# Elastic Wave Scattering From a Strained Region




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# Abstract

This dissertation considers elastic wave scattering from a micro-sphere embedded in a rubber substrate which has been initially strained. Its aim is to ascertain the extent to which the strained region affects the scattering process.

It is proposed that under hydrostatic loading a micro-sphere will compress nonlinearly. This is justified by calculating the compressed radius of a micro-sphere for different forms of the stored energy function corresponding to linear *and* nonlinear elasticity. It is shown that linear elasticity, as used in current TMSL models, predicts that the micro-sphere compresses to a *smaller* radius than that predicted by standard nonlinear elastic models of rubber-like materials.

The initial strain modifies the Lamé moduli and therefore further experimental work is necessary in order to calculate the full equations of motion for small displacements superposed on top of the initial finite strain. Without this further experimental knowledge the equations are correct only to leading order.

The low frequency scattering problem is solved at leading order so that we can ascertain how monopole scattering is affected by the strained region. It is shown that the monopole scattering cross section for scattering from a spherical cavity in a strained region is three orders of magnitude smaller than that for an isotropic region. Hence, the scattering process is significantly affected by the strained region.

Two modifications to the current TMSL model are proposed. Firstly, the prediction of the compressed radius should be made according to nonlinear elasticity. Secondly, scattering effects due to the strained region should be included in the model since they contribute significantly to the scattering process.

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# Chapter 1

# Introduction

This dissertation arises from an in-house theoretical study by Thomson Marconi Sonar Limited (TMSL) into the material properties of composite materials which have been reported on by Dr. Peter Brazier-Smith in [2],[3] and [4].

Composite materials are used in a wide variety of contexts in industry. One particular use is for tile cladding on the exterior of submarines. These tiles have an acoustic function which depends upon their location on the submarine. For example, tiles placed on sonar arrays must allow incoming sound waves to pass through them with as little distortion as possible. In order to achieve this the acoustic impedance of the composite material and sea water must be matched, i.e.

$$(\rho c)_{\text{comp}} \quad = \quad (\rho c)_{\text{sea water}} \tag{1.1}$$

where $\rho$ and $c$ are the density and sound speed respectively in the appropriate medium.

In this example of tile cladding, the composite material is formed from a rubber substrate and therefore sound will travel faster through the tiles than it will through sea water. Hence in order for (1.1) to hold, the density of the composite material must be reduced. This can be achieved by introducing micro-spheres into the rubber substrate. Micro-spheres are gas filled thin polythene shells with an approximate diameter of 35 microns.

This is just one example where it is extremely important to understand the material properties of a composite material.

Due to the introduction of the micro-spheres, an incident small amplitude sound wave will be scattered from the inclusions and because the diameter of the micro-spheres is

much smaller than the wavelength of the incident sound wave, we have low frequency scattering. It is this phenomenon which this dissertation addresses and this analysis allows modified material properties of the composite material to be deduced.

The theory behind the modelling of composite materials including micro-spheres is well developed. In particular, when the material is at atmospheric pressure, existing TMSL models are in good agreement with experimental results. However, due to its use as tile cladding on submarines, the composite material is subjected to a wide range of hydrostatic pressures. Existing models have found it difficult to match experimental results when the material is subjected to such a range of static pressures. In particular [4] shows that for three alternative models over a wide range of static pressures, there is a large discrepancy between the experimental bulk modulus of the material and that predicted by the models.

TMSL have proposed several explanations as to why these discrepancies occur. One explanation was that under static pressure a proportion of the shells buckle, affecting the scattered field differently from unbuckled shells and hence altering the material properties. This has been thoroughly investigated by TMSL in [4] and by Fok in [6]. However, it was found that provided the shells are properly bonded onto the rubber substrate then this is not the primary cause of the discrepancy.

Under static pressure the micro-spheres compress, reduce in size and a strained (anisotropic) region forms around the inclusions. Existing models for wave scattering in the composite material allow for the compression of micro-spheres but not for the existence of a strained region. Therefore another proposed explanation of the discrepancy was that scattering from this strained region should be considered and this is the topic in this dissertation.

Initially we shall review the literature relevant to this problem. This includes a discussion of the best way to model vulcanised rubber and consideration of the present TMSL model. We shall then propose modifications to the model and as such describe any relevant literature in the field of low frequency scattering from a spherical obstacle.

The main aim of this dissertation will then be to ascertain how the strained region will affect the scattered field from a micro-sphere.

# Chapter 2

# Literature Review

## 2.1 Modelling the Static Behaviour of Rubber-like materials

If we treat rubber as an isotropic material then for small strains its behaviour is determined by its Lamé moduli $\lambda$ and $\mu$. These obey $\mu \ll \lambda$ corresponding to the fact that rubber is approximately incompressible. However the elasticity problem we have to solve is one for which the hydrostatic pressure $p$ can be many times greater than the shear modulus $\mu$. It is therefore probable that nonlinear elasticity will have to be used and we describe models for this.

An elastic material is characterised by its stored energy function $W = W(A)$, i.e. the work done in deforming a unit volume of material from the reference state to a state with deformation gradient $A$. In an unstrained state or rigid body rotation, $W = 0$. Assuming that deformations take place either isothermally or adiabatically so that thermodynamic variables can be omitted from the discussion[1], $W$ can be expressed as a function of the principal stretches of the material. These are the singular values of the deformation gradient matrix and completely specify the state of strain of the material. We denote them by $\lambda_i, i = 1, 2, 3$. The volumetric expansion of the material is given by $V = \lambda_1\lambda_2\lambda_3$, and for much of this work it is a good approximation to treat rubber as incompressible so that $V = 1$.[2]

[1] A full account of thermodynamic aspects with a detailed examination of energy and entropy changes in rubber during deformation can be found in Green and Adkins [9] and Treloar [20].

[2] Small volume changes in rubber during stretching have been measured, (see Gee et al [8]) but in general we can assume incompressibility of rubber-like materials.

The form of the stored energy function for vulcanised rubber has been studied in great depth using both theoretical and experimental techniques (see Green and Adkins [9] and Treloar [20]). Both immediately assume that rubber is isotropic and incompressible and they choose to express $W$ in terms of the strain invariants which are defined by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.1}$$

$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \tag{2.2}$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.3}$$

If $A$ is the deformation gradient matrix of an elastic material, then the characteristic polynomial of $A^TA$ is

$$(x-\lambda_1^2)(x-\lambda_2^2)(x-\lambda_3^2) = x^3 - I_1x^2 + I_2x - I_3. \tag{2.4}$$

For an isotropic material, the stored energy function must be a symmetric function in the principal stretches and therefore a function of $I_1, I_2$ and $I_3$. Furthermore, the condition of incompressibility means that $I_3 = 1$, $\lambda_3 = 1/\lambda_1\lambda_2$ and so $W$ is a function of $I_1$ and $I_2$ only which in turn can be expressed in terms of $\lambda_1$ and $\lambda_2$ only.

Note also that for small strains $e_i = \lambda_i - 1$ of an incompressible material,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} = 3 + 2\sum_{i=1}^{3} e_i^2 + O(||e_i||^3), \tag{2.5}$$

$$I_2 = \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 3 + 2\sum_{i=1}^{3} e_i^2 + O(||e_i||^3) \tag{2.6}$$

so that there is only a difference between the strain invariants $I_1$ and $I_2$ for an incompressible material when *large* strains are considered.

The general form of the stored energy function is expressed by Treloar in [20] as a power series in $(I_1 - 3)$ and $(I_2 - 3)$, i.e.

$$W = \frac{\mu}{2}\sum_{i,j=0}^{\infty} C_{ij}(I_1-3)^i(I_2-3)^j. \tag{2.7}$$

Conditions on the real coefficients $C_{ij}$ are $C_{00} = 0$ and $C_{10} + C_{01} = 1$ so that for small strains, $W$ reduces to the correct form for linear elasticity as we shall see later. With this exception the coefficients are determined experimentally. In [9], Green and Adkins simply use a general form of $W$, that being

$$W = W(I_1, I_2). \tag{2.8}$$

Putting $i = 1$ and $j = 0$ in (2.7) gives the form of energy function which is derived from a statistical theory based upon the molecular picture of vulcanised rubber as a network of long chain molecules and is

$$\begin{aligned} W = \frac{\mu}{2} C_{10}(I_1 - 3) &= \frac{\mu}{2} C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) \\ &= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) \end{aligned} \tag{2.9}$$

assuming $C_{10} = 1$, consistent with conditions on the coefficients.

A variation which under certain circumstances gives a better fit to experimental stress-strain curves was given by Mooney in [16] as

$$\begin{aligned} W &= \frac{\mu}{2}\Big(C_{10}(I_1 - 3) + C_{01}(I_2 - 3)\Big) \\ &= \frac{\mu}{2}\Big(C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) + C_{01}(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2 - 3)\Big). \end{aligned} \tag{2.10}$$

This form has been found to be extremely valuable in the study of large deformations of rubber although in [9], Green and Adkins state that it is not an entirely satisfactory form of $W$. Matching this form of $W$ with an experiment involving simple extension would mean that the ratio $C_{01}/C_{10}$ would be approximately 0.8 which contradicts further, more involved experiments which give much smaller values of this ratio. In particular in [20] Treloar gives this ratio as 0.1.

It was suggested by Rivlin and Saunders in [17] and [18] that $W$ should take the form

$$W = \frac{\mu}{2}\Big(C_{10}(I_1 - 3) + f(I_2 - 3)\Big) \tag{2.11}$$

where $f$ is a function whose behaviour can be determined. This seems to agree more closely with several experiments carried out. One suggestion for $f$ is a power series expansion in $(I_2 - 3)$. This would appear to explain why (2.10) is only correct under certain conditions.

A general form of $W$ for an isotropic, elastic material, which includes all of the above as special cases, is

$$W = \sum_m C_m W_m + \frac{1}{2}\lambda(V - 1)^2 F(V) \tag{2.12}$$

where $C_m$ are coefficients such that $\sum_m C_m = 1$,$m \in \mathbb{R}$, $F(V)$ is a function of $V$ such that $F(1) = 1$ and

$$W_m = \frac{\mu}{2m^2}\left(\Big(\sum_{i=1}^{3} \lambda_i^{2m}\Big) - 3 - 2m\log V\right) \qquad m \neq 0, \tag{2.13}$$

$$W_0 \;=\; \lim_{m\to 0} W_m = \mu \sum_{i=1}^{3} \log^2 \lambda_i. \tag{2.14}$$

Notice that whereas in [9] and [20] Green and Adkins and Treloar assume strict incompressibility, $V = 1$, we will take an incompressible material to be the limit of a compressible material as $V \to 1$ and $\lambda \to \infty$.

The reason that (2.12) is chosen as the form of stored energy function is so that under the assumption of small strains $e_i = \lambda_i - 1$, it will reduce to the correct form for linear elasticity as we now show. For small strains,

$$W_0 = \mu \sum_{i=1}^{3} \log^2 \lambda_i \;=\; \mu \sum_{i=1}^{3} e_i^2 + O(||e_i||^3) \tag{2.15}$$

and for $m \neq 0$,

$$\begin{aligned} W_m \;&=\; \frac{\mu}{2m^2}\Big( \Big( \sum_{i=1}^{3} \lambda_i^{2m} \Big) - 3 - 2m \log V \Big) \\ &=\; \frac{\mu}{2m^2} \sum_{i=1}^{3} \Big( \lambda_i^{2m} - 1 - 2m \log \lambda_i \Big) \\ &=\; \mu \sum_{i=1}^{3} e_i^2 + O(||e_i||^3), \qquad \forall m. \end{aligned} \tag{2.16}$$

Finally,

$$\begin{aligned} \frac{1}{2}\lambda (V-1)^2 F(V) \;&=\; \frac{1}{2}\lambda (\lambda_1\lambda_2\lambda_3 - 1)^2 F(\lambda_1\lambda_2\lambda_3)) \\ &=\; \frac{1}{2}\lambda \Big( \sum_{i=1}^{3} e_i \Big)^2 + O(||e_i||^3) \end{aligned} \tag{2.17}$$

and therefore for small strains we have that

$$\begin{aligned} W \;&=\; \sum_m C_m W_m + \frac{1}{2}\lambda (V-1)^2 F(V) \\ &=\; \mu \sum_{i=1}^{3} e_i^2 \Big( \sum_m C_m \Big) + \frac{1}{2}\lambda \Big( \sum_{i=1}^{3} e_i \Big)^2 + O(||e_i||^3). \end{aligned} \tag{2.18}$$

Hence, the condition on the coefficients $C_m$ so that $W$ represents the correct form of the stored energy function for small strains is that $\sum_m C_m = 1$, $m \in \mathbb{R}$. This proves the point made earlier about the chosen form of $W$.

Although the $W_m$ all have the same form for small strains, the value of $m$ affects the large strain behaviour. For instance we shall show later that $m = 1/2$ corresponds to linear elasticity and $m = 1$ for an incompressible material corresponds to the form of stored energy function given in (2.9). Also for $V = 1$, then

$$I_2 \;=\; \sum_{i=1}^{3} \frac{1}{\lambda_i^2} \tag{2.19}$$

so that (2.10) is a linear combination of $W_1$ and $W_{-1}$.

In section 3 where we make use of this theory, we solve the elasticity problem of the compression of a gaseous spherical hole in a rubber substrate considering the different forms of the stored energy function discussed here.

## 2.2 The TMSL Model and Low Frequency Scattering

As we mentioned in the introduction, the wavelength of the incident compressive wave is much longer than the average diameter of the micro-spheres, so that we have low frequency scattering of elastic waves. Hence in considering the scattered field we do not have any of the problems associated with diffraction of waves.

The most important element of the theory in the TMSL reports [2], [3] and [4] is that it includes *multiple scattering* as opposed to many models of the scattering of elastic waves in composite materials which do not (see Baird [1] and Gaunard and Überall [7]). Even for low concentration of micro-spheres, it is expected that secondary scattering is important in calculating the modified material properties. In [3], Brazier-Smith argues that in neglecting the multiple scattering effects, even for low concentration of micro-spheres, the properties of the composite material would then only be correct to leading order. Therefore, modified properties would be the same as those of the substrate material. Therefore, multiple scattering is fundamental to the model.

The method of multiple scattering is well developed and well understood by TMSL and so given the scattering effects from a single, isolated inclusion, the theory developed by TMSL can account for an ensemble of inclusions, characterising the behaviour

of the material as a whole. As such what this dissertation will do is to explain how a sound wave is scattered from a strained region around a single inclusion. TMSL will then be able to incorporate this into a modified, self consistent, multiple scattering model in order to predict the modified material properties.

### 2.2.1 The TMSL Model

We describe the TMSL model and in particular discuss any assumptions made.

At a particular static pressure, the TMSL report [4] states that the inclusions fall into two groups, those that are buckled and those that are not.

The rigidity of the shell of an unbuckled micro-sphere will tend to preserve its spherical form and size and therefore *linear elasticity* is used to calculate the radius of a compressed, unbuckled micro-sphere. This is justified retrospectively by Fok in [6] by showing that strains at the point of buckling remain within the range of linearity.

A buckled micro-sphere will have a somewhat random shape and its shell will lose its rigidity.[3] As such, buckled micro-spheres are modelled as gaseous holes with no shell. Again *linear elasticity* is used to calculate the radius of the compressed gaseous hole but no justification of this is given in [4]. Denoting $a_0$ as the radius of the hole before compression and $a$ as the compressed radius, then experiments show that $(a_0/a)^3 \approx 10$ for buckled micro-spheres and therefore $a < a_0/2$, i.e. the radius of the hole is significantly reduced under hydrostatic pressure. It therefore seems that nonlinear elasticity is more appropriate, particularly since the hydrostatic pressure $p$ can be many times greater than $\mu$.

Under pressure we will assume that the compression is spherically symmetric. It was suggested in [4] that the inclusions may compress into an oblate spheroidal shape due to the way the hydrostatic pressure acts on the exterior of the submarine. However, spherically symmetric compression is acknowledged as a reasonable assumption.

