Technology and Employment: Tasks, Capabilities, and Tastes

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Abstract

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This thesis explores the consequences of 'increasingly capable machines' on earnings and employment. A new literature, the task-based approach, has been developed for this purpose. And this literature presents an optimistic account of the prospects for labour in the 21st century. The central claim in this literature is that 'people tend to overstate the extent of machine substitution for labour and ignore the complementarities'.

This thesis challenges this optimism. I argue that such optimism is based on two assumptions, neither of which is justified. The first is that the supply-side analysis in this literature is based on outdated reasoning about how these machines operate. The result is that the models arbitrarily constrain what machines are capable of doing. The second is that the demand-side analysis in this literature is either altogether missing, or is carried out in a way that is constrained by the arbitrary supply-side assumption.

In this thesis I build a new range of task-based models that are based on more justifiable assumptions. The first set of models show that updated reasoning about how machines operate leads to a pessimistic account of the prospects for labour. The second set of models show that the demand-side has an important role in either strengthening, or weakening, this pessimism that is reached when the supply-side is looked at in isolation. This analysis leads to the identification of an important new 'race' in the labour market.

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Contents

0. Introduction

0.1. Overview

"Demand for commodities is not demand for labour."

John Stuart Mill, *Principles*

This thesis explores the consequences of 'increasingly capable machines' for earnings and employment.[1] Recently there has been a renewed interest in these consequences. Advances in technology have been great. Over the last 150 years, for example, computational power has increased between 1.7 and 76 trillion-fold (Nordhaus 2007). Machines are now capable of doing things that, until recently, most people thought that only human beings could ever do. Most experts anticipate that this progress will continue.

A fascinating new area of the economic literature has developed to explore this phenomenon. This is called the 'task-based approach' to the labour market.[2] The account that emerges from this literature is optimistic about the prospects for labour in the 21st century. Consider Autor (2015), a re-

[1] By 'increasingly capable machines', a term used throughout this thesis, I mean 'capital-saving', or 'capital-biased', technological change. In addition, I use the terms 'machines', 'systems', and 'capital' interchangeably. As a point of style, I use single quotation marks, '...', as traditional scare quotes, and double quotation marks, "..." to highlight quoted text.

[2] Or simply, the 'task approach'.

cent survey of the task-based approach, entitled "Why Are There Still so Many Jobs?". It claims that just as past concerns about automation were misplaced, so too are similar concerns that "have recently regained prominence". The central claim is that people tend to "overstate the extent of machine substitution for labour" and "ignore the strong complementarities".

This thesis explores the origins of that optimism. I take the core set of models in the literature, set out their relationship to one another, and examine the assumptions upon which they rely. I argue that this optimism follows from two limitations. The first is a limitation of the *supply-side* analysis – at present the literature is built upon outdated reasoning about how these systems and machines operate. The result is that the models use a very conservative conception of the capabilities of machines relative to human beings in production. I argue that if this supply-side limitation is taken seriously, the outcome is *pessimistic* – increasingly capable machines drive down relative wages and the labour share of income, and force labour to specialise in a shrinking set of tasks.[3]

The second is a limitation of the *demand-side* analysis – at present the models either have a very limited role for tastes and preferences or a role for tastes and preferences that is constrained by the limitation of the supply-side analysis. I show that if consumer expenditure shifts towards goods that are produced by tasks in which labour retains a comparative advantage over machines, then the outcome may not be as pessimistic as the supply-

[3] I refer to 'optimism' and 'pessimism' throughout this thesis. I do so as a form of short-hand – 'optimistic outcomes' are those that reflect the analytical results in Chapter 3, 'pessimistic outcomes' reflect Chapter 4. I will discuss these results in detail. Put simply, 'pessimistic outcomes' are those that make people feel pessimistic about labour's future, and vice-versa for 'optimistic outcomes'.

side analysis suggests. But if consumer expenditure does not shift, or shifts towards goods that are produced by tasks in which machines, rather than labour, retain a comparative advantage then the outcome may be *even more pessimistic* than the supply-side analysis suggests.

I argue that the character of the 21st century labour market is best understood by exploring this interaction between an analysis of the supply-side that is appropriately revised to reflect the character of recent technological progress, and an analysis of the demand-side that often is either omitted or restricted. I do not claim that the future of the labour market is known. But I do claim that there are reasons for pessimism, not adequately reflected in the existing literature, that undermine the case for the optimism that currently exists.

The set of arguments I developed are captured by John Stuart Mill's claim, "demand for commodities is not demand for labour".[4] The argument in this thesis, in ways that I will make apparent, reflects this.[5] The central idea is this – demand for commodities is not demand for labour, but instead is demand for tasks that must be performed to produce those commodities. The problem is that, at present, many areas of the literature either conflate these two statements, or in different ways make very strong assumptions about the extent to which the tasks that are demanded – or will be demanded in the future – are those in which labour, rather machines, have the comparative advantage. In this thesis I build a set of models that allow me to explore the

[4]Mill (1848) states both that demand for commodities "does not constitute demand for labour" and, separately, that it "is not demand for labour."

[5]Although not in the way Mill intended – that is discussed, for example, in Neisser (1942).

limitations set out and understand the consequences of resolving them.

0.2. Structure of the Thesis

This thesis is composed of four parts and eight chapters. In Part A, Chapters 1 and 2, I provide the foundations for the thesis. In Chapter 1 I set out the origins of the task-based literature and provide a general framework for the task-based approach. This is not an original analytical contribution but it allows me to do three things. First to show why the task-based approach is a useful advance on the traditional literature. Secondly to show how a core set of task-based models are related to each other. And thirdly to show how dependent the existing task-based literature is upon what is known as the 'ALM hypothesis'.

This third point is especially important. The ALM hypothesis is a particular understanding of how machines operate and their resulting capabilities. It has two parts – a distinction between 'routine' and 'non-routine' tasks, and an assumption that while machines substitute for labour that perform 'routine' tasks, machines complement labour that performs 'non-routine' tasks. This is based on the observation that humans struggle to articulate how they perform 'non-routine' tasks, and a claim that it is therefore difficult to compose a set of rules for a machine to follow in order to perform those tasks. An important consequence of this hypothesis is that this core set of task-based models share a property – all goods necessarily require at least one type of labour to produce them.

In Chapter 2 I argue that the ALM hypothesis is mistaken. In practice, we see an increasing range of 'non-routine' tasks that are performed by machines. The problem is that the ALM hypothesis is based on an outdated, 'first-generation' conception of artificial intelligence. Machines in the 'second-generation', supported by brute-force processing power and massive data storage capability, are performing tasks in different ways to human beings. And so the fact that human beings struggle to articulate how *they* perform 'non-routine' tasks is not a constraint – the rules that systems follow must no longer reflect the thinking processes of human beings. Yet this is what the ALM hypothesis supposes.

This final point is central to this thesis. The ALM hypothesis, which is at the core of the task-based literature, and whose spirit dominates informal analysis of the future of work, relies on a constraint on machine capability that no longer holds. I argue that a new concept – 'routinisability' – ought to replace the existing concept of 'routineness' in the literature. There is no boundary between 'non-routine' and 'routine' tasks, and in the limit everything is routinisable at a cost.

In Part B, Chapters 3, 4 and 5, I try to resolve this limitation of the supply-side analysis. Working within the framework set out in Part A, I develop a set of new task-based models to do this. In Chapter 3 the new model is Ricardian with two types of labour – low-skilled and high-skilled. There is a range of tasks ordered in a line from left to right, going from 'simple' to 'complex'. These tasks are performed by labour and the productivity of high-skilled labour relative to low-skilled increases to the right (i.e. as tasks become more 'complex').

In certain respects this is familiar to existing task-based models. But it goes beyond the familiar in two ways. The first departure is that there is a range of goods produced, not just one good, where the production of each good requires its own particular type of task performed by labour. Secondly, to produce a good requires not just a task performed by labour but also a task performed by a particular type of capital – what I call 'traditional' capital. Capital is 'traditional' in the sense that it cannot perform the same type of tasks as labour. This reflects the spirit of the ALM hypothesis.[6] In equilibrium there is a single cut-off – all of the tasks to the left of the cut-off are performed by low-skilled labour and all of the tasks to the right by high-skilled labour. The main lesson from this chapter is that improvements in the capability of traditional capital across the range of tasks have a neutral effect on labour – it causes an equiproportionate rise in the factor prices of capital and both types of labour, and does not affect the cut-off. This is optimism at work.

In Chapter 4 I introduce a second type of capital – 'advanced' capital – alongside traditional capital. Unlike traditional capital, it can perform the type of tasks that are performed by labour. This is a direct challenge to the ALM hypothesis and I use the concept of 'routinisability' developed in Chapter 2 to structure this claim. Furthermore, goods no longer require at least one type of labour to produce them. In equilibrium there are now two cut-offs – to the left of a lower cut-off tasks are performed by low-skilled labour, to the right of an upper cut-off tasks are performed by high-skilled labour, and those in-between are performed by advanced capital. Improvements in the

[6] In the ALM hypothesis, as noted before, those tasks that can only be performed by labour are called 'non-routine' tasks.

capability of advanced capital cause the relative returns to labour to fall, and force labour to specialise in a shrinking set of types of tasks. This analysis is original and it contradicts the main conclusions in the existing task-based literature. This is pessimism at work.

In Chapter 5 I place the static models in a dynamic Ramsey-model setting. An endogenous process of capital accumulation strengthens the optimism of Chapter 3 and the pessimism of Chapter 4. A particularly pessimistic conclusion follows from the case that I examine in which there is advanced capital and continuous technological progress in the use of this advanced capital – the economy must approach a steady-state where labour is entirely driven out the economy by advanced capital. Again, this is pessimism at work.

In Part C, Chapters 6 and 7, I explore the limitation of the demand-side analysis. The supply-side analysis in Part B leads to a pessimistic outcome for labour. Part C explores a possible demand-side argument for optimism. In Chapter 6 I provide an overview of the role that the demand-side performs in the existing task-based literature. First I review the role of demand in the task-based models from Part A – in these, tastes and preferences have a restricted role or no role. I then go on to explain why the demand-side is important – the demand-side has an important role in determining the task composition of the economy, alongside the supply-side.[7] Then I turn to look at Autor and Dorn (2013) in particular. Unlike the other task-based models it has an important role for tastes and preferences. But I show that this role

[7]The term 'task composition' refers to the set of tasks that must be performed in an economy to produce output.

is constrained by the supply-side limitation noted before – because Autor and Dorn (2013) relies upon the ALM hypothesis, the role that the demand-side can perform is limited. I explain why it is not possible simply to 'update' the AD model.

Chapter 7 focuses on one particular demand-side argument for optimism. This is the 'structural change' argument – that technological progress causes expenditure shares across goods to change, and this will benefit labour. The argument for optimism is that labour displaced from producing certain types of goods can instead be employed in producing different goods that are in greater relative demand than before. The main lesson from this chapter is that this is possible but not inevitable. The consequence for labour depends upon whether demand shifts towards goods that require tasks in which labour, rather than capital, holds the comparative advantage. Which direction demand will actually shift in the future is not clear. I make clear in closing this thesis what kind of empirical work is necessarily to understand this effect.

In Part D I point to an important area for further research – a 'race' in task-space. This race is a possible new source of optimism for labour.[8] I use the static analysis in Chapters 3 and 4 of this thesis to show how this race might take place, and discuss a recent unpublished working paper, Acemoglu and Restrepo (2016), that has begun to explore what they also call a race. Both are incomplete, however. Analysing this race in detail is an important

[8]The literature on technology and the labour market is traditionally characterised by a different race, that set out in Tinbergen (1974) between technology and education ("technical progress and the extension of third-level education"). Katz and Goldin (2008) is named after this race. As I will explain, the new race is very different.

task for future work. I present a set of additional research questions and close with a review of the central argument of this thesis.

Part A: Foundations

1. The Task-Based Literature

1.1. Outline

The task-based literature is an exciting new field but it is haphazardly structured – there exists a small and dominant set of different models that are hard to reconcile with one another and which are used to analyse the same set of questions.

In this chapter I describe the origins of the task-based literature and set out a general framework for the task-based approach. This has not been done before in the systematic way that follows. I am able to use this framework to show how the core models in this literature are related. This framework also allows me to show why the task-based approach is more useful than the traditional, non-task-based approach in trying to understand the consequences of technological change on the labour market. This chapter does not make an original analytical contribution. But it is in a sense the most important chapter of the thesis. It demonstrates how reliant the existing task-based literature is upon what is known as the 'ALM hypothesis', a particular understanding of how machines operate and an implied set of assumptions about the capabilities of machines based on that understanding.[1] In turn it pro-

[1]The ALM hypothesis is not only influential in the formal economics literature but also in the informal expert commentary on the future of work. I note this in greater detail

vides the semantic equipment for the next chapter in which I challenge the validity of the ALM hypothesis. It thereby sets the foundations for the whole of the rest of the thesis. It is only after having set up this framework that I am able to build a new set of task-based models that do not rely on the ALM hypothesis.

1.2. Skills-Biased Technological Change

1.2.1. The Concept

Until recently, the 'skills-biased technical change' (SBTC) thesis was the main way that the economic literature understood the relationship between technological change and contemporary trends in earnings and employment. The thesis responded to a "puzzle" – that during the post-war period there was "a large increase in the supply of more educated workers" and, at the same time, "returns to education" had "risen" (Acemoglu 2002).[2] The thesis explained this by claiming that technological change is 'skills-biased' – although the supply of high-skilled workers increased significantly, the thesis argued that technological change raised the demand for high-skilled workers more than proportionately. The consequence was that the relative price of high skilled workers, the skill premium, also rose. As summarised by Ace-

in this chapter. I devote substantial attention to the ALM hypothesis because of this influence.

[2] In most of this literature the 'returns to education' are measured through the 'skill premium', the relative wage of those with a college education, 'high-skilled' workers, to those with only a high school education, 'low-skilled' workers. Acemoglu (2002) is predominantly focused on the US. Beckman, Bound and Machin (1998), for example, shows that the SBTC thesis appeared to hold "in most, if not all, developed countries" in the two decades prior to publication.

moglu (2002):

> "The recent consensus is that technical change favors more skilled workers, replaces tasks previously performed by the unskilled, and exacerbates inequality." (pg. 7)

There is a large literature on the SBTC thesis. For example, Bound and Johnson (1992), Goldin and Katz (1998), Bekman, Bound and Machin (1998), Autor, Katz and Krueger (1998); Card and Lemieux (2001); Acemoglu (2002) noted before; and Katz and Goldin (2008).

1.2.2. The Canonical Model

The SBTC thesis was formalised in what is known as the 'canonical model' (Acemoglu and Autor 2011, 2012). In a basic version of this model there are two factors, low-skilled labour L^L and high-skilled labour L^H. The production function is CES:

$$Y = \left[(A^L L^L)^{\frac{\sigma-1}{\sigma}} + (A^H L^H)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (1.1)$$

where A^L and A^H are factor augmenting technology terms, and $\sigma \in [0, \infty]$. Note that technology in the canonical model is necessarily 'factor-augmenting' – it necessarily improves the productivity of at least one type of labour.[3] Also note that the factors are q-complements – an increase in the quantity or productivity of one factor necessarily raises the marginal product of the other.[4]

[3]This is a particularly problematic feature of the canonical model. Technological progress must always improve the productivity of at least one type of labour. As Acemoglu and Autor (2011) put it, "there are no explicitly skill replacing technologies." This latter possibility is the focus of the new model developed in Part B.

[4]Formally, $\frac{\partial w^L}{\partial \frac{L^H}{L^L}} > 0$, $\frac{\partial w^L}{\partial A^H} > 0$, $\frac{\partial w^L}{\partial A^L} > 0$ – and similarly for w^H.

In this model σ is the elasticity of substitution between high-skilled and low-skilled labour. The value of σ is critical for understanding the relationship between the skill premium and the relative productivities or relative supply of the two types of labour. This follows from the fact that, if firms are perfectly competitive, the assumption that factors are paid their marginal products implies that the skill premium is equal to:[5]

$$\ln\left[\frac{w^H}{w^L}\right] = \frac{\sigma-1}{\sigma}\ln\left[\frac{A^H}{A^L}\right] - \frac{1}{\sigma}\ln\left[\frac{L^H}{L^L}\right] \tag{1.2}$$

And so if $\sigma > 1$, an increase in the relative productivity of high-skilled labour will increase the skill premium, and the level of σ determines the extent to which an increase in the relative supply of high-skilled labour will decrease the skill premium. As a result, a large proportion of the SBTC literature is focused on estimating σ.

1.2.3. The Challenge: Autor et al. (2003)

Autor et al. (2003) represents an important step away from the SBTC thesis. It is motivated by a dissatisfaction with the thesis. The authors note that the SBTC thesis clearly has empirical support:

> "A wealth of quantitative and case-study evidence documents a striking correlation between the adoption of computer-based technologies and the increased use of college-education labor within detailed industries, within firms, and across plants within industries. This robust correlation is frequently interpreted as evidence of skill-biased technical change." (pg. 1279)

[5] This derivation is shown in Acemoglu (2002).

But they are frustrated with how "this interpretation merely labels the correlation without explaining its cause". They write, critically:

> "It fails to answer the question of what it is that computers do – or what it is that people do with computers – that causes educated workers to be relatively more in demand." (pg. 1280)

This is the purpose of Autor et al. (2003) – to provide a deeper explanation for the SBTC thesis and to understand what it is about "technology" that gives it its "skill-biased" property. In doing this Autor et al. (2003) introduce two fundamental innovations. The first is the introduction of what is now known as the 'task-based' approach. The second is a theory about the capabilities of "computer capital". I now look at these in turn.

1.3. The Task-Based Approach

The task-based approach is the first innovation in Autor et al. (2003). It relies on a distinction between five basic components, and a set of two types of functions that set out how these basic components are related. Autor et al. (2003) does not distinguish between these components and functions, and they are rarely distinguished in other applications of the task-based approach. But they are implicit in all task-based models. In what follows I present a general framework that clarifies them.[6]

The Five Basic Components

The task-based approach relies on a distinction between five basic components: factors; skills; tasks; capabilities; and goods:

[6]Goos, Manning and Salomons (2014) perform a similar exercise, describing what follows as a "two-stage setup".

1. **Factors** – e.g. capital K and labour L

2. **Skills** – e.g. 'low'-skilled labour L^L and 'high'-skilled labour L^H

3. **Tasks** – e.g. $z(s,t)$ where s and t are characteristics of the task z, such that task $z(s',t')$ may be 'hammering a nail' and task $z(s'',t'')$ 'solving an equation'[7]

4. **Capabilities** – e.g. L^L can perform task $z(s',t')$ with productivity $a^L(s',t') = 5$ but task $z(s'',t'')$ with productivity $a^L(s'',t'') = 2$ i.e. 1 unit of L^L produces 5 units of $z(s',t')$ but only 2 units of $z(s'',t'')$

5. **Goods** – $y(i)$ where i is the type of good e.g. $y(1)$ are 'widgets' and $y(2)$ are 'gadgets'

Components 1 and 2 state that there are *factors* that may be differentiated by *skill*. This is a traditional distinction. The task-based approach goes further, however, and also distinguishes between *skills* and *tasks*. This is component 3. A *task* is a unit of work activity that produces output. A factor's *skill* indicates that factor's *capability* at performing a given task. This is component 4. Finally, these *tasks* then combine to produce *goods*. This is component 5. In short, *factors* with a given *skill* perform *tasks* with a given *capability*, and those *tasks* are then combined to produce *goods*.

The Two Sets of Functions

The relationship between these five basic components is described by two

[7] Note that here tasks are defined in two dimensions – s and t. The 'task-type space' is two-dimensional. It is of course possible to define tasks in fewer, or more, dimensions. I will turn to this later in the chapter.

distinct sets of functions. The first set of functions describe how *factors* with particular *skills* and *capabilities* combine to perform the different types of *task* – these are the *'factor-based' production functions for tasks.* The second set of functions describe how different types of *tasks* are combined to produce each type of *good* – these are the *'task-based' production functions for goods.* When these two distinct sets of functions are taken together, the resulting aggregate function describes how particular types of *factors* combine to produce different types of *goods.* Note that the canonical model only provides this final aggregate production function, as in (1.1). I now turn to look at the generalised form of these two sets of functions.[8]

1. The 'Factor-Based' Production Functions for Tasks

The first set of functions are the factor-based production functions for tasks. These describe how factors combine to produce each type of task. A general form for these functions is where the output of each type of task $z(s,t)$ is a CES combination of all types of factors allocated to that task. Suppose there are J types of labour, i.e. L^j where $j \in \{0, ..., J\}$, and N types of capital, i.e. K^n where $n \in \{0, ..., N\}$. Then it follows that this generalised function is:[9]

$$z(s,t) = \left[\sum_0^J \left(a^j(s,t) L^j(s,t) \right)^{\gamma(j,s,t)} + \sum_0^N \left(a^n(s,t) K^n(s,t) \right)^{\gamma(n,s,t)} \right]^{\frac{1}{\gamma(s,t)}} \tag{1.3}$$

[8] I continue to assume for simplicity that tasks are only defined in two dimensions. See footnote 7.

[9] This implies that the elasticity of substitution, σ, is equal to $\frac{1}{1-\gamma}$, where $\sigma \in [0, \infty]$. This is an alternative form of a CES function to that in (1.1).

where $a^j(s,t)$ is the productivity of labour type j at performing the particular type of task $z(s,t)$, and $a^n(s,t)$ is the productivity of capital type n at performing task type $z(s,t)$. $L^j(s,t)$ is the allocation of labour type j to performing task type $z(s,t)$, and $K^n(s,t)$ is the allocation of capital type n to performing task type $z(s,t)$. If a factor is used to produce $z(s,t)$, then $\gamma(j,s,t)$ or $\gamma(n,s,t)$ is equal to $\gamma(s,t) \in [-\infty, 1]$. Otherwise the factor does not appear in the expression in (1.3).

2. The 'Task-Based' Production Functions for Goods

The second set of functions are the task-based production functions for goods. These describe the way in which different types of task combine to produce each type of good. A general form for these functions is where the output of each type of good $y(i)$ is a CES combination of all types of tasks $z(s,t)$ performed for that good, where $s \in \{0, ..., S\}$ and $t \in \{0, ..., T\}$:

$$y(i) = \left[\sum_{0}^{S} \sum_{0}^{T} (\beta(s,t,i) z(s,t))^{\mu(s,t,i)} \right]^{\frac{1}{\mu(i)}} \tag{1.4}$$

If a task $z(s,t)$ is used to produce $y(i)$, then $\mu(s,t,i)$ is equal to $\mu(i) \in [-\infty, 1]$ and $\sum\limits_{0}^{S} \sum\limits_{0}^{T} \beta(s,t,i) = 1$. Otherwise the task does not appear in (1.4). If $\mu(i) = 0$ then the task-based production function is Cobb-Douglas:

$$y(i) = \prod_{0}^{S} \prod_{0}^{T} z(s,t)^{\beta(s,t,i)} \tag{1.5}$$

It is helpful in what follows to note the existence of (1.5) as a special case. Also note that I have written (1.4) and (1.5) in discrete form. But it is possible, as I will show, to write this in continuous form as well – where goods are produced by a continuous spectrum of tasks, rather than a discrete bundle of tasks. This is an approach taken in parts of the literature.[10]

Combining the Two Functions

The factor-based production function for tasks and the task-based production function for goods can be combined to form a general aggregate production function where a single function describes how the output of a given type of good i is produced by different types of factors:

$$y(i) =$$
$$\left[\sum_{0}^{T}\sum_{0}^{S}\beta(s,t,i)\left[\left[\sum_{0}^{J}\left(a^j(s,t)L^j(s,t)\right)^{\gamma(j,s,t)}+\sum_{0}^{N}\left(a^n(s,t)K^n(s,t)\right)^{\gamma(n,s,t)}\right]^{\frac{1}{\gamma(s,t)}}\right]^{\mu(s,t,i)}\right]^{\frac{1}{\mu(i)}} \tag{1.6}$$

This is a general production function for the task-based approach – it describes how different factors combine to produce different types of goods. This function is complex but nests many of the most important production functions in economics – both those outside the task-based approach, and those within it. The supply-side in all the models that follow in this thesis use simplified versions of (1.6), based on assumptions about the five basic components and parameter restrictions on the two composite functions. But

[10] Note that both the factor-based production function for tasks, and the task-based production function for goods, are expressed as CES functions with more than two arguments. Most CES functions have only two arguments. This is Arrow et al. (1961). There is an additional literature that explores CES functions with more than two arguments, for example, Uzawa (1962) and Sato (1967). Klump and Preissler (2000) reviews the different varieties of CES functions that are used in the literature, and their role in growth models.

of course this is only the supply-side, and it is necessarily to look at the demand-side as well.

Consumer Tastes

To close a task-based model it is necessary also to describe the tastes and preferences of the consumers. A general form of this is a CES utility function of the form:

$$U = \left[\sum_{0}^{i} \theta(i) x(i)^{\rho} \right]^{\frac{1}{\rho}} \tag{1.7}$$

where $\rho \in [-\infty, 1]$ and $\sum_{0}^{S} \theta(i) = 1$.[11] If $\rho = 0$ then the utility function is Cobb-Douglas:

$$U = \prod_{0}^{I} x(i)^{\theta(i)} \tag{1.8}$$

As with the task-based production for goods, the special Cobb-Douglas case is useful for the models that follow.

1.4. The ALM Hypothesis

1.4.1. The Theory

The second innovation in Autor et al. (2003) is a theory about the capabilities of computer capital. This is known as the 'ALM hypothesis' – after 'Autor, Levy, and Murnane', the authors of Autor et al. (2003). Before discussing

[11] It follows that the elasticity of substitution in consumption, $\sigma = \frac{1}{1-\rho} \in [0, \infty]$.

this particular theory, it is useful to make an observation about terminology. In using the term 'computer capital', Autor et al. (2003) is simply extending the traditional conception of 'capital' to reflect the particular character of the capital that was increasingly prevalent at the turn of the 21th century – namely, some combination of computer hardware and software. Autor et al. (2003) use other interchangeable terms for this as well – for example, "computer-based technologies", "computers", "computer technology". In this thesis I use the terms 'machines' and 'systems' in a similar way.

Autor et al. (2003) argue that a theory of the capabilities of computer capital is important because it explains how "computerisation" – the use of computer capital – changes the task composition of the economy and, in turn, changes the structure of labour demand, since labour will be demanded to perform different types of tasks.[12] Based on "an intuitive set of observations" from economists and others, Autor et al. (2003) make two assumptions:

> "(1) that computer capital substitutes for workers in carrying out a limited and well-defined set of cognitive and manual activities, those that can be accomplished by following explicit rules (what we term "routine tasks"); and (2) that computer capital complements workers in carrying out problem-solving and complex communication activities ("nonroutine tasks")." (pg. 1280)

These two assumptions are critical. Taken seriously they imply that machines can only perform 'routine' tasks, and labour can only perform 'non-routine' tasks. Together, this distinction between 'routine' and 'non-routine' tasks, and these two assumptions with respect to that distinction, are the ALM hypothesis. The 'ALM hypothesis' is a term I use frequently in this thesis.

[12] Again, the term 'task composition' refers to the set of tasks that must be performed in an economy to produce output.

When I do so, it is to refer specifically to this distinction and these two assumptions. In the next chapter, I will focus on why the ALM hypothesis, although hugely influential, is mistaken.

1.4.2. The ALM Model

Autor et. al (2003) contains the first model that reflects the ALM hypothesis. I call this the 'ALM model'. It is important to emphasise the distinction between the 'ALM hypothesis' and the 'ALM model'. The former is a general hypothesis that consists of the distinction between 'routine' and 'non-routine' tasks and the two assumptions about the capabilities of capital with respect to those tasks as set out in section 1.4.1. The latter is one particular way of formally representing the ALM hypothesis. Most of the task-based literature and the accompanying models subscribe to ALM hypothesis – and many of these are explored in this thesis – but do so in a different way to the ALM model.

The ALM model can be derived from the general framework set out in section 1.3. The first set of assumptions are with respect to the five basic components:

1. **Factors** – two factors, L and K
2. **Skills** – no differentiation between types of factors
3. **Tasks** – two types of tasks, 'routine', r, and 'non-routine', nr
4. **Capabilities** – labour can perform 'routine' and 'non-routine' tasks (with productivity 1) but capital can only perform 'routine' tasks

5. **Goods** – a unique final good, Y

Note that the ALM model defines tasks in one dimension, s. This means the variable t that describes the second dimension in which tasks are differentiated in section 1.3 can be suppressed. The ALM model then makes a related set of parameter restrictions for the factor-based production function for tasks in (1.3) and the task-based production function for goods in (1.4). For the factor-based production functions for tasks the ALM model assumes that $\gamma(j,r) = \gamma(j,nr) = \gamma = 1$ and:

$$\gamma(n,s) = \begin{cases} \gamma = 1 \text{ if } s = r \\ \\ \text{factor does not appear if } s = nr \end{cases}$$

The result is that (1.3) collapses to a set of two functions that describe how different factors combine to produce each type of task:

$$\begin{aligned} z(r) &= L(r) + K \\ z(nr) &= L(nr) \end{aligned} \tag{1.9}$$

Where $L(r)$ is the labour input of 'routine' tasks, $L(nr)$ is the labour input of 'non-routine' tasks, $L(r) + L(nr) = L$, and K is the capital input of 'routine' tasks. Turning to the task-based production function for goods, since there is a unique final good Y in the ALM model the i in (1.4) can be suppressed. The ALM model assumes for this function that $\mu(r) = \mu(nr) = \mu = 0$ and $\beta(r) = \beta$, and $\beta(nr) = 1 - \beta$. As a result (1.4) collapses to a function that describes how tasks combine to produce goods:

$$Y = z(r)^{\beta} z(nr)^{1-\beta} \tag{1.10}$$

And so when the factor-based production functions for tasks in (1.9) are combined with the task-based production function for goods in (1.10) the result is an aggregate production function:

$$Y = (L(r) + K)^{\beta} L(nr)^{1-\beta} \tag{1.11}$$

This is the ALM model version of the generalised task-based production function in (1.6). Since there is a unique final good Y in the ALM model there is no need for further specification of the demand-side of the economy.

In the ALM model, capital is supplied perfectly elastically at a price ρ per efficiency unit – and ρ falls exogenously over time due to technological progress. Autor et al. (2003) assume that in equilibrium the wage paid to labour that performs 'routine' tasks, call this w^R, is equal to ρ. This implies that in equilibrium labour and capital are perfect substitutes in performing 'routine' tasks.[13]

Note that the aggregate production function in (1.11) reflects the ALM hypothesis. First, capital is a perfect substitute for labour in performing 'routine' tasks. Secondly, capital cannot perform 'non-routine' tasks and so can only substitute indirectly for labour that does perform 'non-routine' tasks – by performing 'routine' tasks. Thirdly capital is a q-complement to

[13] Autor et al. (2003) recognise that perfect substitutability is a strong assumption, but defend it with the claim that, in fact, "the only substantive requirement for our model is that computer capital is more substitutable for routine than nonroutine tasks."

labour that performs 'non-routine' tasks – an increase in the capital input of 'routine' tasks causes an increase in the marginal productivity of labour that performs the 'non-routine' tasks. Finally, note that capital is also a relative complement to labour that performs 'non-routine' tasks.[14] However also note that as a consequence there is no type of output that is produced only by capital – both 'routine' and 'non-routine' tasks require labour.

Autor et al. (2003) use the ALM model to provide a formal account of the trends in the task composition of work from 1960 to 1998 in the US. They argue that industries that make relatively intensive use of labour input of 'routine' tasks would make relatively large investments in computer capital as its price exogenously declines. The result would be twofold. First, these industries would reduce the labour input of 'routine' tasks, and replace it with capital input of 'routine' tasks (this relies on (1) in the quotation in section 1.4.1). Secondly, these industries would increase their labour input of 'non-routine' tasks, for which capital complements (this relies on (2) in the quotation in section 1.4.1). The thesis is tested in Autor et al. (2003) and not falsified.

1.4.3. Contrast with the Canonical Model

It is informative to compare the task-based ALM model with the non-task-based canonical model set out in section 1.2.2. Although the canonical model is non-task-based it can still be derived from the generalised functions set out in section 1.3. The first set of assumptions are again with respect to the five

[14] Relative to labour that performs 'routine' tasks. It reduces the marginal product of 'routine' labour, but raises the marginal product of 'non-routine' labour.

basic components:

1. **Factors** – one factor, L
2. **Skills** – two skill levels, low-skilled and high-skilled i.e. $j \in \{L, H\}$
3. **Tasks** –two types of tasks, $z(L)$ and $z(H)$ i.e. $s \in \{L, H\}$.
4. **Capabilities** – low-skilled labour has productivity a^L at $z(L)$ and high-skilled labour has productivity a^H at $z(H)$
5. **Goods** – a unique final good, Y

Tasks in the canonical model, as in the ALM model, are defined in one dimension, s, and there are only two types of tasks, L and H. For the factor-based production functions for tasks the canonical model assumes that:

$$\gamma(j, s) = \begin{cases} \gamma = 1 \text{ if } s = j \\ \text{factor does not appear if } s \neq j \end{cases}$$

The result is that (1.3) collapses to a set of two functions that describe how different factors combine to produce each type of task:

$$\begin{aligned} z(L) &= a^L L^L \\ z(H) &= a^H L^H \end{aligned} \tag{1.12}$$

For the task-based production function for goods the canonical model assumes that $\mu(s) = \mu \ \forall s$ and $\beta(s) = 1 \ \forall s$.[15] The result is that (1.4) collapses to a function that describes how different tasks combine to produce

[15] As in the ALM model there is only a unique final good and so the i in (1.3) can be suppressed.

the unique final good:

$$Y = [z(L)^{\mu} + z(H)^{\mu}]^{\frac{1}{\mu}} \tag{1.13}$$

And so when the factor-based production functions for tasks in (1.12) are combined with the task-based production function for goods in (1.13) the result is the canonical production function:

$$Y = \left[\left(a^L L^L\right)^{\mu} + \left(a^H L^H\right)^{\mu}\right]^{\frac{1}{\mu}} \tag{1.14}$$

This is the canonical model version of the general aggregate task-based function in (1.6). There is no need to specify the demand-side of the economy – output again consists of a unique final good Y in the canonical model, and so there is no role for tastes and preferences.

That the canonical model can be derived from (1.6) is illuminating for two reasons. First it shows that the task-based approach can be generalised sufficiently to nest models that do not explicitly follow the task-based approach. Secondly, breaking down the canonical production function into two parts – the assumptions about what factors produce which tasks, and the assumptions about how tasks combine to produce goods – makes clear the *implicit* assumption in the canonical approach. This is the assumption that each type of factor performs its own distinct type of task. Put simply, the type of tasks that factors perform in the canonical approach never change – there are 'labour tasks' and there are 'capital tasks'. There is no distinction between *factors*, their *skills* and the *tasks* they perform – $\gamma(j, s) = 1$ only if j, the type of skill, is equal to s, the type of task.

The ALM model therefore makes two advances on the canonical model. The first is that it explicitly distinguishes between the factor-based production function for tasks and the task-based production function for goods, unlike the canonical model which conflates them. The canonical model is semantically deprived – without the concept of the 'task' and the accompanying concept of a factor's 'capability' at performing that task, it is not possible to have an explicit set of factor-based production functions for tasks. The second advance is that, in making this distinction, the ALM model can support a more nuanced understanding of the consequences of increasingly capable machines on the labour market.[16]

1.5. Further Use of the Task-Based Approach

1.5.1. Multi-Dimensional 'Task-Type' Space

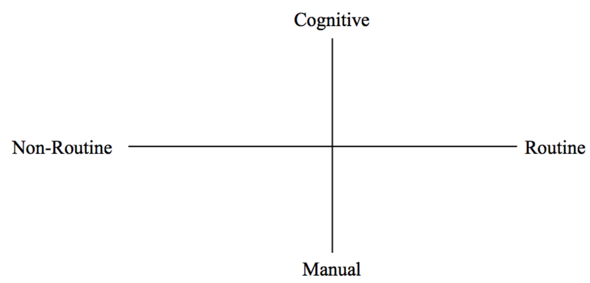
In the ALM model, tasks are only defined in one dimension – they are either 'routine' or 'non-routine'. Elsewhere in the literature, tasks $z(.)$ are often conceptualized as falling in multidimensional task-type space such that $z(s_1, ..., s_n)$ where each s_i is a particularly characteristic of the task. If any given tasks have different s_i those tasks are of a different type.

One example of this is shown in Figure 1.1.[17] This particular grid distinguishes between two dimensions of a task's type – how 'routine' the task is and also the physical character of the task (the former can be thought of as 's_1' and the latter as 's_2'). This is from McDermott (2014). This particular

[16]I first use the term 'increasingly capable machines' in Susskind and Susskind (2015).
[17]See McDermott (2014) and Levy and Murnane (2004), or Table I in Autor et al. (2003).

distinction is useful because it challenges two common, but false, assumptions about tasks: that only 'manual' tasks can be computerised and that 'cognitive' tasks cannot. Figure 1.1 makes clear that it is important not to conflate the features of tasks that make them amenable to computerisation (accordingly to the ALM hypothesis, the 'routine' character of the task) and the physical character of the task (whether it is manual or cognitive).

Figure 1.1: Two-Dimensional Task-Type Space



Note again though that Figure 1.1 represents only one type of task-type space – other economists have used different task-type space.

1.5.2. Labour Market Polarization

A significant use of the task-based approach and the ALM hypothesis is the literature that explores labour market 'polarisation'. Labour market polarisation refers to two recent features of the labour market in the US and elsewhere. The first is 'wage polarisation' – "the simultaneous growth of high and low wages relative to the middle" (Acemoglu and Autor 2011). Earnings growth in the US, for example, having been fairly monotone in the 1970s and 1980s, became notably non-monotone in the 1990s and 2000s. The second is 'job polarization' – "the simultaneous growth of the share of employment in high skill, high wage occupations, and low skill, low wage occupations" (Acemoglu and Autor 2011). Employment growth in the US, for example, having been fairly monotone in the 1980s, became polarised during the 1990s and, during the 2000s, was "heavily concentrated" at the bottom of the occupational skill distribution. These trends were not unique to the US and required an explanation – but the SBTC thesis cannot explain them. In particular it cannot explain the growth in demand for labour at the lower-tail of the skills distribution – the SBTC thesis relies on technological change being 'skills-biased', but in this case it appears to be 'no-skills-biased' as well.

Three key papers form the foundation of the alternative task-based explanation: Goos and Manning (2007) which considers polarization in Britain; Autor, Katz, and Kearney (2006, 2008), polarization in the US; Goos, Manning and Salomons (2009), polarization in the EU. The approach in these papers is similar, and in two parts. First, the papers adopt the principles of the ALM hypothesis. But, secondly, and distinct from the basic ALM hypothesis, the authors also make an empirical claim about the relationship between tasks and skills. Specifically, they show that 'routine' tasks are concentrated in the middle of the earnings and occupational skills distribution. This leads to a more nuanced application of the ALM hypothesis – that increasingly capable machines, with a growing capability in 'routine' tasks, displace labour from the middle of the occupational skills distribution, and drive displaced labour towards the tails. This is the 'hollowing out' or 'routinisation' thesis.

1.5.3. The AA Model

These first papers exploring polarization were predominantly empirical.[18] Acemoglu and Autor (2011) provides a new formal model – I call this the 'AA model'. The AA model is presented as an alternative to the canonical model and builds on the mechanics of Dornbusch et al. (1977).[19] Again, a basic AA model can be derived from the general task-based framework set out in section 1.3.[20] The first set of assumptions are with respect to the five basic components, and in a basic AA model:

1. **Factors** – two factors, L and K
2. **Skills** – low- and high-skilled labour, and no differentiation on types of capital i.e. $j \in \{L, H\}$ and $n = K$
3. **Tasks** – there is a continuous spectrum of tasks $z(s)$ where $s \in [0, 1]$
4. **Capabilities** – $a^j(s)$ is the productivity of labour with skill level j at performing task type s and $a^n(s)$ is the productivity of capital type n at that same type of task
5. **Goods** – a unique final good, Y

[18] They are not entirely empirical. For example Autor, Katz, and Kearney (2006) present a model – the 'AKK model' – where output, Y, is a Cobb-Douglas combination of 'abstract', 'manual', and 'routine' tasks i.e. $Y = A^{\alpha} R^{\beta} M^{\gamma}$ where $\alpha + \beta + \gamma = 1$. This model is also based on the ALM hypothesis – capital can only perform 'routine' tasks, whereas labour can perform all types of tasks. But the AKK model is so similar to the ALM model it does not warrant its own separate exposition.

[19] Dornbusch et al. (1977) is a two-country trade model where a continuum of goods are traded and the result is a Ricardian pattern of specialisation. The Ricardian feature of this model is also a feature of the AA model and the new model I develop in Part B. In ways that I will explain, the new model in Part B draws upon the structure of Dornbusch et al. (1977).

[20] In Acemoglu and Autor (2011), the model has three types of labour and capital.

Like the canonical model and the ALM model, the AA model also defines tasks in one dimension, s. However, rather than there being two types of tasks, there is now a spectrum of types of tasks.[21] In addition, there are no restrictions on the type of tasks that factors can perform. In the canonical model, each factor can only perform its own type of task. In the ALM model, labour can perform all types of tasks but capital cannot. However, in the AA model *all* labour with skill j and capital of type n can perform each task s with productivity $a^j(s)$ and $a^n(s)$.

With respect to the factor-based production function for tasks the AA model assumes that $\gamma(j,s) = \gamma(n,s) = \gamma = 1 \; \forall j,n,s$. The result is that (1.3) collapses to a set of functions that describes how different factors perform different types of tasks:[22]

$$z(s) = a^L(s)L^L(s) + a^H(s)L^H(s) + a^K(s)K(s) \quad \forall s \tag{1.15}$$

For the task-based production function for goods it assumes that $\mu(s) = \mu = 0$ and $\beta(s) = 1 \; \forall s$.[23] The result is that (1.4) collapses to a function that describes how tasks combine to produce goods:

$$Y = \prod_0^1 z(s)^{ds} \tag{1.16}$$

where output Y is the 'geometric' or 'product' integral of $z(s)$, since s is on a continuous spectrum $[0,1]$. Equivalently, (1.16) can be written:

[21] Again, this means the variable t in section 1.3 can be suppressed.
[22] For simplicity, I write K^K as K.
[23] Again since there is a unique final good the i in (1.4) can be suppressed.

$$\ln Y = \int_0^1 \ln z(s)\, ds \tag{1.17}$$

The form in (1.17) is used in Acemoglu and Autor (2011). When the factor-based production function for tasks in (1.15) is combined with the task-based production function for goods in (1.17) the result is the aggregate production function:

$$\ln Y = \int_0^1 \ln \left[a^L(s)L^L(s) + a^H(s)L^H(s) + a^K(s)K(s) \right]\, ds \tag{1.18}$$

(1.18) shows how in the AA model different factors combine to produce the different goods. This is the alternative to (1.11) in the ALM model and (1.14) in the canonical model. Again, since there is a unique final good Y there is no need for further specification of the demand-side of the economy, and no role for consumer tastes and preferences.

The AA model is known as a 'Ricardian' model because in equilibrium there is a clear pattern of factor specialisation across the task spectrum. Unlike the ALM model, where labour that performs 'routine' tasks is paid a wage w^R equal to the rental price of capital ρ,[24] in the AA model the factor prices are not tied down in this way. So although factors are perfect substitutes in performing different types of tasks, as in (1.15), because their factor prices may differ it can be shown that in equilibrium each type of factor performs its own unique set of types of tasks i.e. no two factors perform the

[24] The ALM model is discussed in section 1.4.2.

same task $z(s)$.[25] Formally, for some spectrum of types of tasks $z(s)$ where $s \in [0,1]$, in equilibrium there exists two cut-offs, $\bar{s}$ and $\bar{\bar{s}}$, such that all tasks $s \in [0,\bar{s}]$ are performed by low-skilled workers, all tasks $s \in [\bar{\bar{s}},1]$ are performed by high-skilled workers, and $s \in [\bar{s},\bar{\bar{s}}]$ are performed by capital. There is a clear pattern of specialisation. This means that in equilibrium (1.15) can be re-written as:

$$\begin{aligned} z(s) &= a^L(s)L^L(s) \quad \forall s \in [0,\bar{s}] \\ z(s) &= a^K(s)K(s) \quad \forall s \in [\bar{s},\bar{\bar{s}}] \\ z(s) &= a^H(s)L^H(s) \quad \forall s \in [\bar{\bar{s}},1] \end{aligned} \tag{1.19}$$

Like the ALM model, the AA model is an advance on the canonical model. Since the AA model also uses the task-based approach, it also explicitly distinguishes between the factor-based production function for tasks and the task-based production function for goods. But the AA model is also an advance on the ALM model, in several respects. First the AA model uses a spectrum of tasks – whereas in the ALM model there are only two types of tasks, 'routine' and 'non-routine'. Secondly, in the AA model, factors perform a range of different types of tasks – whereas in the ALM model, labour performs only two types of tasks, and capital only one. And thirdly, in the AA model, factor prices can be different.[26]

Despite these differences, though, the AA model again relies upon the ALM hypothesis. But because the AA model uses a more nuanced conception

[25] Contrast this with the ALM model in section 1.4.2 where labour and capital work together to perform 'routine' tasks.

[26] As noted before, in the ALM model, the labour that performs 'routine' tasks is paid a w^R equal to ρ. See section 1.4.2.

of tasks, it relies upon the ALM hypothesis in a different way. In the ALM model the ALM hypothesis is bluntly introduced – a task is either 'routine' or 'non-routine', and can either be performed by capital or not. But in the AA model the application of the ALM hypothesis is more subtle – the tasks in the middle of the spectrum of tasks are *more* 'routine' than those that surround them, and there is not a sharp demarcation between types of tasks as in the ALM model.[27]

It is also important to note that again, as in the ALM model, there is no output in this model that is produced by capital alone – although labour specialises in performing its own subset of tasks, and capital in its own subset of tasks, the unique output requires that *all* tasks are necessarily performed to produce it.

1.5.4. The AD Model

Autor and Dorn (2013) use the ALM hypothesis to answer a different question. Their focus is on the growth of low-skill service occupations in the US between 1980 and 2005 (having been flat between 1959 and 1979, Autor 2014) – *and* the polarisation of employment and wages that accompanied it. Like the ALM model and the AA model, the model in Autor and Dorn (2013) – I call this the 'AD model' – is also task-based. However the AD model has two final goods and a role for tastes and preferences – unlike the ALM and the AA model which have only a unique final good and so no role for the demand-side.

Again the AD model can be derived from the general framework set out

[27] This difference is important in developing the new model in Part B of the thesis.

before. The first set of assumptions are with respect to the five basic components:

1. **Factors** – two factors, labour L and capital K
2. **Skills** – two skills, low-skilled L^L and high-skilled L^H i.e. $j \in \{L, H\}$
3. **Tasks** – three types of tasks, 'manual', $z(m)$, 'routine', $z(r)$, and 'abstract', $z(a)$
4. **Capabilities** – only high-skilled labour can perform 'abstract' tasks (with productivity $a^H(a) = 1$), only low-skilled labour can perform 'manual' tasks (with productivity $a^L(m)$), but *both* low-skilled labour and computer capital can perform 'routine' tasks (with productivity $a^L(r)$ and $a^K(r)$ respectively)
5. **Goods** – two outputs, 'goods' $y(g)$ and 'services' $y(se)$

The AD model makes a set of parameter assumptions for the factor-based production function for tasks in (1.3).[28] The AD model assumes that $\gamma(m, L) = \gamma(m) = 1$ and since only low-skilled labour can perform 'manual' tasks neither $\gamma(m, H)$ nor $\gamma(m, K)$ appear. This results in a task-based production function for 'manual' tasks:

$$z(m) = a^L L^L(m) \tag{1.20}$$

Secondly, the AD model assumes that $\gamma(a, H) = \gamma(a) = 1$ and since 'abstract'

[28] Again, since tasks are defined only in one dimension, the t from section 1.3 can be suppressed.

tasks are only performed by high-skilled labour neither $\gamma(a, L)$ nor $\gamma(a, K)$ appear. This results in a task-based production function for 'abstract' tasks:

$$z(a) = L^H(a) \tag{1.21}$$

Note that because 'abstract' tasks and 'manual' tasks can only be performed by labour, it follows from the ALM hypothesis that these tasks are understood to be two distinct subsets of 'non-routine' tasks – namely 'non-routine manual' tasks and 'non-routine abstract' tasks. Thirdly, it assumes that $\gamma(r, L) = \gamma(r, K) = \gamma(r) = \gamma$ and since 'routine' tasks are not performed by high-skilled labour $\gamma(r, H)$ does not appear. This results in a task-based production function for 'routine' tasks:

$$z(r) = \left[(a^L(r)L^L(r))^\gamma + (a^K(r)K(r))^\gamma\right]^{\frac{1}{\gamma}} \tag{1.22}$$

'Routine' tasks are only performed by low-skilled labour and capital, where the elasticity of substitution between them depends on γ – by assumption, Autor and Dorn (2013), make this elasticity of substitution greater than 1.[29]

For the task-based production functions the AD model assumes that $\mu(s) = \mu = 0$. This means the functions are of the form in (1.5). In addition, the AD model assumes that if $i = se$ then $\beta(se, m) = 1$. And if $i = g$ then $\beta(g, r) = \beta$ and $\beta(g, a) = 1 - \beta$:

$$\begin{aligned} y(se) &= z(m) \\ y(g) &= z(a)^{1-\beta} z(r)^\beta \end{aligned} \tag{1.23}$$

[29]And so γ is restricted to $\gamma \in (0, 1]$.

And so when the factor-based production function for tasks in (1.20), (1.21) and (1.22) are combined with the task-based production functions for goods in (1.23) the result is the AD model:

$$\begin{aligned} y(se) &= a^L(m)L^L(m) \\ y(g) &= L^H(a)^{1-\beta}\left[(a^L(r)L^L(r))^\gamma + (a^K(r)K(r))^\gamma\right]^{\frac{\beta}{\gamma}} \end{aligned} \tag{1.24}$$

Since there are now two goods, $y(g)$ and $y(se)$, the AD model uses a two-good version of the CES utility function in (1.7) where $\rho(se) = \rho(g) = \rho > 0$ and $\theta(se) = \theta(g) = 1$:

$$U = [y(se)^\rho + y(g)^\rho]^{\frac{1}{\rho}} \tag{1.25}$$

From (1.20)–(1.24) it is clear that the AD model again reflects the ALM hypothesis. First, capital can only perform 'routine' tasks and cannot perform 'non-routine' tasks. Secondly, labour that performs 'non-routine abstract' tasks is q-complemented by capital that performs 'routine' tasks. Thirdly, technological progress is heavily biased – it takes the form of a "secularly falling price of accomplishing *routine* tasks using computer capital"[30] and, no matter the level of technological progress, computer capital is never able to perform 'non-routine' tasks. And finally note that the consequence of these features is that, although there are two types of goods, neither good is produced by capital alone – both goods require at least one type of labour.

There are two outcomes of interest in the AD model – whether there is employment polarisation or wage polarisation. These outcomes depend upon

[30]Emphasis added.

two factor-price ratios – the wage for 'non-routine manual' tasks relative to 'routine' tasks, $\frac{w^m}{w^r}$, and the wage of 'non-routine abstract' tasks relative to 'non-routine manual' tasks, $\frac{w^a}{w^m}$. The level of $\frac{w^m}{w^r}$ determines whether there is employment growth in low-skilled service occupations. The level of both $\frac{w^m}{w^r}$ and $\frac{w^a}{w^m}$ determine whether there is wage polarisation.

To see how *employment* polarisation takes place in the AD model, in particular the increase in employment in low-skilled service sectors, consider that if low-skilled labour performs 'routine' tasks in the goods sector they receive w^r, and if they perform 'non-routine manual' tasks in the services sector they receive w^m. It follows that if $\frac{w^m}{w^r}$ rises, this causes a flow of low-skilled labour from the goods sector into the services sector (and vice-versa for a fall). This is employment polarisation.[31]

To see how *wage* polarisation takes place in the AD model, first note that Autor and Dorn (2013) describe "a monotone increase in inequality" as the case where $\frac{w^a}{w^m}$ rises and $\frac{w^m}{w^r}$ falls. Since Autor and Dorn (2013) also define wage polarisation in contrast to a monotone increase in inequality like this, it follows that if $\frac{w^a}{w^m}$ is either stable or falls, and $\frac{w^m}{w^r}$ rises, this is said to be wage polarisation in the AD model.[32]

An important theoretical insight from Autor and Dorn (2013) is that these polarisation effects – both in employment and wages – depend upon the elasticities of substitution in production and consumption. If the elasticity of substitution in *production* between capital and low-skilled labour in performing 'routine' tasks is large enough relative to the elasticity of substi-

[31]This is set out in equation (12) in Autor and Dorn (2013).
[32]This is set out in equation (12) and (13) in Autor and Dorn (2013).

tution in *consumption* between goods and services, then $\frac{w^m}{w^r}$ rises as technological progress takes place. This causes wage polarisation in the lower-tail, and employment polarisation. Furthermore, if the consumption elasticity is less than or equal to 1, then $\frac{w^a}{w^m}$ is either constant or falling and there is also wage polarisation in the upper-tail. In short, both the demand-side *and* the supply-side matter in understanding the consequences of technological progress on the labour market.

In using the task-based approach the AD model is, like the ALM model and the AA model, an advance on the canonical model. In addition, since the AD model has two goods rather than one good it has a role for tastes and preferences. This is an advance on the ALM model and the AA model. But there is a cost to this advance – the AD model, in introducing a role for tastes and preferences, abandons the task spectrum, the most useful feature of the AA model. In its place, are three distinct types of tasks. But again, as noted before, the AD model relies heavily upon the ALM hypothesis. I will return to this observation in Chapter 6.

1.5.5. Goos, Manning and Salomons (2014)

Goos et al. (2014) look at job polarisation in 16 Western European countries from 1993 - 2010. First they show that job polarisation is pervasive, holding in each country in this larger set. Secondly they consider two competing hypothesis for polarisation – technological change and offshoring – and show the former is "much more important". In their approach technological change is 'routine-biased' – following the ALM hypothesis, it is biased towards replacing labour that performs 'routine' tasks. To do this, Goos et al. (2014)

develop and estimate a very general task-based model without many of the restrictions that characterise the ALM model, the AA model, or the AD model. This model is similar to the general framework I set out before. But again it depends upon the ALM hypothesis. And again, all types of output necessarily require labour.

1.5.6. Wider Literature

There are two observations to make about the wider literature. The first is that its understanding of the operation and capabilities of machines is almost entirely dominated by the ALM hypothesis. Part of this literature makes direct use of it. For example, Autor et al. (2008) uses the ALM hypothesis to reconcile the traditional SBTC thesis with the "revisionist literature" that argues trends in US earnings in equality are explained by one-off, non-market factors (e.g. the falling real value of the minimum wage). Part of this literature makes indirect use of it. For example, Sachs and Kotlikoff (2012) argue that "smart machines substitute for unskilled workers .... [but] are designed and run by skilled workers". This is an argument that relies upon assumptions about the capabilities of machines that are captured in the ALM hypothesis. And the influence of the ALM hypothesis is widespread in expert commentary outside the formal economics literature as well. For example, Frank Levy and Richard Murnane, the 'L' and the 'M' in the 'ALM' hypothesis,[33], co-authored a book entitled "The new Division of Labor" that was described by Brynjolfsson and McAfee (2011; 2014), writing outside the formal economics literature, as "by far the best research and thinking on this

[33] And the 'A' is David Autor.

topic when it came out in 2004". In turn, Remus and Levy (2015) also use the ALM hypothesis to explore the consequences of technological progress on legal work.

The final observation is that criticism of the ALM hypothesis, where it exists, has tended to take place in the expert commentary outside the economics literature. For example, Frey and Osborne (2013) and Brynjolfsson and McAfee (2014) both contain important criticism of the ALM hypothesis.[34]

1.6. Model Index

It is helpful to gather the four models set out in this chapter together, and set them out alongside each other. Call the factor-based production function for tasks F_{F-T}, the task-based production function for goods F_{T-G}, the combined production function F, and the utility function U. Then it follows that:

Canonical Model

F_{F-T}: $z(L) = a^L L^L; \quad z(H) = a^H L^H$

F_{T-G}: $Y = [z(L)^\mu + z(H)^\mu]^{\frac{1}{\mu}}$

F: $Y = \left[\left(a^L L^L\right)^\mu + \left(a^H L^H\right)^\mu\right]^{\frac{1}{\mu}}$

U: N/A

[34] Frey and Osborne (2013) for example note that "computerisation is now spreading to domains commonly defined as non-routine". This observation motivates the analysis in Chapter 2.

ALM Model

F_{F-T}: $z(r) = L(r) + K; \quad z(nr) = L(nr)$

F_{T-G}: $Y = z(r)^{\beta} z(nr)^{1-\beta}$

F: $Y = (L(r) + K)^{\beta} L(nr)^{1-\beta}$

U: N/A

AA Model

F_{F-T}: $z(s) = a^L(s)L^L(s) + a^H(s)L^H(s) + a^K(s)K(s)$

F_{T-G}: $\ln Y = \int_0^1 \ln z(s)\, ds$

F: $\ln Y = \int_0^1 \ln \left[a^L(s)L^L(s) + a^H(s)L^H(s) + a^K(s)K(s)\right]\, ds$

U: N/A

AD Model

F_{F-T}: $z(a) = L^H(a); \quad z(m) = a^L L^L(m); \quad z(r) = \left[(a^L(r)L^L(r))^{\gamma} + (a^K(r)K(r))^{\gamma}\right]^{\frac{1}{\gamma}}$

F_{T-G}: $y(se) = z(m); \quad y(g) = z(a)^{1-\beta} z(r)^{\beta}$

F: $y(se) = a^L(m)L^L(m); \quad y(g) = L^H(a)^{1-\beta}\left[(a^L(r)L^L(r))^{\gamma} + (a^K(r)K(r))^{\gamma}\right]^{\frac{\beta}{\gamma}}$

U: $U = [y(se)^{\rho} + y(g)^{\rho}]^{\frac{1}{\rho}}$

1.7. Conclusion

The task-based literature orginates with Autor et al. (2003) and was a response to a dissatisfaction with the SBTC thesis. The two innovations in Autor et al. (2003) – the task-based approach and the assumptions about the capabilities of capital – have proved popular for two reasons. First they appear to provide a deeper explanation for the SBTC thesis. In short, the ALM hypothesis claims that computers have a comparative advantage in 'routine' tasks. Secondly, models that are based on the ALM hypothesis appear to provide an explanation for labour market trends that the SBTC thesis struggles to explain. This is the job and wage polarization described in this section.

There are three main lessons from this analysis. The first is that only by setting out a general task-based framework, as I have done in this chapter, is it possible to distinguish between the different assumptions that various task-based models make and to show how these models relate to one another. The second is that the 'task-space' is the appropriate domain in which to explore the consequences of increasingly capable machines on the labour market. The reason is that any analysis that does not use this task-space is semantically deprived. This is a problem with the canonical model. It lacks certain concepts – in particular *tasks* and *capabilities* – that are important in explaining why technological change might be biased in the direction of a particular factor.[35] The third is that it is possible to show how reliant the task-based

[35] Recall that this was the nature of Autor et al. (2003)'s dissatisfaction. See section 1.2.3.

literature is upon the ALM hypothesis. An important consequence of this final point is that the core set of task-based models in this chapter share a common property – all goods necessarily require at least one type of labour to produce them.[36] The ALM hypothesis and its problems are the focus of the next chapter.

[36] I explain this property in greater depth in the next chapter, in section 2.6.2.

2. The ALM Hypothesis: The Problem and an Alternative

2.1. Outline

Chapter 1 shows that the ALM hypothesis is at the core of the task-based literature, and dominates formal and informal analysis of the future of work. In this chapter I argue that the ALM hypothesis is mistaken. This is an original contribution of this thesis. The ALM hypothesis has provided a useful account of labour market trends in the second half of the 20th century, but it is unlikely to do so in the 21st century. I claim this is because the ALM hypothesis uses a distinction between 'routine' and 'non-routine' tasks that is of diminishing relevance, relying on a set of assumptions about the capabilities of machines with respect to those tasks that does not reflect the recent character of technological progress. I show that both of these problems follow from the economic literature's reliance on reasoning about the operation and capability of machines that was correct during the 'first generation' of AI but, in the current 'second generation' of AI, is false. This claim gains some support from recent micro-level evidence from the U.S. labour market.

The contribution in this chapter is conceptual rather than analytical –

nevertheless it is an important one. The ALM hypothesis provides the foundation for the existing task-based literature. It is based on the claim that machines can substitute for labour that performs 'routine' tasks but complement labour that performs 'non-routine' tasks.[1] Yet in practice, as I will describe in this chapter, we see an increasing number of 'non-routine' tasks that can be performed by machines. The contribution of this chapter is emphatically *not* a claim that 'non-routine' tasks are in some sense becoming 'routine'. It is a deeper claim about a change in the nature of how machines operate and, as a result, an argument for why the 'routine' vs. 'non-routine' distinction is no longer a useful demarcation between what machines can and cannot do. I set this claim out in detail in this chapter, but a summary is as follows.

The ALM hypothesis is based on two premises. The first is that to perform a given task a machine must be set an explicit procedure, a set of rules, that reflects how a human being performs that task. The second is that while human beings draw on 'explicit' knowledge to perform 'routine' tasks – this is knowledge that a human being can articulate – they instead draw on 'tacit' knowledge to perform 'non-routine' tasks – this is knowledge that a human being cannot articulate.[2] As a result, the argument of the ALM hypothesis is that since human beings cannot articulate their thinking processes when they perform 'non-routine' tasks, they cannot write an explicit procedure to reflect it, and so these tasks cannot be performed by a machine. The prob-

[1]The ALM hypothesis is set out in section 1.4.1. I explore this claim in depth in this chapter.

[2]This distinction between 'tacit' and 'explicit' knowledge originates with Polanyi (1966), and is discussed at length in this chapter.

lem is that this reasoning is based on the mistaken belief that the only way for a machine to perform a task is for it to follow the way a human being performs the task. This is implicit in the first premise. It is correct that this was the only feasible approach in the first generation of AI. But systems in the second generation of AI are performing tasks in fundamentally different ways to human beings, relying on brute-force processing power and massive data-storage capability. As a result, the fact that human beings cannot articulate how they perform certain types of task is no longer a constraint on computerisation. This apparently simple observation has not been recognised in the economics literature, and the result is that this literature relies upon a mistaken understanding of the operation and capabilities of these machines.

In the closing part of chapter, I present and defend an alternative approach – I argue that a new concept, 'routinisability', ought to replace the existing concept of 'routineness' in the literature. One way to interpret the mistake set out before is that the literature, at present, conflates these two concepts. This alternative approach provides new foundations for the new models developed later in this thesis.

2.2. The Basis for the ALM Hypothesis

At the core of the task-based literature is the ALM hypothesis. As set out in section 1.4.1 the ALM hypothesis has two components – a distinction between 'routine' and 'non-routine' tasks and an assumption about the capabilities of capital with respect to those tasks, namely that machines can only perform 'routine' tasks. This section addresses two fundamental questions about this approach. First – what is the basis for the 'routine' vs. 'non-routine' distinction? And secondly – what is the basis for the assumption that machines can only peform 'routine' tasks?

2.2.1. 'Routine' vs. 'Non-Routine' Tasks

Autor et al. (2003) is the first paper to argue formally for and use the distinction between 'routine' and 'non-routine' tasks. The task-based literature has pervasively and uncritically adopted this argument and distinction. And so it is important to address it as it first appears in that original paper.[3]

In Autor et al. (2003), the distinction is based on the work of Michael Polanyi, a philosopher – in particular Polanyi (1966). The authors argue that 'non-routine' tasks are "tasks fitting Polanyi's description", where Polanyi's description is quoted as follows:

> "**We can know more than we can tell** [p. 4] . . . The skill of a driver cannot be replaced by a thorough schooling in the theory of the motorcar; the knowledge I have of my own body differs altogether from the knowledge of its physiology; and the rules of rhyming and prosody do not tell me what a poem told me, **without any knowledge of its rules** [p. 20]." [Emphasis added] (pg. 1283)

The Autor et al. (2003) account therefore implies that 'non-routine' tasks are those that are performed by human beings using knowledge about which "we can know more than we can tell", that human beings can perform without "any knowledge" of the rules they follow. Put simply 'non-routine' tasks

[3] In Autor, Levy, and Murnane (2002) there is a more informal discussion of the ideas that would become the ALM hypothesis in Autor, Levy, and Murnane (2003). The argument in this earlier paper is that "what computers actually do" is "the execution of procedural or "rules-based" logic". This is very similar to the ALM hypothesis and shares its shortcomings.

are those that rely on what Polanyi called ‘tacit’ knowledge – knowledge that people struggle to articulate or set out when called upon to do so. For example, a world-class sportsman may struggle to articulate the set of accumulated heuristics that allow him to throw a ball unrivalled distances. But this is still ‘knowledge’ – it is just not ‘explicit’ knowledge that can readily be articulated, but is ‘tacit’. Autor (2014), a more recent re-statement of the task-based approach, again emphasises this definition of ‘non-routine’ tasks, and again defines them with reference to Polanyi’s concept of ‘tacit’ knowledge. It argues that ‘non-routine’ tasks are those that we only “tacitly understand how to perform”, that involve processes that we “do not explicitly understand”. That there exists tacit knowledge, and ‘non-routine’ tasks, is, Autor (2014) argues, “Polanyi’s Paradox”.

This addresses the first question – what are the grounds for the distinction between ‘routine’ and ‘non-routine’ tasks? Following Autor et al. (2003) and Autor (2014), ‘non-routine’ tasks are those that require tacit knowledge to perform. This understanding is maintained throughout the task-based literature. For example, Levy and Murnane (2004) describe Polanyi’s concept of tacit knowledge as “the more profound limit” on tasks that can be done by machines.

The second question now follows – why the fact that a task is ‘non-routine’, and so requires tacit knowledge, implies that a machine is poorly suited to perform it? And conversely, why the fact that a task is ‘routine’, and so requires explicit knowledge, implies that a machine is well suited to perform it?

2.2.2. 'Computer Capital' and Routine Tasks

Just as Autor et al. (2003) is the first to use the 'routine' vs. 'non-routine' distinction, so too it is the first to argue that machines can only perform 'routine' tasks. Again this assumption has been adopted pervasively and uncritically in the literature that followed. The grounds for this claim are based on a particular understanding of how machines must operate. In Autor et al. (2003) this understanding is that a machine can only perform a task if it can "follow explicit programmed rules". This implies that a machine requires a task to be "exhaustively specified with programmed instructions" and without these "explicit programmed instructions" a machine cannot perform the task. The same point is made in Autor (2014) but with an additional claim:

> "For a computer to accomplish a task, a programmer must first fully understand the sequence of steps required to perform that task, and then must write a program that, in effect, causes the machine to precisely simulate these steps." (pg. 6)

In Autor (2015) this is described as the "traditional" approach to programming. The claim in Autor (2014; 2015) is not simply the original one made in Autor et al. (2003) that a machine must be set an explicit set of programmed rules. It is a stronger claim that those programmed rules must originate with, and precisely reflect, the thinking process of a human being. More exactly, the steps required to perform the task must first be *articulated* by a human being (the one performing the task), must be *understood* by a human being (the programmer), and then, in turn, a human being must *write* a program to simulate precisely how he performs the task. It is from this stronger claim that the case for the assumption that computer capital can only perform 'routine' tasks follows. Autor (2015) puts this formally:

> "But the scope for this kind of substitution is bounded because there are many tasks that people understand tacitly and accomplish effortlessly but for which neither computer programmers nor anyone else can enunciate the explicit "rules" or procedures." (pg. 11)

And Autor (2014) more prosaically:

> "At a practical level, Polanyi's paradox means that many familiar tasks, ranging from the quotidian to the sublime, cannot currently be computerized because we don't know the rules." (pg. 8)

The answer to the second question – why 'non-routine' tasks cannot be performed by machines – now follows. According to this argument, if human beings cannot articulate their thought process in performing a task, if they do not in some sense 'understand' how they perform the task, then they cannot articulate a set of rules for a machine to follow, and the machine will not be able to perform the task. Crucially, those tasks for which human beings cannot articulate their thought processes are the 'non-routine' ones – and so, accordingly to the ALM hypothesis, 'non-routine' tasks cannot be performed by machines. Conversely, those tasks that human beings can articulate their thought processes for are 'routine' tasks, and so those are the tasks that machines can perform.

Almost all the task-based literature has accepted this argument. Consider, for example, in Levy and Murnane (2004):

> "[C]omputers' comparative advantage over people lies in tasks that can be described using rules-based logic: **step-by-step procedures** with an action specified for every contingency." [Emphasis added] (pg. 16)

The need for “step-by-step procedures” in this extract reflects the need for “explicit programmed rules” and “explicit programmed instructions” in Autor et al. (2003) and Autor (2014).

2.2.3. The ‘First Generation’ of AI Research

The premises of the ALM hypothesis strongly reflect the approach of the first generation of AI research, conducted from the 1950s to the 1980s. For example, consider one dominant set of systems and machines built by these AI researchers, known as ‘expert systems’ or ‘knowledge-based systems’ – Winston (1977), Hayes-Roth et al. (1983) and Russell et al. (1995) provide an overview of these approaches.

Building these systems involved a series of steps. First, for any given task a ‘knowledge engineer’, the person responsible for developing the system, had to identify a human expert at that task, the ‘domain specialist’. Then, the knowledge engineer had to ‘extract’ or ‘acquire’ the knowledge from the domain specialist – a task known as ‘knowledge elicitation’ or ‘knowledge acquisition’. In turn, the knowledge engineer then had to ‘articulate’ or ‘represent’ this knowledge in a way that a system could then use – a task known as ‘knowledge articulation’ or ‘knowledge representation’. Often, this representation of the knowledge was done in the form of a set of ‘rules’, hence many of these systems were also known as ‘rules-based systems’. However, in the event that domain specialists were unable to understand how they performed a task, or precisely articulate how they performed it to a knowledge engineer, then an expert system could not be built to perform the task. Most of the early work in building AI systems adopted this approach – extracting and representing knowledge from a human expert who understood and could articulate the knowledge that he draws on, and the reasoning process that he carries out, in performing a given task.

There is a clear correspondence between the distinctions and assumptions that support the ALM hypothesis and this first generation of AI research. The 'routine' vs. 'non-routine' distinction, based on the distinction between those tasks that require explicit and tacit knowledge respectively, is a useful distinction since the sort of tasks that draw on tacit knowledge will also be those that a domain expert will struggle to articulate to a knowledge engineer. Similarly, to the extent that the only systems that can be built are of an 'expert systems' type, then it follows that these systems will not be able to perform 'non-routine' tasks – since a knowledge engineer, unable to elicit the knowledge from a domain specialist, will in turn be unable to represent it for a system. In the first generation AI literature, this is known as the 'knowledge elicitation bottleneck'.[4]

A consequence of the shared principles supporting the ALM hypothesis and the first generation of work in AI is that there is also a shared conception of the limits of these machines. In the ALM hypothesis, as noted before, this limit is identified as a constraint called 'Polanyi's Paradox' – that there are certain tasks that human beings perform without being able to articulate how they do so (they 'know more than they can tell') and so a computer cannot be given a set of rules to follow based on how human beings perform the task. This is very similar to the argument of Hubert Dreyfus, a prominent critic of AI during the first generation of research (see, for example, Dreyfus 1976 and

[4]See Hoffman (1987), for example.

