac{\mu_2}{\bar{\mu}_2} & \frac{\mu_3}{\bar{\mu}_3} & \frac{\mu_4}{\bar{\mu}_4} \\ m_1^2 + \mu_1\bar{\mu}_1 & m_2^2 + \mu_2\bar{\mu}_2 & m_3^2 + \mu_3\bar{\mu}_3 & m_4^2 + \mu_4\bar{\mu}_4 \end{pmatrix}.$$

It can be re-expressed as a function of the external momenta:

$$N_G = 2i \operatorname{Im} \left(q_2\bar{q}_3 m_1^2 - q_3\bar{q}_4 m_2^2 + q_4\bar{q}_1 m_3^2 - q_1\bar{q}_2 m_4^2 + q_1q_3\bar{q}_2\bar{q}_4 \right).$$

In conclusion, we find that

$$I_G(\varpi^3, q, m) = \int_{L\mathcal{M}_{1,4,4}^{\text{trop}}} \varpi^3 = 3N_G \int_{\sigma_G} \frac{\Omega_G}{\Xi_G(q, m)^2}$$

which is proportional to the Feynman integral for the massive box graph (viewed in $D = 4$ spacetime dimensions).

11.2.2. *Examples from $\mathcal{M}_{2,3,4}^{\text{trop}}$* Figure 3 depicts cells of dimension 4 on $\mathcal{M}_{2,3,4}^{\text{trop}}$.

The graph on the right is an example of a 1-vertex join. There is a similar graph in which the central vertex has no incoming momentum, and the left-most vertex has two incoming vertices. In both cases, the graph Laplacian has the form

$$\begin{pmatrix} \alpha_1 + \alpha_2 & 0 & \mu_1\alpha_1 + \mu_2\alpha_2 \\ 0 & \alpha_3 + \alpha_4 & \mu_3\alpha_3 + \mu_4\alpha_4 \\ \bar{\mu}_1\alpha_1 + \bar{\mu}_2\alpha_2 & \bar{\mu}_3\alpha_3 + \bar{\mu}_4\alpha_4 & \sum_i (\mu_i\bar{\mu}_i + m_i^2)\alpha_i \end{pmatrix}$$

and one verifies that ϖ^3 vanishes for this matrix. Therefore the only non-zero contribution is from the dunce's cap, depicted on the left of Fig. 3. It is one of the first examples of a graph with a non-trivial subdivergence (given by the subgraph spanned by edges 3, 4), so the associated Feynman integral diverges. By contrast, the canonical integral is necessarily finite and it is instructive to see why.

Since there are only 3 external legs, we can always assume that the momenta are complex numbers q_1, q_2, q_3 subject to momentum conservation $q_1 + q_2 + q_3 = 0$. For the basis of cycles given by $c_1 = e_1 + e_3 + e_2$, $c_2 = e_4 - e_3$, and the orientations of edges as shown in Fig. 3, the generalised graph Laplacian takes the form:

$$\tilde{\Lambda}_G = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 & -\alpha_3 & \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 \\ -\alpha_3 & \alpha_3 + \alpha_4 & \alpha_4\mu_4 - \alpha_3\mu_3 \\ \bar{\mu}_1\alpha_1 + \bar{\mu}_2\alpha_2 + \bar{\mu}_3\alpha_3 & \bar{\mu}_4\alpha_4 - \bar{\mu}_3\alpha_3 & \sum_{i=1}^4 \alpha_i(m_i^2 + \mu_i\bar{\mu}_i) \end{pmatrix}$$

where (4.2) takes the form:

$$\begin{aligned} \mu_1 - \mu_2 &= q_1 \\ \mu_3 + \mu_4 - \mu_1 &= q_2 \\ \mu_3 + \mu_4 - \mu_2 &= q_1 + q_2 = -q_3. \end{aligned}$$

One finds that

$$\begin{aligned} \Psi_G &= (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_3\alpha_4 \\ \Phi_G(q, m) &= q_1\bar{q}_1(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4) + q_2\bar{q}_2\alpha_1\alpha_3\alpha_4 + q_3\bar{q}_3\alpha_2\alpha_3\alpha_4. \end{aligned}$$

