Quantum toroidal algebras, quantum affine algebras, and their representation theory

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Abstract

In this thesis we investigate the structure and representation theory of quantum algebras. After gathering the necessary preliminaries, we begin by focusing in particular on quantum toroidal algebras. We first construct an action of the extended double affine braid group $\ddot{\mathcal{B}}$ on the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$ in untwisted and twisted types. As a crucial step in the proof, we obtain a finite Drinfeld new style presentation for a broad class of quantum affinizations.

In the untwisted case, using our action and certain involutions of $\ddot{\mathcal{B}}$ we produce automorphisms and anti-involutions of $U_q(\mathfrak{g}_{\mathrm{tor}})$ that exchange its horizontal and vertical subalgebras. Moreover, they switch the central elements C and $k_0^{a_0} \dots k_n^{a_n}$ up to inverse. This can be regarded as the analogue, for these quantum toroidal algebras, of the duality for double affine braid groups utilized by Cherednik to realise the difference Fourier transform in his celebrated proof of the Macdonald evaluation conjectures. Our work generalises existing results in type $A_n^{(1)}$ due to Miki, which have been instrumental in the study of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$.

We proceed by proving certain compatibilities between (anti-)automorphisms on either side of our braid group action. From these identities, we deduce that the central extension of $SL_2(\mathbb{Z})$ – which is isomorphic to the braid group on three strands – acts on $U_q(\mathfrak{g}_{\mathrm{tor}})$ in untwisted types. This provides a counterpart within the quantum setting to the congruence group actions on double affine braid groups and Hecke algebras established by Cherednik and Ion-Sahi.

We conclude our treatment of $U_q(\mathfrak{g}_{\mathrm{tor}})$ by discussing ongoing research related to its representation theory. In particular, we show that twisting Drinfeld's topological coproduct with our anti-involution ψ yields a tensor product structure that is well-defined for ℓ-highest weight modules. Additionally, in the simply laced case, we construct vector representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ that are explicitly described in terms of Young column bases.

In the latter sections of this thesis, we consider crystal bases for representations of quantum affine algebras in types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$. Specifically, we construct Young wall models for the level 1 irreducible highest weight crystals $B(\lambda)$ and Fock space crystals $B(\mathcal{F}(\lambda))$. In both instances the starting point is a perfect crystal of level 1, which we represent in terms of equivalence classes of Young columns stacked within a Young column pattern.

In conjunction with the theory of perfect crystals and energy functions, we then realise the crystals $B(\lambda)$ and $B(\mathcal{F}(\lambda))$ in terms of reduced and proper Young walls respectively. These consist of coloured blocks stacked inside a relevant Young wall pattern, satisfying certain combinatorial conditions. Moreover, the crystal structure in each case is described entirely in terms of adding and removing blocks.

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Chapter 1

Introduction

Quantum groups $U_q(\mathfrak{g})$ were originally defined by Drinfeld [22] and Jimbo [47] as non-trivial deformations of the universal enveloping algebras for Kac-Moody Lie algebras, within the category of Hopf algebras. Their introduction was motivated by work of Faddeev and the Leningrad School developing the quantum inverse scattering method for constructing and solving quantum integrable systems. Quantum groups and their generalisations have since become objects of fundamental importance within mathematics, possessing remarkable connections across a wide range of areas such as algebra and representation theory; algebraic geometry; low-dimensional topology and knot theory; and mathematical physics.

Quantum affinizations and quantum toroidal algebras

In particular, quantum affine algebras occur in this way as the Drinfeld-Jimbo quantum groups $U_q(\hat{\mathfrak{g}})$ associated to affine Kac-Moody algebras. Subsequently, Drinfeld [23] provided an alternative realization of $U_q(\hat{\mathfrak{g}})$ in the untwisted case as a quantum affinization $\widehat{U_q(\mathfrak{g})}$ of the corresponding finite quantum group, as well as a similar realization for twisted types. Proofs of the equivalence of the two presentations were then published in work by Beck [3], Jing and Zhang [50, 52, 53], and Damiani [17, 18]. This new presentation, known as the 'Drinfeld new realization', has played a crucial role in studying the rich representation theory of quantum affine algebras. For example, Chari and Pressley [9–12] classified the finite dimensional representations in terms of Drinfeld polynomials and studied R-matrices on their tensor products, while Frenkel and Jing [27, 49] constructed vertex representations.

Drinfeld's quantum affinization resembles the formation of untwisted affine Lie algebras by adjoining a derivation to a central extension of the loop algebra of a finite dimensional simple Lie algebra. This procedure can more generally be applied to any Kac-Moody algebra, and as a special case takes affine Kac-Moody algebras to 'double affine' or toroidal Lie algebras.

Similarly, the quantum affinization process works for the quantum group of any Kac-Moody algebra, and produces a broad new class of quantum algebras. In particular, from quantum affine algebras we obtain the quantum toroidal algebras $U_q(\mathfrak{g}_{\mathrm{tor}})$, which were first introduced by Ginzburg-Kapranov-Vasserot [29] in their study of Langlands reciprocity for algebraic surfaces, and also by Varagnolo-Vasserot [105] in type $A_n^{(1)}$.

While the theory of quantum affinizations is at this stage far less developed than that of quantum groups, there nevertheless exist many notable successes. Extending results for quantum affine algebras to the general case, Hernandez [35] classified the ℓ-highest weight representations by Drinfeld polynomials, while Jing [51] constructed vertex representations. One major obstacle when dealing with representations of quantum affinizations is the lack of a coproduct or Hopf algebra structure. However, many authors have employed *topological* coproduct constructions in certain types [20, 21, 24, 31, 35, 36, 87], and the existence of such structures in general was very recently proved by Damiani [19].

Nakajima [92] realised arbitrary simply laced quantum affinizations geometrically, via a morphism to the equivariant K-theory convolution algebra of quiver varieties on the underlying Dynkin diagram. Furthermore, by considering fibers of the natural projective morphism from smooth to affine quiver varieties, this construction immediately produces families of representations for these algebras. For the specific case of simply laced quantum toroidal algebras, see also [93, 107]. Moreover, this realization was recently extended to arbitrary types (and indeed to *shifted* quantum loop groups) by Varagnolo-Vasserot [108, 109] using critical K-theory.

Quantum toroidal algebras in particular contain horizontal and vertical subalgebras $\mathcal{U}_h$ and $\mathcal{U}_v$, each isomorphic to a quantum affine algebra. Specifically, $\mathcal{U}_h$ is the natural copy of $U_q(\hat{\mathfrak{g}})$ from which $U_q(\mathfrak{g}_{\mathrm{tor}})$ is formed via quantum affinization, and $\mathcal{U}_v$ is the quantum affinization of the finite quantum group lying inside it. Just as in the quantum affine setting, representations of $U_q(\mathfrak{g}_{\text{tor}})$ are equipped with a level determined by the action of the central elements. Namely, a representation is said to have level (a, b) if $\mathcal{U}_v$ acts with level a and $\mathcal{U}_h$ with level b.

In type $A_n^{(1)}$, Varagnolo and Vasserot [105] established a Schur-Weyl duality between representations of $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ and those of the double affine Hecke algebra. This duality was then used to construct a level $(0, 1)$ action on the q-Fock space [101, 106]. Note that [101] also proves the irreducibility of the representation. Nagao [91] showed that this is isomorphic to Nakajima's geometric representation in type $A_n^{(1)}$ – torus fixed points on the equivariant K-theory side are identified with certain simultaneous eigenvectors in the q-Fock space, defined using non-symmetric Macdonald polynomials.

There is also the level $(1, 0)$ vertex representation of $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ originally due to Saito [100]. Motivated by trying to understand the relationship with the q-Fock space representation, Miki [86] constructed an automorphism of $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ using a toroidal braid group action. In particular, his automorphism exchanges the horizontal and vertical subalgebras and swaps their central elements up to inverse.

Miki [87] then used this automorphism to study the representation theory of $U_q(\mathfrak{sl}_{n+1,\text{tor}})$, obtaining among other things a classification of ℓ-highest weight representations by Drinfeld polynomials, and R-matrices on tensor products of these modules. He also clarified the relationship between the vertex and q-Fock space representations. Furthermore, the Miki automorphism has been applied extensively in papers by many other authors, and has been instrumental for studying $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ and related topics.

Surprisingly, comparatively little is known for quantum toroidal algebras outside of type $A_n^{(1)}$. The primary aim of our work in Chapter 3 is to investigate their structure and representation theory in general.

Crystal bases and Young wall realizations

Let us now return to the realm of Drinfeld-Jimbo quantum groups. The theory of crystal bases pioneered by Kashiwara [65, 66] provides a powerful tool for studying the representation theory of these algebras. Crystal bases can be seen as the $q = 0$ limits of global (canonical) bases, and possess a host of combinatorial features that reflect the internal structures of the representations. On the other hand, Lusztig developed the theory of canonical bases using a more geometric approach [79, 80].

These crystal bases retain much of the information regarding the corresponding representations, and their combinatorial description frequently enables us to reduce challenging and abstract questions in representation theory – such as the characters and tensor decompositions of integrable modules – to far more tractable problems in combinatorics.

Moreover it is often possible to obtain concrete realizations of crystal bases, which shed light on the structure of the representations. Producing such combinatorial models is therefore an important problem in the representation theory of quantum groups. A key goal in this direction is to construct realizations for the crystal bases of irreducible integrable highest weight representations, since these form the connected components of the crystal bases for all integrable representations.

Young tableau and Young wall models provide a particularly intuitive and easy-to-operate class of such realizations. Here, the vertices of a crystal are represented by stackings of coloured blocks within certain patterns, and arrows simply correspond to adding or removing a block. In the case of finite quantum groups, Kashiwara and Nakashima [73, 95] described the crystal bases of finite dimensional $U_q(\mathfrak{g})$-modules in all non-exceptional types in terms of (generalised) Young tableaux. Moreover Kang and Misra [64] gave a similar construction in type G_2, while recent work of Hong-Lee [43] deals with tableau realizations in types E_6 and E_7. Tableaux models for the crystal basis $B(\infty)$ associated to the negative part of the quantum group are considered in [41, 42].

In order to approach the case of quantum affine algebras, Kang et al. [57, 58] used the theory of perfect crystals and energy functions to construct a *path realization* for the irreducible highest weight crystals $B(\lambda)$, known as the Kyoto path model. Here, paths are infinite sequences of elements inside some (finite) perfect crystal which stabilise to a certain ground state sequence.

Using this path realization, Kang [56] obtained Young wall models for the level 1 highest weight crystals of quantum affine algebras in types $A_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$. The remaining non-exceptional affine type $C_n^{(1)}$ was subsequently addressed in work of Hong-Kang-Lee [40]. Furthermore, this programme was later generalised to arbitrary level by Kang and Lee in [62, 63].

As for the exceptional affine types, Fan-Han-Kang-Shin [25] built Young wall models for the level 1 irreducible highest weight crystals in types $D_4^{(3)}$ and $G_2^{(1)}$. Moreover, the author [77] recently constructed such models in types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$. This part of the story was then completed by the author [33] – in collaboration with Han, Jin and Kang – via a treatment of the remaining affine types $E_6^{(2)}$ and $F_4^{(1)}$. The Young walls contained in these models are said to be *reduced*.

Following initial work by Kashiwara, Miwa and Stern [72, 102] in type $A_n^{(1)}$, Kashiwara-Miwa-Petersen-Yung [71] gave a semi-infinite wedge construction of Fock space representations $\mathcal{F}(\lambda)$ for quantum affine algebras. For a $U_q'(\hat{\mathfrak{g}})$-module V satisfying various technical conditions, one considers the semi-infinite limit $\lim_{r\to\infty} \bigwedge^r V_{\text{aff}}$ of the q-exterior powers of its affinization, taken along a certain ground state vector. This space can be endowed with the structure of a $U_q(\hat{\mathfrak{g}})$-module, and is called a Fock space representation.

Among other things, Kashiwara et al. described the crystal bases $B(\mathcal{F}(\lambda))$ for these representations in terms of the crystal basis B of V and its energy function H, and gave the decomposition of $\mathcal{F}(\lambda)$ as a sum of irreducible representations. Kashiwara [70] later proved that certain *good* $U_q'(\hat{\mathfrak{g}})$-modules possess the properties required to produce Fock spaces via this process, and moreover obtained global bases for the ensuing representations.

Furthermore, just as irreducible highest weight crystals $B(\lambda)$ have found realizations in terms of Young walls, so too have Fock space crystals. Indeed, such models for the level 1 Fock space crystals – given in terms of *proper* Young walls – were constructed by Kang-Kwon, Kim-Shin, Misra-Miwa, and Premat in the non-exceptional types $A_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $C_n^{(1)}$ (cf. [59–61, 74, 88, 98]). The author and collaborators then recently addressed the exceptional types $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ [77] and $E_6^{(2)}$, $F_4^{(1)}$ [33].

Moreover, via a similar treatment to [77, §5] or [33, §5], the existing reduced Young wall models for $B(\lambda)$ of level 1 in types $D_4^{(3)}$ and $G_2^{(1)}$ from [25] can be readily adapted to provide proper Young wall models for $B(\mathcal{F}(\lambda))$.

Outline of the thesis

In Chapter 2, we establish our notations and gather the necessary preliminaries for this thesis. We start by recalling the fundamentals of Kac-Moody Lie algebras, with a particular focus on the affine case. Section 2.2 introduces the Drinfeld-Jimbo quantum groups along with their braid group action, and explores fundamental representation theoretic constructs such as crystal and global bases for integrable modules.

In Section 2.3 we focus on the quantum affine algebras, detailing the path realization for highest weight crystals and constructing Fock space representations. Section 2.4 then examines quantum affinizations, defining their triangular decomposition, ℓ-highest weight representations and topological coproducts.

Chapter 3 focuses on the toroidal setting, in particular presenting the author's results from [76] as well as subsequent work in this area. We first define the quantum toroidal algebra, proving that it is generated by its horizontal and vertical subalgebras, and include some natural (anti-)automorphisms. Section 3.1 introduces the extended double affine braid group $\ddot{\mathcal{B}}$ together with its horizontal and vertical subgroups $\mathcal{B}_h$ and $\mathcal{B}_v$, each of which is isomorphic to an extended affine braid group. We moreover recall Coxeter style presentations and congruence group actions for $\ddot{\mathcal{B}}$ due to Ion-Sahi [46].

For a broad class of quantum affinizations, we then prove in Section 3.1.2 a simplified Drinfeld new style presentation involving only finitely many generators and relations (Proposition 3.1.8). In particular, this includes the quantum toroidal algebras $U_q(\mathfrak{g}_{\mathrm{tor}})$ in all types other than $A_1^{(1)}$ and $A_2^{(2)}$, as well as the untwisted quantum affine algebras $U_q(\hat{\mathfrak{g}})$. This allows us to define automorphisms $\mathcal{T}_0, \ldots, \mathcal{T}_n$ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which restrict to the braid automorphisms of Lusztig [81] on both the horizontal and vertical subalgebras (Proposition 3.1.10). We are then able to present an action of $\ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$ in Theorem 3.1.11. The horizontal and vertical subgroups $\mathcal{B}_h$ and $\mathcal{B}_v$ restrict to the extended affine action of Lusztig and Beck on the horizontal and vertical subalgebras $\mathcal{U}_h$ and $\mathcal{U}_v$ respectively.

The braid group $\ddot{\mathcal{B}}$ possesses a natural involution $\mathfrak{t}$ which interchanges its horizontal and vertical subgroups. In Section 3.2, using the action on $U_q(\mathfrak{g}_{\mathrm{tor}})$ we transfer this over to an anti-involution ψ of the quantum toroidal algebra in all untwisted types except $A_1^{(1)}$ and $G_2^{(1)}$ (Theorem 3.2.1). Moreover, ψ exchanges the horizontal and vertical subalgebras and acts on central elements by $C \leftrightarrow (k_0^{a_0} \dots k_n^{a_n})^{-1}$. Composing ψ with a standard anti-automorphism η, we obtain an automorphism Φ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which in type $A_n^{(1)}$ recovers that of Miki[1] [86] (Corollary 3.2.3).

Section 3.3 then proves compatibility relations between various (anti-)automorphisms on either side of our action of $\ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$ (Proposition 3.3.1 and Corollary 3.3.2). From these results we deduce an action of the universal cover of $SL_2(\mathbb{Z})$ on untwisted $U_q(\mathfrak{g}_{\mathrm{tor}})$, which can be seen as a quantum toroidal analogue of the congruence group actions for $\ddot{\mathcal{B}}$.

We finish by discussing some applications and future directions following on from our results. In Section 3.4.1 we define vector representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ in all simply laced types except $E_8^{(1)}$ – written explicitly in terms of Young column bases – which should be useful for obtaining combinatorial Fock space modules, as well as finite dimensional representations at roots of unity. Furthermore, in Section 3.4.2 we prove that twisting the usual topological coproduct for $U_q(\mathfrak{g}_{\mathrm{tor}})$ by our anti-involution ψ gives rise to a tensor product that is well-defined for ℓ-highest weight modules. This provides us with a new method for constructing representations of quantum toroidal algebras, and moreover shall allow us to study the associated R-matrices in future work.

Chapter 4 contains the author's construction from [77] of Young wall models for representations of quantum affine algebras in untwisted type E. In Section 4.1 we introduce level 1 perfect crystals in each of the types $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ and provide descriptions of their energy functions. For types $E_6^{(1)}$ and $E_7^{(1)}$ we use the crystal basis of a level 0 fundamental representation associated to a minuscule node of the affine Dynkin diagram. However, in type $E_8^{(1)}$ we require the uniform construction of Benkart-Frenkel-Kang-Lee [4].

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[1] More specifically, Miki considers a quantum toroidal algebra $U_{q,\kappa}(\mathfrak{sl}_{n+1,\mathrm{tor}})$ involving an extra deformation parameter κ which is not known to exist in other types. Our automorphism Φ in type $A_n^{(1)}$ is equal to that of Miki with κ set to 1.

We then provide Young column realizations for each of these crystals and their affinizations as (equivalence classes of) stackings of blocks inside a corresponding Young column pattern (Propositions 4.1.13 and 4.1.17). These patterns are formed by splitting each cube within an infinite strip of unit cubes into coloured blocks, according to a collection of vertical cuts.

In Section 4.2 we derive Young wall models for the level 1 irreducible highest weight crystals. We define the structure of an affine crystal on a set of *reduced* Young walls (Theorem 4.2.10), which are particular stackings of blocks inside a Young wall pattern that stabilise to a certain ground state wall. In order to better understand their structure, we show that these reduced Young walls satisfy a *right block property* and moreover are built on top of the ground state wall. We then prove in Theorem 4.2.11 that this Young wall crystal is isomorphic to the path realization of $B(\lambda)$, reinterpreting the energy function condition on adjacent elements of a path as a more combinatorial condition on adjacent columns in a Young wall.

Section 4.3 constructs Young wall models for the crystal bases of level 1 Fock space representations $\mathcal{F}(\lambda)$. Using our Young column realizations from Section 4.1, we once again define a particular collection of *proper* Young walls within the relevant Young wall pattern that stabilise to the ground state wall. Endowing this set with an affine crystal structure as in Section 4.2, we obtain a crystal which by Theorem 4.3.3 provides a concrete model for the Fock space crystal $B(\mathcal{F}(\lambda))$.

Throughout Section 4.3.1 we then investigate the structure of these proper Young walls in more detail, proving a right block property (which is slightly weakened in type $E_8^{(1)}$) and giving explicit combinatorial conditions for when a Young wall is proper and thus lies inside our Fock space model. We conclude the chapter by mentioning possible connections of this work to algebraic geometry – in particular regarding quiver varieties, Hilbert schemes and Kleinian singularities – as well as categorification and KLR algebras.

Chapter 2

Preliminaries

2.1 Basic notations

Consider a Kac-Moody Lie algebra $\mathfrak{s}$ with generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ and finite index set I. We shall assume that A is symmetrizable, which is to say that there exists a diagonal matrix $D = \operatorname{diag}(d_i \mid i \in I)$ with relatively prime entries in $\mathbb{Z}_{>0}$ such that the product DA is symmetric. Its Cartan subalgebra $\mathfrak{h}$ contains simple coroots $\alpha_i^\vee$ and fundamental coweights $\Lambda_i^\vee$ for each $i \in I$, as well as $\operatorname{corank}(A)$ scaling elements. The coweight lattice $P^\vee$ is the $\mathbb{Z}$-span of the simple coroots and scaling elements, and moreover contains the coroot lattice $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$.

With the natural pairing $\langle\ ,\ \rangle$ between $\mathfrak{h}$ and its dual space $\mathfrak{h}^*$ we define the weight lattice $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, P^\vee\rangle \subset \mathbb{Z}\}$, simple roots α_i and fundamental weights Λ_i for each $i \in I$. In particular, these must satisfy $\langle \alpha_j, \alpha_i^\vee\rangle = a_{ij}$ and $\langle \Lambda_j, \alpha_i^\vee\rangle = \delta_{ij}$ for all $i, j \in I$. We denote the root lattice $\bigoplus_{i\in I} \mathbb{Z}\alpha_i$ by Q, and let $P^+ = \{\lambda \in P \mid \text{all } \lambda(\alpha_i^\vee) \geq 0\}$ be the set of dominant integral weights. The standard non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ satisfies $(\alpha_i, \alpha_j) = d_i a_{ij}$ for all $i, j \in I$, and induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ which maps each $\alpha_i^\vee \mapsto d_i^{-1}\alpha_i$. Throughout this thesis we shall occasionally identify the elements of $\mathfrak{h}$ with their images under ν without mention.

Let $D(A)$ be the Dynkin diagram associated to our generalized Cartan matrix A, with vertex set I and $a_{ij}a_{ji}$ edges between any distinct $i, j \in I$ that point to j whenever $a_{ij} \geq a_{ji}$. The corresponding braid group $\mathcal{B}$ is defined as the Coxeter group generated by $\{T_i \mid i \in I\}$ subject to the braid relations $T_iT_jT_i \ldots = T_jT_iT_j \ldots$ with $a_{ij}a_{ji} + 2$ factors on each side whenever $a_{ij}a_{ji} \leq 3$. The Weyl group $W = \langle s_i \mid i \in I\rangle$ is the quotient obtained by specifying that each generator is self-inverse, and acts on $P^\vee$ via $s_i(x) = x - \langle \alpha_i, x\rangle \alpha_i^\vee$ for each $i \in I$. Note that both $\mathcal{B}$ and W are constructed independently of the orientation of arrows in $D(A)$, but that the action on $P^\vee$ is not.

Throughout this thesis, every algebra associated to a Cartan datum shall be considered with respect to the field $\Bbbk = \mathbb{Q}(q)$ for an indeterminate q. Setting $q_i = q^{d_i}$ for all $i \in I$, the q_i-integers, q_i-factorials and q_i-binomial coefficients are defined as

$$[s]_i = \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}}, \qquad [s]_i! = \prod_{\ell=1}^{s} [\ell]_i, \qquad \begin{bmatrix} s \\ r \end{bmatrix}_i = \frac{[s]_i!}{[s-r]_i!\,[r]_i!}$$

respectively for all non-negative integers $s \geq r$. More generally, we furthermore let $\{x\}_i = (x - x^{-1})/(q_i - q_i^{-1})$ and

$$\begin{Bmatrix} x \\ r \end{Bmatrix}_i = \frac{\{x\}_i \{q_i^{-1}x\}_i \ldots \{q_i^{1-r}x\}_i}{[r]_i!}$$

for all $x \in \Bbbk$ and $r \in \mathbb{N}$. When our generalized Cartan matrix is symmetric, since all $d_i = 1$ we shall often drop the i subscripts above for simplicity.

For certain elements $x_i^\pm$ and $x_{i,m}^\pm$ of the quantum algebras introduced in later sections, we introduce the divided powers $(x_i^\pm)^{(s)} = (x_i^\pm)^s/[s]_i!$ and $(x_{i,m}^\pm)^{(s)} = (x_{i,m}^\pm)^s/[s]_i!$ for each non-negative integer s. Following Jing [50] we shall also define their twisted commutators $[b_1, \ldots, b_s]_{u_1 \cdots u_{s-1}}$ inductively via $[b_1, b_2]_u = b_1 b_2 - u b_2 b_1$ and

$$[b_1, \ldots, b_s]_{u_1 \cdots u_{s-1}} = [b_1, [b_2, \ldots, b_s]_{u_1 \cdots u_{s-2}}]_{u_{s-1}}.$$

Let us now restrict our focus to the affine case, where our conventions mostly follow Kac [55]. We shall consider an indecomposable affine Kac-Moody algebra $\hat{\mathfrak{g}}$ with Cartan matrix $A = (a_{ij})_{i,j \in I}$ and index set $I = \{0, \ldots, n\}$. Since $\operatorname{corank}(A) = 1$ its Cartan subalgebra $\hat{\mathfrak{h}}$ has a basis consisting of the simple coroots $\alpha_0^\vee, \ldots, \alpha_n^\vee$ together with a unique scaling element d (alternatively, this can be replaced by $\Lambda_0^\vee$). Furthermore, the centre of $\hat{\mathfrak{g}}$ is spanned by a canonical non-divisible element $c \in \bigoplus_{i \in I} \mathbb{Z}_{>0} \alpha_i^\vee$.

On the other hand, the dual space $\hat{\mathfrak{h}}^*$ possesses a basis $\{\Lambda_0, \alpha_0, \ldots, \alpha_n\}$ and the root lattice Q contains a unique standard non-divisible imaginary root δ. Since the natural pairing between $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$ is given by $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, $\langle \Lambda_i, d \rangle = \langle \delta, \alpha_j^\vee \rangle = 0$ and $\langle \delta, d \rangle = 1$, the bilinear form $(\ ,\ )$ is determined by

$$(\alpha_i, \alpha_j) = d_i a_{ij}, \qquad (\alpha_i, \Lambda_0) = d_0 \delta_{i0}, \qquad (\Lambda_0, \Lambda_0) = 0,$$

for all $i, j \in I$ and in particular satisfies $(\delta, \alpha_i) = 0$. The corresponding isomorphism $\nu : \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}^*$ sends $\Lambda_0^\vee \mapsto d_0^{-1}\Lambda_0$. Moreover, we can now express explicitly

- the affine weight lattice $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$,
- the affine coweight lattice $P^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee \oplus \mathbb{Z}d$,
- the set of dominant affine integral weights $P^+ = \bigoplus_{i\in I} \mathbb{N}\Lambda_i \oplus \mathbb{Z}\delta$.

Removing the null root δ produces the classical weight lattice $\overline{P} = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ which can be viewed as both a sublattice and a quotient of P, as well as its subset of dominant classical weights $\overline{P}^+ = \bigoplus_{i\in I} \mathbb{N}\Lambda_i$. Note that the action of the affine Weyl group W on P descends to an action on $\overline{P}$.

Each node $i \in I$ of the affine Dynkin diagram $D(A)$ has a numerical label a_i, and a dual label $a_i^\vee$ coming from the diagram with the same vertex numbering and all arrows reversed. The affine Dynkin diagrams, together with their a_i and $a_i^\vee$ labels, are given in Appendix A – there our choice of vertex numbering matches Bourbaki [7, Plates I–IX] in all untwisted types, and the twisted types are obtained by reversing arrows. The affine Cartan matrix of type $X_n^{(r)}$ is then symmetrized by a positive integer multiple of $\mathrm{diag}(a_0^\vee/a_0, \dots, a_n^\vee/a_n)$. Furthermore, the null root δ equals $\sum_{i\in I} a_i\alpha_i$ with $a_0 = 1$ outside type $A_{2n}^{(2)}$, and the central element c is $\sum_{i\in I} a_i^\vee \alpha_i^\vee$ with $a_0^\vee = 1$. The level of an affine or classical weight λ is given by the pairing $\langle \lambda, c\rangle$ and is invariant under the Weyl group action.

A vertex $i \in I$ is minuscule if it is sent to 0 by some automorphism of the affine Dynkin diagram, and we denote the set of minuscule nodes by $I_{\min} \subset \{i \in I \mid a_i = a_0\}$. An automorphism is inner if it fixes the 0 vertex, and thus restricts to an automorphism of the finite Dynkin diagram. The outer automorphism group Ω is then the quotient of the entire automorphism group by the subgroup of inner automorphisms, and therefore has elements indexed by $I_{\min}$. In particular, for each $i \in I_{\min}$ we let π_i be the corresponding element of Ω, which is uniquely determined by the condition $\pi_i(0) = i$.

In all affine types except $A_{2n}^{(1)}$ we can fix a length function $o : I \to \{\pm 1\}$ satisfying $o(i) = -o(j)$ whenever $a_{ij} < 0$. We shall write $o_{i,j}$ as shorthand for $o(i)/o(j)$. However, in type $A_{2n}^{(1)}$ this is not possible since the affine Dynkin diagram contains an odd length cycle. For our purposes, there are two approximations to a length function to consider in this case: $o(i) = (-1)^i$ and $-o(i) = (-1)^{i+1}$. Furthermore, we define $o_{i,j} = (-1)^{\overline{j-i}}$ for all $i, j \in I$, where $\overline{j-i}$ is the anti-clockwise distance $i \to j$ in the affine Dynkin diagram.

Contained in each affine Lie algebra $\hat{\mathfrak{g}}$ is a corresponding finite dimensional simple Lie algebra $\mathfrak{g}$ with Cartan matrix $(a_{ij})_{i,j\in I_0}$ where $I_0 = \{1, \dots, n\}$. It has simple roots α_i, simple coroots $\alpha_i^\vee$, fundamental weights ω_i, and fundamental coweights $\omega_i^\vee$ for each $i \in I_0$ and we denote its root, coroot, weight and coweight lattices by $\mathring{Q}$, $\mathring{Q}^\vee$, $\mathring{P}$ and $\mathring{P}^\vee$. By mapping each $\omega_i^\vee \mapsto a_0\Lambda_i^\vee - a_i\Lambda_0^\vee$ we can embed $\mathring{P}^\vee$ inside $P^\vee$ at level 0, so that $\langle \delta, \omega_i^\vee \rangle = 0$ for all $i \in I_0$. The image is invariant under the action of the finite Weyl group $W_0 = \langle s_i \mid i \in I_0 \rangle$. Similarly, we can view $\mathring{P}$ inside the affine weight lattice P by sending each $\omega_i \mapsto a_0^\vee \Lambda_i - a_i^\vee \Lambda_0$. In order to simplify our notation in later sections we shall moreover define $\omega_0^\vee = 0$ and $\omega_0 = 0$.

As explained in the general case above, the affine braid group $\mathcal{B}$ has a Coxeter presentation with generators $T_0, \dots, T_n$ satisfying the braid relations for all distinct $i, j \in I$. Since this construction is independent of the orientation of arrows, note that any affine braid group is isomorphic to one of untwisted type. We remark that in types $A_1^{(1)}$ and $A_2^{(2)}$ this is simply the free group generated by T_0 and T_1 since $a_{01}a_{10} = 4$.

However, for affine braid groups in particular there exists a second realization due to Bernstein as follows. In all untwisted and $A_{2n}^{(2)}$ types, let $M = \mathring{Q}^\vee$ and $A_i^\vee = \alpha_i$ for each $i \in I$. Conversely, in the remaining twisted types we define $M = \mathring{Q}$ and all $A_i^\vee = \alpha_i^\vee$. Then in each case, the Bernstein presentation of $\mathcal{B}$ is generated by the finite braid group $\mathcal{B}_0 = \langle T_i \mid i \in I_0 \rangle$ and the lattice $\{X_\beta \mid \beta \in M\}$, with

- $T_i X_\beta = X_\beta T_i$ if $(\beta, A_i^\vee) = 0$,
- $T_i^{-1} X_\beta T_i^{-1} = X_{s_i(\beta)}$ if $(\beta, A_i^\vee) = 1$.

When $M = \mathring{Q}^\vee$ the correspondence between the two presentations is given by $T_0 = X_{\theta^\vee}\Theta^{-1}$ where $\Theta = T_{s_\theta}$ for $\theta = \sum_{i\in I_0} a_i\alpha_i$ the highest root of $\mathfrak{g}$, and $\theta^\vee = \nu^{-1}(a_0^{-1}\theta)$. Otherwise, θ is the short dominant root in $M = \mathring{Q}$ and we instead have $T_0 = X_\theta\Theta^{-1}$. See

[46, Chapter 3] for more details, noting that the Bernstein presentation there is obtained from ours by applying the automorphism of $\mathcal{B}$ which inverts $T_1, \ldots, T_n$ and fixes each X_β.

2.2 Drinfeld-Jimbo quantum groups

For an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{s}$ with generalized Cartan matrix $(a_{ij})_{i,j\in I}$, the corresponding quantum group is given in terms of certain Chevalley style generators as follows.

Definition 2.2.1. *The quantum group $U_q(\mathfrak{s})$ is the unital associative $\Bbbk$-algebra generated by elements q^h for each $h \in P^\vee$ and $x_i^\pm$ for all $i \in I$, subject to the following relations:*

- $q^0 = 1,$
- $q^h q^{h'} = q^{h+h'},$
- $q^h x_j^\pm q^{-h} = q^{\pm\langle\alpha_i,h\rangle} x_j^\pm,$
- $[x_i^+, x_j^-] = \dfrac{\delta_{ij}}{q_i - q_i^{-1}}(t_i - t_i^{-1}),$
- $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s (x_i^\pm)^{(s)} x_j^\pm (x_i^\pm)^{(1-a_{ij}-s)} = 0$ *whenever* $i \neq j,$

where $t_i = q^{d_i\alpha_i^\vee}$ for each $i \in I$.

This is called the Drinfeld-Jimbo realization for $U_q(\mathfrak{s})$, and makes clear a natural $\Bbbk$-algebra anti-involution $\sigma = (q^h \mapsto q^{-h},\ x_i^\pm \mapsto x_i^\pm)$ and $\mathbb{Q}$-algebra involution $\omega = (q \mapsto q^{-1},\ q^h \mapsto q^h,\ x_i^\pm \mapsto x_i^\mp)$.

Remark 2.2.2. It shall often be convenient to employ an alternative notation for $U_q(\mathfrak{s})$ in which the x_i^+, x_i^- and t_i are instead denoted by e_i, f_i and k_i respectively. In particular, while our original presentation is more natural when dealing with quantum affinizations and quantum toroidal algebras, this second option shall be more well-suited to our work on crystal and global bases for representations of quantum groups.

Definition 2.2.3. *A triangular decomposition of an algebra A consists of three subalgebras A^-, A^0 and A^+ such that multiplication $a_- \otimes a_0 \otimes a_+ \mapsto a_-a_0a_+$ provides an isomorphism of vector spaces $A^- \otimes A^0 \otimes A^+ \cong A$.*

It is clear that for any Drinfeld-Jimbo quantum group there exists a natural triangular decomposition $U_q(\mathfrak{s}) \cong U^- \otimes U^0 \otimes U^+$ into negative, zero and positive subalgebras $\langle x_i^- \mid i \in I\rangle$, $\langle q^h \mid h \in P^\vee\rangle$ and $\langle x_i^+ \mid i \in I\rangle$ respectively.

The quantum group $U_q(\mathfrak{s})$ possesses various Hopf algebra structures. Throughout this thesis we shall use the one with coproduct Δ given by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(x_i^+) = x_i^+ \otimes t_i^{-1} + 1 \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes 1 + t_i \otimes x_i^-, \tag{2.2.1}$$

counit ε satisfying $\varepsilon(q^h) = 1$ and $\varepsilon(x_i^\pm) = 0$, and antipode S with

$$S(q^h) = q^{-h}, \quad S(x_i^+) = -x_i^+ t_i, \quad S(x_i^-) = -t_i^{-1} x_i^-.$$

Our choice is the same as for example [57, 70], and is denoted by Δ_- in [71] where alternative commonly-used coproducts are given by

$$\begin{aligned}
&\Delta_+(q^h) = q^h \otimes q^h, && \Delta_+(x_i^+) = x_i^+ \otimes 1 + t_i \otimes x_i^+, && \Delta_+(x_i^-) = x_i^- \otimes t_i^{-1} + 1 \otimes x_i^-,\\
&\overline{\Delta}_+(q^h) = q^h \otimes q^h, && \overline{\Delta}_+(x_i^+) = x_i^+ \otimes 1 + t_i^{-1} \otimes x_i^+, && \overline{\Delta}_+(x_i^-) = x_i^- \otimes t_i + 1 \otimes x_i^-,\\
&\overline{\Delta}_-(q^h) = q^h \otimes q^h, && \overline{\Delta}_-(x_i^+) = x_i^+ \otimes t_i + 1 \otimes x_i^+, && \overline{\Delta}_-(x_i^-) = x_i^- \otimes 1 + t_i^{-1} \otimes x_i^-.
\end{aligned}$$

These are obtained by conjugating $\Delta = \Delta_-$ with ω, $\omega\sigma$ and σ respectively. Both Δ_+ and $\overline{\Delta}_+$ interact well with upper crystal bases, while Δ_- and $\overline{\Delta}_-$ interact well with lower crystal bases.

2.2.1 Braid group action

We briefly recall the action of the braid group $\mathcal{B}$ on the quantum group $U_q(\mathfrak{s})$ due to Lusztig. For each $i \in I$ there exists an automorphism $\mathbf{T}_i$ of $U_q(\mathfrak{s})$ defined by $\mathbf{T}_i(q^h) = q^{s_i(h)}$ for each $h \in P^\vee$ and

$$\mathbf{T}_i(x_i^+) = -x_i^- t_i, \ \mathbf{T}_i(x_j^+) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} (x_i^+)^{(-a_{ij}-s)} x_j^+ (x_i^+)^{(s)} \text{ if } i \neq j,$$

$$\mathbf{T}_i(x_i^-) = -t_i^{-1} x_i^+, \ \mathbf{T}_i(x_j^-) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{s} (x_i^-)^{(s)} x_j^- (x_i^-)^{(-a_{ij}-s)} \text{ if } i \neq j.$$

Its inverse $\mathbf{T}_i^{-1}$ is given by $\mathbf{T}_i^{-1}(q^h) = q^{s_i(h)}$ and

$$\mathbf{T}_i^{-1}(x_i^+) = -t_i^{-1} x_i^-, \ \mathbf{T}_i^{-1}(x_j^+) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} (x_i^+)^{(s)} x_j^+ (x_i^+)^{(-a_{ij}-s)} \text{ if } i \neq j,$$

$$\mathbf{T}_i^{-1}(x_i^-) = -x_i^+ t_i, \ \mathbf{T}_i^{-1}(x_j^-) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{s} (x_i^-)^{(-a_{ij}-s)} x_j^- (x_i^-)^{(s)} \text{ if } i \neq j.$$

In particular, we note that $\mathbf{T}_i(t_j) = \mathbf{T}_i^{-1}(t_j) = t_j t_i^{-a_{ij}}$ for all $j \in I$. A quick check verifies that $\mathbf{T}_i^{-1} = \sigma \mathbf{T}_i \sigma$ for all $i \in I$, where σ is the anti-involution of $U_q(\mathfrak{s})$ defined above.

Theorem 2.2.4. [81] *The braid group $\mathcal{B}$ acts on the quantum group $\mathcal{U}_q(\mathfrak{s})$ via $T_i \mapsto \boldsymbol{T}_i$ for each $i \in I$.*

Throughout this thesis we shall use without comment that $\mathbf{T}_i\mathbf{T}_j(x_i^\pm) = x_j^\pm$ and $\mathbf{T}_i^{-1}\mathbf{T}_j^{-1}(x_i^\pm) = x_j^\pm$ whenever $a_{ij} = a_{ji} = -1$. The short technical proof of this result can be found in [81, Chapter 37].

Remark 2.2.5. The automorphisms $\mathbf{T}_i$ and $\mathbf{T}_i^{-1}$ were first introduced in the general case by Lusztig [81], who denoted them by $T''_{i,1}$ and $T'_{i,-1}$ respectively.

Every automorphism π of the associated Dynkin diagram $D(A)$ gives rise to an automorphism S_π of $U_q(\mathfrak{s})$ which permutes the generators accordingly:

$$S_\pi(x_j^\pm) = x_{\pi(j)}^\pm, \quad S_\pi(q^h) = q^{\pi(h)},$$

where $\pi(h)$ is given by the natural action of Ω on $P^\vee$, extended trivially from the permutation action on the set of simple coroots. We note in particular that each $S_\pi(t_i^{\pm 1}) = t_{\pi(i)}^{\pm 1}$. As we shall see in Section 2.3.3, these automorphisms play a crucial role in extending the braid group action in the affine case.

2.2.2 Representation theory

Here we introduce some of the basic definitions regarding modules for quantum groups. A representation V of $U_q(\mathfrak{s})$ is a weight module if it decomposes as a direct sum $\bigoplus_{\lambda \in P} V_\lambda$ of its weight spaces $V_\lambda = \{u \in V \mid q^h \cdot u = q^{\langle \lambda, h \rangle} u \text{ for all } h \in P^\vee\}$. It is moreover a highest weight module with highest weight $\lambda \in P$ if there exists some non-zero $u_\lambda \in V_\lambda$ such that $V = U_q(\mathfrak{s}) \cdot u_\lambda$ and all $x_i^+ \cdot u_\lambda = 0$.

Example 2.2.6.

- The Verma module $M(\lambda)$ is the quotient of $U_q(\mathfrak{s})$ by the left ideal generated by $\{q^h - q^{\langle \lambda, h \rangle} 1 \mid h \in P^\vee\}$ and $U^+ = \langle x_i^+ \mid i \in I \rangle$. It has the universal property that every highest weight module with highest weight λ is the image of $M(\lambda)$ under the unique homomorphism that sends $1 \mapsto u_\lambda$.

- $M(\lambda)$ possesses a unique maximal submodule, whereby the corresponding quotient $V(\lambda)$ is the unique irreducible highest weight module of highest weight λ up to isomorphism.

A weight module is integrable if all $x_i^{\pm}$ act locally nilpotently, that is for each $u \in V$ we have $(x_i^{\pm})^k \cdot u = 0$ for some $k \geq 0$. An element $u \in V$ is extremal if there exists a set of vectors $\{u_w\}_{w \in W}$ such that

- $u_e = u$,
- if $\langle w\lambda, \alpha_i^\vee \rangle \geq 0$ then $x_i^+ \cdot u_w = 0$ and $(x_i^-)^{(\langle w\lambda, \alpha_i^\vee \rangle)} \cdot u_w = u_{s_i w}$,
- if $\langle w\lambda, \alpha_i^\vee \rangle \leq 0$ then $x_i^- \cdot u_w = 0$ and $(x_i^+)^{(-\langle w\lambda, \alpha_i^\vee \rangle)} \cdot u_w = u_{s_i w}$.

Such a set must be unique, with each u_w spanning $V_{w\lambda}$. In this case, we say that V is an extremal weight module [68]. For each $\lambda \in P$ define $V^{\text{ext}}(\lambda)$ to be the representation of $U_q(\mathfrak{s})$ generated by a non-zero vector u_λ, subject only to the condition that it is an extremal vector of weight λ. In particular, if λ is dominant then $V^{\text{ext}}(\lambda)$ is isomorphic to the irreducible highest weight module $V(\lambda)$.

Let $\mathcal{O}_{\text{int}}^q$ be the category of integrable representations V of $U_q(\mathfrak{s})$ with finite dimensional weight spaces, for which there exist $\mu_1, \ldots, \mu_r \in P$ such that

$$\{\lambda \in P \mid V_\lambda \neq 0\} \subset \bigcup_{j=1}^{r} (\mu_j - Q^+)$$

where $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ is the positive root lattice. Then $\mathcal{O}_{\text{int}}^q$ is closed under finite direct sums and tensor products, and moreover we have the following structural result.

Theorem 2.2.7. [39, Chapter 3] *The category $\mathcal{O}_{\text{int}}^q$ is semisimple, and the indecomposable objects are precisely the irreducible highest weight modules $V(\lambda)$ with $\lambda \in P^+$.*

Therefore in many situations, in order to understand the entire category $\mathcal{O}_{\text{int}}^q$ it is enough to consider those $V(\lambda)$ for which λ is a dominant integral weight.

2.2.3 Crystal bases

The theory of crystal bases pioneered by Kashiwara [65–68] provides a powerful tool for studying the representation theory of quantum groups. Essentially, the crystal basis of a representation is a nice basis at $q = 0$, and captures much of the structural information whilst being a far more straightforward and stripped-back object. For a more detailed introduction to crystal bases we refer the reader to [39,66,69]. Note that throughout this subsection and the next, we shall employ the alternative notations of Remark 2.2.2.

Consider an integrable representation $V = \bigoplus_{\lambda \in P} V_\lambda$ of $U_q(\mathfrak{s})$. For each $i \in I$, we can write any $u \in V_\lambda$ uniquely as a sum

$$u = \sum f_i^{(n)} u_n$$

over integers $n \geq \max\{-\langle \lambda, \alpha_i^\vee \rangle, 0\}$ where each $u_n \in V_{\lambda + n\alpha_i} \cap \ker(e_i)$. The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ are then defined to be the linear endomorphisms of V given by

$$\tilde{e}_i u = \sum f_i^{(n-1)} u_n, \quad \tilde{f}_i u = \sum f_i^{(n+1)} u_n.$$

While these do not in general respect the $U_q(\mathfrak{s})$-module structure on V, they do however commute with all $U_q(\mathfrak{s})$-module homomorphisms. Let A be the subring of functions in $\Bbbk$ that are regular at $q = 0$.

Definition 2.2.8. *A crystal lattice for an integrable $U_q(\mathfrak{s})$-module V is a free A-submodule L of V such that*

- $V \cong \Bbbk \otimes_A L$,
- $L = \bigoplus_{\lambda \in P} L_\lambda$ *where* $L_\lambda = L \cap V_\lambda$,
- $\tilde{e}_i L \subset L$ *and* $\tilde{f}_i L \subset L$ *for all* $i \in I$.

Definition 2.2.9. *A crystal basis for an integrable $U_q(\mathfrak{s})$-module V is a pair (L, B) such that*

- L *is a crystal lattice of* V,
- B *is a basis of* L/qL *as a vector space over* $A/qA \cong \mathbb{Q}$,
- $B = \bigsqcup_{\lambda \in P} B_\lambda$ *where* $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,
- $\tilde{e}_i B \subset B \sqcup \{0\}$ *and* $\tilde{f}_i B \subset B \sqcup \{0\}$ *for all* $i \in I$,
- $\tilde{f}_i b = b'$ *if and only if* $b = \tilde{e}_i b'$ *for all* $b, b' \in B$ *and* $i \in I$.

To avoid confusion, we shall occasionally denote the crystal basis of a representation V by $(L(V), B(V))$. It is clear that if a collection of $U_q(\mathfrak{s})$-modules $\{V_j\}_{j \in J}$ each possesses a crystal basis, then their direct sum $\bigoplus_{j\in J} V_j$ has crystal basis $(\bigoplus L(V_j), \bigoplus B(V_j))$.

Every crystal basis has an associated (I-coloured, directed) crystal graph, formed on the vertex set B by including an i-arrow $b \xrightarrow{i} b'$ whenever $\tilde{f}_i b = b'$. Here, connected components of the spanning subgraph containing only i-arrows are called i-strings. Furthermore, we can define a weight function $\mathrm{wt} : B \to P$ by sending the elements of each B_λ to λ, as well as maps $\varepsilon_i, \varphi_i : B \to \mathbb{N}$ for all $i \in I$ given by

$$\varepsilon_i(b) = \max\{n \mid \tilde{e}_i^n b \neq 0\}, \qquad \varphi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\}. \tag{2.2.2}$$

It was shown in [68] that for any $\lambda \in P$, the irreducible highest weight representation $V(\lambda)$ has a crystal basis $(L(\lambda), B(\lambda))$ which is unique up to isomorphism. In particular, when λ is dominant this has the following description.

Theorem 2.2.10. [66] *For each $\lambda \in P^+$ the lattice $L(\lambda)$ is the smallest A-submodule of $V(\lambda)$ containing u_λ, and $B(\lambda)$ is the set of all non-zero vectors in $L(\lambda)/qL(\lambda)$ of the form $\tilde{f}_{i_\ell} \dots \tilde{f}_{i_1} u_\lambda$.*

In combination with Theorem 2.2.7 and the construction of direct sums of crystals above, this immediately implies the existence and uniqueness of crystal bases for all representations in $\mathcal{O}^q_{\mathrm{int}}$.

We can generalise the idea of these crystal bases for integrable representations of $U_q(\mathfrak{s})$ to the following more abstract notion.

Definition 2.2.11. *An abstract crystal of $U_q(\mathfrak{s})$ is a set B together with maps $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}$ and $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ for each $i \in I$, and a weight function* wt $: B \to P$, *such that*

- $\varphi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$,
- $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ *if* $\tilde{e}_i b \in B$,
- $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ *if* $\tilde{f}_i b \in B$,

- *$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in B$,*
- *$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in B$,*
- *$\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,*
- *if $\varphi_i(b) = -\infty$ then $\tilde{e}_i b = \tilde{f}_i b = 0$,*

for all $b, b' \in B$ and $i \in I$.

Similarly to our definition of extremal vectors for representations of $U_q(\mathfrak{s})$ in Section 2.2.2, an element $b \in B$ is extremal if there exists a subset $\{b_w\}_{w \in W}$ of B such that

- $b_e = b$,
- if $\langle w\lambda, \alpha_i^\vee \rangle \geq 0$ then $\tilde{e}_i b_w = 0$ and $\tilde{f}_i^{\langle w\lambda, \alpha_i^\vee \rangle} b_w = b_{s_i w}$,
- if $\langle w\lambda, \alpha_i^\vee \rangle \leq 0$ then $\tilde{f}_i b_w = 0$ and $\tilde{e}_i^{-\langle w\lambda, \alpha_i^\vee \rangle} b_w = b_{s_i w}$.

An element of a crystal is therefore extremal if and only if it lies at the start or end of every i-string it is contained in.

A crystal morphism $\Psi : B \to B'$ between two crystals is a map $\Psi : B \sqcup \{0\} \to B' \sqcup \{0\}$ satisfying the following conditions for all $b, b' \in B$ and $i \in I$:

- $\Psi(0) = 0$,
- if $\Psi(b) \in B'$ then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$,
- if $\Psi(b), \Psi(b') \in B'$ and $\tilde{f}_i b = b'$ then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$.

Moreover Ψ is an isomorphism if $\Psi : B \sqcup \{0\} \to B' \sqcup \{0\}$ is a bijection.

Two crystals B and B' have a tensor product $B \otimes B'$ defined on the set $B \times B'$ with crystal structure given by

$$\begin{aligned}
\tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases} \\
\tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'), \end{cases} \\
\mathrm{wt}(b \otimes b') &= \mathrm{wt}(b) + \mathrm{wt}(b'), \\
\varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \langle \mathrm{wt}(b), \alpha_i^\vee \rangle), \\
\varphi_i(b \otimes b') &= \max(\varphi_i(b'), \varphi_i(b) + \langle \mathrm{wt}(b'), \alpha_i^\vee \rangle).
\end{aligned} \tag{2.2.3}$$

This endows the category of crystals with the structure of a tensor category.