1992, and Dreyfus and Dreyfus 1986). His analysis of the capabilities and limits of machines also focuses on rules. Consider, Dreyfus (1976) discussing a set of tasks that, at his time of writing, could not be performed by machines:

> "[I]n playing games such as chess, in solving complex problems, in recognising similarities and family resemblances, and in using language metaphorically and in ways we feel to be odd or ungrammatical, human beings do not seem to themselves or to observers **to be following strict rules**." [Emphasis added] (pg. 286)

Dreyfus argues that it is the presence of this "apparently nonrulelike activity" that makes these tasks incapable of being performed by a machine. And this is for the same reason as the ALM hypothesis. Dreyfus believes that the only way to get a machine to perform a given task is to provide that machine with a set of rules that reflect the thought process of a human being performing that task. But since there are no clear rules that can be derived, Dreyfus argues, the machine cannot perform those tasks.

In a similar spirit, consider the work of Joseph Weizenbaum, another researcher involved with the first generation of AI research. In trying to distinguish between those tasks that machines can perform, and those they cannot – if, he reflects, there are indeed any – Weizenbaum (1976) asks:

> "Are all the decisionmaking processes that humans employ reducible to effective procedures and hence amenable to machine computation?" (pg. 67)

Again, this question reflects the same set of assumptions as the ALM hypothesis. A task is amenable to computerisation if and only if the thought-processes of a human being can be reduced to an "effective procedure" for a computer to follow. Though Dreyfus and Weizenbaum disagreed about the answer to this question it is clear that both their approaches reflect a common set of assumptions about how machines operate.[5] This set of assumptions is shared by many of those who were working in the first generation of AI research – and those who use the ALM hypothesis in the economic literature.

2.3. Machines Can Perform 'Non-Routine' Tasks

The ALM hypothesis has a very clear implication – machines cannot perform 'non-routine' tasks. Yet in practice this does not hold. Increasingly, machines perform a wide range of 'non-routine' tasks. This is a fatal challenge to the ALM hypothesis. Particularly important cases of this are the focus of this section.

2.3.1. Driverless Cars

Polanyi (1966) provides a set of examples of tasks that he thought relied particularly heavily upon tacit knowledge. The task of driving a car is one of these. Polanyi writes:

> "The skill of a driver cannot be replaced by a thorough schooling in the theory of the motorcar." (pg. 20)

The intuition behind this claim is clear – if a person were asked how he drives a car he would struggle to articulate his precise thought process when doing

[5] Weizenbaum was more optimistic than Dreyfus, and focused less on the positive question – what machines *could* do – and more on the normative question, what machines *ought* to be permitted to do. This was a source of tension between the researchers – consider Dreyfus (1992), "After these damaging admissions Weizenbaum is left with only the moralistic position that "however intelligent machines may be made to be, there are some acts of thought that ought to be attempted only by humans"" (pg. 65) where these "damaging admissions" reflects a belief that more tasks could be computerized than Dreyfus thought was feasible.

so. Again, this is because driving a car relies upon 'tacit' rather than 'explicit' knowledge, and the former is hard to articulate. Given that the task-based approach relies on Polanyi (1966), it is understandable that Autor et al. (2003) also use the task of driving a car as a definitive case of a 'non-routine' task – if driving a car requires tacit knowledge, then necessarily, given their argument, it must also be a 'non-routine' task. Autor et al. (2003) write:

> "Navigating a car through city traffic or deciphering the scrawled handwriting on a personal check – minor undertakings for most adults – are not routine tasks by our definition. The reason is that these tasks require visual and motor processing capabilities that cannot at present be described in terms of a set of programmable rules." (pg. 1283)

In the task-based literature that followed Autor et al. (2003), the task of driving a car is also used as a representative case of a 'non-routine' task incapable of being performed by a machine. In Levy and Murnane (2004), for example, the authors discuss the task of a "bakery truck driver making a left turn against traffic":

> "The bakery truck driver is processing a constant stream of information from his environment .... To program this behaviour we could begin with a video camera and other sensors to capture the sensory input. But executing a left turn across oncoming traffic involves so many factors that **it is hard to imagine discovering the set of rules that can replicate the driver's behaviour**." (pg. 20) [Emphasis added]

In 2004, this position – that the task of driving a car necessarily required a human being – was consistent with the available evidence. Clearly, though this no longer holds. Within a year of this statement, the task of driving a car had been computerised. Almost all major car companies are now developing driverless cars.[6]

2.3.2. Additional Cases

Driverless cars are only one example of a broader and recent phenomenon – the computerisation of 'non-routine' tasks. Consider, for example, the second task that Autor et al. (2003) suggests is 'non-routine' and so incapable of computerisation, that of "deciphering the scrawled handwriting on a personal check". This task too can now be performed by machines.[7] Indeed, it can be performed in a variety of ways, and is a small part of a much larger field in AI research on 'pattern recognition'.[8]

As another example, consider the case that Polanyi (1966) uses to introduce the idea of tacit knowledge, 'facial recognition':

> "Take an example. We know a person's face, and can recognise it among a thousand, indeed among a million. Yet we usually cannot tell how we recognise a face we know. So most of this knowledge cannot be put into words." (pg. 4)

Given that Polanyi argues that the task of recognising faces is one that requires tacit knowledge, then according to the ALM hypothesis it must also be a 'non-routine' task. And so, given the assumptions about the capabilities of machines under the the ALM hypothesis, it must be a task that cannot be performed by machine. Yet this is not the case. There is a large literature

[6]See Gibbs (2016).

[7]These systems are now ubiquitous. For example, for economists there is a smartphone app called 'Mathpix' which scans handwritten equations and solves them. This is available at https://itunes.apple.com/gb/app/mathpix-solve-graph-math-using/id1075870730?mt=8.

[8]Consider *Pattern Recognition*, published by Elsevier, a journal dedicated to this field.

in 'affective computing' and AI research (again a small part of the pattern recognition field) describing systems capable of performing this task – in cases, outperforming human beings (see Lu and Tang 2014).[9]

In Autor and Dorn (2013) several other 'non-routine' tasks are presented that they argue cannot be performed by machines.[10] Consider three examples that are given – personal care, order-taking, and table-waiting. In the 20th century, each of these may have required a human being. But whether this will continue is unclear. In Japan, for example, where the elderly population is the highest in the world and expected to rise further (23 percent 65 or older, Muramatsu et al 2011), the burden of care is increasingly being shared between human beings and machines. Commerical robots like 'Paro', a therapeutic toy seal, and 'Pepper', a personal robot, are often used.[11] In 2013, the Japanese government allocated £14.3 million to 'develop robots to help with care' (Hudson 2013) and the Ministry of Economy, Trade and Industry chose 24 companies to receive R&D subsidies to develop 'nursing care robot equipment' (Iida 2013). In AI research, the field of 'socially assistive robots' is exploring these problems. As for order-taking and table-waiting, there are now many restaurants and cafes where these processes are performed entirely by machines.[12]

[9]See Calvo et al. (2015) for an overview of the 'affective computing' literature.

[10]In Chapter 6 I look in more depth at Autor and Dorn (2013) and the AD model.

[11]See http://www.parorobots.com/ and https://www.ald.softbankrobotics.com/en/cool-robots/pepper.

[12]For example, in 2013, the US restaurants Chili's and Applebee's announced that they were installing 100,000 tablets to allow customers to order and pay without interacting with a human waiter. See Pudzer (2016).

2.4. The Problem with the ALM Hypothesis

The observation that 'non-routine' tasks can now be performed by machines is not an entirely novel one.[13] However as yet there is no explanation for why the ALM hypothesis has proven mistaken. In this section, I explain this. In the following section I present an alternative.

2.4.1. The General Mistake

To understand why the ALM hypothesis is mistaken it is important to explore how these new systems and machines actually perform the type of tasks that the economics literature concluded they could not. Consider again the driverless car. Autor (2014) provides an account of how the driverless car works:[14]

> "It is sometimes said by computer scientists that the Google car does not drive on roads but rather on maps ... the Google car navigates through the road network primarily by comparing its real-time audio-visual sensor data (collected using LIDAR) against painstakingly hand-curated maps that specify the exact locations of all roads, signals, signage, obstacles, etc. ... while the Google car appears outwardly to be as adaptive and flexible as a human driver, it is in reality more akin to a train running on invisible tracks." (pp. 33-4)

The most important observation here is that the system performs the task of driving a car in a very different way to the way that human beings perform the task. While both a human being and a machine rely upon sensor data to drive

[13] For example, Frey and Osborne (2013) note that computerisation is "spreading to domains commonly defined as non-routine" as pointed out in section 1.5.6.

[14] The Google car is the most well-known driverless car.

the car – though their respective sensors vary – only the machine requires a "pre-specified" map. A human, for example, can drive a car in an area they do not know but a machine cannot at present (in the sense that a machine only 'knows' an area if its software contains a map of it). As Autor (2014) puts it, the machine, unlike a human being, "cannot pilot on an "unfamiliar" road". This observation – that the most capable machines perform tasks in different ways to human beings – has also been made by others. Consider, for example, the chess-playing system Deep Blue, developed by the computer company IBM. In 1997 it defeated the then then chess world-champion, Garry Kasparov, in a six game set. Kasparov (2010) made the following observation about that event:

> "Instead of a computer that thought and played chess like a human, with human creativity and intuition, they got one that played like a machine, systematically evaluating 200 million possible moves on the chess board per second and winning with brute number-crunching force."

Both Autor (2014) and Kasparov (2010) make the important point that human beings and machines perform the same task in different ways. And both draw the same conclusion from this observation – that the way in which the machine performs the task is 'inferior' to the way in which the human being performs the task. Autor (2014) describes how Google's driverless car only "*appears outwardly* to be as adaptive and flexible as a human driver" (emphasis added) but is "*in reality* more akin to a train running on invisible tracks" (emphasis added). The implication is that that the way that a human being drives a car is in some sense 'better' – that the "invisible tracks" create an illusion of human sophistication. In a similar way Kasparov (2010) notes that Deep Blue was "an impressive achievement, of course" but the machine was "only intelligent the way your programmable alarm clock is intelligent." Again, the implication is that the human way of playing chess is in some sense 'better' – the brute-force processing power of the computer again created an illusion of human sophistication.

However, this is the wrong conclusion to draw from this observation for two reasons. The first reason is that, to the extent that these machines can perform these tasks at a higher standard than human beings, there are compelling reasons to think them superior rather than inferior. To focus on the inferiority of these systems relative to human beings based on *how* they perform those tasks, rather than *how well* they perform those tasks, is a disproportionately non-consequentialist position to take.

The second reason is more important and explains why the ALM hypothesis is mistaken. If machines are able to perform a given task differently to the way that human beings perform that same task, this has important consequences for how those machines are designed, programmed, and operated. In particular, if a machine performs a task in a different way to a human being, *then there is no need to understand and articulate the way in which the human being traditionally performed that task.* Yet the assumption that a machine can only perform a task if we understand and can articulate the way in which a human being traditionally performs that task is central to the task-based approach. Indeed, the only purpose of the 'routine' vs. 'non-routine' distinction is to distinguish between those types of tasks for which human beings can articulate their thought processes ('routine' tasks) and those they cannot ('non-routine' tasks). And the basis for the assumption that computer capital can only perform 'routine' tasks is that those are the only tasks for which we can write a specific procedure, since those are the only tasks we can articulate how human beings perform – they rely on explicit rather than tacit knowledge. However, if we do not require an understanding or articulation of the thought process of human beings to get a machine to perform the task, then the ALM hypothesis is far less useful – the 'routine' verses 'non-routine' distinction is of diminishing use, and the assumption that machines can only perform 'routine' tasks increasingly mistaken.

The Levy and Murnane (2004) discussion of the task of driving a car in section 2.3.1 demonstrates this point. They wrote that it is "hard to imagine discovering the set of rules that can replicate the driver's behaviour". And for that reason, it was hard to imagine that a driverless car was feasible. But the mistake was to think that the only way to drive a car was to understand how it is that human beings drive cars, and capture that in a set of rules. A driverless car has indeed been invented, not by capturing a human reasoning process in a set of rules, but in an entirely different way to how human beings drive cars – by following a set of "invisible tracks", and making use of brute force processing power to interpret sensor data. Remus and Levy (2015) repeats the mistake. In setting out their analysis of the likely future consequences of technological change on the legal profession, they claim that "for a computer to automate a lawyer's work, it must be possible to model the lawyer's information processing in a set of instructions". Again, this fails to recognise that machines are increasingly performing tasks in different ways to human beings.[15]

[15]See for example, Katz et al. (2014). This is a system that predicts the outcome of

This analysis does not imply at all that Polanyi's distinction between 'tacit' and 'explicit' knowledge is wrong. But it does imply that Polanyi's distinction is the wrong constraint when thinking about the capabilities of machines *when those machines operate in very different ways to human beings.* Only when the way in which a machine performs a task must exactly reflect the way in which a human being performs that same task does a Polanyi-type constraint bind.

2.4.2. The 'Second Generation' of AI Research

The 'first generation' of AI research assumed that for a machine to perform a given type of task required a representation of how a human being performed that task (i.e. a set of rules) that a machine could then follow. This understanding is, as I have argued, central also to the ALM hypothesis. But, as I have also argued, this position is incorrect. Section 2.3 shows how many non-routine tasks, contrary to the expectations of the ALM hypothesis, can be performed by machines. And section 2.4.1 explains why this mistake has been made – machines increasingly perform tasks in entirely different ways to human beings, and so there is no need to understand and articulate the way in which a human being performs those tasks.

Accordingly, the 'second generation' of AI research no longer relies upon the assumption that, in order for a machine to perform a task, a set of rules must be composed that reflect how a human being performs that task. Reflecting the argument in section 2.4.1, these systems and machines perform tasks in 'unhuman' ways. This shift is captured by Russell and Norvig (1995):

Supreme Court decisions, but does not try to copy the 'judgement' of a traditional lawyer.

> "The quest for artificial flight succeeded when the Wright brothers and others stopped imitating birds and learned about aerodynamics. Aeronautical engineering texts do not define the goal of their field as machines that fly so exactly like pigeons that they can fool even other pigeons." (pg. 3)

To see this in practice, consider a second generation AI that relies upon exploiting brute-force processing power and massive data retrieval and storage capabilities. A good example of this is IBM's chess-playing machine, Deep Blue, discussed briefly in section 2.4.1. This machine did not rely upon a clearly articulated set of rules reflecting an understanding of how human beings play chess. Instead, it relied upon an ability to search up to 330 million chess positions per second (Campbell et al. 2002). This is a very different process to that performed by a human chess player. The discipline of 'machine learning' contains many additional examples of second generation AIs that are not performing tasks by following explicit and fixed rules that reflect how a human being performs those tasks. Mitchell (1997) and Murphy (2012) review this literature. For a recent practical case consider DeepMind, the AI system acquired by Google – its accomplishments are set out in Mnih et al. (2015) and Silver et al. (2016).[16]

2.4.3. Anthropocentrism

Autor (2013), a review of the task-based approach, claims that the problem with the canonical model is that the traditional "production function found in economic models is anthropomorphic – it's built to resemble humans". In

[16] Autor (2015) does discuss the discipline of machine learning in a set of closing reflections on whether 'Polanyi's Paradox' will be overcome in the future. But for the reasons set out in this chapter, and noted in the next section on 'anthropocentrism', Autor (2015) seriously underestimates the significance of the challenge it poses to the ALM hypothesis.

traditional production functions there is necessarily a role for human beings and a role for machines, and those two roles are distinct. It is as if the production function is designed to reflect the image of a human being with a tool in his hand. Autor (2013) argues that the purpose of the task-based approach is to correct this anthropocentrism. In the task-based approach, factors do not have a fixed role in production – there is not a fixed set of "labour tasks" and a fixed set of "capital tasks" – but instead the allocation of factors to different tasks can change.

It may be right that the task-based approach resolves that particular anthropocentrism. But one interpretation of the argument in this chapter is that the ALM hypothesis has introduced a different anthropocentrism. Under a task-based approach the production function itself may no longer resemble a human being – but the assumptions about the capabilities of machines reflected in the ALM hypothesis *are* heavily constrained by the way in which human beings think and behave. Simply put, they rely on the view that if a human being cannot understand and articulate how they perform a particular task, then a machine cannot perform that task. To see how strongly this position is *still* maintained, consider Autor (2014) and his response to the latest wave of technological progress, in particular the use of 'machine learning' techniques:

> "Contemporary computer science seeks to overcome Polanyi's paradox by building machines that learn from human examples, thus inferring the rules that we tacitly apply but do not explicitly understand."

On this account, these increasingly capable systems are only able to perform 'non-routine' tasks by learning the rules that *human beings* follow but cannot articulate. This fails to recognise the central argument of this chapter – that many of these machines perform these tasks in fundamentally different ways to human beings. It is possible to perform these tasks even if the particular rules that human beings follow are never discovered.

2.5. Evidence from the Labour Market

So far this chapter has focused on 'capital'. It has explored how the type of tasks that capital can perform are changing. But an alternative standpoint to take is that of 'labour' – to explore how the type of tasks that labour performs are changing, given that labour performs the tasks that capital does not. The task-based literature has, accordingly, drawn on evidence from the labour market to justify the ALM hypothesis. I turn to this now.

2.5.1. Autor et al. (2003)

In Autor et al. (2003) the authors look at how the tasks performed by labour in the US had changed from 1960-98. Again, this approach uses the 'routine' vs. 'non-routine' distinction, but in addition it also distinguishes between different types of 'routine' task – 'manual' and 'cognitive' – and 'non-routine' task – 'analytical', 'manual', 'interpersonal'. The results are shown in Figure 2.1.[17] Each task variable is given a mean of 50 centiles in 1960. The value of each task variable in the years following 1960 reflect changes in task inputs relative to the 1960 task distribution.[18]

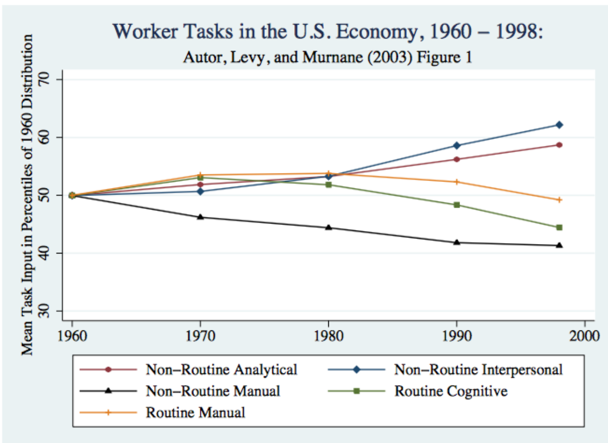
[17]This chart is from Autor and Price 2013.

[18]There is no 'adding-up constraint' across the task variables i.e. total task allocation do not sum to one within particular jobs or time periods. In practice, note Autor et al. (2003), this over-time variation in labour task input is modest.

Autor et al. (2003) make three observations about the data captured in Figure 2.1. The first is that labour input of 'non-routine analytical' and 'non-routine interpersonal' tasks increased substantially over the period. Although both increased during the 1960s and 1970s, the growth in this 'non-routine' labour input accelerated during the 1980s and after. By 1998, 'non-routine analytical' labour task input was 6.8 centiles above its 1970 level, and 'non-routine interpersonal' labour task input 11.5 centiles.

The second observation is that the labour input of 'routine cognitive' tasks and 'routine manual' tasks decreased substantially over the same period. From 1960 to 1970, the labour input of 'routine' tasks in fact rose. But labour input of these tasks, beginning in the 1970s for 'routine cognitive' tasks and in the 1980s for 'routine manual' tasks, started to fall. By 1998, both are below their 1950 level, 'routine cognitive' tasks significantly so.

Figure 2.1: Labour Tasks in US Economy 1960 – 1998



The third observation is that labour input of 'non-routine manual' tasks, unlike the other types of 'non-routine' tasks, fell steadily throughout the sample period.

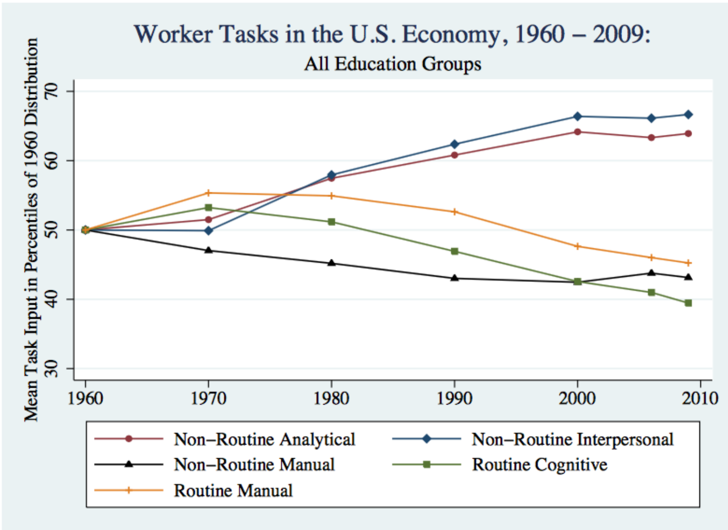
Taking only the first and second observations, it is clear why Autor et al. (2003) argue this evidence supports the "primary macroeconomic implications" of the ALM hypothesis. The ALM hypothesis argues that capital substitutes for 'routine' labour task input and complements 'non-routine' labour task input. And this hypothesis appears to be reflected in those first two observations – there is a decline in 'routine' labour task input, and a rise in 'non-routine' labour task input. And, importantly, both of these trends begin in the 1980s – when significant growth in the capability of computer capital began. Simply put, the data suggests that as machines became increasingly capable from the 1980s onwards, a new division of labour emerged where labour performs more 'non-routine' tasks and machines performed more 'routine' tasks.

However the third observation, the fall in the labour input of 'non-routine manual' tasks, does appear to conflict with the ALM hypothesis. The input of this type of task would be expected to rise, not fall, since these are 'non-routine' tasks that are complemented by increasingly capable machines. Nevertheless, based on the first two observations the argument provides important insights into the possible consequences of technological change on the 20th century labour market.

2.5.2. Autor and Price (2013)

In Autor and Price (2013) the authors use the same empirical approach as Autor et al. (2003) but they extend the data to look at 1960 to 2009. Given that the argument developed in this chapter is that an important change appears to have taken place in the character of technological change in the 21st century, it is important to examine this additional data. Their results are shown in Figure 2.2.

Figure 2.2: Labour Tasks in US Economy 1960 – 2009



The authors make three observations about this updated data. The first is that the first decade of the 21st century saw a continued decrease in labour input of 'routine' tasks – both 'routine manual' and 'routine cognitive' tasks continued on a downward trend. Between the end of the original Autor et al. (2003) sample and 2009, labour input of 'routine cognitive' tasks fell at the same rate as it did in the earlier sample. A similar trend can be observed in the labour input of 'routine manual' tasks.

The second observation is that the unexpected fall in the labour input of 'non-routine manual' tasks that was noted in the more limited 1960-98 data – this was the third observation noted in the previous subsection – appears to have reversed. The updated data show that the labour input of 'non-routine manual' tasks rose "modestly" after 2000.

These initial observations about the first decade of the 21st century appear to support the ALM hypothesis. But the third observation made by Autor and Price (2013) is problematic. In the first decade of the 21st century, the labour input of 'non-routine cognitive' tasks in fact "slows substantially".[19] Indeed, the labour input of 'non-routine analytical' and 'non-routine interpersonal' tasks *declines* between 2000 and 2006. In 2009 the former still remains below its 2000 level and the latter is only very slightly above.

As Autor and Price (2013) note, this final observation is "not expected from the ALM hypothesis". What is expected from the ALM hypothesis is the opposite – that the labour input of these 'non-routine' tasks should increase not decrease. As Autor and Price (2013) put it, the ALM hypothesis argues that if machines "take on 'routine' tasks, they have boosted demand for workers who perform 'nonroutine' tasks that are complementary to the automated activity". This argument is discussed in section 1.4.2. But this data suggests that not only has the labour input of 'routine' tasks fallen, but so too has the labour input of certain types of 'non-routine' tasks as well.

The authors conclude that "this unanticipated pattern merits further study". This is a significant discrepancy. One possible explanation is that the one advanced in this chapter. The 'routine' vs. 'non-routine' distinction, and the assumptions about the capability of capital with respect to those categories, is incorrect. Capital is increasingly substituting for labour in performing both 'routine' *and* 'non-routine' tasks.

[19] 'Non-routine cognitive' tasks combines both 'non-routine analytical' and 'non-routine interpersonal' tasks.

2.6. An Alternative Approach

2.6.1. 'Routinisability' rather than 'Routineness'

If the 'routine' vs. 'non-routine' distinction is of diminishing use, and the assumptions about the capabilities of machines with respect to those tasks do not align with recent technological progress, it is important to describe what ought to replace them.

The problem is that the 'routine' vs. 'non-routine' distinction is made from the standpoint of a *human being.* This is the anthropocentrism discussed in section 2.4.3. A given type of task is either 'routine' or 'non-routine' according to the particular way in which a human being performs that task – a task is 'routine' if a human being can articulate the set of rules that he follows in performing that task, and it is 'non-routine' if not. But given that machines now perform many tasks in very different ways to human beings, this is the wrong standpoint. The correct standpoint to take is that of the *machine* – to ask whether a given type of task has features that make it more or less susceptible to being performed by machines.

Accordingly, the appropriate criteria is not whether a task is 'routine' or 'non-routine', but whether it has features that make it more or less *routinisable.* If a task is routinisable, a routine can be written that allows a machine to perform it – but that routine may not reflect the way in which a human being performs the task. For example, driving a car is a 'non-routine' task from the standpoint of a human being. But as Google and others have shown, it is now possible to routinise that task – but the routine does not resemble the way a human being performs that task. The concept of 'routinisability' is fundamentally different to 'routineness'. The latter asks whether a task has features that make it more or less feasible to articulate how a human being performs it in a set of rules for a machine to follow. The former asks whether a task has features that make it more or less feasible to articulate a set of rules for a machine to follow, *irrespective of whether those rules resemble the way in which a human being performs the task.*

The difficulty in the existing literature is that these two concepts are often conflated. The ALM hypothesis is based on the premise that the routinisability of a task is synonymous with its routineness from the standpoint of a human being. There are moments in the literature when it *appears* that this is not the case. For example, Autor (2013) appears to escape the conflation when, in a discussion of the empirical difficulties of measuring tasks, he notes that "the heart of the difficult is that what is routine from the perspective of human labor is generally not routine from the perspective of machine automation." But this is not as radical a concession as it first appears. What Autor (2013) means here is that when we adopt a traditional understanding of the word 'routine' as to mean "mundane" and "repetitive", those tasks that we call 'routine' are not necessarily the easiest to automate. And observationally this is correct. As Autor (2013) notes, for example, "mopping floors is a mundane repetitive task from the perceptive of janitors" but very hard to automate. Yet Autor (2013)'s understanding of why this is the case is still based on the ALM hypothesis – that janitors find it difficult to articulate the set of rules that they follow in mopping the floor.[20]

[20]'Mopping the floor', in short, is a 'non-routine' task under the ALM hypothesis rather

In summary, I submit that a sounder approach is not to look at a given task and ask what sort of knowledge is required by a human being to perform it – is it 'tacit' or 'explicit', as in the Autor et al. (2003) approach – but instead to look at a given task and ask whether it has features that make it more or less difficult for a machine to perform it. A task must be interrogated and it must be asked whether, given the current state of technological progress, that task is *routinisable.* The set of tasks that were routinisable in the 20th century is likely to be far smaller than the set of tasks that is routinisable in the 21st century.

2.6.2. Implication for this Thesis

The set of task-based models in Chapter 1 are based on the ALM hypothesis. As a result they share two related properties. The first is that the models are based on the existence of a set of tasks that can only be performed by labour – namely 'non-routine' tasks. The second is that the production of all types of goods necessarily require at least one type of labour to produce them. This second property relies on the fact that the production of goods in these models necessarily requires 'non-routine' tasks – and given the first property, this therefore also implies they also necessarily require labour.[21] Importantly, this second property also holds over time – no matter the level

than a 'routine' one. 'Ordering food' is a good example of a task that captures these different distinctions. A waiter might call this task 'routine' in an informal sense (i.e. that is mundane and repetitive). Autor (2013) would instead call this task 'non-routine' in a formal sense since although it is repetitive it relies upon tacit knowledge. *But this task can still nevertheless be computerised* since systems are able to perform this task in different ways to human beings. Its 'non-routine' character is not a constraint on computerisation.

[21]Formally, the task-based production functions for goods all require at least one 'non-routine' task.

of technological progress, goods will always require tasks that necessarily are performed by labour. The consequence is that there is no good in these models that will ever be produced by capital alone.

This consequence is justified in the literature by the ALM hypothesis. But, as argued in this chapter, the ALM hypothesis is mistaken. Capital can increasingly perform 'non-routine' tasks. The reason for this has been set out. This challenges both the existence of a set of tasks that must indefinitely be performed by labour and, in turn, it challenges the consequence that goods in these models will never be produced by capital alone. The new models in this thesis explores this, using the alternative concept of 'routinisability'.

2.7. Conclusion

The task-based literature originated with two innovations in Autor et al. (2003) – the introduction of the task-based approach and a theory about the capabilities of machines known as the ALM hypothesis. This is set out in Chapter 1. This chapter does not challenge the usefulness of the first of these innovations. The 'task-space' is the appropriate setting in which to analyse the consequence of increasingly capable machines on the labour market. This is for the reason that Autor et al. (2003) provides – only by looking at what it is that factors actually do in production, namely the nature of the underlying *tasks* that factors perform, is it possible to understand why technological change may be biased in the direction of a particular factor.[22]

But this chapter does find the second innovation, the ALM hypothesis,

[22] See section 1.2.3 and section 1.3.

problematic. This is because the ALM hypothesis is based on an understanding of how machines operate that held in the first generation of AI, but does not hold in the second generation of AI. It is not that the ALM hypothesis has always been wrong – but that the ALM hypothesis no longer adequately captures the nature and capabilities of capital.

The argument developed in this chapter provides the foundations for the rest of this thesis. In Part B, Chapters 3, 4, and 5, I focus on the supply-side. In Chapter 3, in the spirit of the ALM hypothesis, I maintain a clear demarcation between a set of tasks that are performed by labour, and a set of tasks that are performed by capital. All goods necessarily require the input of one type of labour to produce them. The outcome is *optimistic* for labour. But in Chapter 4 I remove this demarcation. I replace the 'routine' vs. 'non-routine' distinction and the assumptions about the capabilities of machines relative to human beings with the new concept of 'routinisability'. I show that increasingly capable machines have very different consequences for the labour market when the conservative assumptions about the capabilities of machines, reflected in the ALM hypothesis, are relaxed. The outcome is *pessimistic* for labour. Chapter 5 places these static models in a dynamic setting, and this strengthens the optimism of Chapter 3 and the pessimism of Chapter 4. Then in Part C of the thesis, Chapters 6 and 7, I turn to the demand-side and a possible countervailing argument for *optimism*.

Part B: The Supply-Side

3. A New Task-Based Model with 'Traditional Capital'

3.1. Outline

In this chapter I build a simple version of a new task-based model. I call this the 'new model'. The new model is Ricardian with two types of labour – low-skilled labour and high-skilled labour. There is a range of tasks ordered in a line from left to right, going from 'simple' to 'complex'. These tasks are performed by labour, and the relative productivity of high-skilled labour with respect to low-skilled labour increases to the right (i.e. as tasks become more 'complex').

In certain respects this is familiar to the AA model set out in section 1.5.3. But it goes well beyond the familiar in two ways. The first departure is that there is a range of goods produced, not just one good, where the production of each good requires its own particular type of task performed by labour.[1] And second, to produce a good requires not just a task performed by labour but also a task performed by a particular type of capital – what I call 'traditional' capital. Capital is 'traditional' in the sense that it cannot

[1]The variety of goods becomes an important feature of the model in Part C of the thesis. Unlike the AA model, this new model preserves the many-good property of the original Dornbusch et al. (1977) model, *and* also has a variety of tasks.

perform the same types of tasks as labour. Although goods are produced by a combination of tasks that are performed by labour and tasks that are performed by capital, *these sets of tasks do not overlap.* In this sense, the factors' roles in production are distinct.

This is the simplest many-good, many-task model that reflects the ALM hypothesis – although there is no reference to the 'routine' vs. 'non-routine' distinction, each good requires a task that must be performed by labour and that cannot be performed by capital. Traditional capital is always a q-complement to labour in producing each type of good. Importantly, like the models in Chapter 1, the result is that all goods require at least one type of labour to produce them, alongside traditional capital.

The purpose of this new model is to study the effect of technological progress in the use of traditional capital. In equilibrium there is a single cut-off and all of the tasks to the left of the cut-off are performed by low-skilled labour and all of the tasks to the right are performed by high-skilled labour. The main conclusion is that when capital is 'traditional' in this way, with a fixed and distinct role from labour in production, improvements in its capability across the range of tasks have a neutral effect on labour – an increase in the capability of capital causes an equiproportionate rise in the factor prices of capital and both types of labour. This is optimism at work.

In later chapters the assumption that capital and labour have distinct roles is relaxed, and the effects on labour are shown to be far from neutral. Certain goods no longer necessary require any type of labour to produce them. But this simple version is an important first step towards the more complex versions later in the thesis.

3.2. The Model

3.2.1. Consumers

There is a continuum of consumers $j \in [0,1]$ who are either low-skilled workers, high-skilled workers or capitalists. If consumer j is a low-skilled worker he sells his labour l_j^L for a wage $w^L \geq 0$. If he is a high-skilled worker he sells his labour l_j^H for a wage $w^H \geq 0$. If he is a capitalist he rents his capital k_j to to earn $r \geq 0$. There is a continuum of different types of goods where each type of good $x(i)$ is indexed on the interval $i \in [0,1]$. Each consumer j has the same preferences over the goods described by the Cobb-Douglas utility function:

$$u(\boldsymbol{x}) = \int_0^1 \theta(i) \ln x_j(i)\, di \tag{3.1}$$

where $\theta(i) > 0$, $\int_0^1 \theta(i)\, di = 1$, and $x_j(i)$ is consumer j's demand for good i. As a result, consumers face the following optimization problem:

$$\begin{aligned} &\max_{\boldsymbol{x}} \int_0^1 \theta(i) \ln x_j(i)\, di \\ &\text{s.t.} \quad r \cdot k_j + w^L \cdot l_j^L + w^H \cdot l_j^H \geq \int_0^1 p(i) \cdot x_j(i)\, di \end{aligned} \tag{3.2}$$

where $p(i)$ is the price for each type of good i.

3.2.2. Production and Firms

Each type of good $x(i)$ is produced by a Cobb-Douglas combination of two tasks $z_1(i)$ and $z_2(i)$:

$$x(i) = z_1(i)^{\psi} z_2(i)^{1-\psi} \tag{3.3}$$

where $\psi \in (0,1)$. The first set of types of task, $z_1(i)$, are those performed only by labour – either low-skilled or high-skilled. The second set of types of task, $z_2(i)$, are performed only by capital. This is the sense in which labour and capital have distinct roles in production, as described in the outline to this chapter. For each good, firms must decide whether to hire the low-skilled labour of workers, the high-skilled labour of workers, or the traditional capital of the capitalists. The total volume of labour and capital available to these firms is set exogenously, and equal to:

$$L^L = \int_0^{\aleph} l_j^L \, dj \qquad K = \int_{\aleph}^{\beth} k_j \, dj \qquad L^H = \int_{\beth}^{1} l_j^H \, dj \tag{3.4}$$

The individuals are partitioned into three sets. Those who sell their labour and are low-skilled, $j \in [0, \aleph]$, those who own and rent capital, $j \in [\aleph, \beth]$, and those who sell their labour and are high-skilled, $j \in [\beth, 1]$. The low-skilled labour that is assigned to produce good type $x(i)$ is $L^L(i)$, the high-skilled labour that is assigned is $L^H(i)$, and the capital that is assigned is $K(i)$.

The productivity of different factors at performing different types of tasks varies. Call $a^L(i)$ and $a^H(i)$ the factor productivity of low-skilled and high-skilled workers respectively at task $z_1(i)$, and $a^K(i)$ the factor productivity of capital at task $z_2(i)$. For example, if $a^L(i) = 5$, this means that one unit of low-skilled labour will perform five tasks of type $z_1(i)$. These productivities combine to form a 'relative productivity schedule' for low-skilled labour over the task spectrum:

$$A(i) = \frac{a^L(i)}{a^H(i)} \tag{3.5}$$

I now make a first important assumption about the nature of the tasks performed by labour, and their relative productivity at performing these tasks:

ASSUMPTION 1: *$A(i)$ is continuous, $A(i) > 0$, and $A'(i) < 0$.*

The assumption that $A'(i) < 0$ is a 'comparative advantage' assumption.[2] It reflects two principles. First, as I move left to right on the spectrum of goods, and i increases, the task that is performed by labour, $z_1(i)$, to produce the good becomes more 'complex'. And secondly, that high-skilled labour has an increasing comparative advantage over low-skilled labour at performing more complex tasks. This, then, is the sense in which labour is 'low-skilled' and 'high-skilled' – the latter is comparatively better at performing more complex tasks. Note that I have not made an assumption about $A''(i)$ – the main features of equilibrium depend upon only the first derivative. However, for simplicity, in the diagrams and comparative statics that follow, I use an $A(i)$ schedule where $A''(i) = 0$ (i.e. it is linear).

Note that tasks are only defined in one dimension in this model – how 'complex' they are. There is, for example, no mention of the 'routine' vs. 'non-routine' distinction that is crucial in the ALM hypothesis. This feature is important in the new models developed in Chapters 4 and 5 involving

[2]The AA model also uses this approach and assumption. Both that model, and this, are following the structure of Dornbusch et al. (1977). See section 1.5.3.

different types of capital.

3.2.3. Relation to Chapter 1 Framework

This model can also be set out using the general framework in Chapter 1. The factor-based production functions for tasks are:

$$\begin{aligned} z_1(i) &= a^L(i)L^L(i) + a^H(i)L^H(i) \\ z_2(i) &= a^K(i)K(i) \end{aligned} \tag{3.6}$$

And the task-based production function for goods is:

$$x(i) = z_1(i)^{\psi} z_2(i)^{1-\psi} \tag{3.7}$$

It follows then that the aggregate factor-based production function for goods is:

$$x(i) = \left[a^L(i)L^L(i) + a^H(i)L^H(i))\right]^{\psi} \left[a^K(i)K(i)\right]^{1-\psi} \tag{3.8}$$

When set out in this way, the contrast between this new model and the four models set out in Chapter 1 is clear. The model is task-based and so, unlike the canonical model, there is a distinction between the factor-based production functions for tasks and the task-based production functions for goods. But there is a similarity with the canonical model in that labour performs its own type of tasks, and capital performs its own type of tasks.

Focusing on the factor-based production function for tasks, like the AA model and unlike the ALM model and the AD model, there is a spectrum of tasks $z_1(i)$, where $i \in [0, 1]$. And like the AA model the factors (in this case, the two types of labour) are perfect substitutes in performing these tasks. In the ALM model, labour and capital are also perfect substitutes – but only for 'routine' tasks. And in the AD model, labour and capital are again substitutes for 'routine' tasks – but with a variable degree of substitutability.[3]

Focusing on the task-based production function for goods, unlike the other task-based models there is also a spectrum of goods – the ALM and AA models have one good, and the AD model has two. In addition, these goods are produced by a Cobb-Douglas combination of two different types of tasks, $z_1(i)$ and $z_2(i)$. This is the same functional form that is used in the AA model, the ALM model, and the AD model, albeit in those cases for a smaller number of goods.[4] Put simply, these task-based models have different numbers of goods, different factor-based production functions for tasks, but the same task-based production function for goods.

3.2.4. Equilibrium

Factor Demand

The perfectly competitive firms face an assignment problem. They must decide which factors to hire to perform the tasks that will produce each type of good. It is clear that to perform the tasks $z_2(i)$ the firms will need to rent some capital $K(i)$ – capital is the only factor capable of performing those types of task. Less obviously, the firms will hire either low-skilled labour *or* high-skilled labour to carry out the tasks $z_1(i)$ for each good i – but never

[3] The perfect substitutability of factors in performing $z_1(i)$ is necessary to ensure that equilibrium in this new model is 'Ricardian'. To see why perfect substitutability is necessary see the Appendix, section 9.0.1.

[4] And in the AD model, only to produce 'goods', not 'services'.

both types of labour together. This is Lemma 1:

LEMMA 1: *In equilibrium, there exists some type of cut-off good $\tilde{i}$ such that low-skilled labour works with traditional capital to produce goods of type $i \in [0, \tilde{i}]$, and high-skilled labour works with traditional capital to produce the goods of type $i \in [\tilde{i}, 1]$.*

The proof is intuitive.[5] For any w^L and w^H in equilibrium, there will exist a type of good $\tilde{i}$ such that the cost of performing a unit of $z_1(\tilde{i})$ to produce that good i is the same for high-skilled labour and low-skilled labour:

$$\frac{w^L}{a^L(\tilde{i})} = \frac{w^H}{a^H(\tilde{i})} \tag{3.9}$$

For task $z_1(\tilde{i})$, the firm will therefore be indifferent between hiring either low-skilled or high-skilled labour. And since $\frac{a^L(i)}{a^H(i)}$ is strictly decreasing, it follows that it will always cost strictly less to perform types of task $z_1(i)$ to produce good i where $i \in [0, \tilde{i}]$ using low-skilled labour, and $i \in [\tilde{i}, 1]$ using high-skilled labour. More formally, a perfectly competitive firm will hire low-skilled labour rather than high-skilled labour to perform $z_1(i)$ if:

$$\begin{aligned} \frac{w^L}{a^L(i)} &\leq \frac{w^H}{a^H(i)} \\ \frac{a^L(i)}{a^H(i)} &\geq \frac{w^L}{w^H} \\ A(i) &\geq \frac{w^L}{w^H} \end{aligned} \tag{3.10}$$

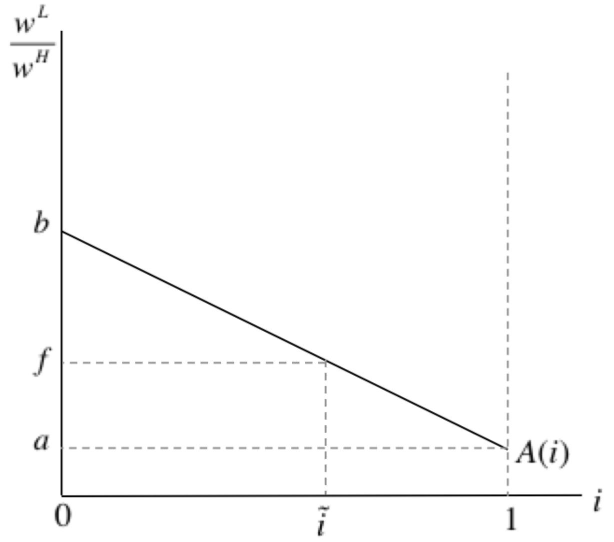
[5]This lemma and the proof are similar to Lemma 1 in Acemoglu and Autor (2011).

which takes place to the left of $\tilde{i}$, given the properties of $A(i)$ that are set out in Assumption 1. The opposite argument applies for the choice of high-skilled labour over low-skilled labour. Given this reasoning and Lemma 1, (3.8) can be re-written:

$$\begin{aligned} x(i) &= \left[a^L(i)L^L(i)\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \ \forall i \in [0,\tilde{i}] \\ x(i) &= \left[a^H(i)L^H(i))\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \ \forall i \in [\tilde{i},1] \end{aligned} \tag{3.11}$$

In equilibrium, each good will be produced by two factors – one type of labour and traditional capital, each performing their own distinct types of tasks – the former, $z_1(i)$, and the latter, $z_2(i)$. For any distribution of the stock of traditional capital K to perform the task $z_2(i)$ across the goods, firms choose the cost-minimising allocation of labour to perform the task $z_1(i)$. This is shown in Figure 3.1. For any $\frac{w^L}{w^H} \in [a,b]$ there is a single 'cut-off' type of good $\tilde{i}$. Given that wage ratio, firms would want to hire low-skilled labour (to work with traditional capital) to produce all goods to the left of $\tilde{i}$ and hire high-skilled labour (to work with traditional capital) to produce all goods to the right of $\tilde{i}$. This is intuitive. $A(i)$ is a factor demand schedule, and its shape is determined by Assumption 1. It describes the wage ratio for each type of good i that would make a firm indifferent between using either of the two types of labour to perform the tasks $z_1(i)$ necessary to produce the type of good i. As I move left to right along the good spectrum, if a firm is to remain indifferent between employing low-skilled or high-skilled labour to perform the tasks $z_1(i)$ then the relative price of low-skilled labour falls. Again, this is because as i increases the comparative advantage of high-skilled labour over low-skilled labour at performing the task $z_1(i)$ increases.[6]

Figure 3.1: The Relative Productivity Schedule for Low-Skilled Labour



Consumer Expenditure

Consumers have Cobb-Douglas preferences. In a discrete setting with a variety of goods this would imply that each good i has an associated constant expenditure share $b(i)$. However in this continuous setting, following Dornbusch et al. (1977), this implies that each good has an associated constant expenditure density:[7]

[6]That only one cut-off is considered reflects the Ricardian property of the model in equilibrium. Factors specialise in performing particular types of tasks, and so producing particular types of goods. The division of labour is very precise.

[7]The expenditure density for the more general CES utility function is derived in section 7.3.1, and shown in (7.7). The literature can refer to these expenditure 'densities' as expenditure 'shares'. This is a mistake, since the spectrum of goods and tasks is continuous.

$$\begin{aligned} b(i) &= \frac{p(i) \cdot x(i)}{Y} \\ &= \theta(i) \end{aligned} \tag{3.12}$$

where $x(i)$ is the aggregate consumption of good i, Y is total income, and $p(i)$ is the price of good type i. Note that because the utility functions are homothetic it is possible to express the model as a single representative consumer with the same Cobb-Douglas utility function.

Market Equilibrium

Take some hypothetic cut-off good $\tilde{i}$. Given that preferences are Cobb-Douglas it follows that the share of total consumer expenditure that is spent on all *goods* of type $i \in [0, \tilde{i}]$ is:

$$\begin{aligned} \alpha(\tilde{i}) &= \int_0^{\tilde{i}} b(i)\, di \\ &= \int_0^{\tilde{i}} \theta(i)\, di \end{aligned} \tag{3.13}$$

And the share of total consumer expenditure on all *goods* of type $i \in [\tilde{i}, 1]$ is:

$$\begin{aligned} 1 - \alpha(\tilde{i}) &= 1 - \int_0^{\tilde{i}} b(i)\, di \\ &= \int_{\tilde{i}}^{1} \theta(i)\, di \end{aligned} \tag{3.14}$$

In turn, given that the task-based production function is also Cobb-Douglas, with exponents ψ and $(1 - \psi)$, it follows that the share of total consumer expenditure that is spent on all the *tasks* performed by low-skilled labour is:[8]

$$\begin{aligned} &\psi \int_0^{\tilde{i}} b(i)\, di \\ &= \psi \int_0^{\tilde{i}} \theta(i)\, di \end{aligned} \tag{3.15}$$

And the share of total expenditure on all the *tasks* performed by high-skilled labour is:

$$\begin{aligned} &\psi \int_{\tilde{i}}^{1} b(i)\, di \\ &= \psi \int_{\tilde{i}}^{1} \theta(i)\, di \end{aligned} \tag{3.16}$$

As firms are perfectly competitive, market equilibrium requires that the total consumer expenditure on the types of tasks that are performed by each factor is equal to the income that the factor receives in return for carrying out those tasks. This is a zero-profit condition. If firms pay factors more than the consumer expenditure share on the tasks they perform, then they will make a loss and leave the market. If firms pay factors less than the consumer expenditure share on the tasks they perform, there is a profitable opportunity for a new entrant (i.e. enter, and pay the factor marginally more than the incumbent firms). As a result, if market equilibrium is to hold it must be the case that the following three conditions hold:[9]

[8] This follows intuitively from the Cobb-Douglas form of the task-based production function for goods. To see this formally, call the implicit 'price' of $z_1(i)$, $p_{z_1}(i)$. Perfectly competitive firms will set this 'price' equal to the marginal revenue product of $z_1(i)$ in producing $x(i)$, which is $p(i) \cdot \psi z_1(i)^{\psi-1} z_2(i)^{1-\psi}$. As a result the $z_1(i)$ share of the expenditure on a particular $x(i)$ is $\frac{p(i) \cdot \psi \cdot z_1(i)^{\psi-1} z_2(i)^{1-\psi} \cdot z_1(i)}{p(i) \cdot x(i)} = \psi$.

[9] The first two are the same condition, repeated for expositional clarity.

$$
\begin{aligned}
w^L L^L &= \psi \cdot \alpha(\tilde{i}) \cdot (w^L L^L + w^H L^H + rK) \\
w^H L^H &= \psi \cdot (1 - \alpha(\tilde{i})) \cdot (w^L L^L + w^H L^H + rK) \\
rK &= (1 - \psi) \cdot (w^L L^L + w^H L^H + rK)
\end{aligned}
\tag{3.17}
$$

From the last of these conditions it follows that:

$$
rK = \frac{(1-\psi)}{\psi} \cdot (w^L L^L + w^H L^H) \tag{3.18}
$$

From (3.18) and the first condition in (3.17) it follows that:

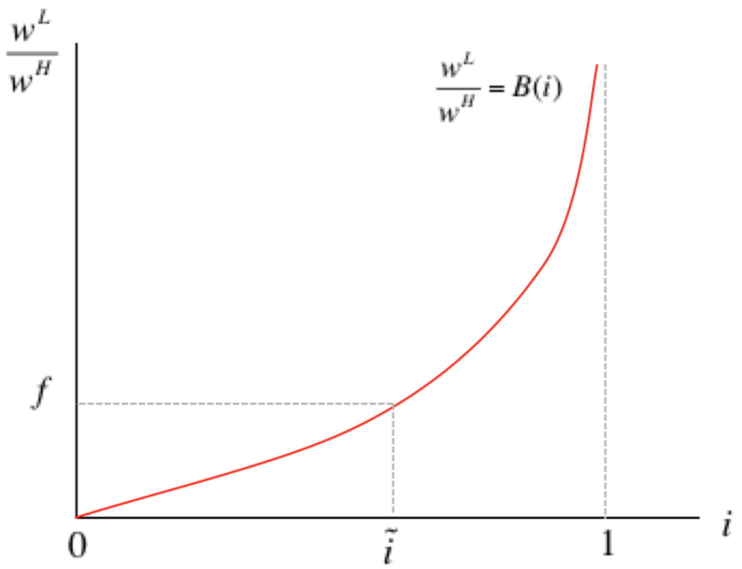
$$
\begin{aligned}
w^L L^L &= \psi \cdot \alpha(\tilde{i}) \cdot \left[w^L L^L + w^H L^H + \frac{(1-\psi)}{\psi} \cdot \left(w^L L^L + w^H L^H \right) \right] \\
w^L L^L &= \psi \cdot \alpha(\tilde{i}) \cdot \left[\frac{1}{\psi} \cdot \left(w^L L^L + w^H L^H \right) \right] \\
\frac{w^L}{w^H} &= \frac{\alpha(\tilde{i})}{1 - \alpha(\tilde{i})} \cdot \frac{L^H}{L^L} \\
\frac{w^L}{w^H} &= B(\tilde{i})
\end{aligned}
\tag{3.19}
$$

This function $B(i)$, where i represents all possible cut-offs (rather than $\tilde{i}$, a particular cut-off) generates a new schedule that represents market equilibrium. This is shown in Figure 3.2. The $B(i)$ schedule describes the relationship between the wage ratio for low-skilled and high-skilled labour and the cut-off good $\tilde{i}$ that maintains market equilibrium. From inspection of Figure 3.2 it is clear that for any $\frac{w^L}{w^H}$ there is a unique cut-off good of type $\tilde{i}$ that ensures the zero profit condition is met and market equilibrium holds. The $B(i)$ schedule has the following properties: $B'(i) > 0$; $B(0) = 0$; and

$\lim_{i \to 1} B(i) = \infty$. That $B'(i) > 0$ since:

$$\begin{aligned} \frac{\partial B(i)}{\partial i} &= \frac{\alpha'(i)(1-\alpha(i)) + \alpha'(i)\alpha(i)}{(1-\alpha(i))^2} \cdot \frac{L^H}{L^L} \\ &= \frac{\alpha'(i)}{(1-\alpha(i))^2} \cdot \frac{L^H}{L^L} > 0 \end{aligned} \tag{3.20}$$

Figure 3.2: The Zero-Profit Schedule



since $\alpha'(i) > 0$ and $(1 - \alpha(i)) \geq 0$ as preferences are Cobb-Douglas.[10] As drawn in Figure 3.2, $B''(i) > 0$. This is not necessarily the case though. $B''(i)$ is:

$$\begin{aligned} \frac{\partial^2 B(i)}{\partial i^2} &= \frac{\alpha''(i)(1-\alpha(i))^2 + 2\alpha'(i)^2(1-\alpha(i))}{(1-\alpha(i))^4} \cdot \frac{L^H}{L^L} \\ &= \frac{\alpha''(i)(1-\alpha(i)) + 2\alpha'(i)^2}{(1-\alpha(i))^3} \cdot \frac{L^H}{L^L} \end{aligned} \tag{3.21}$$

[10] $\alpha'(i) > 0$ since each type of good i has a positive associated expenditure density $b(i)$ and so $\alpha'(i) = b(i) = \theta(i) > 0$, and $(1 - \alpha(i)) \geq 0$ since $\alpha(\tilde{i}) = \int_0^{\tilde{i}} b(i)\, di = \int_0^{\tilde{i}} \theta(i)\, di \geq 0$.

and so $B''(i)$ is only positive when:

$$\alpha''(i) \geq -\frac{2\alpha'(i)^2}{1-\alpha(i)} \tag{3.22}$$

Formally, this means $B''(i) > 0$ when the expenditure densities on each type of good $b(i)$ do not decrease at a faster rate than the term on the right-hand side of (3.22). The main results of this model do not depend upon $B''(i)$, only $B'(i)$. But for now I will assume that the condition in (3.22) holds. A simple way to do this is using Assumption 2:

ASSUMPTION 2: *All goods i have the same fixed expenditure density associated with them such that $\theta(i) = \theta = 1\ \forall i$.*

The consequence of Assumption 2 therefore is that $\alpha''(i) = 0$, and so (3.22) holds. The boundary properties follow from the fact that $\alpha(0) = 0$ and $\alpha(1) = 1$. To see the intuition for the asymptote, consider that if the cut-off good $\tilde{i}$ were at 1, this would mean that only low-skilled labour is hired to work with capital to produce output. But if market equilibrium is to hold, and firms are to make zero profit, then all labour income must also flow to low-skilled labour. So long as there is some high-skilled labour, this can only happen if high-skilled wages, w^H, are equal to zero – this results in a wage ratio $\frac{w^L}{w^H}$ equal to ∞. To see the intuition for why $B(i)$ begins at the origin, consider the opposite argument – if the cut-off good were at $i = 0$, then only high-skilled labour is hired to work with capital to produce output, and so low-skilled wages, w^L must be equal to zero if high-skilled labour is to receive all the labour income – this results in a wage ratio $\frac{w^L}{w^H}$ equal to zero.

Note that it is through the $B(i)$ schedule that the *demand-side* of the economy enters. It is the consumer expenditure density $b(i)$, derived from optimizing their Cobb-Douglas utility function subject to their constraint, that determine $\alpha(i)$ and so the slope of $B(i)$. I will return to this observation in Part C of this thesis.

Combining the Schedules

The $A(i)$ schedule describes the pattern of factor specialisation $\tilde{i}$ for each wage ratio $\frac{w^L}{w^H}$. This is a factor-demand schedule. $B(i)$ describes the wage ratio $\frac{w^L}{w^H}$ that ensures market equilibrium for each pattern of specialisation $\tilde{i}$. This is a zero profit schedule. To derive the equilibrium cut-off type of good $\bar{i}$, rather than a hypothetical cut-off type of good, $\tilde{i}$, these two schedules are combined. This is shown in Figure 3.3 and Proposition 1 follows.[11]

PROPOSITION 1: *Given the properties of $A(i)$ and $B(i)$, there is a unique equilibrium wage ratio, f, and a unique equilibrium cut-off type of good, $\bar{i}$.*

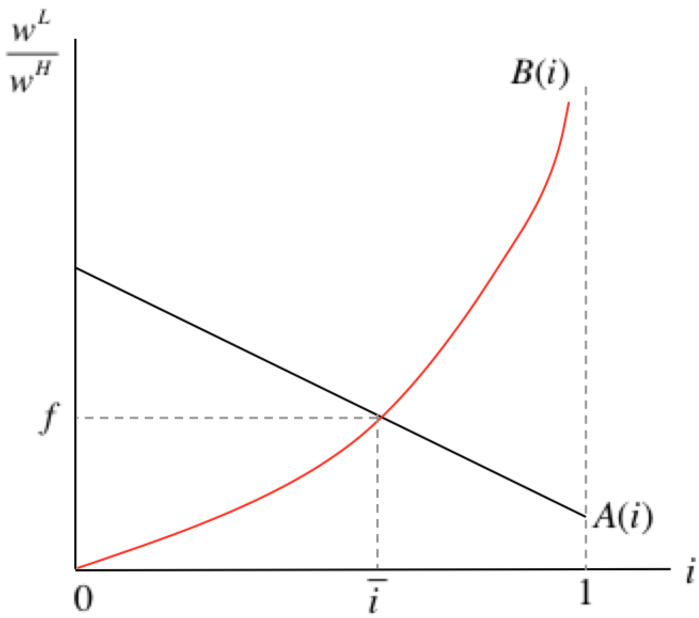
The uniqueness of the equilibrium can be seen from an inspection of Figure 3.3 – since $A(0) > 0$ and $B(0) = 0$, the $A(i)$ schedule must start above the $B(i)$ schedule. As $A'(i) < 0$, $B'(i) > 0$, and $\lim_{i \to 1} B(i) = \infty$ the schedules must cross only once.[12] In equilibrium there is a clear pattern of specialisation. Low-skilled labour specialises in producing the types of goods $i \in [0, \bar{i}]$

[11]Equilibrium here is analytically similar to the equilibrium in Dornbusch et al. (1977), though theirs is an equilibrium in international trade, not the labour market. Acemoglu and Autor (2011) note that their model is 'isomorphic' to Dornbusch et al. (1977). Nevertheless, because of the differences between their approach and the new model in this chapter, the character of their equilibrium is different from that in Figure 3.3.

[12]Note that this result does not depend upon $B''(i)$, although due to Assumption 2, as discussed before, $B''(i) > 0$.

with traditional capital, and high-skilled labour specialises in producing the goods $i \in [\bar{i}, 1]$ with traditional capital. Equilibrium in the labour market is 'Ricardian', with a clear pattern of labour specialisation.

Figure 3.3: Equilibrium in the Labour Market



3.2.5. Prices and Quantities

Relative Factor Prices

From Figure 3.3 it is clear that there is a unique equilibrium wage ratio $f = \frac{w^L}{w^H}$ where:

$$\frac{w^L}{w^H} = \frac{\alpha(\bar{i})}{1 - \alpha(\bar{i})} \cdot \frac{L^H}{L^L} = \frac{a^L(\bar{i})}{a^H(\bar{i})} \tag{3.23}$$

Alongside $\frac{w^L}{w^H}$, the relative factor price for each type of labour with respect to traditional capital, $\frac{w^L}{r}$ and $\frac{w^H}{r}$, can also be derived by considering the marginal products of the factors and the profit-maximising conditions for the perfectly competitive firms.[13] If $i \in [0, \bar{i}]$ then $x(i)$ is produced with low-skilled labour and traditional capital, where the marginal product of low-skilled labour for each $x(i)$ is:[14]

$$\begin{aligned} MPL^L(i) &= a^L(i) \cdot \psi \left[a^L(i)L^L(i)\right]^{\psi-1} \left[a^K(i)K(i)\right]^{1-\psi} \\ &= a^L(i) \cdot \psi \left[a^L(i)L^L(i)\right]^{-1} \cdot x(i) \\ &= \psi \cdot \frac{x(i)}{L^L(i)} \end{aligned} \tag{3.24}$$

And the marginal product of traditional capital for each $x(i)$ is:

$$\begin{aligned} MPK(i) &= a^K(i) \cdot (1-\psi) \cdot \left[a^L(i)L^L(i)\right]^{\psi} \left[a^K(i)K(i)\right]^{-\psi} \\ &= a^K(i) \cdot (1-\psi) \cdot \left[a^K(i)K(i)\right]^{-1} \cdot x(i) \\ &= (1-\psi) \cdot \frac{x(i)}{K(i)} \end{aligned} \tag{3.25}$$

Similarly, if $i \in [\bar{i}, 1]$ then $x(i)$ is produced by high-skilled labour and traditional capital, where the marginal product of high-skilled labour for $x(i)$ is:

$$\begin{aligned} MPL^H(i) &= a^H(i) \cdot \psi \left[a^H(i)L^H(i)\right]^{\psi-1} \left[a^K(i)K(i)\right]^{1-\psi} \\ &= \psi \cdot \frac{x(i)}{L^H(i)} \end{aligned} \tag{3.26}$$

[13] There is a simpler way to derive these relative factor prices that does not use these conditions. I will set this out later in the chapter. Nevertheless, it is instructive in this first instance to see how these relative factor prices can be derived from the marginal conditions of the firms. In the chapters that follow I use the simpler approach.

[14] Recall the production function for goods $x(i)$ from equations (3.8) and (3.10).

And the marginal product of traditional capital for each $x(i)$ is again:

$$\begin{aligned} MPK(i) &= a^K(i) \cdot (1-\psi) \cdot \left[a^H(i)L^H(i)\right]^{\psi} \left[a^K(i)K(i)\right]^{-\psi} \\ &= (1-\psi) \cdot \frac{x(i)}{K(i)} \end{aligned} \tag{3.27}$$

Given perfectly competitive profit-maximising firms, the price of each of these factors – w^L, w^H, and r – must be equal to their respective marginal revenue products as follows:

$$\begin{aligned} w^L &= p(i) \cdot \psi \cdot \frac{x(i)}{L^L(i)} \qquad \forall i \in [0, \bar{i}] \\ w^H &= p(i) \cdot \psi \cdot \frac{x(i)}{L^H(i)} \qquad \forall i \in [\bar{i}, 1] \\ r &= p(i) \cdot (1-\psi) \cdot \frac{x(i)}{K(i)} \qquad \forall i \in [0, 1] \end{aligned} \tag{3.28}$$

Given these conditions must hold for all i in each subset of i,[15] this implies that each factor's marginal revenue product must be the same for all the goods it produces. From these conditions it follows that $\frac{w^L}{r}$ is equal to:

$$\begin{aligned} \frac{w^L}{r} &= \frac{p(i)}{p(i)} \cdot \frac{MPL^L(i)}{MPK(i)} \\ &= \frac{\psi}{1-\psi} \cdot \frac{x(i)}{L^L(i)} \cdot \frac{K(i)}{x(i)} \\ &= \frac{\psi}{1-\psi} \cdot \frac{K(i)}{L^L(i)} \end{aligned} \tag{3.29}$$

Which can be re-arranged so that:

[15] $i \in [0, \bar{i}]$ for w^L, $i \in [\bar{i}, 1]$ for w^H, and $i \in [0, 1]$ for r.

$$\frac{w^L}{r} \cdot L^L(i) = \frac{\psi}{1-\psi} \cdot K(i) \tag{3.30}$$

Integrating over this implies that:

$$\begin{aligned}
\frac{w^L}{r} \int_0^{\bar{i}} L^L(i)di &= \frac{\psi}{1-\psi} \int_0^{\bar{i}} K(i)di \\
\frac{w^L}{r} \cdot L^L &= \frac{\psi}{1-\psi} \int_0^{\bar{i}} K(i)di \\
\frac{w^L}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_0^{\bar{i}} K(i)\, di}{L^L}
\end{aligned} \tag{3.31}$$

Similarly, $\frac{w^H}{r}$ is equal to:

$$\begin{aligned}
\frac{w^H}{r} &= \frac{p(i)}{p(i)} \cdot \frac{MPL^H(i)}{MPK(i)} \\
&= \frac{\psi}{1-\psi} \cdot \frac{K(i)}{L^H(i)}
\end{aligned} \tag{3.32}$$

Which can be re-arranged as:

$$\frac{w^H}{r} \cdot L^H(i) = \frac{\psi}{1-\psi} \cdot K(i) \tag{3.33}$$

Again, integrating over this implies that:

$$\begin{aligned}
\frac{w^H}{r} \int_{\bar{i}}^1 L^H(i)di &= \frac{\psi}{1-\psi} \int_{\bar{i}}^1 K(i)di \\
\frac{w^H}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_{\bar{i}}^1 K(i)\, di}{L^H}
\end{aligned} \tag{3.34}$$

It follows, then, that the full set of relative factor prices in equilibrium are:

$$\begin{aligned}
\frac{w^L}{w^H} &= \frac{\alpha(\bar{i})}{1-\alpha(\bar{i})} \cdot \frac{L^H}{L^L} \\
\frac{w^L}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_0^{\bar{i}} K(i)\, di}{L^L} \\
\frac{w^H}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_{\bar{i}}^1 K(i)\, di}{L^H}
\end{aligned} \tag{3.35}$$

Also note that in equilibrium, since $\frac{w^H}{r}$ and $\frac{w^L}{r}$ are unique, (3.29) and (3.32) imply that the ratio of the allocation of capital to each type of labour in producing the goods in which that type of labour specialises is constant. This follows from the Cobb-Douglas property of the task-based production functions for goods.

The Allocation of Traditional Capital

In the expressions in (3.35) the allocation of traditional capital between those goods that are produced with low-skilled labour and high-skilled labour – $\int_0^{\bar{i}} K(i)\, di$ and $\int_{\bar{i}}^1 K(i)\, di$ – are not defined. There are two ways to derive these expressions – either by integrating over the expression for $K(i)$, derived from the first-order conditions for the perfectly competitive firms in (3.28) and substituting in the relative factor prices above, or by taking the appropriate ratios of (3.17) in equilibrium (i.e. using $\bar{i}$ rather than $\tilde{i}$). For now I do the former.[16] This is shown in full in the Appendix, section 9.0.2. These are

[16] To see the latter, consider the case with low-skilled labour and traditional capital. (3.17) implies that in equilibrium:

$$\begin{aligned}
\frac{w^L L^L}{rK} &= \frac{\psi \cdot \alpha(\bar{i}) \cdot Y}{(1-\psi) \cdot Y} \\
\therefore \quad \frac{w^L}{r} &= \frac{\psi \cdot \alpha(\bar{i})}{(1-\psi)} \cdot \frac{K}{L^L}
\end{aligned}$$

shown to be:

$$\begin{aligned}\int_0^{\bar{i}} K(i)\,di &= \alpha(\bar{i}) \cdot K \\ \int_{\bar{i}}^1 K(i)\,di &= (1-\alpha(\bar{i})) \cdot K\end{aligned} \tag{3.36}$$

The result is intuitive – the allocation of traditional capital follows the allocation of consumer expenditure. Note that this result implies that the relative factor prices in (3.35) can be re-written:

$$\begin{aligned}\frac{w^L}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\alpha(\bar{i}) \cdot K}{L^L} \\ \frac{w^H}{r} &= \frac{\psi}{1-\psi} \cdot \frac{(1-\alpha(\bar{i})) \cdot K}{L^H}\end{aligned} \tag{3.37}$$

Prices of Goods[17]

In Dornbusch et al. (1977), upon which both the new model in this chapter and the AA model build, the absolute prices of the goods are not discussed. That is because the model is static, and the interesting comparative statics can be recovered from relative prices alone. In Chapter 5, I place this static model in a dynamic setting, and the absolute prices have an important role. As a result I derive them here. This can be done by first re-arranging the firms' marginal conditions in (3.28) in the following way:

And so $\int_0^{\bar{i}} K(i)\,di = \alpha(\bar{i}) \cdot K$. This is the simpler approach I use in the rest of the thesis.

[17]To derive the accompanying quantities demanded of each good, note that in equilibrium $x(i) = \theta(i) \cdot \frac{Y}{p(i)}$, where $p(i)$ is as derived in this section.

$$
\begin{aligned}
L^L(i) &= p(i) \cdot \psi \cdot \frac{x(i)}{w^L} \qquad \forall i \in [0, \bar{i}] \\
L^H(i) &= p(i) \cdot \psi \cdot \frac{x(i)}{w^H} \qquad \forall i \in [\bar{i}, 1] \\
K(i) &= p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r} \qquad \forall i \in [0, 1]
\end{aligned} \tag{3.38}
$$

The price $p(i)$ for those goods produced by low-skilled labour with traditional capital, i.e. where $i \in [0, \bar{i}]$, can be found by placing the expressions for $L^L(i)$ and $K(i)$ in (3.38) into the production function in (3.11):

$$
\begin{aligned}
x(i) &= \left[a^L(i) \cdot p(i) \cdot \psi \cdot \frac{x(i)}{w^L}\right]^{\psi} \left[a^K(i) \cdot p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r}\right]^{1-\psi} \\
x(i) &= x(i) \cdot p(i) \left[\frac{\psi \cdot a^L(i)}{w^L}\right]^{\psi} \left[\frac{(1-\psi) \cdot a^K(i)}{r}\right]^{1-\psi} \\
p(i) &= \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi}
\end{aligned} \tag{3.39}
$$

This $p(i)$ applies for all goods $i \in [0, \bar{i}]$. Placing the expression for $L^H(i)$ and $K(i)$ in (3.38) into the production function in (3.11) provides $p(i)$ for those goods $i \in [\bar{i}, 1]$ produced by high-skilled labour with traditional capital:

$$
\begin{aligned}
x(i) &= \left[a^H(i) \cdot p(i) \cdot \psi \cdot \frac{x(i)}{w^H}\right]^{\psi} \left[a^K(i) \cdot p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r}\right]^{1-\psi} \\
p(i) &= \left[\frac{w^H}{\psi \cdot a^H(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi}
\end{aligned} \tag{3.40}
$$

Absolute Factor Prices

Again, in Dornbusch et al. (1977) the focus is on relative factor prices rather than absolute factor prices. However in the dynamic setting in Chapter 5 the absolute factor prices – in particular, the return on capital r – have an important role. As a result, I derive them here. The absolute factor prices can be derived by combining the relative factor prices in (3.35) and (3.37), the expression for absolute prices in (3.39) and (3.40), and an appropriate price normalisation. There are two possible price normalisations – a simplex price normalisation, that $\int_0^1 p(i)di = 1$, or a numeraire price normalisation, that $p(\tilde{i}) = 1$ for some good $\tilde{i}$. To see how this works in practice, consider for example the absolute value of r. First, consider the simplex price normalisation. Note that (3.39) and (3.40) imply that:

$$\begin{aligned}\int_0^1 p(i)\,di = &\int_0^{\bar{i}} \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di \\ &+ \int_{\bar{i}}^1 \left[\frac{w^H}{\psi \cdot a^H(i)}\right]^{\psi} \left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di\end{aligned} \tag{3.41}$$

and so using the simplex normalisation and the expressions for w^L and w^H in terms of r that are implied by (3.37), it follows from (3.41) that:[18]

$$\begin{aligned}1 = r \cdot \Bigg[&\int_0^{\bar{i}} \left[\frac{\frac{\psi}{1-\psi}\cdot \alpha(\bar{i}) \cdot \frac{K}{L^L}}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{1}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di \\ &+ \int_{\bar{i}}^1 \left[\frac{\frac{\psi}{1-\psi}\cdot (1-\alpha(\bar{i})) \cdot \frac{K}{L^H}}{\psi \cdot a^H(i)}\right]^{\psi} \left[\frac{1}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di \Bigg]\end{aligned} \tag{3.42}$$

[18] That $w^L = r \cdot \frac{\psi}{1-\psi} \cdot \frac{\alpha(\bar{i})\cdot K}{L^L}$ and that $w^H = r \cdot \frac{\psi}{1-\psi} \cdot \frac{(1-\alpha(\bar{i}))\cdot K}{L^H}$.

and so under the simplex normalisation r is equal to:

$$r = \Bigg[D_1 \cdot \alpha(\bar{i})^{\psi} \cdot \int_0^{\bar{i}} \left[\frac{1}{a^L(i)}\right]^{\psi} \left[\frac{1}{a^K(i)}\right]^{1-\psi} di \\ + D_2 \cdot (1-\alpha(\bar{i}))^{\psi} \cdot \int_{\bar{i}}^{1} \left[\frac{1}{a^H(i)}\right]^{\psi} \left[\frac{1}{a^K(i)}\right]^{1-\psi} di \Bigg]^{-1} \tag{3.43}$$

where $D_1 = \frac{1}{1-\psi} \cdot \left[\frac{K}{L^L}\right]^{\psi}$ and $D_2 = \frac{1}{1-\psi} \cdot \left[\frac{K}{\cdot L^H}\right]^{\psi}$, and both are both positive constants.