A computation shows that the canonical form of the second kind in degree 3 is

$$\varpi_G^3 = 3(q_1\bar{q}_2 - q_2\bar{q}_1) \left(\alpha_3^2 m_3^2 - \alpha_4^2 m_4^2 \right) \frac{\Omega_G}{\Xi_G^2(q, m)},$$

where we note that the factor $q_1\bar{q}_2 - q_2\bar{q}_1 = q_3\bar{q}_1 - q_1\bar{q}_3$ is indeed symmetric, by momentum conservation. The numerator vanishes at $\alpha_3 = \alpha_4 = 0$, to compensate the pole arising from the vanishing of $\Xi_G(q, m)$ along this locus. Note also that if $m_3 = m_4$ then the integral of ϖ_G^3 will vanish, due to the fact that interchanging edges 3 and 4 induces an odd permutation on the edges of G . In conclusion, for a suitable choice of orientation of G one has

$$\int_{\sigma_G} \varpi_G^3 = 3(q_1\bar{q}_2 - q_2\bar{q}_1) \int_{\sigma_G} \left(\alpha_3^2 m_3^2 - \alpha_4^2 m_4^2 \right) \frac{\Omega_G}{\Xi_G^2(q, m)}.$$

It is interesting to compare this to the renormalised amplitude associated to the graph G , which was discussed in [BK13].

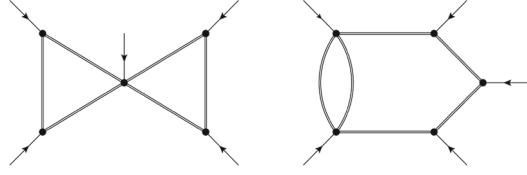


Fig. 4. Some graphs with 6 edges for which ϖ_G^5 vanishes

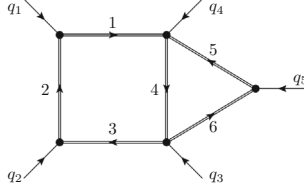


Fig. 5. The box-triangle graph

11.2.3. The space $\mathcal{M}_{3,2,4}^{\text{trop}}$ The associated graphs have at most two external momenta. By momentum conservation, they lie on a real line and so we may choose a real momentum routing for which the generalised Laplacian $\tilde{\Lambda}_G(q, m)$ will be symmetric. Therefore ϖ_G^3 vanishes, and there are no non-zero canonical integrals.

11.3. Canonical amplitudes in degree 5. In degree 5 there exist canonical forms ω^5 and ϖ^5 of both kinds. There is a unique vacuum diagram, namely the wheel with 3 spokes and no external legs, which pairs non-trivially with the former. Its amplitude is $6\zeta(3)$, and is discussed in detail in [Bro21].

Henceforth we shall focus on the form ϖ^5 . Since β^5 vanishes on matrices of rank ≤ 2 , the form ϖ_G^5 is zero unless $h_G \geq 2$. There are several graphs G with 6 edges and two or more loops which could potentially pair non-trivially with ϖ^G , but we shall only focus on a couple of examples for reasons of space.

11.3.1. Examples on $\mathcal{M}_{2,5,6}^{\text{trop}}$ One shows by direct calculation that the two graphs depicted in Fig. 4 satisfy $\varpi_G^5 = 0$.

There is a single potentially non-zero canonical integral on $\mathcal{M}_{2,5,6}^{\text{trop}}$, given by the box-triangle graph depicted in Fig. 5. A computer calculation shows that

$$\varpi_G^5 = \left(\frac{N_1}{\Xi_G(q, m)} - 5N_2 \right) \frac{\Omega_G}{\Xi_G(q, m)^2}$$

where N_2 is a certain polynomial in the m_i^2 and $q_i, \bar{q}_i$, and the polynomial N_1 factorises into a product of two terms

$$N_1 = f(\alpha_1, \alpha_2, \alpha_3, q)g(\alpha_4, \alpha_5, \alpha_6, q, m)$$

where f is a homogeneous linear form in $\alpha_1, \alpha_2, \alpha_3$ which does not depend on the masses and g is of degree 2 in the α_i .

