ON DEFICIENCY GRADIENT OF GROUPS

ADITI KAR AND NIKOLAY NIKOLOV

Abstract. Deficiency gradient is a higher dimensional analogue of rank gradient. In this paper, we give a combinatorial proof that the fundamental group of a simply connected complex of amenable groups has deficiency gradient zero. We apply this to establish the vanishing of deficiency gradient in special linear groups over polynomial rings and number fields, and in Artin groups for which the nerve of the Coxeter matrix is simply connected. This implies that the first and second l^2-Betti numbers vanish for these Artin groups without recourse to the $K(\pi, 1)$ conjecture. We propose a conjecture about the stabilisation of deficiency gradient, which characterises groups with 2-dimensional classifying spaces.

1. Introduction

Throughout the paper, the phrase *a (normal) chain in a group G* will refer to a descending sequence $G > H_1 > H_2 > H_3 > \dots$ of finite index (normal) subgroups in G such that the intersection $\cap_i H_i$ is the residual $\mathfrak{R}(G)$ of G. The residual of a group is the intersection of all its subgroups of finite index and is trivial in any group that is residually finite.

Let G be a finitely presented group. Set $d(G)$ to be the cardinality of a minimal generating set and $\delta(G)$ to be the deficiency of G. More precisely,

$$\delta(G) = \max\{|S| - |R| \mid G := \langle S|R\rangle\}$$

We denote by $b_n(G) = \dim_{\mathbb{Q}} H_n(G, \mathbb{Q}) = b_n(X)$, where $X = K(G, 1)$ is a classifying space for G (i.e. a connected CW-complex X with contractible universal cover and fundamental group G), and $b_n(X)$ denotes the n-th Betti number of X. Note that $\delta(G) \leq b_1(G) \leq d(G)$. Therefore for a finite index subgroup $H \leq G$,

$$\frac{\delta(H) - 1}{[G : H]} \leq \frac{d(H) - 1}{[G : H]} \leq d(G) - 1$$

Starting with the presentation $\langle S \mid R\rangle$ for G one obtains a Schreier presentation for H with $[G : H](|S| - 1) + 1$ generators and $[G : H]|R|$ relations showing that

$$\frac{\delta(H) - 1}{[G : H]} \geq \delta(G) - 1.$$

It is well-known that $\delta(G) \leq b_1(G) - b_2(G)$ as a special case of the Morse inequalities [13, §4, p. 210].

Let G be a finitely presented group and let $\{H_i\}$ be a normal chain in G. The sequence $\{\frac{\delta(H_i)-1}{[G:H_i]}\}$ is non-decreasing and bounded above by $d(G)-1$ and so we make the following definition.

Definition 1. *The deficiency gradient of G with respect to (H_i) is*

$$\Delta(G,(H_i)) = \lim_{i\to\infty} \frac{\delta(H_i)-1}{[G:H_i]}.$$

This notion has received some attention recently. In [1], deficiency gradient is studied using analytic tools while in [8], the authors study asymptotic growth of homology groups of residually free groups. In this note we investigate deficiency gradient with the tools of combinatorial group theory.

Recall that the limit $RG(G,(H_i)) = \lim_{i\to\infty} \frac{d(H_i)-1}{[G:H_i]}$ is called the rank gradient [4] of G with respect to the chain (H_i). Trivially one has $\Delta(G,(H_i)) \le RG(G,(H_i))$.

Let $\beta_i(G)$ denote the i-th l^2-Betti number of the group G. Then Proposition 4 below gives

$$\beta_1(G) - \beta_2(G) \ge \Delta(G,(H_i)) \tag{1}$$

for any chain (H_i) in G. If the chain (H_i) is normal with trivial intersection we could ask by analogy with the similar question for rank gradient whether equality always holds above. Finding any examples where the inequality is strict seems to be a hard problem.

1.1. Groups with two dimensional classifying spaces.

The deficiency gradient is easy to compute for groups G which have finite two-dimensional classifying space $K(G,1)$. Examples of such groups are surface groups or more generally, torsion-free one relator groups and, direct products of two free groups.

Lemma 2. *If a group G has a finite two-dimensional classifying space $K(G,1)$, then $\delta(G) = 1-\chi(G)$ and consequently, $\delta(H)-1 = [G:H](\delta(G)-1)$ for every subgroup H of finite index in G.*

Consequently, $\Delta(F_n \times F_m,(H_i)) = -(n-1)(m-1)$ for any chain (H_i) while the deficiency gradient of a torsion-free one relator group defined on d generators is $d-2$, a number not surprisingly equal to the difference in its first and second l^2-Betti numbers.

We conjecture that the converse of Lemma 2 holds:

Conjecture 3. *Let G be a torsion-free residually finite finitely presented group such that $\delta(H)-1 = [G:H](\delta(G)-1)$ for every subgroup H of finite index in G. Then G has a finite 2-dimensional space $K(G,1)$.*

Conjecture 3 is much in the character of open questions like the Eilenberg-Ganea conjecture and the Relation gap problem in topological group theory. The following basic characterisation of groups with two dimensional classifying spaces may be useful:

Lemma 4. *Let G be an infinite finitely presented group. Then $\delta(G) - 1 \leq \beta_1(G) - \beta_2(G)$ with equality if and only if G has a two dimensional classifying space.*

In particular any counterexample to Conjecture 3 will be a group G with a normal chain (H_i) such that $\Delta(G, (H_i)) < \beta_1(G) - \beta_2(G)$.

1.2. **Economical groups: basic examples.** If the deficiency gradient of an infinite group G with respect to any chain is positive then G has a finite index subgroup H whose deficiency is at least 2. By a result of Baumslag and Pride [5] one deduces that H and hence G is large. Therefore deficiency gradient for amenable groups and torsion groups is at most 0. In fact, deficiency gradient is almost always zero for amenable groups and they satisfy the stronger condition of being *economical*:

Definition 5. *Let (H_i) be a chain of finite index subgroups in an infinite finitely presented group Γ. We say that the chain (H_i) is economical if each H_i has a presentation of d_i generators and r_i relations such that*

$$\lim_{i\to\infty} \frac{d_i}{[\Gamma : H_i]} = \lim_{i\to\infty} \frac{r_i}{[\Gamma : H_i]} = 0.$$

The group Γ is economical if every normal chain in Γ which intersects in $\mathfrak{R}(\Gamma)$ is economical. Note that this implies that $[\Gamma : \mathfrak{R}(\Gamma)] = \infty$.

If Γ is economical then since $\delta(H_i) \geq d_i - r_i$ and the ratios d_i/n_i and r_i/n_i tend to zero we have $\Delta(\Gamma, (H_i)) \geq 0$. Now $0 \leq \Delta(\Gamma, (H_i)) \leq RG(\Gamma, (H_i)) = 0$ and so both the deficiency gradient and the rank gradient of Γ are zero. The *slow* groups defined in [8] are examples of economical groups.

Theorem 6. *Let G be a finitely presented, residually finite amenable group which is infinite. Then G is economical.*

Another class of examples of economical groups comes from the presence of normal subgroups which are also economical.