Alternatively consider the numeraire price normalisation. Take some good $\tilde{i} = 0$ – since $\tilde{i} \in [0, \bar{i}]$ this implies that good $\tilde{i}$ is produced by low-skilled labour and traditional capital. (3.39) implies that:

$$p(0) = \left[\frac{w^L}{\psi \cdot a^L(0)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(0)}\right]^{1-\psi} \tag{3.44}$$

and so using the numeraire price normalisation and the expression for w^L from (3.37), it follows from (3.44) that:[19]

$$r = D_3 \cdot \alpha(\bar{i})^{-\psi} \cdot \left[a^L(0)\right]^{\psi} \left[a^K(0)\right]^{1-\psi} \tag{3.45}$$

where $D_3 = (1-\psi) \cdot \left[\frac{L^L}{K}\right]^{\psi}$. The other absolute factor prices follow from substituting the expression for r either in (3.43) or (3.45) into the relative factor prices in (3.37). In later parts of the thesis where I use the absolute factor prices, I will use the numeraire price normalisation. This is because (3.45) is far more analytically tractable than (3.43). The only caution with

[19] That $w^L = r \cdot \frac{\psi}{1-\psi} \cdot \frac{\alpha(\bar{i}) \cdot K}{L^L}$.

this numeraire approach is to ensure that $\bar{i} \neq 0$ – this implies that the numeraire good $x(0)$ is always produced by low-skilled labour and traditional capital, and so the expression for r derived from (3.44) always holds.[20] I discuss this in Chapter 5.

3.3. Comparative Statics

This model provides three sets of comparative statics for traditional capital. First, consider the consequence of increasing the quantity of traditional capital K, or the productivity of traditional capital $a^K(i)$ on the wage ratio, $\frac{w^L}{w^H}$ and the pattern of labour specialisation.[21] Since the expenditure densities on different types of goods, $\theta(i)$, have not changed the $B(i)$ schedule does not shift. Since the relative productivities of the two types of labour have not changed in the $A(i)$ schedule does not shift. As a result the wage ratio and the division of labour stays the same in Figure 3.3.

Secondly, consider the consequence of increasing the quantity of traditional capital K, or the productivity of traditional capital $a^K(i)$, on the relative wage of each type of labour with respect to the return on traditional capital, $\frac{w^L}{r}$ and $\frac{w^H}{r}$.[22] Increasing K drives up the relative wage i.e. labour does relatively better as capital becomes more abundant. Increasing the productivity of capital does not change the relative wage, and does not leave labour worse off. This can be seen from the fact that the productivity of tra-

[20]If $\bar{i} = 0$ then the numeraire good would be produced by high-skilled labour and traditional capital, and so $p(0)$ would not be equal to the expression in (3.44).

[21]The wage ratio is the first expression in (3.35).

[22]These factor-price ratios are the latter two expressions in (3.35), and re-written in (3.37).

ditional capital does not affect any of the variables in (3.37). Intuitively, this is because capital is a q-complement to both types of labour in the production of all types of goods and, because the task-based production functions for goods are Cobb-Douglas, this implies a rise in the marginal productivity of traditional capital in producing a given good causes an equiproportionate rise in the marginal productivity of labour producing that good. This can be seen from (3.29) and (3.32) – since the relative wages are unique, and do not change as traditional capital becomes more productive, the marginal productivities of labour and capital in producing that good must rise in the same proportion.

Thirdly, consider change in the labour market alone. To perform comparative statics, for simplicity, consider that the relative productivity schedule of labour, $A(\cdot)$ is linear:

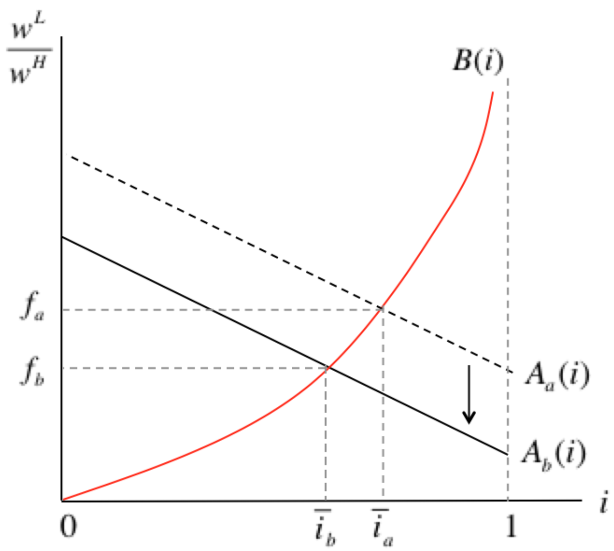
$$A(i) = a - b \cdot i \tag{3.46}$$

where a and b are positive constants.[23] Consider a fall in a – this means that there is an uniform increase in the relative advantage of high-skilled labour over low-skilled labour. This is shown in Figure 3.4. The $A(i)$ schedule shifts down from $A_a(i)$ to $A_b(i)$. The relative wage of low-skilled labour falls from f_a to f_b, the cut-off falls from $\bar{i}_a$ to $\bar{i}_b$, and high-skilled labour produces a wider range of goods with traditional capital.[24]

[23]Note that this linearity is not necessarily for the existence and uniqueness of equilibrium, only for simplicity.

[24]It is possible with the simple linear form of the $A(i)$ schedule in Figure 3.4 that, if a falls enough, this would imply $A(i) < 0$ for certain i. In those cases, it is intuitive to think of $A(i) = 0$ i.e. $A(i) = a + b \cdot i$ if $a + b \cdot i > 0$, else $= 0$.

Figure 3.4: More Productive High-Skilled Workers



3.4. Conclusion

The model in this chapter uses traditional capital. Capital is traditional in the sense that it cannot perform the same type of tasks as labour – both labour and capital perform their own distinct set of tasks. There are two central results. The first is that increasingly capable traditional capital does not change relative wages with respect to the return on capital, and labour necessarily receives a constant share of income, ψ.[25] All types of labour, relative to capital, are no better or worse off. The second result is that increasingly capable traditional capital does not change the division of labour between the two types of labour. This division of labour among tasks performed by labour depends only on the relative productivity of the two types

[25]How that ψ is divided up between the two types of labour is endogenously determined.

of labour in performing the tasks that must be done (by labour) to produce each different type of good. This is optimism at work.

This is not a surprising set of results. Capital in the new model is necessarily a q-complement to labour in performing a distinct set of types of tasks. And because the task-based production functions for goods are Cobb-Douglas, a rise in the marginal productivity of capital causes an equiproportionate rise in the marginal productivity of labour. These features reflect the spirit of the ALM hypothesis. However, when the roles for capital and labour are no longer distinct these conclusions no longer hold. This is the focus on the next chapter.

4. A New Task-Based Model with ‘Advanced Capital’

4.1. Outline

In this chapter I adapt the new model in Chapter 3 so that as well as two types of labour there are two types of capital – ‘traditional’ and ‘advanced’. Unlike traditional capital, advanced capital can perform the type of tasks that are performed by labour. This is a direct challenge to the ALM hypothesis. To determine the extent to which advanced capital can perform these tasks, I use the concept of ‘routinisability’ developed in Chapter 2. As a result either of the two types of labour *or* this new advanced capital performs tasks in combination with traditional capital to produce goods. In all other respects, though, the model in this chapter is the same as in the last. This is an original contribution of the thesis.

The existence of machines that can perform tasks previously performed by human beings is not a recent phenomenon – consider the Industrial Revolution, where Hargreave’s spinning jenny, Arkwright’s roller spinner, and Cartwright’s power loom each displaced labour from performing particular types of tasks. However, what is a recent phenomenon is that an increasing number of tasks that were traditionally considered necessarily human ones –

what the ALM hypothesis would call 'non-routine' tasks – can now be performed by machines. As a result, understanding the consequences of relaxing the ALM hypothesis is increasingly important.

The introduction of advanced capital shows that different types of capital have very different consequences for earnings and employment. The neutral effect of increasingly capable machines on the labour market in the previous chapter where capital is traditional, and not capable of performing the same type of tasks as human beings, no longer holds. Traditional capital may still be a q-complement to labour that performs certain types of tasks. But advanced capital is a perfect substitute for labour in performing those tasks. The outcome is pessimistic for labour – in equilibrium, increasingly capable advanced capital erodes the set of types of tasks in which traditional capital q-complements labour. The relative reward to labour falls. This analysis is new and challenges the existing task-based literature.

The main lesson from this static model is that in thinking about the consequence of technological change on the labour market we need to pay far more attention to the encroachment of increasingly capable machines on those tasks that, until very recently, were considered necessarily human ones.

4.2. The Model

4.2.1. Consumers

Again, there is a continuum of consumers $j \in [0,1]$ who are now either workers – high-skilled or low-skilled – or capitalists – owners of traditional or advanced capital. If consumer j is a low-skilled worker, he sells his labour

l_j^L for a wage $w^L \geq 0$. If he is a high-skilled worker, he sells his labour l_j^H for a wage $w^H \geq 0$. If he is a capitalist who owns traditional capital, he rents his capital k_j to earn $r \geq 0$. If he is a capitalist who owns advanced capital, he rents his capital k_j^A to earn $r^A \geq 0$. Again, there is a continuum of different types of goods, where each type of good $x(i)$ is indexed on the interval $i \in [0,1]$. Each consumer j has the same preferences over the goods described by the Cobb-Douglas utility function:

$$u(\boldsymbol{x}) = \int_0^1 \theta(i) \ln x_j(i)\, di \tag{4.1}$$

where $\theta(i) > 0$, $\int_0^1 \theta(i)\, di = 1$, and $x_j(i)$ is consumer j's demand for good i. As a result, consumers face the following optimization problem:

$$\begin{aligned} &\max_{\boldsymbol{x}} \int_0^1 \theta(i) \ln x_j(i)\, di \\ &\text{s.t.} \quad r \cdot k_j + r^A \cdot k_j^A + w^L \cdot l_j^L + w^H \cdot l_j^H \geq \int_0^1 p(i) \cdot x_j(i)\, di \end{aligned} \tag{4.2}$$

where $p(i)$ is the price for each type of good i.

4.2.2. Production and Firms

Again, each type of good $x(i)$ is produced by a Cobb-Douglas combination of two types of task $z_1(i)$ and $z_2(i)$:

$$x(i) = z_1(i)^{\psi} z_2(i)^{1-\psi} \tag{4.3}$$

where $\psi \in (0,1)$. The second type of task $z_2(i)$, is again performed only by traditional capital. But now the first type of task $z_1(i)$, can be performed by labour – either low-skilled or high-skilled – *and the advanced capital.*

For each good, firms must decide whether to hire the low-skilled labour of workers, the high-skilled labour of workers, the traditional capital of the capitalists, or the advanced capital of the capitalists. The total volume of labour and capital available to these firms is set exogenously, and equal to:

$$L^L = \int_0^{\aleph} l_j^L \, dj \qquad K = \int_{\aleph}^{\beth} k_j \, dj \qquad K^A = \int_{\beth}^{\gimel} k_j^A \, dj \qquad L^H = \int_{\gimel}^{1} l_j^H \, dj \tag{4.4}$$

The individuals are partitioned into four sets. Those who are sell their labour and are low-skilled, $j \in [0, \aleph]$, those who own and rent traditional capital, $j \in [\aleph, \beth]$, those who own and rent advanced capital, $j \in [\beth, \gimel]$, and those who sell their labour and are high-skilled, $j \in [\gimel, 1]$.

Again, the productivity of different factors varies at performing different tasks. Call $a^L(i)$, $a^H(i)$, $a^A(i)$ the factor productivity of low-skilled workers, high-skilled workers, and advanced capital respectively at task $z_1(i)$. Again, call $a^K(i)$ the factor productivity of traditional capital at task $z_2(i)$. Given there are now three factors that can perform the tasks $z_1(i)$, there are two relative productivity schedules. The first is the relative productivity schedule of low-skilled labour and advanced capital:

$$A^L(i) = \frac{a^L(i)}{a^A(i)} \tag{4.5}$$

And the second is the relative productivity schedule of high-skilled labour and advanced capital:

$$A^H(i) = \frac{a^H(i)}{a^A(i)} \tag{4.6}$$

I call these $A^L(i)$ and $A^H(i)$. These reflect the relative productivities of the different factors at performing the type of tasks $z_1(i)$. I now make an important set of assumptions about the relative productivities of the factors at performing these different types of task:

ASSUMPTION 3: *$A^L(i)$ is continuous, $A^L(i) > 0$, and $A^{L\prime}(i) < 0$. $A^H(i)$ is continuous, $A^H(0) > 0$, and $A^{H\prime}(i) > 0$. In addition, $A^H(0) \geq A^L(0)$ and so, given the other properties of $A^H(i)$ and $A^L(i)$, it follows that $A^H(i) \geq A^L(i)\ \forall i$. Finally, $A^{L\prime\prime}(i) = A^{H\prime\prime}(i) = 0$.*

Assumption 3 is, like Assumption 1, an assumption about 'advantage'. Now, it reflects four principles. Again, it reflects the fact that as I move left to right on the spectrum of goods, and i increases, the tasks that are done by labour and advanced capital, $z_1(i)$, to produce the goods become more 'complex'. But now there are additional assumptions about the comparative advantage of machines relative to each type of labour.

The assumptions about $A^L(i)$ reflect the fact that advanced capital has a *decreasing* comparative advantage over low-skilled labour at performing simpler tasks. This assumption reflects 'Moravec's paradox' – a finding in AI research that tasks that seem very simple to human beings are often the hardest for a machine to perform. More precisely, that "it is comparatively easy to make computers exhibit adult level performance on intelligence tests or playing checkers or chess, and it is difficult or impossible to give them the skills of a one-year old when it comes to perception and mobility." (Moravec 1988). This assumption therefore reflects this fact – that the simplest tasks, which often rely on basic perception and mobility, are relatively hard to *routinise*. Note that I am appealing here to the concept of 'routinisability', not the 'routineness' of the task.[1]

The assumptions about $A^H(i)$ reflect the fact that advanced capital has a *decreasing* comparative advantage over high-skilled labour at performing more complex tasks. This assumption reflects the fact that the most complex tasks draw on creative, problem-solving, and interpersonal faculties that as yet are hard to replicate with a machine. These are among the most intractable of what Frey and Osborne (2013) call the "engineering bottlenecks" to computerisation. As a result, the most complex tasks are relatively hard to *routinise*. Again, note that I am appealing here to the concept of 'routinisability', not the 'routineness' of the task.

Alongside these assumptions about *comparative* advantage is a fourth assumption about *absolute* advantage – high-skilled labour is more productive at performing all the tasks $z_1(i)$ than low-skilled labour in absolute terms, however complex.

The final assumption – that both relative productivity schedules are linear – makes the model far more tractable, as the comparative statics will show at the end of this chapter.

[1]See section 2.6.

4.2.3. Relation to Chapter 1 Framework

This model can also be set out using the new framework in Chapter 1. The factor-based production functions for tasks are:

$$\begin{aligned} z_1(i) &= a^L(i)L^L(i) + a^A(i)K^A(i) + a^H(i)L^H(i) \\ z_2(i) &= a^K(i)K(i) \end{aligned} \tag{4.7}$$

And the task-based production function for goods is:

$$x(i) = z_1(i)^{\psi} z_2(i)^{1-\psi} \tag{4.8}$$

It follows then that the aggregate function (i.e. the equivalent to 1.6) is:

$$x(i) = \left[a^L(i)L^L(i) + a^A(i)L^A(i) + a^H(i)L^H(i))\right]^{\psi} \left[a^K(i)K(i)\right]^{1-\psi} \tag{4.9}$$

The most significant change in this new model, relative to the four models set out in Chapter 1, is the role of capital in the factor-based production function for tasks. In the canonical model, capital and labour perform distinct types of tasks. In the ALM model and the AD model, capital can perform the same type of tasks as labour, but *only* 'routine' tasks – 'non-routine' tasks are performed by labour. In the AA model, capital can perform all types of tasks. But in the new model in this chapter there are two types of capital – traditional capital cannot perform the same types of tasks as labour, but advanced capital can. There is both a type of capital that acts as a q-complement to the tasks that are performed by labour – as in the canonical, ALM, and AD models – but there is a different type of capital that is a perfect substitute for those that tasks that are performed by labour – as in the AA model. The capabilities of advanced capital represent a direct challenge to the ALM hypothesis. In turn, unlike the core set of task-based models set out in Chapter 1 that require all goods to be produced by at least one type of labour, this set-up implies that there may be some goods that are produced by capital alone.

Note that the distinction between advanced capital and traditional capital is similar to one proposed in Summers (2013) and discussed in Atkinson and Bourguignon (2015). In these analyses, according to the latter, capital has two roles – "not only directly via the first argument of the production function but also indirectly insofar as it supplants human labor through robotisation". The particular production function considered in both is $F(K_1, AL + BK_2)$, where L is labour, K capital where $K = K_1 + K_2$, and A and B are factor-augmenting terms. One way to interpret the model in this chapter is as a task-based representation of this informal idea.

4.2.4. Equilibrium

Factor Demand

The perfectly competitive firms again face an assignment problem. They must decide which factors to hire to perform the tasks that will produce each type of good. Again it is clear that to perform the tasks $z_2(i)$ the firms will rent traditional capital $K(i)$ – traditional capital is the only factor that performs that type of task. Again, less obviously, the firms will only hire one other factor to carry out the tasks $z_1(i)$ for each good i – but never more than one together. This is Lemma 2:

LEMMA 2: *In equilibrium, there exists some type of cut-off good* $\tilde{i}$ *such that low-skilled labour works with traditional capital to produce goods of type* $i \in [0, \tilde{i}]$*, and there exists some type of cut-off good* $\tilde{\tilde{i}}$ *such that advanced capital works with traditional capital to produce goods of type* $i \in [\tilde{i}, \tilde{\tilde{i}}]$ *and high-skilled labour works with traditional capital to produce the goods of type* $i \in [\tilde{\tilde{i}}, 1]$.

The proof is again intuitive. For any w^L, w^H and r^A, in equilibrium, there will exist some type of good $\tilde{i}$ such that the cost of performing a unit of $z_1(\tilde{i})$ is the same for low-skilled labour and advanced capital:

$$\frac{w^L}{a^L(\tilde{i})} = \frac{r^A}{a^A(\tilde{i})} \tag{4.10}$$

and there will exist some type of good $\tilde{\tilde{i}}$ such that the cost of performing a unit of $z_1(\tilde{\tilde{i}})$ is the same for advanced capital and high-skilled labour:

$$\frac{w^H}{a^H(\tilde{\tilde{i}})} = \frac{r^A}{a^A(\tilde{\tilde{i}})} \tag{4.11}$$

For task $z_1(\tilde{i})$, the firm will therefore be indifferent between hiring either low-skilled labour or advanced capital. And since $\frac{a^L(i)}{a^A(i)}$ is strictly decreasing, it follows that it will always cost strictly less to do types of task $z_1(i)$ where $i \in [0, \tilde{i}]$ using low-skilled labour. For task $z_1(\tilde{\tilde{i}})$, the firm will therefore be indifferent between hiring either advanced capital or high-skilled labour. And since $\frac{a^H(i)}{a^A(i)}$ is strictly increasing, it follows that it will always cost strictly less to do types of task $z_1(i)$ where $i \in [\tilde{\tilde{i}}, 1]$ using high-skilled labour. It follows that the remaining tasks, $z_1(i)$ where $i \in [\tilde{i}, \tilde{\tilde{i}}]$ will be done by advanced capital. Formally, a perfectly competitive firm will hire low-skilled labour and not advanced capital if:

$$\begin{aligned} \frac{w^L}{a^L(i)} &\leq \frac{r^A}{a^A(i)} \\ \frac{a^L(i)}{a^A(i)} &\geq \frac{w^L}{r^A} \\ A^L(i) &\geq \frac{w^L}{r^A} \end{aligned} \tag{4.12}$$

which takes place to the left of $\tilde{i}$, given the properties of $A^L(i)$ that are set out in Assumption 3. The opposite argument applies for the choice of advanced capital over high-skilled labour. Similarly a firm will hire high-skilled labour and not advanced capital if:

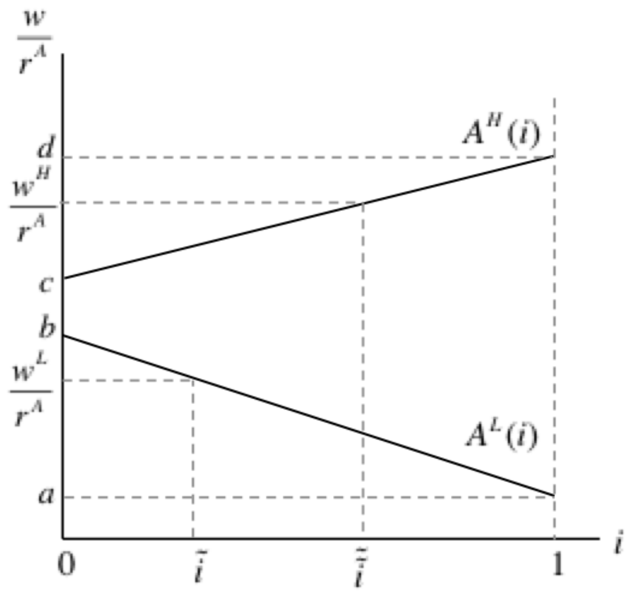
$$\begin{aligned} \frac{w^H}{a^H(i)} &\leq \frac{r^A}{a^A(i)} \\ \frac{a^H(i)}{a^A(i)} &\geq \frac{w^H}{r^A} \\ A^H(i) &\geq \frac{w^H}{r^A} \end{aligned} \tag{4.13}$$

which takes place to the right of $\tilde{\tilde{i}}$, given the properties of $A^H(i)$ that are set out in Assumption 3. The opposite argument applies for the choice of advanced capital over high-skilled labour. Again, for any distribution of traditional capital K across the goods, firms will want to choose the cost-minimising allocation of labour and advanced capital. Given this reasoning and Lemma 2, (4.9) can be re-written:[2]

$$
\begin{aligned}
x(i) &= \left[a^L(i)L^L(i)\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \quad i \in [0, \tilde{i}] \\
x(i) &= \left[a^A(i)K^A(i))\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \quad i \in [\tilde{i}, \tilde{\tilde{i}}] \\
x(i) &= \left[a^H(i)L^H(i)\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \quad i \in [\tilde{\tilde{i}}, 1]
\end{aligned} \tag{4.14}
$$

Given the characteristics of $A^L(i)$ and $A^H(i)$ set out in Assumption 3, this is shown in Figure 4.1.

Figure 4.1: The Relative Productivity Schedules for Labour



That the $A^H(i)$ schedule is above the $A^L(i)$ schedule reflects the absolute advantage claim in Assumption 3. The slope of the schedules reflects the comparative advantage claims in Assumption 3. It is clear from Figure 4.1

[2]Note that this assumes $\tilde{i} < \tilde{\tilde{i}}$. In a moment I show that the opposite is not feasible in equilibrium.

that for any $\frac{w^L}{r^A} \in [a, b]$ there is a 'cut-off' type of good $\tilde{i}$, and for any $\frac{w^H}{r^A} \in [c, d]$ there is a 'cut-off' type of good $\tilde{\tilde{i}}$. This leads to two relative factor prices and two cut-offs. There is a cut-off type of good $\tilde{i}$ that marks the boundary between the tasks that low-skilled labour carries out, with an accompanying relative factor price $\frac{w_L}{r^A}$. Good $\tilde{i}$ is the good for which a perfectly competitive firm is indifferent between using low-skilled labour and advanced capital to work with traditional capital. There is a second cut-off $\tilde{\tilde{i}}$ that marks the boundary between the type of tasks that high-skilled labour carries out, with an accompanying relative factor price $\frac{w_H}{r^A}$. Good $\tilde{\tilde{i}}$ is the good for which a perfectly competitive firm is indifferent between using high-skilled labour and advanced capital to work with traditional capital. Note that there are relative factor price such that $\tilde{i} > \tilde{\tilde{i}}$. In the analysis that follows I show this is impossible in equilibrium.

Consumer Expenditure

Again, since consumers have identical Cobb-Douglas preferences over the goods, it follows that the expenditure density for each type of good i, $b(i)$, is always constant and equal to $\theta(i)$. I maintain Assumption 2 from Chapter 3 – that $\theta(i)$ is identical and equal to 1 for all goods.

Market Equilibrium

Take some hypothetical cut-off goods $\tilde{i}$ and $\tilde{\tilde{i}}$. Given that preferences are Cobb-Douglas, it follows that the share of total consumer expenditure that is spent on all goods of type $i \in [0, \tilde{i}]$ is:

$$
\begin{aligned}
\alpha(\tilde{i}) &= \int_0^{\tilde{i}} b(i)\, di \\
&= \int_0^{\tilde{i}} \theta(i)\, di
\end{aligned}
\tag{4.15}
$$

The share of total expenditure on all goods of type $i \in [\tilde{\tilde{i}}, 1]$ is:

$$
\begin{aligned}
\beta(\tilde{\tilde{i}}) &= \int_{\tilde{\tilde{i}}}^{1} b(i)\, di \\
&= \int_{\tilde{\tilde{i}}}^{1} \theta(i)\, di
\end{aligned}
\tag{4.16}
$$

And the share of total expenditure on all goods of type $i \in [\tilde{i}, \tilde{\tilde{i}}]$ is:

$$
\begin{aligned}
1 - \alpha(\tilde{i}) - \beta(\tilde{\tilde{i}}) &= 1 - \int_0^{\tilde{i}} b(i)\, di - \int_{\tilde{\tilde{i}}}^{1} b(i)\, di \\
&= \int_{\tilde{i}}^{\tilde{\tilde{i}}} \theta(i)\, di
\end{aligned}
\tag{4.17}
$$

In turn, given that the task-based production for goods is also Cobb-Douglas, it follows that the share of total consumer expenditure that is spent on all the *tasks* performed by low-skilled labour is:

$$
\begin{aligned}
&\psi \int_0^{\tilde{i}} b(i)\, di \\
&= \psi \int_0^{\tilde{i}} \theta(i)\, di
\end{aligned}
\tag{4.18}
$$

The share of total expenditure on all the *tasks* performed by advanced capital is:

$$\begin{aligned}
&\psi \int_{\tilde{i}}^{\tilde{\tilde{i}}} b(i)\, di \\
&= \psi \int_{\tilde{i}}^{\tilde{\tilde{i}}} \theta(i)\, di
\end{aligned} \tag{4.19}$$

And the share of total expenditure on all the *tasks* performed by high-skilled labour is:

$$\begin{aligned}
&\psi \int_{\tilde{\tilde{i}}}^{1} b(i)\, di \\
&= \psi \int_{\tilde{\tilde{i}}}^{1} \theta(i)\, di
\end{aligned} \tag{4.20}$$

As firms are perfectly competitive, market equilibrium requires that the total consumer expenditure on the types of tasks that are performed by each factor is equal to the income that the factor receives for carrying out those tasks. Again, this is the zero-profit condition. As a result, in this new setting, if market equilibrium is to hold it must be the case that the following four conditions hold:[3]

$$\begin{aligned}
w^L L^L &= \psi \cdot \alpha(\tilde{i}) \cdot \left[w^L L^L + w^H L^H + r^A K^A + rK\right] \\
w^H L^H &= \psi \cdot \beta(\tilde{\tilde{i}}) \cdot \left[w^L L^L + w^H L^H + r^A K^A + rK\right] \\
r^A K^A &= \psi \cdot (1 - \alpha(\tilde{i})) - \beta(\tilde{\tilde{i}})) \cdot \left[w^L L^L + w^H L^H + r^A K^A + rK\right] \\
rK &= (1 - \psi) \cdot \left[w^L L^L + w^H L^H + r^A K^A + rK\right]
\end{aligned} \tag{4.21}$$

This four conditions lead to two market equilibrium schedules, $B(\tilde{i}, \tilde{\tilde{i}})$ and

[3]The first two conditions imply the third, but they are shown here for expositional simplicity.

$C(\tilde{i}, \tilde{\tilde{i}})$. The first is derived by nothing that from the final condition it follows that:

$$
\begin{aligned}
rK \cdot (1 - 1 + \psi) &= (1 - \psi) \cdot \left[w^L L^L + w^H L^H + r^A K^A\right] \\
rK &= \frac{(1 - \psi)}{\psi} \cdot \left[w^L L^L + w^H L^H + r^A K^A\right]
\end{aligned}
\tag{4.22}
$$

Combining (4.22) with the first equilibrium condition in (4.21) leads to:

$$
\begin{aligned}
& w^L L^L = \psi \cdot \alpha(\tilde{i}) \cdot \left[\frac{\psi}{\psi} + \frac{1 - \psi}{\psi}\right] \cdot \left[w^L L^L + w^H L^H + r^A K^A\right] \\
& w^L L^L = \alpha(\tilde{i}) \cdot \left[w^L L^L + w^H L^H + r^A K^A\right]
\end{aligned}
\tag{4.23}
$$

Combining (4.22) with the second equilibrium condition in (4.21) leads to:

$$
\begin{aligned}
& w^H L^H = \psi \cdot \beta(\tilde{\tilde{i}}) \cdot \left[\frac{\psi}{\psi} + \frac{1 - \psi}{\psi}\right] \cdot \left[w^L L^L + w^H L^H + r^A K^A\right] \\
& w^H L^H = \beta(\tilde{\tilde{i}}) \cdot \left[w^L L^L + w^H L^H + r^A K^A\right]
\end{aligned}
\tag{4.24}
$$

And combining (4.22) with the third equilibrium condition in (4.21) leads to:

$$
\begin{aligned}
& r^A K^A = \psi \cdot (1 - \alpha(\tilde{i}) - \beta(\tilde{\tilde{i}})) \cdot \left[\frac{\psi}{\psi} + \frac{1 - \psi}{\psi}\right] \cdot \left[w^L L^L + w^H L^H + r^A K^A\right] \\
& r^A K^A = (1 - \alpha(\tilde{i}) - \beta(\tilde{\tilde{i}})) \cdot \left[w^L L^L + w^H L^H + r^A K^A\right]
\end{aligned}
\tag{4.25}
$$

It follows from (4.24) that $w^H L^H$ is:

$$
w^H L^H = \frac{\beta(\tilde{\tilde{i}})}{1 - \beta(\tilde{\tilde{i}})} \cdot \left[w^L L^L + r^A K^A\right] \tag{4.26}
$$

And so it follows from (4.23) and (4.26) that:

$$
\begin{aligned}
w^L L^L &= \alpha(\tilde{i}) \cdot \left[1 + \frac{\beta(\tilde{\tilde{i}})}{1-\beta(\tilde{\tilde{i}})}\right] \cdot \left[w^L L^L + r^A K^A\right] \\
w^L L^L &= \left[\frac{\alpha(\tilde{i})}{1-\beta(\tilde{\tilde{i}})}\right] \cdot \left[w^L L^L + r^A K^A\right] \\
\frac{w^L}{r^A} &= \left[\frac{\alpha(\tilde{i})}{1-\alpha(\tilde{i})-\beta(\tilde{\tilde{i}})}\right] \cdot \frac{K^A}{L^L} \\
\frac{w^L}{r^A} &= B(\tilde{i}, \tilde{\tilde{i}})
\end{aligned}
\tag{4.27}
$$

This is the first market equilibrium condition. The second market equilibrium condition follows from the fact that (4.23) can be re-written as:

$$
w^L L^L = \frac{\alpha(\tilde{i})}{1-\alpha(\tilde{i})} \cdot \left[w^H L^H + r^A K^A\right] \tag{4.28}
$$

And so it follows from (4.28) and (4.24) that:

$$
\begin{aligned}
w^H L^H &= \beta(\tilde{\tilde{i}}) \left[1 + \frac{\alpha(\tilde{i})}{1-\alpha(\tilde{i})}\right] \cdot \left[w^H L^H + r^A K^A\right] \\
w^H L^H &= \left[\frac{\beta(\tilde{\tilde{i}})}{1-\alpha(\tilde{i})}\right] \cdot \left[w^H L^H + r^A K^A\right] \\
\frac{w^H}{r^A} &= \frac{\beta(\tilde{\tilde{i}})}{1-\alpha(\tilde{i})-\beta(\tilde{\tilde{i}})} \cdot \frac{K^A}{L^H} \\
\frac{w^H}{r^A} &= C(\tilde{i}, \tilde{\tilde{i}})
\end{aligned}
\tag{4.29}
$$

Note that now there are two schedules that ensure the zero profit condition holds. The first schedule, $B(\cdot)$ determines the lower cut-off, $\tilde{i}$, given some relative factor price $\frac{w^L}{r^A}$ and some upper good-cut off, $\tilde{\tilde{i}}$. The second schedule, $C(\cdot)$ determines the upper cut-off, $\tilde{\tilde{i}}$, given some relative factor price $\frac{w^H}{r^A}$ and some lower good-cut off, $\tilde{i}$. These are shown below in Figure 4.2.

Figure 4.2: The Zero Profit Conditions

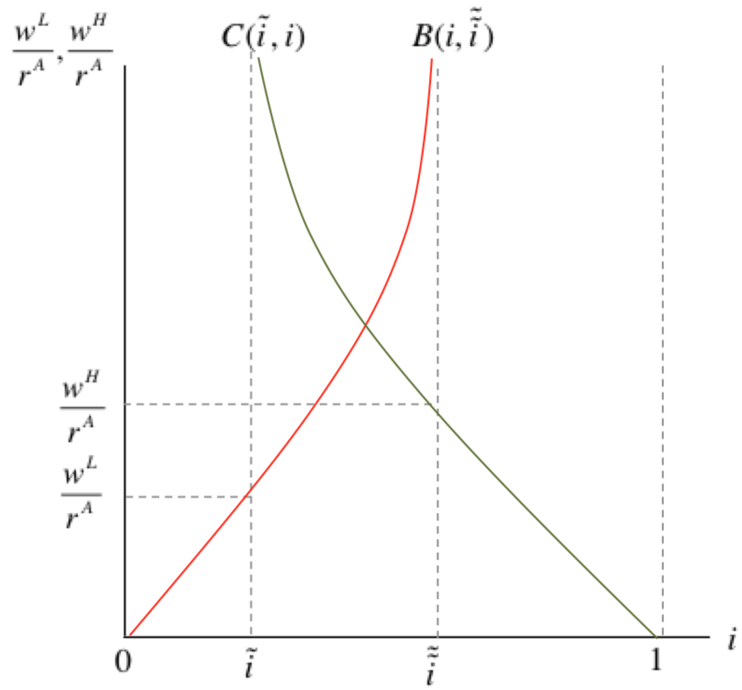


Figure 4.2 is a two-dimensional simplification of the problem. To see this note that the position of $B(i,\tilde{\tilde{i}})$ depends upon the position of $C(\tilde{i},i)$, but the position of $C(\tilde{i},i)$ in turn depends upon $B(i,\tilde{\tilde{i}})$. Nevertheless, this two-dimensional representation is intuitively helpful because it captures the most important features of the problem in one diagram.

The $B(\cdot)$ schedule has the following properties – for a given upper cut-off type of good $\tilde{\tilde{i}}$, $B_1(i,\tilde{\tilde{i}}) > 0$, $B_{11}(i,\tilde{\tilde{i}}) > 0$, $B(0,\tilde{\tilde{i}}) = 0$ and $\lim_{i \to \tilde{\tilde{i}}} B(i,\tilde{\tilde{i}}) = \infty$.[4]

[4] $B_1(\cdot)$ and $B_{11}(\cdot)$ are the first and second derivatives of $B(\cdot)$ with respect to i, respec-

The first two conditions imply that $B(\cdot)$ is strictly increasing in the type of good i. This is for the same reason as in the basic model – demand is Cobb-Douglas, each type of good i has a positive expenditure density $\theta(i)$ on it and, since $\alpha(\tilde{i}) = \int_0^{\tilde{i}} \theta(i)\, di$, it follows that $\alpha'(i) = \theta(i) > 0$, and so $B(i, \tilde{\tilde{i}}) = \left[\frac{\alpha(i)}{1-\alpha(i)-\beta(\tilde{\tilde{i}})}\right] \frac{K^A}{L^L}$ is strictly increasing in i.

The boundary conditions are more complicated. $B(i, \tilde{\tilde{i}})$ starts at the origin since $\alpha(0) = 0$ and so $B(0, \tilde{\tilde{i}}) = 0$. The intuition is the same as in the basic model – if $i = 0$ and low-skilled labour performs no tasks (and so there is no expenditure on tasks performed by low-skilled labour), then the only way that market equilibrium and the zero profit condition can hold is if low-skill wages are equal to zero, and the relative factor price is $\frac{w^L}{r^A}$ equal to zero. To see why the $B(i, \tilde{\tilde{i}})$ schedule is asymptotic to the upper cut-off, note that $\lim_{i \to \tilde{\tilde{i}}} (1 - \alpha(i) - \beta(\tilde{\tilde{i}})) = 0$ and so $\lim_{i \to \tilde{\tilde{i}}} B(i, \tilde{\tilde{i}}) = \infty$. Intuitively, if the lower cut-off is equal to the top cut-off, then advanced capital is not performing any tasks and, for the zero profit condition to hold, r^A must be equal to zero, and the relative factor price $\frac{w^L}{r^A}$ tends to infinity.

The $C(\cdot)$ schedule has the following properties – for a given lower cut-off type of good $\tilde{i}$, $C_2(\tilde{i}, i) < 0$, $C_{22}(\tilde{i}, i) > 0$, $C(\tilde{i}, 1) = 0$ and $\lim_{i \to \tilde{i}} C(\tilde{i}, i) = \infty$. The first two conditions imply that $C(\cdot)$ is strictly decreasing in i. This is for the same reason as before – demand is Cobb-Douglas, each type of good i has a positive expenditure density $\theta(i)$ on it and, since $\beta(\tilde{\tilde{i}}) = \int_{\tilde{\tilde{i}}}^1 \theta(i)\, di$, it follows that $\beta'(i) = -\theta(i) < 0$, and so $\left[\frac{\beta(i)}{1-\alpha(\tilde{i})-\beta(i)}\right] \frac{K^A}{L^H}$ and $C(\tilde{i}, i)$ are strictly decreasing in i.

The limit conditions for the $C(\cdot)$ schedule are opposite to those for the

tively.

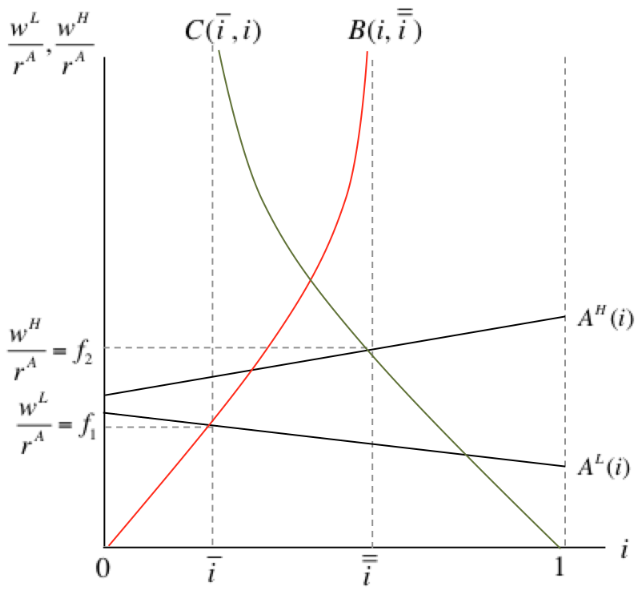
$B(\cdot)$ schedule. $C(\tilde{i}, i)$ intercepts the x-axis at $i = 1$ since $\beta(1) = 0$ and so $C(\tilde{i}, 1) = 0$. The intuition is the same as before – the only way that market equilibrium and the zero profit condition can hold if $i = 1$ and high-skilled labour performs no tasks (and so there is no expenditure on tasks performed by high-skilled labour) is if high-skill wages are equal to zero and the relative factor price $\frac{w^H}{r^A}$ is equal to zero. To see why the $C(\tilde{i}, i) = 0$ schedule is asymptotic to the upper cut-off, note that $\lim_{i \to \tilde{i}}(1 - \alpha(\tilde{i}) - \beta(i)) = 0$ and so $\lim_{i \to \tilde{i}} C(\tilde{i}, i) = \infty$. Again, if the upper cut-off is equal to the lower cut-off, then advanced capital is not performing any tasks and, for the zero profit condition to hold, r^A must be equal to zero and the relative factor price $\frac{w^H}{r^A}$ equal to infinity.

That $\tilde{\tilde{i}} > \tilde{i}$ in equilibrium can be proven by contradiction. Suppose instead that $\tilde{\tilde{i}} < \tilde{i}$. The relative factor price $\frac{w^L}{r^A}$ is determined at the point where the $B(i, \tilde{\tilde{i}})$ schedule meets $\tilde{i}$ (see Figure 4.2). Since the $B(i, \tilde{\tilde{i}})$ schedule starts at the origin, if $\tilde{\tilde{i}} < \tilde{i}$ this would require $B(i, \tilde{\tilde{i}})$ to pass $\tilde{\tilde{i}}$ to meet $\tilde{i}$ in order to determine $\frac{w_L}{r^A}$. But it was previously established that $B(i, \tilde{\tilde{i}})$ must also be asymptotic to $\tilde{\tilde{i}}$. This would require the $B(i, \tilde{\tilde{i}})$ schedule to bend backwards i.e. that $B_1(i, \tilde{\tilde{i}}) < 0$. But I have also shown that $B_1(i, \tilde{\tilde{i}}) > 0$. And so it must be that $\tilde{\tilde{i}} > \tilde{i}$.

Combining the Schedules

The $A^L(i)$ and $A^H(i)$ schedules describe the pattern of factor specialisation $\tilde{i}$ and $\tilde{\tilde{i}}$ for each relative wage $\frac{w^L}{r^A}$ and $\frac{w^H}{r^A}$. These are again factor-demand schedules. $B(\cdot)$ and $C(\cdot)$ describe the relative wages $\frac{w^L}{r^A}$ and $\frac{w^H}{r^A}$ that ensure market equilibrium for each pattern of specialisation $\tilde{i}$ and $\tilde{\tilde{i}}$. These are zero profit schedules. To derive the equilibrium cut-off type of goods $\bar{i}$ and $\bar{\bar{i}}$, rather than a hypothetical cut-off type of good, $\tilde{i}$ and $\tilde{\tilde{i}}$, these two schedules are combined. This is shown in Figure 4.3 and Proposition 2 follows.[5]

Figure 4.3: Equilibrium in the Market for Tasks $z_1(i)$



PROPOSITION 2: *Given the properties of $A^L(i)$, $A^H(i)$, $B(i, \bar{\bar{i}})$, and $C(\bar{i}, i)$ there is a unique equilibrium with low-skilled relative wage f_1, high-skilled relative wage f_2, and two unique cut-offs $\bar{i}$ and $\bar{\bar{i}}$.*

In the model with three factors, two cut-offs now exist. Low-skilled labour specialises in performing the tasks $z_1(i)$ to produce the type of goods $i \in [0, \bar{i}]$ (working with traditional capital), advanced capital specialises in specialises

[5]Again, this figure is a two-dimensional simplification of the problem. The position of $B(i, \bar{\bar{i}})$ depends upon the position of $C(\bar{i}, i)$, but the position of $C(\bar{i}, i)$ in turn depends upon $B(i, \bar{\bar{i}})$. But again, this two-dimensional representation of equilibrium is intuitively helpful because it captures the most important features of equilibrium in one diagram.

in performing the tasks $z_1(i)$ to produce the type of goods $i \in [\bar{i}, \bar{\bar{i}}]$ (working with traditional capital), and high-skilled labour specialises in performing the tasks $z_1(i)$ to produce the type of goods $i \in [\bar{\bar{i}}, 1]$ (working with traditional capital). In equilibrium, there are two unique relative factor prices of labour with respect to advanced capital, $f_1 = \frac{w^L}{r^A}$ and $f_2 = \frac{w^H}{r^A}$. Most importantly, note that the presence of advanced capital now drives a wedge between the two types of labour in the task-space.

4.2.5. Prices and Quantities

Relative Factor Prices

The two unique equilibrium relative wages with respect to advanced capital, $f_1 = \frac{w^L}{r^A}$ and $f_2 = \frac{w^H}{r^A}$, are equal to:

$$\begin{aligned}
\frac{w^L}{r^A} &= \left[\frac{\alpha(\bar{i})}{1-\alpha(\bar{i})-\beta(\bar{\bar{i}})}\right]\frac{K^A}{L^L} = \frac{a^L(\bar{i})}{a^A(\bar{i})} \\
\frac{w^H}{r^A} &= \left[\frac{\beta(\bar{\bar{i}})}{1-\alpha(\bar{i})-\beta(\bar{\bar{i}})}\right]\frac{K^A}{L^H} = \frac{a^H(\bar{\bar{i}})}{a^A(\bar{\bar{i}})}
\end{aligned} \tag{4.30}$$

Alongside $\frac{w^L}{r^A}$ and $\frac{w^H}{r^A}$, the relative factor prices for low-skilled labour, high-skilled labour, and advanced capital with respect to traditional capital – $\frac{w^L}{r}$, $\frac{w^H}{r}$, and $\frac{r^A}{r}$ can also be derived. As in Chapter 3, these can be found by considering the marginal products of the factors and the profit-maximising conditions for the perfectly competitive firms in the following way – if $i \in [0, \bar{i}]$ then $x(i)$ is produced with low-skilled labour and traditional capital, where the marginal product of low-skilled labour for each $x(i)$ is:

$$
\begin{aligned}
MPL^L(i) &= a^L(i) \cdot \psi \left[a^L(i)L^L(i)\right]^{\psi-1} \left[a^K(i)K(i)\right]^{1-\psi} \\
&= a^L(i) \cdot \psi \left[a^L(i)L^L(i)\right]^{-1} \cdot x(i) \\
&= \psi \cdot \frac{x(i)}{L^L(i)}
\end{aligned} \tag{4.31}
$$

And the marginal product of traditional capital for each $x(i)$ is:

$$
\begin{aligned}
MPK(i) &= a^K(i) \cdot (1-\psi) \cdot \left[a^L(i)L^L(i)\right]^{\psi} \left[a^K(i)K(i)\right]^{-\psi} \\
&= a^K(i) \cdot (1-\psi) \cdot \left[a^K(i)K(i)\right]^{-1} \cdot x(i) \\
&= (1-\psi) \cdot \frac{x(i)}{K(i)}
\end{aligned} \tag{4.32}
$$

In turn, if $i \in [\bar{i}, \bar{\bar{i}}]$ then $x(i)$ is produced by advanced capital and traditional capital, where the marginal product of advanced capital for each $x(i)$ is:

$$
\begin{aligned}
MPK^A(i) &= a^A(i) \cdot \psi \left[a^A(i)K^A(i)\right]^{\psi-1} \left[a^K(i)K(i)\right]^{1-\psi} \\
&= \psi \cdot \frac{x(i)}{K^A(i)}
\end{aligned} \tag{4.33}
$$

And the marginal product of traditional capital for each $x(i)$ is:

$$
\begin{aligned}
MPK(i) &= a^K(i) \cdot (1-\psi) \cdot \left[a^A(i)K^A(i)\right]^{\psi} \left[a^K(i)K(i)\right]^{-\psi} \\
&= (1-\psi) \cdot \frac{x(i)}{K(i)}
\end{aligned} \tag{4.34}
$$

And if $i \in [\bar{\bar{i}}, 1]$ then $x(i)$ is produced with high-skilled labour and traditional capital, where the marginal product of high-skilled labour for each $x(i)$ is:

$$
\begin{aligned}
MPL^H(i) &= a^H(i) \cdot \psi \left[a^H(i)L^H(i)\right]^{\psi-1} \left[a^K(i)K(i)\right]^{1-\psi} \\
&= \psi \cdot \frac{x(i)}{L^H(i)}
\end{aligned}
\tag{4.35}
$$

And the marginal product of traditional capital for each $x(i)$ is:

$$
\begin{aligned}
MPK(i) &= a^K(i) \cdot (1-\psi) \cdot \left[a^H(i)L^H(i)\right]^{\psi} \left[a^K(i)K(i)\right]^{-\psi} \\
&= (1-\psi) \cdot \frac{x(i)}{K(i)}
\end{aligned}
\tag{4.36}
$$

Given perfectly competitive profit-maximising firms, the price of each of these factors – w^L, w^H, r^A, and r – must be equal to their respective marginal revenue products as follows:

$$
\begin{aligned}
w^L &= p(i) \cdot \psi \cdot \frac{x(i)}{L^L(i)} \qquad \forall i \in [0, \bar{i}] \\
r^A &= p(i) \cdot \psi \cdot \frac{x(i)}{K^A(i)} \qquad \forall i \in [\bar{i}, \bar{\bar{i}}] \\
w^H &= p(i) \cdot \psi \cdot \frac{x(i)}{L^H(i)} \qquad \forall i \in [\bar{\bar{i}}, 1] \\
r &= p(i) \cdot (1-\psi) \cdot \frac{x(i)}{K(i)} \qquad \forall i \in [0, 1]
\end{aligned}
\tag{4.37}
$$

Given these conditions must hold for all i in each equilibrium subset of i this implies that each factor's marginal revenue product must be the same for all the goods it produces.[6] From these conditions it follows that $\frac{w^L}{r}$ is equal to:

[6]The subsets are $i \in [0, \bar{i}]$ for w^L, $i \in [\bar{i}, \bar{\bar{i}}]$ for r^A, $i \in [\bar{\bar{i}}, 1]$ for w^H, and $i \in [0, 1]$ for r.

$$
\begin{aligned}
\frac{w^L}{r} &= \frac{p(i)}{p(i)} \cdot \frac{MPL^L}{MPK} \\
&= \frac{\psi}{1-\psi} \cdot \frac{x(i)}{L^L(i)} \cdot \frac{K(i)}{x(i)} \\
&= \frac{\psi}{1-\psi} \cdot \frac{K(i)}{L^L(i)}
\end{aligned} \tag{4.38}
$$

Which can be re-arranged so that:

$$
\frac{w^L}{r} \cdot L^L(i) = \frac{\psi}{1-\psi} \cdot K(i) \tag{4.39}
$$

Integrating over this implies that:

$$
\begin{aligned}
\frac{w^L}{r} \int_0^{\bar{i}} L^L(i)di &= \frac{\psi}{1-\psi} \int_0^{\bar{i}} K(i)di \\
\frac{w^L}{r} \cdot L^L &= \frac{\psi}{1-\psi} \int_0^{\bar{i}} K(i)di \\
\frac{w^L}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_0^{\bar{i}} K(i)\, di}{L^L}
\end{aligned} \tag{4.40}
$$

Similarly it follows that $\frac{r^A}{r}$ is equal to:

$$
\begin{aligned}
\frac{r^A}{r} &= \frac{p(i)}{p(i)} \cdot \frac{MPK^A}{MPK} \\
&= \frac{\psi}{1-\psi} \cdot \frac{x(i)}{K^A(i)} \cdot \frac{K(i)}{x(i)} \\
&= \frac{\psi}{1-\psi} \cdot \frac{K(i)}{K^A(i)}
\end{aligned} \tag{4.41}
$$

Which can be re-arranged so that:

$$
\frac{r^A}{r} \cdot K^A(i) = \frac{\psi}{1-\psi} \cdot K(i) \tag{4.42}
$$

Integrating over this implies that:

$$
\begin{aligned}
\frac{r^A}{r}\int_{\bar{i}}^{\bar{\bar{i}}} K^A(i)di &= \frac{\psi}{1-\psi}\int_{\bar{i}}^{\bar{\bar{i}}} K(i)di \\
\frac{r^A}{r}\cdot K^A &= \frac{\psi}{1-\psi}\int_{\bar{i}}^{\bar{\bar{i}}} K(i)di \\
\frac{r^A}{r} &= \frac{\psi}{1-\psi}\cdot\frac{\int_{\bar{i}}^{\bar{\bar{i}}} K(i)\,di}{K^A}
\end{aligned}
\tag{4.43}
$$

Similarly, $\frac{w^H}{r}$ is equal to:

$$
\begin{aligned}
\frac{w^H}{r} &= \frac{p(i)}{p(i)}\cdot\frac{MPL^H}{MPK} \\
&= \frac{\psi}{1-\psi}\cdot\frac{x(i)}{L^H(i)}\cdot\frac{K(i)}{x(i)} \\
&= \frac{\psi}{1-\psi}\cdot\frac{K(i)}{L^H(i)}
\end{aligned}
\tag{4.44}
$$

Which can be re-arranged as:

$$
\frac{w^H}{r}\cdot L^H(i) = \frac{\psi}{1-\psi}\cdot K(i) \tag{4.45}
$$

Again, integrating over this implies that:

$$
\begin{aligned}
\frac{w^H}{r}\int_{\bar{i}}^{1} L^H(i)di &= \frac{\psi}{1-\psi}\int_{\bar{i}}^{1} K(i)di \\
\frac{w^H}{r}\cdot L^H &= \frac{\psi}{1-\psi}\int_{\bar{i}}^{1} K(i)di \\
\frac{w^H}{r} &= \frac{\psi}{1-\psi}\cdot\frac{\int_{\bar{i}}^{1} K(i)\,di}{L^H}
\end{aligned}
\tag{4.46}
$$

It follows, then, that the full set of relative factor prices in equilibrium are:

$$
\begin{aligned}
\frac{w^L}{r^A} &= \frac{\alpha(\bar{i})}{1-\alpha(\bar{i})-\beta(\bar{\bar{i}})} \cdot \frac{K^A}{L^L} \\
\frac{w^H}{r^A} &= \frac{\beta(\bar{i})}{1-\alpha(\bar{i})-\beta(\bar{\bar{i}})} \cdot \frac{K^A}{L^H} \\
\frac{w^L}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_0^{\bar{i}} K(i)\,di}{L^L} \\
\frac{w^H}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_{\bar{\bar{i}}}^1 K(i)\,di}{L^H} \\
\frac{r^A}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\int_{\bar{i}}^{\bar{\bar{i}}} K(i)\,di}{K^A}
\end{aligned}
\tag{4.47}
$$

The Allocation of Traditional Capital

In the expressions in (4.47), the allocation of traditional capital between those goods that are produced with low-skilled labour, advanced capital, and high-skilled labour – $\int_0^{\bar{i}} K(i)\,di$, $\int_{\bar{i}}^{\bar{\bar{i}}} K(i)\,di$ and $\int_{\bar{\bar{i}}}^1 K(i)\,di$ – are not defined. As in Chapter 3 these can be derived either by integrating over the expressions for $K(i)$ derived from the first-order conditions for the perfectly competitive firms and substituting in the relative factor-prices above or by taking the appropriate ratios of (4.21) – for example, $\frac{w^L}{r}$ follows from taking the ratio of the first and final expressions in (4.21), for $\bar{i}$ rather than $\tilde{i}$. The latter, simpler approach, implies that:

$$
\begin{aligned}
\int_0^{\bar{i}} K(i)\,di &= \alpha(\bar{i}) \cdot K \\
\int_{\bar{i}}^{\bar{\bar{i}}} K(i)\,di &= (1-\alpha(\bar{i})-\beta(\bar{\bar{i}})) \cdot K \\
\int_{\bar{\bar{i}}}^1 K(i)\,di &= \beta(\bar{\bar{i}}) \cdot K
\end{aligned}
\tag{4.48}
$$

and so the relative factor prices in (4.47) can be re-written:

$$\begin{aligned}
\frac{w^L}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\alpha(\bar{i}) \cdot K}{L^L} \\
\frac{r^A}{r} &= \frac{\psi}{1-\psi} \cdot \frac{(1-\alpha(\bar{i})-\beta(\bar{\bar{i}})) \cdot K}{K^A} \\
\frac{w^H}{r} &= \frac{\psi}{1-\psi} \cdot \frac{\beta(\bar{\bar{i}}) \cdot K}{L^H}
\end{aligned} \tag{4.49}$$

Prices of Goods[7]

Re-arranging the firms' marginal conditions in (4.37), it follows that:

$$\begin{aligned}
L^L(i) &= p(i) \cdot \psi \cdot \frac{x(i)}{w^L} \qquad \forall i \in [0, \bar{i}] \\
K^A(i) &= p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r^A} \qquad \forall i \in [\bar{i}, \bar{\bar{i}}] \\
L^H(i) &= p(i) \cdot \psi \cdot \frac{x(i)}{w^H} \qquad \forall i \in [\bar{\bar{i}}, 1] \\
K(i) &= p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r} \qquad \forall i \in [0, 1]
\end{aligned} \tag{4.50}$$

The price $p(i)$ for those goods produced by low-skilled labour with traditional capital, i.e. where $i \in [0, \bar{i}]$, can be found by placing the expression for $L^L(i)$ and $K(i)$ in (4.50) into the production function in (4.14):

$$\begin{aligned}
x(i) &= \left[a^L(i) \cdot p(i) \cdot \psi \cdot \frac{x(i)}{w^L}\right]^{\psi} \left[a^K(i) \cdot p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r}\right]^{1-\psi} \\
x(i) &= x(i) \cdot p(i) \left[\frac{\psi \cdot a^L(i)}{w^L}\right]^{\psi} \left[\frac{(1-\psi) \cdot a^K(i)}{r}\right]^{1-\psi} \\
p(i) &= \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi}
\end{aligned} \tag{4.51}$$

[7] Again, the accompanying quantity densities in equilibrium are $x(i) = \theta(i) \cdot \frac{Y}{p(i)}$ where $p(i)$ is as derived in this section.

Similarly, placing the expression for $K^A(i)$ and $K(i)$ in (4.50) into the production function in (4.14) provides the price $p(i)$ for those goods $i \in [\bar{i}, \bar{\bar{i}}]$ that are produced by advanced capital with traditional capital:

$$\begin{aligned} x(i) &= \left[a^A(i) \cdot p(i) \cdot \psi \cdot \frac{x(i)}{r^A}\right]^{\psi} \left[a^K(i) \cdot p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r}\right]^{1-\psi} \\ x(i) &= x(i) \cdot p(i) \left[\frac{\psi \cdot a^A(i)}{r^A}\right]^{\psi} \left[\frac{(1-\psi) \cdot a^K(i)}{r}\right]^{1-\psi} \\ p(i) &= \left[\frac{r^A}{\psi \cdot a^A(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \end{aligned} \tag{4.52}$$

And, finally, by placing the expression for $L^H(i)$ and $K(i)$ in (4.50) into the production function in (4.14) provides the price $p(i)$ for those goods $i \in [\bar{\bar{i}}, 1]$ that are produced by high-skilled labour with traditional capital:

$$\begin{aligned} x(i) &= \left[a^H(i) \cdot p(i) \cdot \psi \cdot \frac{x(i)}{w^H}\right]^{\psi} \left[a^K(i) \cdot p(i) \cdot (1-\psi) \cdot \frac{x(i)}{r}\right]^{1-\psi} \\ x(i) &= x(i) \cdot p(i) \left[\frac{\psi \cdot a^H(i)}{w^H}\right]^{\psi} \left[\frac{(1-\psi) \cdot a^K(i)}{r}\right]^{1-\psi} \\ p(i) &= \left[\frac{w^H}{\psi \cdot a^H(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \end{aligned} \tag{4.53}$$

Absolute Factor Prices

The absolute factor prices follow as in Chapter 3. They can be derived using either a simplex price normalisation or a numeraire price normalisation. To see these different approaches, consider the derivation of r. Given (4.51)-

(4.53) the price of any good $p(i)$ is equal to:

$$\begin{aligned}\int_0^1 p(i)\,di = &\int_0^{\bar{i}} \left[\frac{w^L}{\psi\cdot a^L(i)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di\\ &+\int_{\bar{i}}^{\bar{\bar{i}}} \left[\frac{r^A}{\psi\cdot a^A(i)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di\\ &+\int_{\bar{\bar{i}}}^{1} \left[\frac{w^H}{\psi\cdot a^H(i)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi} di\end{aligned} \tag{4.54}$$

It follows that with a simplex price normalisation, such that $\int_0^1 p(i)\ di = 1$, substituting in the absolute factor prices in terms of r from (4.49) implies that r is equal to:

$$\begin{aligned} r = \Bigg[& D_1\cdot\alpha(\bar{i})^{\psi}\cdot\int_0^{\bar{i}}\left[\frac{1}{a^L(i)}\right]^{\psi}\left[\frac{1}{a^K(i)}\right]^{1-\psi} di\\ &+D_2\cdot\left[1-\alpha(\bar{i})-\beta(\bar{\bar{i}})\right]^{\psi}\cdot\int_{\bar{i}}^{\bar{\bar{i}}}\left[\frac{1}{a^A(i)}\right]^{\psi}\left[\frac{1}{a^K(i)}\right]^{1-\psi} di\\ &+D_3\cdot\beta(\bar{\bar{i}})^{\psi}\cdot\int_{\bar{\bar{i}}}^{1}\left[\frac{1}{a^H(i)}\right]^{\psi}\left[\frac{1}{a^K(i)}\right]^{1-\psi} di\Bigg]^{-1}\end{aligned} \tag{4.55}$$

where $D_1 = \frac{1}{1-\psi}\cdot\left[\frac{K}{L^L}\right]^{\psi}$, $D_2 = \frac{1}{1-\psi}\cdot\left[\frac{K}{K^A}\right]^{\psi}$, and $D_3 = \frac{1}{1-\psi}\cdot\left[\frac{K}{L^H}\right]^{\psi}$, where all are positive constants. Alternatively, to use the numeraire normalisation, consider some good $\tilde{i} = 0$ – since $\tilde{i} \in [0, \bar{i}]$ this implies that good $\tilde{i}$ is produced by low-skilled labour and traditional capital. (4.51) implies that:

$$p(0) = \left[\frac{w^L}{\psi \cdot a^L(0)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(0)}\right]^{1-\psi} \tag{4.56}$$

and so using the numeraire price normalisation and the expression for w^L in terms of r from (4.49), it follows from (4.56) that:

$$r = D_4 \cdot \alpha(\bar{i})^{-\psi} \cdot \left[a^L(0)\right]^{\psi} \left[a^K(0)\right]^{1-\psi} \tag{4.57}$$

where $D_4 = (1-\psi) \cdot \left[\frac{L^L}{K}\right]^{\psi}$. The other absolute factor prices follow from substituting the expression for r either in (4.55) or (4.57) into the relative factor prices in (4.49). Again, in later parts of the thesis where I use the absolute factor prices, I will use the numeraire price normalisation for tractability reasons.

4.2.6. Existence and Uniqueness of Equilibrium

The equilibrium set out in Figure 4.3 and described in Proposition 2 is characterised by two equations in terms of the cut-offs, $\bar{i}$ and $\bar{\bar{i}}$:

$$\begin{aligned} A^L(\bar{i}) &= B(\bar{i}, \bar{\bar{i}}) \\ A^H(\bar{\bar{i}}) &= C(\bar{i}, \bar{\bar{i}}) \end{aligned} \tag{4.58}$$

The proof for the existence and uniqueness of the equilibrium $\bar{i}$ and $\bar{\bar{i}}$ in Figure 4.3 is based on the recognition that these two expressions in (4.58) imply two schedules in $\bar{i}$–$\bar{\bar{i}}$ space that intersect only once – at the 'equilibrium' equilibrium $\bar{i}$ and $\bar{\bar{i}}$, which is shown in Figure 4.3. First note that the linearity of the $A^L(\bar{i})$ and $A^H(\bar{\bar{i}})$ schedules, given Assumption 3, imply that they can be written as:

$$
\begin{aligned}
A^L(\bar{i}) &= k_1 - k_2 \cdot \bar{i} \\
A^H(\bar{\bar{i}}) &= k_3 + k_4 \cdot \bar{\bar{i}}
\end{aligned}
\tag{4.59}
$$

where k_1, k_2, k_3, and k_4 are positive constants. Also note that because of Assumption 2 – that expenditure densities on different goods are constant and equal to 1 – it follows that:

$$
\begin{aligned}
\alpha(\bar{i}) &= \int_0^{\bar{i}} \theta(i) di \\
&= \int_0^{\bar{i}} 1 \; di \\
&= \bar{i}
\end{aligned}
\tag{4.60}
$$

and:

$$
\begin{aligned}
\beta(\bar{\bar{i}}) &= \int_{\bar{\bar{i}}}^{1} \theta(i) di \\
&= \int_{\bar{\bar{i}}}^{1} 1 \; di \\
&= 1 - \bar{\bar{i}}
\end{aligned}
\tag{4.61}
$$

Given (4.60) and (4.61), it follows that the $B(\bar{i}, \bar{\bar{i}})$ and $C(\bar{i}, \bar{\bar{i}})$ schedules can be written as:

$$
\begin{aligned}
B(\bar{i}, \bar{\bar{i}}) &= \frac{\bar{i}}{\bar{\bar{i}} - \bar{i}} \cdot k_5 \\
C(\bar{i}, \bar{\bar{i}}) &= \frac{1 - \bar{\bar{i}}}{\bar{\bar{i}} - \bar{i}} \cdot k_6
\end{aligned}
\tag{4.62}
$$

where k_5 and k_6 are $\frac{K^A}{L^L}$ and $\frac{K^A}{L^H}$ respectively, and are positive constants. It therefore follows from (5.58) to (4.62) that the equilibrium condition $A^L(\bar{i}) = B(\bar{i}, \bar{\bar{i}})$ implies a relationship between $\bar{i}$ and $\bar{\bar{i}}$:

$$\bar{\bar{i}} = \frac{k_5 \cdot \bar{i}}{k_1 - k_2 \cdot \bar{\bar{i}}} + \bar{i} \tag{4.63}$$

where:

$$\frac{d\bar{\bar{i}}}{d\bar{i}} = \frac{k_5}{k_1 - k_2 \cdot \bar{i}} + \frac{k_5 \cdot k_2 \cdot \bar{i}}{(k_1 - k_2 \cdot \bar{i})^2} + 1 > 0 \tag{4.64}$$

and:

$$\frac{d^2\bar{\bar{i}}}{d\bar{i}^2} = -\frac{2 \cdot k_5 \cdot k_1 \cdot k_2}{(k_2 \cdot \bar{i} - k_1)^3} > 0 \tag{4.65}$$

In turn the equilibrium condition $A^H(\bar{\bar{i}}) = C(\bar{i}, \bar{\bar{i}})$ implies a second relationship between $\bar{i}$ and $\bar{\bar{i}}$:

$$\bar{i} = \bar{\bar{i}} - \frac{k_6 \cdot (1 - \bar{\bar{i}})}{k_3 + k_4 \cdot \bar{\bar{i}}} \tag{4.66}$$

where:

$$\frac{d\bar{\bar{i}}}{d\bar{i}} = \left[\frac{d\bar{i}}{d\bar{\bar{i}}}\right]^{-1} = \left[\frac{k_6}{k_4 \cdot \bar{\bar{i}} + k_3} + \frac{k_4 \cdot k_6 \cdot (1 - \bar{\bar{i}})}{(k_4 \cdot \bar{\bar{i}} + k_3)^2} + 1\right]^{-1} > 0 \tag{4.67}$$

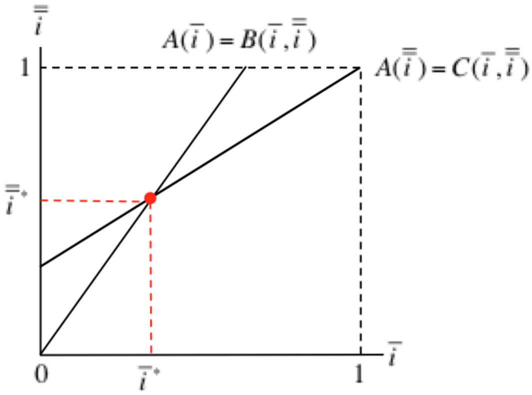
and:[8]

$$\frac{d^2\bar{\bar{i}}}{d\bar{i}^2} = -\frac{d^2\bar{i}}{d\bar{\bar{i}}^2} \cdot \left[\frac{d\bar{\bar{i}}}{d\bar{i}}\right]^3 = \frac{2 \cdot k_4 \cdot (k_4 + k_4) \cdot k_6}{(k_4 \cdot \bar{\bar{i}} + k_3)^3} \cdot \left[\frac{d\bar{\bar{i}}}{d\bar{i}}\right]^3 > 0 \tag{4.68}$$

[8]That $\frac{d^2\bar{\bar{i}}}{d\bar{i}^2} = -\frac{d^2\bar{i}}{d\bar{\bar{i}}^2} \cdot \left[\frac{d\bar{\bar{i}}}{d\bar{i}}\right]^3$ is shown in Sychev (1991), Chapter 2, section 2.1.5.

The schedules implied by the equilibrium conditions in (4.58) can be drawn in $\bar{i}$–$\bar{\bar{i}}$ space. This is shown in Figure 4.4. The $(\bar{i}^*, \bar{\bar{i}}^*)$ corresponds to the equilibrium $(\bar{i}, \bar{\bar{i}})$ in the previous models – the * represents the 'equilibrium' equilibrium cut-off. Both schedules are above the 45° line since $\bar{\bar{i}} > \bar{i}$ – this is proven by contradiction in section 4.2.4. Both schedules have positive first derivatives – as established in (4.64) and (4.67) – and positive second derivatives – as established in (4.65) and (4.68) – though they are drawn as straight lines for simplicity.