The integral corresponding to the term N_2 is the usual Feynman integral in 4 space time dimensions. It would be interesting to know if the canonical integral admits a natural

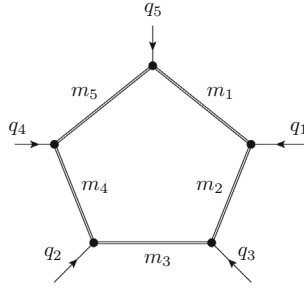


Fig. 6. Massive pentagon diagram

interpretation in terms of Feynman integrals via integration by parts identities, or if it may be obtained from the usual Feynman integral by application of a natural differential operator in the external kinematics.

12. Example of a Stokes Relation for the Massive Box Diagram

Consider the massive pentagon diagram with arbitrary masses $m_1, \dots, m_5 \in \mathbb{R}$ and external particle momenta in $q_1, \dots, q_5 \in \mathbb{R}^2 \cong \mathbb{C}$ subject to momentum conservation (Fig. 6). Contracting each of its edges leads to five massive box diagrams (Fig. 7).

Since ϖ^3 is of compact type for 1-loop graphs, we may apply Corollary 10.4 to deduce a five-term relation for the massive box diagram. Indeed (10.4) implies that

$$\sum_{i=1}^5 I_{G_i}(\varpi^3, q, m) = 0 \quad (12.1)$$

where $G_1, \dots, G_5$ are the five diagrams pictured in Fig. 7. This relation can easily be checked in the massless case using Panzer's Hyperint [Pan15].

Since the canonical integrals for massive boxes are proportional to their Feynman integrals, we deduce a 5-term relation for massive box Feynman integrals. It is known [DD98] that the latter are expressible as a linear combination of a number of dilogarithms. The graphical relation above is presumably equivalent to Abel's 5-term equation for the dilogarithm. This functional equation should, in addition, be compatible with the motivic coaction which was computed in [Tap21].

13. Example of a Canonical Amplitude in the Quaternionic Case

13.1. A canonical amplitude for ϖ^5 . Let G be the one-loop hexagon diagram with 6 external momenta in $\mathbb{R}^4$, and 6 masses. The generalised graph Laplacian is

$$\tilde{\Lambda}_G(\mu, m) = \sum_{i=1}^6 \alpha_i \left(\frac{1}{\mu_i} \middle| \frac{\mu_i}{\mu_i \mu_i + m_i^2} \right)$$

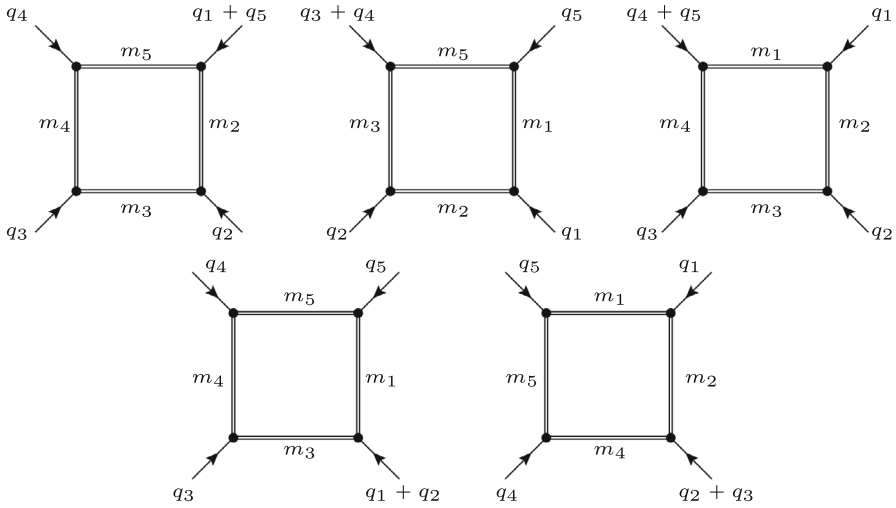


Fig. 7. Five-term relation from pentagon diagram

where $\mu_i \in \mathcal{Q}$ is a quaternionic momentum routing. Its complex adjoint may be represented by