Proposition 2.2.12. *If* (L, B) *and* (L', B') *are the crystal bases of* $U_q(\mathfrak{s})$*-modules* V *and* V' *respectively, then* $(L \otimes_A L', B \otimes B')$ *is a crystal basis for* $V \otimes V'$*.*

Remark 2.2.13. We refer the reader to [71, §2.2] for a nice explanation of how the various coproducts for $U_q(\mathfrak{s})$ and tensor products of crystals relate to one another.

2.2.4 Global bases

While crystal bases can be viewed as the bases at $q = 0$ for representations of quantum groups, this notion can often be globalized to obtain true bases for the modules. Here we shall present the approach to these global bases developed by Kashiwara [66, 67], which builds upon our existing definition of crystal bases. However it is important to note that Lusztig [79] independently obtained a geometric construction in symmetric types, which has been crucial for proving various positivity results in these cases. The equivalence of these definitions in general was given in [30]. We remark that the alternative terminology of *canonical bases* is often used in the literature, in particular when considering this second construction.

Recall that A is the subring of rational functions in $\Bbbk$ that are regular at $q = 0$. Let $-$ be the automorphism of $\Bbbk$ sending $q \mapsto q^{-1}$. Then $A_\infty = \overline{A}$ is the ring of functions in $\Bbbk$ which are instead regular at $q = \infty$.

Definition 2.2.14. *A balanced triple* $(L, L_\infty, V_{\mathbb{Q}})$ *for a* $\Bbbk$*-vector space* V *consists of*

- *an* A*-submodule* L *of* V*,*
- *an* A_∞*-submodule* L_∞ *of* V*,*
- *a* $\mathbb{Q}[q, q^{-1}]$*-submodule* $V_{\mathbb{Q}}$ *of* V*,*

each of which generates V *as a* $\Bbbk$*-vector space, such that the following equivalent conditions are satisfied:*

- *the natural projection* $E \to L/qL$ *is an isomorphism,*
- *the natural projection* $E \to L_\infty/q^{-1}L_\infty$ *is an isomorphism,*

- *$(L \cap V_{\mathbb{Q}}) \oplus (q^{-1}L_\infty \cap V_{\mathbb{Q}}) \to V_{\mathbb{Q}}$ is an isomorphism,*
- *$A \otimes_{\mathbb{Q}} E \to L$, $\overline{A} \otimes_{\mathbb{Q}} E \to L_\infty$, $\mathbb{Q}[q, q^{-1}] \otimes_{\mathbb{Q}} E \to V_{\mathbb{Q}}$ and $\Bbbk \otimes_{\mathbb{Q}} E \to V$ are all isomorphisms,*

where $E = L \cap L_\infty \cap V_{\mathbb{Q}}$.

We shall also denote by $-$ the ring automorphism of $U_q(\mathfrak{s})$ which sends q, q^h, e_i and f_i to q^{-1}, q^{-h}, e_i and f_i respectively, and call this the bar involution of $U_q(\mathfrak{s})$. A bar involution of a $U_q(\mathfrak{s})$-module V is then an involution $-$ satisfying $\overline{a \cdot u} = \overline{a} \cdot \overline{u}$ for all $a \in U_q(\mathfrak{s})$ and $u \in V$.

Let $U_q(\mathfrak{s})_{\mathbb{Q}}$ be the $\mathbb{Q}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{s})$ generated by $\{{q^h \atop n}\}$, $e_i^{(n)}$ and $f_i^{(n)}$ for all $h \in P^\vee$, $i \in I$ and $n \geq 0$.

Definition 2.2.15. *An integrable $U_q(\mathfrak{s})$-module V with crystal basis (L, B) has a global basis if there exists a $U_q(\mathfrak{s})_{\mathbb{Q}}$-submodule $V_{\mathbb{Q}}$ such that*

- *$\overline{V_{\mathbb{Q}}} = V_{\mathbb{Q}}$ and $u - \overline{u} \in (q-1)V_{\mathbb{Q}}$ for all $u \in V_{\mathbb{Q}}$,*
- *$(L, \overline{L}, V_{\mathbb{Q}})$ is a balanced triple.*

In this case, letting $G : L/qL \xrightarrow{\sim} E$ be the inverse of the natural projection, $\{G(b) \mid b \in B\}$ forms a basis for V called the (lower) global basis.

Such bases satisfy the following nice properties, among many others:

- $\overline{G(b)} = G(b)$ for all $b \in B$,
- $G(b) \equiv b \bmod q$ for all $b \in B$,
- for all $i \in I$ and $b \in B$ we have

$$f_i G(b) = [\varepsilon_i(b) + 1]_i G(\tilde{f}_i b) + \sum_{\varphi_i(b') \geq \varphi_i(b)} F^i_{b,b'} G(b'), \tag{2.2.4}$$

$$e_i G(b) = [\varphi_i(b) + 1]_i G(\tilde{e}_i b) + \sum_{\varepsilon_i(b') \geq \varepsilon_i(b)} E^i_{b,b'} G(b'), \tag{2.2.5}$$

where the coefficients satisfy

$$F^i_{b,b'} \in q^{1-\varphi_i(b')}\mathbb{Z}[q] \cup q^{\varphi_i(b')-1}\mathbb{Z}[q^{-1}],$$

$$E^i_{b,b'} \in q^{1-\varepsilon_i(b')}\mathbb{Z}[q] \cup q^{\varepsilon_i(b')-1}\mathbb{Z}[q^{-1}].$$

The conditions on each sum are equivalent to requiring that b' lies in a strictly longer i-string than b.

The following provides a globalized version of Theorem 2.2.10 for irreducible highest weight modules of $U_q(\mathfrak{s})$.

Theorem 2.2.16. [66, 79] *For any* $\lambda \in P^+$ *there exists a unique global basis* $\{G(b) \mid b \in B(\lambda)\}$ *for* $V(\lambda)$.

Furthermore, Kashiwara [68] later extended this result to include all extremal weight modules $V^{\mathrm{ext}}(\lambda)$. For a more detailed introduction to global bases and their basic properties, see for example [67].

2.3 Quantum affine algebras

In this section we specialise to the particular case of quantum affine algebras $U_q(\hat{\mathfrak{g}})$, outlining some of the fundamental results on their representation theory. We include the path realization for their irreducible highest weight crystals, and present the construction of Fock space representations by Kashiwara-Miwa-Petersen-Yung [71]. Finally, we introduce the extended affine braid groups $\dot{\mathcal{B}}$ and give their action on $U_q(\hat{\mathfrak{g}})$ due to Lusztig [81] and Beck [3].

Associated to any affine Kac-Moody algebra $\hat{\mathfrak{g}}$ there exists a quantum affine algebra $U_q(\hat{\mathfrak{g}})$ with Drinfeld-Jimbo presentation provided by Definition 2.2.1. We define $U_q'(\hat{\mathfrak{g}})$ to be the subalgebra generated by all $x_i^{\pm}$ and $t_i^{\pm 1}$, which can alternatively be obtained by replacing the affine coweight lattice $P^\vee$ with the classical coweight lattice $\overline{P}^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$. It is clear that $U_q'(\hat{\mathfrak{g}})$ inherits (among other things) the triangular decomposition, Hopf algebra structure with coproduct Δ, and affine braid group action from the previous subsections.

In addition, the representation and crystal theoretic constructions of Sections 2.2.2–2.2.4 carry over simply by replacing P with $\overline{P}$. In either case, the level of a module is defined as the power of q by which the central element acts. To avoid confusion between (abstract) crystals for $U_q(\hat{\mathfrak{g}})$ and $U_q'(\hat{\mathfrak{g}})$, we shall call them affine and classical crystals respectively. The following procedure demonstrates how given any $U_q'(\hat{\mathfrak{g}})$-module, we can form an associated representation of $U_q(\hat{\mathfrak{g}})$.

Definition 2.3.1. *The affinization of an integrable $U_q'(\hat{\mathfrak{g}})$-module $V = \bigoplus_{\lambda \in \overline{P}} V_\lambda$ is the $U_q(\hat{\mathfrak{g}})$-module $V_{\text{aff}} = \mathbb{C}[z, z^{-1}] \otimes V$ where all $x_i^\pm$ act by $z^{\pm\delta_{i0}} \otimes x_i^\pm$, and the action of each q^h is determined by setting $(V_{\text{aff}})_{\lambda+n\delta} = z^n V_\lambda$ for all $\lambda \in \overline{P}$ and $n \in \mathbb{Z}$.*

Example 2.3.2. Consider the $U_q(\hat{\mathfrak{g}})$-module $V^{\text{ext}}(\omega_i)$ where $\omega_i = \Lambda_i - a_i^\vee \Lambda_0$ is the ith level 0 fundamental weight in P. Since $\omega_i + \mathbb{Z}\delta \subset W\omega_i$ there exists a unique $U_q'(\hat{\mathfrak{g}})$-module automorphism z of $V^{\text{ext}}(\omega_i)$ sending u_{ω_i} to $u_{\omega_i+\delta}$. The quotient

$$W(\omega_i) = V^{\text{ext}}(\omega_i)/(z-1)V^{\text{ext}}(\omega_i)$$

is a finite dimensional irreducible integrable $U_q'(\hat{\mathfrak{g}})$-module, called the ith level 0 fundamental representation (see [70, §5.3] for more details). Then it is easy to see that passing $W(\omega_i)$ through this affinization process recovers the extremal weight module $V^{\text{ext}}(\omega_i)$.

This idea descends naturally to the level of crystals.

Definition 2.3.3. *The affinization B_{aff} of a classical crystal B is the set $\{z^n b \mid n \in \mathbb{Z},\, b \in B\}$, with the structure of an affine crystal as follows:*

- $\mathrm{wt}(z^n b) = \mathrm{wt}(b) + n\delta$,
- $\varepsilon_i(z^n b) = \varepsilon_i(b)$,
- $\varphi_i(z^n b) = \varphi_i(b)$,
- $\tilde{e}_i(z^n b) = z^{n+\delta_{i0}}(\tilde{e}_i b)$,
- $\tilde{f}_i(z^n b) = z^{n-\delta_{i0}}(\tilde{f}_i b)$.

It is clear that given an (iso)morphism $\Psi : B \to B'$ of classical crystals, $\Psi_{\text{aff}}(z^n b) = z^n \Psi(b)$ defines an (iso)morphism $\Psi_{\text{aff}} : B_{\text{aff}} \to B'_{\text{aff}}$ between their affinizations.

We remark that if a $U_q'(\hat{\mathfrak{g}})$-module V has crystal basis (L, B) then its affinization V_{aff} has crystal basis $(L_{\text{aff}}, B_{\text{aff}})$ where L_{aff} is defined similarly. In particular, this applies to the crystal bases of the level 0 fundamental representations $W(\omega_i)$ and extremal weight modules $V^{\text{ext}}(\omega_i)$.

2.3.1 The path realization for highest weight crystals

In order to better understand representations of quantum groups and their associated crystal bases, one often seeks to realise them inside tensor products of more simple objects. For example, in the case of $U_q(\mathfrak{sl}_{n+1})$ every finite dimensional – or equivalently, irreducible highest weight – module V embeds inside some tensor power $V(\Lambda_1)^{\otimes N}$ of the vector representation. On the level of crystals, this means that $B(V)$ occurs as a connected component of $B(\Lambda_1)^{\otimes N}$. This result was used by Kashiwara-Nakashima [73] to obtain their famous realization of highest weight crystals in terms of semi-standard Young tableaux.

In the quantum affine setting, the irreducible highest weight representations $V(\lambda)$ are infinite dimensional and hence the crystal graph of $B(\lambda)$ contains infinitely many vertices and arrows. Thus we can no longer expect both the tensor products and tensor factors in a realization similar to the one above to be finite. Nevertheless, the path realization developed by Kang et al. in [57,58] provides an isomorphism between $B(\lambda)$ and an abstract crystal of paths inside some perfect crystal B. Since a perfect crystal must be finite by definition, these paths are required to be of infinite length.

With this in mind, let us begin by introducing the notion of a perfect crystal. For each element b of a classical crystal B, define associated weights $\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$ inside $\overline{P}$.

Definition 2.3.4. *A perfect crystal of level* $\ell \in \mathbb{Z}_{>0}$ *is a classical crystal* B *such that*

- *there is a finite dimensional irreducible* $U_q'(\hat{\mathfrak{g}})$*-module whose crystal graph is isomorphic to* B*,*
- $B \otimes B$ *is connected,*
- *there exists some* $\mu \in \overline{P}$ *with* $\mathrm{wt}(B) \subset \mu - \mathring{Q}^+$ *and* $\#\{b \in B \mid \mathrm{wt}(b) = \mu\} = 1$*,*
- $\langle \varepsilon(b), c \rangle \geq \ell$ *for all* $b \in B$*,*
- *for any* $\lambda \in \overline{P}^+$ *with* $\langle \lambda, c \rangle = \ell$ *there exist unique* $b^\lambda, b_\lambda \in B$ *with* $\varepsilon(b^\lambda) = \varphi(b_\lambda) = \lambda$*.*

The importance of perfect crystals is then demonstrated by the following results.

Theorem 2.3.5. [57] *Let B be a perfect crystal of level ℓ. If $\lambda \in \overline{P}^+$ has $\langle \lambda, c\rangle = \ell$ then there is a unique isomorphism of classical crystals*

$$\Psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes B$$

given by $u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$, where u_λ is the highest weight element of $B(\lambda)$ and $b_\lambda \in B$ is as in Definition 2.3.4.

Letting $\lambda_0 = \lambda$, $b_r = b_{\lambda_r}$ and $\lambda_{r+1} = \varepsilon(b_{\lambda_r})$ for all $r \in \mathbb{N}$, we obtain a sequence of crystal isomorphisms by applying this theorem repeatedly:

$$\begin{tikzcd}
B(\lambda) \arrow[r] & B(\lambda_1) \otimes B \arrow[r] & B(\lambda_2) \otimes B \otimes B \arrow[r] & \cdots \\
u_\lambda \arrow[r, mapsto] & u_{\lambda_1} \otimes b_0 \arrow[r, mapsto] & u_{\lambda_2} \otimes b_1 \otimes b_0 \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

The infinite sequence $\mathbf{p}_\lambda = (b_r)_{r=0}^\infty = \cdots \otimes b_2 \otimes b_1 \otimes b_0$ is called the ground state sequence of weight λ. The set of λ-paths

$$\mathcal{P}(\lambda) = \{\mathbf{p} = (p_r)_{r=0}^\infty = \cdots \otimes p_2 \otimes p_1 \otimes p_0 \mid \text{all } p_r \in B,\ p_r = b_r \text{ for } r \gg 0\}$$

can be endowed with the structure of a classical crystal as follows. If $\mathbf{p} \in \mathcal{P}(\lambda)$ has $p_r = b_r$ for all $r \geq k$, we set

$$\begin{aligned}
\mathrm{wt}(\mathbf{p}) &= \lambda_k + \mathrm{wt}(\mathbf{p}'),\\
\tilde{e}_i\mathbf{p} &= \cdots \otimes p_{k+1} \otimes \tilde{e}_i(p_k \otimes \cdots \otimes p_0),\\
\tilde{f}_i\mathbf{p} &= \cdots \otimes p_{k+1} \otimes \tilde{f}_i(p_k \otimes \cdots \otimes p_0),\\
\varepsilon_i(\mathbf{p}) &= \max(\varepsilon_i(\mathbf{p}') - \varphi_i(b_k), 0),\\
\varphi_i(\mathbf{p}) &= \varphi_i(\mathbf{p}') + \max(\varphi_i(b_k) - \varepsilon_i(\mathbf{p}'), 0),
\end{aligned} \tag{2.3.1}$$

where $\mathbf{p}' = p_{k-1} \otimes \cdots \otimes p_0$.

Theorem 2.3.6. [57] *There is an isomorphism $\Psi_\lambda : B(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda)$ of classical crystals given by $u_\lambda \mapsto \mathbf{p}_\lambda$.*

This is called the path realization for the irreducible highest weight crystal $B(\lambda)$. In order to upgrade this to an isomorphism of affine crystals, we require the notion of an energy function.

Definition 2.3.7. *An energy function for an affine or classical crystal B is a map $H : B \otimes B \to \mathbb{Z}$ satisfying the condition*

$$H(\tilde{f}_i(b \otimes b')) = \begin{cases} H(b \otimes b') & \text{if } i \neq 0, \\ H(b \otimes b') + 1 & \text{if } i = 0 \text{ and } \tilde{f}_i(b \otimes b') = (\tilde{f}_i b) \otimes b' \neq 0, \\ H(b \otimes b') - 1 & \text{if } i = 0 \text{ and } \tilde{f}_i(b \otimes b') = b \otimes (\tilde{f}_i b') \neq 0. \end{cases}$$

Such a function H is therefore determined up to constant shift provided that $B \otimes B$ is connected, and can often be normalised by specifying that $H(b \otimes b) = 0$ for any extremal element $b \in B$. Moreover the existence of energy functions for every perfect crystal was shown in [57].

Given an energy function H on a classical crystal B, the affine energy function H_{aff} for B_{aff} is defined by

$$H_{\text{aff}}(z^m a \otimes z^n b) = H(a \otimes b) + m - n \tag{2.3.2}$$

for all $m, n \in \mathbb{Z}$ and $a, b \in B$.

Remark 2.3.8. It is crucial to note that while our definitions of energy functions and affine energy functions match those of [70, 77], they are equal to *minus* those of references such as [4, 25, 33, 57, 58, 71].

Lemma 2.3.9. *The affine energy function H_{aff} is constant on each connected component of $B_{\text{aff}} \otimes B_{\text{aff}}$.*

Proof. This is exactly the same as in [25, §3.3] after adjusting for the sign difference mentioned in Remark 2.3.8. □

Returning to the path realization, the existence of an energy function H on the perfect crystal B allows us to endow $\mathcal{P}(\lambda)$ with the structure of an affine crystal by replacing the weight function in (2.3.1) with

$$\text{wt}(\mathbf{p}) = \lambda_k + \text{wt}(\mathbf{p}') + \delta \sum_{r=0}^{\infty} (r+1)(H(p_{r+1} \otimes p_r) - H(b_{r+1} \otimes b_r)). \tag{2.3.3}$$

Theorem 2.3.6 can then be strengthened to an isomorphism of affine crystals by viewing λ inside P^+ rather than $\overline{P}^+$.

2.3.2 Fock space representations

Kashiwara-Miwa-Petersen-Yung [71] constructed Fock space representations for $U_q(\hat{\mathfrak{g}})$ as semi-infinite limits of q-wedge products $\bigwedge^r V_{\mathrm{aff}}$ along a certain ground state vector, where V is a $U_q'(\hat{\mathfrak{g}})$-module satisfying particular assumptions. Kashiwara [70] later showed that these assumptions hold for so-called 'good modules' and further studied the resulting Fock spaces, for example obtaining global bases for the representations. In this subsection we shall recall the definition of a good $U_q'(\hat{\mathfrak{g}})$-module and the construction of the Fock spaces, as well as some fundamental results on their structure.

We say that a subset $J \subset I$ is of finite type if $\mathfrak{g}_J = \langle e_i, f_i, \pm h_i \mid i \in J\rangle$ is a finite dimensional semisimple Lie subalgebra of $\hat{\mathfrak{g}}$. An affine or classical crystal B is regular if for any finite type $J \subset I$, if we remove all arrows with labels in $I \setminus J$ then B is isomorphic to the crystal basis of an integrable $U_q(\mathfrak{g}_J)$-module. By [57, Proposition 2.4.4] this is equivalent to the same condition but with J limited to subsets of order 2.

Definition 2.3.10. *A finite regular classical crystal B whose weights are of level 0 is simple if there exists some $\lambda \in \overline{P}$ such that $|B_\lambda| = 1$, and the weight of any extremal element of B is contained in its Weyl group orbit $W\lambda$.*

We are now ready to define a good $U_q'(\hat{\mathfrak{g}})$-module in the sense of Kashiwara [70].

Definition 2.3.11. *A finite dimensional $U_q'(\hat{\mathfrak{g}})$-module V is good if it has a crystal basis (L, B) with B simple; a bar involution; and a global basis.*

Since a simple crystal is always connected [1], such a module must be irreducible.

Example 2.3.12.

- Any level 0 fundamental representation $W(\omega_i)$ is a good $U_q'(\hat{\mathfrak{g}})$-module [70].
- In Proposition 4.1.4 we prove that the level 1 perfect crystals of Benkart-Frenkel-Kang-Lee [4] are the crystal bases of good modules in all affine types.

Next we shall explain the construction of Fock space representations for quantum affine algebras. Consider a good $U_q'(\hat{\mathfrak{g}})$-module V with crystal basis (L, B). We define a

$U_q(\hat{\mathfrak{g}})$-submodule

$$N = U_q(\hat{\mathfrak{g}})[z^{\pm 1} \otimes z^{\pm 1}, z \otimes 1 + 1 \otimes z](u \otimes u)$$

of $V_{\text{aff}} \otimes V_{\text{aff}}$, which is independent of a choice of extremal vector $u \in V_{\text{aff}}$. Kashiwara-Miwa-Petersen-Yung [71, §3.3] introduced the q-wedge product $\bigwedge^r V_{\text{aff}}$ as the quotient of $V_{\text{aff}}^{\otimes r}$ by

$$N_r = \sum_{k=0}^{r-2} V_{\text{aff}}^{\otimes k} \otimes N \otimes V_{\text{aff}}^{\otimes (r-k-2)},$$

which has the structure of a $U_q(\hat{\mathfrak{g}})$-module coming from the coproduct (2.2.1). For each $v_1, \ldots, v_r \in V_{\text{aff}}$ the image of $v_1 \otimes \cdots \otimes v_r$ in $\bigwedge^r V_{\text{aff}}$ is denoted by $v_1 \wedge \cdots \wedge v_r$.

Remark 2.3.13. An equivalent definition of the submodule N is as $\ker(R-1)$ where R is the action of the R-matrix on $V_{\text{aff}} \otimes V_{\text{aff}}$.

We say that a sequence $(s_k)_{k=1}^r$ in B_{aff} is normally ordered if each $H_{\text{aff}}(s_{k+1} \otimes s_k) > 0$, and call $G(s_r) \wedge \cdots \wedge G(s_1)$ a normally ordered wedge. It was shown in [71] that $\bigwedge^r V_{\text{aff}}$ has a crystal basis with

- $L(\bigwedge^r V_{\text{aff}})$ the image of $L(V_{\text{aff}}^{\otimes r}) = L(V_{\text{aff}}) \otimes_A \cdots \otimes_A L(V_{\text{aff}})$ in $\bigwedge^r V_{\text{aff}}$,
- $B(\bigwedge^r V_{\text{aff}})$ the set of normally ordered sequences in B_{aff}.

Here we identify each element of $B(\bigwedge^r V_{\text{aff}})$ with the image of its associated normally ordered wedge inside $L(\bigwedge^r V_{\text{aff}})/qL(\bigwedge^r V_{\text{aff}})$.

Suppose further that B is a perfect crystal of level ℓ as defined in Section 2.3.1. Recall that for any weight $\lambda \in \overline{P}^+$ with $\langle \lambda, c \rangle = \ell$ the ground state sequence of weight λ is the unique sequence $\mathbf{p}_\lambda = (b_r)_{r=0}^\infty$ in B with $\varphi(b_{r+1}) = \varepsilon(b_r)$ for all $r \geq 0$ and $\varphi(b_0) = \lambda$. Letting $m_r \in \mathbb{Z}$ be such that $H_{\text{aff}}(z^{m_{r+1}} b_{r+1} \otimes z^{m_r} b_r) = 1$ for all $r \geq 0$, we can also define a ground state sequence $\mathbf{s}_\lambda = (g_r)_{r=0}^\infty = (z^{m_r} b_r)_{r=0}^\infty$ in B_{aff}.

Let $\overline{\mathcal{F}(\lambda)} = \lim_{r \to \infty} \bigwedge^r V_{\text{aff}}$ be the inductive limit of the q-exterior powers with respect to the maps $\bigwedge^r V_{\text{aff}} \to \bigwedge^{r+1} V_{\text{aff}}$ which send $v \mapsto G(g_r) \wedge v$. We similarly define $L(\overline{\mathcal{F}(\lambda)})$ to be the inductive limit of the crystal lattices $L(\bigwedge^r V_{\text{aff}})$.

Definition 2.3.14. *The Fock space $\mathcal{F}(\lambda)$ is the quotient $\overline{\mathcal{F}(\lambda)} / \bigcap_{m>0} q^m L(\overline{\mathcal{F}(\lambda)})$.*

Remark 2.3.15. We can endow the Fock space with a separated q-adic topology by letting $\{q^m L(\mathcal{F}(\lambda)) \mid m \geq 0\}$ be the neighbourhood system of 0, where $L(\mathcal{F}(\lambda))$ is the image of $L(\overline{\mathcal{F}(\lambda)})$ in $\mathcal{F}(\lambda)$. Moreover, the expression for the action of x_i^- in Proposition 2.3.16 below converges with respect to this topology.

Any element of $\mathcal{F}(\lambda)$ can be written as a linear combination of vectors $\langle r| \wedge v$ where $v \in \bigwedge^r V_{\text{aff}}$ and we define $\langle r| = \cdots \wedge G(g_{r+1}) \wedge G(g_r)$ for each $r \in \mathbb{N}$.

Proposition 2.3.16. [71] *The Fock space $\mathcal{F}(\lambda)$ is equipped with a well-defined $U_q(\hat{\mathfrak{g}})$-module structure given by*

- $\text{wt}(\langle r| \wedge v) = \lambda_r + \text{wt}(v)$,
- $x_i^+(\langle r| \wedge v) = \langle r| \wedge x_i^+ v$,
- $x_i^-(\langle r| \wedge v) = t_i \langle r| \wedge x_i^- v + \sum_{s>0} t_i \langle r+s| \wedge x_i^- G(g_{r+s-1}) \wedge \cdots \wedge G(g_r) \wedge v$,

for any $v \in \bigwedge^r V_{\text{aff}}$ and $r \in \mathbb{N}$. Here $\text{wt}(v)$ and $x_i^\pm v$ come from the action of $U_q(\hat{\mathfrak{g}})$ on $\bigwedge^r V_{\text{aff}}$ via the coproduct (2.2.1).

We say that an infinite sequence $\mathbf{s} = (s_r)_{r=0}^\infty$ in B_{aff} is normally ordered if all $H_{\text{aff}}(s_{r+1} \otimes s_r) > 0$, and call $G(\mathbf{s}) := \cdots \wedge G(s_2) \wedge G(s_1) \wedge G(s_0)$ a normally ordered wedge. The set $B(\mathcal{F}(\lambda))$ of normally ordered sequences with $s_r = g_r$ for $r \gg 0$ has the structure of an affine crystal via (2.3.1) and (2.3.3), just as for $\mathcal{P}(\lambda)$.

The following are just some of the properties for Fock spaces that are proven in [70,71]:

- $\mathcal{F}(\lambda)$ is an integrable highest weight $U_q(\hat{\mathfrak{g}})$-module with weights contained in $\lambda - Q^+$. In particular, its character is given by

$$\text{ch}(\mathcal{F}(\lambda)) = \text{ch}(V(\lambda)) \cdot \prod_{k>0} (1 - e^{-k\delta})^{-1}$$

  and it is therefore isomorphic to a direct sum of irreducible highest weight modules $V(\lambda - m\delta)$ with $m \in \mathbb{N}$.

- $(L(\mathcal{F}(\lambda)), B(\mathcal{F}(\lambda)))$ is a crystal basis for $\mathcal{F}(\lambda)$ by identifying each $\mathbf{s} \in B(\mathcal{F}(\lambda))$ with the image of $G(\mathbf{s})$ inside $L(\mathcal{F}(\lambda))/qL(\mathcal{F}(\lambda))$.

- The highest weight elements of $B(\mathcal{F}(\lambda))$ are precisely those of the form $\cdots \wedge z^{n_2} g_2 \wedge z^{n_1} g_1 \wedge z^{n_0} g_0$ with $n_0 \leq n_1 \leq n_2 \leq \ldots$ and $n_r = 0$ for $r \gg 0$.

- If $\mathbf{s} \in B(\mathcal{F}(\lambda))$ has $s_k = g_k$ for some $k \in \mathbb{N}$, then $s_r = g_r$ for all $r \geq k$.

- The normally ordered wedges $\{G(\mathbf{s}) \mid \mathbf{s} \in B(\mathcal{F}(\lambda))\}$ form a global basis for $\mathcal{F}(\lambda)$.

It is clear that the submodule generated by the ground state vector $G(\mathbf{s}_\lambda)$ is a copy of the irreducible highest weight representation $V(\lambda)$. On the level of affine crystals, this tells us that the connected component of $\mathbf{s}_\lambda$ in $B(\mathcal{F}(\lambda))$ is isomorphic to $B(\lambda)$. The following proposition provides a more concrete description of this subcrystal.

Proposition 2.3.17. *The connected component of* $\mathbf{s}_\lambda$ *in* $B(\mathcal{F}(\lambda))$ *consists of those sequences* $\mathbf{s} = (s_r)_{r=0}^\infty$ *with all* $H_{\text{aff}}(s_{r+1} \otimes s_r) = 1$.

Proof. It is easy to show that if $n_r = n_{r+1} - 1 + H(p_{r+1} \otimes p_r)$ for all $r \in \mathbb{N}$ and $n_r = m_r$ for $r \gg 0$ then

$$(p_r)_{r=0}^\infty \mapsto (z^{n_r} p_r)_{r=0}^\infty$$

defines a morphism $B(\lambda) \to B(\mathcal{F}(\lambda))$ which bijects onto the desired subcrystal. Since $\mathbf{p}_\lambda$ is sent to $\mathbf{s}_\lambda$ and $B(\lambda)$ is connected, we are done. $\square$

Yet another way to think of this subcrystal is as the set of $\mathbf{s} = (s_r)_{r=0}^\infty \in B(\mathcal{F}(\lambda))$ such that applying z to any entry produces a sequence in B_{aff} which no longer lies in $B(\mathcal{F}(\lambda))$.

2.3.3 Extended affine braid groups

Here we introduce the extended affine braid group $\dot{\mathcal{B}}$ and present its action on $U_q(\hat{\mathfrak{g}})$ due to Lusztig [81] and Beck [3]. For a more complete introduction to extended affine braid groups, the interested reader may wish to consult [82, Chapters 2-3] and [46, Chapter 9].

Recall from Section 2.1 the Coxeter and Bernstein presentations of the affine braid group $\mathcal{B}$. By replacing the lattice M in the latter with a larger lattice N, defined to be $\mathring{P}^\vee$ in all untwisted and $A_{2n}^{(2)}$ types and $\mathring{P}$ otherwise, we obtain a Bernstein presentation for the extended affine braid group.

Definition 2.3.18. *The extended affine braid group* $\dot{\mathcal{B}}$ *is generated by the finite braid group* $\mathcal{B}_0 = \langle T_i \mid i \in I_0 \rangle$ *and the lattice* $\{X_\beta \mid \beta \in N\}$*, subject to the relations*

- $T_i X_\beta = X_\beta T_i$ *if* $(\beta, A_i^\vee) = 0,$ (2.3.4)

- $T_i^{-1} X_\beta T_i^{-1} = X_{s_i(\beta)}$ *if* $(\beta, A_i^\vee) = 1.$ (2.3.5)

There also exists a Coxeter style presentation for $\dot{\mathcal{B}}$. It is clear from the above that $\mathcal{B}_0$ and $\{X_\beta \mid \beta \in M\}$ generate a normal subgroup of $\dot{\mathcal{B}}$ isomorphic to $\mathcal{B}$, and therefore $\dot{\mathcal{B}} \cong (\dot{\mathcal{B}}/\mathcal{B}) \ltimes \mathcal{B}$. When $N = \mathring{P}^\vee$ set $\beta_\theta = \theta^\vee$ and $\beta_i = \omega_i^\vee$ for each $i \in I$, and when $N = \mathring{P}$ set $\beta_\theta = \theta$ and each $\beta_i = \omega_i$. Let $v_i = w_0 w_{0i}$ where w_0 is the longest element[1] of W_0 and w_{0i} is the longest element of the isotropy subgroup $\langle s_j \mid j \neq i \rangle$ of β_i. It was shown in [82, Chapter 2] that $\dot{\mathcal{B}}/\mathcal{B} = \{U_i = X_{\beta_i} T_{v_i}^{-1} \mid i \in I_{\min}\}$, and furthermore that $\dot{\mathcal{B}}/\mathcal{B}$ acts on $\mathcal{B}$ by outer automorphisms of the affine Dynkin diagram. More specifically, $U_i T_j U_i^{-1} = T_{\pi_i(j)}$ for all $i \in I_{\min}$ and $j \in I$ and so we have the following.

Proposition 2.3.19. *The extended affine braid group* $\dot{\mathcal{B}}$ *is isomorphic to the semidirect product* $\Omega \ltimes \mathcal{B}$*.*

The correspondence between the Coxeter and Bernstein presentations of $\dot{\mathcal{B}}$ is given by $T_0 = X_{\beta_\theta} \Theta^{-1}$ and $\pi_i = X_{\beta_i} T_{v_i}^{-1}$ for each $i \in I_{\min}$.

Remark 2.3.20. There is an automorphism of $\dot{\mathcal{B}}$ which inverts $T_0, \dots, T_n$ and fixes each element of Ω. Letting Y_β be the image of X_β for all $\beta \in N$, we obtain an *alternative Bernstein presentation* for $\dot{\mathcal{B}}$ matching that of [46, Proposition 9.1]. In particular, for each $i \in I_0$ and $\beta \in N$ we have the relations

- $T_i Y_\beta = Y_\beta T_i$ if $(\beta, A_i^\vee) = 0,$ (2.3.6)

- $T_i Y_\beta T_i = Y_{s_i(\beta)}$ if $(\beta, A_i^\vee) = 1.$ (2.3.7)

It immediately follows that the Coxeter presentation relates to this alternative Bernstein realization via $T_0 = \Theta^{-1} Y_{-\beta_\theta}$ and $\pi_i = Y_{\beta_i} T_{v_i^{-1}}$ for each $i \in I_{\min}$.

---

[1] For a nice explanation of how to find a reduced expression for any w_0 (and thus w_{0i}) by 2-colouring the Dynkin diagram, see Allen Knutson's answer at `https://mathoverflow.net/questions/54926/longest-element-of-weyl-groups` (last accessed 18th May 2024). Alternatively, [5, Table 1] contains such an expression in each finite type.

Example 2.3.21. We fix natural representatives for π_i in all affine types where there exists a non-trivial automorphism of the corresponding finite Dynkin diagram.

- In type $A_n^{(1)}$ we have $\Omega \cong \mathbb{Z}_{n+1}$ by identifying $\pi_i = (j \mapsto j + i \mod n+1)$ with $i \in \mathbb{Z}_{n+1}$ for each $i \in I$.
- In type $D_{2n}^{(1)}$ we have $\Omega \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with non-trivial elements given by

$$\begin{aligned}
\pi_1 &= (0 \leftrightarrow 1, n-1 \leftrightarrow n),\\
\pi_{n-1} &= (0 \leftrightarrow n-1, 1 \leftrightarrow n),\\
\pi_n &= (0 \leftrightarrow n, 1 \leftrightarrow n-1).
\end{aligned}$$

- In type $D_{2n+1}^{(1)}$ we instead have $\Omega \cong \mathbb{Z}_4$ with

$$\begin{aligned}
\pi_1 &= (0 \leftrightarrow 1, n-1 \leftrightarrow n),\\
\pi_{n-1} &= (0 \mapsto n-1 \mapsto 1 \mapsto n \mapsto 0),\\
\pi_n &= (0 \mapsto n \mapsto 1 \mapsto n-1 \mapsto 0).
\end{aligned}$$

- In type $E_6^{(1)}$ we have $\Omega \cong \mathbb{Z}_3$ and non-trivial elements

$$\begin{aligned}
\pi_1 &= (0 \mapsto 1 \mapsto 6 \mapsto 0),\\
\pi_6 &= (0 \mapsto 6 \mapsto 1 \mapsto 0).
\end{aligned}$$

It is immediate that the affine braid group action from Section 2.2.1 extends as follows.

Theorem 2.3.22. *The extended affine braid group $\dot{\mathcal{B}}$ acts on the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ via $T_i \to \boldsymbol{T}_i$ for each $i \in I$ and $\pi \to S_\pi$ for each $\pi \in \Omega$.*

2.4 Quantum affinizations

In the untwisted case $U_q(\hat{\mathfrak{g}})$ has an alternative *Drinfeld new presentation*, first stated by Drinfeld [23], as the quantum affinization of the corresponding finite quantum group $U_q(\mathfrak{g})$. This realization resembles the loop presentation for untwisted affine Lie algebras, and has been immensely useful for studying the representation theory of quantum affine algebras. In particular, it was implemented by Chari and Pressley in a systematic treatment of the finite dimensional modules and their R-matrices [9–12], as well as by Frenkel and Jing [27, 48] to construct vertex representations.

The quantum affinization procedure applies just as readily to any quantum group associated to a Kac-Moody algebra [51, 92], and yields a broad new class of quantum algebras that includes both the untwisted quantum affine algebras and the quantum toroidal algebras. The representation theory of these quantum affinizations in general has been studied by Hernandez [35, 36], while in the simply laced case work of Nakajima [92] provides a geometric realization in terms of the equivariant K-theory of quiver varieties on the underlying Dynkin diagram. Moreover the existence of vertex representations beyond the case of quantum affine algebras is explored in [13, 51].

In this section we first introduce the quantum affinization $\widehat{U_q(\mathfrak{s})}$ of any Drinfeld-Jimbo quantum group, together with natural (anti-)automorphisms of the algebra which shall prove useful later on. We then focus in particular on the untwisted quantum affine algebras, outlining Jing's isomorphism between the two presentations as well as a loop-style version of the extended affine braid group action from Theorem 2.3.22. Returning to the general case, we provide the triangular decomposition and classification of loop highest weight modules for $\widehat{U_q(\mathfrak{s})}$ due to Hernandez [35], and conclude with a discussion of topological coproduct structures on these algebras.

One may view the quantum affinization of a quantum group $U_q(\mathfrak{s})$ as a deformation quantization of the one-dimensional central extension of the loop algebra $\mathfrak{s}[t, t^{-1}]$. In particular, when $\mathfrak{s}$ is of finite type this is the loop-style realization of the corresponding untwisted affine Kac-Moody algebra $\hat{\mathfrak{s}}$ without derivation. Loosely speaking, the $x^+_{i,m}, x^-_{i,m}, h_{i,r}, q^h$ generators below correspond to the elements $e_i t^m, f_i t^m, h_i t^r, h$ respectively inside $\mathfrak{s}[t, t^{-1}]$, and $C^{\pm 1}$ is identified with the central extension.

Definition 2.4.1. *The quantum affinization of $U_q(\mathfrak{s})$ is the unital associative $\Bbbk$-algebra $\widehat{U_q(\mathfrak{s})}$ with generators $x^\pm_{i,m}$, $h_{i,r}$, q^h, $C^{\pm 1}$ ($i \in I$, $m \in \mathbb{Z}$, $r \in \mathbb{Z}^*$, $h \in P^\vee$) and relations*

- $C^{\pm 1}$ *central,*
- $C^{\pm 1} C^{\mp 1} = q^0 = 1,$
- $q^h q^{h'} = q^{h+h'},$

- $[q^h, h_{j,r}] = 0,$
- $[h_{i,r}, h_{j,s}] = \delta_{r+s,0}\dfrac{[ra_{ij}]_i}{r}\dfrac{C^r - C^{-r}}{q_j - q_j^{-1}},$
- $q^h x^\pm_{j,m} q^{-h} = q^{\pm\langle \alpha_j, h\rangle} x^\pm_{j,m},$
- $[h_{i,r}, x^\pm_{j,m}] = \pm\dfrac{[ra_{ij}]_i}{r} C^{\frac{r\mp|r|}{2}} x^\pm_{j,r+m},$
- $[x^+_{i,m}, x^-_{j,l}] = \dfrac{\delta_{ij}}{q_i - q_i^{-1}}(C^{-l}\phi^+_{i,m+l} - C^{-m}\phi^-_{i,m+l}),$
- $[x^\pm_{i,m+1}, x^\pm_{j,l}]_{q_i^{\pm a_{ij}}} + [x^\pm_{j,l+1}, x^\pm_{i,m}]_{q_i^{\pm a_{ij}}} = 0,$

and whenever $i \neq j$, *for any integers* m *and* $m_1, \dots, m_{a'}$ *where* $a' = 1 - a_{ij}$,

- $\displaystyle\sum_{\pi \in S_{a'}} \sum_{s=0}^{a'} (-1)^s \begin{bmatrix} a' \\ s \end{bmatrix}_i x^\pm_{i,m_{\pi(1)}} \dots x^\pm_{i,m_{\pi(s)}} x^\pm_{j,m} x^\pm_{i,m_{\pi(s+1)}} \dots x^\pm_{i,m_{\pi(a')}} = 0.$

Here each $k_i = q^{d_i \alpha_i^\vee}$ *and the* $\phi^\pm_{i,\pm s}$ *are given by the formula*

$$\sum_{s \geq 0} \phi^\pm_{i,\pm s} z^{\pm s} = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{s' > 0} h_{i,\pm s'} z^{\pm s'}\right)$$

when $s \geq 0$, *and are zero otherwise.*

It is clear that $\widehat{U_q(\mathfrak{s})}$ possesses a natural $\mathbb{Z}$-grading given by

$$\deg_{\mathbb{Z}}(x^\pm_{i,m}) = m, \quad \deg_{\mathbb{Z}}(h_{i,r}) = r, \quad \deg_{\mathbb{Z}}(C^{\pm 1}) = \deg(q^h) = 0, \tag{2.4.1}$$

as well as a finer grading defined by

$$\deg(x^\pm_{i,m}) = m\delta' \pm \alpha_i, \quad \deg(h_{i,r}) = r\delta', \quad \deg(C^{\pm 1}) = \deg(q^h) = 0, \tag{2.4.2}$$

which takes values in the free abelian group on $\{\alpha_i \mid i \in I\} \cup \{\delta'\}$.

Remark 2.4.2.

1. The final set of relations in Definition 2.4.1 are called the affine q-Serre relations.

2. The definition of quantum affinization varies slightly between sources. We use the one found for example in [17, 19, 86] since it is more precise regarding the isomorphism between the two presentations of the quantum affine algebra. The one found in other works such as [3, 37, 50] can then be obtained by adjoining $C^{\pm 1/2}$ and scaling each $x^\pm_{i,m}$ generator by $C^{m/2}$.

It is clear that any quantum affinization $\widehat{U_q(\mathfrak{s})}$ possesses the following natural automorphisms and anti-automorphisms.

- Every automorphism π of the underlying Dynkin diagram gives rise to an automorphism of $P^\vee$ preserving $(\ ,\ )$, extended trivially from the permutation of the simple coroots. This leads to an automorphism $\mathcal{S}_\pi$ of $\widehat{U_q(\mathfrak{s})}$ defined by
$$\mathcal{S}_\pi(x^\pm_{i,m}) = o^m_{i,\pi(i)} x^\pm_{\pi(i),m}, \quad \mathcal{S}_\pi(q^h) = q^{\pi(h)}, \quad \mathcal{S}_\pi(h_{i,r}) = o^r_{i,\pi(i)} h_{\pi(i),r}, \quad \mathcal{S}_\pi(C) = C.$$
- For each $i \in I$ there is an automorphism $\mathcal{X}_i$ given by
$$\mathcal{X}_i(x^\pm_{j,m}) = \upsilon(j)^{\delta_{ij}} x^\pm_{j,m\mp\delta_{ij}}, \quad \mathcal{X}_i(h_{j,r}) = h_{j,r}, \quad \mathcal{X}_i(q^h) = C^{-\langle \Lambda_i, h\rangle} q^h, \quad \mathcal{X}_i(C) = C,$$
where υ is any $\{\pm 1\}$-valued function on I, for example a length function.
- There is also an anti-involution η with
$$\eta(x^\pm_{i,m}) = x^\pm_{i,-m}, \quad \eta(h_{i,r}) = -C^r h_{i,-r}, \quad \eta(q^h) = q^{-h}, \quad \eta(C) = C.$$
- There exists a $\mathbb{Q}$-algebra involution $\mathcal{W}$ sending $q \mapsto q^{-1}$ such that
$$\mathcal{W}(x^\pm_{i,m}) = C^m x^\mp_{i,m}, \quad \mathcal{W}(h_{i,r}) = -h_{i,r}, \quad \mathcal{W}(q^h) = q^h, \quad \mathcal{W}(C) = C^{-1}.$$

Remark 2.4.3. Certain sources add extra generators $D^{\pm 1}$ satisfying the relations
$$D^{\pm 1} D^{\mp 1} = 1, \quad D q^h D^{-1} = q^h, \quad D h_{i,r} D^{-1} = q^r h_{i,r}, \quad D x^\pm_{i,m} D^{-1} = q^m x^\pm_{i,m},$$
to their definition of quantum affinization. In this case, conjugation by D acts as a degree operator (specifically, associated to $\deg_\mathbb{Z}$) and we can extend $\mathcal{S}_\pi$, $\mathcal{X}_i$, η and $\mathcal{W}$ by mapping D to D, $Dq^{\Lambda_0^\vee}$, D and D^{-1} respectively.

2.4.1 Drinfeld new presentation for quantum affine algebras

The relationship between the two presentations of the quantum affine algebra in the untwisted case was first studied by Beck [3], who used an action of the extended affine braid group to construct a morphism from the Drinfeld new realization to the Drinfeld-Jimbo realization. Jing [50] then defined an inverse morphism using q-commutators, while Damiani proved the surjectivity [17] and injectivity [18] of Beck's map.

Let us now present Jing's isomorphism. For each $i_1 \in I_0$ there exist sequences $\underline{i} = (i_1, i_2, \ldots, i_{h-1})$ in I_0 and $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_{h-2})$ in $\mathbb{Q}_{\leq 0}$ such that

$$(\alpha_{i_1} + \cdots + \alpha_{i_s}, \alpha_{i_{s+1}}) = \epsilon_s \text{ for } s = 1, \ldots, h-2, \tag{2.4.3}$$

where $h = \sum_{i \in I} a_i$ is the Coxeter number of $\hat{\mathfrak{g}}$. Then for any such sequences, the following extends to a $\Bbbk$-algebra isomorphism from the Drinfeld-Jimbo realization of $U_q'(\hat{\mathfrak{g}})$ to its Drinfeld new realization as the quantum affinization $\widehat{U_q(\mathfrak{g})}$:

- $x_i^{\pm} \mapsto x_{i,0}^{\pm}$ and $t_i \mapsto k_i$ for each $i \in I_0$,
- $x_0^+ \mapsto \left[x_{i_{h-1},0}^-, \ldots, x_{i_2,0}^-, x_{i_1,1}^-\right]_{q^{\epsilon_1} \ldots q^{\epsilon_{h-2}}} C k_\theta^{-1}$,
- $x_0^- \mapsto a(-q)^{-\epsilon} C^{-1} k_\theta \left[x_{i_{h-1},0}^+, \ldots, x_{i_2,0}^+, x_{i_1,-1}^+\right]_{q^{\epsilon_1} \ldots q^{\epsilon_{h-2}}}$,
- $t_0 \mapsto C k_\theta^{-1}$,

where $k_\theta = k_1^{a_1} \ldots k_n^{a_n}$, $\epsilon = \epsilon_1 + \cdots + \epsilon_{h-2}$, and a is a constant depending on type (in particular $a = 1$ when $\hat{\mathfrak{g}}$ is simply laced). Example sequences in all types can be found in [50, Table 2.1].

Remark 2.4.4. 1. Extending Jing's isomorphism via $q^d \mapsto D$ with D as in Remark 2.4.3 provides a Drinfeld new realization for $U_q(\hat{\mathfrak{g}})$.

2. It is clear in both presentations that $U_q(\hat{\mathfrak{g}})$ and $U_q'(\hat{\mathfrak{g}})$ contain a natural copy of the finite quantum group $U_q(\mathfrak{g})$ – it is the subalgebra generated by $\{x_i^{\pm}, t_i^{\pm 1} \mid i \in I_0\}$ in the Drinfeld-Jimbo, and by $\{x_{i,0}^{\pm}, k_i^{\pm 1} \mid i \in I_0\}$ in the Drinfeld new.

So we see that the quantum affine algebra can be formed from $\mathfrak{g}$ either as the quantum group associated to the corresponding affine Kac-Moody algebra $\hat{\mathfrak{g}}$, or by performing quantum affinization to $U_q(\mathfrak{g})$. This is precisely the commutativity of the following diagram, taken from [37].

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\text{Affinization}"] \arrow[d, "\text{Quantization}"'] & \hat{\mathfrak{g}} \arrow[d, "\text{Quantization}"] \\
U_q(\mathfrak{g}) \arrow[r, "\text{Quantum Affinization}"'] & U_q'(\hat{\mathfrak{g}})
\end{tikzcd}$$

Since quantum affinization is defined for the quantum group of any Kac-Moody algebra, we can apply it to the quantum affine algebra to obtain a sort of 'double affine quantum group'. As we will see in Chapter 3, this is precisely the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$. However it should be noted that $U_q(\mathfrak{g}_{\mathrm{tor}})$ is not the quantum group of any Kac-Moody algebra, and so cannot by further affinized in this way.

Remark 2.4.5. In fact, a generalization of the Drinfeld new realization for $U_q(\hat{\mathfrak{g}})$ which includes all twisted types was also stated by Drinfeld [23]. A morphism from the Drinfeld-Jimbo presentation was defined by Jing and Zhang [52, 53], while the proof that it is an isomorphism was again completed by Damiani in [17, 18]. Furthermore, using this presentation the construction of vertex representations was extended to twisted types in [49]. However, we do not include these cases here since they are not required for our purposes.

For untwisted quantum affine algebras and each $i \in I_0$, we define $\mathbf{X}_i$ to be the automorphism $\mathcal{X}_i$ in the case where v is the restriction to I_0 of some affine length function $o : I \to \{\pm 1\}$ as outlined in Section 2.1. Moreover we shall denote the anti-involution η by η', and note that $\mathbf{T}_i^{-1} = \eta' \mathbf{T}_i \eta'$ for all $i \in I_0$. The following then provides a loop–style analogue of Theorem 2.3.22 with respect to the Bernstein and Drinfeld new presentations.

Theorem 2.4.6. [3] *The extended affine braid group $\dot{\mathcal{B}}$ acts on the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ via $T_i \to \boldsymbol{T}_i$ and $X_{\omega_i^\vee} \to \boldsymbol{X}_i$ for each $i \in I_0$.*

In Section 3.1.2 we wish to define automorphisms $\mathcal{T}_i$ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ for each $i \in I$ which are comparable to the automorphisms $\mathbf{T}_i$ of $U_q(\hat{\mathfrak{g}})$. In particular, this requires us to write the actions of $\mathcal{T}_i$ on certain Drinfeld new style generators.