Figure 4.4: Existence and Uniqueness

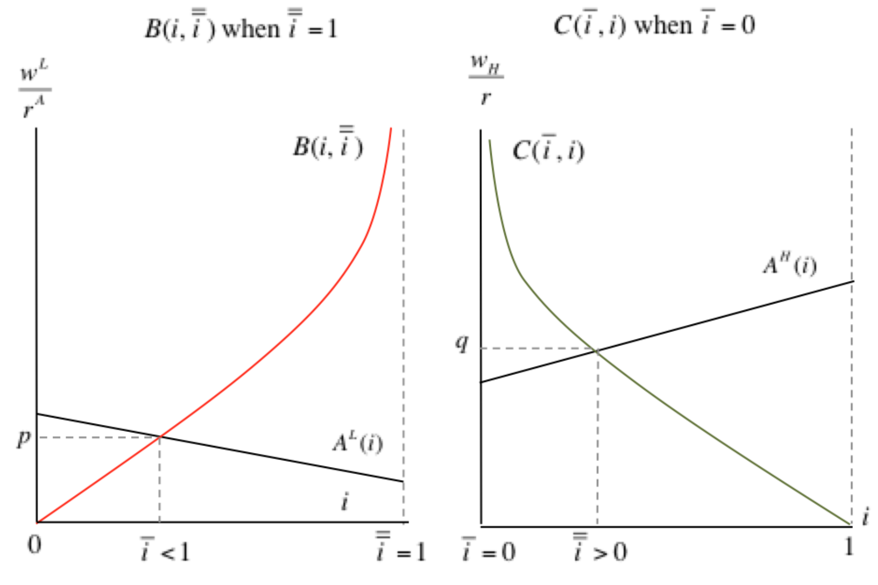


The schedule $A(\bar{i}) = B(\bar{i}, \bar{\bar{i}})$ in $(\bar{i}, \bar{\bar{i}})$ space begins at $(0, 0 + \epsilon)$ since if necessarily $\bar{i} < \bar{\bar{i}}$ then as $\bar{\bar{i}}$ approaches 0, $\bar{i} = 0$.[9] And conversely when $\bar{\bar{i}} = 1$, $\bar{i} < 1$. This follows from the fact that if $\bar{\bar{i}} = 1$, then the analysis collapses

[9] Again, that $\bar{i} < \bar{\bar{i}}$ is proven by contradiction in section 4.2.4. Also note that it is not possible for $\bar{i} = 0$ for positive values of $\bar{\bar{i}}$ i.e. the $A(\bar{i}) = B(\bar{i}, \bar{\bar{i}})$ schedule cannot have a positive intercept with the $\bar{\bar{i}}$ axis in Figure 4.4. This would imply that this schedule has an infinite gradient for those positive values of $\bar{\bar{i}}$ – this gradient would necessarily fall as $\bar{i}$ became non-zero, violating the condition that $\frac{\partial^2 \bar{\bar{i}}}{\partial \bar{i}^2} \geq 0$, or $\bar{i}$ must always be zero which requires a vertical $B(i, \bar{\bar{i}})$ schedule for all possible $\bar{\bar{i}}$, which is infeasible given the assumptions made about expenditure and factor stocks. This can be seen intuitively in Figure 4.3 – $\bar{i} = 0$ only when the $B(i, \bar{\bar{i}})$ schedule is vertical, which requires $\bar{\bar{i}} = 0$.

to a simple case where only two factors compete to perform the tasks $z_1(i)$ – low-skilled labour and advanced capital. Equilibrium in this setting is shown in the left-hand diagram in Figure 4.5. The cut-off $\bar{i}$ is < 1 for finite factor productivities and stocks. Similarly, the schedule $A(\bar{\bar{i}}) = C(\bar{i}, \bar{\bar{i}})$ in $(\bar{i}, \bar{\bar{i}})$ space ends at $(1 - \epsilon, 1)$ since if necessarily $\bar{i} < \bar{\bar{i}}$, then as $\bar{i}$ approaches 1, $\bar{\bar{i}} = 1$.[10] And conversely when $\bar{i} = 0$, $\bar{\bar{i}} > 0$. This follows from the fact that if $\bar{i} = 0$ then the analysis again collapses to a simple case where only two factors compete to perform the tasks $z_1(i)$ – advanced capital and high-skilled labour. Equilibrium in this setting is shown in the right-hand diagram in Figure 4.5. The cut-off $\bar{\bar{i}}$ is > 0 for finite factor productivities and stocks.

Figure 4.5: Determining the Intercepts



[10]Again, that $\bar{i} < \bar{\bar{i}}$ is proven by contradiction in section 4.2.4.

4.3. Comparative Statics

4.3.1. Four-Factor Setting

As in Chapter 3, (4.49) implies that an increase in the productivity of traditional capital has no effect on relative factor prices, nor on the pattern of specialisation, $\bar{i}$ and $\bar{\bar{i}}$. Again, traditional capital q-complements labour and, in addition now, advanced capital. And because the task-based production functions for goods are Cobb-Douglas, this again implies that a rise in the marginal productivity of traditional capital in producing a given good causes an equiproportionate rise in the marginal productivity of either labour or advanced capital in producing that good. However, the comparative statics involving increasingly capable advanced capital, which is a perfect substitute for labour, are different. I now turn to look at these in detail.

In the three-factor setting of Chapter 3, I performed comparative statics by considering the consequence of a uniform increase in the relative productivity of advanced capital – this shifted the $A(i)$ schedule down a uniform amount across the task spectrum.[11] However, in this four-factor setting this is no longer possible – there are two relative productivity schedules, $A^L(i)$ and $A^H(i)$, and an increase in the absolute productivity of advanced capital that causes a uniform decrease in the relative productivity of one type of labour across the task spectrum need not cause a uniform decrease in the relative productivity of the other type of labour across the task spectrum.

Accordingly, in this four-factor setting, to perform comparative statics I have to be more precise about the underlying *absolute* productivity schedules

[11] This is shown in Figure 3.4.

of the different factors. In what follows, I use the simplest possible absolute productivity schedules that generate the relative productivity schedules in Figure 4.1. These are as follows:

$$
\begin{aligned}
a^L(i) &= a - b \cdot i \\
a^H(i) &= a + b \cdot i \\
a^A(i) &= c
\end{aligned}
\tag{4.69}
$$

where a, b, and c are positive constants.[12] Given these schedules, suppose there is a uniform increase in the absolute productivity of advanced capital across the task spectrum – c rises to c'. This will cause a downward shift in the $A^L(i)$ and $A^H(i)$ schedules. The new equilibrium can be solved computationally in $\bar{i}$–$\bar{\bar{i}}$ space. This is shown in Figure 4.6 for a specific parameterisation. Note that it is necessary to parameterise the model to perform comparative statics since it is not possible to solve (4.63) and (4.66) simultaneously to find an explicit expression for $\bar{i}$ or $\bar{\bar{i}}$ in terms of the k's alone, and so to find a general expression for the effect of a change in the productivity of advanced capital on the pattern of specialisation.[13]

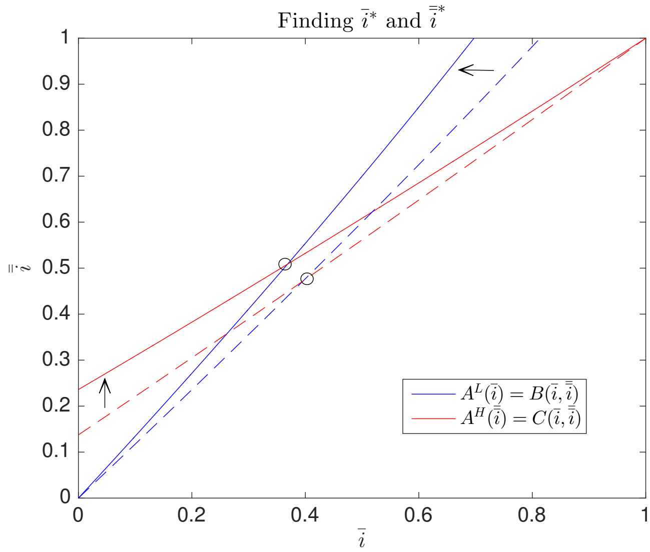
The original schedules (the dashed schedules) in Figure 4.6 reflect Figure 4.4. The result of a uniform increase in the absolute productivity of advanced capital in this parameterisation is a fall in $\bar{i}^*$ and a rise in $\bar{\bar{i}}^*$ – both high-skilled and low-skilled labour are displaced from performing certain types of tasks. The consequence of the fall in $\bar{i}^*$ and the rise in $\bar{\bar{i}}^*$ is that relative

[12]Note that (4.69) therefore implies that in (4.59) $k_1 = \frac{a}{c}$, $k_2 = \frac{b}{c}$, $k_3 = \frac{a}{c}$, and $k_4 = \frac{b}{c}$.

[13]Here $a = 6$, $b = 2$, $c = 1$ and $c' = 2$. The same results hold for a range of other parameters.

wages with respect to both types of capital fall – this follows from inspection of (4.49) since $\alpha(\bar{i})$ falls, $\beta(\bar{\bar{i}})$ falls, and $1 - \alpha(\bar{i}) - \beta(\bar{\bar{i}})$ rises.

Figure 4.6: New Equilibrium

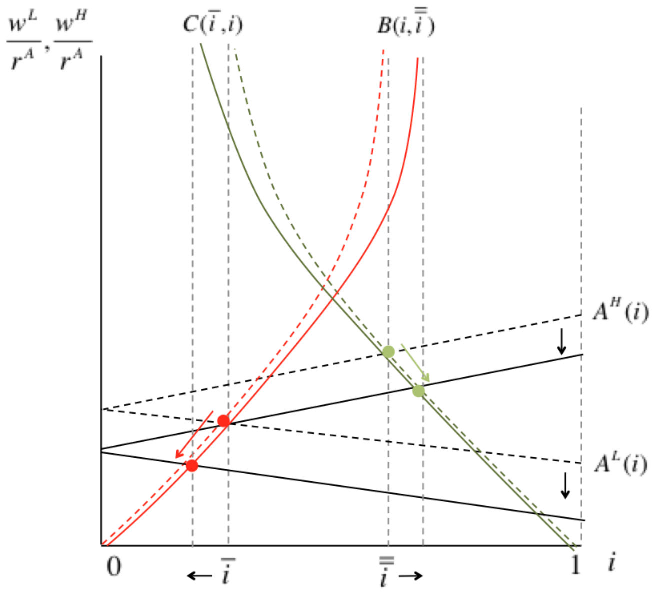


Furthermore, note that goods $i \in [\bar{i}, \bar{\bar{i}}]$ are produced by capital alone – advanced capital and traditional capital working together. This is in contrast to the existing task-based literature, in which goods necessarily require at least one type of labour to produce them. Also note that there are no exogenously determined limits on $\bar{i}$ or $\bar{\bar{i}}$ – they are determined endogenously according to the relative productivity of the factors. This is in contrast to the existing task-based literature, in which the cut-offs are often determined implicitly, and exogenously, by the ALM hypothesis.[14]

These comparative statics can also be shown in the two-dimensional representation used in Figure 4.2 and Figure 4.3. This is shown in Figure 4.7.

[14]The exception in Chapter 1 is the AA model which also uses a task spectrum.

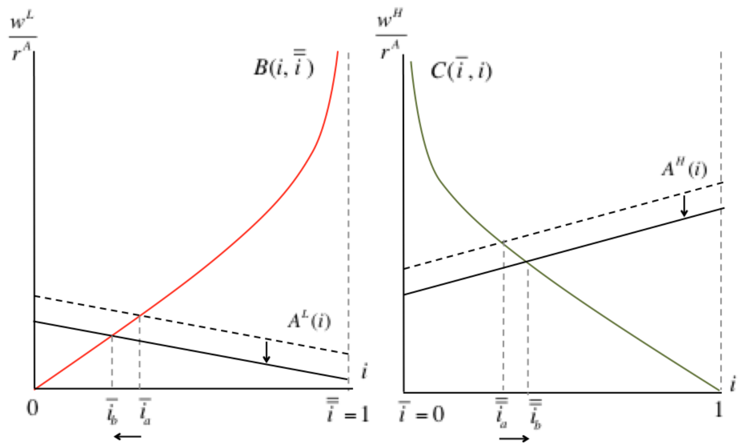
Figure 4.7: More Productive Advanced Capital



The increase in the productivity of advanced capital causes a downward shift in the relative productivity schedules, $A^H(i)$ and $A^L(i)$, and this drives down $\bar{i}$ and drives up $\bar{\bar{i}}$ –in the $\bar{i}$–$\bar{\bar{i}}$ space of Figure 4.6, these are $\bar{i}^*$ and $\bar{\bar{i}}^*$ – and causes the relative wage of both types of labour with respect to advanced capital to fall.

Returning to Figure 4.6, the change in the position of the schedules is intuitive. The increase in the relative productivity of advanced capital causes the intercept between the $A(\bar{i}) = B(\bar{i}, \bar{\bar{i}})$ schedule and the $\bar{\bar{i}} = 1$ line to fall and the intercept between the $A(\bar{\bar{i}}) = C(\bar{i}, \bar{\bar{i}})$ schedule and the $\bar{i} = 0$ line to rise. This follows from Figure 4.5. When the $A(\bar{i}) = B(\bar{i}, \bar{\bar{i}})$ schedule intercepts the $\bar{\bar{i}} = 1$ line, it is as if only low-skilled labour and advanced capital are competing for tasks $z_1(i)$ – in this setting, the result of an increase in the productivity of advanced capital is a fall in the cut-off from $\bar{i}_a$ to $\bar{i}_b$. This is shown in the left-hand diagram of Figure 4.8.

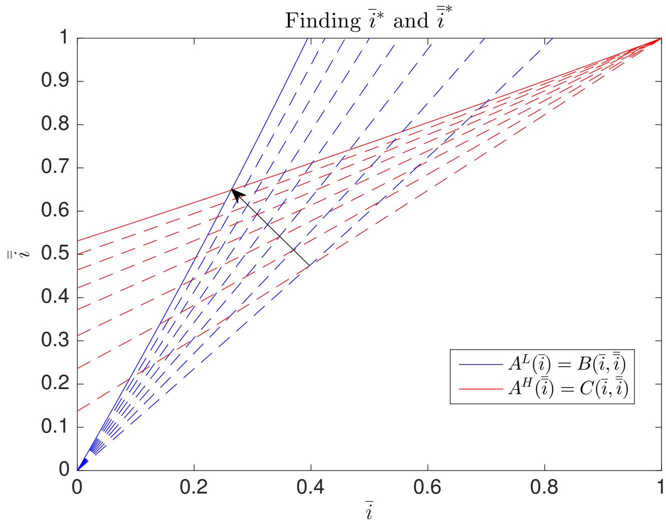
Figure 4.8: $B(i, \bar{\bar{i}})$ when $\bar{\bar{i}} = 1$ (left-hand side) & $C(\bar{i}, i)$ when $\bar{i} = 0$ (right-hand side)



Similarly, when the $A(\bar{\bar{i}}) = C(\bar{i}, \bar{\bar{i}})$ schedule intercepts with the $\bar{i} = 0$ line it is as if only high-skilled labour and advanced capital are competing for tasks $z_1(i)$ – in this setting, the result of an increase in the productivity of advanced capital is a rise in the cut-off from $\bar{\bar{i}}_a$ to $\bar{\bar{i}}_b$. This is shown in the right-hand diagram of Figure 4.8. For the reasons set out before, both the $A(\bar{i}) = B(\bar{i}, \bar{\bar{i}})$ and $A(\bar{\bar{i}}) = C(\bar{i}, \bar{\bar{i}})$ schedules must still originate at (0, 0) and (1, 1) respectively.

Note that, as the absolute productivity of advanced capital continues to rise, it is possible to trace out the evolution of the equilibrium cut-offs in $\bar{i}$–$\bar{\bar{i}}$ space. This is shown in Figure 4.9, where each shift represents a unit increase in c. Note that in these comparative statics the stock of capital remains fixed despite these changes in productivity. In the next chapter, when this static model is placed in a dynamic setting, capital is instead accumulated endogenously.

Figure 4.9: The Evolution of Equilibrium Cut-Offs



4.3.2. Three-Factor Setting

The comparative statics in this four-factor setting are less tractable than the three-factor setting in Chapter 3. As a result, in the comparative statics set out, I explore only one possible parameterisation of the model. In the rest of the thesis I explore the role of advanced capital in a three-factor, rather than a four-factor setting – I assume there is only one-type of labour and two types of capital. This allows me to develop more complex versions of this model – for example, a dynamic model in Chapter 5, a model where demand is not Cobb-Douglas in Chapter 7, and a model with an increasing variety of goods in Chapter 8 – without needing to parameterise the model in a particular way, and so making it possible to draw more general conclusions about the effects of increasingly capable advanced capital.

Accordingly, consider the three-factor case where there is only high-skilled labour and the two types of capital. The right-hand diagram in Figure 4.5 shows what equilibrium looks like in this setting – a three-factor setting without low-skilled labour is equivalent to the four-factor setting with low-skilled labour, but where $\bar{i} = 0$ and low-skilled labour performs no tasks. The right-hand diagram in Figure 4.8 shows the consequences of a uniform increase in the relative productivity of advanced capital in this setting – a fall in the relative wage of high-skilled labour with respect to advanced capital, and a fall in the set of tasks that are performed by high-skilled labour. Although the four-factor setting provides a richer insight into equilibrium, the three-factor setting is more tractable and will be used in the rest of the thesis.

4.4. A Comment on Complementarities

In a survey of analysis of technology and the labour market, Autor (2014) writes:

> "A key observation of the paper is that journalists and expert commentators overstate the extent of machine substitution for human labor and ignore the strong complementarities that increase productivity, raise earnings, and augment demand for skilled labor. The challenges to substituting machines for workers in tasks requiring flexibility, judgment, and common sense remain immense." (pg. 2)

The model in this chapter does not challenge the central claim in this statement – that machines substitute for labour in performing certain types of tasks, and complement labour in performing other types of tasks. This is clear in the case of traditional capital – it cannot perform the same type of tasks as labour, $z_1(i)$, and necessarily q-complements labour. But it is also true in the case of advanced capital – though it substitutes for labour in performing certain types of $z_1(i)$, it also 'complements' labour in performing all other types of $z_1(i)$. In particular, it complements labour by raising demand for the residual types of tasks $z_1(i)$ in which labour retains a comparative advantage. To see this, note that as advanced capital becomes more productive, the total output of the economy will rise.[15] This is intuitive – factors are more productive, and so output must rise. Since output is equal to income in this economy, and expenditure densities on all goods are fixed, this means that actual expenditure will rise on all goods. And because the total expenditure on different types of tasks is a fixed proportion of the expenditure on the good that the task is used to produce, a rise in actual expenditure on all types of goods implies a rise in actual expenditure on all types of tasks, including the residual tasks in which labour retains the comparative advantage.

However, the analysis of this chapter shows that, when expenditure densities on different types of goods are fixed, this complementarity is not sufficient to maintain relative wages with respect to either type of capital. The reason is intuitive – traditional capital may q-complement labour, but as advanced capital become increasingly capable *the set of types of tasks in which labour is q-complemented by traditional capital in – namely, those in which labour retain the comparative advantage over advanced capital – gets smaller in equi-*

[15] Formally r in (4.57) rises as $\alpha(\bar{i})$ falls, and since $K \cdot r$ is a constant share $1 - \psi$ of output, this implies output must also rise.

librium. The result is that the share of total expenditure on tasks that are performed by labour, relative to both types of capital, gets smaller.

A different way to see this claim is to recognise that it is not challenging this statement from Autor (2014):

> "The fact that a task cannot be computerized does not imply that computerization has no effect on that task. On the contrary: tasks that cannot be substituted by computerization are generally complemented by it." (pg. 8)

nor Autor (2015):

> "tasks that cannot be substituted by automation are generally complemented by it ... productivity improvements in one set of tasks almost necessarily increase the economic value of the remaining tasks." (pg. 6)

but it is challenging the assumption that *necessarily* labour, rather than capital, is best placed to perform those "remaining" tasks.[16] Given the analysis in Chapter 1, the reason why Autor (2015) thinks that this must be the case is clear – the models have a 'traditional' conception of capital which complements labour that performs its own set of tasks. But for the reasons also set out in Chapter 2 there is an important case for thinking that many of these "remaining tasks" will be performed no longer by labour but by capital. Traditional capital may q-complement both types of labour and advanced capital, but advanced capital is a substitute for both types of labour. And so, if advanced capital becomes increasingly capable, in equilibrium it

[16]There are two important challenges to this pessimism. The first is that expenditure densities might change over *existing* tasks. The second is that *entirely new* types of tasks might be created in which labour has the comparative advantage. These are discussed in Chapters 7 and 8 respectively.

erodes the set of types of tasks in which traditional capital q-complements labour.

4.5. Conclusion

In Chapter 3 the consequence of increasingly capable machines on the relative wages of labour are neutral when those machines are modelled as traditional capital. However, this conclusion no longer holds when those machines are modelled as advanced capital. This is shown in this chapter. In all cases, increasingly capable advanced capital drives down the relative wages of labour with respect to both types of capital. This is pessimism at work.

In Chapter 3, as in the core set of task-based models in Chapter 1, the production of all types of goods necessarily requires at least one type of labour. Traditionally this property has relied upon the ALM hypothesis – the existence of set of 'non-routine' tasks that only labour can perform, however capable capital becomes. In this chapter I relax this property. There is a set of type of tasks that only traditional capital performs, $z_2(i)$, and a set of type of tasks that only labour *initially* performs, $z_1(i)$. But as advanced capital becomes increasingly capable it displaces labour from more types of $z_1(i)$. There is no exogenous limit to the range of tasks that can be performed by advanced capital. Nor do goods necessarily require at least one type of labour – a growing proportion of goods are produced by capital alone. The result – that if one type of capital (the advanced capital) erodes the set of types of tasks in which labour is q-complemented by the other type of capital (the traditional capital) the outcome is pessimistic for labour – is new and important.

This chapter shows an important sense in which John Stuart Mill's comment, that "demand for commodities is not demand for labour" is astute. Demand for commodities is not demand for labour, but instead is demand for the tasks that must be performed to produce those commodities. The ALM hypothesis assumes that there is some subset of those tasks must be performed by human beings – this makes it more likely that demand for commodities is demand for labour. I have argued in this section of the thesis that this subset is likely to be far smaller than the ALM hypothesis suggests – and the accompanying demand for labour is also smaller. But this is an empirical matter and something which is important to consider how to investigate further. I discuss this in the conclusion of this thesis.

5. A Dynamic Version of the New Model

5.1. Outline

In Chapters 3 and 4, the new model is static and the stock of capital is exogenously fixed. As capital becomes more productive, consumers do not respond to any changes in the rate of return to capital. In this chapter, I place that new model in a dynamic setting and introduce an endogenous process of capital accumulation. First I do this for the new model in Chapter 3 where there is only traditional capital – I consider the case where there is no technological progress, and then the case where there is technological progress. Then I repeat this analysis for the new model in Chapter 4 where there is both traditional and advanced capital.

There are two central results in this chapter. The first is that the endogenous accumulation of capital strengthens the optimism in Chapter 3 and the pessimism in Chapter 4. Of particular interest is the latter case. Chapter 4 shows how advanced capital drives labour out of the centre of the task spectrum, and that a one-off uniform increase in the relative productivity of advanced capital causes labour to be driven out further. In this chapter the endogenous accumulation of advanced capital strengthens this effect – as advanced capital is endogenously accumulated, labour is driven out at an endogenously determined rate. The second central result is that, in the case with advanced capital and technological progress, no steady-state is possible until labour has been entirely driven out the economy by advanced capital – i.e. the economy must approach a steady-state where $\bar{i}(t) = 1$. In short, an optimistic outcome for labour can only be achieved in the dynamic setting with traditional capital as in Chapter 3. With advanced capital as in Chapter 4, the dynamic outcome is remorselessly pessimistic for labour.

The approach to solving all these models is the same, and relies upon the use of a numeraire good. This numeraire good allows me to do two things. First it allows me to reduce the dimensionality of the many-good model and make it tractable in the dynamic setting. As I will show, with Cobb-Douglas preferences across the range of goods, the law of motion for the numeraire good is the same as the law of motion for aggregate consumption across all goods – deriving this particular law of motion allows me to focus on aggregate consumption alone, rather than track the full set of laws of motion for each good. Secondly, with Cobb-Douglas production across the range of goods, the numeraire good allows me to derive analytically tractable expressions for the absolute factor prices, r and r^A. These absolute factor prices perform an important role in this dynamic setting that they do not perform in the static setting of Chapters 3 and 4. I described how to derive them in those chapters. First I solved the static models to find the relative factor prices as in Dornbusch et al. (1977). Then I solved for the absolute prices of the different goods which Dornbusch et al. (1977) did not do. Combining the expression for the relative price of the factors with the expression for the absolute prices of the goods, in the particular case of the numeraire good, allowed me to derive a set of expressions for the absolute factor prices. In what follows I will explain this approach in detail.[1]

Although the model shares many similarities with a traditional Ramsey model, it differs in important ways because it is task-based and involves many goods. I will make these differences clear throughout this chapter.[2]

5.2. Traditional Capital

In this section I place the model in Chapter 3 with traditional capital in a dynamic setting. There is no advanced capital, only traditional capital and two types of labour. First I do this without technological progress. Then I introduce technological progress. For simplicity, there is no depreciation in either setting. In the final part of this chapter I consider the role of depreciation.

5.2.1. No Technological Progress

Consumers

As in Chapter 3, all consumers have the same Cobb-Douglas utility function. As a result the economy can be captured by a representative consumer with Cobb-Douglas preferences. The consumer faces the following dynamic maximisation problem:

[1] See sections 3.2.5 and 4.2.5. Intuitively, in the dynamic setting, Chapter 3 and 4 provide an instantaneous 'snapshot' of equilibrium in any given t.

[2] By 'traditional Ramsey model', I mean the sort set out in Sala-i-Martin and Barro (2003), reflecting Ramsey (1928), Cass (1965), and Koopmans (1965).

$$
\begin{aligned}
&\max_{\boldsymbol{x(t)}} \int_0^{\infty} e^{-\rho t} \int_0^1 \theta(i) \ln x(i,t)\, di\, dt \\
&\text{s.t.} \\
&\dot{K}(t) = r(t) \cdot K(t) + w^L(t) \cdot L^L(t) + w^H(t) \cdot L^H(t) - \int_0^1 x(i,t) \cdot p(i,t)\, di \\
&K(0) = K_0 \\
&K(t) \geq 0
\end{aligned}
\tag{5.1}
$$

The only asset in the economy is traditional capital, $K(t)$, which is accumulated. As a result, the stock of $K(t)$ can change over time. In what follows, I continue to assume that labour supplies are fixed so that $L^L(t) = L^L$ and $L^H(t) = L^H$.

Production

As in Chapter 3 the task-based production function for each good $x(i,t)$ is a Cobb-Douglas combination of two types of tasks, $z_1(i,t)$ and $z_2(i,t)$:[3]

$$x(i,t) = z_1(i,t)^{\psi} z_2(i,t)^{1-\psi} \tag{5.2}$$

With the accompanying factor-based production functions for tasks:

$$
\begin{aligned}
z_1(i,t) &= a^L(i,t) L^L(i,t) + a^H(i,t) L^H(i,t) \\
z_2(i,t) &= a^K(i,t) K(i,t)
\end{aligned}
\tag{5.3}
$$

[3]See equations (3.6) - (3.8).

Again, perfectly competitive firms hire the labour and rent the capital of consumers. Again, labour and traditional capital are each used to perform their own distinct set of tasks. As in Chapter 3, in any equilibrium the division of labour between the two types of labour in performing $z_1(i,t)$ is determined endogenously. For now I assume there is no technological progress – the productivity of traditional capital $a^K(i,t)$ does not change over time.

Dynamic Equilibrium

From the maximisation problem in (5.1), a current-value Hamiltonian follows:

$$\begin{aligned} H = & \int_0^1 \theta(i) \cdot \ln x(i,t) \, di \\ & + \mu(t) \left[r(t) \cdot K(t) + w^L(t) \cdot L^L + w^H(t) \cdot L^H - \int_0^1 x(i,t) \cdot p(i,t) \, di \right] \end{aligned} \tag{5.4}$$

It is possible to solve this Hamiltonian – with one co-state variable $\mu(t)$ and a spectrum of control variables, $x(i,t)$ where $i \in [0,1]$ – in the traditional way. I show this in the Appendix, section 9.1.1. A set of first order conditions follow for each good $x(i,t)$, the control variables:

$$\begin{aligned} H_{x(i)} &= \frac{\theta(i)}{x(i,t)} - \mu(t) \cdot p(i,t) \\ &= 0 \end{aligned} \tag{5.5}$$

And for $K(t)$, the state variable:

$$\begin{aligned} H_K &= \mu(t) \cdot r(t) \\ &= \rho \cdot \mu(t) - \dot{\mu}(t) \end{aligned} \tag{5.6}$$

From (5.5) it follows that:

$$x(i,t) = \frac{\theta(i)}{\mu(t) \cdot p(i,t)} \tag{5.7}$$

And from (5.6) that:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -r(t) + \rho \tag{5.8}$$

Given Assumption 2 in Chapter 3 – that expenditure densities on different types of goods are constant and equal to 1 – it follows from (5.7) that:

$$\dot{x}(i,t) = \frac{-\theta(i) \cdot (\dot{\mu}(t) \cdot p(i,t) + \mu(t) \cdot \dot{p}(i,t))}{(\mu(t) \cdot p(i,t))^2} \tag{5.9}$$

and in turn:

$$\frac{\dot{x}(i,t)}{x(i,t)} = -\frac{\dot{p}(i,t)}{p(i,t)} - \frac{\dot{\mu}(t)}{\mu(t)} \tag{5.10}$$

Given (5.8), it follows from (5.10) that for good $x(i,t)$:

$$\frac{\dot{x}(i,t)}{x(i,t)} = -\frac{\dot{p}(i,t)}{p(i,t)} + r(t) - \rho \tag{5.11}$$

In a traditional Ramsey-growth model, there is only a unique final output and so there is no need to consider how the price of the good changes over time. But (5.11) shows that when there are Cobb-Douglas preferences over a variety of goods the rate of growth of demand for $x(i,t)$ will depend upon how its price changes over time.

To maintain tractability in this many-good setting, I now use a numeraire price normalisation. For some numeraire good $\tilde{i}$, I assume that $p(\tilde{i}, t) = 1$. Given this it follows from (5.11) that for $x(\tilde{i}, t)$:

$$\frac{\dot{x}(\tilde{i}, t)}{x(\tilde{i}, t)} = r(t) - \rho \tag{5.12}$$

Given that the utility function is Cobb-Douglas, and all goods have the same fixed associated expenditure densities, it follows that the law of motion for the numeraire good $x(\tilde{i}, t)$ is the same as the law of motion for total consumption $c(t)$.[4] To see this note that:

$$\begin{aligned} c(t) &= \frac{x(\tilde{i}, t) \cdot p(\tilde{i}, t)}{\theta(\tilde{i})} \\ &= \frac{x(\tilde{i}, t)}{\theta(\tilde{i})} \end{aligned} \tag{5.13}$$

since $p(\tilde{i}, t) = 1$. And so:

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \frac{\dot{x}(\tilde{i}, t)}{x(\tilde{i}, t)} - \frac{\dot{\theta}(\tilde{i})}{\theta(\tilde{i})} \\ &= \frac{\dot{x}(\tilde{i}, t)}{x(\tilde{i}, t)} \\ &= r(t) - \rho \end{aligned} \tag{5.14}$$

as $\theta(\tilde{i})$ is constant. It follows that if $x(\tilde{i}, t)$ reaches a steady-state then total consumption $c(t)$ will also be in steady-state. From now, I use the law of motion in (5.14) and focus on $c(t)$. This is the first use of the numeraire price normalisation noted in the outline to this chapter – it allows me to reduce the dimensionality of the many-good model by moving from tracking

[4] Where $c(t) = \int_0^1 x(i, t) \cdot p(i, t)\, di$.

individual goods $x(i,t)$ to aggregate consumption $c(t)$.

(5.14) is the law of motion for $c(t)$. From (5.1) the law of motion for $K(t)$ is:

$$\dot{K}(t) = Y(t) - c(t) \tag{5.15}$$

Both these laws of motion can be expressed in terms of $c(t)$ and $K(t)$ alone. First, the $\frac{\dot{c}(t)}{c(t)}$ schedule in (5.14) can be expressed in terms of $K(t)$ by finding $r(t)$ in terms of $K(t)$. The numeraire price normalisation also allows me to find this. This is the second use of the numeraire price normalisation noted in the outline to this chapter. This approach was discussed in detail Chapter 3. I now repeat that analysis for clarity.

First I derive the relative wage of low-skilled labour with respect to traditional capital, in the same way that the relative factor price is derived in Dornbusch et al. (1977). This is shown in Chapter 3 in (3.37), and in any period t is equal to:

$$\frac{w^L(t)}{r(t)} = \frac{\psi}{1-\psi} \cdot \frac{\alpha(\bar{i}(t)) \cdot K(t)}{L^L} \tag{5.16}$$

Secondly I derive the absolute prices of those goods that are produced by low-skilled labour and traditional capital i.e. goods $i \in [0, \bar{i}]$. This is not done in Dornbusch et al. (1977). This is shown in Chapter 3 in (3.39), such that for any good $i \in [0, \bar{i}]$ in period t:

$$p(i,t) = \left[\frac{w^L(t)}{\psi \cdot a^L(i,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(i,t)}\right]^{1-\psi} \tag{5.17}$$

Thirdly I use the numeraire price normalisation discussed before. I now make a further assumption about the numeraire good $\tilde{i}$:

ASSUMPTION 4: $\tilde{i} = 0$ *such that the numeraire good is* $x(0,t)$*. Factor productivities and factor stocks are finite such that* $\bar{i}(t) \neq 0\ \forall t$*.*

Since $\tilde{i} \in [0, \bar{i}(t)]$, this implies that $x(0,t)$ is produced by a combination of low-skilled labour and traditional capital. Since $\bar{i}(t) \neq 0\ \forall t$ this implies that $x(0,t)$ is *always* produced by low-skilled labour and traditional capital – this is a weak assumption, but ensures that the numeraire good is always to the left of $\bar{i}(t)$. (5.17) therefore implies that the price of the numeraire good $x(0,t)$ in any period t is always equal to:

$$p(0,t) = \left[\frac{w^L(t)}{\psi \cdot a^L(0,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(0,t)}\right]^{1-\psi} \tag{5.18}$$

And finally, to derive $r(t)$, I now combine these expressions. Substituting the expression for $w^L(t)$ in terms of $r(t)$ from (5.16) into the expression for $p(0,t)$ in (5.18) implies that:

$$p(0,t) = \left[\frac{\frac{\psi}{1-\psi} \cdot \alpha(\bar{i}(t)) \cdot \frac{K(t)}{L^L} \cdot r(t)}{\psi \cdot a^L(0,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(0,t)}\right]^{1-\psi} \tag{5.19}$$

and given Assumption 4, that $p(0,t) = 1$, (5.19) implies that:

$$\begin{aligned} 1 &= \left[\frac{\frac{\psi}{1-\psi} \cdot \alpha(\bar{i}(t)) \cdot \frac{K(t)}{L^L} \cdot r(t)}{\psi \cdot a^L(0,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(0,t)}\right]^{1-\psi} \\ 1 &= r(t) \cdot D_0 \cdot \alpha(\bar{i}(t))^{\psi} \cdot \left[\frac{1}{a^L(0,t)}\right]^{\psi} \left[\frac{1}{a^K(0,t)}\right]^{1-\psi} \end{aligned} \tag{5.20}$$

where $D_0 = \frac{1}{1-\psi} \cdot \left[\frac{K(t)}{L^L}\right]^{\psi}$. Re-arranging (5.20), it follows that $r(t)$ must be equal to:[5]

$$\begin{aligned} r(t) &= K(t)^{-\psi} \cdot (L^L)^{\psi} \cdot (1-\psi) \cdot \alpha(\bar{i}(t))^{-\psi} \cdot \left[a^L(0,t)\right]^{\psi} \left[a^K(0,t)\right]^{1-\psi} \\ &= K(t)^{-\psi} \cdot D_1 \end{aligned} \tag{5.21}$$

where D_1 is a positive constant. Note that in (5.21) both $\alpha(\bar{i}(t))$ and the factor productivities $a^L(0,t)$ and $a^K(0,t)$ are treated as constants, and included in D_1. This is because in the case with no technological progress, neither of the factor productivities change over time. And in addition, since there is no change in the stock of labour, $\bar{i}(t)$, and so $\alpha(\bar{i}(t))$, also do not change over time. This follows from the analysis in Chapter 3.

Now I can express the $\frac{\dot{c}(t)}{c(t)}$ schedule in terms of $c(t)$ and $K(t)$ alone. Substituting the expression for $r(t)$ in (5.21) into (5.14) implies that:

$$\frac{\dot{c}(t)}{c(t)} = K(t)^{-\psi} \cdot D_1 - \rho \tag{5.22}$$

[5] (5.21) implies that the *level* of $r(t)$ depends on the choice of the numeraire good, $\tilde{i}$ – if a different $\tilde{i}$ is chosen, the level of $a^L(\tilde{i},t)$ and $a^K(\tilde{i},t)$ will differ from those in (5.21) where $\tilde{i} = 0$. However, this does not affect the important features of absolute factor prices in equilibrium. In the case where there is a one-off change in the productivity of capital, so long as that change is uniform – as in Dornbusch et al. (1977) – (5.21) implies that $r(t)$ will always move in the same direction regardless of the choice of numeraire. In the case where there is an increase in the growth rate of the productivity of capital, so long as the growth rates remain the same across different tasks, (5.21) implies that $r(t)$ will always increase at the same rate regardless of the choice of numeraire. (And variables in steady-state, as I will show, will grow at the same rate.) This also holds in the case with advanced capital in the next section. To explore non-uniform changes in productivity, or to make the *level* of $r(t)$ invariant to the normalisation, it is necessary to use the simplex price normalisation. But this leads to an implicit, rather than explicit and tractable, solution to the model. This need for uniformity is an interesting limitation of Dornbusch et al. (1977) that was not explored. Complications involving price normalisations are discussed elsewhere in the literature – Dierker and Grodal (1999), for example, on models of imperfect competiton.

The expression in (5.22) is therefore the $\frac{\dot{c}(t)}{c(t)}$ schedule in $K(t)$–$c(t)$ space.

The $\dot{K}(t)$ schedule in (5.15) can also be expressed in terms of $K(t)$ and $c(t)$ alone. This requires that $Y(t)$ is expressed in terms of $K(t)$. To do this, note that the structure of production in (5.2) and (5.3) implies that traditional capital income is:

$$r(t) \cdot K(t) = (1 - \psi) \cdot Y(t) \tag{5.23}$$

and so substituting the expression for $r(t)$ in (5.21) into (5.23) implies that $Y(t)$ is equal to:

$$\begin{aligned} Y(t) &= r(t) \cdot K(t) \cdot \frac{1}{1-\psi} \\ &= K(t)^{1-\psi} \cdot D_1 \cdot \frac{1}{1-\psi} \\ &= K(t)^{1-\psi} \cdot D_2 \end{aligned} \tag{5.24}$$

where D_2 is a positive constant. Substituting this expression for $Y(t)$ back into the $\dot{K}(t)$ schedule in (5.15) results in:

$$\dot{K}(t) = K(t)^{1-\psi} \cdot D_2 - c(t) \tag{5.25}$$

This is the $\dot{K}(t)$ schedule in $K(t)$–$c(t)$ space. This schedule is intuitive. The first term on the right-hand side of (5.25) is output $Y(t)$ which is diminishing in $K(t)$ since $\psi \in (0, 1)$ and, given there is no depreciation, the difference between $Y(t)$ and $c(t)$ is equal to the change in the stock of traditional capital.

Given the $\frac{\dot{c}(t)}{c(t)}$ schedule in (5.22) and the $\dot{K}(t)$ schedule in (5.25) it is now possible to show the $\frac{\dot{c}(t)}{c(t)} = 0$ and $\dot{K}(t) = 0$ schedules in $K(t)$–$c(t)$ space. The intersection of these schedules is the steady-state. Given (5.22) the $\frac{\dot{c}(t)}{c(t)} = 0$ schedule in $K(t)$–$c(t)$ space is:

$$K(t)^{-\psi} \cdot D_1 = \rho \tag{5.26}$$

and given (5.25) the $\dot{K}(t) = 0$ schedule in $K(t)$–$c(t)$ space is:

$$c(t) = K(t)^{1-\psi} \cdot D_2 \tag{5.27}$$

These schedules are shown in $K(t)$–$c(t)$ space in the left-hand diagram of Figure 5.1. The $\frac{\dot{c}(t)}{c(t)} = 0$ schedule is vertical at the level of $K(t)$ such that (5.26) holds. It follows that this is equal to:

$$K(t) = K^* = \left[\rho \cdot \frac{1}{D_1}\right]^{-\frac{1}{\psi}} \tag{5.28}$$

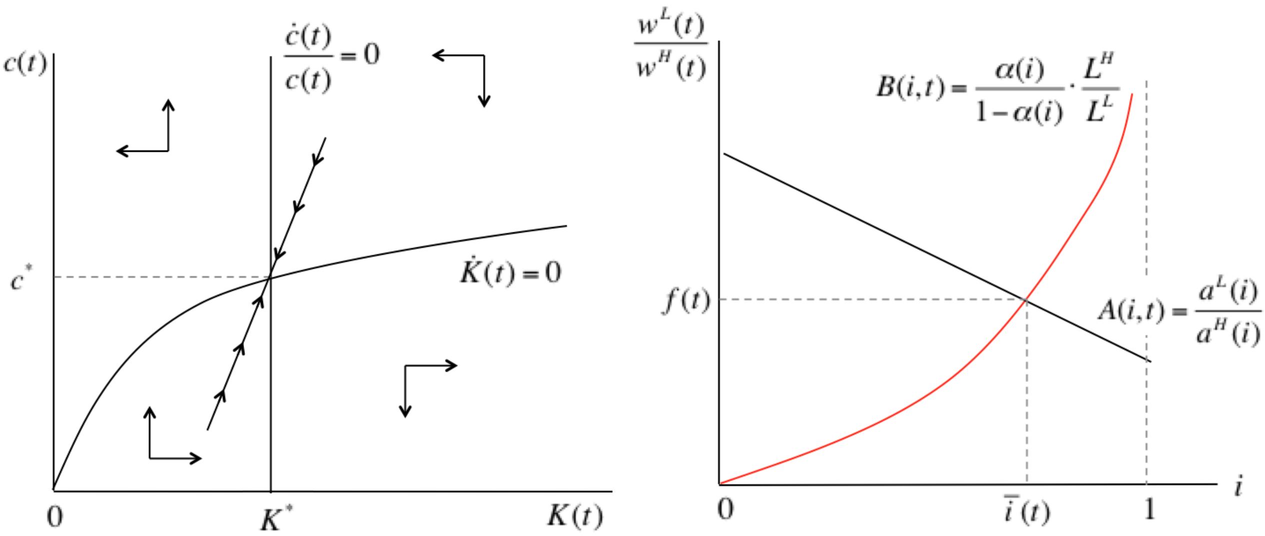
And this level of K^* implies that:

$$r(t) = r^* = \rho \tag{5.29}$$

Put another way, the requirement that $r^* = \rho$, in order that $\frac{\dot{c}(t)}{c(t)} = 0$, pins down a unique steady-state level of traditional capital stock, K^* – namely, the level of $K(t)$ at which the marginal revenue product of traditional capital for all goods is equal to r^*. The expression in (5.27) shows that the $\dot{K}(t) = 0$ schedule is positively sloped, but diminishing in $K(t)$ since $\psi \in (0, 1)$.

The transition dynamics follow from (5.26) and (5.27). (5.26) implies that if $K(t) < K^*$ then $r(t) > \rho$ and $c(t)$ will be increasing. If $K(t) > K^*$ then $r(t) < \rho$ and $c(t)$ will be decreasing. (5.27) implies that, for a given $K(t)$, if $c(t)$ is below the value of $c(t)$ on the $\dot{K}(t) = 0$ schedule then $Y(t) > c(t)$ and $K(t)$ will be increasing and vice-versa if $c(t)$ is above it.

Figure 5.1: Equilibrium



It is possible to find an explicit approximation of the saddle-path in Figure 5.1. As established in (5.22) and (5.25) the system in Figure 5.1 is captured by the following laws of motion:[6]

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= K(t)^{-\psi} \cdot D_2 \cdot (1-\psi) - \rho \\ \frac{\dot{K}(t)}{K(t)} &= K(t)^{-\psi} \cdot D_2 - \frac{c(t)}{K(t)} \end{aligned} \tag{5.30}$$

[6]Note from (5.24) that $D_1 = (1-\psi) \cdot D_2$. Also note that the second schedule in (5.30) is simply (5.25) divided through by $K(t)$.

This system can be re-expressed in terms of the logs of $c(t)$ and $K(t)$:

$$\begin{aligned}
\frac{d\ln c(t)}{dt} &= (1-\psi)\cdot D_2 \cdot e^{-\psi\cdot\ln K(t)} - \rho \\
\frac{d\ln K(t)}{dt} &= D_2 \cdot e^{-\psi\cdot\ln K(t)} - e^{\ln\left[\frac{c(t)}{K(t)}\right]}
\end{aligned} \tag{5.31}$$

(5.31) also implies that in steady-state:

$$\begin{aligned}
&(1-\psi)\cdot D_2 \cdot e^{-\psi\cdot\ln K^*} = \rho \\
&D_2 \cdot e^{-\psi\cdot\ln K^*} - e^{\ln\left[\frac{c^*}{K^*}\right]} = 0
\end{aligned} \tag{5.32}$$

A first-order Taylor expansion of (5.31) around the steady-state provides a linear system. I take each of the schedules in turn. First the $\frac{d\ln c(t)}{dt}$ schedule. Using (5.32), (5.31) implies that a first-order Taylor expansion of $\frac{d\ln c(t)}{dt}$ is:

$$\begin{aligned}
\frac{d\ln c(t)}{dt} &\approx \left[-\frac{(1-\psi)}{K^*}\cdot\psi\cdot D_2\cdot e^{-\psi\cdot\ln K^*}\right]\cdot(K(t)-K^*) \\
&\approx -\psi\cdot\rho\cdot\ln\left[\frac{K(t)}{K^*}\right]
\end{aligned} \tag{5.33}$$

Again using (5.32), (5.31) also implies that a first-order Taylor expansion of $\frac{d\ln K(t)}{dt}$ is:

$$\begin{aligned}
\frac{d\ln K(t)}{dt} &\approx \left[\frac{-\psi}{K^*}\cdot D_2\cdot e^{-\psi\cdot\ln K^*} + \frac{1}{K^*}\cdot e^{\ln\left[\frac{c^*}{K^*}\right]}\right]\cdot(K(t)-K^*) \\
&\quad - \frac{1}{c^*}\cdot e^{\ln\left[\frac{c^*}{K^*}\right]}\cdot(c(t)-c^*) \\
&\approx \left[-\psi\cdot D_2\cdot e^{-\psi\cdot\ln K^*} + e^{\ln\left[\frac{c^*}{K^*}\right]}\right]\cdot\ln\left[\frac{K(t)}{K^*}\right]
\end{aligned} \tag{5.34}$$

$$- e^{\ln\left[\frac{c^*}{K^*}\right]} \cdot \ln\left[\frac{c(t)}{c^*}\right]$$

$$\approx \rho \cdot \ln\left[\frac{K(t)}{K^*}\right] - \left[\frac{\rho}{1-\psi}\right] \cdot \ln\left[\frac{c(t)}{c^*}\right]$$

Given the expressions in (5.33) and (5.34) it follows that the first-order Taylor expansion of the two schedules can be represented as:

$$\begin{bmatrix} \frac{d\ln(K(t))}{dt} \\ \frac{d\ln(c(t))}{dt} \end{bmatrix} = \begin{bmatrix} \rho & -\frac{\rho}{1-\psi} \\ -\psi \cdot \rho & 0 \end{bmatrix} \cdot \begin{bmatrix} \ln\left[\frac{K(t)}{K^*}\right] \\ \ln\left[\frac{c(t)}{c^*}\right] \end{bmatrix} \tag{5.35}$$

The determinant of the matrix in (5.35) is:

$$\det = [\psi \cdot \rho] \cdot \left[-\frac{\rho}{1-\psi}\right] \tag{5.36}$$

And since $\rho > 0$ and $\psi < 1$ the determinant of this matrix is always negative. This implies, formally, that there exists a saddle-path to the steady-state.[7]

Orthogonality

The solution to this new model is characterised by four simultaneous equations – the differential equations in (5.22) and (5.25), and the $A(i,t)$ and $B(i,t)$ schedules from Chapter 3. In effect, Figure 5.1 shows that this model

[7]The maximisation problem also implies a transversality condition, that $\lim_{t\to\infty} e^{-\rho t} \cdot \mu(t) \cdot K(t) = 0$. This implies that the value of the household's stock of traditional capital must approach zero in the limit. Intuitively, the consumers do not want to hold productive assets at the end of time. As shown in Barro and Sala-i-Martin (2004), this transversality condition also implies that the stock of capital must not increase in the limit at a greater rate than the level of r^* (pg. 92). This is clearly satisfied. Given the similarities between the structure of this new model and the traditional Ramsey model, the equivalent transversality conditions are also met in the settings with technological progress that follow. These are shown in Barro and Sala-i-Martin (2004). I do not repeat them here.

can be split into two orthogonal systems – a dynamic system on the left-hand side that captures the differential equations, and a static system on the right-hand side. The orthogonality follows from the fact that the position and shape of the $A(i,t)$ and $B(i,t)$ schedules in the static equilibrium do not change when the model is placed in a dynamic setting. This is for three reasons. First there is no change in relative labour productivities over time, so $A(i,t)$ does not change. Secondly, because the utility functions are Cobb-Douglas, there are no changes in expenditure shares over time that could affect the $B(i,t)$ schedule. Thirdly, because the task-based production functions for goods are Cobb-Douglas, any change in $K(t)$ during transition to the steady-state also does not affect the $B(i,t)$ schedule. These three features are familiar from the analysis in Chapter 3. Note that with more general utility functions and task-based production functions the model will still have a solution, but it will be implicitly rather than explicitly defined.

Comparative Dynamics: Factor Prices and Cut-Off

As noted, neither the $A(i,t)$ schedule nor the $B(i,t)$ schedule on the right-hand side of Figure 5.1 change when the model is placed in a dynamic setting. As a result both the wage ratio $f(t)$ and the pattern of specialisation $\bar{i}(t)$ remain unchanged over time. However, the transition to steady-state along the saddle-path will cause the relatives wage with respect to the return on traditional capital to change. Those relative wages are:[8]

$$\frac{w^L(t)}{r(t)} = \frac{\psi}{1-\psi} \cdot \frac{\alpha(\bar{i}) \cdot K(t)}{L^L} \tag{5.37}$$

[8]These relative factor prices are derived in Chapter 3.

$$\frac{w^H(t)}{r(t)} = \frac{\psi}{1-\psi} \cdot \frac{(1-\alpha(\bar{i})) \cdot K(t)}{L^H}$$

and they adjust because the stock of traditional capital changes during the transition to steady-state. To see this, call the initial stock of traditional capital $K(0)$. If, for example, $K(0) < K^*$ then as the economy moves along the saddle-path, and $K(t)$ is accumulated, (5.37) implies that both relative wages will rise. Note that, given the definition of $r(t)$ in (5.21), when $K(t)$ is being accumulated this implies that $r(t)$ is falling. The converse holds when $K(0) > K^*$.

Comparative Dynamics: One-off Increase in the Productivity of Traditional Capital

Suppose there is a one-off increase in the productivity of traditional capital such that $a^K(i,t)$ rises $\forall i$. This is analogous to the comparative static analysis in Chapter 3. This has consequences for both the $\frac{\dot{c}(t)}{c(t)} = 0$ and the $\dot{K}(t) = 0$ schedules. I take each of these in turn.

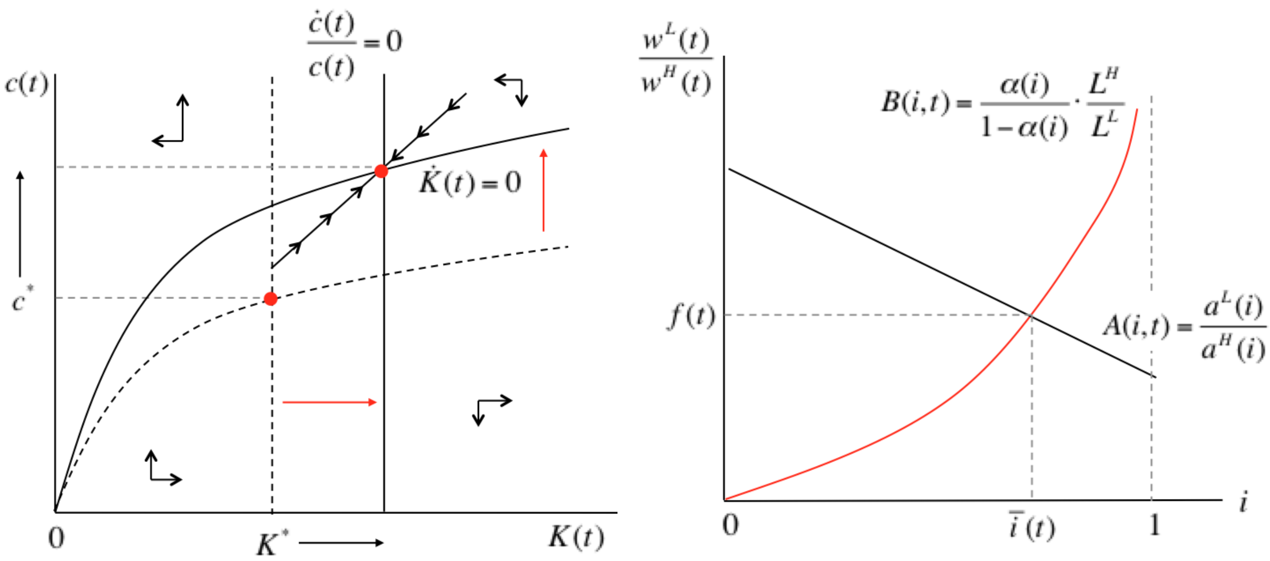
The increase in $a^K(i,t)$ causes D_1 to increase in (5.21). This implies that at any level of $K(t)$, $r(t)$ is higher. As a result (5.26) implies that the $\frac{\dot{c}(t)}{c(t)} = 0$ schedule must shift right – in order for $r(t)$ to remain at $r^* = \rho$, a higher level of $K(t)$ is required. This is intuitive – since $\psi \in (0,1)$ there are diminishing returns to $K(t)$, and so if $K(t)$ becomes more productive and $r(t)$ rises then a higher level of $K(t)$ is required to lower $r(t)$ back to $r^* = \rho$. The increase in $a^K(i,t)$ also implies an increase in D_2 in (5.24). This implies that for any level of $K(t)$, output is also higher. As a result (5.25) implies that the

$\dot{K}(t) = 0$ pivots upwards. This again is intuitive – for any level of $K(t)$, the increase in productivity causes $Y(t)$ to rise and so $c(t)$ must also rise to ensure that $\dot{K}(t) = 0$. These changes are shown in the left-hand diagram of Figure 5.2.

The result of the one-off increase in the productivity of traditional capital is an increase in the steady-state level of both c^* and K^*. The return to traditional capital, $r(t)$, rises initially, but is driven back down by the accumulation of $K(t)$ to r^*. Note that, as in Chapter 3, the one-off increase in the productivity of traditional capital leaves the static equilibrium on the right-hand side unchanged. Again, the orthogonality discussed before is maintained.

In Figure 5.2, the saddle-path to the new equilibrium has been drawn in such a way to imply that $c(0)$ rises following the technology shock, and then rises further to c^* in the new steady-state. This is because utility is Cobb-Douglas, and is shown in the Appendix, section 9.1.2. The ambiguity is intuitive –given a one-off technological shock, there is an income effect and a substitution effect. The former follows from the fact that the consumer is wealthier today, and so will want to raise consumption today. The latter implies that the consumer's return on capital is now higher, and so the return to saving is higher, and so he will want to delay consumption today. In this particular case the income effect dominates the substitution effect. In a more general setting, whether this takes place depends on the particular parameters of the model.

Figure 5.2: New Equilibrium



Comparative Dynamics: Relative Goods Prices

It is possible to describe how relative prices of the different goods change along the transition path and in steady-state. The expressions for the price of any good i in period t were derived in Chapter 3, and shown in (3.39) and (3.40). These imply that the price of good i in period t can be written more concisely as:

$$p(i,t) = \left[\frac{F(i,t)}{\psi \cdot a^F(i,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(i,t)}\right]^{1-\psi} \tag{5.38}$$

where $F(i,t)$ is the factor price for the particular factor that works with traditional capital to produce good i and $a^F(i)$ is that factor's productivity i.e.:

$$F(i,t)=\begin{cases} w^L(t) & \text{if } i\in[0,\bar{i}(t)] \\ w^H(t) & \text{if } i\in[\bar{i}(t),1] \end{cases}$$

And:

$$a^F(i,t)=\begin{cases} a^L(i,t) & \text{if } i\in[0,\bar{i}(t)] \\ a^H(i,t) & \text{if } i\in[\bar{i}(t),1] \end{cases}$$

This reflects the fact that if $i\in[0,\bar{i}(t)]$ then low-skilled labour works with traditional capital to produce $x(i,t)$ and if $i\in[\bar{i}(t),1]$ then high-skilled labour works with traditional capital to produce $x(i,t)$. It follows that the relative price of any two goods $\tilde{i}$ and $\tilde{\tilde{i}}$ is:

$$\frac{p(\tilde{i},t)}{p(\tilde{\tilde{i}},t)}=\left[\frac{F(\tilde{i},t)}{F(\tilde{\tilde{i}},t)}\right]^{\psi}\left[\frac{a^F(\tilde{\tilde{i}},t)}{a^F(\tilde{i},t)}\right]^{\psi}\left[\frac{a^K(\tilde{\tilde{i}},t)}{a^K(\tilde{i},t)}\right]^{1-\psi} \tag{5.39}$$

The rate of change in relative prices is therefore:

$$\frac{\dot{p}(\tilde{i},t)}{p(\tilde{i},t)}-\frac{\dot{p}(\tilde{\tilde{i}},t)}{p(\tilde{i},t)}=$$
$$\psi\cdot\left[\frac{\dot{F}(\tilde{i},t)}{F(\tilde{i},t)}-\frac{\dot{F}(\tilde{\tilde{i}},t)}{F(\tilde{\tilde{i}},t)}\right]+\psi\cdot\left[\frac{\dot{a}^F(\tilde{\tilde{i}},t)}{a^F(\tilde{\tilde{i}},t)}-\frac{\dot{a}^F(\tilde{i},t)}{a^F(\tilde{i},t)}\right]+(1-\psi)\cdot\left[\frac{\dot{a}^K(\tilde{\tilde{i}},t)}{a^K(\tilde{\tilde{i}},t)}-\frac{\dot{a}^K(\tilde{i},t)}{a^K(\tilde{i},t)}\right] \tag{5.40}$$

The expression in (5.40) is intuitive. The first term captures the relative change in the factor price of the types of labour that are used to produce goods $\tilde{i}$ and $\tilde{\tilde{i}}$. If two types of goods are produced by the same type of labour this term is necessarily equal to zero. The second term captures the relative change in the productivities of the labour used to produce each good. The third term captures the relative change in the productivity of traditional capital at producing each good. Intuitively, the zero profit condition ensures that the relative prices of any two goods change to compensate exactly for any changes in the relative price of the labour used to produce each good, any changes in the relative productivity of the labour that is used to produce each good, and any changes in the relative productivity of traditional capital in producing each good.

Given that there is no capital-augmenting technological progress in the model it follows that for all possible $\tilde{i}$ and $\tilde{\tilde{i}}$:

$$\frac{\dot{a}^K(\tilde{\tilde{i}},t)}{a^K(\tilde{\tilde{i}},t)} = \frac{\dot{a}^K(\tilde{i},t)}{a^K(\tilde{i},t)} = 0 \tag{5.41}$$

and given the productivities of both types of labour are constant, it also follows that for all possible $\tilde{i}$ and $\tilde{\tilde{i}}$:

$$\frac{\dot{a}^F(\tilde{\tilde{i}},t)}{a^F(\tilde{\tilde{i}},t)} = \frac{\dot{a}^F(\tilde{i},t)}{a^F(\tilde{i},t)} = 0 \tag{5.42}$$

and so (5.40) collapses to:

$$\frac{\dot{p}(\tilde{i},t)}{p(\tilde{i},t)} - \frac{\dot{p}(\tilde{\tilde{i}},t)}{p(\tilde{i},t)} = \psi \cdot \left[\frac{\dot{F}(\tilde{i},t)}{F(\tilde{i},t)} - \frac{\dot{F}(\tilde{\tilde{i}},t)}{F(\tilde{\tilde{i}},t)}\right] \tag{5.43}$$

This implies that the growth rate in relative prices between two goods $\tilde{i}$ and $\tilde{\tilde{i}}$ depends only upon the difference in the difference in the growth rate of the wages of the type of labour that is used to produce those goods. But in steady-state and on the transition path, as shown before, $\frac{w^L(t)}{w^H(t)}$ is constant – wages of both types of labour grow at the same rate during transition. As a result, this implies that for all possible $\tilde{i}$ and $\tilde{\tilde{i}}$, in all t:

$$\frac{\dot{F}(\tilde{i},t)}{F(\tilde{i},t)} = \frac{\dot{F}(\tilde{\tilde{i}},t)}{F(\tilde{\tilde{i}},t)} \tag{5.44}$$

It follows that in this setting, during both steady-state and transition, relative prices of the goods do not change. This is because there are no changes in relative productivities, or in the wage ratio, for relative prices to offset.

From this analysis Proposition 3 follows:

PROPOSITION 3: *When the static model with traditional capital is placed in a dynamic setting with endogenous capital accumulation but no technological progress, there is a unique steady-state where $r^* = \rho$, $c(t) = c^*$ and $K(t) = K^*$. There is no change in the cut-off $\bar{i}(t)$. Relative goods prices are constant along the saddle-path and in steady-state.*

5.2.2. With Technological Progress

In the model set out before there is no technological progress. I now assume that there is exogenous growth in the productivity of traditional capital:

$$a^K(i,t) = a^K(i,0) \cdot e^{gt} \quad \forall i \tag{5.45}$$

Such that:

$$\frac{\dot{a}^K(i,t)}{a^K(i,t)} = g \quad \forall i \tag{5.46}$$

Immediately it is clear from the analysis in the previous section that in $K(t)$–$c(t)$ space this continuous increase in the level of productivity of traditional capital will now cause a continuous shift in the $\frac{\dot{c}(t)}{c(t)} = 0$ and the $\dot{K}(t) = 0$ schedule in the directions set out in Figure 5.2, rather than the one-off shifts caused by the one-off increase that I considered. As a result steady-state will no longer exist in the $K(t)$–$c(t)$ space.

The traditional approach to finding the alternative steady-state in a Ramsey model with technological progress is to re-define the variables in 'effective' terms, dividing each variable by the prevailing level of labour-augmenting technology. The result is that a steady-state is reached not in the actual variables as before, but instead in these 'effective' variables, the variable 'per efficiency unit of labour'. However in this model, technological progress is capital-augmenting, not labour-augmenting. As a result, this traditional approach does not work directly. Instead, the production functions must first be transformed in such a way that the capital augmenting technological progress I am considering is instead exactly reflected in a process of labour-augmenting technological progress. Consider again a good $x(\tilde{i},t)$ that is produced by low-skilled labour and traditional capital – again I assume that $\tilde{i} = 0$. The transformation of the production function is as follows:

$$\begin{aligned} x(0,t) &= \left[a^L(0,t) \cdot L^L(0,t)\right]^{\psi} \left[a^K(0,t) \cdot K(0,t)\right]^{1-\psi} \\ &= \left[a^K(0,t)^{\frac{1-\psi}{\psi}} \cdot a^L(0,t) \cdot L^L(0,t)\right]^{\psi} \left[K(0,t)\right]^{1-\psi} \end{aligned} \tag{5.47}$$

This transformation is possible because the task-based production functions for goods are Cobb-Douglas. (5.47) implies that a labour-augmenting process of technological change in $a^K(0,t)^{\frac{1-\psi}{\psi}}$ is identical to a capital-augmenting process of technological change $a^K(0,t)$. Given this I now define the variables in the model in 'effective' terms with respect to this new factor i.e. a variable v in effective terms is $\hat{v}$ where:

$$\hat{v} = \frac{v}{a^K(0,t)^{\frac{1-\psi}{\psi}}} \tag{5.48}$$

It follows that $\hat{x}(0,t)$ is:

$$\hat{x}(0,t) = \frac{x(0,t)}{a^K(0,t)^{\frac{1-\psi}{\psi}}} \tag{5.49}$$

And $\hat{K}(t)$:

$$\hat{K}(t) = \frac{K(t)}{a^K(0,t)^{\frac{1-\psi}{\psi}}} \tag{5.50}$$

Dynamic Equilibrium

Dynamic equilibrium can be described in terms of the effective variables. The law of motion for $\hat{x}(0,t)$ follows from (5.46) and (5.49):

$$\begin{aligned} \frac{\dot{\hat{x}}(0,t)}{\hat{x}(0,t)} &= \frac{\dot{x}(0,t)}{x(0,t)} - \frac{1-\psi}{\psi} \cdot \frac{\dot{a}^K(0,t)}{a^K(0,t)} \\ &= r(t) - \rho - \frac{1-\psi}{\psi} \cdot g \end{aligned} \tag{5.51}$$

The analysis in (5.13) and (5.14) implies that the law of motion for $\hat{c}(t)$ is:

$$\begin{aligned}\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \frac{\dot{c}(t)}{c(t)} - \frac{1-\psi}{\psi} \cdot \frac{\dot{a}^K(0,t)}{a^K(0,t)} \\ &= r(t) - \rho - \frac{1-\psi}{\psi} \cdot g\end{aligned} \tag{5.52}$$

In addition, (5.46) and (5.50) imply that the law of motion for $\hat{K}(t)$ is:

$$\begin{aligned}\frac{\dot{\hat{K}}(t)}{\hat{K}(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{1-\psi}{\psi} \cdot \frac{\dot{a}^K(0,t)}{a^K(0,t)} \\ &= \frac{Y(t)-c(t)}{K(t)} - \frac{1-\psi}{\psi} \cdot g \\ &= \frac{\hat{Y}(t)-\hat{c}(t)}{\hat{K}(t)} - \frac{1-\psi}{\psi} \cdot g\end{aligned} \tag{5.53}$$

Both these laws of motion can be expressed in terms of $\hat{K}(t)$ and $\hat{c}(t)$ alone – rather than in the $K(t)$–$c(t)$ space used with no technological progress. The $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)}$ schedule from (5.52) can be shown in this space by finding $r(t)$ in terms of $\hat{K}(t)$. This follows from the expression for $r(t)$ derived in (5.21):

$$\begin{aligned}r(t) &= K(t)^{-\psi} \cdot (L^L)^{\psi} \cdot (1-\psi) \cdot \alpha(\bar{i}(t))^{-\psi} \cdot \left[a^L(0,t)\right]^{\psi} \left[a^K(0,t)\right]^{1-\psi} \\ &= \left[\frac{K(t)}{a^K(0,t)^{\frac{1-\psi}{\psi}}}\right]^{-\psi} \cdot (L^L)^{\psi} \cdot (1-\psi) \cdot \alpha(\bar{i}(t))^{-\psi} \cdot \left[a^L(0,t)\right]^{\psi} \\ &= \hat{K}(t)^{-\psi} \cdot D_3\end{aligned} \tag{5.54}$$

where D_3 is a positive constant. As before, note that $\alpha(\bar{i}(t))^{-\psi}$ is included in the constant D_3. Again, this is because $\bar{i}(t)$ does not change over time with fixed stocks of both types of labour, and productivity growth only in traditional capital. Again, this was shown in Chapter 3. (5.54) can be substituted into the expression for the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)}$ schedule in (5.52):

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \hat{K}(t)^{-\psi} \cdot D_3 - \rho - \frac{1-\psi}{\psi} \cdot g \tag{5.55}$$

(5.55) is therefore the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)}$ schedule in $\hat{K}(t)$-$\hat{c}(t)$ space.

The $\dot{\hat{K}}(t)$ schedule implied by (5.53) can also be expressed in terms of $\hat{c}(t)$ and $\hat{K}(t)$. This requires that $\hat{Y}(t)$ is expressed terms of $\hat{K}(t)$. Given the expression for traditional capital income in (5.23) it follows that:

$$r(t) \cdot \hat{K}(t) = (1-\psi) \cdot \hat{Y}(t) \tag{5.56}$$

and so substituting in the expression for $r(t)$ in terms of $\hat{K}(t)$ derived in (5.54) it follows that:

$$\begin{aligned} \hat{Y}(t) &= r(t) \cdot \hat{K}(t) \cdot \frac{1}{(1-\psi)} \\ &= \hat{K}(t)^{1-\psi} \cdot D_3 \cdot \frac{1}{(1-\psi)} \\ &= \hat{K}(t)^{1-\psi} \cdot D_4 \end{aligned} \tag{5.57}$$

where D_4 is a positive constant. If this expression for $\hat{Y}(t)$ is substituted back into the expression for $\dot{\hat{K}}(t)$ implied by (5.53) the result is:

$$\dot{\hat{K}}(t) = \hat{K}(t)^{1-\psi} \cdot D_4 - \hat{c}(t) - \frac{1-\psi}{\psi} \cdot g \cdot \hat{K}(t) \tag{5.58}$$

(5.58) is therefore the $\dot{\hat{K}}(t)$ schedule in $\hat{K}(t)$-$\hat{c}(t)$ space.