$$\chi_{\tilde{\Lambda}_G(\mu, m)} = \sum_{i=1}^6 \alpha_i \left(\begin{array}{cc|cc} 1 & 0 & \lambda_i & v_i \\ 0 & 1 & -\bar{v}_i & \bar{\lambda}_i \\ \hline \bar{\lambda}_i - v_i & \lambda_i \bar{\lambda}_i + v_i \bar{v}_i + m_i^2 & 0 & \lambda_i \bar{\lambda}_i + v_i \bar{v}_i + m_i^2 \\ \bar{v}_i & \lambda_i & 0 & \lambda_i \bar{\lambda}_i + v_i \bar{v}_i + m_i^2 \end{array} \right)$$

where we write $\mu_i = \lambda_i + jv_i$. We find by computer calculation that

$$\omega_G^5 = 60 N_G \frac{\Omega_G}{\Xi_G(q, m)^3}$$

where N_G is the determinant of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 & \bar{\lambda}_4 & \bar{\lambda}_5 & \bar{\lambda}_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 & \bar{v}_5 & \bar{v}_6 \\ m_1^2 + \|\mu_1\|^2 & m_2^2 + \|\mu_2\|^2 & m_3^2 + \|\mu_3\|^2 & m_4^2 + \|\mu_4\|^2 & m_5^2 + \|\mu_5\|^2 & m_6^2 + \|\mu_6\|^2 \end{pmatrix}$$

where we write $\|\mu_i\|^2 = \lambda_i \bar{\lambda}_i + v_i \bar{v}_i$. Thus the canonical integral associated to the massive hexagon is proportional to the usual Feynman integral (viewed in 6 spacetime dimensions). Stokes' formula applied to a massive quaternionic heptagon diagram leads to a 7-term functional equation for these integrals.

14. Tropical Single-Valued Integration on Curves

In this section, which is not required for the paper, we provide some motivation for the definition of the generalised graph Laplacian by analogy with smooth projective algebraic curves with a finite number of punctures.

14.1. Periods on smooth algebraic curves. Let X be a smooth compact curve over a field $k \subset \mathbb{C}$, and let $\Sigma \subset X(\mathbb{C})$ denote a finite set of points. Consider the Gysin (residue) sequence in Rham cohomology

$$0 \longrightarrow H_{dR}^1(X) \longrightarrow H_{dR}^1(X \setminus \Sigma) \xrightarrow{\text{Res}} H_{dR}^0(\Sigma)(-1) \longrightarrow H_{dR}^2(X)$$

and suppose that we are given an element ('external momentum'):

$$\alpha_q \in \ker \left(H_{dR}^0(\Sigma)(-1) \longrightarrow H_{dR}^2(X) \right) \otimes_k \mathbb{C},$$

which may be interpreted as a divisor of degree 0 on X with coefficients in $\mathbb{C}$ supported on Σ . By pulling back the above sequence along the copy of $\mathbb{C}(-1) \subset H_{dR}^0(\Sigma)(-1) \otimes_k \mathbb{C}$ spanned by α_q , we obtain a simple extension over $\mathbb{C}$:

$$0 \longrightarrow H_{dR}^1(X) \otimes_k \mathbb{C} \longrightarrow \mathcal{E}_X \longrightarrow \mathbb{C}(-1) \longrightarrow 0. \quad (14.1)$$

Since $F^1\mathbb{C}(-1) = \mathbb{C}(-1)$, it splits if and only if the sequence

$$0 \longrightarrow F^1 H_{dR}^1(X) \otimes_k \mathbb{C} \longrightarrow F^1 \mathcal{E}_X \longrightarrow \mathbb{C}(-1) \longrightarrow 0$$

splits. Such a splitting is given by an element ('momentum routing')

$$\mu \in F^1 \mathcal{E}_X \quad (14.2)$$

whose image in $\mathbb{C}(-1)$ is α_q . In other words, $\text{Res } \mu = \alpha_q$. Since $H^0(X, \Omega_X^1(\log \Sigma)) \xrightarrow{\sim} \text{gr}_F^1(H_{dR}^1(X \setminus \Sigma))$, we can assume that μ is represented by a differential of the third kind with only logarithmic poles along Σ . It is uniquely defined only up to adding a linear combination of holomorphic forms:

$$\mu \mapsto \mu + \sum_{i=1}^g \lambda_i \omega_i$$

where $\omega_1, \dots, \omega_g \in H^0(X, \Omega_X^1) \cong F^1 H_{dR}^1(X)$ and $\lambda_i \in \mathbb{C}$.