Proposition 7. *Let G be a finitely presented group which has a normal finitely presented infinite subgroup N. Let (H_i) be a normal chain of G such that the chain $(N \cap H_i)$ is economical. Then the chain (H_i) is also economical. In particular, if a residually finite, finitely presented group G has a normal economical subgroup, then G itself is economical.*

Our main result is a vanishing theorem for deficiency gradient of complexes of economical groups.

1.3. **Complexes of groups.** For definition and detailed properties of complexes of groups we refer to [7, Chapter III.C]. Here we summarise the properties that we need to state and prove Theorem 8.

A scwol P is a pair (V, E) of two sets called vertices and edges together with maps $i, t : E \to V$. For an edge $a \in E$ we refer to $i(a)$ as the origin and to $t(a)$ as the terminus of a. A pair of edges $a, b \in E$ are called composable if $i(a) = t(b)$. In that case there is a third edge $c \in E$, called the product of a and b and denoted by $c = ab$ with the property that $i(c) = i(b)$ and $t(c) = t(a)$. The operation of taking products of edges is associative and in addition no edge a is allowed to have $i(a) = t(a)$. An example which covers all applications in this paper is the case when the scwol P is defined by a poset $(V, <)$. The vertex set of P is V and $E = \{(\sigma, \tau) \in V \times V \mid \sigma < \tau\}$. For $a = (\sigma, \tau) \in E$ we define $i(a) = \tau, t(a) = \sigma$.

A complex of groups C over a scwol P is a quadruple

$$C = (P, (G_\sigma), (f_a), (g_{a,b}))$$

consisting of the following

- a scwol $P = (V, E)$,
- a family of groups G_σ indexed by the elements $\sigma \in V$,
- group monomorphisms $f_a : G_{i(a)} \to G_{t(a)}$ for every edge $a \in E$,
- elements $g_{a,b} \in G_{i(a)}$ for every pair of composable edges $a, b \in E$.

The elements $g_{a,b}$ have to satisfy certain compatibility conditions described in [7, Definition III.C.2.1] (which play no role in our argument).

Every scwol P has a realization $|P|$ as a polyhedral complex with edges E. We assume that $|P|$ is connected and choose a spanning tree T of the edges E of P. The fundamental group $\tilde{G}(C)$ is defined to be group generated by $E \cup (\cup_{\sigma \in V} G_\sigma)$ subject to the following four families of relations R1-R4:

R1: All relations of the groups $G_\sigma, \sigma \in V$,

R2: Relations $ab = g_{a,b}c$ for composable pairs of edges $a, b \in E$ with product $c = ab$.

R3: $f_a(x) = axa^{-1}$ for all $a \in E$ and $x \in G_{i(a)}$,

R4: $a = 1$ for all edges a in T.

A complex C is called *simple* if all elements $g_{a,b}$ are identity. A complex C is *developable* if the inclusion homomorphism $G_\sigma \to \tilde{G}(C)$ is injective for each $\sigma \in V$. In that case we will identify the group G_σ with its image in $\tilde{G}(C)$.

In the special case when C is a simple complex whose scwol P has a simply connected realization $|P|$ the additional identities $a = 1$ hold in $\tilde{G}(C)$ for all edges $a \in E$. Therefore in this case the fundamental group $\tilde{G}(C)$ is just the direct limit (amalgam) of the system $((G_\sigma), (f_a))$.

In later application we will only need to consider simple complexes with simply connected realizations. The last condition is often easier to check when we consider from the start a simply connected simplicial compex L, together with a family of groups G_σ for every simplex $\sigma \subset L$ and homomorphisms $f_{\sigma,\tau} : G_\tau \to G_\sigma$ for every pair $\sigma \subset \tau$. The swol $P = P_L$ is defined by the poset $(V, <)$ of simplices of L ordered by inclusion and the geometric realization $|P|$ is the barycentric subdivision of L. Since L is simply connected, so is $|P|$ and hence the fundamental group of the simple complex of groups $(P, (G_\sigma), (f_{\sigma,\tau}))$ is the direct limit of the system $((G_\sigma), (f_{\sigma,\tau}))$. When there is no ambiquity we will write $\tilde{G}(L)$ for $\tilde{G}(P_L, (G_\sigma), (f_{\sigma,\tau}))$.

Our main result states that the fundamental group of a complex of economical groups is also economical:

Theorem 8. *Let C be a developable complex of groups over a scwol P such that $|P|$ is connected. Let (H_i) be a normal chain in $\Gamma := \tilde{G}(C)$ such that for every vertex $\sigma \in V$ the chain $\{H_i \cap G_\sigma\}$ is economical in G_σ. Then (H_i) is an economical chain of Γ and $\Delta(\Gamma, (H_i)) = 0$. In particular, if each G_σ is economical and $G_\sigma \cap \mathfrak{R}(\Gamma) = \mathfrak{R}(G_\sigma)$ for each $\sigma \in L$, then Γ is economical.*

We note that in general it is hard to check whether a complex of groups is developable. However every complex C gives rise to a developable complex on replacing the vertex groups G_σ with their images in $\tilde{G}(C)$. Therefore if the groups G_σ have the property that all their images are economical (e.g. if G_σ are polycyclic) then the conclusion of Theorem 8 holds without assuming the developability of C.

1.4. **Applications of Theorem 8.** We can use Theorem 8 to show that many special linear groups and Artin groups are economical.

1.4.1. *Steinberg groups and special linear groups.* A unital ring R is finitely presented if R is isomorphic to $\mathbb{Z}\langle X\rangle/J$ where $\mathbb{Z}\langle X\rangle$ denotes the free associative ring on a finite set X, and J is a finitely generated ideal of $\mathbb{Z}\langle X\rangle$.

Recall that the Steinberg group $St_n(R)$ over a unital ring R is defined to be the group presented by generators $\{x_r(a) \mid r \in \Sigma,\ a \in R\}$ where Σ is the root system of SL_n of type A_{n-1}, subject to the following relations (cf. [20], §5).

$$\begin{aligned} &x_r(a)x_r(b) = x_r(a+b) && \\ &[x_r(a), x_v(b)] = 1 && \text{if } r+v \notin \Sigma \cup \{0\} \\ &[x_r(a), x_v(b)] = x_{r+v}(ab) && \text{if } r+v \in \Sigma \end{aligned} \tag{2}$$

for all $r, v \in \Sigma$ and all $a, b \in R$.

These relations imply in particular that for any $r \in \Sigma$ the set $X_r = \{x_r(a) \mid a \in R\}$ is a subgroup of $St_n(R)$ isomorphic to $(R, +)$

Theorem 9. *Let R be a finitely presented ring such that $(R, +)$ is torsion free. Assume that $n \geq 4$ and let (H_i) be a normal chain in $St_n(R)$ such that $X_r \cap \cap_{i=1}^{\infty} H_i = \{1\}$. Then the chain (H_i) is economical. If in addition R is a residually finite ring it follows that $St_n(R)$ is economical.*

In the case when R is a polynomial ring the knowledge of the Milnor K-group $K_2(R)$ allows us to deduce that $SL_n(R)$ is economical when n is sufficiently large.