Scattering from an unbuckled or buckled micro-sphere is therefore calculated in [4] by finding the scattered field from an isotropic region surrounding a compressed spherical shell or gaseous hole respectively. The rubber is strained non uniformly under

[3] The buckling criterion for a critical pressure given by Koiter in [13] is seen as important although in reality the shells buckle stochastically in the range 30-50%. The presence of rubber around the shell is also thought to increase the critical load, see Fok [6].

hydrostatic pressure and therefore becomes anisotropic. However, the model makes the assumption that a strained region around a micro-sphere can be treated as if it was an isotropic material.

### 2.2.2 Modifications to the TMSL model

We now summarise what we have said and indicate how we shall approach the problem.

1. We calculate the scattering effects from an isolated, spherically symmetric inclusion so that this theory can then be used in a multiple scattering model by TMSL.

2. It is accepted that *unbuckled* micro-spheres compress according to linear elasticity. Therefore scattering calculations in the TMSL model for *unbuckled* micro-spheres are assumed to be correct since it seems valid to approximate the strained region as an isotropic region in this *small strain* case.

3. Therefore in this dissertation we only consider scattering from *buckled* micro-spheres. We shall calculate the compressed state of these by applying nonlinear elasticity, having justified this in section 2.2.1 above.

4. We then include the existence of the strained, anisotropic region around a micro-sphere in the scattering calculation.

### 2.2.3 Relevant literature

References given in the TMSL reports [2], [3] and [4] are mostly references to models which include multiple scattering and are therefore not useful to us. We must analyse any relevant literature which focuses on the scattering of elastic waves from a spherical obstacle. There has been a great deal of literature published in this area. However, the majority of the work focuses on scattering from a spherical obstacle in a fluid or high frequency scattering and is therefore not relevant.

The study of low frequency elastic wave scattering has been well developed in the geophysical literature, for example in Gritto [11] which analyses the scattering of

seismic waves by inhomogeneities in the earth's crust. Primarily however, papers such as this all refer back to one of two papers, Ying and Truell [21] and/or Einspruch and Truell [5] who analyse elastic wave scattering from a spherical obstacle in an isotropic, elastic solid. The former considers three forms of spheres, a rigid sphere, an elastic sphere and a spherical cavity. The latter considers a spherical obstacle filled with an inviscid fluid. The difference between each of the above cases is of course the boundary conditions on the surface of the obstacle, i.e. how continuity of stress and displacement are expressed in each case. Also, for an elastic sphere and a fluid filled sphere, expressing the relevant boundary conditions requires knowledge of the waves which propagate inside the sphere itself.

Low and high frequency limits are considered and energy calculations are made. In particular, it is found how the type of obstacle alters the total energy carried away by the scattered wave. This property is characterised by the scattering cross section of the obstacle, being defined by

$$\gamma_{\text{obstacle}} = \frac{\text{Total energy scattered per unit time}}{\text{Total energy per unit area, carried by the incident wave per unit time}}$$

where the amount of energy scattered is measured across a spherical surface concentric with and of radius larger than the obstacle. The energy carried by the incident wave is measured across an area which is perpendicular to the direction of propagation.

It is shown in [4] that the modified material properties are functions of the scattering cross sections relating to monopole, dipole, quadrupole and rotational scattering. Therefore provided the concentration of inclusions is known, together with the scattering cross sections, the modified material properties can be deduced.

Perhaps the best way to see what effect the strained region will have on the scattering process is to find the scattering cross section in the case of a strained region and compare the results with those obtained by Ying and Truell in [21] for a spherical cavity or Einsruch and Truell in [5] for a gas filled sphere.

# Chapter 3

# Compression of a Spherical Hole in an Elastic Solid

We shall now solve the problem of the compression of a buckled micro-sphere under hydrostatic loading in the far field. As explained in section 2.2.2, we model a buckled micro-sphere as a gaseous hole ignoring the presence of the shell so that we are simply considering the compression of a spherical hole in an isotropic substrate. We make the assumption that the compression is spherically symmetric. Therefore a perfect spherical inclusion compresses to a smaller, perfect spherical inclusion. This is reasonable since in the first instance we are approximating the random shape of the buckled micro-sphere by a spherical gaseous hole. A similar static problem to this one has been considered by Green and Zerna in [10].

Because we assume spherical symmetry, the principal axes of strain at any point are the radial direction and any two perpendicular tangential directions. These are also the principal axes of stress and we denote the principal stresses by $\sigma_i, i = 1, 2, 3$.

We use the general form of the stored energy function given in (2.12) from section 2.1,

$$W = \sum_m C_m W_m + \frac{1}{2}\lambda(V-1)^2 F(V) \qquad m \in \mathbb{R} \tag{3.1}$$

where

$$W_m = \frac{\mu}{2m^2}\left(\left(\sum_{i=1}^{3}\lambda_i^{2m}\right) - 3 - 2m\log V\right) \qquad m \neq 0 \tag{3.2}$$

$$W_0 = \mu \sum_{i=1}^{3} \log^2 \lambda_i \tag{3.3}$$

and then we can easily calculate the principal stresses in terms of the principal stretches, for example,

$$\begin{aligned} \sigma_1 &= \frac{1}{\lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_1} \\ &= C_0 \frac{2 \log \lambda_1}{V} + \sum_{m \neq 0} C_m \frac{\mu}{Vm} \Big( \lambda_1^{2m} - 1 \Big) \\ &\qquad + \lambda (V-1) F(V) + \frac{\lambda}{2} (V-1)^2 F'(V). \end{aligned} \tag{3.4}$$

We now represent incompressibility as the limit of a compressible material when $\lambda \to \infty$. In this limit, $V \to 1$ to keep $W$ finite and then the penultimate term in (3.4) tends to a finite limit $\widetilde{\sigma}$. The last term tends to zero since $(V-1)^2$ approaches zero faster than $\lambda$ approaches infinity. Therefore in this limit, $\sigma_1$ becomes

$$\sigma_1 = 2\mu C_0 \log \lambda_1 + \left( \sum_{m \neq 0} C_m \frac{\mu}{m} (\lambda_1^{2m} - 1) \right) + \widetilde{\sigma}.$$

More generally we have

$$\sigma_i = 2\mu C_0 \log \lambda_i + \left( \sum_{m \neq 0} C_m \frac{\mu}{m} (\lambda_i^{2m} - 1) \right) + \widetilde{\sigma}, \qquad i = 1, 2, 3. \tag{3.5}$$

Now that we have set up the relevant theory for the work function and principal stresses we can find the location of the compressed inclusion. In order to do this, consider the spherical annulus $a_0 < r_0 < b_0$ in an unstressed state which deforms into the annulus $a < r < b$ when subjected to pressures $p_{in}$ and $p_{out}$ on the inner and outer surfaces respectively (see figure 3.1).

We must calculate the principal stretches under this compression. There will be a radial stretch and two equal, perpendicular tangential stretches along the principal axes of strain. Therefore, rather than label the stretches and stresses with indices $i = 1, 2, 3$ from now on we shall label them with indices $i = r, t$. The radial and tangential stretches will be

$$\lambda_r = \frac{dr}{dr_0}, \tag{3.6}$$

$$\lambda_t = \frac{r}{r_0}. \tag{3.7}$$

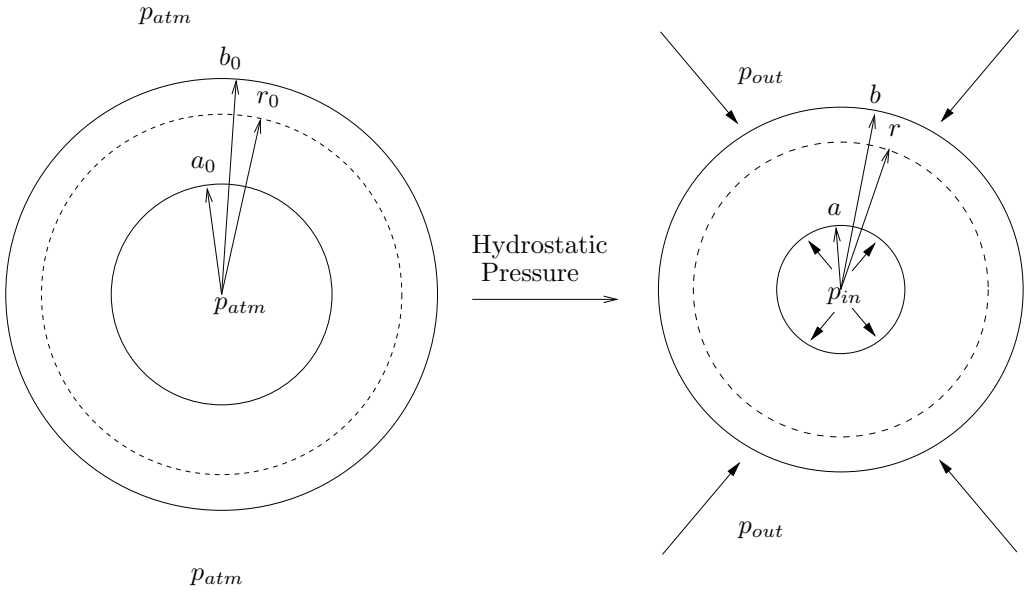



Figure 3.1: Compression of a spherical micro-sphere under hydrostatic pressure.

An important consequence of conservation of mass and incompressibility is the conservation of volume condition,

$$r_0^3 - a_0^3 \;=\; r^3 - a^3 \tag{3.8}$$

and therefore we have that

$$\lambda_r = \frac{dr}{dr_0} = \frac{r_0^2}{r^2}. \tag{3.9}$$

The principal stresses acting in the material, pictured in figure 3.2 can therefore be calculated from (3.5). They are

$$\sigma_r \;=\; 2C_0\mu \log\left(\frac{r_0^2}{r^2}\right) + \left(\sum_{m\neq 0} C_m \frac{\mu}{m}\left((\frac{r_0}{r})^{4m} - 1\right)\right) + \widetilde{\sigma}, \tag{3.10}$$

$$\sigma_t \;=\; 2C_0\mu \log\left(\frac{r}{r_0}\right) + \left(\sum_{m\neq 0} C_m \frac{\mu}{m}\left((\frac{r}{r_0})^{2m} - 1\right)\right) + \widetilde{\sigma}. \tag{3.11}$$

In order to calculate the radius of the compressed micro-sphere we need the equation of static equilibrium. The general form of this in spherical polar coordinates is given by Love in [15], page 91. Alternatively, we can derive it by considering the forces on a hemispherical shell of thickness $\Delta r$ and inner radius $r$ as shown in figure 3.3.

Balancing forces in the vertical direction then gives us

$$\sigma_r|_{r+\Delta r}\pi(r+\Delta r)^2 - \sigma_r|_r \pi r^2 = \sigma_t 2\pi r \Delta r \tag{3.12}$$

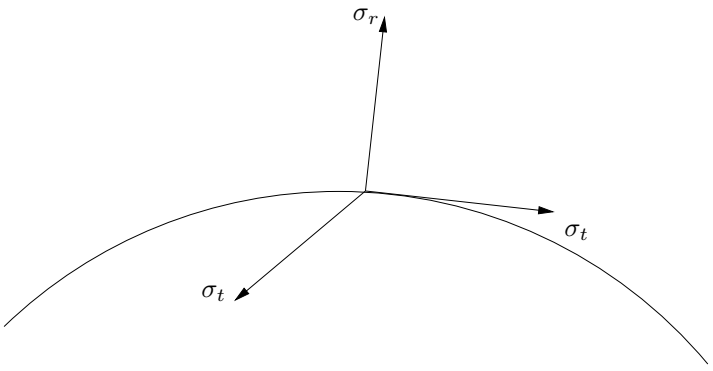



Figure 3.2: The stresses acting in the composite material.

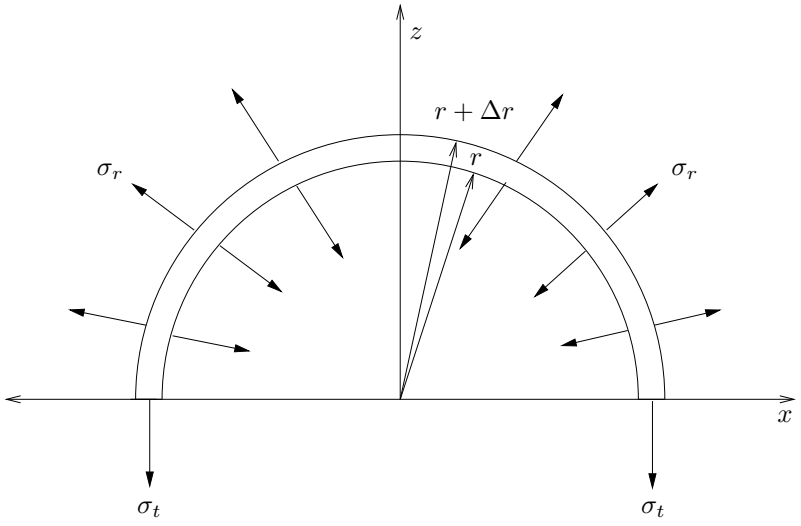



Figure 3.3: Forces on a hemispherical shell.

which when dividing by $\Delta r$ and letting $\Delta r \to 0$ becomes

$$\frac{d}{dr}(r^2\sigma_r) = 2r\sigma_t. \tag{3.13}$$

Expanding the derivative on the left hand side and dividing by $r^2$ this becomes

$$\frac{d\sigma_r}{dr} = \frac{2}{r}(\sigma_t - \sigma_r) \tag{3.14}$$

and we must now solve this subject to the boundary conditions

$$\sigma_r = -p_{in} \quad \text{on} \quad r = a, \tag{3.15}$$

$$\sigma_r = -p_{out} \quad \text{on} \quad r = b. \tag{3.16}$$

Integrating (3.14) between $r = a$ and $r = b$ we get

$$\Big[\sigma_r\Big]_{r=a}^{r=b} = \int_a^b \frac{2\mu}{r}\Big[2C_0 \log\Big(\frac{r^3}{r_0^3}\Big) + \sum_{m\neq 0} \frac{C_m}{m}\Big(\Big(\frac{r}{r_0}\Big)^{2m} - \Big(\frac{r_0}{r}\Big)^{4m}\Big)\Big]dr \tag{3.17}$$

so that on applying the boundary conditions (3.15) and (3.16) we have

$$\frac{p_{out}-p_{in}}{\mu} = 2\int_a^b \Big[2C_0\log\Big(\frac{r_0^3}{r^3}\Big)+\sum_{m\neq 0}\frac{C_m}{m}\Big(\Big(\frac{r_0}{r}\Big)^{4m}-\Big(\frac{r}{r_0}\Big)^{2m}\Big)\Big]\frac{dr}{r}. \quad (3.18)$$

Now make the change of variable $t = r_0/r$ so that by the conservation of volume condition (3.8)

$$t^3 = \Big(\frac{r_0}{r}\Big)^3 = \frac{r^3-a^3+a_0^3}{r^3} = 1+\frac{(a_0^3-a^3)}{r^3} \quad (3.19)$$

and therefore

$$3t^2dt = -3\frac{(a_0^3-a^3)}{r^4}dr \quad (3.20)$$

so that

$$t^2dt = -\frac{dr}{r}(t^3-1). \quad (3.21)$$

Then, (3.18) becomes

$$\begin{aligned}\frac{p_{out}-p_{in}}{\mu} &= -2\int_{a_0/a}^{b_0/b}\Big[2C_0\log(t^3)+\sum_{m\neq 0}\frac{C_m}{m}(t^{4m}-t^{-2m})\Big]\frac{t^2}{(t^3-1)}dt\\ &= 2\int_{b_0/b}^{a_0/a}\Big[6C_0\frac{t^2\log(t)}{(t^3-1)}+\sum_{m\neq 0}\frac{C_m}{m}\frac{(t^{4m}-t^{-2m})t^2}{(t^3-1)}\Big]dt\\ &= 2C_0\Big(G_0\Big(\frac{a_0}{a}\Big)-G_0\Big(\frac{b_0}{b}\Big)\Big)+\\ &\qquad 2\sum_{m\neq 0}C_m\Big(G_m\Big(\frac{a_0}{a}\Big)-G_m\Big(\frac{b_0}{b}\Big)\Big) \quad (3.22)\end{aligned}$$

where

$$G_m(x) = \int_1^x \frac{1}{m}\frac{(t^{4m}-t^{-2m})}{(t^3-1)}t^2dt \qquad m\neq 0, \quad (3.23)$$

$$G_0(x) = \int_1^x \frac{6t^2\log(t)}{(t^3-1)}dt. \quad (3.24)$$

We want to consider an isolated inclusion with the pressure acting in the far field at infinity, so we let $b, b_0 \to \infty$, i.e. $b_0/b \to 1$ and therefore

$$\lim_{b_0/b\to 1} G_m\Big(\frac{b_0}{b}\Big) = 0 \qquad \forall m. \quad (3.25)$$

Therefore given $m$ we can calculate the radius of the compressed micro-sphere by the equation

$$\frac{p_{out} - p_{in}}{\mu} = 2\Big(C_0 G_0\Big(\frac{a_0}{a}\Big) + \sum_{m\neq 0} C_m G_m\Big(\frac{a_0}{a}\Big)\Big). \tag{3.26}$$

Clearly this only gives an *approximation* to the radius $a$ of the equilibrium state because the rubber substrate is not strictly incompressible ($V = 1$) and so we will never have that

$$\lambda_r = \frac{dr}{dr_0} = \frac{r_0^2}{r^2} \tag{3.27}$$

exactly. However the result obtained will be a good approximation to the actual static equilibrium position.