To this end, in the case of untwisted $U_q(\hat{\mathfrak{g}})$ let us derive formulae for some of the $\mathbf{T}_i(x_{j,m}^{\pm})$ when $i, j \in I_0$. It is clear from the definitions that $\mathbf{T}_i$ commutes with $\mathbf{X}_j$ whenever $j \neq i$ and therefore

$$\mathbf{T}_i(x_{j,m}^{+}) = o(j)^m \mathbf{X}_j^{-m} \mathbf{T}_i(x_{j,0}^{+}) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} (x_{i,0}^{+})^{(-a_{ij}-s)} x_{j,m}^{+} (x_{i,0}^{+})^{(s)},$$

$$\mathbf{T}_i(x_{j,m}^{-}) = o(j)^m \mathbf{X}_j^{m} \mathbf{T}_i(x_{j,0}^{-}) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{s} (x_{i,0}^{-})^{(s)} x_{j,m}^{-} (x_{i,0}^{-})^{(-a_{ij}-s)},$$

for all $m \in \mathbb{Z}$. When $i = j$ the $\mathbf{T}_i(x_{j,m}^{\pm})$ are calculated recursively on $|m|$ and expressions quickly become complicated. However, thanks to a simplified presentation for $U_q(\mathfrak{g}_{\mathrm{tor}})$

coming from Proposition 3.1.8, we shall only require the case $m = 0, \mp 1$. Let $\mathcal{U}_i$ be the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $\{x^{\pm}_{i,m}, h_{i,r}, k_i^{\pm 1}, C^{\pm 1} \mid m \in \mathbb{Z},\, r \in \mathbb{Z}^*\}$, and $h_i : U_q(A_1^{(1)}) \xrightarrow{\sim} \mathcal{U}_i$ be the morphism sending

$$
\begin{aligned}
&q \mapsto q_i, \quad k_1 \mapsto k_i, \quad k_0 \mapsto Ck_i^{-1}, \quad x_1^{\pm} \mapsto x^{\pm}_{i,0},\\
&x_0^{+} \mapsto -o(i)Ck_i^{-1}x^{-}_{i,1}, \quad x_0^{-} \mapsto -o(i)x^{+}_{i,-1}k_iC^{-1}.
\end{aligned}
$$

Then by Corollary 3.8 of [3] we have $\mathbf{T}_i \circ h_i = h_i \circ \mathbf{T}_1$ and hence

$$
\begin{aligned}
\mathbf{T}_i(x^{+}_{i,-1}) &= h_i \circ \mathbf{T}_1(-o(i)x_0^{-}k_0) = \sum_{s=0}^{2}(-1)^s q_i^s (x^{-}_{i,0})^{(s)} x^{+}_{i,-1} (x^{-}_{i,0})^{(2-s)} k_i,\\
\mathbf{T}_i(x^{-}_{i,1}) &= h_i \circ \mathbf{T}_1(-o(i)k_0^{-1}x_0^{+}) = k_i^{-1} \sum_{s=0}^{2}(-1)^s q_i^{-s} (x^{+}_{i,0})^{(2-s)} x^{-}_{i,1} (x^{+}_{i,0})^{(s)}.
\end{aligned}
$$

2.4.2 Loop triangular decomposition and ℓ-highest weight modules

Throughout this subsection all representations of $\widehat{U_q(\mathfrak{s})}$ shall be type 1, meaning that the central elements $C^{\pm 1}$ act trivially. It is known [35] that for any quantum affinization there exists a so-called *loop* triangular decomposition $\widehat{U_q(\mathfrak{s})} \cong \widehat{U_q(\mathfrak{s})}^{-} \otimes \widehat{U_q(\mathfrak{s})}^{0} \otimes \widehat{U_q(\mathfrak{s})}^{+}$ into the subalgebras

$$
\langle x^{-}_{i,m} \mid i \in I,\, m \in \mathbb{Z}\rangle, \quad \langle q^h,\, h_{i,r},\, C^{\pm 1} \mid h \in P^{\vee},\, i \in I,\, r \in \mathbb{Z}^*\rangle, \quad \langle x^{+}_{i,m} \mid i \in I,\, m \in \mathbb{Z}\rangle,
$$

respectively. This allows us to define the notion of ℓ-highest weight modules for quantum affinizations, analogously to the constructions of Section 2.2.2 for quantum groups.

Definition 2.4.7. 1. *An ℓ-weight is a pair (λ, Ψ) where $\lambda \in \mathfrak{h}^*$ and $\Psi = (\Psi^{\pm}_{i,\pm s})_{i \in I,\, s \geq 0}$ with all $\Psi^{\pm}_{i,\pm s} \in \mathbb{C}$, satisfying the condition $\Psi^{\pm}_{i,0} = q_i^{\pm\langle \lambda, \alpha_i^{\vee}\rangle}$ for each $i \in I$.*

2. *The set of ℓ-weights is denoted by P_ℓ.*

3. *A $\widehat{U_q(\mathfrak{s})}$-module V is of ℓ-highest weight $(\lambda, \Psi) \in P_\ell$ if there exists some vector $v \in V$ such that $V = \widehat{U_q(\mathfrak{s})} \cdot v$ and moreover all*

$$
x^{+}_{i,m} \cdot v = 0, \quad q^h \cdot v = q^{\langle \lambda, h\rangle} v, \quad \phi^{\pm}_{i,\pm s} \cdot v = \Psi^{\pm}_{i,\pm s} v.
$$

The compatibility between λ and Ψ comes from the fact that $k_i^{\pm 1} = \phi_{i,0}^{\pm}$. Similarly to Section 2.2.2, for each $(\lambda, \Psi) \in P_\ell$ we can define the associated Verma module $M(\lambda, \Psi)$ of ℓ-highest weight (λ, Ψ) as the quotient of $\widehat{U_q(\mathfrak{s})}$ by the left ideal generated by

$$\{x_{i,m}^+,\, q^h - q^{\langle \lambda, h\rangle}1,\, \phi_{i,\pm s}^{\pm} - \Psi_{i,\pm s}^{\pm}1 \mid i \in I,\, m \in \mathbb{Z},\, h \in P^\vee,\, s \in \mathbb{Z}_{\geq 0}\}.$$

Again, this satisfies the universal property that $M(\lambda, \Psi)$ surjects onto any $\widehat{U_q(\mathfrak{s})}$-module of ℓ-highest weight (λ, Ψ), with 1 sent to the ℓ-highest weight vector v in Definition 2.4.7 (3). Moreover, $M(\lambda, \Psi)$ contains a unique maximal submodule and the corresponding quotient $V(\lambda, \Psi)$ is the unique irreducible module of ℓ-highest weight (λ, Ψ) up to isomorphism.

There also exists a notion of integrability for representations of quantum affinizations.

Definition 2.4.8. *A representation of $\widehat{U_q(\mathfrak{s})}$ is integrable if it is integrable as a $U_q(\mathfrak{s})$-module via restriction, with finite dimensional weight spaces.*

Remark 2.4.9. Let $\mathfrak{s}$ be of finite type, so that $\widehat{U_q(\mathfrak{s})}$ is the corresponding untwisted quantum affine algebra.

1. For any ℓ-weight we have that Ψ determines $\lambda = \sum_{i \in I} \langle \lambda, \alpha_i^\vee \rangle \Lambda_i$ uniquely.

2. The notions of highest weight and ℓ-highest weight modules do not coincide.

3. The definition of integrability above differs from that of Section 2.2.2.

For any $\widehat{U_q(\mathfrak{s})}$-module V we can define the weight spaces V_λ exactly as before, so that $x_{i,m}^{\pm} \cdot V_\lambda \subset V_{\lambda \pm \alpha_i}$. It follows that whenever V is integrable, for any $v \in V$ there exists some $k \geq 0$ such that all $(x_{i,m}^{\pm})^k \cdot v = 0$. Furthermore, if V is of ℓ-highest weight (λ, Ψ) then it must be diagonalisable as a representation of $\widehat{U_q(\mathfrak{s})}^0$ and moreover $V = \bigoplus_{\mu \leq \lambda} V_\mu$.

Definition 2.4.10. *The set of ℓ-dominant weights P_ℓ^+ is the collection of $(\lambda, \Psi) \in P_\ell$ for which there exist (Drinfeld) polynomials $P_i(z) \in \mathbb{C}[z]$ with $P_i(0) = 1$ and*

$$\sum_{s \geq 0} \Psi_{i,\pm s}^{\pm} z^{\pm s} = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$$

in $\mathbb{C}[\![z]\!]$ or $\mathbb{C}[\![z^{-1}]\!]$ respectively.

In this case, it follows that every $\langle \lambda, \alpha_i^\vee \rangle = \deg(P_i) \geq 0$ and so λ must be dominant.

Theorem 2.4.11. [35] *An irreducible ℓ-highest weight representation $V(\lambda, \Psi)$ is integrable if and only if $(\lambda, \Psi) \in P_\ell^+$.*

For finite types this is originally due to Chari-Pressley [10, 11], where in fact these modules are precisely the irreducible finite dimensional representations of the quantum affine algebra $U_q'(\hat{\mathfrak{g}})$. In type $A_n^{(1)}$ the result was first proved by Miki [87] via a method which requires a twisted (topological) coproduct of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ defined using the Miki automorphism [86]. Nakajima [92] later addressed all simply laced types by geometric methods involving the equivariant K-theory of quiver varieties on the underlying Dynkin diagram.

Remark 2.4.12. Comparable results for ℓ-lowest weight modules can be deduced simply by twisting every representation by $\mathcal{W}$.

2.4.3 Topological coproducts

Unlike quantum groups, quantum affinizations are not known to possess Hopf algebra or even coproduct structures. Nevertheless, Drinfeld did define in an unpublished note (see also [20, 21]) a *topological* coproduct for $U_q(\widehat{\mathfrak{sl}}_{n+1})$ with respect to the Drinfeld new presentation, taking values in a completion of its tensor square. This was later generalised by Hernandez [35] to a topological coproduct for general quantum affinizations, given in terms of a formal parameter. However, compatibility with the affine q-Serre relations was known only in the finite [24, 31] and simply laced [21] cases.

Recent work of Damiani [19] addresses this issue, proving that there exists a topological coproduct Δ_v of $\widehat{U_q(\mathfrak{s})}$ in the general case. Her method relies upon careful consideration of the specific completion $\widehat{U_q(\mathfrak{s})} \widehat{\otimes} \widehat{U_q(\mathfrak{s})}$ into which Δ_v maps. In particular, she makes use of a different, smaller completion compared to [35]. This provides the correct setting for treating higher tensor powers $\widehat{U_q(\mathfrak{s})}^{\widehat{\otimes} m}$ and proving coassociativity of Δ_v – in contrast to [35, Remark 8] – as well as the existence of a counit ε.

On the other hand, it was shown that there exists neither an antipode $S : \widehat{U_q(\mathfrak{s})} \to \widehat{U_q(\mathfrak{s})}$ nor a Hopf algebra structure associated to Δ_v [19, Remark 7.12]. As discussed in

[31], this would require us to replace the codomain for S with a completion of $\widehat{U_q(\mathfrak{s})}$, and so on.

For simplicity, we shall not dwell on these (important) subtleties here and instead refer the interested reader to [19]. For example, there §3 defines the completions considered, §7 proves the coassociativity and counit properties, and Remark 7.7 discusses differences with [35].

Theorem 2.4.13. [19] *There is a unique algebra morphism $\Delta_v : \widehat{U_q(\mathfrak{s})} \to \widehat{U_q(\mathfrak{s})} \widehat{\otimes} \widehat{U_q(\mathfrak{s})}$ sending*

$$\begin{aligned}
C^{\pm 1} &\mapsto C^{\pm 1} \otimes C^{\pm 1}, \\
q^h &\mapsto q^h \otimes q^h, \\
C^s \phi^+_{i,r} &\mapsto \sum_{k+\ell=r} (C^{s+\ell} \phi^+_{i,k} \otimes C^s \phi^+_{i,\ell}) v^{-\ell}, \\
C^s \phi^-_{i,r} &\mapsto \sum_{k+\ell=r} (C^{s} \phi^-_{i,k} \otimes C^{s+k} \phi^-_{i,\ell}) v^{-\ell}, \\
x^+_{i,m} &\mapsto x^+_{i,m} \otimes 1 + \sum_{\ell \geq 0} (C^{m-\ell} \phi^+_{i,\ell} \otimes x^+_{i,m-\ell}) v^{-(m-\ell)}, \\
x^-_{i,m} &\mapsto (1 \otimes x^-_{i,m}) v^{-m} + \sum_{\ell \leq 0} (x^-_{i,m-\ell} \otimes C^{m-\ell} \phi^-_{i,\ell}) v^{-\ell},
\end{aligned}$$

for all $h \in P^\vee$, $i \in I$ and $r, s, m \in \mathbb{Z}$. This map is injective, and satisfies the coassociativity property

$$(\Delta_v \widehat{\otimes} \mathrm{id}) \circ \Delta_v = (\mathrm{id} \widehat{\otimes} \Delta_v) \circ \Delta_v : \widehat{U_q(\mathfrak{s})} \to \widehat{U_q(\mathfrak{s})}^{\widehat{\otimes} 3}.$$

Moreover Δ_v possesses a counit $\varepsilon : \widehat{U_q(\mathfrak{s})} \to \mathbb{Q}(q)$ given by

$$\varepsilon(C^{\pm 1}) = \varepsilon(q^h) = \varepsilon(\phi^\pm_{i,r}) = 1, \quad \varepsilon(x^\pm_{i,m}) = 0,$$

such that $(\varepsilon \widehat{\otimes} \mathrm{id}) \circ \Delta_v = (\mathrm{id} \widehat{\otimes} \varepsilon) \circ \Delta_v = \mathrm{id}$.

It is worth noting that the power of v in each of the expressions above records *minus* the degree $\deg_{\mathbb{Z}}$ of the second factor. Furthermore, we have that

$$\Delta_v : h_{i,r} \mapsto \begin{cases} h_{i,r} \otimes 1 + (C^r \otimes h_{i,r}) v^{-r} & \text{if } r > 0, \\ h_{i,r} \otimes C^r + (1 \otimes h_{i,r}) v^{-r} & \text{if } r < 0, \end{cases}$$

and hence Δ_v sends $\widehat{U_q(\mathfrak{s})}^0$ into the usual (non-completed) tensor square $\widehat{U_q(\mathfrak{s})}^0 \otimes \widehat{U_q(\mathfrak{s})}^0$ since its generators are mapped to finite sums of elementary tensors.

One can view Δ_v as an affinized version of the coproduct Δ_+ for $U_q(\mathfrak{s})$ from Section 2.2. Similarly, Hernandez' topological coproduct in [35, 36] corresponds to $\overline{\Delta}_+$, and η is like the affinization of σ. Via a simple check we confirm that just as $\overline{\Delta}_+$ is obtained by conjugating Δ_+ by σ, conjugating the topological coproduct Δ_v by η recovers that of Hernandez. Moreover, conjugating each of these by $\mathcal{W}$ produces affinized versions of Δ_- and $\overline{\Delta}_-$ respectively.

Remark 2.4.14. Although Δ_v does not give a well-defined morphism to $\widehat{U_q(\mathfrak{s})} \otimes \widehat{U_q(\mathfrak{s})}$, it can still be used to define tensor products of $\widehat{U_q(\mathfrak{s})}$-modules in certain cases by specialising v to particular elements of $\mathbb{C}^*$. See for example the construction of Fock space representations for $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ by Feigin-Jimbo-Miwa-Mukhin [26].

Chapter 3

Quantum toroidal algebras

This chapter is concerned with investigating the structure and representation theory of quantum toroidal algebras. After discussing the basic definitions and some elementary results, in Section 3.1.1 we introduce the extended double affine braid groups $\ddot{\mathcal{B}}$ as the corresponding toroidal objects within the braid group setting.

We then obtain in Section 3.1.2 an action of $\ddot{\mathcal{B}}$ on the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$ in all untwisted and twisted types other than the rank 1 cases $A_1^{(1)}$ and $A_2^{(2)}$. The construction of certain $\mathcal{T}_i$ automorphisms involved in this action requires a simplified Drinfeld new style presentation of $U_q(\mathfrak{g}_{\mathrm{tor}})$ given in terms of finitely many generators and relations. Moreover, by generalising the notion of $\ddot{\mathcal{B}}$ appropriately, both the finite presentation and the braid group action extend to all quantum affinizations for which the underlying Dynkin diagram has at most triple arrows.

In Section 3.2 we use our braid group action to construct automorphisms and anti-involutions of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which exchange its horizontal and vertical subalgebras. In particular, we are able to extend to all untwisted types except $A_1^{(1)}$ and $G_2^{(1)}$ the remarkable Miki automorphism that has been instrumental for studying the structure and representation theory of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$. These results build upon existing work of the author [76] in the simply laced case.

Section 3.3 proves compatibilities between the action of $\ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$ and various (anti-)automorphisms on either side. Using these we are able to prove in all untwisted types outside $A_1^{(1)}$ and $G_2^{(1)}$ that central extensions of the modular groups $SL_2(\mathbb{Z})$ and $GL_2(\mathbb{Z})$ act on $U_q(\mathfrak{g}_{\mathrm{tor}})$. This provides a quantum toroidal analogue of corresponding results in the braid group setting due to Cherednik [14] and Ion-Sahi [45, 46].

We conclude this chapter by discussing some future directions, as well as applications of our results to the representation theory of quantum toroidal algebras. In Section 3.4.1 we construct vector representations for $U_q(\mathfrak{g}_{\text{tor}})$ in all simply laced types except $E_8^{(1)}$, written explicitly in terms of Young column bases. Furthermore, in Section 3.4.2 we prove that twisting the usual topological coproduct for $U_q(\mathfrak{g}_{\text{tor}})$ by our anti-involution ψ from Section 3.2 gives rise to a tensor product that is well-defined for ℓ-highest weight modules.

As mentioned in Section 2.4.1, the quantum toroidal algebra of type $X_n^{(r)}$ is defined as the quantum affinization of the quantum affine algebra $U_q'(\hat{\mathfrak{g}})$ of type $X_n^{(r)}$. In particular, it follows that all $U_q(\mathfrak{g}_{\text{tor}})$ possess a topological coproduct, loop triangular decomposition, and classification of ℓ-highest weight representations as outlined in Section 2.4.

Definition 3.0.1. *The quantum toroidal algebra* $U_q(\mathfrak{g}_{\text{tor}})$ *is the unital associative* $\Bbbk$-*algebra with generators* $x_{i,m}^{\pm}$, $h_{i,r}$, $k_i^{\pm 1}$, $C^{\pm 1}$ *(*$i \in I$, $m \in \mathbb{Z}$, $r \in \mathbb{Z}^*$*), subject to the following relations:*

- $C^{\pm 1}$ *central,*
- $C^{\pm 1} C^{\mp 1} = k_i^{\pm 1} k_i^{\mp 1} = 1,$
- $[k_i, k_j] = [k_i, h_{j,r}] = 0,$
- $[h_{i,r}, h_{j,s}] = \delta_{r+s,0} \dfrac{[r a_{ij}]_i}{r} \dfrac{C^r - C^{-r}}{q_j - q_j^{-1}},$
- $k_i x_{j,m}^{\pm} k_i^{-1} = q_i^{\pm a_{ij}} x_{j,m}^{\pm},$
- $[h_{i,r}, x_{j,m}^{\pm}] = \pm \dfrac{[r a_{ij}]_i}{r} C^{\frac{r \mp |r|}{2}} x_{j,r+m}^{\pm},$
- $[x_{i,m}^{+}, x_{j,l}^{-}] = \dfrac{\delta_{ij}}{q_i - q_i^{-1}} (C^{-l} \phi_{i,m+l}^{+} - C^{-m} \phi_{i,m+l}^{-}),$
- $[x_{i,m+1}^{\pm}, x_{j,l}^{\pm}]_{q_i^{\pm a_{ij}}} + [x_{j,l+1}^{\pm}, x_{i,m}^{\pm}]_{q_i^{\pm a_{ij}}} = 0,$

and whenever $i \neq j$*, for any integers* m *and* $m_1, \dots, m_{a'}$ *where* $a' = 1 - a_{ij}$*,*

- $\displaystyle\sum_{\pi \in S_{a'}} \sum_{s=0}^{a'} (-1)^s \begin{bmatrix} a' \\ s \end{bmatrix}_i x_{i,m_{\pi(1)}}^{\pm} \dots x_{i,m_{\pi(s)}}^{\pm} x_{j,m}^{\pm} x_{i,m_{\pi(s+1)}}^{\pm} \dots x_{i,m_{\pi(a')}}^{\pm} = 0.$

Here, the $\phi^{\pm}_{i,\pm s}$ are given by the formula

$$\sum_{s\geq 0}\phi^{\pm}_{i,\pm s}z^{\pm s}=k_i^{\pm 1}\exp\left(\pm(q_i-q_i^{-1})\sum_{s'>0}h_{i,\pm s'}z^{\pm s'}\right)$$

when $s \geq 0$, and are zero otherwise.

Remark 3.0.2.

1. Some sources – for example [87, 100, 103] – add horizontal or vertical degree-style generators to their definitions of $U_q(\mathfrak{g}_{\mathrm{tor}})$. With respect to our already established notations these would be written as $q^{\pm d}$ and $D^{\pm 1}$ respectively, and if present satisfy the relations

$$\begin{gathered} q^{\pm d}q^{\mp d}=D^{\pm 1}D^{\mp 1}=1, \qquad [q^d,D]=0, \\ q^dk_iq^{-d}=k_i, \qquad q^dh_{i,r}q^{-d}=h_{i,r}, \qquad q^dx^{\pm}_{i,m}q^{-d}=q^{\pm\delta_{i0}}x^{\pm}_{i,m}, \\ Dk_iD^{-1}=k_i, \qquad Dh_{i,r}D^{-1}=q^rh_{i,r}, \qquad Dx^{\pm}_{i,m}D^{-1}=q^mx^{\pm}_{i,m}, \end{gathered}$$

   with $C^{\pm 1}$ remaining central. In particular, $q^{\pm d}$ corresponds to affinizing $U_q(\hat{\mathfrak{g}})$ rather than $U_q'(\hat{\mathfrak{g}})$, while the inclusion of $D^{\pm 1}$ is as in Remark 2.4.3.

2. In type $A_n^{(1)}$ there is a two-parameter deformation $U_{q,\kappa}(\mathfrak{sl}_{n+1,\mathrm{tor}})$ where some of the relations in Definition 3.0.1 are modified to involve additional central generators $\kappa^{\pm 1}$. The extra parameter κ relates to the rotational symmetry of the Dynkin diagram, and specialising to $\kappa = 1$ recovers the above presentation. However, such a deformation is not known to exist in other types and thus will not be treated in this thesis.

So we see that the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$ of type $X_n^{(r)}$ can be obtained from the corresponding finite quantum group $U_q(\mathfrak{g})$ by affinizing twice on the quantum level. In fact, $U_q(\mathfrak{g}_{\mathrm{tor}})$ contains two natural quantum affine subalgebras. There is a horizontal subalgebra $\mathcal{U}_h$ of type $X_n^{(r)}$ defined as the image of the homomorphism $h : U_q'(X_n^{(r)}) \to U_q(\mathfrak{g}_{\mathrm{tor}})$ sending

$$x_i^{\pm}\mapsto x^{\pm}_{i,0}, \qquad t_i\mapsto k_i,$$

for all $i \in I$. Additionally, there is a vertical subalgebra $\mathcal{U}_v$ of untwisted type $Z_n^{(1)}$, where Z_n is the finite Cartan type of the simple Lie algebra $\mathfrak{g}$. It is the image of the homomorphism $v : U_q'(Z_n^{(1)}) \to U_q(\mathfrak{g}_{\mathrm{tor}})$ given by

$$x_{i,m}^{\pm} \mapsto x_{i,m}^{\pm}, \qquad h_{i,r} \mapsto h_{i,r}, \qquad k_i \mapsto k_i, \qquad C \mapsto C,$$

for all $i \in I_0$, $m \in \mathbb{Z}$ and $r \in \mathbb{Z}^*$. Furthermore, we are able to deduce from the next proposition that $\mathcal{U}_h$ and $\mathcal{U}_v$ together generate the entire quantum toroidal algebra. Figure 3.1 provides a simple illustration of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which highlights its generators and their $\mathbb{Z}$-grading, as well as the horizontal and vertical subalgebras.

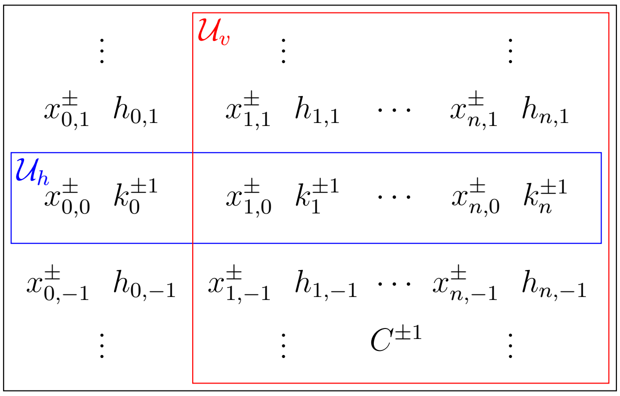


Fig. 3.1 An illustration of $U_q(\mathfrak{g}_{\mathrm{tor}})$ and its quantum affine subalgebras $\mathcal{U}_h$ and $\mathcal{U}_v$

For any $i_1, \dots, i_p \in I$ we define $\mathcal{U}_{i_1 \dots i_p}$ to be the subalgebra of $U_q(\mathfrak{g}_{\mathrm{tor}})$ generated by $\{x_{\ell,m}^{\pm}, h_{\ell,r}, k_\ell^{\pm 1}, C^{\pm 1} \mid \ell = i_1, \dots, i_p,\ m \in \mathbb{Z},\ r \in \mathbb{Z}^*\}$. Then it is clear that each $\mathcal{U}_i \cong U_q(A_1^{(1)})$, and if $i \neq j$ then

$$\mathcal{U}_{ij} \cong \begin{cases} U_q(A_1^{(1)}) \times U_q(A_1^{(1)}) & \text{if } a_{ij}a_{ji} = 0, \\ U_q(A_2^{(1)}) & \text{if } a_{ij}a_{ji} = 1, \\ U_q(C_2^{(1)}) & \text{if } a_{ij}a_{ji} = 2, \\ U_q(G_2^{(1)}) & \text{if } a_{ij}a_{ji} = 3, \\ U_q(\mathfrak{g}_{\mathrm{tor}}) & \text{in types } A_1^{(1)} \text{ and } A_2^{(2)}. \end{cases}$$

Proposition 3.0.3. *For each $i \in I$, the quantum toroidal algebra is generated by $\mathcal{U}_h$, $x_{i,\mp 1}^{\pm}$ and $C^{\pm 1}$.*

Proof. Let $\mathcal{A}$ be the subalgebra of $U_q(\mathfrak{g}_{\mathrm{tor}})$ generated by $\mathcal{U}_h$, $x_{i,\mp 1}^{\pm}$ and $C^{\pm 1}$. Our strategy is to first show that $\mathcal{U}_i$ is contained in $\mathcal{A}$, and then to show that $x_{j,\mp 1}^{\pm} \in \mathcal{A}$ whenever $a_{ij} < 0$, since the result then follows from the connectedness of the Dynkin diagram. Unpacking the formula in Definition 3.0.1, we see that $\phi_{i,0}^{\pm} = k_i^{\pm 1}$, $\phi_{i,\pm r}^{\pm} = 0$ for $r < 0$, and

$$\phi^{\pm}_{i,\pm r} = k_i^{\pm 1} \sum_{\ell=1}^{r} \frac{(\pm 1)^{\ell}(q_i - q_i^{-1})^{\ell}}{\ell!} \sum_{\substack{r_1+\dots+r_\ell = r \\ \text{all } r_j > 0}} h_{i,\pm r_1} \dots h_{i,\pm r_\ell}$$

for $r > 0$. This implies that $h_{i,-1} = C^{-1}k_i[x^+_{i,-1}, x^-_{i,0}]$ and $h_{i,1} = Ck_i^{-1}[x^+_{i,0}, x^-_{i,1}]$. Thus by the relations of the quantum toroidal algebra, all $x^{\pm}_{i,m}$ lie inside $\mathcal{A}$. We also have

$$[x^+_{i,\pm r}, x^-_{i,0}] = \pm \frac{C^{\frac{r \mp r}{2}} k_i^{\pm 1}}{q_i - q_i^{-1}} \sum_{\ell=1}^{r} \frac{(\pm 1)^{\ell}(q_i - q_i^{-1})^{\ell}}{\ell!} \sum_{\substack{r_1+\dots+r_\ell = r \\ \text{all } r_j > 0}} h_{i,\pm r_1} \dots h_{i,\pm r_\ell}$$

when $r > 0$ and so all $h_{i,\pm r} \in \mathcal{A}$ by induction. Therefore $\mathcal{U}_i \subset \mathcal{A}$ and we conclude our proof by noting that $x^{\pm}_{j,\mp 1} = \pm \frac{C^{\pm 1}}{[a_{ij}]_i}[h_{i,\mp 1}, x^{\pm}_{j,0}]$ whenever $a_{ij} < 0$. $\square$

Corollary 3.0.4. *The quantum toroidal algebra is generated by its horizontal and vertical subalgebras.*

Throughout the remainder of this chapter, we shall require the following standard automorphisms and anti-automorphisms of $U_q(\mathfrak{g}_{\mathrm{tor}})$ as introduced in Section 2.4.

- Every outer automorphism $\pi \in \Omega$ of the underlying affine Dynkin diagram gives rise to an automorphism $\mathcal{S}_\pi$ defined by

  $$\mathcal{S}_\pi(x^{\pm}_{i,m}) = o^m_{i,\pi(i)} x^{\pm}_{\pi(i),m}, \quad \mathcal{S}_\pi(k_i) = k_{\pi(i)}, \quad \mathcal{S}_\pi(h_{i,r}) = o^r_{i,\pi(i)} h_{\pi(i),r}, \quad \mathcal{S}_\pi(C) = C,$$

  which restricts to S_π on $\mathcal{U}_h$.

- There is the anti-involution η with

  $$\eta(x^{\pm}_{i,m}) = x^{\pm}_{i,-m}, \quad \eta(h_{i,r}) = -C^r h_{i,-r}, \quad \eta(k_i) = k_i^{-1}, \quad \eta(C) = C,$$

  which restricts to η' on $\mathcal{U}_v$ and σ on $\mathcal{U}_h$.

- For each $i \in I$ there is an automorphism $\mathcal{X}_i$ given by

  $$\mathcal{X}_i(x^{\pm}_{j,m}) = o(j)^{\delta_{ij}} x^{\pm}_{j,m \mp \delta_{ij}}, \quad \mathcal{X}_i(k_j) = C^{-\delta_{ij}} k_j, \quad \mathcal{X}_i(h_{j,r}) = h_{j,r}, \quad \mathcal{X}_i(C) = C,$$

  where $o : I \to \{\pm 1\}$ is some affine length function as outlined in Section 2.1. If $i \in I_0$ then $\mathcal{X}_i$ restricts to $\mathbf{X}_i$ on $\mathcal{U}_v$, while $\mathcal{X}_0$ restricts to the identity.

3.1 Braid group action

3.1.1 Extended double affine braid groups

Just as the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$ is in some sense formed by fusing together its horizontal and vertical quantum affine subalgebras in an appropriate way, we can similarly define the extended double affine braid group $\ddot{\mathcal{B}}$ by combining the Coxeter and Bernstein presentations for $\dot{\mathcal{B}}$.

Recall from Section 2.3.3 that Ω acts naturally on the affine braid group $\mathcal{B} = \langle T_i \mid i \in I\rangle$. There is also a linear action of Ω on $P^\vee$ given by $\pi(\Lambda_i^\vee) = \Lambda_{\pi(i)}^\vee$, which preserves $\mathring{P}^\vee \subset P^\vee$ and thus defines an action on $\{X_\beta \mid \beta \in \mathring{P}^\vee\}$. These actions are compatible with relations (2.3.4) and (2.3.5) for $N = \mathring{P}^\vee$ (extended to all $i \in I$) and so the following is well-defined.

Definition 3.1.1. *The extended double affine braid group $\ddot{\mathcal{B}}$ is generated by the affine braid group $\mathcal{B} = \langle T_i \mid i \in I\rangle$, the lattice $\{X_\beta \mid \beta \in \mathring{P}^\vee\}$ and the group Ω, subject to the relations*

- $T_i X_\beta = X_\beta T_i$ if $(\beta, \alpha_i) = 0$,
- $T_i^{-1} X_\beta T_i^{-1} = X_{s_i(\beta)}$ if $(\beta, \alpha_i) = 1$,
- $\pi T_i \pi^{-1} = T_{\pi(i)}$,
- $\pi X_\beta \pi^{-1} = X_{\pi(\beta)}$.

Remark 3.1.2. 1. The action of W on $\mathring{P}^\vee$ in the definition above is with respect to the embedding $\mathring{P}^\vee \hookrightarrow P^\vee$ of type $X_n^{(r)}$ rather than $Z_n^{(1)}$.

2. Our group $\ddot{\mathcal{B}}$ is the quotient of the X, Y-extended double affine Artin group of Ion and Sahi [46, Chapter 9] by its central element $X_{\frac{1}{m}\delta}$.

It is clear that $\ddot{\mathcal{B}}$ contains two extended affine braid subgroups which together generate the entire group: a horizontal subgroup $\mathcal{B}_h$ of type $X_n^{(r)}$ generated by $\mathcal{B}$ and Ω, and a vertical subgroup $\mathcal{B}_v$ of type $Z_n^{(1)}$ generated by $T_1, \dots, T_n$ and $\{X_\beta \mid \beta \in \mathring{P}^\vee\}$. Figure 3.2 illustrates how these subgroups fit together inside $\ddot{\mathcal{B}}$, as well as indicating a natural $\mathbb{Z}$-grading. We remark that there only exists an isomorphism between $\mathcal{B}_h$ and $\mathcal{B}_v$ which acts by the identity on $\mathcal{B}_0 \cong \mathcal{B}_h \cap \mathcal{B}_v$ in the untwisted case.

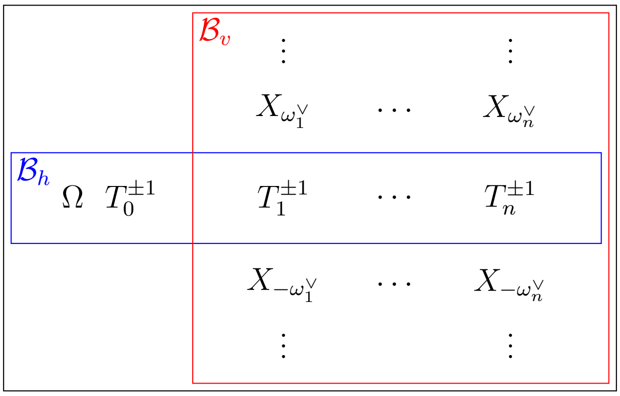


Fig. 3.2 An illustration of $\ddot{\mathcal{B}}$ and its extended affine braid subgroups $\mathcal{B}_h$ and $\mathcal{B}_v$

From Section 2.3.3 we know that $\mathcal{B}_h$ and $\mathcal{B}_v$ each have both Coxeter and Bernstein presentations – Table 3.1 summarises our choice of notation.

	Coxeter generators	Bernstein generators
$\mathcal{B}_h$	$T_1, \dots, T_n$ $T_0 = \Theta^{-1} Y_{-\beta_\theta}$ $\Omega = \{\pi_i = Y_{\beta_i} T_{v_i^{-1}} : i \in I_{\min}\}$	$T_1, \dots, T_n$ $\{Y_\mu : \mu \in N\}$
$\mathcal{B}_v$	$T_1, \dots, T_n$ $T_0^v = X_{\theta^\vee} \Theta^{-1}$ $\Omega^v = \{\rho_i = X_{\omega_i^\vee} T_{v_i}^{-1} : i \in I_{\min}\}$	$T_1, \dots, T_n$ $\{X_\beta : \beta \in \mathring{P}^\vee\}$

Table 3.1 Coxeter and Bernstein generators for $\mathcal{B}_h$ and $\mathcal{B}_v$

In particular, for $\mathcal{B}_h$ we use the alternative Bernstein presentation of Remark 2.3.20 so that while the X_β satisfy relations (2.3.4) and (2.3.5) with $T_0, \dots, T_n$, the Y_μ satisfy relations (2.3.6) and (2.3.7) with $T_0^v, T_1, \dots, T_n$. Note that in all untwisted types, each π_i and ρ_i correspond to the same outer automorphism of the affine Dynkin diagram.

We conclude with several automorphisms of $\ddot{\mathcal{B}}$ which will be important in Sections 3.2 and 3.3. For ease of notation, we restrict to the untwisted case since this is all we shall require.

- There is an involution $\mathfrak{t}$ which inverts $T_1, \dots, T_n$ and interchanges X_β and Y_β for all $\beta \in \mathring{P}^\vee$. It follows that $\mathfrak{t}$ exchanges each π_i and ρ_i, as well as T_0 and $(T_0^v)^{-1}$.

It is equal to the composition of the anti-involution $\mathfrak{e}$ of Ion and Sahi [46, Chapter 9] with the anti-automorphism that inverts every element. When restricted to the natural copy of the (non-extended) double affine braid group inside $\ddot{\mathcal{B}}$, which is generated by $\mathcal{B} = \langle T_0, \dots, T_n \rangle$ and $\{X_\beta \mid \beta \in \mathring{Q}^\vee\}$, this is the involution of Ion [44, Theorem 2.2].

- There exists an involution γ_v inverting $T_0, \dots, T_n$ and all X_β, while fixing each element of Ω. Similarly, there is an involution $\gamma_h = \mathfrak{t} \circ \gamma_v \circ \mathfrak{t}$ which inverts $T_0^v, T_1, \dots, T_n$ and all Y_μ but fixes each element of Ω^v.

3.1.1.1 Coxeter-style presentation

It was recently shown by Ion-Sahi [46] that while the (extended) double affine braid groups are not Coxeter groups themselves, they can be realized as *quotients* of the Coxeter groups associated to so-called 'double affine Coxeter diagrams'. This presentation has numerous applications, including in all affine types $X_n^{(r)}$ a congruence group action of $\Gamma_1(r) \leq SL_2(\mathbb{Z})$ on $\ddot{\mathcal{B}}$ by outer automorphisms. It descends from an action by automorphisms of the central extension $\tilde{\Gamma}_1(r)$, which is isomorphic to the Coxeter group of type A_2, B_2 or G_2 when $r = 1, 2$ or 3 respectively.

This realization of $\ddot{\mathcal{B}}$ provides a finer understanding of its structure that is essential for extending our proof of Theorem 3.2.1 from the simply laced case [76] to all untwisted types (except $A_1^{(1)}$ and $G_2^{(1)}$). Furthermore, as a consequence we obtain actions of $\tilde{\Gamma}_1(r)$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$ which enjoy a compatibility with those on $\ddot{\mathcal{B}}$ and our braid group action from Section 3.1.

We present the Coxeter-style presentation for $\ddot{\mathcal{B}}$ in the untwisted case only, since this is all we shall require for this thesis. Here, the double affine Coxeter diagram of type $\dddot{X}_n$ is formed as follows. First take the affine Dynkin diagram of type $X_n^{(1)}$, and consider the underlying, undirected Coxeter graph. Then replace the 0 vertex with three affine nodes, connected to one other by four edges and to each finite node $i \in I_0$ by $a_{0i}a_{i0}$ edges. We illustrate this process with two examples in Figure 3.3 below.

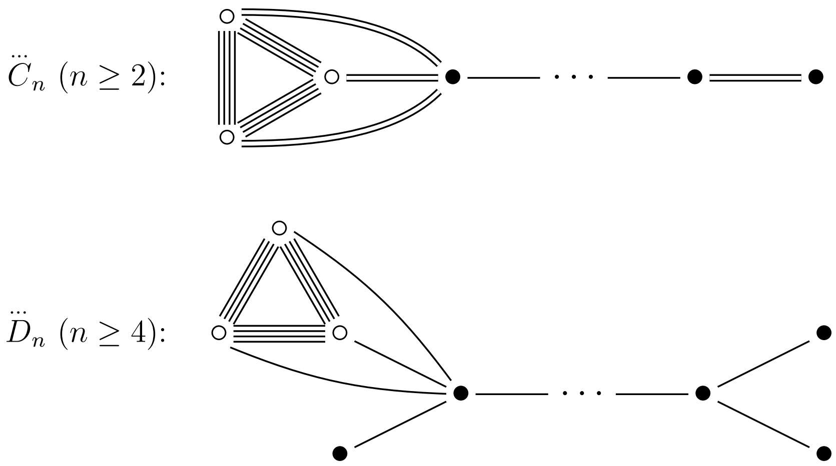


Fig. 3.3 Examples of double affine Coxeter diagrams

The Coxeter group $C(\dddot{X}_n)$ associated to this diagram has affine generators $\Theta_{01}, \Theta_{02}, \Theta_{03}$ and finite generators $T_1, \ldots, T_n$, with braid relations of type $X_n^{(1)}$ on each $\{\Theta_{0i}, T_1, \ldots, T_n\}$. Letting $\mathbf{B}(\dddot{X}_n)$ be its quotient by the relation $\Theta_{01}\Theta_{02}\Theta_{03}\Theta = 1$, as well as $\Theta_{0i}T_1^{-1}\Theta_{0j}T_1 = T_1^{-1}\Theta_{0j}T_1\Theta_{0i}$ for all $i < j$ if $X = C$, the following comes from [46, Theorem 5.19].

Theorem 3.1.3. *There is an isomorphism between $\mathbf{B}(\dddot{X}_n)$ and the (non-extended) double affine braid group of type $X_n^{(1)}$ sending $T_i \mapsto T_i$ for all $i \in I_0$ and*

$$\Theta_{01} \mapsto T_0, \qquad \Theta_{02} \mapsto T_0^{-1}X_{-\theta^\vee}, \qquad \Theta_{03} \mapsto X_{\theta^\vee}\Theta^{-1}.$$

In order to upgrade this to a Coxeter presentation for the *extended* double affine braid group $\ddot{\mathcal{B}}$, we must take the semidirect product of $\mathbf{B}(\dddot{X}_n)$ with two copies of the outer automorphism group Ω of the affine Dynkin diagram. The first, which we shall denote by $\Omega_1 = \{\pi_i \mid i \in I_{\min}\}$, acts naturally by permuting $\Theta_{01}, T_1, \ldots, T_n$ and by

$$\pi_i(\Theta_{02}) = T_{w_i}\Theta^{-1}\Theta_{03}\Theta T_{w_i}^{-1}, \qquad \pi_i(\Theta_{03}) = T_{w_i}\Theta_{01}\Theta_{02}\Theta_{01}^{-1}T_{w_i}^{-1},$$

where w_i is the minimal length element in the finite Weyl group such that $\Theta = T_{w_i^{-1}}T_iT_{w_i}$. In particular, $\pi_i(T_{w_{i^*}^{-1}}) = T_{w_i}$ where i^* is defined by $\pi_{i^*} = \pi_i^{-1}$ and therefore $\pi_i(\Theta) = T_{w_i}\Theta_{01}T_{w_i^{-1}}$. The second copy $\Omega_3 = \{\rho_i \mid i \in I_{\min}\}$ permutes $\Theta_{03}, T_1, \ldots, T_n$ instead, with

$$\rho_i(\Theta_{01}) = T_{w_i^{-1}}^{-1}\Theta_{03}^{-1}\Theta_{02}\Theta_{03}T_{w_i^{-1}}, \qquad \rho_i(\Theta_{02}) = T_{w_i^{-1}}^{-1}\Theta\Theta_{01}\Theta^{-1}T_{w_i^{-1}},$$

as well as $\rho_i(T_{w_{i^*}^{-1}}) = T_{w_i}$ and hence $\rho_i(\Theta) = T_{w_i}\Theta_{03}T_{w_i^{-1}}$.

Theorem 3.1.4. *The previous theorem extends to an isomorphism between* $\Omega_1 \ltimes (\Omega_3 \ltimes \mathbf{B}(\dddot{X}_n))$ *and* $\ddot{\mathcal{B}}$ *by sending* $\pi_i \mapsto \pi_i$ *and* $\rho_i \mapsto X_{\beta_i} T_{v_i}^{-1}$ *for all* $i \in I_{\min}$.

In particular, $\Omega_1 \ltimes \langle \Theta_{01}, T_1, \dots, T_n \rangle$ is sent to the horizontal subgroup $\mathcal{B}_h$, while $\Omega_3 \ltimes \langle \Theta_{03}, T_1, \dots, T_n \rangle$ is identified with the vertical subgroup $\mathcal{B}_v$.

3.1.1.2 Congruence group actions

As mentioned above, Ion-Sahi [46] use their Coxeter-style presentation for $\ddot{\mathcal{B}}$ to prove an action by outer automorphisms of the corresponding congruence group $\Gamma_1(r)$. This moreover descends from an action by automorphisms of its central extension $\tilde{\Gamma}_1(r)$. Again we outline the untwisted types only – where in fact these results are originally due to Cherednik [14] – but refer the reader to [46] for the general case.

Since the congruence groups $\Gamma_1(r)$ are defined by

$$\Gamma_1(r) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod r \right\}$$

for $r \in \{1, 2, 3\}$, in the untwisted case this is simply $SL_2(\mathbb{Z})$. Here we have an alternative presentation

$$\langle \mathfrak{u}_1, \mathfrak{u}_2 \mid \mathfrak{u}_1\mathfrak{u}_2\mathfrak{u}_1 = \mathfrak{u}_2\mathfrak{u}_1\mathfrak{u}_2, (\mathfrak{u}_1\mathfrak{u}_2)^6 = 1 \rangle$$

obtained by identifying $\mathfrak{u}_1$ and $\mathfrak{u}_2$ with $\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ respectively. It follows that there exists a natural surjection $\tilde{\Gamma}_1(1) \twoheadrightarrow SL_2(\mathbb{Z})$ from the braid group of type A_2, which is moreover a central extension since the centre of $\tilde{\Gamma}_1(1)$ is generated by $\mathfrak{c} = (\mathfrak{u}_1\mathfrak{u}_2)^3$.

This morphism extends to a surjection $\tilde{\Xi}_1(1) \twoheadrightarrow GL_2(\mathbb{Z})$ by taking the semidirect product with a cyclic group of order two. On the left, conjugation by the non-trivial element $\mathfrak{r}$ exchanges $\mathfrak{u}_1$ and $\mathfrak{u}_2^{-1}$, whereas in $GL_2(\mathbb{Z})$ this element corresponds to $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

The Coxeter presentation for $\ddot{\mathcal{B}}$ from Theorem 3.1.4 makes clear automorphisms $\mathfrak{a}, \mathfrak{b}$ fixing $T_1, \dots, T_n$ and acting on the affine and outer automorphism generators by

$$\begin{aligned}
&\mathfrak{a}(\Theta_{01}) = \Theta_{02}, \quad \mathfrak{a}(\Theta_{02}) = \Theta_{02}^{-1}\Theta_{01}\Theta_{02}, \quad \mathfrak{a}(\Theta_{03}) = \Theta_{03}, \quad \mathfrak{a}(\pi_i) = X_i\pi_i, \quad \mathfrak{a}(\rho_i) = \rho_i, \\
&\mathfrak{b}(\Theta_{01}) = \Theta_{01}, \quad \mathfrak{b}(\Theta_{02}) = \Theta_{03}, \quad \mathfrak{b}(\Theta_{03}) = \Theta_{03}^{-1}\Theta_{02}\Theta_{03}, \quad \mathfrak{b}(\pi_i) = \pi_i, \quad \mathfrak{b}(\rho_i) = Y_i^{-1}\rho_i.
\end{aligned}$$

Furthermore, there exists an anti-involution $\mathfrak{e}$ which maps $\Theta_{01} \leftrightarrow \Theta_{03}$ and $\pi_i \leftrightarrow \rho_i^{-1}$ but fixes all other generators. It follows that our aforementioned involution $\mathfrak{t}$ inverts $\Theta_{02}, T_1, \dots, T_n$ while swapping $\Theta_{01} \leftrightarrow \Theta_{03}^{-1}$ and each $\pi_i \leftrightarrow \rho_i$. It is easy to verify the relations

$$\mathfrak{a}^{-1} = \mathfrak{e}\mathfrak{b}\mathfrak{e}, \qquad \mathfrak{b}^{-1} = \mathfrak{e}\mathfrak{a}\mathfrak{e}, \qquad \mathfrak{a}\mathfrak{b}\mathfrak{a} = \mathfrak{b}\mathfrak{a}\mathfrak{b}, \tag{3.1.1}$$

and thus obtain the following results [14, 46].

Theorem 3.1.5.

- *There exists an injective homomorphism $\tilde{\Gamma}_1(1) \to \mathrm{Aut}(\ddot{\mathcal{B}})$ given by $\mathfrak{u}_1 \mapsto \mathfrak{a}$ and $\mathfrak{u}_2 \mapsto \mathfrak{b}$.*
- *This extends to a faithful action of $\tilde{\Xi}_1(1)$ where $\mathfrak{r}$ acts by the anti-involution $\mathfrak{e}$.*
- *Moreover, it descends to a morphism $SL_2(\mathbb{Z}) \to \mathrm{Out}(\ddot{\mathcal{B}})$ which is injective in types $A_n^{(1)}, D_{2n}^{(1)}, E_6^{(1)}$ and has kernel $\{\pm \mathrm{I}_2\}$ otherwise.*

Remark 3.1.6. Since the anti-involution which inverts every element of $\ddot{\mathcal{B}}$ must commute with the automorphisms $\mathfrak{a}$ and $\mathfrak{b}$ by definition, it is easy to see that (3.1.1) still holds if we replace $\mathfrak{e}$ with $\mathfrak{t}$. We therefore have an action of $\tilde{\Xi}_1(1)$ on $\ddot{\mathcal{B}}$ entirely by automorphisms by letting $\mathfrak{r}$ act as $\mathfrak{t}$ instead.

Using the automorphisms above we can introduce a *diagonal* extended affine braid subgroup $\mathcal{B}_d$ inside $\ddot{\mathcal{B}}$ which aligns with the second affine node of the double affine Coxeter diagram, in the same way that $\mathcal{B}_h$ and $\mathcal{B}_v$ correspond to the first and third respectively.

In particular, define $\mathcal{B}_d$ to be the subgroup $\mathfrak{a}(\mathcal{B}_h)$ generated by $\Theta_{02}, T_1, \dots, T_n$ and $X_i \pi_i = \rho_i T_{v_i} \pi_i$ for each $i \in I_{\min}$. This is moreover equal to $\mathfrak{b}^{-1}(\mathcal{B}_v)$ since

$$\mathfrak{b}^{-1}(\rho_i) = \mathfrak{b}^{-1}(\rho_{i^*}^{-1}) = \left(\pi_{i^*} T_{v_{i^*}^{-1}}^{-1} \rho_{i^*}\right)^{-1} = \rho_i T_{v_{i^*}^{-1}} \pi_i = \mathfrak{a}(\pi_i),$$

where the final equality holds provided that $v_{i^*}^{-1} = v_i$. Indeed, conjugating by the longest element w_0 of a finite Weyl group permutes the simple reflections according to the unique automorphism of the finite Dynkin diagram that maps $i \mapsto i^*$ for each $i \in I_{\min}$. Note that as in Example 2.3.21, extra care is required regarding the parity of n in type $D_n^{(1)}$. It follows that w_{0i} is sent to w_{0i^*}, and hence $v_i = w_0 w_{0i} = w_{0i^*} w_0 = v_{i^*}^{-1}$ since the longest element of any finite Weyl group is self-inverse [7, p.171].