Given the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)}$ schedule in (5.55) and the $\dot{\hat{K}}(t)$ schedule in (5.58) it is now possible to show the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ and $\dot{\hat{K}}(t) = 0$ schedules in $\hat{K}(t)$–$\hat{c}(t)$ space. The intersection of these schedules is the new steady-state. Given (5.55) the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule in $\hat{K}(t)$-$\hat{c}(t)$ space is:

$$\hat{K}(t)^{-\psi} \cdot D_3 = \rho + \frac{1-\psi}{\psi} \cdot g \tag{5.59}$$

and given (5.58) the $\dot{\hat{K}}(t) = 0$ schedule in $\hat{K}(t)$-$\hat{c}(t)$ space is:

$$\hat{c}(t) = D_4 \cdot \hat{K}(t)^{1-\psi} - \frac{1-\psi}{\psi} \cdot g \cdot \hat{K}(t) \tag{5.60}$$

These schedules are shown in $\hat{K}(t)$-$\hat{c}(t)$ space in the left-hand diagram of Figure 5.3. The $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule is vertical at the level of $\hat{K}(t)$ such that (5.59) holds and $\hat{c}(t)$ is constant. It follows that this is equal to:

$$\hat{K}^* = \left[\left[\rho + \frac{1-\psi}{\psi} \cdot g\right] \cdot \frac{1}{D_3}\right]^{-\frac{1}{\psi}} \tag{5.61}$$

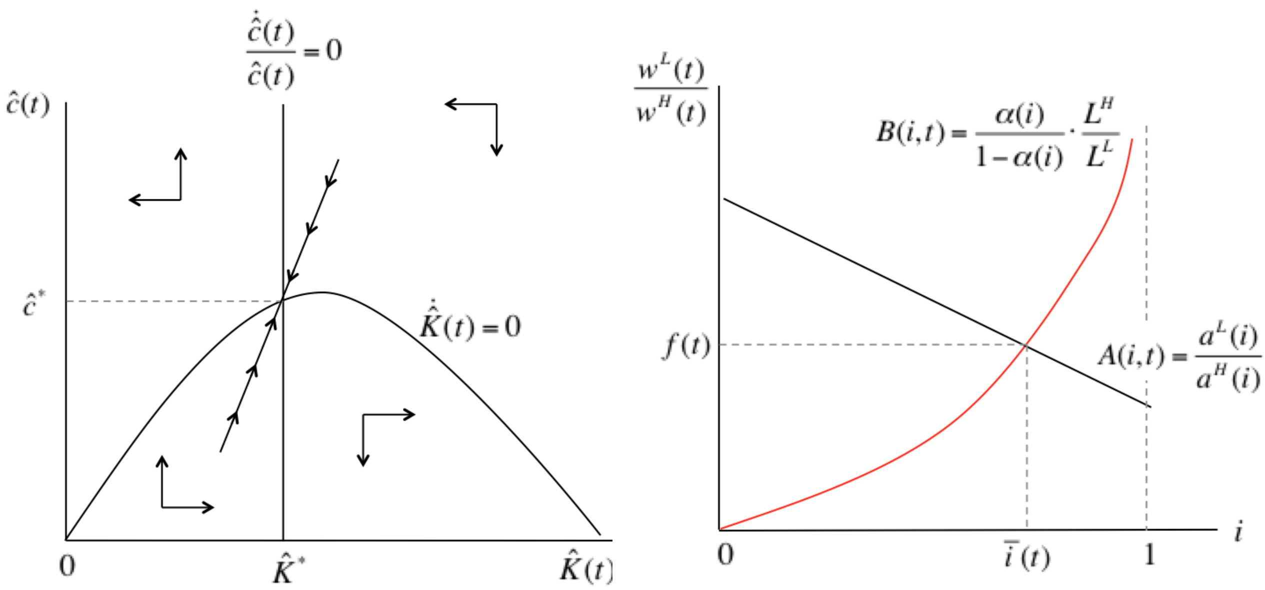
and this level of $\hat{K}^*$ implies that:

$$r(t) = r^* = \rho + \frac{1-\psi}{\psi} \cdot g \tag{5.62}$$

As before, this can be put in the following way – the requirement that $r(t) = r^*$ as in (5.62) pins down a unique steady-state level of effective traditional capital stock, $\hat{K}^*$ – namely, the level of $\hat{K}(t)$ at which the marginal revenue product of effective traditional capital for all goods is equal to r^*. The expression for the $\dot{\hat{K}}(t) = 0$ schedule shows that this is hump-shaped –

this is because of the additional, linear, negative term in the expression for $\dot{\hat{K}}(t) = 0$ that is not present in the expression for $\dot{K}(t) = 0$.[9] Also note that the intersection of the schedules in the left-hand diagram of Figure 5.3 is to the left of the peak of the $\dot{\hat{K}}(t) = 0$ schedule. This is because consumers are impatient, $\rho > 0$, and so the level of effective capital that maximises effective consumption is not the optimal level of effective capital.[10]

Figure 5.3: Equilibrium



The transition dynamics follow from (5.59) and (5.60). (5.59) implies that if $\hat{K}(t) < \hat{K}^*$ then $r(t) > \rho + \frac{1-\psi}{\psi} \cdot g$ and $\hat{c}(t)$ will be increasing. If $\hat{K}(t) > \hat{K}^*$ then $r(t) < \rho + \frac{1-\psi}{\psi} \cdot g$ and $\hat{c}(t)$ will be decreasing. (5.60) implies that, for a given $\hat{K}(t)$, if $\hat{c}(t)$ is below the $\hat{c}(t)$ on the $\dot{\hat{K}}(t) = 0$ schedule then $\hat{K}(t)$ will be increasing and vice-versa if $\hat{c}(t)$ is above it.

[9] This is the $\frac{1-\psi}{\psi} \cdot g \cdot \hat{K}(t)$ term in (5.60).

[10] Differentiating (5.60) shows this is $\hat{K}(t) = \left[\frac{1-\psi}{\psi} \cdot g \cdot \frac{1}{D_3}\right]^{-\frac{1}{\psi}}$, rather than $\hat{K}^*$ in (5.61).

It is again possible to find an explicit expression for the saddle-path in Figure 5.3. As established in (5.55) and (5.58) the dynamic model can be captured by two laws of motion in $\hat{c}(t)$–$\hat{K}(t)$ space:[11]

$$\begin{aligned} \frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \hat{K}(t)^{-\psi} \cdot D_4 \cdot (1-\psi) - \rho - \frac{1-\psi}{\psi} \cdot g \\ \frac{\dot{\hat{K}}(t)}{\hat{K}(t)} &= \hat{K}(t)^{-\psi} \cdot D_4 - \frac{1-\psi}{\psi} \cdot g - \frac{\hat{c}(t)}{\hat{K}(t)} \end{aligned} \tag{5.63}$$

This system can be re-expressed in terms of the logs of $\hat{c}(t)$ and $\hat{K}(t)$:

$$\begin{aligned} \frac{d \ln \hat{c}(t)}{dt} &= (1-\psi) \cdot D_4 \cdot e^{-\psi \cdot \ln \hat{K}(t)} - \rho - \frac{1-\psi}{\psi} \cdot g \\ \frac{d \ln \hat{K}(t)}{dt} &= D_4 \cdot e^{-\psi \cdot \ln \hat{K}(t)} - \frac{1-\psi}{\psi} \cdot g - e^{\ln\left[\frac{\hat{c}(t)}{\hat{K}(t)}\right]} \end{aligned} \tag{5.64}$$

(5.64) also implies that in steady-state:

$$\begin{aligned} (1-\psi) \cdot D_4 \cdot e^{-\psi \cdot \ln \hat{K}^*} &= \rho + \frac{1-\psi}{\psi} \cdot g \\ D_4 \cdot e^{-\psi \cdot \ln \hat{K}^*} - e^{\ln\left[\frac{\hat{c}^*}{\hat{K}^*}\right]} &= \frac{1-\psi}{\psi} \cdot g \end{aligned} \tag{5.65}$$

A first-order expansion of (5.64) around the steady-state provides a linear system. (5.64) and (5.65) imply a first-order Taylor expansion of $\frac{d \ln \hat{c}(t)}{dt}$ is:

$$\begin{aligned} \frac{d \ln \hat{c}(t)}{dt} &\approx \left[-\frac{\psi}{\hat{K}^*} \cdot (1-\psi) \cdot D_4 \cdot e^{-\psi \cdot \ln \hat{K}^*} \right] \cdot (\hat{K}(t) - \hat{K}^*) \\ &\approx -\psi \cdot \left[\rho + \frac{1-\psi}{\psi} \cdot g \right] \cdot \ln \left[\frac{\hat{K}(t)}{\hat{K}^*} \right] \end{aligned} \tag{5.66}$$

[11]Note from (5.57) that $D_3 = D_4 \cdot (1-\psi)$.

(5.64) and (5.65) also imply that a first-order Taylor expansion of $\frac{d\ln\hat{K}(t)}{dt}$ is:

$$\begin{aligned}\frac{d\ln\hat{K}(t)}{dt} &\approx \left[-\frac{\psi}{\hat{K}^*}\cdot D_4\cdot e^{-\psi\cdot\ln\hat{K}^*}+\frac{1}{\hat{K}^*}\cdot e^{\ln\left[\frac{\hat{c}^*}{\hat{K}^*}\right]}\right]\cdot(\hat{K}(t)-\hat{K}^*)\\ &\quad-\frac{1}{\hat{c}^*}\cdot e^{\ln\left[\frac{\hat{c}^*}{\hat{K}^*}\right]}\cdot(\hat{c}(t)-\hat{c}^*)\\ &\approx\left[-\psi\cdot D_4\cdot e^{-\psi\cdot\ln\hat{K}^*}+e^{\ln\left[\frac{\hat{c}^*}{\hat{K}^*}\right]}\right]\cdot\ln\left[\frac{\hat{K}(t)}{\hat{K}^*}\right]\\ &\quad-e^{\ln\left[\frac{\hat{c}^*}{\hat{K}^*}\right]}\cdot\ln\left[\frac{\hat{c}(t)}{\hat{c}^*}\right]\\ &\approx\rho\cdot\ln\left[\frac{\hat{K}(t)}{\hat{K}^*}\right]+\left[\frac{1-\psi}{\psi}\cdot g-\frac{\rho+\frac{1-\psi}{\psi}\cdot g}{1-\psi}\right]\cdot\ln\left[\frac{\hat{c}(t)}{\hat{c}^*}\right]\end{aligned}\tag{5.67}$$

Given the expressions in (5.65)–(5.67) it follows that the first-order Taylor expansion of the two schedules can be represented as:

$$\begin{bmatrix}\frac{d\ln(\hat{K}(t))}{dt}\\ \frac{d\ln(\hat{c}(t))}{dt}\end{bmatrix}=\begin{bmatrix}\rho & \frac{1-\psi}{\psi}\cdot g-\frac{\rho+\frac{1-\psi}{\psi}\cdot g}{1-\psi}\\ -\psi\cdot\left[\rho+\frac{1-\psi}{\psi}\cdot g\right] & 0\end{bmatrix}\cdot\begin{bmatrix}\ln\left[\frac{\hat{K}(t)}{\hat{K}^*}\right]\\ \ln\left[\frac{\hat{c}(t)}{\hat{c}^*}\right]\end{bmatrix}\tag{5.68}$$

The determinant of the matrix in (5.68) is:

$$\det=\left[\psi\cdot\left[\rho+\frac{1-\psi}{\psi}\cdot g\right]\right]\cdot\left[\frac{1-\psi}{\psi}\cdot g-\frac{\rho+\frac{1-\psi}{\psi}\cdot g}{1-\psi}\right]\tag{5.69}$$

And since $\rho>0$ and $\psi<1$ the determinant of this matrix is always negative. A saddle-path must exist.

The growth rate in the actual level of traditional capital in steady-state follows from the definition of $\hat{K}(t)$ in (5.50). Since $\hat{K}(t)$ is constant in steady-state it follows that:

$$\frac{\dot{K}(t)}{K(t)} = \frac{1-\psi}{\psi} \cdot g \tag{5.70}$$

This steady-state condition is intuitive – if steady-state requires a constant $r(t)$ then there must be an endogenous process of capital accumulation at the rate in (5.70) to offset the upward pressure on the interest rate caused by the growth in the productivity of traditional capital. Since in steady-state $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ it also follows from (5.52) that:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1-\psi}{\psi} \cdot g \tag{5.71}$$

In turn, the expression for effective traditional capital income in (5.56) implies that, if $r(t)$ is constant in steady-state, and $K(t)$ grows at the rate in (5.70), then in steady-state:

$$\begin{aligned} \frac{\dot{Y}(t)}{Y(t)} &= \frac{\dot{r}(t)}{r(t)} + \frac{\dot{K}(t)}{K(t)} \\ &= \frac{1-\psi}{\psi} \cdot g \end{aligned} \tag{5.72}$$

In a traditional Ramsey model with labour-augmenting technological progress at rate g the expectation would be that the growth rate in the actual variables in (5.70) - (5.72) is g. But recall that technological progress in this model is capital-augmenting – the $\frac{1-\psi}{\psi}$ reflects the transformation of this capital-augmenting process into an equivalent labour-augmenting process. Finally, note that in steady-state the savings rate with traditional capital is constant since consumption and output grow at the same rate.

Orthogonality

Again in Figure 5.3, as in Figure 5.1, it is possible to split this model into two orthogonal systems – the dynamic system on the left-hand side, that is made tractable by expressing $K(t)$ and $c(t)$ in 'effective' terms, and the static system on the right-hand side. The reason for this is the same as in Figure 5.1. But now, in addition, it is important to note that this orthogonality is not affected by the introduction of an exogenous process of growth in the productivity of traditional capital. This is for the same reason that the one-off increase in the relative productivity of traditional capital does not affect Figure 5.1 – it does not affect either the $A(i,t)$ or the $B(i,t)$ schedule. Again, these properties are familiar from Chapter 3.

Comparative Dynamics: Factor Prices and Cut-Off

Neither the $A(i,t)$ nor the $B(i,t)$ schedule on the right-hand side of Figure 5.3 change when the model is placed in a dynamic setting with technological progress in traditional capital. Again, neither the wage ratio $f(t)$ or the pattern of specialisation $\bar{i}(t)$ changes. However the accumulation of traditional capital does have an implication for the relative wages with respect to the return on traditional capital – both when the economy is in transition as in the case with no technological progress, but also now in the steady-state as well. (5.37) implies that the growth rate in these relative factor prices is:

$$\frac{\dot{w}^L(t)}{w^L(t)} - \frac{\dot{r}(t)}{r(t)} = \frac{\dot{w}^H(t)}{w^H(t)} - \frac{\dot{r}(t)}{r(t)} = \frac{\dot{K}(t)}{K(t)} \tag{5.73}$$

Without technological progress the stock of capital only changes during transition – as a result, as (5.73) implies, the wage of each type of labour relative to the return on traditional capital only changes during transition. In steady-state these relative factor prices were constant. However, with technological progress, capital is now also accumulated in steady-state. As a result, (5.73) now implies that in steady-state the wage of both types of labour relative to the return on traditional capital is increasing over time – the optimistic conclusion that was derived in the static model in Chapter 3 is strengthened in the dynamic setting. This is because in steady-state the return to the scarce factors, the two of type of labour, rises as it becomes increasingly scare relative to the accumulated factor, capital. The rate of return to the produced factor does not rise because this is equal to the rate of time preference and the growth rate in the transformed production function in (5.47). Note that in the setting with traditional capital alone, the labour *share* of income and the capital *share* of income remain constant across time (ψ and $(1-\psi)$).

It is also possible to describe how the wage of each type of labour relative to the return on traditional capital changes during transition. This depends upon which direction the economy approaches the steady-state. If $\hat{K}(0) < \hat{K}^*$ then $\hat{K}(t)$ rises – this implies that, during transition:

$$\frac{\dot{K}(t)}{K(t)} > \frac{1-\psi}{\psi} \cdot g \qquad (5.74)$$

and so from (5.73) $w^L(t)$ relative to $r(t)$ grows at a faster rate during transition than in steady-state. The opposite is the case if $\hat{K}(0) > \hat{K}^*$.

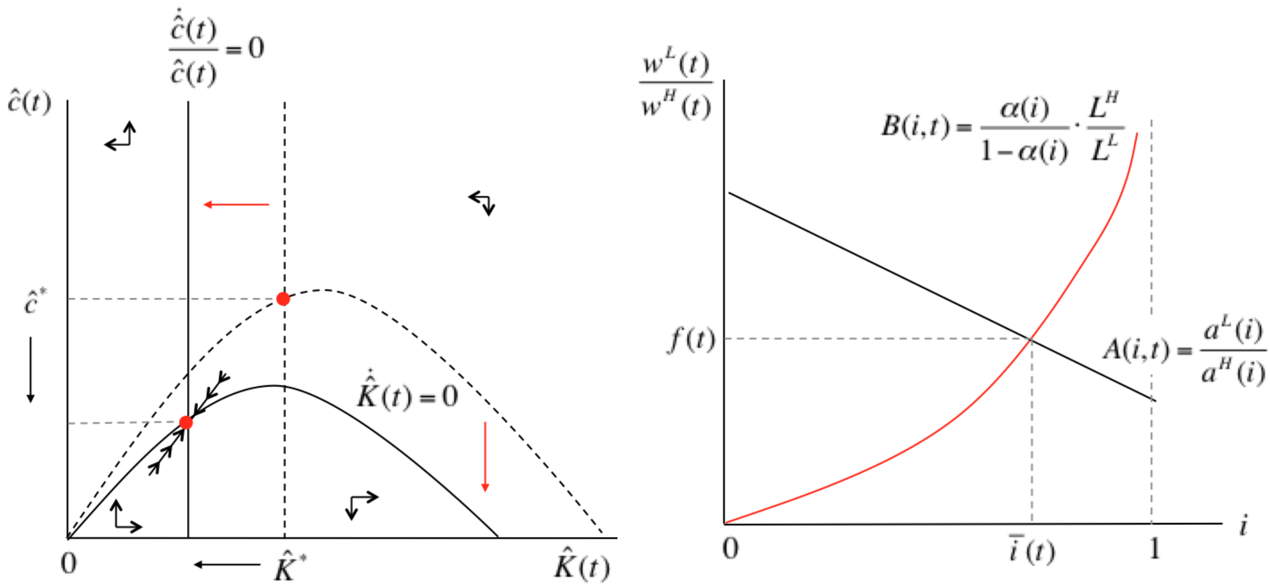
Comparative Dynamics: One-off Increase in the Rate of Growth in the Productivity of Traditional Capital

Suppose there is a one-off increase in the *rate* of growth in the productivity of traditional capital – g rises to g'. This has implications for the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ and $\frac{\dot{\hat{K}}(t)}{\hat{K}(t)} = 0$ schedules. I look at each of these in turn.

The increase in g causes the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule to shift left. This follows from (5.61) – a rise in g requires a fall in $\hat{K}^*$. This is intuitive – the rise in rate of growth of the productivity of traditional capital means that $c(t)$ will also grow at a faster rate over time, and this requires a lower $\hat{K}^*$ and an accompanying higher r^*. This can be seen from (5.52) – this expression implies that $c(t)$ will grow at a rate equal to the rate of growth in $\hat{c}(t)$ and a fixed proportion of the rate of growth of the productivity of advanced capital. Since the former is zero in steady-state, and the latter has increased, it follows that $c(t)$ will grow at a higher rate in steady-state. The increase in g also causes the $\frac{\dot{\hat{K}}(t)}{\hat{K}(t)} = 0$ schedule to fall. This follows from (5.60).

This is shown in Figure 5.4. The result is that a one-off increase in the rate of growth in the productivity of traditional capital is a one-off decrease in the steady-state *level* of both $\hat{c}^*$ and $\hat{K}^*$ – but a one-off increase in the steady-state growth *rate* of both $c(t)$ and $K(t)$. This is analogous to the one-off increase in the steady-state level in both $c(t)$ and $K(t)$ in section 5.2.1 where there was only a one-off increase in the level of the productivity of traditional capital. Also note that there is now a permanent rise in the interest rate r^* – in section 5.2.1 $r(t)$ initially rose, but was driven down in turn by the accumulation of traditional capital. The static equilibrium on the right-hand side in Figure 5.4 is again unchanged.[12]

Figure 5.4: New Equilibrium



Comparative Dynamics: Relative Goods Prices

As in (5.40) the growth rate in the relative price $\frac{p(\tilde{i},t)}{p(\tilde{\tilde{i}},t)}$ of two arbitrary goods $\tilde{i}$ and $\tilde{\tilde{i}}$ is again:

$$\frac{\dot{p}(\tilde{i},t)}{p(\tilde{i},t)} - \frac{\dot{p}(\tilde{\tilde{i}},t)}{p(\tilde{\tilde{i}},t)} =$$

$$\psi \cdot \left[\frac{\dot{F}(\tilde{i},t)}{F(\tilde{i},t)} - \frac{\dot{F}(\tilde{\tilde{i}},t)}{F(\tilde{\tilde{i}},t)} \right] + \psi \cdot \left[\frac{\dot{a}^F(\tilde{\tilde{i}},t)}{a^F(\tilde{\tilde{i}},t)} - \frac{\dot{a}^F(\tilde{i},t)}{a^F(\tilde{i},t)} \right] + (1-\psi) \cdot \left[\frac{\dot{a}^K(\tilde{\tilde{i}},t)}{a^K(\tilde{\tilde{i}},t)} - \frac{\dot{a}^K(\tilde{i},t)}{a^K(\tilde{i},t)} \right] \tag{5.75}$$

Given that, at any time t, $\frac{w^L(t)}{w^H(t)}$ is constant – whether the economy is on the

[12]The analysis of what happens to initial consumption following this shock cannot be determined using the analytical approach for the case with a one-off increase in the productivity of traditional capital, as set out in the Appendix, section 9.1.3. The consequences of similar shocks are simulated and discussed in Carroll and Weil (1994).

transition path, or in steady-state, as shown in Figure 5.3 – the first term is equal to zero. Similarly, given there is no change in the productivities of either type of labour, the second term is also equal to zero. (5.75) therefore collapses to:

$$\frac{\dot{p}(\tilde{i},t)}{p(\tilde{i},t)} - \frac{\dot{p}(\tilde{\tilde{i}},t)}{p(\tilde{i},t)} = (1-\psi)\cdot\left[\frac{\dot{a}^K(\tilde{\tilde{i}},t)}{a^K(\tilde{\tilde{i}},t)} - \frac{\dot{a}^K(\tilde{i},t)}{a^K(\tilde{i},t)}\right] \tag{5.76}$$

In (5.45) I assume that technological progress in traditional capital took place at a constant rate g across all goods. As a result, for any $\tilde{i}$ and $\tilde{\tilde{i}}$ it follows that (5.76) collapses to zero. When technological progress in traditional capital takes place at the same rate for all types of tasks, there is no change in relative prices. As in the previous case with no technological progress in traditional capital, the prices act as slack variables and exactly compensate for relative changes in factor productivities and the relative price of the two types of labour. But given that there is no relative change in the productivity of the factors, or the wage ratio, relative prices do not change either.

From this analysis Proposition 4 follows:

PROPOSITION 4: *When the new model in Chapter 3 is placed in a dynamic setting with endogenous capital accumulation and technological progress there is a unique steady-state where* $r^* = \rho + \frac{1-\psi}{\psi}$, $\hat{K}(t) = \hat{K}^*$, $\hat{c}(t) = \hat{c}^*$, *and* $\hat{Y}(t) = \hat{Y}^*$. *The actual variables grow at rate* $\frac{1-\psi}{\psi}\cdot g$ *in steady-state. The cut-off* $\bar{i}(t)$ *does not change. Relative prices of any two goods do not change during transition and in steady-state.*

5.2.3. Discussion

In the dynamic model with traditional capital, the endogenous accumulation of traditional capital strengthens the optimistic outcome in Chapter 3. In the case with no technological progress, a one-off increase in the productivity of traditional capital results in a new steady-state where c^* and K^* are higher – and during the transition to this steady-state, traditional capital is accumulated and the wages of both types of labour rise relative to the return on traditional capital. This is optimism at work. In the case with technological progress, traditional capital is also accumulated in steady-state. As a result, in steady-state the relative wages of both types of labour continue to rise. This again is optimism at work. The intuition for this is the same in both cases – traditional capital is a q-complement to labour in producing each good and so the return to labour, the scare factor, rises as it becomes increasingly scarce relative to the accumulated factor, traditional capital. In neither case does the equilibrium cut-off $\bar{i}^*(t)$ change. This is for the same reason as in Chapter 3. Just as Chapter 3 is the simplest static model of optimism, this is the simplest dynamic model of optimism.

5.3. Advanced Capital

I now turn to advanced capital and place the static model in Chapter 4 in a dynamic setting. For simplicity, in this section I restrict my attention to the three-factor setting – one type of labour, high-skilled, and two types of capital, traditional and advanced. First I do this without technological progress in advanced capital. Then I introduce technological progress. It is of course possible to explore the four-factor setting computationally. However, the central conclusions can be derived explicitly from the three-factor setting.[13]

I use the same approach to finding equilibrium in both cases. First I derive the laws of motion in the traditional way. However, because changes in the stock and the productivity of advanced capital will now also change the cut-off $\bar{i}(t)$ – an effect that was absent with only traditional capital in section 5.2 – it is necessary to express advanced capital in 'effective' terms that not only capture any technological progress, but also capture the change in $\bar{i}(t)$. I will explain exactly what is meant by this in the analysis that follows.

5.3.1. No Technological Progress

In the model in Chapter 4 there are two types of capital – both traditional capital, $K(t)$, and advanced capital, $K^A(t)$. In this section I assume that the stock of traditional capital is fixed such that $K(t) = K$, as well as L^H as in the previous sections. The return to traditional capital is $r(t)$ and to advanced capital is $r^A(t)$. Since traditional capital is a fixed stock I assume it is non-tradable. As a result the rate of return on the two types of capital may differ.

Consumers

As in Chapter 4, consumers again have Cobb-Douglas utility functions. Again, the economy can be captured by a representative consumer with Cobb-Douglas preferences. The consumer faces the dynamic maximisation prob-

[13] See the discussion in section 4.3.2 where I describe a static equilibrium with advanced capital and only one type of labour.

lem:

$$\max_{\boldsymbol{x(t)}} \int_0^\infty e^{-\rho t} \int_0^1 \theta(i) \ln x(i,t)\, di\ dt$$

s.t.

$$\dot{K}^A(t) = r^A(t) \cdot K^A(t) + r(t) \cdot K + w^H(t) \cdot L^H - c(t) \tag{5.77}$$

$$K^A(0) = K_0^A$$

$$K^A(t) \geq 0$$

Only advanced capital $K^A(t)$ is accumulated. Though the stock of traditional capital is fixed at K, and there is no change in the productivity of traditional capital, the return to traditional capital $r(t)$ can change. This is an important feature of the model.

Production

Again, perfectly competitive firms hire the labour and rent the capital of consumers. The task-based production functions for goods are again Cobb-Douglas in $z_1(i,t)$ and $z_2(i,t)$ as in (5.2) but the factor-based production functions for tasks now allow a role for advanced capital to perform $z_1(i,t)$:[14]

$$\begin{aligned} z_1(i,t) &= a^A(i,t)K^A(i,t) + a^H(i)L^H(i,t) \\ z_2(i,t) &= a^K(i)K(i,t) \end{aligned} \tag{5.78}$$

For the moment I assume that there is no technological progress.

[14]See equations (4.7) - (4.9), without low-skilled labour.

Static Equilibrium

Given there is no low-skilled labour in this model, $\bar{i}(t)$ is now the cut-off that demarcates between those goods that are produced by advanced capital with traditional capital and those that are produced by high-skilled labour with traditional capital. I again use a numeraire good $x(\tilde{i},t)$ and again assume that $\tilde{i}=0$. Now the fact that $\tilde{i}\in[0,\bar{i}(t)]$ implies that advanced capital and traditional capital work together to produce $x(0,t)$.[15] To derive equilibrium in the static model in a given period t I define a new term:

$$\gamma(\bar{i}(t)) = \int_0^{\bar{i}} b(i)di \tag{5.79}$$

where $\gamma(\bar{i}(t))$ is the share of total consumer expenditure on the goods produced by advanced capital and traditional capital together. The derivation of this follows from the analysis in Chapter 4. Following the approach in Chapter 4, equilibrium in this three-factor case is as in Figure 5.5, where $\bar{i}(t)$ is determined by the intersection of the $A^H(i,t)$ schedule:

$$A^H(i,t) = \frac{a^H(i,t)}{a^A(i,t)} \tag{5.80}$$

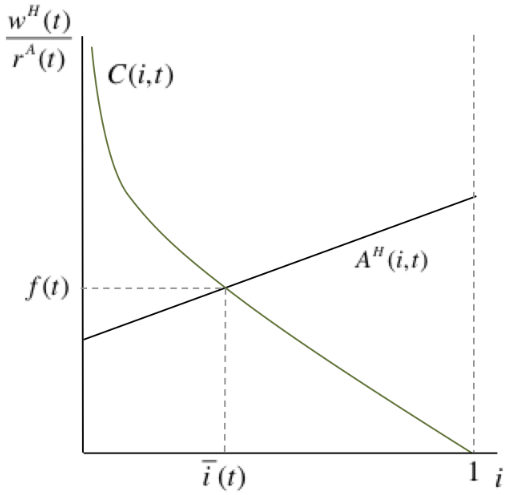
and the $C(i,t)$ schedule:

$$C(i,t) = \frac{1-\gamma(i,t)}{\gamma(i,t)} \cdot \frac{K^A(t)}{L^H} \tag{5.81}$$

[15]In the previous section, with two types of labour and traditional capital, goods $x(i,t)$ where $i\in[0,\bar{i}(t)]$ were produced by low-skilled labour and traditional capital. In this setting, with one type of labour and two types of capital, those goods in that same interval are produced by advanced capital and traditional capital. Assumption 4 again requires that $\bar{i}(t)\neq 0$.

Note that the $C(i,t)$ schedule in (5.81) is expressed differently to the same schedule in Chapter 4 – it is defined in terms of $\gamma(\bar{i}(t))$, the expenditure on goods produced by advanced capital, rather than $\beta(\bar{i}(t))$, the expenditure on goods produced by high-skilled labour. These schedules are the same.[16]

Figure 5.5: Static Equilibrium



Note that given Assumption 2 in Chapter 3 – that expenditure densities on different goods are constant and equal to 1 – it follows that:

$$\begin{aligned}\gamma(\bar{i}(t)) &= \int_0^{\bar{i}(t)} \theta(i)di \\ &= \int_0^{\bar{i}(t)} 1\ di \\ &= \bar{i}(t)\end{aligned} \tag{5.82}$$

The implication of (5.82) is that when preferences are Cobb-Douglas the

[16]See (4.16) for a definition of $\beta(\bar{i}(t))$. Formally, in Chapter 4 with no low-skilled labour $C(i,t) = \frac{\beta(\bar{i}(t))}{1-\beta(\bar{i}(t))} \cdot \frac{K^A(t)}{L^H}$, which is also equal to $\frac{1-\gamma(\bar{i}(t))}{\gamma(\bar{i}(t))} \cdot \frac{K^A(t)}{L^H}$ since $\gamma(\bar{i}(t)) = 1 - \beta(\bar{i}(t))$. As a result, Figure 5.5. is identical to the right-hand side equilibrium in Figure 4.5.

share of total consumer expenditure on goods performed by advanced capital, $\gamma(\bar{i}(t))$, is equal to the cut-off, $\bar{i}(t)$.[17] This is an important relationship that I use throughout the analysis that follows.

Dynamic Equilibrium

The approach to finding the dynamic schedules is the same as in section 5.2.1. Optimisation of the consumer's problem in (5.77) leads to a law of motion for $c(t)$:

$$\frac{\dot{c}(t)}{c(t)} = r^A(t) - \rho \tag{5.83}$$

and a law of motion for $K^A(t)$:

$$\dot{K}^A(t) = Y(t) - c(t) \tag{5.84}$$

Both of these can be expressed in terms of $c(t)$ and $K^A(t)$ alone. The $\frac{\dot{c}(t)}{c(t)}$ schedule in (5.83) can be expressed in terms of $K^A(t)$ by finding $r^A(t)$ in terms of $K^A(t)$. The approach is the same as in section 5.2 with traditional capital where I derived $r(t)$ in terms of $K(t)$. Again it makes use of the numeraire price normalisation.

First I derive the relative factor price between advanced and traditional capital. This follows from the Chapter 4 analysis, such that in any period t:

$$\frac{r^A(t)}{r(t)} = \frac{\psi}{1-\psi} \cdot \frac{\gamma(\bar{i}(t)) \cdot K}{K^A(t)} \tag{5.85}$$

[17] I used this relationship in Chapter 4 to explore the existence and uniqueness of equilibrium and to perform comparative statics. See expressions (4.60) and (4.61).

Secondly I derive the price of the numeraire good $x(0,t)$, produced by advanced capital and traditional capital, which has the same form as in (5.17):

$$p(0,t) = \left[\frac{r^A(t)}{\psi \cdot a^A(0,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(0,t)}\right]^{1-\psi} \tag{5.86}$$

And finally, to derive $r^A(t)$, I now combine these expressions. Substituting the expression for $r(t)$ in terms of $r^A(t)$ from (5.85) into the expression for $p(0,t)$ in (5.86) implies that:

$$p(0,t) = r^A(t) \cdot \left[\frac{1}{\psi \cdot a^A(0,t)}\right]^{\psi} \left[\frac{\frac{1-\psi}{\psi} \cdot \frac{K^A(t)}{\gamma(\bar{i}(t)) \cdot K}}{(1-\psi) \cdot a^K(0,t)}\right]^{1-\psi} \tag{5.87}$$

and given Assumption 4, that $p(0,t) = 1$, (5.87) implies that:

$$1 = r^A(t) \cdot \left[\frac{1}{\psi \cdot a^A(0,t)}\right]^{\psi} \left[\frac{\frac{1-\psi}{\psi} \cdot \frac{K^A(t)}{\gamma(\bar{i}(t)) \cdot K}}{(1-\psi) \cdot a^K(0,t)}\right]^{1-\psi} \tag{5.88}$$

Re-arranging (5.88) it follows that $r^A(t)$ must be equal to:

$$\begin{aligned} r^A(t) &= \left[a^A(0,t)\right]^{\psi} \left[a^K(0,t)\right]^{1-\psi} \cdot [\gamma(\bar{i}(t))]^{1-\psi} \cdot \left[\frac{1}{K^A(t)}\right]^{1-\psi} \cdot K^{1-\psi} \cdot \psi \\ &= [\gamma(\bar{i}(t))]^{1-\psi} \cdot \left[\frac{1}{K^A(t)}\right]^{1-\psi} \cdot D_5 \\ &= [\bar{i}(t)]^{1-\psi} \cdot \left[\frac{1}{K^A(t)}\right]^{1-\psi} \cdot D_5 \end{aligned} \tag{5.89}$$

using the fact from (5.82) that $\gamma(\bar{i}(t)) = \bar{i}(t)$, where D_5 is a positive constant. Note that, unlike the expression for $r(t)$ in (5.21) and (5.54), $\bar{i}(t)$ is no longer included in this constant. This is because the accumulation of advanced capital, unlike the accumulation of traditional capital, does change $\bar{i}(t)$. It can no longer be treated as a constant term. Note also that $r^A(t)$ is decreasing in $K^A(t)$, as in a traditional Ramsey model, but is *also* increasing in $\bar{i}(t)$. This is intuitive – there are diminishing returns to $K^A(t)$ in the production of each type of good and so $r^A(t)$ is decreasing in $K^A(t)$, but as $\bar{i}(t)$ increases a given stock of $K^A(t)$ is spread across the production of a wider range of goods, offsetting those diminishing returns and increasing $r^A(t)$.

$r^A(t)$ in (5.89) can be substituted into the $\frac{\dot{c}(t)}{c(t)}$ schedule in (5.83):

$$\frac{\dot{c}(t)}{c(t)} = [\bar{i}(t)]^{1-\psi} \cdot \left[\frac{1}{K^A(t)}\right]^{1-\psi} \cdot D_5 - \rho \tag{5.90}$$

The expression in (5.90) is therefore the $\frac{\dot{c}(t)}{c(t)}$ schedule in $K^A(t)$–$c(t)$ space.

The $\dot{K}^A(t)$ schedule in (5.84) can also be expressed in terms of $K^A(t)$ and $c(t)$. This requires that $Y(t)$ is expressed in terms of $K^A(t)$. To do this, note that the structure of production in (5.2) and (5.78) implies that advanced capital income in t is equal to:

$$r^A(t) \cdot K^A(t) = \gamma(\bar{i}(t)) \cdot \psi \cdot Y(t) \tag{5.91}$$

and so substituting in $r^A(t)$ from (5.89) it follows that:

$$\begin{aligned} Y(t) &= r^A(t) \cdot K^A(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi} \\ &= [\bar{i}(t)]^{-\psi} \cdot \left[K^A(t)\right]^{\psi} \cdot D_6 \end{aligned} \tag{5.92}$$

where D_6 is a positive constant. Substituting this expression for $Y(t)$ back into the expression for $\dot{K}^A(t)$ in (5.84) implies that:

$$\dot{K}^A(t) = [\bar{i}(t)]^{-\psi} \cdot \left[K^A(t)\right]^{\psi} \cdot D_6 - c(t) \tag{5.93}$$

This is the $\dot{K}^A(t)$ schedule in $K^A(t)$–$c(t)$ space. Given the $\frac{\dot{c}(t)}{c(t)}$ schedule in (5.90) and the $\dot{K}^A(t)$ schedule in (5.93) it is now possible to describe the $\frac{\dot{c}(t)}{c(t)} = 0$ and $\dot{K}^A(t) = 0$ schedules in $K^A(t)$–$c(t)$ space. Given (5.90) the $\frac{\dot{c}(t)}{c(t)} = 0$ schedule is:

$$[\bar{i}(t)]^{1-\psi} \cdot \left[\frac{1}{K^A(t)}\right]^{1-\psi} \cdot D_5 = \rho \tag{5.94}$$

And given (5.93) the $\dot{K}^A(t) = 0$ schedule is:

$$c(t) = [\bar{i}(t)]^{-\psi} \cdot \left[K^A(t)\right]^{\psi} \cdot D_6 \tag{5.95}$$

A critical observation now follows. Unlike the case with traditional capital, the schedules in (5.94) and (5.95) are no longer necessarily stationary over time. This is because as the level of $K^A(t)$ changes, this will also change the level of $\bar{i}(t)$ which appears in both expressions. As a result, if (5.94) and (5.95) are placed in $K^A(t)$–$c(t)$ space they will shift as $K^A(t)$ changes and $\bar{i}(t)$ changes. There is no longer necessarily a steady-state in $K^A(t)$–$c(t)$ space. In order to resolve this, I now define a new 'effective' variable, $\hat{\hat{v}}$:

$$\hat{\hat{v}} = \frac{v}{\bar{i}(t)} \tag{5.96}$$

This is similar to the way in which variables are redefined in 'effective' terms in a traditional Ramsey model with technological progress. This allows me to find steady-state. I do this by re-expressing the schedules in (5.94) and (5.95) in $\hat{\hat{K}}^A(t)$–$c(t)$ space rather than in $K^A(t)$–$c(t)$ space.[18]

This is straightforward for the $\frac{\dot{c}(t)}{\hat{c}(t)} = 0$ schedule. Using the new ‘effective’ definition in (5.96), it follows that the original $\frac{\dot{c}(t)}{\hat{c}(t)}$ schedule in (5.90) can be re-expressed in $\hat{\hat{K}}^A(t)$–$c(t)$ space:

$$\frac{\dot{c}(t)}{c(t)} = \left[\hat{\hat{K}}^A(t)\right]^{\psi-1} \cdot D_5 - \rho \tag{5.97}$$

and a new corresponding $\frac{\dot{c}(t)}{\hat{c}(t)} = 0$ schedule follows:

$$\left[\hat{\hat{K}}^A(t)\right]^{\psi-1} \cdot D_5 = \rho \tag{5.98}$$

The $\dot{\hat{\hat{K}}}^A(t)$ schedule in $\hat{\hat{K}}^A(t)$–$c(t)$ space is more subtle to derive. Given the original expression for the $\dot{K}^A(t)$ schedule in (5.93), it follows from the new ‘effective’ definition in (5.96) that the law of motion of $\hat{\hat{K}}^A(t)$ is:

$$\begin{aligned} \frac{\dot{\hat{\hat{K}}}^A(t)}{\hat{\hat{K}}^A(t)} &= \frac{\dot{K}^A(t)}{K^A(t)} - \frac{\dot{\bar{i}}(t)}{\bar{i}(t)} \\ &= \frac{Y(t) - c(t)}{K^A(t)} - g^{\gamma}(t) \end{aligned} \tag{5.99}$$

where:

$$\frac{\dot{\bar{i}}(t)}{\bar{i}(t)} = g^{\gamma}(t) \tag{5.100}$$

[18]This reflects the fact that $r^A(t)$ depends both upon $K^A(t)$ and $\bar{i}(t)$ in (5.89). This is intuitive, $r^A(t)$ is decreasing in the stock of $K^A(t)$ since there are diminishing returns to the production of any given good, but increasing in the range of goods $\bar{i}(t)$ that $K^A(t)$ is used to produce.

I call this $g^{\gamma}(t)$ since (5.82) implies that $\gamma(\bar{i}(t)) = \bar{i}(t)$. It therefore follows from (5.99) that $\dot{\hat{\hat{K}}}^A(t)$ is equal to:

$$\begin{aligned}
\dot{\hat{\hat{K}}}^A(t) &= \frac{\hat{\hat{K}}^A(t)}{K^A(t)} \cdot [Y(t) - c(t)] - g^{\gamma}(t) \cdot \hat{\hat{K}}^A(t) \\
&= \frac{1}{\bar{i}(t)} \cdot [Y(t) - c(t)] - g^{\gamma}(t) \cdot \hat{\hat{K}}^A(t)
\end{aligned} \tag{5.101}$$

(5.92) shows that $Y(t)$ can also be expressed in terms of $\hat{\hat{K}}^A(t)$ as:

$$Y(t) = \hat{\hat{K}}^A(t)^{\psi} \cdot D_6 \tag{5.102}$$

And so substituting the expression for $Y(t)$ from (5.102) into (5.101) implies:

$$\dot{\hat{\hat{K}}}^A(t) = \frac{1}{\bar{i}(t)} \cdot \left[\hat{\hat{K}}^A(t)^{\psi} \cdot D_6 - c(t)\right] - g^{\gamma}(t) \cdot \hat{\hat{K}}^A(t) \tag{5.103}$$

which can be re-arranged as:

$$c(t) = \hat{\hat{K}}^A(t)^{\psi} \cdot D_6 - \bar{i}(t) \cdot g^{\gamma}(t) \cdot \hat{\hat{K}}^A(t) - \bar{i}(t) \cdot \dot{\hat{\hat{K}}}^A(t) \tag{5.104}$$

(5.104) is a complex expression for the $\dot{\hat{\hat{K}}}^A(t)$ schedule. However it can be simplified further. Noting again from (5.82) that $\gamma(\bar{i}(t)) = \bar{i}(t)$ it follows that $g^{\gamma}(t)$ in (5.104) can be re-expressed as:

$$\begin{aligned}
g^{\gamma}(t) &= \frac{\dot{\bar{i}}(t)}{\bar{i}(t)} \\
&= \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \dot{\hat{\hat{K}}}^A(t) \cdot \frac{1}{\bar{i}(t)}
\end{aligned} \tag{5.105}$$

And so using (5.105) it follows that (5.104) can be re-written as:

$$
\begin{aligned}
c(t) &= \hat{\hat{K}}^A(t)^{\psi} \cdot D_6 - \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \dot{\hat{\hat{K}}}^A(t) \cdot \hat{\hat{K}}^A(t) - \bar{i}(t) \cdot \dot{\hat{\hat{K}}}^A(t) \\
&= \hat{\hat{K}}^A(t)^{\psi} \cdot D_6 - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \hat{\hat{K}}^A(t) + \bar{i}(t) \right] \cdot \dot{\hat{\hat{K}}}^A(t)
\end{aligned}
\tag{5.106}
$$

The $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule now follows. If $\dot{\hat{\hat{K}}}^A(t) = 0$ then (5.106) collapses to:

$$
c(t) = \hat{\hat{K}}^A(t)^{\psi} \cdot D_6 \tag{5.107}
$$

This $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule, like the $\frac{\dot{c}(t)}{c(t)} = 0$ schedule before, is now expressed in terms of $\hat{\hat{K}}^A(t)$ and $c(t)$ alone. As a result both schedules can be shown in $\hat{\hat{K}}^A(t)$–$c(t)$ space. Their intersection is the new steady-state. This is shown in Figure 5.6. The $\frac{\dot{c}(t)}{c(t)} = 0$ schedule is vertical at the level of $\hat{\hat{K}}^A(t)$ such that (5.98) holds. It follows that this is equal to:

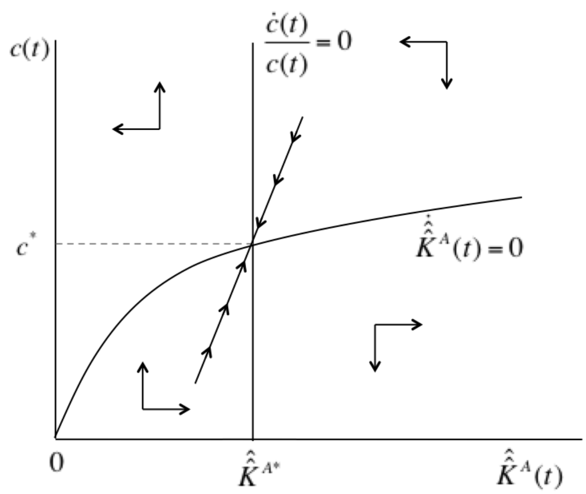
$$
\hat{\hat{K}}^{A*} = \left[\rho \cdot \frac{1}{D_5} \right]^{\frac{1}{\psi - 1}} \tag{5.108}
$$

(5.105) implies that if $\dot{\hat{\hat{K}}}^A(t) = 0$ then $g^{\gamma}(t) = 0$. This implies that $\bar{i}(t)$ is constant in steady-state and, given that $\hat{\hat{K}}^A(t)$ is constant, $K^A(t)$ must also be constant as well. (5.98) implies that this steady-state level of $\hat{\hat{K}}^{A*}$ results in:

$$
r^A(t) = r^{A*} = \rho \tag{5.109}
$$

As in section 5.2.1, the requirement that $r^{A*} = \rho$ pins down a unique steady-state of effective advanced capital $\hat{\hat{K}}^{A*}$. The shape of the $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule follows from (5.107) – since $\psi \in (0, 1)$ this implies that the schedule is concave, the $c(t)$ required to ensure that $\dot{\hat{\hat{K}}}^A(t) = 0$ is increasing at decreasing rate in $\hat{\hat{K}}^A(t)$.

Figure 5.6: Dynamic Equilibrium



The transition dynamics follow from (5.98) and (5.107).[19] (5.98) implies that if $\hat{\hat{K}}^A(t) < \hat{\hat{K}}^*$ then $r^A(t) > \rho$ and $c(t)$ will be increasing. If $\hat{\hat{K}}^A(t) > \hat{\hat{K}}^{A*}$ then $r(t) < \rho$ and $c(t)$ will be decreasing. (5.107) implies that, for a given $\hat{\hat{K}}^A(t)$, if $c(t)$ is below the value of $c(t)$ on the $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule then $\hat{\hat{K}}^A(t)$ will be increasing and vice-versa if $c(t)$ is above. When $\hat{\hat{K}}^A(t)$ is increasing, this is because $K^A(t)$ is increasing at a faster rate than $\bar{i}(t)$ – rather than $K^A(t)$ decreasing at a lower rate than $\bar{i}(t)$.[20]

[19]The formal proof for the existence of the saddle-path in Figure 5.6 is very similar to that in section 5.2.1. I do not repeat it here.

[20]This is intuitive, but to see this formally note that in the Appendix, section 9.1.4, equation (9.39), it is established that, with $g = 0$, $g^{\gamma}(t) = \phi(t) \cdot g^{K^A}(t)$ where $\phi(t) \in [0, 1]$ – this implies that the absolute growth rate in the cut-off is always less than the absolute

Orthogonality

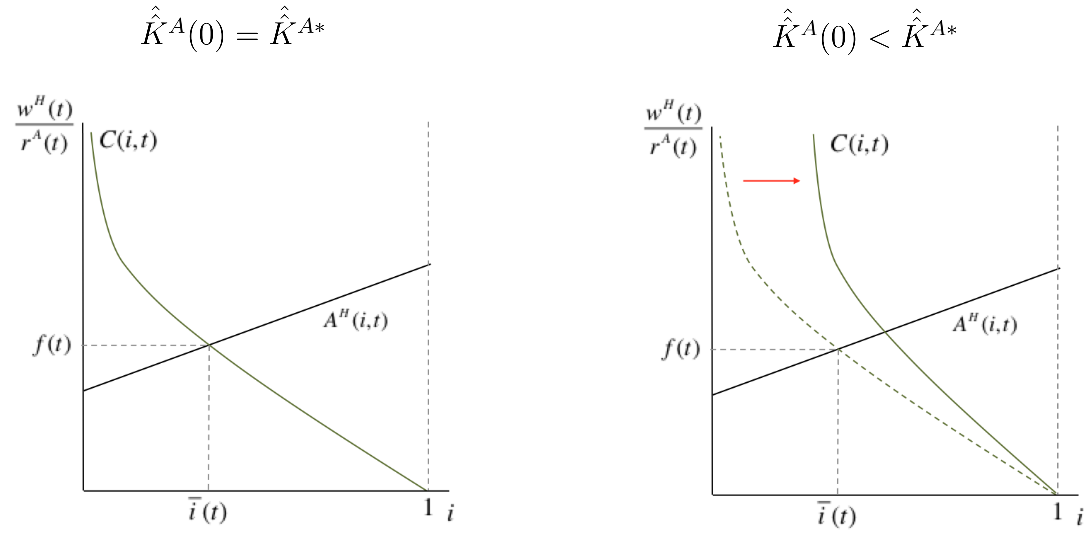
With advanced capital it is no longer possible to analyse the dynamic system in Figure 5.6 orthogonally to the static system in Figure 5.5 as it was in the case with traditional capital. This is for two reasons. First, the accumulation of advanced capital during transition to steady-state does affect the position of the $C(i,t)$ schedule in Figure 5.5. Secondly an increase in the productivity of advanced capital does affect the position of the $A^H(i,t)$ schedule in Figure 5.5. This is familiar from Chapter 4. As a result, unlike the case with traditional capital, there are changes in $f(t)$ and $\bar{i}(t)$ in the static equilibrium. Nevertheless, by using the new 'effective' term in (5.96) the dynamic system remains tractable, despite this loss of orthogonality.

I will consider how changes in relative productivity of advanced capital undermines the orthogonality of the dynamic and static models in a later sub-section. First, however, to see the loss of orthogonality that follows from the accumulation of advanced capital consider two possible initial stocks – one where $\hat{\hat{K}}^A(0)$ begins at the steady-state level $\hat{\hat{K}}^{A*}$ and one where $\hat{\hat{K}}^A(0)$ begins below that steady-state level. The consequence of this for the static model is shown in Figure 5.7. In the former case, there is no change to the static equilibrium – the level of $K^A(t)$ and $\bar{i}(t)$ are constant in steady-state, as noted before. However, in the latter case, this no longer holds. $\hat{\hat{K}}^A(t)$ increases during transition to steady-state – this is implied by Figure 5.6. As a result $K^A(t)$ also rises and causes an outward shift in the $C(i,t)$ schedule, and a rise in $\bar{i}(t)$. Eventually, when $\hat{\hat{K}}^A(t)$ reaches steady-state $K^A(t)$ stops

growth rate in the stock of advanced capital.

increasing, the $C(i,t)$ schedule stops shifting, and $\bar{i}(t)$ stops increasing.

Figure 5.7: Static Equilibrium



Comparative Dynamics: Factor Prices and Cut-Off

The relative factor prices in this model are:

$$\begin{aligned}\frac{r^A(t)}{r(t)} &= \frac{\psi}{1-\psi} \cdot \frac{\gamma(\bar{i}(t)) \cdot K}{K^A(t)} \\ \frac{w^H(t)}{r^A(t)} &= \frac{1-\gamma(\bar{i}(t))}{\gamma(\bar{i}(t))} \cdot \frac{K^A(t)}{L^H}\end{aligned} \tag{5.110}$$

In steady-state the $A^H(i,t)$ and $C(i,t)$ schedules do not shift. Over time there is no change in the relative productivity of advanced capital so the former does not shift, and no change in the factor stocks or the cut-off so the latter does not shift. As a result in steady-state the relative factor prices in (5.110) do not change.

However, during transition to steady-state these relative factor prices do change. This is because during transition the $C(i,t)$ schedule shifts. To see this consider the case on the right-hand side of Figure 5.7 where $\hat{\hat{K}}^A(0) < \hat{\hat{K}}^{A*}$. During transition both $K^A(t)$ and $\bar{i}(t)$ rise. (5.110) implies that this causes the return on advanced capital relative to traditional capital to fall.[21] Figure 5.7 shows that the wage relative to the return on advanced capital rises.

Comparative Dynamics: One-off Increase in the Productivity of Advanced Capital

Suppose there is a one-off uniform increase in the relative productivity of advanced capital, such that $a^A(i,t)$ increases $\forall i$. The increase in $a^A(i,t)$ $\forall i$ implies an increase in $a^A(0,t)$ in particular, and so an increase in D_5 in (5.89). (5.98) implies that this shifts the $\frac{\dot{c}(t)}{c(t)} = 0$ schedule right. This is intuitive – a rise in the productivity of advanced capital, and a rise in D_5 implies that $r^A(t)$ increases. This follows from (5.89). But in order for $\frac{\dot{c}(t)}{c(t)} = 0$ it must be the case that $r^{A*} = \rho$. As a result, since there are diminishing returns to $\hat{\hat{K}}^A(t)$, this requires a higher level of $\hat{\hat{K}}^A(t)$ to drive $r^A(t)$ back to r^{A*}. The increase in $a^A(0,t)$ also causes D_6 to increase in (5.95). As a result (5.107) implies that the $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule pivots upwards. The result in the dynamic model on the left-hand side of Figure 5.8 is a new steady state where both c^* and $\hat{\hat{K}}^{A*}$ are higher. In the static model on the right-hand side of Figure 5.8 the increase in the productivity of advanced capital causes the $A^A(i,t)$ schedule to fall, driving up $\bar{i}(t)$. But in addition, during transition to the new steady-state, $K^A(t)$ is accumulated – as a result, the $C(i,t)$ schedule shifts right, and $\bar{i}(t)$ increases further.

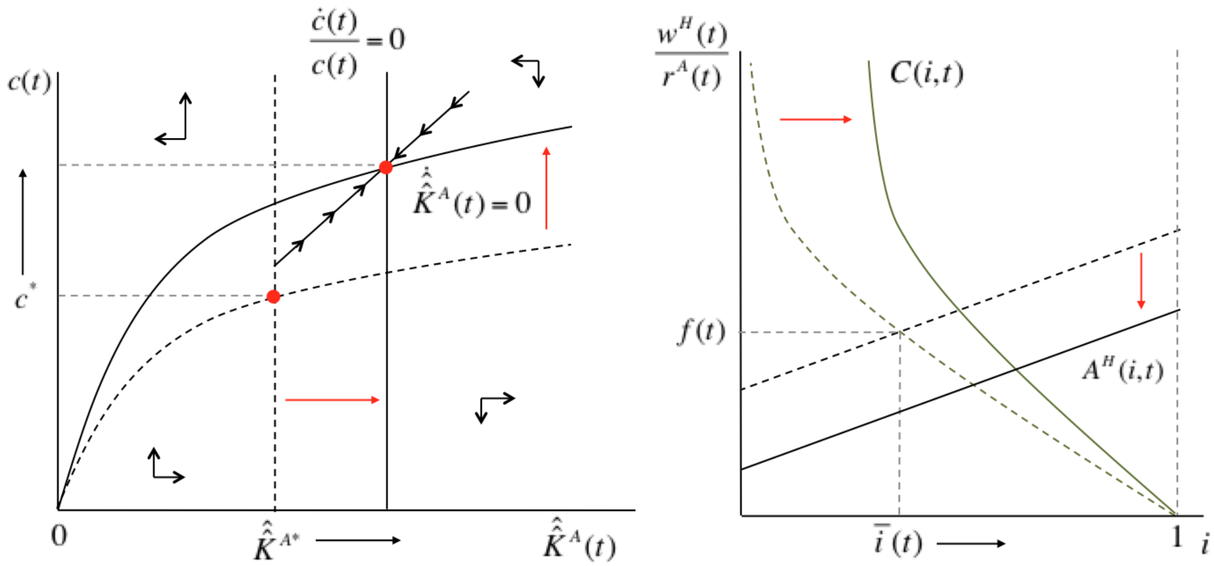
[21] Since the ratio of these returns in (5.110) is inversely proportional to $\hat{K}^A(t)$.

Figure 5.8 shows clearly again how the orthogonality of the dynamic model and the static model no longer holds. The increase in the relative productivity of advanced capital that causes the shift in the schedules in the dynamic model also causes the $A^H(i,t)$ schedule to shift in the static model. And the accumulation of advanced capital that follows in the dynamic model also causes the $C(i,t)$ schedule to shift in the static model.[22]

However, note from Figure 5.8 that the effect of a one-off increase in the relative productivity of advanced capital on the relative wage $f(t)$ is analytically ambiguous. This is clear in the right-hand side diagram – the increase in the productivity of advanced capital causes the $A^H(i,t)$ schedule to fall and this *decreases* the relative wage, but the accumulation of advanced capital causes the $C(i,t)$ schedule to shift which *increases* the relative wage. The first effect was explored in Chapter 4. The second effect is the consequence of the endogenous accumulation of advanced capital following the one-off increase in the the productivity of advanced capital. Without solving the model computationally, it is not possible to resolve this ambiguity in the case with a one-off increase in the relative productivity of advanced capital. However, in the next section where there is continuous growth in the productivity of advanced capital the ambiguity is resolved in the limit. The accumulation of advanced capital cannot lead to optimism. This is an important result.

[22]Note again that $c(0)$ rises after the one-off shock. The proof is the same as for Figure 5.2. See the Appendix, section 9.1.2.

Figure 5.8: New Equilibrium



Comparative Dynamics: Relative Goods Prices

It is again possible to describe how relative prices of the different goods change. As in (5.38) the price of each good $x(i,t)$ is:

$$p(i,t) = \left[\frac{F(i,t)}{\psi \cdot a^F(i,t)}\right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(i,t)}\right]^{1-\psi} \tag{5.111}$$

where $F(i,t)$ is the factor price for the particular factor that works with traditional capital to produce good i and $a^F(i)$ is that factor's productivity, i.e.:

$$F(i,t) = \begin{cases} r^A(t) & \text{if } i \in [0, \bar{i}(t)] \\ w^H(t) & \text{if } i \in [\bar{i}(t), 1] \end{cases}$$

And:

$$a^F(i,t) = \begin{cases} a^A(i,t) & \text{if } i \in [0, \bar{i}(t)] \\ a^H(i,t) & \text{if } i \in [\bar{i}(t), 1] \end{cases}$$

This reflects the fact that if $i \in [0, \bar{i}]$ then advanced capital works with traditional capital to produce $x(i,t)$ and if $i \in [\bar{i}, 1]$ then high-skilled labour works with traditional capital to produce $x(i,t)$. It follows again that the rate of change in the price of any two goods $\tilde{i}$ and $\tilde{\tilde{i}}$ is:

$$\frac{\dot{p}(\tilde{i},t)}{p(\tilde{i},t)} - \frac{\dot{p}(\tilde{\tilde{i}},t)}{p(\tilde{\tilde{i}},t)} =$$
$$\psi \cdot \left[\frac{\dot{F}(\tilde{i},t)}{F(\tilde{i},t)} - \frac{\dot{F}(\tilde{\tilde{i}},t)}{F(\tilde{\tilde{i}},t)}\right] + \psi \cdot \left[\frac{\dot{a}^F(\tilde{\tilde{i}},t)}{a^F(\tilde{\tilde{i}},t)} - \frac{\dot{a}^F(\tilde{i},t)}{a^F(\tilde{i},t)}\right] + (1-\psi) \cdot \left[\frac{\dot{a}^K(\tilde{\tilde{i}},t)}{a^K(\tilde{\tilde{i}},t)} - \frac{\dot{a}^K(\tilde{i},t)}{a^K(\tilde{i},t)}\right] \tag{5.112}$$

Given there is no technological progress (5.112) collapses to:

$$\frac{\dot{p}(\tilde{i},t)}{p(\tilde{i},t)} - \frac{\dot{p}(\tilde{\tilde{i}},t)}{p(\tilde{\tilde{i}},t)} = \psi \cdot \left[\frac{\dot{F}(\tilde{i},t)}{F(\tilde{i},t)} - \frac{\dot{F}(\tilde{\tilde{i}},t)}{F(\tilde{\tilde{i}},t)}\right] \tag{5.113}$$

In steady-state the expression in (5.113) is equal to zero. As a result, relative prices do not change across the goods in steady-state. This is intuitive, and for the same reason that relative goods prices do not change in the setting with traditional capital and no technological progress. There is no change in the relative factor productivities, and there is no change in the relative price of high-skilled labour with respect to advanced capital.

However, during transition, relative goods prices do change. This is because the relative factor price $\frac{w^H(t)}{r^A(t)}$ does change during transition, even though there is no change in the relative factor productivities of those factors. To see this, consider again, for example, the case that $\hat{\hat{K}}^A(0,t) < \hat{\hat{K}}^{A*}$. $K^A(t)$ increases during transition and $\bar{i}(t)$ rises. In period 0 there are three possible pairwise comparisons of goods:

1. $\tilde{i}, \tilde{\tilde{i}} \in [0, \bar{i}(0)]$
2. $\tilde{i}, \tilde{\tilde{i}} \in [\bar{i}(0), 1]$
3. $\tilde{i} \in [0, \bar{i}(0)]$ and $\tilde{\tilde{i}} \in [\bar{i}(0), 1]$

The consequence of this transition for each pairwise comparison is intuitive. Consider the first case where $\tilde{i}, \tilde{\tilde{i}} \in [0, \bar{i}(0)]$ – this implies that both goods are initially produced by advanced capital and traditional capital working together. As $K^A(t)$ increases and $\bar{i}(t)$ rises, these goods will continue to be produced by these factors. Since $r^A(t)$ is the same for the production of any good the first term in (5.112) is zero. Since there is no technological progress in either factor the second two terms are also zero. Relative prices do not change.

Consider the second case where $\tilde{i}, \tilde{\tilde{i}} \in [\bar{i}(0), 1]$ – this implies that both goods are initially produced by high-skilled labour and traditional capital working together. Since $\bar{i}(t)$ rises during transition, it follows that a good to the right of the $\bar{i}(0)$ cut-off, but close to it, may switch to being produced by advanced capital and traditional capital instead. If the good near the cut-off remains produced by high-skilled labour during transition there is no change in relative prices – this is for the same reason as the first case. But if the good does switch from being produced by high-skilled labour to instead being produced by advanced capital, then the relative price of the good that remains produced by high-skilled labour will rise relative to the good that is now produced by advanced capital.[23] This is implied by (5.113) – the wage relative to the return on advanced capital rises during transition, as discussed before, and so the relative factor price of the labour that produces this good increases. If the second good also falls below the cut-off, then relative prices stop changing during transition – both goods are now produced by advanced capital, and $r^A(t)$ is the same across all goods.

Consider the third case where $\tilde{i} \in [0, \bar{i}(0)]$ and $\tilde{\tilde{i}} \in [\bar{i}(0), 1]$ – one good is produced by advanced capital and the other by high-skilled labour. The logic is the same as the previous two cases. During transition, if the goods remain produced by the two different factors then relative prices change. But if the good produced by high-skilled labour becomes produced by advanced capital instead, relative prices become constant.

This analysis leads to Proposition 5.

PROPOSITION 5: *When the static model with advanced capital is placed in a dynamic setting with endogenous advanced capital accumulation but no technological progress there is a unique steady-state where $r^* = \rho$, $c(t) = c^*$, $\bar{i}(t) = \bar{i}^*$, and $K^A(t) = K^{A*}$. The cut-off $\bar{i}(t)$ increases during transition. Relative goods prices can change during transition.*

[23] There is no 'jump' in the price of the good as it switches from being produced by high-skilled labour to being produced by advanced capital since, at the cut-off $\bar{i}(t)$, the firm is indifferent between hiring either of the factors, and so the price of the good is the same whichever factor is chosen. See expression (4.11) in Chapter 4.

5.3.2. With Technological Progress

In the model set out before there is no technological progress. I now assume that there is exogenous growth in the productivity of advanced capital:

$$a^A(i,t) = a^A(i,0) \cdot e^{gt} \; \forall i \tag{5.114}$$

It is important to note that this process of continuous growth in productivity does not correspond to the one-off productivity changes in the previous section. There I considered a uniform increase in the *relative* productivity of advanced capital.[24] This is what is shown in Figure 5.8. However, the process of technological change in (5.114) is a uniform rate of increase in the *absolute* productivity of advanced capital. This is a more appropriate set-up, given that the process of technological change is now continuous.[25]

It is clear from the analysis in the previous section that, in $\hat{\hat{K}}^A(t)$–$c(t)$ space, this continuous increase in the absolute productivity of advanced capital will cause a continuous shift in the $\frac{\dot{c}(t)}{c(t)} = 0$ and $\dot{\hat{\hat{K}}}^A(t) = 0$ schedules in the directions set out in Figure 5.8, rather than the one-off shifts I consid-

[24] Recall that $A^H(i,t) = \frac{a^H(i,t)}{a^A(i,t)}$.

[25] A simple uniform decrease in the *relative* productivity schedule is tractable and intuitive for performing basic comparative statics and dynamics, but no longer works in a setting where there is continuous productivity growth. If the former process took place continuously, for example, it would cause continuous downward shifts in the $A^H(i,t)$ schedule that would eventually result in a relative productivity schedule that is entirely negative. This is deeply unintuitive. The process in (5.114) still causes the $A^H(i,t)$ schedule to shift downwards, but rather than fall below the y-axis, the schedule instead flattens over time and, in the limit, lies directly on-top of the y-axis. In the limit, the relative productivity of labour to advanced capital across all types of tasks is zero. Importantly, this process still ensures that the $A^H(i,t)$ schedule is always weakly positively sloped, reflecting the comparative advantage assumptions discussed in Chapter 4, section 4.2.2. This is shown in the Appendix, section 9.1.3.

ered there. As a result, steady-state will no longer exist in $\hat{\hat{K}}^A(t)$–$c(t)$ space. In order to find the dynamic equilibrium in this setting, I proceed in two stages. First, as in section 5.2.2, I again re-define all variables in the model in 'effective' terms with respect to the technology variable. Secondly, as in section 5.3.1, I again take account of the fact that $\bar{i}(t)$ also changes. First, then, I will consider the technology variable.

In section 5.2.2 with traditional-capital augmenting technological progress, I re-expressed that growth process as a process of labour-augmenting technological progress – and variables were 'effective' with respect to the new technology variable that this transformation implied, as in (5.48). Now I instead re-express the process of advanced-capital augmenting technological change as a process of traditional-capital augmenting change and use the same approach. Consider the production function for the good $x(0,t)$ can be re-written as:

$$\begin{aligned} x(0,t) &= \left[a^A(0,t) \cdot K^A(0,t)\right]^{\psi} \left[a^K(0,t) \cdot K(0,t)\right]^{1-\psi} \\ &= \left[K^A(0,t)\right]^{\psi} \left[a^A(0,t)^{\frac{\psi}{1-\psi}} \cdot a^K(0,t) \cdot K(0,t)\right]^{1-\psi} \end{aligned} \tag{5.115}$$

(5.115) implies that a traditional-capital augmenting process of technological change in $a^A(0,t)^{\frac{\psi}{1-\psi}}$ is identical to the advanced-capital augmenting process of technological change in $a^A(0,t)$ that I am interested in. Given this I first define all variables in model in 'effective' terms with respect to this new factor. For simplicity I re-use '$\hat{..}$' to express this 'effective' variable, though unlike (5.48) $\hat{v}$ is now:

$$\hat{v} = \frac{v}{a^A(0,t)^{\frac{\psi}{1-\psi}}} \tag{5.116}$$

The approach is the same as in section 5.2.2. Optimisation leads to a law of motion for $\hat{c}(t)$:

$$\begin{aligned}\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= r^A(t) - \rho - \frac{\psi}{1-\psi} \cdot \frac{\dot{a}^A(0,t)}{a^A(0,t)} \\ &= r^A(t) - \rho - \frac{\psi}{1-\psi} \cdot g\end{aligned} \tag{5.117}$$

and a law of motion for $\hat{K}^A(t)$:

$$\begin{aligned}\frac{\dot{\hat{K}}^A(t)}{\hat{K}^A(t)} &= \frac{\dot{K}^A(t)}{K^A(t)} - \frac{\psi}{1-\psi} \cdot \frac{\dot{a}^A(0,t)}{a^A(0,t)} \\ &= \frac{Y(t) - c(t)}{K^A(t)} - \frac{\psi}{1-\psi} \cdot g \\ &= \frac{\hat{Y}(t) - \hat{c}(t)}{\hat{K}^A(t)} - \frac{\psi}{1-\psi} \cdot g\end{aligned} \tag{5.118}$$

Both of these can be expressed in terms of $\hat{c}(t)$ and $\hat{K}^A(t)$ alone. The former requires an expression for $r(t)$ in terms of $\hat{K}^A(t)$. The latter requires an expression for $\hat{Y}(t)$ in terms of $\hat{K}^A(t)$. The expression for $r^A(t)$ follows from the one derived in (5.89):

$$\begin{aligned}r^A(t) &= \left[a^A(0,t)\right]^{\psi} \left[a^K(0,t)\right]^{1-\psi} \cdot \left[\frac{\gamma(\bar{i}(t))}{K^A(t)}\right]^{1-\psi} \cdot K^{1-\psi} \cdot \psi \\ &= \left[\bar{i}(t) \cdot \frac{\left[a^A(0,t)\right]^{\frac{\psi}{1-\psi}}}{K^A(t)}\right]^{1-\psi} \cdot \left[a^K(0,t)\right]^{1-\psi} \cdot K^{1-\psi} \cdot \psi \\ &= [\bar{i}(t)]^{1-\psi} \cdot \left[\hat{K}^A(t)\right]^{\psi-1} \cdot D_7\end{aligned} \tag{5.119}$$

where D_7 is a positive constant. This differs from the expression for $r^A(t)$ in (5.89) in that the $a^A(0,t)$ term is not contained in the positive constant. This is because now, with a continuous process of technological change in advanced capital, $a^A(0,t)$ changes over time.