## 3.1 Linear Elasticity

The existing TMSL model [4] calculates the radius of the compressed micro-sphere assuming linear elasticity. If we take $m = 1/2$ only, i.e. $C_{1/2} = 1$ and $C_m = 0$ $\forall m \neq 1/2$ then (3.5) shows that the associated stresses are

$$\sigma_i = 2\mu(\lambda_i - 1) + \widetilde{\sigma}, \quad \text{for } i = r, t. \tag{3.28}$$

i.e. the material behaves linearly if $m = 1/2$. Then we have

$$\begin{aligned} G_{1/2}(x) &= 2\int_1^x \frac{(t^2 - t^{-1})t^2}{(t^3 - 1)} dt \\ &= x^2 - 1. \end{aligned} \tag{3.29}$$

Therefore (3.26) tells us that the radius of the compressed micro-sphere is given by

$$\frac{p_{out} - p_{in}}{\mu} = 2\Big(\Big(\frac{a_0}{a}\Big)^2 - 1\Big). \tag{3.30}$$

This agrees with equation (4.5) in the TMSL report [4] for the compression of a micro-sphere using linear elasticity together with the assumption that $p_{in} = p_{atm}(a_0/a)^3$, according to Boyle's Law.

## 3.2 Nonlinear Elasticity

In section 2.2.1 we gave justification that in fact the elasticity involved in the problem is nonlinear. Here we shall consider the results obtained by using the different forms of stored energy functions relating to nonlinear elasticity given in section 2.1.

We first consider the stored energy function given in (2.9), where $m = 1$ only, i.e. $C_1 = 1$ and $C_m = 0 \quad \forall m \neq 1$. The associated stresses in this case are

$$\sigma_i = \mu(\lambda_i^2 - 1) + \widetilde{\sigma}, \quad \text{for } i = r, t \tag{3.31}$$

and we have

$$\begin{aligned} G_1(x) &= \int_1^x \frac{(t^4 - t^{-2})t^2}{(t^3 - 1)} dt \\ &= \int_1^x (t^3 + 1) dt \\ &= \frac{x^4}{4} + x - \frac{5}{4} \end{aligned} \tag{3.32}$$

and therefore (3.26) tells us that the radius of the compressed state is given by

$$\frac{p_{out} - p_{in}}{\mu} = \frac{1}{2}\Big(\frac{a_0}{a}\Big)^4 + 2\Big(\frac{a_0}{a}\Big) - \frac{5}{2}. \tag{3.33}$$

Another stored energy function which is seen to be an improvement under certain conditions is given by (2.10) where $m = -1, 1$, i.e. $C_{-1}$ and $C_1$ sum to one, are non-zero and $C_m = 0 \quad \forall m \neq \pm 1$. The associated stresses in this case are

$$\sigma_i = C_1\mu(\lambda_i^2 - 1) - C_{-1}\mu(\lambda_i^{-2} - 1) + \widetilde{\sigma}, \quad \text{for } i = r, t. \tag{3.34}$$

$G_1(x)$ is given by (3.32) and we also have that

$$\begin{aligned} G_{-1}(x) &= -\int_1^x \frac{(t^{-4} - t^2)}{(t^3 - 1)} t^2 dt \\ &= -\int_1^x \frac{(1 - t^6)}{t^2(t^3 - 1)} dt \\ &= \int_1^x t + t^{-2} dt \\ &= \frac{x^2}{2} - \frac{1}{x} + \frac{1}{2} \end{aligned} \tag{3.35}$$

and therefore using this modification of the stored energy function gives the following equation for the radius of the compressed micro-sphere.

$$\begin{aligned}\frac{p_{out}-p_{in}}{\mu} &= C_1 G_1\Big(\frac{a_0}{a}\Big) + C_{-1}G_{-1}\Big(\frac{a_0}{a}\Big) \\ &= C_1\bigg(\frac{1}{2}\Big(\frac{a_0}{a}\Big)^4 + 2\Big(\frac{a_0}{a}\Big) - \frac{5}{2}\bigg) + C_{-1}\bigg(\Big(\frac{a_0}{a}\Big)^2 - 2\Big(\frac{a}{a_0}\Big) + 1\bigg) \quad (3.36)\end{aligned}$$

In section 2.1 we discussed the fact that different values for the coefficients $C_1$ and $C_{-1}$ had been given. One value for the ratio $C_{-1}/C_1$ was given as 0.8 which with the condition that coefficients sum to one would mean that

$$C_1 = \frac{5}{9} \qquad \text{and} \qquad C_{-1} = \frac{4}{9} \quad (3.37)$$

However the value of this ratio given by Treloar in [20] was 0.1 which would mean that

$$C_1 = \frac{10}{11} \qquad \text{and} \qquad C_{-1} = \frac{1}{11} \quad (3.38)$$

In order to consider the different forms of stored energy function and the values of $a$ they predict under compression, we shall plot $a_0/a$ as a function of the ratio $(p_{out} - p_{atm})/\mu$.

We shall do this for five different forms of stored energy function $W$,

**(a)** $m = -1$ only, although in [9] Green and Adkins state that this form has not been found to correspond to any known material.

**(b)** $m = 1/2$ only which corresponds to linear elasticity as currently assumed by the TMSL model.

**(c)** $m = \pm 1$ only with $C_1 = 5/9$ and $C_{-1} = 4/9$, the form which Green and Adkins state as unsatisfactory.

**(d)** $m = \pm 1$ only with $C_1 = 10/11$ and $C_{-1} = 1/11$, the more satisfactory form given by Treloar in [20].

**(e)** $m = 1$ only, the form derived by Treloar from a statistical theory.

and for the two cases where,

1. $p_{in} = p_{atm}$=constant - see figure 3.4.

2. $p_{in} = p_{atm}(a_0/a)^3$. This is Boyle's Law, assuming that thermal equilibrium is reestablished after the deformation - see figure 3.5.

Letters in brackets next to each curve on the figures correspond to the form of stored energy function in the list above used in the calculation.

We take the shear modulus for rubber as $\mu = 2 \times 10^6$Pa and $p_{atm} = 10^5$Pa.

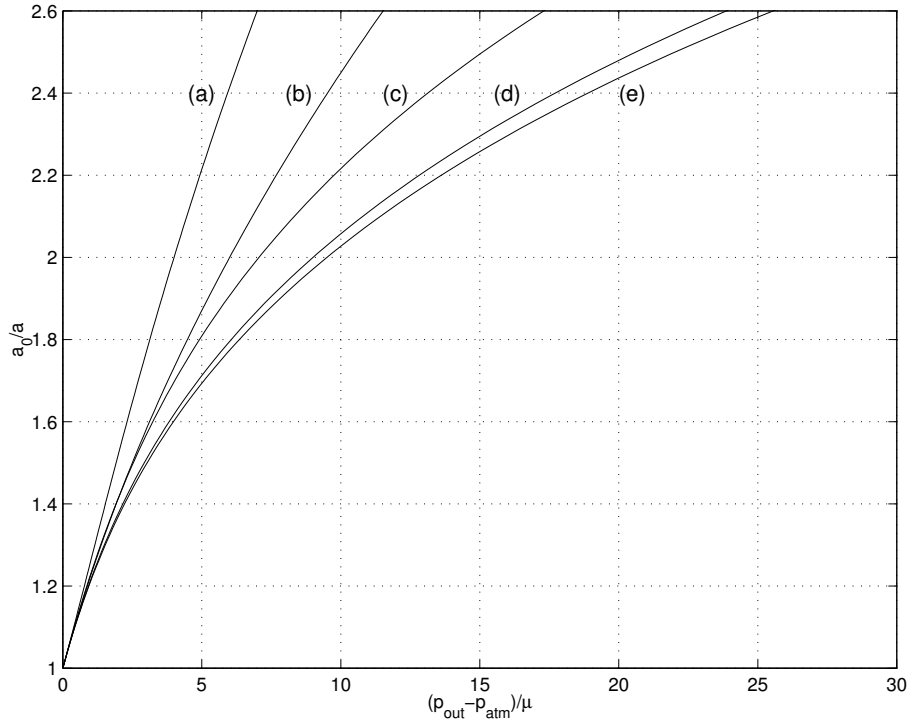



Figure 3.4: Calculation of compressed radius for the different forms of stored energy function when $p_{in}$=$p_{atm}$ =constant.

From this analysis it is clear that the nonlinear elastic behaviour has significant effects. The micro-spheres do not compress as much as linear elasticity predicts and this alone, even without the scattering from the strained region is an important modification that should be made to the current TMSL calculations. Without further experimental measurements on the material, one cannot tell how close to (e) the curve *should* be, although (d) is what the most accurate experiments on similar materials suggest is most applicable.

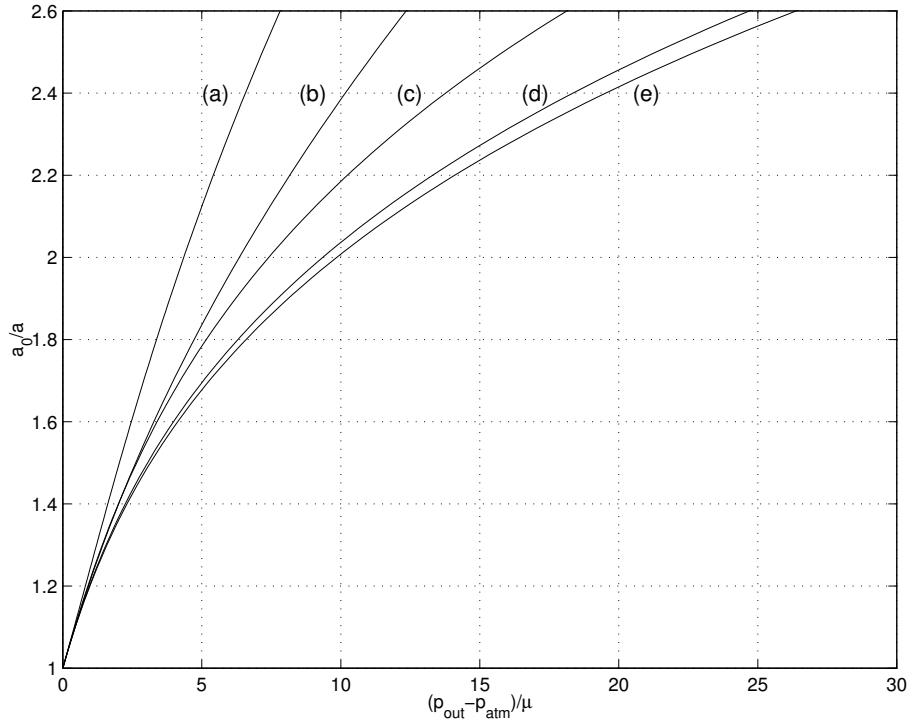



Figure 3.5: Calculation of compressed radius for the different forms of stored energy function when $p_{in} = p_{atm}(a_0/a)^3$.

# Chapter 4

# Small Displacements superposed on the Initial Strain

Now that we have considered how the hole will compress under hydrostatic loading we must derive the equations of motion for an incident compressive wave propagating in the material surrounding the micro-sphere. As such, we should find the equations of motion for small displacements superposed on top of the initial strain. Clearly we cannot immediately use the well known, classical equations of motion for small displacements in an isotropic, elastic material because here the material has initially been strained non-uniformly and is therefore anisotropic.

We can make several assumptions in order to simplify the theory and so initially we present these simplifications and explain how Helmholtz resolution of a vector can be used to represent the additional small displacements in terms of the two types of waves that propagate in solids, compressive and shear waves.

We shall then build up a general framework for the derivation of the equations of motion using a theory derived by Green and Zerna in [10]. Appropriate boundary conditions are discussed and then using a general form of the stored energy function $W$ for rubber, we show how one would derive explicitly, the equations of motion. We then indicate how a numerical solution of these equations is possible, using a spectral method.

## 4.1 Initial Assumptions

We assume that the incident sound wave will be a plane, compressive wave of unit amplitude. We choose the $z$-axis of the reference state so that the incident wave propagates in the positive $z$ direction as in figure 4.1 below.

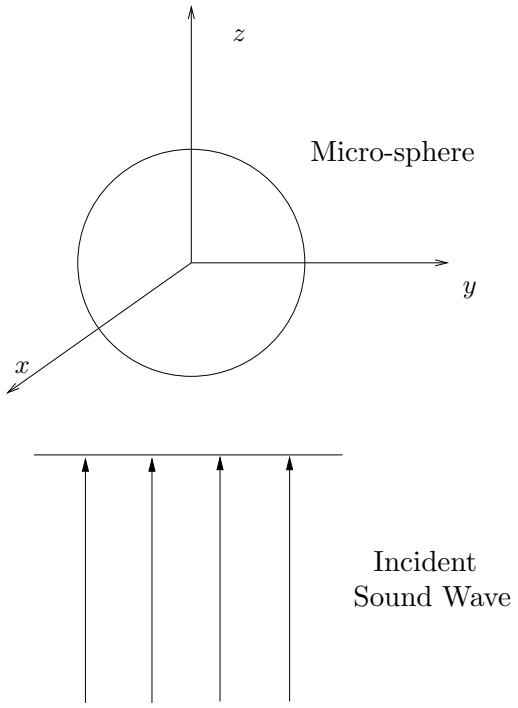



Figure 4.1: The $z$-axis of the reference state is chosen so that the incident wave propagates in the positive $z$ direction.

Of course waves in solids are transmissions of displacements in the material and we denote such displacements by $\underline{\mathbf{U}} = \underline{\mathbf{U}}(r, \theta, \phi, t) = (U_r, U_\theta, U_\phi)$. Assuming that the disturbance is time harmonic we can write this as

$$\underline{\mathbf{U}} \;=\; \underline{\mathbf{u}} e^{i\omega t} \tag{4.1}$$

where $\underline{\mathbf{u}} = \underline{\mathbf{u}}(r, \theta, \phi) = (u_r, u_\theta, u_\phi)$.

Furthermore, because the sound wave is assumed to be plane and incident upon the micro-sphere in the $z$ direction we can assume that there is axisymmetry about the $z$-axis and therefore $u_r$ and $u_\theta$ are independent of $\phi$ and $u_\phi = 0$.

We can represent the displacement as in Ying and Truell [21], using Helmholtz resolution of a vector by writing

$$\underline{\mathbf{u}} = -\nabla\Psi + \nabla \times (\nabla \times (\underline{\hat{\mathbf{r}}} r \Pi)) \tag{4.2}$$

where $\Psi = \Psi(r,\theta)$ and $\Pi = \Pi(r,\theta)$ correspond to compressive and shear waves respectively and $\underline{\hat{\mathbf{r}}}$ is a unit vector in the radial direction. Wherever appropriate we denote the incident and scattered waves by subscripts $i$ and $s$ respectively so that since the incident wave is purely compressive we have that

$$\underline{\mathbf{u}}_i = -\nabla\Psi_i. \tag{4.3}$$

In the case of an isotropic material, Helmholtz decomposition leads to the equations of motion decoupling into two scalar Helmholtz equations for $\Psi$ and $\Pi$ whose solutions in spherical polars are well known. It is then the boundary conditions on the surface of the micro-sphere at $r = a$ which determine the solution.