Corollary 10. *Let $k \geq 0$ and let $R = \mathbb{Z}[x_1, \dots, x_k]$ be the polynomial ring in k variables. Let $n \geq k+4$. The group $SL_n(R)$ is economical and in particular has deficiency gradient zero with respect to any normal chain with trivial intersection.*

When $k = 0$ i.e. $R = \mathbb{Z}$ the above result has been obtained earlier in [1] using measure theoretic methods.

It will be interesting to find out if the group $SL_3(\mathbb{Z})$ is economical. In this connection we note that lattices in some semisimple Lie groups of rank 1 or 2 have strictly negative deficiency gradient, see [19], Theorem 3.

We can apply Theorem 8 to SL_3 over many rings of integers in number fields. In particular, the following Proposition shows that for every integer $m > 1$ the group $SL_3(\mathbb{Z}[\frac{1}{m}])$ is economical.

Proposition 11. *Let k be a number field and for a finite set S of valuations of k including all archimedean ones, let R be the ring of S-integers of k. Assume that the group of units R^* is infinite. Then $SL_3(R)$ is economical.*

1.4.2. *Artin groups.* In [15] we showed that any Artin group A with connected graph has rank gradient and $\beta_1(A)$ equal to zero. Let L be the nerve of the Coxeter matrix of A. Assuming the $K(\pi, 1)$ conjecture for Artin groups it is proved in [10] that $\beta_2(A) = b_1(L)$. One may expect that A has deficiency gradient zero whenever $b_1(L) = 0$. We confirm this when L is simply connected.

Theorem 12. *Let A be an Artin group such that the nerve L of its associated Coxeter matrix is connected and simply connected. Then A is economical.*

When A as above is *right-angled*, its deficiency gradient has been shown to be zero in [1].

Note that Theorem 12 and (1) imply the following, without recourse to the verity of the $K(\pi, 1)$ conjecture for Artin groups.

Corollary 13. *Let A be an Artin group such that the nerve L of its associated Coxeter matrix is connected and simply connected. Then $\beta_1(A) = \beta_2(A) = 0$.*

1.5. **Lattices in $PSL(2, \mathbb{C})$.** A well known open question in this subject asks whether the rank gradient vanishes with respect to any normal chain with trivial intersection for lattices in $PSL(2, \mathbb{C})$, c.f.[3]. From recent developments in 3-manifold theory due to Agol, Wise, et al (see [2] and references therein), we know that lattices in $PSL(2, \mathbb{C})$ are virtually fibered. The non-cocompact lattices in $PSL(2, \mathbb{C})$ are virtually free-by-cyclic i.e. some finite index subgroup is isomorphic to $F_n \rtimes \mathbb{Z}$. The co-compact lattices are virtually $\pi_1(\Sigma_g) \rtimes \mathbb{Z}$, where Σ_g denotes a surface of genus $g \geq 2$. The groups $F_n \rtimes \mathbb{Z}$ are 2-dimensional and we can invoke Lemma 2 to conclude that with respect to any chain, the deficiency gradient of $F_n \rtimes \mathbb{Z}$ is zero. In the co-compact case, the l^2-cohomology is known to be trivial and so the deficiency gradient is at most zero. On the other hand, a cocompact lattice in $PSL(2,\mathbb{C})$ comes with a well-known balanced presentation, arising from the Heegaard decomposition of the hyperbolic 3-manifold. This implies that the deficiency gradient of such a group with respect to any normal chain is at least zero. We conclude then that the deficiency gradient is always zero for both co-compact and non-cocompact lattices in $PSL(2,\mathbb{C})$. One may reformulate the vanishing of the rank gradient conjecture for these groups and ask whether lattices in $PSL(2,\mathbb{C})$ are economical.

1.6. **Free Products.** It is well-known that the l^2-Betti numbers and rank gradient minus 1 are additive under free products. One might wonder if the same is true for deficiency gradient.

Question 14. *Let Γ_1 and Γ_2 be two (torsion-free) residually finite groups and let (H_i) be a normal chain in $\Gamma = \Gamma_1 * \Gamma_2$. Is it always true that*

$$\Delta(\Gamma, (H_i)) = \Delta(\Gamma_1, (H_i \cap \Gamma_1)) + \Delta(\Gamma_2, (H_i \cap \Gamma_2)) + 1?$$

Unfortunately deficiency of groups is not additive under free products: [12] provides examples, e.g. $(C_2 \times C_2) * (C_3 \times C_3)$. The naive approach uses the Bass Serre tree T for the free product. Each H_i is a free product of several copies of $\Gamma_1 \cap H_i$ and $\Gamma_2 \cap H_i$ and possibly a free group which arises as the fundamental group of the graph $H_i \backslash T$. Therefore, starting with a presentation for $\Gamma_1 * \Gamma_2$, we can easily write a presentation for each H_i. As this may not be the optimal presentation, we may conclude only that the following inequality holds:

$$\Delta(\Gamma, (H_i)) \geq \Delta(\Gamma_1, (H_i \cap \Gamma_1)) + \Delta(\Gamma_2, (H_i \cap \Gamma_2)) + 1.$$

Using the inequality (1) we can deduce a positive answer to Question 14 in a special case:

Proposition 15. *Suppose that Γ_j has deficiency gradient equal to $\beta_1(\Gamma_j) - \beta_2(\Gamma_j)$ for any normal chain in Γ_j $(j = 1, 2)$. Then the same holds for $\Gamma_1 * \Gamma_2$.*

The rest of the paper is organised as follows. Lemmas 2 and 4, Theorem 6 and Proposition 7 are proved in Section 2. Our main theorem (Theorem 8) is proved in Section 3. The results 10 and 11 on special linear groups are proved in Section 4. Artin groups are discussed and Theorem 12 is proved in Section 5.

2. Proofs.

Proof of Lemma 2. Let K be the universal cover of the classifying space $X = K(G, 1)$. Thus K is a contractible 2-dimensional complex on which G acts freely with compact quotient $X = K/G$. Let e_i be the number of i-dimensional simplices of X. The Euler characteristic of X is $\chi(X) = e_0 - e_1 + e_2$ while $G = \pi_1(X)$ has a presentation with $e_1 - e_0 + 1$ generators and e_2 relations. We have $b_i(G) = b_i(X)$ for $i = 1, 2$ and $b_0(X) = 1$ since

X is connected. In addition $\delta(G) \leq b_1(G) - b_2(G)$, see for example [13], §4. Thus

$$e_1 - e_0 - e_2 \leq \delta(G) - 1 \leq b_1(X) - 1 - b_2(X) = -\chi(X) = e_1 - e_0 - e_2$$

and we deduce that $\delta(G) - 1 = -\chi(X)$. Similarly we find that $\delta(H) - 1 = -\chi(K/H)$ and the lemma follows from $\chi(K/H) = [G : H]\chi(X)$. □

Proof of Lemma 4. If G has two dimensional classifying space then by Lemma 2 we have $\delta(G) - 1 = -\chi(G) = -\beta_0(G) + \beta_1(G) - \beta_2(G)$ and the result is clear since $\beta_0(G) = 0$.