We wish to consider a Hermitian form Q on $F^1 \mathcal{E}_X$ defined formally by:

$$\begin{aligned} Q : F^1 \mathcal{E}_X \otimes F^1 \mathcal{E}_X &\longrightarrow \mathbb{C} \\ \eta_1 \otimes \eta_2 &\mapsto \frac{1}{2\pi i} \int \eta_1 \wedge \overline{\eta_2} \end{aligned} \quad (14.3)$$

Alas, the integrals on the right are not always convergent. There are three cases:

- (1) $\eta_1 = \omega_i$, and $\eta_2 = \omega_j$ are holomorphic. Then the restriction of Q to $H_{dR}^1(X)$ is given by the convergent integrals

$$\omega_i \otimes \omega_j \mapsto \frac{1}{2\pi i} \int \omega_i \wedge \overline{\omega_j}$$

from which one retrieves the Riemann polarisation form. If k is real, then this actually defines a symmetric quadratic form on $H_{dR}^1(X)$.

- (2) $\eta_1 = \omega_i$ is holomorphic, and $\eta_2 = \mu$ is of the third kind (or vice-versa). Then the singular integral

$$c_i(\mu) = \frac{1}{2\pi i} \int_X \omega_i \wedge \bar{\mu}$$

is convergent since it defines a single-valued period of $H_{dR}^1(X \setminus \Sigma)$. To see this, note that $F^1 H^1(X, \Sigma) \xrightarrow{\sim} F^1 H^1(X)$ is an isomorphism and since ω_i vanishes along Σ it may canonically be viewed as an element of $H^1(X, \Sigma)$. The integral above is an instance of the single-valued pairing [BD21, §6.4]:

$$s : H_{dR}^1(X, \Sigma) \otimes H_{dR}^1(X \setminus \Sigma) \longrightarrow \mathbb{C}$$

- (3) Only in the case $\eta_1 = \eta_2 = \mu$ does one have an ill-defined integral:

$$\frac{1}{2\pi i} \int \mu \wedge \bar{\mu}$$

It is singular in a neighbourhood of a point $\sigma \in \Sigma$, since the integrand may be expressed in local polar coordinates $z = \rho e^{i\theta}$ based at σ in the form

$$\mu \wedge \bar{\mu} = \lambda \frac{d\rho}{\rho} d\theta + \text{regular terms},$$

for some $\lambda \in \mathbb{C}$, which is not ‘polar-smooth’ in the terminology of [BD21]. One can regularise these integrals around each point of Σ by a number of methods (see also [BD21, Remark 3.22]). The method used is unimportant for this discussion but the value of the integral will in general depend upon it. In any case we may simply put

$$\frac{1}{2\pi i} \int^{\text{reg}} \mu \wedge \bar{\mu} := x_m$$

where x_m is any real and positive number of our choosing. The possibility of choosing x_m relates to the presence of masses in Feynman graphs.

Thus, with respect to the basis $\omega_1, \dots, \omega_g, \mu$ of $\mathcal{E}_X$, the form Q is represented by a Hermitian matrix

$$Q = \left(\begin{array}{c|c} Q_0 & \begin{matrix} \vdots \\ c_i(\mu) \\ \vdots \end{matrix} \\ \hline \cdots \overline{c_i(\mu)} \cdots & x_m \end{array} \right)$$

This is precisely the form of the generalised Laplacian matrix (compare (5.4)).