Using (4.2) we can write explicit expressions for the spherical components of the displacement in terms of $\Psi$ and $\Pi$,

$$u_r = -\frac{\partial\Psi}{\partial r} - \frac{1}{r}\Omega\Pi, \tag{4.4}$$

$$u_\theta = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(r\Pi), \tag{4.5}$$

$$u_\phi = 0 \tag{4.6}$$

where

$$\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big). \tag{4.7}$$

$u_\phi = 0$ arises because of the axisymmetry and represents the fact that there is no rotational displacement about the $z$-axis.

For our problem, writing the displacement in terms of the Helmholtz potentials will not lead to the decoupling of the equations of motion. Therefore it does not seem appropriate to express the displacement in the manner of (4.2).

## 4.2 General Theory

### 4.2.1 Equations of Motion

We now derive the equations of motion for the additional small displacements, superposed on top of the initial, possibly nonlinear strain. In order to do this we use

the theory derived by Green and Zerna in [10] for small displacements superposed on top of an initial finite deformation. We have already considered the initial strain in detail in chapter 3.

We shall firstly explain the general theory we shall use, derived by Green and Zerna, following their notation.

Green and Zerna analyse the case in which an unstressed body $B_0$ undergoes a *finite* deformation to a known state $B$. Then additional *small* displacements are made to form a state $B'$. The small deformation from $B$ to $B'$ is expressed as a *linear* perturbation of $B$ but $B$ itself is a *nonlinear* departure from the unstressed state $B_0$.

Therefore in our case, we choose $B_0$ and $B$ to correspond to the material surrounding the micro-sphere before and after being subjected to hydrostatic loading and we choose $B'$ to correspond to the material surrounding the micro-sphere when a small amplitude sound wave passes through the material.

We define the displacement vector $\overrightarrow{P_0P}$ by

$$\underline{\mathbf{v}} = \underline{\mathbf{R}} - \underline{\mathbf{r}}. \tag{4.8}$$

Working in the reference state $(r, \theta, \phi)$ after the initial deformation has taken place, the covariant base vectors at points $P$ of the body $B$ are given by

$$\underline{\mathbf{G}}_1 = \frac{\partial \underline{\mathbf{R}}}{\partial r} = \underline{\mathbf{e}}_r, \tag{4.9}$$

$$\underline{\mathbf{G}}_2 = \frac{\partial \underline{\mathbf{R}}}{\partial \theta} = r\underline{\mathbf{e}}_\theta, \tag{4.10}$$

$$\underline{\mathbf{G}}_3 = \frac{\partial \underline{\mathbf{R}}}{\partial \phi} = r\sin\theta\underline{\mathbf{e}}_\phi, \tag{4.11}$$

where $\underline{\mathbf{e}}_r$, $\underline{\mathbf{e}}_\theta$ and $\underline{\mathbf{e}}_\phi$ are the usual unit vectors in spherical polars. The contravariant base vectors at points $P$ of the body $B$ are given by

$$\underline{\mathbf{G}}^1 = \nabla r = \underline{\mathbf{e}}_r, \tag{4.12}$$

$$\underline{\mathbf{G}}^2 = \nabla\theta = \frac{1}{r}\underline{\mathbf{e}}_\theta, \tag{4.13}$$

$$\underline{\mathbf{G}}^3 = \nabla\phi = \frac{1}{r\sin\theta}\underline{\mathbf{e}}_\phi. \tag{4.14}$$

Therefore the metric tensors $G_{ij}$ and $G^{ij}$ of the deformed state only have non-zero entries along the diagonal, and these are

$$G_{11} = 1, \qquad G_{22} = r^2, \qquad G_{33} = r^2 \sin^2\theta, \tag{4.15}$$

$$G^{11} = 1, \qquad G^{22} = \frac{1}{r^2}, \qquad G^{33} = \frac{1}{r^2 \sin^2\theta}. \tag{4.16}$$

A line element $ds_0$ of the undeformed state can be written in terms of the chosen reference state as

$$(ds_0)^2 \;=\; \Big(\frac{dr_0}{dr}\Big)^2 (dr)^2 + r_0^2 d\theta^2 + r_0^2 \sin^2\theta d\phi^2 \tag{4.17}$$

and so the metric tensors $g_{ij}$ and $g^{ij}$ of the undeformed state are

$$g_{11} = \Big(\frac{dr_0}{dr}\Big)^2, \qquad g_{22} = r_0^2, \qquad g_{33} = r_0^2 \sin^2\theta, \tag{4.18}$$

$$g^{11} = \Big(\frac{dr}{dr_0}\Big)^2, \qquad g^{22} = \frac{1}{r_0^2}, \qquad g^{33} = \frac{1}{r_0^2 \sin^2\theta}. \tag{4.19}$$

Defining the stress vector $\underline{\mathbf{t}}_i$ for each coordinate surface, per unit area of the deformed body $B$, we can write this as

$$\underline{\mathbf{t}}_i \sqrt{G_{ii}} \;=\; \tau^{ij} \underline{\mathbf{G}}_j \tag{4.20}$$

where $\tau^{ij}$ is the contravariant stress tensor. Furthermore we can write the stress vectors as

$$\underline{\mathbf{t}}_i \;=\; \sum_{j=1}^{3} \frac{\sigma_{(ij)} \underline{\mathbf{G}}_j}{\sqrt{G_{jj}}} = \sum_{j=1}^{3} \sigma_{(ij)} \underline{\mathbf{e}}_j \tag{4.21}$$

where

$$\sigma_{(ij)} \;=\; \sqrt{\Big(\frac{G_{jj}}{G^{ii}}\Big)}\, \tau^{ij} \tag{4.22}$$

are the physical components of the stress tensor referred to axes along $\underline{\mathbf{G}}_j$.

Following Green and Zerna, assuming that $W = W(I_1, I_2, I_3)$, we can write the stress tensor $\tau^{ij}$ as

$$\tau^{ij} \;=\; \Phi g^{ij} + \Psi B^{ij} + p G^{ij} \tag{4.23}$$

where

$$\Phi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1}, \qquad \Psi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}, \qquad p = 2\sqrt{I_3} \frac{\partial W}{\partial I_3}, \tag{4.24}$$

and

$$B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs}. \tag{4.25}$$

In this present notation, the strain invariants are

$$I_1 = g^{rs} G_{rs} = \lambda_r^2 + 2\lambda_t^2, \tag{4.26}$$

$$I_2 = g_{rs} G^{rs} I_3 = (\frac{1}{\lambda_r^2} + \frac{2}{\lambda_t^2}) I_3, \tag{4.27}$$

$$I_3 = \frac{G}{g} = (\lambda_r \lambda_t^2)^2. \tag{4.28}$$

By (4.23) we have that the stresses due to the deformation of the body $B_0$ into the body $B$ are,

$$\tau^{11} = \frac{2}{\sqrt{I_3}} \Big( \lambda_r^2 \frac{\partial W}{\partial I_1} + \frac{2I_3}{\lambda_t^2} \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \Big), \tag{4.29}$$

$$\tau^{22} = \frac{2\lambda_t^2}{r^2 \sqrt{I_3}} \Big( \frac{\partial W}{\partial I_1} + (\lambda_r^2 + \lambda_t^2) \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \Big), \tag{4.30}$$

$$\tau^{33} = \frac{2\lambda_t^2}{r^2 \sin^2\theta \sqrt{I_3}} \Big( \frac{\partial W}{\partial I_1} + (\lambda_r^2 + \lambda_t^2) \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \Big), \tag{4.31}$$

$$\tau^{ij} = 0 \qquad \text{if } i \neq j. \tag{4.32}$$

and then the equations of equilibrium for the initial strain can be written as

$$\tau^{ij}||_i = 0 \tag{4.33}$$

where $||_i$ denotes covariant differentiation with respect to the coordinate $\xi_i$ and the metric tensor $G_{mn}$ where $\xi_1 = r, \xi_2 = \theta$ and $\xi_3 = \phi$.

The first of these equilibrium equations should correspond to the equation of equilibrium we derived in (3.14). Indeed if we calculate it, we find that

$$\frac{d\sigma_{(11)}}{dr} = \frac{2}{r}(\sigma_{(22)} - \sigma_{(11)}) \tag{4.34}$$

which is correct.

We now consider small displacements superposed on top of this initial strain. The total displacement vector $\overrightarrow{P_0 P'}$ can be written as

$$\underline{\mathbf{v}}(r, \theta, \phi, t) + \epsilon \underline{\mathbf{w}}(r, \theta, \phi, t) \tag{4.35}$$

where $\epsilon$ is a small constant whose squares and higher powers may be neglected in comparison with $\epsilon$.

The equations of motion for these extra small displacements are obtained in Green and Zerna's theory by making small perturbations of order $\epsilon$ in the other relevant quantities. For example, the covariant base vectors of the coordinate system $(r, \theta, \phi)$ at points $P'$ of the body $B'$ are given by

$$\underline{\mathbf{G}}_i + \epsilon \underline{\mathbf{G}}'_i = \frac{\partial \underline{\mathbf{r}}}{\partial \xi_i} + \frac{\partial \underline{\mathbf{v}}}{\partial \xi_i} + \epsilon \frac{\partial \underline{\mathbf{w}}}{\partial \xi_i}, \tag{4.36}$$

where $\xi_1 = r, \xi_2 = \theta$ and $\xi_3 = \phi$. We can express the displacement vector $\underline{\mathbf{w}}$ using the base vectors in the deformed state so that

$$\underline{\mathbf{w}} = w_m \underline{\mathbf{G}}^m = w^m \underline{\mathbf{G}}_m \tag{4.37}$$

where $w_m$ and $w^m$ are the components of $\underline{\mathbf{w}}$ referred to base vectors at points $P$ of the body $B$.

Therefore we have that

$$\underline{\mathbf{G}}'_i = w_m||_i \underline{\mathbf{G}}^m = w^m||_i \underline{\mathbf{G}}_m \tag{4.38}$$

Similarly, the contravariant base vectors become $\underline{\mathbf{G}}^i + \epsilon \underline{\mathbf{G}}'^i$ where

$$\underline{\mathbf{G}}'^i = G^{ij} \underline{\mathbf{G}}'_j + G'^{ij} \underline{\mathbf{G}}_j \tag{4.39}$$

Perturbing the metric tensors similarly so that the covariant and contravariant metric tensors of the body $B'$ are $G_{ij} + \epsilon G'_{ij}$ and $G^{ij} + \epsilon G'^{ij}$ respectively, we obtain, to first order in $\epsilon$

$$G'_{ij} = w_i||_j + w_j||_i \tag{4.40}$$

$$G'^{ij} = -G^{ir} G^{js} G'_{rs} \tag{4.41}$$

The determinant of the metric tensor components $G_{ij} + \epsilon G'_{ij}$ becomes $G + \epsilon G'$ where

$$G' = G G^{ij} G'_{ij} \tag{4.42}$$

The strain invariants $I_j$ become $I_j + \epsilon I_j{}'$ where

$$I'_1 = g^{rs} G'_{rs} \tag{4.43}$$

$$I'_2 = g_{rs}(G'^{rs} I_3 + G^{rs} I'_3) \tag{4.44}$$

$$I'_3 = G^{ij} G'_{ij} I_3 \tag{4.45}$$

and of course the stored energy function is then altered so that for the body $B'$,

$$W = W(I_1 + \epsilon I_1', I_2 + \epsilon I_2', I_3 + \epsilon I_3') \tag{4.46}$$

The functions $\Phi, \Psi$ and $p$ become $\Phi + \epsilon\Phi'$, $\Psi + \epsilon\Psi'$ and $p + \epsilon p'$ where, using Taylor expansion, to the first order in $\epsilon$, we have

$$\Phi' = AI_1' + FI_2' + EI_3' - \frac{\Phi}{2I_3}I_3' \tag{4.47}$$

$$\Psi' = FI_1' + BI_2' + DI_3' - \frac{\Psi}{2I_3}I_3' \tag{4.48}$$

$$p' = I_3(EI_1' + DI_2' + CI_3') + \frac{p}{2I_3}I_3' \tag{4.49}$$

where

$$A = \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial {I_1}^2}, \qquad B = \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial {I_2}^2}, \qquad C = \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial {I_3}^2}, \tag{4.50}$$

$$D = \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_2 \partial I_3}, \qquad E = \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_3 \partial I_1}, \qquad F = \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_1 \partial I_2}. \tag{4.51}$$

Finally the tensor $B^{ij}$ given in (4.25) becomes $B^{ij} + \epsilon B'^{ij}$ for the strained body $B'$ where

$$B'^{ij} = (g^{ij} - g^{ir}g^{js})G'_{rs} \tag{4.52}$$

Note that the contravariant stress tensor $\tau^{ij}$ becomes $\tau^{ij} + \epsilon\tau'^{ij}$ for the strained body $B'$ so that

$$\tau'^{ij} = g^{ij}\Phi' + B^{ij}\Psi' + B'^{ij}\Psi + G'^{ij}p + G^{ij}p' \tag{4.53}$$

Similarly, since

$$\underline{\mathbf{t}} = \frac{n_i}{\sqrt{G}}\underline{\mathbf{T}}_i \tag{4.54}$$

and because $\underline{\mathbf{T}}_i$ becomes $\underline{\mathbf{T}}_i + \epsilon\underline{\mathbf{T}}_i'$, we use the fact that $\underline{\mathbf{T}}_i = \sqrt{G}\tau^{ij}\underline{\mathbf{G}}_j$ to find that

$$\underline{\mathbf{T}}_i' = \sqrt{G}\lambda^{ij}\underline{\mathbf{G}}_j \tag{4.55}$$

where

$$\lambda^{ij} = \tau'^{ij} + \tau^{im}w^j||_m + \tau^{ij}w^m||_m \tag{4.56}$$

and with these perturbations from rest, the equations of equilibrium given by (4.33) are perturbed to give the equations of motion for the additional small displacements as

$$\lambda^{ij}||_i \;=\; \rho f'^{j} \tag{4.57}$$

where

$$f'^{1} = \frac{\partial^2 u_r}{\partial t^2}, \qquad f'^{2} = \frac{1}{r}\frac{\partial^2 u_\theta}{\partial t^2}, \qquad f'^{3} = \frac{1}{r\sin\theta}\frac{\partial^2 u_\phi}{\partial t^2} \tag{4.58}$$

Now we use the above theory in order to derive the equations of motion in terms of the stored energy function $W$.