We now prove the converse. Assume that G does not have 2-dimensional classifying space and let us choose a presentation $P = (S, R)$ for G with d generators and r relations, such that $d - r = \delta(G)$. Let X_0 denote the 2-complex defined by the Cayley graph of G where we add discs for every relation in P at every vertex of X_0. By assumpition X_0 is not aspherical and therefore $H_2(X_0) \neq \{0\}$. Succesively adding cells in dimension 3 and higher we reach a complex X which is homotopy equivalent to the classifying space $K(G, 1)$ and whose 2-skeleton is X_0. Our group G acts freely on X with $1, d, r$ orbits in dimensions $0, 1, 2$ respectively. Therefore we obtain a free resolution of $\mathbb{R}G$ modules

$$\mathbb{R} \xleftarrow{\gamma_0} \mathbb{R}G \xleftarrow{\gamma_1} (\mathbb{R}G)^d \xleftarrow{\gamma_2} (\mathbb{R}G)^r \xleftarrow{\gamma_3} (\mathbb{R}G)^k \cdots$$

such that the image of γ_3 is non-zero. This resolution induces a complex of l^2G modules

$$0 \xleftarrow{\tilde{\gamma}_0} l^2G \xleftarrow{\tilde{\gamma}_1} (l^2G)^d \xleftarrow{\tilde{\gamma}_2} (l^2G)^r \xleftarrow{\tilde{\gamma}_3} (l^2G)^k \longleftarrow \ldots$$

where each map $\tilde{\gamma}_i$ is induced by the map γ_i. Denoting by $K_i = \ker \tilde{\gamma}_i$ and $J_i = \overline{\mathrm{Im}\tilde{\gamma}_i}$ we have $\beta_i(G) = \dim_{\mathcal{N}} K_i/J_{i+1}$ where for a l^2G module M we denote by $\dim_{\mathcal{N}} M$ its von Neumann dimension. Since G is infinite we have $\beta_0(G) = 0$. The rank-nullity theorem for von Neumann dimension gives that

$$1 - d + r - \dim_{\mathcal{N}} J_3 = \beta_0(G) - \beta_1(G) + \beta_2(G)$$

and since γ_3 is not zero we have $J_3 \neq \{0\}$ and therefore $\dim_{\mathcal{N}} J_3 > 0$. The inequality follows. □

Proof of Theorem 6. We use the following result from [23] (see also [4] for an alternative proof):

Theorem 16. *Let S be a finite generating set for an amenable group G and let (H_i) be an infinite normal chain in G with trivial intersection. For any $\epsilon > 0$ there is some $i \in \mathbb{N}$ and a transversal T_i for H_i in G such that*

$$|\{(t, s) \in T_i \times S \mid ts \notin T_i\}| < \epsilon |T_i|.$$

Now consider a presentation $\langle X, R\rangle$ for an amenable group G and let S be the image of the generating set X in G. Let m be the sum of the lengths of all relations in R as words in X. For any $\epsilon > 0$ let T_i be a transversal for H_i in G as in the conclusion of the above theorem. For an element $a \in G$ write

$\bar{a}$ for the unique element of T_i such that $a \in \bar{a}H_i$. Then H_i has the so-called Schreier presentation $\langle Y_i, W_i \rangle$ where $Y_i = \{e(t,s) \mid t \in T_i, s \in S\}$ and W_i is a collection of $[G : H_i]|R|$ words in the generating set Y_i with the property that any $e(t,s) \in Y_i$ occurs in at most m of the relators W_i. Moreover the image of $e(t,s) \in Y_i$ in H_i is the element $ts(\overline{ts})^{-1}$. Let Y_i^1 be the set of elements from Y_i whose image in H_i is non-trivial and let W_i^1 be the image of the relations from W_i under the homomorphism $F\langle Y_i \rangle \to F\langle Y_i^1 \rangle$ which sends the generators from $Y_i \backslash Y_i^1$ to identity.

We claim that $H_i = \langle Y_i, W_i \rangle$ has presentation $\langle Y_i^1, W_i^1 \rangle$. Let $H_i^1 = \langle Y_i^1, W_i^1 \rangle$ and consider the surjective homomorphism $f : H_i \to H_i^1$ obtained by setting all $Y_i \backslash Y_i^1$ to 1. At the same time since H_i is generated by the images of Y_i^1 and satisfies the relations W_i^1 there is a surjective homomorphism h from H_i^1 to H_i sending Y_i^1 to $Y_i^1 \subseteq Y_i$. Since $f \circ h : H_i^1 \to H_i^1$ is the identity we see that h is injective and hence an isomorphism, proving the claim.

Now observe that $|Y_i^1| < \epsilon[G : H_i]$ by the choice of T_i. At the same time at most $|Y_i^1|m$ of the relations from W_i^1 are non-trivial as words in $F\langle Y_i^1 \rangle$ because each element of Y_i^1 can appear in at most m relations from W_i^1. Hence H_i has presentation with at most $\epsilon[G : H_i]$ generators and at most $m\epsilon[G : H_i]$ relations. Since m depends only on the initial presentation of G while $\epsilon > 0$ could be arbitrary small we deduce that the chain (H_i) is economical. □

Proof of Proposition 7. Let $G, N, (H_i)$ be as in the statement of the Proposition. As N is finitely generated and G is finitely presented, the quotient $Q = G/N$ is also finitely presented. We will show that each H_i has a finite presentation with d_i generators and r_i relators such that $\lim_{i\to\infty} \frac{d_i}{[G:H_i]} = \frac{r_i}{[G:H_i]} = 0$.

For each i, set $N_i = N \cap H_i$. Then the group H_i is an extension of N_i by $Q_i \cong \frac{H_i N}{N}$. Let $x_i = [N : N_i]$ and $y_i = [G : NH_i]$, note that $\lim_{i\to\infty} x_i = \infty$ while $[G : H_i] = x_i y_i$. The chain (N_i) is economical in N and each N_i has a presentation $\langle X_i; R_i \rangle$ with a_i generators and ρ_i relators such that $\lim_{i\to\infty} \frac{a_i}{x_i} = \lim_{i\to\infty} \frac{\rho_i}{x_i} = 0$. Let Q be presented by k generators and l relations. Each Q_i has finite index y_i in Q and so we choose a finite Schreier presentation $\langle Y_i; S_i \rangle$ for each Q_i with $b_i = |Y_i| = y_i(k-1)+1$ generators and $\sigma_i = |S_i| = y_i l$ relators. Observe that $b_i \leq k y_i$ and $\sigma_i = l y_i$ for all i.

The group H_i has a presentation with generating set $X_i \cup Y_i$. The relators are chosen to be $R_i \cup S_i \cup P_i$ where $P_i = \{srs^{-1} = n_{r,s} \mid r \in X_i, s \in Y_i\}$. Therefore, we can take $d_i = a_i + b_i$ and $r_i = \rho_i + \sigma_i + a_i b_i$. It follows that

$$\frac{d_i}{[G:H_i]} = \frac{a_i + b_i}{x_i y_i} \leq \frac{a_i}{x_i} + \frac{k}{x_i}; \quad \frac{r_i}{[G:H_i]} = \frac{\rho_i + \sigma_i + a_i b_i}{x_i y_i} \leq \frac{\rho_i}{x_i} + \frac{l}{x_i} + \frac{k a_i}{x_i}.$$

Using that ρ_i/x_i and a_i/x_i both tend to zero with i, we conclude that $\lim_{i\to\infty} \frac{d_i}{[G:H_i]} = \lim_{i\to\infty} \frac{r_i}{[G:H_i]} = 0$ and the chain (H_i) is economical, as required. □

3. Proof of Theorem 8

Proof. Since Γ is generated by economical groups with infinite intersections we have that $RG(\Gamma,(H_i)) = 0$ by [15]. Therefore $\Delta(\Gamma,(H_i)) \leq 0$. It remains to show that $\Delta(\Gamma,(H_i)) \geq 0$ for which we need to exhibit a suitable presentation for each subgroup $H_i < \Gamma$. By Theorem III.C.3.13 and Corollary III.C.3.15 of [7] there is a universal development scwol $D = (V_D, E_D)$ on which Γ act so that $P = \Gamma\backslash D$ and C is the complex of groups associated to this action (as described in Definition III.C.2.9 (1) of [7]).