It is clear that conjugation by $\mathfrak{a}(\pi_i) = \mathfrak{b}^{-1}(\rho_i)$ permutes $\Theta_{02}, T_1, \ldots, T_n$ according to the relevant outer automorphism of the affine Dynkin diagram, and therefore preserves $\mathcal{B}_d$. Furthermore, $\mathfrak{e}$ sends each

$$\mathfrak{a}(\pi_i) = \rho_i T_{v_i} \pi_i \mapsto \rho_i^{-1} T_{v_i^{-1}} \pi_i^{-1} = \rho_{i^*} T_{v_{i^*}} \pi_{i^*} = \mathfrak{a}(\pi_{i^*}) = \mathfrak{a}(\pi_i)^{-1}$$

and so in particular restricts to an anti-involution on $\mathcal{B}_d$ which fixes $\Theta_{02}, T_1, \ldots, T_n$. Moreover as an immediate consequence we deduce similar results for $\mathfrak{t}$.

3.1.2 Action on quantum toroidal algebras

Here we construct actions of the extended double affine braid groups on the quantum toroidal algebras. We start with a simplified presentation of $U_q(\mathfrak{g}_{\mathrm{tor}})$ involving finitely many generators and relations, which allows us to define automorphisms $\mathcal{T}_i$ for each $i \in I$. Note that our proof relies upon a finite presentation of each subalgebra $\mathcal{U}_{ij}$ that comes from the Drinfeld-Jimbo presentation of the quantum affine algebras. So in fact, our results extend to all quantum affinizations where the underlying Dynkin diagram has at most triple arrows, ie. $a_{ij}a_{ji} \leq 3$ for all distinct $i, j \in I$. In particular, we exclude the quantum toroidal algebras of types $A_1^{(1)}$ and $A_2^{(2)}$ for the remainder of this section.

Lemma 3.1.7. *For X of type A_2, C_2 or G_2, let $(a_{ij})_{i,j=1,2}$ be the corresponding finite type Cartan matrix and take q_1 and q_2 as in Section 2.1. Define $\mathcal{A}_X$ to be the $\Bbbk$-algebra with generators $\hat{x}^{\pm}_{i,0}$, $\hat{x}^{\pm}_{i,\mp 1}$, $\hat{k}_i^{\pm 1}$, $\hat{C}^{\pm 1}$ $(i = 1, 2)$ and relations*

(i) $\hat{C}^{\pm 1}$ *central,*

(ii) $\hat{C}^{\pm 1}\hat{C}^{\mp 1} = \hat{k}_i^{\pm 1}\hat{k}_i^{\mp 1} = 1$,

(iii) $[\hat{k}_i, \hat{k}_j] = 0$,

(iv) $\hat{k}_i \hat{x}^{\pm}_{j,m} \hat{k}_i^{-1} = q_i^{\pm a_{ij}} \hat{x}^{\pm}_{j,m}$,

(v) $[\hat{x}^{+}_{i,0}, \hat{x}^{-}_{j,0}] = \dfrac{\delta_{ij}}{q_i - q_i^{-1}}(\hat{k}_i - \hat{k}_i^{-1})$,

(vi) $[\hat{x}^{+}_{i,-1}, \hat{x}^{-}_{i,1}] = \dfrac{\hat{C}^{-1}\hat{k}_i - \hat{C}\hat{k}_i^{-1}}{q_i - q_i^{-1}}$,

(vii) $[\hat{x}^{+}_{i,0}, \hat{x}^{-}_{j,1}] = [\hat{x}^{+}_{i,-1}, \hat{x}^{-}_{j,0}] = 0$ *whenever* $i \neq j$,

(viii) $[\hat{x}^+_{i,0}, \hat{x}^+_{i,-1}]_{q_i^2} = [\hat{x}^-_{i,1}, \hat{x}^-_{i,0}]_{q_i^{-2}} = 0,$

(ix) $[\hat{x}^+_{1,0}, \hat{x}^+_{2,-1}]_{q_1^{a_{12}}} + [\hat{x}^+_{2,0}, \hat{x}^+_{1,-1}]_{q_1^{a_{12}}} = 0,$

(x) $[\hat{x}^-_{1,1}, \hat{x}^-_{2,0}]_{q_1^{-a_{12}}} + [\hat{x}^-_{2,1}, \hat{x}^-_{1,0}]_{q_1^{-a_{12}}} = 0,$

(xi) $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i y_i^s y_j y_i^{1-a_{ij}-s} = 0$ *whenever* $i \neq j$,

for $(y_i, y_j) = (\hat{x}^\pm_{i,0}, \hat{x}^\pm_{j,0}), (\hat{x}^\pm_{i,\mp1}, \hat{x}^\pm_{j,0}), (\hat{x}^\pm_{i,0}, \hat{x}^\pm_{j,\mp1})$. *Then there is an algebra homomorphism* $U'_q(X^{(1)}) \to \mathcal{A}_X$ *mapping*

$$C \mapsto \hat{C}, \qquad x^\pm_{i,0} \mapsto \hat{x}^\pm_{i,0}, \qquad x^\pm_{i,\mp1} \mapsto \hat{x}^\pm_{i,\mp1}, \qquad k_i \mapsto \hat{k}_i,$$

for $i = 1, 2$.

Proof. Similar to Jing's isomorphism between the two realizations of the quantum affine algebra as outlined in Section 2.4.1, we introduce elements

$$\begin{aligned}
\hat{k}_0 &= \hat{C}\hat{k}_1^{-a_1}\hat{k}_2^{-a_2}, \\
\hat{x}^+_{0,0} &= \left[\hat{x}^-_{i_{h-1},0}, \ldots, \hat{x}^-_{i_2,0}, \hat{x}^-_{i_1,1}\right]_{q^{\epsilon_1}\ldots q^{\epsilon_{h-2}}} \hat{C}\hat{k}_1^{-a_1}\hat{k}_2^{-a_2}, \\
\hat{x}^-_{0,0} &= a(-q)^{-\epsilon}\hat{C}^{-1}\hat{k}_1^{a_1}\hat{k}_2^{a_2}\left[\hat{x}^+_{i_{h-1},0}, \ldots, \hat{x}^+_{i_2,0}, \hat{x}^+_{i_1,-1}\right]_{q^{\epsilon_1}\ldots q^{\epsilon_{h-2}}},
\end{aligned}$$

in $\mathcal{A}_X$ where $\underline{i} = (i_1, \ldots, i_{h-1})$ and $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_{h-2})$ satisfy (2.4.3). By relations (ix)-(x) this is independent of the choice of sequences. Since the Drinfeld-Jimbo presentation of $U'_q(X^{(1)})$ has finitely many relations, it is a finite check to prove that $x^\pm_i \mapsto \hat{x}^\pm_{i,0}$ and $t_i^{\pm1} \mapsto \hat{k}_i^{\pm1}$ for $i = 0, 1, 2$ defines an algebra homomorphism $\xi : U'_q(X^{(1)}) \to \mathcal{A}_X$. Note that it is immediate that ξ preserves all relations with non-zero indices. Then to complete our proof we must express C, $x^\pm_{1,\mp1}$ and $x^\pm_{2,\mp1}$ in terms of the Drinfeld-Jimbo generators of $U'_q(X^{(1)})$ and verify that the images under ξ are equal to $\hat{C}$, $\hat{x}^\pm_{1,\mp1}$ and $\hat{x}^\pm_{2,\mp1}$ respectively (it is trivial that $C = t_0 t_1^{a_1} t_2^{a_2}$ maps to $\hat{C} = \hat{k}_0\hat{k}_1^{a_1}\hat{k}_2^{a_2}$). See Appendix B.1 for more details. □

Using this lemma we are able to prove the following simplified presentation for all quantum affinizations where the underlying Dynkin diagram has at most triple arrows. For notational convenience, we assume that the coweight lattice $P^\vee$ is spanned by the fundamental coweights. However, this result can be extended to include scaling elements simply by adjoining the corresponding $q^{\pm h}$ generators and imposing any relations in Definition 2.4.1 which involve them.

Proposition 3.1.8. *Let* $\mathfrak{s}$ *be a symmetrizable Kac-Moody algebra with generalised Cartan matrix* $(a_{ij})_{i,j\in I}$ *and suppose that* $a_{ij}a_{ji} \leq 3$ *for all distinct* $i, j \in I$*. Then the quantum affinization* $\widehat{U_q(\mathfrak{s})}$ *has a finite presentation with generators* $\{x^{\pm}_{i,0}, x^{\pm}_{i,\mp1}, k_i^{\pm1}, C^{\pm1} \mid i \in I\}$ *and relations*

(i) $C^{\pm1}$ *central,*

(ii) $C^{\pm1}C^{\mp1} = k_i^{\pm1}k_i^{\mp1} = 1,$

(iii) $[k_i, k_j] = 0,$

(iv) $k_i x^{\pm}_{j,m} k_i^{-1} = q_i^{\pm a_{ij}} x^{\pm}_{j,m},$

(v) $[x^{+}_{i,0}, x^{-}_{j,0}] = \dfrac{\delta_{ij}}{q_i - q_i^{-1}}(k_i - k_i^{-1}),$

(vi) $[x^{+}_{i,-1}, x^{-}_{j,1}] = \delta_{ij}\dfrac{C^{-1}k_i - Ck_i^{-1}}{q_i - q_i^{-1}},$

(vii) $[x^{+}_{i,0}, x^{-}_{j,1}] = [x^{+}_{i,-1}, x^{-}_{j,0}] = 0$ *whenever* $i \neq j$*,*

(viii) $[x^{+}_{i,0}, x^{+}_{i,-1}]_{q_i^{2}} = [x^{-}_{i,1}, x^{-}_{i,0}]_{q_i^{-2}} = 0,$

(ix) $[x^{+}_{i,0}, x^{+}_{j,-1}]_{q_i^{a_{ij}}} + [x^{+}_{j,0}, x^{+}_{i,-1}]_{q_i^{a_{ij}}} = 0$ *whenever* $a_{ij} < 0$*,*

(x) $[x^{-}_{i,1}, x^{-}_{j,0}]_{q_i^{-a_{ij}}} + [x^{-}_{j,1}, x^{-}_{i,0}]_{q_i^{-a_{ij}}} = 0$ *whenever* $a_{ij} < 0$*,*

(xi) $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i y_i^s y_j y_i^{1-a_{ij}-s} = 0$ *whenever* $i \neq j$*,*

for $(y_i, y_j) = (x^{\pm}_{i,0}, x^{\pm}_{j,0}), (x^{\pm}_{i,\mp1}, x^{\pm}_{j,0}), (x^{\pm}_{i,0}, x^{\pm}_{j,\mp1})$*.*

Proof. Let $\mathcal{H}$ be the algebra generated by $\{\hat{x}^{\pm}_{i,0}, \hat{x}^{\pm}_{i,\mp1}, \hat{k}_i^{\pm1}, \hat{C}^{\pm1} \mid i \in I\}$, subject to relations (i)-(xi) above with hats over each generator. It is clear that

$$\hat{C} \mapsto C, \qquad \hat{x}^{\pm}_{i,0} \mapsto x^{\pm}_{i,0}, \qquad \hat{x}^{\pm}_{i,\mp1} \mapsto x^{\pm}_{i,\mp1}, \qquad \hat{k}_i \mapsto k_i,$$

defines a homomorphism $f : \mathcal{H} \to U_q(\mathfrak{g}_{\mathrm{tor}})$. Let us build its inverse out of morphisms $\mathcal{U}_{ij} \to \mathcal{H}$. By Lemma 3.1.7, we have for each pair of adjacent nodes $i, j \in I$ an algebra homomorphism $p_{ij} : \mathcal{U}_{ij} \to \mathcal{H}$ given by

$$C \mapsto \hat{C}, \qquad x^{\pm}_{\ell,0} \mapsto \hat{x}^{\pm}_{\ell,0}, \qquad x^{\pm}_{\ell,\mp1} \mapsto \hat{x}^{\pm}_{\ell,\mp1}, \qquad k_\ell \mapsto \hat{k}_\ell,$$

for $\ell = i, j$. Then $p_i = p_{ij}|_{\mathcal{U}_i} = p_{ji}|_{\mathcal{U}_i}$ is well-defined and independent of j, since our proof of Proposition 3.0.3 shows that $\mathcal{U}_i$ is generated by $x^{\pm}_{i,0}$, $x^{\pm}_{i,\mp 1}$, $k_i^{\pm 1}$ and $C^{\pm 1}$. Furthermore, it is immediate from the presentation of $\mathcal{H}$ that $[\mathrm{im}(p_i), \mathrm{im}(p_j)] = 0$ whenever $a_{ij} = 0$. So we see that

$$C \mapsto \hat{C}, \qquad x^{\pm}_{i,0} \mapsto p_i(x^{\pm}_{i,0}), \qquad x^{\pm}_{i,\mp 1} \mapsto p_i(x^{\pm}_{i,\mp 1}), \qquad k_i \mapsto p_i(k_i),$$

for all $i \in I$ defines an algebra homomorphism $g : U_q(\mathfrak{g}_{\mathrm{tor}}) \to \mathcal{H}$, since all relations of $U_q(\mathfrak{g}_{\mathrm{tor}})$ are contained in some $\mathcal{U}_{ij}$. We have $f \circ g = \mathrm{id}$ by checking on Drinfeld-Jimbo generators of each $\mathcal{U}_{ij}$ with $a_{ij} < 0$, and $g \circ f = \mathrm{id}$ by checking on generators of $\mathcal{H}$. $\square$

Remark 3.1.9.

1. This result gives a finite Drinfeld new style presentation for the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$ in all untwisted and twisted types except $A_1^{(1)}$ and $A_2^{(2)}$, as well as for all untwisted quantum affine algebras $U_q(\hat{\mathfrak{g}})$.
2. The relations in Proposition 3.1.8 are a subset of those in the original definition for $\widehat{U_q(\mathfrak{s})}$ which only involve the generators $x^{\pm}_{i,0}$, $x^{\pm}_{i,\mp 1}$, $k_i^{\pm 1}$, $C^{\pm 1}$ for each $i \in I$. In particular, we do not see 'shadows' of other relations appearing in our simplified presentation.

Recall that in Section 2.4.1 we obtained formulae for $\mathbf{T}_i(x^{\pm}_{j,m})$ when $i, j \in I_0$ and $m = 0, \mp 1$. Thanks to Proposition 3.1.8, these are enough to define the remaining automorphisms $\mathcal{T}_i$ that are required for our action of $\ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$.

Proposition 3.1.10. *For each $i \in I$ there exists an automorphism $\mathcal{T}_i$ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ such that*

- $\mathcal{T}_i h = h\mathbf{T}_i$ *for all* $i \in I$,
- $\mathcal{T}_i v = v\mathbf{T}_i$ *for all* $i \in I_0$,
- $\mathcal{T}_i^{-1} = \eta\mathcal{T}_i\eta$ *for all* $i \in I$.

Proof. For each $i \in I$, we define the morphism $\mathcal{T}_i : U_q(\mathfrak{g}_{\text{tor}}) \to U_q(\mathfrak{g}_{\text{tor}})$ by

$$\begin{aligned}
&\mathcal{T}_i(C) = C, \qquad \mathcal{T}_i(k_j) = k_j k_i^{-a_{ij}},\\
&\mathcal{T}_i(x_{i,0}^+) = -x_{i,0}^- k_i, \qquad \mathcal{T}_i(x_{i,0}^-) = -k_i^{-1} x_{i,0}^+,\\
&\mathcal{T}_i(x_{i,-1}^+) = \sum_{s=0}^{2} (-1)^s q_i^s (x_{i,0}^-)^{(s)} x_{i,-1}^+ (x_{i,0}^-)^{(2-s)} k_i,\\
&\mathcal{T}_i(x_{i,1}^-) = k_i^{-1} \sum_{s=0}^{2} (-1)^s q_i^{-s} (x_{i,0}^+)^{(2-s)} x_{i,1}^- (x_{i,0}^+)^{(s)},\\
&\mathcal{T}_i(x_{j,m}^+) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} (x_{i,0}^+)^{(-a_{ij}-s)} x_{j,m}^+ (x_{i,0}^+)^{(s)} \text{ if } i \neq j,\\
&\mathcal{T}_i(x_{j,m}^-) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{s} (x_{i,0}^-)^{(s)} x_{j,m}^- (x_{i,0}^-)^{(-a_{ij}-s)} \text{ if } i \neq j.
\end{aligned}$$

To verify that $\mathcal{T}_i$ is a well-defined homomorphism, we need to show that it preserves every relation in our simplified presentation of $U_q(\mathfrak{g}_{\text{tor}})$. For each $j, \ell \in I$ consider the relations lying inside $\mathcal{U}_{j\ell}$. These are also relations of the subalgebra $\mathcal{U}_{ij\ell}$, to which we may restrict since from the formulae above it is preserved by $\mathcal{T}_i$. Note that $\mathcal{U}_{ij\ell}$ is isomorphic to the quantum affinization of the quantum group associated to the full Dynkin subdiagram $D_{ij\ell}$ on the nodes i, j, ℓ.

If $D_{ij\ell}$ is a subdiagram of some finite Dynkin diagram, then $\mathcal{U}_{ij\ell}$ is isomorphic to a direct product of quantum affine algebras. Furthermore, $\mathcal{T}_i|_{\mathcal{U}_{ij\ell}}$ acts by $\mathbf{T}_i$ on the factor containing $\mathcal{U}_i$ and by the identity on all other factors. Since this is an automorphism of $\mathcal{U}_{ij\ell}$, we see that $\mathcal{T}_i$ preserves all relations lying inside $\mathcal{U}_{j\ell}$.

Otherwise, $D_{ij\ell}$ is the Dynkin diagram of type $A_2^{(1)}$, $C_2^{(1)}$, $G_2^{(1)}$, $A_4^{(2)}$, $D_3^{(2)}$ or $D_4^{(3)}$ and $\mathcal{U}_{ij\ell}$ is isomorphic to the corresponding quantum toroidal algebra. Since by definition $\mathcal{T}_i$ restricts to $\mathbf{T}_i$ on the horizontal subalgebra $(\mathcal{U}_{ij\ell})_h$, it preserves all relations lying inside it. Any other relation can then be obtained from one of these by applying $\mathcal{X}_j$, $\mathcal{X}_\ell$ or $\mathcal{X}_j\mathcal{X}_\ell$, which commute with $\mathcal{T}_i$ since $i \neq j, \ell$.

Hence we may conclude that $\mathcal{T}_i$ respects all relations in our simplified presentation, and is thus an algebra homomorphism. The first two bullet points in the statement of the proposition are then immediate from the formulae for $\mathbf{T}_i$ on the Drinfeld-Jimbo and Drinfeld new generators of $U_q(\hat{\mathfrak{g}})$ in Section 2.4.1.

To show that $\mathcal{T}_i$ is an automorphism with inverse $\eta\mathcal{T}_i\eta$, it suffices to check this on the invariant subspace $\mathcal{U}_{ij}$ for each $j \neq i$. If $a_{ij} < 0$ then $\mathcal{U}_{ij}$ is isomorphic to $U_q(A_2^{(1)})$, $U_q(C_2^{(1)})$ or $U_q(G_2^{(1)})$, and $\mathcal{T}_i$ and η restrict to $\mathbf{T}_i$ and η'. If $a_{ij} = 0$ then $\mathcal{U}_{ij} \cong \mathcal{U}_i \times \mathcal{U}_j$ and $\mathcal{T}_i$ and η restrict to $\mathbf{T}_i \times \mathrm{id}$ and $\eta' \times \eta'$. In either case, since $(\mathbf{T}_i)^{-1} = \eta'\mathbf{T}_i\eta'$ our proof is complete. $\square$

We now have all of the automorphisms required to define our braid group action on $U_q(\mathfrak{g}_{\mathrm{tor}})$. However, in type $A_{2n}^{(1)}$ we are forced to consider a slightly modified version of $\ddot{\mathcal{B}}$. First, we must have that $\pi_1 \in \Omega$ has order $4n+2$ rather than $2n+1$. This is because, as discussed in Section 2.1, there is no length function on the affine Dynkin diagram and so

$$\mathcal{S}_{\pi_1}^{2n+1}(x_{i,m}^{\pm}) = (-1)^m x_{i,m}^{\pm}, \quad \mathcal{S}_{\pi_1}^{2n+1}(k_i) = k_i, \quad \mathcal{S}_{\pi_1}^{2n+1}(h_{i,r}) = (-1)^r h_{i,r}, \quad \mathcal{S}_{\pi_1}^{2n+1}(C) = C,$$

has order two. Let ζ_i be the automorphism mapping each $x_{i,m}^{\pm} \mapsto -x_{i,m}^{\pm}$ and fixing the other generators. Then we have

$$\mathcal{S}_{\pi_1}\zeta_i\mathcal{S}_{\pi_1}^{-1} = \zeta_{\pi_1(i)}, \quad \mathcal{S}_{\pi_1}\mathcal{X}_{2n}\mathcal{S}_{\pi_1}^{-1} = \zeta_0\mathcal{X}_0, \quad \mathcal{T}_0^{-1}\mathcal{X}_0\mathcal{T}_0^{-1} = \zeta_0\mathcal{X}_{2n}\mathcal{X}_0^{-1}\mathcal{X}_1,$$

and we adjust the relations of $\ddot{\mathcal{B}}$ accordingly. The involutions $\mathfrak{t}$, γ_v and γ_h extend naturally to our modified braid group, and our results are not otherwise impacted. The proof of the next theorem is virtually the same as for the other cases (we shall not include the minor differences) and there is only a slight change in Lemma 3.2.8.

Theorem 3.1.11. *The extended double affine braid group* $\ddot{\mathcal{B}}$ *acts on the quantum toroidal algebra* $U_q(\mathfrak{g}_{\mathrm{tor}})$ *via* $T_i \mapsto \mathcal{T}_i$ *for all* $i \in I$, $X_{\omega_i^\vee} \mapsto \mathcal{Z}_{\omega_i^\vee} := \mathcal{X}_i\mathcal{X}_0^{-a_i}$ *for all* $i \in I_0$*, and* $\pi \mapsto \mathcal{S}_\pi$ *for all* $\pi \in \Omega$.

Proof. The relations between $\mathcal{T}_i$ and $\mathcal{Z}_\beta$ follow from the Coxeter relations between $\mathcal{T}_i$ and $\mathcal{X}_j$, namely that $\mathcal{T}_i\mathcal{X}_j = \mathcal{X}_j\mathcal{T}_i$ whenever $i \neq j$ and $\mathcal{T}_i^{-1}\mathcal{X}_i\mathcal{T}_i^{-1} = \mathcal{X}_i \prod_{j\in I} \mathcal{X}_j^{-a_{ij}}$. Commutativity of $\mathcal{T}_i$ and $\mathcal{X}_j$ for $i \neq j$ is clear from the definitions, while the other relation is checked on each $\mathcal{U}_\ell$ by restricting to $\mathcal{U}_{i\ell}$ and applying Theorem 2.4.6, since $\mathcal{U}_{i\ell}$ is a product of quantum affine algebras.

To verify the braid relation between $\mathcal{T}_i$ and $\mathcal{T}_j$ on elements of $\mathcal{U}_\ell$, we restrict to the invariant subalgebra $\mathcal{U}_{ij\ell}$. Similarly to our proof of Proposition 3.1.10, if $D_{ij\ell}$ is a subdiagram of a finite Dynkin diagram then $\mathcal{U}_{ij\ell}$ is a product of quantum affine algebras and we can use the braid relation between $\mathbf{T}_i$ and $\mathbf{T}_j$ from Theorem 2.4.6. Otherwise, $\mathcal{U}_{ij\ell}$ is isomorphic to the quantum toroidal algebra of type $A_2^{(1)}$, $C_2^{(1)}$, $G_2^{(1)}$, $A_4^{(2)}$, $D_3^{(2)}$ or $D_4^{(3)}$. We already have the braid relation on $x^{\pm}_{\ell,0}$ and $k_\ell^{\pm 1}$ using $\mathcal{T}_i h = h\mathbf{T}_i$ and $\mathcal{T}_j h = h\mathbf{T}_j$. Applying $\mathcal{X}_\ell$, which commutes with $\mathcal{T}_i$ and $\mathcal{T}_j$ since $\ell \notin \{i, j\}$, we derive the braid relation on all of $\mathcal{U}_\ell$. The remaining relations follow from the definitions without much difficulty. $\square$

Remark 3.1.12.

1. Our extended double affine braid group action restricts to both an action of $\mathcal{B}_h$ on $\mathcal{U}_h$ and an action of $\mathcal{B}_v$ on $\mathcal{U}_v$, each of which coincides with Lusztig and Beck's action of the extended affine braid group on the quantum affine algebra from Theorems 2.3.22 and 2.4.6 respectively.
2. In their PhD thesis, motivated by trying to obtain a Damiani-Beck style isomorphism on the toroidal level, Mounzer [89] provides a *topological* braid group action on a certain completion of $U_q(\mathfrak{g}_{\mathrm{tor}})$ (verifying the quantum Serre relations is a work in progress in some cases). We note that this action does not restrict to the quantum toroidal algebra and is thus distinct from our results.

It is worth highlighting that these results extend naturally to all quantum affinizations $\widehat{U_q(\mathfrak{s})}$ considered in Proposition 3.1.8, namely those with $a_{ij}a_{ji} \leq 3$ for all distinct $i, j \in I$. In particular, for each $i \in I$ there exists an automorphism $\mathcal{T}_i$ of $\widehat{U_q(\mathfrak{s})}$ defined exactly as in Proposition 3.1.10, with inverse $\mathcal{T}_i^{-1} = \eta\mathcal{T}_i\eta$, which restricts to $\mathbf{T}_i$ on the horizontal copy of $U_q(\mathfrak{s})$. Furthermore, for some appropriate generalisation of $\ddot{\mathcal{B}}$ we obtain a braid group action as in Theorem 3.1.11. In each case, the proofs are the same as above.

Definition 3.1.13. *For any generalised Cartan matrix $(a_{ij})_{i,j\in I}$ we define the affinized braid group $\widehat{\mathcal{B}}$ to be the group generated by $\{T_i, X_i \mid i \in I\}$ and the automorphism group Ω of the associated Dynkin diagram, with relations*

- $T_iT_jT_i \ldots = T_jT_iT_j \ldots$ *whenever $a_{ij}a_{ji} \leq 3$, with $a_{ij}a_{ji} + 2$ factors on each side,*
- $X_iX_j = X_jX_i$*,*
- $T_iX_j = X_jT_i$ *whenever $i \neq j$,*

- $T_i^{-1} X_i T_i^{-1} = X_i \prod_{j \in I} X_j^{-a_{ij}}$,
- $\pi T_i \pi^{-1} = T_{\pi(i)}$,
- $\pi X_i \pi^{-1} = X_{\pi(i)}$,

for all $i, j \in I$ *and* $\pi \in \Omega$.

Note that the extended double affine braid group $\ddot{\mathcal{B}}$ embeds inside the corresponding $\widehat{\mathcal{B}}$ by sending $T_i \mapsto T_i$, $X_{\omega_i^\vee} \mapsto X_i X_0^{-a_i}$ and $\pi \mapsto \pi$ for each $i \in I$ and $\pi \in \Omega$.

When the underlying Dynkin diagram possesses a length function o, by defining $\mathcal{X}_i$ and $\mathcal{S}_\pi$ exactly as for $U_q(\mathfrak{g}_{\mathrm{tor}})$ we obtain the following 'braid group action' on $\widehat{U_q(\mathfrak{s})}$.

Theorem 3.1.14. *The group* $\widehat{\mathcal{B}}$ *acts on the quantum affinization* $\widehat{U_q(\mathfrak{s})}$ *via* $T_i \mapsto \mathcal{T}_i$ *and* $X_i \mapsto \mathcal{X}_i$ *for all* $i \in I$, *and* $\pi \mapsto \mathcal{S}_\pi$ *for all* $\pi \in \Omega$.

If no such o exists and the Dynkin diagram contains an odd length cycle, this should instead hold for a modified version of $\widehat{\mathcal{B}}$ as was the case in type $A_{2n}^{(1)}$.

3.2 Automorphisms and anti-automorphisms of quantum toroidal algebras

We now look to construct certain automorphisms and anti-involutions of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which exchange the horizontal and vertical subalgebras. In [76] we treat the simply laced case, in particular generalising the famous Miki automorphism of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ from [86]. Here we extend our results to all untwisted types except $A_1^{(1)}$ and $G_2^{(1)}$ by using a finer consideration of the extended double affine braid groups involving the Coxeter presentation from Theorem 3.1.4.

For notational simplicity, we will henceforth identify elements of $\ddot{\mathcal{B}}$ with the corresponding automorphisms of $U_q(\mathfrak{g}_{\mathrm{tor}})$ from Theorem 3.1.11. We shall also write X_i for $X_{\omega_i^\vee}$ and Y_i for $Y_{\omega_i^\vee}$ for each $i \in I_0$.

Our approach is roughly as follows. We can in some sense build $U_q(\mathfrak{g}_{\mathrm{tor}})$ out of the copy of the finite quantum group $U_q(\mathfrak{g})$ lying inside $\mathcal{U}_h \cap \mathcal{U}_v$ and the braid group action from Theorem 3.1.11. Twisting the action by certain automorphisms of $\ddot{\mathcal{B}}$ obtains different 'twisted' sets of generators for $U_q(\mathfrak{g}_{\mathrm{tor}})$. Then mapping the standard generators to their twisted counterparts gives our desired (anti-)automorphisms.

More specifically, each generator of our simplified presentation for $U_q(\mathfrak{g}_{\mathrm{tor}})$ from Corollary 3.2.7 (other than $C^{\pm 1}$) can easily be written as $b(z)$ for some $b \in \ddot{\mathcal{B}}$ and $z \in U_q(\mathfrak{g})$. For all $x_{i,0}^{\pm}$ and $k_i^{\pm 1}$ with $i \in I_0$ we may set $b = 1$, and of course $x_{i,\pm 1}^{\pm} = o(i) X_i^{-1}(x_{i,0}^{\pm})$ for each $i \in I_0$. For the other generators we have

- $x_{0,0}^{\pm} = T_\ell T_0(x_{\ell,0}^{\pm}) = T_\ell^{-1} T_0^{-1}(x_{\ell,0}^{\pm})$ for any $\ell \in \tilde{I}$,
- $x_{0,\pm 1}^{\pm} = o(0) T_\ell \Theta_{02}(x_{\ell,0}^{\pm}) = o(0) T_\ell^{-1} \Theta_{02}^{-1}(x_{\ell,0}^{\pm})$ for any $\ell \in \tilde{I}$,
- $k_0^{\pm 1} = T_\ell T_0(k_\ell^{\pm 1}) = T_\ell^{-1} T_0^{-1}(k_\ell^{\pm 1})$ for any $\ell \in \tilde{I}$,

where $\tilde{I}$ is the set of vertices adjacent to 0 in the affine Dynkin diagram, except in type $C_n^{(1)}$ where we instead have

- $x_{0,0}^{\pm} = \pi_n(x_{n,0}^{\pm})$,
- $x_{0,\pm 1}^{\pm} = o(0) \pi_n X_n^{-1}(x_{n,0}^{\pm})$,
- $k_0^{\pm 1} = \pi_n(k_n^{\pm 1})$.

Recall the involution $\mathfrak{t}$ of $\ddot{\mathcal{B}}$ from Section 3.1.1. For each $x_{i,m}^{\pm} = b(z)$ above define $\mathbf{x}_{i,m}^{\pm} = \mathfrak{t}(b)(z)$, and for each $k_i^{\pm 1} = b(z)$ let $\mathbf{k}_i^{\pm 1} = \mathfrak{t}(b)(z^{-1})$. In particular,

$$\mathbf{k}_i^{\pm 1} = k_i^{\mp 1}, \qquad \mathbf{x}_{i,0}^{\pm} = x_{i,0}^{\pm}, \qquad \mathbf{x}_{i,\pm 1}^{\pm} = o(i) Y_i^{-1}(x_{i,0}^{\pm}),$$

for all $i \in I_0$, and outside type $C_n^{(1)}$ we have

$$\begin{aligned}
\mathbf{k}_0^{\pm 1} &= T_\ell^{-1} (T_0^v)^{-1}(k_\ell^{\mp 1}) = T_\ell T_0^v(k_\ell^{\mp 1}), \\
\mathbf{x}_{0,0}^{\pm} &= T_\ell^{-1} (T_0^v)^{-1}(x_{\ell,0}^{\pm}) = T_\ell T_0^v(x_{\ell,0}^{\pm}), \\
\mathbf{x}_{0,\pm 1}^{\pm} &= o(0) T_\ell^{-1} \Theta_{02}^{-1}(x_{\ell,0}^{\pm}) = o(0) T_\ell \Theta_{02}(x_{\ell,0}^{\pm}),
\end{aligned}$$

for any $\ell \in \tilde{I}$, from which we see that $\mathbf{x}_{0,\pm 1}^{\pm} = x_{0,\pm 1}^{\pm}$. For $C_n^{(1)}$ these are replaced by $\mathbf{k}_0^{\pm 1} = \rho_n(k_n^{\mp 1})$, $\mathbf{x}_{0,0}^{\pm} = \rho_n(x_{n,0}^{\pm})$ and

$$\mathbf{x}_{0,\pm 1}^{\pm} = o(0) \rho_n Y_n^{-1}(x_{n,0}^{\pm}) = o(0) X_n T_{v_n}^{-1} T_{v_n^{-1}} \pi_n(x_{n,0}^{\pm}) = o(0) X_n(x_{0,0}^{\pm}) = x_{0,\pm 1}^{\pm},$$

where for the penultimate equality we use the identity $v_{i^*}^{-1} = v_i$ from Section 3.1.1.2. It is immediate that $\mathbf{k}_0^{\pm 1} = C^{\mp 1} k_\theta^{\pm 1}$ in all types.

If we furthermore define $\mathbf{C}^{\pm 1} = k_\delta^{\mp 1}$ then the following theorem shows that mapping generators to their bold counterparts extends to an anti-involution of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which exchanges $\mathcal{U}_h$ and $\mathcal{U}_v$ (up to a twist by σ).

Theorem 3.2.1. *There is an anti-involution ψ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ sending*

$$x_{i,m}^{\pm} \mapsto \mathbf{x}_{i,m}^{\pm}, \qquad k_i^{\pm 1} \mapsto \mathbf{k}_i^{\pm 1}, \qquad C^{\pm 1} \mapsto \mathbf{C}^{\pm 1},$$

for all $i \in I$ and $m = 0, \pm 1$, determined by the conditions $\psi v = h\sigma$ and $\psi h = v\sigma$.

We shall leave the proof to Section 3.2.1, and instead now focus on some immediate consequences of this result.

Figure 3.4 provides simple illustrations of the quantum toroidal algebra containing the two finite generating sets $\{x_{i,0}^{\pm}, x_{i,\pm 1}^{\pm}, k_i^{\pm 1}, C^{\pm 1} \mid i \in I\}$ and $\{\mathbf{x}_{i,0}^{\pm}, \mathbf{x}_{i,\pm 1}^{\pm}, \mathbf{k}_i^{\pm 1}, \mathbf{C}^{\pm 1} \mid i \in I\}$. In particular, in each case they highlight where the generators lie inside $U_q(\mathfrak{g}_{\mathrm{tor}})$ with respect to the horizontal and vertical subalgebras, as well as the $\mathbb{Z}$-grading (except for $C^{\pm 1}$ and $\mathbf{k}_0^{\pm 1}$).

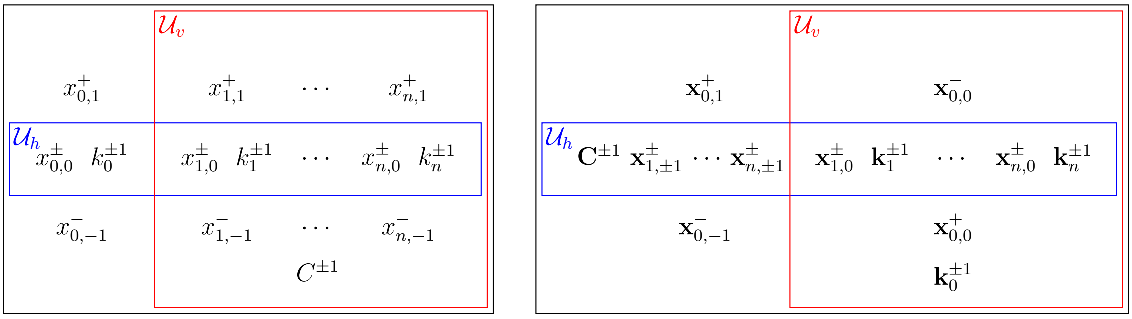


Fig. 3.4 Illustrations of $U_q(\mathfrak{g}_{\mathrm{tor}})$ displaying the two generating sets

We moreover notice that the bold generators in some sense give $U_q(\mathfrak{g}_{\mathrm{tor}})$ as a quantum affinization of its vertical rather than horizontal subalgebra, with $\mathcal{U}_v$ in a Drinfeld-Jimbo presentation and $\mathcal{U}_h$ in a Drinfeld new presentation (although the multiplication is of course reversed).

Our anti-involution ψ indicates the importance of a third quantum affine subalgebra $\mathcal{U}_d$ which we shall call the *diagonal subalgebra*. This is defined as the image of the homomorphism $U'_q(X_n^{(1)}) \to U_q(\mathfrak{g}_{\mathrm{tor}})$ sending

$$x_i^{\pm} \mapsto x_{i,0}^{\pm}, \qquad t_i^{\pm 1} \mapsto k_i^{\pm 1}, \qquad x_0^{\pm} \mapsto x_{0,\pm 1}^{\pm}, \qquad t_0^{\pm 1} \mapsto (Ck_0)^{\pm 1},$$

for each $i \in I_0$, with Ck_δ as its canonical central element. We immediately see that ψ restricts to the anti-involution σ on $\mathcal{U}_d = \mathcal{X}_0^{-1}(\mathcal{U}_h)$, which therefore also equals $\psi\mathcal{X}_0^{-1}\psi(\mathcal{U}_v)$. The diagonal subalgebra corresponds on the braid group side to the diagonal subgroup $\mathcal{B}_d = \mathfrak{a}(\mathcal{B}_h) = \mathfrak{b}^{-1}(\mathcal{B}_v)$ of $\ddot{\mathcal{B}}$. Indeed, $\mathcal{B}_d$ preserves $\mathcal{U}_d$ under our braid group action from Proposition 3.1.11, in particular acting via Lusztig and Beck's affine action.

While it is clear that v is an embedding and hence $\mathcal{U}_v$ is a copy of the quantum affine algebra of type $Z_n^{(1)}$, the analogous horizontal statement is non-obvious. Namely, it could be the case that relations of $U_q(\mathfrak{g}_{\mathrm{tor}})$ involving elements not contained in $\mathcal{U}_h$ might have 'shadows' inside the horizontal subalgebra. However, from Theorem 3.2.1 we can in fact deduce the injectivity of h from that of v. Furthermore, a corresponding diagonal result then follows immediately by composing with $\mathcal{X}_0^{-1}$.

Corollary 3.2.2. *The homomorphism $h : U'_q(X_n^{(1)}) \to U_q(\mathfrak{g}_{\mathrm{tor}})$ is an embedding, and hence $\mathcal{U}_h$ is isomorphic to the quantum affine algebra of type $X_n^{(1)}$.*

By composing ψ with the standard anti-involution η we obtain an automorphism of $U_q(\mathfrak{g}_{\mathrm{tor}})$ which in type $A_n^{(1)}$ is precisely the Miki automorphism [86] (with the extra deformation parameter κ set to 1).

Corollary 3.2.3. *There is an automorphism $\Phi := \eta\psi$ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ with inverse $\Phi^{-1} = \eta\Phi\eta = \psi\eta$, determined by the conditions $\Phi v = h$ and $\Phi h = v\eta'\sigma$.*

In terms of central elements, ψ exchanges C and $(k_0^{a_0} \dots k_n^{a_n})^{-1}$ while Φ maps $C \mapsto k_0^{a_0} \dots k_n^{a_n}$ and $k_0^{a_0} \dots k_n^{a_n} \mapsto C^{-1}$. Twisting level (a, b) representations of $U_q(\mathfrak{g}_{\mathrm{tor}})$ by Φ therefore produces level $(b, -a)$ representations, and in this way we can obtain many new modules for quantum toroidal algebras.

Example 3.2.4.

1. In simply laced types this should relate certain ℓ-highest weight and (future) Fock space representations to vertex representations, since level $(0, 1)$

modules become level $(1, 0)$.

2. To the author's knowledge, outside the simply laced case there do not yet exist representations of $U_q(\mathfrak{g}_{\text{tor}})$ with level $(a, 0)$ for $a \neq 0$, such as vertex representations. The first examples then come from twisting modules with ℓ-highest weight (λ, Ψ) and thus level $(0, \langle \lambda, c \rangle)$.

Remark 3.2.5. 1. Since ψ fixes $x^{\pm}_{0,\pm 1}$ by construction, it follows that $\Phi(x^{\pm}_{0,\pm 1}) = x^{\pm}_{0,\mp 1}$. This was originally shown for $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ in [103, Proposition 2.6(d)] using a type A specific argument.

2. Nevertheless, computing the images under ψ or Φ of arbitrary elements of $U_q(\mathfrak{g}_{\text{tor}})$ is a difficult problem in general. A useful tool in type A has been the situation of $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ within the framework of combinatorially defined *shuffle algebras* through works of Neguţ [96, 97] and Tsymbaliuk [103, 104]. We expect these directions to extend to all simply laced cases and perhaps even beyond, providing new methods for approaching quantum toroidal algebras.

Remark 3.2.6. 1. Type $A^{(1)}_1$ is not included in this section since we do not have the braid group action from Theorem 2.3.22 at this stage.

2. In type $G^{(1)}_2$ our methods successfully verify the vast majority of relations for ψ. However a small number – highlighted throughout our proofs – cannot be accessed. This stems from $\mathring{P}^\vee$ being 'too small' within $P^\vee$ due to the particular a_i labels, and so $\ddot{\mathcal{B}}$ does not reach every relation of $U_q(\mathfrak{g}_{\text{tor}})$ from those lying inside $\mathcal{U}_h$, $\mathcal{U}_v$ or $\mathcal{U}_d$.

Hence we shall not consider these types for the remainder this chapter, and instead leave them for future work.

3.2.1 Proof of Theorem 3.2.1

While our proof uses a finite presentation for $U_q(\mathfrak{g}_{\text{tor}})$ similar to that of Proposition 3.1.8, it also requires the key observation that ψ fixes $x^{\pm}_{0,\pm 1}$. We therefore seek a finite presentation involving these generators, which is obtained by applying the anti-involution η.

Corollary 3.2.7. *The quantum toroidal algebra* $U_q(\mathfrak{g}_{\mathrm{tor}})$ *has a finite presentation with generators* $\{x_{i,0}^{\pm},\ x_{i,\pm1}^{\pm},\ k_i^{\pm1},\ C^{\pm1} \mid i \in I\}$ *and relations*

(i) $C^{\pm1}$ *central,*

(ii) $C^{\pm1}C^{\mp1} = k_i^{\pm1}k_i^{\mp1} = 1,$

(iii) $[k_i, k_j] = 0,$

(iv) $k_i x_{j,m}^{\pm} k_i^{-1} = q_i^{\pm a_{ij}} x_{j,m}^{\pm},$

(v) $[x_{i,0}^{+}, x_{j,0}^{-}] = \dfrac{\delta_{ij}}{q_i - q_i^{-1}}(k_i - k_i^{-1}),$

(vi) $[x_{i,1}^{+}, x_{j,-1}^{-}] = \delta_{ij}\dfrac{Ck_i - C^{-1}k_i^{-1}}{q_i - q_i^{-1}},$

(vii) $[x_{i,0}^{+}, x_{j,-1}^{-}] = [x_{i,1}^{+}, x_{j,0}^{-}] = 0$ *whenever* $i \neq j$*,*

(viii) $[x_{i,1}^{+}, x_{i,0}^{+}]_{q_i^2} = [x_{i,0}^{-}, x_{i,-1}^{-}]_{q_i^{-2}} = 0,$

(ix) $[x_{i,1}^{+}, x_{j,0}^{+}]_{q_i^{a_{ij}}} + [x_{j,1}^{+}, x_{i,0}^{+}]_{q_i^{a_{ij}}} = 0$ *whenever* $a_{ij} < 0$*,*

(x) $[x_{i,0}^{-}, x_{j,-1}^{-}]_{q_i^{-a_{ij}}} + [x_{j,0}^{-}, x_{i,-1}^{-}]_{q_i^{-a_{ij}}} = 0$ *whenever* $a_{ij} < 0$*,*

(xi) $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i y_i^s y_j y_i^{1-a_{ij}-s} = 0$ *whenever* $i \neq j$*,*

for $(y_i, y_j) = (x_{i,0}^{\pm}, x_{j,0}^{\pm}), (x_{i,\pm1}^{\pm}, x_{j,0}^{\pm}), (x_{i,0}^{\pm}, x_{j,\pm1}^{\pm}).$

Next we must verify the expressions $x_{0,\pm1}^{\pm} = b(z)$ outside type $C_n^{(1)}$ stated previously, which imply that $\mathbf{x}_{0,\pm1}^{\pm} = x_{0,\pm1}^{\pm}$ since $\mathfrak{t}$ inverts both T_ℓ and Θ_{02}.

$$\begin{aligned}
T_\ell\Theta_{02}(x_{\ell,0}^{\pm}) &= T_\ell T_0^{-1}\textstyle\prod_{i\in\tilde{I}} X_i^{-1}(x_{\ell,0}^{\pm}) = T_\ell T_0^{-1}\big(\prod_{i\in\tilde{I}} \mathcal{X}_i^{-1}\big)\mathcal{X}_0^{2}(x_{\ell,0}^{\pm}) = T_\ell \mathcal{X}_\ell^{-1} T_0^{-1}(x_{\ell,0}^{\pm}) \\
&= \mathcal{X}_\ell^{-1}\big(\textstyle\prod_{i\in I} \mathcal{X}_i^{a_{\ell i}}\big)T_\ell^{-1}T_0^{-1}(x_{\ell,0}^{\pm}) = \mathcal{X}_\ell^{-1}\big(\prod_{i\in I} \mathcal{X}_i^{a_{\ell i}}\big)(x_{0,0}^{\pm}) \\
&= o(0)x_{0,\pm1}^{\pm} \\
T_\ell^{-1}\Theta_{02}^{-1}(x_{\ell,0}^{\pm}) &= T_\ell^{-1}\big(\textstyle\prod_{i\in\tilde{I}} X_i\big)T_0(x_{\ell,0}^{\pm}) = T_\ell^{-1}\big(\prod_{i\in\tilde{I}} \mathcal{X}_i\big)\mathcal{X}_0^{-2}T_0(x_{\ell,0}^{\pm}) \\
&= \big(\textstyle\prod_{i\in\tilde{I}} \mathcal{X}_i\big)\Big(\prod_{j\in I} \mathcal{X}_j^{-a_{\ell j}}\Big)\mathcal{X}_0^{-2}T_\ell T_0(x_{\ell,0}^{\pm}) \\
&= \big(\textstyle\prod_{i\in\tilde{I}} \mathcal{X}_i\big)\Big(\prod_{j\in I} \mathcal{X}_j^{-a_{\ell j}}\Big)\mathcal{X}_0^{-2}(x_{0,0}^{\pm}) \\
&= o(0)x_{0,\pm1}^{\pm}
\end{aligned}$$

The following alternative expressions for $\mathbf{x}_{0,\pm1}^{\pm}$ shall be useful in calculations.

$$\begin{aligned}
\mathbf{x}_{0,\pm1}^{\pm} &= o(0)\mathfrak{t}(T_\ell\Theta_{02})(x_{\ell,0}^{\pm}) = o(0)\mathfrak{t}(T_\ell T_0^{-1}X_\ell^{-1})(x_{\ell,0}^{\pm}) = o(0)T_\ell^{-1}T_0^{v}Y_\ell^{-1}(x_{\ell,0}^{\pm}) \\
\mathbf{x}_{0,\pm1}^{\pm} &= o(0)\mathfrak{t}(T_\ell^{-1}\Theta_{02}^{-1})(x_{\ell,0}^{\pm}) = o(0)T_\ell Y_\ell(T_0^{v})^{-1}(x_{\ell,0}^{\pm}) \\
&= o(0)Y_{s_\ell(\varpi_\ell^\vee)}T_\ell^{-1}(T_0^{v})^{-1}(x_{\ell,0}^{\pm}) = o(0)Y_{s_\ell(\varpi_\ell^\vee)}T_\ell T_0^{v}(x_{\ell,0}^{\pm})
\end{aligned}$$

A brief technical lemma provides an assortment of identities required for the proof of Theorem 3.2.1. Note that in type $A_{2n}^{(1)}$ we restrict to $\rho = \rho_1$ for (3.2.4).