Furthermore, using the additional, order $\epsilon$ terms of the strain invariants, given by equations (4.43)-(4.45), we can find the additional small stresses due to the incident sound wave. The physical components of the stress tensor $\tau'^{ij}$ are given by

$$\sigma'_{(ij)} \;=\; \sqrt{\Big(\frac{G'_{jj}}{G'^{ii}}\Big)}\,\tau'^{ij} \tag{4.59}$$

and therefore we can write

$$\sigma'_{(ij)} \;=\; c_{ijmn} d_{mn} \tag{4.60}$$

where $d_{mn}$ are the additional small strains given by Love in [15] as

$$d_{rr} \;=\; \frac{\partial U_r}{\partial r} \tag{4.61}$$

$$d_{\theta\theta} \;=\; \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} \tag{4.62}$$

$$d_{\phi\phi} \;=\; \frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{U_\theta}{r}\cot\theta + \frac{U_r}{r} \tag{4.63}$$

$$2d_{r\theta} = 2d_{\theta r} \;=\; \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta} \tag{4.64}$$

$$2d_{r\phi} = 2d_{\phi r} \;=\; \frac{1}{r\sin\theta}\frac{\partial U_r}{\partial \phi} + \frac{\partial U_\phi}{\partial r} - \frac{U_\phi}{r} \tag{4.65}$$

$$2d_{\theta\phi} = 2d_{\phi\theta} \;=\; \frac{1}{r}\frac{\partial U_\phi}{\partial \theta} - \frac{U_\phi}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi} \tag{4.66}$$

and the spherically symmetric coefficients $c_{ijmn}$ can be shown to be

$$c_{1111} \;=\; 2\lambda_r^2 a + \frac{4I_3}{\lambda_t^2} b + 2I_3 c - 2p \tag{4.67}$$

$$c_{1122} = c_{1133} = 2\lambda_t^2 a + 2I_3\Big(\frac{1}{\lambda_r^2} + \frac{1}{\lambda_t^2}\Big)b + 2I_3 c + 2\lambda_r^2\lambda_t^2\Psi \tag{4.68}$$

$$c_{2211} = c_{3311} = 2\lambda_r^2 d + \frac{4I_3}{\lambda_t^2}e + 2I_3 f + 2\lambda_r^2\lambda_t^2\Psi \tag{4.69}$$

$$c_{2222} = c_{3333} = 2\lambda_t^2 d + 2I_3\Big(\frac{1}{\lambda_r^2} + \frac{1}{\lambda_t^2}\Big)e + 2I_3 f - 2p \tag{4.70}$$

$$c_{2233} = c_{3322} = 2\lambda_t^2 d + 2I_3\Big(\frac{1}{\lambda_r^2} + \frac{1}{\lambda_t^2}\Big)e + 2I_3 f + 2\lambda_t^4\Psi \tag{4.71}$$

$$c_{2112} = c_{1212} = c_{3113} = c_{1313} = -2(\lambda_r^2\lambda_t^2\Psi + p) \tag{4.72}$$

$$c_{3223} = c_{2323} = -2(\lambda_t^4\Psi + p) \tag{4.73}$$

with all other coefficients being zero, where

$$a = A\lambda_r^2 + 2F\lambda_r^2\lambda_t^2 + EI_3 \tag{4.74}$$

$$b = F\lambda_r^2 + 2B\lambda_r^2\lambda_t^2 + DI_3 \tag{4.75}$$

$$c = \lambda_r^2\Big(E - \frac{\Phi}{2I_3}\Big) + \lambda_t^2\lambda_r^2\Big(2D - \frac{\Psi}{I_3}\Big) + CI_3 + \frac{p}{2I_3} \tag{4.76}$$

$$d = A\lambda_t^2 + F\lambda_t^2(\lambda_r^2 + \lambda_t^2) + EI_3 \tag{4.77}$$

$$e = F\lambda_t^2 + B\lambda_t^2(\lambda_r^2 + \lambda_t^2) + DI_3 \tag{4.78}$$

$$f = \lambda_t^2\Big(E - \frac{\Phi}{2I_3}\Big) + \lambda_t^2(\lambda_r^2 + \lambda_t^2)\Big(D - \frac{\Psi}{2I_3}\Big) + CI_3 + \frac{p}{2I_3} \tag{4.79}$$

and where $A, B, C, D, E$ and $F$ are defined in (4.50) and (4.51)

Applying the axisymmetry condition means that the small strains $d_{\theta\phi} = d_{\phi\theta} = d_{r\phi} = d_{\phi r} = 0$ and therefore also that the additional small shear stresses $\tau'^{23} = \tau'^{32} = \tau'^{13} = \tau'^{31} = 0$.

We also need the covariant derivatives of the components of displacement $w^i$. On applying the axisymmetry condition and using the relations

$$v^j\Big|\Big|_i = \frac{\partial v^j}{\partial \xi_i} + \Gamma^j_{ki} v^k \tag{4.80}$$

$$v_j\Big|\Big|_i = \frac{\partial v_j}{\partial \xi_i} - \Gamma^k_{ji} v_k \tag{4.81}$$

where $\xi_1 = r, \xi_2 = \theta$ and $\xi_3 = \phi$ and $\Gamma^i_{jk}$ are the Christoffel symbols of the second kind, these are found to be

$$w^1\Big|\Big|_1 = \frac{\partial u_r}{\partial r} = d_{rr} \tag{4.82}$$

$$w^1\Big|\Big|_2 = \frac{\partial u_r}{\partial \theta} - u_\theta \tag{4.83}$$

$$w^1\Big|\Big|_3 = 0 \tag{4.84}$$

$$w^2\Big|\Big|_1 = \frac{1}{r}\frac{\partial u_\theta}{\partial r} \tag{4.85}$$

$$w^2\Big|\Big|_2 = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = d_{\theta\theta} \tag{4.86}$$

$$w^2\Big|\Big|_3 = 0 \tag{4.87}$$

$$w^3\Big|\Big|_1 = 0 \tag{4.88}$$

$$w^3\Big|\Big|_2 = 0 \tag{4.89}$$

$$w^3\Big|\Big|_3 = \frac{u_r}{r} + \frac{\cot\theta}{r}u_\theta = d_{\phi\phi} \tag{4.90}$$

Finally, we require the covariant derivatives the tensor $\lambda^{ij}$ defined in (4.57). Again, on applying the axisymmetry condition and using the relation

$$A^{ij}\Big|\Big|_k = \frac{\partial A^{ij}}{\partial \xi_k} + \Gamma^i_{km}A^{mj} + \Gamma^j_{km}A^{im} \tag{4.91}$$

where $\xi_1 = r, \xi_2 = \theta$ and $\xi_3 = \phi$, these are

$$\lambda^{11}\Big|\Big|_1 = \frac{\partial \lambda^{11}}{\partial r} \tag{4.92}$$

$$\lambda^{21}\Big|\Big|_2 = \frac{\partial \lambda^{21}}{\partial \theta} + \frac{1}{r}\lambda^{11} - r\lambda^{22} \tag{4.93}$$

$$\lambda^{31}\Big|\Big|_3 = \frac{1}{r}\lambda^{11} + \cot\theta\lambda^{21} - r\sin^2\theta\lambda^{33} \tag{4.94}$$

$$\lambda^{12}\Big|\Big|_1 = \frac{\partial \lambda^{12}}{\partial r} + \frac{1}{r}\lambda^{12} \tag{4.95}$$

$$\lambda^{22}\Big|\Big|_2 = \frac{\partial \lambda^{22}}{\partial \theta} + \frac{1}{r}\lambda^{12} + \frac{1}{r}\lambda^{21} \tag{4.96}$$

$$\lambda^{32}\Big|\Big|_3 = \frac{1}{r}\lambda^{12} + \cot\theta\lambda^{22} - \sin\theta\cos\theta\lambda^{33} \tag{4.97}$$

$$\lambda^{13}\Big|\Big|_1 = 0 \tag{4.98}$$

$$\lambda^{23}\Big|\Big|_2 = 0 \tag{4.99}$$

$$\lambda^{33}\Big|\Big|_3 = 0 \tag{4.100}$$

and from this we see that the three equations of motion reduce to two under the axisymmetry condition, as they should do.

The equations of motion are therefore,

$$\lambda^{11}\Big|\Big|_1 + \lambda^{21}\Big|\Big|_2 + \lambda^{31}\Big|\Big|_3 = \rho f'^1 \tag{4.101}$$

$$\lambda^{12}\Big|\Big|_1 + \lambda^{22}\Big|\Big|_2 + \lambda^{32}\Big|\Big|_3 = \rho f'^2 \tag{4.102}$$

Calculating all the necessary parts to these equations leads us to the general equations of motion, general in the sense that any stored energy function can be used in them. They are,

$$A'\frac{\partial^2 u_r}{\partial r^2} + B'\frac{\partial^2 u_r}{\partial \theta^2} + C'\frac{\partial^2 u_\theta}{\partial r \partial \theta} + D'\frac{\partial u_r}{\partial r} + E'\frac{\partial u_r}{\partial \theta} + F'\frac{\partial u_\theta}{\partial r} + H'\frac{\partial u_\theta}{\partial \theta} + J'u_r + K'u_\theta = -\rho\omega^2 u_r \quad (4.103)$$

$$L'\frac{\partial^2 u_\theta}{\partial \theta^2} + M'\frac{\partial^2 u_\theta}{\partial r^2} + N'\frac{\partial^2 u_r}{\partial r \partial \theta} + P'\frac{\partial u_r}{\partial r} + Q'\frac{\partial u_r}{\partial \theta} + R'\frac{\partial u_\theta}{\partial r} + S'\frac{\partial u_\theta}{\partial \theta} + U'u_r + V'u_\theta = -\rho\omega^2 u_\theta \quad (4.104)$$

where, using the fact that the tensor $c_{ijmn}$ is spherically symmetric and that $\sigma_{(22)} = \sigma_{(33)}$ the coefficients reduce to

$$A' = c_{1111} + 2\sigma_{(11)} \quad (4.105)$$

$$B' = \frac{1}{r^2}\Big(\frac{c_{1212}}{2} + \sigma_{(22)}\Big) \quad (4.106)$$

$$C' = \frac{1}{r}\Big(c_{1122} + \frac{c_{1212}}{2} + \sigma_{(11)}\Big) \quad (4.107)$$

$$D' = \Big(\frac{\partial}{\partial r} + \frac{2}{r}\Big)(c_{1111} + 2\sigma_{(11)}) + \frac{2}{r}(c_{1122} - c_{2211} + \sigma_{(11)} - \sigma_{(22)}) \quad (4.108)$$

$$E' = \frac{1}{r^2}\Big(\frac{\partial}{\partial \theta} + \cot\theta\Big)\Big(\frac{c_{1212}}{2} + \sigma_{(22)}\Big) \quad (4.109)$$

$$F' = \Big(\frac{\partial}{\partial \theta} + \cot\theta\Big)\Big(\frac{c_{1212}}{2r}\Big) + \frac{\cot\theta}{r}(c_{1122} + \sigma_{(11)}) \quad (4.110)$$

$$H' = \frac{1}{r}\Big(\frac{\partial}{\partial r} + \frac{1}{r}\Big)(c_{1122} + \sigma_{(11)}) - \frac{1}{r^2}\Big(\frac{c_{1212}}{2} + c_{2233} + c_{2222} + 4\sigma_{(22)}\Big) \quad (4.111)$$

$$J' = \frac{2}{r}\Big(\frac{\partial}{\partial r} + \frac{1}{r}\Big)(c_{1122} + \sigma_{(11)}) - \frac{2}{r^2}\Big(c_{2222} + c_{2233} + 3\sigma_{(22)}\Big) \quad (4.112)$$

$$K' = \frac{\cot\theta}{r}\Big(\frac{\partial}{\partial r} + \frac{1}{r}\Big)(c_{1122} + \sigma_{(11)}) - \frac{1}{r^2}\Big(\frac{\partial}{\partial \theta} + \cot\theta\Big)\Big(\frac{c_{1212}}{2} + \sigma_{(22)}\Big) - \frac{\cot\theta}{r^2}\Big(c_{2233} + c_{3333} + 3\sigma_{(22)}\Big) \quad (4.113)$$

$$L' = \frac{1}{r^2}(c_{2222} + 2\sigma_{(22)}) \quad (4.114)$$

$$M' = \Big(\frac{c_{1212}}{2} + \sigma_{(11)}\Big) \quad (4.115)$$

$$N' = \frac{1}{r}\Big(c_{2211} + \frac{c_{1212}}{2} + \sigma_{(22)}\Big) \quad (4.116)$$

$$P' = \frac{1}{r}\frac{\partial}{\partial \theta}(c_{2211} + \sigma_{(22)}) \quad (4.117)$$

$$Q' = \frac{1}{r^2}(c_{2222} + c_{2233} + 4\sigma_{(22)}) + \frac{1}{2r}\Big(r\frac{\partial}{\partial r} + 4\Big)c_{1212} \quad (4.118)$$

$$R' = \Big(\frac{\partial}{\partial r}+\frac{2}{r}\Big)(c_{1212}+\sigma_{(11)}) \tag{4.119}$$

$$S' = \frac{1}{r^2}\Big(\frac{\partial}{\partial\theta}+\cot\theta\Big)(c_{2222}+2\sigma_{(22)}) \tag{4.120}$$

$$U' = \frac{1}{r^2}\frac{\partial}{\partial\theta}(c_{2222}+c_{2233}+3\sigma_{(22)}) \tag{4.121}$$

$$V' = -\frac{1}{r^2}(c_{2233}+2\sigma_{(22)})-\frac{\cot\theta}{r^2}(c_{2222}+2\sigma_{(22)})-\frac{1}{2r}\Big(\frac{\partial}{\partial r}+\frac{4}{r}\Big)c_{1212}. \tag{4.122}$$

Note that in these equations, we have $\rho=\rho(r,\theta,t)$ but we can use an approximation similar to the Boussinesq approximation in fluids by letting terms algebraic in $\rho$ be constant, but retaining derivatives in $\rho$ (i.e. compressive effects). We take the limit as $V\to 1$, keeping $\lambda$ very large in comparison to $\mu$, but finite so as to allow the sound wave to pass through the material with finite speed. If we took the rubber to be strictly incompressible with $V=1$, then the sound wave would pass through the material with infinite speed. Therefore, $\rho$ on the right hand side of the equations can be taken to be constant.

Under hydrostatic loading, the value of the Lamé moduli $\mu$ and $\lambda$ will become modified to say $\mu_m$ and $\lambda_m$. $\lambda$ will not be affected at its leading order since $\lambda/p=O(10^3)$, however, $\mu$ may be affected at *its* leading order since $\mu/p=O(10)$. Therefore, although the material at infinity will still be isotropic since it is under uniform strain, its Lamé moduli are not the same as that for an unstressed material.

Therefore, as $r, r_0\to\infty$, the equations of motion will tend to the Navier equations of motion for small disturbances from an equilibrium state, but $\mu=\mu_m$ and $\lambda=\lambda_m$ at infinity and hence these equations are

$$(\mu_m+\lambda_m)\nabla(\nabla.\underline{\mathbf{U}})+\mu_m\nabla^2\underline{\mathbf{U}} = \rho\frac{\partial^2\underline{\mathbf{U}}}{\partial t^2} \tag{4.123}$$

which in spherical polar coordinates with axisymmetry about the $z$ axis and considering only time harmonic waves, reduce to the two equations

$$(2\mu_m+\lambda_m)\frac{\partial^2 u_r}{\partial r^2}+\frac{\mu_m}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{(\mu_m+\lambda_m)}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}+\frac{2\mu_m}{r}\frac{\partial u_r}{\partial r}+\frac{\mu_m\cot\theta}{r^2}\frac{\partial u_r}{\partial\theta}$$
$$-\frac{(3\mu_m+\lambda_m)}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\mu_m}{r^2}u_r-\frac{2\mu_m\cot\theta}{r^2}u_\theta=-\rho\omega^2u_r, \tag{4.124}$$

$$\mu_m\frac{\partial^2 u_\theta}{\partial r^2}+(2\mu_m+\lambda_m)\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{(\mu_m+\lambda_m)}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{2\mu_m}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{2\mu_m}{r}\frac{\partial u_\theta}{\partial r}$$
$$+\frac{\mu_m\cot\theta}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\mu_m\csc^2\theta}{r^2}u_\theta=-\rho\omega^2u_\theta. \tag{4.125}$$

### 4.2.2 Form of Incident Wave

The function $\underline{\mathbf{u}}$ which satisfies the equations of motion (4.103) and (4.104) will consist of the incident compressive wave $\underline{\mathbf{u}}_i$ and the scattered field $\underline{\mathbf{u}}_s$, i.e.

$$\underline{\mathbf{u}} = \underline{\mathbf{u}}_i + \underline{\mathbf{u}}_s. \tag{4.126}$$

The incident, time harmonic, compressive wave propagates parallel to the $z$ axis and is incident from infinity. Therefore it will initially satisfy the Navier equations (4.124) and (4.125), being perturbed by $O(\epsilon)$ as in (4.35) by the strained substrate as it approaches the micro-sphere.