From the expression for $Y(t)$ in (5.92) it follows that:

$$\hat{Y}(t) = r^A(t) \cdot \hat{K}^A(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi} \tag{5.120}$$

and so substituting the expression for $r^A(t)$ in (5.119) into the expression for $\hat{Y}(t)$ in (5.120) means that $\hat{Y}(t)$ can be re-expressed in terms of $\hat{K}^A(t)$:

$$\begin{aligned} \hat{Y}(t) &= r^A(t) \cdot \hat{K}^A(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi} \\ &= \left[\bar{i}(t) \cdot \frac{1}{\hat{K}^A(t)}\right]^{1-\psi} \cdot D_7 \cdot \hat{K}^A(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi} \\ &= [\bar{i}(t)]^{-\psi} \cdot \left[\hat{K}^A(t)\right]^{\psi} \cdot D_8 \end{aligned} \tag{5.121}$$

where D_8 is a positive constant. It follows from (5.117) and (5.119) that the law of motion for $\hat{c}(t)$ is:

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = [\bar{i}(t)]^{1-\psi} \cdot \left[\hat{K}^A(t)\right]^{\psi-1} \cdot D_7 - \rho - \frac{\psi}{1-\psi} \cdot g \tag{5.122}$$

It also follows from (5.118) and (5.121) that the law of motion for $\hat{K}^A(t)$ is:

$$\dot{\hat{K}}^A(t) = [\bar{i}(t)]^{-\psi} \cdot \left[\hat{K}^A(t)\right]^{\psi} \cdot D_8 - \hat{c}(t) - \frac{\psi}{1-\psi} \cdot g \cdot \hat{K}^A(t) \tag{5.123}$$

Given the law of motion for $\hat{c}(t)$ in (5.122) and $\hat{K}^A(t)$ in (5.123) it is now possible to describe the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ and $\dot{\hat{K}}^A(t) = 0$ schedules in $\hat{K}^A(t)$–$\hat{c}(t)$ space. Given (5.122) the former is:

$$[\bar{i}(t)]^{1-\psi} \cdot \left[\hat{K}^A(t)\right]^{\psi-1} \cdot D_7 = \rho + \frac{\psi}{1-\psi} \cdot g \tag{5.124}$$

and given (5.123) the latter is:

$$\hat{c}(t) = [\bar{i}(t)]^{-\psi} \cdot \left[\hat{K}^A(t)\right]^{\psi} \cdot D_8 - \frac{\psi}{1-\psi} \cdot g \cdot \hat{K}^A(t) \tag{5.125}$$

Again, as in the advanced capital case with no technological progress, the schedules in (5.124) and (5.125) are no longer stationary over time. Again, this is because as the level of $\hat{K}^A(t)$ changes this will also change the level of $\bar{i}(t)$.[26] Even accounting for technological progress, if (5.124) and (5.125) are placed in $\hat{K}^A(t)$–$\hat{c}(t)$ space they may shift as $\hat{K}^A(t)$ changes and $\bar{i}(t)$ changes – there is no steady-state in $\hat{K}^A(t)$–$\hat{c}(t)$ space either. In order to resolve this I again define a further, new 'effective' variable $\hat{\hat{\hat{v}}}$:

$$\hat{\hat{\hat{v}}} = \frac{v}{a^A(0,t)^{\frac{\psi}{1-\psi}} \cdot \bar{i}(t)} \tag{5.126}$$

which captures *both* technological progress *and* the changing cut-off. I re-express the schedules in (5.122) and (5.123) in such a way that I can place them in $\hat{\hat{\hat{K}}}^A(t)$–$\hat{c}(t)$ space rather than $\hat{K}^A(t)$–$\hat{c}(t)$ space. This approach combines the approach to the process of technological change in section 5.2.2. with the approach to a changing cut-off $\bar{i}(t)$ in section 5.3.1.

The law of motion for $\hat{c}(t)$ in (5.122) implies a $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule in $\hat{\hat{\hat{K}}}^A(t)$–

[26]As in Chapter 4, a change in either $K^A(t)$ or $a^A(0,t)$ must also change $\bar{i}(t)$.

$\hat{c}(t)$ space:

$$\left[\hat{\hat{\hat{K}}}^A(t)\right]^{\psi-1} \cdot D_7 = \rho + \frac{\psi}{1-\psi} \cdot g \tag{5.127}$$

To find the $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule in $\hat{\hat{\hat{K}}}^A(t)$–$\hat{c}(t)$ space, note that:

$$\begin{aligned} \frac{\dot{\hat{\hat{\hat{K}}}}^A(t)}{\hat{\hat{\hat{K}}}^A(t)} &= \frac{\dot{K}^A(t)}{K^A(t)} - \left[g^\gamma(t) + \frac{\psi}{1-\psi} \cdot g\right] \\ &= \frac{\hat{Y}(t) - \hat{c}(t)}{\hat{K}^A(t)} - \left[g^\gamma(t) + \frac{\psi}{1-\psi} \cdot g\right] \end{aligned} \tag{5.128}$$

From the expression for $\hat{Y}(t)$ in (5.121) it follows that $Y(t)$ can be expressed in terms of $\hat{\hat{\hat{K}}}^A(t)$:

$$\hat{Y}(t) = \left[\hat{\hat{\hat{K}}}^A(t)\right]^{\psi} \cdot D_8 \tag{5.129}$$

and substituting this expression into (5.128) it follows that:

$$\dot{\hat{\hat{\hat{K}}}}^A(t) = \frac{1}{\bar{i}(t)} \cdot \left[\hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \hat{c}(t)\right] - \left[g^\gamma(t) + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{\hat{K}}}^A(t) \tag{5.130}$$

which can be re-arranged as:

$$\frac{1}{\bar{i}(t)} \cdot \hat{c}(t) = \frac{1}{\bar{i}(t)} \cdot \hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \left[g^\gamma(t) + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{\hat{K}}}^A(t) - \dot{\hat{\hat{\hat{K}}}}^A(t) \tag{5.131}$$

Note that now $g^\gamma(t)$ is equivalent to:

$$g^\gamma(t) = \frac{\partial \bar{i}(t)}{\partial \hat{\hat{\hat{K}}}^A(t)} \cdot \dot{\hat{\hat{\hat{K}}}}^A(t) \cdot \frac{1}{\bar{i}(t)} \tag{5.132}$$

and substituting this in to (5.131) implies that:[27]

$$\begin{aligned}\hat{c}(t) &= \hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \bar{i}(t) \cdot \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{\hat{K}}}^A(t)} \cdot \dot{\hat{\hat{\hat{K}}}}^A(t) \cdot \frac{1}{\bar{i}(t)} + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{\hat{K}}}^A(t) - \bar{i}(t) \cdot \dot{\hat{\hat{\hat{K}}}}^A(t) \\ &= \hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g \cdot \hat{\hat{\hat{K}}}^A(t) - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{\hat{K}}}^A(t)} \cdot \hat{\hat{\hat{K}}}^A(t) + \bar{i}(t)\right] \cdot \dot{\hat{\hat{\hat{K}}}}^A(t)\end{aligned} \tag{5.133}$$

The $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule in $\hat{\hat{\hat{K}}}^A(t)$–$\hat{c}(t)$ space follows from (5.133). If $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ then:

$$\hat{c}(t) = \hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g \cdot \hat{\hat{\hat{K}}}^A(t) \tag{5.134}$$

(5.134) implies that now, even accounting for the changing cut-off $\bar{i}(t)$ in the definition of 'effective' in the expression for $\hat{\hat{v}}$ in (5.126), the $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule still depends upon the prevailing cut-off, $\bar{i}(t)$. Importantly, because $\bar{i}(t)$ does not affect the terms in (5.134) in a symmetric way, there is no alternative two-dimensional 'effective' space in which the expression in (5.134), suitably transformed, would be stable over time as $\bar{i}(t)$ changes.

The reason that $\bar{i}(t)$ appears in the expression for the $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule in (5.134) is important. This is because in any period t, advanced capital is only used to produce $\bar{i}(t)$ of the goods in the economy. The remaining $1-\bar{i}(t)$ goods are produced by high-skilled labour whose productivity is unaffected by technological progress. However as $\bar{i}(t)$ rises, and more goods are produced by advanced capital rather than high-skilled labour, the production of more

[27] Where, following (5.132), $\frac{\partial \bar{i}(t)}{\partial \hat{\hat{\hat{K}}}^A(t)} = \frac{\dot{\bar{i}}(t)}{\dot{\hat{\hat{\hat{K}}}}^A(t)}$.

goods in the economy is affected by the technological progress. As $\bar{i}(t)$ rises it is as if the 'effective' rate of technological progress – this is $\bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g$ in (5.134) – rises. Indeed, an increase in $\bar{i}(t)$ in this new model with many goods has the same consequence as an increase in the rate of technological progress in a traditional Ramsey model with a unique final good. To see this note that, aside from the role of $\bar{i}(t)$, the schedule in (5.134) is the same as the corresponding 'effective' capital schedule found in a traditional Ramsey model. And in a traditional Ramsey model this schedule implies that, if the rate of technological progress $\frac{\psi}{1-\psi} \cdot g$ were to increase, this would require a lower rate of effective consumption to ensure that the effective capital stock is constant. In the new model with many goods, the effect of an increase in $\bar{i}(t)$ is exactly the same as this increase in $\frac{\psi}{1-\psi} \cdot g$ in a traditional Ramsey model. Intuitively, in a traditional Ramsey model with a unique final good, the economy 'feels the full force' of the technological progress $\frac{\psi}{1-\psi} \cdot g$ whereas in this many-good model, only $\bar{i}(t)$ of the economy does.

Dynamic Equilibrium

Equilibrium in this setting with advanced capital and technological progress is not analytically tractable. The problem requires a non-linear simulation of the following three equations, implied by (5.122), (5.123), and (5.80)-(5.81):[28]

[28]Note from (5.121) that $D_8 = D_7 \cdot \psi$. In (5.135), $\hat{c}(t)$ is a forward-looking state variable, $\hat{\hat{\hat{K}}}^A(t)$ is a backward-looking state variable, and $\bar{i}(t)$ is a simultaneously endogenous non-state variable (i.e. it has no associated differential equation).

$$
\begin{aligned}
\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \hat{\hat{\hat{K}}}^A(t)^{\psi-1} \cdot D_8 \cdot \psi - \rho - \frac{\psi}{1-\psi} \cdot g \\
\frac{\dot{\hat{\hat{\hat{K}}}}^A(t)}{\hat{\hat{\hat{K}}}^A(t)} &= \frac{\hat{\hat{\hat{K}}}^A(t)^{\psi-1} \cdot D_8 - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{\hat{K}}}^A(t)}}{\frac{\partial \bar{i}(t)}{\partial \hat{\hat{\hat{K}}}^A(t)} \hat{\hat{\hat{K}}}^A(t) + \bar{i}(t)} \\
\bar{i}(t) &= f(K^A(t), A^H(\bar{i}(t), t), L^H)
\end{aligned}
\tag{5.135}
$$

Nevertheless it is still possible to identify analytically the unique steady-state that this model approaches and also to provide an intuitive conjecture for transition to this steady-state, without performing the non-linear simulation. That this model cannot be solved analytically is itself interesting and important, as I will explain.

As noted, the steady-state that this model approaches can be identified and is unique. The $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ and $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedules in (5.127) and (5.134) imply that steady-state is only reached in $\hat{\hat{\hat{K}}}^A(t)$–$\hat{c}(t)$ space when $\bar{i}(t) = 1$ and high-skilled labour is entirely driven out of the economy by advanced capital. To see this note that the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule, derived in (5.127), must be vertical at the level of $\hat{\hat{\hat{K}}}^{A*}$ for which r^{A*} is equal to:

$$
r^{A*} = \rho + \frac{\psi}{1-\psi} \cdot g \tag{5.136}
$$

From (5.119) it follows that $r^A(t)$ can be expressed in terms of $\hat{\hat{\hat{K}}}^A(t)$ is:

$$
r^A(t) = \left[\hat{\hat{\hat{K}}}^A(t)\right]^{\psi-1} \cdot D_7 \tag{5.137}
$$

And so it follows that the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule is vertical at a constant $\hat{\hat{\hat{K}}}^{A*}$ equal to:

$$\hat{\hat{\hat{K}}}^{A*} = \left[\left[\rho + \frac{\psi}{1-\psi} \cdot g\right] \cdot \frac{1}{D_7}\right]^{\frac{1}{\psi-1}} \tag{5.138}$$

Now recall that any steady-state must take place where the $\dot{\hat{\hat{\hat{K}}}}^{A}(t) = 0$ and the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedules intersect. Consider the following proof by contradiction. Suppose that the $\dot{\hat{\hat{\hat{K}}}}^{A}(t) = 0$ schedule crosses the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule where $\bar{i}(t) \neq 1$. In this potential steady-state the productivity of advanced capital grows at a rate $\frac{\psi}{1-\psi} \cdot g$. If this is to be a steady-state, then $\bar{i}(t)$ must remain constant. If $\bar{i}(t)$ does not remain constant, then the $\dot{\hat{\hat{\hat{K}}}}^{A}(t) = 0$ schedule will shift. This is implied by (5.134). The static analysis in Chapter 4 implies that $\bar{i}(t)$ will only stay constant if $K^A(t)$ decreases to offset this increase in productivity. But if $K^A(t)$ decreases such that $\bar{i}(t)$ remains constant then $\hat{\hat{\hat{K}}}^{A}(t)$ will fall, violating the condition that $\hat{\hat{\hat{K}}}^{A*}$ is constant at the steady-state, as established in (5.138). $K^A(t)$ must therefore rise to ensure that $\hat{\hat{\hat{K}}}^{A*}$ is constant. However, this growth in the stock of advanced capital increases $\bar{i}(t)$ further. From the definition of the $\dot{\hat{\hat{\hat{K}}}}^{A}(t) = 0$ schedule in (5.134) this will drive down the intersection point of the $\dot{\hat{\hat{\hat{K}}}}^{A}(t) = 0$ and $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule. The steady-state value of $\hat{c}^*$ will be driven down. Repeating this argument implies that if steady-state is to exist in $\hat{\hat{\hat{K}}}^{A}(t)$–$\hat{c}(t)$ space, it must take place when $\bar{i}(t) = 1$.[29]

[29] Note that the static model implies that this steady-state where $\bar{i}(t) = 1$ can only be approached asymptotically. The static model does not provide a closed-form expression for $\bar{i}(t)$, but it is clear from Figure 5.8 that $\bar{i}(t)$ is increasing in both increases in the productivity of advanced capital, which shifts the $A^H(i,t)$ schedule downwards, and increases

It is possible to provide an intuitive conjecture for what the transition path to this steady-state, where $\bar{i}(t) = 1$, will look like. This is shown in Figure 5.9, when the steady-state is approached from the left i.e. $\hat{\hat{\hat{K}}}^A(0) < \hat{\hat{\hat{K}}}^{A*}$. This conjecture for the transition to this steady-state is based on three observations. The first is that the $\frac{\dot{c}(t)}{c(t)} = 0$ schedule is known, given (5.138). The second is that as $\hat{\hat{\hat{K}}}^A(t)$ increases, $\bar{i}(t)$ also increases – this is shown in the Appendix, section 9.1.5. This gives the $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule its hump-shape in Figure 5.9, given (5.134).[30] The third is that, supposing the economy reaches the steady-state where $\bar{i}(t) = 1$, the new model must collapse to have the same general form as a traditional Ramsey model in which there is labour and capital, and a labour-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$ – but rather than there being labour and capital, there is instead only two types of capital, with an advanced capital-augmenting process of technological change taking place at rate g that is equivalent to a traditional capital-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$. As a result, locally to the steady-state, the new model has the same approximated saddle-path as this traditional Ramsey model. This is what is shown in Figure 5.9.[31]

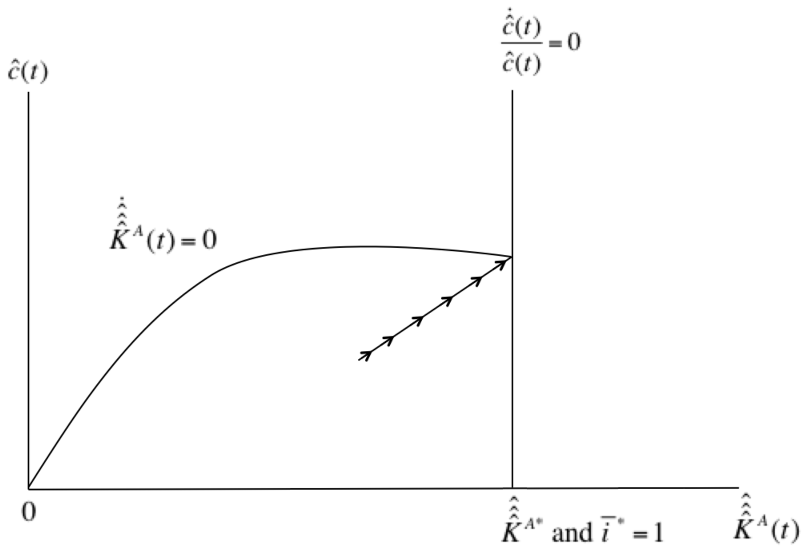
---

in the stock of advanced capital, which shifts the $C(i,t)$ schedule rightwards – but $\bar{i}(t)$ approaches 1 only as the productivity of advanced capital approaches infinity, or the stock of advanced capital approaches infinity. Note that it is not possible for a steady-state where $\bar{i}(t) = 0$ – although $\bar{i}(t)$ is constant, this would imply there is no advanced capital in the economy and so the steady-state condition in (5.138) is violated.

[30]One way to think about this $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule is that it is bounded below by a hypothetical $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule where $\bar{i}(t)$ is fixed at 1 and above by a hypothetical $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule where $\bar{i}(t)$ is fixed at 0.

[31]By 'collapse', I mean that the static three-factor analysis in Chapters 3 and 4 is replaced by a two-factor analysis where there is no labour, and advanced capital combines with traditional capital to produce all goods.

Figure 5.9: Dynamic Equilibrium



The implication of this conjecture is intuitive – during transition $\hat{\hat{\hat{K}}}^A(t)$ increases, driving up $\bar{i}(t)$ as the stock and productivity of $K^A(t)$ increases. However this is only a conjecture and for an important reason – the approximated saddle-path in Figure 5.9 is only correct locally to the steady-state. It is based on the claim that $\bar{i}(t) = 1$ in steady-state, but as soon as $\hat{\hat{\hat{K}}}^A(t)$ deviates from $\hat{\hat{\hat{K}}}^{A*}$ along the saddle-path, $\bar{i}(t) \neq 1$, and this claim no longer holds. In order to find the actual saddle-path to this steady-state, rather than the conjecture in Figure 5.9, it is necessary to perform a non-linear simulation of the three equations in (5.135) as noted before.

The origins of this non-linearity are interesting and important, and follow directly from the previous discussion of why $\bar{i}(t)$ appears in (5.134). Advanced capital during transition must be accumulated not only to offset the increase in its productivity at producing a given good in any period t, but also to offset the fact that it is being used to produce more types of goods as $\bar{i}(t)$ increases over time. As $\bar{i}(t)$ changes over time, what I called the 'effective' growth rate of the economy, $\bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g$, also changes over time. This is the origin of the non-linearity, absent from traditional Ramsey models.

Yet despite the fact that it is only possible to provide an analytical conjecture for transition to steady-state, Proposition 6 follows:

PROPOSITION 6: *When the static model with advanced capital is placed in a dynamic setting with endogenous advanced capital accumulation and technological progress at rate g across the task spectrum, the economy approaches a unique steady-state at $\hat{c}^*$, $\hat{\hat{\hat{K}}}^{A*}$, where $\bar{i}^* = 1$ – in this steady-state, labour is entirely driven out by advanced capital.*

Proposition 6 states that the economy approaches a steady-state where labour is entirely driven out by advanced capital. This is the remorselessly pessimistic outcome identified in the outline to this chapter. Again, it is driven by the fact that steady-state requires advanced capital to be accumulated not only to offset the fact that advanced capital is becoming more productive, but also that it is becoming more prevalent. It is as if the economy is chasing a steady-state that is continually slipping out of its grasp – until $\bar{i}(t) = 1$ and labour is fully driven out.

This result bears an interesting similarity with traditional non-task based models where the elasticity of substitution is greater than 1. Solow (1956) noted that in his model, if the savings rate is above a certain threshold, an elasticity of substitution greater than 1 implies that the "capital-labor ratio increases indefinitely".[32] I conjecture that labour is driven out the economy in this new task-based model for a similar reason – as advanced capital becomes increasingly capable, and as $\bar{i}(t)$ rises, the 'aggregate' elasticity of substitution between labour and capital also increases.[33] By 'aggregate' in this setting, I mean an elasticity that is both an aggregate elasticity *horizontally* across the production of different goods, and also *vertically* across the different levels of production (i.e. across the factor-based production functions for tasks, and the task-based production function for goods). Exploring the relationship between this aggregate elasticity of substitution, and the elasticities of substitution of the production processes taking place at each of these different levels and for different goods, is an important task for future work.[34]

Comparative Dynamics: Introducing Depreciation

In the dynamic models set out so far there has been no depreciation. In

[32]This is noted in Klump and De La Grandville (2000) and Klump and Preissler (2000).

[33]This relates to the Arrow et al. (1961) suggestion that "the process of economic development itself might shift the over-all elasticity of substitution" – as quoted in Klump and De La Granville (2000). Piketty and Zucman (2013) also note how:

> "In the 18th and early 19th century, capital was mostly land ... so that there was limited scope for substituting labor to capital. In the 20th and 21st centuries, by contrast, capital takes many forms, to an extent such that the elasticity of substitution between labor and capital might well be larger than 1."

[34]There is a literature to build upon here. Sato (1967), for example, derives the aggregate elasticities of substitution in production between n factors that combine in a "two-level" CES production function i.e. where output is a CES combination of two distinct bundles of inputs each of which, in turn, are CES combinations of those inputs. However, in Sato (1967) these elasticities are derived for the production of a unique final good, whereas in this setting there is a spectrum of final goods. There is an important task to generalise these findings to a setting with a variety of goods.

the setting with advanced capital and technological progress this may seem like a significant omission, since the depreciation of the existing stock of advanced capital may appear to act as a counterbalance to the accumulation of advanced capital and the displacement of labour. However, this intuition is incorrect. Suppose depreciation takes place at rate δ. The introduction of depreciation changes the expression for $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)}$:[35]

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = r^A(t) - \rho - \delta - \frac{\psi}{1-\psi} \cdot g \tag{5.139}$$

The introduction of depreciation also changes the law of motion for $\hat{\hat{\hat{K}}}^A(t)$. First note that the law of motion for $K^A(t)$ in (5.84) becomes:

$$\dot{K}^A(t) = Y(t) - c(t) - \delta \cdot K^A(t) \tag{5.140}$$

Given that factors are still paid their marginal revenue products, the law of motion for $\hat{\hat{\hat{K}}}^A(t)$ in (5.128) becomes:

$$\begin{aligned}
\frac{\dot{\hat{\hat{\hat{K}}}}^A(t)}{\hat{\hat{\hat{K}}}^A(t)} &= \frac{\dot{K}^A(t)}{K^A(t)} - \left[g^{\gamma}(t) + \frac{\psi}{1-\psi} \cdot g\right] \\
&= \frac{\hat{Y}(t) - \hat{c}(t) - \delta \cdot \hat{K}^A(t)}{\hat{K}^A(t)} - \left[g^{\gamma}(t) + \frac{\psi}{1-\psi} \cdot g\right] \\
&= \frac{\hat{Y}(t) - \hat{c}(t)}{\hat{K}^A(t)} - \left[g^{\gamma}(t) + \delta + \frac{\psi}{1-\psi} \cdot g\right]
\end{aligned} \tag{5.141}$$

[35] This follows from the fact that the budget constraint in (5.77) will therefore contain an additional $-\delta \cdot K^A(t)$ term. An alternative is to consider a decentralised version of the model with financial markets where consumers choose assets $A(t)$ that pay a return $R(t)$, where $R(t) = r^A(t) - \delta$. The outcome is the same.

This can be expressed in the same general form as (5.133):

$$\hat{c}(t) = \hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \bar{i}(t) \cdot \left[\delta + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{\hat{K}}}^A(t) - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{\hat{K}}}^A(t)} \cdot \hat{\hat{\hat{K}}}^A(t) \cdot \bar{i}(t) + \bar{i}(t)\right] \cdot \dot{\hat{\hat{\hat{K}}}}^A(t) \tag{5.142}$$

From (5.139) and the expression for $r^A(t)$ in (5.137) the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule follows:

$$\hat{\hat{\hat{K}}}^A(t) = \left[\left[\rho - \delta - \frac{\psi}{1-\psi} \cdot g\right] \cdot \frac{1}{D_7}\right]^{\frac{1}{\psi - 1}} \tag{5.143}$$

and from (5.142) the $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule follows:

$$\hat{c}(t) = \hat{\hat{\hat{K}}}^A(t)^{\psi} \cdot D_8 - \bar{i}(t) \cdot \left[\delta + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{\hat{K}}}^A(t) \tag{5.144}$$

The implication is that while the introduction of depreciation will change the level of steady-state $\hat{\hat{\hat{K}}}^{A*}$ – implied by (5.143) – it is still the case that the model must approach a steady-state in $\hat{c}(t) - \hat{\hat{\hat{K}}}^A(t)$ space when $\bar{i}(t) = 1$. This is again implied by (5.144).

Comparative Dynamics: Rise in g

Suppose that there is an increase in the rate of technological progress, g. The consequence has the same form as the case with traditional capital and technological progress. A rise in g causes a leftward shift in the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule and a fall in the steady-state level of $\hat{\hat{\hat{K}}}^A(t)$, as implied by (5.138). The rise in g will also cause a change in the $\dot{\hat{\hat{\hat{K}}}}^A(t) = 0$ schedule. Nevertheless, it must still be the case – for the reason set out before – that although the transition to the steady-state can only be conjectured analytically, the model must approach a steady-state where $\bar{i}(t) = 1$.

Comparative Dynamics: Relative Factor Prices and Factor Shares

The conjectured transition path in Figure 5.9 has three important implications. The first is that the return to advanced capital relative to traditional capital falls during transition. This is because the expressions for the relative factor prices in (5.110) imply that:

$$\frac{\dot{r}^A(t)}{r^A(t)} - \frac{\dot{r}(t)}{r(t)} = g^{\gamma}(t) - g^{K^A}(t) \tag{5.145}$$

and since $\hat{\hat{\hat{K}}}^A(t)$ is increasing during transition it follows that $g^{K^A}(t) - g^{\gamma}(t) - \frac{\psi}{1-\psi} \cdot g > 0$ and so the expression in (5.145) is negative.

Secondly, the conjectured transition path implies that, in the limit, the high-skilled wage relative to the return on advanced capital must fall to zero. This resolves the ambiguity in the section 5.3.1 discussion of Figure 5.8 and the relative wage $f(t)$.[36] This follows intuitively from the static model on the right-hand side of Figure 5.8 – as the $A^H(i,t)$ schedule falls and flattens with technological progress, and the $C(i,t)$ schedule shifts right with the endogenous accumulation of advanced capital, the economy approaches an equilibrium in the bottom-right corner of the figure where $f(t) = 0$ and $\bar{i}(t) = 1$. This represents the total immiseration of labour. Formally, the

[36] The ambiguity is that a fall in the $A^H(i,t)$ schedule causes the relative wage to *decrease*, but the accumulation of advanced capital causes the relative wage to *increase*. In the continuous setting this ambiguity is reflected by the fact the high-skilled wage only falls relative to the return on advanced capital when $g^{K^A}(t) < \frac{g}{[\bar{i}(t) - \bar{i}(t)^2] \cdot \frac{A_1^H(\bar{i}(t),t)}{A^H(\bar{i}(t),t)}}$. This is shown in the Appendix, section 9.1.6.

relative wage of labour with respective to advanced capital in the limit is equal to:

$$\begin{aligned} \lim_{t\to\infty} \frac{w^H(t)}{r^A(t)} &= \lim_{t\to\infty} A^H(\bar{i}(t), t) \\ &= \lim_{t\to\infty} \frac{a^H(\bar{i}(t), t)}{a^A(\bar{i}(t), t)} \end{aligned} \tag{5.146}$$

and since $\lim\limits_{t\to\infty} \bar{i}(t) = 1$ it follows that:

$$\begin{aligned} \lim_{t\to\infty} \frac{a^H(\bar{i}(t), t)}{a^A(\bar{i}(t), t)} &= \lim_{t\to\infty} \frac{a^H(1, t)}{a^A(1, t)} \\ &= 0 \end{aligned} \tag{5.147}$$

since $a^H(i, t)$ is constant over time $\forall i$, and there is continuous growth in $a^H(i, t)$ $\forall i$ as set out in (5.114) such that $\lim\limits_{t\to\infty} a^A(1, t) = \infty$.

The third implication is for the factor shares of income:

$$\begin{aligned} \frac{r^A(t) \cdot K^A(t)}{Y(t)} &= \gamma(\bar{i}(t)) \cdot \psi \\ \frac{w^H(t) \cdot L^H(t)}{Y(t)} &= [1 - \gamma(\bar{i}(t))] \cdot \psi \\ \frac{r(t) \cdot K(t)}{Y(t)} &= (1 - \psi) \end{aligned} \tag{5.148}$$

The conjectured transition path implies that $\hat{\hat{\hat{K}}}^A(t)$ rises during transition. This drives up $\bar{i}(t)$, and so $\gamma(\bar{i}(t))$, as noted before.[37] As a result, it also drives up the advanced capital share of income and drives down the labour share of income in (5.148). The result is a rise over time in the capital share of income and a fall in the labour share of income, the latter approaching zero

[37] Again, this is shown in the Appendix, section 9.1.5.

in steady-state. The traditional capital share of income remains constant.

This final observation is an important result. It suggests that the new model in this section could provide a new explanation for the decline in the labour share of income that has been experienced by several advanced economies since the 1980s (see, for example, Piketty and Zucman 2014). This decline is striking given the fact that traditionally it has been assumed that the labour share of income is constant – Keynes called its stability "a bit of miracle" (Keynes 1939) and it was one of Kaldor's stylised facts of growth (Kaldor (1957).

5.3.3. Discussion

In the dynamic model with advanced capital, the endogenous accumulation of advanced capital strengthens the pessimistic outcome in Chapter 4. In the case with no technological progress, a one-off increase in the productivity of advanced capital results in a new steady-state where c^*, $\hat{\hat{K}}^{A*}$, and $\bar{i}^*$ are higher. The rise in $\bar{i}^*$ implies that labour is forced to specialise in a smaller set of types of tasks. This is pessimism at work. Yet the consequence for the relative wage is analytically ambiguous – the increase in the productivity of advanced capital decreases the relative wage, but the endogenous accumulation of advanced capital increases it. However, in the case with continuous technological progress this ambiguity is resolved. An even more pessimistic conclusion follows – the model must approach a steady-state where labour is entirely driven out the economy by advanced capital, and both the relative wage and the labour share of income approach zero.

5.4. Conclusion

In this chapter I placed the static model in Chapters 3 and 4 in a dynamic setting. The endogenous process of capital accumulation strengthens the optimism of Chapter 3 and the pessimism of Chapter 4. A particularly pessimistic result follows from the case with advanced capital and technological progress in the use of that capital – labour is displaced from performing certain types of tasks and the labour share of income falls, both at endogenously determined rates. Furthermore, if technological progress is continuous, no steady-state is possible until labour has been entirely driven out the economy by advanced capital – i.e. when $\bar{i}(t) = 1$.

The reasoning is the same as in the static case. In the case with traditional capital alone, capital is a q-complement to labour. As a result, as traditional capital is accumulated and labour becomes increasingly scarce, the relative return to labour rises. This is optimism at work. However, in the case with advanced capital as well, while traditional capital remains a q-complement to labour, advanced capital is a perfect substitute for those tasks in which labour is q-complemented. As a result, as advanced capital is accumulated, labour performs a shrinking range of those tasks in equilibrium. This is pessimism at work.

This result is new and important. It is the consequence of replacing the ALM hypothesis with an understanding of machine capability that reflects the character of recent technological progress. Recall from Part A that the ALM hypothesis justifies a set of models in which there exists a set of tasks that are performed by labour and that cannot be performed by capital – 'non-routine' tasks. The case with traditional capital in this chapter captures the spirit of this approach – there is a set of tasks that only labour can perform, and in which they are q-complemented by capital. But in practice we see 'non-routine' tasks being automated. This is for the reason set out in Chapter 2. And introducing advanced capital, that can perform those tasks in which labour is q-complemented by traditional capital, reflects this observation. Pessimism for labour follows as a result.

Part C: The Demand-Side

6. An Introduction to Demand

6.1. Outline

In Part B I developed a new task-based model of the labour market to explore the consequences of increasingly capable machines. I showed that replacing the traditional, conservative assumptions about the operation and capabilities of capital – reflected in the ALM hypothesis – leads to a more pessimistic outcome for labour.[1]

However, in these new models the analysis takes place on the supply-side. The role of the demand-side is heavily constrained – although there are many goods in these models, consumer preferences are Cobb-Douglas. The consequence of this is that expenditure shares on different types of goods do not change. Preferences are homothetic so expenditure shares on goods do not respond to changes in income, and they have an elasticity of substitution equal to 1 so expenditure shares do not respond to changes in relative prices. Nor do expenditure shares change in most of the other task-based models in Chapter 1. Indeed, in most of the task-based models I looked at there is no demand-side at all and so the concept of 'expenditure shares' does not have any meaning.

This is problematic for two reasons. The first is that these models are

[1]The ALM hypothesis, and its problems, are set out in Chapter 2.

used to consider the future of the labour market, as well as explain the past. But long-run economic growth is associated with substantial changes in expenditure shares on goods and the structural composition of an economy. As a result these models, with restricted or absent demand-side analyses, are partial accounts of the consequences of technological change in the long-run. The second reason is that arguments for optimism about the consequence of technological change on the labour market often appeal to the demand-side. But without a demand-side, or with a demand-side that keeps expenditure shares on goods fixed, these models cannot interrogate these arguments.

In this chapter, I focus on the first problem. I show that many task-based models cannot capture the structural change described in the broader economic literature because their demand-sides are absent or too restricted. I explore one exception to this – the AD model, developed in Autor and Dorn (2013) and shown in section 1.5.4 – which does have an important role for demand-side analysis. But I show that because the AD model relies so heavily upon the ALM hypothesis, the role that demand does have is diminished by the conservatism of the model's supply-side analysis. I close by showing it is not possible to simply 'update' the AD model with more appropriate assumptions about the capabilities of machines. This chapter provides the foundations for the final chapters of the thesis.

In the next chapter, having established the general importance of the demand-side in this chapter, I will focus on the second problem set out before. I will use the new model developed in Part B to look at one particular argument for optimism that might come from the demand-side. In concluding the thesis, I will point towards an important area for further research – a 'race' in task-space – in which demand-side analysis must have an important role.

6.2. The Role of Demand in the Task-Based Literature

The existing task-based literature that explores the consequences of increasingly capable machines on the labour market is generally characterised by either no demand-side or a restricted demand-side that keeps expenditure shares on goods fixed. In what follows I show this and explain why it is problematic. Chapter 1 terminology is used through out this section to refer to the different task-based models.

6.2.1. The Role of Demand

The canonical model – a CES combination of two types of labour to produce a unique final output – is not a task-based model but, given its role in the history of the task-based literature, it is still interesting to note that it has no demand-side for goods.[2] This is because output consists of a unique type of good, Y. In the absence of any variety of goods, there is no capacity to explore the consequences of consumer tastes and preferences. In a model with a unique final good there is no way to talk about expenditure shares.

The task-based ALM and AA models were designed in response to the canonical model.[3] They are intended to resolve some of its limitations. But both these task-based models also have no demand-side for goods. The reason

[2] For detail on the canonical model, and its role in the history of the task-based literature, see section 1.2.

[3] These are set out in section 1.4.2 and 1.5.3 respectively.

is the same. In the ALM and AA models, as in the canonical model, there is a unique final good. In addition it is interesting to note that because that unique final good is produced using a Cobb-Douglas combination of tasks, for any level of demand for that final good the expenditure shares are fixed across the different tasks required to produce it. This is true both in the ALM model where there are only two types of tasks, and in the AA model where there is a spectrum of types of tasks.[4]

The new model developed in Part B, unlike the ALM and AA models, has a variety of goods, not just a unique good. However, in spite of there being a variety of goods, expenditure shares on those goods are fixed. This is because the demand for goods is Cobb-Douglas. Again, it is interesting to note that because these goods are produced through a Cobb-Douglas combination of two types of tasks, the resulting expenditure shares on tasks are also fixed. The AA model, which also has a spectrum of tasks (but only a unique final good), shares this property.[5]

6.2.2. The Problem with this Approach to Demand

No Account of Structural Change

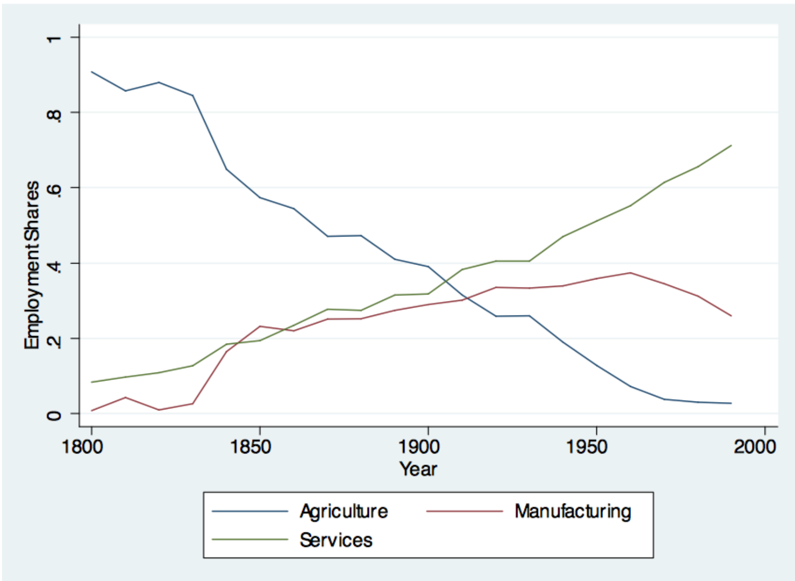
In models where there is no demand-side for goods, it is not possible to talk of expenditure shares. In models where there is a variety of goods, but utility is Cobb-Douglas, expenditure shares on goods are fixed over time. Both cases are problematic given that long-run economic growth is in fact associated

[4]Formally, the task-based production function for goods in these models is Cobb-Douglas. See section 1.3.

[5]Again, the task-based production function for goods is Cobb-Douglas.

with substantial changes in expenditure shares on goods and the structural composition within an economy. This is often described in the broader economic literature by a set of stylised facts called the 'Fisher-Clark-Kuznets' thesis (after Fisher 1939, Clark 1940, and Kuznets 1966 – see Matsuyama 2008). On this account, as growth takes place, production in the economy moves through primary, to secondary, and finally to tertiary industries. This transformation shown in Figure 6.1 (from Acemoglu 2007):

Figure 6.1: Structural Change



I call these the 'structural change' facts, or the 'SC facts' from now. And to repeat, the problem with models in which there is no demand-side or a Cobb-Douglas utility function is that they cannot reflect these SC facts.

Three Explanations

There are three possible explanations for the SC facts. Each of these appeals to the demand-side. First are 'income' effects. With technological progress, incomes rise and this causes a change in the pattern of expenditure across different goods. The identification of this effect originates with Engel (1857). Engel's original observation was that as incomes rise, the expenditure share on food falls (though absolute expenditure on food rises). This idea was generalised by Kindleberger (1989):

> "Engel's law applies to more than food, it is a general law of consumption ... A given item may go through the Engel's consumption cycle of a luxury, with high income elasticity, to a necessity with low income elasticity." (pg. 7)

Different goods or sectors, in short, have different income elasticities of demand. Because these explanations focus on consumption, they are considered 'demand-side' explanations of structural change. As Acemoglu and Guerrieri (2008) note, these explanations are the most widespread explanation of the SC facts – "this literature typically starts by positing nonhomothetic preferences consistent with Engel's law and thus emphasizes the demand-side reasons for nonbalanced growth". See, for example, Matsuyama (1992, 2002) and Foellmi and Zweimüller (2008) for recent contributions.

A second explanation for the SC facts are 'price' effects. With technological progress, changes in the relative productivities of producing certain goods causes changes in relative prices and, in turn, those changes in relative prices can cause changes in patterns of expenditure. Technological progress is in a sense unbalanced – productivity growth is 'uneven' across different sectors. This effect is first described in Baumol (1967) which sets out a model of "unbalanced" growth. As described by Ngai and Pissarides (2007):

> "Baumol divided the economy into two sectors, a "progressive" one that uses new technology and a "stagnant" one that uses labor as the only input. He then claimed that the production costs and prices of the stagnant sector should rise indefinitely, a process known as "Baumol's cost disease," and labor should move in the direction of the stagnant sector." (pg. 429-430)

Here, these differences in relative production productivities – where one set of activities is technologically "progressive" and another set that are technologically "stagnant' – cause changes in relative prices and in relative expenditure. Price effects are often called 'supply-side' effects, for example Comin et al. (2015), but this is misleading. Price effects are caused by a combination both of supply-side effects – a change in relative prices causes firms to change their allocation of resources across production of different goods – and demand-side effects – a change in relative prices causes consumers to change their allocation of income across the consumption of different goods.

Together, the price effect and the income effect are the main way that the broader economic literature explains the SC facts. There is disagreement regarding which explanation is the best.[6] Certainly the income effect is most widely used in the broader literature. It is claimed that there is a particularly strong empirical case for Engel's Law – for example, Houthakker (1987) notes, "of all the empirical regularities observed in economic data, Engel's law is probably the best established."[7] But this is probably too strong a claim

[6]For example, Autor and Dorn (2013) find no role for income effects in the US from 1980 to 2005, and a similar conclusion is reached by Goos et al. (2014) from 1993 to 2010 in 16 Western countries. But there is also contrary evidence to suggest that income effects may be more important that price effects. For example Dennis and Iscan (2009) look at the last two centuries in the US and find significant support for income effects up until 1950, and Comin et al. (2015) find a "predominant role" for income effects in explaining structural change in the postwar period in a large sample of countries.

[7]Quoted in Foellmi and Zweimüller (2008).

– price effects, for example, can also explain the correlation between rising income and structural change.

There is however a third explanation for the SC facts – that the preferences themselves change over time. There is a significant empirical literature that suggests preferences are endogenous (for example Bowles 1998 and Rabin 1998). Yet this is not reflected in the traditional economics literature. This is surprising. It is not that economists lack a variety of possible utility functions. Economists draw on a wide range of different utility functions in different settings or for different individuals. For example, in trying to explain structural change, at different points in the literature economists appeal to CES preferences, Stone-Geary preferences and 'Hierachical Demand Systems'.[8] But there is a reluctance to use a different utility function for the same individual at different times. Consider Stigler and Becker (1977)[9] who in making the case for the "stability of tastes" argue in part that:

> "One does not argue over tastes for the same reason that one does not argue over the Rocky Mountains – both are there, will be there next year, too, and are the same to all men." (pg. 76)

Intuitively, in thinking about the long run – and structural change is a long run phenomenon – the idea that preferences might change seems at least possible. Will I have the same preferences in 10 years? Will my grandchildren have the same preferences as me in 50 years? Neither claim seems obvious.

[8]Consider Ngai and Pissarides (2007), Kongsamut, Rebelo, and Xie (2001), and Matsuyama (2002) respectively.

[9]Quoted in Krackhardt (1998).

6.3. Autor and Dorn (2013)

The exception to the criticism in the previous discussion is the AD model.[10] Of particular importance here is that, unlike the other task-based models set out in section 6.2, the AD model has a less restrictive demand-side such that expenditure shares can change. A culmination of recent work in the task-based literature, it has become an important framework for analysing not only the consequences of increasingly capable machines on the late 20th century labour market – Autor and Dorn (2013) use it to explain trends from 1980 to 2005 – but also what we might expect to happen the 21st century – for example in Autor (2015). However, there is a limitation with the AD model – it relies heavily upon the ALM hypothesis. As a result, in thinking about the future, these insights into the role of the demand-side are limited. In turn it is not possible to simply update the model. I now show this.

6.3.1. AD Model as an Extension of the ALM Model

In the AD model, unlike the other task-based models I have considered, expenditure shares do exist and they are not fixed. The demand-side performs an important role here. To see this role for demand consider that the AD model is in fact an extension of the ALM model, the oldest task-based model.[11] The most important extension is that the AD model has two goods, $y(s)$ and $y(g)$, unlike the ALM model that just has one, Y, and a CES utility function over those goods – this is what creates a new role for demand.

[10] The AD model is set out in section 1.5.4. Here I call 'services' $y(s)$ rather than $y(se)$ as in Chapter 1.

[11] Again, the ALM model is set out in section 1.3.3.

But there are also three further extensions. The first is that the AD model has two types of labour, low-skilled and high-skilled, unlike the ALM model that just has one – this allows technological change to affect different types of labour in different ways. The second is that while the AD model and the ALM model have the same task-based production function for goods $y(g)$ and Y respectively, they have different factor-based production functions for the tasks that combine to produce those goods. And finally the AD model and the ALM model also have different types of tasks – both have 'routine' tasks, but whereas the ALM model has only 'non-routine' tasks the AD model uses a richer conception, distinguishing between 'non-routine manual' tasks and 'non-routine abstract' tasks.[12]

To see these last two extensions more clearly recall that the ALM model consists of a single Cobb-Douglas production function:

$$Y = (L(r) + K)^{\beta} L(nr)^{1-\beta} \tag{6.1}$$

where $\beta \in [0, 1]$, Y is a unique final output, $L(r)$ is the labour input of 'routine' tasks, $L(nr)$ is the labour input of 'non-routine' tasks, and K is the capital input of 'routine' tasks.[13] Again, output is a Cobb-Douglas combination of 'routine' and 'non-routine' tasks. And since it is assumed that the wage earned by labour that performs 'routine' tasks, w^r, is equal to rental price of capital, ρ it follows that capital is a perfect substitute for labour in

[12]In Autor and Dorn (2013) these are simply called 'manual' and 'abstract' tasks. But since they can only be performed by the labour the implication is that, in line with the ALM hypothesis, they must necessarily also be 'non-routine' tasks.

[13]This is shown in section 1.4.2.

performing 'routine tasks'.[14] In contrast the AD model has two final goods, each with the following production functions:

$$
\begin{aligned}
y(s) &= a^L(m)L^L(m) \\
y(g) &= L^H(a)^{1-\beta}\left[(a^L(r)L^L(r))^\gamma + (a^K(r)K(r))^\gamma\right]^{\frac{\beta}{\gamma}}
\end{aligned}
\tag{6.2}
$$

where $y(s)$ is output of 'services', $y(g)$ output of 'goods', $L^L(m)$ low-skilled labour input of 'non-routine manual' tasks with productivity $a^L(m)$, $L^H(a)$ high-skilled labour input of 'non-routine abstract' tasks with productivity 1, $L^L(r)$ low-skilled labour input of 'routine' tasks with productivity $a^L(r)$, and $K(r)$ capital input of 'routine' tasks with productivity $a^K(r)$. As in the ALM model, then, it is again the case that output of $y(g)$ is a Cobb-Douglas combination of 'routine' and 'non-routine' tasks – this is the sense in which they share a task-based production function for certain goods. But note that it is one particular type of 'non-routine' task is used in this production function in the AD model – it is 'non-routine abstract', not just 'non-routine' – and, unlike the ALM model, capital is an imperfect substitute for labour in performing 'routine' tasks, where the degree of substitutability depends upon the parameter γ in (6.2). This is the sense in which they have different factor-based production functions for tasks. Furthermore, note that the additional good in the AD model, $y(s)$, is produced by a second type of 'non-routine' task, namely 'non-routine manual' tasks. This is the sense in which the AD model has a richer conception of 'non-routine' tasks.

However, note that despite the existence of two goods it is still the case

[14] This is also shown in section 1.4.2.

in the AD model, as in the ALM model, that each good requires at least one type of labour to produce it. Unlike the new model developed in Part B, there are no goods produced by capital alone.

6.3.2. New Role for Demand in the AD Model

As a result of these extensions, the AD model has a new channel through which capital can complement labour in performing 'non-routine' tasks – through the demand-side. In the ALM model this is not possible. There, the complementarity between capital and labour that performs 'non-routine' tasks is only through the supply-side – because the task-based production function for Y in the ALM model is Cobb-Douglas, it has a constant elasticity of substitution equal to 1, and capital q-complements labour that performs 'non-routine' tasks.[15] This supply-side channel also exists in the AD model for the production of $y(g)$ – high-skilled labour that performs 'non-routine abstract' tasks is q-complemented by capital because the task-based production function for $y(g)$ is also Cobb-Douglas in 'routine' and 'non-routine abstract' tasks. But in addition, technological progress in the AD model may also cause a relative change in the demand for the second good $y(s)$, and so a relative change in the demand for the 'non-routine manual' tasks performed by low-skilled labour. This is the additional demand-side channel.

The strength of the demand-side effect in the AD model – in particular, whether it is sufficient to cause the wage and employment polarisation that Autor and Dorn (2013) are interested in – depends upon the relative elastic-

[15] An increase in the quantity of capital causes an increase in the marginal product of labour at performing 'non-routine' tasks. See (6.1).

ities of substitution in consumption and production. Wage and employment polarisation only takes place in the AD model if the elasticity of substitution in production between capital and low-skilled labour in performing 'routine' tasks is large enough relative to the elasticity of substitution in consumption between $y(g)$ and $y(s)$, and the elasticity of substitution in consumption is less than or equal to 1. This is discussed in detail in section 1.5.4.

6.3.3. AD Model's Reliance on the ALM Hypothesis

To repeat, in the AD model there are two channels through which capital can complement labour. It can complement high-skilled labour that performs 'non-routine abstract' tasks through q-complementarity and the supply-side channel. Or it can complement low-skilled labour that performs 'non-routine manual' tasks through the new demand-side channel. Put another way, there are two channels that could support an optimistic outcome for labour.

The problem, however, is that both these channels depend upon the ALM hypothesis. In the case of high-skilled labour, the reason is the same as for the ALM model. The output of $y(g)$ is a Cobb-Douglas combination of 'non-routine abstract' tasks and 'routine' tasks – and because of the ALM hypothesis it is assumed that capital can only perform the latter and not the former. As a result, a fixed proportion $(1-\beta)$ of total expenditure on $y(g)$ is necessarily earned by high-skilled labour performing 'non-routine abstract' tasks. This is clear from (6.2). In Chapter 2, I argue that this optimism may be misplaced and in Chapters 4 and 5 show the pessimism that follows from increasingly capable machines that can perform those 'non-routine' tasks. In these models, labour begins with a fixed share of income, but it is eroded by advanced capital.

However, it is also the case that the new demand-side channel depends upon the ALM hypothesis. The second good, $y(s)$, is produced only by 'non-routine manual' tasks, and so only by low-skilled labour. This provides a 'refuge' for low-skilled labour – even if machines become vastly more capable and displace low-skilled labour from performing 'routine' tasks to produce $y(g)$ there may still be work for them to do in performing 'non-routine manual' tasks to produce $y(s)$. Yet whether this 'refuge' will continue to exist in the future is not clear. As set out in Chapter 2, many 'non-routine manual' tasks that Autor and Dorn (2013) have in mind can now be performed by machines despite their 'non-routine' character.[16]

6.4. Why Not Develop the AD Model?

The AD model differs from the other models set out since there is more than one type of output and, in turn, expenditure shares on those outputs are not fixed. It can reflect the SC facts.[17] Given this, a natural question follows – why not simply relax the ALM hypothesis in the AD model, replace it with a less conservative set of assumptions about the capabilities of capital, and use this revised model to explore the role of the demand-side?

The answer is that the AD model lacks the continuous spectrum of types of tasks that is necessary to analyse the consequences of increasingly capable machines. Recall from the last section that the AD model only considers three distinct types of tasks, one 'routine' task and two types of 'non-routine' task

[16]See section 2.3.
[17]See section 6.2.2.

– 'abstract non-routine' and 'manual non-routine' tasks. The AD model and the ALM hypothesis argue that both these types of 'non-routine' tasks can only be performed by labour. I claim this is mistake – there are some types of 'non-routine' tasks that already can be performed by machines, and it is likely in the future that machines will take on *more* types of 'non-routine' tasks as well. Now, suppose that this claim about machines taking on 'non-routine' tasks is right. The problem is that the AD model cannot reflect this. Given there are only two types of 'non-routine' task, the result would necessarily be uninteresting – that machines can perform none, one, or both of the two types of 'non-routine' task. This approach cannot reflect the richness of what we see in practice – a wide range of many types of 'non-routine' task, some of which can be performed by machines, others which can be performed by labour, and the boundary between the two constantly changing. To consider the argument that machines are taking on *more* types of 'non-routine' tasks requires that the approach in the AD model, a discrete bundle of two types of 'non-routine' task, is replaced with a continuous spectrum of many types of task. This is the approach in Part B.

A further question follows – why not then simply introduce the spectrum of types of tasks into the AD model, but retain the rest of the AD model's structure? The difficulty with adding a task spectrum to the AD model is that it involves only two types of good, $y(g)$ and $y(s)$. If it is important to introduce more types of tasks into the model, it also make sense to broaden the variety of goods. Not only does an increase in the variety of goods better reflect the world, but it avoids the non-trivial task of describing how many types of tasks would combine to produce only two goods.[18]

An alternative approach is to retain the form of the task-based production functions for goods in the AD model,[19] and instead increase the number of goods. This is in fact the approach in Part B. In that new model, there are a variety of goods and tasks. Each good is still produced by a Cobb-Douglas combination of two types of tasks as for good $y(g)$ in the AD model, but each type of good is also produced by a different type of task $z_1(i)$. The model is then able to reflect a rich interpretation of tasks (i.e. use a task spectrum) and the completeness of the AD model (i.e. it has more than one good, and a role for tastes and preferences). This is not, then, simply a quest for generality – the new model is able to consider a variety of goods, a variety of tasks, *and* does not rely on the ALM hypothesis to do so.

6.5. Conclusion

In this chapter I have done three things. First I explored the limited role of the demand-side in the task-based literature, looking in particular at the core set of models from Chapter 1. Secondly I looked in detail at the AD model and showed how its analysis of the demand-side depends upon the ALM hypothesis and so is likely to be of diminishing use in thinking about the future. Thirdly I explained why it is not possible to simply update the AD model.

Note that none of this discussion weakens the explanatory power of the

[18] Formally, this is the task of setting out the full set of task-based production functions for the goods.

[19] In particular, that $y(g)$ in the AD model is produced by a Cobb-Douglas combination of two different types of tasks. See the expression for $y(g)$ in (6.2).

AD model in understanding the consequence of technological change on the labour market from 1980 to 2005. The AD model remains an important explanation for the simultaneous growth of low-skill service occupations in the US during that period *and* the polarisation of employment and wages that accompanied it. But there are reasons to think that these limitations are problematic when the AD model is used to think about the future.

In the next chapter, I return to the new task-based model from Part B. This new model – which retains the spectrum of both goods and tasks, but need not rely on the ALM hypothesis – allows me to explore the 'structural change' argument for optimism.

7. The New Model with Structural Change

7.1. Outline

In this chapter I introduce structural change into the new model in Part B. The purpose is to explore a claim that is made by many economists in the formal economics literature and the informal expert commentary on the future of work. This is the 'structural change' argument for optimism. A general form of this argument is as follows:

> **Structural Change Argument**: although machines may replace labour in producing certain types of goods, it is also likely that over time technological progress will cause expenditure shares across different types of goods to change. And so labour displaced from producing certain types of goods can be employed to produce the different types of goods that are now in greater relative demand.

The model in this chapter shows that if consumer expenditure shifts towards goods that require types of tasks in which labour retains a comparative advantage, then the optimism of the structural change argument may be correct. The pessimistic outcome of Chapters 4 and 5, looking only at the supply-side, may be misplaced. However, if consumer expenditure shares do not shift sufficiently, or if expenditure shifts in the opposite direction, away from goods that require tasks in which labour retains the comparative advantage, then the optimism of the structural change argument is incorrect. The outcome is even more pessimistic than in Chapters 4 and 5. In ways that I will explain, this ambiguity is similar to the ambiguity in the AD model – but, unlike the AD model, this ambiguity does not rely on the ALM hypothesis.[1]

Two further lessons follow from this chapter. The first is that traditional explanations of structural change are necessary, but not sufficient, for understanding the consequences of technological progress on the labour market. This analysis is performed in the good-space, and considers how the demand for different goods changes. But the future prospects of labour depend upon the tasks that must be performed to produce those goods. Yet the traditional explanations, as set out in Chapter 1, say nothing about tasks.[2] A common claim, for example, is that just as labour that was displaced during the Agricultural Revolution or the Industrial Revolution was instead employed to produce different types of goods elsewhere in the economy, then so too labour displaced in future periods of structural change will also find work producing different goods elsewhere. The argument of this chapter is that this reasoning is mistaken – during past periods of structural change, labour was successfully re-deployed elsewhere not primarily because expenditure shifted towards different types of *goods*, but only because those goods required *tasks* in which labour, rather than capital, held the comparative advantage.[3]. It

[1]In the AD model, Autor and Dorn (2013) note that whether wage and employment polarisation takes place depends upon the elasticities of substitution in production and consumption. This is set out in section 1.5.4. The AD model, and its results, depend upon the ALM hypothesis. This is set out in section 6.3.

[2]See, for example, the discussion of the canonical model in section 1.2.2.

[3]Consider a more prosaic example – a business that wants to draw up an employment contract for a new employee. A few decades ago, the only way to get this 'good', would

is, however, entirely possible that future episodes of structural change will be characterised by expenditure shifting towards goods that require tasks in which capital, rather than labour, has the comparative advantage.

The second related lesson is that we are used to thinking about whether technological change is 'biased' in *production* – whether it complements or substitutes for a particular factor in supplying output. But we are not used to thinking about whether technological progress is 'biased' in *consumption* – whether it causes demand to change in the direction of a particular factor's comparative advantage.[4] Yet this chapter shows that this is an equally important part of the story. Bias on the demand-side can weaken or strengthen the pessimistic outcome reached when the supply-side is looked at in isolation as in Part B.

7.2. Examples from the Literature

All 'structural change' arguments appeal to the same principle – that expenditure shares on goods might change over time. But different arguments can use different explanations for why this structural change takes place. Re-

have been to go to a traditional lawyer. Today, many businesses use systems like LegalZoom.com, an online automated document preparation system. This system computerises the production of the 'good'. Note that the 'good' has not changed over the decades. What has changed is that the factor that holds the comparative advantage in performing the *tasks* required to produce those goods has switched from labour to capital. The important point is that if expenditure shifts to a different 'good', for example a different type of contract, that does not help labour if that type of contract requires the same type of tasks to be performed as an employment contract.

[4]Consider the 'skills-biased technological change' thesis from Chapter 1. This argues that technological change causes 'demand' to shift towards high-skilled labour. But the canonical model that reflects the SBTC thesis only has one good – this cannot therefore reflect consumer 'demand' in any meaningful sense, only producer demand. There are exceptions to this. For example, Xiang (2002, 2002b).

call from the previous chapter that there are three possible explanations for structural change – price effects, income effects, and endogenous preference effects.[5] Traditionally, only the first two explanations have been used.

In Autor and Dorn (2013), for example, structural change is caused by price effects – it is the interaction of the elasticities in production and consumption that cause employment polarisation.[6] In Goos et al. (2014), structural change is again driven by price effects. An argument for optimism based on these models – that changes in relative prices cause expenditure shares to change – would be a 'structural change' argument that relies on price effects.

Elsewhere in the task-based literature, 'structural change' arguments for optimism instead appeal to income effects. Consider, for example, Autor (2015):

> " ... [T]he output elasticity of demand combined with income elasticity of demand can either dampen or amplify the gains from automation. In the case of agricultural products over the long run, spectacular productivity improvements have been accompanied by declines in the share of household income spent on food. In other cases, such as the health care sector, improvements in technology have led to ever-larger shares of income being spent on health ... Rising income may also spur demand for activities that have nothing to do with the technological vanguard." (pg. 7)

Here the "income elasticity of demand" performs a role, alongside other effects. Arguments of this form are appealing to a traditional income effect – that as income rises expenditure shares across goods change. An argument for optimism based on this type of effect would be a 'structural change' argument that relies on income effects.

[5]See section 6.2.2.
[6]See section 1.5.4.

There is also an argument for optimism in the literature that is based on a different type of income effect – this is a 'rising' income effect, rather than a traditional income effect. This is an argument based on the fact that *total* expenditure rises with technological progress. Consider, as an example, Autor (2015b):

> "I think that people are extremely unduly pessimistic ... we neglect the fact that as we create wealth, which we certainly do through productivity improvements, we create more consumption. People want more experiences; they want more goods and services. Consequently, as people get wealthier, they tend to consume more, so that also creates demand."

And Summers (2013), in his reflections on the past debates about technological change and the labour market:[7]

> "The stupid people thought that automation was going to make all the jobs go away and there wasn't going to be any work to do. And the smart people understood that when more was produced, there would be more income and therefore there would be more demand." (pg. 3)

However this 'rising' income effect cannot result in optimism. Chapters 4 and 5 show why – if the assumptions of the ALM hypothesis are relaxed, rising incomes alone are not sufficient to avoid a pessimistic outcome for labour. This 'rising' income effect argument is not a structural change argument at all – although *total* expenditure rises, expenditure *shares* in fact remain fixed. When I refer to income effects in this chapter, then, it is with respect to the traditional income effect set out before.

[7] Summers (2013) presents this as an account of the debates about technological change and the labour market when he was a graduate student in the 1970s. In Summers (2013) he notes that he is now "no longer sure" it is right.

7.3. Structural Change in the New Model

In the new model in Part B there is no structural change – expenditure densities on each type of good i are fixed and equal to $\theta(i)$. This is a result of three features of the traditional Cobb-Douglas utility function used. First, the utility function is homothetic – changes in income do not affect expenditure densities. There is no traditional income effect. Secondly, this utility function is a CES function with an elasticity of substitution equal to one – changes in relative prices do not affect expenditure densities since a relative change in the MRS between any two goods is offset by an equal relative change in the consumption of those two goods. There is no price effect. Thirdly, this utility function is always the same – and so preferences do not change with income changes. There is no endogenous preferences effect. As incomes rise and prices change with technological progress, expenditure densities remain fixed.

Two broad options follow for introducing a role for the demand-side and exploring the 'structural change' argument for optimism set out before. The first broad option is to abandon the Cobb-Douglas utility function and adopt one of the traditional approaches to introducing structural change – use price effects and a CES utility function, or income effects and a Stone-Geary utility function. I explore the former in detail and set out a possible way to explore the latter in this chapter. These traditional approaches are not tractable in the new model in Part B without solving the model computationally. Both lead to a similar ambiguity to that noted by Autor and Dorn (2013), as discussed in the outline to this chapter.

The second broad option is to adopt an untraditional approach – retain the form of the Cobb-Douglas utility function, but use endogenous preferences and parameterise the expenditure densities in the utility function. Despite its untraditional character, this approach allows me to maintain the tractability of the new model in Part B and provides clear intuition when comparative statics are performed. It also provides an interesting challenge to the traditional approaches. I now follow both these broad options in detail. First I look at price effects and income effects.

7.3.1. Price Effects in the New Model

CES Utility Function

To introduce price effects into the new model I replace the Cobb-Douglas utility function with a CES utility function. The representative consumer faces the following maximisation problem:[8]

$$\begin{aligned} &\max_{\boldsymbol{x}} U = \left[\int_0^1 \theta(i)x(i)^\rho di\right]^{\frac{1}{\rho}} \\ &\text{s.t.} \quad \int_0^1 p(i)\cdot x(i)\,di \leq Y \end{aligned} \tag{7.1}$$

where $\rho \in [-\infty, 1]$ and $\sum_0^S \theta(i) = 1$.[9] A Lagrangian follows:

$$\mathcal{L} = \left[\int_0^1 \theta(i)x(i)^\rho di\right]^{\frac{1}{\rho}} - \lambda\left[Y - \int_0^1 p(i)\cdot x(i)\,di\right] \tag{7.2}$$

with accompanying first order conditions:

[8]Since preferences are homothetic, a representative approach can be used.
[9]And so the elasticity of substitution, $\sigma = \frac{1}{1-\rho} \in [0, \infty]$.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x(i)}: \quad & x(i)^{\rho-1} \cdot \theta(i) = \lambda \cdot p(i) \cdot \left[\int_0^1 \theta(i) x(i)^\rho di\right]^{\frac{\rho-1}{\rho}} \\
\frac{\partial \mathcal{L}}{\partial x(j)}: \quad & x(j)^{\rho-1} \cdot \theta(j) = \lambda \cdot p(j) \cdot \left[\int_0^1 \theta(i) x(i)^\rho di\right]^{\frac{\rho-1}{\rho}}
\end{aligned} \tag{7.3}
$$

And so from (7.3):

$$
\begin{aligned}
\frac{x(i)^{\rho-1} \cdot \theta(i)}{x(j)^{\rho-1} \cdot \theta(j)} &= \frac{p(i)}{p(j)} \\
x(i)^{\rho-1} &= \frac{p(i)}{p(j)} \cdot \frac{\theta(j)}{\theta(i)} \cdot x(j)^{\rho-1} \\
x(i) &= \left[\frac{p(i)}{p(j)} \cdot \frac{\theta(j)}{\theta(i)}\right]^{\frac{1}{\rho-1}} \cdot x(j) \\
x(j) &= \left[\frac{p(i)}{p(j)} \cdot \frac{\theta(j)}{\theta(i)}\right]^{\frac{1}{1-\rho}} \cdot x(i)
\end{aligned} \tag{7.4}
$$

Given the budget constraint it follows from (7.4) that:

$$
\begin{aligned}
Y &= \int_0^1 p(j) \cdot x(j)\, dj \\
&= \int_0^1 p(j) \cdot \left[\frac{p(i)}{p(j)} \cdot \frac{\theta(j)}{\theta(i)}\right]^{\frac{1}{1-\rho}} \cdot x(i)\, dj \\
&= x(i) \cdot \left[\frac{\theta(i)}{p(i)}\right]^{\frac{1}{\rho-1}} \cdot \int_0^1 \left[\frac{p(j)^\rho}{\theta(j)}\right]^{\frac{1}{\rho-1}} dj
\end{aligned} \tag{7.5}
$$

And so:

$$
\frac{x(i)}{Y} = \left[\frac{p(i)}{\theta(i)}\right]^{\frac{1}{\rho-1}} \cdot \left[\int_0^1 \left[\frac{p(j)^\rho}{\theta(j)}\right]^{\frac{1}{\rho-1}} dj\right]^{-1} \tag{7.6}
$$

This implies that expenditure densities $b(i)$ are equal to:

$$b(i) = \frac{x(i) \cdot p(i)}{Y} = \frac{\left[\frac{p(i)^{\rho}}{\theta(i)}\right]^{\frac{1}{\rho-1}}}{\int_0^1 \left[\frac{p(j)^{\rho}}{\theta(j)}\right]^{\frac{1}{\rho-1}} dj} \tag{7.7}$$

Note that if $\rho = 0$ and the elasticity of substitution is equal to 1 then (7.7) collapses to $b(i) = \theta(i)$ – this is the Cobb-Douglas case explored in Part B.[10] With these expenditure densities it is now possible to derive equilibrium.

Equilibrium

Consider the case where low-skilled labour competes with advanced capital to produce goods alongside traditional capital. The structure of this analysis is identical to Chapter 3 where low-skilled labour competes with high-skilled labour – but in this case, there is advanced capital instead of low-skilled labour. There is a relative productivity schedule for low-skilled labour with respect to advanced capital:

$$A^L(i) = \frac{a^L(i)}{a^A(i)} \tag{7.8}$$

and there is a market equilibrium schedule:

$$B(i) = \frac{\alpha(i)}{1 - \alpha(i)} \cdot \frac{K^A}{L^L} \tag{7.9}$$

where:

$$\alpha(\tilde{i}) = \int_0^{\tilde{i}} b(i) di \tag{7.10}$$

[10] Also note that $\frac{\rho}{\rho-1}$ in (7.7) is equal to $1 - \sigma$.

Equilibrium is determined by the intersection of these $A^L(i)$ and $B(i)$ schedules. $A^L(i)$ is decreasing i.e. $A^{L\prime}(i) < 0$ since Assumption 3 holds. And $B(i)$ is weakly increasing i.e. $B'(i) \geq 0$ since $b(i) \geq 0\ \forall i$. As a result, it follows that an equilibrium will exist with a CES utility function.[11]

With a CES utility function the shape of the $B(i)$ schedule depends upon equilibrium prices – this is because $B(i)$ depends upon $\alpha(i)$, as in (7.9), and $\alpha(i)$ depends upon $b(i)$ as in (7.10), and $b(i)$ depends upon $p(i)$ as in (7.7).[12] These prices will have the same form as in Chapter 3, as set out in (3.41) and (3.42).[13] It follows then that in equilibrium the price $p(i)$ of goods produced by low-skilled labour with traditional capital, i.e. where $i \in [0, \bar{i}]$, is:

$$p(i) = \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \tag{7.11}$$

and the price of goods produced by advanced capital with traditional capital, i.e. where $i \in [\bar{i}, 1]$, is:

$$p(i) = \left[\frac{r^A}{\psi \cdot a^A(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \tag{7.12}$$

And so, given (7.7), in equilibrium if $i \in [0, \bar{i}]$ the expenditure density on

[11]It is no longer the case that necessarily $B''(i) > 0$. However, a unique equilibrium will still exist.

[12]Dornbusch et al. (1977) noted this in a footnote – "the construction of the $B(\,)$ schedule relies heavily on the Cobb-Douglas demand structure" and they claim that "In the general homothetic case ... the independence of the $A()$ and $B()$ schedules is obviously lost". Dornbusch et al. (1977) does not explore this. This chapter is doing exactly that.

[13]This follows from the fact that, for a given set of equilibrium factor prices, the goods prices are independent of the form of the utility function. What matters is $\alpha(\tilde{i})$.

good $x(i)$ is:

$$b(i) =$$

$$\frac{\left[\left[\frac{w^L}{\psi\cdot a^L(i)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi}\right]^{\frac{\rho}{\rho-1}}\theta(i)^{\frac{1}{1-\rho}}}{\int_0^{\bar{j}}\left[\left[\frac{w^L}{\psi\cdot a^L(j)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(j)}\right]^{1-\psi}\right]^{\frac{\rho}{\rho-1}}\theta(j)^{\frac{1}{1-\rho}}\,dj+\int_{\bar{j}}^1\left[\left[\frac{r^A}{\psi\cdot a^A(j)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(j)}\right]^{1-\psi}\right]^{\frac{\rho}{\rho-1}}\theta(j)^{\frac{1}{1-\rho}}\,dj}$$

$$=\frac{\left(a^L(i)^{\psi}a^K(i)^{1-\psi}\right)^{\frac{\rho}{1-\rho}}\cdot\theta(i)^{\frac{1}{1-\rho}}}{\int_0^{\bar{j}}\left[(a^L(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj+\left[\frac{r^A}{w^L}\right]^{\frac{\psi\cdot\rho}{\rho-1}}\cdot\int_{\bar{j}}^1\left[(a^A(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj}$$

$$=\frac{\left(a^L(i)^{\psi}a^K(i)^{1-\psi}\right)^{\frac{\rho}{1-\rho}}\cdot\theta(i)^{\frac{1}{1-\rho}}}{\int_0^{\bar{j}}\left[(a^L(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj+\left[\frac{1}{A^L(\bar{i})}\right]^{\frac{\psi\cdot\rho}{\rho-1}}\cdot\int_{\bar{j}}^1\left[(a^A(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj} \tag{7.13}$$

and if $i \in [\bar{i}, 1]$:

$$b(i) =$$

$$\frac{\left[\left[\frac{r^A}{\psi\cdot a^A(i)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(i)}\right]^{1-\psi}\right]^{\frac{\rho}{\rho-1}}\theta(i)^{\frac{1}{1-\rho}}}{\int_0^{\bar{j}}\left[\left[\frac{w^L}{\psi\cdot a^L(j)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(j)}\right]^{1-\psi}\right]^{\frac{\rho}{\rho-1}}\theta(j)^{\frac{1}{1-\rho}}\,dj+\int_{\bar{j}}^1\left[\left[\frac{r^A}{\psi\cdot a^A(j)}\right]^{\psi}\left[\frac{r}{(1-\psi)\cdot a^K(j)}\right]^{1-\psi}\right]^{\frac{\rho}{\rho-1}}\theta(j)^{\frac{1}{1-\rho}}\,dj}$$

$$=\frac{\left(a^A(i)^{\psi}a^K(i)^{1-\psi}\right)^{\frac{\rho}{1-\rho}}\cdot\theta(i)^{\frac{1}{1-\rho}}}{\left[\frac{w^L}{r^A}\right]^{\frac{\psi\cdot\rho}{\rho-1}}\cdot\int_0^{\bar{j}}\left[(a^L(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj+\int_{\bar{j}}^1\left[(a^A(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj}$$

$$=\frac{\left(a^A(i)^{\psi}a^K(i)^{1-\psi}\right)^{\frac{\rho}{1-\rho}}\cdot\theta(i)^{\frac{1}{1-\rho}}}{[A^L(\bar{i})]^{\frac{\psi\cdot\rho}{\rho-1}}\cdot\int_0^{\bar{j}}\left[(a^L(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj+\int_{\bar{j}}^1\left[(a^A(j)^{\psi}a^K(j)^{1-\psi})^{\frac{\rho}{1-\rho}}\cdot\theta(j)^{\frac{1}{1-\rho}}\right]dj} \tag{7.14}$$

These are the expenditure densities for each good i in the general CES case.[14] The shape of the $B(i)$ schedule in equilibrium, given its definition in (7.9) and (7.10), depends upon the expressions for $b(i)$ in (7.13) and (7.14). These general expenditure densities are less tractable than the particular case where utility is Cobb-Douglas and $\rho = 0$.[15] Nevertheless, it is possible to derive equilibrium computationally for the general case. The challenge in determining the $B(i)$ schedule computationally is that it is defined recursively – $B(i)$ determines the endogenous cut-off $\bar{i}$ through its intersection with $A^L(i)$ but the shape of $B(i)$ is itself determined by the endogenous cut-off $\bar{i}$. In particular the shape of the $B(i)$ schedule depends upon $\bar{i}$ in two ways. Recall from (7.9) that $B(i)$ depends upon $\alpha(i)$ which in turn depends upon $b(i)$. The $B(i)$ schedule therefore depends upon $\bar{i}$ both because whether $b(i)$ is defined as in (7.13) or (7.14) for any particular good i depends upon whether $i \in [0, \bar{i}]$ or $i \in [\bar{i}, 1]$, and also because the denominator for both definitions of $b(i)$ depends upon $\bar{i}$ (or, $\bar{j}$, as written in 7.13 and 7.14).