Lemma 3.2.8. · *$Y_i(\mathbf{x}_{j,0}^{\pm}) = \mathbf{x}_{j,0}^{\pm}$ and $Y_i(\mathbf{k}_j^{\pm1}) = \mathbf{k}_j^{\pm1}$ for all distinct $i, j \in I_0$,* (3.2.1)

- · *$\mathbf{x}_{i,m}^{\pm} = h\sigma(x_{i,m}^{\pm})$, $\mathbf{k}_i^{\pm1} = h\sigma(k_i^{\pm1})$ and $\mathbf{C}^{\pm1} = h\sigma(C^{\pm1})$ for all $i \in I_0$ and $m = 0, \pm1$,* (3.2.2)

- · *$\mathbf{x}_{i,0}^{\pm} = v\sigma(x_i^{\pm})$ and $\mathbf{k}_i^{\pm1} = v\sigma(t_i^{\pm1})$ for all $i \in I$,* (3.2.3)

- · *$\rho(\mathbf{x}_{i,m}^{\pm}) = o_{i,\rho(i)}^{m}\mathbf{x}_{\rho(i),m}^{\pm}$ and $\rho(\mathbf{k}_i^{\pm1}) = \mathbf{k}_{\rho(i)}^{\pm1}$ for all $i \in I$, $m = 0, \pm1$ and $\rho \in \Omega^{v}$.*(3.2.4)

Proof. We know from Proposition 3.1.10 that $T_i h = h\mathbf{T}_i = h\sigma\mathbf{T}_i^{-1}\sigma$ for all $i \in I$, and it is immediate from the definitions that $\pi h = hS_\pi = h\sigma S_\pi\sigma$ for each $\pi \in \Omega$. Each Y_β can be written as $\pi T_{i_1}^{\pm1} \dots T_{i_s}^{\pm1}$ and so as σ^2 is the identity,

$$Y_\beta h = h\sigma S_\pi \mathbf{T}_{i_1}^{\mp1} \dots \mathbf{T}_{i_s}^{\mp1}\sigma = h\sigma\mathbf{X}_\beta\sigma. \tag{3.2.5}$$

Note that (3.2.2) is trivial for $\mathbf{x}_{i,0}^{\pm}$, $\mathbf{k}_i^{\pm1}$ and $\mathbf{C}^{\pm1}$, and using (3.2.5) we get

$$\mathbf{x}_{i,\pm1}^{\pm} = o(i)Y_i^{-1}(x_{i,0}^{\pm}) = o(i)Y_i^{-1}h(x_{i,0}^{\pm}) = o(i)h\sigma\mathbf{X}_i^{-1}(x_{i,0}^{\pm}) = h\sigma(x_{i,\pm1}^{\pm}),$$

and so our proof of (3.2.2) is complete. Fixing distinct $i, j \in I_0$ we have from (3.2.5) that

$$\begin{aligned}
Y_i(\mathbf{x}_{j,0}^{\pm}) = Y_i(x_{j,0}^{\pm}) = Y_i h(x_{j,0}^{\pm}) = h\sigma\mathbf{X}_i\sigma(x_{j,0}^{\pm}) = h(x_{j,0}^{\pm}) = x_{j,0}^{\pm} = \mathbf{x}_{j,0}^{\pm}, \\
Y_i(\mathbf{k}_j^{\pm1}) = Y_i(k_j^{\mp1}) = Y_i h(t_j^{\mp1}) = h\sigma\mathbf{X}_i\sigma(t_j^{\mp1}) = h(t_j^{\mp1}) = k_j^{\mp1} = \mathbf{k}_j^{\pm1},
\end{aligned}$$

which verifies (3.2.1). Note that (3.2.3) is trivial when $i \in I_0$, and moreover since $\mathcal{B}_v$ acts on $\mathcal{U}_v$ via Lusztig and Beck's affine action, outside type $C_n^{(1)}$ we have

$$\begin{aligned}
\mathbf{x}_{0,0}^{\pm} = T_\ell T_0(x_{\ell,0}^{\pm}) = T_\ell T_0 v(x_\ell^{\pm}) = v\mathbf{T}_\ell\mathbf{T}_0(x_\ell^{\pm}) = v(x_0^{\pm}) = v\sigma(x_0^{\pm}), \\
\mathbf{k}_0^{\pm1} = T_\ell T_0(k_\ell^{\mp1}) = T_\ell T_0 v(t_\ell^{\mp1}) = v\mathbf{T}_\ell\mathbf{T}_0(t_\ell^{\mp1}) = v(t_0^{\mp1}) = v\sigma(t_0^{\pm1}).
\end{aligned}$$

In type $C_n^{(1)}$ this is replaced with

$$\begin{aligned}\mathbf{x}_{0,0}^{\pm} &= \rho_n(x_{n,0}^{\pm}) = \rho_n v(x_n^{\pm}) = vS_{\rho_n}(x_n^{\pm}) = v(x_0^{\pm}) = v\sigma(x_0^{\pm}),\\ \mathbf{k}_0^{\pm 1} &= \rho_n(k_n^{\mp 1}) = \rho_n v(t_n^{\mp 1}) = vS_{\rho_n}(t_n^{\mp 1}) = v(t_0^{\mp 1}) = v\sigma(t_0^{\pm 1}),\end{aligned}$$

completing the proof of (3.2.3). For all $\rho \in \Omega^v$ we then have that

$$\begin{aligned}\rho(\mathbf{x}_{i,0}^{\pm}) &= \rho v(x_i^{\pm}) = vS_\rho(x_i^{\pm}) = v(x_{\rho(i)}^{\pm}) = \mathbf{x}_{\rho(i),0}^{\pm},\\ \rho(\mathbf{k}_i^{\pm 1}) &= \rho v(t_i^{\mp 1}) = vS_\rho(t_i^{\mp 1}) = v(t_{\rho(i)}^{\mp 1}) = \mathbf{k}_{\rho(i)}^{\pm 1},\end{aligned}$$

using (3.2.3). The equality $\rho(\mathbf{x}_{i,\pm 1}^{\pm}) = o_{i,\rho(i)}\mathbf{x}_{\rho(i),\pm 1}^{\pm}$ is trivial if either $\rho = \mathrm{id}$ or we are in type $C_n^{(1)}$, so we shall henceforth assume otherwise. If $i, \rho(i) \neq 0$ then $\rho Y_i^{-1}\rho^{-1} = Y_{\rho(i)}^{-1}Y_{\rho(0)}^{a_i}$ and therefore

$$\begin{aligned}\rho(\mathbf{x}_{i,\pm 1}^{\pm}) &= o(i)\rho Y_i^{-1}(x_{i,0}^{\pm}) = o(i)Y_{\rho(i)}^{-1}Y_{\rho(0)}^{a_i}\rho(x_{i,0}^{\pm}) = o(i)Y_{\rho(i)}^{-1}Y_{\rho(0)}^{a_i}(x_{\rho(i),0}^{\pm})\\ &= o_{i,\rho(i)}\mathbf{x}_{\rho(i),\pm 1}^{\pm}\end{aligned}$$

by (3.2.1) since $\rho(i), \rho(0) \in I_0$ are distinct. If $i = 0$ then $(\rho(s_\ell(\varpi_\ell^\vee)), \alpha_{\rho(0)}) = (s_\ell(\varpi_\ell^\vee), \alpha_0) = -1$ and we have

$$\begin{aligned}\rho(\mathbf{x}_{0,\pm 1}^{\pm}) &= o(0)\rho Y_{s_\ell(\varpi_\ell^\vee)}T_\ell T_0^v(x_{\ell,0}^{\pm}) = o(0)Y_{\rho(s_\ell(\varpi_\ell^\vee))}T_{\rho(\ell)}T_{\rho(0)}\rho(x_{\ell,0}^{\pm})\\ &= o(0)Y_{\rho(s_\ell(\varpi_\ell^\vee))}T_{\rho(\ell)}T_{\rho(0)}(x_{\rho(\ell),0}^{\pm}) = o_{0,\rho(0)}o(\rho(0))Y_{\rho(s_\ell(\varpi_\ell^\vee))}(x_{\rho(0),0}^{\pm})\\ &= o_{0,\rho(0)}\mathbf{x}_{\rho(0),\pm 1}^{\pm}\end{aligned}$$

where we again make use of (3.2.1). Outside type $A_{2n}^{(1)}$ the case $\rho(i) = 0$ then follows immediately since

$$\rho(\mathbf{x}_{i,\pm 1}^{\pm}) = \rho(\mathbf{x}_{\rho^{-1}(0),\pm 1}^{\pm}) = \rho\left(o_{0,\rho^{-1}(0)}^{-1}\rho^{-1}(\mathbf{x}_{0,\pm 1}^{\pm})\right) = o_{\rho^{-1}(0),0}\mathbf{x}_{0,\pm 1}^{\pm} = o_{i,\rho(i)}\mathbf{x}_{\rho(i),\pm 1}^{\pm}.$$

Type $A_{2n}^{(1)}$ requires more care, but from the other identities it suffices to show that $\rho_1^2(\mathbf{x}_{2n,\pm 1}^{\pm}) = \mathbf{x}_{1,\pm 1}^{\pm}$. From the equality $\rho_1^2 Y_{2n}^{-1}Y_{2n-1} = \zeta_0\zeta_1 Y_1^{-1}\rho_1^2$ inside the braid group we see that $\rho_1^2(\mathbf{x}_{2n,\pm 1}^{\pm}) = \zeta_0\zeta_1(\mathbf{x}_{1,\pm 1}^{\pm})$. Then using (3.2.2) and $\mathbf{X}_1 = S_{\pi_1}\mathbf{T}_n \ldots \mathbf{T}_1$ we can obtain an explicit expression for $\mathbf{x}_{1,\pm 1}^{\pm}$ in terms of $\{x_{i,0}^{\pm}, k_i^{\pm 1} \mid i \in I\}$. From this we can deduce that $\zeta_0\zeta_1(\mathbf{x}_{1,\pm 1}^{\pm}) = \mathbf{x}_{1,\pm 1}^{\pm}$, and so our proof of (3.2.4) is complete. $\square$

A second technical lemma gives information about how certain $Y_\beta \in \ddot{\mathcal{B}}$ act on the twisted generators $\mathbf{x}_{0,0}^{\pm}$ and $\mathbf{x}_{0,\pm 1}^{\pm}$.

Lemma 3.2.9. *Our action of $\ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$ satisfies the following relations.*

(β,α_0)	(β,α_ℓ)	$Y_\beta(\mathbf{x}^{\pm}_{0,0})$	$Y_\beta(\mathbf{x}^{\pm}_{0,\pm1})$
-1	-2	$o(0)\mathbf{x}^{\pm}_{0,\pm1}$	
-1	-1	$o(0)\mathbf{x}^{\pm}_{0,\pm1}$	
-1	0	$o(0)\mathbf{x}^{\pm}_{0,\pm1}$	
-1	1	$o(0)\mathbf{x}^{\pm}_{0,\pm1}$	
0	-1	$\mathbf{x}^{\pm}_{0,0}$	$\mathbf{x}^{\pm}_{0,\pm1}$
0	0	$\mathbf{x}^{\pm}_{0,0}$	$\mathbf{x}^{\pm}_{0,\pm1}$
0	1	$\mathbf{x}^{\pm}_{0,0}$	$\mathbf{x}^{\pm}_{0,\pm1}$
1	-1		$o(0)\mathbf{x}^{\pm}_{0,0}$
1	0		$o(0)\mathbf{x}^{\pm}_{0,0}$
1	1		$o(0)\mathbf{x}^{\pm}_{0,0}$
1	2		$o(0)\mathbf{x}^{\pm}_{0,0}$

Table 3.2 Actions of Y_β on $\mathbf{x}^{\pm}_{0,m}$

Proof. We start by noting that the first five rows of the table follow immediately from the last five. Moreover the proofs in type $C_n^{(1)}$ are easily deduced from

$$\begin{aligned} Y_\beta(\mathbf{x}^{\pm}_{0,0}) &= Y_\beta\rho_n(\mathbf{x}^{\pm}_{n,0}) = \rho_n Y_{\rho_n(\beta)}(\mathbf{x}^{\pm}_{n,0}), \\ Y_\beta(\mathbf{x}^{\pm}_{0,\pm1}) &= o(0)Y_\beta\rho_n Y_n^{-1}(\mathbf{x}^{\pm}_{n,0}) = o(0)\rho_n Y_{\rho_n(\beta)}Y_n^{-1}(\mathbf{x}^{\pm}_{n,0}), \end{aligned}$$

together with (3.2.1) and (3.2.4), and so we may restrict to the other types from now on. In the following we shall freely use without mention the various expressions for $\mathbf{x}^{\pm}_{0,\pm1}$ already presented, equation (3.2.1), and the relations of $\ddot{\mathcal{B}}$.

If $(\beta,\alpha_0)=0$ and $(\beta,\alpha_\ell)=0$ then

$$\begin{aligned} Y_\beta(\mathbf{x}^{\pm}_{0,0}) &= Y_\beta T_\ell T_0(x^{\pm}_{\ell,0}) = T_\ell T_0 Y_\beta(x^{\pm}_{\ell,0}) = T_\ell T_0(x^{\pm}_{\ell,0}) \\ &= \mathbf{x}^{\pm}_{0,0}, \\ o(0)Y_\beta(\mathbf{x}^{\pm}_{0,\pm1}) &= Y_\beta T_\ell^{-1}T_0^v Y_\ell^{-1}(x^{\pm}_{\ell,0}) = T_\ell^{-1}T_0^v Y_\ell^{-1}Y_\beta(x^{\pm}_{\ell,0}) = T_\ell^{-1}T_0^v Y_\ell^{-1}(x^{\pm}_{\ell,0}) \\ &= o(0)\mathbf{x}^{\pm}_{0,\pm1}. \end{aligned}$$

If $(\beta,\alpha_0)=0$ and $(\beta,\alpha_\ell)=1$ then

$$\begin{aligned}
Y_\beta(\mathbf{x}^\pm_{0,0}) &= Y_\beta T_\ell T^v_0(x^\pm_{\ell,0}) = T_\ell^{-1}Y_{s_\ell(\beta)}T^v_0(x^\pm_{\ell,0})\\
&= T_\ell^{-1}(T^v_0)^{-1}Y_{s_0s_\ell(\beta)}(x^\pm_{\ell,0}) = T_\ell^{-1}(T^v_0)^{-1}(x^\pm_{\ell,0})\\
&= \mathbf{x}^\pm_{0,0},\\
o(0)Y_\beta(\mathbf{x}^\pm_{0,\pm1}) &= Y_\beta Y_{s_\ell(\varpi^\vee_\ell)}T_\ell T^v_0(x^\pm_{\ell,0}) = Y_{s_\ell(\varpi^\vee_\ell)}T_\ell^{-1}Y_{s_\ell(\beta)}T^v_0(x^\pm_{\ell,0})\\
&= Y_{s_\ell(\varpi^\vee_\ell)}T_\ell^{-1}(T^v_0)^{-1}Y_{s_0s_\ell(\beta)}(x^\pm_{\ell,0}) = Y_{s_\ell(\varpi^\vee_\ell)}T_\ell^{-1}(T^v_0)^{-1}(x^\pm_{\ell,0})\\
&= o(0)\mathbf{x}^\pm_{0,\pm1}.
\end{aligned}$$

If $(\beta,\alpha_0)=1$ and $(\beta,\alpha_\ell)=-1$ then

$$\begin{aligned}
o(0)Y_\beta(\mathbf{x}^\pm_{0,\pm1}) &= Y_\beta T_\ell^{-1}T^v_0Y_\ell^{-1}(x^\pm_{\ell,0}) = T_\ell Y_{s_\ell(\beta)}T^v_0Y_\ell^{-1}(x^\pm_{\ell,0}) = T_\ell T^v_0 Y_{s_\ell(\beta)-\varpi^\vee_\ell}(x^\pm_{\ell,0})\\
&= \mathbf{x}^\pm_{0,0}.
\end{aligned}$$

If $(\beta,\alpha_0)=1$ and $(\beta,\alpha_\ell)=0$ then

$$\begin{aligned}
o(0)Y_\beta(\mathbf{x}^\pm_{0,\pm1}) &= Y_\beta T_\ell^{-1}T^v_0Y_\ell^{-1}(x^\pm_{\ell,0}) = T_\ell^{-1}Y_\beta T^v_0Y_\ell^{-1}(x^\pm_{\ell,0}) = T_\ell^{-1}(T^v_0)^{-1}Y_{s_0(\beta)-\varpi^\vee_\ell}(x^\pm_{\ell,0})\\
&= \mathbf{x}^\pm_{0,0}.
\end{aligned}$$

If $(\beta,\alpha_0)=1$ and $(\beta,\alpha_\ell)=1$ then

$$\begin{aligned}
o(0)Y_\beta(\mathbf{x}^\pm_{0,\pm1}) &= Y_{\beta+s_\ell(\varpi^\vee_\ell)}T_\ell T^v_0(x^\pm_{\ell,0}) = T_\ell T^v_0 Y_{\beta+s_\ell(\varpi^\vee_\ell)}(x^\pm_{\ell,0}) = T_\ell T^v_0(x^\pm_{\ell,0})\\
&= \mathbf{x}^\pm_{0,0}.
\end{aligned}$$

If $(\beta,\alpha_0)=1$ and $(\beta,\alpha_\ell)=2$ then

$$\begin{aligned}
o(0)Y_\beta(\mathbf{x}^\pm_{0,\pm1}) &= Y_{\beta+s_\ell(\varpi^\vee_\ell)}T_\ell T^v_0(x^\pm_{\ell,0}) = T_\ell^{-1}Y_{s_\ell(\beta+s_\ell(\varpi^\vee_\ell))}T^v_0(x^\pm_{\ell,0})\\
&= T_\ell^{-1}(T^v_0)^{-1}Y_{s_0s_\ell(\beta+s_\ell(\varpi^\vee_\ell))}(x^\pm_{\ell,0}) = T_\ell^{-1}(T^v_0)^{-1}(x^\pm_{\ell,0})\\
&= \mathbf{x}^\pm_{0,0}.
\end{aligned}$$

□

We are now ready to prove Theorem 3.2.1 in all untwisted types other than $A_1^{(1)}$ and $G_2^{(1)}$. Since our $\ddot{\mathcal{B}}$ action and simplified presentations exist for type $G_2^{(1)}$, and we can therefore define how ψ *should* act on each generator in this case as well, we shall make clear precisely which relations are not covered by our methods.

Proof of Theorem 3.2.1. To show that ψ is an anti-homomorphism, we must check that the relations of Corollary 3.2.7 still hold if we reverse the order of multiplication and replace each generator with its image under ψ. Denote these modified relations by **(i)**-**(xi)**.

Every relation with indices in I_0 follows immediately from the Drinfeld new presentation of $\mathcal{U}_h$ using (3.2.2). Moreover, relations involving only $\mathbf{x}_{i,0}^{\pm}$ and $\mathbf{k}_i^{\pm 1}$ terms follow from the Drinfeld-Jimbo presentation of $\mathcal{U}_v$ by (3.2.3). Furthermore, all of the relations containing only $\mathbf{x}_{0,\pm 1}^{\pm}$, $\mathbf{x}_{i,0}^{\pm}$ and $\mathbf{k}_i^{\pm 1}$ with $i \in I_0$ are verified by the Drinfeld-Jimbo presentation for $\mathcal{U}_d$ since ψ acts by σ on these generators. We shall now address the remaining relations not already covered by these arguments.

(iv) Only the $i = 0$, $m = \pm 1$ cases remain, which are verified as follows with $j \neq 0$.

$$\begin{aligned}
\mathbf{k}_0 \mathbf{x}_{0,\pm 1}^{\pm} \mathbf{k}_0^{-1} &= C k_\theta^{-1} x_{0,\pm 1}^{\pm} k_\theta C^{-1} = k_\delta k_\theta^{-1} x_{0,\pm 1}^{\pm} k_\theta k_\delta^{-1} = k_0 x_{0,\pm 1}^{\pm} k_0^{-1} = q_0^{\pm a_{00}} x_{0,\pm 1}^{\pm} \\
&= q_0^{\pm a_{00}} \mathbf{x}_{0,\pm 1}^{\pm} \\
\mathbf{k}_0 \mathbf{x}_{j,\pm 1}^{\pm} \mathbf{k}_0^{-1} &= C k_\theta^{-1} \mathbf{x}_{j,\pm 1}^{\pm} k_\theta C^{-1} = k_\delta k_\theta^{-1} \mathbf{x}_{j,\pm 1}^{\pm} k_\theta k_\delta^{-1} = k_0 \mathbf{x}_{j,\pm 1}^{\pm} k_0^{-1} \\
&= h\sigma(t_0 x_{j,\pm 1}^{\pm} t_0^{-1}) = h\sigma(C k_\theta^{-1} x_{j,\pm 1}^{\pm} k_\theta C^{-1}) = h\sigma \left(\textstyle\prod_{i \in I_0} (q_i^{\mp a_{ij}})^{a_i} x_{j,\pm 1}^{\pm} \right) \\
&= q^{\mp \sum_{i \in I_0} a_i d_i a_{ij}} h\sigma(x_{j,\pm 1}^{\pm}) = q^{\pm a_0 d_0 a_{0j}} h\sigma(x_{j,\pm 1}^{\pm}) \\
&= q_0^{\pm a_{0j}} \mathbf{x}_{j,\pm 1}^{\pm}
\end{aligned}$$

(vi) By Lemma 3.2.9 all $[\mathbf{x}_{j,-1}^{-}, \mathbf{x}_{0,1}^{+}] = 0$ and $[\mathbf{x}_{0,-1}^{-}, \mathbf{x}_{j,1}^{+}] = 0$ with $j \in I_0$ are obtained by applying some Y_β with $(\beta, \alpha_0) = (\beta, \alpha_j) = -1$ and $-2 \leq (\beta, \alpha_\ell) \leq 1$ to the corresponding relations in **(v)**. *In type* $G_2^{(1)}$ *this fails for* $j = 2$.

(vii) Using (3.2.1) and Lemma 3.2.9, every $[\mathbf{x}_{0,0}^{-}, \mathbf{x}_{j,1}^{+}] = 0$ and $[\mathbf{x}_{j,-1}^{-}, \mathbf{x}_{0,0}^{+}] = 0$ with $j \in I_0$ can be reached via one of the following. *In type* $G_2^{(1)}$ *this fails for* $j = 1$.

- Apply Y_β with $(\beta, \alpha_0) = 1$, $(\beta, \alpha_j) = -1$ and $-1 \leq (\beta, \alpha_\ell) \leq 1$ to $[\mathbf{x}_{0,-1}^{-}, \mathbf{x}_{j,0}^{+}] = 0$ and $[\mathbf{x}_{j,0}^{-}, \mathbf{x}_{0,1}^{+}] = 0$ respectively.
- Apply Y_β with $(\beta, \alpha_0) = 1$, $(\beta, \alpha_j) = 0$ and $-1 \leq (\beta, \alpha_\ell) \leq 1$ to the corresponding relation in **(v)**.

(viii) Combining (3.2.3) with Jing's isomorphism between the presentations of $U_q'(\hat{\mathfrak{g}})$ gives

$$\mathbf{x}_{0,0}^{+} = v(x_0^{+}) = [x_{i_{h-1},0}^{-}, \dots, x_{i_2,0}^{-}, x_{i_1,1}^{-}]_{q^{\epsilon_1} \dots q^{\epsilon_{h-2}}} C k_\theta^{-1},$$

so by centrality of k_δ and relation 7 of Definition 3.0.1 we have

$$\begin{aligned}
\mathbf{x}^+_{0,0}\mathbf{x}^+_{0,1} &= [x^-_{i_{h-1},0}, \dots, x^-_{i_2,0}, x^-_{i_1,1}]_{q^{\epsilon_1}\dots q^{\epsilon_{h-2}}} C k_\theta^{-1} x^+_{0,1} \\
&= [x^-_{i_{h-1},0}, \dots, x^-_{i_2,0}, x^-_{i_1,1}]_{q^{\epsilon_1}\dots q^{\epsilon_{h-2}}} C k_0 x^+_{0,1} k_0^{-1} k_\theta^{-1} \\
&= [x^-_{i_{h-1},0}, \dots, x^-_{i_2,0}, x^-_{i_1,1}]_{q^{\epsilon_1}\dots q^{\epsilon_{h-2}}} q_0^2 x^+_{0,1} C k_\delta^{-1} \\
&= q_0^2 x^+_{0,1} [x^-_{i_{h-1},0}, \dots, x^-_{i_2,0}, x^-_{i_1,1}]_{q^{\epsilon_1}\dots q^{\epsilon_{h-2}}} C k_\delta^{-1} \\
&= q_0^2 \mathbf{x}^+_{0,1}\mathbf{x}^+_{0,0}
\end{aligned}$$

and thus $[\mathbf{x}^+_{0,0}, \mathbf{x}^+_{0,1}]_{q_0^2} = 0$. The relation $[\mathbf{x}^-_{0,-1}, \mathbf{x}^-_{0,0}]_{q_0^{-2}} = 0$ is proved similarly.

(ix) Outside type $C_n^{(1)}$ we obtain $[\mathbf{x}^+_{\ell,0}, \mathbf{x}^+_{0,1}]_{q_0^{-1}} + [\mathbf{x}^+_{0,0}, \mathbf{x}^+_{\ell,1}]_{q_0^{-1}} = 0$ by taking $\beta, \gamma \in \mathring{P}^\vee$ such that

$$(\beta, \alpha_0) = 0, \quad (\beta, \alpha_\ell) = -1, \quad (\gamma, \alpha_0) = -1, \quad (\gamma, \alpha_\ell) = 1,$$

and applying both sides of $o(\ell) Y_\beta T_0^v Y_{s_0(\gamma)} T_0^v T_\ell = -o(0) Y_\beta Y_\gamma T_\ell$ to $\mathbf{x}^+_{0,0}$ as follows.

$$\begin{gathered}
\mathbf{x}^+_{0,0} \xmapsto{T_\ell} [\mathbf{x}^+_{\ell,0}, \mathbf{x}^+_{0,0}]_{q^{-1}} \xmapsto{o(0)Y_{\beta+\gamma}} [\mathbf{x}^+_{\ell,0}, \mathbf{x}^+_{0,1}]_{q^{-1}} \\
\mathbf{x}^+_{0,0} \xmapsto{T_0^v T_\ell} \mathbf{x}^+_{\ell,0} \xmapsto{Y_{s_0(\gamma)}} \mathbf{x}^+_{\ell,0} \xmapsto{T_0^v} [\mathbf{x}^+_{0,0}, \mathbf{x}^+_{\ell,0}]_{q^{-1}} \xmapsto{o(\ell)Y_\beta} [\mathbf{x}^+_{0,0}, \mathbf{x}^+_{\ell,1}]_{q^{-1}}
\end{gathered}$$

For $C_n^{(1)}$ we instead apply ρ_n to the corresponding relation with indices $n-1$ and n. *In type $G_2^{(1)}$ this remains to be shown, since such $\beta, \gamma \in \mathring{P}^\vee$ do not exist and Ω^v is trivial.*

(x) We prove $[\mathbf{x}^-_{\ell,-1}, \mathbf{x}^-_{0,0}]_{q_0} + [\mathbf{x}^-_{0,-1}, \mathbf{x}^-_{\ell,0}]_{q_0} = 0$ in the same manner as **(ix)**, except with $\mathbf{x}^+_{0,0}$ replaced by $\mathbf{x}^-_{0,0}$. *Again, type $G_2^{(1)}$ is not covered by our methods.*

(xi) It remains to check the affine q-Serre relations with $(y_i, y_j) = (\mathbf{x}^\pm_{0,0}, \mathbf{x}^\pm_{r,\pm 1}), (\mathbf{x}^\pm_{r,\pm 1}, \mathbf{x}^\pm_{0,0})$ for each $r \in I_0$, which by (3.2.1) and Lemma 3.2.9 can be verified via one of the following. *In type $G_2^{(1)}$ this fails for $j = 1$.*

- Apply Y_β with $(\beta, \alpha_0) = 1$, $(\beta, \alpha_r) = -1$ and $-1 \le (\beta, \alpha_\ell) \le 2$ to the affine q-Serre relations with $(y_i, y_j) = (\mathbf{x}^\pm_{0,\pm 1}, \mathbf{x}^\pm_{r,0}), (\mathbf{x}^\pm_{r,0}, \mathbf{x}^\pm_{0,\pm 1})$.
- Apply Y_β with $(\beta, \alpha_0) = 0$, $(\beta, \alpha_r) = -1$ and $-1 \le (\beta, \alpha_\ell) \le 1$ to the affine q-Serre relations with $(y_i, y_j) = (\mathbf{x}^\pm_{0,0}, \mathbf{x}^\pm_{r,0}), (\mathbf{x}^\pm_{r,0}, \mathbf{x}^\pm_{0,0})$.

We have therefore verified that ψ is an anti-homomorphism. The conditions $\psi v = h\sigma$ and $\psi h = v\sigma$ are then immediate from (3.2.2) and (3.2.3), and moreover determine ψ uniquely since $\mathcal{U}_h$ and $\mathcal{U}_v$ generate $U_q(\mathfrak{g}_{\mathrm{tor}})$. Furthermore, it also follows that $\psi^2 = \mathrm{id}$ on both $\mathcal{U}_h$ and $\mathcal{U}_v$ and so ψ is in fact an anti-involution. $\square$

Remark 3.2.10. In many types, our proof can be streamlined using (3.2.4). In particular, when $|\Omega^v| > 2$ all relations are obtained applying non-trivial ρ_i to those with indices in I_0. Moreover if $|\Omega^v| = 2$ then applying these elements to relations either lying inside $\mathcal{U}_d$ or with indices in I_0 reaches almost all other relations. Nevertheless, we have opted to detail the arguments above since they are effective in a more general situation.

3.3 Compatibilities and congruence group actions

The following results provide compatibilities between our braid group action from Theorem 3.1.11 and various (anti-)automorphisms of $U_q(\mathfrak{g}_{\mathrm{tor}})$, which can therefore be viewed as quantum toroidal analogues of the corresponding involutions of $\ddot{\mathcal{B}}$. Throughout this subsection we remain in the untwisted case and outside types $A_1^{(1)}$ and $G_2^{(1)}$ since the existence of our anti-involution ψ is required.

Proposition 3.3.1. *For all $b \in \ddot{\mathcal{B}}$ we have*

- $\psi \circ b = \mathfrak{t}(b) \circ \psi,$ (3.3.1)
- $\eta \circ b = \gamma_v(b) \circ \eta,$ (3.3.2)
- $\mathcal{X}_0^{-1} \circ b = \mathfrak{a}(b) \circ \mathcal{X}_0^{-1}.$ (3.3.3)

Proof. First note that in each case, the relation for some $b \in \ddot{\mathcal{B}}$ immediately implies the relation for b^{-1} by taking inverses and then composing on either side by the relevant (anti-)automorphism of $U_q(\mathfrak{g}_{\mathrm{tor}})$. Moreover since ψ and $\mathfrak{t}$ (resp. η and γ_v) are self-inverse, the compatibility relation also follows for $\mathfrak{t}(b)$ (resp. $\gamma_v(b)$) by composing with ψ (resp. η) on either side.

We start with the first identity. By Proposition 3.1.10 and Theorem 3.2.1,

$$\psi T_i h = v\sigma \mathbf{T}_i = v\mathbf{T}_i^{-1}\sigma = T_i^{-1}\psi h,$$

$$\psi T_i v = h\sigma \mathbf{T}_i = h\mathbf{T}_i^{-1}\sigma = T_i^{-1}\psi v,$$

for all $i \in I_0$, which gives the $b = T_i$ relation since $\mathcal{U}_h$ and $\mathcal{U}_v$ generate $U_q(\mathfrak{g}_{\mathrm{tor}})$. For $b = \pi_j$ we have that $\psi\pi_j(C^{\pm 1}) = \mathbf{C}^{\pm 1} = \rho_j\psi(C^{\pm 1})$ follows from $\psi Y_\beta(C^{\pm 1}) = \mathbf{C}^{\pm 1} = X_\beta\psi(C^{\pm 1})$ and the corresponding equality for $T_1^{\pm 1}, \dots, T_n^{\pm 1}$. From (3.2.4),

$$\begin{aligned}\psi\pi_j(x_{i,m}^{\pm}) &= \mathbf{x}_{\rho_j(i),m}^{\pm} = \rho_j\psi(x_{i,m}^{\pm}),\\ \psi\pi_j(k_i^{\pm 1}) &= \mathbf{k}_{\rho_j(i)}^{\pm 1} = \rho_j\psi(k_i^{\pm 1}),\end{aligned}$$

for all $i \in I$ and $m = 0, \pm 1$, and therefore $\psi\pi_j = \rho_j\psi$. When Ω is non-trivial our proof is complete since $\mathcal{B}_0$, Ω and Ω^v generate $\ddot{\mathcal{B}}$. Otherwise, it remains to check that $T_0\psi = \psi(T_0^v)^{-1}$. First note that $T_0\psi v = h\mathbf{T}_0\sigma = h\sigma\mathbf{T}_0^{-1} = \psi(T_0^v)^{-1}v$ so we are done on $\mathcal{U}_v$. Furthermore,

$$\begin{aligned}T_0\psi(k_0^{\pm 1}) &= T_0(C^{\mp 1}k_\theta^{\pm 1}) = C^{\mp 1}k_0^{\pm 2}k_\theta^{\pm 1}\\ &= \psi(C^{\mp 2}k_0^{\pm 1}k_\theta^{\pm 2}) = \psi\Theta(C^{\mp 2}k_0^{\pm 1})\\ &= \psi\Theta X_{-\theta^\vee}(k_0^{\pm 1}) = \psi(T_0^v)^{-1}(k_0^{\pm 1}),\end{aligned}$$

and picking $\beta \in \mathring{P}^\vee$ such that

$$(\beta, \alpha_0) = 0, \qquad (\beta, \alpha_i) = -1, \qquad -1 \leq (\beta, \alpha_\ell) \leq 1,$$

for some $i \in I_0$, we have from Lemma 3.2.9 that

$$\begin{aligned}T_0\psi(\mathbf{x}_{i,-1}^-) &= \sum_{s=0}^{-a_{0i}}(-1)^s q_0^s (x_{0,0}^-)^{(s)} x_{i,-1}^- (x_{0,0}^-)^{(-a_{0i}-s)}\\ &= \psi\left(\sum_{s=0}^{-a_{0i}}(-1)^s q_0^s (\mathbf{x}_{0,0}^-)^{(-a_{0i}-s)} \mathbf{x}_{i,-1}^- (\mathbf{x}_{0,0}^-)^{(s)}\right)\\ &= o(i)\psi Y_\beta\left(\sum_{s=0}^{-a_{0i}}(-1)^s q_0^s (\mathbf{x}_{0,0}^-)^{(-a_{0i}-s)} \mathbf{x}_{i,0}^- (\mathbf{x}_{0,0}^-)^{(s)}\right)\\ &= o(i)\psi Y_\beta(T_0^v)^{-1}(\mathbf{x}_{i,0}^-) = o(i)\psi(T_0^v)^{-1}Y_\beta(\mathbf{x}_{i,0}^-)\\ &= (T_0^v)^{-1}(\mathbf{x}_{i,-1}^-),\end{aligned}$$

and similarly for $\mathbf{x}_{i,1}^+$. This verifies $T_0\psi = \psi(T_0^v)^{-1}$ on $\mathcal{U}_h$ and so we have proved (3.3.1).

We immediately have by Proposition 3.1.10 that $\eta T_i = T_i^{-1}\eta$ for all $i \in I$, and moreover (3.3.2) holds for elements of Ω since they are fixed by γ_v and commute with η.

Moving to the final identity, if $b \in \mathcal{B}_v$ then $\mathfrak{a}(b) = b$ and we are done since $\mathcal{X}_0^{-1}$ clearly commutes with the actions of $T_1^{\pm 1}, \dots, T_n^{\pm 1}$ and all X_β. Each $\pi_i \in \Omega$ has $\mathfrak{a}(\pi_i) = X_i \pi_i$ and thus

$$\mathfrak{a}(\pi_i) \circ \mathcal{X}_0^{-1} = \mathcal{X}_i \mathcal{X}_0^{-1} \pi_i \mathcal{X}_0^{-1} = \mathcal{X}_i \mathcal{X}_0^{-1} \mathcal{X}_i^{-1} \pi_i = \mathcal{X}_0^{-1} \pi_i = \mathcal{X}_0^{-1} \circ \pi_i.$$

If $b = T_0 = \Theta_{01}$ then $\mathfrak{a}(b) = \Theta_{02} = T_0^{-1} \prod_{j \in \tilde{I}} X_j^{-1}$ and it follows that

$$\mathfrak{a}(b) \circ \mathcal{X}_0^{-1} = T_0^{-1} \left(\textstyle\prod_{j \in \tilde{I}} \mathcal{X}_j^{-1} \right) \mathcal{X}_0 = \left(\textstyle\prod_{j \in \tilde{I}} \mathcal{X}_j^{-1} \right) \mathcal{X}_0^{-1} \left(\textstyle\prod_{j \in \tilde{I}} \mathcal{X}_j \right) T_0 = \mathcal{X}_0^{-1} T_0 = \mathcal{X}_0^{-1} \circ b,$$

which completes our proof. □

Taking certain compositions of these identities provides us with further compatibilities involving some of the other (anti-)automorphisms of $\ddot{\mathcal{B}}$ and $U_q(\mathfrak{g}_{\mathrm{tor}})$ that we have encountered thus far.

Corollary 3.3.2. *For all $b \in \ddot{\mathcal{B}}$ we have*

- $\psi\eta\psi \circ b = \gamma_h(b) \circ \psi\eta\psi,$ (3.3.4)
- $\Phi \circ b = \gamma_v(\mathfrak{t}(b)) \circ \Phi,$ (3.3.5)
- $\Phi^{-1} \circ b = \gamma_h(\mathfrak{t}(b)) \circ \Phi^{-1},$ (3.3.6)
- $(\psi \mathcal{X}_0 \psi) \circ b = \mathfrak{b}(b) \circ (\psi \mathcal{X}_0 \psi).$ (3.3.7)

Proof. These follow immediately from Proposition 3.3.1 together with identities on the braid group side such as $\mathfrak{t} \circ \gamma_v = \gamma_h \circ \mathfrak{t}$ and $\mathfrak{b}^{-1} = \mathfrak{t} \circ \mathfrak{a} \circ \mathfrak{t}$. □

Furthermore, we are now able to deduce quantum toroidal versions of the congruence group actions from Theorem 3.1.5.

Theorem 3.3.3.

- *There is an action of $\tilde{\Gamma}_1(1)$ by automorphisms on $U_q(\mathfrak{g}_{\mathrm{tor}})$ given by $\mathfrak{u}_1 \mapsto \mathcal{X}_0^{-1}$ and $\mathfrak{u}_2 \mapsto \psi \mathcal{X}_0 \psi$, which fixes $\mathcal{U}_h \cap \mathcal{U}_v \cong U_q(\mathfrak{g})$ pointwise.*
- *This extends to an action of $\tilde{\Xi}_1(1)$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$ by letting $\mathfrak{r}$ act via the anti-involution ψ.*

Equations (3.3.3) and (3.3.7) provide a compatibility between the action of $\ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$, and the action of $\tilde{\Gamma}_1(1)$ on each side. This allows us to extend our braid group action to obtain a combined action of $\tilde{\Gamma}_1(1) \ltimes \ddot{\mathcal{B}}$ on $U_q(\mathfrak{g}_{\mathrm{tor}})$. How should we think of this semidirect product?

On the one hand, $\mathfrak{u}_1^{-1}$ commutes with $\mathcal{U}_v$ and we can easily verify that

$$T_0^{-1}\mathfrak{u}_1^{-1}T_0^{-1} = \mathfrak{u}_1^{-1}X_{-\alpha_0^\vee}, \qquad T_i\mathfrak{u}_1^{-1} = \mathfrak{u}_1^{-1}T_i, \qquad \pi_i\mathfrak{u}_1^{-1}\pi_i^{-1} = \mathfrak{u}_1^{-1}X_i. \tag{3.3.8}$$

These are precisely the relations in Definition 3.1.1 that an element $X_{\Lambda_0^\vee}$ would satisfy. Hence the inclusion of $\mathfrak{u}_1^{-1}$ can be viewed as enlarging the finite coweight lattice $\mathring{P}^\vee$ indexing the $X_\beta \in \ddot{\mathcal{B}}$ to the entire affine coweight lattice $P^\vee$.

Crucially, this is done in a way that preserves the horizontal-vertical symmetry afforded by $\mathfrak{e}$, which extends to an anti-involution of $\tilde{\Gamma}_1(1) \ltimes \ddot{\mathcal{B}}$ simply by swapping $\mathfrak{u}_1$ and $\mathfrak{u}_2$. Indeed, $\mathfrak{u}_2$ commutes with all Y_β and satisfies comparable relations to (3.3.8), except involving T_0^v and ρ_i instead.

This symmetry is reflected in the action on $U_q(\mathfrak{g}_{\mathrm{tor}})$. While $\mathcal{X}_0$ fixes $\mathcal{U}_v$ and shifts generators $x_{0,m}^{\pm} \mapsto o(0)x_{0,m\mp1}^{\pm}$ in column 0 of Figure 3.1 *up and down* the $\mathbb{Z}$-grading, correspondingly $\psi\mathcal{X}_0\psi$ fixes $\mathcal{U}_h$ and maps each $\mathbf{x}_{0,m}^{\pm} \mapsto o(0)\mathbf{x}_{0,m\mp1}^{\pm}$. Note that the above discussion further extends to $\tilde{\Xi}_1(1)$ provided that we let $\mathfrak{e}$ fix the additional element $\mathfrak{r}$.

3.4 Applications and future directions

Braid group actions provide a powerful tool for investigating the structure of quantum algebras. Indeed, we can often translate existing phenomena on the braid group side *across the action* to obtain analogous results in the quantum setting. For instance, the Drinfeld new presentation for quantum affine algebras essentially follows in [3, 17, 18] by combining the Bernstein realization for $\dot{\mathcal{B}}$ with its action on $U_q(\hat{\mathfrak{g}})$.

On the toroidal level, several results in this thesis already demonstrate the potential of our braid group action. Section 3.3 shows that our anti-involution ψ, automorphism Φ and congruence group actions for $U_q(\mathfrak{g}_{\mathrm{tor}})$ are compatible with corresponding results for $\ddot{\mathcal{B}}$. Ongoing work by the author aims to further exploit this philosophy. For example, the results of Ion-Sahi [46] indicate extensions of ψ and Φ to twisted types (potentially without bijectivity) and suggest explicit realizations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ as quotients of the quantum groups $U_q(\dddot{X}_n)$.

The horizontal-vertical symmetry afforded by anti-involution ψ and automorphism Φ should have a range of applications pertaining to the representation theory of untwisted $U_q(\mathfrak{g}_{\mathrm{tor}})$, and in Section 3.4.2 we begin an exploration of one such direction. Namely, by conjugating the topological coproduct Δ_v from Section 2.4.3 with ψ we are able to construct tensor products of ℓ-highest weight modules. Furthermore, we propose conjectures regarding their associated R-matrices that extend the work of Miki [87] in type $A_n^{(1)}$.

Another possible direction in simply laced types concerns the relationship between the level $(1,0)$ vertex representations of $U_q(\mathfrak{g}_{\mathrm{tor}})$ due to Saito and Jing [51, 100]; combinatorially defined Fock space representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$; and modules constructed geometrically by Nakajima [92, 93] using the equivariant K-theory of quiver varieties on the affine Dynkin diagrams. The expectation is that twisting these vertex representations by Φ should be isomorphic to (the dual of) a representation on $\mathcal{F}(\Lambda_0)$, which can in turn be identified with a geometric module.

Remark 3.4.1. Work of Frenkel-Jing-Wang [28] supports these ideas by constructing for every finite subgroup $G \leq SL_2(\mathbb{Z})$ a group-theoretic realization of the vertex representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ in the corresponding ADE type. As discussed in Section 4.4.1, these G also arise naturally on the geometric side via the McKay Correspondence.

Of course, these results are all well-known in type $A_n^{(1)}$ [87, 91, 103] and moreover provide an instance of the celebrated boson-fermion correspondence from mathematical physics. Here, identifications between representations allow us to relate various geometric features on the quiver variety side with more algebraic or combinatorial elements on the other – see Section 4.4.1 for further discussion. Indeed, interesting affine phenomena have already turned out to be meaningful in the geometric setting, for example in work of Nagao [90, 91].

However we note that in other simply laced types, Fock space representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ have only been described geometrically at this stage. In particular, there does not yet exist a definition involving the q-wedge construction due to Kashiwara-Miwa-Petersen-Yung [71], nor is there a combinatorial construction in terms of Young walls.

For $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ such a module was originally obtained by Saito-Takemura-Uglov [101] and Varagnolo-Vasserot [106], and was later connected to both the geometric [91] and vertex [87, 103] representations. A crucial ingredient for their proofs is a Schur-Weyl duality [105] with the type A double affine Hecke algebra. Unfortunately, analogues of this result do not exist in other types and hence we require alternative methods – ideas for future work are discussed in Section 3.4.3.

Work of Feigin-Jimbo-Miwa-Mukhin [26] suggests one such possibility. The starting point is a vector representation for $U_q(\mathfrak{sl}_{n+1,\text{tor}})$, defined explicitly with respect to a basis of Young columns. The topological coproduct Δ_v gives rise to actions on its tensor powers, which descend or restrict to the *exterior* powers for particular specialisations of v. Taking the semi-infinite limit along the ground state vector results in a Fock space representation that is expressed combinatorially in terms of Young walls.

In Section 3.4.1 we generalise the first step of this process to the quantum toroidal algebras in all simply laced types except $E_8^{(1)}$, defining vector representations in terms of a Young column basis in each case. We then mention potential connections to extremal weight modules and finite dimensional modules at roots of unity.

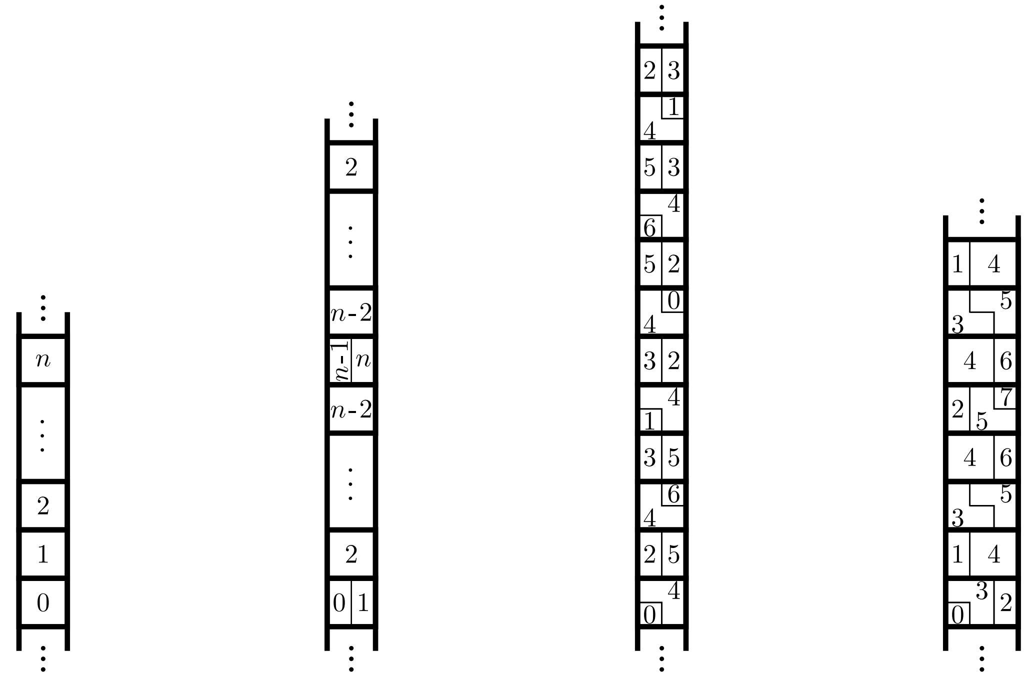
3.4.1 Vector representations and extremal weight modules

Vector representations for Lie algebras and quantum groups in finite and affine types are fundamentally important, both in their own right and as the building blocks for more complex representations. They are finite dimensional in these cases, however it is known [110] that quantum toroidal algebras do not possess finite dimensional representations when q is not a root of unity. The appropriate analogue for quantum toroidal algebras should be the affinization V_{aff} of the vector representation V for $U_q'(\hat{\mathfrak{g}})$. How do we define an action of $U_q(\mathfrak{g}_{\text{tor}})$ on this space?

In this subsection we shall answer this question in types $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$, generalising an existing construction for $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ due to Feigin-Jimbo-Miwa-Mukhin

[26] and Mansuy [83]. Our action is defined in terms of Young column models for the crystal basis B_{aff} of V_{aff} in each type. As detailed in Section 4.1.3, vertices of the crystal correspond to Young columns – stackings of blocks inside the relevant pattern from Figure 3.5 that are bounded above and have no free space below any block. Furthermore, each i-arrow simply corresponds to the addition of an i-block.

In types $A_n^{(1)}$ and $D_n^{(1)}$ these models are originally due to Kang [56], while for $E_6^{(1)}$ and $E_7^{(1)}$ they form part of the author's work in [77] and are moreover included in Chapter 4 of this thesis. See Figure 4.4 for our diagrammatic conventions.



(a) Type $A_n^{(1)}$ **(b)** Type $D_n^{(1)}$ **(c)** Type $E_6^{(1)}$ **(d)** Type $E_7^{(1)}$

Fig. 3.5 Young column patterns for types $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$

Definition 3.4.2.

- *An i-block in a Young column y is removable if removing it from y gives another Young column.*
- *An i-block in the Young column pattern that is not in y is addable if adding it to y gives another Young column.*
- *The i-signature of y is the sequence* $\mathrm{sign}_i(y) = \underbrace{-\cdots-}_{\varepsilon_i(y)}\underbrace{+\cdots+}_{\varphi_i(y)}$.

It is clear that in the cases we consider here, $\mathrm{sign}_i(y)$ is always $-$, empty or $+$, and can thus be identified with the corresponding $\langle \mathrm{wt}(y), \alpha_i^\vee \rangle = \varphi_i(y) - \varepsilon_i(y)$ value of -1, 0 or 1 respectively. In other words, every i-string has length at most 1 and so from (2.2.4)–(2.2.5) the crystal and global bases for V_{aff} in some sense coincide. Furthermore, for any Young column y and each $i \in I$ we may therefore define $y \pm \boxed{i}$ to be the unique Young column obtained by adding or removing an i-block if this exists, and 0 otherwise.

Lemma 3.4.3. *For any Young column y and each $i, j \in I$ we have that* $\mathrm{sign}_i(y \pm \boxed{j}) = \mathrm{sign}_i(y) \mp a_{ij}$ *whenever $y \pm \boxed{j}$ is non-zero.*

Proof. This follows by combining $\mathrm{sign}_i(y) = \langle \mathrm{wt}(y), \alpha_i^\vee \rangle$ with $\mathrm{wt}(y \pm \boxed{j}) = \mathrm{wt}(y) \mp \alpha_j$. ☐

Key to our work is the notion of the *height* of a block inside the Young column pattern. Setting the height of some arbitrary fixed block to be 0, every block is defined to have height one plus that of any block lying directly below it. We then let $h_i(y)$ equal the height of the unique addable or removable i-block for y if it exists.

It shall be convenient to collect the generators from Definition 3.0.1 into power series

$$X_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m}^\pm z^m, \qquad K_i^\pm(z) = \sum_{m \in \mathbb{Z}} \phi_{i,m}^\pm z^m,$$

for each $i \in I$. Ignoring the rank 1 cases $A_1^{(1)}$ and $A_2^{(2)}$ for simplicity, all of the original relations are then either encoded in or generated by the following (see for example Definition 2.10 and Remark 2.11 of [19]):

$$\cdot \ C^{\pm 1} \text{ central}, \tag{3.4.1}$$

$$\cdot \ C^{\pm 1} C^{\mp 1} = k_i^{\pm 1} k_i^{\mp 1} = 1, \tag{3.4.2}$$

$$\cdot \ K_i^\pm(z) K_j^\pm(w) = K_j^\pm(w) K_i^\pm(z), \tag{3.4.3}$$

$$\cdot \ (w - q_i^{\pm a_{ij}} z) K_i^\pm(z) X_j^\pm(w) = (q_i^{\pm a_{ij}} w - z) X_j^\pm(w) K_i^\pm(z), \tag{3.4.4}$$

$$\cdot \ (q_i^{\pm a_{ij}} w - Cz) K_i^\pm(z) X_j^\mp(w) = (w - q_i^{\pm a_{ij}} Cz) X_j^\mp(w) K_i^\pm(z), \tag{3.4.5}$$

$$\cdot \ \big[X_i^+(z), X_j^-(w)\big] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \big(K_i^+(z)\delta(Cz/w) - K_i^-(w)\delta(Cw/z)\big), \tag{3.4.6}$$

$$\cdot \ \sum_{\pi \in S_{a'}} \big[\ldots \big[\big[X_j^\pm(z), X_i^\pm(z_{\pi(1)})\big]_{q_i^{-a_{ij}}}, X_i^\pm(z_{\pi(2)})\big]_{q_i^{-a_{ij}-2}}, \ldots, X_i^\pm(z_{\pi(a')})\big]_{q_i^{a_{ij}}} = 0, \tag{3.4.7}$$

where for the final identity we assume that $i \neq j$ and let $a' = 1 - a_{ij}$. The next theorem provides vector representations for the quantum toroidal algebras in all simply laced types except $E_8^{(1)}$. For simplicity of notation, we introduce a function $\psi(x) = \frac{q-q^{-1}x}{1-x}$ which in particular satisfies $\psi(1/x) = \psi(q^2x)^{-1}$.