Let $Y_j = (V_j, E_j)$ be the scwol $H_j\backslash D$ and let $\mathcal{Y}_j$ be the complex of groups associated to the action of H_j on D which we describe below. By Corollary III.C.3.15 of [7] $H_j \simeq \tilde{G}(\mathcal{Y}_j)$.

Let $p_j : D \to Y_j$ be the natural projection morphism. Following Definition III.C.2.9 (1) of [7] we now describe the complex $\mathcal{Y}_j = (Y_j, H_{j,\sigma}, \psi_{j,a}, g_{j,a,b})$.

For each $\sigma \in V_j$ choose a vertex $\bar{\sigma} \in V_D$ such that $p_j(\bar{\sigma}) = \sigma$. For every edge $a \in E_j$ of Y_j there is a unique edge $\bar{a} \in E_D$ such that $p_j(\bar{a}) = a$ and $i(\bar{a}) = \overline{i(a)}$. Choose $h_{j,a} \in \Gamma$ such that $h_{j,a}t(\bar{a}) = \overline{t(a)}$. Let $H_{j,\sigma} = \mathrm{Stab}_{H_i}(\bar{\sigma})$ and define $\psi_{j,a} : H_{j,i(a)} \to H_{j,t(a)}$ be the injective homomorphism

$$\psi_{j,a}(g) = h_{j,a} g h_{j,a}^{-1} \quad \forall g \in H_{j,i(a)}.$$

For pairs a, b of composable edges of E_i define

$$g_{j,a,b} = h_{j,a} h_{j,b} h_{j,c}^{-1}$$

where $c = ab$ is the product ot a and b in E_j.
The complex of groups $\mathcal{Y}_j$ is the quadruple $(Y_j, (H_{j,\sigma}), (\psi_{j,a}), (g_{j,a,b}))$.

Choose a spanning tree T_j on the edge set E_j of Y_j. Then H_j is isomorphic to the fundamental group $\tilde{G}(\mathcal{Y}_j)$ of $\mathcal{Y}_j$ defined as follows:

Generators of H_j are $\{H_{j,\sigma} \mid \sigma \in Y_j\} \cup E_j$. The relations of H_j are of the following four types:

R1. All relations in the groups $H_{j,\sigma}$,
R2. $ab = g_{j,a,b}c$ for all composable pairs of edges $a, b \in E_j$ with $c = ab$.
R3. $\psi_{j,a}(x) = axa^{-1}$ for all $x \in H_{j,i(a)}$ and all $a \in E_j$.
R4. $a = 1$ for all $a \in T_j$.

We are now ready to prove that $\lim_{j\to\infty} \frac{\delta(H_j)-1}{[\Gamma:H_j]} \geq 0$. Consider the presentation of H_j above. Let $n_j = [\Gamma : H_j]$. Denote by π the morphism of scwols $Y_j = H_j\backslash D \to \Gamma\backslash D = P$. Recall that V and E denote the vertex and edge sets of P. For any vertex $\sigma \in V$ of P choose a vertex $\sigma_0 \in V_j$ of Y_j such that $\sigma_0 \in \pi^{-1}(\sigma)$ and moreover choose $\bar{\sigma} \in V_D$ to be equal to $\bar{\sigma}_0$. Now $|\pi^{-1}(\sigma)| = [\Gamma : H_j G_\sigma] = n_j / l_{j,\sigma}$ where $l_{j,\sigma} = [G_\sigma : H_{j,\sigma_0}]$. Moreover since H_j is normal in Γ we have that the isomorphism class of $H_{j,\sigma_0} = \mathrm{Stab}_{H_j}(\overline{\sigma_0}) = H_j \cap G_\sigma$ does not depend on the choice of $\sigma_0 \in \pi^{-1}(\sigma)$.

Since each $H_j \cap G_\sigma = H_{j,\sigma_0}$ is an economical chain in G_σ we can choose a presentation $\langle X_{j,\sigma} \mid R_{j,\sigma}\rangle$ for the group $H_{j,\sigma_0} \leq G_\sigma$ such that

$$\frac{|X_{j,\sigma}|}{l_{j,\sigma}} \to 0, \quad \frac{|R_{j,\sigma}|}{l_{j,\sigma}} \to 0 \text{ and } l_{j,\sigma} \to \infty \text{ as } j \to \infty.$$

Now H_j is generated by

$$d_j := \sum_{\sigma \in V} |X_{j,\sigma}| \frac{n_j}{l_{j,\sigma}} + |E_j|$$

elements. The number $|E_j|$ of edges $a \in E_j$ of Y_i is at most $|V_j||E|$: There are at most $|V_j|$ choices for the initial vertex $x = i(a)$ of a in Y_j and then at most $|E|$ choices for a because for any $x \in V_j$ the morphism π induces a bijection between $\{a \in E_j \mid i(a) = x\}$ and $\{a \in E \mid i(a) = \pi(x)\}$. Now

$$|V_j| = \sum_{\sigma \in V} |\pi^{-1}(\sigma)| = \sum_{\sigma \in V} \frac{n_j}{l_{j,\sigma}}$$

which together with $l_{j,\sigma} \to \infty$ gives $|Y_j|/n_j \to 0$ with $i \to \infty$. Therefore $d_j/n_j \to \infty$. We now have to count the relations of H_j.

The number of relations in R1 is $r_{j,1} := \sum_{\sigma \in V} |R_{j,\sigma}| n_j / l_{j,\sigma}$ and so $r_{j,1}/n_j \to 0$ as $j \to \infty$.

The number $r_{j,2}$ of relations in R2 is the number of composable pairs $a, b \in E_j$. There are $|Y_j|$ choices for $i(b)$, then at most $|E|$ choices for b which determines $t(b) = i(a)$ and then at most $|E|$ choices for a. Therefore $r_{j,2} \leq |Y_j||E|^2$ and again $r_{j,2}/n_j \to 0$ with j.

The number $r_{j,3}$ of relations in R3 is at most

$$\sum_{a \in E_j} |X_{j,i(a)}| \leq \sum_{\tau \in V} |E| \frac{n_j}{l_{j,\tau}} |X_{j,\tau}|$$

which shows that again $r_{j,3}/n_j \to 0$ with $j \to \infty$.

Finally $|T_j| \leq |E_j| \leq |P||Y_j|$ which shows that the number of relations R4 also grows sub-linearly in n_j.