To find $\bar{i}$ and the equilibrium shape of $B(i)$ I adopt the following two-stage approach. In the first stage I generate the set of all possible $B(i)$ schedules – i.e. the $B(i)$ schedule implied for each possible candidate cut-off. I call these candidate cut-offs $\tilde{\bar{i}}$, and each candidate schedule $B(i|\tilde{\bar{i}})$. I do this by discretising the continuous goods spectrum $i \in [0, 1]$ and constructing a square n by n matrix, called BB, where $n = 500$. The horizontal dimension of BB captures the range of different candidate cut-offs $\tilde{\bar{i}}$, where $\tilde{\bar{i}} \in \{0, \ ... \ , 1\}$ and

[14]Note that in both (7.13) and (7.14) the final step is made by recognising that $\frac{r^A}{w^L} = \frac{1}{A^L(\bar{i})}$. This follows from the fact that in equilibrium $A^L(\bar{i}) = B(\bar{i}) = \frac{w^L}{r^A}$.

[15]Note that if $\rho = 0$, the expenditure densities in (7.13) and (7.14) again collapse to the Cobb-Douglas case.

the vertical dimension captures the different i, where $i \in \{0, \ ... \ , 1\}$. Each entry in the matrix is therefore the expenditure density $b(i)$ for a particular good i given the candidate cut-off $\tilde{\bar{i}}$ – where $b(i)$ is defined as in (7.13) and (7.14). For example, entry $BB(x, y)$ is the expenditure density for good x given the candidate cut-off y – I write this as $b(x|y)$. The BB matrix is shown below:

$$BB = \begin{pmatrix} b(i_1|\,\tilde{\bar{i}}_1) & b(i_1|\,\tilde{\bar{i}}_2) & \cdots & (i_1|\,\tilde{\bar{i}}_n) \\ b(i_2|\,\tilde{\bar{i}}_1) & b(i_2|\,\tilde{\bar{i}}_2) & \cdots & b(i_2|\,\tilde{\bar{i}}_n) \\ \vdots & \vdots & \ddots & \vdots \\ b(i_n|\,\tilde{\bar{i}}_1) & b(i_n|\,\tilde{\bar{i}}_2) & \cdots & b(i_n|\,\tilde{\bar{i}}_n) \end{pmatrix} \tag{7.15}$$

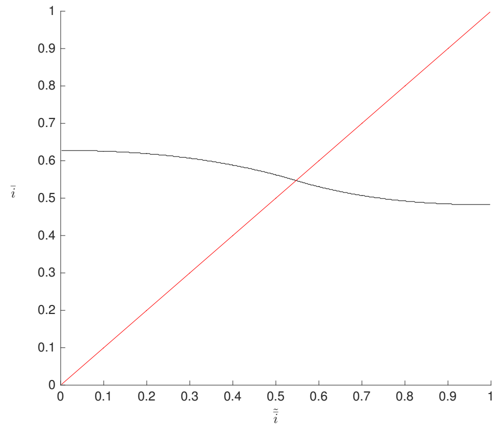
The sum of the entries in each column of this matrix is equal to 1 – this is the sum of expenditure densities across all goods, for a given candidate cut-off $\tilde{\bar{i}}$. Each column in this matrix therefore represents the distribution of expenditure densities across different goods, $b(i)$, for a given $\tilde{\bar{i}}$. As a result, the set of $b(i|\tilde{\bar{i}})$ in each column of BB generates a $B(i)$ schedule for each $\tilde{\bar{i}}$. This is $B(i|\tilde{\bar{i}})$. Put another way, each $\tilde{\bar{i}}$ implies a $B(i|\tilde{\bar{i}})$ schedule.

In the second stage I then find the actual $B(i)$ in this set of $B(i|\tilde{\bar{i}})$. Each $B(i|\tilde{\bar{i}})$ schedule intersects the $A^L(i)$ schedule and generates an actual intersection point $\bar{i}$. This intersection point is necessarily unique – given that $b(i|\bar{i}) \geq 0 \ \forall i$ it follows that all $B(i|\bar{i})$ are weakly positively sloped and so, given $A(i)$ is downward sloping, there is only one intersection between the two schedules.[16] The actual $B(i)$ schedule is therefore the $B(i|\tilde{\bar{i}})$ schedule such that the $\bar{i}$ implied by the intersection of $B(i|\tilde{\bar{i}})$ and $A^L(i)$ is equal to $\tilde{\bar{i}}$. Figure

[16] See section 3.2.4 for a discussion of this uniqueness property.

7.1 shows this for one particular parameterisation.[17] The red 45° line is where $\tilde{\bar{i}} = \bar{i}$. The black plot represents the $\bar{i}$ generated by the intersection between the $B(i|\tilde{\bar{i}})$ schedule, based on $\tilde{\bar{i}}$, and the $A^L(i)$ schedule. The intersection point in Figure 7.1 is the actual cut-off – where $\tilde{\bar{i}}$ is equal to the actual $\bar{i}$.

Figure 7.1: Finding $\tilde{\bar{i}} = \bar{i}$



Comparative Statics

I now consider the consequence of an increase in the relative productivity of advanced capital on equilibrium with price effects. To perform these comparative statics I must specify the underlying absolute productivity schedules for the different factors. In the simulations that follow I assume that:

$$\begin{aligned} a^L(i) &= a - b \cdot i \\ a^A(i) &= c \\ a^K(i) &= d \end{aligned} \tag{7.16}$$

[17] $\rho = 0.9$. The other parameters are those used in the comparative statics that follow.

where a, b, c and d are positive constants. These schedules imply that both types of capital have a constant absolute productivity across the spectrum of tasks, and low-skilled labour has a diminishing absolute productivity. These schedules are a simple way to generate the downward sloping relative productivity schedule $A^L(i)$ between low-skilled labour and advanced capital that was set out in Part B. The particular parameterisation is as follows – initially, $a = 10$, $b = 5$, $c = 10$, and $d = 1$, and the comparative static exercise is an increase in c to 20, reflecting a uniform increase in the absolute productivity of advanced capital. In addition, I assume that $\psi = 0.5$, $\theta(i) = 1 \ \forall i$, and the stock of factors are all constant and equal to one another. The results that follow are robust to more complex specifications of the schedules. I adopt those in (7.16) for simplicity.

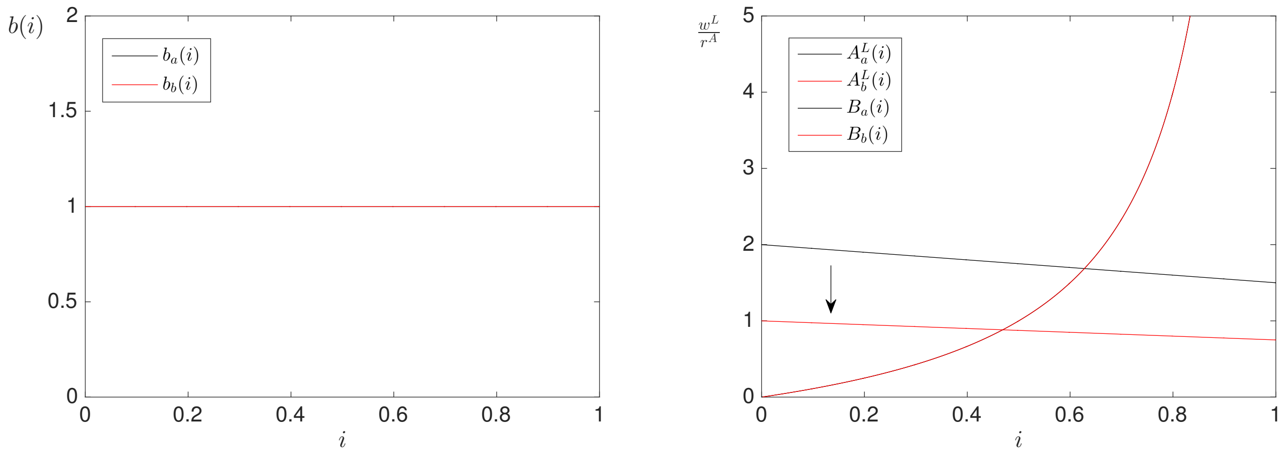
There are three cases of particular interest. The first is the benchmark Cobb-Douglas case in Part B – where $\rho = 0$ and so the elasticity of substitution is $\sigma = 1$, and there are no price effects. The second is the case where $\rho \in (0, 1]$ and so the elasticity of substitution is $\sigma \in (1, \infty]$, and there are price effects. The third is the case where $\rho \in [-\infty, 0)$ and so the elasticity of substitution is $\sigma \in [0, 1)$, and there are price effects. Consider first the benchmark case where the elasticity of substitution $\sigma = 1$.[18] The consequence of a uniform increase in the absolute productivity of advanced capital on expenditure densities and equilibrium is shown in Figure 7.2.[19] The left-hand side diagram shows the distribution of expenditure densities $b(i)$ across the goods before and after the uniform increase in the absolute productivity of

[18]Recall again that the elasticity of substitution is $\sigma = \frac{1}{1-\rho} \in [0, \infty]$.

[19]Again I consider the case where c increases from 10 to 20.

advanced capital. The right-hand side diagram shows the $B(i)$ and $A^L(i)$ schedules before and after the uniform increase in the absolute productivity of advanced capital.[20]

Figure 7.2: Cobb-Douglas case where $\sigma = 1$



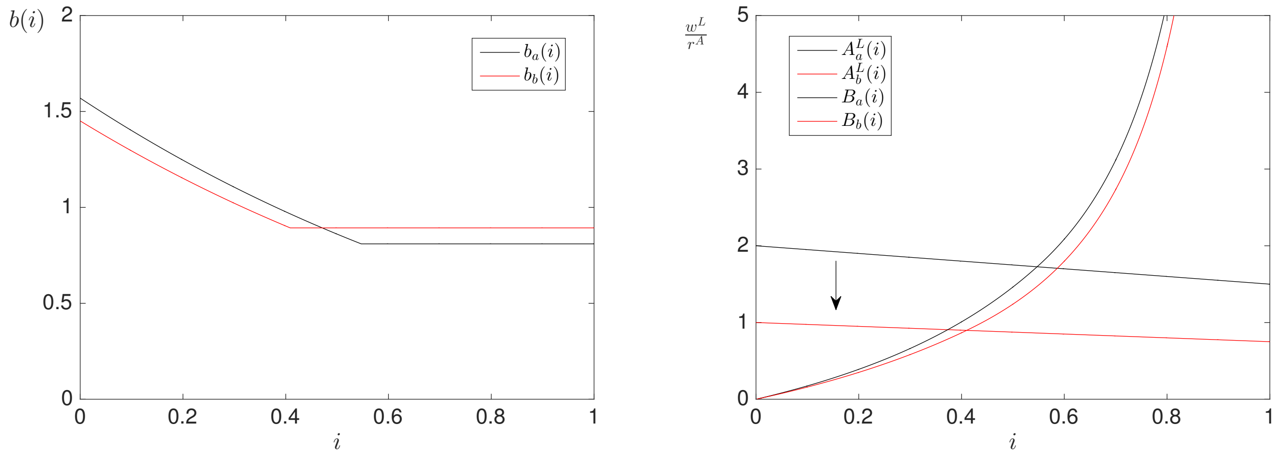
As in Part B, there are no demand-side effects, only supply-side effects. The increase in the relative productivity of advanced capital does not affect expenditure densities across the different goods – $b_a(i)$ and $b_b(i)$ in the left-hand side diagram are the same. Given this, the $B(i)$ schedule in the right-hand side diagram is unchanged – $B_a(i)$ is the same as $B_b(i)$. The fall in the $A^L(i)$ schedule from $A^L_a(i)$ to $A^L_b(i)$ causes the relative wage of low-skilled labour to fall, and $\bar{i}$ to fall. This is the pessimistic outcome in Part B.

However, now consider the second case where $\rho \in (0, 1]$ and the elasticity of substitution $\sigma \in (1, \infty]$. This is shown in Figure 7.3.[21]

[20] Note that because the increase in the productivity of advanced capital is now an absolute increase, rather than a relative increase, the shape of the $A^L(i)$ schedules changes.

[21] In this particular parameterisation I use $\rho = 0.9$.

Figure 7.3: Elastic case where $\sigma > 1$



There are two important features to note about the distribution of expenditure densities $b(i)$ in the left-hand side diagram of Figure 7.3. The first is that both $b_a(i)$ and $b_b(i)$ are no longer uniform as in the Cobb-Douglas case.[22] The second feature is that, after the increase in the productivity of advanced capital, price effects cause expenditure densities to shift towards goods that are produced with more complex tasks i.e. those that are produced by advanced capital – this is the shift from $b_a(i)$ to $b_b(i)$ in the left-hand diagram of Figure 7.3. For any $\tilde{i}$, there is a greater mass of expenditure on goods $i > \tilde{i}$ for $b_b(i)$ than for $b_a(i)$. Unlike the Cobb-Douglas case technological progress now has both supply-side effects *and* demand-side effects.

Again the supply-side effect is reflected by the fall in $A^L_a(i)$ schedule to

[22]The particular shape of the $b(i)$ schedules in Figure 7.3 follows from (7.16). Expenditure shares are constant on all goods produced by advanced capital since it has the same productivity at all tasks – and so the price of all goods produced by advanced capital is the same for any given r^A and r. Expenditure shares are decreasing on goods produced by low-skilled labour as i increases since the absolute productivity of low-skilled labour falls as i increases – and so the price of goods produced by low-skilled labour and traditional capital rises as i increases.

$A_b^L(i)$ in the right-hand side diagram of Figure 7.3, showing equilibrium. This drives down the relative wage and causes $\bar{i}$ to fall. Labour is forced to specialise in a smaller set of types of tasks, at a lower relative wage. But in addition, the shift in expenditure towards goods in which advanced capital has the comparative advantage causes the $B(i)$ schedule in the right-hand side diagram to shift outward – from $B_a(i)$ to $B_b(i)$. This is the additional demand-side effect, and it drives the relative wage down even further. The outcome is even more pessimistic for labour. Note, however, that while the demand-side effect puts further downward pressure on the relative wage, it puts upward pressure on $\bar{i}$. This follows from the fact that for a given set of absolute productivities, if expenditure densities shift towards goods produced by advanced capital then the return to advanced capital rises relative to the return on labour, making labour relatively cheaper and, as a result, labour is dragged into performing a larger range of types of tasks.

These result is intuitive – not only does the supply-side effect cause the balance of comparative advantage to shift in favour of advanced capital but, in addition, a demand-side effect causes the pattern of expenditure to shift from the types of tasks in which labour has the comparative advantage, towards the sort of tasks that advanced capital has the comparative advantage in. Both these effects lead to a pessimistic outcome for labour.

It is possible to also consider the third case where $\rho \in [-\infty, 0)$ and the elasticity of substitution $\sigma \in [0, 1)$. I do not show this case here, but again there are price effects – and now they operate in the opposite direction to the previous case. Expenditure shares shifts towards goods in which low-skilled labour, rather than advanced capital, has the comparative advantage. The result is that now the demand-side effect offsets, rather than strengths, the pessimistic outcome reached when the supply-side is looked at in isolation in Part B.

Implications for the Structural Change Argument

The 'structural change' argument for optimism set out in the outline to this chapter states that technological progress may displace labour from producing certain types of goods, but it will also cause structural change and a change in the distribution of expenditure across different types of goods – and labour can instead be employed in producing those goods. The results in this section show that when structural change is driven by price effects, this result is possible but not inevitable. In fact, in the case set out where the elasticity of substitution $\sigma > 1$ the result is the opposite – expenditure shifts towards goods produced by advanced capital not by low-skilled labour, and the pessimistic outcome set out in Chapter 4 and 5, which looked at the supply-side in isolation, is worsened.

This ambiguity about the effect of the demand-side is similar to the ambiguity in the AD model in Autor and Dorn (2013). In the AD model, the effect of the demand-side on relative wages and the distribution of employment across the production of different goods depends upon the size of both the elasticities of substitution in production and consumption.[23] In this model it is the consumption elasticity that drives the ambiguity. However, the most important difference is that, unlike the AD model, this result does

[23]The AD model is shown in section 1.5.4.

not rely on the ALM hypothesis.[24]

7.3.2. Income Effects in the New Model

It is also possible to introduce income effects into the new model. However, because of the agent heterogeneity in the model – different agents own different factor stocks with different returns – the analysis is far more complex. Suppose for simplicity that the economy consists of a single consumer who owns all the factors.[25] The traditional way to model income effects in this setting is to replace the Cobb-Douglas utility function with a Stone-Geary utility function:

$$U = \int_0^1 \theta(i) \ln\left[x(i) - \gamma(i)\right] di \tag{7.17}$$

where $\gamma(i)$ is the subsistence level of each good i. It follows that the expenditure density on each good is equal to:

$$b(i) = \frac{\gamma(i) \cdot p(i) + \theta(i)\left[Y - \int_0^1 p(i)\gamma(i)di\right]}{Y} \tag{7.18}$$

If consumers have Stone-Geary utility functions, they first spend $\int_0^1 p(i)\gamma(i)di$ on the subsistence level of each good, and then spend their remaining income according to the weights they attach to each good, $\theta(i)$. Again, using the prices in (7.11) and (7.12) it follows that if $i \in [0, \bar{i}]$ the expenditure density

[24]The AD model's reliance on the ALM hypothesis is set out in section 6.3.3.

[25]This is clearly a strong simplification – a representative approach like this ignores the underlying agent heterogeneity in the economy.

on good $x(i)$ is:

$$
\begin{aligned}
b(i) = {} & \frac{\gamma(i)}{Y} \cdot \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} + \theta(i) \\
& - \frac{\theta(i)}{Y} \cdot \left[\int_0^{\bar{i}} \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \gamma(i) di\right] \\
& - \frac{\theta(i)}{Y} \cdot \left[\int_{\bar{i}}^{1} \left[\frac{r^A}{\psi \cdot a^A(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \gamma(i) di\right]
\end{aligned} \tag{7.19}
$$

and if $i \in [\bar{i}, 1]$ the expenditure density on good $x(i)$ is:

$$
\begin{aligned}
b(i) = {} & \frac{\gamma(i)}{Y} \cdot \left[\frac{r^A}{\psi \cdot a^A(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} + \theta(i) \\
& - \frac{\theta(i)}{Y} \cdot \left[\int_0^{\bar{i}} \left[\frac{w^L}{\psi \cdot a^L(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \gamma(i) di\right] \\
& - \frac{\theta(i)}{Y} \cdot \left[\int_{\bar{i}}^{1} \left[\frac{r^A}{\psi \cdot a^A(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi} \gamma(i) di\right]
\end{aligned} \tag{7.20}
$$

Now the shape of $B(i)$ in equilibrium follows from (7.19) and (7.20). Again, given these expenditure densities it is possible to solve the model computationally. I do not do this here, and in concluding this thesis I note this as an interesting area for further work. However, given the analysis of the price effects in the previous sub-section, the expected results are intuitive. It would be possible to partition goods into necessities and luxuries, where technological progress and rising incomes causes expenditure shares to fall on the former and rise on the latter. In this new task-based model, the consequence of this technological progress for labour would depend upon the complexity of the tasks that must be performed to produce those goods – for example, if luxuries require more complex tasks in which advanced capital, rather than low-skilled labour has the comparative advantage, this will likely worsen the pessimistic outcome that follows from the supply-side analysis in Part B. Again, this is an interesting area for further work.

7.4. An Alternative Approach

I now turn to the second broad option set out in section 7.3. This provides far more tractable insights into the role of the demand-side than those offered by exploring traditional price or income effects, but as I will show also provides the same central insight.

7.4.1. Technology-Dependent Preferences

Both the traditional approaches to introducing structural change involve moving away from the Cobb-Douglas utility function used in Part B – either through price effects with a CES utility function or through income effects with a Stone-Geary utility function. But these traditional approaches are far less tractable and intuitive than the Cobb-Douglas approach.

An alternative approach that preserves this intuition is to *retain* the Cobb-Douglas utility function as an account of aggregate demand in the economy – and so maintain the tractability of the model in Part B – but with the additional feature that the expenditure density associated with each good depends upon a parameter a, where a that captures how the economic environment changes over time. In the next sub-section I will define a more exactly. This utility function is as follows:

$$U(x,a) = \int_0^1 \theta(i,a) \ln x(i)\, di \tag{7.21}$$

The income invariant parameter $\theta(i)$ from Part B is replaced by $\theta(i,a)$. One way to interpret this utility function is as taking seriously the third explanation of structural change in Chapter 6 – that tastes and preferences are endogenous, and change over time as the parameter a changes. And although this is untraditional, the tractability of a traditional Cobb-Douglas utility function is retained – $\int_0^1 \theta(i,a)\, di = 1\ \forall a$ and expenditure densities are not affected by changes in relative prices or income. On this approach, demand for good i is equal to:

$$x(i) = \frac{Y}{p(i)} \cdot \theta(i,a) \tag{7.22}$$

This parameterised utility function and associated demand function are clearly an untraditional way to motivate structural change. But there are three reasons why it is sensible to adopt it. First, although the notion of endogenous preferences is untraditional, it has an intuitive explanatory appeal. It is an equally compelling explanation of long-run structural change as those traditional explanations that rely on income and price effects.[26] Secondly, the gain in tractability is very useful and the main lessons of this chapter are far easier to understand. Finally, and most importantly, these main lessons, although derived in the case where endogenous preferences cause structural change, in fact also apply to structural change that is caused instead by price

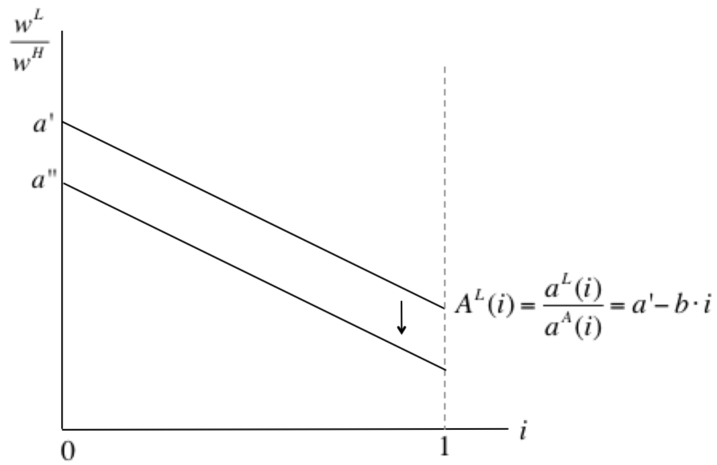
[26]See the discussion of Stigler and Becker (1977) in section 6.2.2.

and income effects. It provides a general insight about structural change. I will explain this in greater depth at the end of this chapter.

7.4.2. Interpreting the a Parameter

In what follows, I assume for convenience that the parameter a depends upon the relative productivity of labour and advanced capital – a can be interpreted as the y-intercept of the relative productivity schedule for labour $A(i)$.[27] A *fall* in a implies a uniform increase in the relative productivity of advanced capital over labour. This is shown in Figure 7.4 for low-skilled labour and advanced capital. I refer to a fall in a as 'technological progress'.[28]

Figure 7.4: Technological Progress: a' to a''



[27]See the Chapter 3 comparative statics where a is the y-intercept of the $A(i)$ schedule.

[28]Note that because the productivity of labour $a^L(i)$ does not change, a fall in a reflects an increase in the productivity of advanced capital. This is why calling a fall in a 'technological progress' is intuitive.

7.4.3. Properties of the $\theta(i, a)$ Function

In a model with utility functions as in (7.21), the expenditure densities, $b(i)$, are equal to $\theta(i, a)$. For the purposes of exploring the 'structural change' argument for optimism, two features of the $\theta(i, a)$ function are important. The first is the value of the partial derivative $\theta_2(i, a)$ – i.e. how technological progress affects the expenditure density on a particular good i. This partial derivative is about expenditure on goods, and so is set in the good-space. The second feature is the value of the cross-partial derivative $\theta_{21}(i, a)$ – i.e. how the value of $\theta_2(i, a)$ changes as i increases. Since a larger i implies that good i requires a more complex task $z_1(i)$ to produce it, this cross-partial describes whether, as technological progress takes place, expenditure densities shift towards goods that require more, or less, complex tasks. Importantly, this cross-partial derivative is about both goods *and* tasks – it is set in both the good-space and the task-space, reaching across the two. I now turn to look at each of these derivatives in detail.

Partial Effect: The Value of $\theta_2(i, a)$

Given that the expenditure density on a given type of good i is $\theta(i, a)$ it follows that the sensitivity of the expenditure density on some type of good i with respect to technological progress is:[29]

$$\frac{\partial b(i, a)}{\partial a} = \theta_2(i, a) \tag{7.23}$$

[29] Recall that expenditure densities are defined in general as $b(i)$ in Part B. And so with a traditional Cobb-Douglas $b(i) = \theta(i)$. That is why I use $b(i, a)$ again here.

If $\exists i \in [0,1]$ *s.t.* $\theta_2(i,a) \neq 0$ then expenditure densities are *dependent* on technological progress. But if $\theta_2(i,a) = 0 \, \forall i \in [0,1]$ then expenditure densities are *independent* of technological progress.[30] Given that $\int_0^1 \theta(i,a)\,di = 1$ it must also be the case that:

$$\int_0^1 \theta_2(i,a)\,di = 0 \tag{7.24}$$

(7.24) says that, after technological progress and a change in a, the total change in the values of $\theta(i,a)$ must sum to zero so the total value of all $\theta(i,a)$ is again equal to 1. As a result the set of all goods i can be partitioned into two subsets – those where $\theta_2(i,a) > 0$ and expenditure densities on them *fall* with technological progress, and those where $\theta_2(i,a) < 0$ and expenditure densities on them *rise* with technological progress.[31]

Cross-Partial Effect: The Value of $\theta_{21}(i,a)$

The second important feature of $\theta(i,a)$ is $\theta_{21}(i,a)$. This captures the relationship between the value of $\theta_2(i,a)$ and the complexity of the task $z_1(i)$ that must be done to produce that particular good i. Formally this is:

$$\frac{\partial^2 b(i,a)}{\partial a\,\partial i} = \theta_{21}(i,a) \tag{7.25}$$

As noted, whereas the value of $\theta_2(i,a)$ is set in the good-space, the derivative in (7.25) also draws on the task-space. As a result, a non-task-based approach cannot consider this cross-partial derivative. A non-task-based account of

[30] This case is captured by a traditional Cobb-Douglas utility function, as in Part B.
[31] Recall that technological progress is capture by a *fall* in a.

structural change can only consider goods – in particular, how technological progress causes the pattern of expenditure on goods changes i.e. $\theta_2(i,a)$. But by using a task-based approach it is possible to consider the tasks associated with those goods as well. It follows that there are three, simple, possible cases for the value of (7.25).

In Case 1, $\theta_{21}(i,a) < 0 \ \forall i$. This implies that as technological progress takes place and a falls, the change in expenditure densities $\theta_2(i,a)$ across goods is *increasing* in i. Put another way, as technological progress takes place, expenditure shifts towards the type of goods that require more complex tasks to produce them i.e. those with a higher i, in which high-skilled labour has the comparative advantage. This is the case of *high-skilled labour bias* in demand.

In Case 2, $\theta_{21}(i,a) > 0 \ \forall i$. This implies that as technological progress takes place and a falls, the change in expenditure densities $\theta_2(i,a)$ across goods is *decreasing* in i. Put another way, as technological progress takes place, expenditure shifts towards the type of goods that require simpler tasks to produce them i.e. those with a lower i, in which low-skilled labour has the comparative advantage. This is the case of *low-skilled labour bias* changes in demand.

In Case 3, $\theta_{21}(i,a) = 0 \ \forall i$. This implies that as technological progress takes place and a falls, the change in expenditure densities $\theta_2(i,a)$ across goods is *invariant* to i. This is the case that characterises the model in Part B. This is the case of *no bias* in demand. I now consider these cases in turn.

7.5. The Three Cases

To simplify the analysis, I again only consider the consequences when one type of labour – first low-skilled labour, then high-skilled labour – and advanced capital compete to perform tasks with traditional capital.

7.5.1. Case 1: High-Skilled Labour Bias

Preliminaries

In Case 1, $\theta_{21}(i,a) < 0$ – with technological progress, expenditure densities shift towards goods that require more complex tasks $z_1(i)$ to produce them. For simplicity, I assume that:

$$\theta_{21}(i,a) = -k_1 \tag{7.26}$$

Where k_1 is some positive constant. This implies that as technological progress takes place the changes in expenditure densities across the goods, equal to $\theta_2(i,a)$ for a given good i, are increasing at a constant rate as i increases. What I want to do is find an expression for $\theta(i,a)$ that results in a constant cross-partial derivative $\theta_{21}(i,a) = -k_1$ as in (7.26). This is shown in detail in the Appendix, section 9.2.1, and the brief account is as follows. This restriction on the cross-partial derivative $\theta_{21}(i,a) = -k_1$ puts restrictions on $\theta(i,a)$, but does not uniquely define it – there are two constants of integration that result in the expression for $\theta(i,a)$. But, given that (7.24) must hold, one constant of integration is pinned down. To pin down the other constant of integration, I choose a particular form for $\theta(i,a)$ such that I can compare $\theta(i,a)$ to an initial value $\theta(i,a^*)$ where a^* is the level of technology such that $\theta(i,a) = 1$ – this is the case in Part B, where all goods have the same level of expenditure on them. The expression for $\theta(i,a)$ is then as follows:

$$\theta(i,a) = -k_1 \cdot (a - a^*) \cdot \left[i - \frac{1}{2}\right] + 1 \tag{7.27}$$

where a^* is some level of technological progress at which expenditure densities on each type of good $\theta(i,a)$ are constant and equal to θ.[32] As a result, as a falls below a^*, the expenditure schedule $\theta(i,a)$ rotates counter-clockwise around the point $i = \frac{1}{2}$.[33] The type of good $i = \frac{1}{2}$ marks the boundary between goods that have a falling share of expenditure and those that have a growing share of expenditure:

1. $\theta_2(i,a) = -k_1(i - \frac{1}{2}) > 0 \quad \forall i \in [0, \frac{1}{2}]$
2. $\theta_2(i,a) = -k_1(i - \frac{1}{2}) = 0 \quad i = \frac{1}{2}$
3. $\theta_2(i,a) = -k_1(i - \frac{1}{2}) < 0 \quad \forall i \in [\frac{1}{2}, 1]$

The New $B(i)$ Schedule

In this simple set-up low-skilled labour $L^L(i)$ again competes with advanced capital $K^A(i)$ to produce goods alongside traditional capital. The $B(i)$ sched-

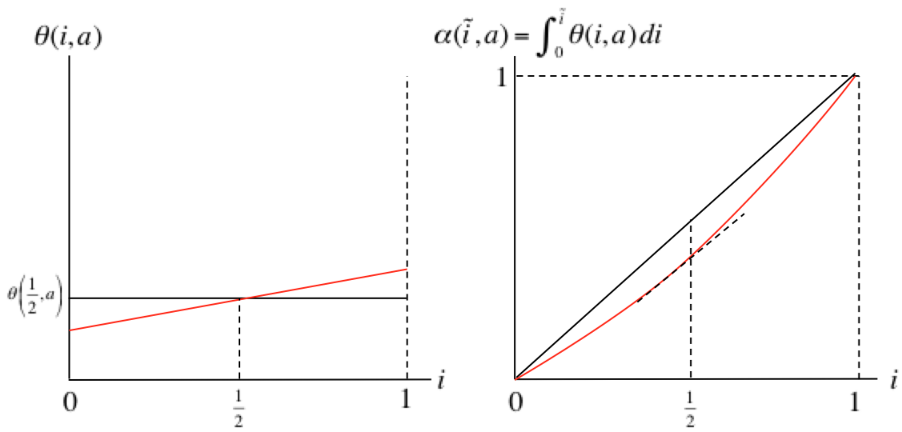
[32] Recall that in Part B this was the case, given Assumption 2.

[33] Note that the simple linear representation of $\theta(i,a)$ in (7.27) only applies for a bounded set of a e.g. for a given k_1, if the change in a is too large then $\theta(i,a)$ might be negative – which is not sensible). A more complex distribution of $\theta(i,a)$ can accommodate this (for example, a 'Kumaraswamy distribution' that becomes increasingly left skewed as m rises, see Jones 2009). It can be thought of as an approximation.

ule is defined as in (7.9). Equilibrium is again determined by the intersection of this $B(i)$ schedule with the relative productivity schedule, $A^L(i)$, defined in (7.8). The model has exactly the same structure as in Chapter 4, but without high-skilled labour.

In Case 1, as technological progress takes place, the $\theta(i,a)$ schedule rotates around the point $(\theta(\frac{1}{2}), \frac{1}{2})$. This follows from the three conditions set out before i.e. that for $i \in [0, \frac{1}{2}]$ $\theta_2(i,a) > 0$, $i \in [\frac{1}{2}, 1]$ $\theta_2(i,a) < 0$, and for $i = \frac{1}{2}$ $\theta_2(i,a) = 0$. As technological progress takes place, consumers spend a greater proportion of their income on goods that require more complex tasks to produce them. This is shown in the left-hand digram of Figure 7.5 for a given fall in a below a^*.

Figure 7.5: Case 1: Change in $\theta(i,a)$ for a Fall in a



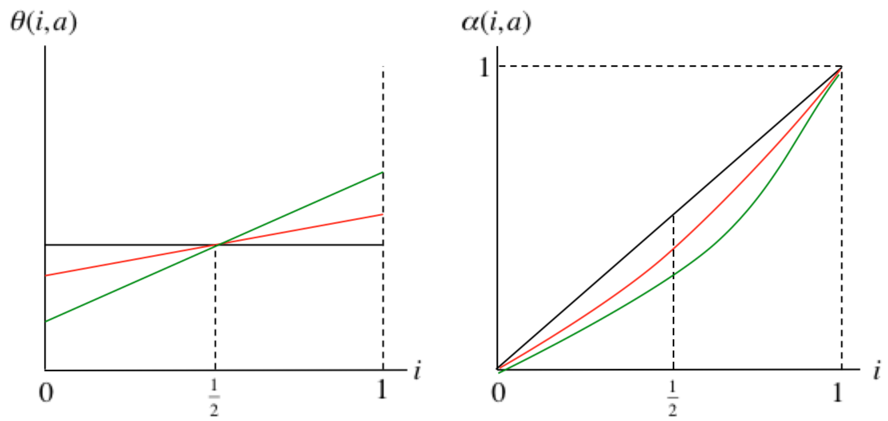
The slope of the new schedule (in red) on the left-hand side is constant, and equal to $-k_1(a - a^*)$.[34] This follows from (7.27). In turn, this means the $\alpha(i,a)$ schedule becomes more convex, as the new schedule (also in red)

[34]Note that $-k_1(a - a^*) > 0$ here because technological progress is associated with a fall in a below a^*.

on the right-hand side of Figure 7.5 shows. This follows from the fact that $\alpha(\tilde{i}, a) = \int_0^{\tilde{i}} \theta(i, a)\, di$ and so, for a given i, the *gradient* of the new schedule on the right-hand side is equal to the *value* of the new schedule on the left-hand side – and so the gradient of $\alpha(i, a)$ is initially lower, but increases in i. Note that since $\theta(\frac{1}{2}, a)$ does not change with technological progress, both the new and old schedule have the same gradient at $\alpha(\frac{1}{2}, a)$.

A further increase in technological progress, and a further fall in a, causes a further rotation in the $\theta(i, a)$ schedule around the point $(\theta(\frac{1}{2}), x(\frac{1}{2}))$. The $\alpha(i, a)$ schedule becomes more convex, but again all of the schedules have the same gradient at $i = \frac{1}{2}$. This is shown in Figure 7.6.

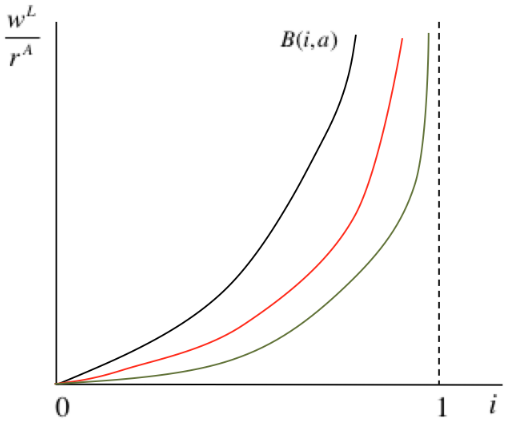
Figure 7.6: Case 1: Change in $\theta(i, a)$ and $\alpha(i, a)$ for a Further Fall in a



Importantly, the changes in expenditure densities shown Figure 7.5 and 7.6 now affect the shape of the $B(i)$ schedule.[35] This is shown in Figure 7.7.

[35] I define this schedule now as $B(i, a)$ to reflect the influence of the level technological progress, a, on the shape of this schedule.

Figure 7.7: Case 1: $B(i, a)$ as a falls



The shape of the $B(i, a)$ schedules follow from (7.9). First, in all three cases, $B(0, a) = 0$ and $\lim_{i \to 1} B(i, a) = \infty$. Secondly, $B_{black}(i, a) > B_{red}(i, a) > B_{green}(i, a)$ – since $\alpha_{black}(i, a) > \alpha_{red}(i, a) > \alpha_{green}(i, a)$. Thirdly, the schedules are positively sloped since:

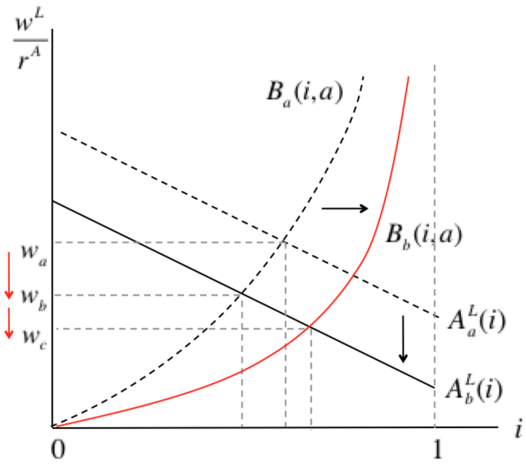
$$\begin{aligned} \frac{\partial B(i,a)}{\partial i} &= \frac{\theta(i,a)}{(1-\alpha(i,a))^2} \cdot \frac{K^A}{L^L} \\ &> 0 \end{aligned} \tag{7.28}$$

and have an increasing positive slope:

$$\begin{aligned} \frac{\partial^2 B(i,a)}{\partial i^2} &= \frac{\alpha''(i,a)(1-\alpha(i,a))^2 + 2\alpha'(i,a)^2(1-\alpha(i,a))}{(1-\alpha(i,a))^4} \cdot \frac{K^A}{L^L} \\ &= \frac{2\theta(i,a)^2}{(1-\alpha(i,a))^3} \cdot \frac{K^A}{L^L} \\ &> 0 \end{aligned} \tag{7.29}$$

In Case 1, technological progress has two effects. The first is the supply-side effect, and it shifts the $A(i)$ schedule. This was explored in Part B. But in addition, it also has a demand-side effect. This is shown in Figure 7.8.

Figure 7.8: Case 1: Technological Change and Low-Skilled Labour



In Case 1, the outcome for low-skilled labour is more pessimistic than in Part B. As in Part B, an increase in the relative productivity of advanced capital causes the $A^L(i)$ schedule to fall, pushing down the relative wage from w_a to w_b. But it also causes structural change, expenditure densities shift towards goods that require more complex tasks to produce them, and the $B(i,a)$ schedule shifts from $B_a(i,a)$ to $B_b(i,a)$. The factor-price ratio falls further to w_c. This result is again intuitive – both the supply-side effect and the demand-side effect have pessimistic consequences for labour. The former causes the comparative advantage of advanced capital over low-skilled labour to increase, the latter causes expenditure to shift towards goods in which advanced capital has the comparative advantage.

The New $C(i)$ Schedule

The case with high-skilled labour follows in exactly the same way. The structure of the model is the same as in Chapter 4, but without low-skilled labour. Equilibrium is determined by the intersection of the $C(i,t)$ schedule:

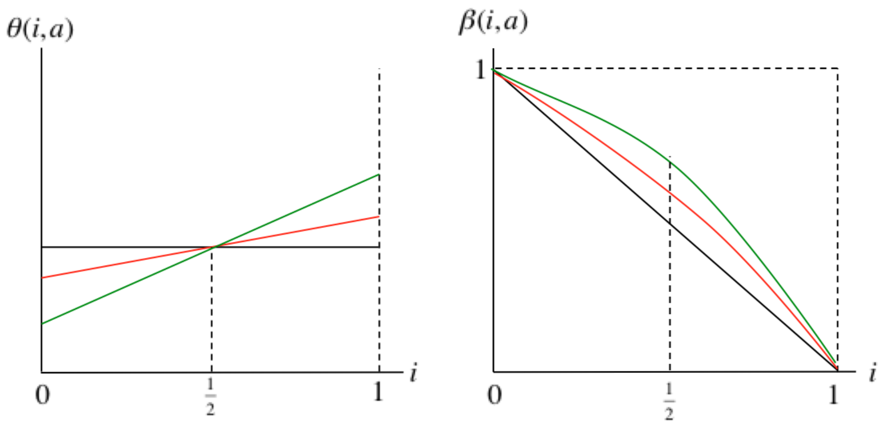
$$C(i,a) = \frac{\beta(i,a)}{1-\beta(i,a)} \cdot \frac{K^A}{L^H} \tag{7.30}$$

where:

$$\beta(\tilde{i},a) = \int_{\tilde{i}}^{1} b(i,a)di = \int_{\tilde{i}}^{1} \theta(i,a)di \tag{7.31}$$

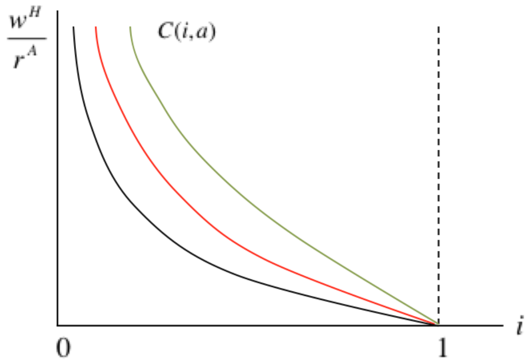
with the relative productivity schedule between high-skilled labour and advanced capital, $A^H(i)$, as in (4.6). In Case 1, the consequences of technological progress for $\theta(i,a)$ and $\beta(i,a)$ are shown in Figure 7.9.

Figure 7.9: Case 1: $\theta(i,a)$ and $\beta(i,a)$ as a Falls



In turn, these changes in the pattern of expenditure in the economy also affect the $C(i,a)$ schedule as in Figure 7.10.

Figure 7.10: Case 1: $C(i,a)$ as a Falls



The shape of the $C(i,a)$ schedule follows from (7.30) and (7.31). First, in all three cases, $\lim_{i \to 0} C(i,a) = \infty$ and $C(1,a) = 0$. Secondly, $C_{black}(i,a) < C_{red}(i,a) < C_{green}(i,a)$ – since $\beta_{black}(i,a) < \beta_{red}(i,a) < \beta_{green}(i,a)$. Thirdly, the schedules are negatively sloped since:

$$\begin{aligned} \frac{\partial C(i,a)}{\partial i} &= \frac{-\theta(i,a)}{(1-\beta(i,a))^2} \cdot \frac{K^A}{L^H} \\ &< 0 \end{aligned} \tag{7.32}$$

and have a decreasing negative slope:

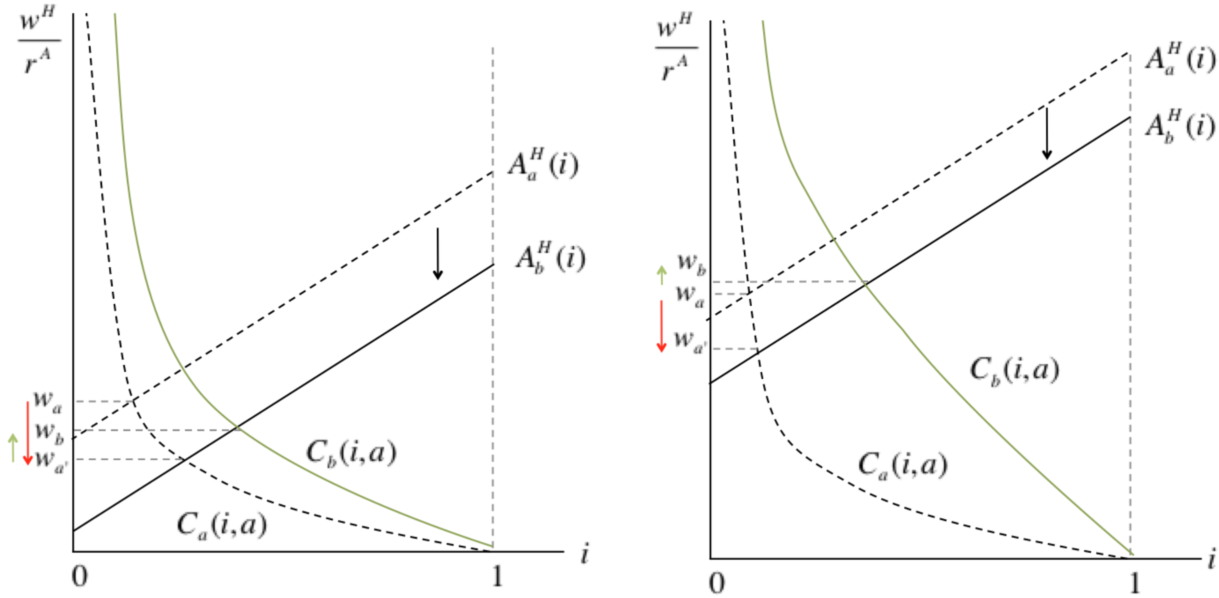
$$\begin{aligned} \frac{\partial^2 C(i,a)}{\partial i^2} &= \frac{\beta''(i,a)(1-\beta(i,a))^2 + 2\beta'(i,a)^2(1-\beta(i,a))}{(1-\beta(i,a))^4} \cdot \frac{K^A}{L^H} \\ \frac{\partial^2 C(i,a)}{\partial i^2} &= \frac{2\theta(i,a)^2}{(1-\beta(i,a))^3} \cdot \frac{K^A}{L^H} \\ &> 0 \end{aligned} \tag{7.33}$$

When structural change is of the Case 1 type, the consequence for high-skilled labour is ambiguous. The increased relative productivity of advanced capital causes the $A^H(i)$ schedule to fall, pushing down the factor-price ratios fall from w_a *to* $w_{a'}$. This is as in Part B. But in addition this technological progress again causes structural change, changing the pattern of expenditure in the economy and causing the $C(i, a)$ schedule to shift. As a result, the factor-price ratio is pushed up from $w_{a'}$ *to* w_b. This countervailing pressure on the factor-price ratio is the source of the ambiguity – the supply-side effect drives down factor-price ratios, but the demand-side effect pushes them back up. In the diagram on the left-hand side of Figure 7.11 the technological progress causes factor-price ratios to decrease from w_a to $w_{a'}$ due to the supply-side effect. But the demand-side effect counterbalances this, causing factor-price ratios to increase to w_b. In this particular case, it is not a large enough increase to outweigh the initial fall. However, if:

$$k_1 > \frac{2 \cdot \theta(\bar{i}(a), a)}{b \cdot [\bar{i}(a) - \bar{i}(a)^2]} \tag{7.34}$$

the demand-side effect is substantial enough to overturn the negative supply-side effect, such that $w_b > w_a$. The derivation of (7.34) is shown in the Appendix, section 9.2.2. This result is shown on the right-hand side of Figure 7.11.

Figure 7.11: Demand-Side Effects



Note again that this runs contrary to the 'structural change' argument set out at the start of this chapter. That argument says that changing expenditure densities on goods will have an positive effect on the labour market. This analysis suggests that while this is possible – as Figure 7.11 shows – it is not inevitable.

Case 1: Summary

In Case 1, technological progress leads to a pessimistic outcome for low-skilled labour – both the supply-side effect and the demand-side effect put downward pressure on factor-price ratios. However, in the case of high-skilled labour, technological progress leads to an ambiguous outcome – the supply-side effect puts downward pressure on the factor-price ratio, but this is offset by the demand-side effect. Both these results challenge the optimism of the structural change argument.

7.5.2. Cases 2 and 3: Low-Skilled Labour Bias and No-Bias

Case 2 and Case 3 can be considered by repeating the analysis of Case 1. For Case 2, again for simplicity, I assume that:

$$\theta_{21}(i,a) = k_1 \tag{7.35}$$

This is in contrast to (7.26) where $\theta_{21}(i,a) = -k_1$. This now implies that as technological progress takes place, the changes in expenditure density across the goods, $\theta_2(i,a)$ for a given good i, are decreasing at a constant rate as i increases. The intuition for Case 2 is the opposite of Case 1 – as technological progress takes place, the type of goods that people spend a greater proportion of their income on are the type of goods that require simpler tasks to produce them. Following the approach in the last section, it follows that the expression for $\theta(i,a)$ is now:

$$\theta(i,a) = k_1 \cdot (a - a^*) \cdot \left[i - \frac{1}{2}\right] + 1 \tag{7.36}$$

where the only difference with (7.26) is the sign of k_1. As a result, as a falls below a^*, the expenditure schedule $\theta(i,a)$ rotates clockwise around the point $(1, \frac{1}{2})$. The type of good $i = \frac{1}{2}$ again marks out a boundary between those goods whose expenditure densities increases, and those whose expenditure density falls:

1. $\theta_2(i,a) = k_1 \cdot (a - a^*) < 0 \quad \forall i \in [0, \frac{1}{2}]$

2. $\theta_2(i,a) = k_1 \cdot (a - a^*) = 0 \quad i = \frac{1}{2}$

3. $\theta_2(i,a) = k_1 \cdot (a - a^*) > 0 \quad \forall i \in [\frac{1}{2}, 1]$

The consequence of (7.35) and (7.36) for the $B(i,a)$ and $C(i,a)$ schedules is the exact opposite of Case 1. And so in Case 2 the results are also the opposite of those in Case 1. The consequences are instead ambiguous for low-skilled labour and pessimistic for high-skilled labour. I do not repeat this symmetric analysis here. In Case 3, where:

$$\theta_{21}(i,a) = 0 \tag{7.37}$$

The analysis collapses to that in Part B. Expenditure densities on different goods are fixed. There is no structural change and no demand-side effect, only a supply-side effect.

7.5.3. Analysis

The analysis of Case 1, 2 and 3 leads to Proposition 7:[36]

PROPOSITION 7: *If $\theta_{21}(i,a) = -k_1$, structural change offsets the decline in relative wages for high-skilled labour but compounds the decline in wages for low-skilled labour. If $\theta_{21}(i,a) = k_1$, structural change offsets the decline in relative wages for low-skilled labour but compounds the decline in wages for high-skilled labour. If $\theta_{21}(i,a) = 0$ there are no demand-side effects.*

From the standpoint of labour, the consequence of structural change and a

[36]These results are shown formally in the Appendix, section 9.2.2 for Case 1.

shift in expenditure densities across different goods depends upon the type of tasks that must be performed to produce those goods. If these are tasks in which that type of labour retains the comparative advantage the consequence is ambiguous. If these are tasks in which that type of labour does not retain the comparative advantage the outcome is even more pessimistic than when the supply-side is looked at in isolation as in Part B.

Proposition 7 again shows why the 'structural change' argument for optimism is possible but not inevitable. It is clear that structural change is characterised by changes in expenditure densities on different types of goods – this is a claim in the good-space about $\theta_2(i, a)$. But it does not follow from this that necessarily labour, rather than capital, will be best placed to perform the tasks required to produce those goods – this is a further claim in the task-space about $\theta_{21}(i, a)$. With traditional price effects in section 7.3, the ambiguity followed from uncertainty about the value of the elasticity of consumption. Now, with un-traditional technology-dependent preferences, a similar ambiguity follows from uncertainty about the value of $\theta_{21}(i, a)$.

7.6. Conclusion

In this chapter I explore the 'structure change' argument for optimism. An optimistic outcome is possible but not inevitable. Optimism can only follow structural change that is characterised by a shift in the task composition of the economy in the direction of labour's comparative advantage. In practice, however, it is entirely possible that the task composition of the economy shifts in a different direction. This is clear from both the traditional approach that uses price effects in section 7.3 and the non-traditional approach that uses technology-dependent preferences in section 7.4.

The difficulty with traditional non-task-based models is that any analysis of structural change must take place in the good-space not the task-space. This means that traditional explanations of structural change are necessary but not sufficient for understanding the consequences of technological change. They are necessary because they describe how expenditure shares change across goods. But they are insufficient because they do not describe what tasks must be performed to produce those goods. As a result, they are silent both on how expenditure shares change across tasks and on which factors are best placed to perform those tasks. Only task-based models with a variety of goods can capture the ambiguity set out in this chapter. And the particular contribution of the new models in this chapter is that, in different ways, they generate this ambiguity – but without relying on the ALM hypothesis.[37]

More generally, we are accustomed to the idea that technological progress might exhibit a bias on the *production* side of the economy. But we are less accustomed to the idea that it might exhibit a bias on the *consumption* side of the economy. The argument of this chapter is that this is an important part of the story. This again reflects the spirit of John Stuart Mill's observation at the start of this thesis. Demand for different commodities implies a demand for further tasks to be performed. But it does not necessarily imply that these are tasks in which labour has the comparative advantage. In concluding this thesis, I will note this as an important area for further empirical work.

[37]The AD model, as set out in Chapter 6 also generates this ambiguity – but as noted there, and in this chapter, that result depends upon the ALM hypothesis.

Part D: Conclusion

8. Further Work and Review

This final chapter is composed of two parts. The first is the identification of an important new 'race' in task-space.[1] This race is important since it presents a possible new source of optimism for labour. I use the static analysis in this thesis to show how this 'race' might take place, and its consequences. I discuss a recent working paper by Acemoglu and Restrepo (2016) that has begun to explore what they also call a race. I distinguish between two versions of this race - a 'demand-side' race and a 'supply-side' race. And finally I close with a set of questions for further research, and a review of the central argument of the thesis.

8.1. Further Work: The Race in 'Task-Space'

In all the task-based models in this thesis, both traditional and new, the variety of tasks has been fixed. In models with a discrete set of tasks, that set does not expand – consider, for example, the ALM model and the AD model.[2] In models with a continuous spectrum of tasks, that spectrum has fixed upper and lower bounds – consider, for example, the AA model and the new model

[1]This is a 'new' race, in contrast to the 'old' race noted in the outline to this thesis – that set out in Tinbergen (1974) between technology and education ("technical progress and the extension of third-level education").

[2]See section 1.4.2.

in Part B.[3] However, Acemoglu and Restrepo (2016), a recent unpublished working paper, makes an important claim that in practice the variety of tasks appears to change over time – for example, in 2000, about 70 percent of software developers in 2000 held "new job titles".[4] In Acemoglu and Restrepo (2016), this increasing variety of tasks is presented as an argument for optimism. The spirit of their argument is that labour displaced from performing 'old' types of tasks can be employed in performing the entirely 'new' types of tasks that are created. They develop a complex dynamic model that shows one way in which this can take place. However, it is submitted that the new model developed in Part B of this thesis shows a more intuitive static account of the race.[5]

To see this, consider again the case where high-skilled labour and advanced capital compete to perform tasks with traditional capital. The relative productivity schedule $A^H(i)$ and the market clearing schedule $C(i)$ are as in Chapter 5, section 5.3. However, I now assume that the spectrum of goods is $i \in [0, c]$, with a variable upper-bound c – rather than $i \in [0, 1]$ as in the rest of this thesis. A rise in c therefore implies an increase in the variety of goods and, since each good requires its own unique tasks, a rise in the variety of tasks as well.[6] If Assumption 2 again holds – such that the expenditure densities $\theta(i)$ on each good are fixed and identical – this implies that now $\theta(i) = \frac{1}{c}$. It follows that the share of expenditure on goods performed

[3]The AA model is set out in section 1.5.3.

[4]A new "job title" does not, however, necessarily imply entirely new types of tasks are performed in that job. I will return to this observation later in this section.

[5]This static account is a 'demand-side' race rather than a 'supply-side' race. I will explain this distinction in the following sections.

[6]See the task-based production function for goods in (4.8).

by advanced capital is again:

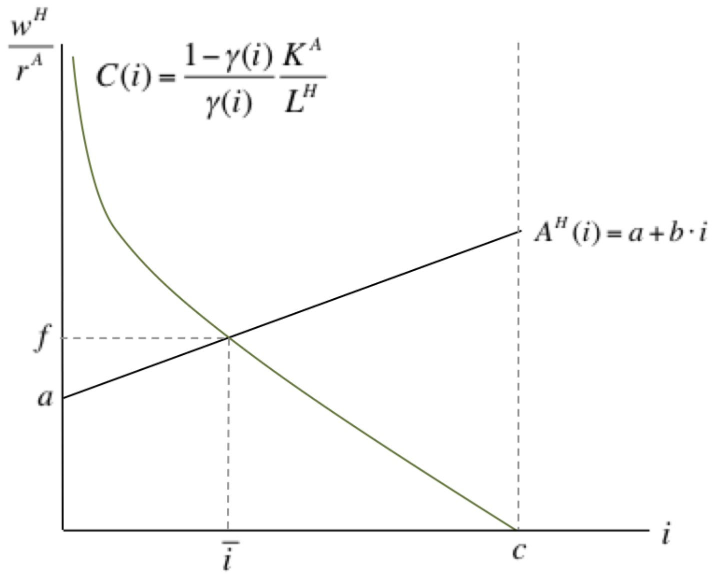
$$\gamma(\tilde{i}) = \int_0^{\tilde{i}} \theta(i)\, di \tag{8.1}$$

but the share of expenditure on goods performed by high-skilled labour is:

$$1 - \gamma(\tilde{i}) = \int_{\tilde{i}}^{c} \theta(i)\, di \tag{8.2}$$

with an upper-bound of c in the integral, rather than 1 as in the rest of this thesis. Suppose the $A^H(i)$ schedule has the same simple linear form in Chapter 5. As a result, equilibrium is as in Figure 8.1. $C(i)$ is defined as in Figure 5.5, but where $\gamma(i)$ is as in (8.1) and (8.2).

Figure 8.1: Equilibrium



By performing comparative statics on the equilibrium in Figure 8.1, it is possible to identify the 'race'. These comparative statics consist of comparing the impact of an increase in the productivity of advanced capital on the one hand, and an increase in the variety of goods on the other. I now consider each of these in turn, and consider their combined effect.

First, consider the consequence of a uniform increase in the relative productivity of advanced capital. This is a *fall* in a in Figure 8.1. For a given upper-bound c this causes a *decrease* in the factor-price ratio f:

$$\left.\frac{\partial f}{\partial a}\right|_c > 0 \tag{8.3}$$

This is shown in the Appendix, section 9.3.1, but is clear from inspection of Figure 8.1. A downward shift in the $A^H(t)$ schedule causes the factor-price ratio f to fall. This is pessimism at work.

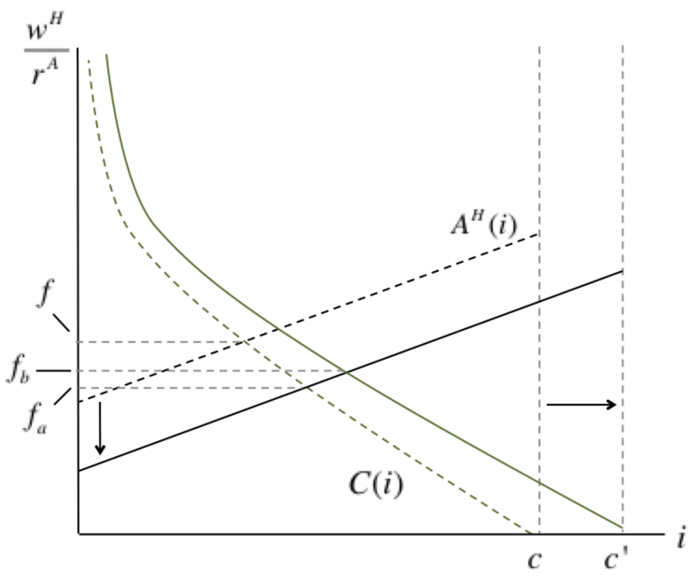
Secondly, consider the consequence of an increase in the variety of goods in the economy. This is a *rise* in c in Figure 8.1. For a given level of a this causes an *increase* in the factor-price ratio f:

$$\left.\frac{\partial f}{\partial c}\right|_a > 0 \tag{8.4}$$

This is shown in the Appendix, section 9.3.2, and is also clear from inspection of Figure 8.1. This is optimism at work.

The combined result, then, of a fall in a and a rise in c is shown in Figure 8.2. The increase in a causes a fall in f to f_a. But the rise in c offsets this and drives f_a to f_b. Note that in this simple comparative static exercise the outcome for labour is ambiguous – a large enough shift in c can offset the effect of a given shift in a, although it does not in Figure 8.2. But the result is intuitive – advanced capital displaces labour from performing certain tasks, but the increase in the upper-bound of the task-spectrum creates a new set of tasks in which labour rather than advanced capital has the comparative advantage.

Figure 8.2: Decrease in a, Increase in c



This is the spirit of what I call the 'race' – it is a race between the displacement of labour from old tasks by advanced capital on the one hand, and the creation of new tasks for labour to perform on the other. In order to understand the outcome of this race it is necessarily to understand the possible processes through which the set of types of tasks can change. In this simple static model, it is through the exogenous creation of new goods that require new tasks to produce them. This is a demand-side account of the race. In Acemoglu and Restrepo (2016) the race is on the supply-side. I now turn to explain this distinction in more detail.

8.1.1. The 'Supply-Side' Race

Acemoglu and Restrepo (2016) focus on the supply-side version of the race. I call the model in this paper the 'AR model'. Their central claim is that labour displaced from performing certain types of tasks can instead be employed in performing entirely new types of tasks that are created by technological progress:

> "The importance of these new complex tasks is well illustrated by the technological and organisational changes during the Second Industrial Revolution, which not only involved the replacement of the stagecoach by the railroad, sailboats by steamboats, and of manual dock workers by cranes, but also the creation of new labour-intensive tasks. These new tasks generated jobs ..." (pg. 1)

The AR model explores this central claim – but through the supply-side, not the demand-side. There is no role for tastes and preferences in the AR model – output consists of a unique final good that is produced by a Cobb-Douglas combination of a unit spectrum of tasks, as in the AA model. Instead, technology has two distinct roles on the supply-side – it can either automate existing non-automated tasks, or it can create entirely new types of tasks. The balance between these two effects is determined endogenously in the AR model – profit-seeking entrepreneurs allocate an inelastic supply of 'scientists' across either automation or the creation of new tasks, given the relative return to these two uses of technology.

There are two particularly important features of the AR model. The first is the role of 'directed technological change' (see Acemoglu 1998 and

2002b) – in the AR model, technological progress causes labour to become cheaper relative to capital, and this creates an incentive for entrepreneurs to create new tasks for labour to perform. This feature is critical – innovation is 'biased' towards the use of labour. The second important innovation in the AR model is that while a task-spectrum of unit length is used, as in the AA model and the new model in this thesis, the spectrum is no longer defined as $i \in [0, 1]$ but $i \in [N-1, N]$ – and N can change over time. The task-spectrum has the same length, but it 'shifts'. Simple tasks are 'destroyed' at the bottom of the task-spectrum, and new complex tasks are 'created' at the top. The AR model has a powerful central result – that in steady-state the economy is self-correcting, the displacement of labour from performing certain types of tasks is exactly offset by the creation of new types of tasks for labour to perform.

One initial problem with the AR model is that the conclusions that follow from it appear to be in direct conflict with the intuition that Acemoglu and Restrepo (2016) use to motivate the AR model in the first place. Specifically, Acemoglu and Restrepo (2016) motivate the AR model by discussing Leontief (1952) and his mistaken claim that labour in the 21st century, just like horses in the 20th century, "will become less and less important ... More and more workers will be replaced by machines." Acemoglu and Restrepo (2016) claim that the AR model explains why Leontieff (1952) was mistaken – "the difference between human labor and horses is that humans have a comparative advantage in new and more complex tasks. Horses did not." But this raises a fundamental question – why does the AR model not then also apply equally well to horses? If technological change displaced horses and drove down their relative factor price, why was there not a surge in the creation of new types of tasks in the 20th century in which horses had the comparative advantage, as the AR model would suggest? An important part of the account appears to be missing.

8.1.2. The 'Demand-Side' Race

A more substantive problem with the AR model is that it is a supply-side account of the race. Producers create new tasks to produce a unique final good. This is problematic because while the idea of the 'race' is often informally appealed to in the economics literature, it is far more often conceptualised as taking place through the demand-side, rather than supply-side. In these accounts it is not *producers* who create new types of tasks to produce existing goods, but *consumers* who demand new goods that in turn require new types of tasks to produce them. For example, consider Mokyr et al. (2015):

> "The future will surely bring **new products** that are currently barely imagined, but will be viewed as necessities by the citizens of 2050 or 2080. These product innovations will combine with new occupations and services that are currently not even imagined." (pg. 45) [Emphasis added]

Autor and Dorn (2013b):

> "Labour saving technological change necessarily displaces workers in accomplishing specific job tasks, which is where the productivity gains come from, but in the long run, these same advances generate **new products and services** that raise national income and increase overall demand for labour in the economy." (pg. 2) [Emphasis added]

and Autor (2014):

> "The green revolution displaced labor from farming... it is unlikely, however, that farmers at the turn of the twentieth century could foresee that one hundred years later, health care, finance, information technology, consumer electronics, hospitality, leisure and entertainment would employ far more workers than agriculture." (pg. 38)

These are not supply-side claims about the creation of new types of tasks to produce a given type of good, but are demand-side claims about the creation of entirely new types of goods that in turn require new types of tasks to produce them. This was the spirit of the simple static version of the race I set out in section 8.1. New tasks are not determined by production technologies, but by consumer tastes and preferences. At present there is no reason to favour one particular version of the race. The evidence that Acemoglu and Restrepo (2016) present to justify the supply-side account – that new tasks appear to have been created – can equally be used to justify a demand-side account.

There is an existing literature that builds on the observation that the variety of goods in an economy can change over time. For example, this is the approach taken in 'Schumpeterian' growth models, such as Grossman and Helpman (1991) and Aghion and Howitt (1992). As yet there is no literature exploring the relationship between the variety of goods in the economy and the variety of tasks that must be performed to produce those goods. This would require a more careful understanding of the task-based production functions for different goods.[7]

Interestingly, there appear to be two new sources of ambiguity in the demand-side version of the race. The first is that it may be that new varieties

[7] See Chapter 1.

of goods are created, but it does not follow that these goods necessarily require entirely new tasks to produce them. For example, old types of tasks might be combined in different ways to produce new types of goods. The second is that it may be that new varieties of goods are created, and new tasks are created in turn, but it does not follow that these tasks are those in which labour, rather than capital, has the comparative advantage. This second ambiguity is similar to that in Chapter 7. There the focus was on expenditure shares that shift across *existing* goods, here the focus is on expenditure shares that shift to entirely *new* goods. But in both cases what matters is whether the tasks that must be performed to produce these goods are those in which labour, rather than machines, has the comparative advantage. Note that in Figure 8.2 this is the case – new tasks are necessarily created in the direction of labour's comparative advantage. But this need not be the case. The simple account of the demand-side race in section 8.1 does not capture these ambiguities. They require further theoretical and empirical work.

8.1.3. Further Research

The New Race

The new race set out in this chapter raises a set of important questions for further research. The first is a theoretical task. It is clear from this initial analysis that to understand properly the race requires a more general task-based model that reflects both the supply-side analysis and the demand-side analysis in this thesis. The promise of this new model is that it will provide a tractable way to analyse the demand-side race, complementing Acemoglu and Restrepo (2016) that at present only considers a unique final good and the supply-side race.

Two related empirical tasks follow. First, it is also necessary to look in greater depth at the type of new tasks that are created, and what determines that type. The optimism of Acemoglu and Restrepo (2016) depends upon the fact that the new tasks that are created are necessarily created in the direction of labour's comparative advantage. The discussion of the demand-side race in the previous section suggests it is conceivable that this property may not hold – new tasks may in fact be those in which advanced capital, rather than labour, has the comparative advantage.

The second empirical task is to explore more carefully the argument that new tasks are indeed being created. As noted before, in Acemoglu and Restrepo (2016) the argument that new tasks are being created is based on the growth in "new job titles".[8] But this risks conflating the novelty of a given 'job' with the novelty of the 'tasks' that must be performed in that job. It does not follow that because a job has a new title necessarily implies that entirely new types of tasks are involved – it is, for example, conceivable that a new job simply requires a different combination of existing types of tasks in which labour still retains the comparative advantage. This also raises a further difficulty – whether 'new' tasks created are genuinely new tasks, or are simply new *classifications* of old tasks. For example, is writing software code a genuinely distinct task from other types of creative writing – or is it simply a certain type of creative task but performed in a different language? This is not immediately obvious, and requires further empirical research.

[8] This draws on data in Lin (2011).