Therefore, the incident wave takes the form (see for example, Jones [12] or Ying and Truell [21])

$$\underline{\mathbf{U}}_i = \underline{\mathbf{u}}_i e^{i\omega t} \tag{4.127}$$

where

$$\underline{\mathbf{u}}_i = \hat{\underline{\mathbf{z}}} e^{-ik_p z} \tag{4.128}$$

$$= (\hat{\underline{\mathbf{r}}} \cos\theta - \hat{\underline{\theta}} \sin\theta) e^{-ik_p r\cos\theta} \tag{4.129}$$

where $\hat{\underline{\mathbf{z}}}, \hat{\underline{\mathbf{r}}}$ and $\hat{\underline{\theta}}$ are unit vectors in the $z, r$ and $\theta$ directions and $k_p$ is the wavenumber of the incoming compressive wave. The incident wave can also be written in spherical coordinates in terms of a Helmholtz potential $\Psi_i$ by writing

$$\underline{\mathbf{u}}_i = -\nabla\Psi_i \tag{4.130}$$

where

$$\Psi_i = \frac{1}{k_p}\sum_{m=0}^{\infty}(-i)^{m+1}(2m+1)j_m(k_p r)P_m(\cos\theta). \tag{4.131}$$

Here $j_m$ denotes spherical Bessel functions of the first kind of order $m$ and $P_m$ denotes Legendre polynomials of degree $m$. Furthermore, for the compressive and shear wavenumbers respectively, we have the relations

$$k_p = \frac{\omega}{c_p} \tag{4.132}$$

$$k_s = \frac{\omega}{c_s} \tag{4.133}$$

where $c_p$ and $c_s$ are the compressive and shear wave speeds respectively. In an isotropic material, these are given by

$$c_p = \sqrt{\left(\frac{2\mu+\lambda}{\rho}\right)} \tag{4.134}$$

$$c_s = \sqrt{\left(\frac{\mu}{\rho}\right)}. \tag{4.135}$$

For the isotropic material at infinity in our case, $\mu = \mu_m$ and $\lambda = \lambda_m$.

### 4.2.3 Boundary Conditions

The micro-sphere imposes conditions on stress and displacement in the rubber substrate at $r = a$. The boundary conditions can be stated in various forms, depending upon how we treat the interior of the micro-sphere.

#### 4.2.3.1 Spherical Cavity (Vacuum)

We could simplify the problem and treat the interior of the sphere as a vacuum as in Ying and Truell [21]. Clearly then we have a known incident compressive wave represented by the Helmholtz potential $\Psi_i$ given by (4.131). Mode conversion at $r = a$ means that both compressive and shear waves are scattered from the micro-sphere. Therefore we have two types of unknown scattered waves. If we were using Helmholtz potentials we could represent these by $\Psi_s$ and $\Pi_s$ but since we are not using this approach we must simply consider the scattered field $\underline{\mathbf{u}}_s$. Since we have a vacuum then no waves will propagate inside the micro-sphere.

Therefore the required boundary conditions are zero stress conditions on the surface of the micro-sphere at $r = a$, i.e.

$$\sigma'_{(rr)_i} + \sigma'_{(rr)_s} = 0, \tag{4.136}$$

$$\sigma'_{(r\theta)_i} + \sigma'_{(r\theta)_s} = 0. \tag{4.137}$$

Because we consider a vacuum inside the sphere then there are no restrictions on the displacement at $r = a$.

#### 4.2.3.2 Gas Filled Sphere

A more correct formulation, but more complicated, is the problem considered by Einspruch and Truell in [5] of a gas filled sphere. Again we have the known incident compressive wave represented by the Helmholtz potential $\Psi_i$ in (4.131) and unknown compressive and shear scattered waves which would be represented by the Helmholtz potentials $\Psi_s$ and $\Pi_s$ if this approach were being followed. In this case compressive waves represented by a Helmholtz potential $\Psi_g$ will also propagate inside the sphere. Shear waves cannot be supported in a fluid and so $\Pi_g = 0$. Therefore the appropriate boundary conditions are continuity of stress for both mediums *and* continuity of radial displacement in the solid with radial displacement in the fluid.

Representing the displacement field in the gas by $\underline{\mathbf{U}}_g = \underline{\mathbf{u}}_g e^{i\omega t}$, where $\underline{\mathbf{u}}_g = (u_{rg}, u_{\theta g})$ and since the waves are time harmonic, we have that the velocity is given by

$$\underline{\mathbf{V}}_g = \underline{\mathbf{v}}_g e^{i\omega t} = \frac{\partial}{\partial t}(\underline{\mathbf{u}}_g e^{i\omega t}) \tag{4.138}$$

so that the boundary conditions at $r = a$ are,

$$\sigma'_{(rr)_i} + \sigma'_{(rr)_s} = \sigma'_{(rr)_g}, \tag{4.139}$$

$$\sigma'_{(r\theta)_i} + \sigma'_{(r\theta)_s} = 0, \tag{4.140}$$

$$u_{ri} + u_{rs} = u_{rg}. \tag{4.141}$$

The velocity in the fluid can be stated in terms of a velocity potential $\Psi_g$ where $\underline{\mathbf{v}}_g = -\nabla\Psi_g$.

Assuming the waves are time harmonic, this velocity potential is governed by Helmholtz equation,

$$(\nabla^2 + \kappa^2)\Psi_g = 0 \tag{4.142}$$

where $\kappa$ is the wavenumber in the gas. Waves inside the sphere are not of direct interest to us here but we need the solution to (4.142) in order to calculate the scattered waves. The solution to (4.142) with axisymmetry about the $z$ axis is well known as

$$\Psi_g = \sum_{m=0}^{\infty} A_m j_m(\kappa r) P_m(\cos\theta) \tag{4.143}$$

where $j_m$ is the spherical Bessel function of the first kind of order $m$, $P_m$ is the Legendre polynomial of degree $m$ and $A_m$ are constants to be determined by the boundary conditions. Therefore we find that the velocity field in the gas is given by

$$\underline{\mathbf{V}}_g = i\omega \underline{\mathbf{u}}_g e^{i\omega t} \tag{4.144}$$

where

$$u_{rg} = -\frac{\partial \Psi_g}{\partial r}, \tag{4.145}$$

$$u_{\theta g} = -\frac{1}{r}\frac{\partial \Psi_g}{\partial \theta}. \tag{4.146}$$

#### 4.2.3.3 Summary

Note that two boundary conditions are required for a spherical cavity, three for a gas-filled sphere and if we had an elastic sphere then four boundary conditions would be needed for then shear waves could also propagate inside the sphere. The extra boundary condition in this case would be continuity of the $\theta$ component of displacement $u_\theta$ on $r = a$.

### 4.2.4 Radiation Conditions

Radiation conditions are necessary to guarantee the uniqueness of the solution. They insist that the scattered displacement field must behave like an outgoing wave at infinity (as $r, r_0 \to \infty$). In the case of Helmholtz equation, the radiation condition merely decides the choice of Bessel function required for the scattered field to be outgoing at infinity.

If $P = pe^{i\omega t}$ is the pressure in the material, then at infinity $p$ will satisfy Helmholtz equation, i.e.

$$\nabla^2 p + k^2 p = 0 \tag{4.147}$$

and the *Sommerfeld radiation conditions* insist (see Jones [12]) that for some finite constant $K$,

$$|rp| < K \tag{4.148}$$

$$r\Big(\frac{\partial p}{\partial r} + ikp\Big) \to 0 \tag{4.149}$$

uniformly with respect to direction as $r \to \infty$.

We need radiation conditions on the scattered displacement $\underline{\mathbf{u}}_s$. Since at infinity $\underline{\mathbf{u}}_s$ satisfies the Navier equations given by (4.124) and (4.125), we know that $\nabla.\underline{\mathbf{u}}_s$ satisfies a scalar Helmholtz equation at infinity with $k = k_p$. Furthermore, $\nabla \times \underline{\mathbf{u}}_s$ will satisfy a vector Helmholtz equation at infinity with $k = k_s$.

Therefore, the radiation conditions on the scattered field are that

$$|ru_{rs}| < K_1 \tag{4.150}$$

$$|ru_{\theta s}| < K_2 \tag{4.151}$$

$$r\Big(\frac{\partial \nabla.\underline{\mathbf{u}}_s}{\partial r} + ik_p \nabla.\underline{\mathbf{u}}_s\Big) \;\; \to \;\; 0 \tag{4.152}$$

$$r\Big(\frac{\partial \nabla \times \underline{\mathbf{u}}_s}{\partial r} + ik_s \nabla \times \underline{\mathbf{u}}_s\Big) \;\; \to \;\; 0 \tag{4.153}$$

as $r, r_0 \to \infty$.

## 4.3 Choice of Stored Energy Function $W$

We have considered several forms of stored energy function in previous chapters. In particular in chapter 3 we showed how the choice of $W$ significantly affected the radius of the compressed micro-sphere in equilibrium. Recall from chapter 2 that the general form of $W$ is,

$$W \;\; = \;\; \sum_m C_m W_m + \frac{1}{2}\lambda(V-1)^2 F(V) \tag{4.154}$$

where

$$W_m \;\; = \;\; \frac{\mu}{2m^2}\Big(\sum_{i=1}^{3} \lambda_i^{2m} - 3 - 2m \log V\Big) \qquad\qquad m \neq 0 \tag{4.155}$$

$$W_0 \;\; = \;\; \mu \sum_{i=1}^{3} \log^2 \lambda_i. \tag{4.156}$$

Thinking again in terms of the general problem and then taking our situation as a special case, we need to calculate $s_m = \lambda_1^{2m} + \lambda_2^{2m} + \lambda_3^{2m}$ in terms of the strain invariants $I_1, I_2$ and $I_3$ where $\lambda_i^2$ are the roots of the characteristic polynomial

$$|xI - A^T A| \;\; = \;\; x^3 - I_1 x^2 + I_2 x - I_3 = 0. \tag{4.157}$$

We can use Newton's equations to deduce $s_m$ and therefore express $W$ in terms of the strain invariants $I_1, I_2$ and $I_3$. Newton's equations are,

$$\begin{aligned} s_1 - I_1 &= 0 && (4.158)\\ s_2 - I_1 s_1 + 2I_2 &= 0 && (4.159)\\ s_3 - I_1 s_2 + I_2 s_1 - 3I_3 &= 0 && (4.160)\\ s_k - I_1 s_{k-1} + I_2 s_{k-2} - I_3 s_{k-3} &= 0 \qquad \forall k \geq 3. && (4.161) \end{aligned}$$

and we also identify that

$$s_{-1} = \frac{I_2}{I_3}. \tag{4.162}$$

Consider the classical form of stored energy function $W$, using equation (4.154) with $m = 1$ and $m = -1$ only, which is

$$\begin{aligned} W &= \frac{\mu}{2}\Big(C_1(s_1 - 3) + C_{-1}(s_{-1} - 3) + 2(C_{-1} - C_1)\log V\Big) \\ &\quad + \frac{1}{2}\lambda(V-1)^2 F(V) \end{aligned} \tag{4.163}$$

which on using (4.158) and (4.162) and the fact that $I_3 = V^2$ becomes

$$\begin{aligned} W &= \frac{\mu}{2}\Big(C_1(I_1 - 3) + C_{-1}\Big(\frac{I_2}{V^2} - 3\Big) + 2(C_{-1} - C_1)\log V\Big) \\ &\quad + \frac{1}{2}\lambda(V-1)^2 F(V). \end{aligned} \tag{4.164}$$

Note also that the condition on the coefficients $C_1$ and $C_{-1}$ is

$$C_1 + C_{-1} = 1. \tag{4.165}$$

We can now use this stored energy function to derive the coefficients $c_{ijmn}$ relating the physical components of stress to strain in (4.60). We can then check that these tend to the correct values as $r, r_0 \to \infty$. We also check that the physical components of stress due to the initial strain derived by this theory correspond to the physical stresses given by (3.34) in chapter 3.

Using equations (4.29)-(4.32) we obtain the physical components of the stress tensor as

$$\begin{aligned} \sigma_{(11)} &= \mu C_1(\lambda_r^2 - 1) - \mu C_{-1}(\lambda_r^{-2} - 1) + \tilde{\sigma}, && (4.166)\\ \sigma_{(22)} = \sigma_{(33)} &= \mu C_1(\lambda_t^2 - 1) - \mu C_{-1}(\lambda_t^{-2} - 1) + \tilde{\sigma}, && (4.167)\\ \sigma_{(ij)} &= 0 \qquad \text{if } i \neq j && (4.168) \end{aligned}$$

where, as in chapter 3

$$\tilde{\sigma} = \lim_{\lambda\to\infty, V\to 1} \lambda(V-1). \tag{4.169}$$

Note that these stresses correspond to those calculated in (3.34) in chapter 3, denoted by $\sigma_r$ and $\sigma_t$.

The values of $A, B, C, D, E$ and $F$ expressed in (4.50) and (4.51) in the limit $\lambda \to \infty$ and $V \to 1$ are

$$\begin{aligned} A &= B = E = F = 0 && (4.170)\\ C &= 2\mu C_{-1}\Big(\frac{1}{\lambda_r^2} + \frac{2}{\lambda_t^2}\Big) - \mu(C_{-1} - C_1) - \frac{\tilde{\sigma}}{2} + \frac{\tilde{\tilde{\sigma}}}{2} + \frac{\lambda}{2} && (4.171)\\ D &= -C_{-1}\mu && (4.172) \end{aligned}$$

where

$$\tilde{\tilde{\sigma}} = 2\tilde{\sigma}F'(1). \tag{4.173}$$

Also, $\Phi$, $\Psi$ and $p$, given by (4.24) in the limit $\lambda \to \infty$ and $V \to 1$ are

$$\begin{aligned} \Phi &= C_1\mu, && (4.174)\\ \Psi &= C_{-1}\mu, && (4.175)\\ p &= \mu C_{-1}\Big(\frac{1}{\lambda_r^2} + \frac{2}{\lambda_t^2}\Big) + \mu(C_{-1} - C_1) + \tilde{\sigma}. && (4.176) \end{aligned}$$

These then mean we can calculate $a, b, c, d, e$ and $f$ from (4.74)-(4.79) which are,

$$\begin{aligned} a &= d = 0 && (4.177)\\ b &= e = -C_{-1}\mu && (4.178)\\ c &= \frac{C_1\mu}{2}(1-\lambda_r^2) + \frac{C_{-1}\mu}{2}(3\lambda_r^{-2} - 1) + \frac{\lambda}{2} + \frac{\tilde{\tilde{\sigma}}}{2} && (4.179)\\ f &= \frac{C_1\mu}{2}(1-\lambda_t^2) + \frac{C_{-1}\mu}{2}(3\lambda_t^{-2} - 1) + \frac{\lambda}{2} + \frac{\tilde{\tilde{\sigma}}}{2} && (4.180) \end{aligned}$$

and these then mean that we can evaluate the coefficients $c_{ijmn}$ given in equations (4.67)-(4.73) which are,

$$\begin{aligned} c_{1111} &= C_1\mu(3-\lambda_r^2) + C_{-1}\mu(5\lambda_r^{-2} - 3) + \lambda + \tilde{\tilde{\sigma}} - 2\tilde{\sigma} && (4.181)\\ c_{1122} &= c_{1133} = C_1\mu(1-\lambda_r^2) + C_{-1}\mu(\lambda_r^{-2} - 1) + \lambda + \tilde{\tilde{\sigma}} && (4.182) \end{aligned}$$

$$c_{2211} = c_{3311} = c_{2233} = c_{3322} = C_1\mu(1-\lambda_t^2) + C_{-1}\mu(\lambda_t^{-2}-1) + \lambda + \tilde{\tilde{\sigma}} \quad (4.183)$$

$$c_{2222} = c_{3333} = C_1\mu(3-\lambda_t^2) + C_{-1}\mu(5\lambda_t^{-2}-3) + \lambda + \tilde{\tilde{\sigma}} - 2\tilde{\sigma} \quad (4.184)$$

$$c_{2112} = c_{1212} = 2\mu(\lambda_r^{-2}+\lambda_t^{-2}) - 2\mu(C_{-1}-C_1) - 2\tilde{\sigma} \quad (4.185)$$

$$c_{3223} = c_{2323} = 4\mu C_{-1}\lambda_t^{-2} - 2\mu(C_{-1}-C_1) - 2\tilde{\sigma} \quad (4.186)$$

and notice that as $r, r_0 \to \infty$ using the condition (4.165) on the coefficients $C_1$ and $C_{-1}$ , we find that

$$c_{1111} = c_{2222} = c_{3333} \rightarrow 2\mu + \lambda + \tilde{\tilde{\sigma}} - 2\tilde{\sigma} \quad (4.187)$$

$$c_{2211} = c_{3311} = c_{2233} = c_{3322} \rightarrow \lambda + \tilde{\tilde{\sigma}} \quad (4.188)$$

$$c_{2112} = c_{1212} = c_{3223} = c_{2323} \rightarrow 2\mu - 2\tilde{\sigma} \quad (4.189)$$

so that they tend to the isotropic values at infinity where the terms $\tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$ are the modifications of $O(p)$ to the Lamé moduli due to the initial strain.