This completes the proof that (H_j) is economical and Theorem 8 follows. □

4. Special Linear Groups

Proof of Theorem 9. Let Σ be the root system A_{n-1} with Weyl group $\mathrm{Sym}(n)$. The root system Σ has the realization

$$\Sigma = \{e_i - e_j \mid 1 \leq i, j \leq n, \ i \neq j\}$$

where $e_1, \ldots, e_n$ is the standard orthonormal basis ot $\mathbb{R}^n$. Let $\mathcal{S}$ be the set of proper non-zero subspaces of $\mathbb{R}^n$ whose basis is a subset of $\{e_1, \ldots, e_n\}$. The Coxeter complex $\mathfrak{C}$ of $\mathrm{Sym}(n)$ is the simplicial complex with vertices $\mathcal{S}$ and simplices the collection F all flags from $\mathcal{S}$:

$$F = \{\sigma = \{V_1, V_2, \ldots, V_k\} \mid V_1 \subset V_2 \subset \cdots \subset V_k, \ V_j \in \mathcal{S}, \ 1 \leq j \leq k\}.$$

The set of maximal flags of F is denoted by $F_{\max}$. These correspond to permutations of $\{e_1, \dots, e_n\}$ and define the simplices of maximal dimension $n-2$ of $\mathfrak{C}$. The Coxeter complex $\mathfrak{C}$ of $\mathrm{Sym}(n)$ is a triangulation of the sphere S^{n-2} and therefore $\mathfrak{C}$ is simply connected when $n \geq 4$.

Let $(F, <)$ be the poset of the flags in F ordered by reverse inclusion, i.e. for flags $\sigma, \tau \in F$ we declare $\sigma < \tau$ whenever $\tau \subset \sigma$. The geometric realization $|F|$ of F is the derived complex (barycentric subdivision) of $\mathfrak{C}$.

For a flag $\sigma = \{V_1, V_2, \dots, V_k\} \in F$ let $r(\sigma)$ be the set of roots

$$s(\sigma) = \{e_i - e_j \mid e_i \in V_l, e_j \notin V_l \text{ for some } l \in \{1, \dots, k\}\}$$

Clearly if $\tau \subset \sigma$ then $s(\tau) \subset s(\sigma)$. If $\sigma \in F_{\max}$ is a maximal flag then $s(\sigma)$ is a set of positive roots $\{e_{\pi(a)} - e_{\pi(b)} \mid 1 \leq a < b \leq n\}$ with respect to some permutation $e_{\pi(1)}, \dots, e_{\pi(n)}$ of $e_1, \dots, e_n$. More generally the roots $s(\sigma)$ correspond to those elementary matrices in SL_n which are strictly upper triangular with respect to the flag σ.

Let Y be a finite generating set of the ring R over $\mathbb{Z}$ such that $1 \in Y$.

For a flag $\sigma \in F$ let G_σ be the subgroup of $St_n(R)$ generated by $\{x_r(a) \mid r \in s(\sigma), a \in R\}$. It is well known that when $\sigma \in F_{\max}$ then G_σ is isomorphic to the group $U_n(R)$ of upper uni-triangular $n \times n$ matrices with entries in R. The difficulty in applying Theorem 8 directly is that G_σ is not finitely generated. We will resolve this by exhibiting G_σ (and consequently $St_n(R)$) as a direct limit of finitely presented groups which eventually stabilizes because $St_n(R)$ is finitely presented.

For an integer $m \in \mathbb{N}$ let $Y_m = Y \cdot Y \cdots Y \subset R$ be the set of all products of some m elements of Y and note that $Y_m \subseteq Y_{m+1}$ since $1 \in Y$.

For $\sigma \in F$ define $G_{\sigma,m}$ to be the subgroup of $St_n(R)$ generated by $\{x_r(a) \mid r \in s(\sigma),\ a \in Y_m\}$ and note that $G_{\sigma,m}$ is a finitely generated nilpotent group. If $\tau \subset \sigma$ are two flags in F then $s(\tau) \subset s(\sigma)$ and therefore $G_{\tau,m}$ is contained in $G_{\sigma,m}$.

Choose and fix a natural number m for the moment. Now define a simple complex of groups

$$\mathcal{C}_m = (\mathcal{F}, (G_{\sigma,m})_{\sigma \in F}, (f_{a,m})_{a \in E})$$

over the scwol $\mathcal{F} = (F, E)$ of the poset $(F, <)$. Recall that the edge set E of $\mathcal{F}$ is $E = \{(\sigma, \tau) \mid \sigma < \tau, \sigma, \tau \in F\}$.

For an edge $a = (\sigma, \tau) \in E$ of $\mathcal{F}$ let $f_{a,m} : G_{\tau,m} \to G_{\sigma,m}$ be the inclusion map.

Now define $\Gamma_m = \tilde{G}(\mathcal{C}_m)$ to be the the fundamental group of the simple complex of groups C_m. The geometric realization $|\mathcal{F}|$ is the derived complex of the Coxeter complex $\mathfrak{C}$ which is simply connected because $n \geq 4$. It follows that Γ_m is in fact the direct limit

$$\varinjlim ((G_{\sigma,m})_{\sigma \in F}, (f_{a,m})_{a \in E}) .$$

There is a homomorphism $i_m : \Gamma_m \to St_n(R)$ which restricts to an injection on each $G_{\sigma,m}$ into $St_n(R)$ by the definition of $G_{\sigma,m}$ as a subgroup of $St_n(R)$.

Therefore the inclusion homomorphism $i_{\sigma,m} : G_{\sigma,m} \to \Gamma_m$ is injective and in particular the complex $\mathcal{C}_m$ is developable.

The following proposition allows us to identify each generator $x_r(a)$ of $St_n(R)$ with its image in Γ_n under $i_{\sigma,m}$ irrespective of the choice of flag σ.

Proposition 17. *Let $g = x_r(a) \in St_n(R)$, $r \in \Sigma, a \in Y_m$ and σ, σ' be two flags in F such that $r \in s(\sigma) \cap s(\sigma')$. Then $i_{\sigma,m}(g) = i_{\sigma',m}(g)$.*

Proof. The collection of flags $\tau \in F$ such that $r \in s(\tau)$ is a convex subcomplex of $\mathfrak{C}$, in fact this is a hemisphere with boundary the great circle perpendicular to r. Therefore we may find a sequence $\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_l = \sigma'$ of simplices of $\mathfrak{C}$ (i.e. flags in F) such that $r \in \cap_{i=1}^{l} s(\sigma_i)$ and $\sigma_i \cap \sigma_{i+1} \neq \emptyset$ for all $i = 1, \ldots, k-1$. Therefore it is sufficient to prove the proposition in the case when the flag $\mu = \sigma \cap \sigma'$ is non empty. Let $a = (\sigma, \mu), a' = (\sigma', \mu)$ be the two edges of $\mathcal{F}$ and recall the inclusion maps $f_{a,m} : G_{\mu,m} \to G_{\sigma,m}$ and $f_{a',m} : G_{\mu,m} \to G_{\sigma',m}$ We have $f_{a,m}(g) = g = f_{a',m}(g)$ from the relations R3 in the definition of $\tilde{G}(\mathcal{C}_m)$ and the fact that $\mathcal{C}_m$ is simply connected. Therefore

$$i_{\sigma,m}(g) = i_{\sigma,m} \circ f_{a,m}(g) = i_{\sigma',m} \circ f_{a',m}(g) = i_{\sigma',m}(g)$$

as required. □

From now on we shall identify the groups $G_{\sigma,m}$ with their images in Γ_m under $i_{\sigma,m}$.