14.2. Tropical analogy. Let G be a connected graph (with no external half edges in the first instance). Choose an orientation on its edges. According to [MZ08], a differential form on G is a linear combination

$$\omega = \sum_{e \in E_G} \lambda_e de$$

where $\lambda_e \in \mathbb{Z}$ such that for all vertices v one has $\sum_{s(e)=v} \lambda_e = \sum_{t(e)=v} \lambda_e$. Thus the space of differential forms Ω_G^1 on G lies in an exact sequence

$$0 \longrightarrow \Omega_G^1 \longrightarrow \mathbb{Z}^{E_G} \longrightarrow \mathbb{Z}^{V_G} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and may be identified with $H_1(G; \mathbb{Z})$. The inner product on $\mathbb{Z}^{E_G}$ is given by

$$\langle de_i, de_j \rangle = x_i \delta_{ij},$$

and restricts to a positive definite quadratic form on Ω_G^1 .

Now suppose that G has external half-edges, and that we are given an element $\mathbf{q} \in \ker(\mathbb{Z}^{V_G} \rightarrow \mathbb{Z}) \otimes \mathbb{C}$. As in §5.1, we deduce an extension $\tilde{\Omega}_G^1$ satisfying

$$0 \longrightarrow \Omega_G^1 \otimes \mathbb{C} \longrightarrow \tilde{\Omega}_G^1 \longrightarrow \mathbf{q} \mathbb{C} \longrightarrow 0$$

and the above inner product (extended $\mathbb{C}$ -linearly) restricts to a Hermitian form

$$\begin{aligned} \tilde{\Omega}_G^1 \otimes_{\mathbb{R}} \tilde{\Omega}_G^1 &\longrightarrow \mathbb{C} \\ \eta_1 \otimes \eta_2 &\mapsto \langle \eta_1, \bar{\eta}_2 \rangle \end{aligned} \quad (14.4)$$

which is a tropical analogue of (14.3). The generalised graph Laplacian matrix $\tilde{\Lambda}_G(q, 0)$ for zero masses is precisely the matrix of this Hermitian form with respect to a splitting of $\tilde{\Omega}_G^1$, which is given by any differential form

$$v = \sum_e \mu_e de$$

which satisfies the ‘residue’ equation $\sum_{s(e)=v} \lambda_e - \sum_{t(e)=v} \lambda_e = \mathbf{q}_v$ for all vertices v . The presence of masses corresponds, as above, to modifying only the value of $\langle v, \bar{v} \rangle$ in the bottom-right hand corner, and from this perspective, we see that the generalised graph Laplacian (5.4) is a tropical analogue of the regularised single-valued integration pairing discussed in §14.1.

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Declarations

Competing interests The author has no relevant financial or non-financial interests to disclose, nor any competing interests to declare that are relevant to the content of this article.