On substituting the values of $c_{ijmn}$ and the stresses calculated in (4.166) and (4.167) into the formulae for the coefficients $A' \to V'$ in (4.105) to (4.122) of the equations of motion, it can be shown that as $r, r_0 \to \infty$, they tend to the correct values to leading order, of the coefficients of the Navier Equations in (4.124) and (4.125). That is, terms of $O(\lambda)$ are correct, but modifications to $\lambda$ and $\mu$ at $O(p)$ due to the initial strain are encompassed in the terms $\tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$ which, without further experimental work, cannot be calculated. For example, $A'$ is given by

$$A' = C_1\mu(\lambda_r^2+1) + C_{-1}(3\lambda_r^{-2}-1) + \lambda + \tilde{\tilde{\sigma}} \quad (4.190)$$

and therefore as $r, r_0 \to \infty$,

$$A' \rightarrow \lambda + 2\mu + \tilde{\tilde{\sigma}} \quad (4.191)$$

$$= \lambda_m + 2\mu_m \quad (4.192)$$

which is the correct value of the coefficient of $\partial^2 u_r/\partial r^2$ in (4.124) with the modification to $\lambda$ and $\mu$ of $O(p)$ being $\tilde{\tilde{\sigma}}$.

## 4.4 Summary

We therefore have a set of equations, correct to leading order. We could solve these leading order equations numerically in order to find the leading order part of the scattered field $\underline{\mathbf{u}}_s$ and this would give us a good idea of the scattered compressive waves.

That is, that part of the scattered field which corresponds to monopole scattering in the TMSL report [4]. We indicate how we could do this using a spectral method in section 4.6. However, in order to solve the low frequency scattering problem at leading order, we use a different method. This is investigated in Chapter 5.

## 4.5 Typical Values of Parameters and the Modified Bulk Modulus due to the Initial Strain

Typical values of parameters in this problem are

$$\begin{aligned} K &= 4\times 10^9 \text{Pa} && (4.193)\\ \mu &= 2\times 10^6 \text{Pa} && (4.194)\\ \lambda &\approx K = 2\times 10^3 \mu && (4.195)\\ \rho &= 1000\text{kg/m}^3 && (4.196)\\ \omega &= 20 \rightarrow 2\times 10^4 \text{Hz} && (4.197)\\ p_{atm} &= 10^5 \text{Pa} && (4.198)\\ c_p &= 2000\text{m/s} && (4.199)\\ c_s &= 45\text{m/s} && (4.200) \end{aligned}$$

where $K$ is the bulk modulus of rubber and these material properties are for an unstressed isotropic material.

As previously explained, under hydrostatic pressure, $\mu$ and $\lambda$ will be modified to $\mu_m$ and $\lambda_m$ and therefore the bulk modulus $K$ will be modified.

Consider a purely volumetric expansion of a rubber-like material, so that $\lambda_1 = \lambda_2 = \lambda_3$ and so $V = \lambda_i^3$. Therefore, the stored energy function given in (2.12) can be written as

$$W = \frac{3\mu}{2}\sum_{m\neq 0}\frac{C_m}{m^2}\phi(V^{2m/3}) + 3C_0\mu\log^2 V^{1/3} + \frac{\lambda}{2}(V-1)^2F(V) \qquad (4.201)$$

where $\phi(x) = x - 1 - \log x$ and therefore, $\phi(1) = 0, \phi'(1) = 0$ and $\phi''(1) = 1$.

Under hydrostatic loading, we have

$$-p = \sigma_{(ii)} = \frac{\partial W}{\partial V} = \mu\sum_{m\neq 0}\frac{C_m}{m}\phi'(V^{2m/3})V^{2m/3-1} + \frac{2C_0\mu}{3V}\log V$$

$$+\lambda(V-1)F(V)+\frac{\lambda}{2}(V-1)^2F'(V) \qquad (4.202)$$

where $(ii)=(11),(22)$ or $(33)$. Now, we have

$$\lambda = \frac{\mu}{\epsilon} \qquad (4.203)$$

$$p = \bar{p}\mu \qquad (4.204)$$

where $\epsilon = O(10^{-3})$ and $\bar{p} = O(10)$ and then we have

$$\begin{aligned} -\bar{p} &= \sum_{m\neq 0}\phi'(V^{2m/3})V^{2m/3-1}+\frac{2C_0\log V}{3V} \\ &\quad +\frac{(V-1)F(V)}{\epsilon}+\frac{(V-1)^2F'(V)}{2\epsilon} \end{aligned} \qquad (4.205)$$

Now, since the volumetric expansion is very small, we can express it in a power series in $\epsilon$, i.e.

$$V = 1+v_1\epsilon+O(\epsilon^2) \qquad (4.206)$$

and substituting this into (4.205), the leading order term implies that

$$v_1 = -\bar{p} \qquad (4.207)$$

and therefore

$$V = 1-\epsilon\bar{p}+O(\epsilon^2) \qquad (4.208)$$

Then, the modified bulk modulus due to the initial strain will be

$$\begin{aligned} K_m &= \frac{d\sigma_{(ii)}}{dV/V} = V\frac{d^2W}{dV^2} \\ &= \sum_{m\neq 0}C_m\Big(\frac{2\mu}{3}\phi''(V^{2m/3})(V^{2m/3-1})^2V+\frac{\mu}{m}\phi'(V^{2m/3})(\frac{2m}{3}-1)V^{2m/3-2}V\Big) \\ &\quad +\frac{2\mu C_0(1-\log V)}{3V}+\lambda VF(V) \\ &\quad +2\lambda V(V-1)F'(V)+\frac{\lambda}{2}V(V-1)^2F''(V) \end{aligned} \qquad (4.209)$$

and therefore on using (4.208), we find that, correct to $O(\epsilon)$, we have

$$K_m = \frac{2\mu}{3}\Big(\sum_m C_m\Big)+\lambda-p(1+3F'(1)) \qquad (4.210)$$

and from the discussion following (2.12), we have that $\sum_m C_m = 1$ and therefore,

$$K_m = \frac{2\mu}{3} + \lambda - p(1 + 3F'(1)) \tag{4.211}$$
$$= K - p(1 + 3F'(1)) \tag{4.212}$$

where $K$ is the bulk modulus for an unstressed isotropic material.

This indicates what we said earlier regarding modifications to the Lamé moduli $\mu$ and $\lambda$. In order to find correct modifications to these, further experimental work would be required in order to find values for $F'(1)$.

## 4.6 Numerical Solution Using a Spectral Method

Since we do not pursue the issue of solving the resulting equations of motion, we simply indicate here how a numerical solution could be possible.

We have an axisymmetric coordinate system with domain $r \geq a$, $0 \leq \theta \leq \pi$, and therefore the geometry is simple. Hence a spectral method is particularly suited to solving the equations of motion.

We can nondimensionalise the equations of motion by letting $s = r/r_0$ and then the infinite domain outside the micro-sphere is reduced to the half annulus $a/a_0 \leq s < 1$, $0 \leq \theta \leq \pi$ where $s \to 1$ corresponds to $r, r_0 \to \infty$.

In order to discretise the domain, we could use a Chebyshev grid for the discretisation of the radial coordinate $s$ and a periodic grid for the discretisation of the angular coordinate $\theta$. Of course the periodic grid would be defined on $0 \leq \theta \leq 2\pi$ and therefore the functions $u_r$ and $u_\theta$ would have to be extended appropriately onto the annulus $a/a_0 \leq s < 1$, $0 \leq \theta \leq 2\pi$ in order for this method to work.

The Chebyshev grid would be set up on a domain with grid points on the boundary at $s = a/a_0$ and $s = 1$. This would allow us to specify the boundary conditions on the micro-sphere and the radiation conditions at infinity. Therefore the Chebyshev differentiation matrix is of size $(N_r + 2)\times(N_r + 2)$ where $N_r$ is an even integer.

Exploiting the symmetry condition we can reduce this to a matrix of size $(N_r+2)/2 \times N_r + 2$.

The standard periodic differentiation matrices would be used, say of size $N_\theta$, (see for example Trefethen [19]).

Then we would have a system of two equations for $u_r$ and $u_\theta$, say

$$L_1 u_r + M_1 u_\theta = 0 \tag{4.213}$$

$$L_2 u_\theta + M_2 u_\theta = 0 \tag{4.214}$$

where $L_1, L_2, M_1$ and $M_2$ are linear operators and since $\underline{\mathbf{u}} = \underline{\mathbf{u}}_i + \underline{\mathbf{u}}_s$, where $\underline{\mathbf{u}}_i$ is known, this gives us a non-zero right hand side, so that we have

$$L_1 u_{rs} + M_1 u_{\theta s} = f(r, r_0, \theta) \tag{4.215}$$

$$L_2 u_{rs} + M_2 u_{\theta s} = g(r, r_0, \theta). \tag{4.216}$$

with each equation being satisfied at the $((N_r + 2)/2 - 2) \times N_\theta$ internal points of the domain. The radiation condition at $s = 1$ will take the form,

$$S_1 u_{rs} + T_1 u_{\theta s} = 0 \tag{4.217}$$

$$S_2 u_{rs} + T_2 u_{\theta s} = 0 \tag{4.218}$$

and boundary conditions on the micro-sphere at $s = a/a_0$ will be,

$$P_1 u_{rs} + Q_1 u_{\theta s} = h(a, a_0, \theta) \tag{4.219}$$

$$P_2 u_{rs} + Q_2 u_{\theta s} = j(a, a_0, \theta) \tag{4.220}$$

where $h$ and $j$ are known functions. Therefore, on letting

$$\mathbf{A} = \begin{pmatrix} S_1 & T_1 \\ S_2 & T_2 \\ L_1 & M_1 \\ L_2 & M_2 \\ P_1 & Q_1 \\ P_2 & Q_2 \end{pmatrix} \tag{4.221}$$

$$\underline{\mathbf{u}}_s = \begin{pmatrix} u_{rs} \\ u_{\theta s} \end{pmatrix} \tag{4.222}$$

and

$$\underline{\mathbf{b}} = \begin{pmatrix} 0 \\ 0 \\ f(r, r_0, \theta) \\ g(r, r_0, \theta) \\ h(a, a_0, \theta) \\ j(a, a_0, \theta) \end{pmatrix} \tag{4.223}$$

we can write the system as

$$\mathbf{A}\underline{\mathbf{u}}_s = \underline{\mathbf{b}} \tag{4.224}$$

and then we could solve for the scattered displacement $\underline{\mathbf{u}}_s$.

# Chapter 5

# Low Frequency Scattering

## 5.1 The Scattering Problem

In order to see the effect that the strained region has on scattering from the micro-sphere we shall compare the results for scattering from a spherical cavity in a *strained* region with those for scattering from a spherical cavity in an *isotropic* material using the results obtained by Ying and Truell in [21]. To do this we shall analyse the scattering cross section of the obstacle in the material as discussed in chapter 2. This is defined as

$$(\gamma)_{\text{obstacle}} = \frac{\text{Total energy scattered per unit time}}{\text{Total energy per unit area, carried by the incident wave per unit time}}$$

where the amount of energy scattered is measured across a spherical surface concentric with and of radius larger than the obstacle. The energy carried by the incident wave is measured across an area which is perpendicular to the direction of propagation.

We shall calculate the leading order term of the scattering cross section of the material (i.e. the monopole scattering cross section) by letting $\lambda/\mu \to \infty$. In order to do this, consider two regions,

1. An inner region around the micro-sphere. This region includes all the stress induced anisotropy, but is still small compared with the wavelength $\Lambda$ of the incident wave.

2. An outer region in which the material is effectively isotropic.

We could for instance regard the boundary between the regions as being at $r = \sqrt{(a_0\Lambda)}$ so that for $\Lambda \gg a_0$, both requirements hold.

Now, consider an incident, time harmonic compressive wave represented by

$$p = p_s + p_1 \cos(\omega t - kz) \tag{5.1}$$

where $p_s$ is the steady static pressure and $p_1$ is the acoustic amplitude. Then for low frequencies, the inner region will be subjected to a slowly varying static pressure and since the inner region is small compared with $\Lambda$, it responds quasistatically, i.e. the radius of the micro-sphere alters according to

$$a = a_s + a_1 \cos(\omega t) \tag{5.2}$$

where $a_s$ is the compressed radius of the micro-sphere under static pressure $p_s$. For an isotropic material, since there is no initial strain, $a_s = a_0$. Furthermore,

$$a_1 \;=\; \Big(\frac{da}{dp}\Big|_{p_s}\Big) p_1 \tag{5.3}$$

and it is small so that $O(a_1^2)$ can be neglected in comparison with $a_1$. $da/dp$ is calculated quasistatically using the analysis of chapter 3.

The inner region therefore has a volume fluctuation of

$$V(t) \;=\; 4\pi a_s^2 a_1 \cos(\omega t). \tag{5.4}$$

Since the outer region is essentially isotropic, it responds like an acoustic medium to this volume change and therefore the radiation of energy is (see Lighthill [14])

$$E_R \;=\; \frac{\rho(\ddot{V})^2}{4\pi c_p}. \tag{5.5}$$

To obtain the scattered energy we take the time average of $E_R$ over a time interval $[0, 2\pi/\omega]$. Furthermore, on using (5.3), we find that

$$E_S \;=\; \langle E_R \rangle = \frac{2\pi\rho a_s^4 \omega^4 p_1^2}{c_p} \Big(\frac{da}{dp}\Big|_{p_s}\Big)^2. \tag{5.6}$$

Now the energy flux density in the incident field is given by

$$E_I \;=\; \frac{p_1^2}{2\rho c_p}. \tag{5.7}$$

Therefore on taking the limit $\lambda \gg \mu$ (i.e. the leading order limit), an approximation of the monopole scattering coefficient for low frequency scattering from a sphere in an elastic medium is

$$(\gamma_m)_{\text{sphere}} \sim 4\pi\rho^2 a_s^4 \omega^4 \Big( \frac{da}{dp}\Big|_{p_s} \Big)^2. \tag{5.8}$$

Consequently, the difference between the isotropic and anisotropic cases will enter into the calculation in the values of $a_s$ and $da/dp|_{p_s}$.

### 5.1.1 Isotropic material - results of Ying and Truell

For an initially unstressed isotropic material, we have that

$$\frac{da}{dp} = -\frac{a_0}{4\mu}. \tag{5.9}$$

Therefore (5.8) tells us that the monopole scatter cross section for scattering from a spherical cavity in an isotropic material is

$$(\gamma_m)_{\text{spherical cavity}} \sim \frac{\pi a_0^6 \omega^4 \rho^2}{4\mu^2}. \tag{5.10}$$

In order to confirm this result, consider the full scatter cross section for a spherical cavity in an isotropic material in a low frequency limit, obtained by Ying and Truell in [21] as

$$(\gamma)_{\text{spherical cavity}} = \frac{4\pi}{9} g_c \frac{1}{k_p^2} (k_p a_0)^6 \tag{5.11}$$

where on letting $q = k_s/k_p$,

$$g_c = \frac{4}{3} + 40\Big( \frac{2+3q^5}{(4-9q^2)^2} \Big) - \frac{3}{2}q^2 + \frac{2}{3}q^3 + \frac{9}{16}q^4 \tag{5.12}$$

and note that

$$q = \frac{k_s}{k_p} = \sqrt{\Big( \frac{\lambda}{\mu} + 2 \Big)}. \tag{5.13}$$

We check that the result we obtained in (5.10) corresponds to the result obtained by Ying and Truell when $\lambda/\mu \to \infty$ in (5.11). In this limit,

$$q \sim \sqrt{\Big( \frac{\lambda}{\mu} \Big)}, \tag{5.14}$$

$$k_p^4 = \frac{\omega^4}{c_p^4} \sim \frac{\omega^4 \rho^2}{\lambda^2} \tag{5.15}$$

and therefore,

$$g_c \sim \frac{9\lambda^2}{16\mu^2}. \tag{5.16}$$

Therefore, the monopole contribution to (5.11) is

$$\gamma_{m_{\text{spherical cavity}}} \sim \frac{\pi a_0^6 \omega^4 \rho^2}{4\mu^2}. \tag{5.17}$$

Note that this is the same result that we obtained in (5.10). Since we are considering the leading order problem, we can use the values of $\lambda$, $\mu$, etc given in section 4.5 for an unstressed isotropic material. Assuming the initial radius $a_0$ of the micro-sphere is 17 microns we calculate the scattering cross sections. In figure 5.1 we see the total scattering cross section in curve (a) and then the monopole scattering cross section in curve (b). Note that the monopole scatter cross section provides the main contribution to the total scattering cross section. Lower order scattering effects such as dipole and quadrupole make up the remaining contribution.