Note that the inclusions $Y_m \subset Y_{m+1}$ give rise to inclusions $G_{\sigma,m} \leq G_{\sigma,m+1}$ which, by the universal property of direct limits give rise to a homomorphism $F_m : \Gamma_m \to \Gamma_{m+1}$ such that F_m restricted to $G_{\sigma,m}$ is the inclusion map of $G_{\sigma,m}$ into $G_{\sigma,m+1} \leq \Gamma_{m+1}$.

For every integer $m \geq 1$ let

$$A_m = \langle G_{\sigma,1} \mid \sigma \in F_{\max} \rangle \leq \Gamma_m.$$

and note that $A_{m+1} \subseteq \mathrm{Im}(F_m)$.

Claim 18. $A_m = \Gamma_m$.

Proof of claim: Let $\sigma \in F_{\max}$. It is sufficient to show that for any integer $k \leq m$ the element $x_r(a) \in \Gamma_m$ with $a \in Y_k$ and $r \in \Sigma$ belongs to A_m. We prove this statement by induction on k, the case $k = 1$ being clear from the definition of A_m.

Let us write $a = a_1 a_2$ with $a_1 \in Y_{k-1}$ and $a_2 \in Y$. Using that the Weyl group $\mathrm{Sym}(n)$ acts transitively on the set Σ of all roots we can find a maximal flag $\sigma' \in F_{\max}$ and roots $r_1, r_2 \in s(\sigma')$ such that $r = r_1 + r_2$. Since $r \in s(\sigma')$ Proposition 17 gives $x_r(a) \in G_{\sigma',m}$. Now $x_{r_1}(a_1)$ and $x_{r_2}(a_2)$ belong to A_m by the induction hypothesis and the identity $x_r(a) = [x_{r_1}(a_1), x_{r_2}(a_2)]$ in $G_{\sigma',m}$ gives that $x_r(a) \in A_m$. This completes the induction step and the claim follows. □

Since $\Gamma_{m+1} = A_{m+1} \subseteq F_m(\Gamma_m)$ we deduce that the maps $F_m : \Gamma_m \to \Gamma_{m+1}$ are surjective for each m. The Steinberg group $St_n(R)$ is a direct limit of of the directed system $\Gamma_1 \to \Gamma_2 \to \cdots$ because for every pair of roots $r_1, r_2 \neq -r_1$ and every $a, b \in R$ there is some $k \in \mathbb{N}$ and a fundamental set of roots $s(\sigma)$ for some $\sigma \in F_{\max}$ such that $r_1, r_2 \in s(\sigma)$ and $x_r(a), x_r(b) \in G_{\sigma,k}$. Now we use the following two results to deduce that $St_n(R) = \Gamma_m$ for some integer m.

Theorem 19 ([16], Theorem 3)**.** *Suppose that R is a unital finitely presented ring and $n \geq 4$. Then the Steinberg group $St_n(R)$ is finitely presented.*

Proposition 20. *Let $B_1 \to B_2 \to \cdots$ be a directed system of surjective group homomorphisms $B_i \to B_{i+1}$ of groups B_i such that B_1 is finitely generated while the direct limit $D = \varinjlim B_i$ is finitely presented. Then the natural homomorphism $B_m \to D$ is an isomorphism for some integer $m \in \mathbb{N}$.*

Proof. Let N_i be the kernel of the composition map $B_1 \to B_i$. We have $\{1\} = N_1 \leq N_2 \leq \cdots$ and $\cup_{i=1}^{\infty} N_i = H$ where $B_1/H = D$. Since D is finitely presented, H is generated by finitely many elements as a normal subgroup of B_1 and therefore $N_m = H$ for some m. In particular the homomorphism from $B_m = B_1/N_m$ to B_1/H is an isomorphism. □

Hence the natural homomorphism $i_m : \Gamma_m \to St_n(R)$ is an isomorphism for some integer m.

For any flag $\sigma \in F$ we have $G_{\sigma,m}$ contains the nontrivial elements $x_r(a)$ for any $a \in Y$ and any root $r \in s(\sigma)$. In particular the centre of $G_{\sigma,m}$ contains an infinite cyclic subgroup $Z_{\sigma,m} < X_r$ for the highest root of $s(\sigma)$. Since by assumption $X_r \cap (\cap_i H_i) = \{1\}$ we deduce $[Z_{\sigma,m} : H_i \cap Z_{\sigma,m}] \to \infty$. By Proposition 7 with $G = G_{\sigma,m}$ and $N = Z_{\sigma,m}$ the chain $H_i \cap G_{\sigma,m}$ is economical in $G_{\sigma,m}$. Theorem 9 now follows from Theorem 8.

□

Proof of Corollary 10. Let $R = \mathbb{Z}[x_1, \dots, x_k]$. Consider the homomorphism $\pi : St_n(R) \to SL_n(R)$ which sends the generator $x_r(a) \in St_n(R)$ with root $r = e_i - e_j$ to the elementary matrix $E_{i,j}(a)$. By Suslin's theorem [22] $SL_n(R)$ is generated by elementary matrices and therefore π is surjective. The group $\ker \pi$ is denoted by $K_2(n, R)$ and since R has Krull dimension $k + 1$ [14] gives that $K_2(n, R) \simeq K_2(R)$ and does not depend on n when $n \geq k + 4$.

In turn [21], p.122, Theorem 8 gives that $K_2(R) = K_2(\mathbb{Z}) = \{\pm 1\}$ and therefore $SL_n(R)$ is a factor group of $St_n(R)$ by a central subgroup $\langle a \rangle$ of order 2. Let H_i be a chain with trivial intersection in $SL_n(R)$ and consider the chain $(\pi^{-1}(H_i))$ in $St_n(R)$. Theorem 9 gives that $(\pi^{-1}(H_i))$ is economical. Adding the single extra relation $a = 1$ to the presentation of each $\pi^{-1}(H_i)$ now shows that (H_i) is economical. □

Proof of Proposition 11. The group $SL_3(R)$ has 6 root subgroups $E_1, \dots E_6$ isomorphic to $(R, +)$ which are in correspondence with the roots $r_1, \dots r_6$

labelled clockwise on the realization of the root system $\Sigma = A_2$ as a regular hexagon. Let D be the subgroup of $SL_3(R)$ of diagonal matrices. We have that $D \simeq R^* \times R^*$ is finitely generated by Dirichlet's unit theorem. For $i = 1, \ldots, 6$ let $T_i = D \ltimes \langle E_i E_{i+1} E_{i+2} \rangle$. Each T_i is isomorphic to the subgroup $T \leq SL_3(R)$ of upper triangular matrices. Let $Tr_3(R)$ be the subgroup of upper triangular matrices in $GL_3(R)$ and let Z be its subgroup of scalar matrices. Now $Z \simeq R^*$ is finitely generated while $Tr_3(R)$ is finitely presented by [18, Proposition 11.2.8]. Since $T \cap Z = 1$ we see that T embeds as a subgroup of finite index in $Tr_3(R)/Z$ and hence is finitely presented. Similarly we can see that each $T_i \cap T_{i+1}$ is finitely presented.