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References

- [ACP22] Allcock, D., Corey, D., Payne, S.: Tropical moduli spaces as symmetric Δ -complexes. *Bull. Lond. Math. Soc.* **54**(1), 193–205 (2022)
- [AHHM22] Arkani-Hamed, N., Hillman, A., Mizera, S.: Feynman polytopes and the tropical geometry of UV and IR divergences (2022)
- [AHT14] Arkani-Hamed, N., Trnka, J.: The amplituhedron. *J. High Energy Phys.* **30**, 1029–8479 (2014)
- [BBC+20] Brandt, M., Bruce, J., Chan, M., Melo, M., Moreland, G., Wolfe, C.: On the top-weight rational cohomology of $\mathcal{A}_g$ (2020)
- [BBH20] Beekveldt, R., Borinsky, M., Herzog, F.: The Hopf algebra structure of the R^* -operation. *J. High Energy Phys.* **2020**(7), 1–36 (2020)
- [BD21] Brown, F., Dupont, C.: Single-valued integration and double copy. *J. Reine Angew. Math.* **775**, 145–196 (2021)
- [BEK06] Bloch, S., Esnault, H., Kreimer, D.: On motives associated to graph polynomials. *Commun. Math. Phys.* **267**(1), 181–225 (2006)
- [BK10] Bloch, S., Kreimer, D.: Feynman amplitudes and Landau singularities for one-loop graphs. *Commun. Number Theory Phys.* **4**(4), 709–753 (2010)
- [BK13] Brown, F., Kreimer, D.: Angles, scales and parametric renormalization. *Lett. Math. Phys.* **103**(9), 933–1007 (2013)
- [BK20] Berghoff, M., Kreimer, D.: Graph complexes and Feynman rules. [arXiv:2008.09540](https://arxiv.org/abs/2008.09540) (2020)
- [BM19] Berghoff, M., Mühlbauer, M.: Moduli spaces of colored graphs. *Topol. Appl.* **268**, 106902 (2019)
- [BMV11] Brannetti, S., Melo, M., Viviani, F.: On the tropical Torelli map. *Adv. Math.* **226**(3), 2546–2586 (2011)
- [Bor74] Borel, A.: Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup.* **4**(7), 235–272 (1975)
- [Bro10] Brown, F. C. S.: On the periods of some Feynman integrals. [arXiv:0910.0114](https://arxiv.org/abs/0910.0114) (2010)
- [Bro17] Brown, F.: Feynman amplitudes, coaction principle, and cosmic Galois group. *Commun. Number Theory Phys.* **11**(3), 453–556 (2017)
- [Bro21] Brown, F.: Invariant differential forms on complexes of graphs and Feynman integrals. *SIGMA Symmetry Integrability Geom. Methods Appl.* **17**, 103–154 (2021)
- [BS21] Borinsky, M., Schnetz, O.: Graphical functions in even dimensions. [arXiv:2105.05015](https://arxiv.org/abs/2105.05015) (2021)
- [BY11] Brown, F., Yeats, K.: Spanning forest polynomials and the transcendental weight of Feynman graphs. *Commun. Math. Phys.* **301**(2), 357–382 (2011)
- [CV86] Culler, M., Vogtmann, K.: Moduli of graphs and automorphisms of free groups. *Invent. Math.* **84**(1), 91–119 (1986)
- [DD98] Davydychev, A.I., Delbourgo, R.: A geometrical angle on Feynman integrals. *J. Math. Phys.* **39**(9), 4299–4334 (1998)
- [DR22] Dresden, A., Richardson, R.G.D.: The twenty-eighth annual meeting of the American Mathematical Society. *Bull. Am. Math. Soc.* **28**(4), 148–164 (1922)
- [GK16] Gürdoğan, Ö., Kazakov, V.: New integrable 4D quantum field theories from strongly deformed planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. *Phys. Rev. Lett.* **117**(20), 201602 (2016) [Addendum: *Phys. Rev. Lett.* **117**, 259903 (2016)]
- [Gol19] Golz, M.: Dodgson polynomial identities. *Commun. Number Theory Phys.* **13**(4), 667–723 (2019)
- [GPS17] Golz, M., Panzer, E., Schnetz, O.: Graphical functions in parametric space. *Lett. Math. Phys.* **107**(6), 1177–1192 (2017)
- [MZ08] Mikhalkin, G., Zharkov, I.: Tropical curves, their Jacobians and theta functions. In: *Curves and Abelian Varieties*, volume 465 of *Contemporary Mathematics*, pp. 203–230. American Mathematical Society, Providence, RI (2008)
- [Pan15] Panzer, E.: Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals. *Comput. Phys. Commun.* **188**, 148–166 (2015)
- [Sch10] Schnetz, O.: Quantum periods: a census of ϕ^4 -transcendentals. *Commun. Number Theory Phys.* **4**(1), 1–47 (2010)

- [Tap21] Tapušković, M.: Motivic Galois coaction and one-loop Feynman graphs. *Commun. Number Theory Phys.* **15**(2), 221–278 (2021)
- [Tig99] Tignol, J.-P.: Pfaffiens et déterminant de E. H. Moore. *Bull. Belg. Math. Soc. Simon Stevin* **6**(4), 537–539 (1999)
- [Zam80] Zamolodchikov, A.B.: ‘Fishnet’ diagrams as a completely integrable system. *Phys. Lett. B* **97**, 63–66 (1980)
- [Zha97] Zhang, F.: Quaternions and matrices of quaternions. *Linear Algebra Appl.* **251**, 21–57 (1997)

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