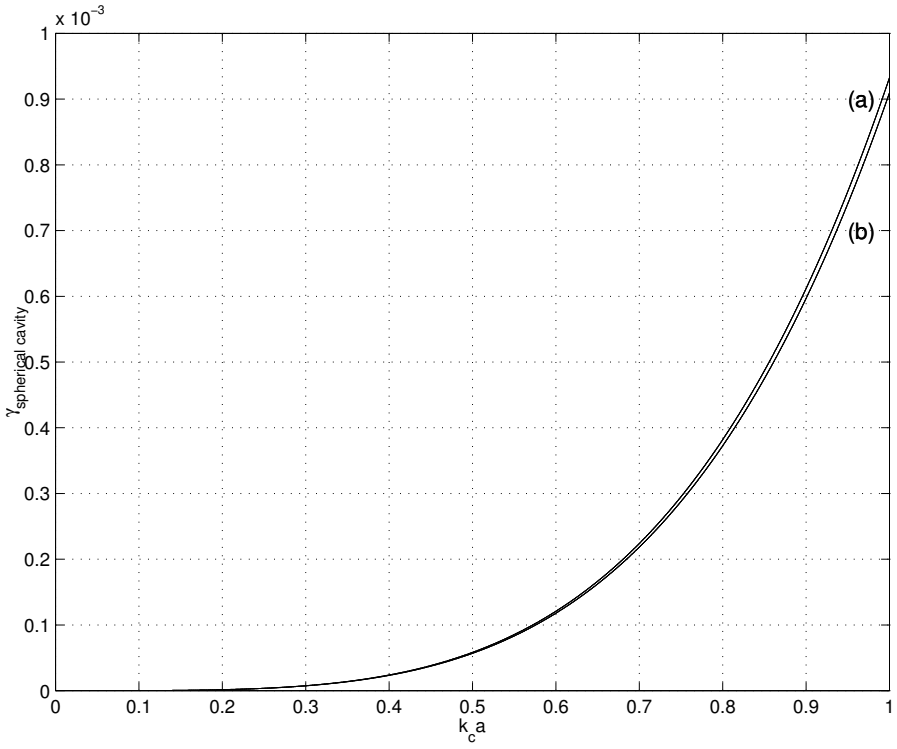



Figure 5.1: Calculation of (a) the total scattering cross section and (b) the monopole scattering cross section, corresponding to scattering from a micro-sphere in an isotropic material.

### 5.1.2 Anisotropic material - results of the model

We now calculate the scattering cross section from a spherical cavity in an anisotropic medium, using the analysis from chapter 3 in order to make a correct prediction for the compressed radius $a_s$ and for the value of $da/dp$ which we calculate quasistatically. We made the point in chapter 3 that the most suitable form of stored energy function $W$, providing most accurate results, was that used to plot curve $(d)$ in figure 3.4. Using this information, in figure 5.2 we plot the radius of the compressed micro-sphere $a_s$, varying with static pressure $p$ for a spherical cavity (i.e. $p_{in} = 0$). Then in figure 5.3 we show how $da/dp$ varies with $p$.

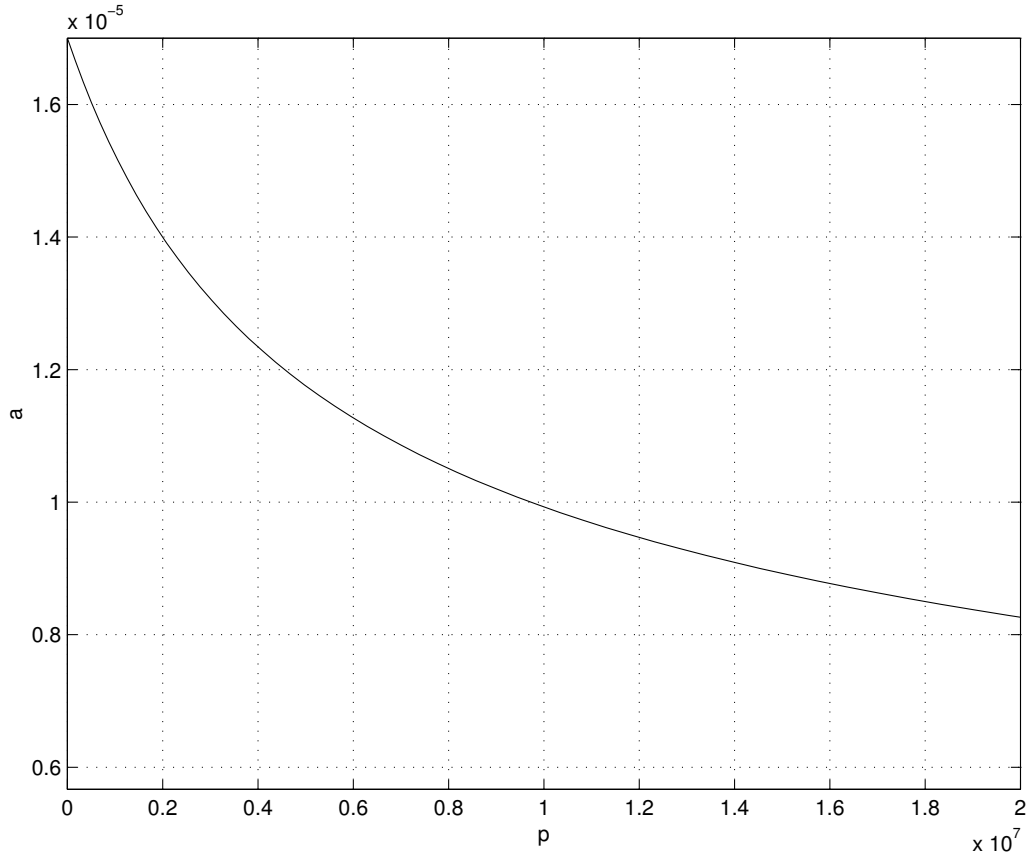



Figure 5.2: Showing how the compressed radius $a_s$ varies with static pressure.

We interpret from these graphs that with more static pressure, the material around the hole effectively stiffens up and is less capable of "breathing". That is, when initially strained, the micro-sphere will not respond as strongly to the incident wave as it does when there is no initial strain. Therefore, we expect smaller values of the monopole scatter cross section for an anisotropic material.

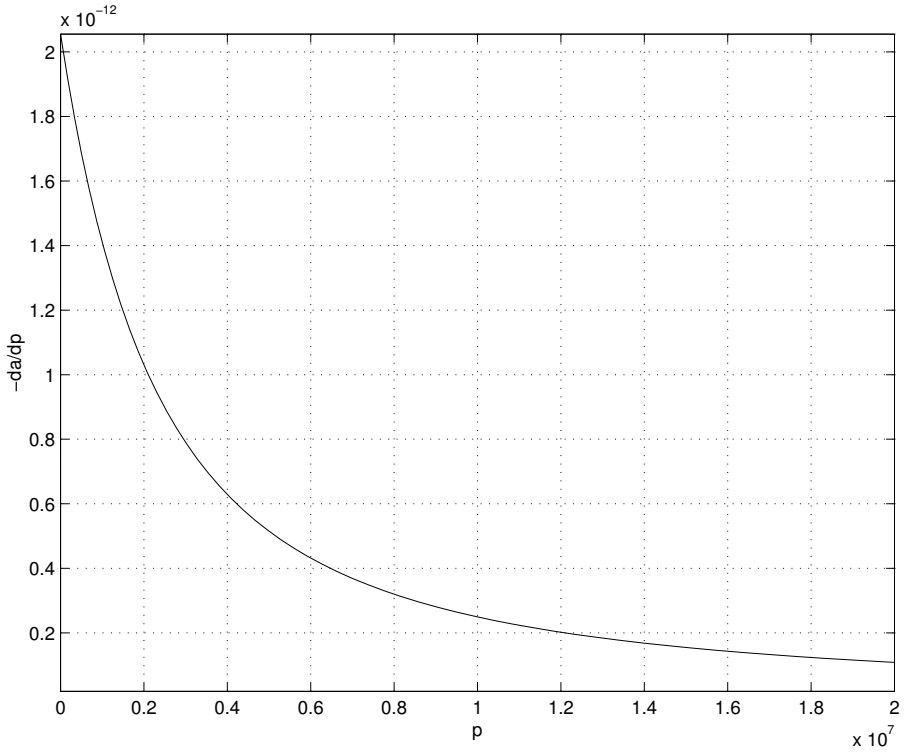



Figure 5.3: Showing how -$da/dp$ varies with static pressure.

Choosing a value of the static pressure as $p_s = 5\mu = 10^7$Pa and an initial radius of $a_0 = 17$ microns, from figures 5.2 and 5.3 we find that that

$$a_s \quad = \quad 10 \text{ microns} \tag{5.18}$$

and

$$\frac{da}{dp}\Big|_{p_s} \quad = \quad -2.5 \times 10^{-13} \text{microns/Pa} \tag{5.19}$$

Using this information, we calculate the monopole scatter cross section for a spherical cavity in an anisotropic material. This is plotted in figure 5.4. As predicted, the scattering cross section is significantly smaller due to the initial strain. In fact it is three orders smaller. This is due to the radius almost halving under a static pressure of $p_s = 5\mu$ and also due to the fact that at that static pressure, $da/dp$ is an order smaller than when no initial strain is applied.

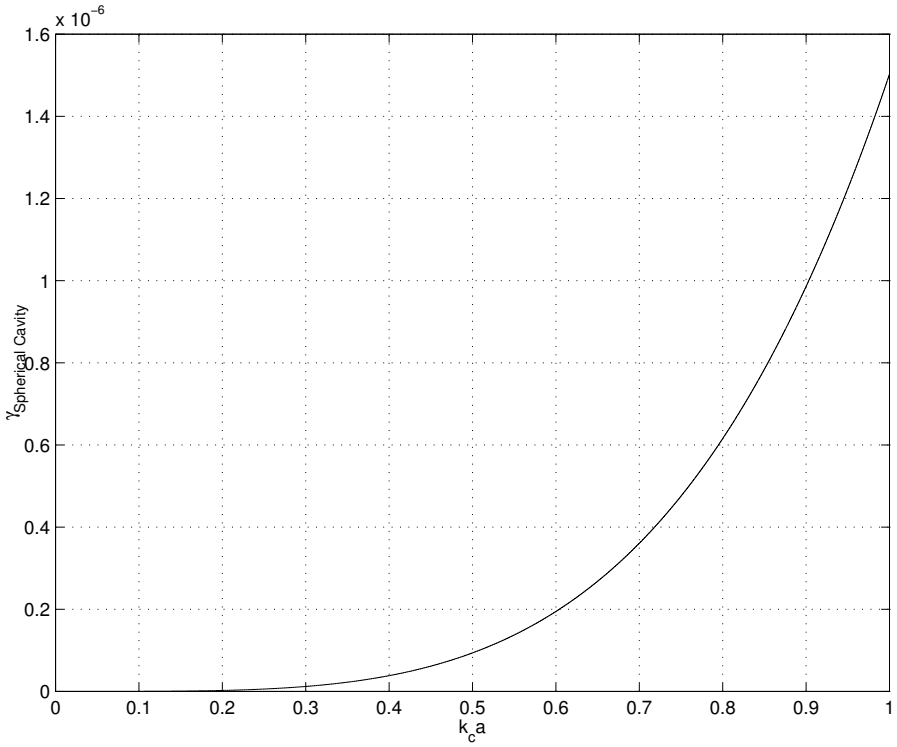



Figure 5.4: Calculation of the monopole scattering cross section corresponding to scattering from a micro-sphere in an anisotropic material.

## 5.2 Discussion

Simply by calculating the leading order (monopole) scattering cross section, we have shown that the strained region affects the scattering properties of a micro-sphere significantly. The problem we solved was for a spherical cavity, however it would not be difficult to extend this to the case of a gas filled sphere using the analysis in Chapter 3 and figure 3.5 which calculates the value of the compressed radius $a_s$ using Boyle's Law for the pressure inside the sphere. These results could then be compared with those of Einspruch and Truell who in [5] carried out this problem in an isotropic material.

Note that there is a high dependency of the scattering cross section on the radius of the micro-sphere and thus for accurate results, this radius must be calculated as accurately as possible.

In the TMSL model, the scattering cross section due to multiple scattering is split up into four parts. Scattering due to monopole, dipole, quadrupole and rotary ef-

fects. Furthermore, the modified material properties are functions of these scattering coefficients and therefore they can be deduced once scattering coefficients have been calculated.

Lower order effects could be calculated, although the purpose of this dissertation was to ascertain whether the strained region would have a significant effect on the scattering process. The fact that the monopole scattering cross section is reduced by three orders of magnitude indicates that quite clearly it does. Furthermore, in addition to the modification of the bulk modulus due to the introduction of the micro-spheres, the initial strain also provides a modification as was shown in section 4.5.

This information can now be used in a modified TMSL model including the effects of the strained field. Using their self consistent theory, multiple scattering effects can also be deduced.

# Chapter 6

# Conclusions

The aim of this dissertation was to ascertain what effect a strained region has on scattering from a micro-sphere in a composite material. The form of stored energy function for a rubber-like material has been discussed in detail and a general form has been used in order to calculate the compressed radius of a micro-sphere after a finite, nonlinear deformation to equilibrium under hydrostatic loading.

Experimental results for values of the radius of the micro-sphere before and after the hydrostatic loading, together with the fact that $p = O(\mu)$, implied that nonlinear elasticity was appropriate for calculating the stressed state. This was justified retrospectively by comparing the results for linear and nonlinear elasticity for several stored energy functions. In fact, linear elasticity predicts that the micro-sphere compresses to a *smaller* radius than that predicted by standard nonlinear elastic models of rubber-like materials.

We used a theory derived by Green and Zerna in [10] in order to find the equations of motion for small displacements superposed on top of the initial finite deformation. By specifying a particular choice of stored energy function it was then shown that in order to explicitly calculate the equations of motion, further experimental work would have to be carried out so that modifications to the Lamé moduli due to the initial strain would be known. Without knowing these modifications, the equations are only correct to leading order. We showed how the bulk modulus is affected by the strained region due to the modifications to the Lamé moduli.

Boundary conditions and radiation conditions were discussed in detail and it was

indicated how if necessary, a numerical solution of the equations of motion would be possible using a spectral method.

We solved the low frequency scattering problem at leading order by letting $\lambda/\mu \to \infty$ in order to calculate the monopole scattering cross section for scattering from a spherical cavity in an anisotropic material. This was then compared with the scattering cross section obtained by Ying and Truell in [21] for scattering from a spherical cavity in an isotropic region. This enabled a direct comparison to be made between scattering from a micro-sphere in a strained and an isotropic material.

In the scattering calculations, the strained region was taken into account by using the analysis of chapter 3 for the value of the compressed radius $a_s$ and the value of $da/dp$. This was possible because the strained region is small compared with the wavelength of the incident wave and therefore responds quasistatically.

The effect of the nonlinear deformation is to effectively stiffen up the rubber surrounding the micro-sphere so that its monopole response to an incident wave in an anisotropic material is significantly smaller than that in an isotropic material. Consequently, the monopole scattering coefficient is three orders of magnitude smaller in the anisotropic case.

Hence we have indicated two corrections which should be made to the current TMSL model. Firstly, the need to use nonlinear elasticity to predict the value of the compressed radius more accurately. Furthermore, the prediction of a significant reduction in the monopole scattering cross section due to the initial strain indicates that including the effects of the strained region in the model is essential.

Even though the monopole scattering cross section has been shown to significantly reduce due to the initial strain, this does not imply anything about the lower order scattering cross sections such as the dipole or quadrupole scattering effects. In order to calculate these, the modifications to the Lamé moduli must be taken into account and this would require further experimental work.

If the modifications to the Lamé moduli *were* made we could derive the full set of equations of motion. These could then be solved numerically using a spectral method as described in section 4.6.

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