Let $\Gamma := St_3(R)$. Recall the definition of the subgroups $X_r < \Gamma$ $(r \in \Sigma)$. Let $\pi : \Gamma \to SL_3(R)$ be the surjective homomorphism which sends X_{r_i} to E_i. The kernel $K = \ker \pi = K_2(3, R) \simeq K_2(R)$ is a finite central subgroup of Γ by [11]. Therefore the groups $\tilde{T}_i := \pi^{-1}(T_i)$ and $\tilde{T}_i \cap \tilde{T}_{i+1} = \pi^{-1}(T_i \cap T_{i+1})$ are finitely presented for $i = 1, \ldots, 6$.

Let L be the simply connected hexagonal 2-complex with six vertices r_i, six edges (r_i, r_{i+1}) for $i = 1, \ldots, 6$ and one disc B. Define $G_{r_i} = \tilde{T}_i, G_{(r_i, r_{i+1})} = \tilde{T}_i \cap \tilde{T}_{i+1}$ and $G_B = \pi^{-1}(D)$. Let C be the resulting complex of groups. One checks that the canonical inclusion map $\tilde{T}_i \to \Gamma$ induces an isomorphism between $\tilde{G}(C)$ and Γ.

Let (H_i) be a chain in $SL_3(R)$ with trivial intersection. Define $\tilde{H}_i = \pi^{-1}(H_i)$. Since K is finite it is sufficient to prove that $(\tilde{H}_i)$ is economical in Γ and by Theorem 8 we just need to verify that $(G_\sigma \cap \tilde{H}_i)$ is economical in G_σ for each cell σ of L. Again using that $K = \ker \pi$ is finite it is sufficient to show that the chain $(\pi(G_\sigma \cap \tilde{H}_i)) = (H_i \cap \pi(G_\sigma))$ is economical inside the finitely presented infinite soluble group $\pi(G_\sigma) < SL_3(R)$. Since $\cap_i H_i = \{1\}$ we are in position to apply Theorem 6. □

5. Artin Groups

Let $K := (V_K, E_K)$ be a graph such that every edge $e = (v, w)$ carries a weight $m_e \in \{2, 3, \ldots\}$. The Coxeter matrix associated to K is the symmetric matrix $M_K = (m_{(v,w)})$ of size $|V|$. We define the Artin group A_K and the Coxeter group W_K associated to K as follows:

$$A_K = \langle\ V \mid \underbrace{vwvw\ldots}_{m_{v,w}} = \underbrace{wvwv\ldots}_{m_{v,w}} \quad \forall\ (v, w) \in E_K \rangle$$

$$W_K = \langle\ V \mid v^2 = 1\ \forall\ v \in V; (vw)^{m_{v,w}} = 1\ \forall\ (v, w) \in E_K \rangle$$

It is clear that A_K maps onto W_K. If W_K is finite, then A_K is called an Artin group of finite type. Artin groups of finite type were shown to have infinite cyclic centre by Bierskorn and Saito [6]. Cohen and Wales proved that they are linear [9]. Our Proposition 7 thus implies that an Artin group of finite type, by virtue of being a residually finite group with an economical normal subgroup, is economical.

By a standard parabolic subgroup of A_K or W_K, we mean any subgroup generated by a (possibly empty) subset of the standard generating set V_K. For $s \subseteq V_K$, let $K(s)$ denote the full subgraph of K spanned by s. The inclusion of s into V induces a natural homomorphism $h_{s,V} : A_{K(s)} \to A_K$. A classical theorem by van der Lek [17] says that for any s, the map $h_{s,V}$ is an isomorphism onto its image.

The simplicial complex L, called the nerve of the Coxeter matrix M_K, is defined as follows: its vertex set is V and a subset $s \subset V$ spans a simplex iff $W_{K(s)}$ is finite. Davis and Leary [10] computed the l^2-cohomology of L and established that the i-th ℓ^2-Betti number of A_K is equal to the $(i-1)$-th ordinary reduced Betti number of L, provided the $K(\pi, 1)$ conjecture for Artin groups holds. In particular, if L is connected and simply connected then both the first and second ℓ^2-Betti numbers of A_K should be zero.

Consider the partial order on the set $\mathcal{V} := \{\emptyset \neq s \subseteq V \ : \ |W_{K(s)}| < \infty\}$ given by reverse inclusion that is $s_1 < s_2$ whenever $s_2 \subset s_1$. Let P be the scwol associated to this partial order. For $s \in \mathcal{V}$ let $G_s = A_{K(s)}$. For an edge $a = (s_1, s_2)$ of P with $s_1 < s_2$ define the monomorphism $f_a : G_{s_2} \to G_{s_1}$ to be the inclusion map h_{s_2,s_1}.

Let $\mathcal{C}$ be the simple complex of groups $(P, (G_s), f_a)$. The geometric realization $|P|$ of P is the derived complex of the nerve L, equal to the barycentric subdivision of L. Therefore $|P|$ is simply connected and $\tilde{G}(\mathcal{C})$ is the direct limit of the system $((G_s)_{s\in\mathcal{V}}, (f_a))$ which is just A_K. We are now in a position to prove Theorem 12.

Proof of Theorem 12. Suppose that L is connected and simply connected so that A_K is the fundamental group of the complex of groups over $\mathcal{C}$ as described above. Let (H_i) be a chain of normal finite index subgroups of A_K so that $\cap H_i$ is the radical $\mathfrak{R}(A_K)$. We will show that the chain (H_i) is economical.

To apply Theorem 8, we need to check that for each simplex $\sigma \subset L$, the normal chain $(G_\sigma \cap H_i)$ is economical. We argue that $[Z_\sigma : H_i \cap Z_\sigma]$ is unbounded, where Z_σ denotes the centre of G_σ. To do this we consider the homomorphism $p : A_K \to \mathbb{Z}$ that sends each $v \in V$ to the generator of $\mathbb{Z}$. Clearly, $\ker p \geq \mathfrak{R}(A_K)$. Each G_σ is of finite type and therefore by [6], Z_σ is an infinite cyclic group generated by a suitable power of the Garside element $\prod_{v\in V(\sigma)} v$ (where $V(\sigma)$ is the set of vertices of the simplex $\sigma \subset L$). This implies that $p(Z_\sigma)$ is non-trivial and so, $Z_\sigma \cap \mathfrak{R}(A_K) = 1$. We conclude that $\lim_{i\to\infty}[Z_\sigma : Z_\sigma \cap H_i] = \infty$ and by Proposition 7, the chain $(G_\sigma \cap H_i)$ is economical, for each $\sigma \subset L$. We invoke Theorem 8 to conclude that the chain (H_i) is economical in A_K. □

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Mathematical Institute, University of Oxford, Andrew Wiles Building, Oxford OX2 6GG, England.

E-mail address: Aditi.Kar@maths.ox.ac.uk (Corresponding author)

E-mail address: Nikolay.Nikolov@maths.ox.ac.uk