Theorem 3.4.4. *In types $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$, for each $u \in \mathbb{C}^*$ there exists a representation of $U_q(\mathfrak{g}_{\mathrm{tor}})$ on V_{aff} given in terms of the basis of Young columns by*

$$X_i^{\pm}(z) \cdot y = \delta(q^{h_i(y)}z/u)(y \mp \boxed{i}),$$

$$K_i^{\pm}(z) \cdot y = \kappa_{i,y}(z)y = \begin{cases} \psi(q^{h_i(y)}z/u)y & \text{if } \mathrm{sign}_i(y) = +, \\ \psi(u/q^{h_i(y)}z)y & \text{if } \mathrm{sign}_i(y) = -, \\ y & \text{if } \mathrm{sign}_i(y) \text{ is empty,} \end{cases}$$

with $C^{\pm 1}$ acting as the identity. Note that the scalars $\kappa_{i,y}(z)$ are regarded as power series in $z^{\pm 1}$ by expanding around $z = 0$ and $z = \infty$ respectively.

Proof. We may consider $u = 1$ without loss of generality, since the general case then follows by twisting with the automorphism of $U_q(\mathfrak{g}_{\mathrm{tor}})$ that scales any x by $u^{-\deg_{\mathbb{Z}}(x)}$. Relations (3.4.1)–(3.4.3) are trivially verified, noting that $k_i^{\pm 1}$ scales y by $q^{-\mathrm{sign}_i(y)}$. Both sides of (3.4.4) are zero unless $\mathrm{sign}_j(y) = \mp$, where it suffices to show that

$$(q^{-h_j(y)} - q^{\pm a_{ij}}z)\kappa_{i,y\mp\boxed{j}}(z) = (q^{-h_j(y)\pm a_{ij}} - z)\kappa_{i,y}(z) \tag{3.4.8}$$

since $w\delta(aw) = a^{-1}\delta(aw)$. This is done on a case-by-case basis:

- If $a_{ij} = 0$ this is trivial.
- If $a_{ij} = -1$ and $\kappa_{i,y}(z) = 1$ then $h_i(y \mp \boxed{j}) = h_j(y) \mp 1$ and (3.4.8) becomes
$$(q^{-h_j(y)} - q^{\mp 1}z)\psi((q^{h_j(y)\mp 1}z)^{\mp 1}) = (q^{-h_j(y)\mp 1} - z).$$
- If $a_{ij} = -1$ and $\kappa_{i,y\mp\boxed{j}}(z) = 1$ then $h_i(y \mp \boxed{j}) = h_j(y) \pm 1$ and (3.4.8) becomes
$$(q^{-h_j(y)} - q^{\mp 1}z) = (q^{-h_j(y)\mp 1} - z)\psi((q^{h_j(y)\pm 1}z)^{\pm 1}).$$
- If $a_{ij} = 2$ then $h_i(y \mp \boxed{j}) = h_j(y)$ and (3.4.8) becomes
$$(q^{-h_j(y)} - q^{\pm 2}z)\psi((q^{h_j(y)}z)^{\pm 1}) = (q^{-h_j(y)\pm 2} - z)\psi((q^{h_j(y)}z)^{\mp 1}).$$

Relation (3.4.5) is verified via similar reasoning. For (3.4.6), when $i \neq j$ it is clear from the Young column patterns in Figure 3.5 that $X_i^+(z)X_j^-(w) \cdot y = X_j^-(w)X_i^+(z) \cdot y = 0$ unless $a_{ij} = 0$, $\mathrm{sign}_i(y) = -$ and $\mathrm{sign}_j = +$. Here $h_i(y) = h_j(y) = h_i(y + \boxed{j}) = h_j(y - \boxed{i})$ and we instead have

$$X_i^+(z)X_j^-(w) \cdot y = X_j^-(w)X_i^+(z) \cdot y = \delta(q^{h_i(y)}z)\delta(q^{h_i(y)}w)(y - \boxed{i} + \boxed{j}).$$

For $i = j$ we must consider three cases, noting that $h_i(y) = h_i(y \pm \boxed{i})$ whenever both sides are well-defined:

- If $\mathrm{sign}_i(y)$ is empty the result follows from $C^{\pm 1}$ acting by the identity and

$$K_i^+(z) \cdot y = K_i^-(w) \cdot y = y, \qquad X_i^+(z) \cdot y = X_i^-(w) \cdot y = 0.$$

- If $\mathrm{sign}_i(y) = +$ then it suffices to check that

$$\delta(q^{h_i(y)}z)\delta(q^{h_i(y)}w) = \frac{1}{q - q^{-1}} \left(\psi(q^{h_i(y)}z)\delta(z/w) - \psi(q^{h_i(y)}w)\delta(w/z)\right)$$

  by expanding around $z = 0$ and $w = \infty$ on the right hand side.

- The $\mathrm{sign}_i(y) = -$ case is proved similarly.

It is clear from the Young column patterns that if $a_{ij} = -1$ then each summand of (3.4.7) sends any y to 0. On the other hand, when $a_{ij} = 0$ one can easily see that the actions of $X_i^\pm(z)$ and $X_j^\pm(w)$ commute, and our proof is complete. □

Remark 3.4.5. 1. The restriction of this representation to the horizontal subalgebra $\mathcal{U}_h$ coincides with the original action of $U_q(\hat{\mathfrak{g}})$ on V_{aff}.

2. These vector representations provide examples of modules which are neither ℓ-highest weight nor ℓ-lowest weight.

The question arises as to whether such vector representations can be constructed in other types. Appropriate Young column models are known to exist [25, 33, 40, 56, 77], however difficulties arise from the existence of i-strings in B_{aff} of length greater than 1. This results in the crystal and global bases not coinciding on the nose, and there is more work to do. The author hopes to address these issues in later work.

Furthermore, the $U_q(\hat{\mathfrak{g}})$ module V_{aff} provides an example of an extremal weight module $V^{\text{ext}}(\lambda)$ for which $\lambda = \varpi_i = \Lambda_i - \Lambda_0$ with $i \in I_{\min} \setminus \{0\}$ is neither dominant nor anti-dominant. It is natural to ask whether other extremal weight modules for quantum groups possess actions of the corresponding quantum affinizations. Moreover by considering particular subquotients of such modules, this should be closely linked to the existence of finite dimensional representations for quantum toroidal algebras and quantum affinizations when q is a root of unity. Indeed, similar problems were considered by Mansuy [83, 84] for $U_q(\mathfrak{sl}_{n+1,\text{tor}})$. Once again, the author plans to return to these questions in the future.

3.4.2 Tensor products of ℓ-highest weight representations

Recall from Section 2.4 the topological coproduct Δ_v and classification of integrable ℓ-highest weight representations for a quantum affinization $\widehat{U_q(\mathfrak{s})}$. It is easy to see that in general, Δ_v fails to produce a well-defined tensor product on ℓ-highest weight modules. Roughly speaking, this is because both Δ_v and the triangular decomposition for $\widehat{U_q(\mathfrak{s})}$ are *infinite with respect to the vertical direction.* As a consequence, $\text{im}(\Delta_v)$ contains infinite sums whose actions on various elements of a tensor product may not converge after specialising v.

However, in the special case of untwisted quantum toroidal algebras, we can overcome this issue by exploiting the horizontal-vertical symmetry afforded by our anti-involution ψ. In particular, conjugating Δ_v by ψ gives rise to a topological coproduct which is instead infinite in the *horizontal* direction, and produces a tensor product that is well-defined for ℓ-highest weight modules. Moreover, one should then be able to prove the existence of R-matrices – solutions to the Yang-Baxter equation in physics – that exchange the tensor factors, depending on a spectral parameter.

Remark 3.4.6. Such work has already been done for $U_q(\mathfrak{sl}_{n+1,\text{tor}})$ by Miki [87], but conjugating by $\mathcal{X}_0^{-1}\Phi$ instead. We have chosen to use ψ here since it acts more symmetrically with respect to the finer grading of $U_q(\mathfrak{g}_{\text{tor}})$ from (2.4.2).

As mentioned in Section 2.4, $U_q(\mathfrak{g}_{\text{tor}})$ possesses a $(Q \oplus \mathbb{Z}\delta')$-grading given by

$$\deg(x_{i,m}^{\pm}) = \pm\alpha_i + m\delta', \qquad \deg(h_{i,r}) = r\delta', \qquad \deg(C^{\pm 1}) = \deg(k_i^{\pm 1}) = 0,$$

which results in a decomposition $U_q(\mathfrak{g}_{\text{tor}}) = \bigoplus_{\mu \in Q \oplus \mathbb{Z}\delta'} \mathcal{U}_\mu$. Just as $\delta \in Q$ is associated to the horizontal subalgebra $\mathcal{U}_h$, one can think of δ' as an imaginary root $\sum_{i \in I} a_i \alpha_i'$ for $\mathcal{U}_v$ where we identify $\alpha_i' = \alpha_i$ for each $i \in I_0$. Then for example by considering the generating set $\{x_{0,\pm 1}^{\pm},\, x_{i,0}^{\pm},\, k_i^{\pm 1},\, C^{\pm 1} \mid i \in I\}$ for $U_q(\mathfrak{g}_{\text{tor}})$, it is clear that

$$\psi : \mathcal{U}_{\beta + k\delta, \ell\delta'} \to \mathcal{U}_{\beta + \ell\delta, k\delta'} \tag{3.4.9}$$

for any $\beta \in \mathring{Q}$ and $k, \ell \in \mathbb{Z}$. By conjugating Δ_v with ψ, we obtain a new (horizontally infinite) topological coproduct

$$\Delta_v^\psi = (\psi \otimes \psi) \circ \Delta_v \circ \psi$$

for $U_q(\mathfrak{g}_{\text{tor}})$. Where does Δ_v^ψ send a graded piece $\mathcal{U}_{\beta + k\delta, \ell\delta'}$? From (3.4.9) we have that ψ sends elements of $\mathcal{U}_{\beta + k\delta, \ell\delta'}$ to elements of $\mathcal{U}_{\beta + \ell\delta, k\delta'}$, which can of course be expressed as linear combinations of products in the $x_{i,m}^{\pm}$, $h_{i,r}$, $k_i^{\pm 1}$ and $C^{\pm 1}$ generators. Then using the formulae in Theorem 2.4.13, any such expression is mapped by Δ_v into

$$\sum_{\substack{\mu \in \mathring{Q} \\ n \in \mathbb{Z}}} \sum_{r \in \mathbb{Z}} \left(\mathcal{U}_{\beta - \mu + (\ell - n)\delta, (k - r)\delta'} \otimes \mathcal{U}_{\mu + n\delta, r\delta'} \right) v^{-r}$$

where the sum over μ and n is finite, but that over r may be infinite. Finally, applying $\psi \otimes \psi$ lands us in

$$\sum_{\substack{\mu \in \mathring{Q} \\ n \in \mathbb{Z}}} \sum_{r \in \mathbb{Z}} \left(\mathcal{U}_{\beta - \mu + (k - r)\delta, (\ell - n)\delta'} \otimes \mathcal{U}_{\mu + r\delta, n\delta'} \right) v^{-r}. \tag{3.4.10}$$

Theorem 3.4.7. *The coproduct Δ_v^ψ defines the structure of an integrable $U_q(\mathfrak{g}_{\text{tor}})$-module on the tensor product of irreducible integrable ℓ-highest weight representations.*

Proof. Consider irreducible integrable ℓ-highest weight representations V, W with ℓ-weights $(\lambda_V, \Psi_V), (\lambda_W, \Psi_W) \in P_\ell^+$ as defined in Section 2.4.2. We know that $V = \bigoplus_{\gamma \leq \lambda_V} V_\gamma$ and $W = \bigoplus_{\tau \leq \lambda_W} W_\tau$ decompose as direct sums of finite dimensional weight spaces, and moreover

$$\mathcal{U}_{\beta + k\delta, \ell\delta'} \cdot V_\gamma \subset V_{\gamma + \beta + k\delta}, \qquad \mathcal{U}_{\beta + k\delta, \ell\delta'} \cdot W_\tau \subset W_{\tau + \beta + k\delta},$$

for all $\beta \in \mathring{Q}$, $k, \ell \in \mathbb{Z}$, $\gamma \leq \lambda_V$ and $\tau \leq \lambda_W$. It follows that

$$\begin{aligned} \left(\mathcal{U}_{\beta - \mu + (k - r)\delta, (\ell - n)\delta'} \otimes \mathcal{U}_{\mu + r\delta, n\delta'} \right) \cdot (V_\gamma \otimes W_\tau) &\subset V_{\gamma + \beta - \mu + (k - r)\delta} \otimes W_{\tau + \mu + r\delta} \\ &= \begin{cases} V_{\gamma + \beta - \mu + (k - r)\delta} \otimes \{0\} & \text{for } r \gg 0, \\ \{0\} \otimes W_{\tau + \mu + r\delta} & \text{for } r \ll 0, \end{cases} \end{aligned}$$

is zero for $|r| \gg 0$, and hence by (3.4.10) every element of $\mathrm{im}(\Delta_v^\psi)$ has a well-defined action on $V \otimes W$. Furthermore, a quick check verifies that $\Delta_v^\psi(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}$ for all $i \in I$ so every weight space

$$(V \otimes W)_\mu = \sum_{\substack{\gamma+\tau=\mu \\ \gamma \leq \lambda_V \\ \tau \leq \lambda_W}} V_\gamma \otimes W_\tau$$

has only finitely many non-zero summands and is therefore finite dimensional. Since $\Delta_v^\psi(C^{\pm 1}) = C^{\pm 1} \otimes C^{\pm 1}$ we have that $V \otimes W$ is again a type 1 representation. □

The following results are work in progress, and due to time constraints are only presented as conjectures at this stage. However, they are expected to hold via analogous reasoning to Miki's work [87] for $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$.

Conjecture 3.4.8. *The tensor product of irreducible integrable ℓ-highest weight modules should again be an irreducible integrable ℓ-highest weight module, with Drinfeld polynomials equal to the product of those for its factors.*

Conjecture 3.4.9. *For all irreducible integrable ℓ-highest weight modules V and W, there should exist a unique rational function $R_{V,W}(x)$ that intertwines $V \otimes W$ and $W \otimes V$ in a particular way. Furthermore, these $R_{V,W}(x)$ should satisfy various properties after specialising x to certain values, including the Yang-Baxter equation.*

3.4.3 Fock space representations for quantum toroidal algebras

As mentioned earlier, outside type $A_n^{(1)}$ there does not yet exist an algebraic or combinatorial realization of Fock space representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ involving the q-wedge construction of Kashiwara et al. [70, 71] or its associated (crystal or global) basis of Young walls. The work of Feigin-Jimbo-Miwa-Mukhin [26] described towards the beginning of this section suggests the following approach, where at each step everything written with respect to a Young column or Young wall basis:

1. Define a vector representation V_{aff} for $U_q(\mathfrak{g}_{\mathrm{tor}})$.
2. Use the topological coproduct Δ_v to deduce an action on each tensor power $V_{\mathrm{aff}}^{\otimes N}$.
3. Specialise the spectral parameters of Δ_v to obtain the exterior powers $\bigwedge^N V_{\mathrm{aff}}$ as quotients or submodules.

4. Take the limit as $N \to \infty$ along the ground state wall to produce a representation on the Fock space $\mathcal{F}(\lambda)$.

Of course, Section 3.4.1 addresses the first step in this process for all simply laced types except $E_8^{(1)}$. Moreover Δ_v does indeed define an action on $V_{\text{aff}}^{\otimes N}$, however issues arise due to the existence of poles for various expressions. Namely, certain specialisations of the spectral parameters are required for $\bigwedge^N V_{\text{aff}}$ to occur as a quotient or submodule, but in type D (and perhaps also type E) these coincide with the poles.

Hernandez' fusion product [35, 36] for representations of quantum affinizations might be able to solve this problem, since in some sense it functions by removing poles from tensor products. Alternatively, replacing Δ_v with our topological coproduct Δ_v^ψ from Section 3.4.2 could perhaps circumvent the issue entirely as it is *horizontally infinite* while V_{aff} is *vertically infinite*. We plan to return to these ideas in future work.

It is worth noting that Kashiwara [70, §12.3] lists many known properties for Fock spaces representations of $U_q(\hat{\mathfrak{g}})$, such as character formulae and decompositions into irreducible components. If one could sufficiently develop this understanding, it might even be possible to define actions of $U_q(\mathfrak{g}_{\text{tor}})$ directly. Indeed, the quantum toroidal algebras – as affinizations of quantum affine algebras – feel like the 'correct' objects to act on Fock spaces, which decompose as the tensor product of an irreducible highest weight module for $U_q(\hat{\mathfrak{g}})$ with a *bosonic Fock space* acted on naturally by its associated Heisenberg algebra [71, §4.4–§4.5].

Conversely, one might seek to prove that the restrictions to $\mathcal{U}_h$ of Nakajima's geometric representations from [92] satisfy enough of Kashiwara's properties to characterise them as Fock spaces.

Conjecture 3.4.10. *The Fock spaces constructed by Kashiwara-Miwa-Petersen-Yung should possess actions of $U_q(\mathfrak{g}_{\text{tor}})$, at least in the level 1 case. Moreover they should arise geometrically via the work of Nakajima.*

Our automorphism Φ could prove useful since for any level $(0, b)$ representation V of $U_q(\mathfrak{g}_{\text{tor}})$ with $b > 0$, the restriction to $\mathcal{U}_h$ is 'easy' – positive level representations of quantum affine algebras are entirely determined by their characters. On the other hand, twisting by Φ produces a level $(b, 0)$ module and here the action of $\mathcal{U}_v$ is more simple. But V and its twist have a lot in common, such as the underlying vector space and the action of $\mathcal{U}_h \cap \mathcal{U}_v \cong U_q(\mathfrak{g})$, so we should have plenty of control over its structure.

Remark 3.4.11. Any extension of this work past the simply laced case would on the geometry side likely use the recent generalisation of [92] by Varagnolo-Vasserot [108, 109].

Chapter 4

Young wall realizations for crystals of quantum affine algebras in untwisted type E

This chapter focuses on constructing Young wall models for the level 1 irreducible highest weight and Fock space crystals of quantum affine algebras in types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$. In Section 4.1 we introduce a level 1 perfect crystal in each case, together with a description of its energy function. Moreover we confirm that these occur as the crystal bases of good $U_q'(\hat{\mathfrak{g}})$-modules, and can therefore be used to build Fock space crystals $B(\mathcal{F}(\lambda))$ for any level 1 weight $\lambda \in \overline{P}^+$. Finally, we represent each level 1 perfect crystal in terms of equivalence classes of columns stacked inside certain Young column patterns.

Section 4.2 then uses the path realization to identify the crystal bases of all level 1 irreducible highest weight representations as abstract crystals of *reduced* Young walls. Similarly, in Section 4.3 we obtain *proper* Young wall models for the corresponding Fock space crystals. In either case, the relevant energy function condition on adjacent entries of a path is rephrased as a combinatorial condition on adjacent columns in the Young wall. Furthermore, Section 4.3.1 provides some results regarding the structure of these proper and reduced Young walls. We then conclude by discussing some possible applications and future directions following on from this work.

4.1 Level 1 perfect crystals

Figure 4.1 gives the type E untwisted affine Dynkin diagrams, with vertices numbered as in Bourbaki [7, Plates V–VII]. Recall that the canonical central element c is equal to

$\sum_{i\in I} a_i^\vee \alpha_i^\vee$ and hence the level 1 dominant weights in $\overline{P}^+$ are precisely the fundamental weights Λ_i with $i \in I_{\min}$, namely

- Λ_0, Λ_1 and Λ_6 in type $E_6^{(1)}$,
- Λ_0 and Λ_7 in type $E_7^{(1)}$,
- Λ_0 in type $E_8^{(1)}$.

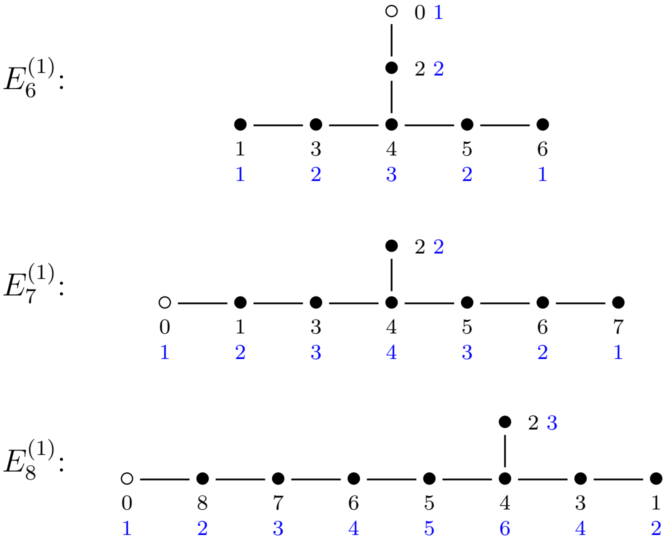


Fig. 4.1 The type E untwisted affine Dynkin diagrams – black labels are vertex numbers, and blue labels are the numerical values $a_i = a_i^\vee$

Let us establish a colouring convention for the arrows in our crystal graphs and the blocks in our Young columns and Young walls. This should make these structures easier to understand and analyse, and their patterns and symmetries easier to recognise. Each label $i \in I$ shall correspond to a particular colour, uniformly across types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$:

0 is black; 1 is red; 2 is yellow; 3 is green; 4 is purple;
5 is blue; 6 is orange; 7 is pink; 8 is brown. (4.1.1)

Let B_6 be the crystal basis of the level 0 fundamental representation $W(\omega_1)$ in type $E_6^{(1)}$, and B_7 be that of $W(\omega_7)$ in type $E_7^{(1)}$. Figures 4.2 and 4.3 contain the associated crystal graphs, with arrows coloured according to (4.1.1). It is easy to verify directly that they are level 1 perfect crystals. (Alternatively, this is a consequence of [38, Theorem 5.1] since they are examples of Kirillov-Reshetikhin crystals $B^{r,s}$ with r minuscule.) So via the path realization of Theorem 2.3.6 these crystals can be used to construct the crystal basis $B(\lambda)$ for each level 1 weight $\lambda \in \overline{P}^+$. Furthermore, Kashiwara [70] proved that level 0 fundamental representations are good, and hence they can also be used to construct the Fock space crystals $B(\mathcal{F}(\lambda))$ as in Section 2.3.2.

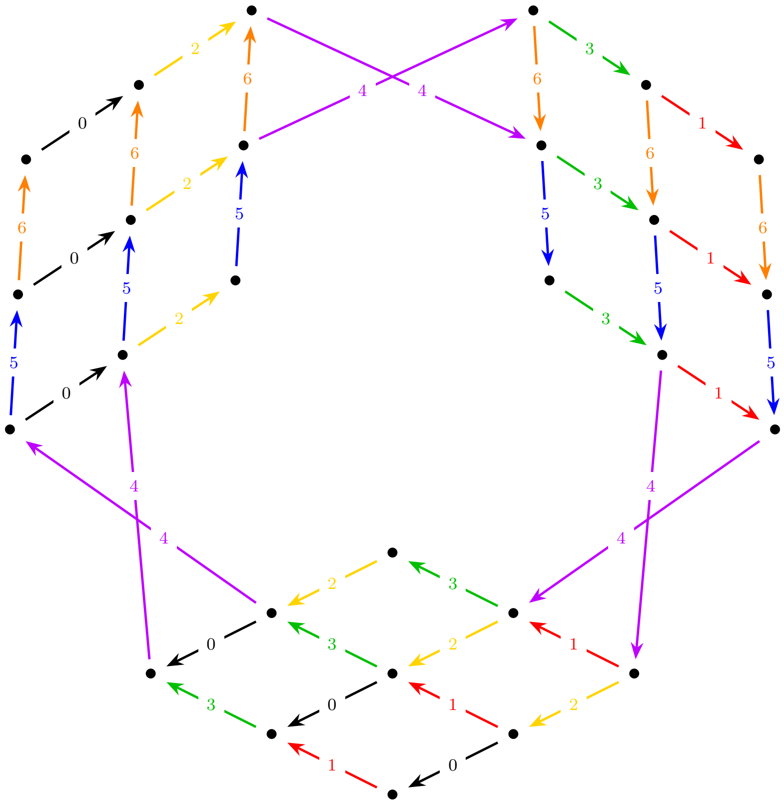


Fig. 4.2 The crystal graph of B_6

The basis elements of B_6 and B_7 can be labelled according to their incident edges as follows, precisely as in Jones-Schilling [54]. In each case, the weight function is injective and all i-strings have length at most 1, so we may denote any element b by $\overline{i_1} \ldots \overline{i_m} j_1 \ldots j_n$ where an $\overline{i}$ (resp. i) records an i-arrow into (resp. out of) b. Without loss of generality we shall always take $i_1 < \ldots < i_m$ and $j_1 < \ldots < j_n$.

Recall from Definition 2.3.4 that in a level 1 perfect crystal, for each $\lambda \in \overline{P}^+$ with $\langle \lambda, c \rangle = 1$ we denote by b^λ and b_λ the unique elements with $\varepsilon(b^\lambda) = \varphi(b_\lambda) = \lambda$.

- For B_6 these are $b^{\Lambda_6} = b_{\Lambda_0} = \overline{6}0$, $b^{\Lambda_1} = b_{\Lambda_6} = \overline{1}6$ and $b^{\Lambda_0} = b_{\Lambda_1} = \overline{0}1$.
- For B_7 these are $b^{\Lambda_7} = b_{\Lambda_0} = \overline{7}0$ and $b^{\Lambda_0} = b_{\Lambda_7} = \overline{0}7$.

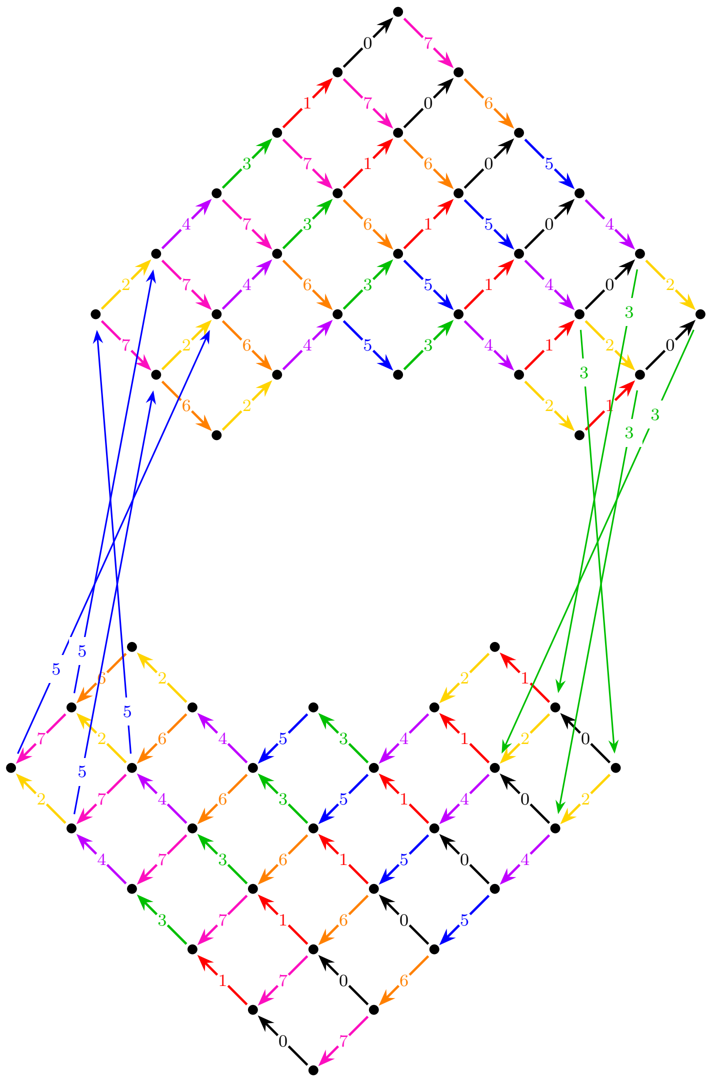


Fig. 4.3 The crystal graph of B_7

Remark 4.1.1.

- The outer automorphism group Ω in type $E_6^{(1)}$ can be seen in the rotational symmetry of B_6 in Figure 4.2. Indeed, twisting $W(\omega_1)$ by the automorphism S_σ of $U_q'(\hat{\mathfrak{g}})$ induced from an anticlockwise rotation $\sigma \in \Omega$ corresponds on the crystal level to the anticlockwise rotation which sends each b_{Λ_i} to $b_{\Lambda_{\sigma(i)}}$.
- Similarly, twisting $W(\omega_7)$ by the automorphism of $U_q'(\hat{\mathfrak{g}})$ induced from the reflection of the $E_7^{(1)}$ Dynkin diagram corresponds to rotating B_7 by 180° around the centre of Figure 4.3.

4.1.1 The level 1 perfect crystals of Benkart-Frenkel-Kang-Lee

Since in type $E_8^{(1)}$ there are no non-zero minuscule vertices in I, our level 1 perfect crystal must be formed in a different manner. Benkart-Frenkel-Kang-Lee [4] provide a uniform construction of a level 1 perfect crystal for all affine types, which we include here (in the untwisted case only). Denote the sets of roots, positive roots and negative roots of the corresponding finite Lie algebra $\mathfrak{g}$ by Φ, Φ^+ and Φ^- respectively, and recall that $\theta = \sum_{i \in I_0} a_i \alpha_i$ is the highest root in Φ.

Theorem 4.1.2. [4] *The classical crystal B with elements*

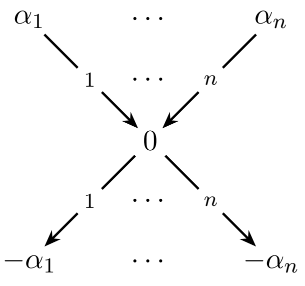
$$\{x_\alpha \mid \alpha \in \Phi^+ \sqcup \Phi^-\} \sqcup \{y_i \mid i \in I_0\} \sqcup \{\emptyset\}$$

and functions wt, $\tilde{e}_i$, $\tilde{f}_i$, ε_i *and* φ_i *given by the crystal graph structure*

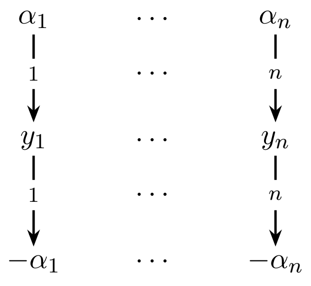
- $x_\alpha \xrightarrow{i} x_\beta$ *whenever* $\alpha - \alpha_i = \beta$ *for some* $\alpha, \beta \in \Phi^+ \sqcup \Phi^-$ *and* $i \in I_0$,
- $x_{\alpha_i} \xrightarrow{i} y_i \xrightarrow{i} x_{-\alpha_i}$ *for all* $i \in I_0$,
- $x_\alpha \xrightarrow{0} x_\beta$ *whenever* $\alpha + \theta = \beta$ *for some* $\alpha, \beta \in (\Phi^+ \sqcup \Phi^-) \setminus \{\pm\theta\}$,
- $x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta$,

is a level 1 *perfect crystal with* $b^{\Lambda_0} = b_{\Lambda_0} = \emptyset$.

Note that if we forget all 0-arrows then we are left with the crystal basis $B(0) \sqcup B(\theta)$ for the finite quantum group $U_q(\mathfrak{g})$. So in all untwisted types, the crystal graph of B is essentially formed by taking the Hasse diagram of the root poset, separating out



into



and adding $\emptyset$ and the 0-arrows. Let $\tilde{I}$ be the set of vertices adjacent to 0 in the affine Dynkin diagram as before, and define

$$\Phi_k^\pm = \{\alpha \in \Phi^\pm \mid \text{the coefficients of } \{\alpha_i\}_{i \in \tilde{I}} \text{ in } \alpha \text{ sum to } \pm k\}$$

for each $k \in \mathbb{N}$. Then there is a decomposition $\Phi^\pm = \Phi_0^\pm \sqcup \Phi_1^\pm \sqcup \Phi_2^\pm$ and the 0-arrows in B can be described as follows.

Lemma 4.1.3. *The* 0*-strings in* B *are precisely* $x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta$ *together with* $x_\alpha \xrightarrow{0} x_{\alpha+\theta}$ *for every* $\alpha \in \Phi_1^-$ *(equivalently* $x_{\beta-\theta} \xrightarrow{0} x_\beta$ *for every* $\beta \in \Phi_1^+$*).*

Proof. This follows from the fact that every 0-string other than $x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta$ is of length at most 1 by construction, and $\langle \alpha_i, \alpha_0^\vee \rangle = -\delta_{i \in \tilde{I}}$ for all $i \in I_0$. ☐

Denote the crystal B in type $E_8^{(1)}$ by B_8, which is in particular equal to the Kirillov-Reshetikhin crystal $B^{8,1}$ of the level 0 fundamental representation $W(\omega_8)$. We include its crystal graph in Appendix C to give an idea of its structure, but to avoid cluttering we remove all vertex and edge labels, as well as all 0-arrows other than $x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta$. However, edges are coloured according to our conventions (4.1.1) and the 0-arrows can easily be deduced from Lemma 4.1.3, together with the vertex icons:

- x_α with $\alpha \in \Phi_0^\pm$ are ✖
- x_α with $\alpha \in \Phi_1^\pm$ are ●
- x_α with $\alpha \in \Phi_2^\pm$ are ★
- $\emptyset$ and y_i with $i \in I_0$ are ■

Next we shall verify that B is the crystal basis of a good $U_q'(\hat{\mathfrak{g}})$ module, and thus in particular can be used to construct the Fock space crystal $B(\mathcal{F}(\Lambda_0))$. We present the untwisted simply laced case only, but the same proof works for all affine types with only minor edits.

As part of their proof that B is a level 1 perfect crystal, Benkart-Frenkel-Kang-Lee [4] show that the space

$$V = \left(\bigoplus_{\beta \in \Phi^+ \sqcup \Phi^-} \mathbb{C}(q) x_\beta \right) \oplus \left(\bigoplus_{j \in I_0} \mathbb{C}(q) y_j \right) \oplus \mathbb{C}(q)\emptyset$$

has the structure of an integrable $U_q'(\hat{\mathfrak{g}})$-module via

$$\begin{aligned}
&q^h \cdot x_\beta = q^{\langle \alpha, h\rangle} x_\beta, \quad q^h \cdot y_j = y_j, \quad q^h \cdot \emptyset = \emptyset,\\
&e_i \cdot x_\beta = \begin{cases} [\varphi_i(x_\beta)+1] x_{\beta+\alpha_i} & \text{if } \beta+\alpha_i \in \Phi^+ \sqcup \Phi^- \\ y_i & \text{if } \beta = -\alpha_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \neq 0),\\
&f_i \cdot x_\beta = \begin{cases} [\varepsilon_i(x_\beta)+1] x_{\beta-\alpha_i} & \text{if } \beta-\alpha_i \in \Phi^+ \sqcup \Phi^- \\ y_i & \text{if } \beta = \alpha_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \neq 0),\\
&e_i \cdot \emptyset = 0, \quad e_i \cdot y_j = [\langle \alpha_i, \alpha_j^\vee \rangle] x_{\alpha_i} \quad (i \neq 0),\\
&f_i \cdot \emptyset = 0, \quad f_i \cdot y_j = [\langle \alpha_i, \alpha_j^\vee \rangle] x_{-\alpha_i} \quad (i \neq 0), \qquad (4.1.2)\\
&e_0 \cdot x_\beta = \begin{cases} x_{\beta-\theta} & \text{if } \beta-\theta \in \Phi^+ \sqcup \Phi^-, \\ \emptyset & \text{if } \beta = \theta, \\ 0 & \text{otherwise,} \end{cases}\\
&f_0 \cdot x_\beta = \begin{cases} x_{\beta+\theta} & \text{if } \beta+\theta \in \Phi^+ \sqcup \Phi^-, \\ \emptyset & \text{if } \beta = -\theta, \\ 0 & \text{otherwise,} \end{cases}\\
&e_0 \cdot \emptyset = [2] x_{-\theta}, \quad e_0 \cdot y_j = 0,\\
&f_0 \cdot \emptyset = [2] x_{-\theta}, \quad f_0 \cdot y_j = 0,
\end{aligned}$$

and moreover that V has crystal basis (L, B) with L the free A-submodule of V generated by $B_V := \{x_\beta, y_j, \emptyset \mid \beta \in \Phi^+ \sqcup \Phi^-, j \in I_0\} \subset V$. As the crystal basis of a finite dimensional integrable $U_q'(\hat{\mathfrak{g}})$-module, B is moreover a *regular* finite classical crystal (see for example [34, p.117]). Note that here we use the notation of Remark 2.2.2 for convenience.

Since an element of a crystal is extremal if and only if it lies at the start or end of every i-string it is contained in, the extremal elements of B are precisely the elements x_β. Then as $\Phi \setminus \{0\}$ is a single Weyl group orbit, by picking $\lambda = \theta$ in Definition 2.3.10 we see that B is simple.

Since $\overline{[n]} = [n]$ for all $n \in \mathbb{Z}$ there is a bar involution $-$ of V which sends $q \to q^{-1}$ and fixes each element of B_V. Then letting $V_\mathbb{Q}$ be the $U_q'(\hat{\mathfrak{g}})_\mathbb{Q}$-submodule of V generated by B_V, it is clear that $\overline{V_\mathbb{Q}} = V_\mathbb{Q}$. From (4.1.2) any $u \in V_\mathbb{Q}$ is a linear combination of elements of B_V where each coefficient is some $f \in \mathbb{Q}[q, q^{-1}]$ multiplied by $[n]^{\pm 1}$ factors. So as $f - \overline{f} \in (q-1)\mathbb{Q}[q, q^{-1}]$ we have $u - \overline{u} \in (q-1)V_\mathbb{Q}$.

It remains to show that $(L, \overline{L}, V_{\mathbb{Q}})$ is a balanced triple. First note that $A \cap \overline{A} \cap \mathbb{Q}[q, q^{-1}] = \mathbb{Q}$ since an element of $\mathbb{Q}[q, q^{-1}]$ is regular at both $q = 0$ and $q = \infty$ if and only if it is constant. Then since $\overline{L}$ is the free $\overline{A}$-module generated by B_V we have that $E = L \cap \overline{L} \cap V_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of B_V.

Now, qL is the qA-span of B_V and qA is the set of $f(q) \in \mathbb{Q}(q)$ without a pole at 0 and with $f(0) = 0$. So L/qL is isomorphic to the (A/qA)-span of B_V. But $A/qA \cong \mathbb{Q}$ via the evaluation-at-0 map, so $E \to L/qL$ via inclusion into L and then projection to L/qL is an isomorphism, and hence $(L, \overline{L}, V_{\mathbb{Q}})$ is balanced.

The preimage of $B \subset L/qL$ in E is a global basis for V, and is precisely equal to B_V. So we may conclude the following.

Proposition 4.1.4. *The level* 1 *perfect crystal of Benkart-Frenkel-Kang-Lee is the crystal basis of a good* $U'_q(\hat{\mathfrak{g}})$*-module in all affine types.*

4.1.2 Energy functions

A quick check verifies that for both B_6 and B_7 the energy function H admits the following nice description.

Lemma 4.1.5. *For all* $a, b \in B_6$ *(resp.* $a, b \in B_7$*),* $H(b \otimes a)$ *is equal to the minimum number of* 0*-arrows in a path* $a \to \cdots \to b$ *in* B_6 *(resp.* B_7*), or equivalently the number of* 0*-arrows in a minimal length path* $a \to \cdots \to b$*.*

In [4] the authors give a description of the energy function for their uniform level 1 perfect crystal. However, in the process of writing the paper [77] we noticed some inaccuracies in their results. Here we provide an updated description in the untwisted case, which in particular includes B_8. We have moreover communicated proofs of these corrections to the authors.

Remark 4.1.6. Aside from the aforementioned differences, our energy function H is obtained from that of [4] by first multiplying by -1 (see Remark 2.3.8), and then adding 2 to renormalise so that $H(b \otimes b) = 0$ for any extremal element $b \in B$.

In all untwisted types, let B be the uniform level 1 perfect crystal of Theorem 4.1.2.

Call an element $b_1 \otimes b_2$ of $B \otimes B$ maximal if $\tilde{e}_i(b_1 \otimes b_2) = 0$ for all $i \in I_0$. We denote by $\mathcal{C}(b_1 \otimes b_2)$ its *classical connected component* in $B \otimes B$ after removing all 0-arrows, on which the energy function H is constant by definition.

Theorem 4.1.7. *The maximal vectors in $B \otimes B$ and their energy functions are as follows:*

- $\emptyset \otimes \emptyset$ *with* $H = 2$ *and classical connected component* $\{\emptyset \otimes \emptyset\}$,
- $x_\theta \otimes x_{-\theta}$ *with* $H = 2$ *and classical connected component* $\{x_\theta \otimes x_{-\theta}\}$,
- $\emptyset \otimes x_\theta$ *with* $H = 1$ *and classical connected component* $\{\emptyset \otimes b \mid b \neq \emptyset\}$,
- $x_\theta \otimes \emptyset$ *with* $H = 1$ *and classical connected component* $\{b \otimes \emptyset \mid b \neq \emptyset\}$,
- $x_\theta \otimes y_i$ *for each* $i \in \tilde{I}$, *with* $H = 2$ *and classical connected components as described below,*
- $x_\theta \otimes x_{\theta - \alpha_i}$ *with* $H = 1$ *for each* $i \in \tilde{I}$,
- $x_\theta \otimes x_\theta$ *with* $H = 0$,
- $x_\theta \otimes x_\beta$ *with* $H = 2$ *for each maximal* $\beta \in \Lambda_0^+$,

and in type $C_n^{(1)}$, since a double arrow connects 0 *to its adjacent vertex,*

- $x_\theta \otimes x_{-\alpha_i}$ *with* $H = 2$ *for* $i \in \tilde{I}$.

We do not provide a complete description of every classical connected component since this will not be required for our work, but do include the case $\mathcal{C}(x_\theta \otimes y_i)$ – for further details and its crystal structure, see the proof of [4, Proposition 5.3]. To this end, define the support of any $\gamma \in \Phi$ to be $\operatorname{supp}(\gamma) = \{j \in I_0 \mid \langle \Lambda_j, \gamma \rangle \neq 0\}$.

Proposition 4.1.8. *Outside types $A_n^{(1)}$ and $C_n^{(1)}$ the component $\mathcal{C}(x_\theta \otimes y_i)$ consists of the elements:*

- $x_\theta \otimes y_i$ *and* $y_i \otimes x_{-\theta}$,
- $x_{\theta-\alpha} \otimes x_{-\beta}$ *and* $x_\beta \otimes x_{-\theta+\alpha}$ *for all* $\gamma \in \Phi^+ \setminus \{\theta\}$, *with* $\alpha = \alpha_{j_1} + \cdots + \alpha_{j_t}$ *and* $\beta = \theta - \gamma - \alpha$, *where* $j_1 \sim \cdots \sim j_t = i$ *is the shortest sequence of vertices in* $I_0 \setminus \operatorname{supp}(\gamma)$ *connecting* $\operatorname{supp}(\gamma)$ *to* i,

- $x_{\theta-\alpha_{j_1}-\cdots-\alpha_{j_t}} \otimes x_{-\theta+\alpha_{j_1}+\cdots+\alpha_{j_t}}$ *for all* $j \in I_0$*, where* $i = j_1 \sim \cdots \sim j_t = j$ *is the shortest sequence of vertices connecting* i *to* j.

In type $A_n^{(1)}$ *the elements of* $\mathcal{C}(x_\theta \otimes y_i)$ *are:*

- $x_\gamma \otimes y_i$ *and* $y_i \otimes x_{-\gamma}$ *for all* $\gamma \in \Phi^+$ *with* $i \in \operatorname{supp}(\gamma)$,
- $x_{\theta-\alpha} \otimes x_{-\beta}$ *and* $x_\beta \otimes x_{-\theta+\alpha}$ *for all* $\gamma \in \Phi^+$ *with* $i \notin \operatorname{supp}(\gamma)$*, with* $\alpha = \alpha_{j_1} + \cdots + \alpha_{j_t}$ *and* $\beta = \theta - \gamma - \alpha$*, where* $j_1 \sim \cdots \sim j_t = i$ *is the shortest sequence of vertices in* $I_0 \setminus \operatorname{supp}(\gamma)$ *connecting* $\operatorname{supp}(\gamma)$ *to* i,
- $y_i \otimes y_i$ *and* $x_{\theta-\alpha_{j_1}-\cdots-\alpha_{j_t}} \otimes x_{-\theta+\alpha_{j_1}+\cdots+\alpha_{j_t}}$ *for all* $j \in I_0 \setminus \{i\}$*, where* $n+1-i = j_1 \sim \cdots \sim j_t = j$ *is the shortest sequence of vertices connecting* $n+1-i$ *to* j.

In type $C_n^{(1)}$ *the elements of* $\mathcal{C}(x_\theta \otimes y_i)$ *are:*

- $x_\gamma \otimes y_i$ *and* $y_i \otimes x_{-\gamma}$ *for all* $\gamma \in \Phi^+$ *with* $i \in \operatorname{supp}(\gamma)$,
- $x_{\gamma+\alpha} \otimes x_{-\alpha}$ *and* $x_\beta \otimes x_{-\theta+\alpha}$ *for all* $\gamma \in \Phi^+$ *with* $i \notin \operatorname{supp}(\gamma)$*, with* $\alpha = \alpha_{j_1} + \cdots + \alpha_{j_t}$ *and* $\beta = \theta - \gamma - \alpha$*, where* $j_1 \sim \cdots \sim j_t = i$ *is the shortest sequence of vertices in* $I_0 \setminus \operatorname{supp}(\gamma)$ *connecting* $\operatorname{supp}(\gamma)$ *to* i,
- $y_i \otimes y_i$ *and* $x_{\alpha_{j_1}+\cdots+\alpha_{j_t}} \otimes x_{-\alpha_{j_1}-\cdots-\alpha_{j_t}}$ *for all* $j \in I_0 \setminus \{i\}$*, where* $i = j_1 \sim \cdots \sim j_t = j$ *is the shortest sequence of vertices connecting* i *to* j.

We do however note that B is a Kirillov-Reshetikhin crystal in each type $X_n^{(1)} \neq A_n^{(1)}$. In particular, it equals $B^{i,s} = B(s\omega_i)$ where i is the unique element in $\tilde{I}$ and $s = -a_{0i}$. Therefore the maximal vectors, together with their energy functions and classical connected components, can easily be computed using SageMath [99].

```
sage: K=crystals.kirillov_reshetikhin.LSPaths(['X',n,1],i,s)
sage: H=K.local_energy_function(K)
sage: K2=crystals.TensorProduct(K,K)
sage: hw=K2.classically_highest_weight_vectors()
sage: for b in hw:
          print("({},{}) {}".format(b[1],b[0],H(b)))
sage: C=K2.subcrystal(generators=[hw[k]],index_set=[1,...,n])
```

Here we reverse the factors `b[0]` and `b[1]` of `b` in the penultimate line of code since by default SageMath uses an alternative tensor crystal structure, where tensor factors are swapped compared to (2.2.3). Type $A_n^{(1)}$ can be dealt with similarly, by using the fact that $B = B(\omega_1 + \omega_n)$ is a connected component of $B^{1,1} \otimes B^{n,1}$.

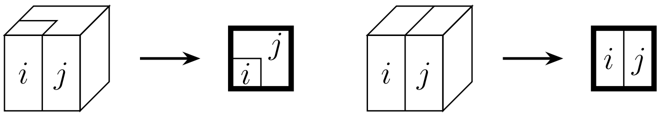
4.1.3 Young column realizations

We are now ready to present for each of our level 1 perfect crystals B_6, B_7 and B_8 a new combinatorial model in terms of *Young columns*. In each case, a Young column pattern splits the infinite vertical column of unit cubes into a particular arrangement of I-coloured blocks. Vertices in our finite crystals then correspond to certain valid stackings of blocks inside this pattern, up to an equivalence relation, and i-arrows correspond to adding / removing an i-coloured block (i-block).

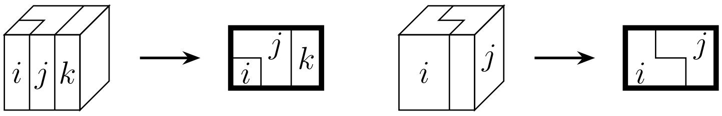
An essential condition for all Young columns (except for two in type $E_8^{(1)}$) is that there is no empty space below any block, which can be thought of as a *stable under gravity* condition. So it is important to make clear exactly how each unit cube is cut into blocks. We therefore introduce the following diagrammatic conventions.

Our unit cubes shall always be split via a collection of vertical cuts, and hence all of the information is contained in the horizontal cross-section at any point. So we shall represent each Young column pattern as a vertical strip formed by lining up the cross-sections of each unit cube, one on top of another. Thicker lines separate the cross-sections of cubes, and thinner lines indicate how each cube is cut.

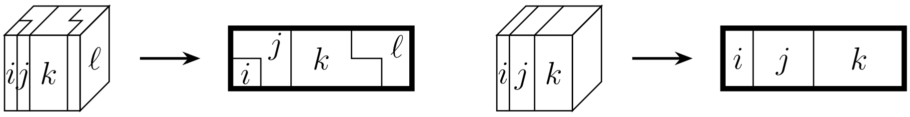
In types $E_7^{(1)}$ and $E_8^{(1)}$ we shall also stretch the cross-section diagrams horizontally by factors of 1.5 and 3 respectively, in order to make our diagrams more clear. This is all demonstrated in Figure 4.4, which contains examples of this process.



(a) Type $E_6^{(1)}$



(b) Type $E_7^{(1)}$



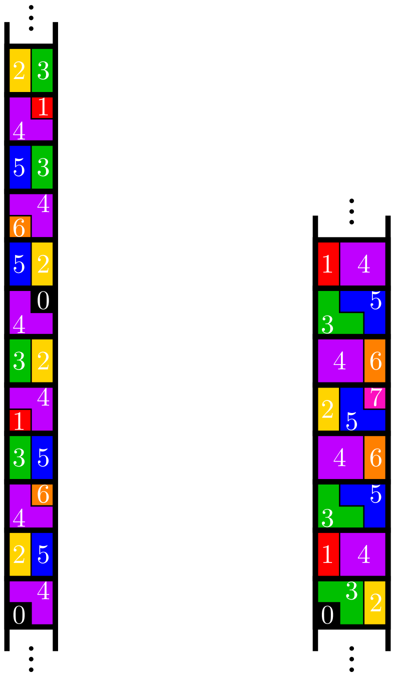
(c) Type $E_8^{(1)}$

Fig. 4.4 Planar notation for split unit cubes

Moreover, blocks in our diagrams shall be coloured according to (4.1.1) when they are contained in the relevant Young column or Young wall, and coloured white if they are not.