Other Areas: Theoretical

This thesis also raises additional, important areas for future work beyond the race. The first is a set of theoretical tasks. The first of these is to perform the non-linear simulation of the model with advanced capital and technological progress in Chapter 5. Although the pessimistic outcome in Chapter 5 does not depend upon this simulation, this will make it possible to display the nature of the transition path to this steady-state.

The second theoretical task is to explore the consequences of using more complex production functions in the new models. At present the new models rely upon a particular set of factor-based production functions for tasks and task-based production functions for goods. This provides explicit solutions in Part B and C and allows me to challenge the existing literature and its reliance on the ALM hypothesis in an analytically tractable way. More complex production functions are likely to only provide implicit solutions, but these would be interesting to explore.

This leads to a third related task, that was noted in section 5.3.2. This is to understand the way in which the structure of the different production functions affects how the 'aggregate' elasticity of substitution of labour and capital changes over time.[9] It was conjectured in section 5.3.2 that labour is driven out the economy because, as advanced capital becomes increasingly capable and $\bar{i}(t)$ rises, the 'aggregate' elasticity of substitution between labour and capital also increases. This aggregate elasticity of substitution depends both upon the 'vertical' elasticities of substitution in producing a

[9]Recall from Chapter 1 that in the task-based approach there are both factor-based production functions for tasks, and task-based production functions for goods.

particular type of good – reflecting both the task-based production function for that good, and the factor-based production functions for those tasks –and the 'horizontal' elasticities across different types of goods. Understanding the relationship between these underlying elasticities of substitution and the aggregate elasticity of substitution is an interesting area for future work.

A final theoretical task is to explore the role of agent heterogeneity in this model, and the dynamics of inequality between different types of factor owners. Autor (2015) notes that "a key takeaway is that rapid automation may create distributional challenges". This is straightforward with CES utility functions, but in the case with income effects in Chapter 6 it was shown that this heterogeneity makes the model analytically intractable. Nevertheless, it is interesting and important to consider these distributional questions further. In turn, this will allow me to explore the income effects set out in Chapter 7.

Other Areas: Empirical

Alongside these theoretical tasks that extend the new model in this thesis, there is a set of empirical tasks that explore the new model as it currently is. The first task is to explore how to measure the concept of 'routinisability'. Autor (2013) sets out the difficulties of measuring the concept of 'routineness' – for example, the difficulty of establishing a clear and consistent taxonomy for different types of tasks. It is likely that many of these difficulties will repeat themselves with this new measure. One interesting possibility to explore is the use of classification algorithms, as in Frey and Osborne (2013).[10]

[10]For example, to determine whether a task is routinisable or not based on its similarity

A clear difference between the new model in this thesis and traditional task-based models that rely upon the ALM hypothesis is that in the former there are goods that are produced by capital alone, whereas in the latter goods always require at least one type of labour to produce them. A clear prediction of this model is that, as advanced capital becomes increasingly productive, more goods are produced by capital alone. There are many anecdotal cases that appear to support this result. Testing this more formally is an interesting and important task.

The two empirical tasks set out explore the consequences of supply-side analysis of the new model in this thesis. There is also an empirical task that explore the demand-side analysis in Chapters 6 and 7. There it was claimed that the demand-side effects can weaken or strengthen the pessimistic outcome reached in Part B, depending upon the direction in which demand shifts – and the way in which demand will shift over time is not clear. Exploring this argument requires a clearer understanding of how the task-composition of different economies have changed over time. As noted in Chapter 6, there is a large literature that explores structural change in terms of industries or goods. But there is not a literature that explores structural change in terms of the underlying tasks that must be performed in those industries and to produce those goods. This is an interesting area for further work.

with other tasks that are known to be routinisable or not. Frey and Osborne (2013) use this approach to determine whether a particular occupation can be computerised. They first hand-label 70 occupations that a set of experts were certain could either be computerised or non-computerised, and use that as a training set for a classification algorithm that allocates a further 632 occupations their likelihood of computerisation, based on the tasks that they have in common with the 70 occupations in the training set.

8.2. A Review of the Argument

In Part A of this thesis I show how dependent the existing task-based literature is upon the ALM hypothesis and why this is mistake. In Part B, I consider the consequence of relaxing the ALM hypothesis. In the spirit of the ALM hypothesis, traditional capital is a q-complement to labour in performing certain types of tasks. But advanced capital is a perfect substitute for labour in performing those tasks. And so, as advanced capital becomes increasingly capable, traditional capital remains a q-complement to labour – but for a shrinking set of types of tasks in equilibrium. The relative return to labour falls. This pessimistic outcome contrasts with the optimism that is supported by the ALM hypothesis.

If the set of types of tasks for which traditional capital is a q-complement to labour gets smaller, two possible arguments for optimism remain – either expenditure increases on that residual set of existing types of tasks, or expenditure shifts to entirely new tasks that are created in which labour holds the comparative advantage. The first possibility is the demand-side argument set out in Part C. The second argument for optimism is the 'race' that is described in this chapter.

The challenge for optimists is to explain why either of these possibilities must take place. The analysis in Chapter 7 shows that the first case for optimism is possible but not inevitable. And so too for the second case. In the AR model's supply-side version of the race, the optimism is built in – new tasks are necessarily created in the direction of labour's comparative advantage. But the alternative demand-side version of the race set out in this chapter suggests that this need not hold – it is plausible that new goods require new tasks, but tasks in which capital rather than labour has the comparative advantage. The different versions of this race are not properly understood. This is an important task for future research.

This thesis began with a quotation from John Stuart Mill, that “demand for commodities is not demand for labour”. This captures my central argument. The demand for commodities is not demand for labour, but demand for tasks that must be performed to produce those commodities. The existing literature is optimistic about the proportion of these tasks that labour rather than capital have the comparative advantage. I have argued that this particular optimism for the future is misplaced – the models on which it is based rely upon a supply-side analysis that does not reflect the character of recent technological progress, and a demand-side analysis that often is either omitted or restricted.

In this thesis I have not claimed that the future is known. However, I have claimed that there are compelling reasons for pessimism that are not reflected in the existing literature. I submit that these represent an important challenge to the case for optimism that currently exists. This presents a set of important theoretical and empirical problems for future work.

9. Appendix

This chapter is the Appendix. It contains the derivations and proofs that are referred to in the main body of the thesis.

9.0.1. Necessity of Perfect Substitutes

In Chapter 3 the factor-based production function for $z_1(i)$ are:

$$z_1(i) = a^L(i)L^L(i) + a^H(i)L^H(i) \tag{9.1}$$

Factors are 'perfect substitutes'. Suppose instead that factors combine in a CES way to perform $z_1(i)$:

$$z_1(i) = \left[\left(a^L(i)L^L(i)\right)^\rho + \left(a^H(i)L^H(i)\right)^\rho\right]^{\frac{1}{\rho}} \tag{9.2}$$

This is the functional form AD model uses in its factor-based production functions for 'routine' tasks.[1] As in Chapter 3, assume there is a wage that is paid to low-skilled labour, w^L, and a wage that is paid to high-skilled labour, w^H. A firm will be indifferent between hiring low-skilled labour and high-skilled labour to perform $z_1(i)$ when:

[1]See equation (1.22). This is the same form as here, but where $z(r) = z_1(i)$ and the two factors are instead low-skilled labour and capital.

$$w^L \cdot \frac{\partial z_1(i)}{\partial L^L} = w^H \cdot \frac{\partial z_1(i)}{\partial L^H} \tag{9.3}$$

Or, re-arranging this, if:[2]

$$\begin{aligned} \frac{w^L}{w^H} &= \frac{\frac{\partial z_1(i)}{\partial L^H}}{\frac{\partial z_1(i)}{\partial L^L}} \\ &= \frac{a^H(i)^\rho L^H(i)^{\rho-1}}{a^L(i)^\rho L^L(i)^{\rho-1}} \end{aligned} \tag{9.4}$$

This is a CES generalisation of the $A(\cdot)$ schedule in Chapter 3.[3] First suppose $\rho \neq 1$ – that the factors are *imperfect* substitutes in performing $z_1(i)$. In addition, also suppose that I impose the Ricardian equilibrium property on this CES generalisation, that each type of task $z_1(i)$ is either performed by high-skilled labour or low-skilled labour, but never both together – formally, $\forall i$ if $L^L(i) > 0$ then $L^H(i) = 0$ and if $L^H(i) > 0$ then $L^L(i) = 0$. It follows from (9.4) that this cannot take place – for those $z_1(i)$ that only low-skilled labour performs the factor price ratio that supports this equilibrium is zero, but for those $z_1(i)$ that only high-skilled labour performs the factor price that supports this equilibrium factor price ratio is infinite. These two equilibrium features are not compatible – it implies an $A(\cdot)$ schedule that simply consists of a discontinuous leap from $\frac{w^L}{w^H} = 0$ to $\frac{w^L}{w^H} = \infty$. However, if $\rho = 1$ and

[2]The marginal products $\frac{\partial z_1(i)}{\partial L^L}$ and $\frac{\partial z_2(i)}{\partial L^H}$ are:

$$\begin{aligned} \frac{\partial z_1(i)}{\partial L^L} &= a^L(i)\left(a^L(i)L^L(i)\right)^{\rho-1}\left[\left(a^L(i)L^L(i)\right)^\rho + \left(a^H(i)L^H(i)\right)^\rho\right]^{\frac{1-\rho}{\rho}} \\ \frac{\partial z_1(i)}{\partial L^H} &= a^H(i)\left(a^H(i)L^H(i)\right)^{\rho-1}\left[\left(a^L(i)L^L(i)\right)^\rho + \left(a^H(i)L^H(i)\right)^\rho\right]^{\frac{1-\rho}{\rho}} \end{aligned}$$

[3]See (3.10).

factors are *perfect* substitutes in performing $z_1(i)$ – as they are in the new model in Part B – then (9.4) collapses to:

$$\frac{w^L}{w^H} = \frac{a^H(i)}{a^L(i)} \tag{9.5}$$

Which is the $A(\cdot)$ schedule in Chapter 3.[4] And now, for the reasons set out in Part B, it is not only possible for the Ricardian equilibrium property to hold but it must hold. In short, the new model in Part B assumes that $\rho = 1$ in the factor-based production functions for tasks and the AD model does not. Yet for the reasons I have set out – that this allows me to maintain the task spectrum and a clear Ricardian pattern of specialisation in equilibria – it is an important simplifying assumption to maintain.

9.0.2. Derivation of $\int_0^{\bar{i}} K(i)\, di$ and $\int_{\bar{i}}^1 K(i)\, di$

From the FOCs for the firms we know that:

$$K(i) = p(i) \cdot (1 - \psi) \cdot \frac{x(i)}{r} \tag{9.6}$$

And so it follows that:

$$\begin{aligned} \int_0^{\bar{i}} K(i)\, di &= \int_0^{\bar{i}} p(i) \cdot x(i)\; di \cdot \frac{(1-\psi)}{r} \\ \int_{\bar{i}}^1 K(i)\, di &= \int_{\bar{i}}^1 p(i) \cdot x(i)\; di \cdot \frac{(1-\psi)}{r} \end{aligned} \tag{9.7}$$

For simplicity, call $\int_0^{\bar{i}} K(i)\, di$, A, and $\int_{\bar{i}}^1 K(i)\, di$, B. Then it follows that:

[4] Again, see (3.10).

$$
\begin{aligned}
A &= \alpha(\bar{i}) \cdot \left[w^L L^L + w^H L^H + rK\right] \frac{(1-\psi)}{r} \\
&= \alpha(\bar{i}) \cdot \left[\frac{w^L}{r} L^L + \frac{w^H}{r} L^H + K\right] \cdot (1-\psi) \\
&= \alpha(\bar{i}) \cdot \left[\frac{\psi}{1-\psi} \cdot A + \frac{\psi}{1-\psi} \cdot B + K\right] \cdot (1-\psi) \\
&= \alpha(\bar{i}) \cdot [\psi A + \psi B + (1-\psi) \cdot K]
\end{aligned}
\tag{9.8}
$$

And similarly it follows that B is equal to:

$$
B = [1-\alpha(\bar{i})] \cdot [\psi A + \psi B + (1-\psi) \cdot K] \tag{9.9}
$$

From the definition of A it follows that:

$$
A = \frac{\alpha(\bar{i})\psi}{1-\alpha(\bar{i})\psi} \cdot B + \frac{\alpha(\bar{i}) \cdot (1-\psi)}{1-\alpha(\bar{i})\psi} \cdot K \tag{9.10}
$$

Substituting this expression for A into the expression for B:

$$
\begin{aligned}
B &= [1-\alpha(\bar{i})] \cdot \left[\frac{\psi}{1-\alpha(\bar{i})\psi} \cdot B + \frac{\psi\alpha(\bar{i}) \cdot (1-\psi)}{1-\alpha(\bar{i})\psi} \cdot K + (1-\psi) \cdot K\right] \\
B \cdot \left[1 - \frac{(1-\alpha(\bar{i})) \cdot \psi}{1-\alpha(\bar{i})\psi}\right] &= (1-\alpha(\bar{i})) \cdot \frac{1-\psi}{1-\psi\alpha(\bar{i})} \cdot K \\
B \cdot \left[\frac{1-\psi}{1-\alpha(\bar{i})\psi}\right] &= (1-\alpha(\bar{i})) \cdot \frac{1-\psi}{1-\psi\alpha(\bar{i})} \cdot K \\
B &= [1-\alpha(\bar{i})] \cdot K
\end{aligned}
\tag{9.11}
$$

In a similar way, note that from the definition of B it follows that:

$$B = \frac{\psi(1-\alpha(\bar{i}))}{1-\psi(1-\alpha(\bar{i}))} \cdot A + \frac{(1-\psi)\cdot(1-\alpha(\bar{i}))}{1-\psi(1-\alpha(\bar{i}))} \cdot K \tag{9.12}$$

Substituting this expression for B into the expression for A:

$$\begin{aligned}
A &= \alpha(\bar{i}) \cdot \left[\frac{\psi}{\psi\alpha(\bar{i}) - \psi + 1} \cdot A + \frac{\psi(1-\psi)\cdot(1-\alpha(\bar{i}))}{1-\psi(1-\alpha(\bar{i}))} \cdot K + (1-\psi)\cdot K \right] \\
A &= \frac{\alpha(\bar{i})\psi}{\psi\alpha(\bar{i}) - \psi + 1} \cdot A + \frac{\alpha(\bar{i})(1-\psi)}{\psi\alpha(\bar{i}) - \psi + 1} \cdot K \\
A \cdot \frac{1-\psi}{\psi\alpha(\bar{i}) - \psi + 1} &= \frac{\alpha(\bar{i})(1-\psi)}{\psi\alpha(\bar{i}) - \psi + 1} \cdot K \\
A &= \alpha(\bar{i}) \cdot K
\end{aligned} \tag{9.13}$$

And so it follows that:

$$\begin{aligned}
A &= \int_0^{\bar{i}} K(i)\, di = \alpha(\bar{i}) \cdot K \\
B &= \int_{\bar{i}}^1 K(i)\, di = (1-\alpha(\bar{i})) \cdot K
\end{aligned} \tag{9.14}$$

9.1. Chapter 5

9.1.1. Deriving the Hamiltonian

To find this steady-state, I used a Hamiltonian. But this is an unconventional Hamiltonian – there is one state variable, $K(t)$, one co-state variable, $\mu(t)$, and a range of choice variables, $x(i,t)$. To show that the traditional Hamiltonian approach applies in this setting, I can derive the Hamiltonian and accompanying first order conditions explicitly. The aim is to maximise:

$$\int_0^\infty e^{-\rho t}\left[\int_0^1 \theta(i)\ln x(i,t)\,di + \mu(t)\left[Y(t) - \int_0^1 x(i,t)\cdot p(i,t)\,di - \dot{K}(t)\right]\right]\,dt \tag{9.15}$$

where $Y(t)$ is defined as before. Integrating by parts implies:

$$\begin{aligned}\int_0^\infty e^{-\rho t}\cdot\mu(t)\cdot\dot{K}(t)dt &= \left[e^{-\rho t}\cdot\mu(t)\cdot K(t)\right]_{t=0}^{t=T} - \int_0^\infty K(t)\frac{\partial\left[\mu(t)\cdot e^{-\rho t}\right]}{\partial t}\,dt\\ &= e^{-\rho T}\cdot\mu(T)\cdot K(T) - \mu(0)\cdot K(0) - \int_0^\infty K(t)\cdot e^{-\rho t}\left[\dot{\mu}(t) - \rho\cdot\mu(t)\right]\,dt\end{aligned} \tag{9.16}$$

and since:

$$A = e^{-\rho T}\cdot\mu(T)\cdot K(T) - \mu(0)\cdot K(0) \tag{9.17}$$

where A is a constant, it follows that (9.15) can be re-written:

$$\int_0^\infty e^{-\rho t}\left[\int_0^1 \theta(i)\ln x(i,t)\,di + \mu(t)\left[Y(t) - \int_0^1 x(i,t)\cdot p(i,t)\,di\right] + K(t)\cdot\left[\dot{\mu}(t) - \rho\cdot\mu(t)\right]\right]\,dt \\ -A \tag{9.18}$$

It follows that to maximise (9.18) over time the following expression must be maximised for each given t:

$$\int_0^1 \theta(i)\ln x(i,t)\,di + \mu(t)\left[Y(t) - \int_0^1 x(i,t)\cdot p(i,t)\,di\right] + K(t)\cdot\left[\dot{\mu}(t) - \rho\cdot\mu(t)\right] \tag{9.19}$$

or more concisely the following must be maximised in each period t:

$$H + K(t)\cdot\left[\dot{\mu}(t) - \rho\cdot\mu(t)\right] \tag{9.20}$$

where H is the current value Hamiltonian. And so, in the traditional way, to maximise the expressions in (9.19) and (9.20) a traditional set of first order conditions follow:

$$\begin{aligned} \frac{\partial H}{\partial x(i,t)} &= 0 \quad \forall i \\ \frac{\partial H}{\partial K(t)} + \dot{\mu}(t) - \rho \cdot \mu(t) &= 0 \end{aligned} \tag{9.21}$$

These are the first order conditions that I use in Chapter 5. This approach is set out in Wren-Lewis (2012).

9.1.2. Effect on Initial Consumption

This property follows from a comparison of the gradient of the saddle-path and the gradient of the locus of possible steady-states. These gradients must both be derived.

First, the gradient of the saddle-path. The saddle-path can be interpreted as a 'policy-function' – an expression $c(K)$ that shows the optimal choice of the control c for a given level of the state K. The slope of the policy-function $c(K)$ is:

$$\frac{d[c(K(t))]}{dK(t)} = \frac{\frac{d[c(t)]}{dt}}{\frac{d[K(t)]}{dt}} = \frac{\dot{c}(t)}{\dot{K}(t)} \tag{9.22}$$

Expressions for $\dot{c}(t)$ and $\dot{K}(t)$ are given in (5.30). These and (9.22) imply that, as the economy approaches steady-state and $K(t)$ approaches K^*, the slope of the policy-function is equal to:

$$\frac{d[c(K)]}{dK} = \lim_{K \to K^*} \frac{(K^{-\psi} \cdot D_2 \cdot (1-\psi) - \rho) \cdot c(K)}{K^{1-\psi} \cdot D_2 - c(K)} \tag{9.23}$$

Both the numerator and denominator in (9.23) are equal to zero in the limit. However, L'Hôpital's Rule implies that (9.23) is equal to:

$$\frac{d[c(K)]}{dK} = \lim_{K \to K^*} \frac{(-\psi \cdot K^{-\psi-1} \cdot D_2 \cdot (1-\psi)) \cdot c(K) + c'(K) \cdot (K^{-\psi} \cdot D_2 \cdot (1-\psi) - \rho)}{(1-\psi) \cdot K^{-\psi} \cdot D_2 - c'(K)} \tag{9.24}$$

(5.30) shows that in steady-state $K^{-\psi} \cdot D_2 \cdot (1-\psi) = \rho$ and so (9.22) is equal to:[5]

$$\frac{d[c(K)]}{dK} = \frac{(-\psi \cdot (K^*)^{-\psi-1} \cdot D_2 \cdot (1-\psi)) \cdot c(K^*)}{\rho - c'(K^*)} \tag{9.25}$$

This leads to a polynomial in $c'(K^*)$:

$$-(c'(K^*))^2 + \rho \cdot c'(K^*) + (\psi \cdot (K^*)^{-\psi-1} \cdot D_2 \cdot (1-\psi)) \cdot c(K^*) = 0 \tag{9.26}$$

With the solution:

$$c'(K^*) = \frac{\rho \pm \left[\rho^2 + 4 \cdot \psi \cdot (K^*)^{-\psi-1} \cdot D_2 \cdot (1-\psi) \cdot c^*\right]^{\frac{1}{2}}}{2} \tag{9.27}$$

Given that the saddle-path is positively sloped, (9.27) implies that the gradient of the saddle-path at steady-state is equal to:

$$c'(K^*) = \frac{\rho + \left[\rho^2 + 4 \cdot \psi \cdot (K^*)^{-\psi-1} \cdot D_2 \cdot (1-\psi) \cdot c^*\right]^{\frac{1}{2}}}{2} \tag{9.28}$$

[5] Note from (5.24) that $D_1 = (1-\psi) \cdot D_2$.

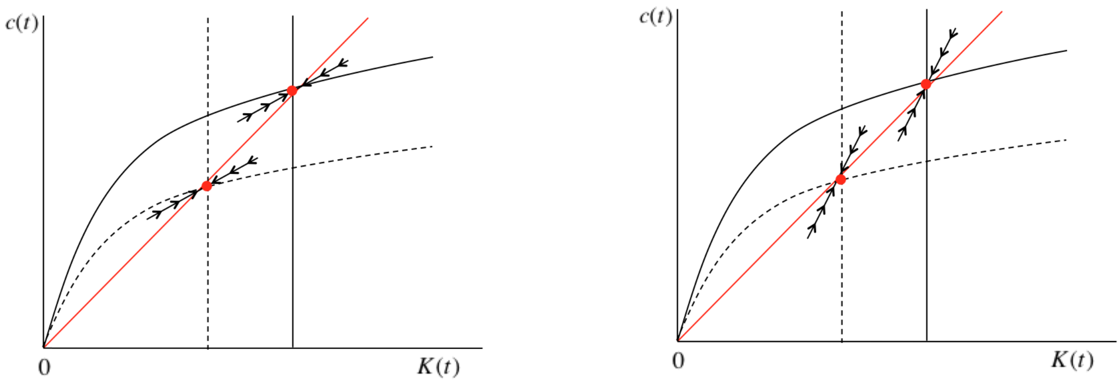
The second task is to find the gradient of the locus of all feasible steady-states. In this model the relationship between $c^*(t)$ and $K^*(t)$ is linear. To see this note that, given the level of c^* implied by (5.27) and the level of K^* implied by (5.28), it follows that:

$$
\begin{aligned}
\frac{c^*}{K^*} &= \frac{(K^*)^{1-\psi} \cdot D_2}{K^*} \\
&= (K^*)^{-\psi} \cdot D_2 \qquad (9.29) \\
&= \frac{\rho}{1-\psi}
\end{aligned}
$$

As a result, the locus of possible steady-state c^* and K^* is a straight-line through the origin with a gradient equal to (9.29).

Given the steady-state slope of the policy-function in (9.28), and the slope of the locus of steady-states derived in (9.29), it is now possible to show that $c(0)$ must rise following the one-off increase in the productivity of traditional capital. There are two possible cases shown in Figure 9.1 – on the left-hand side $c(0)$ rises, and on the right-hand side $c(0)$ falls.

Figure 9.1: Left-hand side: $c(0)$ rises, Right-hand side: $c(0)$ falls



To see that $c(0)$ must rise, and the left-hand side case must hold, I show that the case where $c(0)$ falls is not feasible. As in Figure 9.1, this requires that:

$$c'(K^*) > \frac{c^*}{K^*} \tag{9.30}$$

and so given (9.28) and (9.29) that:

$$\frac{\rho + \left[\rho^2 + 4 \cdot \psi \cdot (K^*)^{-\psi-1} \cdot D_2 \cdot (1-\psi) \cdot c^*\right]^{\frac{1}{2}}}{2} > \frac{\rho}{1-\psi} \tag{9.31}$$

Given (9.29), the expression for $c'(K^*)$ on the left-hand side of (9.31) can be simplified:

$$\begin{aligned} c'(K^*) &= \frac{\rho + \left[\rho^2 + 4 \cdot \psi \cdot (K^*)^{-\psi} \cdot D_2 \cdot (1-\psi) \cdot \frac{c^*}{K^*}\right]^{\frac{1}{2}}}{2} \\ &= \frac{\rho + \left[\rho^2 + 4 \cdot \frac{\psi}{1-\psi} \cdot \rho^2\right]^{\frac{1}{2}}}{2} \\ &= \frac{\rho + \rho \cdot \left[1 + 4 \cdot \frac{\psi}{1-\psi}\right]^{\frac{1}{2}}}{2} \end{aligned} \tag{9.32}$$

And so returning to (9.31), for $c(0)$ to fall it must be the case that:

$$\begin{aligned} \frac{\rho + \rho \cdot \left[1 + 4 \cdot \frac{\psi}{1-\psi}\right]^{\frac{1}{2}}}{2} &> \frac{\rho}{1-\psi} \\ \frac{1 + \left[1 + 4 \cdot \frac{\psi}{1-\psi}\right]^{\frac{1}{2}}}{2} &> \frac{1}{1-\psi} \\ (1-\psi) \cdot \left[1 + 4 \cdot \frac{\psi}{1-\psi}\right]^{\frac{1}{2}} &> 1 + \psi \\ \left[(1-\psi)^2 + 4 \cdot \psi \cdot (1-\psi)\right]^{\frac{1}{2}} &> 1 + \psi \end{aligned} \tag{9.33}$$

which implies that:

$$0 > 4\psi^2 \tag{9.34}$$

which is impossible, since $\psi \in (0, 1)$. As a result, with Cobb-Douglas utility, the only possible case is that $c(0)$ rises.[6]

9.1.3. Slope of the $A^H(i, t)$ Schedule over Time

It is important that the $A^H(i, t)$ schedule remains weakly positively sloped over time, even with technological progress g in $a^A(i, t)$ $\forall i$. This is because this slope reflects the comparative advantage assumption in Assumption 3 in Chapter 4 – that as i increases, the relative productivity of high-skilled labour with respect to advanced capital increases. This holds such that if $A_1^H(i, 0) \geq 0$ $\forall i$ it will remain so $\forall t$.

To see that this is the case, take two arbitrary levels of i such that $\tilde{i} < \tilde{\tilde{i}}$. Suppose $A^H(i, t)$ begins weakly positively sloped, so that at $t = 0$:

$$A^H(\tilde{i}, 0) \leq A^H(\tilde{\tilde{i}}, 0) \tag{9.35}$$

Given the definition of $A^H(i, t)$, it follows that if $A^H(i, 0)$ is to remain weakly positively sloped it must be that for any $\tilde{i} < \tilde{\tilde{i}}$:

[6]See Moll (2016) for the more general case where utility is not Cobb-Douglas.

$$\begin{aligned}\frac{a^H(\tilde{i},0)}{a^A(\tilde{i},0)} &\leq \frac{a^H(\tilde{\tilde{i}},0)}{a^A(\tilde{\tilde{i}},0)} \\ \frac{a^H(\tilde{i},0)}{a^H(\tilde{\tilde{i}},0)} &\leq \frac{a^A(\tilde{i},0)}{a^A(\tilde{\tilde{i}},0)}\end{aligned} \tag{9.36}$$

(9.36) will hold $\forall t$ so long as the growth rate of the term on the left-hand side is less than or equal to the growth rate of the term on the right-hand side:

$$\frac{\dot{a}^H(\tilde{i},t)}{a^H(\tilde{i},t)} - \frac{\dot{a}^H(\tilde{\tilde{i}},t)}{a^H(\tilde{\tilde{i}},t)} \leq \frac{\dot{a}^A(\tilde{i},t)}{a^A(\tilde{i},t)} - \frac{\dot{a}^A(\tilde{\tilde{i}},t)}{a^A(\tilde{\tilde{i}},t)} \tag{9.37}$$

and given there is no growth in the productivity of high-skilled labour this implies that the following must hold if $A(i,t)$ remains weakly positively sloped:

$$\frac{\dot{a}^A(\tilde{i},t)}{a^A(\tilde{i},t)} - \frac{\dot{a}^A(\tilde{\tilde{i}},t)}{a^A(\tilde{\tilde{i}},t)} \geq 0 \tag{9.38}$$

The condition in (9.38) clearly holds when the productivity of advanced capital grows at a constant rate g across all tasks – the expression on the left-hand side of (9.38) is equal to zero. (9.38) is intuitive – the positive slope of the $A^H(i,t)$ schedule reflects a relative advantage assumption, that high-skilled labour has a greater relative productivity in producing goods that require more complex tasks and, given that the absolute productivity of high-skilled labour does not change over time, if the absolute productivity of advanced capital were to increase at a faster rate for higher i, then eventually this would overturn labour's relative advantage.

9.1.4. Relationship between Growth Rates

The following relationship between the growth rates in $\bar{i}(t)$, $K^A(t)$ and g holds in the model in Chapter 5:

$$g^{\gamma}(t) = \phi(t)\left[g^{K^A}(t) + g\right] \tag{9.39}$$

where:

$$\phi(t) = \frac{1-\bar{i}(t)}{[\bar{i}(t)-\bar{i}(t)^2]\cdot y(t)+1} \quad \text{and} \quad y(t) = \left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} - \frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right] \tag{9.40}$$

and $\phi(t)$ has the three properties: $\phi(t) \geq 0$; $\phi(t) \leq 1$; and $\lim_{\bar{i}(t)\to 1}\phi(t) = 0$.

The expression in (9.40) implies that the growth rate in the equilibrium cut-off, $g^{\gamma}(t)$, must always be proportional to the sum of the growth rate in the advanced capital stock $g^{K^A}(t)$ and the productivity of $K^A(t)$, $\frac{\psi}{1-\psi}\cdot g$ where the constant of proportionality $\phi(t)$ is time dependent with the three features set out above. I now derive this expression for $g^{\gamma}(t)$ and the three features of $\phi(t)$.[7]

Deriving $g^{\gamma}(t)$

[7] Note that $y(t)$ is equal to the derivative of $\ln A^H(\bar{i}(t),t)$ with respect to $\bar{i}(t)$. This follows from the fact that:

$$\ln A^H(\bar{i}(t),t) = \ln a^H(\bar{i}(t),t) - \ln a^A(\bar{i}(t),t)$$

and so the gradient of $\ln A^H(\bar{i}(t),t)$ is equal to:

$$\frac{A_1^H(\bar{i}(t),t)}{A^H(\bar{i}(t),t)} = \frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} - \frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)} = y(t)$$

To derive the expression for $g^{\gamma}(t)$, note that in any period t of the static model it must be the case that:

$$
\begin{aligned}
A^H(\bar{i}(t), t) &= C(\bar{i}(t), t) \\
\frac{a^H(\bar{i}(t), t)}{a^A(\bar{i}(t), t)} &= \frac{1-\gamma(\bar{i}(t))}{\gamma(\bar{i}(t))} \cdot \frac{K^A(t)}{L^H}
\end{aligned}
\tag{9.41}
$$

This is the static equilibrium set out in Figure 5.5 – $\bar{i}(t)$ is determined by the intersection of the $A(i,t)$ and $C(i,t)$ schedules. Since $\gamma(\bar{i}(t)) = \bar{i}(t)$ it follows that (9.41) be re-arranged as:

$$
a^H(\bar{i}(t), t) \cdot \bar{i}(t) = [1 - \bar{i}(t)] \cdot a^A(\bar{i}(t), t) \cdot \frac{K^A(t)}{L^H} \tag{9.42}
$$

This condition must hold in any t. Taking time derivatives of (9.42) implies a further condition that must hold in any t:

$$
\begin{aligned}
&\left[\dot{\bar{i}}(t) \cdot a_1^H(\bar{i}(t), t) + a_2^H(\bar{i}(t), t)\right] \cdot \bar{i}(t) + a^H(\bar{i}(t), t) \cdot \dot{\bar{i}}(t) \\
&= -\dot{\bar{i}}(t) \cdot a^A(\bar{i}(t), t) \cdot \frac{K^A(t)}{L^H} + \left[\dot{\bar{i}}(t) \cdot a_1^A(\bar{i}(t), t) + a_2^A(\bar{i}(t), t)\right] \cdot [1 - \bar{i}(t)] \cdot \frac{K^A(t)}{L^H} \\
&\qquad + [1 - \bar{i}(t)] \cdot a^A(\bar{i}(t), t) \cdot \frac{\dot{K}^A(t)}{L^H}
\end{aligned}
\tag{9.43}
$$

where $a_1^H(\bar{i}(t), t)$ is the change in $a^H(\bar{i}(t), t)$ from a change in the cut-off $\bar{i}(t)$, and $a_2^H(\bar{i}(t), t)$ is any change in $a^H(\bar{i}(t), t)$ due to technological progress in advanced capital.[8] This can be re-arranged as:

$$\dot{\bar{i}}(t)\cdot\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}+\frac{a_2^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}+\frac{\dot{\bar{i}}(t)}{\bar{i}(t)}=-\frac{\dot{\bar{i}}(t)}{[1-\bar{i}(t)]}+\dot{\bar{i}}(t)\cdot\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}+\frac{a_2^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}+\frac{\dot{K}^A(t)}{K^A(t)} \tag{9.44}$$

and substituting in the expression for the growth rates implies that:

$$\begin{aligned} g^{\gamma}(t)\cdot\bar{i}(t)\cdot\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}+g^{\gamma}(t) &= -\frac{g^{\gamma}(t)\cdot\bar{i}(t)}{[1-\bar{i}(t)]}+g^{\gamma}(t)\cdot\bar{i}(t)\cdot\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}+g+g^{K^A}(t)\\ g^{\gamma}(t)\left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}-\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right]+\frac{1}{[1-\bar{i}(t)]}\right] &= g+g^{K^A}(t) \end{aligned} \tag{9.45}$$

and so:

$$g^{\gamma}(t)=\phi(t)\left[g^{K^A}(t)+g\right] \tag{9.46}$$

where:

$$\begin{aligned} \phi(t) &= \left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}-\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right]+\frac{1}{[1-\bar{i}(t)]}\right]^{-1}\\ &= \frac{1-\bar{i}(t)}{[\bar{i}(t)-\bar{i}(t)^2]\cdot y(t)+1} \end{aligned} \tag{9.47}$$

and:

$$y(t)=\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}-\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)} \tag{9.48}$$

[8] The former is a horizontal move along a fixed $a^H(i,t)$ schedule as $\bar{i}(t)$ changes, the later is a vertical move to a different $a^H(i,t)$ schedule for a given i as the productivity of advanced capital changes.

Deriving the Three Properties of $\phi(t)$

First to see that $\phi(t) \geq 0$ note that the weakly positive slope of the $A^H(i,t)$ schedule implies $\forall i$:[9]

$$A_1^H(i,t) \geq 0 \tag{9.49}$$

which, given the definition of $A^H(i,t)$, can be re-written as:

$$\frac{a_1^H(i,t) \cdot a^A(i,t) - a_1^A(i,t) \cdot a^H(i,t)}{[a^A(i,t)]^2} \geq 0 \tag{9.50}$$

Since the denominator of the expression in (9.50) is always positive, this implies that the following must hold:

$$\begin{aligned} a_1^H(i,t) \cdot a^A(i,t) &\geq a_1^A(i,t) \cdot a^H(i,t) \\ \frac{a_1^H(i,t)}{a^H(i,t)} &\geq \frac{a_1^A(i,t)}{a^A(i,t)} \end{aligned} \tag{9.51}$$

Therefore so long as the $A^H(i,t)$ remains weakly positively sloped, $y(t)$ remains weakly positive, and the denominator in the expression for $\phi(t)$ in (9.47) is always positive, and so $\phi(t)$ is also always weakly positive. That $A^H(i,t)$ does remain weakly positively sloped is shown in section 9.1.3.

To see the second feature, that $\phi(t) \leq 1$, consider the following proof by contradiction. Suppose I assume instead that $\phi(t) > 1$. Then given this

[9]Again, as in Chapter 4, the positive slope reflects the idea that high-skilled labour is relatively more productive than advanced capital at performing the task required to produce goods with higher i.

assumption, the expression for $\phi(t)$ in (9.47) implies:

$$\begin{aligned}\bar{\imath}(t)\cdot\left[\frac{a_1^H(\bar{\imath}(t),t)}{a^H(\bar{\imath}(t),t)}-\frac{a_1^A(\bar{\imath}(t),t)}{a^A(\bar{\imath}(t),t)}\right]+\frac{1}{[1-\bar{\imath}(t)]}&<1\\ \bar{\imath}(t)\cdot[1-\bar{\imath}(t)]\cdot y(t)&<1-\bar{\imath}(t)-1\\ \bar{\imath}(t)\cdot y(t)-\bar{\imath}(t)^2\cdot y(t)&<-\bar{\imath}(t)\end{aligned}\tag{9.52}$$

But since $\bar{\imath}(t)\in[0,1]$ this condition cannot hold. If $\bar{\imath}(t)=0$, then the result is a contradiction since (9.52) demands that $0<0$. Similarly if $\bar{\imath}(t)>0$ then this condition requires that:

$$y(t)<-\frac{1}{1-\bar{\imath}(t)}<0\tag{9.53}$$

which is not possible since $\bar{\imath}(t)\in[0,1]$ and $y(t)\geq 0$.

The third property, that $\lim_{\bar{\imath}(t)\to 1}\phi(t)=0$, follows from the fact that:

$$\lim_{\bar{\imath}(t)\to 1}\left[\frac{1}{1-\bar{\imath}(t)}\right]=\infty\tag{9.54}$$

and that:

$$\lim_{\bar{\imath}(t)\to 1}\left[\bar{\imath}(t)\cdot\left[\frac{a_1^H(\bar{\imath}(t),t)}{a^H(\bar{\imath}(t),t)}-\frac{a_1^A(\bar{\imath}(t),t)}{a^A(\bar{\imath}(t),t)}\right]\right]\geq 0\tag{9.55}$$

And so $\lim_{\bar{\imath}(t)\to 1}\phi(t)=0$, given that this limit is equal to the reciprocal of the sum of the terms in (9.54) and (9.55).

9.1.5. Proof that $\bar{\imath}(t)$ Increases as $\hat{\hat{\hat{K}}}^A(t)$ Increases

The proof that $\frac{d\bar{\imath}(t)}{d\hat{\hat{\hat{K}}}^A(t)} \geq 0$ during the conjectured transition in Figure 5.9 follows from two sets of relationships between the growth rates in $\bar{\imath}(t)$, the stock of $K^A(t)$, and the productivity of $K^A(t)$ that are implied by the model. The first relationship is implied by the fact that if $\hat{\hat{\hat{K}}}^A(t)$ is increasing then, given the definition of $\hat{\hat{\hat{K}}}^A(t)$ implied by (5.126), it must be the case that:

$$g^{K^A}(t) > g^{\gamma}(t) + \frac{\psi}{1-\psi} \cdot g \tag{9.56}$$

where $g^{K^A}(t)$ is the growth rate in the stock of advanced capital and $g^{\gamma}(t)$ is again the growth rate in the equilibrium cut-off $\bar{\imath}(t)$. The second relationship is that set out in (9.46), where again $\phi(t)$ has the property that $\phi(t) \in [0,1]$ and $\lim_{\bar{\imath}(t)\to 1} \phi(t) = 0$. Combining the two sets of relationships between the growth rates in (9.56) and (9.46) implies that if $\hat{\hat{\hat{K}}}^A(t)$ is increasing then:

$$\begin{aligned} g^{K^A}(t) &> g^{\gamma}(t) + \frac{\psi}{1-\psi} \cdot g \\ \frac{1}{\phi(t)} \cdot g^{\gamma}(t) - g &> g^{\gamma}(t) + \frac{\psi}{1-\psi} \cdot g \\ \frac{1-\phi(t)}{\phi(t)} \cdot g^{\gamma}(t) &> \frac{1}{1-\psi} \cdot g \end{aligned} \tag{9.57}$$

And so:

$$g^{\gamma}(t) > \frac{\phi(t)}{(1-\psi)\cdot(1-\phi(t))} \cdot g \tag{9.58}$$

Given the properties of $\phi(t)$ noted in section 9.1.4, and the fact that $g > 0$, this implies that if $\hat{\hat{\hat{K}}}^A(t)$ is increasing then:

$$g^{\gamma}(t) > 0 \tag{9.59}$$

i.e. the cut-off $\bar{i}(t)$ is also increasing. Note that (9.58) also implies that $\lim\limits_{\bar{i}(t)\to 1} g^{\gamma}(t) = 0$ given the properties of $\phi(t)$ noted in section 9.1.4.

9.1.6. Relative High-Skilled Wage During Transition

From the definition of $\frac{w^H(t)}{r^A(t)}$ in (5.110) it follows that:

$$\frac{\dot{w}^H(t)}{w^H(t)} - \frac{\dot{r}^A(t)}{r^A(t)} = -\frac{\dot{\bar{i}}(t)}{1-\bar{i}(t)} - \frac{\dot{\bar{i}}(t)}{\bar{i}(t)} + g^{K^A}(t) \tag{9.60}$$

Given (9.46) it follows that:

$$\frac{\dot{\bar{i}}(t)}{\bar{i}(t)} = \phi(t) \cdot \left[g^{K^A}(t) + g\right] \tag{9.61}$$

From (9.61) it also follows that:

$$\frac{\dot{\bar{i}}(t)}{1-\bar{i}(t)} = \frac{\bar{i}(t)}{1-\bar{i}(t)} \cdot \phi(t) \cdot \left[g^{K^A}(t) + g\right] \tag{9.62}$$

And so (9.60)–(9.62), and the definition of $\phi(t)$ in (9.47), imply that:

$$
\begin{aligned}
&\frac{\dot{w}^H(t)}{w^H(t)} - \frac{\dot{r}^A(t)}{r^A(t)} \\
&= \left[g^{K^A}(t) + g\right] \cdot \left[-\frac{\bar{i}(t)}{[\bar{i}(t) - \bar{i}(t)^2] \cdot y(t) + 1} - \frac{1 - \bar{i}(t)}{[\bar{i}(t) - \bar{i}(t)^2] \cdot y(t) + 1}\right] + g^{K^A}(t) \\
&= -\frac{\left[g^{K^A}(t) + g\right]}{[\bar{i}(t) - \bar{i}(t)^2] \cdot y(t) + 1} + g^{K^A}(t) \\
&= \frac{g^{K^A}(t) \cdot [\bar{i}(t) - \bar{i}(t)^2] \cdot y(t) - g}{[\bar{i}(t) - \bar{i}(t)^2] \cdot y(t) + 1}
\end{aligned}
\tag{9.63}
$$

And (9.63) is negative when:

$$
g^{K^A}(t) < \frac{g}{[\bar{i}(t) - \bar{i}(t)^2] \cdot y(t)} \tag{9.64}
$$

9.2. Chapter 7

9.2.1. Derivation of $\theta(i, a)$

Suppose that $\theta_{21}(i, a) = k_1$. To derive $\theta(i, a)$, first note that:

$$
\begin{aligned}
\theta_2(i, a) &= \int_0^1 \theta_{21}(i, a)\, di \\
&= k_1 \cdot i + k_2'(a)
\end{aligned}
\tag{9.65}
$$

And so:

$$
\begin{aligned}
\theta(i, a) &= \int_0^1 \theta_2(i, a)\, da \\
&= k_1 \cdot i \cdot a + k_2(a) + k_3(i)
\end{aligned}
\tag{9.66}
$$

In addition, since $\int_0^1 \theta_2(i,a)\,di = 0$ it follows that:

$$\int_0^1 \left[k_1 \cdot i + k_2'(a)\right]\, di = 0 \tag{9.67}$$

And so:

$$\begin{aligned}\left[\frac{1}{2}k_1 \cdot i^2\right]_0^1 + k_2'(a) = 0 \\ k_2'(a) &= -\frac{1}{2}k_1 \\ k_2(a) &= -\frac{1}{2}k_1 \cdot a + k_4(a)\end{aligned} \tag{9.68}$$

Combining these implies that:

$$\begin{aligned}\theta(i,a) &= k_1 \cdot i \cdot a - \frac{1}{2}k_1 \cdot a + k_4(a) + k_3(i) \\ \theta(i,a) &= k_1 \cdot a \cdot \left[i - \frac{1}{2}\right] + k_5(i,a)\end{aligned} \tag{9.69}$$

Where $k_5(i,a) = k_3(i) + k_4(a)$. To find a sensible value for this coefficient of integration, recall in Chapter 3 and 4 the expenditure density on each type of good, $\theta(i)$, was constant and equal to θ – this was Assumption 2. Call the level of income at which expenditure densities on each type of good are constant in this way a^*. Then it follows that:

$$\begin{aligned}k_1 \cdot a^* \cdot \left[i - \frac{1}{2}\right] + k_5(i,a^*) = \theta \\ k_5(i,a^*) &= 1 - k_1 \cdot a^* \cdot \left[i - \frac{1}{2}\right]\end{aligned} \tag{9.70}$$

Since $\theta = 1$ (the distribution of expenditure densities across the goods $x(i)$

where $i \in [0,1]$ is a standard uniform distribution if $\theta(i)$ is constant).

9.2.2. Proof of the Ambiguity

In this section I consider Case 1 in Chapter 7. As noted in Chapter 7, Case 2 follows symmetrically with opposite results, and Case 3 is the same as in Part B of this thesis.

Low-Skilled Labour and Advanced Capital

In the case of low-skilled labour and increasingly capable advanced capital, there is no ambiguity – a uniform increase in the relative productivity of advanced capital across tasks causes the factor price ratio, f, to fall. Recall that:

$$f = B(\bar{i}, a) = \frac{\int_0^{\bar{i}(a)} \theta(i,a)di}{1 - \int_0^{\bar{i}(a)} \theta(i,a)di} \cdot \frac{K^A}{L^L} \tag{9.71}$$

It follows that:

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial \int_0^{\bar{i}(a)} \theta(i,a)di} \cdot \frac{\partial \int_0^{\bar{i}(a)} \theta(i,a)di}{\partial a} \tag{9.72}$$

The first term on the right-hand side of (9.72) is:

$$\frac{\partial f}{\partial \int_0^{\bar{i}(a)} \theta(i,a)di} = \frac{1}{\left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2} \cdot \frac{K^A}{L^L} > 0 \tag{9.73}$$

The second term on the right-hand side of (9.72) is:

$$
\begin{aligned}
\frac{\partial \int_0^{\bar{i}(a)} \theta(i,a)di}{\partial a} &= \int_0^{\bar{i}(a)} \frac{\partial \theta(i,a)}{\partial a} di + \theta(\bar{i}(a),a) \cdot \frac{\partial \bar{i}(a)}{\partial a} + \theta(0,a) \cdot 0 \\
&= \int_0^{\bar{i}(a)} \left[\frac{\partial \theta(i,a)}{\partial i} \cdot \frac{\partial i}{\partial a} + \frac{\partial \theta(i,a)}{\partial a} \right] di + \theta(\bar{i}(a),a) \cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= \int_0^{\bar{i}(a)} \left[-k_1 \cdot \left[i - \frac{1}{2} \right] \right] di + \theta(\bar{i}(a),a) \cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= -k_1 \cdot \left[\frac{1}{2} i^2 - \frac{1}{2} i \right]_0^{\bar{i}(a)} + \theta(\bar{i}(a),a) \cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= \frac{k_1}{2} \cdot (\bar{i}(a) - \bar{i}(a)^2) + \theta(\bar{i}(a),a) \cdot \frac{\partial \bar{i}(a)}{\partial a}
\end{aligned} \tag{9.74}
$$

The sign of the final expression in (9.74) is not immediately clear. It is clear that:

$$
\frac{k_1}{2} \cdot (\bar{i}(a) - \bar{i}(a)^2) > 0 \tag{9.75}
$$

Because $\bar{i}(a)^2 < \bar{i}(a) \; \forall i$. Call the expression in (9.75) A, where $A > 0$. In order to find the sign of (9.74), it is necessary to find $\theta(\bar{i}(a),a) \cdot \frac{\partial \bar{i}(a)}{\partial a}$. The value of $\frac{\partial \bar{i}(a)}{\partial a}$ can be found from equilibrium noting that in equilibrium:

$$
\begin{aligned}
A^L(\bar{i}(a)) &= B(\bar{i}(a),a) \\
a + b \cdot \bar{i}(a) &= \frac{\int_0^{\bar{i}(a)} \theta(i,a)di}{1 - \int_0^{\bar{i}(a)} \theta(i,a)di} \cdot \frac{K^A}{L^L}
\end{aligned} \tag{9.76}
$$

And so:

$$\frac{\partial A(\bar{i}(a))}{\partial a} = \frac{\partial B(\bar{i}(a), a)}{\partial a}$$

$$1 + b\frac{\partial \bar{i}(a)}{\partial a} = \frac{\frac{k_1}{2} \cdot (\bar{i}(a) - \bar{i}(a)^2) + \theta(\bar{i}(a), a) \cdot \frac{\partial \bar{i}(a)}{\partial a}}{\left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2} \cdot \frac{K^A}{L^L}$$

$$\frac{\partial \bar{i}(a)}{\partial a}\left[b - \frac{\theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}}{\left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2}\right] = \frac{\frac{k_1}{2} \cdot (\bar{i}(a) - \bar{i}(a)^2) \cdot \frac{K^A}{L^L}}{\left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2} - 1$$

$$\frac{\partial \bar{i}(a)}{\partial a}\left[b\left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2 - \theta(\bar{i}(a), a)\frac{K^A}{L^L}\right] = \frac{k_1}{2}(\bar{i}(a) - \bar{i}(a)^2)\frac{K^A}{L^L} - \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2$$

$$\frac{\partial \bar{i}(a)}{\partial a} = \frac{\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) \cdot \frac{K^A}{L^L} - \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]} \tag{9.77}$$

Given (9.77), it follows from (9.74) that:

$$\frac{\partial \int_0^{\bar{i}(a)} \theta(i, a) di}{\partial a} = \frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) + \theta(\bar{i}(a), a) \cdot \frac{\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) \cdot \frac{K^A}{L^L} - \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]} \tag{9.78}$$

which is equal to:

$$
\begin{aligned}
&= A + \theta(\bar{i}(a), a) \cdot \frac{A \cdot \frac{K^A}{L^L} - \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]} \\
&= A \cdot \left[1 + \frac{\theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]}\right] - \\
&\frac{\theta(\bar{i}(a), a) \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]} \\
&= A \cdot \left[\frac{b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]}\right] - \\
&\frac{\theta(\bar{i}(a), a) \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]}
\end{aligned}
\tag{9.79}
$$

(9.79) can be written more concisely as:

$$
\frac{A \cdot b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 - \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^L}\right]} \tag{9.80}
$$

which implies that:

$$
\frac{\partial \int_0^{\bar{i}(a)} \theta(i,a)di}{\partial a} > 0 \tag{9.81}
$$

This is because both the denominator and numerator of (9.80) are < 0, given that $b < 0$ and $A > 0$. It follows then from (9.81), (9.73), and (9.72) that:

$$\frac{\partial f}{\partial a} > 0 \tag{9.82}$$

High-Skilled Labour and Advanced Capital

In the case of high-skilled labour and increasingly capable advanced capital, there is ambiguity – a uniform increase in the relative productivity of advanced capital across tasks could cause the factor price ratio, f, to rise or fall. Recall that:

$$f = C(\bar{i}, a) = \frac{\int_{\bar{i}(a)}^{1} \theta(i,a)di}{1 - \int_{\bar{i}(a)}^{1} \theta(i,a)di} \cdot \frac{K^A}{L^H} \tag{9.83}$$

Then:

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial \int_{\bar{i}(a)}^{1} \theta(i,a)di} \cdot \frac{\partial \int_{\bar{i}(a)}^{1} \theta(i,a)di}{\partial a} \tag{9.84}$$

The first term on the right-hand side of (9.84) is:

$$\begin{aligned} \frac{\partial f}{\partial \int_{\bar{i}(a)}^{1} \theta(i,a)di} &= \frac{1}{\left[1 - \int_{\bar{i}(a)}^{1} \theta(i,a)di\right]^2} \cdot \frac{K^A}{L^H} \\ &> 0 \end{aligned} \tag{9.85}$$

The second term on the right-hand side of (9.84) is:

$$
\begin{aligned}
\frac{\partial \int_{\bar{i}(a)}^{1} \theta(i,a)di}{\partial a} &= \int_{\bar{i}(a)}^{1} \frac{\partial \theta(i,a)}{\partial a} di + \theta(1,a)\cdot 0 - \theta(\bar{i}(a),a)\cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= \int_{\bar{i}(a)}^{1} \left[\frac{\partial \theta(i,a)}{\partial i} \cdot \frac{\partial i}{\partial a} + \frac{\partial \theta(i,a)}{\partial a} \right] di - \theta(\bar{i}(a),a)\cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= \int_{\bar{i}(a)}^{1} \left[-k_1 \cdot \left[i - \frac{1}{2} \right] \right] di + \theta(\bar{i}(a),a)\cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= -k_1 \left[\frac{1}{2} i^2 - \frac{1}{2} i \right]_{\bar{i}(a)}^{1} - \theta(\bar{i}(a),a)\cdot \frac{\partial \bar{i}(a)}{\partial a} \\
&= \frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) - \theta(\bar{i}(a),a)\cdot \frac{\partial \bar{i}(a)}{\partial a}
\end{aligned}
\tag{9.86}
$$

The sign of this is not immediately clear. It is clear that:

$$
\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) < 0 \tag{9.87}
$$

since $\bar{i}(a)^2 < \bar{i}(a)\ \forall i$. In order to find the sign of (9.86), it is necessary to find $\theta(\bar{i}(a),a)\cdot \frac{\partial \bar{i}(a)}{\partial a}$. The value of $\frac{\partial \bar{i}(a)}{\partial a}$ can be found from equilibrium noting that in equilibrium:

$$
\begin{aligned}
A^H(\bar{i}(a)) &= C(\bar{i}(a),a) \\
a + b \cdot \bar{i}(a) &= \frac{\int_{\bar{i}(a)}^{1} \theta(i,a)di}{1 - \int_{\bar{i}(a)}^{1} \theta(i,a)di} \cdot \frac{K^A}{L^H}
\end{aligned}
\tag{9.88}
$$

Where now, with advanced capital, $b > 0$. It follows that:

$$\frac{\partial A(\bar{i}(a))}{\partial a} = \frac{\partial C(\bar{i}(a), a)}{\partial a}$$

$$1 + b\frac{\partial \bar{i}(a)}{\partial a} = \frac{\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) - \theta(\bar{i}(a), a) \cdot \frac{\partial \bar{i}(a)}{\partial a}}{\left[1 - \int_{\bar{i}(a)}^{1} \theta(i, a) di\right]^2} \cdot \frac{K^A}{L^H}$$

$$\frac{\partial \bar{i}(a)}{\partial a}\left[b + \frac{\theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}}{\left[1 - \int_{\bar{i}(a)}^{1} \theta(i, a) di\right]^2}\right] = \frac{\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) \cdot \frac{K^A}{L^H}}{\left[1 - \int_{\bar{i}(a)}^{1} \theta(i, a) di\right]^2} - 1$$

$$\frac{\partial \bar{i}(a)}{\partial a}\left[b\left[1 - \int_{\bar{i}(a)}^{1} \theta(i, a) di\right]^2 + \theta(\bar{i}(a), a)\frac{K^A}{L^H}\right] = \frac{k_1}{2}(\bar{i}(a)^2 - \bar{i}(a))\frac{K^A}{L^H} - \left[1 - \int_{\bar{i}(a)}^{1} \theta(i, a) di\right]^2$$

$$\frac{\partial \bar{i}(a)}{\partial a} = \frac{\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) \cdot \frac{K^A}{L^H} - \left[1 - \int_{0}^{\bar{i}(a)} \theta(i, a) di\right]^2}{\left[b \cdot \left[1 - \int_{0}^{\bar{i}(a)} \theta(i, a) di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}\right]} < 0 \tag{9.89}$$

Given (9.87) and (9.89) it follows that the sign of:

$$\frac{\partial \int_{\bar{i}(a)}^{1} \theta(i, a) di}{\partial a} \tag{9.90}$$

in (9.86) is ambiguous. It follows from (9.84)–(9.89) that $\frac{\partial f}{\partial a} < 0$ – i.e. a uniform rise in the productivity of advanced capital (a fall in a) will cause the factor-price ratio f to rise – only if:

$$\frac{\partial \int_{\bar{i}(a)}^{1} \theta(i,a)di}{\partial a} < 0$$

$$\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) - \theta(\bar{i}(a), a) \cdot \frac{\partial \bar{i}(a)}{\partial a} < 0$$

$$\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) < \theta(\bar{i}(a), a) \cdot \frac{\partial \bar{i}(a)}{\partial a}$$

$$\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) < \theta(\bar{i}(a), a) \cdot \frac{\frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) \cdot \frac{K^A}{L^H} - \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}\right]} \tag{9.91}$$

Call:

$$A = \frac{k_1}{2} \cdot (\bar{i}(a)^2 - \bar{i}(a)) \tag{9.92}$$

Then (9.91) requires that:

$$A < \theta(\bar{i}(a), a) \cdot \frac{A \cdot \frac{K^A}{L^H} - \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{\left[b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}\right]} \tag{9.93}$$

And so requires that:

$$A \left[1 - \frac{\theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}}{b \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}}\right] < -\frac{\theta(\bar{i}(a), a) \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{b \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}}$$

$$A \left[\frac{b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}}\right] < -\frac{\theta(\bar{i}(a), a) \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2}{b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i,a)di\right]^2 + \theta(\bar{i}(a), a) \cdot \frac{K^A}{L^H}} \tag{9.94}$$

$$A < -\frac{\theta(\bar{i}(a), a) \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2}{b \cdot \left[1 - \int_0^{\bar{i}(a)} \theta(i, a) di\right]^2}$$

$$-\frac{k_1}{2} \cdot (\bar{i}(a) - \bar{i}(a)^2) < -\frac{\theta(\bar{i}(a), a)}{b}$$

(9.94) therefore implies that, for $\frac{\partial f}{\partial a} < 0$, it must be the case that:

$$k_1 > \frac{2 \cdot \theta(\bar{i}(a), a)}{b \cdot [\bar{i}(a) - \bar{i}(a)^2]} \tag{9.95}$$

where $\frac{2 \cdot \theta(\bar{i}(a), a)}{b \cdot [\bar{i}(a) - \bar{i}(a)^2]} > 0$, since with advanced capital $b > 0$. If (9.95) holds then a uniform increase in the productivity of advanced capital causes an *increase* in the relative wage of high-skilled labour. If not then, as in the low-skilled labour case, an increase in the productivity of advanced capital will cause a *decrease* in the relative wage of high-skilled labour.

Note however that the analysis in Chapter 7 does imply an upper limit to k_1 – to ensure that all expenditure shares are positive, (7.27) implies that the following must hold:

$$k_1 < \frac{2}{a^* - a} \tag{9.96}$$

This ensures that the expenditure share on good 0 is not zero, and so all other expenditure shares are not zero. Clearly, as a continues to fall below a^* this inequality cannot hold for a fixed k_1 – this was noted in Chapter 7, and is why this model is only an approximation. Nevertheless it is important to establish that in those cases when this approximation does make intuitive sense – namely those cases where k_1 is such that (9.96) holds – it is also feasible that (9.95) can hold for some of those cases of k_1.

To see that this is possible, consider initially that $a = a^*$ and expenditure shares are constant and equal to one for all goods. In equilibrium the analysis in Chapter 7 implies that the following must hold:

$$a + b \cdot \bar{i} = \frac{1 - \bar{i}}{\bar{i}} \cdot c \tag{9.97}$$

where $c = \frac{K^A}{L^H}$ and $\beta(\bar{i}) = 1 - \bar{i}$. This can be re-expressed as a polynomial:

$$b \cdot \bar{i}^2 + (a + c) \cdot \bar{i} - c = 0 \tag{9.98}$$

with the following solutions for $\bar{i}$:

$$\bar{i} = \frac{-(a + c) \pm [(a + c)^2 + 4 \cdot b \cdot c]^{\frac{1}{2}}}{2b} \tag{9.99}$$

There are two solutions for $\bar{i}$ – though I am interested in the positive solution, rather than the negative since $\bar{i} \in [0, 1]$. Given that $a = a^*$ (9.95) implies that for a unit increase in the relative productivity of advanced capital to cause a rise in the relative wage of high-skilled labour it must be the case that:

$$k_1 > \frac{2}{b \cdot [\bar{i} - \bar{i}^2]} \tag{9.100}$$

and (9.96) implies that for there to be no negative expenditure shares it must also be the case that:

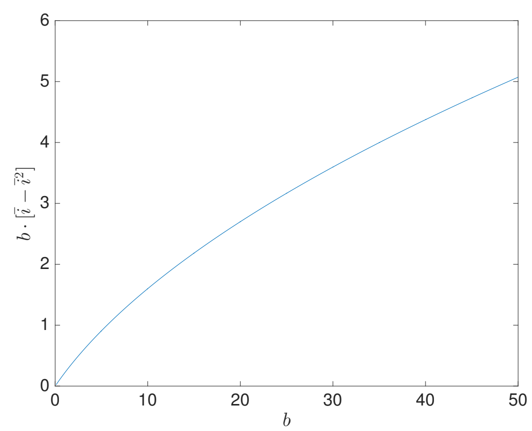
$$k_1 < 2 \tag{9.101}$$

As a result (9.100) and (9.101) together imply that it must be the case that:

$$b \cdot \left[\bar{i} - \bar{i}^2\right] > 1 \tag{9.102}$$

if a uniform increase in the relative productivity of advanced capital is to cause a rise in the relative wage of high-skilled labour, and all expenditure shares remain positive. To see that this is possible, consider a case where $a = 2$, $c = 1$ and $b \in [0, 50]$. Figure 9.2 shows the value of the expression on the left-hand side of (9.102) for these values and the $\bar{i}$ implied by (9.99):

Figure 9.2: Demonstration that (9.102) Can Hold



This shows that there are values of a, b and c such that the expression in (9.102) is satisfied and technological progress causes relative high-skilled wages to increase.

9.3. Chapter 8

9.3.1. An Increase in Advanced Capital Productivity – a Falls

An increase in the productivity of machines at all tasks – a *decrease* in a, holding b and c constant – causes a *decrease* in the factor-price ratio f:

$$-\left.\frac{\partial f}{\partial a}\right|_c < 0 \tag{9.103}$$

The proof is as follows. In equilibrium:

$$\begin{aligned} f &= D(\bar{i}) \\ &= \frac{1-\gamma(\tilde{i})}{\gamma(\tilde{i})} \cdot \frac{K^A}{L^H} \\ &= \frac{\int_{\bar{i}}^{c} \theta(i)\, di}{1 - \int_{\bar{i}}^{c} \theta(i)\, di} \cdot \frac{K^A}{L^H} \end{aligned} \tag{9.104}$$

And so the sensitivity of the factor-price ratio f to an increase in the relative productivity of machines a is:

$$\left.\frac{\partial f}{\partial a}\right|_c = \frac{\partial f}{\partial \int_{\bar{i}}^{c} \theta(i)\, di} \cdot \frac{\partial \int_{\bar{i}}^{c} \theta(i)\, di}{\partial a} \tag{9.105}$$

The first partial derivative on the right-hand side of (9.105) is:

$$\begin{aligned}
\frac{\partial f}{\partial \int_{\bar{i}}^{c} \theta(i)\, di} &= \frac{\left[1 - \int_{\bar{i}}^{c} \theta(i)\, di\right] - \left[- \int_{\bar{i}}^{c} \theta(i)\, di\right]}{\left[1 - \int_{\bar{i}}^{c} \theta(i)\, di\right]^2} \cdot \frac{K^A}{L^H} \\
&= \frac{1}{\left[1 - \int_{\bar{i}}^{c} \theta(i)\, di\right]^2} \cdot \frac{K^A}{L^H} \\
&= \frac{1}{\left[1 - \frac{(c-\bar{i})}{c}\right]^2} \cdot \frac{K^A}{L^H} \\
&= \frac{c^2}{\bar{i}^2} \cdot \frac{K^A}{L^H}
\end{aligned} \tag{9.106}$$

which is > 0. The second partial derivative on the right-hand side of (9.105) is:

$$\begin{aligned}
\frac{\partial \int_{\bar{i}}^{c} \theta(i)\, di}{\partial a} &= \int_{\bar{i}}^{c} \frac{\partial \theta(i)}{\partial a}\, di + \theta(c) \cdot 0 - \theta(\bar{i}) \cdot \frac{\partial \bar{i}}{\partial a} \\
&= -\frac{1}{c} \cdot \frac{\partial \bar{i}}{\partial a}
\end{aligned} \tag{9.107}$$

The full expression for $\frac{\partial f}{\partial a}$ can be found by deriving $\frac{\partial \bar{i}}{\partial a}$. In equilibrium:

$$\begin{aligned}
A(\bar{i}) &= D(\bar{i}) \\
a + b \cdot \bar{i} &= \frac{\int_{\bar{i}}^{c} \theta(i)\, di}{1 - \int_{\bar{i}}^{c} \theta(i)\, di} \cdot \frac{K^A}{L^H}
\end{aligned} \tag{9.108}$$

And so:

$$
\begin{aligned}
\frac{\partial A(\bar{i})}{\partial a} &= \frac{\partial D(\bar{i})}{\partial a} \\
1 + b \cdot \frac{\partial \bar{i}}{\partial a} &= -\frac{c^2}{\bar{i}^2} \cdot \frac{1}{c} \cdot \frac{\partial \bar{i}}{\partial a} \cdot \frac{K^A}{L^H} \\
\frac{\partial \bar{i}}{\partial a} \left[b + \frac{c}{\bar{i}^2} \cdot \frac{K^A}{L^H} \right] &= -1 \\
\frac{\partial \bar{i}}{\partial a} &= -\frac{\bar{i}^2}{b \cdot \bar{i}^2 + c \cdot \frac{K^A}{L^H}}
\end{aligned}
\tag{9.109}
$$

It follows then from (9.104) - (9.109) that:

$$
\begin{aligned}
\left.\frac{\partial f}{\partial a}\right|_c &= \frac{c^2}{\bar{i}^2} \cdot \frac{K^A}{L^H} \cdot \frac{1}{c} \cdot \frac{\bar{i}^2}{b \cdot \bar{i}^2 + c \cdot \frac{K^A}{L^H}} \\
&= \frac{c}{b \cdot \bar{i}^2 + c \cdot \frac{K^A}{L^H}} \cdot \frac{K^A}{L^H} \\
&> 0
\end{aligned}
\tag{9.110}
$$

9.3.2. An Increase in the Variety of Tasks – c Rises

An increase in the number of types of tasks – an increase in c, holding a constant – causes an *increase* in the factor-price ratio f:

$$
\left.\frac{\partial f}{\partial c}\right|_a > 0 \tag{9.111}
$$

The proof is as follows. In equilibrium:

$$
f = \frac{\int_{\bar{i}}^{c} \theta(i)\, di}{1 - \int_{\bar{i}}^{c} \theta(i)\, di} \cdot \frac{K^A}{L^H} \tag{9.112}
$$

And so the sensitivity of the factor-price ratio f to an increase in the range of goods (and tasks) c is:

$$\left.\frac{\partial f}{\partial c}\right|_a = \frac{\partial f}{\partial \int_{\bar{i}}^{c} \theta(i)\, di} \cdot \frac{\partial \int_{\bar{i}}^{c} \theta(i)\, di}{\partial c} \tag{9.113}$$

The first partial derivative on the right-hand side of (9.113) is as in (9.106) and again > 0. The second partial derivative on the right-hand side of (9.113) is:

$$\begin{aligned}
\frac{\partial \int_{\bar{i}}^{c} \theta(i)\, di}{\partial c} &= \int_{\bar{i}}^{c} \frac{\partial \theta(i)}{\partial c}\, di + \theta(c) \cdot 1 - \theta(\bar{i}) \cdot \frac{\partial \bar{i}}{\partial c} \\
&= -\int_{\bar{i}}^{c} \frac{1}{c^2}\, di + \frac{1}{c} - \frac{1}{c} \cdot \frac{\partial \bar{i}}{\partial c} \\
&= \frac{1}{c} \cdot \left[1 - \frac{\partial \bar{i}}{\partial c}\right] - \int_{\bar{i}}^{c} \frac{1}{c^2}\, di \\
&= \frac{1}{c} \cdot \left[1 - \frac{\partial \bar{i}}{\partial c}\right] - \frac{c - \bar{i}}{c^2} \\
&= \frac{\bar{i}}{c^2} - \frac{\partial \bar{i}}{\partial c} \cdot \frac{1}{c}
\end{aligned} \tag{9.114}$$

And so it follows that:

$$\begin{aligned}
\left.\frac{\partial f}{\partial c}\right|_a &= \frac{c^2}{\bar{i}^2} \cdot \frac{K^A}{L^H} \cdot \left[\bar{i} \cdot \frac{1}{c^2} - \frac{\partial \bar{i}}{\partial c} \cdot \frac{1}{c}\right] \\
&= \left[\frac{1}{\bar{i}} - \frac{c}{\bar{i}^2} \cdot \frac{\partial \bar{i}}{\partial c}\right] \cdot \frac{K^A}{L^H}
\end{aligned} \tag{9.115}$$

The full expression for $\frac{\partial f}{\partial c}$ can be found by deriving $\frac{\partial \bar{i}}{\partial c}$. In equilibrium:

$$\begin{aligned} A(\bar{i}) &= D(\bar{i}) \\ a + b \cdot \bar{i} &= \frac{\int_{\bar{i}}^{c} \theta(i)\, di}{1 - \int_{\bar{i}}^{c} \theta(i)\, di} \end{aligned} \tag{9.116}$$

And so:

$$\begin{aligned} \frac{\partial A(\bar{i})}{\partial c} &= \frac{\partial D(\bar{i})}{\partial c} \\ b \cdot \frac{\partial \bar{i}}{\partial c} &= \left[\frac{1}{\bar{i}} - \frac{c}{\bar{i}^2} \cdot \frac{\partial \bar{i}}{\partial c} \right] \cdot \frac{K^A}{L^H} \\ \frac{\partial \bar{i}}{\partial c} \cdot \left[b + \frac{c}{\bar{i}^2} \cdot \frac{K^A}{L^H} \right] &= \frac{1}{\bar{i}} \cdot \frac{K^A}{L^H} \\ \frac{\partial \bar{i}}{\partial c} &= \frac{\bar{i}^2}{b \cdot \bar{i}^2 + c \cdot \frac{K^A}{L^H}} \cdot \frac{1}{\bar{i}} \cdot \frac{K^A}{L^H} \end{aligned} \tag{9.117}$$

It follows then that:

$$\begin{aligned} \left. \frac{\partial f}{\partial c} \right|_a &= \frac{1}{\bar{i}} - \frac{1}{b \cdot \bar{i}^2 + c \cdot \frac{K^A}{L^H}} \cdot \frac{c}{\bar{i}} \cdot \frac{K^A}{L^H} \\ &= \frac{b \cdot \bar{i}}{b \cdot \bar{i}^2 + c \cdot \frac{K^A}{L^H}} \\ &> 0 \end{aligned} \tag{9.118}$$

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