4.1.3.1 Types $\mathbf{E}_6^{(1)}$ and $\mathbf{E}_7^{(1)}$

Figure 4.5 contains the Young column patterns for types $E_6^{(1)}$ and $E_7^{(1)}$.



(a) Type $E_6^{(1)}$ **(b)** Type $E_7^{(1)}$

Fig. 4.5 Young column patterns for types $E_6^{(1)}$ and $E_7^{(1)}$

We now define the notion of a Young column inside each pattern, as well as an equivalence relation on these Young columns.

Definition 4.1.9. *In types $E_6^{(1)}$ and $E_7^{(1)}$ a Young column is an collection of blocks inside the Young column pattern such that*

- *the height of the blocks is bounded above,* (4.1.3)
- *there is no empty space below any block.* (4.1.4)

Example 4.1.10. Figure 4.6 contains some examples and non-examples of Young columns. In each type, the first two diagrams represent valid stackings of blocks inside the relevant Young column pattern, while the rightmost column fails (4.1.4).

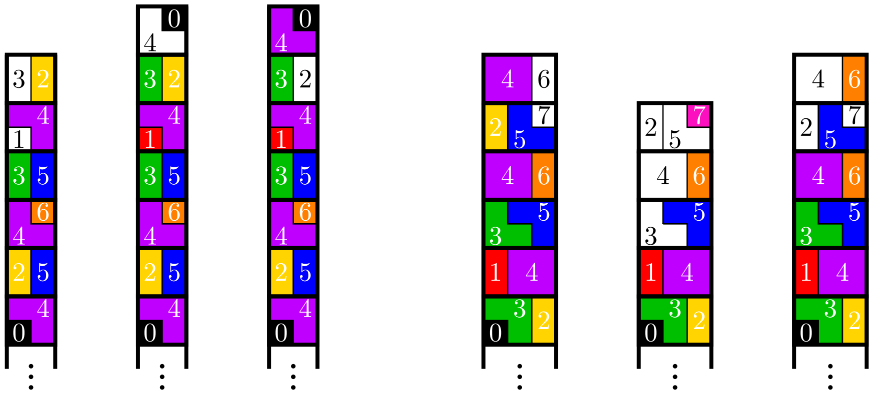

(a) Type $E_6^{(1)}$ **(b)** Type $E_7^{(1)}$

Fig. 4.6 Examples and non-examples of Young columns in types $E_6^{(1)}$ and $E_7^{(1)}$

Definition 4.1.11. *Young columns are equivalent if they can be obtained from one another via vertical shift, or rotation by* 180° *around the vertical axis.*

Note that from Figure 4.5, rotation shall only really be relevant in type $E_6^{(1)}$ at this stage. The set of (equivalence classes of) Young columns can be endowed with a crystal structure as follows.

Definition 4.1.12.
- *An i-block in a Young column y is removable if removing it from y gives another Young column.*
- *An i-block in the Young column pattern that is not in y is addable if adding it to y gives another Young column.*

A quick check then verifies that the following endows the set of Young columns inside each pattern with the structure of an affine crystal.

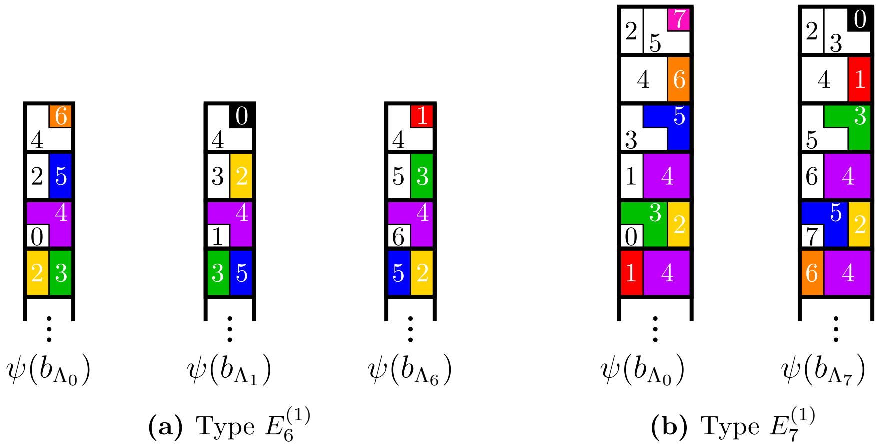
- The $\tilde{f}_i$ act on a Young column y by adding an addable i-block if this exists, and sending to 0 otherwise.
- Similarly, $\tilde{e}_i$ removes a removable i-block from y if possible, and maps to 0 otherwise.
- As in (2.2.2) let $\varphi_i(y) = \max\{n \mid \tilde{f}_i^n y \neq 0\}$ and $\varepsilon_i(y) = \max\{n \mid \tilde{e}_i^n y \neq 0\}$.
- Specify the weight of a particular Young column whose only addable block is a 0-block to be Λ_0. The weight function is then determined by extending via $\mathrm{wt}(\tilde{f}_i y) = \mathrm{wt}(y) - \alpha_i$ if $\tilde{f}_i y \neq 0$ and $\mathrm{wt}(\tilde{e}_i y) = \mathrm{wt}(y) + \alpha_i$ if $\tilde{e}_i y \neq 0$.

Furthermore, by projecting the weights to $\overline{P}$ this descends to a classical crystal structure on the set of equivalence classes of Young columns, which we denote by C_6 and C_7 in types $E_6^{(1)}$ and $E_7^{(1)}$ respectively. It is clear that the affinizations $(C_6)_{\mathrm{aff}}$ and $(C_7)_{\mathrm{aff}}$ recover the sets of Young columns together with the affine crystal structure above.

From Figures 4.2 and 4.3 we see that equivalence classes of Young columns inside our patterns provide us with combinatorial models for our level 1 perfect crystals B_6 and B_7.

Proposition 4.1.13. *There are isomorphisms* $\psi : B_6 \to C_6$ *and* $\psi : B_7 \to C_7$ *of classical crystals.*

Example 4.1.14. Figure 4.7 gives the equivalence classes $\psi(b_{\Lambda_i})$ for all $i \in I_{\min}$ in each type, from which it is easy to deduce the remaining values of ψ simply by adding and removing blocks.



(a) Type $E_6^{(1)}$ **(b)** Type $E_7^{(1)}$

Fig. 4.7 Equivalence classes $\psi(b_{\Lambda_i})$ of Young columns in types $E_6^{(1)}$ and $E_7^{(1)}$

4.1.3.2 Type $\mathbf{E_8^{(1)}}$

Figure 4.8 contains the Young column pattern for type $E_8^{(1)}$.

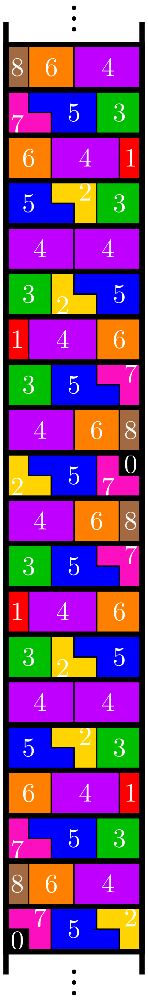


Fig. 4.8 Young column pattern for type $E_8^{(1)}$

The Young columns – together with their equivalence relation, addable blocks and removable blocks – are defined exactly as in Section 4.1.3.1, except that we need to add two extra valid equivalence classes of Young columns. These are displayed in Figure 4.9, and shall correspond to $x_{\pm\alpha_2}$.

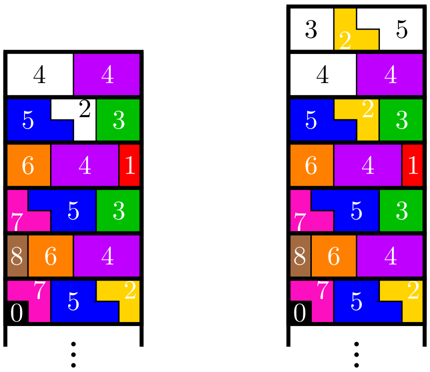


Fig. 4.9 The additional equivalence classes of Young columns in type $E_8^{(1)}$

As in Section 4.1.3.1 the set of Young columns has the structure of an affine crystal, but here we need to be more careful with our definitions of $\tilde{e}_i$ and $\tilde{f}_i$. This is to account for the fact that our B_8 crystal has a more complex structure than B_6 or B_7, for example with i-strings of length greater than 1.

Definition 4.1.15. *The operator $\tilde{f}_i$ acts on a Young column by adding an addable i-block if it exists and mapping to 0 otherwise, with the following caveats.*

- *If there are two addable i-blocks then $\tilde{f}_i$ adds the higher one for $i \neq 4$, and the one lying on top of a 5-block for $i = 4$.*
- *If a Young column y is obtained by adding an i-block to some other Young column under the rules above, and moreover has an addable i-block itself, then $\tilde{f}_j(y) = \tilde{e}_j(y) = 0$ for all $j \neq i$.*

Similarly, $\tilde{e}_i$ acts on a Young column by removing a removable i-block if it exists and mapping to 0 otherwise, with the following caveats.

- *If there are two removable i-blocks then $\tilde{e}_i$ removes the lower one for $i \neq 4$, and the one lying on top of a 3-block for $i = 4$.*
- *If a Young column y is obtained by removing an i-block from some other Young column under the rules above, and moreover has a removable i-block itself, then $\tilde{f}_j(y) = \tilde{e}_j(y) = 0$ for all $j \neq i$.*

Remark 4.1.16. Fan-Han-Kang-Shin similarly require extra conditions for their Young column models in types $D_4^{(3)}$ and $G_2^{(1)}$, given in Definition 3.1 (2) and Remark 3.4 of [25].

As in types $E_6^{(1)}$ and $E_7^{(1)}$ this affine crystal structure descends to a classical crystal structure on the set C_8 of equivalence classes of Young columns, by projecting the weights to $\overline{P}$. And once again, the affinization $(C_8)_{\mathrm{aff}}$ recovers the original affine crystal.

From Appendix C and the definitions above, we see that the equivalence classes of Young columns provide a combinatorial realization of the level 1 perfect crystal B_8.

Proposition 4.1.17. *There is an isomorphism $\psi : B_8 \to C_8$ of classical crystals.*

Notice that Lemma 4.1.3 can therefore be concretely seen in the Young column pattern of Figure 4.8 by looking at the dependency between the 0-blocks and the 1-blocks.

Remark 4.1.18. The caveats in Definition 4.1.15, together with the additional Young columns in Figure 4.9, ensure in particular that our C_8 crystal:

- has well-defined $\tilde{e}_i$ and $\tilde{f}_i$ maps,
- contains i-strings of length 2 corresponding to $x_{\alpha_i} \xrightarrow{i} y_i \xrightarrow{i} x_{-\alpha_i}$ and $x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta$ with no other arrows incident to y_i and $\emptyset$,
- accurately models the B_8 section in Figure 4.10a with Figure 4.11a.

We conclude this section with some example pieces of B_8 and their images under ψ, from which all values of ψ can be derived using the crystal structure on C_8 and Appendix C. These examples should also make clear the role of the additional Young columns from Figure 4.9 within the crystal, as well as the ordering rules for adding and removing blocks.

$$\begin{tikzcd}
 & x_{\alpha_2} \arrow[r, "2"] & y_2 \arrow[r, "2"] & x_{-\alpha_2} \arrow[dr, "4"] & \\
x_{\alpha_2+\alpha_4} \arrow[ur, "4"] \arrow[dr, "2"] & & & & x_{-\alpha_2-\alpha_4} \\
 & x_{\alpha_4} \arrow[r, "4"] & y_4 \arrow[r, "4"] & x_{-\alpha_4} \arrow[ur, "2"] &
\end{tikzcd}$$

(a)

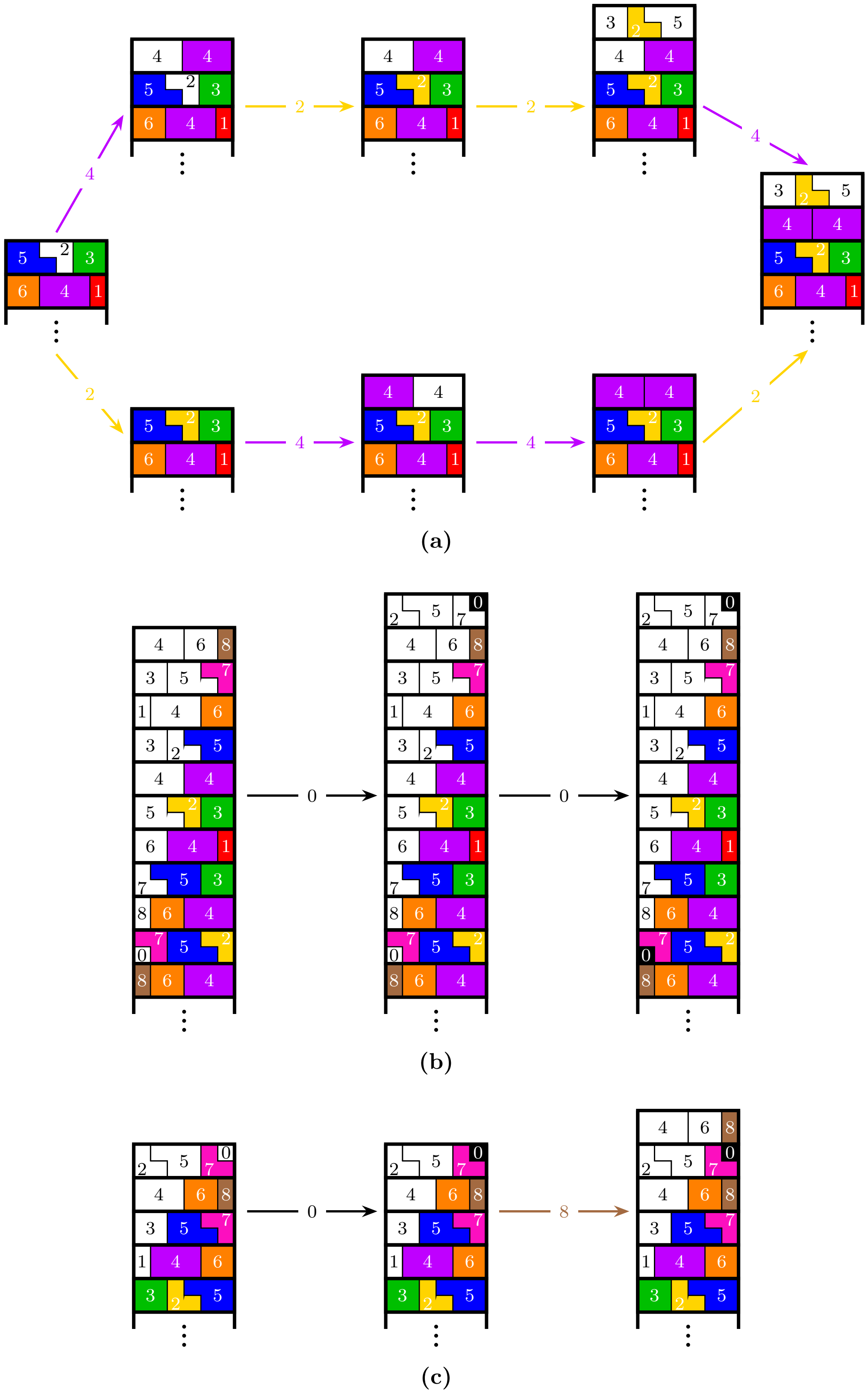
$$\begin{tikzcd}
x_{-\theta} \arrow[r, "0"] & \emptyset \arrow[r, "0"] & x_\theta
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
x_{\beta+\alpha_8-\theta} \arrow[r, "0"] & x_{\beta+\alpha_8} \arrow[r, "8"] & x_\beta
\end{tikzcd}$$

(c)

Fig. 4.10 Example sections of B_8



(a) (b) (c)

Fig. 4.11 Example sections of C_8

4.2 Level 1 irreducible highest weight crystals

Recall the path realization for the level 1 irreducible highest weight crystals from Section 2.3.1, in which the highest weight element $u_\lambda \in B(\lambda)$ corresponds to a ground state sequence $\mathbf{p}_\lambda \in \mathcal{P}(\lambda)$. Table 4.1 gives the sequence $\mathbf{p}_\lambda$ in each case, with entries in the relevant level 1 perfect crystal B_6, B_7 or B_8 from Section 4.1. It also contains the values of the energy function on consecutive factors, which is required for both the affine weight formula (2.3.3) on $\mathcal{P}(\lambda)$ and the ensuing Young wall realization.

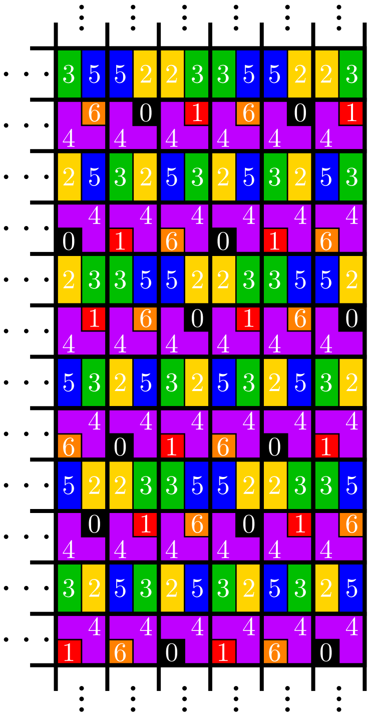
Type	$\lambda \in \overline{P}^+$ of level 1	Ground state sequence $\mathbf{p}_\lambda = (b_r)_{r=0}^{\infty}$	$(H(b_{r+1} \otimes b_r))_{r=0}^{\infty}$
$E_6^{(1)}$	Λ_0	$\cdots \otimes \overline{0}1 \otimes \overline{1}6 \otimes \overline{6}0 \otimes \overline{0}1 \otimes \overline{1}6 \otimes \overline{6}0$	$(\ldots, 0, 2, 2, 0, 2, 2)$
	Λ_1	$\cdots \otimes \overline{1}6 \otimes \overline{6}0 \otimes \overline{0}1 \otimes \overline{1}6 \otimes \overline{6}0 \otimes \overline{0}1$	$(\ldots, 2, 2, 0, 2, 2, 0)$
	Λ_6	$\cdots \otimes \overline{6}0 \otimes \overline{0}1 \otimes \overline{1}6 \otimes \overline{6}0 \otimes \overline{0}1 \otimes \overline{1}6$	$(\ldots, 2, 0, 2, 2, 0, 2)$
$E_7^{(1)}$	Λ_0	$\cdots \otimes \overline{0}7 \otimes \overline{7}0 \otimes \overline{0}7 \otimes \overline{7}0 \otimes \overline{0}7 \otimes \overline{7}0$	$(\ldots, 0, 3, 0, 3, 0, 3)$
	Λ_7	$\cdots \otimes \overline{7}0 \otimes \overline{0}7 \otimes \overline{7}0 \otimes \overline{0}7 \otimes \overline{7}0 \otimes \overline{0}7$	$(\ldots, 3, 0, 3, 0, 3, 0)$
$E_8^{(1)}$	Λ_0	$\cdots \otimes \emptyset \otimes \emptyset \otimes \emptyset \otimes \emptyset \otimes \emptyset \otimes \emptyset$	$(\ldots, 2, 2, 2, 2, 2, 2)$

Table 4.1 Ground state sequences for level 1 irreducible highest weight crystals

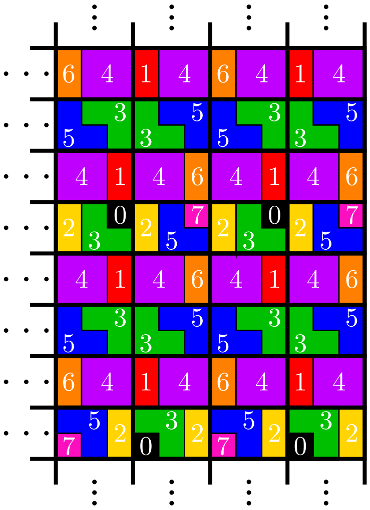
Remark 4.2.1. In type $E_6^{(1)}$ we see that $\mathbf{p}_{\Lambda_6}$ and $\mathbf{p}_{\Lambda_1}$ are just $\mathbf{p}_{\Lambda_0}$ without the first one or two entries respectively, while removing the first entry of $\mathbf{p}_{\Lambda_0}$ in type $E_7^{(1)}$ gives $\mathbf{p}_{\Lambda_7}$. It follows that Young wall models for all level 1 irreducible highest weight crystals $B(\Lambda_i)$ can be obtained by simply removing the first one or two columns from a Young wall model for $B(\Lambda_0)$. For the remainder of this section we shall therefore only deal with the $B(\Lambda_0)$ case, unless stated otherwise.

As shorthand we shall often denote each of the crystals B_6, B_7 and B_8 by B, provided that it is clear which type(s) we are referring to. Similarly, we shall denote each of the crystals C_6, C_7 and C_8 of equivalence classes of Young columns by C.

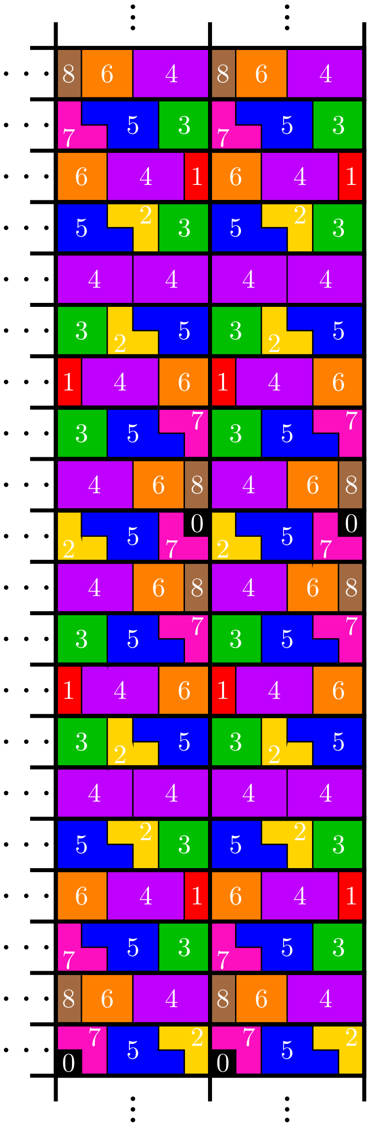
Recall that equivalence classes of Young columns inside the relevant pattern from Figures 4.5 and 4.8 provide a combinatorial model for B, and removing the equivalence relation gives a model for B_{aff}. Pick some representative Young column in the equivalence class $\psi(b_r)$ corresponding to each entry of $\mathbf{p}_{\Lambda_0}$ (given in Figures 4.7 and 4.11b). Then lining them up at the same height and orientation – so they occupy the same spaces within each vertical strip of unit cubes – we obtain the following *Young wall patterns* and *ground state walls.*



(a) Type $E_6^{(1)}$

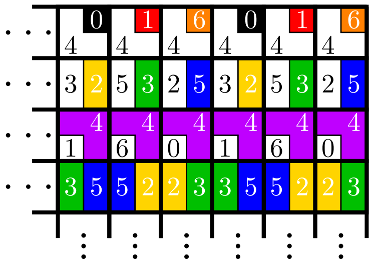


(b) Type $E_7^{(1)}$

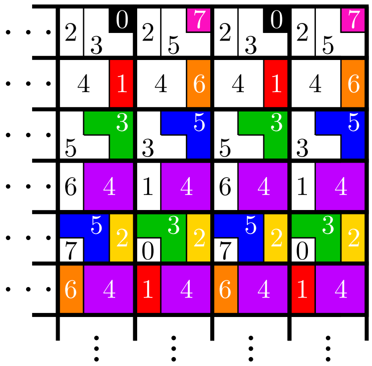


(c) Type $E_8^{(1)}$

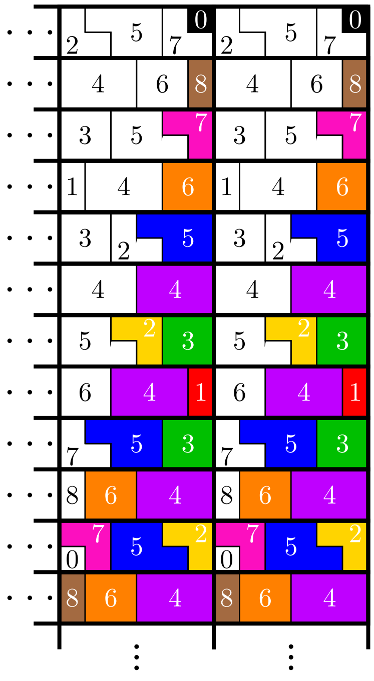
Fig. 4.12 Young wall patterns for types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$



(a) Type $E_6^{(1)}$



(b) Type $E_7^{(1)}$



(c) Type $E_8^{(1)}$

Fig. 4.13 Ground state walls for types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$

Definition 4.2.2. *In each type, a Young wall is a collection of blocks stacked inside the Young wall pattern such that*

- *it differs from the ground state wall in finitely many blocks,*
- *each column of the wall is a Young column.*

Many authors assume two further conditions on their Young walls, the first of which we shall call the *right block property*:

- if a Young wall contains a block, then it must contain the block occupying the same position in the column to the right, (4.2.1)
- a Young wall must be built on top of the ground state wall. (4.2.2)

We have chosen to omit these conditions from our definition since it is not immediately clear that they should should be satisfied by the walls in our models for $B(\lambda)$ and $B(\mathcal{F}(\lambda))$.

Indeed, we shall see in Section 4.3 that they do not hold for every proper Young wall inside our type $E_8^{(1)}$ Fock space model. Note that this is also the case for $E_6^{(2)}$ and $F_4^{(1)}$

[33, §5], as well as for Fock space models in types $G_2^{(1)}$ and $D_4^{(3)}$ that one can derive from the results in [25].

Nevertheless, with Proposition 4.2.5 and Corollary 4.2.6 respectively, we prove that conditions (4.2.1) and (4.2.2) are in fact satisfied by the *reduced* Young walls which form our models for $B(\lambda)$.

Remark 4.2.3. In non-exceptional affine types, the definitions of reduced and proper Young walls given in this thesis produce precisely those of papers such as [56, 60, 61] when written combinatorially. Moreover, conditions (4.2.1) and (4.2.2) do indeed follow in these cases – for both proper and reduced Young walls – as a *consequence* of our definitions (rather than being assumed from the start), and no changes are required to these works.

In this section we write each Young wall Y as a sequence $(y_r)_{r=0}^{\infty}$ of Young columns, considered only up to equivalence class as elements of C. We also define $|y_r|_0$ (resp. $|y_r|$) to be the difference in the number of 0-blocks (resp. blocks) between Y and the ground state wall in column r.

Definition 4.2.4. *A pair of adjacent columns (y_{r+1}, y_r) in a Young wall $Y = (y_r)_{r=0}^{\infty}$ is reduced if*

$$H(y_{r+1} \otimes y_r) - |y_{r+1}|_0 + |y_r|_0 = H(b_{r+1} \otimes b_r). \tag{4.2.3}$$

The Young wall Y is reduced if (y_{r+1}, y_r) is reduced for every $r \in \mathbb{N}$.

In particular, if a pair of adjacent columns (y_{r+1}, y_r) is reduced then there is exactly one option for the integer $|y_r|_0 - |y_{r+1}|_0$. Denote the set of reduced Young walls by $\mathcal{Y}(\Lambda_0)$.

Proposition 4.2.5. *If a Young wall is reduced then it must satisfy the right block property (4.2.1).*

Proof. This follows from Propositions 4.3.7 and 4.3.8 since $\mathcal{Y}(\Lambda_0)$ can be viewed as the set of $(z^{n_r} y_r)_{r=0}^{\infty} \in \mathcal{Z}(\Lambda_0)$ with every $H_{\text{aff}}(z^{n_{r+1}} y_{r+1} \otimes z^{n_r} y_r) = 1$. In particular, a Young wall $(y_r)_{r=0}^{\infty}$ in $\mathcal{Y}(\Lambda_0)$ is precisely the same as the Young wall $(z^{m_r - |y_r|_0} y_r)_{r=0}^{\infty}$ in $\mathcal{Z}(\Lambda_0)$, where the m_r are given by the ground state sequence $\mathbf{s}_{\Lambda_0} = (z^{m_r} b_r)_{r=0}^{\infty}$ for the Fock space. □

Corollary 4.2.6. *Every reduced Young wall is built on top of the ground state wall.*

Proof. Since a Young wall differs from the ground state wall in finitely many blocks and thus matches it in all columns sufficiently far to the left, this condition follows from the right block property. □

Proposition 4.2.7. *If a pair of adjacent columns* (y_{r+1}, y_r) *in a Young wall is reduced then* $|y_r| - |y_{r+1}|$ *is a fixed non-negative integer.*

Proof. It is clear from Definition 4.2.4 that if we specify the columns y_{r+1} and y_r (up to equivalence), then there is precisely one value of $|y_r|_0 - |y_{r+1}|_0$ which makes (y_{r+1}, y_r) reduced. By looking at the Young column patterns from Figures 4.5 and 4.8 we see that this in turn fixes $|y_r| - |y_{r+1}|$. The right block property of Proposition 4.2.5 implies that this number must be non-negative. □

It therefore follows that up to vertical shift, there are precisely $|B|^2$ pairs of reduced adjacent columns (y_{r+1}, y_r) for each $r \in \mathbb{N}$, one for each choice of equivalence class for both y_{r+1} and y_r. However, it is important to note that in types $E_6^{(1)}$ and $E_7^{(1)}$ the options change with r since from Table 4.1 we see that the sequence $(H(b_{r+1} \otimes b_r))_{r=0}^{\infty}$ is not constant. Figure 4.14 demonstrates these phenomena with some examples in type $E_6^{(1)}$. The first wall is reduced, but the second is not since it does not have the required value of $|y_r|_0 - |y_{r+1}|_0$ (or equivalently $|y_r| - |y_{r+1}|$). The third wall shows how the reduced adjacent pairs (y_{r+1}, y_r) depend on r.

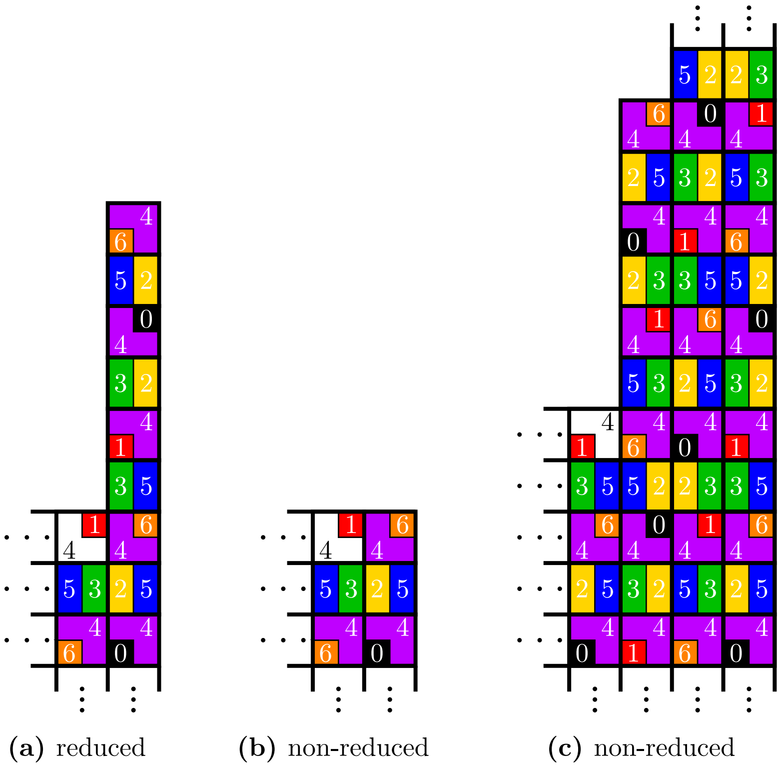

(a) reduced **(b)** non-reduced **(c)** non-reduced

Fig. 4.14 Examples of Young walls in type $E_6^{(1)}$

Next we define the structure of an affine crystal on the set $\mathcal{Y}(\Lambda_0)$ of reduced Young walls. Recall that $\varphi_i(y)$ (resp. $\varepsilon_i(y)$) is by definition the maximum number of i-blocks which can be added to (resp. removed from) a Young column y sequentially, while still remaining a Young column.

Definition 4.2.8. *The i-signature of y is the sequence* $\operatorname{sign}_i(y) = \underbrace{-\cdots-}_{\varepsilon_i(y)}\underbrace{+\cdots+}_{\varphi_i(y)}$.

For a Young wall $Y = (y_r)_{r=0}^{\infty}$ we define the pre-i-signature to be the (possibly infinite) sequence

$$\text{pre-sign}_i(Y) = \ldots \operatorname{sign}_i(y_2)\operatorname{sign}_i(y_1)\operatorname{sign}_i(y_0)$$

of $+$'s and $-$'s. Cancelling every $+-$ pair leaves a finite number of $-$'s followed by a finite number of $+$'s, which we call the i-signature $\operatorname{sign}_i(Y)$ of Y.

Let $\tilde{e}_i(Y)$ be the Young wall obtained from Y by applying $\tilde{e}_i$ to the column containing the rightmost $-$ in $\operatorname{sign}_i(Y)$ if it exists, and 0 otherwise. Conversely, $\tilde{f}_i(Y)$ is defined by applying $\tilde{f}_i$ to the column containing the leftmost $+$ in $\operatorname{sign}_i(Y)$ if it exists, and is 0 otherwise.

Proposition 4.2.9. *For any $Y \in \mathcal{Y}(\Lambda_0)$ we have $\tilde{e}_i(Y), \tilde{f}_i(Y) \in \mathcal{Y}(\Lambda_0) \sqcup \{0\}$.*

Proof. Let $Y = (y_r)_{r=0}^{\infty}$ be a reduced Young wall. If $\tilde{f}_i(Y) = 0$ then we are done, so instead suppose that $\tilde{f}_i(Y) = (\ldots, y_{k+1}, z_k, y_{k-1}, \ldots, y_0)$ where z_k is obtained by adding an i-block to y_k. Both $\text{pre-sign}_i(Y)$ and $\text{sign}_i(Y)$ have at least one $+$ in column k, and no $+$ in column $k+1$. In $\text{pre-sign}_i(Y)$ there are strictly fewer $-$'s in column $k-1$ than $+$'s in column k, and hence there is no $-$ in column $k-1$ of $\text{sign}_i(Y)$. From the formulae (2.2.3) for the tensor product of crystals, we therefore have

$$\tilde{f}_i(z^{-|y_{k+1}|_0} y_{k+1} \otimes z^{-|y_k|_0} y_k) = z^{-|y_{k+1}|_0} y_{k+1} \otimes z^{-|z_k|_0} z_k,$$
$$\tilde{f}_i(z^{-|y_k|_0} y_k \otimes z^{-|y_{k-1}|_0} y_{k-1}) = z^{-|z_k|_0} z_k \otimes z^{-|y_{k-1}|_0} y_{k-1}.$$

By Lemma 2.3.9 it follows that

$$H_{\text{aff}}(z^{-|y_{k+1}|_0} y_{k+1} \otimes z^{-|z_k|_0} z_k) = H_{\text{aff}}(z^{-|y_{k+1}|_0} y_{k+1} \otimes z^{-|y_k|_0} y_k),$$
$$H_{\text{aff}}(z^{-|y_k|_0} y_k \otimes z^{-|y_{k-1}|_0} y_{k-1}) = H_{\text{aff}}(z^{-|z_k|_0} z_k \otimes z^{-|y_{k-1}|_0} y_{k-1}).$$

From equation (2.3.2) which defines H_{aff} we can deduce that since (y_{k+1}, y_k) and (y_k, y_{k-1}) are reduced, so are (y_{k+1}, z_k) and (z_k, y_{k-1}). All other pairs of adjacent columns in $\tilde{f}_i(Y)$ are the same as those in Y and thus satisfy (4.2.3), hence $\tilde{f}_i(Y)$ is a reduced Young wall. The case of $\tilde{e}_i(Y)$ is similar. □

If we further define

- $\varepsilon_i(Y) =$ number of $-$'s in $\text{sign}_i(Y)$
- $\varphi_i(Y) =$ number of $+$'s in $\text{sign}_i(Y)$
- $\text{wt}(Y) = \Lambda_0 - \sum_{i \in I} \kappa_i \alpha_i$

where κ_i is the number of i-blocks in Y added to the ground state wall, then a routine check proves the following.

Theorem 4.2.10. *The maps $\tilde{e}_i, \tilde{f}_i : \mathcal{Y}(\Lambda_0) \to \mathcal{Y}(\Lambda_0) \sqcup \{0\}$, $\varepsilon_i, \varphi_i : \mathcal{Y}(\Lambda_0) \to \mathbb{Z} \cup \{-\infty\}$ and $\text{wt} : \mathcal{Y}(\Lambda_0) \to P$ above endow $\mathcal{Y}(\Lambda_0)$ with the structure of an affine crystal.*

Note that it follows from Corollary 4.2.6 and the definitions of $\tilde{e}_i$ and $\tilde{f}_i$ that the ground state wall is the highest weight element of $\mathcal{Y}(\Lambda_0)$. As an example, we have included the top part of the crystal $\mathcal{Y}(\Lambda_0)$ in type $E_6^{(1)}$ in Appendix D.

Theorem 4.2.11. *There is an isomorphism of affine crystals* $\mathcal{Y}(\Lambda_0) \xrightarrow{\sim} B(\Lambda_0)$ *sending the ground state wall to the highest weight element* u_{Λ_0}.

Proof. From the path realization, we only need to prove that $\mathcal{Y}(\Lambda_0)$ and $\mathcal{P}(\Lambda_0)$ are isomorphic. Define a map $\Psi : \mathcal{Y}(\Lambda_0) \to \mathcal{P}(\Lambda_0)$ by $(y_r)_{r=0}^{\infty} \mapsto (\psi^{-1}(y_r))_{r=0}^{\infty}$ where $\psi : B \xrightarrow{\sim} C$ is as in Propositions 4.1.13 and 4.1.17. The ground state wall is clearly sent to the ground state sequence $\mathbf{p}_{\Lambda_0}$, and by comparing the crystal structure of $\mathcal{Y}(\Lambda_0)$ with (2.2.3) it is easy to see that Ψ commutes with the actions of all $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i and wt.

If reduced Young walls $(y_r)_{r=0}^{\infty}$ and $(z_r)_{r=0}^{\infty}$ in $\mathcal{Y}(\Lambda_0)$ are mapped by Ψ to the same sequence in $\mathcal{P}(\Lambda_0)$, then y_r and z_r must represent the same Young column equivalence class for all $r \in \mathbb{N}$. But since both Young walls are reduced, all $|y_r| - |y_{r+1}| = |z_r| - |z_{r+1}|$ by Proposition 4.2.7. So as $|y_r| = |z_r| = 0$ for $r \gg 0$ it follows that $|y_r| = |z_r|$ for all $r \in \mathbb{N}$ and hence Ψ is injective.

Given a path $\mathbf{p} = (p_r)_{r=0}^{\infty}$ in $\mathcal{P}(\Lambda_0)$ let k be maximal with $p_k \neq b_k$. We can form a reduced Young wall $(y_r)_{r=0}^{\infty}$ by first specifying that it matches the ground state wall after column k, and then recursively from $r = k$ down to $r = 0$ choosing y_r so that (y_{r+1}, y_r) is reduced and $\psi^{-1}(y_r) = p_r$. Such a choice exists and is unique by Proposition 4.2.7. Moreover, Ψ maps this Young wall to $\mathbf{p}$ by construction. So Ψ is surjective and our proof is complete. □

Remark 4.2.12. As mentioned in Remark 4.2.1, we have analogues of all of these results for the other level 1 dominant weights Λ_i simply by removing one or two columns from the Young wall pattern and ground state wall. The only other edit is to replace Λ_0 with Λ_i in the definition of the weight function.

4.3 Level 1 Fock space crystals

Recall the following preliminaries from Section 2.3.2.

- Starting with a good $U_q'(\hat{\mathfrak{g}})$-module with level 1 perfect crystal basis B, we can construct a Fock space representation $\mathcal{F}(\Lambda_i)$ of $U_q(\hat{\mathfrak{g}})$ for each level 1 dominant integral weight $\Lambda_i \in \overline{P}^{+}$.

- There is an associated ground state sequence $\mathbf{s}_{\Lambda_i} = (g_r)_{r=0}^{\infty} = (z^{m_r} b_r)_{r=0}^{\infty}$ in B_{aff}, where $\mathbf{p}_{\Lambda_i} = (b_r)_{r=0}^{\infty}$ is the ground state sequence for $B(\Lambda_i)$ and the $m_r \in \mathbb{Z}$ are chosen so that all $H_{\mathrm{aff}}(z^{m_{r+1}} b_{r+1} \otimes z^{m_r} b_r) = 1$.

- A sequence $\mathbf{s} = (s_r)_{r=0}^{\infty}$ in B_{aff} is normally ordered if all $H(s_{r+1} \otimes s_r) > 0$.

- The crystal basis $B(\mathcal{F}(\Lambda_i))$ consists of the normally ordered sequences $\mathbf{s} = (s_r)_{r=0}^{\infty}$ in B_{aff} with $s_r = g_r$ for $r \gg 0$, and has an affine crystal structure given by (2.3.1) and (2.3.3).

As in Section 4.2 we will often denote the crystals B_6, B_7 and B_8 by B and the crystals C_6, C_7 and C_8 by C, provided it is clear which type(s) we are referring to. From the ground state sequences $\mathbf{p}_\lambda$ and energy function values in Table 4.1, together with equation (2.3.2), we derive ground state sequences $\mathbf{s}_\lambda$ in each case.

Type	$\lambda \in \overline{P}^+$ of level 1	Ground state sequence $\mathbf{s}_\lambda = (g_r)_{r=0}^{\infty}$
$E_6^{(1)}$	Λ_0	$\cdots \otimes z^{-3}\overline{0}1 \otimes z^{-2}\overline{1}6 \otimes z^{-1}\overline{6}0 \otimes z^{-2}\overline{0}1 \otimes z^{-1}\overline{1}6 \otimes z^{0}\overline{6}0$
	Λ_1	$\cdots \otimes z^{-3}\overline{1}6 \otimes z^{-2}\overline{6}0 \otimes z^{-3}\overline{0}1 \otimes z^{-2}\overline{1}6 \otimes z^{-1}\overline{6}0 \otimes z^{-2}\overline{0}1$
	Λ_6	$\cdots \otimes z^{-2}\overline{6}0 \otimes z^{-3}\overline{0}1 \otimes z^{-2}\overline{1}6 \otimes z^{-1}\overline{6}0 \otimes z^{-2}\overline{0}1 \otimes z^{-1}\overline{1}6$
$E_7^{(1)}$	Λ_0	$\cdots \otimes z^{-4}\overline{0}7 \otimes z^{-2}\overline{7}0 \otimes z^{-3}\overline{0}7 \otimes z^{-1}\overline{7}0 \otimes z^{-2}\overline{0}7 \otimes z^{0}\overline{7}0$
	Λ_7	$\cdots \otimes z^{-3}\overline{7}0 \otimes z^{-4}\overline{0}7 \otimes z^{-2}\overline{7}0 \otimes z^{-3}\overline{0}7 \otimes z^{-1}\overline{7}0 \otimes z^{-2}\overline{0}7$
$E_8^{(1)}$	Λ_0	$\cdots \otimes z^{-5}\emptyset \otimes z^{-4}\emptyset \otimes z^{-3}\emptyset \otimes z^{-2}\emptyset \otimes z^{-1}\emptyset \otimes z^{0}\emptyset$

Table 4.2 Ground state sequences for level 1 Fock space crystals

Remark 4.3.1. As in Remark 4.2.1, Young wall models for all other Fock space crystals $B(\mathcal{F}(\Lambda_i))$ can be obtained by removing the first one or two columns from the model for $B(\mathcal{F}(\Lambda_0))$, and so we only need to consider $B(\mathcal{F}(\Lambda_0))$ for the remainder of this section.

Recall that Young columns stacked inside the patterns of Figures 4.5 and 4.8 provide combinatorial models for B_{aff}. Lining up each Young column $\psi_{\mathrm{aff}}(g_r)$ at the same height and orientation (so they occupy the same spaces within each vertical strip) we obtain the *Young wall pattern* and *ground state wall* for $\mathcal{F}(\Lambda_0)$ in each type, which are precisely the same as those in Figures 4.12 and 4.13.

Young walls in the Fock space situation are defined exactly as in Definition 4.2.2. But here we write them as sequences $(z^{n_r}y_r)_{r=0}^{\infty}$ where all $y_r \in C$ and $n_r \in \mathbb{Z}$, and hence each $z^{n_r}y_r$ is a Young column (not up to equivalence) in C_{aff}. Let $\mathcal{Z}(\Lambda_0)$ be the set of Young walls $(z^{n_r}y_r)_{r=0}^{\infty}$ with all $H_{\mathrm{aff}}(z^{n_{r+1}}y_{r+1} \otimes z^{n_r}y_r) > 0$, which we shall call *proper* Young walls. It is clear that every $\mathbf{s} = (s_r)_{r=0}^{\infty}$ in $B(\mathcal{F}(\Lambda_0))$ can be represented by an element of $\mathcal{Z}(\Lambda_0)$ since each $s_r \in B_{\mathrm{aff}}$ and only finitely many $s_r \neq g_r$.

Moreover $\mathcal{Z}(\Lambda_0)$ has an affine crystal structure defined exactly as for $\mathcal{Y}(\Lambda_0)$ in Section 4.2, using the same notions of pre-i-signatures and i-signatures:

- $\tilde{e}_i$ acts on the column corresponding to the rightmost $-$ in $\mathrm{sign}_i(Y)$
- $\tilde{f}_i$ acts on the column corresponding to the leftmost $+$ in $\mathrm{sign}_i(Y)$
- $\varepsilon_i(Y) =$ number of $-$'s in $\mathrm{sign}_i(Y)$
- $\varphi_i(Y) =$ number of $+$'s in $\mathrm{sign}_i(Y)$
- $\mathrm{wt}(Y) = \Lambda_0 - \sum_{i \in I} \kappa_i \alpha_i$

where κ_i is the difference in the number of i-blocks between Y and the ground state wall. A very similar proof to Proposition 4.2.9 shows that $\tilde{e}_i$ and $\tilde{f}_i$ either map to 0 or preserve the values of $H_{\mathrm{aff}}(z^{n_{r+1}}y_{r+1} \otimes z^{n_r}y_r)$, and thus $\tilde{e}_i, \tilde{f}_i : \mathcal{Z}(\Lambda_0) \to \mathcal{Z}(\Lambda_0) \sqcup \{0\}$. Furthermore, the same routine check as for $\mathcal{Y}(\Lambda_0)$ verifies the following.

Theorem 4.3.2. *The maps* $\tilde{e}_i, \tilde{f}_i : \mathcal{Z}(\Lambda_0) \to \mathcal{Z}(\Lambda_0) \sqcup \{0\}$, $\varepsilon_i, \varphi_i : \mathcal{Z}(\Lambda_0) \to \mathbb{Z} \cup \{-\infty\}$ *and* $\mathrm{wt} : \mathcal{Z}(\Lambda_0) \to P$ *above endow* $\mathcal{Z}(\Lambda_0)$ *with the structure of an affine crystal.*

And indeed, this provides a combinatorial Young wall model for the Fock space crystal $B(\mathcal{F}(\Lambda_0))$.

Theorem 4.3.3. *There is an isomorphism of affine crystals* $\mathcal{Z}(\Lambda_0) \xrightarrow{\sim} B(\mathcal{F}(\Lambda_0))$ *sending the ground state wall to the ground state sequence* $\mathbf{s}_{\Lambda_0}$.

Proof. Let $\psi_{\mathrm{aff}} : B_{\mathrm{aff}} \xrightarrow{\sim} C_{\mathrm{aff}}$ be the isomorphism of affine crystals coming from the isomorphism $\psi : B \xrightarrow{\sim} C$ in Proposition 4.1.13 or 4.1.17. From the definition of $\mathcal{Z}(\Lambda_0)$ it is clear that $(s_r)_{r=0}^{\infty} \mapsto (\psi_{\mathrm{aff}}(s_r))_{r=0}^{\infty}$ is a bijection from $B(\mathcal{F}(\Lambda_0))$ to $\mathcal{Z}(\Lambda_0)$ which sends the ground state sequence $\mathbf{s}_{\Lambda_0}$ to the ground state wall. Furthermore, by comparing the crystal structure of $\mathcal{Z}(\Lambda_0)$ with (2.3.1) and (2.3.3) it is easy to see that this commutes with all $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i and wt. $\square$

The next proposition rephrases the normally ordered condition for elements of $B(\mathcal{F}(\Lambda_0))$ as a combinatorial condition on the set of Young walls.

Proposition 4.3.4. *A Young wall $Y = (z^{n_r} y_r)_{r=0}^{\infty}$ is proper if and only if*

$$|z^{n_r} y_r|_0 > |z^{n_{r+1}} y_{r+1}|_0 - H(y_{r+1} \otimes y_r) + m_r - m_{r+1} \tag{4.3.1}$$

for all $r \in \mathbb{N}$, where $|z^{n_r} y_r|_0$ is the difference in the number of 0-blocks between $z^{n_r} y_r$ and column r of the ground state wall.

Proof. Since adding a 0-block to a Young column corresponds to applying $\tilde{f}_0$ we see that $|z^{n_r} y_r|_0 = m_r - n_r$. Putting this into the normally ordered condition and applying (2.3.2) exactly gives condition (4.3.1). $\square$

Recall that Proposition 2.3.17 embeds $B(\Lambda_0)$ into $B(\mathcal{F}(\Lambda_0))$ as the set of $(s_r)_{r=0}^{\infty}$ with all $H_{\text{aff}}(s_{r+1} \otimes s_r) = 1$. In the combinatorial language above, since $m_r - m_{r+1} = H(b_{r+1} \otimes b_r) - 1$ by definition of $\mathbf{s}_{\Lambda_0}$ this precisely becomes the condition (4.2.3) for a Young wall to be reduced. So in the Young wall setting this embedding is simply the identity.

The other interpretation of $B(\Lambda_0)$ as those $(s_r)_{r=0}^{\infty} \in B(\mathcal{F}(\Lambda_0))$ such that applying z to any entry produces a sequence which no longer lies in $B(\mathcal{F}(\Lambda_0))$ can also be seen combinatorially.

Definition 4.3.5. *A δ-column is a continuous piece of the Young column pattern consisting of a_i many i-blocks for each $i \in I$.*

By continuous we mean that if any part of some block lies vertically between two blocks inside the δ-column, then it must also be part of the δ-column. It is clear that removing a δ-column from a Young column precisely corresponds to applying z.

Definition 4.3.6. *A δ-column lying inside a Young wall $Y \in \mathcal{Z}(\Lambda_0)$ is removable if removing it from Y produces another Young wall in $\mathcal{Z}(\Lambda_0)$.*

Therefore $B(\Lambda_0)$ can further be viewed as those elements of $\mathcal{Z}(\Lambda_0)$ without a removable δ-column.

4.3.1 Analysing the structure of proper Young walls

The purpose of this subsection is to investigate in more detail the structure of those Young walls lying inside the crystal $\mathcal{Z}(\Lambda_0)$. In particular, we prove the following propositions.

Proposition 4.3.7. *In types $E_6^{(1)}$ and $E_7^{(1)}$ a proper Young wall must satisfy the right block property (4.2.1).*

A slightly weaker result holds in type $E_8^{(1)}$. Consider the following *local right block property* for a pair of adjacent columns in a Young wall:

· if the Young wall contains a block in column $r+1$ then it contains the block occupying the same position in column r. (4.3.2)

Proposition 4.3.8. *In type $E_8^{(1)}$ a Young wall $(z^{n_r}y_r)_{r=0}^{\infty} \in \mathcal{Z}(\Lambda_0)$ satisfies condition (4.3.2) on columns $r+1$ and r whenever $H_{\mathrm{aff}}(z^{n_{r+1}}y_{r+1} \otimes z^{n_r}y_r) \neq 2$.*

Furthermore, we can describe exactly which possible pairs of adjacent columns in a proper Young wall *do* in fact fail condition (4.3.2). The proof requires the following technical lemma.

Lemma 4.3.9. *In type $E_8^{(1)}$ there is a path $z^n a \to \cdots \to z^{n-1}a$ in B_{aff} if and only if $a \neq \emptyset, x_{\pm\theta}$.*

Proof. For $a = x_\alpha$ with $\alpha \in \Phi_1^+$ we can take

$$z^n x_\alpha \to \cdots \to z^n x_{\alpha_8} \to z^n y_8 \to z^n x_{-\alpha_8} \xrightarrow{0} z^{n-1}x_{\theta-\alpha_8} \to \cdots \to z^{n-1}x_\alpha$$

by Lemma 4.1.3, and similarly for $\alpha \in \Phi_1^-$. If $\alpha \in \Phi_0^\pm$ this is just a quick check using the crystal graph of B_8 in Appendix C, while each $a = y_i$ follows from the case $a = x_{\alpha_i}$. □

Proposition 4.3.10. *In type $E_8^{(1)}$ a proper Young wall $(z^{n_r}y_r)_{r=0}^{\infty} \in \mathcal{Z}(\Lambda_0)$ fails condition (4.3.2) on columns $r+1$ and r if and only if $(z^{n_{r+1}}y_{r+1}, z^{n_r}y_r)$ corresponds to one of $(z^n x_\emptyset, z^{n-1}x_\emptyset)$, $(z^n x_\emptyset, z^{n-2}x_\theta)$, $(z^n x_{-\theta}, z^{n-2}x_\emptyset)$ or $(z^n x_{-\theta}, z^{n-3}x_\theta)$ for some $n \in \mathbb{Z}$.*

Proof. Denote $\psi^{-1}(y_r)$ and $\psi^{-1}(y_{r+1})$ by $a, b \in B$ respectively. We see from the proof of Proposition 4.3.8 that if condition (4.3.2) fails on columns $r+1$ and r then $n_r = n_{r+1} + H(b \otimes a) - 2$ and there does not exist a path $z^0 b \to \cdots \to z^{H(b\otimes a)-3} a$ in B_{aff}. However, there is always a path $z^0 b \to \cdots \to z^{H(b\otimes a)-2} a$ in B_{aff} and so by Lemma 4.3.9 we must have $a, b \in \{\emptyset, \pm\theta\}$. It is then trivial to confirm that the options for $b \otimes a$ are precisely $\emptyset \otimes \emptyset$, $\emptyset \otimes x_\theta$, $x_{-\theta} \otimes \emptyset$, $x_{-\theta} \otimes x_\theta$ and the result follows. ☐

In each type, similarly to Corollary 4.2.6 we can easily deduce the following.

Corollary 4.3.11. *In types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ every proper Young wall is built on top of the ground state wall.*

Proof. This follows from Propositions 4.3.7 and 4.3.8, together with Proposition 4.3.10, since a Young wall differs from the ground state wall in finitely many blocks, and thus matches it in all columns sufficiently far to the left. ☐

Let us conclude this section with the proofs of Propositions 4.3.7 and 4.3.8. In order to do this, we fix $r \in \mathbb{N}$ and introduce the following automorphism of C_{aff} as an *unlabelled digraph*. Each Young column is viewed inside column $r+1$ of the Young wall pattern, and sent to the column with blocks in the same positions but in column r. For example, the ground state column $\psi_{\mathrm{aff}}(g_{r+1})$ is sent to $\psi_{\mathrm{aff}}(g_r)$. By inspecting the Young wall patterns in Figure 4.12 it is clear that this map is independent of r.

Conjugating by ψ_{aff} gives an automorphism σ of B_{aff} which acts on edge labels by the elements π_1, π_7 and $\pi_0 = \mathrm{id}$ of Ω in types $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ respectively.

While σ simply acts by z in type $E_8^{(1)}$, to describe σ for $E_6^{(1)}$ and $E_7^{(1)}$ we let $z^{n+p}c := \sigma(z^n b)$ for all $z^n b \in B_{\mathrm{aff}}$. It is clear that if $b = \overline{i_1} \dots \overline{i_m} j_1 \dots j_n$ then $c = \overline{\pi(i_1)} \dots \overline{\pi(i_m)} \pi(j_1) \dots \pi(j_n)$ where π equals π_1 and π_7 respectively. For completeness, we list all values of $p \in \mathbb{Z}$ and $c \in B$ in Appendix B.2.

Proof of Proposition 4.3.7. It suffices to show that if $H_{\mathrm{aff}}(z^n b \otimes z^m a) > 0$ then there is a directed path $\sigma(z^n b) = z^{n+p} c \to \cdots \to z^m a$ in B_{aff}, as $\psi_{\mathrm{aff}}(z^m a)$ can then be obtained by adding blocks to $\psi_{\mathrm{aff}}(\sigma(z^n b))$.

Since σ commutes with the action of z on B_{aff} we may without loss of generality take $n = 0$, so $m < H(b \otimes a)$ by equation (2.3.2). And as there is always a path $z^{k+1}a \to \cdots \to z^k a$ we can further restrict to the case $m = H(b \otimes a) - 1$.

Lemma 4.1.5 tells us that $H(a \otimes c)$ is the minimum number of 0-arrows in a path $c \to \cdots \to a$ in B. Combining this with the existence of paths $z^{k+1}a \to \cdots \to z^k a$ it remains to verify that $p + 1 - H(b \otimes a) \geq H(a \otimes c)$, which is a finite check using Table B.2 in Appendix B.2. □

Proof of Proposition 4.3.8. Similarly to the proof above, it is enough to verify that if $H_{\text{aff}}(z^n b \otimes z^m a) > 0$ and $H_{\text{aff}}(z^n b \otimes z^m a) \neq 2$ then there is a directed path $\sigma(z^n b) = z^{n+1}b \to \cdots \to z^m a$ in B_{aff}. We may without loss of generality take $n = -1$, whereby $m \leq H(b \otimes a) - 2$ and $m \neq H(b \otimes a) - 3$ by equation (2.3.2). Furthermore it is clear that there always exist paths $z^m a \to \cdots \to z^{m-2}a$ and $z^m a \to \cdots \to z^{m-3}a$ in B_{aff}, so it is enough to consider $m = H(b \otimes a) - 2$. The existence of a path $z^0 b \to \cdots \to z^{H(b \otimes a)-2}a$ in B_{aff} is clear for all $b \otimes a$ inside any of the connected components

$$\mathcal{C}(\emptyset \otimes \emptyset), \qquad \mathcal{C}(\emptyset \otimes x_\theta), \qquad \mathcal{C}(x_\theta \otimes \emptyset), \qquad \mathcal{C}(x_\theta \otimes x_\theta), \qquad \mathcal{C}(x_\theta \otimes x_{-\theta}),$$

and $\mathcal{C}(x_\theta \otimes y_8)$ is a quick check using Proposition 4.1.8. For example, for elements of the form $x_{\theta-\alpha} \otimes x_{-\beta}$ and $x_\beta \otimes x_{-\theta+\alpha}$ there exist paths

$$\begin{gathered} z^0 x_{\theta-\alpha} \to z^0 x_{\alpha_8} \to z^0 y_8 \to z^0 x_{-\alpha_8} \to z^0 x_{-\beta} \\ z^0 x_\beta \to z^0 x_{\alpha_8} \to z^0 y_8 \to z^0 x_{-\alpha_8} \to z^0 x_{-\theta+\alpha} \end{gathered}$$

since $\theta - \alpha, \beta \in \Phi_1^+$. So we are left with $\mathcal{C}(x_\theta \otimes x_{\theta-\alpha_8})$ and $\mathcal{C}(x_\theta \otimes x_\beta)$ where β is the maximal element of Φ_0^+, which can be checked computationally with the help of SageMath [99] as outlined in Appendix B.2. □

As a final remark, it is important to note that in each type, not every Young wall satisfying the relevant right block property is proper. Indeed, Figure 4.14b provides an example of such a wall. In types $E_6^{(1)}$ and $E_7^{(1)}$ a pair of adjacent columns $(z^n\psi(b), z^m\psi(a))$ satisfies condition (4.3.2) precisely when $m \leq n + p - H(a \otimes c)$. On the other hand, $H_{\text{aff}}(z^n\psi(b) \otimes z^m\psi(a)) > 0$ when $m \leq n - 1 + H(b \otimes a)$. Hence $p + 1 - H(b \otimes a) - H(a \otimes c)$ is the difference in the number of 0-blocks between the lowest column $z^m\psi(a)$ such that

$(z^n\psi(b), z^m\psi(a))$ satisfies condition (4.3.2), and the lowest such that $(z^n\psi(b), z^m\psi(a))$ could be adjacent columns in an element of $\mathcal{Z}(\Lambda_0)$. In type $E_8^{(1)}$ this difference is given by $2 - H(b \otimes a)$ plus the minimum number of 0-arrows in a path $b \to \cdots \to a$ in B_8.

4.4 Applications and future directions

4.4.1 Hilbert schemes for Kleinian singularities

The McKay Correspondence [85] provides an *ADE* type classification for the finite subgroups $G \leq SL_2(\mathbb{C})$ and their associated Kleinian singularities $\mathbb{C}^2/G$. The moduli space of finite colength ideals in $\mathcal{O}(\mathbb{C}^2/G) = \mathbb{C}[x,y]^G$ is called the *coarse Hilbert scheme* of points, and decomposes into quasiprojective but singular components

$$\mathrm{Hilb}(\mathbb{C}^2/G) = \bigsqcup_{m \in \mathbb{N}} \mathrm{Hilb}^m(\mathbb{C}^2/G)$$

indexed by the codimension m of the ideal. On the other hand, the *orbifold Hilbert scheme* – defined as the moduli space of G-invariant finite colength subschemes of $\mathbb{C}^2$ – decomposes as

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{\rho \in \mathrm{Rep}(G)} \mathrm{Hilb}^\rho([\mathbb{C}^2/G])$$

where $\mathrm{Rep}(G)$ consists of all finite dimensional representations and each $\mathrm{Hilb}^\rho([\mathbb{C}^2/G]) = \{I \lhd \mathbb{C}[x,y] \mid G \cdot I = I, \mathbb{C}[x,y]/I \cong_G \rho\}$ is a smooth quasiprojective variety. Intersecting any ideal with the ring of G-invariant functions $\mathbb{C}[x,y]^G$ gives a morphism $\mathrm{Hilb}([\mathbb{C}^2/G]) \to \mathrm{Hilb}(\mathbb{C}^2/G)$.

The diagonal action of $T = \mathbb{C}^*$ on $\mathbb{C}^2$ lifts to induced actions on both the coarse and orbifold Hilbert schemes, whereby we have the following result due to Gyenge-Némethi-Szendrői [32].

Theorem 4.4.1. *In types A and D there exist decompositions*

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{Y \in \mathcal{Z}(\Lambda_0)} \mathrm{Hilb}([\mathbb{C}^2/G])_Y, \qquad \mathrm{Hilb}([\mathbb{C}^2/G])^T = \bigsqcup_{Y \in \mathcal{Z}(\Lambda_0)} \mathrm{Hilb}([\mathbb{C}^2/G])^T_Y,$$

into locally closed strata indexed by the sets of proper Young walls defined by Kang [56]. The strata in each case are isomorphic to non-empty affine spaces of varying dimensions, and in particular have Euler characteristic 1. Moreover, $\mathrm{Hilb}([\mathbb{C}^2/G])_Y$ *is a trivial vector bundle over* $\mathrm{Hilb}([\mathbb{C}^2/G])^T_Y$ *and equals its attracting locus under the torus action.*

Labelling the irreducibles in $\mathrm{Rep}(G)$ by $\rho_0, \ldots, \rho_n$, the topological Euler characteristics of the various Hilbert schemes above can be encoded in the following *coarse* and *orbifold* generating series:

$$Z_{\mathbb{C}^2/G}(q) = \sum_{m \in \mathbb{N}} \chi(\mathrm{Hilb}^m(\mathbb{C}^2/G))q^m,$$
$$Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) = \sum_{m_0, \ldots, m_n \in \mathbb{N}} \chi(\mathrm{Hilb}^{m_0\rho_0 + \cdots + m_n\rho_n}([\mathbb{C}^2/G]))q_0^{m_0} \ldots q_n^{m_n}.$$

One recovers from Theorem 4.4.1 explicit formulae for $Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n)$ originally given by Nakajima [93] in terms of the corresponding finite type Cartan matrix. Furthermore, Gyenge-Némethi-Szendrői then used the combinatorics of Young walls in types A and D to prove a *substitution formula* for $Z_{\mathbb{C}^2/G}(q)$ as a particular specialisation of $Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n)$ involving certain roots of unity.

They moreover conjectured the corresponding result in type E, which was later verified by Nakajima [94] via alternative methods. In particular, his arguments make use of the realization of any $\mathrm{Hilb}^\rho([\mathbb{C}^2/G]) \to \mathrm{Hilb}^m(\mathbb{C}^2/G)$ as an instance of the natural morphism $\mathcal{M}_\zeta(\underline{v}, \underline{w}) \to \mathcal{M}_{\zeta^\bullet}(\underline{v}, \underline{w})$ between smooth and degenerate quiver varieties on the affine Dynkin diagram. This result was proved in all ADE types by Craw-Gammelgaard-Gyenge-Szendrői [15].

It is therefore natural to expect results in type E analogous to those of [32], connecting algebraic geometry and combinatorics and in particular involving the Young wall models from [77] as outlined in this chapter. For example, Lukas Bertsch and Balázs Szendrői working on the geometry side have encountered similar combinatorial pictures to those of the author.

Conjecture 4.4.2. *In type E there ought to exist decompositions*

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{Y \in \mathcal{Z}(\Lambda_0)} \mathrm{Hilb}([\mathbb{C}^2/G])_Y, \qquad \mathrm{Hilb}([\mathbb{C}^2/G])^T = \bigsqcup_{Y \in \mathcal{Z}(\Lambda_0)} \mathrm{Hilb}([\mathbb{C}^2/G])^T_Y,$$

into locally closed strata indexed by the proper Young walls constructed by the author [77]. One hopes that each stratum would be isomorphic to an affine space, and thus have Euler characteristic 1*. Furthermore,* $\mathrm{Hilb}([\mathbb{C}^2/G])_Y$ *should be the attracting locus of* $\mathrm{Hilb}([\mathbb{C}^2/G])^T_Y$*.*

Remark 4.4.3. Some of this discussion has already been generalised to Quot schemes for Kleinian orbifolds in all ADE types. In particular, they are isomorphic to quiver varieties on the relevant affine Dynkin diagram with various degenerate stability conditions [16], and were recently shown to possess similar substitution formulae by Bertsch-Gyenge-Szendrői [6].

The results above were already well-known in type A, where proper Young walls correspond to partitions and $\mathrm{Hilb}([\mathbb{C}^2/G])$ has isolated torus fixed points. In this case, the connection between algebraic geometry and the combinatorics of Young walls can be more deeply understood through work of Nagao [91].

Namely, on the one hand there exists the Fock space representation for $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ defined algebraically by Saito-Takemura-Uglov [101] and Varagnolo-Vasserot [106]. Their approach uses a Schur-Weyl duality with the double affine Hecke algebra [105], as well as the semi-infinite wedge construction of $\mathcal{F}(\Lambda_0)$ due to Kashiwara-Miwa-Petersen-Yung [71]. Here Nagao demonstrated that there exists a basis of simultaneous eigenvectors for the $k_i^{\pm 1}$ and $h_{i,r}$ generators which is indexed by partitions.

Conversely, Nakajima [92] constructed in all simply laced types a morphism

$$U_q(\mathfrak{g}_{\mathrm{tor}}) \;\to\; K^{G_{\underline{w}}\times\mathbb{C}^*}(\mathcal{M}_\zeta(\underline{w}) \times_{\mathcal{M}_0(\underline{w})} \mathcal{M}_\zeta(\underline{w})) \tag{4.4.1}$$

to the equivariant K-theory convolution algebra of quiver varieties on the affine Dynkin diagram. One can then immediately produce ℓ-highest weight representations for $U_q(\mathfrak{g}_{\mathrm{tor}})$ by considering fibers of the morphism $\mathcal{M}_\zeta(\underline{w}) \to \mathcal{M}_0(\underline{w})$.

In type A, localising with respect to the torus action induced from the diagonal scaling $(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^2$ gives the classes of all fixed point loci as a basis (of simultaneous eigenvectors) for one of these modules. Nagao exhibited that the identification of fixed points with partitions extends to an isomorphism of representations.

Conjecture 4.4.4. *There should exist in all simply laced types a $U_q(\mathfrak{g}_{\mathrm{tor}})$-module isomorphism between a Fock space representation – written in terms of a basis of proper Young walls – and a geometric representation coming from (4.4.1). Each Young wall should be* *mapped to the class in equivariant K-theory of the corresponding fixed point locus from Theorem 4.4.1 or Conjecture 4.4.2.*

4.4.2 Categorification and KLR algebras

In non-exceptional affine types, it is possible to *globalise* the Young wall models for $B(\mathcal{F}(\lambda))$ in the level 1 case to obtain global bases $G(\mathcal{F}(\lambda))$ of Young walls for Fock space representations. In particular, one can define in purely combinatorial terms an action of $U_q(\hat{\mathfrak{g}})$ on the $\mathbb{Q}(q)$-span of proper Young walls, such that the resulting module is isomorphic to $\mathcal{F}(\lambda)$. Moreover the crystal basis of proper Young walls is then recovered in the limit $q \to 0$.

Since the highest weight module $V(\lambda)$ sits naturally inside $\mathcal{F}(\lambda)$ as the irreducible submodule containing the ground state vector, one may therefore ask whether we can describe its global basis $G(\lambda)$ using proper Young walls. The (generalised) Lascoux-Leclerc-Thibon (LLT) algorithm provides an effective way to compute such expressions. More precisely, the global basis element $G(Y) \in G(\lambda)$ associated to a reduced Young wall Y is given as a $\mathbb{Z}[q]$-linear combination $\sum G_{Y,Z}(q)Z$ of proper Young walls, which satisfies some unitriangular condition.

These results are originally due to Lascoux, Leclerc and Thibon in types $A_n^{(1)}$ and $A_{2n}^{(2)}$ [75, 78] and were later shown for $C_n^{(1)}$ in work of Kim-Shin [74], before being generalised to all non-exceptional affine types by Kang-Kwon [60, 61].

It is known that the standard (Specht) and irreducible representations for the Hecke algebras of type A_N ($N \geq 0$) at primitive nth roots of unity are parameterised by the type $A_n^{(1)}$ proper and reduced Young walls respectively. Addressing a conjecture of Lascoux-Leclerc-Thibon from [75], Ariki [2] proved that the multiplicities of irreducible modules inside standard modules coincide with the evaluation at $q = 1$ of the corresponding coefficients $G_{Y,Z}(q)$.

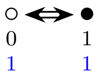
It is natural to expect that in all affine types, there should exist some interesting algebraic structures whose irreducible and 'standard' modules are indexed by the reduced and proper Young walls. Indeed, Brundan and Kleshchev [8] have shown that irreducible representations of the Hecke-Clifford superalgebras at primitive $(2n+1)$th roots of unity are parameterised by the reduced Young walls of type $A_{2n}^{(2)}$. Furthermore, one would hope that the decomposition numbers of standards into irreducibles are computed by the values of $G_{Y,Z}(1)$ in each type. The correct setting to answer these questions in general is conjectured to be that of KLR algebras and cyclotomic Hecke algebras.

Constructing Young wall models for the level 1 irreducible highest weight and Fock space crystals is therefore an important first step towards obtaining such results, and has been recently completed by the work contained in [25, 33, 77]. Using these models we expect to define Fock space representations combinatorially and further generalise the LLT algorithm to exceptional affine types, before investigating potential connections to categorification.

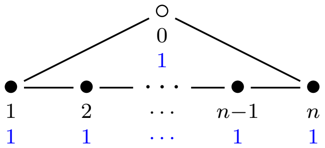
Appendices

A The affine Dynkin diagrams

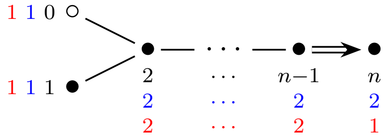
$A_1^{(1)}$:



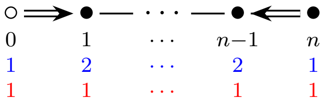
$A_n^{(1)}$ $(n \geq 2)$:



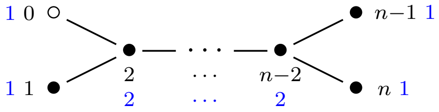
$B_n^{(1)}$ $(n \geq 3)$:



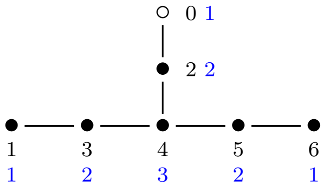
$C_n^{(1)}$ $(n \geq 2)$:



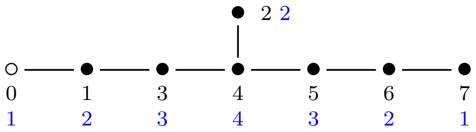
$D_n^{(1)}$ $(n \geq 4)$:



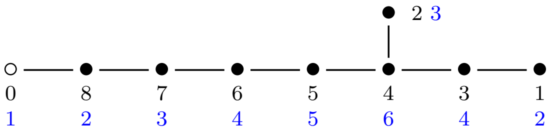
$E_6^{(1)}$:



$E_7^{(1)}$:



$E_8^{(1)}$:



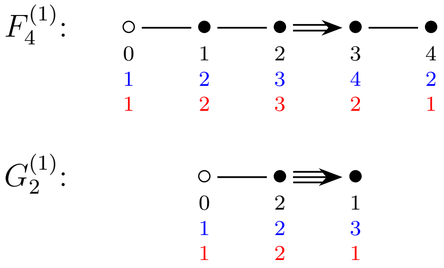


Fig. A.1 The untwisted affine Dynkin diagrams – black labels are vertex numbers, blue labels are a_i values, and in the non-symmetric cases $a_i^\vee$ values are in red

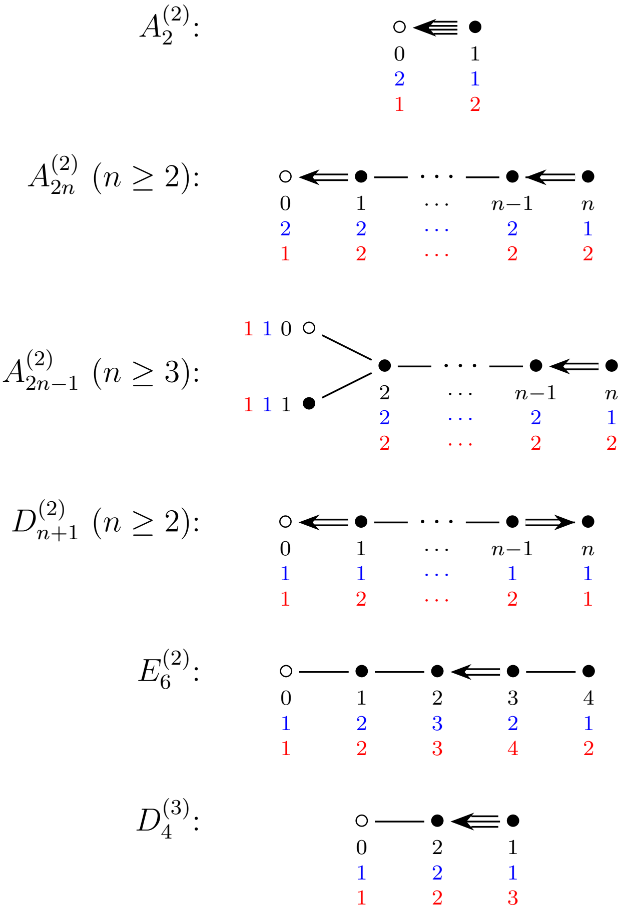


Fig. A.2 The twisted affine Dynkin diagrams – black labels are vertex numbers, blue labels are a_i values, and red labels are $a_i^\vee$ values

B Technical proofs

B.1 Lemma 3.1.7

Here we provide some details regarding the proof of Lemma 3.1.7. Table B.1 contains example sequences $(i_1, i_2, \ldots, i_{h-1})$ in I_0 and $(\epsilon_1, \ldots, \epsilon_{h-2})$ in $\mathbb{Q}_{\leq 0}$ for $i_1 = 1, 2$, which allow us to write expressions for $\hat{x}^{\pm}_{0,0}$ and $\hat{k}_0^{\pm 1}$.

Type A_2	Type C_2	Type G_2
$\alpha_1 \xrightarrow{-1} \alpha_2$	$\alpha_1 \xrightarrow{-1} \alpha_2 \xrightarrow{0} \alpha_1$	$\alpha_1 \xrightarrow{-1} \alpha_2 \xrightarrow{-1/3} \alpha_2 \xrightarrow{0} \alpha_1 \xrightarrow{-2/3} \alpha_2$
$\alpha_2 \xrightarrow{-1} \alpha_1$	$\alpha_2 \xrightarrow{-1} \alpha_1 \xrightarrow{0} \alpha_1$	$\alpha_2 \xrightarrow{-1} \alpha_1 \xrightarrow{-1/3} \alpha_2 \xrightarrow{0} \alpha_1 \xrightarrow{-2/3} \alpha_2$

Table B.1 Sequences $\alpha_{i_1} \xrightarrow{\epsilon_1} \cdots \xrightarrow{\epsilon_{h-2}} \alpha_{i_{h-1}}$ for $i_1 = 1, 2$ in types $A_2^{(1)}$, $C_2^{(1)}$ and $G_2^{(1)}$

We first need to confirm that the Drinfeld-Jimbo relations of $U_q'(X^{(1)})$ involving $x_0^{\pm}$ and $k_0^{\pm 1}$ are preserved under the map $\xi : U_q'(X^{(1)}) \to \mathcal{A}_X$ that sends $x_i^{\pm} \mapsto \hat{x}^{\pm}_{i,0}$ and $t_i^{\pm 1} \mapsto \hat{k}_i^{\pm 1}$ for $i = 0, 1, 2$. We outline a method for each below, referencing which relations (i)-(xi) in $\mathcal{A}_X$ should be applied at each stage. The author notes that many of the relations are easily checked with the help of a computer algebra package such as Magma.

- Both $\hat{k}_0^{\pm 1}\hat{k}_0^{\mp 1} = 1$ and $[\hat{k}_0, \hat{k}_j] = 0$ are trivial by (i)-(iii).
- All $\hat{k}_0 \hat{x}^{\pm}_{\ell,0} \hat{k}_0^{-1} = q_0^{\pm a_{0\ell}} \hat{x}^{\pm}_{\ell,0}$ and $\hat{k}_\ell \hat{x}^{\pm}_{0,0} \hat{k}_\ell^{-1} = q_0^{\pm a_{\ell 0}} \hat{x}^{\pm}_{0,0}$ are deduced from (i)-(iv).
- The relation $[\hat{x}^{+}_{0,0}, \hat{x}^{-}_{0,0}] = \frac{\hat{k}_0 - \hat{k}_0^{-1}}{q_0 - q_0^{-1}}$ is proved as follows:
  1. Input the expressions for $\hat{x}^{\pm}_{0,0}$ with $i_1 = 1$ and expand everything out.
  2. Factor the $\hat{C}^{\pm 1}$, $\hat{k}_1^{\pm 1}$ and $\hat{k}_2^{\pm 1}$ terms using (i) and (iv), then cancel them.
  3. Move any $\hat{x}^{-}_{i,m}$ terms in front of all $\hat{x}^{+}_{i,m}$ terms with (v)-(vii), then pull any $\hat{C}^{\pm 1}$, $\hat{k}_1^{\pm 1}$ and $\hat{k}_2^{\pm 1}$ factors created out to one side with (i) and (iv).
  4. Cancel all remaining summands other than $\frac{\hat{k}_0 - \hat{k}_0^{-1}}{q_0 - q_0^{-1}}$ using the relations (viii) and (xi) which involve $\hat{x}^{\pm}_{1,0}$, $\hat{x}^{\pm}_{1,\pm 1}$ and $\hat{x}^{\pm}_{2,0}$.
- For $[\hat{x}^{\pm}_{0,0}, \hat{x}^{\mp}_{\ell,0}] = 0$ with $\ell \in I_0$:
  1. Input the expression for $\hat{x}^{\pm}_{0,0}$ with $i_1 \neq \ell$ and expand everything out.
  2. Factor the $\hat{C}^{\pm 1}$, $\hat{k}_1^{\pm 1}$ and $\hat{k}_2^{\pm 1}$ terms using (i) and (iv).

3. Cancel everything by (viii) and (xi) with $y_i = \hat{x}^{\mp}_{\ell,0}$ and $y_j = \hat{x}^{\mp}_{i_1,0}, \hat{x}^{\mp}_{i_1,\pm 1}$.

- For the quantum Serre relations between $\hat{x}^{\pm}_{0,0}$ and $\hat{x}^{\pm}_{\ell,0}$ with $\ell \in I_0$:

  1. Input the expression for $\hat{x}^{\pm}_{0,0}$ with $i_1 \neq \ell$ and expand everything out.
  2. Factor the $\hat{C}^{\pm 1}$, $\hat{k}^{\pm 1}_1$ and $\hat{k}^{\pm 1}_2$ terms using (i) and (iv).
  3. Move all $\hat{x}^{\pm}_{\ell,0}$ terms to one side using relations (v) and (vii).
  4. Pull any $\hat{k}^{\pm 1}_{\ell}$ factors created in the previous step out to one side with (iv).
  5. Cancel everything by (viii).

Next we must give $x^{\pm}_{1,\mp 1}$ and $x^{\pm}_{2,\mp 1}$ in terms of the Drinfeld-Jimbo generators of $U'_q(X^{(1)})$. This can be done using Beck's extended affine braid group action from Theorem 2.4.6. In particular, writing each $X_{\omega_i^\vee}$ in the Coxeter presentation of $\dot{\mathcal{B}}$ allows us to present its action with respect to the Drinfeld-Jimbo realization of $U'_q(X^{(1)})$. Then since $o(i)X_{\omega_i^\vee}$ sends $x^{\pm}_{i,0}$ to $x^{\pm}_{i,\mp 1}$ we can obtain the desired expressions, thus allowing us to find the images of $x^{\pm}_{1,\mp 1}$ and $x^{\pm}_{2,\mp 1}$ under ξ. To complete the proof we check that these are equal to $\hat{x}^{\pm}_{1,\mp 1}$ and $\hat{x}^{\pm}_{2,\mp 1}$ respectively by inserting the definitions of $\hat{x}^{\pm}_{0,0}$ and $\hat{k}^{\pm 1}_0$ in terms of the generators of $\mathcal{A}_X$ and applying the relevant relations.

B.2 Propositions 4.3.7 and 4.3.8

b	c	p
$\overline{0}1$	$\overline{1}6$	1
$\overline{01}3$	$\overline{16}5$	1
$\overline{03}4$	$\overline{15}4$	1
$\overline{04}25$	$\overline{14}23$	1
$\overline{05}26$	$\overline{12}03$	1
$\overline{2}5$	$\overline{3}2$	1
$\overline{06}2$	$\overline{01}3$	0
$\overline{25}46$	$\overline{23}04$	1
$\overline{26}4$	$\overline{03}4$	0
$\overline{4}36$	$\overline{4}05$	1
$\overline{46}35$	$\overline{04}25$	0
$\overline{3}16$	$\overline{5}06$	1
$\overline{5}3$	$\overline{2}5$	0
$\overline{36}15$	$\overline{05}26$	0
$\overline{1}6$	$\overline{6}0$	1
$\overline{35}14$	$\overline{25}46$	0
$\overline{16}5$	$\overline{06}2$	0
$\overline{4}12$	$\overline{4}36$	0
$\overline{15}4$	$\overline{26}4$	0
$\overline{2}01$	$\overline{3}16$	0
$\overline{14}23$	$\overline{46}35$	0
$\overline{12}03$	$\overline{36}15$	0
$\overline{3}2$	$\overline{5}3$	0
$\overline{23}04$	$\overline{35}14$	0
$\overline{4}05$	$\overline{4}12$	0
$\overline{5}06$	$\overline{2}01$	0
$\overline{6}0$	$\overline{0}1$	-1

(a) Type $E_6^{(1)}$

b	c	p
$\overline{0}7$	$\overline{7}0$	1
$\overline{07}6$	$\overline{07}1$	0
$\overline{06}5$	$\overline{17}3$	0
$\overline{05}4$	$\overline{37}4$	0
$\overline{04}23$	$\overline{47}25$	0
$\overline{03}12$	$\overline{57}26$	0
$\overline{02}3$	$\overline{27}5$	0
$\overline{1}2$	$\overline{6}2$	0
$\overline{023}14$	$\overline{257}46$	0
$\overline{12}4$	$\overline{26}4$	0
$\overline{04}15$	$\overline{47}36$	0
$\overline{14}35$	$\overline{46}35$	0
$\overline{05}16$	$\overline{37}16$	0
$\overline{3}5$	$\overline{5}3$	0
$\overline{15}36$	$\overline{36}15$	0
$\overline{06}17$	$\overline{17}06$	0
$\overline{35}46$	$\overline{35}14$	0
$\overline{16}37$	$\overline{16}05$	0
$\overline{07}1$	$\overline{07}6$	-1
$\overline{4}26$	$\overline{4}12$	0
$\overline{36}47$	$\overline{15}04$	0
$\overline{17}3$	$\overline{06}5$	-1
$\overline{2}6$	$\overline{2}1$	0
$\overline{46}257$	$\overline{14}023$	0
$\overline{37}4$	$\overline{05}4$	-1
$\overline{26}57$	$\overline{12}03$	0
$\overline{5}27$	$\overline{3}02$	0
$\overline{47}25$	$\overline{04}23$	-1
$\overline{25}47$	$\overline{23}04$	0
$\overline{27}5$	$\overline{02}3$	-1
$\overline{57}26$	$\overline{03}12$	-1
$\overline{4}37$	$\overline{4}05$	0
$\overline{257}46$	$\overline{023}14$	-1
$\overline{6}2$	$\overline{1}2$	-1
$\overline{3}17$	$\overline{5}06$	0
$\overline{47}36$	$\overline{04}15$	-1
$\overline{26}4$	$\overline{12}4$	-1
$\overline{1}07$	$\overline{6}07$	0
$\overline{37}16$	$\overline{05}16$	-1
$\overline{46}35$	$\overline{14}35$	-1
$\overline{17}06$	$\overline{06}17$	-1
$\overline{36}15$	$\overline{15}36$	-1
$\overline{5}3$	$\overline{3}5$	-1
$\overline{16}05$	$\overline{16}37$	-1
$\overline{35}14$	$\overline{35}46$	-1
$\overline{15}04$	$\overline{36}47$	-1
$\overline{4}12$	$\overline{4}26$	-1
$\overline{14}023$	$\overline{46}257$	-1
$\overline{2}1$	$\overline{2}6$	-1
$\overline{3}02$	$\overline{5}27$	-1
$\overline{12}03$	$\overline{26}57$	-1
$\overline{23}04$	$\overline{25}47$	-1
$\overline{4}05$	$\overline{4}37$	-1
$\overline{5}06$	$\overline{3}17$	-1
$\overline{6}07$	$\overline{1}07$	-1
$\overline{7}0$	$\overline{0}7$	-2

(b) Type $E_7^{(1)}$

Table B.2 Describing the function $\sigma : z^n b \mapsto z^{n+p} c$ in types $E_6^{(1)}$ and $E_7^{(1)}$

Finishing the proof of Proposition 4.3.8. As mentioned before, SageMath can be used to show that the Young walls in our crystal $\mathcal{Z}(\Lambda_0)$ in type $E_8^{(1)}$ satisfy the weakened right block property of Proposition 4.3.8. In particular, we can check that for any $b \otimes a$ in $\mathcal{C}(x_\theta \otimes x_{\theta-\alpha_8})$ or $\mathcal{C}(x_\theta \otimes x_\beta)$ there exists a path $z^0 b \to \cdots \to z^{H(b \otimes a)-2} a$ in B_{aff}.

Indeed, since our level 1 perfect crystal B_8 is equal to the crystal basis of the level 0 fundamental representation $W(\varpi_8)$ of $U_q'(\hat{\mathfrak{g}})$, the following produces a list of maximal vectors in $B_8 \otimes B_8$.

```
sage: K=crystals.kirillov_reshetikhin.LSPaths(['E',8,1],8)
sage: K2=crystals.TensorProduct(K,K)
sage: hw=K2.classically_highest_weight_vectors()
```

The second and sixth entries are $x_\theta \otimes x_{\theta-\alpha_8}$ and $x_\theta \otimes x_\beta$ respectively, where we note that SageMath uses an alternative tensor crystal structure in which tensor factors are reversed compared to (2.2.3). Substituting `n` below with `1` and `5` therefore gives the components $\mathcal{C}(x_\theta \otimes x_{\theta-\alpha_8})$ and $\mathcal{C}(x_\theta \otimes x_\beta)$ respectively.

```
sage: C=K2.subcrystal(generators=[hw[n]],
      index_set=[1,2,3,4,5,6,7,8])
sage: C.digraph().vertices()
```

The next step is to weight the edges of B_8 so that 0-arrows have weight 1 and all other arrows have weight 1000. To this end, we start by obtaining the list of edges in B_8.

```
sage: K.digraph().edges()
```

With a simple ‘find and replace’ procedure we can turn this into a list `E` of *weighted* edges, where for technical reasons we must also replace any `Lambda[j]` with `Lj`. The following then defines the desired weighted digraph, and computes the minimal weight of a path between any two vertices.

```
sage: var('L0 L1 L2 L3 L4 L5 L6 L7 L8')
sage: from sage.graphs.base.boost_graph
      import floyd_warshall_shortest_paths
sage: D=DiGraph(E, weighted=True)
sage: floyd_warshall_shortest_paths(D)
```

The final digit of the minimal weight of a path from b to a in our weighted digraph is equal to the minimum number of 0-arrows in a path from b to a in B. Alternatively, this is the minimal k for which there is a path $z^0 b \to \cdots \to z^{-k} a$ in B_{aff}. For every $b \otimes a \in \mathcal{C}(x_\theta \otimes x_\beta)$ this is 0 as desired. For $b \otimes a \in \mathcal{C}(x_\theta \otimes x_{\theta-\alpha_8})$ it is either 1 – in which case we are done – or it is 0. Since $b \neq x_\theta, \emptyset$ or $a \neq x_{-\theta}, \emptyset$ our proof is complete by Lemma 4.3.9. □

C The crystal graph of B_8

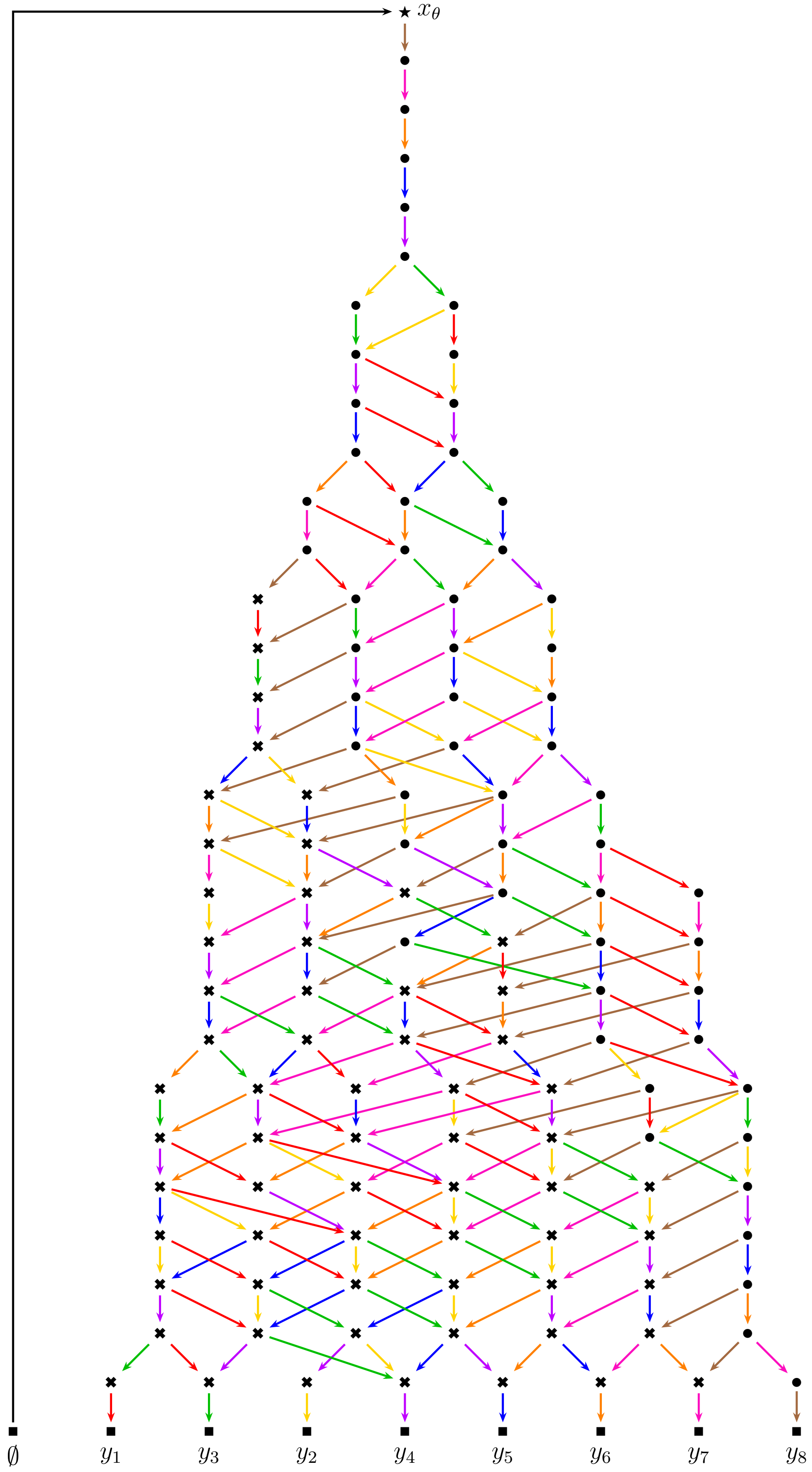


Fig. C.1 The top half of the crystal graph of B_8

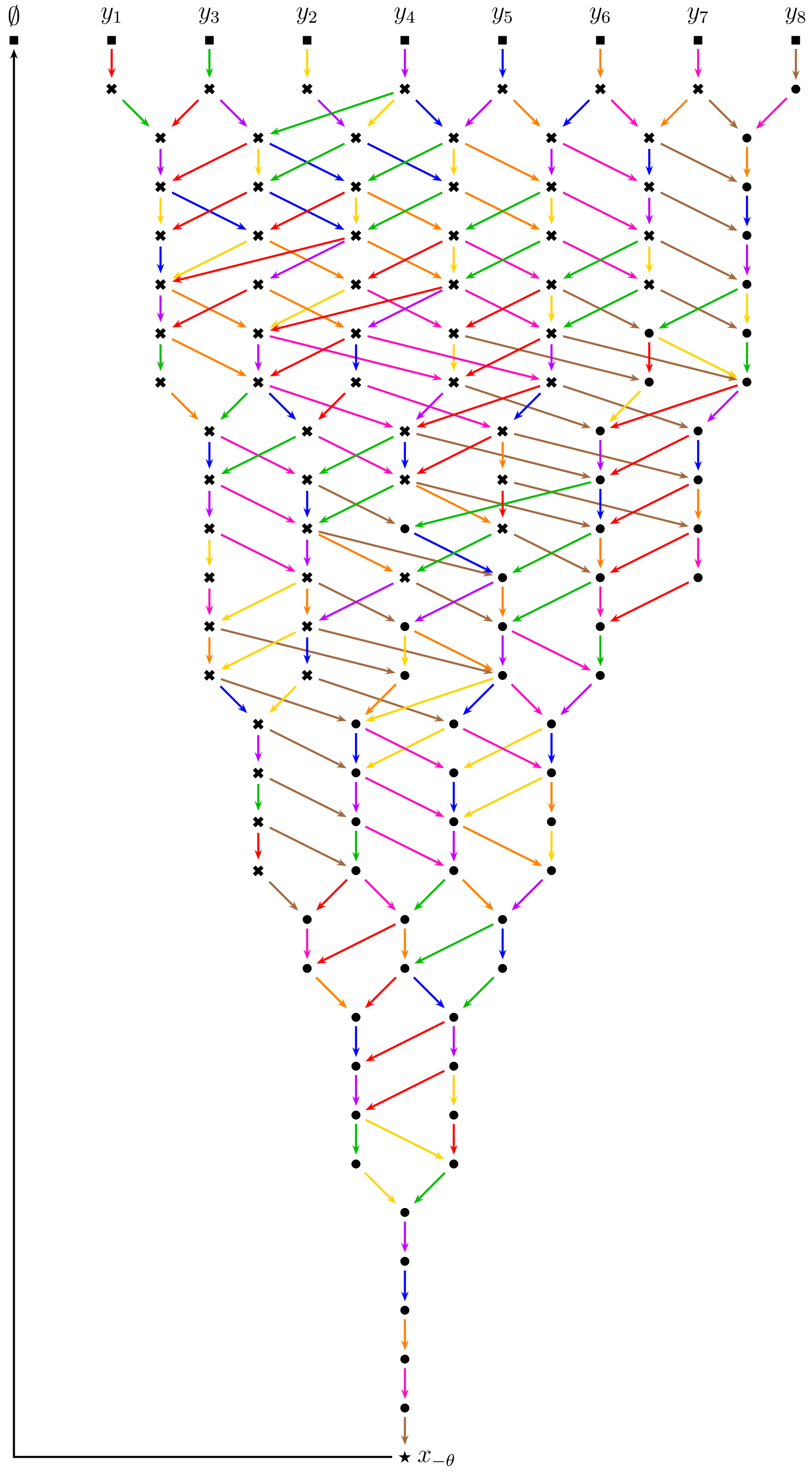


Fig. C.2 The bottom half of the crystal graph of B_8

D Young wall crystals in type $\mathbf{E}_6^{(1)}$

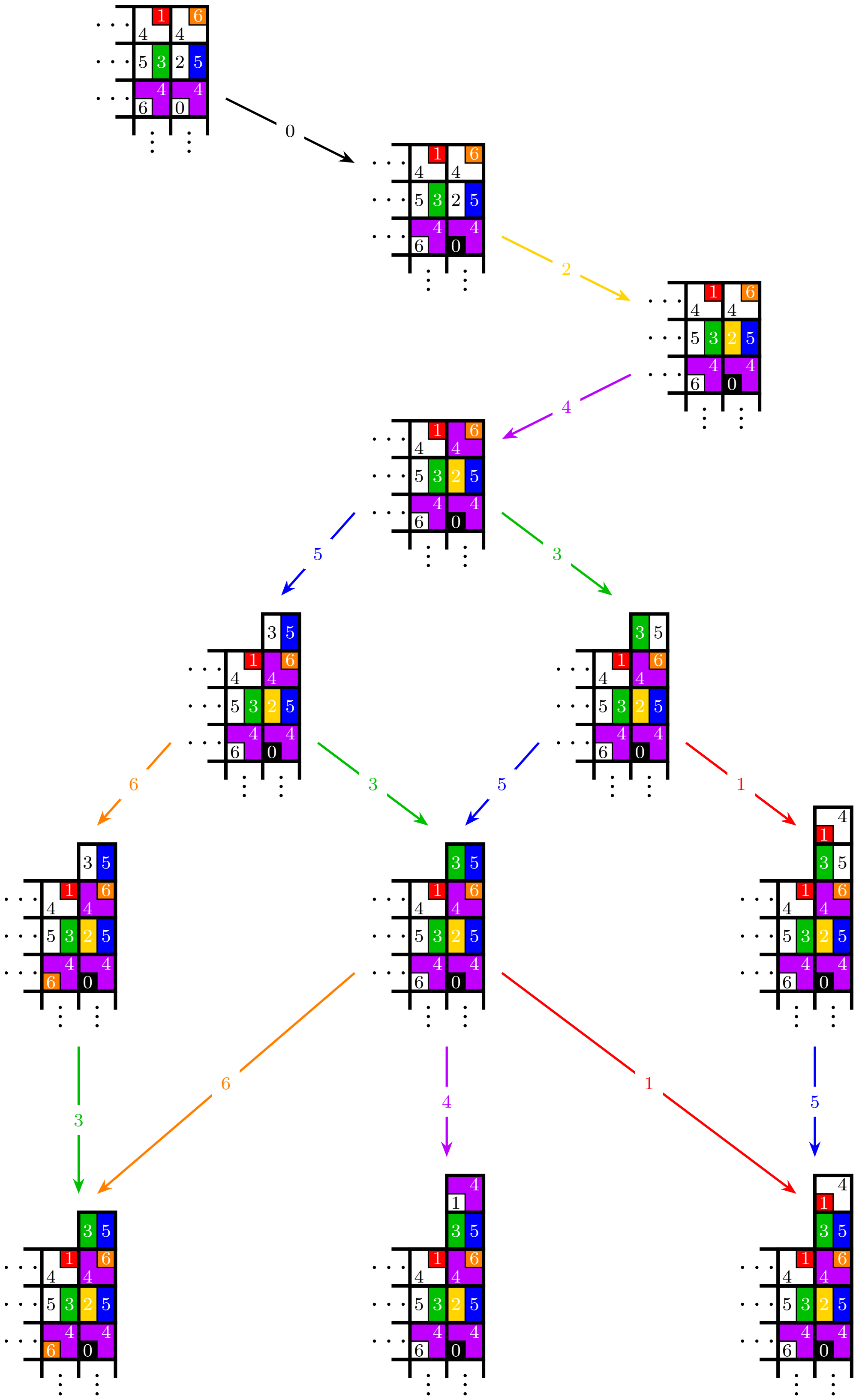


Fig. D.1 The top part of the crystals $\mathcal{Y}(\Lambda_0)$ and $\mathcal{Z}(\Lambda_0)$ in type $E_6^{(1)}$

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